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„Continuous and Lipschitz selections of set-valued maps“

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Marlene Hangl, BSc

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## Zusammenfassung

Die Frage, unter welchen Bedingungen stetige Auswahlfunktionen von mengenwertigen Abbildungen existieren, wurde erstmals von Michael in den 1950er-Jahren behandelt. Das klassische Resultat befasst sich mit Abbildungen von parakompakten Räumen in die Familie der abgeschlossenen konvexen Teilmengen eines Banachraums, die eine Form der Stetigkeit erfüllen.

Der Hauptteil dieser Arbeit besteht aus dem Beweis eines Theorems von Shvartsman über die Existenz von Lipschitz-stetigen Auswahlfunktionen von Abbildungen von pseudometrischen Räumen in die Familie der endlichdimensionalen affinen Teilräume eines Banachraums. Der Beweis führt über die Existenz eines Selektors, der jeder endlichdimensionalen kompakten konvexen Teilmenge eines Banachraums einen Punkt in dieser Menge zuordnet und einige Regularitätseigenschaften besitzt. Dies basiert wiederum auf einer Lipschitz-artigen Eigenschaft des Schwerpunktes in Banachräumen.

## Abstract

The question of what conditions are necessary and sufficient for the existence of continuous selections of set-valued maps was first considered by Michael in the 1950s. The classical result deals with maps from paracompact spaces to the family of closed convex subsets of a Banach space that are continuous in a certain sense.

The main part of this thesis consists of proving a theorem of Shvartsman about the existence of Lipschitz continuous selections of maps from pseudometric spaces to the family of finite-dimensional affine subspaces of a Banach space. The proof involves showing the existence of a selector that assigns points to finite-dimensional compact convex subsets of a Banach space and satisfies some regularity properties. This result, on the other hand, is based on a Lipschitz-like property of the barycenter in Banach spaces.

# Contents

# Chapter 1

# Introduction

Let $T$ and $X$ be topological spaces. Assume we are given a map $F$ on $T$ that takes nonempty subsets of $X$ as values. A *selection* of $F$ is then a function $f : T \to X$ such that $f(u)$ is an element of $F(u)$ for all $u \in T$. It is interesting to know under what conditions such a selection can be chosen to be continuous or have even stronger regularity properties. The first important result of this kind goes back to Ernest Michael [8]. We state and prove this theorem in Chapter 2.

In Chapter 3, the main result is a theorem about the existence of Lipschitz continuous selections which is due to Pavel Shvartsman [13, Theorem 1.1]. Let $M$ be a pseudometric space and $X$ a real Banach space. We denote by $\mathcal{A}_m(X)$ the set of all affine subspaces of $X$ that have dimension at most $m$. The theorem can be stated as follows.

**Theorem 3.1.** *Let $F : M \to \mathcal{A}_m(X)$ be a map such that, for every finite subset $N$ of $M$ with cardinality $|N|$ at most $2^{m+1}$, the restriction $F|_N$ of $F$ to $N$ has a Lipschitz continuous selection with Lipschitz constant $1$. Then $F$ has a Lipschitz continuous selection with a Lipschitz constant $\kappa(m)$ depending only on $m$.*

In general, the theorem does not hold if one replaces $2^{m+1}$ with a smaller number, see [12, Remark 5.4].

This selection theorem is related to the problem of characterizing trace spaces of classes of real-valued functions on $\mathbb{R}^n$, see [3] for example. Let $Y$ be a space of functions from $\mathbb{R}^n$ to $\mathbb{R}$ with norm $\|\cdot\|_Y$ and let $C$ be a closed subset of $\mathbb{R}^n$. The *trace* of $Y$ to $C$ is the space $Y|_C := \{f|_C : f \in Y\}$ of functions in $Y$ restricted to $C$.

We say that the trace space $Y|_C$ has the *finiteness property* if there exists a number $K \in \mathbb{N}$ such that the following holds: If $f : C \to \mathbb{R}$ is a function and there exists a collection

$$\{f_{C'} : C' \subseteq C,\, |C'| = K\} \subseteq Y$$

such that $\sup_{C'} \|f_{C'}\|_Y < \infty$ and each $f_{C'}$ coincides with $f$ on $C'$, then $f$ is an element of $Y$.

Let $K_C[Y]$ be the smallest such integer and set

$$K[Y] := \sup_{C \subseteq \mathbb{R}^n,\ C \text{ closed}} K_C[Y].$$

A generalized version of Theorem 3.1 with $X = \mathbb{R}^n$ can then be used to show that $K[C^{1,\omega}(\mathbb{R}^n)] = 3 \cdot 2^{n-1}$. Here, $\omega$ is a modulus of continuity and $C^{1,\omega}(\mathbb{R}^n)$ a subspace of $C^1(\mathbb{R}^n)$, see [3] for the precise definition.

# Chapter 2

# The classical selection theorem by E. Michael

Let $T$ and $X$ again be topological spaces. The set of all nonempty subsets of $X$ is denoted by $2^X$. We say that the map $F : T \to 2^X$ is *lower semi-continuous* if for all open subsets $V$ of $X$ the set $\{u \in T : F(u) \cap V \neq \emptyset\}$ is open in $T$. Another way of expressing this is given in the proposition below [8, Proposition 2.1].

**Proposition 2.1.** *For a map $F : T \to 2^X$ the following are equivalent:*

(i) *$F$ is lower semi-continuous.*

(ii) *If $x$ is an element of $F(u)$ for some $u \in T$ and if $V$ is a neighborhood of $x$ in $X$, then there exists a neighborhood $U$ of $u$ in $T$ such that $F(u') \cap V \neq \emptyset$ for all $u' \in U$.*

*Proof.* To see that (i) implies (ii), assume that $F$ is lower semi-continuous, $x$ is in $F(u)$ for some $u \in T$ and $V$ is a neighborhood of $x$ in $X$. Without loss of generality $V$ is open, otherwise take an open subset containing $x$. Then $U := \{u' \in T : F(u') \cap V \neq \emptyset\}$ is open by assumption and satisfies all requirements.

For the other direction we take some open $V$ in $X$ and show that $U' := \{u \in T : F(u) \cap V \neq \emptyset\}$ is open in $T$. By assumption every $u \in U'$ has a neighborhood $U$ such that $F(u') \cap V \neq \emptyset$ for all $u' \in U$. It follows that $U \subseteq U'$ and hence $U'$ is open. $\square$

In a Banach space $(X, \|\cdot\|)$ the open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$.

The classical theorem of Michael [8, Theorem 3.2”] appeared in 1956; in the same year Michael also published a simpler version of the proof that (i) implies (ii) in the following Theorem 2.2 in [9]. Definitions of the topological properties used here can be found in Appendix A.

**Theorem 2.2** (Michael selection theorem)**.** *For a $T_1$ space $T$ the following are equivalent:*

(i) *$T$ is paracompact.*

(ii) *If $F$ is a lower semi-continuous map from $T$ to the set of all nonempty closed convex subsets of a Banach space $X$, then $F$ has a continuous selection.*

We first prove two auxiliary lemmas.

**Lemma 2.3.** *If $T$ is a paracompact space, $X$ a Banach space and $F$ a lower semi-continuous map from $T$ to the set of all nonempty convex subsets of $X$, then for every $r > 0$ the map $G$ defined by $G(u) = F(u) + B(0, r)$ for all $u \in T$ has a continuous selection.*

*Proof.* For $x \in X$ set

$$U_x := \{u \in T : \, x \in G(u)\} = \{u \in T : \, F(u) \cap B(x, r) \neq \emptyset\}.$$

The collection $\{U_x\}_{x\in X}$ covers $T$ and since $F$ is lower semi-continuous, every $U_x$ is open in $T$. From the paracompactness of $T$ and Proposition A.1 in Appendix A it follows that there is a locally finite partition of unity $\{p_i\}_{i\in I}$, where $I$ is some index set, that is subordinated to the cover $\{U_x\}_{x\in X}$. This implies that for every $i \in I$ there exists a point $x(i)$ in $X$ such that $p_i$ vanishes outside of $U_{x(i)}$. The function $f : T \to X$ defined by

$$f(u) = \sum_{i\in I} p_i(u)x(i)$$

for all $u \in T$ is continuous since every $p_i$ is continuous and the collection $\{p_i\}_{i\in I}$ is locally finite. It remains to show that $f$ is a selection for $G$. For a fixed $u \in T$ the set $J := \{j \in I : \, p_j(u) > 0\}$ is finite. Since $u$ is in $U_{x(j)}$ for all $j \in J$, from the definition of $U_{x(j)}$ it follows that all $x(j)$ are elements of $G(u)$. Hence $f(u)$ is a convex combination of elements of the convex set $G(u)$ and therefore itself contained in $G(u)$. □

**Lemma 2.4.** *If $F : T \to 2^X$ is lower semi-continuous, $f : T \to X$ continuous and $r > 0$ such that $G(u) := F(u) \cap (f(u) + B(0, r))$ is nonempty for all $u \in T$, then $G : T \to 2^X$ is lower semi-continuous.*

*Proof.* Let $V$ be an open subset of $X$. We have to show that the set $U := \{u \in T : G(u) \cap V \neq \emptyset\}$ is open in $T$. It suffices to show that every point $u$ in $U$ has a neighborhood that is contained in $U$. We fix $u \in U$, then there is an $s < r$ such that $F(u) \cap (f(u) + B(0,s)) \cap V \neq \emptyset$. We set

$$U_1 := \Big\{u' \in T : F(u') \cap \Big((f(u) + B(0,s)) \cap V\Big) \neq \emptyset\Big\}$$

and

$$U_2 := \{u' \in T : \|f(u) - f(u')\| < r - s\}.$$

By the lower semi-continuity of $F$ and the continuity of $f$ the sets $U_1$ and $U_2$ are open in $T$. For all $u' \in U_1 \cap U_2$ we have

$$\emptyset \neq F(u') \cap \Big(f(u) + B(0,s)\Big) \cap V \subseteq F(u') \cap \Big(f(u') + B(0,r)\Big) \cap V,$$

so that $U_1 \cap U_2 \subseteq U$. The open set $U_1 \cap U_2$ clearly contains $u$, so that it is the required neighborhood. $\square$

*Proof of Theorem 2.2.* (i) implies (ii): Let $T$ be a paracompact space, $X$ a Banach space and $F$ a map as in (ii). We claim that a continuous selection for $F$ can be approximated by a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions from $T$ to $X$ with the following properties:

$$f_n(u) \in F(u) + B(0, 2^{-n}) \qquad \text{for } n \geq 1, \text{ and} \tag{2.1}$$

$$\|f_n(u) - f_{n-1}(u)\| < 2^{-n+2} \qquad \text{for } n \geq 2 \tag{2.2}$$

for all $u \in T$. The sequence $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy: For a given $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $2^{-N+1} < \varepsilon$. Then by (2.2) for all $i, j \in \mathbb{N}$ such that $N < i < j$ and all $u \in T$ we have

$$\begin{aligned}\|f_i(u) - f_j(u)\| &\leq \|f_i(u) - f_{i+1}(u)\| + \cdots + \|f_{j-1}(u) - f_j(u)\| \\ &\leq 2^{-i+1} + \cdots + 2^{-j+2} \leq 2^{-N+1} < \varepsilon.\end{aligned}$$

Since $X$ is complete, the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to a continuous function $f : T \to X$. From the closedness of $F(u)$ and (2.1) it follows that $f(u) \in F(u)$ for every $u \in T$. Thus, the function $f$ constructed in this way is the required selection.

We show the existence of the functions $f_n$ by induction. By Lemma 2.3 there is an $f_1$ satisfying (2.1). Assume that for some $n \in \mathbb{N}$ we have already found $f_1, \ldots, f_n$ satisfying (2.1) and (2.2). Define

$$F_{n+1}(u) := F(u) \cap \Big(f_n(u) + B(0, 2^{-n})\Big)$$

for all $u \in T$. Then by the induction hypothesis $F_{n+1}(u)$ is nonempty and the map $F_{n+1} : T \to 2^X$ is lower semi-continuous by Lemma 2.4. Since the values of $F_{n+1}$ are convex sets we can apply Lemma 2.3 to get a continuous function $f_{n+1} : T \to X$ such that

$$f_{n+1}(u) \in F_{n+1}(u) + B(0, 2^{-(n+1)})$$

for all $u \in T$. Then $f_{n+1}(u) \in F(u) + B(0, 2^{-(n+1)})$ for all $u$ and hence (2.1) is satisfied. Moreover,

$$f_{n+1}(u) \in f_n(u) + B(0, 2^{-n}) + B(0, 2^{-(n+1)}) \subseteq f_n(u) + B(0, 2^{-n+1})$$

and (2.2) follows.

(ii) implies (i): We assume that the $T_1$ space $T$ satisfies property (ii). In order to show that $T$ is paracompact, by Proposition A.1 in the appendix it is sufficient to prove that every open cover of $T$ has a partition of unity subordinated to it.

Let $\mathcal{U}$ be an open cover of $T$. For a real-valued function $x : \mathcal{U} \to \mathbb{R}$ we define

$$\|x\| := \sum_{U \in \mathcal{U}} |x(U)|,$$

so that the space $\ell_1(\mathcal{U}) := \{x : \mathcal{U} \to \mathbb{R} : \|x\| < \infty\}$ together with the norm $\| \cdot \|$ is a Banach space.

We set

$$C := \Big\{x \in \ell_1(\mathcal{U}) : x(U) \geq 0 \text{ for all } U \in \mathcal{U}, \sum_{U \in \mathcal{U}} x(U) = 1\Big\},$$

then the set $C$ is convex and closed in $\ell_1(\mathcal{U})$. We define a map $F : T \to 2^{\ell_1(\mathcal{U})}$ by setting

$$F(u) := C \cap \{x \in \ell_1(\mathcal{U}) : x(U) = 0 \text{ for all } U \in \mathcal{U} \text{ such that } u \notin U\}$$

for all $u \in T$. Clearly, the values of $F$ are nonempty, closed and convex in $\ell_1(\mathcal{U})$. In order to use assumption (ii), we have to check that $F$ is also lower semi-continuous. By Proposition 2.1 it is sufficient to show the following: If $u \in T$, $x \in F(u)$ and $V$ is a neighborhood of $x$, then there is a neighborhood $U$ of $u$ such that $F(u') \cap V \neq \emptyset$ for all $u' \in U$. Without loss of generality we assume that $V$ is an open ball $B(x, \varepsilon)$ around $x$ for some $\varepsilon > 0$, so that it suffices to show the existence of a point $x' \in F(u') \cap B(x, \varepsilon)$ for all $u'$ in some suitable neighborhood $U$ of $u$.

We pick finitely many sets $U_1, \ldots, U_m \in \mathcal{U}$ such that $x(U_i) > 0$ for all $i \in \{1, \ldots, m\}$ and

$$\delta := x(U_1) + \cdots + x(U_m) > 1 - \frac{\varepsilon}{2}.$$

From the definition of $F$ it follows that $u \in U_i$ for all $i$, so that the set

$$U := U_1 \cap \cdots \cap U_m,$$

is a neighborhood of $u$. For $U \in \mathcal{U}$ we define

$$x'(U) := \begin{cases} x(U) + 1 - \delta & \text{if } U = U_1, \\ x(U) & \text{if } U \in \{U_2, \ldots, U_m\}, \\ 0 & \text{else.} \end{cases}$$

It follows that $x' \in C$ and

$$\|x - x'\| = x'(U_1) - x(U_1) + \sum_{U \in \mathcal{U} \setminus \{U_1, \ldots, U_m\}} x(U) = 1 - \delta + 1 - \delta < \varepsilon.$$

Moreover, $x' \in F(u')$ for all $u' \in U$, which proves the claim that $F$ is lower semi-continuous.

By assumption (ii) of the theorem there exists a continuous selection $f : U \to \ell_1(\mathcal{U})$ of $F$. For every $U \in \mathcal{U}$ we define a function $f_U : T \to [0, 1]$ by setting $f_U(u) := f(u)(U)$ for all $u \in T$. By construction, the $f_U$ are continuous and $\{f_U\}_{U \in \mathcal{U}}$ is a partition of unity on $T$ that is subordinated to the cover $\mathcal{U}$. $\square$

# Chapter 3

# Lipschitz selections of maps taking affine subspaces as values

This chapter closely follows the original paper by Shvartsman [13] from 2004. Unless stated otherwise, all results can be found there.

Let $M$ be some set. A *pseudometric* on $M$ is a map $\rho : M \times M \to [0, +\infty]$ that satisfies

$$\begin{aligned}&\rho(v,v) = 0,\\&\rho(v,w) = \rho(w,v), \quad \text{and}\\&\rho(v,w) \le \rho(v,u) + \rho(u,w)\end{aligned}$$

for all $u, v, w \in M$. This means that $\rho$ is a distance function that may allow two different points to have distance zero from each other. In our case the pseudometric can also assume the value infinity.

In the following, $(X, \|\cdot\|)$ is a real Banach space. A function $f : M \to X$ is *Lipschitz continuous* with Lipschitz constant $\kappa$ if

$$\|f(v) - f(w)\| \le \kappa\, \rho(v,w) \quad \text{for all } v, w \in M.$$

If $L$ is a linear subspace of $X$ and $x$ is a point in $X$, then $\tilde{L} := x + L$ is an *affine subspace* of $X$ that has the same dimension as $L$. The set of all affine subspaces of $X$ with dimension at most $m$ is denoted by $\mathcal{A}_m(X)$ for $m \in \mathbb{N}_0$.

We can now state the main theorem.

**Theorem 3.1.** *Let $F : M \to \mathcal{A}_m(X)$ be a map such that, for every finite subset $N$ of $M$ with cardinality $|N|$ at most $2^{m+1}$, the restriction $F|_N$ of $F$ to $N$ has a Lipschitz continuous selection with Lipschitz constant* 1*. Then $F$*

*has a Lipschitz continuous selection with a Lipschitz constant $\kappa(m)$ depending only on $m$.*

The proof of this theorem involves showing some interesting results about the collection of finite-dimensional compact convex subsets of a Banach space. We introduce some basic notation before outlining the contents of the following sections.

If $\tilde{X}$ is a Banach space containing $X$ as a subspace, then we can clearly replace $\mathcal{A}_m(X)$ with $\mathcal{A}_m(\tilde{X})$ in the statement of the theorem. We make use of this in the following way: Let $U := \{x' \in X' : \|x'\| \leq 1\}$ be the unit ball of the continuous dual space $X'$ of $X$, and let $\ell_\infty(U)$ be the space of bounded real-valued functions on $U$ with the supremum norm. Then $X$ can be identified with a subspace of $\ell_\infty(U)$ by the linear embedding $j : X \to \ell_\infty(U)$ with $j(x)(u) = u(x)$ for all $x \in X$ and all $u \in U$. This embedding is isometric, since

$$\|j(x)\| = \sup_{u \in U} |j(x)(u)| = \|x\|.$$

Therefore, in the proof of the theorem we assume without loss of generality that $X$ is a space of bounded functions. This is used in Lemma 3.15 and Lemma 3.27.

In the following, $\bar{B}(x, r)$ stands for the closed ball with center $x$ and radius $r$ for $x \in X$ and $r \geq 0$. The distance $\operatorname{dist}(x, C)$ of a point $x \in X$ to a subset $C \subseteq X$ is defined as

$$\operatorname{dist}(x, C) = \inf \{\|x - y\| : y \in C\}.$$

The diameter $\operatorname{diam}(C)$ of a set $C$ is given by

$$\operatorname{diam}(C) = \sup_{x, y \in C} \|x - y\|.$$

The *affine span* $\operatorname{affspan}(C)$ of a set $C \subseteq X$ is the smallest affine subspace containing $C$. We define the *dimension* $\dim(C)$ of $C$ to be the dimension of the affine span of $C$.

For $m \in \mathbb{N}_0$, the collection of all nonempty compact convex subsets of $X$ with dimension at most $m$ is denoted by $\mathcal{K}_m(X)$. The collection of all finite-dimensional compact convex subsets of $X$ is $\mathcal{K}(X) := \bigcup_{m=0}^{\infty} \mathcal{K}_m(X)$. For two sets $C_1, C_2 \in \mathcal{K}(X)$ we define their Hausdorff distance

$$\begin{aligned} d_H(C_1, C_2) &:= \inf \left\{ r \geq 0 : C_2 \subseteq C_1 + \bar{B}(0, r),\ C_1 \subseteq C_2 + \bar{B}(0, r) \right\} \\ &= \max \left\{ \sup_{x \in C_2} \operatorname{dist}(x, C_1),\ \sup_{x \in C_1} \operatorname{dist}(x, C_2) \right\}. \end{aligned}$$

Then $(\mathcal{K}(X), d_H)$ is a metric space: For any two sets $C_1, C_2 \in \mathcal{K}(X)$ satisfying $d_H(C_1, C_2) = 0$ it follows that $C_1 = C_2$ by closedness of the sets. The other properties of a metric follow directly from the definition.

A *selector* is a map $S : \mathcal{K}(X) \to X$ such that $S(C) \in C$ for all $C \in \mathcal{K}(X)$.

For $C \in \mathcal{K}(X)$ we can define a Lebesgue measure $\lambda$ on $\operatorname{affspan}(C)$; see section B.1 in the appendix for more details. The measure $\lambda$ is unique up to multiplication by a positive constant. Among all balls with centers in $\operatorname{affspan}(C)$ that contain $C$, we choose a ball $\bar{B}(x, r)$ with minimal radius $r$. Since $C$ is compact and hence bounded, such a ball exists. The *regularity coefficient* $\delta_C$ of $C$ is then defined as

$$\delta_C = \frac{\lambda\Big(\bar{B}(x, r) \cap \operatorname{affspan}(C)\Big)}{\lambda(C)}. \tag{3.1}$$

The regularity coefficient does not depend on the choice of the Lebesgue measure or the choice of the ball $\bar{B}(x, r)$. The definition implies $\delta_C \geq 1$, and for zero-dimensional sets $C$ (one-point subsets of $X$) we always have $\delta_C = 1$.

The following definition makes use of the Bochner integral, which is defined in section B.2 of the appendix. The *barycenter* (center of mass) of a set $C \in \mathcal{K}(X)$ is given by

$$b(C) := \frac{1}{\lambda(C)} \int_C x \,\mathrm{d}\lambda(x),$$

where again $\lambda$ is a Lebesgue measure on $\operatorname{affspan}(C)$. The barycenter is an element of $C$ and does not depend on the choice of the Lebesgue measure. Furthermore, the barycentric map $b(\cdot)$ commutes with one-to-one affine maps; this means that

$$b(AC) = Ab(C)$$

for each $C \in \mathcal{K}(X)$ and every map $A : \operatorname{affspan}(C) \to X$ that is the composition of a linear map and a translation and satisfies $\dim(AC) = \dim(C)$.

Section 3.1 contains a series of geometrical results involving Kolmogorov widths and ellipsoids. These results are used in Sections 3.2 and 3.3.

In general, the barycentric map $b : (\mathcal{K}_m(X), d_H) \to (X, \|\cdot\|)$ is not continuous for $m \geq 2$. (Discontinuities can occur when approximating a set by sets of higher dimensions.) In Section 3.2, however, we show that the barycenter is Lipschitz continuous on the subset $\{C \in \mathcal{K}_m(X) : \delta_C \leq \alpha\}$ for every $\alpha \geq 1$.

This result is used in Section 3.3 to show the existence of a selector $S_X$ on $\mathcal{K}_m(X)$ that is Lipschitz continuous and commutes with translations and multiplications by real scalars.

Section 3.4 concludes the proof of Theorem 3.1. We construct a suitable map $G : M \to \mathcal{K}_m(X)$ such that $G(v) \subseteq F(v)$ for every $v \in M$. Then we use the selector $S_X$ from the previous section to define the required selection $f : M \to X$ by setting $f(v) := S_X(G(v))$ for all $v \in M$.

## 3.1 Kolmogorov widths and ellipsoids

Many of the following proofs make use of *Kolmogorov n-widths*. For $n \in \mathbb{N}_0$ and $C \in \mathcal{K}(X)$ the Kolmogorov $n$-width of $C$ in $X$ is given by

$$d_n(C;X) := \inf_{L \in \mathcal{A}_n(X)} \inf\{r \geq 0 : \, C \subseteq L + \bar{B}(0,r)\}. \tag{3.2}$$

The definition implies that $d_n(C) \geq d_{n+1}(C)$ for all $n$ and $d_n(C) = 0$ whenever $n \geq \dim(C) = \dim(\operatorname{affspan}(C))$. If $Y$ is an affine subspace of $X$ and $C \subseteq Y$, then by abuse of notation we define the Kolmogorov $n$-width of $C$ in $Y$ as

$$d_n(C;Y) = \inf_{\substack{L \in \mathcal{A}_n(X) \\ L \subseteq Y}} \inf\{r \geq 0 : \, C \subseteq L + \bar{B}(0,r)\}.$$

**Proposition 3.2.** *For all $C \in \mathcal{K}(X)$ and all $n$ we have*

$$d_n(C;X) = \inf\{d_H(C,K) : \, K \in \mathcal{K}(X), \, \dim(K) = n\}.$$

*Proof.* Assume that we have $L \in \mathcal{A}_n(X)$ and $r \geq 0$ such that $C \subseteq L + \bar{B}(0,r)$. Without loss of generality we can assume that $\dim(L) = n$. For a given $\varepsilon > 0$ we set

$$K := L \cap \Big(C + \bar{B}(0, r+\varepsilon)\Big).$$

The set $K$ is closed, bounded and finite-dimensional and hence compact. It is also convex and thus an element of $\mathcal{K}(X)$. To show that the dimension of $K$ is $n$ we show that $K$ has an $n$-dimensional subset. By the definition of $L$ and $r$ there exists a point $x$ in $L \cap \Big(C + \bar{B}(0,r)\Big)$. We have $\bar{B}(x,\varepsilon) \subseteq C + \bar{B}(0, r+\varepsilon)$ and therefore

$$\bar{B}(x,\varepsilon) \cap L \subseteq K.$$

The set $\bar{B}(x,\varepsilon) \cap L$ has dimension $n$ and hence also $K$ has dimension $n$.

If $y \in C$ then again by the definition of $L$ and $r$ there is a $y' \in L$ such that $\|y - y'\| \leq r$. This implies that $y'$ is an element of $K$ and thus $C \subseteq K + \bar{B}(0,r)$. By the definition of $K$ we have that $K \subseteq C + \bar{B}(0, r+\varepsilon)$, and hence $d_H(C,K) \leq r + \varepsilon$. This holds for all choices of $\varepsilon$, therefore

$$\inf\{d_H(C,K) : \, K \in \mathcal{K}(X), \, \dim(K) = n\} \leq d_n(C;X).$$

To show the other direction we take a $K \in \mathcal{K}(X)$ with $\dim(K) = n$ and set $r := d_H(C, K)$. Then $L := \operatorname{affspan}(K)$ has dimension $n$ and $C \subseteq L + \bar{B}(0, r)$. From this

$$d_n(C; X) \leq \inf\{d_H(C, K) : \, K \in \mathcal{K}(X), \, \dim(K) = n\}.$$

follows. □

**Corollary 3.3.** *The Kolmogorov $n$-width $d_n(\,\cdot\,; X)$ is Lipschitz continuous on $(\mathcal{K}(X), d_H)$. More precisely, we have*

$$|d_n(C_1; X) - d_n(C_2; X)| \leq d_H(C_1, C_2)$$

*for all $C_1, C_2 \in \mathcal{K}(X)$.*

*Proof.* If $C_1, C_2 \in \mathcal{K}(X)$ and $K \in \mathcal{K}(X)$ with dimension $\dim(K) = n$, then by Proposition 3.2 and the triangle inequality for the metric $d_H$ we have

$$d_n(C_1; X) \leq d_H(C_1, K) \leq d_H(C_1, C_2) + d_H(C_2, K).$$

Since this holds for all such $K$, it follows that

$$\begin{aligned} d_n(C_1; X) &\leq d_H(C_1, C_2) + \inf\{d_H(C_2, K) : \, K \in \mathcal{K}(X), \, \dim(K) = n\} \\ &= d_H(C_1, C_2) + d_n(C_2; X). \end{aligned}$$

The statement is symmetric in $C_1$ and $C_2$, so we can conclude that

$$|d_n(C_1; X) - d_n(C_2; X)| \leq d_H(C_1, C_2)$$

holds. □

If $C \in \mathcal{K}(X)$, $x \in \operatorname{affspan}(C)$ and $t > 0$, then we denote by $t \circ_x C$ the dilation of $C$ with respect to the center $x$ by the factor $t$, that is,

$$t \circ_x C = t\,(C - x) + x.$$

In the Euclidean space $\mathbb{R}^m$, we denote the closed ball with center $x \in \mathbb{R}^m$ and radius $r \geq 0$ by $\bar{B}_{\mathbb{R}^m}(x, r)$. If $Y$ is an affine subspace of $X$, $y \in Y$ and $r \geq 0$, then we set $\bar{B}_Y(y, r) := \bar{B}(y, r) \cap Y$.

Let $Y$ be an $m$-dimensional affine subspace of $X$ and $A : \mathbb{R}^m \to Y$ be an invertible affine map. We call

$$E := A\bar{B}_{\mathbb{R}^m}(0, 1)$$

an $m$-dimensional *ellipsoid* with center $A0$.

There are two important results for finite-dimensional compact convex sets that we use. The first one was originally shown by Minkowski and can be found in the book by Bonnesen and Fenchel [2].

**Proposition 3.4.** *Let $C$ be an $m$-dimensional compact convex subset of $\mathbb{R}^m$. The barycenter $b(C)$ of $C$ satisfies*

$$\bar{B}_{\mathbb{R}^m}\Big(b(C), \frac{2}{m+1}\, d_{m-1}(C;\mathbb{R}^m)\Big) \subseteq C \subseteq \bar{B}_{\mathbb{R}^m}\Big(b(C), \frac{m}{m+1}\operatorname{diam}(C)\Big).$$

The second important result is due to John [6] and can also be found in [5, Theorem 12.11]. It was originally shown for subsets of $\mathbb{R}^m$, but the result translates directly to finite-dimensional subsets of a Banach space.

**Proposition 3.5.** *For every $C \subseteq \mathcal{K}_m(X)$ with $\dim(C) = m$ there is an ellipsoid $E$ such that*

$$E \subseteq C \subseteq m \circ_z E,$$

*where $z$ is the center of the ellipsoid $E$. If $C$ is symmetric with respect to one of its points, then the center of $C$ is also the center of the ellipsoid $E$, and the dilation factor $m$ can be replaced with $\sqrt{m}$.*

**Corollary 3.6.** *Let $Y$ be an $m$-dimensional subspace of $X$. There is a linear isomorphism $A : \mathbb{R}^m \to Y$ such that*

$$A\bar{B}_{\mathbb{R}^m}(0,1) \subseteq \bar{B}_Y(0,1) \subseteq A\bar{B}_{\mathbb{R}^m}(0,\sqrt{m}).$$

*In particular, there is a linear isomorphism between $\mathbb{R}^m$ and $Y$ satisfying $\|A\| \leq 1$ and $\|A^{-1}\| \leq \sqrt{m}$.*

*Proof.* Since $\bar{B}_Y(0,1)$ is symmetric, by Proposition 3.5 there is an ellipsoid $E \subseteq Y$ with center $0 \in Y$ such that

$$E \subseteq \bar{B}_Y(0,1) \subseteq \sqrt{m}E.$$

The ellipsoid $E$ is the image of $\bar{B}_{\mathbb{R}^m}(0,1)$ under an affine map $A : \mathbb{R}^m \to Y$; since the center of $E$ is the origin, the map $A$ is linear and the inclusion relations follow. The bounds for the norms of $A$ and $A^{-1}$ are an immediate consequence. $\square$

**Proposition 3.7.** *For every $C$ in $\mathcal{K}(X)$ with dimension $m \geq 1$ we have*

$$\bar{B}\Big(b(C), \frac{1}{\alpha(m)}\, d_{m-1}(C;X)\Big) \cap \operatorname{affspan}(C) \subseteq C,$$

*where $\alpha(m) \geq 1$ is a constant that depends only on $m$.*

*Proof.* Without loss of generality $Y := \operatorname{affspan}(C)$ is a linear subspace of $X$ with origin $0 = b(C)$. By Corollary 3.6 there is a linear isomorphism $A : \mathbb{R}^m \to Y$ such that

$$A\bar{B}_{\mathbb{R}^m}(0, r) \subseteq \bar{B}_Y(0, r) \subseteq A\bar{B}_{\mathbb{R}^m}(0, \sqrt{m}\, r) \tag{3.3}$$

for all $r \geq 0$.

We set $\tilde{C} := A^{-1}C$, then $b(\tilde{C}) = A^{-1}b(C) = 0$. From Proposition 3.4 it hence follows that

$$\bar{B}_{\mathbb{R}^m}\Big(0, \frac{2}{m+1}\, d_{m-1}(\tilde{C}; \mathbb{R}^m)\Big) \subseteq \tilde{C}. \tag{3.4}$$

If there are $L \in \mathcal{A}_{m-1}(\mathbb{R}^m)$ and $r \geq 0$ such that $\tilde{C} \subseteq L + \bar{B}_{\mathbb{R}^m}(0, r)$, then from (3.3) it follows that $C = A\tilde{C} \subseteq AL + \bar{B}_Y(0, r)$. Therefore,

$$d_{m-1}(C; Y) \leq d_{m-1}(\tilde{C}; \mathbb{R}^m).$$

Thus, by (3.3) and (3.4), this leads to

$$\begin{aligned} A^{-1}\bar{B}_Y\Big(0, \frac{1}{\sqrt{m}} \frac{2}{m+1}\, d_{m-1}(C; Y)\Big) &\subseteq A^{-1}\bar{B}_Y\Big(0, \frac{1}{\sqrt{m}} \frac{2}{m+1}\, d_{m-1}(\tilde{C}; \mathbb{R}^m)\Big) \\ &\subseteq \bar{B}_{\mathbb{R}^m}\Big(0, \frac{2}{m+1}\, d_{m-1}(\tilde{C}; \mathbb{R}^m)\Big) \\ &\subseteq \tilde{C}. \end{aligned}$$

Clearly, $d_{m-1}(C; X) \leq d_{m-1}(C; Y)$, so by applying $A$ we get

$$\bar{B}_Y\Big(0, \frac{1}{\alpha(m)}\, d_{m-1}(C; X)\Big) \subseteq C$$

for $\alpha(m) = \sqrt{m}(m + 1)/2$. $\square$

**Proposition 3.8.** *Assume that we have a compact convex set $C \in \mathcal{K}(X)$ with dimension $\dim(C) = m$. Then there is an $m$-dimensional ellipsoid $E_b$ with center $b(C)$ such that*

$$E_b \subseteq C \subseteq \beta(m) \circ_{b(C)} E_b.$$

*The constant $\beta(m)$ depends only on $m$.*

*Proof.* By Proposition 3.5, there exists an $m$-dimensional ellipsoid $E \subseteq X$ with center $z \in E$ such that

$$E \subseteq C \subseteq m \circ_z E.$$

There is an affine isomorphism $A : \mathbb{R}^m \to \operatorname{affspan}(C)$ such that $A\bar{B}_{\mathbb{R}^m}(0,1) = E$. Thus, for $\tilde{C} := A^{-1}C$ we have

$$\bar{B}_{\mathbb{R}^m}(0,1) \subseteq \tilde{C} \subseteq \bar{B}_{\mathbb{R}^m}(0,m),$$

and hence $1 \leq d_{m-1}(\tilde{C};\mathbb{R}^m)$ and $\operatorname{diam}\tilde{C} \leq 2m$. From Proposition 3.4 it follows that

$$\bar{B}_{\mathbb{R}^m}\Big(b(\tilde{C}), \frac{2}{m+1}\Big) \subseteq \tilde{C} \subseteq \bar{B}_{\mathbb{R}^m}\Big(b(\tilde{C}), \frac{2m^2}{m+1}\Big).$$

Therefore, the ellipsoid $E_b := A\bar{B}_{\mathbb{R}^m}\big(b(\tilde{C}), 2(m+1)^{-1}\big)$ with center $Ab(\tilde{C}) = b(C)$ is as required with $\beta(m) = m^2$. $\square$

Assume that the points $y_0, \dots, y_m \in X$ are affinely independent, meaning that $\dim(\operatorname{affspan}\{y_0, \dots, y_m\}) = m$. We call the convex hull $D := \operatorname{conv}(\{y_0, \dots, y_m\})$ an $m$-dimensional *simplex* with vertices $y_0, \dots, y_m$. The barycenter of this simplex is

$$b(D) = \frac{1}{m+1}\,(y_0 + \cdots + y_m).$$

**Proposition 3.9.** *Let $E$ be an $m$-dimensional ellipsoid in $X$, where $m \geq 1$, and let $z$ be the center of $E$. There is an $m$-dimensional simplex $D$ such that*

$$\frac{1}{m} \circ_z E \subseteq D \subseteq E$$

*and the barycenter of $D$ is $z$.*

*Proof.* Let $A : \mathbb{R}^m \to \operatorname{affspan}(E)$ be an invertible affine map such that $A\bar{B}_{\mathbb{R}^m}(0,1) = E$. We pick points $y_0, \dots, y_m$ in the boundary of $\bar{B}_{\mathbb{R}^m}(0,1)$ such that their convex hull $\operatorname{conv}(\{y_0, \dots, y_m\})$ is a regular simplex with its barycenter in the origin. For a regular $m$-dimensional simplex in $\mathbb{R}^m$, the ratio of the radius of the minimal circumscribed ball to the radius of the maximal inscribed ball is $m$, therefore we have

$$\bar{B}_{\mathbb{R}^m}\Big(0, \frac{1}{m}\Big) \subseteq \operatorname{conv}(\{y_0, \dots, y_m\}) \subseteq \bar{B}_{\mathbb{R}^m}(0,1).$$

We set $D := A\operatorname{conv}(\{y_0, \dots, y_m\}) = \operatorname{conv}(\{Ay_0, \dots, Ay_m\})$ and get

$$\frac{1}{m} \circ_z E = A\bar{B}_{\mathbb{R}^m}\Big(0, \frac{1}{m}\Big) \subseteq D \subseteq A\bar{B}_{\mathbb{R}^m}(0,1) = E$$

as required. Moreover, the barycenter of $D$ is

$$b(D) = \frac{1}{m+1}(Ay_0 + \cdots + Ay_m) = A\Big(\frac{1}{m+1}(y_0 + \cdots + y_m)\Big) = A0 = z,$$

where $z$ is the center of $E$. $\square$

**Proposition 3.10.** *Assume that $C \in \mathcal{K}(X)$ with dimension $\dim(C) = m$ for some $m \geq 1$. There is a positive constant $\xi(m)$ depending only on $m$ such that the regularity coefficient of $C$ can be estimated by*

$$\frac{1}{\xi(m)}\,\delta_C \leq \frac{(\operatorname{diam}(C))^m}{\prod_{n=0}^{m-1} d_n(C;X)} \leq \xi(m)\,\delta_C.$$

*Proof.* We first prove the proposition for an $m$-dimensional ellipsoid $E \subseteq X$. Without loss of generality the center of $E$ is the origin, so that $E = -E$. Let $n \in \mathbb{N}_0$ and assume that there are an affine subspace $L \in \mathcal{A}_n(X)$ and a scalar $r \geq 0$ such that $E \subseteq L + \bar{B}(0,r)$. Let $y$ be a point in $L$, then $L - y$ is a linear subspace of $X$ and we therefore have $L - y = -L + y$. Moreover, it follows that

$$E = -E \subseteq -L + \bar{B}(0,r) = L - 2y + \bar{B}(0,r).$$

Thus, with the linear subspace $\tilde{L} := L - y$ we get

$$E \subseteq \Big(L + \bar{B}(0,r)\Big) \cap \Big(L - 2y + \bar{B}(0,r)\Big) \subseteq \tilde{L} + \bar{B}(0,r).$$

It is therefore sufficient to consider only linear subspaces of $X$ in the definition of the Kolmogorov $n$-width of $E$.

We set $Y := \operatorname{affspan}(E)$. By the Kadets-Snobar theorem (see [1, Theorem 13.1.7], for example) there is a linear projection $P_Y$ of $X$ onto $Y$ with norm $\|P_Y\| \leq \sqrt{m}$. If $r \geq 0$ and $L$ is a linear subspace of $X$ with dimension at most $m$ such that $E \subseteq L + \bar{B}(0,r)$, then $L' := P_Y L \subseteq Y$ and $E \subseteq L' + \bar{B}(0, \sqrt{m}r)$. Thus,

$$d_n(E;X) \leq d_n(E;Y) \leq \sqrt{m}\, d_n(E;X) \tag{3.5}$$

for all $n$ and it suffices to consider the finite-dimensional space $Y$.

In the next step we show that we get equivalent values if we replace $E$ with a suitable ellipsoid $\tilde{E}$ in the Euclidean space $\mathbb{R}^m$. By Corollary 3.6 there is a linear isomorphism $A : \mathbb{R}^m \to Y$ such that

$$A\bar{B}_{\mathbb{R}^m}(0,r) \subseteq \bar{B}_Y(0,r) \subseteq A\bar{B}_{\mathbb{R}^m}(0, \sqrt{m}r) \tag{3.6}$$

for all $r \geq 0$. We put $\tilde{E} := A^{-1}E$. Since we assume that $E$ has its center in the origin, the same follows for $\tilde{E}$. Using (3.6), it can be calculated that

$$\frac{1}{\sqrt{m}}\, d_n(\tilde{E};\mathbb{R}^m) \leq d_n(E;Y) \leq d_n(\tilde{E};\mathbb{R}^m) \tag{3.7}$$

if $0 \leq n \leq m-1$, and

$$\frac{1}{\sqrt{m}} \operatorname{diam}(\tilde{E}) \leq \operatorname{diam}(E) \leq \operatorname{diam}(\tilde{E}). \tag{3.8}$$

We define a Lebesgue measure $\lambda_Y$ on $Y$ by setting $\lambda_Y(C) := \lambda_{\mathbb{R}^m}(A^{-1}C)$ for all Borel sets $C \subseteq Y$. By the definition of the regularity coefficient (3.1), (3.6) and (3.7) we have

$$\begin{aligned}\delta_{\tilde{E}} &= \frac{\lambda_{\mathbb{R}^m}\Big(\bar{B}_{\mathbb{R}^m}(0, d_0(\tilde{E}; \mathbb{R}^m))\Big)}{\lambda_{\mathbb{R}^m}(\tilde{E})} \le \frac{\lambda_{\mathbb{R}^m}\Big(\bar{B}_{\mathbb{R}^m}(0, \sqrt{m}\, d_0(E;Y))\Big)}{\lambda_{\mathbb{R}^m}(\tilde{E})} \\ &\le \frac{\lambda_Y\Big(\bar{B}_Y(0, \sqrt{m}\, d_0(E;Y))\Big)}{\lambda_Y(E)} = \sqrt{m}^m\, \delta_E.\end{aligned}$$

In a similar way, it follows that $\delta_E \le \sqrt{m}^m\, \delta_{\tilde{E}}$, so that

$$\frac{1}{\sqrt{m^m}}\, \delta_{\tilde{E}} \le \delta_E \le \sqrt{m^m}\, \delta_{\tilde{E}}. \tag{3.9}$$

By the principal axis theorem, the principal axes of an ellipsoid in $\mathbb{R}^m$ are perpendicular, so that after the application of a suitable rotation the ellipsoid $\tilde{E}$ is given by

$$\tilde{E} = \Big\{x = (x_1, \dots, x_m) \in \mathbb{R}^m : \sum_{n=1}^{m} \frac{x_n^2}{s_n^2} \le 1\Big\},$$

where $s_1 \ge \cdots \ge s_m > 0$. It can be shown that $d_n(\tilde{E}; \mathbb{R}^m) = s_{n+1}$ if $0 \le n \le m-1$ (see [14, Section 8.1.4] for more details). Moreover, $\tilde{E}$ has diameter $2s_1$ and volume $\lambda_{\mathbb{R}^m}(\tilde{E}) = s_1 \cdots s_m \lambda_{\mathbb{R}^m}(\bar{B}_{\mathbb{R}^m}(0,1))$. The ball $\bar{B}_{\mathbb{R}^m}(0, s_1)$ is the ball with minimal radius containing $\tilde{E}$. Hence,

$$\delta_{\tilde{E}} = \frac{\lambda_{\mathbb{R}^m}\Big(\bar{B}_{\mathbb{R}^m}(0, s_1)\Big)}{\lambda_{\mathbb{R}^m}(\tilde{E})} = \frac{s_1^m \lambda_{\mathbb{R}^m}\Big(\bar{B}_{\mathbb{R}^m}(0,1)\Big)}{s_1 \cdots s_m \lambda_{\mathbb{R}^m}\Big(\bar{B}_{\mathbb{R}^m}(0,1)\Big)} = \frac{2^{-m}(\mathrm{diam}(\tilde{E}))^m}{\prod_{n=0}^{m-1} d_n(\tilde{E}; \mathbb{R}^m)}.$$

For the ellipsoid $E \subseteq Y$ by (3.7), (3.8) and (3.9) we hence get

$$\begin{aligned}\frac{2^m}{m^m}\, \delta_E &\le \frac{2^m}{\sqrt{m^m}}\, \delta_{\tilde{E}} = \frac{1}{\sqrt{m^m}} \frac{(\mathrm{diam}(\tilde{E}))^m}{\prod_{n=0}^{m-1} d_n(\tilde{E}; \mathbb{R}^m)} \le \frac{(\mathrm{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E;Y)} \\ &\le \sqrt{m^m} \frac{(\mathrm{diam}(\tilde{E}))^m}{\prod_{n=0}^{m-1} d_n(\tilde{E}; \mathbb{R}^m)} = 2^m \sqrt{m^m}\, \delta_{\tilde{E}} \le 2^m m^m\, \delta_E.\end{aligned}$$

From (3.5) it now follows that

$$\frac{2^m}{m^m} \delta_E \le \frac{(\mathrm{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E;X)} \le 2^m m^m \sqrt{m^m}\, \delta_E, \tag{3.10}$$

which proves the proposition for ellipsoids.

We now look at the general case of an $m$-dimensional set $C \in \mathcal{K}(X)$. By Proposition 3.5 there is an ellipsoid $E$ such that

$$E \subseteq C \subseteq m \circ_z E,$$

where $z$ is the center of $E$. Since $\operatorname{diam}(m \circ_z E) = m \operatorname{diam}(E)$, we have

$$\operatorname{diam}(E) \leq \operatorname{diam}(C) \leq m \operatorname{diam}(E).$$

Moreover, since $d_n(m \circ_z E; X) = m\, d_n(E; X)$ for all $n$, it follows that

$$d_n(E; X) \leq d_n(C; X) \leq m\, d_n(E; X).$$

Thus,

$$\frac{1}{m^m} \frac{(\operatorname{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E; X)} \leq \frac{(\operatorname{diam}(C))^m}{\prod_{n=0}^{m-1} d_n(C; X)} \leq m^m \frac{(\operatorname{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E; X)}. \tag{3.11}$$

Furthermore, for every Lebesgue measure $\lambda$ on $\operatorname{affspan}(C)$ we have

$$\lambda(E) \leq \lambda(C) \leq \lambda(m \circ_z E) = m^m \lambda(E).$$

Let $\bar{B}(x, r)$ be a ball with center $x$ in $\operatorname{affspan}(C)$ and minimal radius $r$ such that $C \subseteq \bar{B}(x, r)$ and let $s$ be the minimal radius such that $E \subseteq \bar{B}(z, s)$, then $s \leq r \leq ms$ and by the definition of the regularity coefficient we get

$$\begin{aligned} \frac{1}{m^m} \delta_E = \frac{1}{m^m} \frac{\lambda\Big(\bar{B}(z, s) \cap \operatorname{affspan}(C)\Big)}{\lambda(E)} &\leq \frac{\lambda\Big(\bar{B}(x, r) \cap \operatorname{affspan}(C)\Big)}{\lambda(C)} \\ = \delta_C \leq \frac{\lambda\Big(\bar{B}(z, ms) \cap \operatorname{affspan}(C)\Big)}{\lambda(E)} &= m^m \delta_E. \end{aligned}$$

Hence, by (3.10) and (3.11),

$$\begin{aligned} \frac{2^m}{m^{3m}}\, \delta_C \leq \frac{2^m}{m^{2m}}\, \delta_E \leq \frac{1}{m^m} \frac{(\operatorname{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E; X)} &\leq \frac{(\operatorname{diam}(C))^m}{\prod_{n=0}^{m-1} d_n(C; X)} \\ \leq m^m \frac{(\operatorname{diam}(E))^m}{\prod_{n=0}^{m-1} d_n(E; X)} \leq 2^m m^{5m/2}\, \delta_E &\leq 2^m m^{7m/2}\, \delta_C. \end{aligned}$$

This implies the required result for $\xi(m) = 2^m m^{7m/2}$. □

## 3.2 The barycentric map

**Theorem 3.11.** *For every $m \in \mathbb{N}_0$ there is a constant $\gamma(m)$ depending only on $m$ such that*

$$\|b(C_1) - b(C_2)\| \leq \gamma(m)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2) \tag{3.12}$$

*for all $C_1, C_2 \in \mathcal{K}_m(X)$.*

For the proof of this theorem we need two lemmas.

**Lemma 3.12.** *Let $(Y, \|\cdot\|_Y)$ be a Banach space of dimension $n > 0$ and $\lambda$ a Lebesgue measure on $Y$. Then for every nonempty compact convex set $C \in \mathcal{K}(Y)$ and every closed ball $\bar{B}_Y(x, s)$ containing $C$ with radius $s > 0$ we have*

$$\lambda\Big(C + \bar{B}_Y(0, r)\Big) - \lambda(C) \leq \bigg(\Big(1 + \frac{r}{s}\Big)^n - 1\bigg)\, \lambda\Big(\bar{B}_Y(x, s)\Big)$$

*for every $r \geq 0$.*

*Proof.* This proof makes use of the theory of mixed volumes. More about this topic can be found in the book by Schneider [11, Chapter 5]. The mixed volume is defined as the symmetric function $V : (\mathcal{K}(Y))^n \to \mathbb{R}$ satisfying

$$\lambda(t_1 K_1 + \cdots + t_m K_m) = \sum_{i_1, \ldots, i_n = 1}^{m} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} \cdots t_{i_n} \tag{3.13}$$

for all $m \in \mathbb{N}$, all $K_i \in \mathcal{K}(Y)$ and $t_i \geq 0$ with $1 \leq i \leq m$. The mixed volume is multilinear and invariant under translation:

$$V(t_1 K_1 + t_1' K_1', K_2, \ldots, K_n) = t_1 V(K_1, \ldots, K_n) + t_1' V(K_1', \ldots, K_n) \tag{3.14}$$

for all $K_1, K_1', \ldots K_n \in \mathcal{K}(Y)$, $t_1, t_1', \ldots, t_n \geq 0$; and

$$V(K_1 + y, K_2, \ldots, K_n) = V(K_1, K_2, \ldots, K_n) \tag{3.15}$$

for every $y \in Y$. Furthermore, $V$ is increasing with respect to inclusion:

$$V(K_1, K_2, \ldots, K_n) \leq V(K_1', K_2, \ldots, K_n) \tag{3.16}$$

if $K_1 \subseteq K_1'$.

For an integer $k$ with $1 \leq k \leq n$ and sets $K_1, K_2 \in \mathcal{K}(Y)$ we set

$$V(K_1[k], K_2[n-k]) := V(\underbrace{K_1, \ldots, K_1}_{k \text{ times}}, \underbrace{K_2, \ldots, K_2}_{n-k \text{ times}}).$$

Now assume that $C \in \mathcal{K}(Y)$ and $C \subseteq \bar{B}_Y(x,s)$ with $s > 0$. By (3.13) for $r \geq 0$ we get

$$\lambda\Big(C + \bar{B}_Y(0,r)\Big) = \lambda\Big(C + r\bar{B}_Y(0,1)\Big) = \sum_{k=0}^{n} \binom{n}{k} V\Big(C[n-k], \bar{B}_Y(0,1)[k]\Big) r^k.$$

The definition of the mixed volume also implies $\lambda(C) = V(C, C, \ldots, C)$, and hence

$$\lambda\Big(C + \bar{B}_Y(0,r)\Big) - \lambda(C) = \sum_{k=1}^{n} \binom{n}{k} V\Big(C[n-k], \bar{B}_Y(0,1)[k]\Big) r^k.$$

By assumption $C \subseteq \bar{B}_Y(x,s)$, so from property (3.16) it follows that

$$V\Big(C[n-k], \bar{B}_Y(0,1)[k]\Big) \leq V\Big(\bar{B}_Y(x,s)[n-k], \bar{B}_Y(0,1)[k]\Big).$$

Properties (3.14) and (3.15) then yield

$$\begin{aligned} V\Big(\bar{B}_Y(x,s)[n-k], \bar{B}_Y(0,1)[k]\Big) &= \frac{1}{s^k}\, V\Big(\bar{B}_Y(x,s)[n-k], \bar{B}_Y(0,s)[k]\Big) \\ &= \frac{1}{s^k}\, V\Big(\bar{B}_Y(x,s)[n-k], \bar{B}_Y(x,s)[k]\Big) \\ &= \frac{1}{s^k}\, \lambda\Big(\bar{B}_Y(x,s)\Big). \end{aligned}$$

Putting everything together, we finally get

$$\begin{aligned} \lambda\Big(C + \bar{B}_Y(0,r)\Big) - \lambda(C) &\leq \left(\sum_{k=1}^{n} \binom{n}{k} \frac{r^k}{s^k}\right) \lambda\Big(\bar{B}_Y(x,s)\Big) \\ &= \left(\left(1 + \frac{r}{s}\right)^n - 1\right) \lambda\Big(\bar{B}_Y(x,s)\Big), \end{aligned}$$

which finishes the proof of the lemma. □

**Lemma 3.13.** *Assume that two linear subspaces $L_1$ and $L_2$ of $X$ with dimension $\dim(L_1) = \dim(L_2) = m$ are given for some $m \geq 0$. There exist a linear map $A$ from $L_1$ onto $L_2$ and a positive constant $\zeta$ depending only on $m$ that satisfy*

$$\|A\| \leq \zeta, \quad \|A^{-1}\| \leq \zeta,$$

*and*

$$\|x - Ax\| \leq \zeta \operatorname{dist}(x, L_2)$$

*for all $x \in L_1$.*

*Proof.* For the proof it is sufficient to consider only the subspace $Y := L_1+L_2$ of $X$ which has dimension at most $2m$. By Corollary 3.6 there is a linear isomorphism $B$ from $(Y, \|\cdot\|)$ to $(\mathbb{R}^{dim(Y)}, \|\cdot\|_2)$ such that the operator norms of $B$ and its inverse $B^{-1}$ satisfy $\|B\| \leq \sqrt{2m}$ and $\|B^{-1}\| \leq 1$.

Therefore, it suffices to prove the result for the Euclidean space $\mathbb{R}^{\dim(Y)}$: If there exist an invertible linear map

$$\hat{A} : BL_1 \to BL_2$$

and a constant $\hat{\zeta}$ depending on $m$ such that the operator norms $\|\hat{A}\|$ and $\|\hat{A}^{-1}\|$ are bounded by $\hat{\zeta}$ and $\|\hat{x} - \hat{A}\hat{x}\|_2 \leq \hat{\zeta} \operatorname{dist}(\hat{x}, BL_2)$ for all $\hat{x} \in BL_1$, then

$$A := B^{-1}\hat{A}B$$

is as required with $\zeta := \|B\| \, \hat{\zeta} \, \|B^{-1}\| \leq \sqrt{2m}\hat{\zeta}$.

Hence we assume without loss of generality that $Y$ is $\mathbb{R}^{\dim(Y)}$ with the inner product denoted by $\langle \cdot, \cdot \rangle$. We show by induction on $m$ that for every pair of linear subspaces $L_1$ and $L_2$ of $Y$ of dimension $m \geq 0$ there is a linear map $A$ from $L_1$ onto $L_2$ such that $A$ is an isometry and for all $x \in L_1$ the inequality

$$\|x - Ax\| \leq 5^m \operatorname{dist}(x, L_2) \tag{3.17}$$

holds.

For $m = 0$ the subspaces are equal to the origin of $Y$ and the map $A$ is trivial. Now we assume that the result is shown for some $m \geq 0$ and we want to prove it for $m + 1$. In the following for every linear subspace $L \subseteq Y$ the orthogonal projection onto $L$ is denoted by $P_L$; by $S := \{x \in Y : \|x\| = 1\}$ we denote the unit sphere in $Y$.

For fixed subspaces $L_1, L_2 \subseteq Y$ of dimension $m + 1$ we choose a point $e_1 \in S \cap L_1$ such that the distance $\operatorname{dist}(e_1, L_2)$ to $L_2$ is minimal. If $P_{L_2}e_1 \neq 0$ we define the point $e_2 \in S \cap L_2$ by $e_2 = \|P_{L_2}e_1\|^{-1}P_{L_2}e_1$, otherwise we pick any point of $S \cap L_2$ as $e_2$. Then for the case $P_{L_2}e_1 \neq 0$ by the Pythagorean theorem we have

$$\begin{aligned}
\|e_1 - e_2\|^2 &= \|e_1 - P_{L_2}e_1\|^2 + \left\| P_{L_2}e_1 - \frac{1}{\|P_{L_2}e_1\|} P_{L_2}e_1 \right\|^2 \\
&= \|e_1 - P_{L_2}e_1\|^2 + \left(\|P_{L_2}e_1\| - 1\right)^2 .
\end{aligned}$$

Since $\|P_{L_2}e_1\| \geq \|P_{L_2}e_1\|^2$ and $1 = \|e_1\|^2 = \|P_{L_2}e_1\|^2 + \|e_1 - P_{L_2}e_1\|^2$ it follows that

$$\begin{aligned}
\|e_1 - e_2\|^2 &\leq \|e_1 - P_{L_2}e_1\|^2 + 1 - \|P_{L_2}e_1\|^2 \\
&= 2\,\|e_1 - P_{L_2}e_1\|^2 \\
&= 2\left(\operatorname{dist}(e_1, L_2)\right)^2 .
\end{aligned}$$

For the case $P_{L_2}e_1 = 0$ we directly get

$$\|e_1 - e_2\|^2 = \|e_1\|^2 + \|e_2\|^2 = 2 = 2\operatorname{dist}(e_1, L_2),$$

and hence in both cases it follows that

$$\|e_1 - e_2\| \leq \sqrt{2}\operatorname{dist}(e_1, L_2). \tag{3.18}$$

If we define $\tilde{L}_i$ as the orthogonal complement of the span of $e_i$ in $L_i$ for $i \in \{1, 2\}$, then $\dim \tilde{L}_i = m$ and by the induction assumption there is a linear isometry $\tilde{A} : \tilde{L}_1 \to \tilde{L}_2$ such that

$$\|x - \tilde{A}x\| \leq 5^m \operatorname{dist}(x, \tilde{L}_2)$$

for all $x \in \tilde{L}_1$. We can extend $\tilde{A}$ to a map $A : L_1 \to L_2$ by setting $Ae_1 = e_2$ and using linear extension.

Every point $x \in L_1$ can be written as $x = x' + x''$, where $x'$ is in the span of $e_1$ and $x''$ is in the orthogonal complement $\tilde{L}_1$. Since $\tilde{A}$ is isometric, from the Pythagorean theorem it follows that

$$\|Ax\|^2 = \|Ax'\|^2 + \|\tilde{A}x''\|^2 = \|x'\|^2 + \|x''\|^2 = \|x\|^2,$$

and hence $A$ is an isometry. Now we only have to check that (3.17) holds.

For $x = 0$ this is obvious. Thus we fix $x \in L_1$ not equal to the origin. By definition of $e_1$ we have that

$$\operatorname{dist}(e_1, L_2) \leq \operatorname{dist}\left(\frac{x}{\|x\|}, L_2\right) = \frac{1}{\|x\|}\operatorname{dist}(x, L_2). \tag{3.19}$$

Hence for the orthogonal projection $x'$ of $x$ onto the span of $e_1$ by (3.18) it follows that

$$\begin{aligned}\|x' - Ax'\| &= \|x'\| \cdot \|e_1 - Ae_1\| = \|x'\| \cdot \|e_1 - e_2\| \leq \sqrt{2}\,\|x'\|\operatorname{dist}(e_1, L_2)\\ &\leq \sqrt{2}\frac{\|x'\|}{\|x\|}\operatorname{dist}(x, L_2) \leq \sqrt{2}\operatorname{dist}(x, L_2).\end{aligned}$$

By the induction assumption we know that

$$\|x'' - Ax''\| = \|x'' - \tilde{A}x''\| \leq 5^m \operatorname{dist}(x'', \tilde{L}_2)$$

for $x'' = P_{\tilde{L}_1}x$.

Now we want to estimate $\operatorname{dist}(x'', \tilde{L}_2)$ in terms of $\operatorname{dist}(x, L_2)$. Since $P_{L_2}x'' = P_{\tilde{L}_2}x'' + \langle e_2, x''\rangle e_2$, we get

$$\begin{aligned}\operatorname{dist}(x'', \tilde{L}_2) &= \|x'' - P_{\tilde{L}_2}x''\|\\ &\leq \|x'' - P_{\tilde{L}_2}x'' - \langle e_2, x''\rangle e_2\| + \|\langle e_2, x''\rangle e_2\|\\ &= \operatorname{dist}(x'', L_2) + |\langle e_2, x''\rangle|.\end{aligned}$$

On the one hand, since $x'' = x - x'$, we get

$$\operatorname{dist}(x'', L_2) = \|x - x' - P_{L_2}x + P_{L_2}x'\| \leq \operatorname{dist}(x, L_2) + \operatorname{dist}(x', L_2),$$

but by (3.19) we have

$$\operatorname{dist}(x', L_2) = \|x'\| \operatorname{dist}(e_1, L_2) \leq \frac{\|x'\|}{\|x\|} \operatorname{dist}(x, L_2) \leq \operatorname{dist}(x, L_2).$$

On the other hand, using that $\langle e_1, x'' \rangle = 0$ and (3.18) and again (3.19), we obtain

$$\begin{aligned} |\langle e_2, x'' \rangle| = |\langle e_1 - e_2, x'' \rangle| &\leq \|e_1 - e_2\| \cdot \|x''\| \leq \sqrt{2}\|x''\| \operatorname{dist}(e_1, L_2) \\ &\leq \sqrt{2}\frac{\|x''\|}{\|x\|} \operatorname{dist}(x, L_2) \leq \sqrt{2} \operatorname{dist}(x, L_2). \end{aligned}$$

Putting these estimates together yields

$$\operatorname{dist}(x'', \tilde{L_2}) \leq \left(2 + \sqrt{2}\right) \cdot \operatorname{dist}(x, L_2).$$

Hence

$$\begin{aligned} \|x - Ax\| &\leq \|x' - Ax'\| + \|x'' - Ax''\| \\ &\leq \left(\sqrt{2} + \left(2 + \sqrt{2}\right) \cdot 5^m\right) \operatorname{dist}(x, L_2) \leq 5^{m+1} \operatorname{dist}(x, L_2), \end{aligned}$$

which is the required estimate (3.17) for dimension $m + 1$. $\square$

Now we are ready to prove Theorem 3.11.

*Proof of Theorem 3.11.* Let $C_1$ and $C_2$ be elements of $\mathcal{K}_m(X)$ for some $m \in \mathbb{N}_0$. We have to show that there is a constant $\gamma(m)$ depending only on $m$ such that

$$\|b(C_1) - b(C_2)\| \leq \gamma(m)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2).$$

We split the proof up into four steps.

*Step 1.* We consider the case $\operatorname{affspan}(C_1) = \operatorname{affspan}(C_2)$. Here we can without loss of generality assume that $Y := \operatorname{affspan}(C_1)$ is a linear subspace of $X$ of dimension $m$. We also assume that $\operatorname{diam}(C_1) \leq \operatorname{diam}(C_2)$. We write $r := d_H(C_1, C_2)$ for the Hausdorff distance of the sets and set $\bar{B}_Y(x, s) := \bar{B}(x, s) \cap Y$ for all $x \in Y$ and $s \geq 0$.

By the definition of the Hausdorff distance we have

$$C_2 \subseteq C_1 + \bar{B}_Y(0, r).$$

The barycenters are elements of the sets, so there is an $x \in C_1$ such that $\|x - b(C_2)\| \leq r$. If we have $\operatorname{diam}(C_1) \leq r$, this gives

$$\begin{aligned}\|b(C_1) - b(C_2)\| &\leq \|b(C_1) - x\| + \|x - b(C_2)\| \\ &\leq \operatorname{diam}(C_1) + r \leq 2r.\end{aligned}$$

Since the regularity coefficients are bounded from below by 1, this implies the required inequality (3.12).

If $r < \operatorname{diam}(C_1)$ we first look at the special case $C_1 \subseteq C_2$. For a fixed $y \in C_1$ and some Lebesgue measure $\lambda$ on $Y$ we calculate

$$\begin{aligned}\|b(C_1) - b(C_2)\| &= \left\| \frac{1}{\lambda(C_1)} \int_{C_1} (x - y) \,\mathrm{d}\lambda(x) - \frac{1}{\lambda(C_2)} \int_{C_2} (x - y) \,\mathrm{d}\lambda(x) \right\| \\ &\leq \left\| \frac{1}{\lambda(C_1)} \int_{C_1} (x - y) \,\mathrm{d}\lambda(x) - \frac{1}{\lambda(C_1)} \int_{C_2} (x - y) \,\mathrm{d}\lambda(x) \right\| \\ &\quad + \left\| \frac{1}{\lambda(C_1)} \int_{C_2} (x - y) \,\mathrm{d}\lambda(x) - \frac{1}{\lambda(C_2)} \int_{C_2} (x - y) \,\mathrm{d}\lambda(x) \right\|.\end{aligned}$$

By the properties of the Bochner integral

$$\begin{aligned}\|b(C_1) - b(C_2)\| \leq& \frac{1}{\lambda(C_1)} \int_{C_2 \setminus C_1} \|x - y\| \,\mathrm{d}\lambda(x) \\ &+ \left( \frac{1}{\lambda(C_1)} - \frac{1}{\lambda(C_2)} \right) \int_{C_2} \|x - y\| \,\mathrm{d}\lambda(x) \\ \leq& \operatorname{diam}(C_2) \left( \frac{\lambda(C_2) - \lambda(C_1)}{\lambda(C_1)} + \frac{\lambda(C_2)}{\lambda(C_1)} - \frac{\lambda(C_2)}{\lambda(C_2)} \right),\end{aligned}$$

and hence

$$\begin{aligned}\|b(C_1) - b(C_2)\| \leq& 2 \operatorname{diam}(C_2) \frac{\lambda(C_2) - \lambda(C_1)}{\lambda(C_1)} \\ \leq& 2 \operatorname{diam}(C_2) \frac{\lambda\Big(C_1 + \bar{B}_Y(0, r)\Big) - \lambda(C_1)}{\lambda(C_1)}.\end{aligned} \tag{3.20}$$

We pick a ball $\bar{B}(x, s)$ in $X$ containing $C_1$ with its center $x$ in $Y = \operatorname{affspan}(C_1)$ and minimal radius $s$. Then by the definition of the regularity coefficient (3.1) we have $\lambda\Big(\bar{B}_Y(x, s)\Big) = \delta_{C_1} \lambda(C_1)$. Since we are considering the case $r < \operatorname{diam}(C_1)$, it follows that $s > 0$ and the dimension $m$ of $Y$ is not zero, and hence we can apply Lemma 3.12 to obtain

$$\begin{aligned}\lambda\Big(C_1 + \bar{B}_Y(0, r)\Big) - \lambda(C_1) &\leq \left( \left(1 + \frac{r}{s}\right)^m - 1 \right) \lambda\Big(\bar{B}_Y(x, s)\Big) \\ &= \left( \left(1 + \frac{r}{s}\right)^m - 1 \right) \delta_{C_1} \lambda(C_1).\end{aligned}$$

Using that $\binom{m}{k} \leq m\binom{m-1}{k-1}$ for $1 \leq k \leq m$, we calculate

$$\left(1+\frac{r}{s}\right)^m - 1 = \sum_{k=1}^{m} \binom{m}{k} \frac{r^k}{s^k} \leq m\frac{r}{s} \sum_{k=1}^{m} \binom{m-1}{k-1} \frac{r^{k-1}}{s^{k-1}} = m\frac{r}{s}\left(1+\frac{r}{s}\right)^{m-1}.$$

Putting everything back into (3.20) yields

$$\|b(C_1) - b(C_2)\| \leq 2\,\mathrm{diam}(C_2) m \frac{r}{s}\left(1+\frac{r}{s}\right)^{m-1} \delta_{C_1}.$$

Since $C_2 \subseteq C_1 + \bar{B}_Y(0,r)$ and $\mathrm{diam}(C_1) \leq \mathrm{diam}\left(\bar{B}_Y(x,s)\right) = 2s$ and by assumption $r < \mathrm{diam}(C_1)$, it follows that

$$\mathrm{diam}(C_2) \leq \mathrm{diam}(C_1) + 2r \leq 6s.$$

Hence

$$\|b(C_1) - b(C_2)\| \leq 12m3^{m-1}\delta_{C_1} r,$$

and the required inequality is shown for the case $r < \mathrm{diam}(C_1)$ and $C_1 \subseteq C_2$.

For the case $r < \mathrm{diam}(C_1)$ with arbitrary $C_1, C_2 \in \mathcal{K}(X)$ that have the same affine span we define

$$C_3 := C_1 + \bar{B}_Y(0,r),$$

where still $r = d_H(C_1, C_2)$. Then $d_H(C_1, C_3) \leq r$ and $d_H(C_2, C_3) \leq 2r$, and from the previous results it follows that

$$\begin{aligned} \|b(C_1) - b(C_2)\| &\leq \|b(C_1) - b(C_3)\| + \|b(C_3) - b(C_2)\| \\ &\leq 12m3^{m-1}(\delta_{C_1} + 2\delta_{C_2})r \leq \gamma_1(m)(\delta_{C_1} + \delta_{C_2})r. \end{aligned}$$

This gives inequality (3.12) with the constant $\gamma_1(m) = 24m3^{m-1}$.

*Step 2.* We show the theorem for the case $\dim(C_1) = \dim(C_2) = m$ and

$$\mathrm{affspan}(C_1) \cap \mathrm{affspan}(C_2) \neq \emptyset.$$

By shifting the origin to a point in the intersection, we can regard $\mathrm{affspan}(C_1)$ and $\mathrm{affspan}(C_2)$ as linear subspaces of $X$. By Lemma 3.13 there are a constant $\zeta$ depending only on $m$ and a linear map $A$ from $\mathrm{affspan}(C_1)$ onto $\mathrm{affspan}(C_2)$ such that the operator norms $\|A\|$ and $\|A^{-1}\|$ are bounded by $\zeta$ and

$$\|x - Ax\| \leq \zeta\,\mathrm{dist}(x, \mathrm{affspan}(C_2))$$

for all $x \in \mathrm{affspan}(C_1)$.

For every $x \in C_1$ we have

$$\|x - Ax\| \le \zeta \operatorname{dist}(x, \operatorname{affspan}(C_2)) \le \zeta \operatorname{dist}(x, C_2) \le \zeta\, d_H(C_1, C_2). \tag{3.21}$$

In particular, since the barycentric map commutes with $A$ and since $b(C_1) \in C_1$, we get

$$\|b(C_1) - b(AC_1)\| = \|b(C_1) - Ab(C_1)\| \le \zeta\, d_H(C_1, C_2). \tag{3.22}$$

Moreover, inequality (3.21) implies that $d_H(C_1, AC_1) \le \zeta\, d_H(C_1, C_2)$. By the triangle inequality for the metric $d_H$ we get

$$d_H(AC_1, C_2) \le d_H(C_1, AC_1) + d_H(C_1, C_2) \le (\zeta + 1)\, d_H(C_1, C_2).$$

Thus, since $\operatorname{affspan}(C_2) = \operatorname{affspan}(AC_1)$, from the result for the special case considered in Step 1 it follows that

$$\begin{aligned} \|b(AC_1) - b(C_2)\| &\le \gamma_1(m)(\delta_{AC_1} + \delta_{C_2})\, d_H(AC_1, C_2) \\ &\le \gamma_1(m)(\zeta + 1)(\delta_{AC_1} + \delta_{C_2})\, d_H(C_1, C_2). \end{aligned} \tag{3.23}$$

Now we want to estimate the regularity coefficient $\delta_{AC_1}$ in terms of $\delta_{C_1}$. We choose Lebesgue measures $\lambda_1$ and $\lambda_2$ on $L_1 := \operatorname{affspan}(C_1)$ and $L_2 := \operatorname{affspan}(C_2)$, respectively, such that

$$\lambda_1\Big(\bar{B}(x_1, r) \cap L_1\Big) = \lambda_2\Big(\bar{B}(x_2, r) \cap L_2\Big). \tag{3.24}$$

for all $x_1 \in L_1$, $x_2 \in L_2$ and $r \ge 0$. We write $\bar{B}_{L_1}(x_1, r)$ for $\bar{B}(x_1, r) \cap L_1$ and $\bar{B}_{L_2}(x_2, r)$ for $\bar{B}(x_2, r) \cap L_2$. Since $\|A^{-1}\| \le \zeta$, we have the inclusion $A^{-1}\bar{B}_{L_2}(0, 1) \subseteq \zeta \bar{B}_{L_1}(0, 1)$, and hence

$$\lambda_1\Big(A^{-1}\bar{B}_{L_2}(0, 1)\Big) \le \lambda_1\Big(\zeta \bar{B}_{L_1}(0, 1)\Big) = \zeta^m \lambda_1\Big(\bar{B}_{L_1}(0, 1)\Big) = \zeta^m \lambda_2\Big(\bar{B}_{L_2}(0, 1)\Big).$$

The Lebesgue measure is determined by one constant, so this inequality holds for all measurable sets in $L_2$; in particular we have

$$\lambda_1(C_1) = \lambda_1(A^{-1}AC_1) \le \zeta^m \lambda_2(AC_1). \tag{3.25}$$

We choose $x \in L_1$ and $r \ge 0$ such that $C_1$ is a subset of $\bar{B}(x, r)$ and $r$ is minimal. By $\|A\| \le \zeta$ we get

$$AC_1 \subseteq A\bar{B}_{L_1}(x, r) \subseteq \bar{B}_{L_2}(Ax, \zeta r).$$

Thus, by the definition of the regularity coefficient (3.1) and properties (3.24) and (3.25), we have

$$\delta_{AC_1} \le \frac{\lambda_2\Big(\bar{B}_{L_2}(Ax, \zeta r)\Big)}{\lambda_2(AC_1)} \le \frac{\lambda_1\Big(\bar{B}_{L_1}(x, \zeta r)\Big)}{\zeta^{-m}\lambda_1(C_1)} = \zeta^{2m} \frac{\lambda_1\Big(\bar{B}_{L_1}(x, r)\Big)}{\lambda_1(C_1)} = \zeta^{2m} \delta_{C_1}.$$

Combining this with inequalities (3.22) and (3.23), we finally get

$$\begin{aligned}\|b(C_1) - b(C_2)\| &\leq \|b(C_1) - b(AC_1)\| + \|b(AC_1) - b(C_2)\| \\ &\leq \Big(\zeta + \gamma_1(m)(\zeta + 1)(\zeta^{2m}\delta_{C_1} + \delta_{C_2})\Big)\, d_H(C_1, C_2).\end{aligned}$$

Since $\zeta^{2m}$ and the regularity coefficients are bounded from below by 1, it follows that

$$\begin{aligned}\|b(C_1) - b(C_2)\| &\leq \Big(\zeta + \gamma_1(m)(\zeta + 1)\zeta^{2m}\Big)\, (\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2) \\ &= \gamma_2(m)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2).\end{aligned}$$

This finishes the proof of Step 2.

*Step 3.* We look at the case $\dim(C_1) = \dim(C_2) = m$ with arbitrary positions of the affine spans of $C_1$ and $C_2$. We choose two points $x_1 \in C_1$ and $x_2 \in C_2$ such that $\|x_1 - x_2\| \leq d_H(C_1, C_2)$. If we consider $\tilde{C}_2 := C_2 + x_1 - x_2$, then we have $\operatorname{affspan}(C_1) \cap \operatorname{affspan}(\tilde{C}_2) \neq \emptyset$. From the result of Step 2 we obtain

$$\|b(C_1) - b(\tilde{C}_2)\| \leq \gamma_2(m)(\delta_{C_1} + \delta_{\tilde{C}_2})\, d_H(C_1, \tilde{C}_2).$$

Since $\delta_{\tilde{C}_2} = \delta_{C_2}$ and

$$d_H(C_1, \tilde{C}_2) \leq d_H(C_1, C_2) + \|x_1 - x_2\| \leq 2d_H(C_1, C_2),$$

this yields

$$\|b(C_1) - b(\tilde{C}_2)\| \leq 2\gamma_2(m)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2).$$

Moreover, since $b(\tilde{C}_2) = b(C_2) + x_1 - x_2$, we have

$$\|b(\tilde{C}_2) - b(C_2)\| = \|x_1 - x_2\| \leq d_H(C_1, C_2).$$

Thus, putting the inequalities together, we obtain

$$\begin{aligned}\|b(C_1) - b(C_2)\| &\leq \|b(C_1) - b(\tilde{C}_2)\| + \|b(\tilde{C}_2) - b(C_2)\| \\ &\leq (2\gamma_2(m) + 1)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2).\end{aligned}$$

This finishes the proof if $C_1$ and $C_2$ have the same dimension.

*Step 4.* We consider the case that $C_1$ and $C_2$ do not have the same dimension, so without loss of generality we assume

$$l := \dim(C_2) < m := \dim(C_1).$$

From the definition of the Kolmogorov widths (3.2) it follows that

$$\operatorname{diam}(C_1) \geq d_0(C_1; X) \geq d_1(C_1; X) \geq \cdots \geq d_{m-1}(C_1; X) > 0,$$

and Proposition 3.2 implies that

$$d_H(C_1, C_2) \geq d_l(C_1; X) \geq d_{m-1}(C_1; X).$$

Hence,

$$\operatorname{diam}(C_1) \leq \frac{\operatorname{diam}(C_1)}{d_{m-1}(C_1; X)}\, d_H(C_1, C_2) \leq \frac{(\operatorname{diam}(C_1))^m}{\prod_{n=0}^{m-1} d_n(C_1; X)}\, d_H(C_1, C_2),$$

and from Proposition 3.10 we get

$$\operatorname{diam}(C_1) \leq \xi(m)\, \delta_{C_1} d_H(C_1, C_2).$$

Since $b(C_2) \in C_2$, we can pick an $x \in C_1$ such that $\|x - b(C_2)\| \leq d_H(C_1, C_2)$, so we get

$$\begin{aligned}\|b(C_1) - b(C_2)\| &\leq \|b(C_1) - x\| + \|x - b(C_2)\| \leq \operatorname{diam}(C_1) + d_H(C_1, C_2)\\ &\leq (\xi(m)\, \delta_{C_1} + 1)\, d_H(C_1, C_2) \leq \xi(m)(\delta_{C_1} + \delta_{C_2})\, d_H(C_1, C_2),\end{aligned}$$

and the proof of Theorem 3.11 is complete. □

## 3.3 A Lipschitz continuous selector

A selector $T : \mathcal{K}(X) \to X$ is *dilation-shift equivariant*, if

$$T(tC + x) = tT(C) + x$$

for all $t \in \mathbb{R}$, $C \in \mathcal{K}(X)$ and $x \in X$.

**Theorem 3.14.** *There is a dilation-shift equivariant selector $S_X : \mathcal{K}(X) \to X$ with the following properties: For every $m \in \mathbb{N}_0$ there is a constant $\theta_1(m)$ such that*

$$\|S_X(C_1) - S_X(C_2)\| \leq \theta_1(m)\, d_H(C_1, C_2) \tag{3.26}$$

*for all $C_1, C_2 \in \mathcal{K}_m(X)$; moreover, there is a constant $\theta_2(m)$ such that for every $C \in \mathcal{K}_m(X)$ there is an ellipsoid $E_C$ with center $S_X(C)$ that satisfies*

$$E_C \subseteq C \subseteq \theta_2(m) \circ_{S_X(C)} E_C. \tag{3.27}$$

*The constants $\theta_1(m)$ and $\theta_2(m)$ depend only on $m$.*

On the collection of the subsets of $X$ that consist only of one point, $\mathcal{K}_0(X)$, the definition of $S_X$ has to be $S_X(\{x\}) = x$ for all $x \in X$. By induction on

$m$ we extend the selector to subsets of higher dimension. We assume that for some fixed $m \in \mathbb{N}_0$ there are constants $\theta_1(m)$ and $\theta_2(m)$ and a selector

$$S_X{}' : \mathcal{K}_m(X) \to X$$

such that $S_X{}'$ is dilation-shift equivariant and Lipschitz continuous with Lipschitz constant $\theta_1(m)$, and that for every $C \in \mathcal{K}_m(X)$ there is an ellipsoid $E_C$ such that

$$E_C \subseteq C \subseteq \theta_2(m) \circ_{S_X{}'(C)} E_C.$$

We have to show that $S_X{}'$ can be extended to a dilation-shift equivariant selector $S_X : \mathcal{K}_{m+1}(X) \to X$ that satisfies (3.26) and (3.27) with constants $\theta_1(m+1)$ and $\theta_2(m+1)$. The proof of this is again built on several lemmas.

**Lemma 3.15.** *If $T$ is a Lipschitz continuous dilation-shift equivariant map from $(\mathcal{K}_m(X), d_H)$ to $(X, \|\cdot\|)$ with Lipschitz constant $\kappa$, then $T$ has an extension $\tilde{T} : \mathcal{K}_{m+1}(X) \to X$ that is also dilation-shift equivariant and Lipschitz continuous with the same Lipschitz constant.*

*Proof.* As mentioned in the introduction to this chapter, in this proof we use that $X$ can be regarded as the space $\ell_\infty(U)$ of bounded real-valued functions on some set $U$. The norm of every $x \in X$ is then given by $\|x\| = \sup_{u \in U} |x(u)|$. For every set $C \in \mathcal{K}_{m+1}(X)$ we define the function $\tilde{T}(C) \in \ell_\infty(U)$ by setting

$$\begin{aligned}\tilde{T}(C)(u) := &\frac{1}{2} \inf_{K \in \mathcal{K}_m(X)} \Big(T(K)(u) + \kappa\, d_H(K, C)\Big) \\ &+ \frac{1}{2} \sup_{K \in \mathcal{K}_m(X)} \Big(T(K)(u) - \kappa\, d_H(K, C)\Big)\end{aligned}$$

for all $u \in U$.

We first show that both the infimum and the supremum in the definition are finite. Let $C$ be an element of $\mathcal{K}_{m+1}(X)$. By the triangle inequality for $d_H$ and the Lipschitz continuity of $T$ on $\mathcal{K}_m(X)$ we have

$$\begin{aligned}T(K)(u) + \kappa\, d_H(K, C) &\geq T(K)(u) + \kappa\Big(d_H(\{0\}, K) - d_H(\{0\}, C)\Big) \\ &\geq T(K)(u) + T(\{0\})(u) - T(K)(u) - \kappa\, d_H(\{0\}, C) \\ &= -\kappa\, d_H(\{0\}, C)\end{aligned}$$

for every $K \in \mathcal{K}_m(X)$ and $u \in U$. Thus, we get

$$\inf_{K \in \mathcal{K}_m(X)} \Big(T(K)(u) + \kappa\, d_H(K, C)\Big) \geq -\kappa\, d_H(\{0\}, C).$$

Similarly, it follows that

$$\sup_{K \in \mathcal{K}_m(X)} \Big( T(K)(u) - \kappa\, d_H(K, C) \Big) \leq \kappa\, d_H(\{0\}, C).$$

We now verify that $\tilde{T}$ is an extension of $T$, that is, $\tilde{T}(C) = T(C)$ for all $C \in \mathcal{K}_m(X)$. To this end, we show that

$$\inf_{K \in \mathcal{K}_m(X)} \Big( T(K)(u) + \kappa\, d_H(K, C) \Big) = T(C)(u) \tag{3.28}$$

for all $u \in U$. Clearly, $T(C)(u)$ is greater than or equal to the infimum. We assume that there are $K \in \mathcal{K}_m(X)$ and $u \in U$ such that

$$T(K)(u) + \kappa\, d_H(K, C) < T(C)(u).$$

This implies

$$\kappa\, d_H(K, C) < |T(C)(u) - T(K)(u)| \leq \|T(C) - T(K)\|,$$

which is a contradiction to the definition of the Lipschitz constant $\kappa$. Thus, equality (3.28) holds. In the same way it can be shown that

$$\sup_{K \in \mathcal{K}_m(X)} \Big( T(K)(u) - \kappa\, d_H(K, C) \Big) = T(C)(u),$$

hence the definition of $\tilde{T}$ coincides with $T$ on $\mathcal{K}_m(X)$.

In order to show the Lipschitz continuity of $\tilde{T}$, we have to prove that

$$\sup_{u \in U} \Big| \tilde{T}(C_1)(u) - \tilde{T}(C_2)(u) \Big| \leq \kappa\, d_H(C_1, C_2) \tag{3.29}$$

for all $C_1, C_2 \in \mathcal{K}_{m+1}(X)$. We fix $u \in U$ and show that

$$\begin{aligned} \left| \frac{1}{2} \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C_1) \Big) - \frac{1}{2} \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C_2) \Big) \right| & \\ \leq \frac{1}{2} \kappa\, d_H(C_1, C_2). & \end{aligned} \tag{3.30}$$

For a given $\varepsilon > 0$, we pick a set $K' \in \mathcal{K}_m(X)$ such that

$$T(K')(u) + \kappa\, d_H(K', C_2) \leq \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C_2) \Big) + \varepsilon.$$

It follows that

$$\begin{aligned} &\frac{1}{2} \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C_1) \Big) - \frac{1}{2} \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C_2) \Big) \\ &\qquad \leq \frac{1}{2} \Big( T(K')(u) + \kappa\, d_H(K', C_1) - T(K')(u) - \kappa\, d_H(K', C_2) + \varepsilon \Big) \\ &\qquad\qquad \leq \frac{1}{2} \Big( \kappa\, d_H(C_1, C_2) + \varepsilon \Big). \end{aligned}$$

The same holds if we interchange $C_1$ and $C_2$, so inequality (3.30) follows. Similarly, an analogous inequality can be shown for the term with the supremum in the definition of $\tilde{T}$. By putting these inequalities together we get (3.29).

It remains to verify the dilation-shift equivariance of $\tilde{T}$. Let $t \in \mathbb{R}$, $C \in \mathcal{K}_{m+1}(X)$ and $x \in X = \ell_\infty(U)$ be given. We need to show that $\tilde{T}(tC + x) = t\tilde{T}(C) + x$. For all $u \in U$ we have

$$\begin{aligned}
\frac{1}{2} \inf_K \Big( T(K)(u) + \kappa\, d_H(K, tC + x) \Big) + \frac{1}{2} \sup_K \Big( T(K)(u) - \kappa\, d_H(K, tC + x) \Big) \\
= \frac{1}{2} \inf_K \Big( T(tK + x)(u) + \kappa\, d_H(tK + x, tC + x) \Big) \\
+ \frac{1}{2} \sup_K \Big( T(tK + x)(u) - \kappa\, d_H(tK + x, tC + x) \Big).
\end{aligned}$$

Furthermore, $d_H(tK + x, tC + x) = |t|\, d_H(K, C)$ for all $K \in \mathcal{K}_m(X)$. Consequently, using the dilation-shift equivariance of $T$, in the case $t \geq 0$ it follows that

$$\begin{aligned}
\tilde{T}(tC + x)(u) = \frac{1}{2} t \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C) \Big) + \frac{1}{2} x(u) \\
+ \frac{1}{2} t \sup_K \Big( T(K)(u) - \kappa\, d_H(K, C) \Big) + \frac{1}{2} x(u).
\end{aligned}$$

If $t < 0$, then

$$\inf_K \Big( tT(K)(u) + |t| \kappa\, d_H(K, C) \Big) = t \sup_K \Big( T(K)(u) - \kappa\, d_H(K, C) \Big)$$

and

$$\sup_K \Big( tT(K)(u) - |t| \kappa\, d_H(K, C) \Big) = t \inf_K \Big( T(K)(u) + \kappa\, d_H(K, C) \Big),$$

so in both cases we get

$$\tilde{T}(tC + x)(u) = t\tilde{T}(C) + x(u)$$

as required. □

The map ${S_X}' : \mathcal{K}_m(X) \to X$ is dilation-shift equivariant and Lipschitz continuous with Lipschitz constant $\theta_1(m)$. By Lemma 3.15 there is a dilation-shift equivariant extension $\tilde{S} : \mathcal{K}_{m+1}(X) \to X$ such that

$$\|\tilde{S}(C_1) - \tilde{S}(C_2)\| \leq \theta_1(m)\, d_H(C_1, C_2) \tag{3.31}$$

for all $C_1, C_2 \in \mathcal{K}_{m+1}(X)$. We use $\tilde{S}$ for the definition of the extension $S_X : \mathcal{K}_{m+1}(X) \to X$, but first we set

$$\eta := 8m\theta_2(m)\,,$$

and for every $C$ in $\mathcal{K}_{m+1}(X)$ we define

$$r_m(C) := 8\eta^{m+1}\theta_1(m)\, d_m(C;X)$$

and

$$\tilde{C} := C \cap \bar{B}\Big(\tilde{S}(C), r_m(C)\Big). \tag{3.32}$$

For $C \in \mathcal{K}_{m+1}(X)$ we now define $S_X(C)$ to be the barycenter of $\tilde{C}$;

$$S_X(C) := b(\tilde{C}).$$

From now on we abbreviate $d_m(C;X)$ to $d_m(C)$ for all $C \in \mathcal{K}(X)$.

**Lemma 3.16.** *The map $S_X$ defined above is an extension of ${S_X}'$ and a dilation-shift equivariant selector on $\mathcal{K}_{m+1}(X)$.*

*Proof.* If $\dim(C) \leq m$ we have $d_m(C) = 0$ and hence $\tilde{C} = \{\tilde{S}(C)\} = \{{S_X}'(C)\}$. Therefore, on $\mathcal{K}_m(X)$ the map $S_X$ coincides with the already defined map ${S_X}'$ and hence is an extension.

If $\dim(C) = m+1$, we have to check that $\tilde{C}$ is nonempty. Since $d_m(C) > 0$ by Proposition 3.2 there is a set $K \in \mathcal{K}_m(X)$ such that $d_H(K, C) \leq 2d_m(C)$. Since $\tilde{S}(K) \in K$, by the definition of the Hausdorff distance there is an $x \in C$ such that

$$\|x - \tilde{S}(K)\| \leq d_H(K, C).$$

Thus, by the Lipschitz continuity of $\tilde{S}$ we get

$$\begin{aligned}\|x - \tilde{S}(C)\| &\leq \|x - \tilde{S}(K)\| + \|\tilde{S}(K) - \tilde{S}(C)\| \\ &\leq d_H(K, C) + \theta_1(m)\, d_H(K, C) \leq 2(1 + \theta_1(m))\, d_m(C).\end{aligned}$$

Noting that all the involved constants are greater than or equal to 1, this implies $\|x - \tilde{S}(C)\| \leq r_m(C)$, and hence $\tilde{C}$ is nonempty. Moreover, since the barycentric map is a selector, we see that

$$S_X(C) = b(\tilde{C}) \in \tilde{C} \subseteq C$$

and $S_X$ is a selector.

Finally, we show that $S_X$ is dilation-shift equivariant. Let $t \in \mathbb{R}$, $C \in \mathcal{K}_{m+1}(X)$ and $y \in X$ be given. The definition of the Kolmogorov $m$-width implies $d_m(tC+y) = |t|d_m(C)$ and hence also $r_m(tC+y) = |t|r_m(C)$. This and the dilation-shift equivariance of $\tilde{S}$ show that

$$\begin{aligned}\bar{B}(\tilde{S}(tC+y), r_m(tC+y)) &= \bar{B}(t\tilde{S}(C)+y, |t|r_m(C)) \\ &= t\bar{B}(\tilde{S}(C), r_m(C)) + y.\end{aligned}$$

Therefore,

$$\begin{aligned}(tC+y)\breve{} &= (tC+y) \cap \bar{B}(\tilde{S}(tC+y), r_m(tC+y)) \\ &= (tC+y) \cap \Big(t\bar{B}(\tilde{S}(C), r_m(C)) + y\Big) \\ &= t\Big(C \cap \bar{B}(\tilde{S}(C), r_m(C))\Big) + y = t\tilde{C} + y.\end{aligned}$$

The barycentric map commutes with invertible affine maps, in particular it is dilation-shift equivariant. Therefore,

$$S_X(tC+y) = b((tC+y)\breve{}) = b(t\tilde{C}+y) = tb(\tilde{C}) + y = tS_X(C) + y,$$

and the dilation-shift equivariance of $S_X$ follows. $\square$

Our next goal is to show that for every $C \in \mathcal{K}_{m+1}(X)$ there is an ellipsoid $E_C$ such that the inclusion relations (3.27) are satisfied with some constant $\theta_2(m+1)$. For $C \in \mathcal{K}_{m+1}(X)$ with $\dim(C) \leq m$ this already follows from the induction hypothesis. For this part of the proof we hence assume that

$$\dim(C) = m + 1.$$

Some of the following lemmas are used again to show the Lipschitz continuity of $S_X$. First we consider the case $d_0(C) \leq \eta^m d_m(C)$.

**Lemma 3.17.** *If $d_0(C) \leq \eta^m d_m(C)$, then*

$$C \subseteq \bar{B}\Big(\tilde{S}(C), \frac{1}{2} r_m(C)\Big)$$

*and the regularity coefficient $\delta_{\tilde{C}}$ is bounded by a constant that depends only on $m$. Furthermore, there is an ellipsoid $E_C$ with center $S_X(C)$ such that*

$$E_C \subseteq C \subseteq \beta \circ_{S_X(C)} E_C,$$

*where the constant $\beta$ depends only on $m$.*

*Proof.* By the Lipschitz continuity of $\tilde{S}$ we see that for all $x \in C$

$$\|x - \tilde{S}(C)\| = \|\tilde{S}(\{x\}) - \tilde{S}(C)\| \leq \theta_1(m)\, d_H(\{x\}, C) \leq \theta_1(m) \operatorname{diam}(C).$$

By assumption $d_0(C) \le \eta^m d_m(C)$ and from the definition of the Kolmogorov 0-width it follows that $\operatorname{diam}(C) \le 2d_0(C)$, so

$$\begin{aligned} C &\subseteq \bar{B}\Big(\tilde{S}(C), \theta_1(m) \operatorname{diam}(C)\Big) \subseteq \bar{B}\Big(\tilde{S}(C), 2\theta_1(m)\eta^m d_m(C)\Big) \\ &\subseteq \bar{B}\Big(\tilde{S}(C), \frac{1}{2} r_m(C)\Big). \end{aligned}$$

This implies $\tilde{C} = C$ and hence $\delta_{\tilde{C}} = \delta_C$. The Kolmogorov widths satisfy $d_0(C) \ge d_1(C) \ge \cdots \ge d_m(C)$, so by Proposition 3.10 we get

$$\begin{aligned} \delta_{\tilde{C}} &\le \xi(m+1) \, \frac{(\operatorname{diam}(C))^{m+1}}{\prod_{n=0}^m d_n(C)} \\ &\le \xi(m+1) \, \frac{(2\eta^m d_m(C))^{m+1}}{(d_m(C))^{m+1}} = 2^{m+1} \xi(m+1) \, \eta^{m(m+1)}. \end{aligned}$$

Thus, the regularity coefficient $\delta_{\tilde{C}}$ is bounded from above by a constant that depends only on $m$.

Since $S_X(C) = b(\tilde{C}) = b(C)$, we can use Proposition 3.8 to show the existence an ellipsoid $E_C$ centered at $S_X(C)$ satisfying the required inclusion relations with constant $\beta(m+1)$. □

For the rest of this part of the proof we assume that

$$d_0(C) > \eta^m d_m(C).$$

Since

$$\eta^m < \frac{d_0(C)}{d_m(C)} = \frac{d_0(C)}{d_1(C)} \frac{d_1(C)}{d_2(C)} \cdots \frac{d_{m-1}(C)}{d_m(C)},$$

there is at least one index $k \in \{0, \ldots, m-1\}$ such that $\dfrac{d_k(C)}{d_{k+1}(C)} > \eta$. We set

$$n := \max\Big\{k : \frac{d_k(C)}{d_{k+1}(C)} > \eta, \, 0 \le k \le m-1\Big\} \tag{3.33}$$

and

$$\mu := m\beta(m) \, \theta_2(m),$$

where $\beta(m)$ is the constant from Proposition 3.8.

**Lemma 3.18.** *There is an ellipsoid $\tilde{E}$ of dimension $n+1$ such that*

$$\|\tilde{z} - \tilde{S}(C)\| \le \frac{1}{2} r_m(C),$$

*where $\tilde{z}$ is the center of $\tilde{E}$, and*

$$\tilde{E} \subseteq C \subseteq \mu \circ_{\tilde{z}} \tilde{E} + \bar{B}(0, \eta^{m+1} d_m(C)).$$

*Proof.* Since $d_{n+1}(C) > 0$, by Proposition 3.2 there is a convex set $K \in \mathcal{K}_{n+1}(X)$ of dimension $n+1$ such that

$$d_H(K, C) \leq 2d_{n+1}(C). \tag{3.34}$$

We have $n+1 \leq m$, so by the induction assumption there is an ellipsoid $E_K$ satisfying

$$E_K \subseteq K \subseteq \theta_2(m) \circ_{S_X{}'(K)} E_K. \tag{3.35}$$

By Proposition 3.9 there is a simplex $D_K$ with vertices $y_0, \dots, y_{n+1} \in E_K$ such that

$$\frac{1}{m} \circ_{S_X{}'(K)} E_K \subseteq D_K \subseteq E_K \tag{3.36}$$

and

$$b(D_K) = \frac{1}{n+2}(y_0 + \dots + y_{n+1}) = S_X{}'(K).$$

By the definition of the Hausdorff distance there are points $z_0, \dots, z_{n+1} \in C$ such that $\|z_k - y_k\| \leq d_H(K, C)$ for all $k \in \{0, \dots, n+1\}$. The set $D_C := \operatorname{conv}(\{z_0, \dots, z_{n+1}\})$ by (3.34) then satisfies

$$d_H(D_C, D_K) \leq \max_{0 \leq k \leq n+1} \|z_k - y_k\| \leq d_H(K, C) \leq 2d_{n+1}(C). \tag{3.37}$$

We now show that $D_C$ has dimension $n+1$. The Lipschitz continuity of the Kolmogorov widths (Corollary 3.3), inequality (3.34) and the definition of $n$ (3.33) give

$$d_n(K) \geq d_n(C) - d_H(K, C) \geq d_n(C) - 2d_{n+1}(C) \geq \eta d_{n+1}(C) - 2d_{n+1}(C).$$

Since $2 \leq \frac{\eta}{2}$, we get

$$d_n(K) \geq \eta d_{n+1}(C) - \frac{\eta}{2} d_{n+1}(C) = \frac{\eta}{2} d_{n+1}(C),$$

and from (3.35) and (3.36) and since $\eta = 8m\theta_2(m)$ it then follows that

$$d_{n+1}(C) \leq \frac{2}{\eta} d_n(K) \leq \frac{2\theta_2(m)}{\eta} d_n(E_K) \leq \frac{2m\theta_2(m)}{\eta} d_n(D_K) = \frac{1}{4} d_n(D_K).$$

Consequently, by (3.37) and the properties of the Kolmogorov widths we have

$$\begin{aligned} d_n(D_C) &\geq d_n(D_K) - d_H(D_C, D_K) \\ &\geq 4d_{n+1}(C) - 2d_{n+1}(C) = 2d_{n+1}(C) \geq 2d_m(C). \end{aligned}$$

We have $\dim(C) = m+1$ by assumption, so $d_m(C) > 0$ and hence also $d_n(D_C) > 0$. Thus, $D_C$ is an $(n+1)$-dimensional simplex.

We set

$$\tilde{z} := b(D_C) = \frac{1}{n+2}(z_0 + \cdots + z_{n+1}).$$

The definitions of the points $y_k$ and $z_k$ yield

$$\begin{aligned}\|\tilde{z} - S_X{}'(K)\| &= \left\| \frac{1}{n+2} \sum_{k=0}^{n+1} (z_k - y_k) \right\| \\ &\leq \frac{1}{n+2} \sum_{k=0}^{n+1} \|z_k - y_k\| \leq d_H(K, C).\end{aligned}$$

Since $K \in \mathcal{K}_m(X)$, we have $S_X{}'(K) = \tilde{S}(K)$, so (3.31) implies

$$\|S_X{}'(K) - \tilde{S}(C)\| \leq \theta_1(m)\, d_H(K, C),$$

and from (3.34) it now follows that

$$\begin{aligned}\|\tilde{z} - \tilde{S}(C)\| &\leq \|\tilde{z} - S_X{}'(K)\| + \|S_X{}'(K) - \tilde{S}(C)\| \\ &\leq 2(1 + \theta_1(m)) d_{n+1}(C) \leq 4\theta_1(m)\, d_{n+1}(C).\end{aligned}$$

By the definition of $n$ it holds that

$$\frac{d_{n+1}(C)}{d_m(C)} = \frac{d_{n+1}(C)}{d_{n+2}(C)} \cdots \frac{d_{m-1}(C)}{d_m(C)} \leq \eta^{m-(n+1)} \leq \eta^m, \tag{3.38}$$

and hence the required inequality

$$\|\tilde{z} - \tilde{S}(C)\| \leq 4\eta^m \theta_1(m) d_m(C) \leq \frac{1}{2} r_m(C)$$

follows.

By Proposition 3.8 there is an ellipsoid $\tilde{E}$ of dimension $n+1$ centered at $\tilde{z}$ satisfying

$$\tilde{E} \subseteq D_C \subseteq \beta(m) \circ_{\tilde{z}} \tilde{E}. \tag{3.39}$$

Clearly, $\tilde{E}$ is a subset of $C$. By (3.35), (3.36) and (3.37) we get the inclusions

$$\begin{aligned}K &\subseteq \theta_2(m) \circ_{S_X{}'(K)} E_K \subseteq \Big(m\theta_2(m)\Big) \circ_{S_X{}'(K)} D_K \\ &= m\theta_2(m)\Big(D_K - S_X{}'(K)\Big) + S_X{}'(K) \\ &\subseteq m\theta_2(m)\Big(D_C + \bar{B}(0, 2d_{n+1}(C))\Big) - (m\theta_2(m) - 1) S_X{}'(K).\end{aligned}$$

Furthermore, by (3.39) we have

$$\begin{aligned}K &\subseteq m\theta_2(m)\Big(\beta(m)(\tilde{E} - \tilde{z}) + \tilde{z}\Big) + (1 - m\theta_2(m)) S_X{}'(K) \\ &\quad + \bar{B}(0, 2m\theta_2(m) d_{n+1}(C)) \\ &= \Big(m\beta(m)\theta_2(m)\Big) \circ_{\tilde{z}} \tilde{E} + (m\theta_2(m) - 1)\Big(\tilde{z} - S_X{}'(K)\Big) \\ &\quad + \bar{B}(0, 2m\theta_2(m) d_{n+1}(C)).\end{aligned}$$

Since $\|\tilde{z} - S_X{}'(K)\| \leq d_H(K, C) \leq 2d_{n+1}(C)$, this yields

$$K \subseteq \Big(m\beta(m)\theta_2(m)\Big) \circ_{\tilde{z}} \tilde{E} + \bar{B}\Big(0, (4m\theta_2(m) - 2)d_{n+1}(C)\Big),$$

and from (3.34), (3.38) and the definitions of $\eta$ and $\mu$ it now follows that

$$\begin{aligned} C &\subseteq K + \bar{B}(0, 2d_{n+1}(C)) \subseteq \Big(m\beta(m)\theta_2(m)\Big) \circ_{\tilde{z}} \tilde{E} + \bar{B}(0, 4m\theta_2(m)d_{n+1}(C)) \\ &\subseteq \mu \circ_{\tilde{z}} \tilde{E} + \bar{B}(0, \eta^{m+1}d_m(C)), \end{aligned}$$

which finishes the proof. □

**Lemma 3.19.** *Let $\tilde{z}$ be the point from Lemma 3.18. There is a point $w \in C$ such that*

$$\|w - \tilde{z}\| \leq \eta^{m+1}d_m(C)$$

*and*

$$\bar{B}\Big(w, \frac{1}{2\mu\alpha(m+1)}d_m(C)\Big) \cap \operatorname{affspan}(C) \subseteq C,$$

*where $\alpha(m+1)$ is the constant from Proposition 3.7.*

*Proof.* The ellipsoid $\tilde{E}$ from Lemma 3.18 satisfies $\tilde{E} \subseteq C$. Moreover, there is a point $u \in \mu \circ_{\tilde{z}} \tilde{E}$ such that

$$\|b(C) - u\| \leq \eta^{m+1}d_m(C).$$

We consider the line through $u$ and $\tilde{z}$. If $u = \tilde{z}$, we pick any line in $\operatorname{affspan}(\tilde{E})$ passing through $u$. This line intersects the boundary of $\tilde{E}$ in $\operatorname{affspan}(\tilde{E})$ in two points $v$ and $v'$. Without loss of generality we assume that $\|v' - u\| \leq \|v - u\|$. Then $\tilde{z}$ is contained in the closed line segment with endpoints $u$ and $v$, and since $u \in \mu \circ_{\tilde{z}} \tilde{E}$ it follows that

$$\|v - u\| = \|v - \tilde{z}\| + \|\tilde{z} - u\| \leq \|v - \tilde{z}\| + \mu\|v - \tilde{z}\|.$$

Thus, $t := \frac{\|v-\tilde{z}\|}{\|v-u\|}$ satisfies

$$1 \geq t \geq \frac{\|v - \tilde{z}\|}{(1+\mu)\|v - \tilde{z}\|} \geq \frac{1}{2\mu}$$

and

$$\tilde{z} = (1-t)v + tu.$$

We set $w := (1-t)v + tb(C)$, then by the definition of $u$ we have

$$\|w - \tilde{z}\| = t\|b(C) - u\| \leq \|b(C) - u\| \leq \eta^{m+1}d_m(C).$$

By Proposition 3.7 there is a constant $\alpha(m+1)$ depending only on the dimension $m+1$ of $C$ such that

$$\bar{B}\Big(b(C), \frac{1}{\alpha(m+1)} d_m(C)\Big) \cap \operatorname{affspan}(C) \subseteq C.$$

Since $C$ is convex and $v \in C$, this yields

$$\begin{aligned}
&\bar{B}\Big(w, \frac{t}{\alpha(m+1)} d_m(C)\Big) \cap \operatorname{affspan}(C) \\
&\qquad = \Big((1-t)v + t\bar{B}\Big(b(C), \frac{1}{\alpha(m+1)} d_m(C)\Big)\Big) \cap \operatorname{affspan}(C) \subseteq C,
\end{aligned}$$

and from $(2\mu)^{-1} \leq t$ the result follows. □

**Lemma 3.20.** *The set $\tilde{C}$ defined by* (3.32) *has dimension $m+1$ and its regularity coefficient $\delta_{\tilde{C}}$ is bounded by a constant that depends only on $m$. Furthermore, we have $d_m(C) \leq \nu d_m(\tilde{C})$, where the constant $\nu$ only depends on $m$.*

*Proof.* Let $w$ be the point from Lemma 3.19 and define the set

$$K := \bar{B}\Big(w, \frac{1}{2\mu\alpha(m+1)} d_m(C)\Big) \cap \operatorname{affspan}(C).$$

By the definition of $w$ and Lemma 3.18 for every $x \in K$ we have

$$\begin{aligned}
\|x - \tilde{S}(C)\| &\leq \|x - w\| + \|w - \tilde{z}\| + \|\tilde{z} - \tilde{S}(C)\| \\
&\leq \frac{1}{2\mu\alpha(m+1)} d_m(C) + \eta^{m+1} d_m(C) + \frac{1}{2} r_m(C),
\end{aligned}$$

and since by definition $r_m(C) = 8\eta^{m+1}\theta_1(m) d_m(C)$ it thus follows that

$$\|x - \tilde{S}(C)\| \leq (1 + \eta^{m+1}) d_m(C) + \frac{1}{2} r_m(C) \leq r_m(C).$$

Hence, $K \subseteq \bar{B}(\tilde{S}(C), r_m(C))$, and by Lemma 3.19 we also have $K \subseteq C$, so $K \subseteq \tilde{C}$. Since

$$m+1 = \dim(\operatorname{affspan}(C)) = \dim(K) \leq \dim(\tilde{C}) \leq \dim(C) = m+1,$$

we have $\dim(\tilde{C}) = m+1$. By definition (3.1)

$$\begin{aligned}
\delta_{\tilde{C}} &\leq \frac{\lambda\Big(\bar{B}(\tilde{S}(C), r_m(C)) \cap \operatorname{affspan}(C)\Big)}{\lambda\Big(\bar{B}(w, \frac{1}{2\mu\alpha(m+1)} d_m(C)) \cap \operatorname{affspan}(C)\Big)} \\
&= \Big(16\mu\alpha(m+1)\eta^{m+1}\theta_1(m)\Big)^{m+1},
\end{aligned}$$

so $\delta_{\tilde{C}}$ is bounded by a constant that depends only on $m$.

We know that $d_m(\tilde{C}) \le \cdots \le d_0(\tilde{C}) \le \operatorname{diam}(\tilde{C})$, so Proposition 3.10 gives us

$$(\operatorname{diam}(\tilde{C}))^{m+1} \le \xi(m+1)\delta_{\tilde{C}} d_m(\tilde{C})(\operatorname{diam}(\tilde{C}))^m.$$

Since $K \subseteq \tilde{C}$, we have $\frac{1}{\mu\alpha(m+1)} d_m(C) \le \operatorname{diam}(\tilde{C})$. Thus,

$$d_m(C) \le \mu\alpha(m+1) \operatorname{diam}(\tilde{C}) \le \mu\alpha(m+1)\xi(m+1)\delta_{\tilde{C}} d_m(\tilde{C}),$$

and from the boundedness of $\delta_{\tilde{C}}$ the result follows. □

**Lemma 3.21.** *There is an ellipsoid $E_C$ with center $S_X(C)$ such that*

$$E_C \subseteq C \subseteq \beta_1 \circ_{S_X(C)} E_C,$$

*where $\beta_1$ is a constant depending only on $m$.*

*Proof.* Let $\tilde{E}$ be the ellipsoid with center $\tilde{z}$ from Lemma 3.18. From the definitions of $\tilde{C}$ and $S_X(C)$ it follows that $\|S_X(C) - \tilde{S}(C)\| \le r_m(C)$, thus

$$\begin{aligned} \|S_X(C) - \tilde{z}\| &\le \|S_X(C) - \tilde{S}(C)\| + \|\tilde{S}(C) - \tilde{z}\| \\ &\le r_m(C) + \frac{1}{2} r_m(C) \le 2r_m(C) \le \rho d_m(C), \end{aligned} \tag{3.40}$$

where $\rho = 16\eta^{m+1}\theta_1(m)$. From Lemma 3.20 we know that $\dim(\tilde{C}) = m+1$, so $\operatorname{affspan}(\tilde{C}) = \operatorname{affspan}(C)$ and by Proposition 3.7 we have

$$\bar{B}\Big(b(\tilde{C}), \frac{1}{\alpha(m+1)} d_m(\tilde{C})\Big) \cap \operatorname{affspan}(C) \subseteq \tilde{C}.$$

Since $d_m(C) \le \nu d_m(\tilde{C})$ by Lemma 3.20 and $b(\tilde{C}) = S_X(C)$, this implies

$$B := \bar{B}\Big(S_X(C), \frac{1}{\nu\alpha(m+1)} d_m(C)\Big) \cap \operatorname{affspan}(C) \subseteq \tilde{C}.$$

We consider the line passing through $\tilde{z}$ and $S_X(C)$, if $\tilde{z} \ne S_X(C)$, or any line through $\tilde{z}$ in $\operatorname{affspan}(C)$ otherwise. There are two points $v$ and $v'$ in the intersection of this line with the boundary of $B$ in $\operatorname{affspan}(C)$. Without loss of generality the closed line segment with end points $v$ and $\tilde{z}$ contains $S_X(C)$. From the definition of $B$ and (3.40) it follows that

$$\|v - \tilde{z}\| = \|v - S_X(C)\| + \|S_X(C) - \tilde{z}\| \le \frac{1}{\nu\alpha(m+1)} d_m(C) + \rho d_m(C),$$

so that $t := \frac{\|v - S_X(C)\|}{\|v - \tilde{z}\|}$ satisfies

$$1 \geq t \geq \frac{\frac{1}{\nu\alpha(m+1)}}{\frac{1}{\nu\alpha(m+1)} + \rho} = \frac{1}{1 + \rho\nu\alpha(m+1)}. \tag{3.41}$$

Moreover, we have

$$S_X(C) = (1-t)v + t\tilde{z}. \tag{3.42}$$

The ellipsoid

$$E' := (1-t)v + t\tilde{E}$$

then has its center at $S_X(C)$, and since $v \in B \subseteq C$ and $\tilde{E} \subseteq C$, by the convexity of $C$ also $E' \subseteq C$. Thus the set $K$ defined by

$$K = \frac{1}{2}E' + \frac{1}{2}B$$

also is contained in $C$. Furthermore, $K$ is symmetric with respect to $S_X(C)$, so that $b(K) = S_X(C)$. By Proposition 3.8 there is an ellipsoid $E_C$ centered at $S_X(C)$ that satisfies

$$E_C \subseteq K \subseteq \beta(m+1) \circ_{S_X(C)} E_C. \tag{3.43}$$

We show that $E_C$ fulfills the requirements of the lemma.

It immediately follows that $E_C \subseteq K \subseteq C$. By the definition of $E'$ and (3.42) we get

$$\tilde{E} = t^{-1}(E' - (1-t)v) = t^{-1}(E' - S_X(C)) + \tilde{z},$$

and the properties of $\tilde{E}$ from Lemma 3.18 then yield

$$\begin{aligned} C &\subseteq \mu \circ_{\tilde{z}} \tilde{E} + \bar{B}(0, \eta^{m+1} d_m(C)) \\ &= \mu t^{-1}(E' - S_X(C)) + \tilde{z} + \bar{B}(0, \eta^{m+1} d_m(C)) \\ &= (\mu t^{-1}) \circ_{S_X(C)} E' + (\tilde{z} - S_X(C)) + \bar{B}(0, \eta^{m+1} d_m(C)). \end{aligned}$$

By (3.40) we obtain

$$C \subseteq (\mu t^{-1}) \circ_{S_X(C)} E' + \bar{B}\Big(0, (\rho + \eta^{m+1}) d_m(C)\Big),$$

and since $t^{-1} \leq 1 + \rho\nu\alpha(m+1)$ by (3.41), it follows that

$$\begin{aligned} C &\subseteq \Big(\mu(1 + \rho\nu\alpha(m+1))\Big) \circ_{S_X(C)} E' + \bar{B}\Big(0, (\rho + \eta^{m+1}) d_m(C)\Big) \\ &\subseteq \frac{\tau}{2} \circ_{S_X(C)} E' + \bar{B}\Big(0, \frac{\tau}{2} \frac{1}{\nu\alpha(m+1)} d_m(C)\Big) \end{aligned}$$

with $\tau := 2\mu(1+\rho\nu\alpha(m+1))+2(\rho+\eta^{m+1})\nu\alpha(m+1)$. Clearly, $C \subseteq \operatorname{affspan}(C)$, so by the definition of $B$ we hence get

$$\begin{aligned} C &\subseteq \frac{\tau}{2}(E' - S_X(C)) + S_X(C) + \frac{\tau}{2}(B - S_X(C)) \\ &= \tau\Big(\frac{1}{2}E' + \frac{1}{2}B - S_X(C)\Big) + S_X(C) = \tau \circ_{S_X(C)} K. \end{aligned}$$

Finally, from (3.43) it follows that $C \subseteq (\tau\beta(m+1)) \circ_{S_X(C)} E_C$, which is the required inclusion for $\beta_1 := \tau\beta(m+1)$. □

This lemma finishes the proof of the existence of the ellipsoid $E_C$; to complete the proof of Theorem 3.14 we need to show that $S_X$ is Lipschitz continuous. For this we need two more auxiliary lemmas. The proof of the following lemma is adapted from [10, Lemma 5.3].

**Lemma 3.22.** *Assume that the set $C \subseteq X$ is convex, $x \in X$ and $r \geq 0$ such that the intersection of $C$ with $\bar{B}(x,r)$ is nonempty. Then*

$$\Big(C + \bar{B}(0,s)\Big) \cap \Big(\bar{B}(x,2r) + \bar{B}(0,s)\Big) \subseteq C \cap \bar{B}(x,2r) + \bar{B}(0,7s)$$

*for all $s \geq 0$.*

*Proof.* If $r = 0$, then $x \in C$ by assumption and

$$\Big(C + \bar{B}(0,s)\Big) \cap \Big(x + \bar{B}(0,s)\Big) = x + \bar{B}(0,s) \subseteq x + \bar{B}(0,7s)$$

for all $s \geq 0$. To show the lemma for the case $r > 0$ we can without loss of generality assume that $x = 0$. Thus, it suffices to prove

$$\Big(C + \bar{B}(0,s)\Big) \cap \bar{B}(0,2r+s) \subseteq C \cap \bar{B}(0,2r) + \bar{B}(0,7s).$$

Let $y$ be an element of $\Big(C + \bar{B}(0,s)\Big) \cap \bar{B}(0,2r+s)$. There is a point $v \in C$ such that $\|v - y\| \leq s$. If $\|v\| \leq 2r$, then $v \in C \cap \bar{B}(0,2r)$ and hence

$$y \in C \cap \bar{B}(0,2r) + \bar{B}(0,s) \subseteq C \cap \bar{B}(0,2r) + \bar{B}(0,7s).$$

We from now on assume that $\|v\| > 2r$. By the hypothesis of the lemma there is a point $v' \in C \cap \bar{B}(0,r)$. We choose the parameter $t$, where $0 < t < 1$, so that the point

$$v_t := tv' + (1-t)v$$

satisfies $\|v_t\| = 2r$. From the convexity of $C$ it follows that $v_t \in C$. The definition of $v$ implies

$$\|v\| \leq \|y\| + s \leq 2r + 2s, \tag{3.44}$$

so we get the estimate

$$2r = \|v_t\| \leq t\|v'\| + (1-t)\|v\| \leq tr + (1-t)(2r+2s),$$

which yields

$$t \leq \frac{2s}{r+2s}.$$

Thus, by the definition of $v_t$ and (3.44),

$$\begin{aligned}\|v_t - v\| = t\|v - v'\| &\leq \frac{2s}{r+2s}(\|v\| + \|v'\|) \leq \frac{2s}{r+2s}(2r+2s+r) \\ &\leq \frac{2s(3r+6s)}{r+2s} = 6s,\end{aligned}$$

and

$$\|v_t - y\| \leq \|v_t - v\| + \|v - y\| \leq 6s + s = 7s.$$

Since $v_t \in C \cap \bar{B}(0, 2r)$, this implies $y \in C \cap \bar{B}(0, 2r) + \bar{B}(0, 7s)$. □

**Lemma 3.23.** *Let $C_1$ and $C_2$ be convex subsets of $X$, and let $x_1, x_2 \in X$ and $r_1, r_2 \geq 0$. If $C_1 \cap \bar{B}(x_1, r_1) \neq \emptyset$ and $C_2 \cap \bar{B}(x_2, r_2) \neq \emptyset$, then*

$$\begin{aligned}&d_H\Big(C_1 \cap \bar{B}(x_1, 2r_1), C_2 \cap \bar{B}(x_2, 2r_2)\Big) \\ &\qquad\qquad \leq 7\Big(d_H(C_1, C_2) + \|x_1 - x_2\| + 2|r_1 - r_2|\Big).\end{aligned}$$

*Proof.* We define $s$ by

$$s = d_H(C_1, C_2) + \|x_1 - x_2\| + 2|r_1 - r_2|.$$

From the definition of the Hausdorff distance it follows that

$$C_2 \subseteq C_1 + \bar{B}(0, d_H(C_1, C_2)) \subseteq C_1 + \bar{B}(0, s).$$

Furthermore, since for all $y \in \bar{B}(x_2, 2r_2)$ we have

$$\begin{aligned}\|y - x_1\| &\leq \|y - x_2\| + \|x_2 - x_1\| \leq 2r_2 + \|x_2 - x_1\| \\ &\leq 2r_1 + 2|r_2 - r_1| + \|x_2 - x_1\| \leq 2r_1 + s,\end{aligned}$$

we get

$$\bar{B}(x_2, 2r_2) \subseteq \bar{B}(x_1, 2r_1 + s) = \bar{B}(x_1, 2r_1) + \bar{B}(0, s).$$

These inclusion relations together with Lemma 3.22 then yield

$$\begin{aligned}C_2 \cap \bar{B}(x_2, 2r_2) &\subseteq \Big(C_1 + \bar{B}(0, s)\Big) \cap \Big(\bar{B}(x_1, 2r_1) + \bar{B}(0, s)\Big) \\ &\subseteq C_1 \cap \bar{B}(x_1, 2r_1) + \bar{B}(0, 7s).\end{aligned}$$

Clearly, this also implies that

$$C_1 \cap \bar{B}(x_1, 2r_1) \subseteq C_2 \cap \bar{B}(x_2, 2r_2) + \bar{B}(0, 7s),$$

so

$$d_H\Big(C_1 \cap \bar{B}(x_1, 2r_1), C_2 \cap \bar{B}(x_2, 2r_2)\Big) \leq 7s$$

as claimed. $\square$

**Lemma 3.24.** *There is a constant* $\theta_1(m+1)$ *depending only on* $m$ *such that*

$$\|S_X(C_1) - S_X(C_2)\| \leq \theta_1(m+1) d_H(C_1, C_2)$$

*for all* $C_1, C_2 \in \mathcal{K}_{m+1}(X)$.

*Proof.* Let $C$ be an element of $\mathcal{K}_{m+1}(X)$. If $\dim(C) \leq m$, then $\tilde{S}(C) = S_X(C) \in C$. If $\dim(C) = m+1$, then by Lemma 3.17 in the case $d_0(C) \leq \eta^m d_m(C)$ we have $C \subseteq \bar{B}(\tilde{S}(C), 2^{-1} r_m(C))$, and in the case $d_0(C) > \eta^m d_m(C)$ by Lemma 3.18 there is a point $\tilde{z} \in C$ such that $\|\tilde{z} - \tilde{S}(C)\| \leq 2^{-1} r_m(C)$. Hence in all cases it follows that

$$\bar{B}\Big(\tilde{S}(C), \frac{1}{2} r_m(C)\Big) \cap C \neq \emptyset.$$

By Lemma 3.23 for all $C_1, C_2 \in \mathcal{K}_{m+1}(X)$ we have

$$\begin{aligned} d_H(\tilde{C_1}, \tilde{C_2}) &= d_H\Big(C_1 \cap \bar{B}\Big(\tilde{S}(C_1), r_m(C_1)\Big), C_2 \cap \bar{B}\Big(\tilde{S}(C_2), r_m(C_2)\Big)\Big) \\ &\leq 7\Big(d_H(C_1, C_2) + \|\tilde{S}(C_1) - \tilde{S}(C_2)\| + |r_m(C_1) - r_m(C_2)|\Big). \end{aligned}$$

Inequality (3.31) gives us

$$\|\tilde{S}(C_1) - \tilde{S}(C_2)\| \leq \theta_1(m) d_H(C_1, C_2),$$

and from Corollary 3.3 it follows that

$$|r_m(C_1) - r_m(C_2)| = 8\eta^{m+1}\theta_1(m)|d_m(C_1) - d_m(C_2)| \leq 8\eta^{m+1}\theta_1(m) d_H(C_1, C_2).$$

Thus, from the definition of $S_X$ and Theorem 3.11 we get

$$\begin{aligned} \|S_X(C_1) - S_X(C_2)\| = \|b(\tilde{C_1}) - b(\tilde{C_2})\| &\leq \gamma(m+1)(\delta_{\tilde{C_1}} + \delta_{\tilde{C_2}}) d_H(\tilde{C_1}, \tilde{C_2}) \\ &\leq \gamma(m+1)(\delta_{\tilde{C_1}} + \delta_{\tilde{C_2}}) 70\eta^{m+1}\theta_1(m) d_H(C_1, C_2). \end{aligned}$$

For every $C \in \mathcal{K}_{m+1}(X)$ the regularity coefficient $\delta_{\tilde{C}}$ is bounded by a constant depending only on $m$. If $\dim(C) = m+1$, this follows from Lemma 3.17 in the case $d_0(C) \leq \eta^m d_m(C)$ and Lemma 3.20 otherwise. For sets $C$ with dimension less than or equal to $m$ we have $\tilde{C} = \{S_X(C)\}$, so $\delta_{\tilde{C}} = 1$. $\square$

This lemma finishes the proof of Theorem 3.14.

## 3.4 Proof of the main theorem

We are now ready to prove Theorem 3.1. We recall that $(M, \rho)$ is a pseudometric space, where the pseudometric $\rho$ takes values in $[0, +\infty]$; by $\mathcal{A}_m(X)$ we denote the collection of affine subspaces of the Banach space $X$ that have dimension at most $m$. The theorem states that a map $F : X \to \mathcal{A}_m(X)$ has a Lipschitz continuous selection with Lipschitz constant $\kappa(m)$ if every restriction of $F$ to a finite subset of $M$ consisting of at most $2^{m+1}$ points has a Lipschitz continuous selection with Lipschitz constant 1.

The proof works by induction on $m$. For a map $F : M \to \mathcal{A}_0(X)$ satisfying the theorem's hypothesis for $m = 0$, the selection is trivial and has Lipschitz constant $\kappa(0) = 1$. We now assume that the theorem is shown up to dimension $m \in \mathbb{N}_0$.

Let $F : M \to \mathcal{A}_{m+1}(X)$ be a map such that every restriction $F|_N$ to a finite set $N \subseteq M$ with at most $2^{m+2}$ points has a Lipschitz continuous selection with Lipschitz constant 1. We have to show that $F$ has a Lipschitz selection that has a Lipschitz constant $\kappa(m + 1)$ depending only on $m$.

Let $v$ and $w$ be two distinct points in $M$. By assumption the restriction $F|_{\{v,w\}}$ has a Lipschitz selection with constant 1, that is, there are points $a_1(v, w) \in F(v)$ and $a_2(v, w) \in F(w)$ such that

$$\|a_1(v, w) - a_2(v, w)\| \leq \rho(v, w). \tag{3.45}$$

The set

$$C(v, w) := F(v) \cap \Big(F(w) + a_1(v, w) - a_2(v, w) + \bar{B}(0, 2\rho(v, w))\Big) \tag{3.46}$$

is then nonempty, closed, convex and symmetric with respect to $a_1(v, w)$.

**Lemma 3.25.** *The set $C(v, w)$ defined in (3.46) can be represented as*

$$C(v, w) = \bigcap_{i \in I(v,w)} \Big(F(v) \cap (L_i + \bar{B}(0, r_i))\Big),$$

*where $I(v, w)$ is a suitable index set and for all $i \in I(v, w)$ we have $r_i \in [0, +\infty]$ and $L_i \in \mathcal{A}_m(X)$ such that $L_i \subseteq F(v)$ and $a_1(v, w) \in L_i$.*

*Proof.* If $\dim(F(v)) = 0$, then $C(v, w) = F(v) = \{a_1(v, w)\}$ and the representation is trivial. In the case $\dim(F(v)) \geq 1$ we assume without loss of generality that $F(v)$ is a linear subspace of $X$ with origin $a_1(v, w) = 0$. We choose a set of indices $I(v, w)$ and affine subspaces $L_i$ such that

$$\{L_i : i \in I(v, w)\} = \{L \in \mathcal{A}_m(X) : 0 \in L \text{ and } L \subseteq F(v)\}.$$

Moreover, for every $i \in I(v,w)$ we define $r_i \in [0,+\infty]$ by

$$r_i = \inf\{r \geq 0 : \, C(v,w) \subseteq L_i + \bar{B}(0,r)\}.$$

Clearly, those definitions imply

$$C(v,w) \subseteq \bigcap_{i \in I(v,w)} \Big(F(v) \cap (L_i + \bar{B}(0,r_i))\Big),$$

To prove the opposite inclusion let $x$ be a point in $F(v) \setminus C(v,w)$. We have to show that there is at least one index $i$ such that $L_i + \bar{B}(0,r_i)$ does not contain $x$.

We consider an inner product $\langle \cdot, \cdot \rangle$ on the finite-dimensional space $F(v)$. By the hyperplane separation theorem (see [11, Theorem 1.3.4] for example), there are a vector $z \in F(v) \setminus \{0\}$ and a scalar $s \in \mathbb{R}$ such that

$$\langle x, z \rangle > s, \quad \text{and} \quad \langle y, z \rangle < s \quad \text{for all} \quad y \in C(v,w). \tag{3.47}$$

There is an index $i \in I(v,w)$ such that

$$L_i = \{y \in F(v) : \, \langle y, z \rangle = 0\}.$$

The affine hyperplane $H \subseteq F(v)$ defined by

$$H = \{y \in F(v) : \, \langle y, z \rangle = s\}$$

is parallel to $L_i$. Hence, there is a real number $r > 0$ such that for all $y \in H$ we have $\operatorname{dist}(y, L_i) = r$, where the distance is calculated with respect to the original norm on $X$.

Since $C(v,w)$ is symmetric with respect to the origin, (3.47) implies that $|\langle y, z \rangle| < s$ for all $y \in C(v,w)$. It follows that $\operatorname{dist}(y, L_i) < r$ for all $y \in C(v,w)$ and $\operatorname{dist}(x, L_i) > r$. Thus, $r_i \leq r$ and the point $x$ is not contained in $L_i + \bar{B}(0,r_i)$. $\square$

We define the set

$$\tilde{M} := \{\tilde{v} = (v,w,i) : \, v,w \in M, \, i \in I(v,w)\}$$

and put $p_1(\tilde{v}) := v$, $p_2(\tilde{v}) := w$ and $i(\tilde{v}) := i$ for $\tilde{v} = (v,w,i) \in \tilde{M}$. For two distinct points $\tilde{v}$ and $\tilde{v}'$ in $\tilde{M}$ we define their distance

$$\tilde{\rho}(\tilde{v}, \tilde{v}') := \rho(p_1(\tilde{v}), p_1(\tilde{v}')) + r_{i(\tilde{v})} + r_{i(\tilde{v}')}$$

and we set $\tilde{\rho}(\tilde{v}, \tilde{v}) := 0$, making $(\tilde{M}, \tilde{\rho})$ a pseudometric space.

Moreover, we define the map $\tilde{F} : \tilde{M} \to \mathcal{A}_m(X)$ by setting

$$\tilde{F}(\tilde{v}) := L_{i(\tilde{v})} \tag{3.48}$$

for all $\tilde{v} \in \tilde{M}$.

**Lemma 3.26.** *The map $\tilde{F}$ defined by* (3.48) *has a Lipschitz continuous selection $\tilde{f} : (\tilde{M}, \tilde{\rho}) \to (X, \|\cdot\|)$ with Lipschitz constant $\kappa(m)$, where $\kappa(m)$ is the constant from Theorem 3.1.*

*Proof.* We can use the induction assumption for $m$ if we show that $\tilde{F} : \tilde{M} \to \mathcal{A}_m(X)$ satisfies the hypothesis of Theorem 3.1. Let $\tilde{N}$ be a subset of $\tilde{M}$ with $|\tilde{N}| \leq 2^{m+1}$. We have to show that there exists a selection $\tilde{f}_{\tilde{N}}$ of $\tilde{F}|_{\tilde{N}}$ with Lipschitz constant 1.

The set $N := p_1(\tilde{N}) \cup p_2(\tilde{N})$ is a subset of $M$ satisfying $|N| \leq 2^{m+2}$. By the assumptions on the map $F$, its restriction $F|_N$ has a Lipschitz continuous selection $f_N : N \to X$ with Lipschitz constant 1. For all $v, w \in N$ we have, by (3.45),

$$\begin{aligned} &\|f_N(v) - f_N(w) - a_1(v,w) + a_2(v,w)\| \\ &\qquad\qquad \leq \|f_N(v) - f_N(w)\| + \|a_1(v,w) - a_2(v,w)\| \leq 2\rho(v,w), \end{aligned}$$

so the definition (3.46) of $C(v,w)$ implies that $f_N(v) \in C(v,w)$. Hence, by Lemma 3.25 and the definition of $\tilde{F}$, for every $\tilde{v} \in \tilde{M}$ it follows that

$$f_N(p_1(\tilde{v})) \in L_{i(\tilde{v})} + \bar{B}(0, r_{i(\tilde{v})}) = \tilde{F}(\tilde{v}) + \bar{B}(0, r_{i(\tilde{v})}).$$

In order to define a selection $\tilde{f}_{\tilde{N}} : \tilde{N} \to X$, for every $\tilde{v} \in \tilde{N}$ we can thus choose a point $\tilde{f}_{\tilde{N}}(\tilde{v}) \in \tilde{F}(\tilde{v})$ such that

$$\left\|\tilde{f}_{\tilde{N}}(\tilde{v}) - f_N(p_1(\tilde{v}))\right\| = \operatorname{dist}\left(f_N(p_1(\tilde{v})), \tilde{F}(\tilde{v})\right) \leq r_{i(\tilde{v})}.$$

Then for all $\tilde{v}, \tilde{v}' \in \tilde{N}$ we have

$$\begin{aligned} \left\|\tilde{f}_{\tilde{N}}(\tilde{v}) - \tilde{f}_{\tilde{N}}(\tilde{v}')\right\| &\leq \left\|f_N(p_1(\tilde{v})) - f_N(p_1(\tilde{v}'))\right\| + r_{i(\tilde{v})} + r_{i(\tilde{v}')} \\ &\leq \rho(p_1(\tilde{v}), p_1(\tilde{v}')) + r_{i(\tilde{v})} + r_{i(\tilde{v}')} = \tilde{\rho}(\tilde{v}, \tilde{v}'), \end{aligned}$$

which shows the required Lipschitz continuity of $\tilde{f}_{\tilde{N}}$. □

In the following lemma we use that the Banach space $X$ is a space $\ell_\infty(U)$ of bounded real-valued functions on some set $U$ (see the explanation after the statement of Theorem 3.1 at the beginning of the chapter). The proof of the lemma is based on [12, Lemma 3.2].

**Lemma 3.27.** *Let $K$ be a map from a pseudometric space $(M, \rho)$ to the collection of all closed balls in $X = \ell_\infty(U)$ with radius $r \in [0, +\infty]$. Assume that for all $v, w \in M$ the restriction $K|_{\{v,w\}}$ has a Lipschitz continuous selection with constant* 1. *Then the map $K$ also has a Lipschitz continuous selection with Lipschitz constant* 1.

*Proof.* We first prove the lemma for the case $X = \mathbb{R}$, where $K$ is a map from $M$ to $\mathcal{K}(\mathbb{R}) \cup \{\mathbb{R}\}$. For $v \in M$, we set

$$a(v) := \inf\{t : t \in K(v)\} \in \mathbb{R} \cup \{-\infty\}$$

and

$$b(v) := \sup\{t : t \in K(v)\} \in \mathbb{R} \cup \{+\infty\},$$

so that we either have $K(v) = [a(v), b(v)]$ or $K(v) = \mathbb{R}$.

By assumption, for all $v, w \in M$ there are $s \in K(v)$ and $t \in K(w)$ such that $|s - t| \leq \rho(v, w)$. Thus, it follows that

$$a(v) \leq s \leq \rho(v, w) + t \leq \rho(v, w) + b(w). \tag{3.49}$$

We define

$$\tilde{k}(v) := \inf_{w \in M} (b(w) + \rho(v, w)),$$

then $\tilde{k}(v) \leq b(v)$ and by (3.49) we also have $a(v) \leq \tilde{k}(v)$.

We show that $\tilde{k}(v) > -\infty$ for all $v$. If $\tilde{k}(v) < +\infty$, then there is a point $v' \in M$ such that $b(v') + \rho(v, v') < +\infty$. This implies $\rho(v, v') < +\infty$, $b(v') < +\infty$ and hence also $a(v') > -\infty$. Thus, by the triangle inequality and (3.49), for all $w \in M$ we have

$$b(w) + \rho(w, v) \geq b(w) + \rho(v', w) - \rho(v, v') \geq a(v') - \rho(v, v') > -\infty,$$

which proves the claim.

Since it is possible that $\tilde{k}(v) = +\infty$, we define the function $k : M \to \mathbb{R}$ by setting

$$k(v) := \begin{cases} \tilde{k}(v) & \text{if } \tilde{k}(v) \in \mathbb{R}, \\ 0 & \text{if } \tilde{k}(v) = +\infty \end{cases}$$

for all $v \in M$. If $v \in M$ such that $\tilde{k}(v) = +\infty$, then $K(v) = \mathbb{R} \ni 0$, otherwise $\tilde{k}(v) \in [a(v), b(v)]$, so $k$ is indeed a selection of $K$. It remains to show that $k : M \to \mathbb{R}$ is Lipschitz continuous.

Let $v$ and $w$ be points in $M$. If $\tilde{k}(v) = \tilde{k}(w) = +\infty$, then $|k(v) - k(w)| = |0 - 0| \leq \rho(v, w)$. If $\tilde{k}(v) < +\infty$ and $\tilde{k}(w) = +\infty$, then there is a point $v' \in M$ such that

$$b(v') + \rho(v, v') < +\infty, \quad \text{but} \quad b(v') + \rho(w, v') = +\infty.$$

This implies $\rho(v, v') < +\infty$ and $\rho(w, v') = +\infty$. Therefore, we can conclude that $|k(v) - k(w)| \leq +\infty = \rho(v, w)$.

In the case $\tilde{k}(v), \tilde{k}(w) \in \mathbb{R}$, for a given $\varepsilon > 0$ we pick $w' \in M$ such that

$$b(w') + \rho(w, w') \leq k(w) + \varepsilon.$$

Then, again using the triangle inequality, we obtain

$$k(v) - k(w) \le b(w') + \rho(v, w') - b(w') - \rho(w, w') + \varepsilon \le \rho(v, w) + \varepsilon.$$

Similarly, we can show that $k(w) - k(v) \le \rho(v, w) + \varepsilon$ for all $\varepsilon > 0$, which implies $|k(v) - k(w)| \le \rho(v, w)$. Hence, $k$ is the required selection with Lipschitz constant 1.

Now we consider the general case $X = \ell_\infty(U)$. For every $v \in M$ the image of $v$ under $K$ is then of the form

$$K(v) = \bar{B}(f, r) = \{g \in \ell_\infty(U) : |g(u) - f(u)| \le r \,\forall u \in U\}$$

for some $f \in \ell_\infty(U)$ and $r \in [0, +\infty]$. For $u \in U$ we set

$$K_u(v) := \bar{B}(f(u), r) \subseteq \mathbb{R},$$

then $K_u : M \to \mathcal{K}(\mathbb{R}) \cup \{\mathbb{R}\}$ satisfies the hypotheses of the lemma with $X = \mathbb{R}$. By the result of the first part of the proof there is a selection $k_u$ of $K_u$ such that

$$|k_u(v) - k_u(w)| \le \rho(v, w)$$

for all $v, w \in M$. We define $k(v)(u) := k_u(v)$ for all $v \in M$, so that $k : M \to \ell_\infty(U)$ is the required selection. $\square$

**Lemma 3.28.** *Let $\tilde{f} : \tilde{M} \to X$ be the function from Lemma 3.26. There is a Lipschitz continuous function $g : M \to X$ with Lipschitz constant $\kappa(m)$ such that*

$$\left\| g(v) - \tilde{f}(\tilde{v}) \right\| \le \kappa(m) r_{i(\tilde{v})}$$

*for all $\tilde{v} \in \tilde{M}$ with $p_1(\tilde{v}) = v$.*

*Proof.* For $\tilde{v}, \tilde{v}' \in \tilde{M}$ we set

$$\hat{\rho}(\tilde{v}, \tilde{v}') := \kappa(m)\, \rho\Big(p_1(\tilde{v}), p_1(\tilde{v}')\Big),$$

then $\hat{\rho} : \tilde{M} \to [0, +\infty]$ is a pseudometric. We define the map $K$ on $\tilde{M}$ by putting

$$K(\tilde{v}) := \bar{B}\Big(\tilde{f}(\tilde{v}), \kappa(m) r_{i(\tilde{v})}\Big)$$

for all $\tilde{v} \in \tilde{M}$. By the Lipschitz continuity of $\tilde{f}$ on $(\tilde{M}, \tilde{\rho})$, for all $\tilde{v}, \tilde{v}' \in \tilde{M}$ we have

$$\left\| \tilde{f}(\tilde{v}) - \tilde{f}(\tilde{v}') \right\| \le \kappa(m)\, \tilde{\rho}(\tilde{v}, \tilde{v}') = \kappa(m)\Big(\rho\Big(p_1(\tilde{v}), p_1(\tilde{v}')\Big) + r_{i(\tilde{v})} + r_{i(\tilde{v}')}\Big).$$

Thus, there are points $x \in K(\tilde{v})$ and $x' \in K(\tilde{v}')$ such that

$$\|x - x'\| \leq \kappa(m)\, \rho\Big(p_1(\tilde{v}), p_1(\tilde{v}')\Big) = \hat{\rho}(\tilde{v}, \tilde{v}').$$

The map $K$ hence satisfies the hypotheses of Lemma 3.27, so there is a selection $\tilde{g} : (\tilde{M}, \hat{\rho}) \to X$ that satisfies

$$\Big\|\tilde{g}(\tilde{v}) - \tilde{g}(\tilde{v}')\Big\| \leq \hat{\rho}(\tilde{v}, \tilde{v}') = \kappa(m)\, \rho\Big(p_1(\tilde{v}), p_1(\tilde{v}')\Big)$$

for all $\tilde{v}, \tilde{v}' \in \tilde{M}$. If $p_1(\tilde{v}) = p_1(\tilde{v}')$, we therefore have $\tilde{g}(\tilde{v}) = \tilde{g}(\tilde{v}')$. For a given $v \in M$ we pick $\tilde{v} \in \tilde{M}$ such that $p_1(\tilde{v}) = v$ and set

$$g(v) := \tilde{g}(\tilde{v}).$$

The map $g : (M, \rho) \to X$ is then well-defined, Lipschitz continuous with Lipschitz constant $\kappa(m)$, and satisfies

$$\Big\|g(v) - \tilde{f}(\tilde{v})\Big\| = \Big\|\tilde{g}(\tilde{v}) - \tilde{f}(\tilde{v})\Big\| \leq \kappa(m) r_{i(\tilde{v})}$$

for every $v \in M$ and $\tilde{v} \in \tilde{M}$ with $p_1(\tilde{v}) = v$. $\square$

With the help of the function $g$ from Lemma 3.28 we define a map $G : M \to \mathcal{K}_{m+1}(X)$ by setting

$$G(v) := F(v) \cap \bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v))\Big) \tag{3.50}$$

for every $v \in M$. We show that $G : (M, \rho) \to (\mathcal{K}_{m+1}(X), d_H)$ is Lipschitz continuous. In order to do this we need another technical lemma.

**Lemma 3.29.** *For all $v, w \in M$ we have*

$$G(v) \subseteq F(w) + \bar{B}\Big(0, 11\kappa(m)\rho(v, w)\Big);$$

*moreover, it holds that*

$$\Big|\operatorname{dist}(g(v), F(v)) - \operatorname{dist}(g(w), F(w))\Big| \leq 12\kappa(m)\rho(v, w).$$

*Proof.* For fixed $v$ and $w$ in $M$ we set

$$r := \inf\{r_i : i \in I(v, w)\},$$

where $I(v, w)$ is the index set from Lemma 3.25. We pick a point $x \in F(v)$ such that

$$\|g(v) - x\| = \operatorname{dist}(g(v), F(v)).$$

Assume that $i \in I(v,w)$ and set $\tilde{v} := (v,w,i) \in \tilde{M}$. The function $\tilde{f}$ from Lemma 3.26 is a selection of $\tilde{F}$; we thus have $\tilde{f}(\tilde{v}) \in L_i \subseteq F(v)$. From Lemma 3.28 it follows that

$$\operatorname{dist}(g(v), F(v)) \leq \|g(v) - \tilde{f}(\tilde{v})\| \leq \kappa(m) r_i. \tag{3.51}$$

Since this inequality holds for all $i \in I(v,w)$, we have

$$\|g(v) - x\| = \operatorname{dist}(g(v), F(v)) \leq \kappa(m) r.$$

This yields

$$\bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v))\Big) \subseteq \bar{B}(g(v), 2\kappa(m)r) \subseteq \bar{B}(x, 3\kappa(m)r). \tag{3.52}$$

We recall the points $a_1(v,w) \in F(v)$ and $a_2(v,w) \in F(w)$ defined in (3.45). We have $a_1(v,w) \in L_i$ for all $i \in I(v,w)$. From (3.51) it follows that

$$\|x - \tilde{f}(\tilde{v})\| \leq 2\kappa(m) r_i.$$

Thus, since $r \leq r_i$ for all $i \in I(v,w)$ and $L_i - a_1(v,w)$ is a linear subspace, we have

$$\begin{aligned}
\bar{B}(x, 3\kappa(m)r) &\subseteq \bar{B}\Big(\tilde{f}(\tilde{v}), 5\kappa(m)r_i\Big) \subseteq L_i + \bar{B}(0, 5\kappa(m)r_i) \\
&= 5\kappa(m)(L_i - a_1(v,w)) + a_1(v,w) + \bar{B}(0, 5\kappa(m)r_i) \\
&= 5\kappa(m) \circ_{a_1(v,w)} \Big(L_i + \bar{B}(0, r_i)\Big).
\end{aligned}$$

Note that $\kappa(m) \geq 1 > 0$. Thus,

$$\begin{aligned}
\bar{B}(x, 3\kappa(m)r) \cap F(v) &\subseteq 5\kappa(m) \circ_{a_1(v,w)} \bigg( \bigcap_{i \in I(v,w)} \Big( (L_i + \bar{B}(0, r_i)) \cap F(v) \Big) \bigg) \\
&= 5\kappa(m) \circ_{a_1(v,w)} C(v,w).
\end{aligned}$$

By (3.46), this implies

$$\begin{aligned}
&\bar{B}(x, 3\kappa(m)r) \cap F(v) \\
&\qquad \subseteq 5\kappa(m) \circ_{a_1(v,w)} \Big( F(w) + a_1(v,w) - a_2(v,w) + \bar{B}(0, 2\rho(v,w)) \Big)
\end{aligned}$$

and hence, since $F(w) - a_2(v,w)$ is a linear subspace of $X$ and by (3.45),

$$\begin{aligned}
\bar{B}(x, 3\kappa(m)r) \cap F(v) &\subseteq F(w) + a_1(v,w) - a_2(v,w) + \bar{B}(0, 10\kappa(m)\rho(v,w)) \\
&\subseteq F(w) + \bar{B}(0, 11\kappa(m)\rho(v,w)).
\end{aligned}$$

Together with (3.52) this shows the inclusion relation. Moreover, since $x \in F(v)$, this implies that there is a point $y \in F(w)$ such that $\|x - y\| \leq 11\kappa(m)\rho(v,w)$. Thus,

$$\begin{aligned}\operatorname{dist}(g(w), F(w)) &\leq \|g(w) - y\| \leq \|g(w) - g(v)\| + \|g(v) - x\| + \|x - y\| \\ &\leq \kappa(m)\rho(v,w) + \operatorname{dist}(g(v), F(v)) + 11\kappa(m)\rho(v,w) \\ &= \operatorname{dist}(g(v), F(v)) + 12\kappa(m)\rho(v,w).\end{aligned}$$

The same inequality holds if we interchange $v$ and $w$, so the required inequality follows. $\square$

**Lemma 3.30.** *The map $G$ defined by* (3.50) *satisfies*

$$d_H(G(v), G(w)) \leq 175\kappa(m)\rho(v,w)$$

*for all $v, w \in M$.*

*Proof.* We fix $v$ and $w$ in $M$ and set $s := 25\kappa(m)\rho(v,w)$. From Lemma 3.29 it follows that

$$G(w) \subseteq F(v) + \bar{B}\Big(0, 11\kappa(m)\rho(v,w)\Big) \subseteq F(v) + \bar{B}(0,s),$$

and by definition (3.50), the Lipschitz continuity of $g$ and again Lemma 3.29 we have

$$\begin{aligned}G(w) &\subseteq \bar{B}\Big(g(w), 2\operatorname{dist}(g(w), F(w))\Big) \\ &\subseteq \bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v)) + 24\kappa(m)\rho(v,w)\Big) + \bar{B}(0, \kappa(m)\rho(v,w)) \\ &= \bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v))\Big) + \bar{B}(0,s).\end{aligned}$$

Thus, by Lemma 3.22 we have

$$\begin{aligned}G(w) &\subseteq \Big(F(v) + \bar{B}(0,s)\Big) \cap \Big(\bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v))\Big) + \bar{B}(0,s)\Big) \\ &\subseteq F(v) \cap \bar{B}\Big(g(v), 2\operatorname{dist}(g(v), F(v))\Big) + \bar{B}(0,7s) = G(v) + \bar{B}(0,7s).\end{aligned}$$

In the same way it follows that $G(v) \subseteq G(w) + \bar{B}(0,7s)$, so we can conclude that

$$d_H(G(v), G(w)) \leq 7s = 175\kappa(m)\rho(v,w)$$

as claimed. $\square$

Using the selector $S_X$ from Theorem 3.14, we now put

$$f(v) := S_X(G(v))$$

for all $v \in M$. This function $f : M \to X$ is clearly a selection of $F$; to finish the proof of Theorem 3.1 we have to show that $f$ is Lipschitz continuous with a Lipschitz constant depending only on $m$. By the Lipschitz continuity of $S_X$ on $\mathcal{K}_{m+1}(X)$ and the previous lemma we have

$$\begin{aligned}\|f(v) - f(w)\| = \|S_X(G(v)) - S_X(G(w))\| &\le \theta_1(m+1)\, d_H(G(v), G(w)) \\ &\le 175\kappa(m)\theta_1(m+1)\rho(v,w) = \kappa(m+1)\rho(v,w)\end{aligned}$$

for all $v, w \in M$ with $\kappa(m+1) := 175\kappa(m)\theta_1(m+1)$. This concludes the proof of Theorem 3.1. $\square$

# Appendix A

# Paracompact spaces

In this chapter we recall some basic topological properties and state an important result that characterizes paracompactness. In the following, $T$ is a topological space.

We say that $T$ is a $T_1$ *space* if for every pair of distinct points $u_1$ and $u_2$ of $T$ there exists a neighborhood $U_1$ of $u_1$ such that $u_2$ is not an element of $U_1$. By interchanging $u_1$ and $u_2$ this implies that there is also a neighborhood $U_2$ of $u_2$ not containing $u_1$.

The space $T$ is a *Hausdorff space* if for every pair of distinct points $u_1, u_2 \in T$ there are neighborhoods $U_1$ of $u_1$ and $U_2$ of $u_2$ such that $U_1 \cap U_2 = \emptyset$.

A collection $\mathcal{U}$ of subsets of $T$ is a *cover* of $T$ if $T \subseteq \bigcup_{U \in \mathcal{U}} U$. The cover is an *open cover* if all sets $U \in \mathcal{U}$ are open. The cover is *locally finite* if every point $u \in T$ has some neighborhood $V \subseteq T$ such that $V \cap U = \emptyset$ for all but finitely many $U$ in $\mathcal{U}$.

A cover $\mathcal{V}$ of $T$ is a *refinement* of the cover $\mathcal{U}$ if for every set $V \in \mathcal{V}$ there is a set $U \in \mathcal{U}$ such that $V \subseteq U$.

We say that a collection $\mathcal{P}$ of continuous functions $p : T \to [0, 1]$ is a *partition of unity* if

$$\sum_{p \in \mathcal{P}} p(u) = 1$$

for all $u \in T$. The partition of unity is *locally finite* if for every $u \in T$ there is a neighborhood $V$ of $u$ such that all but finitely many $p \in \mathcal{P}$ vanish on $V$. The partition of unity $\mathcal{P}$ is *subordinated* to the cover $\mathcal{U}$ if for every $p \in \mathcal{P}$ there is an element $U \in \mathcal{U}$ such that $p$ vanishes outside of $U$.

Finally, a Hausdorff space is *paracompact* if every open cover of it has a locally finite open refinement. The following proposition goes back to Michael [7, Proposition 2]; more on the topic can be found in the book by Engelking [4, Chapter 5].

**Proposition A.1.** *For a $T_1$ space $T$ the following properties are equivalent:*

(i) *$T$ is paracompact.*

(ii) *Every open cover of $T$ has a locally finite partition of unity subordinated to it.*

(iii) *Every open cover of $T$ has a partition of unity subordinated to it.*

# Appendix B

# Finite-dimensional Banach spaces

## B.1 Lebesgue measure on finite-dimensional Banach spaces

This section is based on Albiac and Kalton [1, Section 13.1]. We assume that $Y$ is an $n$-dimensional Banach space for some $n \geq 0$ and $A : \mathbb{R}^n \to Y$ is a linear isomorphism. By $\lambda^n$ we denote the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. Then we can define a measure $\lambda_A$ on the Borel $\sigma$-algebra of $Y$ via

$$\lambda_A(C) := \lambda^n\Big(A^{-1}(C)\Big)$$

for every Borel set $C \subseteq Y$. We call such a measure a *Lebesgue measure* on $Y$. For two isomorphisms $A, \tilde{A} : \mathbb{R}^n \to Y$ it follows that

$$\lambda_{\tilde{A}}(C) = \lambda^n\Big(\tilde{A}^{-1}AA^{-1}(C)\Big) = |\det(\tilde{A}^{-1}A)|\, \lambda_A(C)$$

for every Borel set $C$, and hence two such measures are equal up to multiplication by a positive constant. This implies that for two measurable sets $C_1, C_2 \subseteq Y$ the ratio

$$\frac{\lambda_A(C_1)}{\lambda_A(C_2)}$$

does not depend on the choice of the isomorphism $A$.

The Lebesgue measure $\lambda_A$ satisfies

$$\lambda_A(tC + x) = t^n \lambda_A(C)$$

for all $t \geq 0$, $x \in Y$ and measurable $C \subseteq Y$. In the case $n = 0$, when $Y$ consists only of a point, the Lebesgue measure is a positive constant.

## B.2 The Bochner integral

The Bochner integral is a way to define an integral for functions that take values in Banach spaces, see [1, Appendix K] for example. In our case we look at an $n$-dimensional Banach space $(Y, \|\cdot\|)$, where $n \in \mathbb{N}_0$, equipped with a Lebesgue measure $\lambda$ on the $\sigma$-algebra of all Borel sets in $Y$.

Assume we have a measurable function $f : Y \to Y$. Then $\|f\| : Y \to \mathbb{R}$ is measurable and we can consider

$$\|f\|_1 := \int_Y \|f(x)\| \,\mathrm{d}\lambda(x).$$

If we identify functions that are equal almost everywhere, then $\|\cdot\|_1$ is a norm on the space of measurable functions $f : Y \to Y$ satisfying $\|f\|_1 < \infty$. This space is called the space of *Bochner integrable* functions and denoted by $L_1(\lambda; Y)$.

A *simple function* is a function $s : Y \to Y$ of the form

$$s = \sum_{i=1}^{n} x_i \chi_{K_i},$$

where $n \in \mathbb{N}$, $x_i \in Y$ and $K_i$ are Borel sets with finite measure for $1 \leq i \leq n$. The space of simple functions $S(\lambda; Y)$ is a dense subspace of $L_1(\lambda; Y)$. For a simple function $s = \sum_{i=1}^{n} x_i \chi_{K_i}$ we define

$$I(s) := \sum_{i=1}^{n} x_i \lambda(K_i).$$

The map $I : S(\lambda; Y) \to Y$ is a bounded linear operator with norm one and can hence be uniquely extended to $L_1(\lambda; Y)$. For $f \in L_1(\lambda; Y)$ we say that $I(f)$ is the *Bochner integral* of $f$ and use the notation

$$I(f) = \int_Y f(x) \,\mathrm{d}\lambda(x).$$

The boundedness implies

$$\left\| \int_Y f(x) \,\mathrm{d}\lambda(x) \right\| \leq \int_Y \|f(x)\| \,\mathrm{d}\lambda(x)$$

for all $f \in L_1(\lambda; Y)$.

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