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„Kinetic theory applied to skiers"

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Ai miei genitori e Vera,
per avermi mostrato cosa voglia dire amare
in modo incondizionato una figlia e una sorella.
Grazie, perchè senza di voi non sarei ciò che sono
e non sarei mai arrivata fin qui.
Vi devo tutto.

Abstract

Emergent phenomena are appearances of large-scale structure from underlying microscopic dynamics. At the microscopic level, particles interact according to some rules, which can not be clearly seen from the macroscopical point of view. Despite the widespread of emergence in nature (think of bird flocks, ant trials, opinion dynamics, tumor growth), it is only in the last few years that scientists started understanding it. The mathematical tools applied in order to study this emergence derives from kinetic theory. Originally, this theory was developed to study gas dynamics in Mathematical Physics. The application of these tools to explore questions coming from biology, poses many new interesting challenges at the level of the modelling and mathematical analysis. In this paper, the main goal consists of studying a particular type of emergent phenomena, which involves a large crowd of skiers who line up at the end of a slope while waiting for their turn on ski lifts.

Zusammenfassung

Entstehende Phänomene sind Erscheinungen einer großflächigen Struktur aus der zugrunde liegenden mikroskopischen Dynamik. Auf mikroskopischer Ebene interagieren Partikel nach einigen Regeln, die aus makroskopischer Sicht nicht deutlich sichtbar sind. Trotz des weit verbreiteten Auftretens in der Natur (Vogelherden, Ameisenstudien, Meinungsdynamik, Tumorwachstum) haben die Wissenschaftler erst in den letzten Jahren damit begonnen, es zu verstehen. Die mathematischen Werkzeuge, die zur Untersuchung dieser Entstehung verwendet werden, stammen aus der kinetischen Theorie. Ursprünglich wurde diese Theorie entwickelt, um die Gasdynamik in der Mathematischen Physik zu untersuchen. Die Anwendung dieser Werkzeuge zur Erforschung von Fragen aus der Biologie, bringt viele neue interessante Herausforderungen in der Modellierung und mathematischen Analyse hervor. In diesem Beitrag besteht das Hauptziel in der Untersuchung einer bestimmten Art von Emergenz-Phänomenen, die eine große Menge von Skifahrern involviert, die sich am Ende einer Piste aufstellen und warten, bis sie am Skilift an der Reihe sind.

Acknowledgements: I would like to thank my Supervisor, Professor Christian Schmeiser, who helped me creating the model, always shared encouraging words and understood what I meant when I arrived to his office in February 2019 and wanted to model skiers at Italian ski resorts, which are far outdated compared to the Austrian ones.

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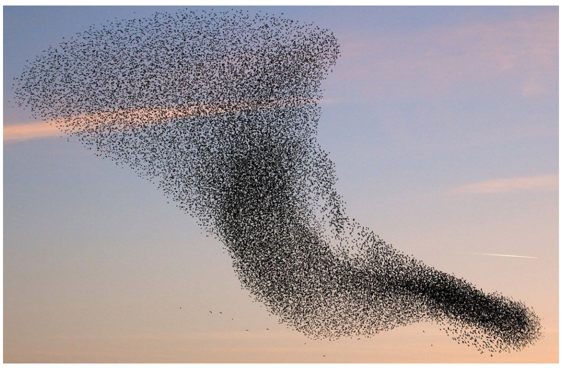
1 Introduction

I would like to start by introducing a question that can straightforwardly clarify the difference between a microscopic and a macroscopic point of view. In the book "Sapiens: a Brief History of Humankind" by Yuval Noah Harari, the author arises the following question: "Human cultures are continuously changing. But are these changes random or do they follow a particular model? In other words, does history have a direction to follow?". Indeed, it does and to be precise it follows the direction of creating a unity. To observe this phenomenon, it is required to look at history in order of millenniums, since if we consider it in the scale of decades or even centuries, we will not see this tendency towards unity. We will later identify the millenniums as the macroscopic scale, while decades and centuries as microscopic. A small example is now presented in order to clear up this concept. This will be based on Christianity and the Latin language. The presence of both increased drastically in a little amount of time: in fact, one converted hundreds of millions of people around the world, and the other became diffused in the whole central and western Europe. But for each of them, it arrived a time where the growth stopped and got split into smaller subgroups, as Christianity shattered in many sects, while Latin divided in many dialects that later became national languages, such as Italian, French or Spanish.
Thus, looking at a scale of centuries, we could establish that for both cases the tendency is to grow, become very big and then split in many small pieces. So what about the convergence towards unity we mentioned before? In order to analyse this aspect, we have to leave the microscopic time scale and use the macroscopic one, which, as previously discussed, uses millenniums instead of centuries. Furthermore, to sense the general direction of history, the author suggests to count the number of the so called "human separated worlds" coexisting at a certain time on our planet. Harari says that around 10.000BC we could count around thousands of them, while in 2.000BC they were only hundreds and in 1450AD even less. From this point of view, "it is crystal clear", says Harari, "that history is moving towards unity; the split of Christianity and of the Latin language are just small slowdowns in the highway of history". In order to understand some concepts that will be discussed later on this paper, it is essential to keep in mind this difference between the two above discussed time scales.

Related to the previously stated concepts, we now introduce the definition of emergence, which refers to the fact that local interaction rules between some agents (e.g. birds, fish, bacteria, humans, skiers), like attraction, alignment, repulsion, etc., give rise to a large scale collective behaviour. Usually this large scale structure is not recognisable from the microscopic point of view. Typical examples of emergence phenomena are pattern formation, e.g. cells that are organised and form a tissue, coordination, e.g. any kind of collective motion (birds forming bird flocks or fish forming fish schools) and self-organisation, e.g. pedestrian dynamics (people lining up forming lanes in a crowded street).



(a) Bird flocks creating astonishing patterns.

(b) Pedestrian dynamics in a busy station.

Figure 1: Examples of some emergent phenomena.

As already mentioned before, it is difficult to explain how the macroscopic, also called observable, dynamics emerge from the microscopic ones. Imagine seeing a very large bird flock in the sky: you will be fascinated by the various forms the flock assumes over time, but you will not be able to define to behaviour of each single bird. The only way to study emergence phenomena, hence the arising of bird flocks from the underlying dynamics of each single bird, is by applying kinetic theory. What steps have to be undertaken in order to achieve this?

At first, we study the particle model, which is a good tool to reproduce the observed phenomena. However, the number of agents involved is really high and makes simulations computationally costly, considering the amount of detail contained at this level. Therefore, by letting the number of particles tend to infinity and by using the mean field limit, we obtain a kinetic description of our phenomena. By applying some rescaling to our so obtained kinetic equations, we will be able to see that in front of some terms forming it will appear parameters and hence, letting the so obtained parameters go to zero and with the use of the hydrodynamic limit, we attain the macroscopic description, which is what we are interested in finding since hydrodynamic (macroscopic) models are typically easier to simulate and analyse. At the end, validating a model means that it holds on every kind of scale.

Before moving on, I would like to shortly point out a few considerations. On one hand, moving from the particle description to the macroscopic one through the kinetic model, causes information loss step by step. On the other hand, the advantage of starting from a particle description consists of it being the natural place to model: at this level, we know everything about the dynamics of each single particle that composes the phenomena at every time step. This means that we have a differential equation describing how each particle moves. The problem arises as soon as the number of particles is substantial: in this case, we will have a consistently large system of differential equations, which will be computationally hard to solve and thus it would be tortuous to undertake a stability analysis. Moreover, when the number of particles is large, it is impossible to know the exact position of each them. A solution to this problem regarding untreatability, consists, as we already said before, of using the mean field limit in order to reach the kinetic description. A mean-field equation, or kinetic equation, is a model that describes the evolution of a typical particle subject to the collective interaction created by a large number N of other particles. The state of the particle is given by its space density. The force field exerted by the N other particles on a single one is approximated by the average, with respect to the space density, of the force field exercised on that particle from each point in the space. Therefore, at the kinetic description we have a partial differential equation, which is easier to study than a system of ODEs. Two negative aspects arise from this description: on one hand, we are still modelling on the microscopic scale (thus not the one where we see the emergence phenomena), while on the other this approximation is only valid in case of a large number of particles composing the system. To overcome the first problem, we use the hydrodynamic limit to reach the macroscopic level. At this point, the complete information of the model obtained at the particle level is all averaged. Note that now we are at the right level of description to observe emergent phenomena and it is possible to perform stability analysis.

Usually, the process of moving from the particle to the kinetic description is relatively easy, which means that we have a sort of recipe describing each step. But passing from the kinetic to the macroscopic level is considered to be complicated and not always possible. Mathematically, one of the main challenges is the lack of conserved quantities, which is typically a key ingredient in the computation of the macroscopic limit. This lack of conservation is frequent in biological systems, which do not conserve momentum. One strategy to overcome this is to use the concept of generalised collision invariants (GCI), which was introduced for the first time in [1] to derive a hydrodynamic model of the Vicsek dynamics.

The kind of emergence phenomenon we are going to study involves skiers. Indeed, once a skier arrives at the end of the slope, he desires to reach the turnstile as soon as possible, so that he can take the chairlift and ski again. The emergence arises as soon as the ski resort is particularly crowded: especially during weekends or under Christmas holidays, it happens to queue for a long period of time to reach the turnstile. It also needs to be underlined that skiers arrive at the queuing area from different parts of the slope. In order to model the skiers at the microscopic level, we assume that they follow some precise rules (which we will analyse in more detail in the next chapter): for example, if two skiers are not too far away and there is a probability that their skis are going to intersect in a short period of time, they will both turn in such a way that avoids interaction. On the contrary, if intersection between the skiers cannot be avoided by changing the skis' direction due to the closeness of the two individuals, then, other that turning, they also change the position of the skis. Note that we do not see these rules from macroscopic structures and, therefore, it is a challenge to explain how the observable dynamics emerge from the microscopic ones. Thus, we have that every skier shares the same goal, the turnstile (hence the desired common direction is the chairlift), but each one comes from different positions of the slopes. As an aerial photo would show, during a crowded day at a ski resort, a creation of clusters inside the group will be likely to take place. Each one of these differs from the other in terms of orientation of the skis: a skier belongs to a particular cluster, if the direction of his skis is the same as the one of the cluster itself.

We can see the similarities of the emergence phenomenon we are interested in and the movement towards unity that Harari is writing about. At the beginning, if we look at a small time scale, we can notice that in the group of skiers waiting to reach the turnstile, there are small subgroups who share the same direction, which can be compared to all the smaller dialects that were born from Latin language or the numerous sects that spread from Christianity. But, if we consider a bigger time scale, we see that all the skiers from each subgroup share the same direction: they are all pointing to get to the same position. In this way, we can assert that in a macroscopic scale they all move towards unity, since they all share the same destination. And this is exactly the message that the book "Sapiens: a Brief History of Humankind" is giving.

In the figures below we can see two examples of the emergence phenomena we are interested in modelling and studying. They are both areal views of skiers lining up in two different ski resorts. The picture on the left, which, compared to the second one, represents less precisely the emergence phenomenon we are interested in studying, has been taken at the ski resort where my family and I always go skiing during winter break. Also, it is the location where the idea of creating and studying this model came to mind. The picture on the right, instead, explains clearly the situation we are interested in. Indeed, on the left of this picture, we can see a number of skiers lining up. There is a fence delimiting the queuing area, which is a technique adopted by big ski resorts to be able to keep everything in order. However, the interesting part of the picture consists of the right-hand side. As a matter of fact, we can clearly see that the skiers reach the queue from every point of the slope, to then line up and wait until they arrive to the lift and are able to take it to climb the mountain up again.

(a) Queuing in Champoluc. (b) Lining up at two ski lifts.

Figure 2: Examples of the emergent phenomenon we are interested in.

The rest of the paper is structured as follows: in Section (2) we are going to describe the particle model and present some simulations we did on Octave. From this microscopic description we will derive the nondimensionalised kinetic equation in section (3). Moreover, in this section we will obtain some important results, in particular approximations of the earlier derived equations, that will be useful while determining the macroscopic equations, which we will do at the end of the document with section (4). In this last section we will use a definition of generalised collision invariant which differs from the one proposed by Pierre Degond in [1], since ours is more general and applies nicely in this work.

2 Microscopic model

Before heading to the model, some assumptions regarding the skis have to be made. First of all, we assume that every skier has just a single ski, meaning that a skier in this model would resemble more a snowboarder in reality. Furthermore, the shape of every skier's ski is simplified to be a rectangle with length ℓ and of width w.
The skis can move in every direction in the two dimensional plane: they can, therefore, move both sideways and frontways. Moreover, each of them will have its own direction, depending on its position in the queuing area. The direction of a ski determines where its head and tail are: the head is where it is headed, while the tail is opposite of the head. This fact implies that we have nematic alignment, which means that, depending on the angle two skis intersect, they will turn in such a way that at the end they are parallel or antiparallel to each other: if the angle between two skis' heads is acute, then they will slowly turn to become parallel, while if the angle is obtuse, then they become antiparallel. This nematic alignment rule in physics typically results from volume exclusion and can be understood as an inelastic collision between polar particles.
Note that making two skis antiparallel is reasonable since we assume that there are two different ski lifts, one in the opposite direction of the other. This is motivated by the fact that if a skier happens to be in the wrong direction compared to all the others, then there are not much possibilities to turn completely around, i.e. doing a 180 degree turn, to get to the lift where everyone else is headed.

In order to start modelling, we consider N skiers in the two dimensional plane $\mathbb{R}^2$, that are labeled by i, with $i = 1, ..., N$. Each skier, and in particular the i-th skier, has a center of mass, which position is denoted by $x_i(t) \in \mathbb{R}^2$, and the orientation of his velocity is given by $\varphi_i(t) \in \mathbb{T}$. Moreover, for each skier, we have a direction he wants to go to, which is in our case the turnstile, and we denote that by $\phi(x_i)$.
We also assume that each skier can be seen as a self-propelled particle, meaning that the velocity of each individual is different and depends on the difference between the skier's actual direction and the direction he wants to go to, i.e. the speed S of the i-th skier depends on φ_i and $\phi(x_i)$ respectively. This can be explained in the following way: if, on one hand, a skier has nobody around, then he directly goes in the direction of the turnstile and therefore the difference between the two angles is small. He will choose the most direct route and since there is no one around his speed is high and hence he will get to the lift quickly. On the other hand, if a skier has many other skiers around, then the route chosen by him will be the one consisting of possibly avoiding the others, thus bringing him to decrease his own velocity. We can notice that if the argument of S is small, S will be a large value while if the difference between the two angles is big, the value of the function S is small.

As we will see in the next section, the particle model which we will analyse is composed by two differential equation for each skier: one describes the changes in the positions of the center of mass of the skis, while the other expresses the changes in the orientations over time. Both differential equations are formed by two terms: a so called active term, which contributes to the change in position and orientation directly, and a interaction term, which arises as soon as two skis intersect.
Interactions between skiers are binary, meaning that there can only be interaction between a pair of skiers. The interaction terms have the following meaning: if the center point of the skis are far enough apart, then the skiers will just turn leaving the center points exactly where they are and hence the interaction term is equal to zero. However, if the center points are too close to one another, then by only changing their direction we will not obtain a substantial difference, meaning that they still would be intersecting. Thus, other than turning, they have to move their center points sideways in order to not cross anymore and this is expressed by the interaction term.

2.1 Individual based model

We will now introduce the model we are interested in studying. It is a system of ordinary differential equations, which for every $i = 1, \dots, N$ is composed by two equations. These two describe the change of the positions of the center of masses of the skis and of the velocities' orientation of each skier over time. Therefore, the particle dynamics of ski i, for $i = 1, \dots, N$, reads:

$$\dot{x}_i = \nu_{max1} S(d_{torus}(\varphi_i, \phi(x_i)))\omega(\varphi_i) + \sum_{j \neq i} \psi_x(B^x_{ij}) \frac{x_i - x_j}{|x_i - x_j|} \tag{2.1.1}$$

$$\dot{\varphi}_i = k_1 \frac{d_{torus}(\phi(x_i), \varphi_i)}{\pi} \operatorname{Sign}(\sin(\phi(x_i) - \varphi_i)) + \sum_{j \neq i} \psi_\varphi(B^\varphi_{ij}) \operatorname{Sign}(\sin 2(\varphi_i - \varphi_j)) \tag{2.1.2}$$

where we use the notation such that for every angle φ_i, $\omega(\varphi_i) = \omega_i = (\cos \varphi_i, \sin \varphi_i)$ and where ν_{max1} is the maximal forward speed of a skier; we put this factor in front because of dimensional issues. Similarly, the factor k_1, which is in front of the active term of the second differential equation, represents the maximal angular velocity. Moreover, we can notice that each of the two equations is formed by two parts: the first terms on the right hand side are the active terms, while the second ones are those arising from interactions between skiers. Each of these terms has its own magnitude and direction it operates towards. For instance, the interaction term of the angles has magnitude $\psi_\varphi(B^\varphi_{ij})$ and direction $\operatorname{Sign}(\sin 2(\varphi_i - \varphi_j))$.

As we already said before, we can immediately see that the first term on the right hand side of (2.1.1) is the active part and in particular it is a self-propelling force of the skis, which has magnitude $\nu_{max1} S(d_{torus}(\varphi_i, \phi(x_i)))$ in the direction of the ski, i.e. $\omega(\varphi_i)$. We will define S in such a way that if a skier is in the wrong direction and with many others around then he will not move at all until he has room to slowly start turning. We could therefore say that the skier could stop already when he is in the semicircle of wrong direction; that would mean when the difference to the desired goal is bigger than $\pi/2$. Therefore, what we are looking for could be expressed in the following way:

$$S(d_{torus}(\varphi_i, \phi(x_i))) = \left(1 - \frac{d_{torus}(\varphi_i, \phi(x_i))}{\pi/2}\right)_+ = \max\left\{0, \left(1 - \frac{d_{torus}(\varphi_i, \phi(x_i))}{\pi/2}\right)\right\}.$$

Thus, he slows down when he is away from optimal direction and when he is orthogonal to optimal direction he stops completely. Note that we used d_{torus} and remember that the distance between two angles is not easily definable on a torus $\mathbb{T}$. Hence, how do we write it? How do we compute it?

At first sight we could say that $d_{torus}(\varphi, \phi) = |\varphi - \phi - 2k\pi|$, where k is chosen to be the minimum over all k; therefore:

$$d_{torus}(\varphi, \phi) = \min_k |\varphi - \phi - 2k\pi|.$$

If $d_{torus}(\varphi, \phi) \leqslant \pi/2$, then the distance between the two angles is between $[0, \pi/2]$, where $\pi/2$ represents complete orthogonality between the two skiers. In the case when the distance between desired and actual direction of the ski is at most $\pi/2$ (meaning that the skier's actual orientation is not that bad) we have that his speed is the fastest (which means equal to 1 when the difference is zero). As the distance gets nearer and nearer to $\pi/2$, the velocity $S(d_{torus}(\varphi_i, \phi(x_i)))$ decreases, and when the distance is exactly $\pi/2$, the skier stops and continues staying exactly where he is, which means not moving his center point at all, as long as d_{torus} is bigger than $\pi/2$.

From the skiers point of view, this means exactly what we already said above, i.e. when a skier, whose distance between the two directions is at least $\pi/2$ (meaning that he is not looking toward his goal), is surrounded by other skiers, he does neither move nor turn at all. As soon as no one is around anymore, he will not move his center of mass, but instead he just turns (i.e. changes φ_i and therefore decreases the d_{torus}) until the point when his actual direction and the desired direction differ by an angle which is smaller than $\pi/2$. At this point, in addition to turning his skis, he also slowly start moving his center of mass and hence moving.

The second term of (2.1.1) gives the interaction force, that is given by binary short ranged repulsion $\psi_x(B^x_{ij})$ in the normalised direction $x_i - x_j/|x_i - x_j|$. The function ψ_x tells how much the neighbours of skier i influence him: if he is close to his neighbours, then they are going to influence each other more than if they would be far away from him. Hence, if we have that $B^x_{ij} := 1 - |x_i - x_j|/w > 0$, then we have a bad situation (therefore we call the term B^x_{ij} badness of the center of masses), which means that ski i is intersecting ski j. In other words, when $w > |x_i - x_j|$ the two center of masses x_i and x_j are too close to avoid the intersection of the skis only by turning; they also have to move their centers away. Note that when B^x_{ij} is exactly zero, it means that the skis are exactly side by side. Therefore, $\psi_x(B^x_{ij})$ can be defined in the following way:

$$\psi_x(B^x_{ij}) = \psi_x\left(1 - \frac{|x_i - x_j|}{w}\right) = v_{\max 2}\left(1 - \frac{|x_i - x_j|}{w}\right)_+ = v_{\max 2}\max\left\{0, 1 - \frac{|x_i - x_j|}{w}\right\},$$

where $v_{\max 2}$ is again a maximal velocity. We put this factor in front because $\left(1 - |x_i - x_j|/w\right)_+$ is dimensionless. The meaning of ψ_x is the following: when two skis are not too near (i.e. $w < |x_i - x_j|$) then they do not intersect and therefore the interaction term should have the value of zero. Otherwise, if $w > |x_i - x_j|$, then the strength of the interaction term becomes more and more important as the difference between w and $|x_i - x_j|$ becomes bigger (which means that the skis become nearer and nearer) and the probability to cross with the skis gets higher.

Equation (2.1.2) describes the time evolution of the direction of skier i. Regarding the active term, and as we said before, we have to put a factor of k_1 in front because all the other terms composing the active part are dimensionless. Indeed, the term $d_{\text{torus}}(\phi(x_i) - \varphi_i)$ is normalised by π, which is the maximal value d_{torus} can have. Therefore the whole fraction is normalised to maximum 1.

The second term, as for equation (2.1.1), gives the interaction force, but this time it is a bit more complex to derive. In order to do that, we need to measure how strong we have violated the constraint, which means how much of a ski is above another. This involves not only the φ_i's but also the x_i's. The basic idea is that we have two lines, the skis, and we have to check if they intersect and, if they do, we are interested in knowing exactly where they do and how much of a ski is crossing the other one's. At this point, for each ski $i = 1, \ldots, N$, we have to consider the smaller part it is divided into during an intersection; this is what we are going to call badness of the orientations, i.e. B^φ_{ij}. Note that while finding the orientation badness' we do not really consider the skis to be rectangles. Indeed, if we want to be precise and consider the skis as rectangles, we should consider the overlapping area, meaning that B^φ_{ij} should be an area. In order to simplify this matter, the overlapping length will instead be taken into account. Hence, we just draw a line from the top to the bottom of the ski exactly in the center of it, passing through the center of mass. Let us call this segment the core of the ski. Thus, we say that two skis intersect if their cores do. Note that, as we can see from figure (3), the two areas of the skis can overlap, but it does not imply that the two cores do. This means that, in our case, we do not consider this as intersection.

So, for skier i, we can parametrize the core of his ski as $x_i + \lambda_i\omega(\varphi_i)$. We can proceed in the same way to parametrize the skis of the j-th skier: $x_j + \lambda_j\omega(\varphi_j)$. Now, we know that there will be an intersection of the skis if there exist λ_i, λ_j such that:

$$x_i + \lambda_i\omega(\varphi_i) = x_j + \lambda_j\omega(\varphi_j). \tag{2.1.3}$$

At this point we can define the badness of the orientations as the minimum of the four part of the two skis that an intersection determines, i.e. $\min\left\{\frac{\ell}{2} - \lambda_i, \frac{\ell}{2} + \lambda_i, \frac{\ell}{2} - \lambda_j, \frac{\ell}{2} + \lambda_j\right\}$, which can be written in a normalised and more compact way as:

$$B^\varphi_{ij} = \min\left\{\frac{1}{2} - \frac{|\lambda_i|}{\ell}, \frac{1}{2} - \frac{|\lambda_j|}{\ell}\right\}.$$

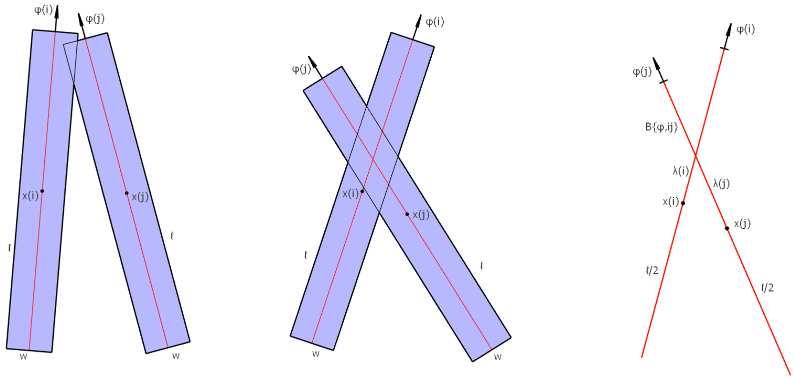


Figure 3: On the left, skis that are not intersecting. In the middle, two skis that cross. On the right, intersecting cores.

Remark 1. *Note that $\frac{\ell}{2} - \lambda_i$ is the distance from the front of the ski to the intersection, and $\frac{\ell}{2} + \lambda_i$ is the distance from back side of the ski. Hence, we can say that:*

- *If λ_i and λ_j are both positive, then the intersection will be for both skis in the front half of them;*
- *If λ_i and λ_j are one positive and one negative, then the intersection will be between the front part of one skier's ski and the back part of the other's ski;*
- *If λ_i and λ_j are both negative, then the intersection will be for both skis in the back half of them.*

At this point we can say that for every pair of skiers', meaning two skis in total, there can be defined a badness B_{ij}^{φ}. Note that negative badness means actual goodness because if one of the λs is bigger than $\ell/2$, then the two ski do not interact with each other, or better, their cores cross far away, i.e. computing the intersection point of the two lines, we see that the crossing point is far apart and thus we do not have to worry about it, since they will not intersect.

In order to compute the badness, we need to write λ_i and λ_j explicitly. To this aim, we introduce the following notation: $\omega(\varphi_i)^{\perp} = \omega_i^{\perp} = (-\sin\varphi_i, \cos\varphi_i)^T$. Then, we see that if we multiply (2.1.3) by $\omega_j^{\perp}$ we get λ_i, while in the case when we multiply it by $\omega_i^{\perp}$ we obtain λ_j. Specifically, we have that:

$$\lambda_i = \frac{(x_j - x_i) \cdot \omega_j^{\perp}}{\omega_i \cdot \omega_j^{\perp}} \qquad \lambda_j = \frac{(x_i - x_j) \cdot \omega_i^{\perp}}{\omega_j \cdot \omega_i^{\perp}} = \frac{(x_j - x_i) \cdot \omega_i^{\perp}}{\omega_i \cdot \omega_j^{\perp}}.$$

Obviously, the two formulas for the λs work only if the two skis are not parallel: in that case the scalar product which appears at the denominator of both λ_i and λ_j would be zero.
Using the definitions of the λs, we can write the badness as function of the ski center's position and orientation, i.e. $B_{ij}^{\varphi}(x_i, x_j, \varphi_i, \varphi_j)$. One can note that since in the formula for B_{ij}^{φ} we take the absolute value of λs, it is symmetric in i and j. Now, we do not want that when the badness turns into goodness the skiers move in the opposite direction; we want them to follow their paths. Hence, we do not want to have directly the badness of the orientations, but a function of it, and thus we introduce $\psi_{\varphi}(B_{ij}^{\varphi})$. As before, we want to have that when the badness B_{ij}^{φ} is big, the function ψ_{φ} is large, and when the badness becomes negative, ψ_{φ} just goes to zero. This is because a skier does not want to touch the tips of the others and therefore we want the interaction term to be zero when there is no intersection. So, what we obtain at the end is a function which has the same shape of ψ_x, i.e.

$$\psi_{\varphi}(B_{ij}^{\varphi}) = k_2 (B_{ij}^{\varphi})_+ = k_2 \max\{0, B_{ij}^{\varphi}\},$$

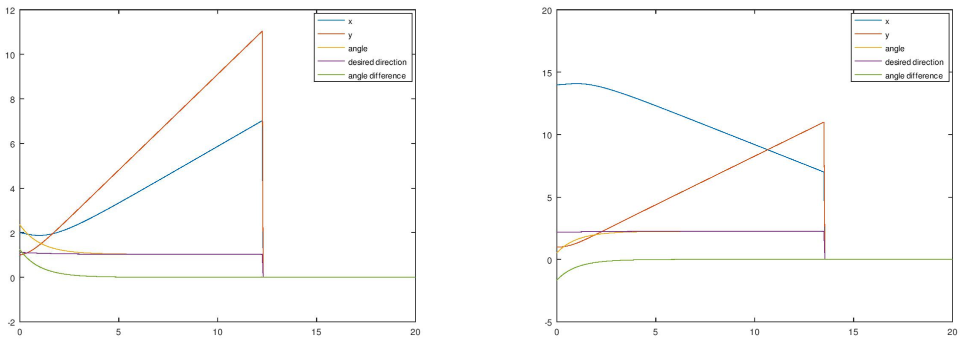
where k_2 has the dimension of an angular velocity, since B_{ij}^{φ} is dimensionless. To get to the final formula, we still have to write the direction in which the orientation's badness works. To do that, we need to see how parallel or antiparallel the two skis are and what they have to achieve total parallelism or antiparallelism. Hence, we say that when the product between the cos and the sin of the difference between the two angles φ_i and φ_j is positive, then φ_i gets smaller, i.e. the two skis become parallel, while if the product is negative, φ_i gets bigger, meaning that they become antiparallel. Since we need the product of cos and sin, we can use the duplication formula of the sin. Thus, finally, we have come to explain equation (2.1.2).

In conclusion, we could say that the first of the two equations, i.e. (2.1.1), describing the particle model expresses the spatial motion of the particle i. Moreover, it is composed by two parts: the first one describes the velocity generated by self propulsion, which has magnitude $v_{max1}S(d_{torus}(\varphi_i, \phi(x_i)))$ in direction $\omega(\varphi_i)$. The second term is a repulsive force.
The second equation, i.e. (2.1.2), describes the time evolution of the orientation of a skier's ski. Its first term simply describes the active change of orientation, while the second one describes the changes of the orientations due to interaction with other skiers.

2.2 Simulations

In the following section we are going to present some simulations we did of the model we studied above, i.e. equations (2.1.1) and (2.1.2). Firstly, we will present the case where we just have one skier. Afterwards, we show simulations regarding a pair of skiers, followed by one of four skier and at the very end we will see a figure regarding seven skiers.

From the figures below, we can see that when we have only one skier, he goes directly towards his goal. At the beginning, when the two directions are differing from each other, his center of mass does not change its position a lot, while his direction is. As soon as the direction of the skier gets near to the desired one, he stars speeding up (which we can see by the increasing steepness of the lines describing the evolution of x and y, that the two components of the center of mass x_1), until he gets to the chairlift, where at the end he stops. In both cases the goal is $(7, 11)$, $h = 0.05$, $T = 20$, where T is the final time and h is the temporal step size.

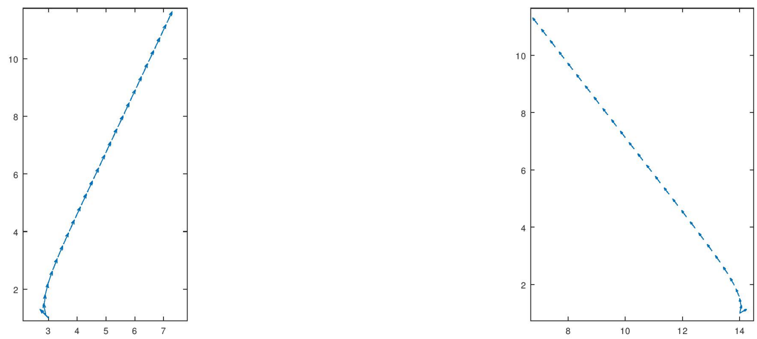


(a) $x = 2$, $y = 1$ and $\varphi_1 = 3\pi/4$.

(b) $x = 14$, $y = 1$ and $\varphi_1 = \pi/6$.

Figure 4: Simulations of the case when we only have one skier. Two different initial conditions are shown.

If we want to have a different and more clear look at what happens, it might be useful to look at the next figures. Note that every parameter remains unvaried exception made for h, which we took to be equal to 0.5 instead of 0.05. Therefore, the temporal step size is bigger. Moreover, notice that the actual values of center of masses $x_= (x, y)$ are the one starting from the bottom of the arrow and not at the top; hence the arrows start from the center of the ski and represent the front half of it. This, together with the fact that the step size is bigger, can be explained by an argument of convenience: in the case the arrows are bigger than the actual distance between the center point in consecutive times, we could not appreciate the real positions of the center of masses over time.

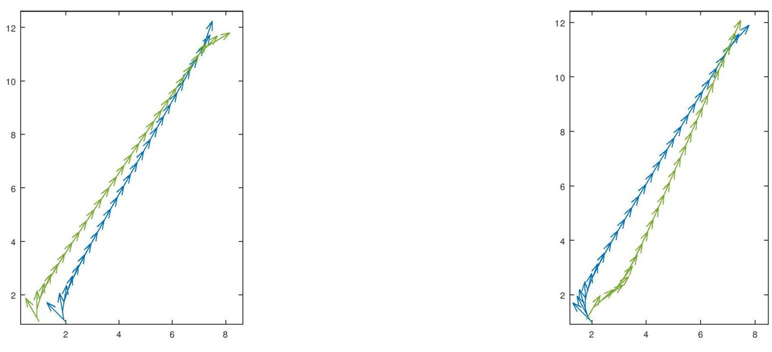


(a) $x_1 = (2, 1)$ and $\varphi_1 = 3\pi/4$.

(b) $x_1 = (14, 1)$ and $\varphi_1 = \pi/6$.

Figure 5: Paths of a skier with two different initial conditions.

Regarding the two next figures, we can see simulations regarding two skiers. On one hand, in the one on the left we can that they want to reach the same goal and during their whole itinerary they never intersect. On the other hand, in the figure on the right, we have that initially the two skiers' skis cross. As a consequence, the interaction terms are not zero and due to that we can see that they modify their paths. In both cases the goal is $(7, 11)$, while the other parameters are $w = 0.1$, $\ell = 1.8$, $h = 0.5$ and $T = 20$. In the picture on the left $x_1 = (2, 1)$, $\varphi_1 = 3\pi/4$ for skier 1 (the blue one) $x_2 = (1, 1)$, $\varphi_2 = 2\pi/3$ for skier 2 (the green one), while for the picture on the right we changed the initial conditions only for skier 2, which are now $x_2 = (1.8, 1.1)$, $\varphi_2 = \pi/3$, leaving the initial conditions of skier 1 untouched. Both paths intersect at their goal, i.e. $(7, 11)$, as it should be.



(a) Case with no interactions.

(b) Case with initial intersection.

Figure 6: Simulations of the case when we have two skiers.

Since the two pictures are a bit small, it is not very clear see that in the second case the two skis are intersecting at the beginning. Thus, in Figure (7), we present an enlargement of the initial condition of Figure (6b).

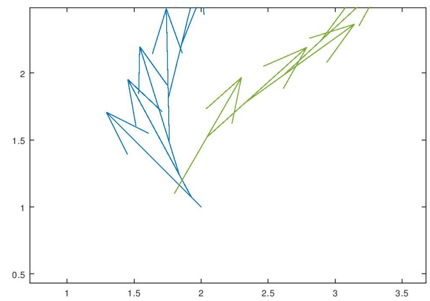


Figure 7: Initial conditions are intersecting.

As we can see from Figure (6a), skier 1 starts with the same initial conditions as in Figure (5a) and since he does not meet skier 2, the evolution over time of his position and orientation is the same as in one skier case.
How come the two skis are not intersecting at the end, when they are both near to the chairlift? Since they are very near to each other, they should avoid it. However, this does not happen because they do not reach their shared goal at the same time. We can clearly see that from the picture below: on the abscissa we have the time, while on the ordinate axis we have the coordinates of the skiers' positions. The graph has to be read in the following way: the position over time of skiers 1 is given by both the blue and red line. The first one represents the evolution over time of the x-coordinate composing the position of skier 1, while the second one is the y-coordinate. We can see that he reaches his goal exactly when the blue line arrives at 7 and the red one at 11, which is exactly his final goal. Similarly, we can do the same for skier 2. Now, we can clearly see that skier 1 reaches the final destination in 12.5 units of time, while skier 2 take a little more, 13. Hence, the intersection terms do not act.

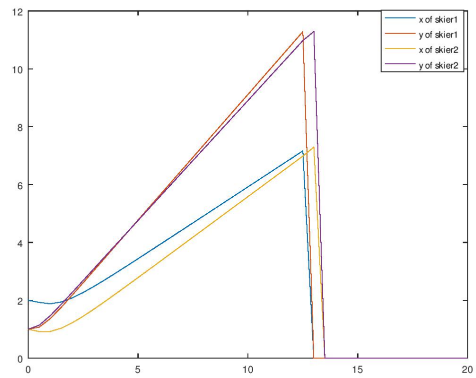


Figure 8: Graph showing the time it takes the two skiers to reach the goal.

The next figure shows four paths of four different skiers who reach their goal, which is again $(7, 11)$. Again, similarly to the case when we had two skiers, we have $h = 0.5$, $T = 11$, $w = 0.1$ and $\ell = 1.8$. The initial conditions are the following:

- skier 1, who is magenta, starts from $x_1 = (2, 2)$ with orientation $\varphi_1 = \pi/3$;
- skier 2, who is blue, starts from $x_2 = (3.2, 3.5)$ with orientation $\varphi_2 = 280°$;
- skier 3, who is green, starts from $x_3 = (4, 2)$ with orientation $\varphi_3 = \pi/3$;

- skier 4, who is black, starts from $x_4 = (5, 3)$ with orientation $\varphi_4 = 280°$.

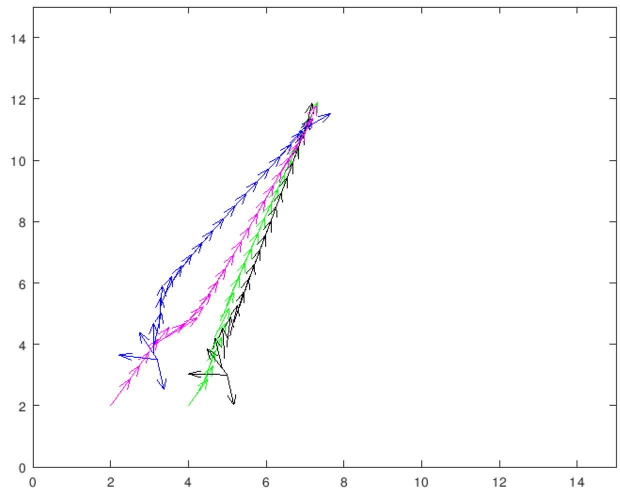


Figure 9: Paths of four skiers reaching the shared goal.

We can see that regarding the interaction between skier 1 and skier 2, since they are farther apart than skier 3 and skier 4, only the orientations' interaction appears. But for the other two skiers also the interaction term of the center of masses is used and we clearly see that by the jumps that appear in the position of skier 4 between the third and fourth time step.

Lastly, we will present the case of a group of seven skiers who reach their goal. As in the cases above, the turnstile is in position $(7, 11)$ and we have that $h = 0.5$ and $T = 11$. Moreover, as before, the length of the ski is $\ell = 1.8$ and the width is $w = 0.1$.

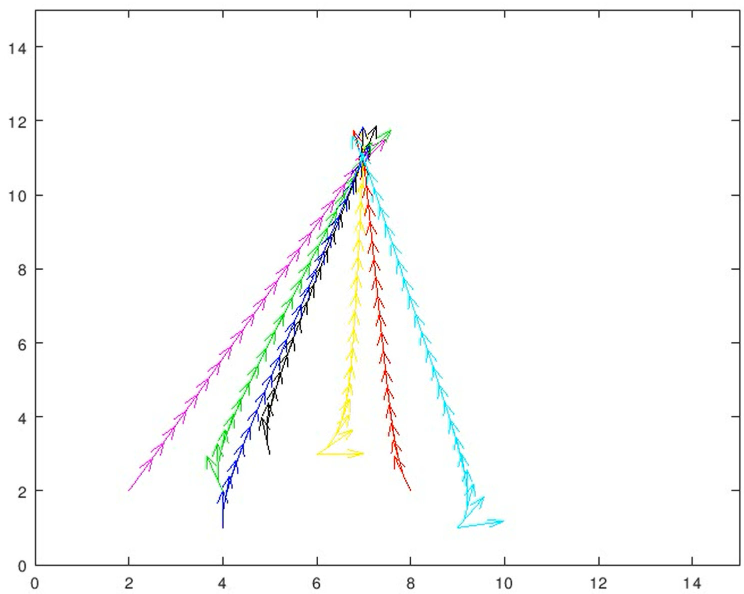


Figure 10: Paths of seven skiers reaching the shared goal.

The initial conditions of the skiers are:

- skier 1 starts from $x_1 = (2, 2)$ with orientation equal to $\varphi_1 = \pi/3$; his arrow is magenta;
- skier 2 starts from $x_2 = (4, 1)$ with orientation equal to $\varphi_2 = \pi/2$; his arrow is blue;
- skier 3 starts from $x_3 = (4, 2)$ with orientation equal to $\varphi_3 = 110°$; his arrow is green;
- skier 4 starts from $x_4 = (5, 3)$ with orientation equal to $\varphi_4 = 100°$; his arrow is black;
- skier 5 starts from $x_5 = (6, 3)$ with orientation equal to $\varphi_5 = 0$; his arrow is yellow;
- skier 6 starts from $x_6 = (9, 1)$ with orientation equal to $\varphi_6 = 10°$; his arrow is cyan;
- skier 7 starts from $x_7 = (8, 2)$ with orientation equal to $\varphi_7 = 110°$; his arrow is red.

As we could see from a movie showing the paths formation, the skiers' time to reach the turnstile is different for every single one of them. At the beginning, we can see that skier 2 and skier 3 are pretty close to each other and to avoid future interaction between their skis they change their direction.

3 Mean field kinetic model

Since we have the particle model, now we want to write the kinetic equation, because it simplifies the mathematical analysis as it corresponds to a partial differential equation. Therefore, we will have only one equation for the whole system, which is much nicer than the system of ODEs we had in the previous chapter. As we already said in the introduction chapter, the kinetic model is a good approximation of the particle dynamics since the number of skiers in the system is large. However, the kinetic formulation will still correspond to a microscopic description, and hence it is not the right level of description to observe emergent properties.

In order to write the kinetic equations we have to define:

- the empirical distribution function which depends on time t, position $x \in \mathbb{R}^2$ and orientation $\varphi \in \mathbb{T}$ of the skier, which we will call $f^N(t, x, \varphi)$;
- the test function g such that $\langle g, f^N \rangle = \frac{1}{N} \sum_{i=1}^{N} g(x_i(t), \varphi_i(t))$.

At this point we are able to write the following:

$$
\begin{aligned}
\frac{d}{dt}\langle g, f^N \rangle &= \frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N} g(x_i(t), \varphi_i(t))\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[(\nabla_x g)(x_i(t), \varphi_i(t)) \cdot \frac{dx_i}{dt} + (\partial_\varphi g)(x_i(t), \varphi_i(t)) \cdot \frac{d\varphi_i}{dt}\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[(\nabla_x g \cdot v_f)(x_i(t), \varphi_i(t)) + (\partial_\varphi g \cdot F_f)(x_i(t), \varphi_i(t))\right] \\
&= \langle \nabla_x g \cdot v_f, f^N \rangle + \langle \partial_\varphi g \cdot F_f, f^N \rangle \\
&= -\langle g, \nabla_x \cdot (v_f f^N) \rangle - \langle g, \partial_\varphi \cdot (F_f f^N) \rangle \\
&= \langle g, (-\nabla_x \cdot (v_f f^N) - \partial_\varphi \cdot (F_f f^N)) \rangle,
\end{aligned}
$$

where in the first equality we simply applied the definition of g, in the second we used the chain rule and then we used (2.1.1) and (2.1.2), meaning that

$$
v_f(x, \varphi) = v_{\max\, 1} S(d_{\text{torus}} \varphi, \phi(x)) \omega(\varphi) + \int \psi_x \left(1 - \frac{|x - x'|}{w}\right) \frac{x - x'}{|x - x'|} f(x', \varphi') dx' d\varphi' \tag{3.0.1}
$$

and

$$
\begin{aligned}
F_f(x, \varphi) = {} & k_1 \frac{d_{\text{torus}}(\phi(x), \varphi)}{\pi} \operatorname{Sign}(\sin(\phi(x) - \varphi)) \\
& + \int \psi_\varphi(B^\varphi(x, x', \varphi, \varphi')) \operatorname{Sign}(\sin 2(\varphi - \varphi')) f(x', \varphi') dx' d\varphi'.
\end{aligned} \tag{3.0.2}
$$

At this point we can see that:

$$
\frac{d}{dt}\langle g, f^N \rangle = \langle g, \partial_t f^N \rangle
$$

and thus, putting all the terms we derived together, we have that:

$$
\langle g, \partial_t f^N \rangle = \langle g, (-\nabla_x \cdot (v_f f^N) - \partial_\varphi \cdot (F_f f^N)) \rangle.
$$

When we let the total number of skiers $N \to \infty$, we have to introduce the distribution function $f^N(t, x, \varphi) \to f(t, x, \varphi)$. At this point, the evolution of f can be written as:

$$
\partial_t f + \nabla_x \cdot (v_f f) + \partial_\varphi (F_f f) = 0, \tag{3.0.3}
$$

where v_f and F_f are given, respectively, in (3.0.1) and (3.0.2). This is the kinetic equation we were interested in deriving.

3.1 Scaling and nondimensionalization

Before performing an asymptotic analysis of the kinetic equation, it is best to write the equation in its dimensionless form with the introduction of new dimensionless and macroscopic variables. Hence, we are interested in writing the partial differential equation without units so that the number of constants in it decreases. In this way, we will also be able to highlight the role of the various terms composing our kinetic equation and thus we will be able to say which ones are more relevant. At the end, what we expect to have is a large parameter in front of both the interaction terms, meaning that on the long term, when we have a very large number of skiers, those parts will dominate over the active ones. Therefore, the interaction factor will determine the whole dynamics of the skiers.

In order to write the system in dimensionless form we need to:

- replace the length x with $x_0 x_s$, where x_0 has a dimension and x_s is dimensionless. At this point we call x_s again x and thus, at the end, x will be replaced by $x_0 x$. The same applies for x', i.e. $x' \to x_0 x'$.
  Note that x_0 is called typical length.
- rescale time t in the same way we did for the length, i.e. $t \to t_0 t$;
- when we rescale f in (3.0.3), nothing changes, since the equation is linear and homogeneous in f. However, f also appears in the interaction terms of (3.0.1) and (3.0.2) and in that case the rescaling ends up in new terms and hence we rescale $f \to f_0 f$.

Note that φ is an angle, which is already dimensionless, and therefore we do not need to rescale it. At this point we can define the natural characteristic velocity, since we can get that from the above rescaled quantities:

$$\nu_0 = \frac{x_0}{t_0}.$$

Moving on, we can see that the functions ψ_x and ψ_φ are not dimensionless. Indeed, they have to be chosen in such a way that the dimension of the whole integral they appear in is of the same dimension of the other components. For instance, the term ν_f is a velocity, which has a dimension of length over time, therefore ψ_x has to be such that $\int \psi_x (1 - \frac{|x-x'|}{w}) \frac{x-x'}{|x-x'|} f(x', \varphi') dx' d\varphi'$ is a velocity as well. Now, $f(x', \varphi')$ is the number of skiers per length and per angle: this means that $f dx' d\varphi'$ is a number, i.e. the number of skiers. Hence, ψ_x has the scale dimension of velocity/number of skiers. Similarly for ψ_φ: since F_f is an angular speed (i.e. angle per time and therefore 1/s), ψ_φ is of the dimension of angles per time per number of skiers, i.e. angular velocity per number of skiers.

At this point we still have to take care of the other quantities and in order to do that we proceed with the following rescaling:

- we rescale ν_f , which is also a velocity, as follows: $\nu_f \to \nu_0 \nu_f$;
- we write the speed ν_{max1} in the following way: $\nu_{max1} \to \nu_0 \nu_{max1}$;
- we write the speed ν_{max2} in the following way: $\nu_{max2} \to \nu_0 \nu_{max2}$;
- we write F_f, which is an angular velocity, as $F_f \to \frac{1}{t_0} F_f$;
- we write k_1, which is an angular velocity, as $k_1 \to \frac{1}{t_0} k_1$;
- we write k_2, which is an angular velocity, as $k_2 \to \frac{1}{t_0} k_2$;
- we rescale in the following way $\psi_x \to \psi_{x,0} \psi_x$, where $\psi_{x,0}$ has the dimension of order of velocity/number of skiers;

- similarly for $\psi_\varphi \to \psi_{\varphi,0}\psi_\varphi$, where $\psi_{\varphi,0}$ has units of 1/time/number of skiers.

As we already noticed above while discussing the rescaling of f, if we look at (3.0.3), the scaling of f itself does not change anything, i.e.

$$\text{rescaling f}: \ f_0\partial_t f + f_0\nabla_x \cdot (\nu_f f) + f_0\partial_\varphi(F_f f) = 0$$

and dividing both the sides by f_0 what we obtain is the same equation. Moreover, scaling time in the very same equation leaves everything unchanged since from the first term, using the chain rule, we get a term of $1/t_0$ in front, from the second with the chain rule we have $1/x_0$ but also a ν_0 from the scaling of the ν_f, and therefore simplifying we also have a term of order of $1/t_0$ in front. We also get a $1/t_0$ in front of the third term from the scaling of F_f. Thus, we obtain:

$$\text{rescaling time}: \frac{1}{t_0}\partial_t f + \frac{1}{x_0}\nu_0\nabla_x \cdot (\nu_f f) + \frac{1}{t_0}\partial_\varphi(F_f f) = 0.$$

and by multiplying by t_0, we get the same equation as the original one.

Now we have to proceed by rescaling the two equations describing ν_f and F_f, i.e. (3.0.1) and (3.0.2) respectively. Starting from the first one, rescaling and dividing by ν_0 both terms leads us to:

$$\nu_f(x,\varphi) = \nu_{\max 1}\, S(d_{\text{torus}}\varphi, \phi(x))\omega(\varphi) + \frac{1}{\epsilon}\int \psi_x\left(1 - \frac{|x-x'|}{w}\right)\frac{x-x'}{|x-x'|} f(x',\varphi')dx'd\varphi'$$

where

$$\frac{1}{\epsilon} = \frac{f_0\psi_{x,0}x_0}{\nu_0} = f_0\psi_{x,0}t_0.$$

As we said before, f_0 is the density of skiers, $\psi_{x,0}$ describes how the skiers interact microscopically and ν_0 is the speed scale. When $1/\epsilon$ is large, then we have a crowded situation.
In an analogous way, even if this time we multiply by t_0 both terms instead of dividing by ν_0, for the second equation we end up with:

$$\begin{aligned} F_f(x,\varphi) &= k_1\frac{d_{\text{torus}}(\phi(x),\varphi)}{\pi}\,\text{Sign}(\sin(\phi(x)-\varphi)) \\ &\quad + \frac{\mu}{\epsilon}\int \psi_\varphi\left(B^\varphi(x,x',\varphi,\varphi')\right)\text{Sign}(\sin 2(\varphi-\varphi'))f(x',\varphi')dx'd\varphi' \end{aligned}$$

where we call

$$\frac{\mu}{\epsilon} = t_0 f_0\psi_{\varphi,0}x_0$$

which we also want to be big. At this point we made all our equations dimensionless (note that the terms composing it still have a dimension), which is exactly what we wanted to do from the beginning.

We can see that, exception made for the interaction parts, all the other terms remained unchanged after the scaling. However, both the interaction terms ended up being multiplied by two constants. These constants are exactly what will make the interaction terms dominate over the active terms in the long time.

Remark 2. *If the rescaling we did until now is correct, then both ϵ and μ, and hence also μ/ϵ, should be dimensionless. Let us check that:*

$$\frac{1}{\epsilon} = f_0\psi_{x,0}t_0 = \frac{\textit{number of skiers}}{\textit{angle*length}}\,\frac{\textit{length}}{\textit{time*number of skiers}}\,\textit{time}$$

everything cancels out and at the end we will remain only with the angle, which is dimensionless. Therefore ϵ is dimensionless. Analogously for μ/ϵ:

$$\frac{\mu}{\epsilon} = t_0 f_0\psi_{\varphi,0}x_0 = \textit{time}\,\frac{\textit{number of skiers}}{\textit{angle*length}}\,\frac{\textit{angle}}{\textit{time*number of skiers}}\,\textit{length}$$

which is again dimensionless. Moreover, to be clear, let us write down the explicit formula of μ*:*

$$\mu = \frac{t_0 x_0 \psi_{\varphi,0} f_0}{f_0 \psi_{x,0} t_0} = x_0 \frac{\psi_{\varphi,0}}{\psi_{x,0}}$$

which is dimensionless since:

$$\mu = \textit{length} \frac{\textit{angle}}{\textit{time*number of skiers}} \left(\frac{\textit{length}}{\textit{time*number of skiers}} \right)^{-1}.$$

At this point we still have to see what happens to the components defining both the badnesses B^x and B^φ during this rescaling process. We start with looking at B^φ: we can rescale both λs by the same factor of ℓ, i.e. $\lambda \to \ell\lambda$ and $\lambda' \to \ell\lambda'$. In this way the definition of B^φ becomes:

$$B^\varphi = \min \left\{ \frac{1}{2} - |\lambda|, \frac{1}{2} - |\lambda'| \right\}.$$

Since we rescaled both values of the λs, we also have to look at what happens to the equation describing them once we do this nondimensionalization process. The equation that defines them, rescaling also the x and x' by x_0, becomes: $x_0 x + \ell\lambda\omega(\varphi) = x_0 x' + \ell\lambda'\omega(\varphi')$. Dividing both sides by x_0, we obtain a new parameter that we call

$$\delta_2 = \frac{\ell}{x_0},$$

so that $x + \delta_2\lambda\omega(\varphi) = x' + \delta_2\lambda'\omega(\varphi')$. Note that basically what we do while defining δ_2 is rescaling the length of the ski ℓ with the typical length x_0.
In a similar way, regarding B^x, we rescale the width by calling

$$\delta_1 = \frac{w}{x_0}$$

so that we end up with $\psi_x(\delta_1 - |x - x'|)$. In this case, the rescaling of B^x is the following: $B^x \to w/v_0 B^x$.

At this point we have rescaled all the quantities composing the kinetic equations. To sum up, the dimensionless kinetic equations are:

$$\partial_t f + \nabla_x \cdot (v_f f) + \partial_\varphi (F_f f) = 0 \tag{3.1.1}$$

$$v_f(x, \varphi) = v_{\text{active}} + \frac{1}{\epsilon} \int \psi_x(\delta_1 - |x - x'|) \frac{x - x'}{|x - x'|} f(x', \varphi') dx' d\varphi' \tag{3.1.2}$$

with $v_{\text{active}} = v_{\max 1} S(d_{\text{torus}}(\varphi, \phi(x)))\omega(\varphi)$ and where $\delta_1 = w/x_0$ is the rescaled width, while

$$F_f(x, \varphi) = F_{\text{active}} + \frac{\mu}{\epsilon} \int \psi_\varphi \left(\frac{B^\varphi}{\delta_2} \right) \text{Sign}(\sin 2(\varphi - \varphi')) f(x', \varphi') dx' d\varphi' \tag{3.1.3}$$

where $F_{\text{active}} = k_1 \frac{d_{\text{torus}}(\phi(x), \varphi)}{\pi} \text{Sign}(\sin(\phi(x) - \varphi))$ and $\delta_2 = \ell/x_0$ is the rescaled length.
When we are going to derive the macroscopic equations we will let $\epsilon \to 0$; in this way the second terms of (3.1.2) and (3.1.3) will dominate over the active ones, i.e. v_{active} and F_{active}.

3.2 Important results

Let us focus on the integral:

$$\int \psi_x(\delta_1 - |x - x'|) \frac{x - x'}{|x - x'|} f(x', \varphi') dx' d\varphi'.$$

We can immediately see that the integral with respect to φ' only applies to $f(x', \varphi')$ and hence, defining $\rho_f(x') := \int f(x', \varphi')d\varphi'$, we obtain:

$$\int_{x' \in B_{\delta_1}(x)} (\delta_1 - |x - x'|)_+ \frac{x - x'}{|x - x'|} \rho_f(x') dx'.$$

After the rescaling, δ_1 will be a small parameter because w, the width of the ski, is small compared to the characteristic (or typical) length x_0. Therefore, since we have the integral over $x' \in B_{\delta_1}(x)$ (note we integrate over that ball because if we are outside of it we do not have interactions, i.e. ψ_x would be zero), where the ball has a small radius, we have that x' is very close to x. To make this visible, we introduce a new variable $x - x' = \delta_1 \nu$, where δ_1 is simply the distance, and thus a number, while $\nu \leqslant 1$ is the directional vector. At this point we can write:

$$\begin{aligned}\int_{x' \in B_{\delta_1}(x)} (\delta_1 - |x - x'|)_+ \frac{x - x'}{|x - x'|} \rho_f(x') dx' &= \delta_1^2 \int_{|\nu|<1} (\delta_1 - \delta_1|\nu|)_+ \frac{\delta_1 \nu}{\delta_1 |\nu|} \rho_f(x - \delta_1 \nu) d\nu \\ &= \delta_1^3 \int_{\nu \in B_1} (1 - |\nu|) \frac{\nu}{|\nu|} \rho_f(x - \delta_1 \nu) d\nu\end{aligned}$$

where the term δ_1^2 appearing in front of the first integral in the second equality comes from the change of coordinates (and remembering that we are in two dimensions), while B_1 is the ball with radius one. Furthermore, we can see that the only term containing δ_1 is ρ_f. Therefore, since x is close to x', we can Taylor expand the term ρ_f.
When Taylor expanding, the first approximation would be to set $\delta_1 = 0$. In this way, ρ_f only depends on x, which means that we can take it out of the integral. Hence, the integrand remains $(1 - |\nu|)\nu/|\nu|$, which is an odd function of ν and considering that we integrate over the whole ball, the integral will be zero. To sum up, the first approximation is equal to zero.
Continuing Taylor expanding, with the second approximation we end up with:

$$\delta_1^3 \int_{\nu \in B_1} (1 - |\nu|) \frac{\nu}{|\nu|} \rho_f(x - \delta_1 \nu) d\nu \sim \delta_1^3 (-\delta_1) \int_{\nu \in B_1} (1 - |\nu|) \frac{\nu}{|\nu|} \nu \cdot \nabla_x \rho_f(x) d\nu.$$

Using the notation $\nu \otimes \nu$ denoting the tensor product (or outer since ν is a vector), we can get the term $\nabla_x \rho_f(x)$ out of the integral, which is exactly what we do now:

$$= -\delta_1^4 \int_{\nu \in B_1} \frac{(1 - |\nu|)}{|\nu|} \nu \otimes \nu d\nu \nabla_x \rho_f(x).$$

Focusing on the integral:

$$\int_{\nu \in B_1} \frac{(1 - |\nu|)}{|\nu|} \nu \otimes \nu d\nu$$

we can see that it gives a matrix. In particular, if we look at $\nu \otimes \nu$, we know that it is a rank one, positive semidefinite matrix. Indeed, if we consider ν in its components $\nu = (\nu_1, \nu_2)^T$ we have that

$$D := \int_{\sqrt{\nu_1^2 + \nu_2^2} < 1} \frac{(1 - \sqrt{\nu_1^2 + \nu_2^2})}{\sqrt{\nu_1^2 + \nu_2^2}} \begin{pmatrix} \nu_1^2 & \nu_1 \nu_2 \\ \nu_2 \nu_1 & \nu_2^2 \end{pmatrix} d\nu_1 d\nu_2$$

$$= \begin{pmatrix} \int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}} \nu_1^2 d\nu_1 d\nu_2 & \int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}} \nu_1 \nu_2 d\nu_1 d\nu_2 \\ \int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}} \nu_1 \nu_2 d\nu_1 d\nu_2 & \int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}} \nu_2^2 d\nu_1 d\nu_2 \end{pmatrix}.$$

On one hand, we notice that the terms not on the diagonal are zero because the integrands, i.e. $\frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}} \nu_1 \nu_2$, in those cases are odd functions and therefore integrating over the unit ball we

obtain zero.
On the other hand, both terms on the diagonal give the same value, which is:

$$\int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}}\nu_1^2 d\nu_1 d\nu_2 = \int_{\sqrt{\nu_1^2+\nu_2^2}<1} \frac{(1-\sqrt{\nu_1^2+\nu_2^2})}{\sqrt{\nu_1^2+\nu_2^2}}\nu_2^2 d\nu_1 d\nu_2 = \frac{\pi}{12}.$$

Thus, $D = \frac{\pi}{12}\mathrm{Id}$ which means that D is a multiple of the identity matrix. Moreover, using Sylvester's criterion, we get that D is a symmetric and positive definite matrix.

Hence, at the end, we have computed an approximation of the second term composing ν_f for small δ_1 and obtained:

$$\int_{x'\in B_{\delta_1}(x)} (\delta_1 - |x-x'|)_+ \frac{x-x'}{|x-x'|}\rho_f(x')dx' \sim -\delta_1^4 D\nabla_x \rho_f(x)$$

where D, which is a number, is:

$$D = \int_{B_1} \frac{(1-|\nu|)}{|\nu|}\nu_i^2 d\nu = \frac{1}{2}\int_{B_1}(1-|\nu|)|\nu|d\nu = \frac{\pi}{12}$$

where, as we already mentioned before, ν_i is one of the component of ν.

Considering that now we know that a factor of δ_1^4 is popping out of the integral forming the interaction part of the velocities, we can let ϵ assume exactly that value. In this way our equations simplifies and looks much nicer. Note that both ϵ and δ_1 are dimensionless quantities. Indeed we proved it in the section before that ϵ does not have dimension, while δ_1 is the rescaled width, meaning that it is the original width w divided by the typical length x_0. Thus, we assume $\epsilon = \delta_1^4$ and have that:

$$\frac{1}{\epsilon}\int_{x'\in B_{\delta_1}(x)} (\delta_1 - |x-x'|)_+ \frac{x-x'}{|x-x'|}\rho_f(x')dx' \sim -D\nabla_x \rho_f(x),$$

hence we obtain:

$$\nabla_x \cdot (\nu_f f) \sim \nabla_x \cdot [\nu_{\text{active}} - Df\nabla_x \rho_f(x)]\,.$$

Moving on, let us focus on the integral that appears in the interaction part of the orientations, i.e.

$$\int \psi_\varphi\left(\frac{B^\varphi(x,x',\varphi,\varphi')}{\delta_2}\right)\mathrm{Sign}(\sin 2(\varphi-\varphi'))f(x',\varphi')dx'd\varphi'.$$

Small reminder of what all the quantities involved are:

$$\lambda = \frac{(x'-x)\cdot\omega'^{\perp}}{\omega\cdot\omega'^{\perp}} \qquad \text{and} \qquad \frac{B^\varphi}{\delta_2} = \min\left\{\frac{1}{2}-\frac{|\lambda|}{\delta_2}, \frac{1}{2}-\frac{|\lambda'|}{\delta_2}\right\} = \frac{1}{2} - \max\left\{\frac{|\lambda|}{\delta_2}, \frac{|\lambda'|}{\delta_2}\right\}.$$

In a completely analogous way as before, we now call $x - x' = \delta_2 \nu$, so that we have:

$$\lambda' = \delta_2 \frac{\nu\cdot\omega'^{\perp}}{\omega\cdot\omega'^{\perp}} \qquad \text{i.e.} \qquad \frac{\lambda'}{\delta_2} = \frac{\nu\cdot\omega'^{\perp}}{\omega\cdot\omega'^{\perp}}.$$

Therefore, substituting x' with $x - \delta_2\nu$ and remembering again that x' has two coordinates and thus the integral over dx' is in two dimensions, we have in front of the integral the constant term δ_2^2, i.e.

$$\delta_2^2 \int \left(\frac{B^\varphi}{\delta_2}\right)_+ \mathrm{Sign}(\sin 2(\varphi-\varphi'))f(x-\delta_2\nu,\varphi')d\nu\varphi'.$$

The above formula can not be simplified further and hence we keep it the way it is. Again, for small δ_2, i.e. $\delta_2 \to 0$, we have that:

$$\partial_\varphi(F_f f) \sim \partial_\varphi\left[F_{\text{active}} + \frac{f\mu}{\epsilon}\left(\delta_2^2\int\left(\frac{B^\varphi}{\delta_2}\right)_+ \mathrm{Sign}(\sin 2(\varphi-\varphi'))f(x-\delta_2\nu,\varphi')d\nu\varphi'\right)\right].$$

If we now concentrate on

$$\frac{\mu}{\epsilon}\delta_2^2\int\left(\frac{1}{2}-\max\{|\lambda|,|\lambda'|\}\right)_+\mathrm{Sign}(\sin 2(\varphi-\varphi'))f(x-\delta_2\nu,\varphi')d\nu\varphi',$$

we can see that in equilibrium we have that $f(t,x,\varphi)$ times the above mentioned integral should be zero (see for more detail lemma (1) in the next chapter). This can be achieved by allowing $f(\varphi)f(\varphi')$ to be nonzero only when $\varphi=\varphi'$ or $\varphi=\varphi'+\pi$. As a consequence f has to have the following form:

$$f(t,x,\varphi)=\rho_+(t,x)\delta(\varphi-\varphi_+(t,x))+\rho_-(t,x)\delta(\varphi-\varphi_+(t,x)-\pi),$$

where $\delta(\varphi-\varphi_+(t,x))$ is the Dirac delta in $\varphi-\varphi_+(t,x)$. This basically means that when we are at the equilibrium, the whole group of skiers can be divided into two subgroups which have opposite orientations: the ones that have direction φ_+ and the others that have direction $\varphi_++\pi$. In the subgroup that has direction φ_+ there are ρ_+ skiers, while the other subgroup is composed by ρ_- skiers.

Remark 3. *Note that the way we derive the equilibrium value for* f *is formal and a bit mediocre. Indeed also when we have* $f=$ *constant, we obtain an equilibrium, which is independent of* φ*. However, if we only take the rotation term, it can be expected to be unstable and therefore we do not consider it here.*

Now we want the divergence of the φ term to dominate over the divergence term of the x. In other words, we want $\delta_2^2\mu/\epsilon$ to dominate the term δ_1^4/ϵ. To this aim, we have to remember that δ_1 is the rescaled width while δ_2 is the rescaled length and thus we have that δ_2 is bigger than δ_1. At this point we keep the ratio of these two rescaled lengths fixed, so that we have:

$$a:=\frac{w}{l}=\frac{\delta_1}{\delta_2}$$

where a is called aspect ratio. Then we have that:

$$\delta_2^2=\frac{\delta_1^2}{a^2}$$

and using the fact that $\epsilon=\delta_1^4$ we get that:

$$\frac{\delta_2^2\mu}{\epsilon}=\frac{\mu}{a^2\delta_1^2}.$$

In order to have the orientation term dominating over the position term, we want the ratio above to be big. We know that the aspect ratio is rather small, but the important point is that δ_1 is definitely assumed to be small, therefore their product is a product of two small things, which means that the inverse is a big quantity. Thus, if we assume $\mu=(\delta_2^2)^{-1}$ and obviously $\epsilon=\delta_1^4$, we have the following approximation of equation (3.1.1)

$$\partial_t f+\nabla_x\cdot(f(\nu_{\text{active}}-D\nabla_x\rho_f))+\partial_\varphi\left(f\left(F_{\text{active}}+\frac{\mu}{a^2\delta_1^2}\mathcal{B}(f)\right)\right)=0,$$

where $\mathcal{B}(f)$ is defined as the operator created from the interaction term of the change of directions, i.e.

$$\mathcal{B}(f):=\int\left(\frac{1}{2}-\max\{|\lambda|,|\lambda'|\}\right)_+\mathrm{Sign}(\sin(2(\varphi-\varphi')))f(x,\varphi')d\varphi'dx'. \tag{3.2.1}$$

Remark 4. *Note that the interaction term of the angles contains the* Sign *function, but the whole argument of the integral does not have jumps since when* ψ_φ *goes to zero,* φ *goes to* φ'*.*

Remark 5. *We can notice that now we use the calculations we did before for finding an explicit equation for* f *when* $\epsilon\to 0$*, i.e. at equilibrium. When* $\epsilon\to 0$*, we have that* $\mathcal{B}(f)=0$ *if and only if*

$$f(x,\varphi)=\rho_+(x)\delta(\varphi-\varphi_+(x))+\rho_-(x)\delta(\varphi-\varphi_+(x)-\pi).$$

4 Macroscopic model

As we already said in the introduction, the goal of kinetic theory applied to biology is to study emergence. This means that we are interested in seeing how the large scale dynamics appear from the particle dynamics. In order to do that, we need to derive the macroscopic equations. Hence, we need less information about the particle system we are studying and we do that by taking averages and variances of f; basically, we do statistical mechanics.
The aim is now to derive the macroscopic equations, which we obtain by taking the limit $\epsilon \to 0$. We know that $f = f(t, x, \varphi)$ is the function that gives the evolution of the distribution of a skier in space and angle, whose dynamics are given by a kinetic equation. In other words, f is the number of skiers in position x, with orientation φ at time t; therefore $f(t, x, \varphi)dxd\varphi$ is the number of skiers in a volume $dxd\varphi$ centred at (x, φ) at time t.

As mentioned at the end of the last section, we are interested in deriving the macroscopic equations describing ρ_+, ρ_- and φ_+, which have not particular meaning since, on contrary to classical physics, the skiers (or particles) in our case are active. In classical physics, meaning the physics of passive particles, the classical macroscopic quantities, for which we are interested in deriving conservation laws, are mass (or density), momentum and energy.

To derive the macroscopic equations we need to focus on the rescaled kinetic equations, which we will write again here for convenience:

$$\partial_t f + \nabla_x \cdot (\nu_f f) + \partial_\varphi (F_f f) = 0$$

with

$$\nu_f(x, \varphi) = \nu_{\text{active}} + \frac{1}{\epsilon} \int \psi_x(\delta_1 - |x - x'|) \frac{x - x'}{|x - x'|} f(x', \varphi') dx' d\varphi'$$

and

$$F_f(x, \varphi) = F_{\text{active}} + \frac{\mu}{\epsilon} \int \psi_\varphi \left(\frac{B\varphi}{\delta_2} \right) \text{Sign}(\sin 2(\varphi - \varphi')) f(x', \varphi') dx' d\varphi'$$

where ν_{active} is the velocity that comes from the active part of the velocity term, i.e.

$$\nu_{\text{active}} = \nu_{\max 1} S(d_{\text{torus}}(\varphi, \phi(x)))\omega(\varphi)$$

and F_{active} comes from the active part of the changes of angle, i.e.

$$F_{\text{active}} = k_1 \frac{d_{\text{torus}}(\phi(x), \varphi)}{\pi} \text{Sign}(\sin(\phi(x) - \varphi)).$$

We are now going to define two important concepts that will be very useful to derive the macroscopic equations. These definitions are the ones of collision invariant and general collision invariant.

4.1 Collision invariants

Definition 1. *A Collision Invariant (CI) for a collision operator $\mathcal{Q}(f)$ is any function $\Psi = \Psi(\varphi)$ such that*

$$\int \mathcal{Q}(f) \Psi d\varphi = 0 \qquad \forall f.$$

We can clearly see that if $\Psi =$ constant, in particular if it is equal to 1, then Ψ is a collision invariant.

Lemma 1. *For $\mathcal{Q}(f) = \partial_\varphi(f\mathcal{B}(f))$, with $\mathcal{B}(f)$ the velocity defined in* (3.2.1), *$\Psi = 1$ and $\Psi = \varphi$ are collision invariants.*

Proof. We can see that the proof of $\Psi = 1$ is trivial.
Therefore, let us focus on proving that $\Psi = \varphi$ is a CI. In this case we have to prove that the integral

$$\int \mathcal{Q}(f)\varphi d\varphi = \int \varphi \partial_{\varphi}(f\mathcal{B}(f))d\varphi$$

is equal to zero. Integration by parts gives:

$$\int \mathcal{Q}(f)\varphi d\varphi = \varphi f\mathcal{B}(f) - \int f\mathcal{B}(f)d\varphi. \tag{4.1.1}$$

We are now going to prove that the integral on the right hand side of the equation above is equal to zero. If we write down $\mathcal{B}(f)$:

$$\mathcal{B}(f) = \int \left(\frac{1}{2} - \max\{|\lambda|, |\lambda'|\}\right)_+ \mathrm{Sign}(\sin(2(\varphi - \varphi')))f(x, \varphi')d\varphi' d\nu,$$

reminding us that λ and λ' are:

$$\lambda = \frac{\nu \cdot \omega'^{\perp}}{\omega \cdot \omega'^{\perp}} \qquad \lambda' = \frac{\nu \cdot \omega^{\perp}}{\omega' \cdot \omega^{\perp}} = \frac{\nu \cdot \omega^{\perp}}{\omega \cdot \omega'^{\perp}},$$

we immediately see that the only term that applies to the integral with respect to ν is the first term. Hence we can call:

$$\gamma = \int \left(\frac{1}{2} - \max\{|\lambda|, |\lambda'|\}\right)_+ d\nu.$$

We is the argument of the integral defining γ equal to zero? It is never zero. Indeed, if ν is big, meaning that the center point of the skis are not close, then the skiers are far away and therefore the integrand is zero, since there is no chance that the two skiers hit. In other words, the interaction term is zero. Having a big value of ν means that both λs are larger than $1/2$ which means that $\omega \cdot \omega'^{\perp} = 0$, i.e. $\omega' = \omega$ and consequently $\omega'^{\perp} = \omega^{\perp}$. Hence, we have that $\varphi = \varphi'$ or $\varphi = \varphi' + \pi$. Otherwise, whenever $\omega \cdot \omega'^{\perp}$ is finite, we have a small value of ν, which makes both λ and λ' smaller than $1/2$. Therefore, the two skis intersect and thus the integrand is not zero.
So, now, it is time to consider the actual integral we are interested in, i.e.

$$\int \gamma\, \mathrm{Sign}(\sin(2(\varphi - \varphi')))f(x, \varphi')f(x, \varphi)d\varphi' d\varphi$$

and to prove that it is zero. We can see that when we interchange φ and φ' the product $f(x, \varphi)f(x, \varphi')$ does not change, while the Sign of $\sin 2(\varphi - \varphi')$ does. Hence, integrating with respect to φ and φ', we get $\int f\mathcal{B}(f) = 0$.
At this point, a function whose integral is zero has to have a zero by continuity assumptions (or can also be zero everywhere); therefore since we know that $\int f\mathcal{B}(f)d\varphi = 0$, it is clear that $f\mathcal{B}(f)$ has to have at least one zero. Actually, since we are on the torus, $f\mathcal{B}(f)$ has at least two zeros that are $\varphi_{j,f}$:

$$\int f\mathcal{B}(f)d\varphi = 0 \Longrightarrow \exists \varphi_{j,f} : (f\mathcal{B}(f))(\varphi_{j,f}) = 0 \tag{4.1.2}$$

with $j = 0$ or 1. Note that $\varphi_{j,f}$ depend on f because they are zeros of $f\mathcal{B}(f)$. Since we have showed that $f\mathcal{B}(f)$ has zeros, we can choose $\Psi = \varphi$ in such a way that its jumps are at the points where these zeros are. Thus, we have proved that (4.1.1) is equal to zero and hence that $\Psi = \varphi$ is a collision operator. ∎

Formally and assuming that all the functions are smooth, if me multiply equation (3.0.3) by the collision operator Ψ and integrate with respect to φ, we obtain the general conservation law

$$\partial_t \int \Psi f d\varphi + \nabla_x \cdot \int \Psi f \nu_f d\varphi + \int \Psi \partial_{\varphi} f F_{\mathrm{active}} d\varphi = 0,$$

which, if we integrate by parts the last term, can be rewritten as:

$$\partial_t \int \Psi f d\varphi + \nabla_x \cdot \int \Psi f \nu_f d\varphi + \sum (fF_{\text{active}})([\Psi]) - \int (\partial_\varphi \Psi) fF_{\text{active}} d\varphi = 0, \qquad (4.1.3)$$

where $[\Psi]$ means that we have jumps in Ψ: if Ψ is not periodic, it will jump in at least one point, while if it is not there will be no jumps, i.e. $fF_{\text{active}}[\Psi] = 0$. Moreover, we put the sum in front of that term just to remember that Ψ can have more jumps and hence we have to sum over all jumps. For (4.1.3), if we substitute the values of our collision invariants $\Psi = 1$ and $\Psi = \varphi$, we have the following two conservation laws:

- choosing $\Psi = 1$, the last two terms of (4.1.3) vanish, and hence we end up with:

$$\partial_t \int f d\varphi + \nabla_x \cdot \int f \nu_f d\varphi = 0,$$

  which is in particular the mass conservation law. It is called continuity equation.

- choosing $\Psi = \varphi$ with jumps at $\varphi_{0,f}$, down by 2π, we obtain:

$$\partial_t \int \varphi f d\varphi + \nabla_x \cdot \int \varphi f \nu_f d\varphi - 2\pi (fF_{\text{active}})(\varphi_{0,f}) - \int fF_{\text{active}} d\varphi = 0.$$

  This means that we start with φ, we go up and when we arrive at $\varphi_{0,f}$, we jump down by 2π and then we continue going up again. This is an evolution equation.

4.2 Generalised Collision Invariant

To sum up, as it is explained in paper [2], the the collision invariants $\Psi = 1$ and $\Psi = \varphi$ span a two-dimensional function space, while the set of equilibria is a three-dimensional manifold. Thus, we need to find some substitute to the notion of collision invariant, otherwise the problem will be under-determined. To this aim we introduce the concept of generalised collision invariant.

Definition 2. *A Generalised Collision Invariant (GCI) is a function Ψ_f which depends on $f = f(t, x, \varphi)$ and for which, when we multiply the the collision operator by the GCI and integrate, we get zero, i.e.*

$$\int \mathcal{Q}(f) \Psi_f d\varphi = 0 \qquad \forall f.$$

So, now, how do we find a GCI? We test the collision operator against some GCI $\Psi_f = \Psi_f(\varphi)$, obtaining:

$$\int \Psi_f \mathcal{Q}(f) d\varphi = \int \Psi_f \partial_\varphi (f \mathcal{B}(f)) d\varphi. \qquad (4.2.1)$$

Lemma 2. *With $\varphi_{0,f}$ and $\varphi_{1,f}$ defined as in* (4.1.2)*, $\Psi_f = \mathbb{1}_{[\varphi_{0,f}, \varphi_{1,f}]}$ is a GCI.*

Proof. Integrating by parts the right hand side of (4.2.1), we get:

$$\int \Psi_f \mathcal{Q}(f) d\varphi = f\mathcal{B}(f)[\Psi_f] - \int f\mathcal{B}(f) \partial_\varphi \Psi_f d\varphi,$$

where the notation $[\Psi_f]$ means that there are jumps in Ψ_f.
If now we plug in as Ψ_f the value of the indicator function $\mathbb{1}_{[\varphi_{0,f}, \varphi_{1,f}]}$, we can see that the first term at the right hand side is equal to zero and so is the second one since the derivative in φ of one is zero. Hence, $\mathbb{1}_{[\varphi_{0,f}, \varphi_{1,f}]}$ is a GCI. ∎

In a very similar way as we did above for the collision invariants, and hence obtaining an equation which is exactly like (4.1.3) but with Ψ_f instead of Ψ, we can now write the general macroscopic equation:

- when we choose $\Psi_f = \mathbb{1}_{[\varphi_{0,f},\varphi_{1,f}]}(\varphi)$, we have that the jumps are at $\varphi_{0,f}$ and $\varphi_{1,f}$, therefore, the last term of (4.1.3) vanishes and we end up with the following equation:

$$\partial_t \int_{\varphi_{0,f}}^{\varphi_{1,f}} f d\varphi + \nabla_x \cdot \int_{\varphi_{0,f}}^{\varphi_{1,f}} f v_f d\varphi + (fF_{\text{active}})(\varphi_{0,f}) - (fF_{\text{active}})(\varphi_{1,f}) = 0.$$

  The last two terms are due to the fact that $(fv_{\varphi,0})$ has jump up at $\varphi_{0,f}$ (that is the reason why we have a plus sign) by 1 and a jump down at $\varphi_{1,f}$ (that is why we have a minus sign) by 1. Moreover, it is an evolution equation.

Note that $\Psi_f = \mathbb{1}_{[\varphi_{0,f},\varphi_{1,f}]}(\varphi)$ is a GCI in a strange way since it depends on f through the zeros of $f\mathcal{B}(f)$.

4.3 Macroscopic equations

Until now, we have only prepared calculations in order to be able to write the macroscopic equations in terms of ρ_+, ρ_- and φ_+, which is our final goal. In this section it is finally time to put everything together and derive them.
Let us briefly remember our original equations one last time with the approximations we did in the previous chapter:

$$\partial_t f + \nabla_x \cdot (v_f f) + \partial_\varphi (F_f f) = 0$$

with

$$\begin{aligned} v_f(x,\varphi) &= v_{\text{active}} + \frac{1}{\epsilon}\int \psi_x(\delta_2 a - |x-x'|)\frac{x-x'}{|x-x'|} f(x',\varphi')dx'd\varphi' \\ &\sim v_{\text{active}} - D\nabla_x \rho_f \end{aligned}$$

and

$$\begin{aligned} F_f(x,\varphi) &= F_{\text{active}} + \frac{\mu}{\epsilon}\int \psi_\varphi\left(\frac{B^\varphi}{\delta_2}\right)\text{Sign}(\sin 2(\varphi-\varphi'))f(x',\varphi')dx'd\varphi' \\ &= F_{\text{active}} + \frac{\mu}{\epsilon}\mathcal{B}(f). \end{aligned}$$

In order to write the macroscopic equations, we let $\epsilon \to 0$ and hence equation (4.1.3) becomes:

$$\begin{aligned} \partial_t \int \Psi_f f d\varphi + \nabla_x \cdot \int \Psi_f (f(v_{\text{active}} - D\nabla_x \rho_f))d\varphi \\ + \sum (fF_{\text{active}})([\Psi_f]) - \int (\partial_\varphi \Psi_f) f F_{\text{active}} d\varphi = 0. \end{aligned} \tag{4.3.1}$$

Thus, equation (4.3.1) is the macroscopic equation which holds for both the collision invariants Ψ and for the generalised collision invariant Ψ_f. Hence, now we put in these values and, since we know that when $\epsilon \to 0$ our f goes to equilibrium, instead of using a general f we write:

$$f(x,\varphi) = \rho_+(x)\delta(\varphi-\varphi_+) + \rho_-(x)\delta(\varphi-\varphi_+^\downarrow)$$

where, for convenience, we denote $\varphi_+^\downarrow = \varphi_+ + \pi$. We can immediately see that:

$$\rho_f = \int f d\varphi = \int \rho_+(x)\delta(\varphi-\varphi_+)d\varphi + \int \rho_-\delta(\varphi-\varphi_+^\downarrow)d\varphi = \rho_+ + \rho_-.$$

The last equality holds because ρ_+ and ρ_- do not depend on φ, and therefore we can take them out of the integral, while the integral of a delta distribution is equal to one.

- the equation we get for $\Psi = 1$ is

$$\partial_t \rho_f + \nabla_x \cdot \left(\int f v_{\text{active}} d\varphi - D\rho_f \nabla_x \rho_f \right) = 0.$$

The second term can be written in the following way:

$$\begin{aligned} \int f v_{\text{active}} d\varphi &= \int \rho_+(x)\delta(\varphi - \varphi_+) v_{\text{active}} d\varphi + \int \rho_-(x)\delta(\varphi - \varphi_+^{\downarrow}) v_{\text{active}} d\varphi \\ &= \rho_+(x) v_{\text{active}}(\varphi_+) + \rho_-(x) v_{\text{active}}(\varphi_+^{\downarrow}). \end{aligned}$$

Hence, putting everything together we obtain:

$$\partial_t \rho_f + \nabla_x \cdot \left(\rho_+(x) v_{\text{active}}(\varphi_+) + \rho_-(x) v_{\text{active}}(\varphi_+^{\downarrow}) - D\rho_f \nabla_x \rho_f \right) = 0.$$

- taking $\Psi = \varphi$ we have:

$$\partial_t \int \varphi f d\varphi + \nabla_x \cdot \int \varphi (f(v_{\text{active}} - D\nabla_x \rho_f)) d\varphi - 2\pi (f F_{\text{active}})(\varphi_{0,f}) - \int f F_{\text{active}} d\varphi = 0.$$

Plugging in the well known equation for f in the first term we find that:

$$\begin{aligned} \partial_t \int \varphi f d\varphi &= \partial_t \left[\rho_+ \int \varphi \delta(\varphi - \varphi_+) d\varphi + \rho_- \int \varphi \delta(\varphi - \varphi_+^{\downarrow}) d\varphi \right] \\ &= \partial_t (\rho_+ \varphi_+ + \rho_- \varphi_+^{\downarrow}). \end{aligned}$$

The last equality holds since we integrate over any interval of length 2π; hence we can consider as limit of integration 0 and 2π.
Regarding the next term, which is

$$\nabla_x \cdot \int \varphi (f(v_{\text{active}} - D\nabla_x \rho_f)) d\varphi,$$

we have that it is equal to

$$\begin{aligned} = \nabla_x \cdot \Bigg[\int \varphi \delta(\varphi - \varphi_+) \rho_+ v_{\text{active}} d\varphi + \int \varphi \delta(\varphi - \varphi_+^{\downarrow}) \rho_- v_{\text{active}} d\varphi \\ - D\nabla_x \rho_f \int \varphi \rho_+ \delta(\varphi - \varphi_+) d\varphi - D\nabla_x \rho_f \int \varphi \rho_- \delta(\varphi - \varphi_+^{\downarrow}) d\varphi \Bigg]. \end{aligned}$$

Thus, at the end, we obtain that the second term becomes:

$$\nabla_x \cdot [\varphi_+ \rho_+ v_{\text{active}}(\varphi_+) + \varphi_+^{\downarrow} \rho_- v_{\text{active}}(\varphi_+^{\downarrow}) - \varphi_+ D\rho_+ \nabla_x \rho_f - \varphi_+^{\downarrow} D\rho_- \nabla_x \rho_f].$$

For the third term we obtain:

$$2\pi (f F_{\text{active}})(\varphi_{0,f}) = 2\pi \rho_+ F_{\text{active}}(\varphi_{0,f}) \delta(\varphi_{0,f} - \varphi_+) + 2\pi \rho_- F_{\text{active}}(\varphi_{0,f}) \delta(\varphi_{0,f} - \varphi_+^{\downarrow}).$$

Lastly, we have that:

$$\begin{aligned} \int f F_{\text{active}} d\varphi &= \int \rho_+ \delta(\varphi - \varphi_+) F_{\text{active}} d\varphi + \int \rho_- \delta(\varphi - \varphi_+^{\downarrow}) F_{\text{active}} d\varphi \\ &= \rho_+ F_{\text{active}}(\varphi_+) + \rho_- F_{\text{active}}(\varphi_+^{\downarrow}). \end{aligned}$$

- regarding the case of the generalised collision invariant $\Psi_f = \mathbb{1}_{[\varphi_{0,f},\varphi_{1,f}]}(\varphi)$ we have that the general macroscopic equation (4.3.1) becomes:

$$\partial_t \int_{\varphi_{0,f}}^{\varphi_{1,f}} f d\varphi + \nabla_x \cdot \int_{\varphi_{0,f}}^{\varphi_{1,f}} (f(\nu_{\text{active}} - D\nabla_x \rho_f))d\varphi \\ + (fF_{\text{active}})(\varphi_{0,f}) - (fF_{\text{active}})(\varphi_{1,f}) = 0.$$

  Plugging in the value of f at the equilibrium, we can write the first term in the following way:

$$\partial_t \int_{\varphi_{0,f}}^{\varphi_{1,f}} f d\varphi = \partial_t \int_{\varphi_{0,f}}^{\varphi_{1,f}} \rho_+ \delta(\varphi - \varphi_+) d\varphi = \partial_t \rho_+.$$

  For the second term we obtain

$$\nabla_x \cdot \int_{\varphi_{0,f}}^{\varphi_{1,f}} (\rho_+ \delta(\varphi - \varphi_+)(\nu_{\text{active}} - D\nabla_x \rho_f))d\varphi,$$

  and thus, at the end, it gets:

$$\nabla_x \cdot (\rho_+(\nu_{\text{active}}(\varphi_+) - D\nabla_x \rho_f)).$$

  The last two terms are $(fF_{\text{active}})(\varphi_{0,f})$ and $(fF_{\text{active}})(\varphi_{1,f})$, which can be written in the following way:

$$(fF_{\text{active}})(\varphi_{0,f}) = \rho_+ \delta(\varphi_{0,f} - \varphi_+) F_{\text{active}}(\varphi_{0,f})$$

  and

$$(fF_{\text{active}})(\varphi_{1,f}) = \rho_+ \delta(\varphi_{1,f} - \varphi_+) F_{\text{active}}(\varphi_{1,f})$$

Remark 6. *It can be shown that the values of $\varphi_{0,f}$ and $\varphi_{1,f}$ in terms of ρ_+, ρ_- and φ_+ are:*

$$\varphi_{0,f} = \varphi_+ - \frac{\pi}{2} \text{ and } \varphi_{1,f} = \varphi_+ + \frac{\pi}{2}.$$

Thus, the macroscopic equations we were interested in deriving from the beginning read:

$$\partial_t \rho_f + \nabla_x \cdot \Big(\rho_+(x)\nu_{\text{active}}(\varphi_+) + \rho_-(x)\nu_{\text{active}}(\varphi_+^{\downarrow}) - D\rho_f \nabla_x \rho_f\Big) = 0,$$

which is the one that we get from the collision invariant $\Psi = 1$

$$\partial_t(\rho_+\varphi_+ + \rho_-\varphi_+^{\downarrow}) + \nabla_x \cdot [\varphi_+\rho_+\nu_{\text{active}}(\varphi_+) + \varphi_+^{\downarrow}\rho_-\nu_{\text{active}}(\varphi_+^{\downarrow}) - \varphi_+ D\rho_+\nabla_x\rho_f - \varphi_+^{\downarrow} D\rho_-\nabla_x\rho_f] \\ - 2\pi\rho_+F_{\text{active}}(\varphi_{0,f})\delta(\varphi_{0,f} - \varphi_+) + 2\pi\rho_-F_{\text{active}}(\varphi_{0,f})\delta(\varphi_{0,f} - \varphi_+^{\downarrow}) \\ + \rho_+F_{\text{active}}(\varphi_+) + \rho_-F_{\text{active}}(\varphi_+^{\downarrow}) = 0,$$

from the collision invariant $\Psi = \varphi$, and lastly

$$\partial_t \rho_+ + \nabla_x \cdot (\rho_+(\nu_{\text{active}}(\varphi_+) - D\nabla_x \rho_f)) \\ + \rho_+ \delta(\varphi_{0,f} - \varphi_+) F_{\text{active}}(\varphi_{0,f}) - \rho_+ \delta(\varphi_{1,f} - \varphi_+) F_{\text{active}}(\varphi_{1,f}) = 0,$$

that we obtain from the generalised collision invariant $\Psi_f = \mathbb{1}_{[\varphi_{0,f},\varphi_{1,f}]}(\varphi)$.

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