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Lozenge Tilings of Holey Hexagons

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I, Won Hyok Kim, declare that this thesis titled, "*Lozenge Tilings of Holey Hexagons*" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Date:

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Contents

Chapter 1

Introduction

Let $G = (V, E)$ be a graph. Then a *matching* M in G is defined as a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share a common vertex. A *perfect matching* is a matching such that each vertex is covered by an edge.

A *plane partition* on the other hand is a two-dimensional array of non-negative integers $\pi_{i,j}$ (with positive integer indices i and j) that is non-increasing in both indices. This means that

$$\pi_{i,j} \geq \pi_{i,j+1} \quad \textit{and} \quad \pi_{i,j} \geq \pi_{i+1,j}$$

for all i and j. A plane partition can be represented as piles of unit cubes, i.e., the number of cubes on the position (i, j) is $\pi_{i,j}$. The following is an example of a plane partition which is in a given box of size $a \times b \times c$, where a, b, c are positive integers.

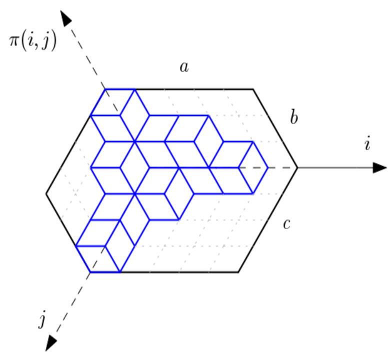

FIGURE 1.1: A Plane Partition with a=5, b=3, c=4.

The main combinatorial object in my thesis - *lozenge tiling* is quite related to perfect matching and plane partition. So what is the lozenge tiling problem of holey hexagons?

We are working on the triangular lattice plane. Every triangle is an equilateral triangle with side lengths equal to 1. If we are given a polygonal area bounded by sides along these lattices, we can consider the tiling problem of this area by lozenges. A *lozenge* is a tile that consists of two triangles which are sharing an edge.

Since every lozenge has one up-pointing triangle and one down-pointing triangle, if a bounded area in our plane is tilable by lozenges, it must have same number of these two kinds of triangles. In my thesis, I mostly deal with lozenge tiling problem of hexagon-type area.

A *convex* hexagon without any defect inside and on its boundary can be tiled with lozenges if and only if it is a semi-regular hexagon. (Throughout the whole thesis, we only think about the convex hexagon with all six inner angles equal to 120°.) It follows from the simple observation below.

Let the side lengths of a hexagon be a, b, c, d, e, f in cyclic order (starting from the north side), which are all integers. If the side lengths a, b, c, d are already given, then we can know the two remaining lengths from formulas below :

$$e = a + b - d\,;\ f = c + d - a\,.$$

If we put $d = a + m$ for some integer m, then we can get $e = b - m$ and $f = c + m$. In such a hexagon, the number of up-pointing triangles and the number of down-pointing triangles always differ by $|m|$. Thus, if $m = 0$, then $a = d$, $b = e$, and $c = f$, so the hexagon is a semi-regular hexagon (see e.g. Figure 1.2) and tilable by lozenges. Using the same argument, we can say that if $m \neq 0$, then the hexagon is not tilable (Figure 1.3 for example).

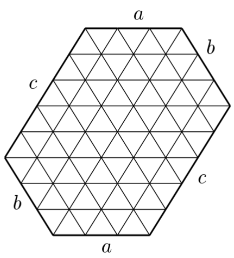

FIGURE 1.2: A Semi-regular Hexagon with a=3, b=3, c=5.

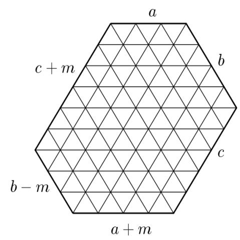

FIGURE 1.3: A Hexagon with a=3, b=4, c=5, m=1.

Figure 1.4 shows one example of a lozenge tiling of a semi-regular hexagon with side lengths a , b , c are 3, 2 and 4 respectively. In the picture we can also observe that one lozenge tiling corresponds to one perfect matching in a honeycomb graph. (If we regard each triangle as a vertex in a honeycomb graph and every lozenge as an edge in a matching, then this correspondence is a bijection.)

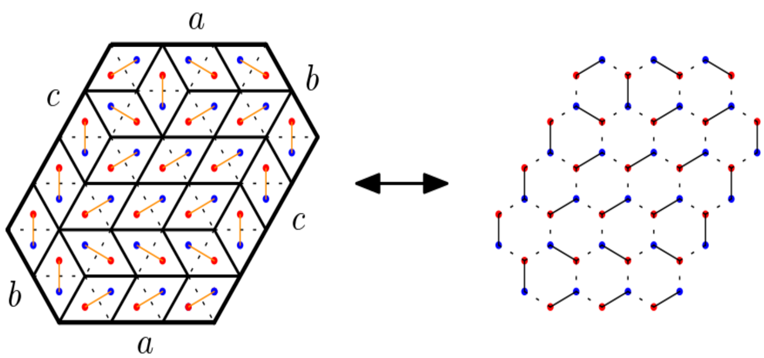

FIGURE 1.4: A lozenge tiling of a semi-regular hexagon and the corresponding perfect matching in a honeycomb graph.

In this thesis, I will mainly focus on the enumeration of lozenge tilings in hexagonal area on the triangluar lattice in the plane. From observations above, we can divide these problems in two cases; one case is for the semi–regular hexagons and another one is for non-semiregular hexagon with some defects.

In the case of semi-regular hexagons (see Figure 1.2), the lozenge tiling problem coincides with plane partition problem. There is a very nice enumeration formula for it. Let $N(a, b, c)$ be the number of plane partitions in a given $a \times b \times c$ box with a, b, c all positive integers. Then,

Theorem 1.0.1. *MacMahon,1916 [22]*
The number of plane partitions in a given $a \times b \times c$ box is :

$$N(a,b,c) = \prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \prod_{k=0}^{c-1} \frac{i+j+k+2}{i+j+k+1}. \tag{1.1}$$

This equation can be rewritten with hyperfactorial form :

$$N(a,b,c) = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)}, \tag{1.2}$$

where $H(x)$ stands for the "hyperfactorial" equal to $\prod_{k=0}^{x-1} k!$.

Many results of lozenge tiling enumeration are the generalizations of this MacMahon's theorem.

In the non-semiregular hexagon cases, since the hexagon itself can not be tiled by lozenges from the observation above, we delete some triangles from it, so that the remaining area is lozenge-tilable. So far there are a lot of enumeration results of such *holey-hexagons* and I will introduce some of them in the following chapters.

It is natural to ask for the number of tilings of given regions (it can be regarded as the enumeration of perfect matchings). There have been so many valuable works on these methods and the *Kasteleyn-Wilson* matrix [17] is one of them.

Consider a finite subgraph G of the infinite square lattice with as many white as black vertices, where the infinite square lattice is colored like a checkerboard. Let $(v_1, \ldots, v_n)$ be its white vertices and $(w_1, \ldots, w_n)$ its black vertices with an arbitrary ordering from 1 to n. The Kasteleyn-Wilson matrix

$$K(G) = K((v_1, \ldots, v_n), (w_1, \ldots, w_n)) \tag{1.3}$$

is defined to be $(a_{i,j})_{1 \le i,j \le n}$, where $a_{i,j}$ is

- 0, if $\{v_i, w_j\}$ is not an edge of G;
- 1, if $\{v_i, w_j\}$ is a horizontal edge of G;
- $(-1)^k$, if $\{v_i, w_j\}$ is a vertical edge going from row l to row $l+1$ and there are k vertices in row l to the left of the edge.

Then the number of perfect matchings of G is equal to the absolute value of the determinant of $K(G)$ [17]. (This method can also be used for every planar graph.)

For example, let us consider the enumeration of perfect matchings in the following graph :

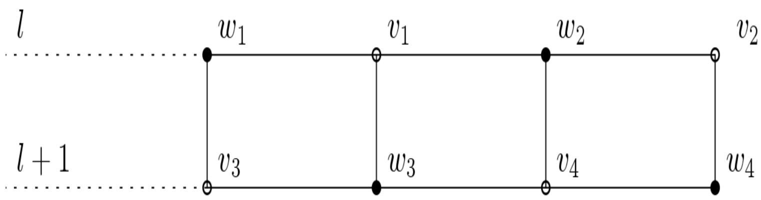

FIGURE 1.5: A $2 \times n$ rectangular graph on the square lattice, where $n = 4$.

It is the well-known fact that the number of perfect matchings in such a graph is equal to the Fibonacci number $F(n+1)$ with $F(1) = F(2) = 1$.

With the settings above, we should calculate the absolute value of the following determinant :

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix},$$

which is equal to 5, which is equal to $F(5)$ as desired.

Using this method, it is very simple to construct a determinant, however, we will have to deal with the determinant of a relatively large matrix.

In general, we use two methods to enumerate lozenge tilings of such areas. They are namely, *bijection* with *non-intersecting lattice paths* and *Kuo's graphical condensation* method. These two methods will be introduced with applications and examples later.

In the last chapter, I will introduce my result on the enumeration of lozenge tilings in a semi-regular hexagon with one *"bow-tie"* removed from the special point. I conjectured the enumeration formulas for such a case and provided some approaches to prove them.

Chapter 2

Plane Partitions and Non-Intersecting Lattice Paths

In Chapter 1, we saw that the correspondence between lozenge tilings of a semi-regular hexagon and plane partitions in a given box is a bijection. (The side lengths of the hexagon are corresponding to the side lengths of the box.) The number of tilings in a semi-regular hexagon with side lengths a, b, c can be enumerated with the Theorem 1.0.1. What about symmetry classes of tilings? There are already important results on the symmetry classes of plane partitions. Since these two combinatorial objects are in bijective correspondence, we can use the same formula to enumerate lozenge tilings and we introduce these formulas in this chapter. Then we deal with Mihai Ciucu 's factorization theorem and the Gessel-Viennot-Linderström theorem, which are playing important roles in this area of lozenge tiling enumeration.

2.1 Lozenge Tilings of Semi-Regular Hexagons : Symmetry Classes of Plane Partitions

Definition 2.1.1. A *plane partition* in an $a \times b \times c$ box is a subset $\pi \subseteq \{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\} \times \{1, 2, \ldots, c\}$ with $(i, j, k) \in \pi \implies (i', j', k') \in \pi$ for $\forall (i', j', k') \leq (i, j, k)$.

It will be convenient, if we denote a box by $B(a, b, c) = [0, a] \times [0, b] \times [0, c]$. We define three actions [21] on the set of pairs $(\pi, B(a, b, c))$, where $\pi \subseteq B(a, b, c)$:

1. Define $\tau(\pi)$, the transpose of π, to be $\{(x, y, z) | (y, x, z) \in \pi\}$, and $\tau(\pi, B(a, b, c)) = (\tau(\pi), B(b, a, c))$.
2. Define $\rho(\pi)$, the rotation of π, to be $\{(x, y, z) | (y, z, x) \in \pi\}$, and $\rho(\pi, B(a, b, c)) = (\rho(\pi), B(c, a, b))$.
3. Define $\kappa(\pi, B(a, b, c)) = (\pi', B(a, b, c))$, where π' is the complement of π in $B(a, b, c)$.

Let T be a group generated by τ, ρ, κ. For each subgroup G of T, we define $N_G(a, b, c)$ to be the number of pairs $(\pi, B(a, b, c))$ fixed by G. Then there are 10 inequivalent choices $G_1, \ldots, G_{10}$, which are introduced by Stanley in [24]. The table below [21] is giving us the definition of each 10 groups, together with the corresponding enumeration formulas of $N_i(a, b, c) = N_{G_i}(a, b, c)$.

Group	Formula	Symmetry
$G_1 = \langle e \rangle$	$N_1(a,b,c) = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)}$	Any
$G_2 = \langle \tau \rangle$	$N_2(a,a,b) = \frac{H_2(2a+b+1)H(a)H_2(b)}{H_2(2a+1)H(a+b)}$	Symmetric
$G_3 = \langle \rho \rangle$	$N_3(a,a,a) = \frac{H_3(3a+2)H(a)}{H(2a)F_3(3a-2)}$	Cyclically Symmetric
$G_4 = \langle \tau, \rho \rangle$	$N_4(a,a,a) = \frac{H_2(a)H_6(3a+5)}{H_2(2a+1)F_6(3a-2)}$	Totally Symmetric
$G_5 = \langle \kappa \rangle$	$N_5(2a,2b,2c) = N_1(a,b,c)^2$ $N_5(2a,2b,2c+1) = N_1(a,b,c)N_1(a,b,c+1)$ $N_5(2a+1,2b+1,2c) = N_1(a+1,b,c)N_1(a,b+1,c)$	Self Complementary
$G_6 = \langle \kappa\tau \rangle$	$N_6(a,a,2b) = \frac{H_2(2b+1)H_2(2b+a)H(a)}{H(2b+a)H_2(2a)}$	Transpose Complementary
$G_7 = \langle \kappa, \tau \rangle$	$N_7(2a,2a,2b) = N_1(a,a,b)$ $N_7(2a+1,2a+1,2b) = N_1(a,a+1,b)$	Symmetric and Self-Complementary
$G_8 = \langle \kappa\tau, \rho \rangle$	$N_8(2a,2a,2a) = \frac{F_3(3a-2)H_6(6a)H_2(2a)}{H_4(4a+1)H_4(4a)}$	Cyclically Symmetric and Transpose Complementary
$G_9 = \langle \kappa, \rho \rangle$	$N_9(2a,2a,2a) = \frac{H_3(3a+1)^2 H(a)^2}{H(2a+1)^2}$	Cyclically Symmetric and Self Complementary
$G_{10} = \langle \kappa, \rho \rangle$	$N_{10}(2a,2a,2a) = \frac{H_3(3a+1)H(a)}{H(2a+1)}$	Totally Symmetric and Self Complementary

TABLE 2.1: Enumeration Formulas for Symmetric Plane Partitions.

In the Table 2.1, $H_k(n) := \prod_{i=1}^{\lfloor \frac{n}{k} \rfloor}(n-ik)!$ and $F_k(n)$ stands for the "*staggered factorial*" $\prod_{i=0}^{\lfloor \frac{n}{k} \rfloor}(n-ik)$ for positive integers n and k. In particular, if $k=1$ then $H_1(n)$ is just the hyperfactorial $H(n) = \prod_{i=0}^{n-1} i!$.

The following pictures show examples of lozenge tilings which are symmetric, cyclically symmetric and self complementary respectively.

The proves of these enumeration formulas can be found in [24], [21], [9] and [3]. We can directly use the formulas in the Table 2.1 to enumerate lozenge tilings having corresponding symmetries, since plane partitions and lozenge tilings in a semi-regular hexagon are in a bijective relation.

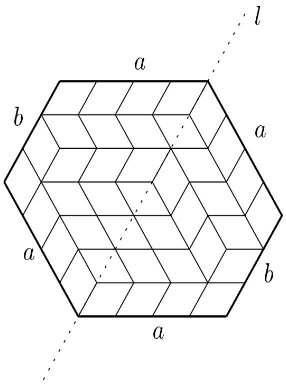

FIGURE 2.1: A symmetric lozenge tiling in a hexagon with side lengths a=4, b=3.

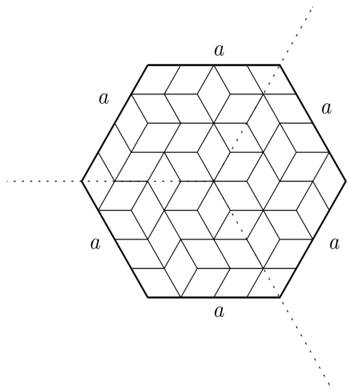

FIGURE 2.2: A cyclic lozenge tiling in a regular hexagon with side lengths a=4.

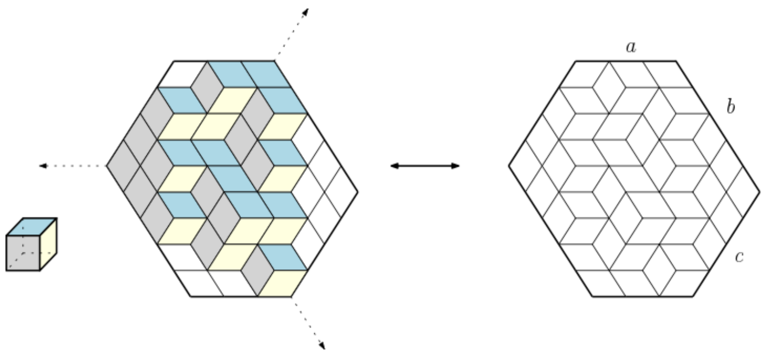

FIGURE 2.3: A self complementary plane partition and corresponding lozenge tiling.

2.2 Factorization Theorem of Mihai Ciucu

A planar graph is called *symmetric* if it is invariant under the reflection across some straight line. In this section, I will provide a very important result of Mihai Ciucu, which is introduced in [2], which is giving a result that expresses the number of perfect matchings of a large class of symmetric graphs in terms of the product of the number of matchings of two subgraphs.

Let G be a symmetric plane graph. Figure 2.4 shows an example of a symmetric graph. We can simply observe that the symmetry axis should contain even number of vertices, if this graph has a perfect matching. If not, the total number of vertices of the graph is odd, hence the graph has no perfect matching.

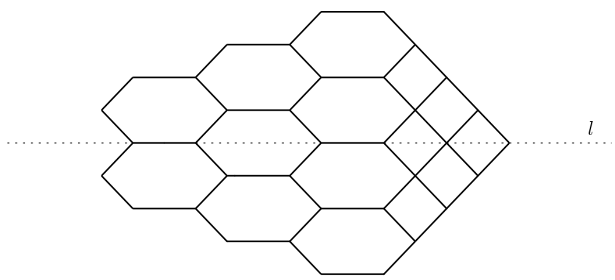

FIGURE 2.4: An example of a symmetric graph.

We define *weighted* symmetric graph as a symmetric graph equipped with a symmetric weight function on the edges. We denote the width of a symmetric graph G by $w(G)$. Here, $w(G)$ is the half the number of vertices of G, which are located on the symmetry axis. Let G be a weighted symmetric graph with symmetry axis l, which is horizontal (w.l.o.g) in the picture. Let $a_1, b_1, a_2, b_2, \dots, a_{w(G)}, b_{w(G)}$ be the vertices positioned on l, in the order from left to right. A *reduced* subgraph of G is a graph obtained from G by deleting at each vertex a_i either all incident edges above l (we refer to this operation for short as "cutting above l") or all incident edges below l ("cutting below l"). Figure 2.5 shows a reduced subgraph of the graph presented in Figure 2.4 (the deleted edges of the original graph are represented by dotted lines).

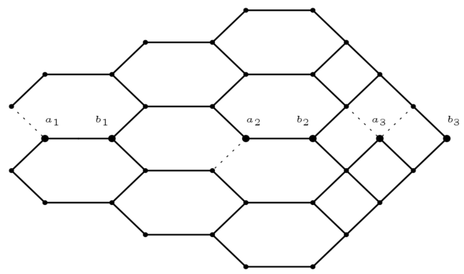

FIGURE 2.5: A reduced subgraph of the graph in Figure 2.4.

The weight of a matching μ is defined to be the product of the weights of the edges contained in μ. The matching generating function of a weighted graph G, denoted $M(G)$, is the sum of the weights of all matchings of G. Since the matching generating function is multiplicative with respect to disjoint unions of graphs, we only consider the cases of connected graphs [2].

Lemma 2.2.1. All $2^{w(G)}$ reduced subgraphs of a weighted symmetric graph G have the same matching generating function.

Proof. See [2] Lemma 1.1. □

Let G be a weighted symmetric bipartite graph. We call the set of vertices a *cutting set*, if removing these vertices disconnects the graph. Suppose the set of vertices lying on l is such a set. In such a case we say that l *separates* G. We color vertices in the two bipartition classes black and white.(W.l.o.g, we choose the leftmost vertex on l to be white.) We disconnect G with two parts G^+ and G^- as follows. Perform cutting operations above all white a_i's and black b_i's and below all black a_i's and white b_i's. This procedure yields cuts of the same kind at the endpoints of each edge lying on l. We reduce the weight of each such edge by half; leave all other weights unchanged. Since l separates G, the graph produced by the above procedure is disconnected into one component lying above l, which we denote by G^+, and one below l, denoted by G^- [2].

Figure 2.6 shows the procedure above for the graph in Figure 2.4.

Theorem 2.2.2. ***Factorization Theorem of Mihai Ciucu***
Let G be a bipartite weighted symmetric graph separated by its symmetry axis. Then

$$M(G) = 2^{w(G)} M(G^+) M(G^-). \tag{2.1}$$

Proof. See [2] Theorem 1.2. □

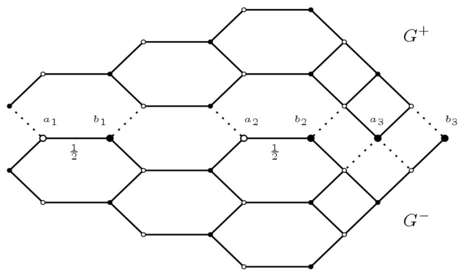

FIGURE 2.6: Disconnecting the graph in Figure 2.4 into two components.

2.3 The Lindström-Gessel-Viennot Theorem

Definition 2.3.1. A *directed acyclic graph* (*DAG*) is a finite directed graph with no directed cycles. That is, it consists of finitely many vertices and edges, with each edge directed from one vertex to another, such that there is no way to start at any vertex v and follow a consistently-directed sequence of edges that eventually loops back to v again.

We let G be a locally finite directed acyclic graph. We consider one set of starting vertices $A = \{a_1, \ldots, a_n\}$ and another set of ending vertices $B = \{b_1, \ldots, b_n\}$. And we also assign a weight w_e to each directed edge e. In general, these weights are elements of a commutative ring. For each directed path P between two vertices, we define $\omega(P)$ as the product of the weights of the edges lying in P. For any two vertices a and b, $e(a,b)$ stands for the sum $e(a,b) = \sum_{P:a\to b} \omega(P)$ over all paths from a to b. If there are only finitely many paths between any two vertices, this sum is well-defined. But it can also be well-defined in the general case under some additional circumstances [16]. (In this thesis, we only consider finite cases.) If we assign the weight 1 to each edge, then $e(a,b)$ exactly is the number of paths from a to b.

Under this setting, we define an $n \times n$ matrix M as follows:

$$M = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}. \tag{2.2}$$

Definition 2.3.2. An *n-tuple* of *non-intersecting* paths from A to B is an n-tuple $(P_1, \ldots, P_n)$ of paths in G with the following properties:

1. There exists a permutation σ of $\{1, 2, \ldots, n\}$ such that, for every i, the path P_i is a path from a_i to $b_{\sigma(i)}$.

2. Whenever $i \neq j$, the paths P_i and P_j have no two vertices in common (not even endpoints).

If we are given such an n-tuple $(P_1, \ldots, P_n)$, we denote by $\sigma(P)$ the permutation of σ from the first condition.

Theorem 2.3.3. ***Lindström-Gessel-Viennot***
With definitions above, the determinant of M is equal to the signed sum over all n-tuples $(P_1, \ldots, P_n)$ of non-intersecting paths from A to B:

$$\det(M) = \sum_{(P_1,\ldots,P_n):A\to B} sign(\sigma(P)) \prod_{i=1}^{n} w(P_i). \tag{2.3}$$

I.e., the determinant of M counts the weights of all n-tuples of non-intersecting paths starting at A and ending at B, each affected with the sign of the corresponding permutation of $(1, 2, \ldots, n)$, where P_i is taking a_i to $b_{\sigma(i)}$.

Proof. See [16] Theorem 1. □

Remark 2.3.4. In particular, if every n-tuple of non-intersecting paths from A to B takes a_i to b_i for each i and we take the weights to be 1, then $\det(M)$ is exactly the number of non-intersecting n-tuples of paths starting at A and ending at B.

The following example explains why lozenge tilings and non-intersecting lattice paths are related and how MacMahon's theorem (Theorem 1.0.1) can be proven using the Theorem 2.3.3.

Example 2.3.5. The Proof of MacMahon's Theorem
Let $H(a, b, c)$ denote the semi-regular hexagon of which side lengths are a,b,c,a,b,c in a cyclic order (starting from the northern side).

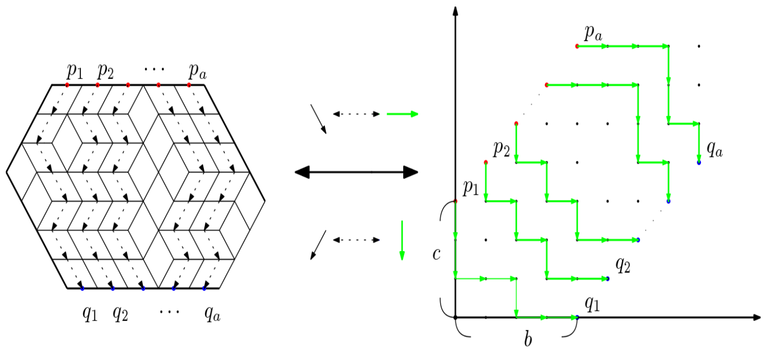

FIGURE 2.7: One lozenge tiling of a semi-regular hexagon and relation with non-intersecting lattice paths in $\mathbb{Z}^2$.

There is a standard bijection between lozenge tilings and families of non-intersecting lattice paths. Figure 2.7 shows this bijection. We mark the midpoints of the edges along the north and south sides of lengths a with red and blue respectively. Denote these points by p_i's and q_j's, where i and j are running from 1 to a [8]. Now, we connect these points by paths which are following along the lozenges of the tiling, as in the Figure 2.7. It is obvious that the resulting paths are non-intersecting. And if we transform the underlying lattice as in the picture, we can get orthogonal paths in the lattice $\mathbb{Z}^2$ with steps in direction $(1, 0)$ and $(0, -1)$. In the square lattice (right hand side of the Figure 2.7), each p_i and q_j has the following coordinate values:

$$\begin{aligned} p_i &= (i-1, c+i-1), & i &= 1, 2, \ldots, a\,, \\ q_j &= (b+j-1, j-1), & j &= 1, 2, \ldots, a\,. \end{aligned}$$

We should compute the determinant of M, which is defined as in (2.2). Each entry $e(p_i, q_j)$ is the number of possible paths only with two steps in direction $(1, 0)$ and $(0, -1)$ from p_i to q_j in the square lattice plane. In general, the number of such paths from the point (x_1, y_1) to (x_2, y_2) is equal to $\binom{x_2 - x_1 + y_2 - y_1}{x_2 - x_1}$ if the ending point is located in the south-east of the starting point. Otherwise, the number of such paths is equal to zero, since there is no directed path only with steps in two kinds connecting two points. From the Theorem 2.3.3 and the Remark 2.3.4, the following determinant is exactly the number of n-tuples of non-intersecting lattice paths from $p_1, p_2, \ldots, p_a$ to $q_1, q_2, \ldots, q_a$, hence the number of lozenge tilings of the semi-regular hexagon $H(a,b,c)$:

$$\#\,\text{Lozenge Tilings of } H(a,b,c) = \det M_a(b,c) = \det_{1\le i,j\le a}\left(\binom{b+c}{b-i+j}\right). \tag{2.4}$$

The determinant in (2.4) can be easily evaluated with the Jacobi-Desnanot identity

$$\det A \cdot \det A_{1,n}^{1,n} = \det A_1^1 \cdot \det A_n^n - \det A_1^n \cdot \det A_n^1 \quad , \tag{2.5}$$

where A is an $n \times n$ matrix and the submatrix of A in which rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ are omitted is denoted by $A_{i_1,i_2,\ldots,i_k}^{j_1,j_2,\ldots,j_k}$.

We can use this identity to obtain the following recursion [18],

$$\begin{aligned}
\big(M_a(b,c)\big)_a^a &= M_{a-1}(b,c)\,,\\
\big(M_a(b,c)\big)_1^1 &= M_{a-1}(b,c)\,,\\
\big(M_a(b,c)\big)_a^1 &= M_{a-1}(b+1,c-1)\,,\\
\big(M_a(b,c)\big)_1^n &= M_{a-1}(b-1,c+1)\,,\\
\big(M_a(b,c)\big)_{1,a}^{1,a} &= M_{a-2}(b,c)\,.
\end{aligned} \tag{2.6}$$

Since $\det M_a(b,c)$ is expressed in terms of quantities of the form $\det M_{a-1}(\cdot)$ and $\det M_{a-2}(\cdot)$, the only things we need to check are the initial cases in the induction with $a = 0$ and $a = 1$. This will give us the exact result which is described in the Theorem 1.0.1.

2.4 Cored Hexagon

In this section, I will provide results of enumerations of lozenge tilings of a hexagon with side lengths a, $b+m$, c, $a+m$, b, $c+m$ (starting from the north side), where an equilateral triangle of side length m has been removed from its centre. These results are given by M.Ciucu, T.Eisenkölbl, C.Krattenthaler and D.Zare and introduced in [8].

The removed triangle is called the *core*. The remaining region is denoted by $C_{a,b,c}(m)$ and called a *cored* hexagon.

First, we need to identify the centre, i.e., the position of the core. Let s be a side of the core and let u and v be the sides of the hexagon parallel to it. The most natural definition of the centre is: the distance between s and u should be the same as the distance between v and the vertex of the core opposite s, for all three choices of s.

But the sides of the core will be along lines of the triangular lattice only if a, b and c have the same parity (Figure 2.8 is a case that a, b and c have the same parity); in this case, we define this to be the position of the core. On the other hand, if for example a has the different parity from b and c, two sides of the triangle which is satisfying the above conditions would not be along the lattice lines. These two sides will be distanced half a unit from two consecutive lattice lines. In this case, we move this central triangle half a unit towards the side of the hexagon of length b along the direction parallel to the side of length a so that all the three sides of this triangle are underlying on the lattice lines. We then define this to be the position of the core (the position of the core in this case can be seen from Figure 2.9) [8].

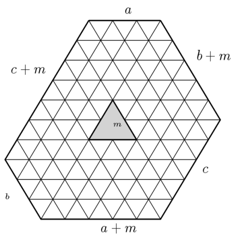

FIGURE 2.8: Cored Hexagon with a=3, b=3, c=5, m=2.

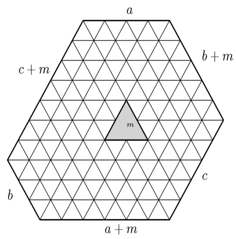

FIGURE 2.9: Cored Hexagon with a=4, b=3, c=5, m=2.

The theorems below provide explicit formulas for the total number of lozenge tilings of such a cored hexagon [8]. These results can be expressed as quotients of products of hyperfactorials. It is necessary to extend the definition of hyperfactorials to half-integers :

$$H(n) := \begin{cases} \prod_{k=0}^{n-1} \Gamma(k+1), & \text{for } n \text{ an integer}, \\ \prod_{k=0}^{n-\frac{1}{2}} \Gamma(k+\frac{1}{2}), & \text{for } n \text{ a half-integer}. \end{cases} \tag{2.7}$$

Let $L(R)$ be the number of lozenge tilings of the region R. For a, b and c having the same parity, the following theorem gives the precise enumeration formula for lozenge tilings of $C_{a,b,c}(m)$.

Theorem 2.4.1. *Let a, b, c, m be non-negative integers, a, b, c having the same parity. The number of lozenge tilings of a hexagon with sides a, $b+m$, c, $a+m$, b, $c+m$, with an equilateral triangle of side m removed from its center (see Figure 2.8 for an example) is given by*

$$\begin{aligned} &L(C_{a,b,c}(m)) = \\ &\frac{H(a+m)H(b+m)H(c+m)H(a+b+c+m)}{H(a+b+m)H(a+c+m)H(b+c+m)} \frac{H(m+\lceil\frac{a+b+c}{2}\rceil)H(m+\lfloor\frac{a+b+c}{2}\rfloor)}{H(\frac{a+b}{2}+m)H(\frac{b+c}{2}+m)H(\frac{a+c}{2}+m)} \\ &\times \frac{H(\lceil\frac{a}{2}\rceil)H(\lceil\frac{b}{2}\rceil)H(\lceil\frac{c}{2}\rceil)H(\lfloor\frac{a}{2}\rfloor)H(\lfloor\frac{b}{2}\rfloor)H(\lfloor\frac{c}{2}\rfloor)}{H(\frac{m}{2}+\lceil\frac{a}{2}\rceil)H(\frac{m}{2}+\lceil\frac{b}{2}\rceil)H(\frac{m}{2}+\lceil\frac{c}{2}\rceil)H(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)H(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)H(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)} \\ &\times \frac{H(\frac{m}{2})^2 H(\frac{a+b+m}{2})^2 H(\frac{b+c+m}{2})^2 H(\frac{a+c+m}{2})^2}{H(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)H(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)H(\frac{a+b}{2})H(\frac{a+c}{2})H(\frac{b+c}{2})}. \end{aligned} \tag{2.8}$$

Outline of the Proof. The proof of this theorem can be done by using the bijection of lozenge tilings with non-intersecting paths.

Once we find a bijection (see Figure 2.10), then by the Lindström-Gessel-Viennot theorem (Theorem 2.3.3) it is equivalent to prove that the determinant

$$\det_{1\le i,j\le a+m} \begin{pmatrix} \binom{b+c+m}{b-i+j} & 1\le i\le a \\ \binom{\frac{b+c}{2}}{\frac{b+a}{2}-i+j} & a+1\le i\le a+m \end{pmatrix} \tag{2.9}$$

is equal to the product formula (2.8). See the Chapter 7 in [8] for the determinant evaluation part. ☐

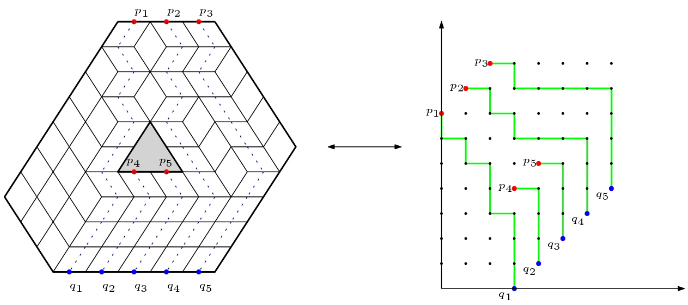

FIGURE 2.10: One lozenge tiling of a cored hexagon in the Figure 2.8 and the corresponding non-intersecting lattice paths in $\mathbb{Z}^2$.

The corresponding result for the case when a, b and c do not have the same parity reads as follows.

Theorem 2.4.2. *Let a, b, c, m be non-negative integers, with a of parity different from the parity of b and c. The number of lozenge tilings of a hexagon with sides a, $b+m$, c, $a+m$, b, $c+m$, with the "central" triangle of side m removed (see Figure 2.9 for an example) is given by*

$$
\begin{aligned}
&L(C_{a,b,c}(m)) = \\
&\frac{H(a+m)H(b+m)H(c+m)H(a+b+c+m)}{H(a+b+m)H(a+c+m)H(b+c+m)} \frac{H(m+\lceil\frac{a+b+c}{2}\rceil)H(m+\lfloor\frac{a+b+c}{2}\rfloor)}{H(\lfloor\frac{a+c}{2}\rfloor+m)H(\frac{b+c}{2}+m)H(\lceil\frac{a+b}{2}\rceil+m)} \\
&\times \frac{H(\frac{m}{2})^2 H(\lceil\frac{a}{2}\rceil)H(\lceil\frac{b}{2}\rceil)H(\lceil\frac{c}{2}\rceil)H(\lfloor\frac{a}{2}\rfloor)H(\lfloor\frac{b}{2}\rfloor)H(\lfloor\frac{c}{2}\rfloor)}{H(\frac{m}{2}+\lceil\frac{a}{2}\rceil)H(\frac{m}{2}+\lceil\frac{b}{2}\rceil)H(\frac{m}{2}+\lceil\frac{c}{2}\rceil)H(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)H(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)H(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)} \\
&\times \frac{H(\lceil\frac{a+b}{2}\rceil+\frac{m}{2})H(\lceil\frac{a+c}{2}\rceil+\frac{m}{2})H(\lfloor\frac{a+b}{2}\rfloor+\frac{m}{2})H(\lfloor\frac{a+c}{2}\rfloor+\frac{m}{2})H(\frac{b+c}{2}+\frac{m}{2})^2}{H(\lceil\frac{a+b+c}{2}\rceil+\frac{m}{2})H(\lfloor\frac{a+b+c}{2}\rfloor+\frac{m}{2})H(\lfloor\frac{a+b}{2}\rfloor)H(\lceil\frac{a+c}{2}\rceil)H(\frac{b+c}{2})}.
\end{aligned}
\tag{2.10}
$$

Outline of the Proof. The proof is analogous to the case with all three side lengths having the same parities. In this case, we again need to prove that the following determinant

$$
\det_{1\le i,j\le a+m} \begin{pmatrix} \binom{b+c+m}{b-i+j} & 1\le i\le a \\ \binom{\frac{b+c}{2}}{\frac{b+a+1}{2}-i+j} & a+1\le i\le a+m \end{pmatrix} \tag{2.11}
$$

can be evaluated as the product formula (2.10). See the Chapter 7 in [8] for the determinant evaluation part. □

Remark 2.4.3. The formulas (2.8) and (2.10) above reduce to MacMahon's formula (Theorem 1.0.1) for $m = 0$.

2.5 Symmetric Hexagon with a "Bow-Tie" removed on its Symmetry Axis

In [12], T.Eisenkölbl enumerated the number of lozenge tilings of the symmetric hexagon with two triangles touching in one vertex ("bow-tie") removed on the symmetry axis.

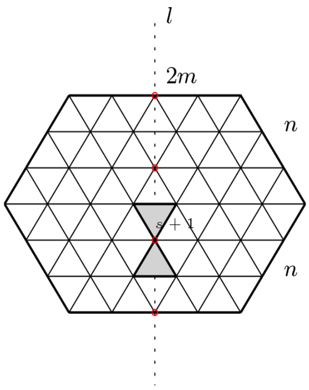

FIGURE 2.11: A hexagon with sides $2m$,n,n,$2m$,n,n and missing one "bow-tie" missing at the position $s+1$ (counted from the top), where $m=2$, $n=3$, $s=2$.

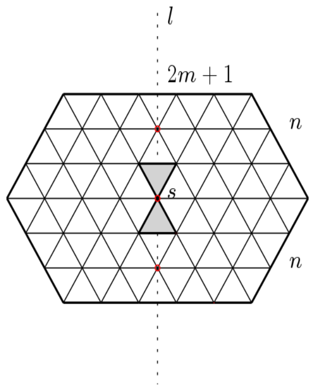

FIGURE 2.12: A hexagon with sides $2m+1$, n, n, $2m+1$, n, n and missing one "bow-tie" missing at the position s (counted from the top), where $m=2$, $n=3$, $s=2$.

Theorem 2.5.1. *The number of lozenge tilings of a hexagon with sides $2m$, n, n, $2m$, n, n (starting from the north side) and two missing triangles on the vertical symmetry axis sharing the $(s+1)$-th vertex on the axis (see Figure 2.11) equals*

$$\frac{(2m-1)\binom{2m-2}{m-1}\binom{2n-2s}{n-s}\binom{2s}{s}}{\binom{2m+2n}{m+n}} \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{2m} \frac{i+j+k-1}{i+j+k-2}. \tag{2.12}$$

Outline of the proof. Since the region is symmetric, we can use Mihai Ciucu's factorization theorem (Theorem 2.2.2). To do this, we color up-pointing triangles with red and down-pointing triangles with blue. If we consider every triangle as a colored vertex, then the whole area is a bipartite honeycomb graph.

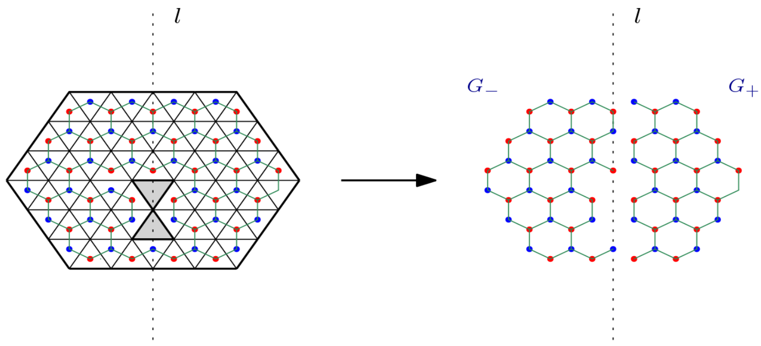

FIGURE 2.13: The region shown in Figure 2.11 as a honeycomb graph and the usage of Mihai Ciucu's factorization theorem.

We divide this area into two parts as in Figure 2.13. We enumerate perfect mathings in both parts separately. These can be done using the Lindström-Gessel-Viennot theorem (Theorem 2.3.3) to enumerate non-intersecting paths of corresponding graphs (see the Example 2.3.5 for converting lattice from triangular one to square one). Now we evaluate two determinants and use Theorem 2.2.2 to get the desired result (2.12). See [12] for determinant evaluation parts.

□

Theorem 2.5.2. *The number of lozenge tilings of a hexagon with sides $2m+1, n, n, 2m+1, n, n$ and two missing triangles on the symmetry axis sharing the s–th vertex on the axis (see Figure 2.12) equals*

$$\frac{(2m+1)\binom{2m}{m}\binom{2n-2s}{n-s}\binom{2s-2}{s-1}}{\binom{2m+2n}{m+n}} \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{2m+1} \frac{i+j+k-1}{i+j+k-2}. \tag{2.13}$$

Outline of the proof. The proof of this theorem is analogous to the proof of Theorem 2.5.1. See [12] for the whole proof. □

2.6 Semi-strict Gelfand Pattern and Cohn-Larsen-Propp's Theorem

In previous sections, we saw that combinatorial objects such as perfect matchings in a honeycomb graph, plane partitions in a given box and non-intersecting lattice paths on the square lattice are in bijective correspondence with lozenge tilings in a semi-regular hexagon. I will provide one more combinatorial object which is also bijective with lozenge tiling, namely, semi-strict *Gelfand Patterns*.

Definition 2.6.1. A *semi-strict Gelfand pattern* is an isosceles triangular array of integers, with the property that the entries increase weakly along diagonals running down and to the right, and the entries increase strictly along diagonals running up and to the right.

It is quite obvious that one lozenge tiling of a semi-regular hexagon is determined by the locations of its vertical lozenges. We can use semi-strict Gelfand pattern to keep track of these locations [10].

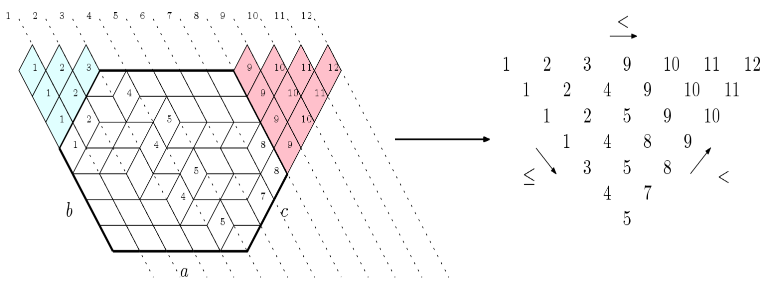

FIGURE 2.14: One lozenge tiling of a semi-regular hexagon and the corresponding semi-strict Gelfand pattern.

Figure 2.14 shows a lozenge tiling of a hexagon with sides a,b,c,a,b,c (in the cyclic order from the north side) and numbering its vertical lozenges inside, where a,b and c are 5,4 and 3 respectively. To do this, we add $\frac{c(c+1)}{2}$ vertical lozenges on the left (colored by light blue) and $\frac{b(b+1)}{2}$ vertical lozenges on the right (pink). Then we get a trapezoid of upper base $a+b+c$ and lower base a, with some triangular protrusions along its upper border, as in Figure 2.14 [10]. Now we associate with each vertical lozenge in the tiling the horizontal distance from its right corner to the left border of the trapezoid, which we call its *trapezoidal position*. If we index the locations of the vertical lozenges with their trapezoidal positions, then we can obtain the semi-strict Gelfand pattern as shown in the right part of the Figure 2.14.

It is easy to see that there is a simple bijection between semi-strict Gelfand patterns with top row $1, 2, \ldots, a+b+c$ and lozenge tilings of a semi-regular hexagon with side lengths a,b,c,a,b,c. The following theorem is a very important result [10] which is enumerating the specific semi-strict Gelfand patterns, hence lozenge tilings.

Theorem 2.6.2. ***Cohn-Larsen-Propp***
There are exactly

$$\prod_{1\le i<j\le n} \frac{x_j - x_i}{j-i} \tag{2.14}$$

semi-strict Gelfand patterns with top row consisting of integers $x_1, x_2, \ldots, x_n$ *such that* $x_1 < x_2 < \cdots < x_n$.

Proof. See Proposition 2.1 in [10] for the proof. □

We can use this theorem whenever we encounter lozenge tiling enumeration problem of regions which have defects on one of their *boundaries*. We will see the specific applications of this theorem in the next chapter.

2.7 Procter's Theorem and Hexagons with "Staircases" missing from Corners

In the previous part of my thesis, we saw that a plane partition contained in a $a \times b \times c$ box can be regarded as a lozenge tiling of a semi-regular hexagon with side lengths a,b,c,a,b,c (in cyclic order).

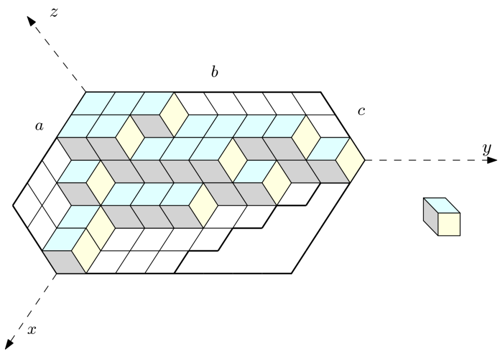

FIGURE 2.15: A plane partition for which its projection on one of the coordinate planes fits in the "maximal stair case"

Proctor considered the problem of enumerating those plane partitions π contained in an $a \times b \times c$ box for which the projection of π on one of the coordinate planes, on Oxy for example, fits in the "maximal staircase" $\lambda = (b, b-1, ..., b-a+1)$ (when λ is viewed as the corresponding Ferrers diagram) contained in the $a \times b$ basis rectangle[23]. Without loss of generality, we assume that $a \le b$. Figure 2.15 is an example of such a plane partition with $a = 5, b = 8, c = 3$. Then the number of such plane partitions is given by following theorem.

Theorem 2.7.1. ***Procter***
The number of plane partitions above is:

$$\prod_{i=1}^{a}[\prod_{j=1}^{b-a+1}\frac{c+i+j-1}{i+j-1}\prod_{j=b-a+2}^{b-a+i}\frac{2c+i+j-1}{i+j-1}]. \tag{2.15}$$

From the bijection between lozenge tilings and plane partitions, this problem is equivalent to counting the lozenge tilings of a semi-regular hexagon with side lengths a,b,c with a maximal "staircase of lozenges" removed from a corner at which edges of lengths a and b.

Based on this result, Ciucu and Krattenthaler gave one more restriction to the plane partition π in [7]. In this case, they required that the projection of π on *two* of the coordinate planes be contained in the corresponding staircases. It means that this enumeration problem is equivalent to counting the number of tilings of a semi-regular hexagon with side lengths a, b, c with *two* maximal staircases removed, from vertices that are non-adjacent and non-opposite[7].

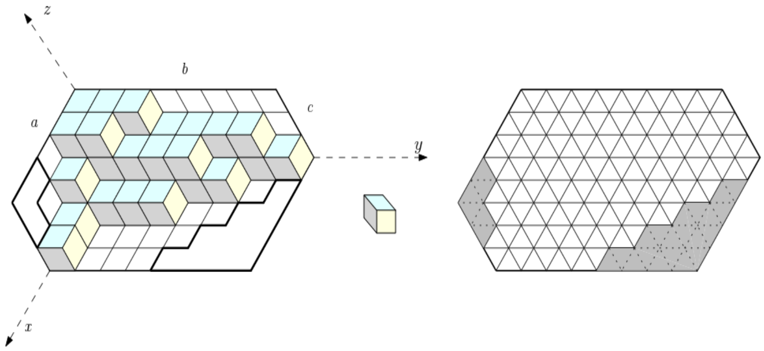

FIGURE 2.16: A plane partition of which projection on two coordinate planes contained in maximal staircases and one example of the corresponding triangular lattice region for lozenge tiling.

Figure 2.16 shows such a plane partition and the corresponding 2-dimensional region for the lozenge tiling. Actually there are six cases to consider in the lozenge tiling enumeration problem, because we can get non-equivalent regions depending on the relative orderings of side lengths a, b and c. Ciucu and Krattenthaler provided explicit product formulas for these enumerations of lozenge tilings and gave the proof by using the Lindström-Gessel-Viennot theorem (Theorem 2.3.3) in [7].

Chapter 3

Kuo's Graphical Condensation Method and Its Applications

3.1 Kuo's Graphical Condensation Method

Eric H.Kuo provided a very strong way to enumerate perfect matchings in a plane bipartite graph in [20]. This method is called *Kuo's Graphical Condensation* and it is very efficient to solve lozenge tiling enumeration problems, since these problems can be regarded as enumeration problems of perfect matchings in their *dual* graphs.

In this section, I will introduce this method together with its sketched proof. The very first version of this condensation was for the plane bipartite graph.

Let $G = (V_1, V_2, E)$ be the plane bipartite graph in which V_1,V_2 are disjoint sets of vertices in G and every edge in E connects a vertex in V_1 to a vertex in V_2. If U is a subset of vertices in G, then $G - U$ is the subgraph of G obtained by deleting the vertices in U and all edges incident to those vertices. If a is a vertex in G, then $G - a = G - \{a\}$. Let $M(G)$ be the number of perfect matchings of G, and $\mathcal{M}(G)$ be the set of all perfect matchings of G[20].

Theorem 3.1.1. ***Kuo's Graphical Condensation***
Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c, and d appear in a cyclic order on a face of G (see Figure 3.1). If $a, c \in V_1$ and $b, d \in V_2$, then

$$\begin{aligned} M(G)M(G - \{a,b,c,d\}) = M(G - \{a,b\})M(G - \{c,d\}) \\ + M(G - \{a,d\})M(G - \{b,c\}). \end{aligned} \tag{3.1}$$

Outline of the Proof. It will be enough for us to show that :

$$\begin{aligned} &|\mathcal{M}(G) \times \mathcal{M}(G - \{a,b,c,d\})| \\ &= |(\mathcal{M}(G - \{a,b\}) \times \mathcal{M}(G - \{c,d\})) \cup (\mathcal{M}(G - \{a,d\})\mathcal{M}(G - \{b,c\}))|. \end{aligned}$$

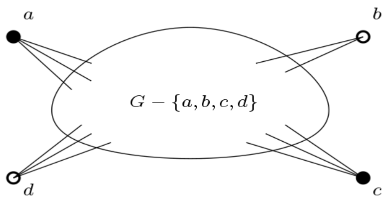

FIGURE 3.1: An illustration of Kuo's graphical condensation method.

By the superimposing a matching of $G - \{a,b,c,d\}$ onto a matching of G, we can get a new graph. If one edge belongs to both matchings, then we replace these two edges by a doubled edge. It is easy to see that only 4 vertices a, b, c and d will have degree 1, while every other vertex in the united matching has degree 2. With the same argument, every vertex in the united matching of a matching of $G - \{a,b\}$ and a matching of $G - \{c,d\}$ has the degree 2 except for the vertices a, b, c and d having degree 1. We can get the similar results from superimposing a matching of $G - \{a,d\}$ onto a matching of $G - \{b,c\}$ [20].

We define $\mathcal{H}$ to be the set of multigraphs on the vertices of G in which every vertex except for a, b, c, and d has degree 2. Then from the observations above, every pair of graphs in $\mathcal{M}(G) \times \mathcal{M}(G - \{a,b,c,d\})$, $\mathcal{M}(G - \{a,b\}) \times \mathcal{M}(G - \{c,d\})$ and $\mathcal{M}(G - \{a,d\}) \times \mathcal{M}(G - \{b,c\})$ can be united together to form a multigraph in $\mathcal{H}$.

For every multigraph H in $\mathcal{H}$, it is always possible to partition this graph into two graphs so that one of them is a matching of G and another one is a matching of $G - \{a,b,c,d\}$. At the same time, it is also possible to partition H into either one matching of $G - \{a,b\}$ and one matching of $G - \{c,d\}$ or one matching of $G - \{a,d\}$ and one matching of $G - \{b,c\}$. But never both.

There are 2^k possible ways to partition H into two matchings for both cases above. Here k is the number of cycles in H. See [20] for the precise proof.

In conclusion, we can prove that

$$\begin{aligned}&|\mathcal{M}(G) \times \mathcal{M}(G - \{a,b,c,d\})| = |\mathcal{H}| \\ &= |(\mathcal{M}(G - \{a,b\}) \times \mathcal{M}(G - \{c,d\})) \cup (\mathcal{M}(G - \{a,d\})\mathcal{M}(G - \{b,c\}))|.\end{aligned}$$

□

In 2006, Kuo generalized this 4-point condensation to plane graphs that are not necessarily bipartite in [19].

Theorem 3.1.2. *For any plane graph G and any four vertices a, b, c, d that appear in cyclic order on some face of G, one has that*

$$\begin{aligned} M(G)M(G - \{a,b,c,d\}) = {} & M(G - \{a,b\})M(G - \{c,d\}) \\ & -M(G - \{a,c\})M(G - \{b,d\}) \\ & +M(G - \{a,d\})M(G - \{b,c\}). \end{aligned} \tag{3.2}$$

The equation (3.2) can be rewritten in the pfaffian form as follows :

$$M(G)M(G - \{a,b,c,d\}) =$$
$$Pf \begin{bmatrix} 0 & M(G - \{a,b\}) & M(G - \{a,c\}) & M(G - \{a,d\}) \\ -M(G - \{a,b\}) & 0 & M(G - \{b,c\}) & M(G - \{b,d\}) \\ -M(G - \{a,c\}) & -M(G - \{b,c\}) & 0 & M(G - \{c,d\}) \\ -M(G - \{a,d\}) & -M(G - \{b,d\}) & -M(G - \{c,d\}) & 0 \end{bmatrix} \tag{3.3}$$

This pfaffian form of graphical condensation gives us an idea to generalize the condensation to $2k$ points for $(k \geq 3)$. A *weighted* graph is a graph with weights (that could be considered indeterminates) on its edges. For a weighted graph G, $M(G)$ denotes the sum of the weights of the perfect matchings of G, where the weight of a perfect matching is taken to be the product of the weights of its constituent edges[1].

Then, the generalization of Kuo's condensation (see [19] and [1]) is as the following :

Theorem 3.1.3. *Let G be a planar graph with the vertices $a_1, \ldots, a_{2k}$ appearing in a cyclic order on a face of G. Consider the skew-symmetric matrix $A = (a_{ij})_{1\le i,j\le 2k}$ with entries given by*

$$a_{ij} = \begin{cases} M(G - \{a_i, a_j\}), & \text{if} \quad i < j, \\ -M(G - \{a_i, a_j\}), & \text{if} \quad i > j. \end{cases}$$

Then we have that

$$[M(G)]^{k-1} M(G - \{a_1, \ldots a_{2k}\}) = Pf(A). \tag{3.4}$$

Proof. See [1]. □

3.2 Hexagons with Arbitrary Dents

3.2.1 A Hexagon with Three Unit Dents along Alternating Sides

In [11], Theresia Eisenkölbl proved the following formula for the number of lozenge tilings of a hexagon with three unit dents along alternating sides. In the paper, she proved this result by using the bijection with non-intersecting lattice paths and determinant evaluation. As an example of the application of Kuo's graphical method, Mihai Ciucu provided the conceptual proof of this theorem in [1] as follows.

Recall that the Pochhammer symbol $(a)_k$ is defined by

$$(a)_k = a(a+1)\cdots(a+k-1),$$

and let $M(R)$ be the number of lozenge tilings of the region R.

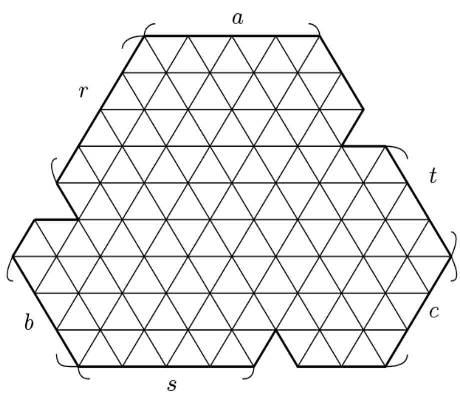

FIGURE 3.2: $H^{4,4,3}_{4,3,3}$

Theorem 3.2.1. *Let $H^{r,s,t}_{a,b,c}$ be the region obtained from the hexagon of side lengths $a, b+3, c, a+3, b, c+3$ (clockwise, starting with the northern side) by deleting three up-pointing unit triangles from along its boundary (see Figure 3.2 for the indication of variables). Then* *we have*

$$M(H^{r,s,t}_{a,b,c}) = (r+1)_b(s+1)_c(t+1)_a(a+3-r)_c(b+3-s)_a(c+3-t)_b$$
$$\times \frac{\prod_{k=0}^{a} k! \prod_{k=0}^{b} k! \prod_{k=0}^{c} k! \prod_{k=0}^{a+b+c+2} k!}{\prod_{k=0}^{b+c+2} k! \prod_{k=0}^{a+c+2} k! \prod_{k=0}^{a+b+2} k!}$$
$$\times [(a+1)(b+1)(c+1)(a+2-r)(b+2-s)(c+2-t) + (a+1)(b+1)(c+1)rst$$
$$-(a+2-r)(b+2-s)(c+2-t)rst + (a+1)(c+1)(b+2-s)(c+2-t)rs$$
$$+(b+1)(a+1)(a+2-r)(c+2-t)st + (c+1)(b+1)(a+2-r)(b+2-s)rt]\,. \tag{3.5}$$

Proof. We denote by $H^*_{a,b,c}$ the region obtained from $H^{r,s,t}_{a,b,c}$ by filling back the three unit dents along its sides, and adding three additional unit triangles sticking out next to the bottom left, right and top left corners as indicated in the left part of Figure 3.3.

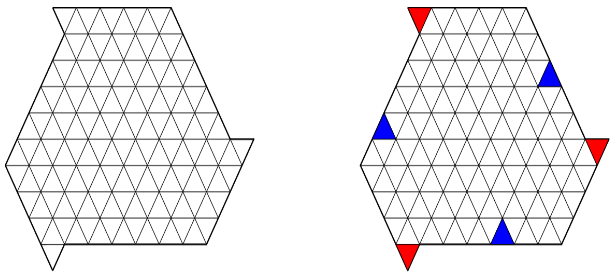

FIGURE 3.3: $H^*_{4,3,3}$ and choosing the vertices for the condensation.

We apply the Theorem 3.1.3 to the plane *dual* graph[1] G of $H^*_{a,b,c}$, with $k = 3$, and choose three dents stated in the theorem and three extra unit triangles we added to be the six removed vertices (see the right part of Figure 3.3). Here, some of r, s, t are allowed to be equal to 0. Let a_1, a_2 and a_3 be the dents along the sides of lengths $a+3$, $b+3$ and $c+3$ (blue), respectively, and let b_1, b_2 and b_3 be the unit triangles that stick out from the corresponding edges(red). Then $b_1, a_1, b_2, a_2, b_3, a_3$ occur in cyclic order along the unbounded face of G. By the Theorem 3.1.3, we obtain that $[M(G)]^2 M(G \setminus \{b_1, a_1, b_2, a_2, b_3, a_3\})$ is equal to the Pfaffian of the matrix

$$\begin{bmatrix} 0 & M(G_{b_1,a_1}) & M(G_{b_1,b_2}) & M(G_{b_1,a_2}) & M(G_{b_1,b_3}) & M(G_{b_1,a_3}) \\ -M(G_{b_1,a_1}) & 0 & M(G_{a_1,b_2}) & M(G_{a_1,a_2}) & M(G_{a_1,b_3}) & M(G_{a_1,a_3}) \\ -M(G_{b_1,b_2}) & -M(G_{a_1,b_2}) & 0 & M(G_{b_2,a_2}) & M(G_{b_2,b_3}) & M(G_{b_2,a_3}) \\ -M(G_{b_1,a_2}) & -M(G_{a_1,a_2}) & -M(G_{b_2,a_2}) & 0 & M(G_{a_2,b_3}) & M(G_{a_2,a_3}) \\ -M(G_{b_1,b_3}) & -M(G_{a_1,b_3}) & -M(G_{b_2,b_3}) & -M(G_{a_2,b_3}) & 0 & M(G_{b_2,a_3}) \\ -M(G_{b_1,a_3}) & -M(G_{a_1,a_3}) & -M(G_{b_2,a_3}) & -M(G_{a_2,a_3}) & -M(G_{b_2,a_3}) & 0 \end{bmatrix}, \tag{3.6}$$

where $G_{x,y}$ stands for $G \setminus \{x, y\}$. Since G is bipartite with the same number of vertices in the two color classes, all entries $M(G_{a_i,a_j})$'s and $M(G_{b_i,b_j})$'s in the matrix 3.6 are zero [1].

Thus we obtain

[1]The plane dual graph of a region R on the triangular lattice is the graph whose vertices are the unit triangles in R, and whose edges connect vertices corresponding to unit triangles that share an edge. See Figure 1.4 for example.

$$[M(G)]^2 M(G \setminus \{b_1, a_1, b_2, a_2, b_3, a_3\}) =$$
$$Pf \begin{bmatrix} 0 & M(G_{b_1,a_1}) & 0 & M(G_{b_1,a_2}) & 0 & M(G_{b_1,a_3}) \\ -M(G_{b_1,a_1}) & 0 & M(G_{a_1,b_2}) & 0 & M(G_{a_1,b_3}) & 0 \\ 0 & -M(G_{a_1,b_2}) & 0 & M(G_{b_2,a_2}) & 0 & M(G_{b_2,a_3}) \\ -M(G_{b_1,a_2}) & 0 & -M(G_{b_2,a_2}) & 0 & M(G_{a_2,b_3}) & 0 \\ 0 & -M(G_{a_1,b_3}) & 0 & -M(G_{a_2,b_3}) & 0 & M(G_{b_2,a_3}) \\ -M(G_{b_1,a_3}) & 0 & -M(G_{b_2,a_3}) & 0 & -M(G_{b_2,a_3}) & 0 \end{bmatrix}. \tag{3.7}$$

We can reorder rows and columns (each simultaneous interchange of two rows and the corresponding two columns results in a sign change for the Pfaffian) to obtain the following result from (3.7) :

$$[M(G)]^2 M(G \setminus \{b_1, a_1, b_2, a_2, b_3, a_3\}) = -Pf \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}, \tag{3.8}$$

where

$$B = \begin{bmatrix} M(G \setminus \{b_1, a_1\}) & M(G \setminus \{b_1, a_2\}) & M(G \setminus \{b_1, a_3\}) \\ -M(G \setminus \{b_2, a_1\}) & M(G \setminus \{b_2, a_2\}) & M(G \setminus \{b_2, a_3\}) \\ -M(G \setminus \{b_3, a_1\}) & M(G \setminus \{b_3, a_2\}) & M(G \setminus \{b_3, a_3\}) \end{bmatrix}. \tag{3.9}$$

Since for any $n \times n$ matrix C we have

$$Pf \begin{bmatrix} 0 & C \\ -C^T & 0 \end{bmatrix} = (-1)^{\frac{n(n-1)}{2}} \det(C) \quad , \tag{3.10}$$

we obtain from 3.8 that

$$[M(G)]^2 M(G \setminus \{b_1, a_1, b_2, a_2, b_3, a_3\}) =$$
$$\det \begin{bmatrix} M(G \setminus \{b_1, a_1\}) & M(G \setminus \{b_1, a_2\}) & M(G \setminus \{b_1, a_3\}) \\ -M(G \setminus \{b_2, a_1\}) & M(G \setminus \{b_2, a_2\}) & M(G \setminus \{b_2, a_3\}) \\ -M(G \setminus \{b_3, a_1\}) & M(G \setminus \{b_3, a_2\}) & M(G \setminus \{b_3, a_3\}) \end{bmatrix}. \tag{3.11}$$

By our settings up, $M(G \setminus \{b_1, a_1, b_2, a_2, b_3, a_3\})$ is precisely the left hand side of (3.5), which we want to calculate. So to do this calculation, we should get the number of perfect matchings $M(G)$ and $M(G \setminus \{b_i, a_j\})$'s.

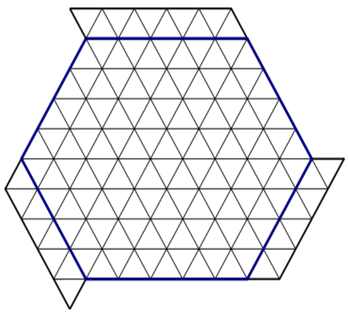

FIGURE 3.4: Removing the forced tiles from $H^*_{a,b,c}$.

From the Figure 3.4, it is obvious that after removing forced tiles, the region is exactly an semi-regular hexagon with side lengths $a+1$, $b+1$, $c+1$, $a+1$, $b+1$ and $c+1$ (from the north side):

$$M(G) = M(H^*_{a,b,c}) = M(H_{a+1,b+1,c+1}),$$

and this number can be easily calculated using MacMahon's formula (Theorem 1.0.1).

For $M(G \setminus \{b_i, a_j\})$'s, we can use Cohn-Larsen-Propp's theorem (Theorem 2.6.2).

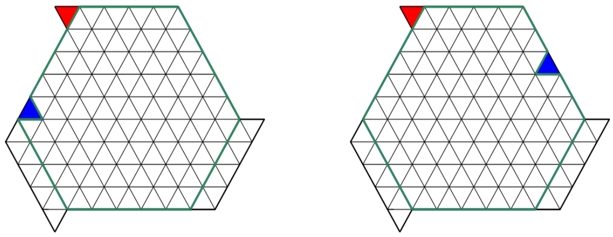

FIGURE 3.5: Removing the forced tiles from $H^*_{a,b,c} \setminus \{a_i, b_j\}$'s.

Figure 3.5 shows that the region corresponding to $G \setminus \{a_i, b_j\}$ is a hexagon with a single unit dent along one of its sides for all $i, j \in \{1, 2, 3\}$. Then the number of lozenge tilings of this region will be the same of the region in Figure 3.6, which we can calculate using Cohn-Larsen-Propp's theorem (Theorem 2.6.2).

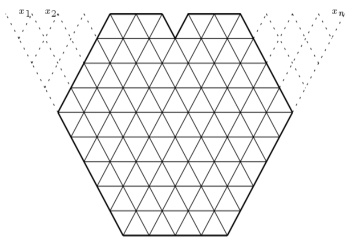

FIGURE 3.6: The equivalent region with the region in the right part in Figure 3.5 .

Now we substitute all values we know in (3.11) so that we can get the desired result (3.5). □

3.2.2 Hexagons with Arbitrary Dents

In [5], Mihai Ciucu and Ilse Fischer generalized Eisenkölbl's result (3.5) to the case when an arbitrary set of unit triangles is removed from along the boundary of the hexagon. Let $H^k_{a,b,c}$ be the hexagon on the triangular lattice whose sides have lengths $a, b+k, c, a+k, b, c+k$ (clockwise from the north side). From the simple observation we can know that $H^k_{a,b,c}$ has k more up-pointing unit triangles than down-pointing unit triangles. Therefore, in order to create a region that can be tiled by lozenges by removing unit triangles from along the boundary, we must remove k more up-pointing ones than down-pointing ones [5]. There are precisely $a+b+c+3k$ up-pointing unit lattice triangles in it that share an edge with the boundary — $a+k$, $b+k$, resp. $c+k$ along the southern, northeastern, resp. northwestern sides. Choose $n+k$ of them, and denote them by $\alpha_1, \ldots, \alpha_{n+k}$ (dents of type α, colored with blue in Figure 3.7). We also choose n unit triangles from the $a+b+c$ down-pointing ones that share an edge with the boundary, which are denoted by $\beta_1, \ldots, \beta_k$ (dents of type β, colored with red in Figure 3.7). The generalization of Eisenkölbl's regions is the family of regions of type $H^k_{a,b,c} \setminus \{\alpha_1, \ldots, \alpha_{n+k}, \beta_1, \ldots, \beta_k\}$.

Let $\bar{H}^k_{a,b,c}$ be the region obtained from $H^k_{a,b,c}$ by augmenting it with one string of k contiguous down-pointing unit triangles along its bottom as shown in the Figure

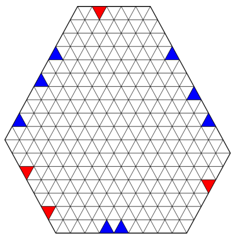

FIGURE 3.7: The region $H^4_{5,7,6}$ with 8 up-pointing and 4 down-pointing unit triangles removed from along the boundary.

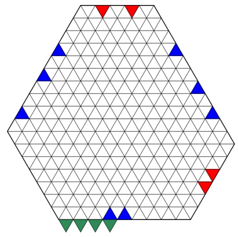

FIGURE 3.8: $\bar{H}^4_{5,7,6}$ and a choice of unit triangles along five of its sides.

3.8 and denote these k down-pointing unit triangles in this string by $\gamma_1, \dots, \gamma_k$. For a skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le 2k}$, denote its Pfaffian by $Pf[(a_{ij})_{1 \le i,j \le 2k}]$[5].

Theorem 3.2.2. *Assume that one of the three sides on which dents of type β can occur does not actually have any dents on it. W.l.o.g, suppose this is the southwestern side. Let $\delta_1, \dots, \delta_{2n+2k}$ be the elements of the set $\{\alpha_1, \dots, \alpha_{n+k}\} \bigcup \{\beta_1, \dots, \beta_n\} \bigcup \{\gamma_1, \dots, \gamma_k\}$ in a cyclic order. Then we have*

$$
\begin{aligned}
&M(H^k_{a,b,c} \setminus \{\alpha_1, \dots, \alpha_{n+k}, \beta_1, \dots, \beta_n\}) \\
&= \frac{1}{[M(\bar{H}^k_{a,b,c})]^{n+k-1}} Pf[(M(\bar{H}^k_{a,b,c} \setminus \{\delta_i, \delta_j\}))_{1 \le i < j \le 2n+2k}],
\end{aligned}
\tag{3.12}
$$

where the quantities on the right hand side are given by explicit formulas: $M(\bar{H}^k_{x,y,z})$ by MacMahon's theorem (Theorem 1.0.1), $M(\bar{H}^k_{a,b,c} \setminus \{\alpha_i, \beta_j\})$'s, $M(\bar{H}^k_{a,b,c} \setminus \{\alpha_i, \delta_j\})$'s by Cohn-Larson-Propp theorem (Theorem 2.6.2) and $M(\bar{H}^k_{x,y,z} \setminus \{\alpha_i, \alpha_j\}) = M(\bar{H}^k_{x,y,z} \setminus \{\beta_i, \beta_j\}) = M(\bar{H}^k_{x,y,z} \setminus \{\gamma_i, \gamma_j\}) = M(\bar{H}^k_{x,y,z} \setminus \{\beta_i, \delta_j\}) = 0$.

Proof. See [1] and [5]. □

3.3 A Hexagon with a "Shamerock" missing - "S-Cored Hexagon"

In [6], Ciucu and Krattenthaler presented a dual part of MacMahon's formula (Theorem 1.0.1), corresponding to the exterior of a concave hexagon obtained by turning 120° after drawing each side. (MacMahon's hexagon is obtained by turning 60° after each step.) Then one obtains a shape of the type shown in the Figure 3.9; we call such a shape a *"shamrock"* [6].

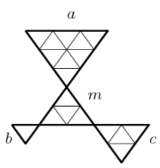

FIGURE 3.9: *Shamerock*$(3,1,2,2)$.

We denote the shamrock whose central equilateral triangle has side length m, whereas its top, bottom left and bottom right lobes are equilateral triangles of side lengths a, b and c by $S(a,b,c,m)$.

We have already considered the case, in which one triangle is removed from the center of the hexagon on the triangular lattice (Section 2.4). Compared with this, we will deal with the enumeration formula for the hexagon, which has one "shamerock" missing from the centre instead of one triangle in this section.

First, we let $C_{x,y,z}(m)$ be a cored hexagon with side lengths x, $y+m$, z, $x+m$, y, $z+m$ (in clockwise order, starting from top), with an equilateral triangle of side m removed from its center. (The position of the centre is precisely defined in the Section 2.4.) Given non-negative integers a, b and c, construct our region as follows: start with the cored hexagon $C_{x,y,z}(m)$, and "push out" the six lattice lines containing its sides as follows: the top, southwestern, and southeastern sides, a, b, and c units, respectively, and the bottom, northeastern, and northwestern sides, $b+c$, $a+c$, and $a+b$ units, respectively. Also enlarge the core by adding to it down-pointing equilateral triangles of sides a, b, and c that touch the original core at its top, left, and right vertex, respectively. The hexagon formed by the pushed-out edges, with the enlarged, shamrock-shaped core taken out of it, is called an *"S-cored hexagon"* and is denoted by $SC_{x,y,z}(a,b,c,m)$[6]. Figure 3.10 and 3.11 show the examples of S-cored hexagons depending on parities of x, y and z.

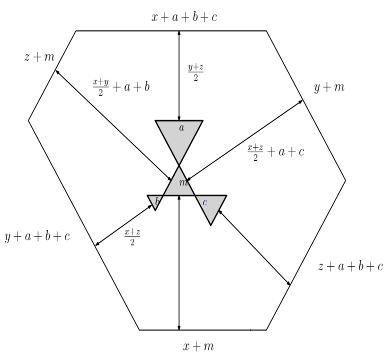

FIGURE 3.10: Distances within the S-cored hexagon $SC_{x,y,z}(a,b,c,m)$ when x, y, z have the same parity.

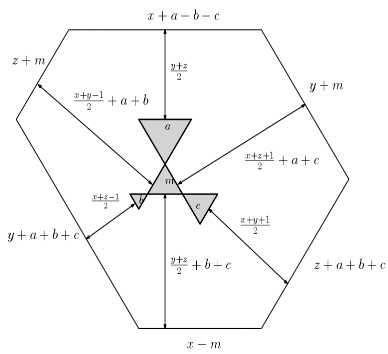

FIGURE 3.11: Distances within the S-cored hexagon $SC_{x,y,z}(a,b,c,m)$, where x has the different parity from y and z.

Let $M(SC_{x,y,z}(a,b,c,m))$ be the number of lozenge tilings of this S-cored hexagon. The following two theorems give us explicit product formulas for the enumeration of lozenge tilings in such regions :

Theorem 3.3.1. *Let x, y, z, a, b, c and m be non-negative integers. If x, y and z have the same parity, we have*

$$
\begin{aligned}
M(SC_{x,y,z}(a,b,c,m)) &= \frac{H(m)^3H(a)H(b)H(c)}{H(m+a)H(m+b)H(m+c)}\\
&\times \frac{H(\frac{x+y}{2}+m+a+b)H(\frac{x+z}{2}+m+a+c)H(\frac{y+z}{2}+m+b+c)}{H(\frac{x+y}{2}+m+c)H(\frac{x+z}{2}+m+b)H(\frac{y+z}{2}+m+a)}\\
&\times \frac{H(\frac{x+y}{2}+c)H(\frac{x+z}{2}+b)H(\frac{y+z}{2}+a)H(x+y+z+m+a+b+c)}{H(\frac{x+y}{2}+a+b)H(\frac{x+z}{2}+a+c)H(\frac{y+z}{2}+b+c)}\\
&\times \frac{H(x+m+a+b+c)H(y+m+a+b+c)H(z+m+a+b+c)}{H(x+y+m+a+b+c)H(x+z+m+a+b+c)H(y+z+m+a+b+c)}\\
&\times \frac{H(\lceil\frac{x+y+z}{2}\rceil+m+a+b+c)H(\lfloor\frac{x+y+z}{2}\rfloor+m+a+b+c)}{H(\frac{x+y}{2}+m+a+b+c)H(\frac{x+z}{2}+m+a+b+c)H(\frac{y+z}{2}+m+a+b+c)}\\
&\times \frac{H(\lceil\frac{x}{2}\rceil)H(\lceil\frac{y}{2}\rceil)H(\lceil\frac{z}{2}\rceil)}{H(\lceil\frac{x}{2}\rceil+\frac{m+a+b+c}{2})H(\lceil\frac{y}{2}\rceil+\frac{m+a+b+c}{2})H(\lceil\frac{z}{2}\rceil+\frac{m+a+b+c}{2})}\\
&\times \frac{H(\lfloor\frac{x}{2}\rfloor)H(\lfloor\frac{y}{2}\rfloor)H(\lfloor\frac{z}{2}\rfloor)}{H(\lfloor\frac{x}{2}\rfloor+\frac{m+a+b+c}{2})H(\lfloor\frac{y}{2}\rfloor+\frac{m+a+b+c}{2})H(\lfloor\frac{z}{2}\rfloor+\frac{m+a+b+c}{2})}\\
&\times \frac{H(\frac{m+a+b+c}{2})^2H(\frac{x+y}{2}+\frac{m+a+b+c}{2})^2H(\frac{x+z}{2}+\frac{m+a+b+c}{2})^2H(\frac{y+z}{2}+\frac{m+a+b+c}{2})^2}{H(\lceil\frac{x+y+z}{2}\rceil+\frac{m+a+b+c}{2})H(\lfloor\frac{x+y+z}{2}\rfloor+\frac{m+a+b+c}{2})H(\frac{x+y}{2})H(\frac{x+z}{2}H(\frac{y+z}{2}))}.
\end{aligned}
\tag{3.13}
$$

Outline of proof. Using Kuo's graphical condensation method (Theorem 3.1.1) (the choice of 4 points is indicated in [6]), we can get the following recursion :

$$
\begin{aligned}
&M(SC_{x,y,z}(a,b,c,m)) \times M(SC_{x,z-1,y-1}(a,c,b,m))\\
&= M(SC_{z-1,x,y}(c,a,b,m)) \times M(SC_{y-1,z,x}(b,c,a,m))\\
&+ M(SC_{x-1,z,y}(a,c,b,m)) \times M(SC_{x+1,y-1,z-1}(a,b,c,m)).
\end{aligned}
\tag{3.14}
$$

After verifying the initial case $M(SC_{0,0,0}(a,b,c,m))$ and applying the induction on the quantity $x+y+z$, we can get the desired result. The complete proof can be found in [6]. □

Theorem 3.3.2. *Let x, y, z, a, b, c and m be non-negative integers. If x has parity different from the parity of y and z, we have*

$$
\begin{aligned}
M(SC_{x,y,z}(a,b,c,m)) &= \frac{H(m)^3H(a)H(b)H(c)}{H(m+a)H(m+b)H(m+c)}\\
&\times \frac{H(\lfloor\frac{x+y}{2}\rfloor+m+a+b)H(\lceil\frac{x+z}{2}\rceil+m+a+c)H(\frac{y+z}{2}+m+b+c)}{H(\lceil\frac{x+y}{2}\rceil+m+c)H(\lfloor\frac{x+z}{2}\rfloor+m+b)H(\frac{y+z}{2}+m+a)}\\
&\times \frac{H(\lceil\frac{x+y}{2}\rceil+c)H(\lfloor\frac{x+z}{2}\rfloor+b)H(\frac{y+z}{2}+a)H(x+y+z+m+a+b+c)}{H(\lfloor\frac{x+y}{2}\rfloor+a+b)H(\lceil\frac{x+z}{2}\rceil+a+c)H(\frac{y+z}{2}+b+c)}\\
&\times \frac{H(x+m+a+b+c)H(y+m+a+b+c)H(z+m+a+b+c)}{H(x+y+m+a+b+c)H(x+z+m+a+b+c)H(y+z+m+a+b+c)}\\
&\times \frac{H(\lceil\frac{x+y+z}{2}\rceil+m+a+b+c)H(\lfloor\frac{x+y+z}{2}\rfloor+m+a+b+c)}{H(\lfloor\frac{x+y}{2}\rfloor+m+a+b+c)H(\lceil\frac{x+z}{2}\rceil+m+a+b+c)H(\frac{y+z}{2}+m+a+b+c)}\\
&\times \frac{H(\lceil\frac{x}{2}\rceil)H(\lceil\frac{y}{2}\rceil)H(\lceil\frac{z}{2}\rceil)}{H(\lceil\frac{x}{2}\rceil+\frac{m+a+b+c}{2})H(\lceil\frac{y}{2}\rceil+\frac{m+a+b+c}{2})H(\lceil\frac{z}{2}\rceil+\frac{m+a+b+c}{2})}\\
&\times \frac{H(\lfloor\frac{x}{2}\rfloor)H(\lfloor\frac{y}{2}\rfloor)H(\lfloor\frac{z}{2}\rfloor)}{H(\lfloor\frac{x}{2}\rfloor+\frac{m+a+b+c}{2})H(\lfloor\frac{y}{2}\rfloor+\frac{m+a+b+c}{2})H(\lfloor\frac{z}{2}\rfloor+\frac{m+a+b+c}{2})}\\
&\times \frac{H(\frac{m+a+b+c}{2})^2H(\lfloor\frac{x+y}{2}\rfloor+\frac{m+a+b+c}{2})H(\lceil\frac{x+y}{2}\rceil+\frac{m+a+b+c}{2})}{H(\lceil\frac{x+y+z}{2}\rceil+\frac{m+a+b+c}{2})H(\lfloor\frac{x+y+z}{2}\rfloor+\frac{m+a+b+c}{2})}\\
&\times \frac{H(\lfloor\frac{x+z}{2}\rfloor+\frac{m+a+b+c}{2})H(\lceil\frac{x+z}{2}\rceil+\frac{m+a+b+c}{2})H(\frac{y+z}{2}+\frac{m+a+b+c}{2})^2}{H(\lfloor\frac{x+y}{2}\rfloor)H(\lceil\frac{x+z}{2}\rceil)H(\frac{y+z}{2})}.
\end{aligned}
\tag{3.15}
$$

Outline of the proof. The proof is analogous to the previous case. We can again get the following recursive relation by using Kuo's graphical condensation method :

$$
\begin{aligned}
&M(SC_{x,y,z}(a,b,c,m)) \times M(SC_{x,y-1,z-1}(a,b,c,m))\\
&= M(SC_{y,x,z-1}(b,a,c,m)) \times M(SC_{z,y-1,x}(c,b,a,m))\\
&+ M(SC_{x-1,y,z}(a,b,c,m)) \times M(SC_{x+1,y-1,z-1}(a,b,c,m)).
\end{aligned}
\tag{3.16}
$$

Again, after verifying the initial case and applying the induction on the quantity $x+y+z$, we can get the desired result. See [6] for the complete proof. □

3.4 A Hexagon with "Fern" missing - "F-Cored Hexagon"

In [4], Mihai Ciucu introduced another dual form of MacMahon's formula (Theorem 1.0.1). He introduced a new structure, called a *"fern"*, consists of an arbitrary number of equilateral triangles of alternating orientations lined up along a lattice line. He provided the enumeration formula for the hexagon with such a fern removed at the special position inside the hexagon.

Let $s(b_1, b_2, \ldots, b_l)$ denote the number of lozenge tilings of the semi-hexagonal region $S(b_1, b_2, \ldots, b_l)$ with the leftmost b_1 up-pointing unit triangles on its base removed, the next segment of length b_2 staying unchanged, the following b_3 removed, and so on (see an illustration in Figure 3.12. By the Cohn-Larsen-Propp interpretation of the Gelfand-Tsetlin result[14] we have that

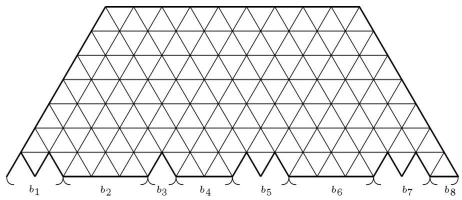

FIGURE 3.12: $S(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ with $b_1 = 2$, $b_2 = 3$, $b_3 = 1$, $b_4 = 2$, $b_5 = 2$, $b_6 = 3$, $b_7 = 2$ and $b_8 = 1$.

$$s(b_1, b_2, \ldots, b_{2l}) = s(b_1, b_2, \ldots, b_{2l-1}) = \frac{\prod_{1 \leq i < j \leq 2l-1, j-i+1 odd} (b_i + b_{i+1} + \cdots + b_j)}{\prod_{1 \leq i < j \leq 2l-1, j-i+1 even} (b_i + b_{i+1} + \cdots + b_j)}. \tag{3.17}$$

For non-negative integers $a_1, \ldots, a_k$, define the fern $F(a_1, \ldots, a_k)$ to be a string of k lattice triangles lined up along a horizontal lattice line, touching at their vertices, alternately oriented up and down and having sizes $a_1, \ldots, a_k$ as encountered from left to right (with the leftmost oriented up, without loss of generality). Figure 3.13 shows such an example for $F(1, 2, 3, 2, 2)$.

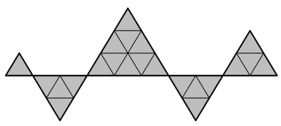

FIGURE 3.13: Fern with $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 2$ and $a_5 = 2$.

The regions we are considering are defined as follows. Set

$$o := a_1 + a_3 + a_5 + \cdots, \tag{3.18}$$

and

$$e := a_2 + a_4 + a_6 + \cdots, \tag{3.19}$$

and let H be the hexagon of side-lengths $x + e$, $y + o$, $z + e$, $x + o$, $y + e$, $z + o$ (clockwise from top). We call the hexagon $\bar{H}$ of side-lengths x, y, z, x, y, z (clockwise from top) that fits in the western corner of H the *auxiliary hexagon* (in Figures. 3.14, 3.15, 3.16 and 3.17 the auxiliary hexagon is indicated by a thick dotted blue line). Our regions are obtained by taking out from H a certain translation of the fern that places its base point either at the very center of the *auxiliary* hexagon (when this center is a lattice point), or as close to it as possible[14].

The details, which depend on the relative parities of x, y and z, are as follows.

The center of $\bar{H}$ is a lattice point precisely when x, y and z have the same parity. In this case, we put the fern $F(a_1, \ldots, a_k)$ inside the hexagon H so that the base of the fern is at the center of the auxiliary hexagon $\bar{H}$ (see Figure 3.14)[14].

Denote by $FC^{\odot}_{x,y,z}(a_1, \ldots, a_k)$ the region obtained from H by removing the fern placed in this position.

If x has opposite parity to y and z, draw the fern $F(a_1, \ldots, a_k)$ inside H so that the base of the fern is half a lattice spacing to the west of the center of the auxiliary hexagon (this center is marked by a white dot in Figure 3.15)[14].

Denote by $FC^{\leftarrow}_{x,y,z}(a_1, \ldots, a_k)$ the region obtained by taking out from H this placement of the fern; the arrow at the superscript records the relative position of the base versus the center of the auxiliary hexagon.

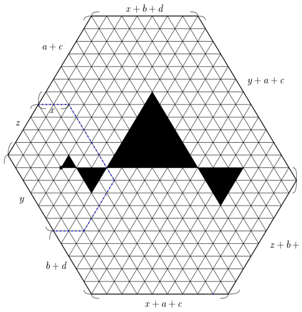

FIGURE 3.14: The F-cored hexagon $FC^{\odot}_{2,6,4}(1,2,6,3)$.

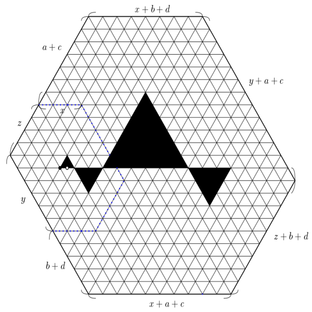

FIGURE 3.15: The F-cored hexagon $FC^{\leftarrow}_{3,6,4}(1,2,6,3)$

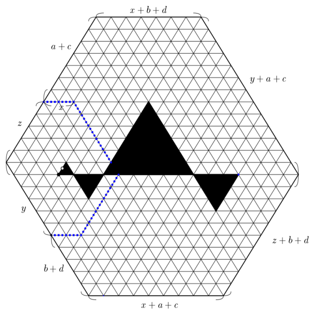

FIGURE 3.16: The F-cored hexagon $FC^{\swarrow}_{2,6,5}(1,2,6,3)$.

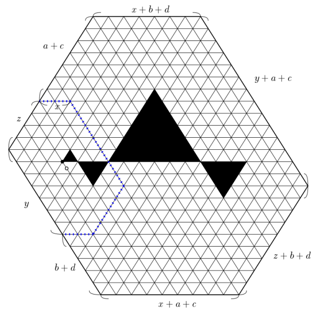

FIGURE 3.17: The F-cored hexagon $FC^{\nwarrow}_{2,7,4}(1,2,6,3)$

If z has parity opposite to x and y, we place the fern so that its base is half a lattice spacing to the southwest of the center of $\bar{H}$ and denote the resulting region by $FC^{\swarrow}_{x,y,z}(a_1,\dots,a_k)$ (see Figure 3.16)[14]. Finally, if y has parity opposite to x and z, place the fern so that its base is half a lattice spacing to the the northwest of the center of $\bar{H}$, and denote the resulting region by $FC^{\nwarrow}_{x,y,z}(a_1,\dots,a_k)$ (see Figure 3.17)[14].

Remark 3.4.1. We compare F-cored hexagons with S-cord hexagons which were introduced in the Section 3.3. Then it is not hard to observe that F-cored hexagons are specializations of the S-cored hexagons $SC_{x,y,z}(a,b,c,m)$ when the fern has only two lobes:

$$
\begin{aligned}
FC^{\odot}_{x,y,z}(a,b) &= SC_{x,y,z}(0,0,b,a)\,,\\
FC^{\leftarrow}_{x,y,z}(a,b) &= SC_{x,y,z}(0,0,b,a)\,,\\
FC^{\swarrow}_{x,y,z}(a,b) &= SC_{z,y,x}(a,0,0,b)\,,\\
FC^{\nwarrow}_{x,y,z}(a,b) &= SC_{y,z,x}(0,a,0,b)\,.
\end{aligned}
$$

We define the *F-cored hexagon* $FC_{x,y,z}(a_1,\dots,a_k)$ by

$$
FC_{x,y,z}(a_1,\dots,a_k) := \begin{cases} FC^{\odot}_{x,y,z}(a_1,\dots,a_k), & \text{if } x,\, y,\, z \text{ same parity}\,,\\ FC^{\leftarrow}_{x,y,z}(a_1,\dots,a_k), & \text{if } x \text{ has parity opposite to } y,\, z\,,\\ FC^{\swarrow}_{x,y,z}(a_1,\dots,a_k), & \text{if } z \text{ has parity opposite to } x,\, y\,,\\ FC^{\nwarrow}_{x,y,z}(a_1,\dots,a_k), & \text{if } y \text{ has parity opposite to } x,\, z\,. \end{cases} \tag{3.20}
$$

Then the result of the lozenge tilings of *F-Cored* hexagon is the following.

Theorem 3.4.2. *Let x, y, z and $a_1, \dots, a_k$ be non-negative integers. Then the number of lozenge tilings of the F-cored hexagon $FC_{a,b,c}(a_1, \dots, a_k)$ is given by*

$$
\begin{aligned}
\frac{M(FC_{x,y,z}(a_1,\dots,a_k))}{M(FC_{x,y,z}(a_1+a_3+a_5+\cdots, a_2+a_4+a_6+\cdots))} &= s(a_1,\dots,a_{k-1})s(a_2,\dots,a_k)\\
&\times \frac{H(\lfloor\frac{x+z}{2}\rfloor + a_1+a_3+\cdots)}{H(\lfloor\frac{x+y}{2}\rfloor + a_1+a_3+\cdots)}\frac{H(\lceil\frac{x+z}{2}\rceil + a_2+a_4+\cdots)}{H(\lceil\frac{x+y}{2}\rceil + a_2+a_4+\cdots)}\\
&\times \prod_{1\le 2i+1\le k}\frac{H(\lfloor\frac{x+y}{2}\rfloor + a_1+\cdots+a_{2i+1})}{H(\lfloor\frac{x+z}{2}\rfloor + a_1+\cdots+a_{2i+1})}\frac{H(\lceil\frac{x+y}{2}\rceil + \overline{a_1+\cdots+a_{2i+1}})}{H(\lceil\frac{x+z}{2}\rceil + \overline{a_1+\cdots+a_{2i+1}})}\\
&\times \prod_{1<2i<k}\frac{H(\lfloor\frac{x+z}{2}\rfloor + a_1+\cdots+a_{2i})}{H(\lfloor\frac{x+y}{2}\rfloor + a_1+\cdots+a_{2i})}\frac{H(\lceil\frac{x+z}{2}\rceil + \overline{a_1+\cdots+a_{2i}})}{H(\lceil\frac{x+y}{2}\rceil + \overline{a_1+\cdots+a_{2i}})},
\end{aligned}
\tag{3.21}
$$

where H is the hyperfactorial function, s is defined by (3.17) and $\overline{a_1+\cdots+a_i}$ stands for $a_{i+1}+\cdots+a_k$.

Outline of the proof. Again, using Kuo's condensation method (Theorem 3.1.1; see [4] for the choice of four points), we can get the following recursive relations :

1. If x, y, z have the same parity,

$$
\begin{aligned}
&M(FC^{\odot}_{x,y,z}(a_1,\dots,a_k))M(FC^{\leftarrow}_{x,y,z}(a_1,\dots,a_k)) =\\
&M(FC^{\nwarrow}_{x,y-1,z}(a_1,\dots,a_k))M(FC^{\swarrow}_{x,y,z-1}(a_1,\dots,a_k))\\
&+ M(FC^{\leftarrow}_{x-1,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x+1,y-1,z-1}(a_1,\dots,a_k)).
\end{aligned}
\tag{3.22}
$$

2. If x has parity opposite to y and z,

$$
\begin{aligned}
&M(FC^{\leftarrow}_{x,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x,y-1,z-1}(a_1,\dots,a_k)) =\\
&M(FC^{\nwarrow}_{x,y,z-1}(a_1,\dots,a_k))M(FC^{\swarrow}_{x,y-1,z}(a_1,\dots,a_k))\\
&+ M(FC^{\leftarrow}_{x+1,y-1,z-1}(a_1,\dots,a_k))M(FC^{\odot}_{x-1,y,z}(a_1,\dots,a_k)).
\end{aligned}
\tag{3.23}
$$

3. If z has the different parity from x and y,

$$
\begin{aligned}
&M(FC^{\swarrow}_{x,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x-1,y-1,z}(a_1,\dots,a_k)) =\\
&M(FC^{\leftarrow}_{x,y-1,z}(a_1,\dots,a_k))M(FC^{\nwarrow}_{x-1,y,z}(a_k,\dots,a_1))\\
&+ M(FC^{\odot}_{x,y,z-1}(a_1,\dots,a_k))M(FC^{\swarrow}_{x-1,y-1,z+1}(a_1,\dots,a_k)),
\end{aligned}
\tag{3.24}
$$

for the case that the number of lobes k is even and

$$
\begin{aligned}
&M(FC^{\swarrow}_{x,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x-1,y-1,z}(a_1,\dots,a_k)) =\\
&M(FC^{\leftarrow}_{x,y-1,z}(a_1,\dots,a_k))M(FC^{\swarrow}_{x-1,z,y}(a_k,\dots,a_1))\\
&+ M(FC^{\odot}_{x,y,z-1}(a_1,\dots,a_k))M(FC^{\swarrow}_{x-1,y-1,z+1}(a_1,\dots,a_k)),
\end{aligned}
\tag{3.25}
$$

for the case that the number of lobes k is odd.

4. As the last case, if the parity of y is opposite to x and z,

$$\begin{aligned} &M(FC^{\nwarrow}_{x,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x-1,y,z-1}(a_1,\dots,a_k)) = \\ &M(FC^{\leftarrow}_{x,y,z-1}(a_1,\dots,a_k))M(FC^{\swarrow}_{x-1,y,z}(a_k,\dots,a_1)) \\ &+ M(FC^{\odot}_{x,y-1,z}(a_1,\dots,a_k))M(FC^{\nwarrow}_{x-1,y+1,z-1}(a_1,\dots,a_k)), \end{aligned} \tag{3.26}$$

for the case the number of lobes k is even and

$$\begin{aligned} &M(FC^{\nwarrow}_{x,y,z}(a_1,\dots,a_k))M(FC^{\odot}_{x-1,y,z-1}(a_1,\dots,a_k)) = \\ &M(FC^{\leftarrow}_{x,y,z-1}(a_1,\dots,a_k))M(FC^{\nwarrow}_{x-1,z,y}(a_k,\dots,a_1)) \\ &+ M(FC^{\odot}_{x,y-1,z}(a_1,\dots,a_k))M(FC^{\nwarrow}_{x-1,y+1,z-1}(a_1,\dots,a_k)), \end{aligned} \tag{3.27}$$

for the case the number of lobes k is odd.

All six recurrences (3.22) - (3.27), in which all kinds of regions are involved, are well-defined for $x, y, z \geq 1$. Based on these recursions, we apply the induction on the quantity $x + y + z$ after verifying the initial cases to get the desired result. See [4] for the complete proof. □

3.5 A Short Remark on the Lozenge Tiling Enumeration Problems

In the previous chapters and sections, I tried to introduce all the *simple* enumeration results of lozenge tilings in various hexagonal regions on the triangular lattice plane. Based on these results, we can think about more complicated regions and tiling-enumeration formulas of them. So far there are already a lot of successful results of lozenge tilings generalizing the previous ones. Except for the main results and examples which I introduced before, I want to make a short list of the generalized problems.

- The lozenge tiling enumeration of the semi-regular hexagon with hole(s) in it.
- Symmetry classes of lozenge tilings of *holey*-hexagons.
- The lozenge tiling enumeration of the hexagon with more than one *shamerocks* removed in it.
- The lozenge tiling enumeration of the hexagon with more than one *ferns* removed in it.
- The lozenge tiling enumeration of the hexagon with some combinations of *defects* in it. For example, we can think about the tiling enumeration problem of a hexagon having *dents* on boundary and some *holes* inside at the same time. We can also think about different positions of these holes, since I just stated hexagons whose holes are located at special points.

Chapter 4

The Lozenge Tiling of a Semi-Regular Hexagon with One "Bow-Tie" removed at the Special Point

In this chapter, as the main work of my thesis, I will introduce the lozenge tiling enumeration of a semi-regular hexagon with one *"bow-tie"* removed from a *"special"* point inside the hexagon. Here, "special" point means a point that which is having the distances of a,b,c units (one unit is $\frac{\sqrt{3}}{2} \times$ length of an equilateral triangle) from alternating sides with lengths a,b,c respectively (see Figure 4.1).

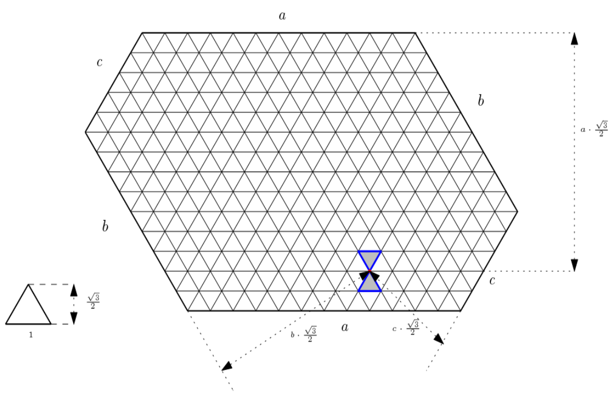

FIGURE 4.1: A semi-regular hexagon with side lengths a, b, c and one bow-tie removed from the point which is distanced $\frac{\sqrt{3}}{2} \cdot a$, $\frac{\sqrt{3}}{2} \cdot b$, resp. $\frac{\sqrt{3}}{2} \cdot c$ from the north, southwest,resp.southeast side.

Before we start, we note that this "special" point (colored with red in the Figure 4.1) is inside the hexagon if and only if a,b and c satisfy the triangle inequality, i.e.

$$a < b + c; \quad b < a + c; \quad c < a + b. \tag{4.1}$$

Throughout the remaining part, w.l.o.g, we assume two edges of two triangles of the bow-tie which are not intersecting to be parallel to the sides of the hexagon having lengths a (see the Figure 4.1). Denote the number of lozenge tilings of the hexagon with a "bow-tie" removed from this point by $BT(a,b,c)$. Then there is one more basic observation we can get that $BT(a,b,c) = BT(a,c,b)$, since two hexagons with side lengths a, b, c resp. a, c, b with one bow-tie removed from the corresponding point are equivalent (see the Figure 4.2).

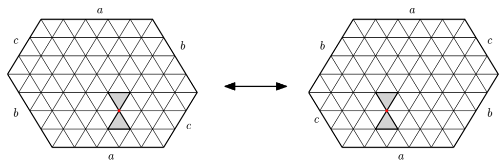

FIGURE 4.2: The symmetry on sides b and c.

By this symmetry, in the following part, I will assume that $b \geq c$ without loss of generality.

4.1 Guessing the Enumeration Formula

4.1.1 From Lozenge Tilings to Non-Intersecting Lattice Paths and Determinants

I used the bijection between lozenge tilings of a semi-regular hexagon with one "bow-tie" removed (this bow-tie can be located on the arbitrary position inside the hexagon) and non-intersecting paths in $\mathbb{Z}^2$ with steps in direction $(1,0)$ and $(0,-1)$ (see Figure 4.3).

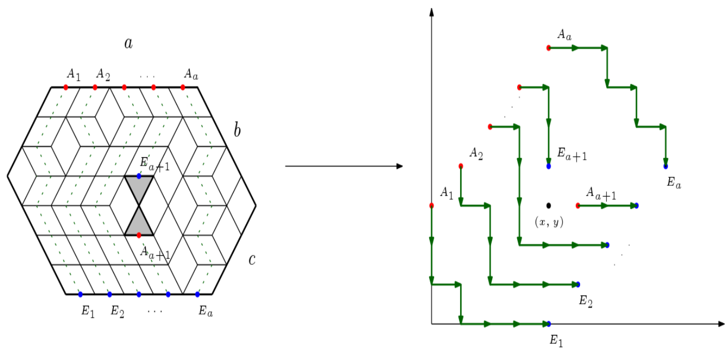

FIGURE 4.3: One lozenge tiling in a semi-regular hexagon with one bow-tie removed inside and the corresponding non-intersecting lattice paths on $\mathbb{Z}^2$.

The initial points A_i's and the destination points E_i's of the paths in Figure 4.3 are the centres of the sides of the lozenges that form the two sides of the hexagon with side lengths a.

Figure 4.3 shows the bijection between lozenge tilings of a hexagon with side lengths a,b,c,a,b,c and non-intersecting lattice paths with initial points

$$A_i = (i-1, c+i-1) \tag{4.2}$$

and destination points

$$E_i = (b+i-1, i-1), \tag{4.3}$$

for $i = 1, 2, \ldots, a$ on the square lattice.

For the "bow-tie" I removed, under the settings above, I add one extra initial point and destination point, i.e.,

$$A_{a+1} = (x+1, y) \qquad \text{and} \qquad E_{a+1} = (x, y+1). \tag{4.4}$$

(I set the coordinate of the point which is located one unit below than E_{a+1} as (x, y).)

Then, by using Lindström-Gessel-Viennot theorem (Theorem 2.3.3), we can see that the following holds.

Lemma 4.1.1. Let a,b and c be positive integers and (x, y) be an integer point such that $0 \leq x \leq a+b$ and $0 \leq y \leq a+c$. Then the number of lozenge tilings of a hexagon with side lengths a,b,c,a,b,c which contains the fixed "bow-tie" that corresponds to the point (x, y) in the bijection described above equals

$$-\det_{1\le i,j\le a+1} \begin{array}{cc} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y-1}{y-i+1} \\ \hline \binom{c+x-y-1}{x-j+1} & 0 \end{array}\right) & \begin{array}{c} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{4.5}$$

Proof. The construction of the matrix is analogous to the procedure as I introduced in the Section 2.4. The entries in $a \times a$ submatrix of the matrix (4.5) (top-left part of the matrix) is corresponding to the number of paths from the starting point A_i to the end point E_j for $1 \leq i, j \leq a$. The entries in the last row are the numbers of paths from A_{a+1} to the end points E_j's while the entries in the last column are corresponding to the numbers of paths from A_i's to the end point E_{a+1}. The entry in the position of $(a+1, a+1)$ is zero, because we are unable to make the path from A_{a+1} to E_{a+1} with steps $(1, 0)$ and $(0, -1)$. $\square$

4.1.2 From the Determinant to a Triple Sum

In the previous section, I claimed that the determinant in (4.5) is equal to the number of lozenge tilings of the semi-regular hexagon with one bow-tie removed at the point corresponding to x and y. We describe this determinant in the triple sum form to be able to do further work, because the determinant itself is not easy to handle. The methodology I used here is exactly the same as Ilse Fischer did in her paper [13] to enumerate lozenge tilings of a hexagon with one lozenge removed from its center.

Lemma 4.1.2. The determinant in (4.5) equals

$$\begin{aligned} &\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c} \\ &\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{c+x-y+n-2}{x}\binom{b-x+y+s-2}{y}\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\right. \\ &\left.\binom{m-1}{s-1}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right]. \end{aligned} \tag{4.6}$$

Proof. First, we perform elementary row and column operations that transform the 'homogeneous' $a \times a$ submatrix $\left(\binom{b+c}{c-i+j}\right)_{1\le i,j\le a}$ of the matrix underlying the determinant in 4.5 into an upper triangular matrix.

We begin with some elementary column operations: By using the symmetry of the binomial coefficient (i.e., the identity $\binom{n}{k} = \binom{n}{n-k}$) in the last row of the matrix underlying the determinant in (4.5) we observe that the determinant in (4.5) is equal to

$$-\det_{1\le i,j\le a+1}\begin{array}{c} \begin{array}{cc} 1\le j\le a & j=a+1\end{array}\\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y-1}{y-i+1}\\ \hline \binom{c+x-y-1}{c-y+j-2} & 0\end{array}\right) \end{array} \begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.7}$$

Add the $(j-1)$-th column to the j-th column, $j=a,a-1,\ldots,2$, in that order. The entries of the changed matrix will be equal to

$$\binom{b+c}{c-i+j}+\binom{b+c}{c-i+j-1}=\binom{b+c+1}{c-i+j}$$

for $i=1,2,\ldots,a$ and $j=2,3,\ldots,a$, and

$$\binom{c+x-y-1}{c-y+j-2}+\binom{c+x-y-1}{c-y+j-3}=\binom{c+x-y}{c-y+j-2}$$

for $i=a+1$ and $j=2,3,\ldots,a$. The other entries remain unchanged. Thus, the following determinant is equal to the determinant in (4.7):

$$-\det_{1\le i,j\le a+1}\begin{array}{c} \begin{array}{ccc} j=1 & 2\le j\le a & j=a+1\end{array}\\ \left(\begin{array}{c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+j} & \binom{b-x+y}{y-i+1}\\ \hline \binom{c+x-y-1}{c-y-1} & \binom{c+x-y}{c-y+j-2} & 0\end{array}\right) \end{array} \begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.8}$$

We repeat the procedure, i.e., we add the $(j-1)$-th column to the j-th column, $j=a,a-1,\ldots,3$, in that order. Thus, we obtain that the following determinant is equal to the determinant in (4.8):

$$-\det_{1\le i,j\le a+1}\begin{array}{c} \begin{array}{cccc} j=1 & j=2 & 3\le j\le a & j=a+1\end{array}\\ \left(\begin{array}{c|c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+2} & \binom{b+c+2}{c-i+j} & \binom{b-x+y-1}{y-i+1}\\ \hline \binom{c+x-y-1}{c-y-1} & \binom{c+x-y}{c-y} & \binom{c+x-y+1}{c-y+j-2} & 0\end{array}\right) \end{array} \begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}.$$

We repeat this procedure of adding successive columns from right to left each time stopping one column earlier than before, totally $a-1$ times including the two steps described above. Since the upper parameter of the binomial entry increases by one every time it is involved and every entry in the j-th column participates $j-1$ times exactly, for $1\le j\le a$, after this procedure we can get the following determinant:

$$
-\det_{1\le i,j\le a+1}\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\binom{b+c+j-1}{c-i+j} & \binom{b-x+y-1}{y-i+1}\\
\hline
\binom{c+x-y+j-2}{c-y+j-2} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.9}
$$

Now we apply some elementary row operations to (4.9). In fact these row operations are analogous to the column operations we just applied to (4.7). In order to do so, we first use the symmetry of the binomial coefficient for the first a rows of the determinant in (4.9) and note that the determinant (4.10) is equal to the determinant in (4.9).

$$
-\det_{1\le i,j\le a+1}\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\binom{b+c+j-1}{b+i-1} & \binom{b-x+y-1}{b+i-x-1}\\
\hline
\binom{c+x-y+j-2}{c-y+j-2} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.10}
$$

Now we add the $(i-1)$-th row of the determinant in (4.10) to the i-th row, starting at $i=a$ and stopping at $i=2$. Next we do the same with the resulting determinant, starting at $i=a$ but stopping at $i=3$. After repeating this procedure $a-1$ times we obtain that the following determinant is equal to the determinant in (4.10).

$$
-\det_{1\le i,j\le a+1}\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\binom{b+c+i+j-2}{b+i-1} & \binom{b-x+y+i-1}{b+i-x-2}\\
\hline
\binom{c+x-y+j-2}{c-y+j-2} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.11}
$$

Then we take the factor $(-1)^{b+c+i-1}(b+c+i-1)!/(b+i-1)!$ out from the i-th row of the determinant in (4.11), $i=1,2,\dots,a$. We get

$$
-\prod_{i=1}^{a}(-1)^{b+c+i-1}\frac{(b+c+i-1)!}{(b+i-1)!}
$$
$$
\times\det_{1\le i,j\le a+1}\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\frac{(j-1)!}{(c+j-1)!}\binom{-j}{b+c+i-1} & (-1)^{b+c+i-1}\frac{(b+i-1)!}{(b+c+i-1)!}\binom{b-x+y+i-2}{b+i-x-2}\\
\hline
\binom{c+x-y+j-2}{c-y+j-2} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.12}
$$

Finally we apply the elementary row operations we just applied to the determinant in (4.10) once again. An analysis of the elementary row operations that we applied to the determinant in (4.10)[13] yields that the determinant in (4.11) was produced from the determinant in (4.10) by replacing the i-th row by

$$\sum_{s=1}^{i}\binom{i-1}{s-1}A(s),$$

for $1\le i\le a$, where $A(s)$ denotes the s-th row of the determinant in (4.10).

We perform this replacement of the entries in the determinant in(4.12) and obtain

$$-\prod_{i=1}^{a}(-1)^{b+c+i-1}\frac{(b+c+i-1)!}{(b+i-1)!}$$
$$\times\det_{1\le i,j\le a+1}\left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!}\binom{i-j-1}{b+c+i-1} & \sum_{s=1}^{i}\binom{i-1}{s-1}(-1)^{b+c+s-1}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-2}{b+s-x-2} \\ \hline \binom{c+x-y+j-2}{c-y+j-2} & 0 \\ 1\le j\le a & j=a+1 \end{array}\right)\begin{array}{l}1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.13}$$

The entry in the i-th row and j-th column, $1\le i\le a$ and $1\le j\le a$, develops from the corresponding entry in (4.12) by using Vandermonde's summation formula:

$$\sum_{s=1}^{i}\binom{i-1}{s-1}\frac{(j-1)!}{(c+j-1)!}\binom{-j}{b+c+s-1}=\frac{(j-1)!}{(c+j-1)!}\binom{i-j-1}{b+c+i-1}. \tag{4.14}$$

Next we apply the elementary identity

$$\binom{n}{k}=\binom{-n+k-1}{k}(-1)^k \tag{4.15}$$

to this entry in the i-th row and j-th column of the matrix underlying the determinant in (4.13), $1\le i\le a$ and $1\le j\le a$, and then use the symmetry of the binomial coefficient.

Finally we take the factor $(-1)^{b+c+i-1}$ out of the i-th row, $1\le i\le a$, and obtain that the expression in (4.13) is equal to

$$-\prod_{i=1}^{a}\frac{(b+c+i-1)!}{(b+i-1)!}$$
$$\times\det_{1\le i,j\le a+1}\left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!}\binom{b+c+j-1}{j-i} & \sum_{s=1}^{i}\binom{i-1}{s-1}(-1)^{i-s}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-2}{b+s-x-2} \\ \hline \binom{c+x-y+j-2}{c-y+j-2} & 0 \\ 1\le j\le a & j=a+1 \end{array}\right)\begin{array}{l}1\le i\le a\\ \\ i=a+1\end{array}. \tag{4.16}$$

We can find that the $a\times a$ submatrix induced by the first a rows and first a columns of the matrix underlying the determinant in (4.16) is an upper triangular matrix, since the binomial coefficient $\binom{n}{k}$ is equal to zero if $k < 0$ for any indeterminant n by its definition.

Next we expand (4.16) along the last row and then along the last column, to obtain

$$-\prod_{i=1}^{a}\frac{(b+c+i-1)!}{(b+i-1)!}\sum_{n=1}^{a}(-1)^{a+1+n}\binom{c+x-y+n-2}{c-y+n-2}$$
$$\times\sum_{m=1}^{a}(-1)^{a+m}\sum_{s=1}^{m}\binom{m-1}{s-1}(-1)^{m-s}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-2}{b+s-x-2}$$
$$\times\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\frac{(j-1)!}{(c+j-1)!}\binom{b+c+j-1}{j-i}\right). \tag{4.17}$$

Then we take the factor $(j-1)!/(c+j-1)!$ out of the j-th column of the remaining determinant. This gives

$$\prod_{i=1}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}$$
$$\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\binom{c+x-y+n-2}{x}\binom{b-x+y+s-2}{y}\binom{m-1}{s-1}$$
$$\times\frac{(b+s-1)!}{(b+c+s-1)!}\frac{(c+n-1)!}{(n-1)!}\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{4.18}$$

Now we compute

$$\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{4.19}$$

This is a determinant of an upper triangular matrix, where the m-th row and n-th column were deleted. Hence, it is only different from zero if $n \le m$. We just assume $n \le m$. Expansion of the determinant along the first $(n-1)$ columns and along the last $(a-m)$ rows yields

$$\det_{\substack{n+1\le j\le m\\ n\le i\le m-1}}\left(\binom{b+c+j-1}{j-i}\right)=\det_{\substack{n\le j\le m-1\\ n\le i\le m-1}}\left(\binom{b+c+j}{j-i+1}\right) \tag{4.20}$$

for the determinant in (4.19).

Now we reduce this determinant to Vandermonde's determinant. To do this, we take $(b+c+j)!/(j-n+1)!$ out of the jth column, $1 \le j \le a$, and $1/(b+c+i-1)!$ out of the ith row, $1 \le i \le a$. Thus, we obtain that the determinant in (4.20) is equal to

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\det_{\substack{1\le j\le m-n\\ 1\le i\le m-n}}\left((j-i+2)_{i-1}\right). \tag{4.21}$$

The entries of this determinant are monic polynomials in j of degree $i-1$, where i and j denote as usual the index of the row and the column of the entry. It is now straightforward to reduce this determinant by appropriate row operations to Vandermonde's determinant,

$$\det_{\substack{1\le i\le m-n\\1\le j\le m-n}}\left((j-i+2)_{i-1}\right)=\det_{\substack{1\le i\le m-n\\1\le j\le m-n}}\left(j^{i-1}\right)=\prod_{i=1}^{m-n-1} i!. \tag{4.22}$$

The last equation was obtained by applying the well-known formula for Vandermonde's determinant to $\det_{1\le i\le m-n,1\le j\le m-n}\left(j^{i-1}\right)$ [13].

Using (4.22) in (4.21), we obtain that the determinant in (4.19) equals

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\prod_{i=1}^{m-n-1} i! = \frac{(b+c+n)_{m-n}}{(m-n)!}.$$

We replace the determinant in (4.18) by its value $(b+c+n)_{m-n}/(m-n)!$. Thus, we obtain the following triple sum for the determinant in (4.5):

$$\begin{aligned}&\left(\prod_{i=2}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}\right)\\ &\quad\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{c+x-y+n-2}{x}\binom{b-x+y+s-2}{y}\binom{m-1}{s-1}\right.\\ &\qquad\qquad\left.\times\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right].\end{aligned}$$

Since

$$\prod_{i=2}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}=\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c},$$

we can get the desired result.

□

4.1.3 The Conjectured Product Formula

The results above can be simplified through computer experiments. I used the *simplify* function in *Maple*, which can simplify complicated equations involving sums and products into simpler forms. If we evaluate $x=c-1$, $y=a-b+c-1$ in the triple sum (see the Lemma 4.1.2) of the last section, we can get a triple sum which counts the number of lozenge tilings of the semi-regular hexagon with side lengths a, b, c, a, b, c in the cyclic order with one *bow-tie* removed at the "special" point. Let $\overline{BT}(a,b,c)$ be the triple sum which is stated in the Lemma 4.1.2 with $x=c-1$ and $y=a-b+c-1$ evaluated. I fixed a and simplified $\overline{BT}(a,b,c)$ in Maple. Using data for $a=1,2,\ldots,7$, I was able to get the following result.

We first define $F(a,b,c)$ as follows when the triangle inequality (4.1) is fulfilled.

$$\begin{aligned}F(a,b,c):=&\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}\frac{bc}{(b+c)!}\frac{(b+c-a-1)!}{(a+b-c-1)!(a-b+c-1)!}\\ &\times\frac{1}{(b+c+1)_{a-1}},\end{aligned}$$

where $H(n)$ denotes the hyperfactorial of n as always.

From the experimental data, I was able to know that the expression $\frac{\overline{BT}(a,b,c)}{F(a,b,c)}$ was always some polynomial in b and c.

For example, for $a = 2$, $\frac{\overline{BT}(a,b,c)}{F(a,b,c)} = b^4 - 2b^2c^2 + c^4 + b^2 + c^2 - 2$ and for $a = 3$, $\frac{\overline{BT}(a,b,c)}{F(a,b,c)} = b^8 - 4b^6c^2 + 6b^4c^4 - 4b^2c^6 + c^8 - 2b^6 + 2b^4c^2 + 2b^2c^4 - 2c^6 + b^4 + 46b^2c^2 + c^4 - 48b^2 - 48c^2 + 48$.

Then I could get a conjecture that $P_a(b,c) := \frac{\overline{BT}(a,b,c)}{F(a,b,c)}$ would be a polynomial in two variables b and c and the degree of this polynomial is $4(a-1)$.

In general, polynomials $P_a(b,c)$'s in two variables can not be factorized.

Example 4.1.3. If we set $a = 3$,

$$\begin{aligned} P_3(b,c) =& b^8 - 4b^6c^2 + 6b^4c^4 - 4b^2c^6 + c^8 - 2b^6 + 2b^4c^2 + 2b^2c^4 - 2c^6 \\ &+ b^4 + 46b^2c^2 + c^4 - 48b^2 - 48c^2 + 48. \end{aligned} \tag{4.23}$$

This polynomial can not be factorized in variables b and c. But we have the restriction on these values : the triangle inequality of a, b, c (4.1). I.e, b can only take one value from $\{c-2, c-1, c, c+1, c+2\}$. And I also assumed $b \geq c$. So if we evaluate $b = c$, $b = c+1$ and $b = c+2$ in (4.23), then we get

$$\begin{aligned} P_3(c,c) &= 48(c-1)^2(c+1)^2\,, \\ P_3(c+1,c) &= 48(c-1)c(c+1)(c+2)\,, \\ P_3(c+2,c) &= 240c(c+1)^2(c+2)\,, \end{aligned}$$

which can be nicely factorized in one variable c.

These experiments led me to change the polynomial $P_a(b,c)$ into a polynomial in only one variable c. To do this, I fixed a and new variable $n := b - c$ such that n is strictly smaller than a. Then the polynomial will be of the form in only one variable c. And it was not too hard identify linear factors in c of degree 1. The hard part was guessing coefficients of polynomials which are depending on a and n. Fortunately, in Theorem 2.5.1 and Theorem 2.5.2, Theresia established the number of lozenge tilings of semi-regular hexagon with one of which side lengths b and c are equal and one bow-tie is missing from the symmetry axis. If we put $s = \frac{a}{2}$ for a even and $s = \frac{a+1}{2}$ for a odd in Theorems 2.5.1 and 2.5.2 respectively, we can get the coefficient of $P_a(c,c)$'s which are dependent on a. From the experimental result I get, I could guess all results in the following Lemma.

Lemma 4.1.4. The polynomial $P_a(b,c) =: P_{a,n}(c)$ is equal to following product formula :

1. a is odd, n is even:

$$\begin{aligned} P_{a,n}(c) = a(a-1)!^2 \binom{a-1}{\frac{a-1}{2}}^2 \left[\frac{(\frac{a}{2})_{\frac{n}{2}}}{(\frac{a-n+2}{2})_{\frac{n}{2}}}\right]^2 \frac{a+n}{a-n} \\ \times \frac{1}{c(c+n)} \prod_{i=\frac{n-a+1}{2}}^{\frac{a+n-1}{2}} (c+i)^2. \end{aligned} \tag{4.24}$$

2. a is odd, n is odd:

$$\begin{aligned} P_{a,n}(c) = a(a-1)!^2 \binom{a-1}{\frac{a-1}{2}}^2 \left[\frac{(\frac{a}{2}+1)_{\frac{n-1}{2}}}{(\frac{a-n+1}{2})_{\frac{n-1}{2}}} \right]^2 \\ \times \frac{1}{c(c+n)(c+\frac{n-a}{2})(c+\frac{a+n}{2})} \prod_{i=\frac{n-a}{2}}^{\frac{a+n}{2}} (c+i)^2. \end{aligned} \tag{4.25}$$

3. a is even, n is even:

$$\begin{aligned} P_{a,n}(c) = (a-1)(a-1)!^2 \binom{a-2}{\frac{a-2}{2}} \binom{a}{\frac{a}{2}} \left[\frac{(\frac{a+1}{2})_{\frac{n}{2}}}{(\frac{a-n+1}{2})_{\frac{n}{2}}} \right]^2 \\ \times \frac{1}{c(c+n)(c+\frac{n-a}{2})(c+\frac{a+n}{2})} \prod_{i=\frac{n-a}{2}}^{\frac{a+n}{2}} (c+i)^2. \end{aligned} \tag{4.26}$$

4. a is even, n is odd:

$$\begin{aligned} P_{a,n}(c) = (a-1)(a-1)!^2 \binom{a-2}{\frac{a-2}{2}} \binom{a}{\frac{a}{2}} \left[\frac{(\frac{a+1}{2})_{\frac{n-1}{2}}}{(\frac{a-n}{2})_{\frac{n-1}{2}}} \right]^2 \frac{a+n}{a-n} \\ \times \frac{1}{c^2} \prod_{i=\frac{n-a+1}{2}}^{\frac{a+n-1}{2}} (c+i)^2. \end{aligned} \tag{4.27}$$

Let $BT(a,b,c)$ be the number of lozenge tilings of a semi-regular hexagon with one bow-tie removed from the "special" point. Then our conjecture can be written as

$$\begin{aligned} BT(a,b,c) = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)} \frac{bc}{(b+c)!} \\ \times \frac{(b+c-a-1)!}{(a+b-c-1)!(a-b+c-1)!} \frac{P_{a,n}(c)}{(b+c+1)_{a-1}}, \end{aligned} \tag{4.28}$$

where $n = b-c < a$.

4.2 An Approach to the Proof of the Product Formula

If we evaluate $x = c-1$, $y = a-b+c-1$ in the determinant (4.5), then the determinant which we should evaluate is :

$$- \det_{1\le i,j\le a+1} \begin{array}{cc} \begin{array}{cc} 1 \le j \le a & j = a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{a-1}{b-c+i+1} \\ \hline \binom{b+c-a-1}{b-a+j-1} & 0 \end{array} \right) & \begin{array}{c} 1 \le i \le a \\ i = a+1 \end{array} \end{array}. \tag{4.29}$$

In the previous section, I used the upper triangulation of the homogeneous submatrix of size $a \times a$. In this section, I will state the result which I got through the diagonalization of the homogeneous submatrix of our main matrix. The reason I used another method to evaluate the matrix is that in this way I could prove that many factors appearing in my conjectured product formulas are actually the factors. I will state this more precisely in the later part.

4.2.1 Diagonalization of the Homogeneous Submatrix

Lemma 4.2.1. If we fix a and $n := b - c$, then the determinant in (4.29) is equal to

$$-\frac{H(b)H(c)}{H(a+b)H(a+c)}(b+c)!^{a-1}(b+c-a-1)! \times \det_{1\le i,j\le a+1} \begin{array}{cc} \left(\begin{array}{c|c} (c+n+1+i-j)_{a-i}(c-i+j+1)_{i-1} & \binom{a-1}{n+i-1} \\ \hline (c+n-a+j)_{a-n}(c-j+1)_{a+n} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1\end{array} \\ \begin{array}{cc} \quad 1\le j\le a \qquad\qquad & j=a+1\end{array} & \end{array}, \tag{4.30}$$

where $H(n)$ stands for the "hyperfactorial" $\prod_{i=0}^{n-1} i!$.

Proof. First we evaluate $b = c + n$ in the determinant (4.29). Then we get :

$$-\det_{1\le i,j\le a+1} \begin{array}{cc} \begin{array}{cc} 1\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{2c+n}{c-i+j} & \binom{a-1}{n+i+1} \\ \hline \binom{2c+n-a-1}{c+n-a+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1\end{array} \end{array}. \tag{4.31}$$

Since $\binom{2c+n}{c-i+j} = \frac{(2c+n)!}{(c-i+j)!(c+n+i-j)!}$, we can factor $\frac{(2c+n)!}{(c-1+j)!(c+n+a-j)!}$ out from j-th column($j = 1..a$). Then the entries in the homogeneous submatrix will be $(c+n+i-j+1)_{a-i}(c-i+j+1)_{i-1}$, because

$$\binom{2c+n}{c-i+j} = \frac{(2c+n)!}{(c-i+j)!(c+n+a-j)!}(c+n+i-j+1)_{a-i}(c-i+j+1)_{i-1}.$$

The entries in the last row will become $(c+n-a+j)_{a-n}(c-j+1)_{a+n}$, after we take out common factor $\frac{(2c+n-a-1)!}{(2c+n)!}$, since

$$\begin{aligned}\binom{2c+n-a-1}{c+n-a+j-1} &= \frac{(2c+n-a-1)!}{(c+n-a+j-1)!(c-j)!} \\ &= \frac{(2c+n)!}{(c-1+j)!(c+n+a-j!)} \cdot \frac{(2c+n-a-1)!}{(2c+n)!} \\ &\quad \times (c+n-a+j)_{a-n} \cdot (c-j+1)_{a+n}\,.\end{aligned}$$

We can easily check that the product of all common factors which we took from the matrix is equal to $\frac{H(b)H(c)}{H(a+b)H(a+c)}(b+c)!^{a-1}(b+c-a-1)!$ and the proof is complete.

$\square$

Remark 4.2.2. The purpose of the Lemma 4.2.1 is to make the matrix which has binomial coefficients as its entries into the matrix which has entries of which forms are polynomials in only one variable c. Since we fixed a and n as constants which are independent of c, the entries in the last column of the matrix in (4.30) are just constants. Remaining entries are polynomials in c. And the degree of the polynomials in the homogeneous submatrix is $a-1$, while the degree of the polynomials in the last row is $2a$. Throughout remaining part of my thesis, I will denote the following matrix in (4.30) by $K(a,n)$:

$$K(a,n) \quad := \quad \begin{array}{c} \begin{array}{cc} \scriptstyle 1\le j\le a & \scriptstyle j=a+1 \end{array} \\ \left(\begin{array}{c|c} (c+n+1+i-j)_{a-i}(c-i+j+1)_{i-1} & \binom{a-1}{n+i-1} \\ \hline (c+n-a+j)_{a-n}(c-j+1)_{a+n} & 0 \end{array}\right) \end{array} \begin{array}{c} \scriptstyle 1\le i\le a \\ \\ \scriptstyle i=a+1 \end{array} . \tag{4.32}$$

Lemma 4.2.3. Let n_i 's and m_i's be polynomials in one variable c defined as follows:

$$n_i := n_i(c) := \frac{(c-i+1)_i}{i!}, \quad \text{for} i \geq 1,$$

and

$$m_i := m_i(c) := (-1)^{i-1}\frac{(c)_i}{i!}, \quad \text{for } i \geq 0.$$

Then for an arbitrary integer $k \geq 1$, the following holds :

$$\sum_{i=1}^{k} n_i m_{k-i} = m_k. \tag{4.33}$$

Proof. From the definition of n_i 's and m_i's, the sum in the equation (4.33) can be also written with the terms in binomial coefficients as follows :

$$\sum_{i=1}^{k} n_i m_{k-i} = \sum_{i=1}^{k}(-1)^{k-i+1}\binom{c}{i}\binom{c+k-i-1}{k-i}. \tag{4.34}$$

We use the elementary identity

$$(-1)^k\binom{z}{k} = \binom{-z+k-1}{k}$$

to the right hand side of (4.34). Then it turns out:

$$\begin{aligned}\sum_{i=1}^{k}(-1)^{k-i+1}\binom{c}{i}\binom{c+k-i-1}{k-i} &= -\sum_{i=1}^{k}\binom{c}{i}\binom{-c-k+i+1+k-i-1}{k-i} \\ &= -\sum_{i=1}^{k}\binom{c}{i}\binom{-c}{k-i} \\ &= -\sum_{i=0}^{k}\binom{c}{i}\binom{-c}{k-i}+\binom{c}{0}\binom{-c}{k}.\end{aligned} \tag{4.35}$$

From the Chu-Vandermonde identity, the sum $\sum_{i=0}^{k}\binom{c}{i}\binom{-c}{k-i}$ is equal to zero. Since $\binom{c}{0}=1$ and $\binom{-c}{k}=(-1)^k\binom{c+k-1}{k}=m_k$ from the definition, the statement holds. □

Theorem 4.2.4. *The determinant of $K(a,n)$ is equal to following determinant:*

$$\det(K(a,n)) = \det\left(\begin{array}{cccccc|c} d_1 & 0 & 0 & \cdots & 0 & h_1 \\ 0 & d_2 & 0 & \cdots & 0 & h_2 \\ 0 & 0 & d_3 & \cdots & 0 & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_a & h_a \\ \hline r_1 & r_2 & r_3 & \cdots & r_a & 0 \end{array}\right). \tag{4.36}$$

Here, for $1 \le i,j \le a$, the entries d_i , h_i and r_j 's are the followings respectively :

$$d_i := d_i(a,n,c) := (i-1)!(2c+n+1)_{a-i}\,, \tag{4.37}$$

$$h_i := h_i(a,n) := \binom{2a-i-1}{a+n-1},$$

and

$$r_j := r_j(a,n,c) := \sum_{k=1}^{a-j+1} m_k q_{j+k-1}\,, \tag{4.38}$$

where

$$p_i := p_i(a,n,c) := (c+n-a+i)_{a-n}(c-i+1)_{a+n}\,, \tag{4.39}$$

$$q_i := q_i(a,n,c) := \sum_{j=1}^{i}(-1)^{i-j}\binom{i-1}{j-1}p_j\,, \tag{4.40}$$

$$m_i := m_i(c) := \frac{(-1)^{i-1}(c)_{i-1}}{(i-1)!}\,. \tag{4.41}$$

Proof. For the diagonailzation of the homogeneous submatrix of the matrix in (4.32), we perform elementary row and column operations. The strategy is almost the same as making the matrix into a triangular matrix as I did in the Lemma 4.1.2. On the first step, we add the $(i+1)$-th row to i-th row, in the order of $i = 1, 2, \ldots, a-1$. Then except for the a-th and $(a+1)$-th row unchanged, remaining rows will be as follows :

$$
\begin{array}{cl}
\begin{array}{cc} \qquad\qquad 1\le j\le a \qquad\qquad & j=a+1 \end{array} & \\
\left(\begin{array}{c|c}
(2c+n+1)(c+n+2+i-j)_{a-i-1}(c-i+j+1)_{i-1} & \binom{a}{n+i} \\ \hline
(c-a+j+1)_{a-1} & \binom{a-1}{n+a-1} \\ \hline
(c+n-a+j)_{a-n}(c-j+1)_{a+n} & 0
\end{array}\right). & \begin{array}{l} 1\le i\le a-1 \\ i=a \\ i=a+1 \end{array}
\end{array} \tag{4.42}
$$

We do this procedure (adding $(i+1)$-th row to i-th row) again for i from 1 to $a-2$. Then the last 3 rows are unchanged and the first $a-2$ rows will change as follows :

$$
\begin{array}{cl}
\begin{array}{cc} \qquad\qquad 1\le j\le a \qquad\qquad & j=a+1 \end{array} & \\
\left(\begin{array}{c|c}
\begin{array}{c}(2c+n+1)(2c+n+2)(c+n+3+i-j)_{a-i-2} \\ \times(c-i+j+1)_{i-1}\end{array} & \binom{a+1}{n+i+1} \\ \hline
(2c+n+1)(c-a+j+2)_{a-2} & \binom{a}{n+a-1} \\ \hline
(c-a+j+1)_{a-1} & \binom{a-1}{n+a-1} \\ \hline
(c+n-a+j)_{a-n}(c-j+1)_{a+n} & 0
\end{array}\right). & \begin{array}{l} 1\le i\le a-2 \\ i=a-1 \\ i=a \\ i=a+1 \end{array}
\end{array} \tag{4.43}
$$

We repeat this procedure of adding successive rows from bottom to top each time starting one row earlier than before, totally $a-1$ times including the two steps above. Then after this procedure, our matrix will be :

$$
\begin{array}{cl}
\begin{array}{cc} \qquad\qquad 1\le j\le a \qquad\qquad & j=a+1 \end{array} & \\
\left(\begin{array}{c|c}
(2c+n+1)_{a-i}(c-i+j+1)_{i-1} & \binom{2a-i-1}{a+n-1} \\ \hline
(c+n-a+j)_{a-n}(c-j+1)_{a+n} & 0
\end{array}\right). & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}
\end{array} \tag{4.44}
$$

So far we only performed elementary row operations to first a rows so that the entries in the last row remained unchanged. Now we start elementary column operations with first a columns. If we let i equal to 1, it is easy to see that the first a entries in the first row are all equal. Therefore, except for the first and last entry, we make all entries in the first row into zeroes by subtracting $(j-1)$-th column from j-th column for $j = a, a-1, \ldots, 2$ in the decreasing order. We first denote the entries in the last row of the matrix (4.44) by p_j's $(j = 1 \ldots a)$:

$$p_j := p_j(a,n,c) := (c+n-a+j)_{a-n}(c-j+1)_{a+n}, \quad 1 \le j \le a. \tag{4.45}$$

Then the matrix turns out to be of the following form:

$$\begin{array}{cc} \begin{array}{ccc} j=1 & 2\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c|c} (2c+n+1)_{a-i}(c-i+2)_{i-1} & \begin{matrix}(i-1)(2c+n+1)_{a-i}\\ \times(c-i+j+1)_{i-2}\end{matrix} & \binom{2a-i-1}{a+n-1} \\ \hline p_1 & p_j - p_{j-1} & 0 \end{array}\right) & \begin{array}{c} 1\le i\le a \\ \\ i=a+1 \end{array} \end{array}. \tag{4.46}$$

We perform this procedure again, i.e. subtract the $(j-1)$-th column from the j-th column in the decreasing order of j from a to 3. Then we will get:

$$\begin{array}{cc} \begin{array}{cccc} j=1 & j=2 & 3\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c|c|c} \begin{matrix}(2c+n+1)_{a-i}\\ \times(c-i+2)_{i-1}\end{matrix} & \begin{matrix}(i-1)(2c+n+1)_{a-i}\\ \times(c-i+3)_{i-2}\end{matrix} & \begin{matrix}(i-1)(i-2)(2c+n+1)_{a-i}\\ \times(c-i+j+1)_{i-3}\end{matrix} & \binom{2a-i-1}{a+n-1} \\ \hline p_1 & p_2 - p_1 & p_j - 2p_{j-1} + p_{j-2} & 0 \end{array}\right) & \begin{array}{c} 1\le i\le a \\ \\ i=a+1 \end{array} \end{array}. \tag{4.47}$$

We repeat this procedure of subtracting successive columns from left in the decreasing order of j each time stopping one column earlier than before , totally $a-1$ times including the two steps above.

After these procedures, we will get the following matrix whose homogeneous submatrix of size a is lower-triangular:

$$\left(\begin{array}{c|c} d_{i,j} & h_i \\ \hline q_j & 0 \end{array}\right)_{1\le i,j\le a}, \tag{4.48}$$

where

$$d_{i,j} := \begin{cases} \frac{(i-1)!}{(j-1)!}(2c+n+1)_{a-i}(c-i+j+1)_{i-j} & i \ge j\,, \\ 0 & i < j, \end{cases} \tag{4.49}$$

$$q_j := \sum_{k=1}^{j} (-1)^{j-k} \binom{j-1}{k-1} p_k\,, \tag{4.50}$$

and

$$h_i := \binom{2a-i-1}{a+n-1}.$$

In order to make the homogeneous submatrix of the matrix (4.48) into the diagonal matrix, we perform multiplication of i-th column by some polynomial and subtraction from the j-th column for each j, for $1 \leq j < i$.

First, we let $d_{a,j}$'s ($1 \leq j < a$) be zeroes, which means we should subtract $\frac{1}{(a-j)!}(c-a+j+1)_{a-j}a$-th column from the j-th column. Next, we subtract $\frac{1}{(a-1-j)!}(c-a+j+2)_{a-j-1}(a-1)$-th column from the j-th column, to make $d_{a-1,j}$'s ($1 \leq j \leq a-2$) into zeroes. We keep doing this procedure, i.e. at the k-th step, subtract $\frac{1}{(a-k+1-j)!}(c-a+j+k)_{a-j-k+1}(a-k+1)$-th column from the j-th column to make $d_{a-k+1,j}$'s ($1 \leq j \leq a-k$) into zeroes. Including above 2 steps, we do this procedure $a-1$ times with the decreasing order of k from a to 2.

After all these operations have been done, there will be only diagonal entries in the homogeneous submatrix while all the remaining entries are zeroes and the matrix will be of the form (4.36).

During the column operations above, the entries in the last row have been changed. We denote final entries in the last row by r_j's ($1 \leq j \leq a$). These entries are the linear combinations of q_j's as follows:

$$r_a := q_a \quad ,$$

$$r_{a-1} := q_{a-1} - c r_a \quad ,$$

$$\begin{aligned} r_{a-2} &:= q_{a-2} - c r_{a-1} - \frac{c(c-1)}{2} r_a \\ &= q_{a-2} - c q_{a-1} + \frac{c(c+1)}{2} q_a \quad , \end{aligned}$$

$$\begin{aligned} r_{a-3} &:= q_{a-3} - c r_{a-2} - \frac{c(c-1)}{2} r_{a-1} - \frac{c(c-1)(c-2)}{6} r_a \\ &= q_{a-3} - c q_{a-2} + \frac{c(c+1)}{2} q_{a-1} - \frac{c(c+1)(c+2)}{6} q_a \\ &\dots \end{aligned}$$

From the Lemma 4.2.3, we can easily see that

$$r_j = \sum_{k=1}^{a-j+1} m_k q_{j+k-1} \,, \tag{4.51}$$

where

$$m_j := \frac{(-1)^{j-1}(c)_{j-1}}{(j-1)!} \,.$$

□

Remark 4.2.5. Our purpose is to prove that the determinant (4.29) is equal to the product formula (4.28). I.e.,

$$\begin{aligned} BT(a,b,c) &= \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)} \frac{bc}{(a+b+c)!} \frac{(b+c-a-1)!P_{a,n}(c)}{(a+b-c-1)!(a-b+c-1)!} \\ &= -\frac{H(b)H(c)}{H(a+b)H(a+c)}(b+c)!^{a-1}(b+c-a-1)! \det(K_{a,n}(c)). \end{aligned} \tag{4.52}$$

After cancellation from the both sides, we should prove that

$$\det(K(a,n)) = g(a,n)t_{a,n}(c)s_{a,n}(c)\prod_{i=1}^{a-1}(2c+n+i)^{a-i-1}\,, \tag{4.53}$$

where $g(a,n)$ is the constant term depending on a and n, $t_{a,n}$ is a polynomial in c with degree a and $s_{a,n}$ is also some polynomial in c. From the determinant with diagonalized submatrix (4.36), we can observe that there are $a-i$ rows having the linear factor $(2c+n+i)$ in the homogeneous $a\times a$ submatrix. It means that there are $a-i-1$ linearly independent linear combinations of rows which are zero, when we evaluate $c = \frac{-(n+i)}{2}$. Therefore, $\det(K_{a,n})$ has $(2c+n+i)^{a-i-1}$ as its factor as the polynomial form.

In the equation (4.53), $t_{a,n}(c)$ is the common factor of p_i's $(= (c+n-a+i)_{a-n}(c-i+1)_{a+n})$ for $1\le i\le a$. Since these entries are lying in the same row, when we evaluate the determinant, we can just factor it out from the last row. $t_{a,n}(c)$ is depending on parities of a and n as follows :

$$t_{a,n}(c) = \begin{cases} \prod_{j=-\frac{a-1}{2}}^{\frac{a-1}{2}}(c+\frac{n}{2}+j)\,, & \text{for } a \text{ is odd and } n \text{ is even}\,,\\ \prod_{j=-\frac{a-1}{2}}^{\frac{a-1}{2}}(c+\frac{n-1}{2}+j)\,, & \text{for } a \text{ is odd and } n \text{ is odd}\,,\\ \prod_{j=-\frac{a}{2}}^{\frac{a}{2}-1}(c+\frac{n}{2}+j)\,, & \text{for } a \text{ is even and } n \text{ is even}\,,\\ \prod_{j=-\frac{a}{2}+1}^{\frac{a}{2}}(c+\frac{n-1}{2}+j)\,, & \text{for } a \text{ is even and } n \text{ is odd}\,. \end{cases}$$

The remaining part of my conjecture is dealing with $s_{a,n}(c)$. And I was not able to prove this part. From my conjecture, I can know that

$$s_{a,n}(c) = \begin{cases} \prod_{j=-\frac{a-1}{2}}^{\frac{a-1}{2}}(c+\frac{n}{2}+j)\,, & \text{for } a \text{ is odd and } n \text{ is even}\,,\\ \prod_{j=-\frac{a-1}{2}}^{\frac{a-1}{2}}(c+\frac{n-1}{2}+j+1)\,, & \text{for } a \text{ is odd and } n \text{ is odd}\,,\\ \prod_{j=-\frac{a}{2}}^{\frac{a}{2}-1}(c+\frac{n}{2}+j+1)\,, & \text{for } a \text{ is even and } n \text{ is even}\,,\\ \prod_{j=-\frac{a}{2}+1}^{\frac{a}{2}}(c+\frac{n-1}{2}+j)\,, & \text{for } a \text{ is even and } n \text{ is odd}\,. \end{cases}$$

Remark 4.2.6. Since $t_{a,n}$ is the common factor of p_i's, q_i's are linear combinations of p_i's and r_i's are linear combinations of q_i's, we can define polynomials (p_i')'s, (q_i')'s and (r_i')'s in polynomial in c as follows:

$$p_i' := p_i'(c) := \frac{p_i(c)}{t_{a,n}(c)}\,, \text{ for } \quad 1\le i\le a\,,$$

$$q_i' := q_i'(c) := \frac{q_i(c)}{t_{a,n}(c)}\,, \text{ for } \quad 1\le i\le a\,,$$

$$r_i' := r_i'(c) := \frac{r_i(c)}{t_{a,n}(c)}\,, \text{ for } \quad 1\le i\le a\,.$$

Example 4.2.7. In this example, I will illustrate the procedure thorough which I diagonalize the homogeneous submatrix of $K(3,1)$.

$$K(3,1) = \left(\begin{array}{ccc|c} (c+2)(c+3) & (c+1)(c+2) & c(c+1) & 2 \\ c(c+3) & (c+1)(c+2) & (c+1)(c+2) & 1 \\ (c-1)c & c(c+1) & (c+1)(c+2) & 0 \\ \hline \frac{(c-1)c^2(c+1)}{(c+2)(c+3)} & \frac{(c-1)c^2}{(c+1)^2(c+2)} & \frac{(c-2)(c-1)c}{(c+1)^2(c+2)} & 0 \end{array}\right).$$

As my definitions in the proof of the Theorem 4.2.4, we define polynomials p_i's to be the i-th element in the last row for $1 \leq i \leq 3$, i.e.,

$$\begin{aligned} p_1 &= (c-1)c^2(c+1)(c+2)(c+3) \\ p_2 &= (c-1)c^2(c+1)^2(c+2) \\ p_3 &= (c-2)(c-1)c(c+1)^2(c+2). \end{aligned} \tag{4.54}$$

p_i's have the common factor $(c-1)c(c+1)$. (Actually, there is one more common factor $(c+2)$, but here we follow the definition of $t_{a,n}(c)$ in the Remark 4.2.5). We define this common factor as $t_{3,1}(c)$ and factor it out of the last row. And define $p_i' := p_i/t_{3,1}(c)$.

After we do row operations as we did in the Theorem 4.2.4, then we get

$$K(3,1) = t_{3,1}(c)\left(\begin{array}{ccc|c} (2c+2)(2c+3) & (2c+2)(2c+3) & (2c+2)(2c+3) & 4 \\ (2c+2)c & (2c+2)(c+1) & (2c+2)(c+2) & 1 \\ (c-1)c & c(c+1) & (c+1)(c+2) & 0 \\ \hline p_1' & p_2' & p_3' & 0 \end{array}\right). \tag{4.55}$$

Now we perform column operations with first 3 columns to make the homogeneous submatrix into the lower triangular form. And the resulted matrix will be the following :

$$K(3,1) = t_{3,1}(c)\cdot\left(\begin{array}{ccc|c} (2c+2)(2c+3) & 0 & 0 & 4 \\ (2c+2)c & (2c+2) & 0 & 1 \\ (c-1)c & 2c & 2 & 0 \\ \hline q_1' & q_2' & q_3' & 0 \end{array}\right), \tag{4.56}$$

where $q_i' = q_i/t_{3,1}(c)$ and (q_i)'s are defined as in the Theorem 4.2.4. We again do elementary column operations as in the last part of the Theorem 4.2.4 to obtain the diagonalized form of the homogeneous submatrix :

$$K(3,1) = t_{3,1}(c)\cdot\left(\begin{array}{ccc|c} (2c+2)(2c+3) & 0 & 0 & 4 \\ 0 & (2c+2) & 0 & 1 \\ 0 & 0 & 2 & 0 \\ \hline r_1' & r_2' & r_3' & 0 \end{array}\right), \tag{4.57}$$

where $r_i' = r_i/t_{3,1}(c)$ and (r_i)'s are defined as in the Theorem 4.2.4. In this example, $r_1' = 2c(c+1)(c+2)$, $r_2' = 0$ and $r_3' = -2$ respectively.

4.2.2 Discussion on the Polynomial r_1' and Super ballot Numbers

In this last section, I want to present the polynomial $r_1'(c)$ which we define in the Theorem 4.2.4 and the Remark 4.2.6. From the definition of p_i's, q_i's and r_i's, we can observe that the degree of the polynomial $r_1(c)$ is equal to $2a$ and the degree of the polynomial $t_{a,n}$ is a, which means that the degree of $r_1'(c)$ is $2a - a = a$.

This polynomial has the nice form when we factorize it as follows.

Theorem 4.2.8. *The polynomial $r_1'(c)$ in the matrix $K(a,n)$ after its homogeneous submatrix is diagonalized (see (4.36)) is of the following form :*

$$r_1'(c) = l(a,n)(c)_a \,, \tag{4.58}$$

where $l(a,n)$ is the constant which is depending on a and n.

Proof. We recall the definitions of polynomials p_i's, q_i's and r_i's for $1 \leq i \leq a$.

$$p_i := p_i(a,n,c) := (c+n-a+i)_{a-n}(c-i+1)_{a+n} \,,$$

$$q_i := q_i(a,n,c) := \sum_{j=1}^{i} (-1)^{i-j} \binom{i-1}{j-1} p_j \,,$$

$$r_j := r_j(a,n,c) := \sum_{k=1}^{a-j+1} m_k q_{j+k-1} \,,$$

where

$$m_i := m_i(c) := \frac{(-1)^{i-1}(c)_{i-1}}{(i-1)!} \,.$$

From these definitions, we can see that

$$r_1 = \sum_{k=1}^{a} m_k q_k \,, \tag{4.59}$$

and

$$r_1' = \sum_{k=1}^{a} m_k q_k' \,. \tag{4.60}$$

If we expand this we can observe that

$$r_1' = q_1' + (-c)q_2' + \left(\frac{c(c+1)}{2}\right)q_3' + \left(-\frac{c(c+1)(c+2)}{6}\right)q_4' + \cdots \,. \tag{4.61}$$

If we want to prove r_1' has the polynomial factor $(c+i-1)$, then it is equivalent to prove that $r_1'(-i+1) = 0$ for $1 \leq i \leq a$. Actually it is not hard to see that

$$r_1'(-i+1) = \sum_{k=1}^{i} q_k' = p_i' \,,$$

from the definitions of (m_k)'s, (q_k)'s and (p_k)'s.

Since $p_i' = \frac{p_i}{t(a,n)}$ and from the definitions of (p_i)'s, it is obvious that $p_i'(c)$ has a factor $(c+i-1)$, which means $p_i(-i+1) = 0$, hence $r_1'(-i+1) = 0$. Because the degree of $r_1'(c)$ is a and $r_1'(-i+1)$ is equal to zero for each i from 1 to a, the equation (4.58) holds.

□

$a \setminus n$	0	1	2	3	4	5	6	7	8	9	$\cdots$
1	1										
2	1	3									
3	-2	2	10								
4	-3	-5	5	35							
5	6	-6	-14	14	126						
6	10	14	-14	-42	42	462					
7	-20	20	36	-36	-132	132	1716				
8	-35	-45	45	99	-99	-429	429	6135			
9	70	-70	-110	110	286	-286	-1430	1430	24310		
10	126	154	-154	-286	286	858	-858	-4862	4862	92378	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 4.1: Coefficients of the polynomial $r_1'(c)$ which are depending on a and n.

As the last result of my thesis, I want to argue that the coefficients of $r_1'(c)$: $l(a,n)$'s have remarkable properties. After some computer experiments I could get these numbers as in the table above. If we read entries in the first diagonal ($n = a-1$, colored with red), then these numbers are exactly the binomial coefficients $\binom{2k+1}{k+1}$ from $k = 0$. The entries in the second diagonal (blue ones) are the *Catalan* numbers : $\frac{1}{k+1}\binom{2k}{k}$ from $k = 1$. The fifth (green) and sixth diagonals (purple) respectively are corresponding to the *super ballot* numbers $6\frac{(2k)!}{(k)!(k+2)!}$ ($k \geq 3$) and $60\frac{(2k)!}{k!(k+3)!}$ ($k \geq 3$) respectively. In general, super ballot numbers are defined as $f(r)\frac{(2k)!}{k!(k+r)!}$ for non-negative integer r. Here $f(r)$ is the least positive integer which makes $f(r)\frac{(2k)!}{k!(k+r)!}$ (for arbitrary $k \geq 0$) into integers, since in general $\frac{(2k)!}{k!(k+r)!}$'s are not integers. For example, $f(2) = 6$ and $f(3) = 60$. See [15] for the precise interpretation for super ballot numbers. From computer experiments with increasing a and n, I could get a conjecture that we can find all families of super ballot numbers in the table above (diagonals with positive entries).

4.3 Conclusion

Since I was unable to prove my conjecture completely, I want to achieve following results in my further work.

- Proving the conjecture that coefficients of (r_1')'s are actually the super ballot numbers and looking for the combinatorial meanings of these numbers, if there exist.
- Proving my main conjecture (4.52). This can be done by turning triple sum to hypergeometric summation form and using hypergeometric transformation methods.
- Based on my determinant (4.5) and the triple sum (4.6), maybe I can find more positions to remove a bow-tie so that the triple sum can have a nice form in the product formula. Furthermore, we can also generalize this case to the semi-regular hexagon with a fixed bow-tie which has the size more than one. (Increasing the side lengths of two equilateral triangles sharing one vertex.)

Abstract

In combinatorics, there are many tiling problems as regions and type of tiles can vary. In general, whenever we are given the region to tile and types of tiles, there are 3 things to consider.

1. The tilability of the region : Can one tile the given region with given shapes of tiles?
2. Enumeration of tilings : How many tilings are there?
3. Asymptotic behavior of tilings : What is the asymptotic behavior of the number of tilings, when the region becomes larger?

In this thesis, I mainly focused on the second topic - "*Enumeration problem*". Actually the regions we are dealing with in this thesis, are finite hexagonal regions on the triangular lattice plane for which tilability is guaranteed. There are combinatorial objects such as *plane partition*, *non-intersecting lattice paths*, *perfect matchings* and *semi-strict Gelfand patterns* which are all in bijective correspondence with lozenge tilings. I.e., we can use the enumeration formulas of these objects directly in order to enumerate corresponding lozenge tilings. Furthermore, based on these results, we can also think about hexagonal regions with some holes in it. This thesis deals with tiling problems of such regions and the methods to enumerate them.

Zusammenfassung

In der Kombinatorik gibt es unzählige Kachelungsprobleme, je nach Form des Gebietes und den Typen von Kacheln. Im Allgemeinen, wenn Region und Kacheln gegeben sind, kann man die folgenden drei Aspekte betrachten:

1. Existenz: Gibt es überhaupt eine Kachelung?
2. Abzählung der Kachelungen: Wie viele Kachelungen gibt es?
3. Asymptotisches Verhalten: Was ist das asymptotische Verhalten der Abzählfunktion, wenn die Region größer wird?

In dieser Masterarbeit beschäftige ich mich hauptsächlich mit dem zweiten Thema, der Abzählung. Genauer gesagt beschränken wir uns auf sechseckige Gebiete auf dem Dreiecksgitter für die es immer Kachelungen gibt. Wir geben einen Überblick über einige solche Regionen für die die Abzählung gelungen ist.

Darüber hinaus gibt es verschiedene kombinatorische Objekte wie Plane Partitions, nicht-überschneidende Gitterpunktwege, perfekte Matchings und semi-strikte Gelfand-Tsetlin-Muster, die in Bijektion zu den Tilings stehen. Deshalb ist es daher möglich Resultate über diese verwandten Objekte auf die Tilings zu übertragen. Eine Verallgemeinerung besteht nun darin, dass man auch sogenannte Defekte, das sind Löcher in der Region, betrachtet. Auch auf diese Verallgemeinerung wird in dieser Arbeit eingegangen.

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