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„Optimal Blowup Stability for the Energy Critical Wave Equation“

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Quite a beautiful place, really, isn't it? Almost like a world on its own.

— *The Prisoner*

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Abstract

The blowup behaviour of nonlinear partial differential equations is a well-known topic in mathematical research. This means that smooth compactly supported initial data can lead to a breakdown of the solution because of the nonlinear effect. In this thesis, we are concerned with the blowup behaviour of wave equations of power nonlinearity.

This thesis consists of two parts. The first part is an introduction. We give an overview of the study of wave equations with power nonlinearity, including basic tools and methods. We also present some known results. The second part is an original result. We have shown that, in 5 dimensions, the ODE blowup of the energy critical wave equation is asymptotically stable under small perturbation in the energy space. This work was submitted for publication.

More precisely, we investigate the stability of the ODE blowup in the lightcone for the energy critical wave equation in the energy space, which is the optimal space for well-posedness. In order to work with the time-dependent potential, we introduce similarity coordinates. The linear evolution is then studied using semigroup theory. However, because the energy space is the scale invariant Sobolev space for this equation, no decay of the semigroup can be obtained. Instead, we construct the representation of the semigroup explicitly using Volterra iterations. With this explicit representation, we then use oscillatory integral techniques to establish Strichartz estimates and improved energy bounds. With these bounds, we turn to the nonlinear problem proved the stated result via a fixed point argument.

Zusammenfassung

Das Blow-up Verhalten nichtlinearer partieller Differentialgleichungen ist ein bekanntes Thema in mathematischer Forschung. Das bedeutet, glatte Anfangswerte mit kompaktem Träger führen zu einer Singularität einer Lösung wegen nichtlinearer Effekte. In dieser Doktorarbeit beschäftigen wir uns mit dem Blow-up von Wellengleichungen mit Potenznichtlinearität.

Diese Doktorarbeit besteht aus zwei Teilen. Der erste Teil ist die Einführung. Wir geben einen Überblick über das Studium nichtlinearer Wellengleichungen, einschließlich grundlegender Werkzeuge und Methoden. Wir präsentieren auch einige bekannte Ergebnisse. Im zweiten Teil stellen wir unser Ergebnis vor. Wir beweisen, dass der *ODE Blow-up* der energiekritischen Wellengleichung in 5 Dimensionen unter kleinen Störungen im Energieraum asymptotisch stabil ist. Dieser Teil wird zur Veröffentlichung eingereicht.

Genauer gesagt, wir untersuchen die Stabilität des ODE Blow-ups der energiekritischen Wellengleichung in dem Energieraum des Lichtkegels. Das ist der optimale Sobolev-Raum für Well-posedness. Wir führen ein selbstähnliches Koordinatensystem ein, damit wir mit dem zeitabhängigen Potenzial arbeiten können. Die lineare Evolution wird mittels Halbgruppentheorie untersucht. Weil der Energieraum der skaleninvariante Sobolev-Raum ist, hat die Halbgruppe keinen Abfall. Stattdessen konstruieren wir einen expliziten Ausdruck mittels Volterra-Iterationen. Mithilfe dieser expliziten Darstellung können wir Strichartz-Abschätzungen und eine verbesserte Energieabschätzung erreichen. Dann wenden wir uns dem nichtlinearen Problem zu und beweisen unser Resultat unter Zuhilfenahme des Banachschen Fixpunktsatzes.

Contents

Chapter 1

Overview

The theory of nonlinear partial differential equations has far reaching applications to all of science. In particular, the study of nonlinear wave equations has attracted great interest in recent years. A prototypical example is the wave equation with power nonlinearity,

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = \pm|u(t,x)|^{p-1}u(t,x) \\ u[0] = (f,g) \end{cases} \tag{1.1$^\pm$}$$

for $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$ an interval with $0 \in I$, and $u[t] := (u(t,\cdot), \partial_t u(t,\cdot))$.

This equation admits the conserved energy

$$E(u[t]) = \int_{\mathbb{R}^d} \left(\frac{1}{2}(\partial_t u(t,x))^2 + \frac{1}{2}|\nabla u(t,x)|^2 \mp \frac{1}{p+1}|u(t,x)|^{p+1} \right) dx. \tag{1.2}$$

Symmetries of the equation include spacetime translation, time reversal symmetry, and space rotation. One of the most important symmetries is the scaling symmetry. This says that, given a (classical) solution u to $(1.1)^\pm$,

$$u_\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$$

is also a solution with initial data now given by

$$u_\lambda[0] = (f_\lambda, g_\lambda) = \left(\lambda^{-\frac{2}{p-1}} f\left(\frac{\cdot}{\lambda}\right), \lambda^{-\frac{2}{p-1}-1} g\left(\frac{\cdot}{\lambda}\right)\right)$$

for all $\lambda > 0$. As will be discussed later, it turns out that the behaviour of solutions to $(1.1)^\pm$ is heavily related to the scaling property. If we measure the rescaled solution u_λ in the Sobolev space $\dot{H}^s(\mathbb{R}^d)$, we have

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-\frac{2}{p-1}} \left\| u\left(\frac{t}{\lambda}, \frac{\cdot}{\lambda}\right)\right\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-\frac{2}{p-1}+\frac{d}{2}-s} \left\| u\left(\frac{t}{\lambda}, \cdot\right)\right\|_{\dot{H}^s(\mathbb{R}^d)}.$$

We see that when $s = s_c := -\frac{2}{p-1} + \frac{d}{2}$, the $\dot{H}^s$ norm is invariant under the scaling. In general we would expect ill-posedness in the space $\dot{H}^s$ if $s < s_c$.

A special case is when $s = 1$. In this case we are looking at the *energy space* $\dot{H}^1 \times L^2(\mathbb{R}^d)$. For a nonlinearity with $p = \frac{d+2}{d-2}$ and $d > 2$, the energy of a solution is invariant under scaling, and we call the equation *energy critical.* Moreover, we call the equation *subcritical* if $1 < p < \frac{d+2}{d-2}$, and *supercritical* if $p > \frac{d+2}{d-2}$.

Another notable symmetry is the conformal symmetry in the case $p = \frac{d+3}{d-1}$. Given a solution u, the conformal transformation

$$\tilde{u}(t,x) = (-t^2 + |x|^2)^{-\frac{d-1}{2}} u \left(\frac{t}{-t^2 + |x|^2}, \frac{x}{-t^2 + |x|^2} \right)$$

is also a solution. We call this case *conformal.* Similarly, we call the case $1 < p < \frac{d+3}{d-1}$ *subconformal*, and the case $p > \frac{d+3}{d-1}$ *superconformal.*

In this introduction, we will discuss some aspects of $(1.1)^{\pm}$. We start by introducing some properties of the *linear* wave equation, including how one can solve the free equation in low regularity, and what bounds one can have for the solution. With such bounds for the linear evolution, we will turn to the nonlinear problem, where we use a fixed point argument to obtain solutions that satisfy Duhamel's principle. To give concrete examples, we discuss some well-posedness results for the cubic (subcritical) and quintic (critical) case with $d = 3$.

Then we will discuss blowup solutions and blowup stability. We outline the method of showing blowup stability developed in [12, 13, 15]. Finally, we turn to the energy critical case. We introduce the two types of blowup, type I and type II. We describe the method and the difficulties of showing the stability of the ODE blowup in the energy critical equation in energy space.

Eventually, we will approach the main result of this thesis, which states that in the focusing *energy critical* case in $d = 3$ ([8]) and $d = 5$ ([10]), there exists an open set of radial data in the *energy space* that lead to the ODE blowup. In other words, the ODE blowup is stable in the energy topology. The $d = 5$ case is the original result of this thesis and is submitted for publication. The strategy for proving this result is described in Section 1.4.2, and the detailed proof of the result for $d = 5$ is presented in the second part of the thesis.

1.1 Notations and definitions

First of all, we use $(\cdot)$ as a placeholder for the variable of a function. For example, if we write $g = (\cdot)f$, we mean that $g(x) = xf(x)$.

We use the letter C to denote generic independent constants. For example, for two functions f and g, when we write $|f(x)| \leq C|g(x)|$, we mean that the constant $C > 0$ is independent of the functions themselves and the variable x (unless otherwise stated). In this case we also write $|f(x)| \lesssim |g(x)|$.

We use bold letters such as $\mathbf{u}, \mathbf{f}$ to denote functions of two components, and we write $[\mathbf{u}]_1$ to mean the first component. For example, $\mathbf{u} = (u_1, u_2)$ and $[\mathbf{u}]_1 = u_1$.

We write $\mathbb{B}^d_R := \{x \in \mathbb{R}^d : |x| < R\} \subset \mathbb{R}^d$, and $\mathbb{B}^d := \mathbb{B}^d_1$.

The Japanese bracket notation $\langle x \rangle := \sqrt{1 + |x|^2}$ is also used.

For a radial function f on $\mathbb{B}^d_R$, we identify f with the function $\hat{f}$ on $[0, R)$ where $\hat{f}(|x|) = f(x)$. In the following, both functions are denoted by f.

For a function f defined on $I \times \Omega$ where $I \subset \mathbb{R}$ is an interval and $\Omega \subset \mathbb{R}^d$, the mixed Lebesgue norm is defined by

$$\|f\|_{L^p(I;L^q(\Omega))} := \left(\int_I \left(\int_\Omega |f(t,x)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}.$$

Denote by $\mathscr{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$. For a function $f \in \mathscr{S}(\mathbb{R}^d)$, we define the *Fourier transform* and *inverse Fourier transform* by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi.$$

And we indeed have $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$. One can make sense of these transforms for $f \in L^2(\mathbb{R}^d)$.

Using the Fourier transform, we define the *homogeneous Sobolev spaces* $\dot{H}^s(\mathbb{R}^d)$, for $s \in \mathbb{R}$, as the completion of $\mathscr{S}(\mathbb{R}^d)$ under the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^{\frac{d}{2}}} \| |\cdot|^s \mathcal{F} f\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |\mathcal{F}(f)(\xi)|^2 |\xi|^{2s} d\xi \right)^{\frac{1}{2}}.$$

By properties of the Fourier transform, we see that when s is a nonnegative integer, the norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d)}$ coincides with the usual definition of the homogenous Sobolev norm given by derivatives. In particular, we have $\dot{H}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

Finally, functions of symbol type are denoted by $\mathcal{O}$. This is defined in Section 2.3.1, and some properties we used in Chapter 2 are derived in Appendix A.

1.2 Linear properties

In order to study nonlinear wave equations, the first step is to understand the linear problem. We start with the free wave equation,

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = 0 \\ u[0] = (f,g), \end{cases} \tag{1.3}$$

for $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$ an interval with $0 \in I$. This equation has the conserved energy

$$E_{Lin}(u[t]) = \int_{\mathbb{R}^d} \left(\frac{1}{2}(\partial_t u(t,x))^2 + \frac{1}{2}|\nabla u(t,x)|^2 \right) dx. \tag{1.4}$$

An important property of (1.3) is the *finite speed of propagation*. This means that, if we have initial conditions with $u[0] = 0$ on some ball $\mathbb{B}_r^d$ of radius $r > 0$, then the solution satisfies $u[t] = 0$ on $\mathbb{B}_{r-|t|}^d$ for time $|t| < r$. This is shown by considering the local energy

$$E_{loc}(u[t]) = \int_{\mathbb{B}_{r-|t|}^d} \left(\frac{1}{2}(\partial_t u(t,x))^2 + \frac{1}{2}|\nabla u(t,x)|^2 \right) dx, \tag{1.5}$$

and using the divergence theorem one can show that (for classical solutions) $\frac{d}{dt} E_{loc}(u[t]) \leq 0$. So if $u[0] = 0$ on $\mathbb{B}_r$, then $E_{loc}(u[t]) = 0$.

1.2.1 Solution via Fourier transform

One way to find solutions to this equation is by using the Fourier transform. Suppose we start with initial conditions $f, g \in \mathscr{S}(\mathbb{R}^d)$. Taking the Fourier transform in the spatial variable of (1.3), we obtain a simple ODE in t, given by

$$\begin{cases} (\partial_t^2 + |\xi|^2)\mathcal{F}(u)(t,\xi) = 0 \\ \mathcal{F}(u)[0] = (\mathcal{F}(f), \mathcal{F}(g)). \end{cases}$$

Solving this ODE and taking the inverse Fourier transform, we find that

$$u(t,x) = \cos(t|\nabla|)f(x) + \frac{\sin(t|\nabla|)}{|\nabla|}g(x), \tag{1.6}$$

where the *wave propagators* are defined by

$$\cos(t|\nabla|)f := \mathcal{F}^{-1}\left(\cos(t|\cdot|)\mathcal{F}(f)\right) \quad \text{and} \quad \frac{\sin(t|\nabla|)}{|\nabla|}g := \mathcal{F}^{-1}\left(\frac{\sin(t|\cdot|)}{|\cdot|}\mathcal{F}(g)\right).$$

From energy conservation, we get the following bounds on the wave propagators,

$$\|\cos(t|\nabla|)f\|_{\dot{H}^1(\mathbb{R}^d)} \leq \|f\|_{\dot{H}^1(\mathbb{R}^d)} \quad \text{and} \quad \left\|\frac{\sin(t|\nabla|)}{|\nabla|}g\right\|_{\dot{H}^1(\mathbb{R}^d)} \leq \|g\|_{L^2(\mathbb{R}^d)} \tag{1.7}$$

for all $t \in I$. In fact, by Plancherel, we also have

$$\|\cos(t|\nabla|)f\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|f\|_{\dot{H}^s(\mathbb{R}^d)} \quad \text{and} \quad \left\|\frac{\sin(t|\nabla|)}{|\nabla|}g\right\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|g\|_{\dot{H}^{s-1}(\mathbb{R}^d)}$$

for all $t \in I$ and $s \geq 0$.

These already provide very nice bounds on the solution. In particular, we see that the solution (1.6) in fact makes sense for initial conditions $(f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. In this case, u is called a *strong solution*.

By combining these bounds with energy conservation, we could already approach many nonlinear problems. The energy critical case (see later) is however a bit more subtle, and we will need some decay (in some sense) of the solution.

1.2.2 Dispersion

A key feature of (1.3) is its dispersive nature for $d > 1$. Physically, dispersion means that waves with different frequencies propagate at different velocities. Let us consider the ansatz of a plane wave, $u(t,x) = e^{i(k\cdot x - \omega t)}$, where $k \in \mathbb{R}^d$ and $w \in \mathbb{R}_+$. Plugging this into the free wave equation (1.3), we obtain the relation

$$\omega = |k|.$$

Hence the phase velocity of a plane wave of spatial frequency k is given by $\frac{\omega}{|k|}\hat{k} = \hat{k}$, where $\hat{k}$ is the unit vector in the direction k. So we see that for $d > 1$, the direction of the wave propagation is determined by its frequency k. The same computation shows that for $d = 1$, waves at all frequencies travel at the same velocity. Hence the 1-dimensional free wave equation is not dispersive.

This phenomenon does not occur for dissipative equations (e.g. the heat equation), for which waves do not propagate but instead decay to 0 in time.

The dispersive nature of the free wave equation is reflected by the *Strichartz estimates.* These estimates were originally discovered by Strichartz in [47] when studying Fourier restriction theory. They are space-time estimates of solutions to dispersive equations. In [28, 33, 22], Strichartz estimates were established for the wave operators, and existence results for semilinear wave equations with low regularity were established using these estimates, see Section 1.3 for some examples. Subsequently, in [26] Strichartz estimates were established for wave and Schrödinger equations with a large range of exponents including the endpoint cases.

In the case of the free wave equation, we have the following homogeneous Strichartz estimates.

Theorem 1.2.1 (see [48, Theorem 2.6])**.** *Let $s \in [0, \infty)$, $p \in [2, \infty]$, and $q \in [2, \infty)$ be such that $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s$. Then we have the estimates,*

$$\begin{aligned} \| \cos(t|\nabla|)f\|_{L^p_t(I;L^q(\mathbb{R}^d)} &\lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \\ \left\| \frac{\sin(t|\nabla|)}{|\nabla|} g \right\|_{L^p_t(I;L^q(\mathbb{R}^d))} &\lesssim \|g\|_{\dot{H}^{s-1}(\mathbb{R}^d)}, \end{aligned}$$

for all $(f, g) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$, provided that

$$\frac{1}{p} + \frac{d-1}{2q} \le \frac{d-1}{4}.$$

Solutions given by (1.6) to the wave equation gain no regularity. Nevertheless the above estimates describe a kind of space-time improvement of the solution. For example, setting $s = 0$, we can see that, starting with L^2 data, solutions to the wave equation actually gain integrability in space, but only in an 'average in time' sense.

1.2.3 The inhomogeneous problem

The second step towards understanding nonlinear problems of the type $(1.1)^{\pm}$ is to understand the inhomogeneous problem,

$$\begin{cases} (\partial_t^2 - \Delta)u(t, x) = F(t, x) \\ u[0] = (f, g). \end{cases} \tag{1.8}$$

Duhamel's principle tells us that solutions to this problem are given by

$$u(t, \cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g + \int_0^t \frac{\sin((t - s)|\nabla|)}{|\nabla|} F(s, \cdot) ds. \tag{1.9}$$

Hence, as long as we have estimates on the *linear* part (i.e. the wave propergators), we can use the them to bound the inhomogeneity.

For example, in the case when $I = [0, T]$, we estimate the last term in (1.9) using the homogeneous Strichartz estimates. By Minkowski's inequality,

$$
\begin{aligned}
&\left\| \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} F(s,\cdot) ds \right\|_{L_t^p((0,T);L^q(\mathbb{R}^d))} \\
\leq &\int_0^\infty \left(\int_0^\infty 1_{\mathbb{R}_+}(t-s) 1_{[0,T]}(t) \left\| \frac{\sin((t-s)|\nabla|)}{|\nabla|} F(s,\cdot) \right\|_{L^q(\mathbb{R}^d)}^p dt \right)^{\frac{1}{p}} ds \\
\leq &\int_0^T \left(\int_0^T \left\| \frac{\sin(t|\nabla|)}{|\nabla|} F(s,\cdot) \right\|_{L^q(\mathbb{R}^d)}^p dt \right)^{\frac{1}{p}} ds \\
\leq &\int_0^T \left\| \frac{\sin(t|\nabla|)}{|\nabla|} F(s,\cdot) \right\|_{L^p((0,T);L^q(\mathbb{R}^d))} ds \\
\lesssim &\int_0^T \| F(s,\cdot) \|_{\dot{H}^{s-1}(\mathbb{R}^d)} ds,
\end{aligned}
$$

if p, q, s satisfy the condition given in Theorem 1.2.1. [1]

In the next section, we shall see how these can help with solving nonlinear problems.

1.3 The nonlinear problem

Let us finally turn to the nonlinear problem $(1.1)^\pm$.

When we have a negative sign of the nonlinear forcing term on the right hand side, which we refer to as $(1.1)^-$, this term works *with* the dispersive effect and is called *defocusing*. When we have a positive sign, which we refer to as $(1.1)^+$, it works *against* the dispersion and is called *focusing*. In the defocusing case, the energy is non-negative, thus the conservation of energy provides good bounds for u. However, in the focusing case, although the energy is still conserved, the free energy and the nonlinear part can grow towards ∞ and $-\infty$ respectively. Hence in this case energy does not provide good control of the solution.

In order to study the properties of solutions to $(1.1)^\pm$, we have to first define what a solution means. Motivated by Duhamel's principle, we say that a function u is a *mild solution* to $(1.1)^\pm$ if

$$
u(t,\cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g \pm \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(|u(s,\cdot)|^{p-1} u(s,\cdot) \right) ds \tag{1.10}
$$

holds continuously in t on a certain space. Unlike (1.6) and (1.9), this does not give a formula for the solution, since the function u appears on both sides. Instead, we define the operator

$$
K_{(f,g)}(u(t,\cdot)) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g \pm \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(|u(s,\cdot)|^{p-1} u(s,\cdot) \right) ds, \tag{1.11}
$$

[1] In fact, there are more general Strichartz estimates for the inhomogeneity which admit a wide range of exponents. But we do not need this for our application.

and our task is to find a fixed point of (1.11) in some suitable space. This fixed point is then a solution to (1.10)

A solution to $(1.1)^{\pm}$ still has finite speed of propagation. This can be shown in two ways. We can first show that classical solutions have finite speed of propagation as in [48, Proposition 3.3], then use a density argument. For this, we consider again the time derivative of the local energy (1.5). Suppose we have $u[0] = 0$ on $\mathbb{B}^d_r$, using the equation $(1.1)^{\pm}$ and Grönwall's inequality, we can show that $E_{loc}(u[t]) = 0$. The other way is to consider the fixed point argument as in [27, Remark 1.10]. Indeed, the fixed point is obtained via an iteration of the map $K_{(f,g)}$. We start with

$$u_0(t,\cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g,$$

so that the initial conditions are satisfied. Then the fixed point is obtained by taking the limit $n \to \infty$ of the sequence $\{u_n(t,\cdot)\}$ defined by $u_n(t,\cdot) := K_{(f,g)}(u_{n-1}(t,\cdot))$. At every step, $u_n[t] = 0$ on $\mathbb{B}^d_{r-|t|}$, hence the limit is also 0 on $\mathbb{B}^d_{r-|t|}$.

1.3.1 Well-posedness theory

We first discuss some well-posedness theory which we illustrate by some examples for $d = 3$.

Example 1.3.1 (Cubic NLW)**.** As a first example, let us briefly discuss some well-posedness properties for the case $p = 3$ on $\mathbb{R}^3$ in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^3)$. This equation is conformal. Notice that this problem is subcritical, since the scale invariant Sobolev space in this case is $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$.

The goal here is to start with data $(f,g) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$, and solve (1.10) on $t \in [0,T]$ for some small $T > 0$, using the Banach fixed point theorem on (1.11). This can be done via bounds on the propagators. We need to first find a complete metric space on which $K_{(f,g)}$ maps to itself, and then show that $K_{(f,g)}$ is a contraction on this space.

We would like to control the $\dot{H}^1(\mathbb{R}^3)$ norm of $K_{(f,g)}(u)$. Using the bounds (1.7) on the wave propagators, we compute

$$\begin{aligned}
&\|K_{(f,g)}(u)\|_{\dot{H}^1(\mathbb{R}^3)} \\
&= \left\| \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g \pm \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(|u(s,\cdot)|^{p-1}u(s,\cdot)\right) ds \right\|_{\dot{H}^1(\mathbb{R}^3)} \\
&\le \|\cos(t|\nabla|)f\|_{\dot{H}^1(\mathbb{R}^3)} + \left\|\frac{\sin(t|\nabla|)}{|\nabla|}g\right\|_{\dot{H}^1(\mathbb{R}^3)} + \left\|\int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(u(s,\cdot)^3\right) ds\right\|_{\dot{H}^1(\mathbb{R}^3)} \\
&\le \|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} + \left\|\int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(u(s,\cdot)^3\right) ds\right\|_{\dot{H}^1(\mathbb{R}^3)}.
\end{aligned}$$

Hence the task is to treat the nonlinear Duhamel term. Estimating it in $\dot{H}^1(\mathbb{R}^3)$ and using the bound on the sine propagator and the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we have

$$\begin{aligned}\left\|\int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(u(s,\cdot)^3\right)ds\right\|_{\dot{H}^1(\mathbb{R}^3)} &\leq \int_0^t \left\|u(s,\cdot)^3\right\|_{L^2(\mathbb{R}^3)}ds\\ &= \int_0^t \|u(s,\cdot)\|^3_{L^6(\mathbb{R}^3)}ds\\ &\leq C\int_0^t \|u(s,\cdot)\|^3_{\dot{H}^1(\mathbb{R}^3)}ds\\ &\leq TC\sup_{t\in[0,T]}\|u(t,\cdot)\|^3_{\dot{H}^1(\mathbb{R}^3)},\end{aligned}$$

for some constant $C>0$ coming from the Sobolev embedding.

So the natural space we would consider is

$$X_T = C([0,T],\dot{H}^1(\mathbb{R}^3))$$

equipped with the norm

$$\|u\|_{X_T} = \sup_{t\in[0,T]}\|u(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Now, let $R = 2\|(f,g)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)}$. We look at the closed ball $B_R^T = \{u\in X_T : \|u\|_{X_T}\leq R\}$. Observe that for $u\in B_R^T$, we have the bound

$$\|K_{(f,g)}(u)\|_{X_T} \leq \frac{R}{2} + TCR^3.$$

So if T is chosen small enough, $K_{(f,g)}$ maps B_R^T into itself.

The contraction property is similar. For two functions $u,v\in B_R^T$, we compute

$$\begin{aligned}&\|K_{(f,g)}(u)(t,\cdot) - K_{(f,g)}(v)(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)}\\ &\leq \int_0^t \left\|\frac{\sin((t-s)|\nabla|)}{|\nabla|}\left(u(s,\cdot)^3 - v(s,\cdot)^3\right)\right\|_{\dot{H}^1(\mathbb{R}^3)}ds\\ &\leq \int_0^t \|u(s,\cdot)^3 - v(s,\cdot)^3\|_{L^2(\mathbb{R}^3)}ds\\ &\leq C\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{L^6(\mathbb{R}^3)}\left(\|u(s,\cdot)\|^2_{L^6(\mathbb{R}^3)} + \|v(s,\cdot)\|^2_{L^6(\mathbb{R}^3)}\right)ds\\ &\leq C\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)}\left(\|u(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|v(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)}\right)ds\\ &\leq TCR^2\|u-v\|_{X_T}.\end{aligned}$$

Here we have used the elemenary algebraic identity $(x^3-y^3) = (x-y)(x^2+xy+y^2)$, and Hölder's inequality. Again, choosing T small enough, $K_{(f,g)}$ is a contraction on B_R^T.

Therefore, by the Banach fixed point theorem, there exists a unique fixed point u of $K_{(f,g)}$ on B_R^T. This u solves (1.10). Continuous dependence on the initial data can also be shown by a similar computation. Uniqueness on X_T (instead of only in B_R^T) can be shown by Grönwall's inequality.

Note that this procedure works for both the focusing and defocusing case, and it does not assume smallness of the initial data.

In the the defocusing case $(1.1)^-$, if f is in addition in $L^4(\mathbb{R}^3)$, we even have global well-posedness. To see this, we note that because the conserved energy has a sign, so the $\dot{H}^1 \times L^2(\mathbb{R}^3)$ norm of $u[t]$ does not increase with the evolution. Hence when the evolution gets to time T, we can again use the same procedure to solve the Cauchy problem with initial condition $u[T]$ on $[T, 2T]$, and we can iterate this procedure. Unfortunately, this argument does not apply in the focusing case $(1.1)^+$ because the energy does not have a sign.

Increasing the power of the nonlinearity creates difficulties. In particular, the procedure in Example 1.3.1 will not work, as we no longer have the Sobolev embedding. The way to get around this is by using the Strichartz estimates to control the excess nonlinearity.

Staying in the space $\dot{H}^1 \times L^2(\mathbb{R}^3)$, Strichartz estimates require p, q to satisfy

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}.$$

Example 1.3.2 (Quintic NLW)**.** Let us demonstrate with the case when $p = 5$. This is the *energy critical* case. Here, the Duhamel term is estimated by

$$\begin{aligned}
\left\| \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(u(s,\cdot)^5\right) ds \right\|_{\dot{H}^1(\mathbb{R}^3)} &\leq \int_0^t \left\| u(s,\cdot)^5 \right\|_{L^2(\mathbb{R}^3)} ds \\
&= \int_0^t \| u(s,\cdot) \|_{L^{10}(\mathbb{R}^3)}^5 ds \\
&\leq \| u \|_{L^2(\mathbb{R}_+; L^{10}(\mathbb{R}^3))}^5.
\end{aligned}$$

Notice that the space $L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))$ is Strichartz admissible. Hence we consider the space

$$X = C(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3)) \cap L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))$$

with the norm

$$\| u \|_X = \sup_{t \in \mathbb{R}_+} \| u(t,\cdot) \|_{\dot{H}^1(\mathbb{R}^3)} + \| u \|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))}.$$

In order to close the argument for the fixed point theorem, we also need to measure this term in $L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))$. This is done in a similar way as the inhomogeneous case in Section 1.2.3. By the Strichartz estimates, we have

$$\begin{aligned}
\left\| \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(u(s,\cdot)^5\right) ds \right\|_{L_t^5((\mathbb{R}_+; L^{10}(\mathbb{R}^3))} &\leq C \int_0^\infty \| u(s,\cdot)^5 \|_{L^2(\mathbb{R}^3)} ds \\
&= \int_0^\infty \| u(s,\cdot) \|_{L^{10}(\mathbb{R}^3)}^5 ds \\
&= \| u \|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))}^5.
\end{aligned}$$

Hence we obtain the bound

$$\| K_{(f,g)}(u) \|_X \leq C \left(\| f \|_{\dot{H}^1(\mathbb{R}^3)} + \| g \|_{L^2(\mathbb{R}^3)} + \| u \|_X^5 \right). \tag{1.12}$$

Showing the contraction property is similar to before, but we have to bound both the $\dot{H}^1(\mathbb{R}^3)$ norm and the $L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))$ norm. Given two functions u, v, we compute

$$\begin{aligned}
&\|K_{(f,g)}(u)(t,\cdot) - K_{(f,g)}(v)(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)} \\
&\le \int_0^t \left\| \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(u(s,\cdot)^5 - v(s,\cdot)^5\right) \right\|_{\dot{H}^1(\mathbb{R}^3)} ds \\
&\le C \int_0^t \|u(s,\cdot)^5 - v(s,\cdot)^5\|_{L^2(\mathbb{R}^3)} ds \\
&\le C \int_0^t \|u(s,\cdot) - v(s,\cdot)\|_{L^{10}(\mathbb{R}^3)} \left(\|u(s,\cdot)\|^4_{L^{10}(\mathbb{R}^3)} + \|v(s,\cdot)\|^4_{L^{10}(\mathbb{R}^3)} \right) ds \\
&\le C \|u - v\|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \left(\|u\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} + \|v\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \right),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\|K_{(f,g)}(u) - K_{(f,g)}(v)\|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \\
&\le C \int_0^\infty \|u(s,\cdot)^5 - v(s,\cdot)^5\|_{L^2(\mathbb{R}^3)} ds \\
&\le C \|u - v\|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \left(\|u\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} + \|v\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \right).
\end{aligned}$$

Again we used the algebraic factorisation of $(x^5 - y^5)$ and Hölder's inequality. Hence

$$\|K_{(f,g)}(u) - K_{(f,g)}(v)\|_X \le C \|u - v\|_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \left(\|u\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} + \|v\|^4_{L^5(\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \right). \tag{1.13}$$

Now, let $\delta > 0$ be so small such that $C\delta^4 \le \frac{1}{2}$. If we choose initial data with

$$\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \le \frac{\delta}{2C},$$

then we see that $K_{(f,g)}$ is a contraction mapping on $B_\delta = \{u \in X : \|u\|_X \le \delta\}$ from (1.12) and (1.13).

So we see that with the help of Strichartz estimates, we can actually obtain global well-posedness for small data, and this works for both the focusing and defocusing case.

Example 1.3.3 (Quintic NLW, again)**.** Let us consider the quintic case again. Now, if we would like to look at any initial data $(f, g) \in \dot{H} \times L^2(\mathbb{R}^3)$ without size restriction, the above method will not apply. Instead, we can consider the evolution in the mixed Lebesgue space up to time T only, namely

$$X_T = L^5((0,T); L^{10}(\mathbb{R}^3)).$$

This might seem less intuitive, but we are able to do so because of the Strichartz estimates.

Estimating $K_{(f,g)}(u)$ in X_T, we have

$$\begin{aligned}
&\|K_{(f,g)}(u)\|_{X_T} \\
&\le \left\| \cos(\cdot|\nabla|) f + \frac{\sin(\cdot|\nabla|)}{|\nabla|} g \right\|_{X_T} + \left\| \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} \left(u(s,\cdot)^5\right) ds \right\|_{L^5_t((\mathbb{R}_+; L^{10}(\mathbb{R}^3))} \\
&\le \left\| \cos(\cdot|\nabla|) f + \frac{\sin(\cdot|\nabla|)}{|\nabla|} g \right\|_{X_T} + C\|u\|^5_{X_T},
\end{aligned} \tag{1.14}$$

where we used the Strichartz estimates in the second inequality.

For two functions u and v, we have

$$\|K_{(f,g)}(u) - K_{(f,g)}(v)\|_{X_T} \leq C\|u - v\|_{X_T} \left(\|u\|_{X_T}^4 + \|v\|_{X_T}^4\right). \tag{1.15}$$

Hence we choose $C\delta^4 \leq \frac{1}{2}$ and look at the closed ball $B_\delta^T = \{u \in X_T : \|u\|_{X_T} \leq \delta\}$. Observe that as $T \to 0$, the X_T norm of the linear evolution also goes to 0. Hence for any $\delta > 0$, we can choose $T > 0$ so small that

$$\left\|\cos(\cdot|\nabla|)f + \frac{\sin(\cdot|\nabla|)}{|\nabla|}g\right\|_{X_T} \leq \frac{\delta}{2}.$$

Then from (1.14) and (1.15), we see that $K_{(f,g)}(u)$ is a contraction map from B_δ^T into itself. Therefore, we can obtain short time existence in the space $L^5((0,T); L^{10}(\mathbb{R}^3))$.

Different from the cubic case, which is also an example of local well-posedness, the existence time here depends not only on the size of the data (f, g), but the whole linear evolution $\cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$. In this case, we do not have a way to extend the existence time T from energy conservation.

Furthermore, for solutions of this type, we do not have control of the $\dot{H}^1 \times L^2$ norm. So in principle there could be blowup solution where the X_T norm blows up while the $\dot{H}^1 \times L^2$ norm stays bounded. This is the notion of Type II blowup, which we will discuss in the next section.

1.3.2 Ill-posedness below scaling

When we study a problem like $(1.1)^{\pm}$, we would naturally want to study it at low regularity. But this regularity cannot be pushed arbitrarily low.

Indeed, for the focusing energy critical case, i.e. $p = \frac{d+2}{d-2}$, the scale invariant Sobolev space $\dot{H}^1 \times L^2(\mathbb{R}^d)$ is already the minimal regularity required for well-posedness. This can be seen from a simple scaling argument described in [33]. Let us consider this evolution in the space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$ where $s < 1$. As we will see in the next section, this equation admits finite time blowup solutions. Let u^T be such a solution that blows up at time $T \in (0, \infty)$, with initial data (f, g). Then the rescaling

$$u_\lambda^T = \lambda^{-\frac{d-2}{2}} u^T\left(\frac{\cdot}{\lambda}, \frac{\cdot}{\lambda}\right)$$

is also a solution, with initial data

$$(f_\lambda, g_\lambda) = \lambda^{-\frac{d-2}{2}}\left(f\left(\frac{\cdot}{\lambda}\right), g\left(\frac{\cdot}{\lambda}\right)\right).$$

Now u_λ^T blows up at time λT. On the other hand, the $\dot{H}^s(\mathbb{R}^d)$ norm of the initial data is also scaled by

$$\|(f_\lambda, g_\lambda)\|_{\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)} = \lambda^{1-s}\|(f, g)\|_{\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)}.$$

Since $s < 1$, the $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$ size of (f_λ, g_λ) becomes small when we take λ to be small. This shows that there exists data that are arbitrarily small, but the corresponding existence time is also arbitrarily small. Hence this problem is in fact *ill-posed.*

In general, we would expect ill-posedness in spaces with regularity below the scaling of the equation. Therefore, it is sensible to at least require the data to be in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}(\mathbb{R}^d)$ for $(1.1)^{\pm}$.

1.3.3 Description of blowup solutions

One of the key aspects in the study of nonlinear wave equations is their blowup behaviour. Blowup is the phenomenon of singularity formation from smooth initial data. This means that the solution 'breaks down' in some sense at some finite time $T > 0$. The first question we can ask is, when does blowup occur? Indeed this does not always happen. In Example 1.3.1, we see that the defocusing cubic equation in $d = 3$ is globally well-posed. For the focusing case, [32] showed that initial data with negative energy lead to solutions that blow up in finite time. This is done by showing that the L^2 norm of the solution has to become infinite in finite time by showing the convexity of a certain functional. However, this proof does not give descriptions of the blowup profile.

The most basic explicit blowup solution is the ODE blowup. Consider the ansatz $u^T(t,x) = C_p(T-t)^{-\alpha}$, where $T > 0$ is a parameter. Plugging it into $(1.1)^+$, we have

$$\partial_t^2 u^T = (u^T)^p.$$

Solving this ODE, we obtain

$$\alpha = \frac{2}{p-1}, \quad C_p = \left(\frac{2(1+p)}{(1-p)^2}\right)^{\frac{1}{p-1}},$$

so the solution is

$$u^T(t,x) = C_p(T-t)^{-\frac{2}{p-1}}.$$

This solution becomes singular as $t \to T$. Note that although u^T is homogeneous in space, using finite speed of propagation we can truncate it to obtain finitely supported data which lead to a solution that blows up at time T.

A solution u to $(1.1)^{\pm}$ is called *self-similar* if $u(t,x) = (T-t)^{-\frac{2}{p-1}}\varphi\left(\frac{x}{T-t}\right)$ for some function φ. The function φ is called the self-similar *profile*. The ODE blowup is also the simplest example of a self-similar solution, where φ in this case is constant.

Many questions can be asked about the properties of blowup solutions. For example, for a particular equation, we would like to know what kind of blowup solutions can occur, and what kind of initial data will lead to a blowup solution. Moreover, when blowup occurs, in what sense does the solution break down: do they always break down in the same or similar way, or are there different kinds of blowup behaviours?

If certain initial data lead to a certain blowup solution, we would like to know whether or not this blowup pattern is generic. In other words, we want to know whether such a blowup solution occurs by accident of choosing particular initial data, or it happens in general. For this, one can ask the question: do other 'nearby' initial data also lead to the same type of blowup solutions? This is the concept of blowup stability. In a sense, the study of local well-posedness is like the study of stability of the zero solution, although it is not a blowup solution.

Merle and Zaag [34, 35] showed for the subconformal and conformal case, i.e., when $p \leq \frac{d+3}{d-1}$, *any* blowup solution blows up at the rate of the ODE blowup in the energy space.

In a series of papers [12, 13, 15], Donninger and Schörkhuber studied the ODE blowup stability in the backwards lightcone

$$\Gamma_T := \{(t,x) : t \in [0,T), |x| \leq T-t\} \subset \mathbb{R}_+ \times \mathbb{R}^d.$$

They showed that for radially symmetric data, the ODE blowup is stable. More precisely, there exists an open set of initial data which have solutions converging to the ODE blowup in the lightcone, although in the superconformal case, a stronger topology, namely $H^{\frac{d+1}{2}} \times H^{\frac{d-1}{2}}(\Gamma_T)$, is required. Moreover, the radial symmetry assumption was removed in [14] for the $d = 3$ case, and by Chatzikaleas and Donninger [3] for the cubic case in odd dimensions.

1.3.4 ODE blowup stability with higher regularity

Let us briefly discuss the method established in [12, 13, 15]. As it turns out, this method is also successful in showing stability of self-similar blowup solutions for other equations, see for example Yang-Mills [7] and wave maps [4], although more complicated spectral analysis is needed.

This approach is perturbative. Let us start by naively plugging the ansatz

$$u(t,x) = u^T(t,x) + v(t,x)$$

into $(1.1)^+$. Then we obtain an equation for v which is a nonlinear wave equation with self-similar potential,

$$\left(\partial_t^2 - \Delta - (T-t)^{-2} V\left(\frac{x}{T-t}\right)\right) v(t,x) = \text{Nonlinearity}.$$

In other words, we take a linearisation around the ODE blowup u^T, where v is regarded as a small perturbation. The potential term in the equation of v comes from this linearisation procedure. The function V is constant in this case, since the profile of u^T is constant. The challenge now is that this potential is *time dependent*.

Here we are interested in the ODE blowup at the origin, and we will only look at the *radial case* and set $r = |x|$. By finite speed of propagation, we can restrict our attention to the backwards lightcone. Because of the self-similar nature of the ODE blowup, the time-dependent potential is also self-similar. Hence we introduce similarity coordinates on Γ_T, given by

$$\tau := \log \frac{T}{T-t}, \quad \rho := \frac{r}{T-t}. \tag{1.16}$$

Under this coordiante transformation, Γ_T is mapped into the infinite cylinder $[0,\infty) \times \overline{\mathbb{B}^d}$. The blow-up behaviour at $(T,0) \in \mathbb{R}_+ \times \mathbb{R}^d$ is transformed into the asymptotic behaviour as $\tau \to \infty$.

For a classical solution $u \in C^\infty(\Gamma_T)$ to $(1.1)^+$, we define $\Psi(\tau)(\rho) = (\psi_1(\tau,\rho), \psi_2(\tau,\rho))$ where

$$\begin{aligned}\psi_1(\tau,\rho) &:= (T-t)^{\frac{2}{p-1}} u(t,r),\\ \psi_2(\tau,\rho) &:= (T-t)^{\frac{2}{p-1}+1} \partial_t u(t,r).\end{aligned}$$

Then $(1.1)^+$, when restricted to the backwards lightcone, is transformed into an evolution problem

$$\partial_\tau \Psi(\tau) = \widetilde{\mathbf{L}_0} \Psi(\tau) + \mathbf{F}(\Psi(\tau)), \tag{1.17}$$

where $\widetilde{\mathbf{L}_0}$ is a linear differential operator on $\mathbb{B}^d$, and $\mathbf{F}$ is some nonlinearity. Now the ODE blowup u^T corresponds to the static solution

$$\Psi^T(\tau, \rho) = \left(C_p, \frac{2}{p-1} C_p\right)$$

to (1.17). If we take a perturbation around Ψ^T, i.e., plugging into (1.17) the ansatz

$$\Psi(\tau, \rho) = \Psi^T(\tau, \rho) + \Phi(\tau),$$

we obtain an evolution problem of the form

$$\partial_\tau \Phi(\tau) = (\widetilde{\mathbf{L}_0} + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)). \tag{1.18}$$

The point is that because Ψ^T itself is time independent, the linearisation procedure produces a potential term $\mathbf{L}'$ which does not depend on time.

Of course, now we have to say what we mean by a 'solution' to the equation (1.18). Indeed, we solve (1.18) via semigroup theory on a suitably chosen Sobolev space $\mathcal{H}$ of functions on $\mathbb{B}^d$ of enough regularity. In fact, this procedure is very similar to that of showing well-posedness. One first establishes suitable bounds for the linear evolution, and uses them to control the nonlinearity in the Duhamel term.

First, we show that the operator $\widetilde{\mathbf{L}_0}$, originally only defined for functions with enough regularity, is closable with closure $\mathbf{L}_0$ that generates a semigroup $\mathbf{S}_0$. For this, we use the Lumer–Phillips theorem.

Theorem 1.3.4 ([21, p.83 Theorem 3.15 and p.88 Proposition]). *For a densely defined operator $(A, D(A))$ on a Hilbert space H, if*

(a) $\operatorname{Re}(Ax|x) \leq w_0\|x\|$ *for every* $x \in D(A)$ *and some* $w_0 \in \mathbb{R}$*, and*

(b) $\operatorname{rg}(\lambda - A) \subset H$ *is dense in* X *for some (hence all)* $\lambda > w_0$*,*

then the closure $\bar{A}$ *of* A *generates a strongly continuous semigroup* $S(t)$ *with*

$$\|S(t)\| \leq e^{w_0 t} \text{ for all } t \geq 0.$$

Then, by the bounded perturbation theorem we show that $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}'$ generates a semigroup $\mathbf{S}(\tau)$ on $\mathcal{H}$. Now, the natural way to define a solution to (1.18) would be via Duhamel's principle,

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}(\tau - \sigma)\mathbf{N}(\Phi(\sigma))d\sigma,$$

where $\Phi(0) = \mathbf{u}$ is the initial condition.

If we want to have control on the nonlinearity in the Duhamel term, we would need some decay of the semigroup. For this, we would like the generator $\mathbf{L}$ to have spectrum contained in some half plane to the left of the imaginary axis. This is however not the case, as $\mathbf{L}$ actually has an unstable eigenvalue 1. This unstable eigenvalue reflects the time translation symmetry of the original problem — in general, perturbation will change the blowup time, hence we cannot expect convergence to the original ODE blowup u^T, but instead to some other $u^{T'}$, where T' is close to T.

To handle this, we show that we have a projection $\mathbf{P}$ onto this unstable eigenspace, which decomposes the space $\mathcal{H} = \ker \mathbf{P} \oplus \operatorname{rg} \mathbf{P}$. Then we have the desired growth bound of $\mathbf{S}(\tau)$ on $\ker \mathbf{P}$. This bound is obtained via the Gearhart–Prüss theorem by establishing a resolvent estimate.

Theorem 1.3.5 ([21, p.302 Theorem 1.11])**.** *For a strongly continuous semigroup $S(t)$ on a Hilbert space H with generator $(A, D(A))$, if we have*

(a) $\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 0\} \subset \rho(A)$, *and*

(b) $\sup_{\operatorname{Re}\lambda > 0} \|R_A(\lambda)\| < \infty$,

then there exists a constant $C > 1$ such that $\|S(t)\| \leq C$.

We also remark here that computing the spectrum of $\mathbf{L}$ for equations with more complicated nonlinearities is potentially not as straightforward, see for example [4] for the wave maps equation.

Finally, we can turn to the nonlinear problem. The difference here is that, in order to get rid of the instability given by the translation symmetry, we have to calibrate the blowup time of the perturbed data. With the bound on $\mathbf{S}$, we look at the modified equation

$$\mathbf{K}_{\mathbf{U(T,v)}}(\Phi)(\tau) = \mathbf{S}(\tau)\mathbf{U}(T, \mathbf{v}) + \int_0^\tau \mathbf{S}(\tau - \sigma)\mathbf{N}(\Phi(\sigma))d\sigma - e^\tau \mathbf{C}(\Phi, \mathbf{U}(T, \mathbf{v})).$$

We show that for small $\mathbf{v}$, we are able to find a blowup time T^* close to T, so that there exists a unique fixed point $\Phi = \mathbf{K}_{\mathbf{U}(T^*,\mathbf{v})}(\Phi)$ with $\mathbf{C}(\Phi, \mathbf{U}(T^*, \mathbf{v})) = 0$. Translating back into the physical coordinates, this means that, starting with a small perturbation of the initial data that leads to the ODE blowup u^T, we obtain a solution that converges to the ODE blowup u^{T^*}, where T^* is close to T.

1.4 Blowup of the energy critical equation

We would like to turn our attention to the focusing energy critical problem $p = \frac{d+2}{d-2}$,

$$(\partial_t^2 - \Delta)u(t, x) = |u(t, x)|^{\frac{4}{d-2}} u(t, x). \tag{1.19}$$

We have already seen that, with the help of Strichartz estimates, global well-posedness can be obtained in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^d)$ for small data, and this is the optimal result as any lower regularity will lead to ill-posedness.

Let us first look at the ODE blowup, which in this case is given by

$$u^T(t, x) = c_d(T - t)^{\frac{-(d-2)}{2}}, \; c_d = \left(\frac{d(d-2)}{4}\right)^{\frac{d-2}{4}}.$$

We see that (after truncation) $\|u^T(t, \cdot), \partial_t u^T(t, \cdot)\|_{\dot{H}^1 \times L^2} \to \infty$ as $t \to T$. A blowup solution of which the $\dot{H}^1 \times L^2$ norm goes to ∞ is called a *Type I blowup.*

There is another type of blowup solutions whose existence time $T > 0$ is finite, such that their $\dot{H}^1 \times L^2$ norm stays bounded. A blowup with this property is called a *Type II blowup*. In fact, Type II blowups are closely related to another important explicit solution to $(1.1)^+$, namely the *soliton solution*, given by

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}.$$

This solution is independent of t and solves the equation $\Delta W + W^{\frac{d+2}{d-2}} = 0$. Notice that the solution $(W, 0)$ is in the energy space for all $d > 2$.

Duyckaerts, Kenig, and Merle in [17] showed that for $d = 3$, in the radial case, any Type II blowup solution decomposes into the sum of (time-dependent) rescalings of W, a radiation term, and a term that goes to 0 as $t \to T$. In [19] they extended the result to the non-radial case in $d = 3$ and $d = 5$, where in the decomposition, we would have rescaling of the Lorentz transform of W instead of W itself.

1.4.1 Constructions of Type II blowups

A priori, it is unclear whether or not Type II blowup solutions really exist. In this section we describe some explicit constructions of Type II blowup solutions. Indeed they are not as simple and straightforward as the ODE blowup. The first example of a Type II blowup was constructed in Krieger, Schlag, and Tataru in [31] for the $d = 3$ case. With a completely different approach, Hillairet and Raphaël constructed Type II blowups for $d = 4$ in [23].

Both constructions give energy class solutions of the form described in [17], namely, as $t \to T$, we have the decomposition

$$(u(t,r), \partial_t u(t,r)) = \big(W_{\lambda(t)}(r), 0\big) + (v_0, v_1) + o(1)$$

in $\dot{H}^1 \times L^2(\mathbb{R}^d)$, and $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ is small. Moreover, both constructions admit initial data with energy arbitrarily close to $E(W, 0)$. Indeed, while we would expect the time-dependent rescaling $W_{\lambda(t)}(r) = \lambda(t)^{-\frac{d-2}{2}} W\left(\frac{r}{\lambda(t)}\right)$ to dominate the blowup behaviour, $W_{\lambda(t)}$ itself is no longer a solution to (1.19), since it has nonzero time derivatives. Hence we would need to first construct approximate profiles as a perturbation of $W_{\lambda(t)}$ inside the lightcone, then decompose the evolution as the approximate profile plus an error term.

The two completely different methods of construction lead to differences in the results too. In [31] and later [30], the blowup they constructed admits a continuous spectrum of blowup speeds, given by $\lambda(t) = (T - t)^{\nu}$ for all $\nu > 1$, but the blowup profile is not smooth at the lightcone. Moreover, using this method, Type II blowups with an uncountable family of oscillatory blowup rates are constructed in [9]. On the other hand, the blowup solution obtained in [23] is C^∞ with blowup speed given by $\lambda(t) = (T - t)e^{-\sqrt{|\log(T-t)|}(1+o(1))}$.

In [31] and [30] the $d = 3$ case is studied, where the blowup speed is fixed to $\lambda(t) = (T - t)^{\nu}$ for $\nu > 1$ at the start. This means that this type of blowup is faster than, but can be arbitrarily close to, self-similarity. Their approximate profile is constructed iteratively.

Starting from $W_{\lambda(t)}$, they add correction terms by alternating the approximations inside the lightcone for small r, and near the lightcone $r \approx T - t$. This results in a profile u_k after k steps, such that when plugging into (1.19), the error decays like $(T-t)^N$ for any N, provided k is large enough. Then they linearise (1.19) around u_k by considering the ansatz $u(t,r) = u_k(t,r) + \epsilon(t,r)$. In this analysis, similar to the idea of using similarity coordinates, adapted coordinates (s,y) for this problem are introduced, given by

$$\frac{ds}{dt} = \frac{1}{\lambda(t)},\ y = \frac{r}{\lambda(t)}. \tag{1.20}$$

The linearised (s-independent) operator L has continuous spectrum on $[0,\infty)$ where 0 is a resonance, and one negative eigenvalue. A distorted Fourier transform $\mathcal{F}$ is defined which is adapted to the spectral measure of L. For large enough k, applying $\mathcal{F}$ to the equation and solving backwards with the data $\epsilon = 0$ at $s = \infty$ (which is the time when the blowup happens), they use a fixed point argument to obtain a finite time blowup of which the leading order behaviour is prescribed by $W_{\lambda(t)}$ near the blowup time. Moreover, for ν sufficiently close to 1, [2] and [29] showed that the blowup constructed here is codimension 1 stable. This stability is optimal because of the presence of the unstable eigenvalue.

The method of [23] is completely different. It is much more in the spirit of the pioneering work [44] (and later [43]) on the wave maps equation. Here the $d = 4$ case is studied, and the blowup rate $\lambda(t)$ is a priori unknown. Although adapted coordinates like (1.20) are occasionally also used, in this case an explicit expression for s cannot be immediately obtained. The approximation is done by looking for an approximate self-similar profile P which is a perturbation of W. Similar to above, they then consider the ansatz $u(t,r) = P(t,r) + \epsilon(t,r)$. Again, the linearised operator has one negative eigenvalue and a resonance at 0. However, instead of studying the linearisation first, they look at the full equation for ϵ directly in the original coordinates (t,r). The negative eigenvalue and the resonance are handled by imposing certain orthogonality conditions. Bounds on the energy for the full evolution are established, which give enough control for a bootstrap argument. The blowup rate λ is then computed from an ODE which is naturally derived in order to satisfy the bootstrap bounds. This type of blowup is also of codimension 1, hence is optimally stable.

1.4.2 ODE blowup stability in the energy space

Finally, we turn to the the main part of this thesis. This is the study of stability of the ODE blowup at the lowest possible regularity, namely in the *energy space*. Again, we restrict ourselves to the radial case. The study of the $d = 5$ case is given in Chapter 2 of this thesis, which is self-contained. Substantial differences between the $d = 5$ case and the $d = 3$ case are discussed there too.

Let us follow the procedure in Section 1.3.4 and see what goes wrong in this case. As usual, we introduce similarity coordinates given by (1.16) on Γ_T, which maps Γ_T into $[0,\infty) \times \overline{\mathbb{B}^d}$. The ODE blowup then cooresponds to a static solution, perturbing around which yields the evolution (1.18). From the Lumer-Phillips theorem and the bounded perturbation theorem, we obtain a semigroup $\mathbf{S}(\tau)$ generated by $\mathbf{L}$ on the *energy space* $\mathcal{H} = H^1 \times L^2(\mathbb{B}^d)$. Again, the time translation symmetry introduces an unstable eigenspace with eigenvalue 1. We take the projection $\mathbf{P}$ onto this unstable eigenspace, and look at the evolution on $\ker \mathbf{P}$. So far this is exactly the same as before.

The first problem comes in when we try to employ the Gearhart–Prüss theorem. The continuous spectrum of $\mathbf{L}$ actually goes all the way to the imaginary axis, and the required resolvent estimate is only true for every $\epsilon > 0$. Hence we are only able to obtain the bound, for any $\epsilon > 0$

$$\|\mathbf{S}(\tau)\tilde{\mathbf{f}}\|_{\mathcal{H}} \leq C_\epsilon e^{\epsilon\tau} \|\tilde{\mathbf{f}}\|_{\mathcal{H}}$$

for $\tilde{\mathbf{f}} \in \ker \mathbf{P}$. The constant C_ϵ depends on ϵ and we have no control on it as $\epsilon \to 0$.

In fact, even if we could establish such bound of the semigroup by the Gearhart-Prüss theorem, we would still not be able to control the nonlinearity purely from this bound. In order to establish the stability results in Section 1.3.4, enough *decay* from the free wave semigroup has to be obtained to control the nonlinearity. Since a stronger topology (than scaling of the equation) is used, one can obtain the decay needed purely from scaling. Indeed, from scaling we see that, if we measure the function φ_1 in $\dot{H}^k(\mathbb{B}^d)$, we have

$$\begin{aligned}
\|\varphi_1(\tau, \cdot)\|_{\dot{H}^k(\mathbb{B}^d)} &= (T-t)^{\frac{2}{p-1}} \left\| u\left(t, (\cdot)(T-t)\right)\right\|_{\dot{H}^k(\mathbb{B}^d)} \\
&= (T-t)^{\frac{2}{p-1} - \frac{d}{2} + k} \|u(t, \cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} \\
&= (T-t)^{-s_c + k} \|u(t, \cdot)\|_{\dot{H}^k(\mathbb{B}^d_{T-t})}.
\end{aligned}$$

Of course, since on $\mathbb{B}^d$ the homogeneous norms are only seminorms, one must consider instead the full norm H^k. In [15], equivalent norms which admit the 'right' scaling were introduced, so the desired decay can be obtained. This is however not the case if we want to study an equation at the critical level. In our case, the energy space is the scale invariant Sobolev space for the energy critical equation. Hence no decay can be obtained in such a way.

Motivated by the well-posedness theory in Example 1.3.2 for critical equations, the answer is to use *Strichartz estimates*. This is done the first time in [8] for the $d = 3$ case. We remark that this is also the first time that Strichartz estimates were established for wave equations with self-similar potential. Subsequently, in [10] we were able to establish this for $d = 5$ also.

The Strichartz estimates and the energy estimates were obtained by directly working on the explicit representation of the semigroup. The first step is to obtain such a representation. By Laplace inversion ([21, p.234 Corollary 5.15]) we have the expression

$$\mathbf{S}(\tau)\tilde{\mathbf{f}}(\rho) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\epsilon - iN}^{\epsilon + iN} e^{\lambda\tau} \mathbf{R}_{\mathbf{L}}(\lambda)\tilde{\mathbf{f}} d\lambda$$

for any $\epsilon > 0$, where $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2) \in \ker \mathbf{P}$ is sufficiently regular. From the definition of the resolvent, if $\mathbf{u} = \mathbf{R}_{\mathbf{L}}(\lambda)\tilde{\mathbf{f}}$, then $(\lambda - \mathbf{L})\mathbf{u} = \tilde{\mathbf{f}}$. Writing out this spectral ODE, we see that the first component of $\mathbf{u}$ satisfies

$$\begin{aligned}
-(1-\rho^2)u_1''(\rho) + &\left(-\frac{d-1}{\rho} + \rho(2\lambda + d)\right) u_1'(\rho) \\
&+ \left(\lambda + \frac{d}{2}\right)\left(\lambda + \frac{d-1}{2}\right) u_1(\rho) + V(\rho)u_1(\rho) = F_\lambda(\rho),
\end{aligned} \tag{1.21}$$

where $V(\rho) = -\frac{d(d+2)}{4}$ and $F_\lambda(\rho) = \left(\lambda + \frac{d}{2}\right)\tilde{f}_1(\rho) + \rho \tilde{f}_1'(\rho) + \tilde{f}_2(\rho)$. Denote by G the Green function of the ODE (1.21). Then we have the representation of the resolvent,

$$[\mathbf{R}_{\mathbf{L}}(\lambda)\tilde{\mathbf{f}}]_1(\rho) = \int_0^1 G(\rho, s; \lambda) F_\lambda(s) ds.$$

Using this, we arrive at the expression of the semigroup

$$[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1(\rho) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\epsilon - iN}^{\epsilon + iN} e^{\lambda\tau} \int_0^1 G(\rho, s; \lambda) F_\lambda(s) ds d\lambda. \tag{1.22}$$

The point is now to construct G explicitly. This is done perturbatively. We start with the fundamental system $\{\varphi_0, \varphi_1\}$ of the spectral ODE associated to $\mathbf{L}_0$, which is given by (1.21) with $V = 0$. This fundamental system is explicitly computed. The function φ_0 is regular at 0 but singular at 1, while the function φ_1 is regular at 1 but singular at 0. Then, from the variation of constants formula we turn the (homogeneous) ODE with potential V into a Volterra equation, which we solve by Volterra iterations.

Lemma 1.4.1 (Volterra iterations. See [45, Lemma 2.4])**.** *Consider the equation*

$$f(x) = g(x) + \int_c^x K(x, s) f(s) ds \tag{1.23}$$

on an interval (a, b)*, where* $g \in L^\infty(a, b)$ *and* $c \in [a, b]$*. If*

$$\mu := \int_c^b \sup_{x \in (c,s)} |K(x, s)| ds < \infty,$$

then (1.23) *has a unique solution satisfying the bound*

$$\|f\|_{L^\infty(a,b)} \leq e^\mu \|g\|_{L^\infty(a,b)}.$$

Using this, we are able to construct a solution u_0 near 0 which is a perturbation of φ_0, and similarly a solution u_1 near 1 which is a perturbation of φ_1. Then we extend them on the whole interval $(0, 1)$ to a fundamental system $\{u_0, u_1\}$ for (1.21). This construction is technical but rather robust. With this, we can finally write out the expression of the Green function G that has enough regularity. Moreover, this gives a decomposition $G(\rho, s; \lambda) = G_0(\rho, s; \lambda) + \tilde{G}(\rho, s; \lambda)$, where G_0 is the Green function of the *free equation*. See Section 2.3 for this construction.

With the decomposition of G, we decompose also the representation of the semigroup $\mathbf{S}(\tau) = \mathbf{S}_0(\tau) + \mathbf{T}(\tau)$. Here $\mathbf{S}_0(\tau)$ is the semigroup of the *free evolution*, and $\mathbf{T}(\tau)$ is perturbative and has extra decay in λ. Strichartz estimates for $\mathbf{S}_0(\tau)$ are obtained by a scaling argument from the standard Strichartz estimates. For the perturbative part $\mathbf{T}(\tau)$, we perform very delicate analysis on the oscillatory kernel $\tilde{G}$. As it turns out, the terms that have more decay in λ are also more singular in ρ. Roughly speaking, we encounter oscillatory integrals of the form

$$\int_{-\infty}^{\infty} e^{i\omega\phi(\tau,s)} \mathcal{O}\left(\langle\omega\rangle^{-m} \rho^{-n}\right) d\omega,$$

for some exponents m and n, where $\omega = \operatorname{Im}\lambda$. Hence there is a tradeoff between the decay as $|\operatorname{Im}\lambda| \to \infty$ (to ensure convergence of the ω integral) and the singularity as $\rho \to 0$ (to ensure the integrability of the resulting function in ρ). This is perhaps the most difficult and delicate part of this work. In fact, as dimension increases, we would obtain a fundamental system that has more decay for large $|\operatorname{Im}\lambda|$, but is more singular for ρ close to 0. Hence in the $d = 5$ case, finding the right balance is significantly more difficult than the $d = 3$ case. With this, we finally obtain the desired Strichartz estimates in Section 2.4. The energy bound is obtained in a similar way in Section 2.5.

These bounds on $\mathbf{S}(\tau)$ finally enable us to control the nonlinearity in the Duhamel term in Section 2.6. The idea is similar to that in Section 1.3.4.

Our main result is as follows.

Theorem 1.4.2 ([8, 10])**.** *Let $d \in \{3, 5\}$ and $p = \frac{d+2}{d-2}$. There exist $\delta, M > 0$ such that the following holds. For all radial functions f, g on $\mathbb{R}^d$ with*

$$\|(f, g) - u^1[0]\|_{H^1 \times L^2(\mathbb{B}^d_{1+\delta})} \leq \frac{\delta}{M},$$

there exists a $T \in [1-\delta, 1+\delta]$ such that the solution u to (1.1)$^+$ exists in the backwards lightcone $\Gamma_T := \{(t, x) \in [0, T) \times \mathbb{R}^d : |x| \leq T - t\}$ and satisfies

$$\int_0^T \frac{\|u(t, \cdot) - u^T(t, \cdot)\|^2_{L^q(\mathbb{B}^d_{T-t})}}{\|u^T(t, \cdot)\|^2_{L^q(\mathbb{B}^d_{T-t})}} \frac{dt}{T - t} \lesssim \delta^2,$$

where $q = \infty$ for $d = 3$ and $q = 5$ for $d = 5$. In particular, u blows up at the tip of the cone Γ_T.

Finally, we point out that this gives insights for other critical problems (for example wave maps in the optimal space of well-posedness), as well as problems in higher dimensions. We believe that, with the techniques on the oscillatory kernel used in the $d = 5$ case, we shall be able obtain the desired estimates for this perturbative part in higher dimensions too.

Chapter 2

Blowup Stability at Optimal Regularity for the Critical Wave Equation

This chapter contains the proof of the original result of this thesis. The content of this chapter was submitted as a paper for publication with my advisor Roland Donninger as a co-author. He introduced me to this problem and provided some guidance. Subsequently, I carried out the work on the problem independently.

2.1 Introduction

We consider the 5 dimensional energy-critical wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = |u(t,x)|^{\frac{4}{3}}u(t,x) \\ u[0] = (f,g) \end{cases} \tag{2.1}$$

for $u : I \times \mathbb{R}^5 \to \mathbb{R}, I \subset \mathbb{R}, 0 \in I$, and $u[t] := (u(t,\cdot), \partial_t u(t,\cdot))$.

For studying well-posedness of this equation in low regularity, the suitable solution concept is given by *Duhamel's formula*

$$u(t,\cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\Big(|u(s,\cdot)|^{\frac{4}{3}}u(s,\cdot)\Big)\,ds, \tag{2.2}$$

where $\cos(t|\nabla|)$ and $\frac{\sin(t|\nabla|)}{|\nabla|}$ are the standard wave propagators. We say that a function u is a solution to (2.1) if it satisfies the integral equation (2.2).

It is known that this equation is locally well-posed in $\dot{H}^1 \times L^2(\mathbb{R}^5)$ and this is the minimal regularity required, see [33] and [46]. The proof relies on the celebrated *Strichartz estimates*. These estimates capture the dispersive nature of the equation by describing a space-time improvement of the solution. This gives enough control of the nonlinearity.

One of the most important features of (2.1) is the possibility of singularity formation (or *blowup*) in finite time from smooth initial data. This is evidenced by the explicit solution

$$u^T(t,x) := c_5(T-t)^{-\frac{3}{2}}, \quad c_5 = \left(\frac{15}{4}\right)^{\frac{3}{4}},$$

where $T > 0$ is a free parameter. The solution u^T is homogeneous in space but can be truncated by finite speed of propagation. This leads to an explicit example of singularity formation from smooth compactly supported data.

In order to see whether or not this blowup is generic, we need to study its stability. Ideally one would like to study the stability at optimal regularity, which requires certain Strichartz estimates. More precisely, as shown in [8], this involves establishing Strichartz estimates for wave equations with self-similar potentials, since perturbing around this solution produces an equation of the form

$$\left(\partial_t^2 - \Delta_x + (T-t)^{-2} V\left(\frac{x}{T-t}\right)\right) u(t,x) = \text{nonlinear terms}.$$

Our goal here is to establish such estimates in 5 dimensions, which will also give insight into other critical models such as the wave maps equation. Compared to the 3 dimensional case treated in [8], the 5 dimensional case gives rise to a number of new substantial difficulties.

To be more precise, we study the stability of the blowup solution u^T locally in the backwards lightcone in the energy space. Since $f \in \dot{H}^1(\mathbb{R}^5)$ implies $f \in H^1_{loc}(\mathbb{R}^5)$, it is natural to consider perturbations in the space $H^1 \times L^2$. We restrict ourselves to radial data, in which case (2.1) effectively reduces to

$$\begin{cases} (\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r) u(t,r) = |u(t,r)|^{\frac{4}{3}} u(t,r) \\ u[0] = (f,g). \end{cases} \tag{2.3}$$

Our notion of solutions in the lightcone is given by the evolution semigroup in similarity coordinates, see Section 2.2.1. Our main result is the following.

Theorem 2.1.1. *There exist $\delta, M > 0$ such that the following holds. For all radial functions f, g on $\mathbb{R}^5$ with*

$$\|(f,g) - u^1[0]\|_{H^1 \times L^2(\mathbb{B}^5_{1+\delta})} \leq \frac{\delta}{M},$$

there exists a $T \in [1-\delta, 1+\delta]$ such that the solution u to (2.1) exists in the backwards lightcone $\Gamma_T := \{(t,x) \in [0,T) \times \mathbb{R}^5 : |x| \leq T - t\}$ and satisfies

$$\int_0^T \frac{\|u(t,\cdot) - u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^5_{T-t})}}{\|u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^5_{T-t})}} \frac{dt}{T-t} \lesssim \delta^2.$$

In particular, u blows up at the tip of the cone Γ_T.

2.1.1 Remarks

- Naively, one might think that stability of the ODE blowup solution u^1 means that small initial perturbations of u^1 lead to a solution that converges back to u^1. This is, however, wrong in general as the perturbation might change the blowup time. Consequently, Theorem 2.1.1 says that if we slightly perturb the initial data for the ODE blowup solution u^1, the corresponding solution u approaches u^T as $t \to T-$ for some T close to 1. In particular, the blowup time depends continuously on the perturbation.

- Theorem 2.1.1 shows that there exists an *open* set in the energy space of initial data that lead to the ODE blowup. In terms of regularity, this result is optimal.

- The convergence of the solution u to u^T is normalized to the blowup behaviour of u^T and measured in a Strichartz norm. Observe that $\int_0^T \frac{dt}{T-t} = \infty$ and thus, the integrand

$$\frac{\|u(t,\cdot) - u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^5_{T-t})}}{\|u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^5_{T-t})}}$$

must go to zero in an averaged sense as $t \to T-$.

2.1.2 Related results

Singularity formation in critical wave equations has attracted a lot of interest in recent years. There are different types of blowup. In type I blowup, the energy norm of the solution becomes unbounded in finite time (e.g. the aforementioned ODE blowup). On the other hand, type II blowup solutions have bounded energy norm. Blowup solutions of type II were constructed in [31], [30], [23], and [24]. Substantial progress has been made in the study of type II blowup. In [17, 19, 18, 20], type II blowup profiles were classified completely. It is worth mentioning that all type II blowup solutions are unstable. However, in recent breakthrough work, a family of type II blowup solutions that is "as stable as it can possibly be" has been identified, see [29] and [2].

Less is known for the type I case. For subcritical equations, there is the remarkable series of papers [34, 35, 36, 37, 40, 38, 39]. Unfortunately, the methods developed there do not extend to the critical case. A numerical study in [1] suggests that type I blowup is generic. The stability of the ODE blowup in the lightcone was established in a stronger topology in [13, 14, 15]. Subsequently, [8] was able to prove this stability in the energy norm by establishing suitable Strichartz estimates in 3 dimensions.

Finally, our results may also be interesting from the point of view of Strichartz estimates for wave equations with variable coefficients. This is a very active area of current research, see e.g. [5, 41].

2.1.3 Outline

We follow the approach from [8], but we will point out some major differences in this work. As is well known, the 3 dimensional radial wave equation is equivalent to the 1 dimensional wave equation with a Dirichlet condition at the origin. This special structure was heavily exploited in [8]. As a consequence, the present paper is far from being a straightforward adaptation of the methods from [8].

Following the machinery from [12, 6, 16], we first introduce similarity coordinates in the lightcone Γ_T by

$$\tau := \log \frac{T}{T-t}, \qquad \rho := \frac{r}{T-t}. \tag{2.4}$$

Notice that this coordinate transformation depends on the parameter T, but we drop this dependence in the notation for simplicity. Under the transformation $(t,r) \mapsto (\tau,\rho)$, Γ_T

is mapped to the infinite cylinder $\mathbb{R}_+ \times \overline{\mathbb{B}^5}$. We rewrite (2.3) as an evolution problem of the form

$$\partial_\tau \Psi(\tau) = \tilde{\mathbf{L}}_0 \Psi(\tau) + \mathbf{F}(\Psi(\tau)),$$

where $\tilde{\mathbf{L}}_0$ is a spatial differential operator and $\mathbf{F}$ is the nonlinearity. The ODE blowup u^T in this formulation is given by the constant solution $(c_5, \frac{2}{3}c_5)$. Perturbing around this solution by inserting the ansatz $\Psi(\tau) = (c_5, \frac{2}{3}c_5) + \Phi(\tau)$ yields

$$\partial_\tau \Phi(\tau) = (\tilde{\mathbf{L}}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)).$$

Here we obtain a potential term $\mathbf{L}'$ from the linearisation of $\mathbf{F}$, and $\mathbf{N}$ is the remaining nonlinearity. We show that the closure of $\tilde{\mathbf{L}}_0$, denoted by $\mathbf{L}_0$, generates a semigroup $\mathbf{S}_0(\tau)$ on $\mathcal{H} := H^1 \times L^2(\mathbb{B}^5)$. For the first component of this semigroup, we have the Strichartz estimates

$$\|[\mathbf{S}_0(\cdot)\mathbf{f}]_1\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|\mathbf{f}\|_{\mathcal{H}}, \quad \frac{1}{p} + \frac{5}{q} = \frac{3}{2}$$

for $p \in [2, \infty]$ and $q \in [\frac{10}{3}, 5]$. This follows from the standard Strichartz estimates for the wave operator via a scaling argument.

From the bounded perturbation theorem, we also have that $\mathbf{L}_0 + \mathbf{L}'$ generates a semigroup $\mathbf{S}(\tau)$ on $\mathcal{H}$. The operator $\mathbf{L}_0 + \mathbf{L}'$ has one unstable eigenvalue $\lambda = 1$ coming from the time translation symmetry of the equation, and we use the Riesz projection $\mathbf{P}$ to remove this unstable subspace. An application of the Gearhart-Prüss theorem yields the bound $\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\|_{\mathcal{B}(\mathcal{H})} \leq C_\epsilon e^{\epsilon\tau}$ for any (fixed) $\epsilon > 0$. In order to obtain Strichartz estimates for the semigroup $\mathbf{S}(\tau)$, we find an explicit representation from Laplace inversion. For sufficiently regular $\mathbf{f}$, let $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2) := (\mathbf{I} - \mathbf{P})\mathbf{f}$. Then we have

$$[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1(\rho) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau} \int_0^1 G(\rho, s; \lambda) F_\lambda(s) ds d\lambda,$$

where G is the Green function of the spectral ODE associated to $\mathbf{L}$ and $F_\lambda(s) := (\lambda + \frac{5}{2})\tilde{f}_1(s) + s\tilde{f}_1'(s) + \tilde{f}_2(s)$.

The Green function G is constructed by considering a perturbation of the free equation by a potential, and this is done by Volterra iterations. Then, we obtain a decomposition $G(\rho, s; \lambda) = G_0(\rho, s; \lambda) + \tilde{G}(\rho, s; \lambda)$, where G_0 is the Green function of the free equation. Hence the representation of $\mathbf{S}(\tau)$ also decomposes into a free part $\mathbf{S}_0(\tau)$ and a perturbation $\mathbf{T}(\tau)$ accordingly. The perturbative part has an extra decay as $|\operatorname{Im}\lambda| \to \infty$, which is ultimately what we exploit in establishing the Strichartz estimates. Due to the singular behaviour of the fundamental system, the construction of G here is considerably more difficult than in the 3 dimensional case.

Next, we turn to proving Strichartz estimates for $\mathbf{T}(\tau)$ by carefully examining the oscillatory kernels. An extra difficulty comes in, compared to the 3 dimensional case, from the more complicated form of the free fundamental solutions, one of which is given by

$$\varphi_1(\rho; \lambda) = \rho^{-3}(1+\rho)^{\frac{1}{2}-\lambda}(2 + \rho(-1 + 2\lambda)).$$

One part of this solution has growth of order $O(\langle\lambda\rangle)$, which destroys the nice extra decay of $\tilde{G}$. However, this part is also less singular in ρ near 0 (ρ^{-2} instead of ρ^{-3}). Balancing the undesired growth in λ and the singularity in ρ is a delicate procedure, which requires additional work. With this we prove Strichartz estimates for the full semigroup

$$\|[\mathbf{S}(\cdot)\mathbf{f}]_1\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|\mathbf{f}\|_{\mathcal{H}}, \quad \frac{1}{p}+\frac{5}{q}=\frac{3}{2}$$

for $p \in [2,\infty]$ and $q \in [\frac{10}{3},5]$. With the same techniques we also improve the growth bound from the Gearhart-Prüss theorem to $\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\|_{\mathcal{B}(\mathcal{H})} \lesssim 1$.

Using Strichartz estimates and the improved growth bound, we are able to control the nonlinearity as in [8] and finish the proof of Theorem 2.1.1.

2.1.4 Notation

We write $\mathbb{B}^5_R := \{x \in \mathbb{R}^5 : |x| < R\}$ and $\mathbb{B}^5 := \mathbb{B}^5_1$. A function $f \in C^1([0,1])$ with $f'(0) = 0$ gives rise to a radial function on $\overline{\mathbb{B}^5}$. Hence we define

$$\|f\|^p_{L^p(\mathbb{B}^5_R)} := \int_0^R |f(r)|^p r^4 dr,$$

and

$$\|f\|^2_{H^1(\mathbb{B}^5_R)} := \int_0^R |f'(r)|^2 r^4 dr + \int_0^R |f(r)|^2 r^4 dr.$$

Bold letters denote 2-component functions, for example, $\mathbf{f} = (f_1, f_2)$. We write $f(x) = O(g(x))$ if $|f(x)| \lesssim |g(x)|$, and $f \sim g$ as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. We also write $f \simeq g$ if $f \lesssim g \lesssim f$. Lastly, we use the Japanese bracket notation $\langle x \rangle := \sqrt{1+|x|^2}$.

2.2 Similarity coordinates and semigroup theory

2.2.1 Similarity coordinates

As mentioned before, similarity coordinates are defined by (2.4) on Γ_T. For a solution $u \in C^\infty(\Gamma_T)$ to (2.3), we define

$$\psi(\tau,\rho) = (Te^{-\tau})^{\frac{3}{2}} u(T - Te^{-\tau}, Te^{-\tau}\rho),$$

and we have $\psi \in C^\infty(\mathbb{R}_+ \times \overline{\mathbb{B}^5})$. Then, by setting

$$\begin{aligned} \psi_1(\tau,\rho) &:= \psi(\tau,\rho), \\ \psi_2(\tau,\rho) &:= \left(\partial_\tau + \rho\partial_\rho + \frac{3}{2}\right)\psi(\tau,\rho), \end{aligned} \tag{2.5}$$

we obtain the first order system from (2.3),

$$\left\{ \begin{array}{l} \partial_\tau\psi_1 = -\rho\partial_\rho\psi_1 - \frac{3}{2}\psi_1 + \psi_2 \\ \partial_\tau\psi_2 = \partial_\rho^2\psi_1 + \frac{4}{\rho}\partial_\rho\psi_1 - \rho\partial_\rho\psi_2 - \frac{5}{2}\psi_2 + |\psi_1|^{\frac{4}{3}}\psi_1 \\ \psi_1(0,\rho) = T^{\frac{3}{2}} f(T\rho), \quad \psi_2(0,\rho) = T^{\frac{5}{2}} g(T\rho). \end{array} \right. \tag{2.6}$$

The ODE blowup solution u^T corresponds to the constant solution $(\psi_1^T, \psi_2^T) = (c_5, \frac{3}{2}c_5)$. We take a perturbation around this solution by the ansatz

$$(\psi_1, \psi_2) = \left(c_5, \frac{3}{2}c_5\right) + (\varphi_1, \varphi_2). \tag{2.7}$$

This gives the system

$$\begin{cases} \partial_\tau \varphi_1 = -\rho\partial_\rho\varphi_1 - \frac{3}{2}\varphi_1 + \varphi_2 \\ \partial_\tau\varphi_2 = \partial_\rho^2\varphi_1 + \frac{4}{\rho}\partial_\rho\varphi_1 - \rho\partial_\rho\varphi_2 - \frac{5}{2}\varphi_2 + F(\varphi_1) \\ \varphi_1(0,\rho) = \psi_1(0,\rho) - c_5, \quad \varphi_2(0,\rho) = \psi_2(0,\rho) - \frac{3}{2}c_5 \end{cases} \tag{2.8}$$

where $F(x) = -c_5^{\frac{7}{3}} + |c_5 + x|^{\frac{4}{3}}(c_5 + x)$. We take the Taylor expansion of F around zero: $F(x) = F(0) + F'(0)x + N(x)$. We see that $F(0) = 0$, and

$$F'(0) = \frac{7}{3}|c_5 + x|^{\frac{4}{3}}\Big|_{x=0} = \frac{7}{3}\cdot\frac{15}{4} = \frac{35}{4}.$$

This linearisation produces a potential term. The nonlinearity is then given by

$$N(x) = F(x) - F(0) - F'(0)x = |c_5 + x|^{\frac{4}{3}}(c_5 + x) - c_5^{\frac{7}{3}} - \frac{35}{4}x.$$

Hence we define the formal differential operators

$$\tilde{\mathbf{L}}_0\mathbf{u}(\rho) := \begin{pmatrix} -\rho u_1'(\rho) - \frac{3}{2}u_1(\rho) + u_2(\rho) \\ u_1''(\rho) + \frac{4}{\rho}u_1'(\rho) - \rho u_2'(\rho) - \frac{5}{2}u_2(\rho) \end{pmatrix},$$

$$\mathbf{L}'\mathbf{u}(\rho) := \begin{pmatrix} 0 \\ \frac{35}{4}u_1(\rho) \end{pmatrix},$$

and

$$\mathbf{N}(\mathbf{u})(\rho) := \begin{pmatrix} 0 \\ N(u_1(\rho)) \end{pmatrix}.$$

Let $\Phi(\tau)(\rho) = (\varphi_1(\tau,\rho), \varphi_2(\tau,\rho))$. We can then write the system (2.8) as

$$\begin{cases} \partial_\tau\Phi(\tau) = (\tilde{\mathbf{L}}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)) \\ \Phi(0)(\rho) = (\varphi_1(0,\rho), \varphi_2(0,\rho)). \end{cases} \tag{2.9}$$

We study this evolution on the Banach space

$$\mathcal{H} := \{\mathbf{f} \in H^1 \times L^2(\mathbb{B}^5) : \mathbf{f} \text{ radial}\},$$

with the norm

$$\|\mathbf{f}\|_{\mathcal{H}}^2 := \|f_1\|_{H^1(\mathbb{B}^5)}^2 + \|f_2\|_{L^2(\mathbb{B}^5)}^2$$

for $\mathbf{f} = (f_1, f_2)$. In next parts of this section, we will see that the closure of $\tilde{\mathbf{L}}_0 + \mathbf{L}'$ with a suitable domain generates a semigroup $\mathbf{S}(\tau)$ on $\mathcal{H}$, which leads to the concept of energy-class strong lightcone solutions.

Definition 2.2.1. We say that $u : \Gamma_T \to \mathbb{C}$ is an *energy-class solution to* (2.3) *in the lightcone* if the corresponding $\Phi : [0,\infty) \to \mathcal{H}$ is in $C([0,\infty);\mathcal{H})$ and satisfies

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}(\tau - \sigma)\mathbf{N}(\Phi(\sigma))d\sigma$$

for all $\tau > 0$.

2.2.2 Semigroup of the free evolution

For the linear differential operator $\tilde{\mathbf{L}}_0$, we define its domain to be

$$\mathcal{D}(\tilde{\mathbf{L}}_0) := \{\mathbf{f} = (f_1, f_2) \in C^2 \times C^1([0,1]) : f_1'(0) = 0\}.$$

So now $\tilde{\mathbf{L}}_0$ becomes a densely defined (unbounded) linear operator on $\mathcal{H}$. We would like to show that the closure of $\tilde{\mathbf{L}}_0$ generates a semigroup. We start with introducing an equivalent inner product on the dense subspace $\mathcal{D}(\tilde{\mathbf{L}}_0)$ given by

$$(\mathbf{f}|\mathbf{g})_E := \int_0^1 f_1'(r)\overline{g_1'(r)}r^4 dr + \int_0^1 f_2(r)\overline{g_2(r)}r^4 dr + f_1(1)\overline{g_1(1)},$$

and we denote by $\|\cdot\|_E$ the norm defined by $(\cdot|\cdot)_E$.

Lemma 2.2.2. *The norms $\|\cdot\|_E$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms on $\mathcal{D}(\tilde{\mathbf{L}}_0)$. In other words, we have*

$$\|\mathbf{f}\|_E \simeq \|\mathbf{f}\|_{\mathcal{H}}$$

for all $\mathbf{f} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$.

In particular, taking the completions of $\mathcal{D}(\tilde{\mathbf{L}}_0)$ with respect to both the norms $\|\cdot\|_E$ and $\|\cdot\|_{\mathcal{H}}$, we conclude that they are also equivalent norms on $\mathcal{H}$.

Proof. Here we need to show

$$\int_0^1 |f'(r)|^2 r^4 dr + |f(1)|^2 \lesssim \|f\|^2_{H^1(\mathbb{B}^5)} \lesssim \int_0^1 |f'(r)|^2 r^4 dr + |f(1)|^2 \tag{2.10}$$

for all $f \in C^2([0,1])$. Since $\|f\|_E^2 = \|\operatorname{Re} f\|_E^2 + \|\operatorname{Im} f\|_E^2$, we only need to consider the case when f is real-valued.

By the Sobolev embedding $H^1(\frac{1}{2}, 1) \hookrightarrow L^\infty(\frac{1}{2}, 1)$, we have

$$|f(1)|^2 \lesssim \|f\|^2_{H^1(\frac{1}{2},1)} \lesssim \|f\|^2_{H^1(\mathbb{B}^5)},$$

which gives the first inequality of (2.10).

Next, we need to control the L^2 part of the H^1 norm of f. Since $r \in [0,1]$, we can write

$$\begin{aligned}
|f(r)|^2 r^4 &= \left| \int_0^r \frac{d}{ds}\left(f(s)s^2\right) ds \right|^2 \\
&\leq \int_0^1 \left| f'(s)s^2 + 2f(s)s \right|^2 ds \\
&= \int_0^1 |f'(s)|^2 s^4 ds + 4\int_0^1 |f(s)|^2 s^2 ds + 2\int_0^1 \left(\frac{d}{ds}|f(s)|^2\right) s^3 ds \\
&= \int_0^1 |f'(s)|^2 s^4 ds + 2\left(|f(s)|^2 s^3\right)\Big|_{s=0}^{s=1} - 6\int_0^1 |f(s)|^2 s^2 ds \\
&\lesssim \int_0^1 |f'(s)|^2 s^4 ds + |f(1)|^2.
\end{aligned}$$

The right hand side is independent of r, so we can integrate over r and obtain

$$\int_0^1 |f(r)|^2 r^4 dr \lesssim \int_0^1 |f'(s)|^2 s^4 ds + |f(1)|^2,$$

and we also have the second inequality of (2.10). $\square$

With this set up, we now turn to the operator $\tilde{\mathbf{L}}_0$.

Proposition 2.2.3. *The operator $\tilde{\mathbf{L}}_0 : \mathcal{D}(\tilde{\mathbf{L}}_0) \subset \mathcal{H} \to \mathcal{H}$ is closable and its closure $\mathbf{L}_0$ generates a strongly-continuous one-parameter semigroup $\{\mathbf{S}_0(\tau) : \tau \geq 0\}$ on $\mathcal{H}$, with the estimate*

$$\|\mathbf{S}_0(\tau)\mathbf{f}\|_{\mathcal{H}} \lesssim \|\mathbf{f}\|_{\mathcal{H}},$$

for all $\tau \geq 0$ and $\mathbf{f} \in \mathcal{H}$.

Proof. We use the Lumer-Philips theorem ([21, p.83 Theorem 3.15 and p.88 Proposition 3.23]) with the inner product $(\cdot|\cdot)_E$ and norm $\|\cdot\|_E$. Here we have to show two things:

- $\mathrm{Re}(\tilde{\mathbf{L}}_0\mathbf{u}|\mathbf{u})_E \leq 0$ for all $\mathbf{u} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$;
- range of $(\lambda - \tilde{\mathbf{L}}_0)$ is dense for some $\lambda > 0$.

For the first part, let $\mathbf{u} = (u_1, u_2) \in \mathcal{D}(\tilde{\mathbf{L}}_0)$. We compute

$$\begin{aligned}
\mathrm{Re}(\tilde{\mathbf{L}}_0\mathbf{u}|\mathbf{u})_E =\ & \mathrm{Re}\left(\int_0^1 \left(-\frac{5}{2}|u_1'(\rho)|^2 - \rho u_1''(\rho)\overline{u_1'(\rho)} + u_2'(\rho)\overline{u_1'(\rho)}\right)\rho^4 d\rho\right) \\
& + \mathrm{Re}\left(\int_0^1 \left(u_1''(\rho)\overline{u_2(\rho)} + \frac{4}{\rho}u_1'(\rho)\overline{u_2(\rho)} - \rho u_2'(\rho)\overline{u_2(\rho)} - \frac{5}{2}|u_2(\rho)|^2\right)\rho^4 d\rho\right) \\
& - \mathrm{Re}\left(u_1'(1)\overline{u_1(1)} - \frac{3}{2}|u_1(1)|^2 + u_2(1)\overline{u_1(1)}\right).
\end{aligned}$$

We first collect the terms containing u_1 only and use integration by parts,

$$\begin{aligned}
I_1 &= -\frac{5}{2}\int_0^1 |u_1'(\rho)|^2\rho^4 d\rho - \frac{1}{2}\int_0^1 \frac{d}{d\rho}\left(|u_1'(\rho)|^2\right)\rho^5 d\rho - \mathrm{Re}\left(u_1'(1)\overline{u_1(1)}\right) - \frac{3}{2}|u_1(1)|^2 \\
&= -\frac{1}{2}|u_1'(1)|^2 - \mathrm{Re}\left(u_1(1)\overline{u_1'(1)}\right) - \frac{3}{2}|u_1(1)|^2.
\end{aligned}$$

Similarly we collect the terms containing u_2 only,

$$I_2 = -\frac{1}{2}\int_0^1 \frac{d}{d\rho}\left(|u_2(\rho)|^2\right)\rho^5 d\rho - \frac{5}{2}\int_0^1 |u_2(\rho)|^2\rho^4 d\rho = -\frac{1}{2}|u_2(1)|^2.$$

And the rest of the terms are

$$\begin{aligned}
I_3 =& \mathrm{Re}\left(\int_0^1 \left(u_2'(\rho)\overline{u_1'(\rho)}\rho^4 + 4u_1'(\rho)\overline{u_2(\rho)}\rho^3\right)d\rho + \int_0^1 \frac{d}{d\rho}\left(u_1'(\rho)\right)\overline{u_2(\rho)}\rho^4 d\rho + u_2(1)\overline{u_1(1)}\right) \\
=& \mathrm{Re}\left(u_1'(1)\overline{u_2(1)}\right) + \mathrm{Re}\left(u_2(1)\overline{u_1(1)}\right).
\end{aligned}$$

Summing up all terms, we get

$$
\begin{aligned}
&\operatorname{Re}(\tilde{\mathbf{L}}_0\mathbf{u}|\mathbf{u})_E \\
= {} & I_1 + I_2 + I_3 \\
= {} & -\frac{1}{2}|u_1'(1)|^2 - \frac{3}{2}|u_1(1)|^2 - \frac{1}{2}|u_2(1)|^2 + \operatorname{Re}\left(u_1'(1)\overline{u_2(1)}\right) + \operatorname{Re}\left(u(1)\overline{(u_2(1) - u_1'(1))}\right) \\
\leq {} & -\frac{1}{2}|u_1'(1)|^2 - \frac{3}{2}|u_1(1)|^2 - \frac{1}{2}|u_2(1)|^2 + \operatorname{Re}\left(u_1'(1)\overline{u_2(1)}\right) + \frac{1}{2}|u_1(1)|^2 + \frac{1}{2}|u_2(1) - u_1'(1)|^2 \\
= {} & -|u_1(1)|^2 \leq 0.
\end{aligned}
$$

Next, in order to show that the range of $(\lambda - \tilde{\mathbf{L}}_0)$ is dense, we show that the system of ODEs

$$(\lambda - \tilde{\mathbf{L}}_0)\mathbf{u} = \mathbf{f}$$

admits a solution $\mathbf{u} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$ for every $\mathbf{f} \in C^\infty \times C^\infty([0,1])$, which is a dense subspace of $\mathcal{H}$. Writing $\mathbf{f} = (f_1, f_2) \in C^\infty \times C^\infty([0,1])$, we need to solve the system

$$
\begin{aligned}
\lambda u_1(\rho) + \rho u_1'(\rho) + \frac{3}{2}u_1(\rho) - u_2(\rho) &= f_1(\rho), \\
\lambda u_2(\rho) - u_1''(\rho) - \frac{4}{\rho}u_1'(\rho) + \rho u_2'(\rho) + \frac{5}{2}u_2(\rho) &= f_2(\rho).
\end{aligned}
$$

From the first equation we see that

$$u_2(\rho) = \left(\lambda + \frac{3}{2}\right) u_1(\rho) + \rho u_1'(\rho) - f_1(\rho). \tag{2.11}$$

Inserting this into the second one, we obtain a single ODE

$$\left(-1 + \rho^2\right) u_1''(\rho) + \left(-\frac{4}{\rho} + \rho\,(2\lambda + 5)\right) u_1'(\rho) + \left(\lambda + \frac{5}{2}\right)\left(\lambda + \frac{3}{2}\right) u_1(\rho) = F_\lambda(\rho), \tag{2.12}$$

where $F_\lambda(\rho) := \left(\lambda + \frac{5}{2}\right) f_1(\rho) + \rho f_1'(\rho) + f_2$. By definition we have $F_\lambda \in C^\infty([0,1])$. We choose $\lambda = 1$ and use the variation of constants method to solve (2.12). The homogeneous equation (i.e., $F_\lambda = 0$) admits a fundamental system

$$\phi_1(\rho) = \frac{2+\rho}{\rho^3(1+\rho)^{\frac{1}{2}}}, \qquad \phi_0(\rho) = \frac{1}{\rho^3}\left(\frac{2+\rho}{(1+\rho)^{\frac{1}{2}}} - \frac{2-\rho}{(1-\rho)^{\frac{1}{2}}}\right).$$

The Wronskian is given by

$$W(\rho) := W(\phi_0, \phi_1)(\rho) = \frac{3}{\rho^4(1-\rho^2)^{3/2}}.$$

A solution to the non-homogeneous problem is then given by

$$
\begin{aligned}
u_1(\rho) &= \phi_0(\rho)\int_\rho^1 \frac{\phi_1(s)}{W(s)}\frac{F_1(s)}{-1+s^2}ds + \phi_1(\rho)\int_0^\rho \frac{\phi_0(s)}{W(s)}\frac{F_1(s)}{-1+s^2}ds \\
&= -\frac{\phi_0(\rho)}{3}\int_\rho^1 s(2+s)(1-s)^{\frac{1}{2}}F_1(s)ds \\
&\qquad - \frac{\phi_1(\rho)}{3}\int_0^\rho s\left((2+s)(1-s)^{\frac{1}{2}} - (2-s)(1+s)^{\frac{1}{2}}\right)F_1(s)ds
\end{aligned}
$$

By definition, we have $u_1 \in C^2(0,1)$, hence we only need to check the behaviour at the boundary points $\rho = 0$ and $\rho = 1$. Using l'Hospital's rule we find that u_1 is $C^1([0,1])$ with $u_1'(0) = 0$. The second derivative of u_1 is given by

$$u_1''(\rho) = -\frac{\phi_0''(\rho)}{3}\int_\rho^1 s(2+s)(1-s)^{\frac{1}{2}} F_1(s)ds \\ - \frac{\phi_1''(\rho)}{3}\int_0^\rho s\left((2+s)(1-s)^{\frac{1}{2}} - (2-s)(1+s)^{\frac{1}{2}}\right) F_1(s)ds + \frac{F_1(\rho)}{(-1+\rho^2)}.$$

Again an application of l'Hospital's rule shows that at u_1'' is not singular at $\rho = 0$. At $\rho = 1$, observe that the second term is not singular. In the first term, we have $\phi_0''(\rho) \sim -\frac{3}{4}(1-\rho)^{-5/2}$ and $\int_0^\rho s\left((2+s)(1-s)^{1/2}\right) F_1(s)ds \sim \frac{2}{3}(1-\rho)^{3/2}F_1(1)$ as $\rho \to 1$. So

$$-\frac{\phi_0''(\rho)}{3}\int_\rho^1 s(2+s)(1-s)^{1/2}F_1(s)ds \sim \frac{1}{2}(1-\rho)^{-1}F_1(1) \text{ as } \rho \to 1.$$

This cancels out exactly with the singularity in the last term, since $F_1(\rho)(-1+\rho^2)^{-1} \sim -\frac{1}{2}F(1)(1-\rho)^{-1}$ as $\rho \to 1$. Thus u_1'' is indeed continuous at $\rho = 1$, and $u_1 \in C^2([0,1])$. From (2.11) we also see that $u_2 \in C^1([0,1])$. Hence we can conclude that the range of $(1 - \tilde{\mathbf{L}}_0)$ is dense.

Applying the Lumer-Philips theorem we see that $\tilde{\mathbf{L}}_0$ is closable, its closure $\mathbf{L}_0$ generates a C_0 semigroup $\{\mathbf{S_0}(\tau) : \tau \geq 0\}$ on $\mathcal{H}$ satisfying the bound

$$\|\mathbf{S_0}(\tau)\mathbf{f}\|_E \leq \|\mathbf{f}\|_E$$

for all $\tau \geq 0$ and $\mathbf{f} \in \mathcal{H}$. But since the norms $\|\cdot\|_E$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent, the conclusion follows. $\square$

This gives the growth bound $\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \lesssim 1$, and we also have that the spectrum $\sigma(\mathbf{L_0})$ satisfies

$$\sigma(\mathbf{L}_0) \subset \{z \in \mathbb{C} | \operatorname{Re} z \leq 0\}$$

by [21, p.55 Theorem 1.10].

2.2.3 The free Strichartz estimates

For a solution to (2.3) that blows up at $r = 0, t = T$, due to the finite speed of propagation, this blow up is influenced only by initial conditions supported in $\mathbb{B}^5_T$.

Solutions to the free wave equation with initial conditions supported in $\mathbb{B}^5_T$, when restricted to the backwards lightcone, correspond to the semigroup $\{\mathbf{S}_0(\tau) : \tau > 0\}$ in similarity coordinates. The standard Strichartz estimates imply Strichartz estimates in similarity coordinates by a scaling argument.

Proposition 2.2.4. *Let $p \in [2,\infty]$ and $q \in [\frac{10}{3}, 5]$ such that $\frac{1}{p} + \frac{5}{q} = \frac{3}{2}$. Then we have the bound*

$$\|[\mathbf{S}_0(\cdot)\mathbf{f}]_1\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|\mathbf{f}\|_{\mathcal{H}}$$

for all $\mathbf{f} \in \mathcal{H}$. And hence we also have

$$\left\|\int_0^\tau [\mathbf{S}_0(\tau-\sigma)\mathbf{h}(\sigma,\cdot)]_1 \, d\sigma\right\|_{L^p_\tau(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|\mathbf{h}\|_{L^1(\mathbb{R}_+,\mathcal{H})},$$

for all $\mathbf{h} \in C([0,\infty),\mathcal{H}) \cap L^1(\mathbb{R}_+,\mathcal{H})$.

Proof. Let $\mathbf{f} = (f_1, f_2) \in C^2 \times C^1([0,1])$ with $f'(0) = 0$. Then (f_1, f_2) can be extended to radial functions $(\tilde{f}_1, \tilde{f}_2) \in C^2 \times C^1(\mathbb{R}^5)$ by Sobolev extension, in such a way that

$$\|\tilde{f}_1\|_{\dot{H}^1(\mathbb{R}^5)} \lesssim \|f_1\|_{H^1(\mathbb{B}^5)}, \text{ and } \|\tilde{f}_2\|_{L^2(\mathbb{R}^5)} \lesssim \|f_2\|_{L^2(\mathbb{B}^5)}.$$

By the change of coordinates (2.4) with $T = 1$, the free evolution $\mathbf{S}_0(\cdot)\mathbf{f}$ is given by the classical solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$, restricted to the backwards lightcone Γ_1, of the free radial wave equation

$$\begin{cases} (\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r)u(t,r) = 0 \\ u[0] = (\tilde{f}_1, \tilde{f}_2). \end{cases} \tag{2.13}$$

In other words, we have

$$[\mathbf{S}_0(\tau)\mathbf{f}]_1(\rho) = e^{-\frac{3}{2}\tau} u(1 - e^{-\tau}, e^{-\tau}\rho)$$

for $(\tau, \rho) \in \mathbb{R}_+ \times [0,1]$ by (2.5). Now we have

$$\begin{aligned}
\| [\mathbf{S}_0(\tau)\mathbf{f}]_1 \|_{L^5(\mathbb{B}^5)} &= \left(\int_0^1 | [\mathbf{S}_0(\tau)\mathbf{f}]_1 (\rho)|^5 \rho^4 d\rho \right)^{\frac{1}{5}} \\
&= \left(\int_0^1 \left| e^{-\frac{3}{2}\tau} u(1 - e^{-\tau}, e^{-\tau}\rho) \right|^5 \rho^4 d\rho \right)^{\frac{1}{5}} \\
&= e^{-\frac{1}{2}\tau} \left(\int_0^{e^{-\tau}} \left| u(1 - e^{-\tau}, r) \right|^5 r^4 dr \right)^{\frac{1}{5}} \\
&\le e^{-\frac{1}{2}\tau} \|u(1 - e^{-\tau}, \cdot)\|_{L^5(\mathbb{R}^5)},
\end{aligned}$$

then

$$\begin{aligned}
\| [\mathbf{S}_0(\cdot)\mathbf{f}]_1 \|_{L^2(\mathbb{R}_+; L^5(\mathbb{B}^5))} &\le \left(\int_0^\infty \left(e^{-\frac{1}{2}\tau} \|u(1 - e^{-\tau}, \cdot)\|_{L^5(\mathbb{R}^5)} \right)^2 d\tau \right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \|u(t, \cdot)\|^2_{L^5(\mathbb{R}^5)} dt \right)^{\frac{1}{2}} \\
&\le \|u\|_{L^2(\mathbb{R}_+; L^5(\mathbb{R}^5))}.
\end{aligned}$$

By the Strichartz estimates for wave equations ([48, Theorem 2.6] with $d = 5, s = 1, q = 2, r = 5$), we have

$$\|u\|_{L^2(\mathbb{R}_+, L^5(\mathbb{R}^5))} \lesssim \|(\tilde{f}_1, \tilde{f}_2)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)},$$

hence we obtain

$$\| [\mathbf{S}_0(\cdot)\mathbf{f}]_1 \|_{L^2(\mathbb{R}_+; L^5(\mathbb{B}^5))} \lesssim \|(\tilde{f}_1, \tilde{f}_2)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)} \lesssim \|(f_1, f_2)\|_{H^1 \times L^2(\mathbb{B}^5)} = \|\mathbf{f}\|_{\mathcal{H}}.$$

By density this holds for all $\mathbf{f} \in \mathcal{H}$.

The other endpoint comes from the Sobolev embedding $H^1(\mathbb{B}^5) \hookrightarrow L^{\frac{10}{3}}(\mathbb{B}^5)$ and Proposition 2.2.3,

$$\| [\mathbf{S}_0(\tau)\mathbf{f}]_1 \|_{L^{\frac{10}{3}}(\mathbb{B}^5)} \lesssim \| [\mathbf{S}_0(\tau)\mathbf{f}]_1 \|_{H^1(\mathbb{B}^5))} \lesssim \|\mathbf{S}_0(\tau)\mathbf{f}\|_{\mathcal{H}} \lesssim \|\mathbf{f}\|_{\mathcal{H}}.$$

Hence $\| [\mathbf{S}_0(\cdot)\mathbf{f}]_1 \|_{L^\infty(\mathbb{R}_+;L^{\frac{10}{3}}(\mathbb{B}^5))} \lesssim \|\mathbf{f}\|_{\mathcal{H}}$.

For other pairs (p,q) we can interpolate, for all $q \in [\frac{10}{3}, 5]$,

$$\|[\mathbf{S}_0(\tau)\mathbf{f}]_1\|_{L^q(\mathbb{B}^5)} \le \|[\mathbf{S}_0(\tau)\mathbf{f}]_1\|_{L^{\frac{10}{3}}(\mathbb{B}^5)}^{\frac{10}{q}-2} \|[\mathbf{S}_0(\tau)\mathbf{f}]_1\|_{L^5(\mathbb{B}^5)}^{-\frac{10}{q}+3}.$$

Then using Hölder's inequality we have

$$\|[\mathbf{S}_0(\cdot)\mathbf{f}]_1\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \le \|[\mathbf{S}_0(\cdot)\mathbf{f}]_1\|_{L^\infty(\mathbb{R}_+;L^{\frac{10}{3}}(\mathbb{B}^5))}^{\frac{10}{q}-2} \|[\mathbf{S}_0(\cdot)\mathbf{f}]_1\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))}^{-\frac{10}{q}+3} \lesssim \|\mathbf{f}\|_{\mathcal{H}},$$

provided $(-\frac{10}{q}+3)p = 2$, which gives $\frac{1}{p} + \frac{5}{q} = \frac{3}{2}$.

The inhomogeneous estimate is given by Minkowski's inequality as in [8]. $\square$

2.2.4 The linearisation

The operator $\mathbf{L}' : \mathcal{H} \to \mathcal{H}$ is bounded and also compact. We define $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}'$ with $\mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$. Then by the bounded perturbation theorem, we have that $\mathbf{L}$ generates a semigroup $\mathbf{S}(\tau)$.

Lemma 2.2.5. *The spectrum of* $\mathbf{L}$ *satisfies*

$$\sigma(\mathbf{L}) \subset \{z \in \mathbb{C} | \operatorname{Re} z \le 0\} \cup \{1\}.$$

Proof. Let $\lambda \in \sigma(\mathbf{L})$. If $\operatorname{Re}\lambda \le 0$, then the assertion is satisfied. Thus we consider the case when $\operatorname{Re}\lambda > 0$. In this case, $\lambda \in \sigma(\mathbf{L})\backslash\sigma(\mathbf{L}_0)$, so the resolvent $\mathbf{R}_{\mathbf{L}_0}(\lambda)$ exists. Then from $(\lambda - \mathbf{L}) = [1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda - \mathbf{L}_0)$, we see that $1 \in \sigma(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda))$. But since $\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$ is compact, we have $1 \in \sigma_p(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda))$, so there exists $\mathbf{f} \in \mathcal{H}$ with $\mathbf{f} = \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$. Defining $\mathbf{u} := \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$, we have $(\lambda - \mathbf{L})\mathbf{u} = 0$. Hence $\lambda \in \sigma_p(\mathbf{L})$, and there exists a nonzero $\mathbf{u} \in \mathcal{D}(\mathbf{L})$ such that $(\lambda - \mathbf{L})\mathbf{u} = 0$. Following a similar computation as in Proposition 2.2.3, the spectral equation implies

$$-(1-\rho^2)u''(\rho) + \left(-\frac{4}{\rho} + \rho(2\lambda+5)\right)u'(\rho) + \left(\lambda + \frac{5}{2}\right)\left(\lambda + \frac{3}{2}\right)u(\rho) - \frac{35}{4}u(\rho) = 0,$$

where $u = u_1$. We define $h(\rho^2) = u(\rho)$ and set $z = \rho^2$, then this ODE is transformed into the hypergeometric differential equation

$$z(1-z)h''(z) + (c - (a+b+1)z)\,h'(z) - abh(z) = 0, \tag{2.14}$$

where $a = \frac{\lambda}{2} + \frac{5}{2}$, $b = \frac{\lambda}{2} - \frac{1}{2}$, $c = \frac{5}{2}$. In the case when $\operatorname{Re}(c-a-b) = \frac{1}{2} - \operatorname{Re}\lambda$ is not a non-positive integer, a fundamental system at $z = 1$ is given by

$$h_1(z;\lambda) = {}_2F_1(a,b;a+b+1-c;1-z),$$
$$\tilde{h}_1(z;\lambda) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-z),$$

see [42]. When $\frac{1}{2} - \operatorname{Re}\lambda = -n$ for $n \in \mathbb{N}_0$, one solution is still given by h_1, while the second solution is given by

$$\tilde{h}_1(z;\lambda) = ch_1(z;\lambda)\log(1-z) + (1-z)^{-n}g(z;\lambda)$$

where g is analytic around $z = 1$ with $g(1; \lambda) \neq 0$ and c is a constant which can be zero unless $n = 0$. Since $\operatorname{Re} \lambda > 0$, the requirement that $u_1 \in H^1(\mathbb{B}^5)$ excludes the solution $\tilde{h}_1$, so a solution h must be a multiple of h_1. A fundamental system at $z = 0$ is given by

$$\begin{aligned} h_0(z; \lambda) &= {}_2F_1(a, b; c; z), \\ \tilde{h}_0(z; \lambda) &= z^{1-c}{}_2F_1(a - c + 1, b - c + 1; 2 - c; z). \end{aligned}$$

From the connection formula, we have

$$h_1(z; \lambda) = \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)} h_0(z; \lambda) + \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} \tilde{h}_0(z; \lambda). \quad (2.15)$$

Since $\rho \mapsto \tilde{h}_0(\rho^2; \lambda)$ does not belong to $H^1(\mathbb{B}^5_{1/2})$, we need to have

$$\frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} = 0.$$

This happens only when

$$-a = -\frac{\lambda}{2} - \frac{5}{2} \in \mathbb{N}_0 \text{ or } -b = -\frac{\lambda}{2} + \frac{1}{2} \in \mathbb{N}_0,$$

and the only possibility for $\operatorname{Re} \lambda > 0$ is when $\lambda = 1$. □

Lemma 2.2.6. *For every $\epsilon > 0$, there exist constants $C_\epsilon, R_\epsilon > 0$, such that*

$$\|\mathbf{R}_{\mathbf{L}}(\lambda)\|_{\mathcal{H}} \leq C_\epsilon$$

for all λ with $\operatorname{Re} \lambda > \epsilon$ and $|\lambda| > R_\epsilon$.

Proof. Fix $\epsilon > 0$. From Proposition 2.2.3, we have

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}} \lesssim \frac{1}{\operatorname{Re} \lambda}$$

by [21, p.55 Theorem 1.10]. From $(\lambda - \mathbf{L}) = [1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda - \mathbf{L}_0)$, $\mathbf{R}_{\mathbf{L}}(\lambda)$ exists if and only if $[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}$ exists. Now let $\mathbf{u} = \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$, then $\mathbf{u}$ and $\mathbf{f}$ satisfy

$$\left(\lambda + \frac{3}{2}\right) u_1(\rho) = f_1(\rho) - \rho u_1'(\rho) + u_2(\rho).$$

Recall that

$$\mathbf{L}'\mathbf{u}(\rho) := \begin{pmatrix} 0 \\ \frac{35}{4} u_1(\rho) \end{pmatrix},$$

then

$$\begin{aligned} \left|\lambda + \frac{3}{2}\right| \|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}\|_{\mathcal{H}} = \left|\lambda + \frac{3}{2}\right| \|\mathbf{L}'\mathbf{u}\|_{\mathcal{H}} &\lesssim \left|\lambda + \frac{3}{2}\right| \|u_1\|_{L^2(\mathbb{B}^5)} \\ &\leq \|f_1\|_{L^2(\mathbb{B}^5)} + \|(\cdot)u_1'\|_{L^2(\mathbb{B}^5)} + \|u_2\|_{L^2(\mathbb{B}^5)} \\ &\lesssim \|\mathbf{f}\|_{\mathcal{H}} + \|\mathbf{u}\|_{\mathcal{H}} \\ &= \|\mathbf{f}\|_{\mathcal{H}} + \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}\|_{\mathcal{H}} \\ &\lesssim \|\mathbf{f}\|_{\mathcal{H}}. \end{aligned}$$

So $\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}} \lesssim \left|\lambda + \frac{3}{2}\right|^{-1}$, and for $|\lambda|$ large enough, we obtain $\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}} < 1$. Therefore the Neumann series for $[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}$ converges, and we have

$$\begin{aligned}
\|\mathbf{R}_{\mathbf{L}}(\lambda)\|_{\mathcal{H}} &= \|\mathbf{R}_{\mathbf{L}_0}(\lambda)[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}]^{-1}(\lambda)\|_{\mathcal{H}} \\
&\leq \|\mathbf{R}_{\mathbf{L}_0}\|_{\mathcal{H}} \sum_{k=0}^{\infty} \|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}}^k \\
&\leq C_\epsilon
\end{aligned}$$

as claimed. ☐

Lemma 2.2.7. *The geometric multiplicity of the eigenvalue* $1 \in \sigma_p(\mathbf{L})$ *is 1, and its geometric eigenspace is spanned by*

$$\mathbf{g}(\rho) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Proof. First we note that $\mathbf{g} \in \mathcal{D}(\mathbf{L})$. A straightforward computation shows that $(1 - \mathbf{L})\mathbf{g} = 0$. Now suppose that $\tilde{\mathbf{g}}$ is another eigenfunction with eigenvalue 1, then $[\tilde{\mathbf{g}}]_1$ also satisfies the transformed ODE (2.14) with $\lambda = 1$. In this case $[\tilde{\mathbf{g}}]_1(\rho) = c \cdot {}_2F_1(a, b, c; \rho^2)$ for $b = \frac{\lambda}{2} - \frac{1}{2} = 0$ and some constant $c \in \mathbb{C}\backslash\{0\}$. Therefore $[\tilde{\mathbf{g}}]_1(\rho)$ is a constant, and we must have $[\tilde{\mathbf{g}}]_1 = c'[\mathbf{g}]_1$ for some constant $c' \in \mathbb{C}\backslash\{0\}$. The second component is uniquely determined by the first component, hence we must have $\tilde{\mathbf{g}} = c'\mathbf{g}$. ☐

The next lemma shows that the algebraic multiplicity of the eigenvalue $\lambda = 1$ is also 1, so we can use the Riesz projection to remove the unstable subspace.

Lemma 2.2.8. *There exists a bounded projection* $\mathbf{P} : \mathcal{H} \to \langle \mathbf{g} \rangle$ *such that* $\mathbf{P}\mathbf{S}(\tau) = \mathbf{S}(\tau)\mathbf{P}$. *In particular, we have* $\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^\tau \mathbf{P}\mathbf{f}$ *for all* $\tau \geq 0$ *and* $\mathbf{f} \in \mathcal{H}$.

Proof. Since $\lambda = 1$ is an isolated point in the spectrum, we define the projection

$$\mathbf{P} = \frac{1}{2\pi i} \int_\gamma \mathbf{R}_{\mathbf{L}}(\lambda) d\lambda,$$

where γ is a curve in $\rho(\mathbf{L})$ enclosing the point 1, such that no other points of $\sigma(\mathbf{L})$ lie in it. Then $\mathbf{P}$ commutes with the operator $\mathbf{L}$ (this means $\mathbf{PL} \subset \mathbf{LP}$) and the semigroup $\mathbf{S}(\tau)$. We have a decomposition of the Hilbert space $\mathcal{H} = \ker \mathbf{P} \oplus \operatorname{rg} \mathbf{P}$. Then we have the parts of the operator $\mathbf{L}$, $\mathbf{L}_{\ker \mathbf{P}}$ and $\mathbf{L}_{\operatorname{rg} \mathbf{P}}$, as operators on the spaces $\ker \mathbf{P}$ and $\operatorname{rg} \mathbf{P}$ respectively, with spectra $\sigma(\mathbf{L}_{\ker \mathbf{P}}) = \sigma(\mathbf{L})\backslash\{1\}$ and $\sigma(\mathbf{L}_{\operatorname{rg} \mathbf{P}}) = \{1\}$. It is obvious that $\langle \mathbf{g} \rangle \subset \operatorname{rg} \mathbf{P}$, so we need to show the other inclusion.

First of all, notice that $\dim \operatorname{rg} \mathbf{P} < \infty$. Otherwise, 1 would be in the essential spectrum of $\mathbf{L}$, which is impossible since $\mathbf{L}'$ is compact and $1 \notin \sigma(\mathbf{L}_0)$([25, p.244 Theorem 5.35]). Hence there is a minimal $k \in \mathbb{N}$ such that $(1 - \mathbf{L}_{\operatorname{rg} \mathbf{P}})^k \mathbf{u} = 0$ for all $\mathbf{u} \in \operatorname{rg} \mathbf{P} \cap \mathcal{D}(\mathbf{L})$. If $k = 1$ then we are done. Now we argue by contradiction. If $k \geq 2$, then there exists $\mathbf{u} \in \mathcal{D}(\mathbf{L})$ such that $(1 - \mathbf{L})\mathbf{u} = \mathbf{g}$. This is the same as saying that there exists a solution to the inhomogeneous ODE

$$-(1 - \rho^2)u''(\rho) + \left(-\frac{4}{\rho} + 7\rho\right) u'(\rho) = 12,$$

where $u = u_1$. For the homogeneous problem, we already know that one solution is given by the constant solution $g_1(\rho) = 2$. Another linearly independent solution to the homogeneous problem is given by

$$\tilde{g}_1(\rho) = \frac{1 + 4\rho^2 - 8\rho^4}{\rho^3(1-\rho^2)^{\frac{1}{2}}}.$$

The Wronskian is

$$W(g_1, \tilde{g}_1)(\rho) = \frac{6}{\rho^4(1-\rho^2)^{\frac{3}{2}}}.$$

Solutions to the inhomogeneous problem are given by

$$\begin{aligned} u(\rho) &= C_1 + C_2\tilde{g}_1(\rho) + 24\int_0^\rho \frac{\tilde{g}_1(s)}{W(g_1,\tilde{g}_1)(s)}\frac{1}{1-s^2}ds + 24\tilde{g}_1(\rho)\int_\rho^1 \frac{1}{W(g_1,\tilde{g}_1)(s)}\frac{1}{1-s^2}ds \\ &= C_1 + C_2\tilde{g}_1(\rho) + 24\int_0^\rho s(1+4s^2-8s^4)ds + \frac{24(1+4\rho^2-8\rho^4)}{\rho^3(1-\rho^2)^{\frac{1}{2}}}\int_\rho^1 s^4(1-s^2)^{\frac{1}{2}}ds \end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. At $\rho = 1$, the requirement that $u \in H^1(\mathbb{B}^5)$ gives $C_2 = 0$ and by looking at the asymptotic behavior at $\rho = 0$, we see that we must have

$$\int_0^1 s^4(1-s^2)^{\frac{1}{2}}ds = 0.$$

However, this clearly fails because the integrand is positive for all $s \in (0,1)$. $\square$

Next, we apply the Gearhart–Prüss theorem to obtain a growth bound for the semigroup.

Lemma 2.2.9. *For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}} \leq C_\epsilon e^{\epsilon\tau}\|(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}$$

for all $\tau \geq 0$ and $\mathbf{f} \in \mathcal{H}$.

Proof. From Lemma 2.2.6 and the fact that $\sigma(\mathbf{L}_{\ker \mathbf{P}}) \subset \{z \in \mathbb{C} | \operatorname{Re} z \leq 0\}$, we have that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|\mathbf{R}_{\mathbf{L}}(\lambda)(\mathbf{I}-\mathbf{P})\|_{\mathcal{H}} \leq C_\epsilon$$

for all λ with $\operatorname{Re}\lambda > \epsilon$. Then from the Gearhart–Prüss theorem ([21, p.302 Theorem 1.11]) we obtain the stated growth bound. $\square$

2.2.5 Explicit representation of the semigroup

Let $\mathbf{f} \in C^2 \times C^1([0,1]) \cap \mathcal{D}(\mathbf{L})$ and $\tilde{\mathbf{f}} := (\mathbf{I}-\mathbf{P})\mathbf{f}$. Then $\tilde{\mathbf{f}} \in C^2 \times C^1([0,1]) \cap \mathcal{D}(\mathbf{L}_0)$. By Laplace inversion ([21, p.234 Corollary 5.15]), we have

$$\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f} = \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}\mathbf{R}_{\mathbf{L}}(\lambda)\tilde{\mathbf{f}}d\lambda$$

for any $\epsilon > 0$. Hence we would like to find an explicit expression for $\mathbf{R_L}(\lambda)$. Since $\mathbf{u} = \mathbf{R_L}(\lambda)\tilde{\mathbf{f}}$ means $(\lambda - \mathbf{L})\mathbf{u} = \tilde{\mathbf{f}}$, following the exact same computation as in Proposition 2.2.3 we get the ODE

$$-(1-\rho^2)u''(\rho) + \left(-\frac{4}{\rho} + \rho(2\lambda+5)\right)u'(\rho) + \left(\lambda + \frac{5}{2}\right)\left(\lambda + \frac{3}{2}\right)u(\rho) - \frac{35}{4}u(\rho) = F_\lambda, \tag{2.16}$$

where $u = u_1$ and

$$F_\lambda(\rho) = \left(\lambda + \frac{5}{2}\right)\tilde{f}_1(\rho) + \rho\tilde{f}_1'(\rho) + \tilde{f}_2(\rho).$$

Thus

$$[\mathbf{R_L}(\lambda)\tilde{\mathbf{f}}]_1(\rho) = \int_0^1 G(\rho, s; \lambda)F_\lambda(s)ds,$$

where G is the Green function of Equation (2.16). Then we have the expression

$$[\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f}]_1(\rho) = \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}\int_0^1 G(\rho, s; \lambda)F_\lambda(s)dsd\lambda \tag{2.17}$$

for any $\epsilon > 0$.

2.3 Constructing the Green function

In this section we construct the Green function for (2.16). Most of our construction here works for the equation

$$-(1-\rho^2)u''(\rho) + \left(-\frac{4}{\rho} + \rho(2\lambda+5)\right)u'(\rho) + \left(\lambda + \frac{5}{2}\right)\left(\lambda + \frac{3}{2}\right)u(\rho) + V(\rho)u(\rho) = F_\lambda \tag{2.18}$$

with any $V \in C^\infty([0,1])$.

First of all, we set

$$v(\rho) := \rho^2\left(1-\rho^2\right)^{\frac{1}{4}+\frac{\lambda}{2}}u(\rho).$$

Then (2.18) with $F_\lambda = 0$ is equivalent to

$$v''(\rho) + \frac{-2 + \frac{1}{4}\rho^2(11 + 4\lambda - 4\lambda^2)}{\rho^2\left(1-\rho^2\right)^2}v(\rho) = \frac{V(\rho)}{1-\rho^2}v(\rho), \tag{2.19}$$

but now (2.19) has no first order terms. Notice that if $v(\cdot;\lambda)$ is a solution to (2.19), then so is $v(\cdot; 1-\lambda)$.

2.3.1 Symbol calculus

Definition 2.3.1. For a function $f : I \subset \mathbb{R} \to \mathbb{C}$, $a \in \overline{I}$, and $\alpha \in \mathbb{R}$, we write $f(x) = \mathcal{O}((x-a)^\alpha)$ if

$$\left|\partial_x^j f(x)\right| \leq C_j|x-a|^{\alpha-j}$$

for all $x \in I$ and $j \in \mathbb{N}_0$. Similarly, for a function $f : \mathbb{R} \to \mathbb{C}$ and $\alpha \in \mathbb{R}$, we write $f(x) = \mathcal{O}(\langle x\rangle^\alpha)$ if

$$\left|\partial_x^j f(x)\right| \leq C_j\langle x\rangle^{\alpha-j}$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{N}_0$.

We say that these functions *are of symbol type*, or *behave like symbols*.

Functions of symbol type have some nice properties. For example, we can easily check that if $f(x) = \mathcal{O}(x^\alpha)$ and $g(x) = \mathcal{O}(x^\alpha)$, then $f(x) + g(x) = \mathcal{O}(x^\alpha)$. Some further properties of functions of symbol type are discussed in Appendix A.

The same notation is also used for functions with more than one variables. For example, for $f : I \times \mathbb{R} \to \mathbb{C}$, we write $f(x, y) = \mathcal{O}(x^\alpha \langle y \rangle^\beta)$, if

$$\left|\partial_x^j \partial_y^k f(x, y)\right| \leq C_{j,k} |x|^{\alpha - j} \langle y \rangle^{\beta - k}$$

for all $(x, y) \in U$ and $j, k \in \mathbb{N}_0$.

Remark 2.3.2. In the following, we work with functions of three variables ρ, s, and ω. We always have $\rho, s \in [0, 1]$ and $\omega \in \mathbb{R}$. We write $\mathcal{O}_o$ and $\mathcal{O}_e$ to indicate oddness and evenness with respect to the variable ω. For example, $f(\rho, \omega) = \mathcal{O}_o(\rho \langle \omega \rangle^{-1})$ means that $f(\rho, \cdot) : \mathbb{R} \to \mathbb{C}$ is an odd function for each ρ.

2.3.2 The homogeneous equation

On solving (2.19) with $V = 0$, we obtain a fundamental system

$$\begin{aligned}
\psi_1(\rho; \lambda) &= \frac{1}{\rho}(1 - \rho)^{\frac{1}{4} + \frac{\lambda}{2}}(1 + \rho)^{\frac{3}{4} - \frac{\lambda}{2}}(2 + \rho(-1 + 2\lambda)), \\
\tilde{\psi}_1(\rho; \lambda) &= \psi_1(\rho; 1 - \lambda) = \frac{1}{\rho}(1 + \rho)^{\frac{1}{4} + \frac{\lambda}{2}}(1 - \rho)^{\frac{3}{4} - \frac{\lambda}{2}}(2 + \rho(1 - 2\lambda)).
\end{aligned}$$

We define another solution,

$$\psi_0(\rho; \lambda) := \psi_1(\rho; \lambda) - \tilde{\psi}_1(\rho; \lambda).$$

Notice that ψ_0 is not singular at $\rho = 0$, since the singularities of ψ_1 and $\tilde{\psi}_1$ cancel out.

Since (2.19) has no first order terms, the Wronskian of any two solutions is independent of ρ. Some useful Wronskians between the above three solutions are

$$\begin{aligned}
W(\lambda) := W(\psi_1(\cdot; \lambda), \tilde{\psi}_1(\cdot; \lambda)) &= W(\psi_0(\cdot; \lambda), \psi_1(\cdot; \lambda)) = W(\psi_0(\cdot; \lambda), \tilde{\psi}_1(\cdot; \lambda)) \\
&= (3 - 2\lambda)(1 + 2\lambda)(-1 + 2\lambda).
\end{aligned}$$

We can define another solution as follows.

Lemma 2.3.3. *Let $\lambda = \epsilon + i\omega$. There exists $\delta_0 > 0$, such that*

$$\tilde{\psi}_0(\rho; \lambda) := \psi_0(\rho; \lambda) \int_\rho^{\delta_0 \langle \omega \rangle^{-1}} \psi_0(s; \lambda)^{-2} ds$$

is also a solution to (2.19) *with $V = 0$ for $\rho \in (0, \delta_0 \langle \omega \rangle^{-1}]$. Moreover, it satisfies*

$$\tilde{\psi}_0(\rho; \lambda) = -2\rho^{-1} W(\lambda)^{-1}[1 + \mathcal{O}(\rho \langle \lambda \rangle)]$$

for all $\rho \in (0, \delta_0 \langle \omega \rangle^{-1}]$, $\omega \in \mathbb{R}$, and $\epsilon \in [0, \frac{1}{4}]$. We also have

$$W(\psi_0(\cdot; \lambda), \tilde{\psi}_0(\cdot; \lambda)) = -1.$$

Proof. Taking the Taylor expansion we have $\psi_0(\rho;\lambda) = -\frac{1}{6}\rho^2 W(\lambda)[1+\mathcal{O}(\rho\langle\lambda\rangle)]$. Hence for sufficiently small $\delta_0 > 0$, we have $\psi_0(\rho;\lambda) \neq 0$ for $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$, and thus $\psi_0(\rho;\lambda)^{-1} = -6\rho^{-2}W(\lambda)^{-1}[1+\mathcal{O}(\rho\langle\lambda\rangle)]$. Hence $\tilde{\psi}_0$ is well-defined for $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. Differentiating $\tilde{\psi}_0$ with respect to ρ we can see that it is indeed a solution to (2.19) for $V = 0$.

Now compute

$$\begin{aligned}
\frac{\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds}{12\rho^{-3}W(\lambda)^{-2}} &= \frac{1}{12}\rho^3 W(\lambda)^2 \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds \\
&= \frac{1}{12}\rho^3 W(\lambda)^2 \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \left(-6s^{-2}W(\lambda)^{-1}\right)^2 [1+\mathcal{O}(s\langle\lambda\rangle)]ds \\
&= 3\rho^3 \int_\rho^{\delta_0\langle\omega\rangle^{-1}} s^{-4}[1+\mathcal{O}(s\langle\lambda\rangle)]ds \\
&= 3\rho^3 \left[-\frac{1}{3}s^{-3}\right]_{s=\rho}^{s=\delta_0\langle\omega\rangle^{-1}} + \mathcal{O}(\rho\langle\lambda\rangle) \\
&= 1 + \mathcal{O}(\rho\langle\lambda\rangle),
\end{aligned}$$

for $\lambda = \epsilon + i\omega$ with $\epsilon \in [0, \frac{1}{4}]$. The value of the Wronskian follows from a straightforward computation. □

2.3.3 The fundamental system

In the following, we use Volterra iterations to construct solutions for (2.19) (see [45, Lemma 2.4]).

Lemma 2.3.4. *Let $\lambda = \epsilon + i\omega$. There exists $\delta_0 > 0$ such that (2.19) has a solution of the form*

$$v_0(\rho;\lambda) = \psi_0(\rho;\lambda)\left[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)\right]$$

for all $\rho \in \left[0, \delta_0\langle\omega\rangle^{-1}\right]$, $\omega \in \mathbb{R}$, $\epsilon \in [0, \frac{1}{4}]$.

Proof. We use Volterra iterations as in [8]. Using the fundamental system $\{\psi_0, \psi_1\}$, the variation of constants formula suggests that a solution v_0 should satisfy the integral equation

$$\begin{aligned}
v_0(\rho;\lambda) = \psi_0(\rho;\lambda) &- \frac{\psi_0(\rho,\lambda)}{W(\lambda)} \int_0^\rho \psi_1(s;\lambda)\frac{V(s)}{1-s^2}v_0(s;\lambda)ds \\
&+ \frac{\psi_1(\rho,\lambda)}{W(\lambda)} \int_0^\rho \psi_0(s;\lambda)\frac{V(s)}{1-s^2}v_0(s;\lambda)ds.
\end{aligned}$$

For sufficiently small $\delta_0 > 0$, we have $\psi_0(\rho;\lambda) \neq 0$ for $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. So we can set $h_0 := \frac{v_0}{\psi_0}$, and we obtain the equation

$$h_0(\rho;\lambda) = 1 + \int_0^\rho K(\rho,s;\lambda)h_0(s;\lambda)ds, \tag{2.20}$$

where

$$K(\rho,s;\lambda)=\frac{1}{W(\lambda)}\left[\frac{\psi_1(\rho;\lambda)}{\psi_0(\rho;\lambda)}\psi_0(s;\lambda)^2-\psi_0(s;\lambda)\psi_1(s;\lambda)\right]\frac{V(s)}{1-s^2}.$$

As mentioned before, by Taylor expansion we have $\psi_0(\rho;\lambda)^{-1} = -6\rho^{-2}W(\lambda)^{-1}[1+\mathcal{O}(\rho\langle\lambda\rangle)]$. Since $s\leq\rho$, we have,

$$\left|\frac{1}{W(\lambda)}\frac{\psi_1(\rho;\lambda)}{\psi_0(\rho;\lambda)}\psi_0(s;\lambda)^2\right|\lesssim\psi_1(\rho;\lambda)\rho^{-2}s^4\lesssim\psi_1(\rho;\lambda)s^2,$$

and similarly

$$\left|\frac{1}{W(\lambda)}\psi_0(s;\lambda)\psi_1(s;\lambda)\right|\lesssim\psi_1(s;\lambda)s^2\lesssim\psi_1(s;\lambda)s^2.$$

Since $|\psi_1(\rho;\lambda)|\lesssim\rho^{-1}$ for $\rho\leq\delta_0\langle\omega\rangle^{-1}$, we conclude that

$$|K(\rho,s;\lambda)|\lesssim(\rho^{-1}+s^{-1})s^2\leq s \tag{2.21}$$

for all $0\leq s\leq\rho\leq\delta_0\langle\omega\rangle^{-1}$. Thus we have

$$\int_0^{\delta_0\langle\omega\rangle^{-1}}\sup_{\rho\in\left(0,\delta_0\langle\omega\rangle^{-1}\right)}|K(\rho,s;\lambda)|ds\lesssim\langle\omega\rangle^{-2}.$$

Hence (2.20) has a solution h_0 that satisfies

$$|h_0(\rho;\lambda)-1|\lesssim\int_0^\rho|K(\rho,s;\lambda)|ds\lesssim\rho^2.$$

Re-inserting this into the equation we find

$$h_0(\rho;\lambda)=1+O(\rho^2).$$

Since every function behaves like symbol under differentiation with respect to both ρ and ω, we indeed have $h_0(\rho;\lambda)=1+\mathcal{O}(\rho^2\langle\omega\rangle^0)$ (see for example [11, Appendix B]). □

Similar to before, we can define a second solution for small ρ.

Lemma 2.3.5. *Let $\lambda=\epsilon+i\omega$. There exists $\delta_0>0$ such that*

$$\tilde{v}_0(\rho;\lambda):=v_0(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}}v_0(s;\lambda)^{-2}ds$$

is also a solution to (2.19) *on $\rho\in(0,\delta_0\langle\omega\rangle^{-1}]$. Moreover, $\tilde{v}_0$ is of the form*

$$\tilde{v}_0(\rho;\lambda)=\tilde{\psi}_0(\rho;\lambda)\left[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)\right],$$

for all $\rho\in\left(0,\delta_0\langle\omega\rangle^{-1}\right]$, $\omega\in\mathbb{R}$, $\epsilon\in[0,\frac{1}{4}]$, where $\tilde{\psi}_0$ is from Lemma 2.3.3. We also have

$$W(v_0(\cdot;\lambda),\tilde{v}_0(\cdot;\lambda))=-1.$$

Proof. We choose $\delta_0 > 0$ so small that v_0, ψ_0, and $\tilde{\psi}_0$ have no zeros on $\rho \in [0, \delta_0 \langle\omega\rangle^{-1}]$. Then $\tilde{v}_0$ is well-defined on $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$. We can directly verify that $\tilde{v}_0$ indeed solves (2.19).

Now notice that at $\rho = \delta_0 \langle\omega\rangle^{-1}$, both $\tilde{v}_0$ and $\tilde{\psi}_0$ are 0, so the statement of the lemma holds. Then using Lemmas 2.3.3 and 2.3.4 we compute

$$\begin{aligned}
&\frac{\tilde{v}_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} - 1 \\
&= \frac{1}{\tilde{\psi}_0(\rho;\lambda)} \left(\tilde{v}_0(\rho;\lambda) - \tilde{\psi}_0(\rho;\lambda)\right) \\
&= \frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} \left(\left[1 + O(\rho^2 \langle\omega\rangle^0)\right] \int_\rho^{\delta_0\langle\omega\rangle^{-1}} v_0(s;\lambda)^{-2} ds - \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} ds \right) \\
&= \frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} \times \\
&\quad \left(\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} \mathcal{O}(s^2 \langle\omega\rangle^0) ds + \mathcal{O}(\rho^2 \langle\omega\rangle^0) \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} \left[1 + \mathcal{O}(s^2 \langle\omega\rangle^0)\right] ds \right).
\end{aligned}$$

For $\delta_0 > 0$ small enough, we also have $\tilde{\psi}_0(\rho;\lambda)^{-1} = -\frac{1}{2}\rho W(\lambda)[1 + \mathcal{O}(\rho\langle\omega\rangle)]$ for $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$, from Lemma 2.3.3. So using the asymptotics of ψ_0 and $\tilde{\psi}_0$, we find

$$\frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} = \mathcal{O}(\rho^2 \langle\omega\rangle^3)\mathcal{O}(\rho\langle\omega\rangle^3) = \mathcal{O}(\rho^3 \langle\omega\rangle^6).$$

Moreover,

$$\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} \mathcal{O}(s^2 \langle\omega\rangle^0) ds = \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \mathcal{O}(s^{-4} \langle\omega\rangle^{-6}) \mathcal{O}(s^2 \langle\omega\rangle^0) ds = \mathcal{O}(\rho^{-1} \langle\omega\rangle^{-6}),$$

and

$$\begin{aligned}
&\mathcal{O}(\rho^2 \langle\omega\rangle^0) \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} \left[1 + \mathcal{O}(s^2 \langle\omega\rangle^0)\right] ds \\
&= \mathcal{O}(\rho^2 \langle\omega\rangle^0) \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \mathcal{O}(s^{-4} \langle\omega\rangle^{-6}) + \mathcal{O}(s^{-2} \langle\omega\rangle^{-6}) ds = \mathcal{O}(\rho^{-1} \langle\omega\rangle^{-6}).
\end{aligned}$$

So we obtain

$$\frac{\tilde{v}_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} - 1 = \mathcal{O}(\rho^2 \langle\omega\rangle^0).$$

The value of the Wronskian follows from a straightforward computation. □

Next, we construct perturbed solutions from ψ_1 and $\tilde{\psi}_1$.

Lemma 2.3.6. *Let $\lambda = \epsilon + i\omega$. There exists $\delta_1 > 0$, such that* (2.19) *has solutions of the form*

$$\begin{aligned}
v_1(\rho;\lambda) &= \psi_1(\rho;\lambda) \left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)} a_1(\rho) + \mathcal{O}(\rho^0 (1-\rho) \langle\omega\rangle^{-2})\right], \\
\tilde{v}_1(\rho;\lambda) &= \tilde{\psi}_1(\rho;\lambda) \left[1 - \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)} a_1(\rho) + \mathcal{O}(\rho^0 (1-\rho) \langle\omega\rangle^{-2})\right],
\end{aligned}$$

or alternatively we can also write

$$\begin{aligned}
v_1(\rho;\lambda) &= \psi_1(\rho;\lambda)\left[1+\mathcal{O}_o(\langle\omega\rangle^{-1})a_1(\rho)+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})\right],\\
\tilde{v}_1(\rho;\lambda) &= \tilde{\psi}_1(\rho;\lambda)\left[1+\mathcal{O}_o(\langle\omega\rangle^{-1})a_1(\rho)+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})\right],
\end{aligned}$$

for all $\rho\in\left[\delta_1\langle\omega\rangle^{-1},1\right], \omega\in\mathbb{R}, \epsilon\in\left[0,\frac{1}{4}\right]$, *where*

$$a_1(\rho)=-\int_\rho^1 V(s)ds.$$

Proof. This also follows from Volterra iterations, but is technically a little more complicated.

Existence of solution

Motivated by the variation of constants formula, solutions to (2.19) should satisfy the integral equation

$$\begin{aligned}
v_1(\rho;\lambda)=\psi_1(\rho;\lambda)&+\frac{\psi_1(\rho,\lambda)}{W(\lambda)}\int_\rho^{\rho_1}\tilde{\psi}_1(s;\lambda)\frac{V(s)}{1-s^2}v_1(s;\lambda)ds\\
&-\frac{\tilde{\psi}_1(\rho,\lambda)}{W(\lambda)}\int_\rho^{\rho_1}\psi_1(s;\lambda)\frac{V(s)}{1-s^2}v_1(s;\lambda)ds
\end{aligned}$$

for some constant ρ_1. Since $\operatorname{Re}\lambda\geq 0$, $|\psi_1(\rho;\lambda)|>0$, so we can set $h_1:=\frac{v_1}{\psi_1}$ and obtain the Volterra equation

$$h_1(\rho;\lambda)=1+\int_\rho^{\rho_1}K(\rho,s;\lambda)h_1(s;\lambda)ds, \tag{2.22}$$

where

$$K(\rho,s;\lambda)=\frac{1}{W(\lambda)}\left(\psi_1(s;\lambda)\tilde{\psi}_1(s;\lambda)-\frac{\tilde{\psi}_1(\rho;\lambda)}{\psi_1(\rho;\lambda)}\psi_1(s;\lambda)^2\right)\frac{V(s)}{1-s^2}.$$

We first compute

$$\psi_1(s;\lambda)\tilde{\psi}_1(s;\lambda)=\frac{(1-s^2)\left(4-s^2(1-2\lambda)^2\right)}{s^2},$$

so

$$\begin{aligned}
|\psi_1(s;\lambda)\tilde{\psi}_1(s;\lambda)| &\lesssim \frac{1-s}{s^2}\left|4-s^2(1-2\lambda)^2\right|\\
&\lesssim \frac{1-s}{s^2}+(1-s)\langle\lambda\rangle^2\\
&\leq (1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\left(\frac{1}{s^2}+\langle\lambda\rangle^2\right),
\end{aligned}$$

for all $0\leq\rho\leq s\leq 1$ and $\operatorname{Re}(\lambda)\in[0,\frac{1}{4}]$. Moreover,

$$\frac{\tilde{\psi}_1(\rho;\lambda)}{\psi_1(\rho;\lambda)}\psi_1(s;\lambda)^2=\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}\frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}(2+s(-1+2\lambda))^2,$$

so

$$\left|\frac{\tilde{\psi}_1(\rho;\lambda)}{\psi_1(\rho;\lambda)}\psi_1(s;\lambda)^2\right| \lesssim \frac{(1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}}{s^2}\left|\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}\right| |2+s(-1+2\lambda)|^2$$
$$\lesssim (1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\left(\frac{1}{s^2}+\langle\lambda\rangle^2\right),$$

for all $0 \le \rho \le s \le 1$ and $\mathrm{Re}(\lambda) \in [0, \frac{1}{4}]$. Hence, when restricted to $\delta_1 \langle\omega\rangle^{-1} \le \rho \le s < 1$ for some small $\delta_1 > 0$, we obtain the bound for the kernel

$$|K(\rho, s;\lambda)| \lesssim \frac{1}{W(\lambda)}\frac{V(s)}{1-s^2}(1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\left(\frac{1}{s^2}+\langle\lambda\rangle^2\right)$$
$$\lesssim (1-\rho)^{\frac{1}{2}}(1-s)^{-\frac{1}{2}}\langle\omega\rangle^{-1}.$$

Now we can choose $\rho_1 = 1$ and have

$$\int_{\delta_1\langle\omega\rangle^{-1}}^1 \sup_{\rho\in\left(\delta_1\langle\omega\rangle^{-1},1\right)} |K(\rho, s;\lambda)|ds \lesssim \langle\omega\rangle^{-1},$$

thus the Volterra (2.22) has a solution h_1 that satisfies

$$|h_1(\rho;\lambda) - 1| \lesssim \int_\rho^1 |K(\rho, s;\lambda)|ds \lesssim (1-\rho)\langle\omega\rangle^{-1} \tag{2.23}$$

for all $\rho \in \left[\delta_1 \langle\omega\rangle^{-1}, 1\right]$. Re-inserting this into the equation we find

$$h_1(\rho;\lambda) = 1 + \int_\rho^1 K(\rho, s;\lambda)ds + O((1-\rho)^2\langle\omega\rangle^{-2}).$$

Form of the integral of the kernel
Now we look at the term

$$\int_\rho^1 K(\rho, s;\lambda)ds$$
$$= \frac{1}{W(\lambda)}\left(\int_\rho^1 \frac{V(s)}{s^2}\left(4-s^2(1-2\lambda)^2\right)ds - \left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}I(\rho;\lambda)\right)$$
$$=: \frac{1}{W(\lambda)}(A+B)$$

where

$$I(\rho;\lambda) = \int_\rho^1 \frac{V(s)}{s^2}\left(\frac{1-s}{1+s}\right)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda))^2 ds.$$

We first look at the term A,

$$\int_\rho^1 \frac{V(s)}{s^2}\left(4-s^2(1-2\lambda)^2\right)ds = 4\int_\rho^1 \frac{V(s)}{s^2} - \int_\rho^1 V(s)ds(1-2\lambda)^2.$$

Here we have

$$\left|\int_\rho^1 \frac{V(s)}{s^2}ds\right| \leq \|V\|_{L^\infty}\int_\rho^1 \frac{1}{s^2}ds = \|V\|_{L^\infty}\left(-1+\rho^{-1}\right) \lesssim \rho^{-1}(1-\rho) \lesssim (1-\rho)\langle\omega\rangle\,,$$

where the last inequality follows because $\rho \in \left[\delta_1 \langle\omega\rangle^{-1}, 1\right]$.

For the term B, we insert the identity

$$\left(\frac{1-s}{1+s}\right)^{-\frac{1}{2}+\lambda}\frac{(2+s(-1+2\lambda))}{s^2} = -s\,\partial_s\left(\frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}\right)$$

into $I(\rho;\lambda)$. Integrating by parts once, we get

$$\begin{aligned}
I(\rho;\lambda) = &-\int_\rho^1 V(s)(2s+s^2(-1+2\lambda))\partial_s\left(\frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}\right)ds \\
&= V(\rho)(2+\rho(-1+2\lambda))\frac{(1-\rho)^{\frac{1}{2}+\lambda}(1+\rho)^{\frac{3}{2}-\lambda}}{\rho} \\
&\quad + \int_\rho^1 \frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}\partial_s\left[V(s)(2s+s^2(-1+2\lambda))\right]ds.
\end{aligned}$$

So we have now

$$\begin{aligned}
&\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}I(\rho;\lambda) \\
&= (1-\rho^2)\left(\frac{2V(\rho)}{\rho}+V(\rho)(1-2\lambda)\right) \\
&\quad + \left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}\int_\rho^1 \frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}\partial_s\left[V(s)(2s+s^2(-1+2\lambda))\right]ds.
\end{aligned}$$

We estimate the boundary term

$$\left|(1-\rho^2)\left(\frac{2V(\rho)}{\rho}+V(\rho)(1-2\lambda)\right)\right| \lesssim \rho^{-1}(1-\rho)+(1-\rho)\langle\omega\rangle \lesssim (1-\rho)\langle\omega\rangle\,.$$

For remaining term, notice that for $\operatorname{Re}\lambda \in [0,\frac{4}{1}]$,

$$\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)}$$

is bounded, so we only need to look at the integral. When the derivative falls on V, the whole term is of $O((1-\rho)\langle\omega\rangle)$ which is already good, thus we only need to consider when the derivative falls on the rest of the term. We look at

$$\begin{aligned}
&\int_\rho^1 \frac{(1-s)^{\frac{1}{2}+\lambda}(1+s)^{\frac{3}{2}-\lambda}}{s^2}V(s)\partial_s\left[(2s+s^2(-1+2\lambda))\right]ds \\
&= \int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda}(1+s)^2\left(\frac{2V(s)}{s^2}+\frac{2V(s)}{s}(-1+2\lambda)\right)ds \\
&= 2\int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda}(1+s)^2\frac{V(s)}{s^2}ds + (-2+4\lambda)\int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda}(1+s)^2\frac{V(s)}{s}ds.
\end{aligned}$$

In the first term, similar to before, we have, for $\operatorname{Re}\lambda \in [0, \frac{1}{4}]$,

$$\begin{aligned}\left|\int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda} (1+s)^2 \frac{V(s)}{s^2} ds\right| \leq& \int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\operatorname{Re}\lambda} (1+s)^2 \left|\frac{V(s)}{s^2}\right| ds \\ \lesssim& \|V\|_{L^\infty} \int_\rho^1 \frac{1}{s^2} ds \\ \lesssim& \rho^{-1}(1-\rho) \lesssim (1-\rho)\langle\omega\rangle .\end{aligned}$$

For the next part we use

$$\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda} = -\frac{(1+s)^2}{3+2\lambda} \partial_s \left(\frac{1-s}{1+s}\right)^{\frac{3}{2}+\lambda},$$

and integrate by parts again. Then we have

$$\begin{aligned}&(-2+4\lambda) \int_\rho^1 \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}+\lambda} (1+s)^2 \frac{V(s)}{s} ds \\ =& \frac{1-2\lambda}{3+2\lambda} \int_\rho^1 \frac{V(s)(1+s)^4}{s} \partial_s \left(\frac{1-s}{1+s}\right)^{\frac{3}{2}+\lambda} ds \\ =& \frac{1-2\lambda}{3+2\lambda} \left(\frac{V(\rho)}{\rho}(1+\rho)^4 \left(\frac{1-\rho}{1+\rho}\right)^{\frac{3}{2}+\lambda} - \int_\rho^1 \partial_s \left[\frac{V(s)(1+s)^4}{s}\right] \left(\frac{1-s}{1+s}\right)^{\frac{3}{2}+\lambda}\right).\end{aligned}$$

Now

$$\left|\frac{1-2\lambda}{3+2\lambda} \frac{V(\rho)}{\rho}(1+\rho)^4 \left(\frac{1-\rho}{1+\rho}\right)^{\frac{3}{2}+\lambda}\right| \lesssim \rho^{-1}(1-\rho) \lesssim (1-\rho)\langle\omega\rangle .$$

And for the integral, as before, it is only better when derivative falls on V, so we only need to compute

$$\begin{aligned}\left|\int_\rho^1 \partial_s \left[\frac{(1+s)^4}{s}\right] V(s) \left(\frac{1-s}{1+s}\right)^{\frac{3}{2}+\lambda} ds\right| \leq& \int_\rho^1 \frac{|(3s-1)(1+s)^3|}{s^2} |V(s)| \left(\frac{1-s}{1+s}\right)^{\frac{3}{2}+\operatorname{Re}\lambda} ds \\ \lesssim& \int_\rho^1 \frac{1}{s^2} ds \lesssim \rho^{-1}(1-\rho) \lesssim (1-\rho)\langle\omega\rangle .\end{aligned}$$

Hence we obtain

$$\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda} \frac{2+\rho(1-2\lambda)}{2+\rho(-1+2\lambda)} I(\rho;\lambda) = O((1-\rho)\langle\omega\rangle).$$

Putting everything back together we see that

$$W(\lambda) \int_\rho^1 K(\rho, s; \lambda) ds = (1-2\lambda)^2 \int_\rho^1 V(s) ds + O((1-\rho)\langle\omega\rangle).$$

Finally, observe that we have

$$\frac{(1-2\lambda)^2}{W(\lambda)} = \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)} = \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})$$

as stated. The second solution is given by $\tilde{v}_1(\rho;\lambda) := v_1(\rho; 1-\lambda)$, so again we have

$$\frac{1-2(1-\lambda)}{(3-2(1-\lambda))(1+2(1-\lambda))} = \frac{-1+2\lambda}{(1+2\lambda)(3-2\lambda)} = \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2}).$$

Estimating the derivatives
The fact that h_1 is indeed of symbol type is shown in Appendix B. $\square$

Since δ_1 is arbitrary, we can now choose $\delta_1 \leq \delta_0$. Then we can use the fundamental system $\{v_0, \tilde{v}_0\}$ to construct v_1 and $\tilde{v}_1$ on the interval $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$. These happen to be perturbations of ψ_1 and $\tilde{\psi}_1$.

Lemma 2.3.7. *The solutions $v_1, \tilde{v}_1$ can be extended to $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$, and are of the form*

$$\begin{aligned} v_1(\rho;\lambda) &= \psi_1(\rho;\lambda)[1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0 \langle\omega\rangle^{-2})], \\ \tilde{v}_1(\rho;\lambda) &= \tilde{\psi}_1(\rho;\lambda)[1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0 \langle\omega\rangle^{-2})] \end{aligned}$$

for all $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$, $\omega \in \mathbb{R}$, $\epsilon \in [0, \frac{1}{4}]$.

Proof. On $\rho \in (0, \delta_0 \langle\omega\rangle^{-1}]$, there exist $a(\lambda)$ and $b(\lambda)$ such that

$$v_1(\rho;\lambda) = a(\lambda) v_0(\rho;\lambda) + b(\lambda)\tilde{v}_0(\rho;\lambda),$$

where

$$\begin{aligned} a(\lambda) &= \frac{W(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda))}{W(v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda))} = -W(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)) \\ b(\lambda) &= -\frac{W(v_1(\cdot;\lambda), v_0(\cdot;\lambda))}{W(v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda))} = W(v_1(\cdot;\lambda), v_0(\cdot;\lambda)). \end{aligned}$$

So far we only have valid expressions of v_1 and $\tilde{v}_1$ on $\rho \in [\delta_1 \langle\omega\rangle^{-1}, 1]$ from Lemma 2.3.6. Since we have chosen $\delta_1 \leq \delta_0$, these expressions are valid at the point $\rho = \delta_0 \langle\omega\rangle^{-1}$. In order to compute $W(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda))$ and $W(v_1(\cdot;\lambda), v_0(\cdot;\lambda))$, we evaluate at $\rho = \delta_0 \langle\omega\rangle^{-1}$. Using Lemmas 2.3.6 and 2.3.5, we have

$$\begin{aligned} &W(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)) \\ &= W(\psi_1(\cdot;\lambda), \tilde{\psi}_0(\cdot;\lambda)) \left[1 + \mathcal{O}(\langle\omega\rangle^{-2})\right] \left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)} a_1(\delta_0 \langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right] \\ &\quad + \tilde{\psi}_0(\delta_0 \langle\omega\rangle^{-1};\lambda)\psi_1(\delta_0 \langle\omega\rangle^{-1};\lambda)\mathcal{O}(\langle\omega\rangle^{-1}) \\ &= W(\psi_1(\cdot;\lambda), \tilde{\psi}_0(\cdot;\lambda)) \left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)} a_1(\delta_0 \langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right] + \mathcal{O}(\langle\omega\rangle^{-2}), \end{aligned}$$

where the second line follows from $\tilde{\psi}_0(\delta_0 \langle\omega\rangle^{-1};\lambda) = \mathcal{O}(\langle\omega\rangle^{-2})$ and $\psi_1(\delta_0 \langle\omega\rangle^{-1};\lambda) = \mathcal{O}(\langle\omega\rangle)$. The function a_1 is from Lemma 2.3.6. Since ψ_1 and $\tilde{\psi}_0$ are explicitly defined, we can compute that

$$
\begin{aligned}
W(\psi_1(\rho;\lambda),\tilde{\psi}_0(\rho;\lambda)) &= \psi_1(\rho;\lambda)\tilde{\psi}_0'(\rho;\lambda) - \psi_1'(\rho;\lambda)\tilde{\psi}_0(\rho;\lambda) \\
&= \psi_1(\rho;\lambda)\psi_0'(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds - \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1} \\
&\quad - \psi_1'(\rho;\lambda)\psi_0(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds \\
&= W(\psi_1(\rho;\lambda),\psi_0(\rho;\lambda))\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds - \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1} \\
&= -W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds - \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1},
\end{aligned}
$$

which is a constant in ρ by the structure of the ODE. Evaluating at $\rho = \delta_0\langle\omega\rangle^{-1}$, we see that

$$
W(\psi_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)) = \psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)^{-1} = \mathcal{O}(\langle\omega\rangle^0).
$$

But for now we do not evaluate it, instead we use that the expression valid for all $\rho \in (0,\delta_0\langle\omega\rangle^{-1}]$, not just at one point. So we obtain

$$
\begin{aligned}
a(\lambda) &= -W(v_1(\rho;\lambda),\tilde{v}_0(\rho;\lambda)) \\
&= \left(W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds + \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}\right) \\
&\quad \times \left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right]
\end{aligned}
$$

for $\rho \in (0,\delta_0\langle\omega\rangle^{-1}]$. On the other hand, evaluating at $\rho = \delta_0\langle\omega\rangle^{-1}$ we have

$$
\begin{aligned}
b(\lambda) &= W(v_1(\cdot;\lambda),v_0(\cdot;\lambda)) \\
&= W(\psi_1(\cdot;\lambda),\psi_0(\cdot;\lambda))\left[1+\mathcal{O}(\langle\omega\rangle^{-2})\right]\left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right] \\
&\quad + \psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)\psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)\mathcal{O}(\langle\omega\rangle^{-1}) \\
&= W(\psi_1(\cdot;\lambda),\psi_0(\cdot;\lambda))\left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right] \\
&= -W(\lambda)\left[1 + \frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right],
\end{aligned}
$$

where in the second line we used $\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda) = \mathcal{O}(\langle\omega\rangle)$, $\psi_1(\delta_0\langle\omega\rangle^{-1};\lambda) = \mathcal{O}(\langle\omega\rangle)$, and $W(\lambda) = \mathcal{O}(\langle\omega\rangle^3)$. Hence

$$\begin{aligned}
v_1(\rho;\lambda) &= v_0(\rho;\lambda)\left(W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}}\psi_0(s;\lambda)^{-2}ds + \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}\right)\\
&\quad\times\left[1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&\quad - \tilde{v}_0(\rho;\lambda)W(\lambda)\left[1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&= W(\lambda)\tilde{\psi}_0(\rho;\lambda)[1+\mathcal{O}(\rho^0\langle\omega\rangle^{-2})]\\
&\quad + \psi_1(\rho;\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]\left[1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&\quad - W(\lambda)\tilde{\psi}_0(\rho;\lambda)[1+\mathcal{O}(\rho^0\langle\omega\rangle^{-2})]\\
&= W(\lambda)\tilde{\psi}_0(\rho;\lambda)\left[\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right]\\
&\quad + \psi_1(\rho;\lambda)\left[1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right].
\end{aligned}$$

Then we use $\tilde{\psi}_0(\rho;\lambda) = \mathcal{O}(\rho^{-1}\langle\omega\rangle^{-3})$ and $\psi_1(\rho;\lambda)^{-1} = \mathcal{O}(\rho\langle\omega\rangle^0)$ for small ρ, and see that

$$\begin{aligned}
\frac{v_1(\rho;\lambda)}{\psi_1(\rho;\lambda)} &= \frac{\tilde{\psi}_0(\rho;\lambda)}{\psi_1(\rho;\lambda)}\mathcal{O}(\langle\omega\rangle^3)\left[\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right]\\
&\quad + \left[1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right]\\
&= 1+\frac{1-2\lambda}{(3-2\lambda)(1+2\lambda)}a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0\langle\omega\rangle^{-2})\\
&= 1+\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0\langle\omega\rangle^{-2})
\end{aligned}$$

Here we have used that $a_1(\delta_0\langle\omega\rangle^{-1}) = \mathcal{O}(\langle\omega\rangle^0)$ is an even function in ω. The second solution is again given by $\tilde{v}_1(\rho;\lambda) = v_1(\rho;1-\lambda)$ as in the last part of the proof of Lemma 2.3.6. □

In the next step, we express v_0 in terms of the fundamental system $\{v_0, \tilde{v}_0\}$ on the interval $\rho \in [\delta_1\langle\omega\rangle^{-1}, 1]$.

Lemma 2.3.8. *The solution v_0 has the representation*

$$v_0(\rho;\lambda) = [1+\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})]v_1(\rho;\lambda) - [1+\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})]\tilde{v}_1(\rho;\lambda)$$

for all $\rho \in [\delta_1\langle\omega\rangle^{-1}, 1]$, $\omega \in \mathbb{R}$, $\epsilon \in [0, \frac{1}{4}]$.

Proof. Since the Wronskian is constant, we can evaluate at $\rho = 1$ and get

$$\begin{aligned}
&W(v_1(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda))\\
&= W(\psi_1(\cdot;\lambda), \tilde{\psi}_1(\cdot;\lambda))\left[1+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-1})\right] + \psi_1(1;\lambda)\tilde{\psi}_1(1;\lambda)\mathcal{O}(\rho^{-1}(1-\rho)^0\langle\omega\rangle^{-1})\\
&= W(\psi_1(\cdot;\lambda), \tilde{\psi}_1(\cdot;\lambda)) = W(\lambda).
\end{aligned}$$

So there exist $a(\lambda)$ and $b(\lambda)$ such that

$$v_0(\rho;\lambda) = a(\lambda)v_1(\rho;\lambda) + b(\lambda)\tilde{v}_1(\rho;\lambda),$$

given by

$$\begin{aligned} a(\lambda) &= \frac{W(v_0(\cdot;\lambda),\tilde{v}_1(\cdot;\lambda))}{W(v_1(\cdot;\lambda),\tilde{v}_1(\cdot;\lambda))} = \frac{W(v_0(\cdot;\lambda),\tilde{v}_1(\cdot;\lambda))}{W(\lambda)}, \\ b(\lambda) &= -\frac{W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))}{W(v_1(\cdot;\lambda),\tilde{v}_1(\cdot;\lambda))} = -\frac{W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))}{W(\lambda)}. \end{aligned}$$

By using Lemma 2.3.6 and evaluating at $\rho = \delta_0 \langle\omega\rangle^{-1}$, we have

$$\begin{aligned} W(v_0(\cdot;\lambda),\tilde{v}_1(\cdot;\lambda)) &= W(\psi_0(\cdot;\lambda),\tilde{\psi}_1(\cdot;\lambda))\left[1 + \mathcal{O}_o(\langle\omega\rangle^{-1})a_1(\delta_0\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})\right] \\ &\quad + \tilde{\psi}_1(\delta_0\langle\omega\rangle^{-1};\lambda)\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)\mathcal{O}(\langle\omega\rangle^{-1}) \\ &= W(\lambda)[1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})], \end{aligned}$$

hence

$$a(\lambda) = [1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})].$$

Similarly,

$$W(v_0(\cdot;\lambda),v_1(\cdot;\lambda)) = W(\lambda)[1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})],$$

so

$$b(\lambda) = -[1 + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})].$$

□

2.3.4 The Wronskian

The transformation

$$u_j(\rho) = \rho^{-2}(1-\rho^2)^{-\frac{1}{4}-\frac{\lambda}{2}}v_j(\rho)$$

now gives solutions $u_j, j = 0,1$ to (2.18) with $F_\lambda = 0$. Near $\rho = 0$ we have

$$\lim_{\rho\to 0^+} u_0(\rho;\lambda) = \lim_{\rho\to 0^+}\frac{v_0(\rho;\lambda)}{\rho^2} = \lim_{\rho\to 0^+}\frac{\psi_0(\rho;\lambda)}{\rho^2} = -\frac{1}{6}W(\lambda) = O(\langle\lambda\rangle^3),$$

and we see that $u_0(\cdot;\lambda) \in C([0,1)) \cap C^\infty(0,1)$. Taking its derivative with respect to ρ, we have, for $\rho \in [0, \delta_0\langle\omega\rangle^{-1}], \lambda = \epsilon + i\omega$,

$$|u_0'(\rho;\lambda)| \lesssim \rho^{-1}\langle\omega\rangle^3.$$

Hence $u_0(\cdot;\lambda) \in H^1(\mathbb{B}^5_{1-\delta})$ for all $\delta \in (0,1)$. On the other hand, near $\rho = 1$, from the transformation we have

$$\begin{aligned} &\lim_{\rho\to 1^-} u_1(\rho;\lambda) \\ &= \lim_{\rho\to 1^-}\left[\rho^{-3}(1+\rho)^{\frac{1}{2}-\lambda}(2+\rho(-1+2\lambda))\left[1 + \mathcal{O}_0(\langle\omega\rangle^{-1})a_1(\rho) + \mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})\right]\right] \\ &= 2^{\frac{1}{2}-\lambda}(1+2\lambda) = O(\langle\lambda\rangle), \end{aligned}$$

and its derivative is

$$\lim_{\rho\to 1^-} u_1'(\rho;\lambda) = O(\langle\lambda\rangle^2).$$

Hence $u_1(\cdot;\lambda)\in C^1(0,1]\cap C^\infty(0,1)$, so $u_1\in H^1(\mathbb{B}^5\backslash\{0\})$.

Another two solutions to (2.18) are given by, for $j=0,1$,

$$\tilde{u}_j(\rho) = \rho^{-2}(1-\rho^2)^{-\frac{1}{4}-\frac{\lambda}{2}}\tilde{v}_j(\rho).$$

The next step is to construct the Wronskian of the fundamental system $\{u_0,u_1\}$. Here we turn our attention to the specific potential $V(\rho) = -\frac{35}{4}$, i.e., we are looking at (2.16).

Lemma 2.3.9. *We have*

$$W(u_0(\cdot;\lambda),u_1(\cdot;\lambda)) = W(\lambda)w_0(\lambda)\rho^{-4}(1-\rho^2)^{-\frac{1}{2}-\lambda}$$

where $w_0(\epsilon+i\omega) = 1+\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\langle\omega\rangle^{-2})$ *for* $\rho\in(0,1)$, $\epsilon\in[0,\frac{1}{4}]$, $\omega\in\mathbb{R}$. *Moreover,* $|w_0(\lambda)|\gtrsim 1$ *for all* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda\in[0,\frac{1}{4}]$.

Proof. From the transformation we compute directly

$$\begin{aligned}W(u_0(\cdot;\lambda),u_1(\cdot;\lambda)) &= W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))\rho^{-4}(1-\rho^2)^{-\frac{1}{2}-\lambda}\\ &= W(\lambda)\left[1+\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\langle\omega\rangle^{-2})\right]\rho^{-4}(1-\rho^2)^{-\frac{1}{2}-\lambda},\end{aligned}$$

which is in the stated form.

Next we show that $|w_0(\lambda)|\neq 0$. Since

$$w_0(\lambda) = \frac{W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))}{W(\lambda)},$$

so $w_0\neq 0$ if and only if $W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))\neq 0$. Moreover, since

$$W(v_0(\cdot;\lambda),v_1(\cdot;\lambda)) = \rho^4(1-\rho^2)^{\frac{1}{2}+\lambda}W(u_0(\cdot;\lambda),u_1(\cdot;\lambda)),$$

$W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))\neq 0$ if and only if $W(u_0(\rho;\lambda),u_1(\rho;\lambda))\neq 0$ for $\rho\in(0,1)$. So we define $h(\rho^2)=u(\rho)$ and set $z=\rho^2$, then (2.18) with $F_\lambda=0$ is transformed to the hypergeometric differential equation (2.14). This equation was studied in Lemma 2.2.5. By comparing the asymptotics as $\rho\to 0^+$ we see that

$$\begin{aligned}u_0(\rho;\lambda) &= h_0(\rho^2;\lambda)O(\langle\omega\rangle^3),\\ \tilde{u}_0(\rho;\lambda) &= \tilde{h}_0(\rho^2;\lambda)O(\langle\omega\rangle^{-3}).\end{aligned}$$

and

$$u_1(\rho;\lambda) = h_1(\rho^2;\lambda).$$

So the Wronskian $W(u_0(\rho;\lambda),u_1(\rho;\lambda))\neq 0$ if and only if $h_0(\cdot;\lambda)$ and $h_1(\cdot;\lambda)$ are linearly independent. From (2.15), $h_0(\cdot;\lambda)$ and $h_1(\cdot;\lambda)$ are linearly dependent only when

$$-a = -\frac{\lambda}{2}-\frac{5}{2}\in\mathbb{N}_0 \text{ or } -b = -\frac{\lambda}{2}+\frac{1}{2}\in\mathbb{N}_0.$$

But this never happens for $\lambda=\epsilon+i\omega$, $\epsilon\in[0,\frac{1}{4}]$. Therefore we obtain

$$|w_0(\lambda)| = \left|\frac{W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))}{W(\lambda)}\right|\gtrsim 1$$

as stated. $\square$

This shows that $\{u_0(\cdot;\lambda, u_1(\cdot;\lambda)\}$ is a fundamental system for (2.16) with $F_\lambda = 0$.

Corollary 2.3.10. *We have*

$$\frac{1}{w_0(\lambda)} = 1 + \mathcal{O}_o\left(\langle\omega\rangle^{-1}\right) + \mathcal{O}\left(\langle\omega\rangle^{-2}\right)$$

for the function w_0 from Lemma 2.3.9, where $\lambda = \epsilon + i\omega$, $\epsilon \in [0, \frac{1}{4}]$, $\omega \in \mathbb{R}$.

Proof. We write

$$\begin{aligned}\frac{1}{w_0(\lambda)} &= \frac{\overline{w_0(\lambda)}}{|w_0(\lambda)|^2} \\ &= \left[1 + \mathcal{O}_o\left(\langle\omega\rangle^{-1}\right) + \mathcal{O}\left(\langle\omega\rangle^{-2}\right)\right]\left[1 + \mathcal{O}\left(\langle\omega\rangle^{-1}\right)\right] \\ &= 1 + \mathcal{O}_o\left(\langle\omega\rangle^{-1}\right) + O\left(\langle\omega\rangle^{-2}\right).\end{aligned}$$

□

2.3.5 Decomposition of the Green function

We first set

$$\begin{aligned}\varphi_1(\rho;\lambda) &:= \rho^{-2}(1-\rho^2)^{-\frac{1}{4}-\frac{\lambda}{2}}\psi_1(\rho;\lambda) = \rho^{-3}(1+\rho)^{\frac{1}{2}-\lambda}(2+\rho(-1+2\lambda)), \\ \tilde{\varphi}_1(\rho;\lambda) &:= \rho^{-2}(1-\rho^2)^{-\frac{1}{4}-\frac{\lambda}{2}}\tilde{\psi}_1(\rho;\lambda) = \rho^{-3}(1-\rho)^{\frac{1}{2}-\lambda}(2+\rho(1-2\lambda)),\end{aligned}$$

and

$$\varphi_0(\rho;\lambda) := \varphi_1(\rho;\lambda) - \tilde{\varphi}_1(\rho;\lambda).$$

Then $\{\varphi_0(\cdot;\lambda), \varphi_1(\cdot;\lambda)\}$ is a fundamental system of (2.18) with $F_\lambda = V = 0$. The Green function is given by

$$G_0(\rho, s;\lambda) = \frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{(3-2\lambda)(1+2\lambda)(1-2\lambda)}\begin{cases}\varphi_0(\rho;\lambda)\varphi_1(s;\lambda) \text{ if } \rho \le s \\ \varphi_0(s;\lambda)\varphi_1(\rho;\lambda) \text{ if } \rho \ge s\end{cases},$$

which is the Green function that gives the semigroup $\mathbf{S}_0$ using Laplace inversion.

General solutions to (2.16) are given by

$$\begin{aligned}u(\rho;\lambda) &= c_0(\lambda)u_0(\rho;\lambda) + c_1(\lambda)u_1(\rho;\lambda) \\ &\quad - \int_0^\rho \frac{u_0(s;\lambda)u_1(\rho;\lambda)}{W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds - \int_\rho^1 \frac{u_0(\rho;\lambda)u_1(s;\lambda)}{W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds,\end{aligned}$$

where $c_0(\lambda), c_1(\lambda)$ are arbitrary constants. But since neither $u_0(\cdot;\lambda)$ nor $u_1(\cdot;\lambda)$ belongs to $H^1(\mathbb{B}^5)$, we choose c_0 and c_1 to be identically 0. Then there is a unique solution in $H^1(\mathbb{B}^5)$ given by

$$u(\rho;\lambda) = -\int_0^\rho \frac{u_0(s;\lambda)u_1(\rho;\lambda)}{W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds - \int_\rho^1 \frac{u_0(\rho;\lambda)u_1(s;\lambda)}{W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds.$$

So the Green function for (2.16) is given by

$$G(\rho, s; \lambda) = \frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{(3-2\lambda)(1+2\lambda)(1-2\lambda)w_0(\lambda)} \begin{cases} u_0(\rho;\lambda)u_1(s;\lambda) \text{ if } \rho \le s \\ u_1(\rho;\lambda)u_0(s;\lambda) \text{ if } \rho \ge s \end{cases},$$

which is the Green function in the integral kernel of (2.17).

We define a smooth cut-off $\chi : \mathbb{R} \to [0,1]$ with $\chi(x) = 1$ for $|x| \le \delta_1$ and $\chi(x) = 0$ for $|x| \ge \delta_0$, where δ_0 and δ_1 are chosen from before. We decompose G in the following way.

Lemma 2.3.11. *We have*

$$G(\rho, s; \lambda) = G_0(\rho, s; \lambda) + \sum_{n=1}^{6} G_n(\rho, s; \lambda),$$

where

$$\begin{aligned}
G_1(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(s-\rho)\chi(\rho\langle\omega\rangle)s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\varphi_0(\rho;\lambda)\varphi_1(s;\lambda)\gamma_1(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)} \\
G_2(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(s-\rho)\left[1-\chi(\rho\langle\omega\rangle)\right]s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\varphi_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_2(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)} \\
G_3(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(s-\rho)\left[1-\chi(\rho\langle\omega\rangle)\right]s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\tilde{\varphi}_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_3(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)} \\
G_4(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\varphi_1(\rho;\lambda)\varphi_0(s;\lambda)\gamma_4(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)} \\
G_5(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(\rho-s)\left[1-\chi(s\langle\omega\rangle)\right]s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\varphi_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_5(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)} \\
G_6(\rho, s; \lambda) &= 1_{\mathbb{R}_+}(\rho-s)\left[1-\chi(s\langle\omega\rangle)\right]s^4(1-s^2)^{-\frac{1}{2}+\lambda}\frac{\varphi_1(\rho;\lambda)\tilde{\varphi}_1(s;\lambda)\gamma_6(\rho,s;\lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)},
\end{aligned}$$

and

$$\gamma_n(\rho, s; \lambda) = \mathcal{O}(\rho^0 s^0)O_0(\langle\omega\rangle^{-1}) + \mathcal{O}((1-\rho)^0 s^0\langle\omega\rangle^{-2}) + \mathcal{O}(\rho^0(1-s)\langle\omega\rangle^{-2}),$$

for all $\rho, s \in (0,1)$, $\lambda = \epsilon + i\omega$, $\epsilon \in [0, \frac{1}{4}]$, $\omega \in \mathbb{R}$, $n \in \{1, 2, \dots, 6\}$.

Proof. From Lemmas 2.3.6 and 2.3.7, we have

$$\begin{aligned}
u_1(\rho;\lambda) &= \chi(\rho\langle\omega\rangle)u_1(\rho;\lambda) + [1-\chi(\rho\langle\omega\rangle)]u_1(\rho;\lambda) \\
&= \chi(\rho\langle\omega\rangle)\varphi_1(\rho;\lambda)[1+\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0\langle\omega\rangle^{-2})] \\
&\quad + [1-\chi(\rho\langle\omega\rangle)]\varphi_1(\rho;\lambda)[1+\mathcal{O}_o(\langle\omega\rangle^{-1})a_1(\rho) + \mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})] \\
&= \varphi_1(\rho;\lambda)[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})]. \qquad (2.24)
\end{aligned}$$

On the other hand, using Lemmas 2.3.4, 2.3.6, and 2.3.8, we have

$$
\begin{aligned}
&u_0(\rho;\lambda) = \chi(\rho\langle\omega\rangle)u_0(\rho;\lambda) + [1-\chi(\rho\langle\omega\rangle)]\,u_0(\rho;\lambda)\\
&= \chi(\rho\langle\omega\rangle)\varphi_0(\rho;\lambda)\left[1+\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right]\\
&\quad + [1-\chi(\rho\langle\omega\rangle)]\,u_1(\rho;\lambda)\left[1+\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&\quad - [1-\chi(\rho\langle\omega\rangle)]\,\tilde{u}_1(\rho;\lambda)\left[1+\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&= \chi(\rho\langle\omega\rangle)\varphi_0(\rho;\lambda)\left[1+\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\right]\\
&\quad + [1-\chi(\rho\langle\omega\rangle)]\,\varphi_1(\rho;\lambda)\left[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})+\mathcal{O}(\langle\omega\rangle^{-2})\right]\\
&\quad - [1-\chi(\rho\langle\omega\rangle)]\,\tilde{\varphi}_1(\rho;\lambda)\left[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})+\mathcal{O}(\langle\omega\rangle^{-2})\right].
\end{aligned}
\tag{2.25}
$$

Then we obtain the stated decomposition from Corollary 2.3.10. □

2.3.6 The semigroup

Using Lemma 2.3.11, the representation of the semigroup in (2.17) is given by

$$
[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1(\rho) = [\mathbf{S}_0(\tau)\tilde{\mathbf{f}}]_1(\rho) + \frac{1}{2\pi i} + \sum_{n=1}^{6} \lim_{N\to\infty} \int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau} \int_0^1 G_n(\rho,s;\lambda)F_\lambda(s)dsd\lambda \tag{2.26}
$$

for any $\epsilon > 0$, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2) \in \operatorname{rg}(\mathbf{I}-\mathbf{P}) \cap C^2 \times C^1([0,1])$, and

$$
F_\lambda(s) = s\tilde{f}_1'(s) + \left(\lambda + \frac{5}{2}\right)\tilde{f}_1(s) + \tilde{f}_2(s).
$$

So we define, for $n \in \{1,2,\dots,6\}$, $f \in C([0,1])$, the operators

$$
\begin{aligned}
T_{n,\epsilon}(\tau)f(\rho) &:= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}\int_0^1 G_n(\rho,s;\lambda)f(s)dsd\lambda\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\epsilon+i\omega)\tau}\int_0^1 G_n(\rho,s;\epsilon+i\omega)f(s)dsd\omega
\end{aligned}
$$

for $\tau \geq 0$ and $\rho \in (0,1)$. We would like to take the limit $\epsilon \to 0^+$. Indeed, for all $\rho \in (0,1]$, $\epsilon \in [0,\frac{1}{4}]$, and $\omega \in \mathbb{R}$ we have $|\varphi_1(\rho;\epsilon+i\omega)| + |\tilde{\varphi}_1(\rho;\epsilon+i\omega)| \lesssim \rho^{-3} + \rho^{-2}\langle\omega\rangle$, so $|G_n(\rho,s;\epsilon+i\omega)| \lesssim \rho^{-3}s(1-s)^{-\frac{1}{2}+\epsilon}\langle\omega\rangle^{-2}$. By dominated convergence and Fubini, we have

$$
\begin{aligned}
T_n(\tau)f(\rho) := \lim_{\epsilon\to 0+} T_{n,\epsilon}(\tau)f(\rho) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\int_0^1 G_n(\rho,s;\omega)f(s)dsd\omega\\
&= \frac{1}{2\pi}\int_0^1\int_{-\infty}^{\infty} e^{i\omega\tau}G_n(\rho,s;\omega)d\omega f(s)ds.
\end{aligned}
$$

2.3.7 Representations of φ_0

In this section we apply the fundamental theorem of calculus to $\varphi_0(\rho;\lambda)\rho^3$ to obtain two ways of writing this term. These representations will be useful when we estimate the oscillatory integrals. Since $\lim_{\rho\to 0}\varphi_0(\rho)\rho^3 = 0$, we have

$$\begin{aligned}\varphi_0(\rho;\lambda)\rho^3 &= -\frac{1}{2}(3-2\lambda)(1-2\lambda)\int_0^\rho \left((1+t_1)^{-\frac{1}{2}-\lambda} - (1-t)^{-\frac{1}{2}-\lambda}\right) t_1 dt_1 \\ &= -\frac{\rho^2}{2}(3-2\lambda)(1-2\lambda)\int_{-1}^1 (1+\rho t_1)^{-\frac{1}{2}-\lambda} t_1 dt_1.\end{aligned}$$

Then we have

$$\varphi_0(\rho;\lambda) = \frac{1}{2\rho}(3-2\lambda)(-1+2\lambda)\int_{-1}^1 (1+\rho t_1)^{-\frac{1}{2}-\lambda} t_1 dt_1 \tag{2.27}$$

Notice that $(1+\rho t_1)^{-\frac{1}{2}-\lambda} t_1|_{\rho=0} = t_1$, so we can write

$$\begin{aligned}(1+\rho t_1)^{-\frac{1}{2}-\lambda} t_1 &= \int_0^\rho \frac{d}{dt_2}(1+t_1 t_2)^{-\frac{1}{2}-\lambda} t_1 dt_2 + t_1 \\ &= -\frac{\rho(1+2\lambda)}{2}\int_0^1 (1+\rho t_1 t_2)^{-\frac{3}{2}-\lambda} t_1^2 dt_2 + t_1\end{aligned}$$

Then

$$\begin{aligned}\int_{-1}^1 (1+\rho t_1)^{-\frac{1}{2}-\lambda} t_1 dt_1 &= -\frac{\rho(1+2\lambda)}{2}\int_{-1}^1\int_0^1 (1+\rho t_1 t_2)^{-\frac{3}{2}-\lambda} t_1^2 dt_2 dt_1 + \int_{-1}^1 t_1 dt_1 \\ &= -\frac{\rho(1+2\lambda)}{2}\int_{-1}^1\int_0^1 (1+\rho t_1 t_2)^{-\frac{3}{2}-\lambda} t_1^2 dt_2 dt_1.\end{aligned}$$

Hence we can further reduce φ_0 in the form

$$\varphi_0(\rho;\lambda) = -\frac{(3-2\lambda)(-1+2\lambda)(1+2\lambda)}{4}\int_{-1}^1\int_0^1 (1+\rho t_1 t_2)^{-\frac{3}{2}-\lambda} t_1^2 dt_2 dt_1. \tag{2.28}$$

2.4 Strichartz estimates of the full semigroup

2.4.1 Preliminary

We first introduce some results that will be useful. We start with recalling the result on oscillatory integrals from [8].

Lemma 2.4.1 ([8, Lemma 4.2])**.** *We have*

$$\left|\int_{\mathbb{R}} e^{ia\omega}[\mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2})]d\omega\right| \lesssim \langle a\rangle^{-2}$$

for all $a\in\mathbb{R}\backslash\{0\}$. □

The next result is in the same spirit as Lemma 2.4.1.

Lemma 2.4.2. *Let $\alpha\in(0,1)$. Then we have*

$$\left|\int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-\alpha})d\omega\right| \lesssim |a|^{-1+\alpha}\langle a\rangle^{-2}$$

for all $a\in\mathbb{R}\backslash\{0\}$.

Proof. Without loss of generality assume $a > 0$. We decompose

$$\begin{aligned}
&\int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-\alpha})d\omega \\
&= \int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-\alpha})\chi(a\omega)d\omega + \int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-\alpha})[1-\chi(a\omega)]d\omega \\
&=: I_1(a) + I_2(a).
\end{aligned}$$

So we have

$$I_1(a) = \int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\omega^{-\alpha})\chi(a\omega)d\omega = a^{-1}\int_{\mathbb{R}} e^{i\omega}\mathcal{O}\left(\left(\frac{\omega}{a}\right)^{-\alpha}\right)\chi(\omega)d\omega = O\left(a^{-1+\alpha}\right),$$

and an integration by parts gives

$$\begin{aligned}
I_2(a) &= a^{-1}\int_{\mathbb{R}}[1-\chi(a\omega)]e^{ia\omega}\mathcal{O}\left(\omega^{-\alpha-1}\right)d\omega + \int_{\mathbb{R}}\chi'(a\omega)e^{ia\omega}\left(\omega^{-\alpha}\right)d\omega \\
&= a^{-2}\int_{\mathbb{R}}[1-\chi(\omega)]e^{i\omega}\mathcal{O}\left(\left(\frac{\omega}{a}\right)^{-\alpha-1}\right)d\omega + a^{-1}\int_{\mathbb{R}}\chi'(\omega)e^{i\omega}\left(\left(\frac{\omega}{a}\right)^{-\alpha}\right)d\omega \\
&= O\left(a^{-1+\alpha}\right).
\end{aligned}$$

Moreover, for $a \geq 1$, we can do integration by parts as many times as we want and get

$$\int_{\mathbb{R}} e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-\alpha})d\omega = O(\langle a\rangle^{-2}).$$

This gives the stated bound. □

We also have two results for the non-oscillatory case.

Lemma 2.4.3. *We have*

$$\left|\rho^{-n}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-n-1})d\omega\right| \lesssim \langle a\rangle^{-2},$$

for all $n \geq 1$, $\rho \in (0,1)$, and $a \in \mathbb{R}$.

Proof. We compute

$$\begin{aligned}
\rho^{-n}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,\mathcal{O}(\langle\omega\rangle^{-n-1})d\omega &= \rho^{-n}\int_{\mathbb{R}}\chi(|\omega|)\,[1-\chi(\rho\langle\omega\rangle)]\,\mathcal{O}(\langle\omega\rangle^{-n-1})d\omega \\
&\quad + \rho^{-n}\int_{\mathbb{R}}[1-\chi(|\omega|)]\,[1-\chi(\rho|\omega|)]\,\mathcal{O}(|\omega|^{-n-1})d\omega \\
&= \int_{\mathbb{R}}\chi(|\omega|)\,[1-\chi(\rho\langle\omega\rangle)]\,\mathcal{O}(\langle\omega\rangle^{-1})d\omega \\
&\quad + \int_{\mathbb{R}}\left[1-\chi\left(\left|\frac{\omega}{\rho}\right|\right)\right][1-\chi(|\omega|)]\,\mathcal{O}(\rho^0|\omega|^{-n-1})d\omega \\
&= \mathcal{O}(\rho^0).
\end{aligned}$$

Then the statement follows from two integrations by parts. □

The next lemma is similar.

Lemma 2.4.4. *We have*

$$\left|\rho^{-n}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-n})d\omega\right| \lesssim |a|^{-1}\langle a\rangle^{-2},$$

for $n\geq 2$, $\rho\in(0,1)$, and $a\in\mathbb{R}\backslash\{0\}$.

Proof. Without loss of generality we assume $a>0$. Integrating by parts once we get

$$\begin{aligned}
&\rho^{-n}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-n}))d\omega\\
&=a^{-1}\rho^{-n}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-n-1})d\omega+a^{-1}\rho^{-n+1}\int_{\mathbb{R}}\chi'(\rho\langle\omega\rangle)e^{ia\omega}\mathcal{O}(\langle\omega\rangle^{-n})d\omega\\
&=a^{-1}(I_1(\rho,a)+I_2(\rho,a)),
\end{aligned}$$

where both I_1 and I_2 are $\mathcal{O}(\rho^0)$ by a similar computation as in Lemma 2.4.3. Then, when $a\geq 1$, we integrate by parts twice more to obtain the statement. ☐

2.4.2 Kernel estimates

Our goal is now to estimate $\|T_n(\tau)f\|_{L^5(\mathbb{B}^5)}$. Applying Minkowski's inequality we get

$$\begin{aligned}
\|T_n(\tau)f\|_{L^5(\mathbb{B}^5)} &\cong \left\|\int_0^1 f(s)\int_{\mathbb{R}}e^{i\omega\tau}G_n(\rho,s;i\omega)d\omega ds\right\|_{L^5_\rho(\mathbb{B}^5)}\\
&\leq \int_0^1|f(s)|\left\|\int_{\mathbb{R}}e^{i\omega\tau}G_n(\rho,s;i\omega)d\omega\right\|_{L^5_\rho(\mathbb{B}^5)}ds.
\end{aligned}$$

We have the following result.

Proposition 2.4.5. *We have the bounds*

$$\left\|\int_{\mathbb{R}}e^{i\omega\tau}G_n(\rho,s;\omega)d\omega\right\|_{L^5_\rho(\mathbb{B}^5)}\lesssim s^2(1-s)^{-\frac{1}{2}}\left|\tau+\log(1-s)\right|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-1}$$

for all $\tau\geq 0$, $s\in(0,1)$, and $n\in\{1,2,\dots,6\}$.

Proof. For G_1 we have

$$\begin{aligned}
&\int_{\mathbb{R}}e^{i\omega\tau}G_1(\rho,s;\omega)d\omega\\
&=\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega\tau}1_{\mathbb{R}_+}(s-\rho)\frac{s(1-s)^{-\frac{1}{2}+i\omega}(2+s(-1+2i\omega))\varphi_0(\rho;i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)}\gamma_1(\rho,s;i\omega)d\omega\\
&=2\cdot 1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)}\gamma_1(\rho,s;i\omega)d\omega\\
&\quad-1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(1+2i\omega)}\gamma_1(\rho,s;i\omega)d\omega\\
&=:2I_1(\rho,s,\tau)+I_2(\rho,s,\tau).
\end{aligned}$$

For I_1 we use (2.28), then we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)}\gamma_1(\rho,s;i\omega)d\omega\right| \\
\cong &\left|\int_{\mathbb{R}} \chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s))}\left(\int_{-1}^1\int_0^1 (1+\rho t_1t_2)^{-\frac{3}{2}-i\omega}t_1^2dt_2dt_1\right)\gamma_1(\rho,s;i\omega)d\omega\right| \\
= &\left|\int_{-1}^1\int_0^1 (1+\rho t_1t_2)^{-\frac{3}{2}}t_1^2\left(\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s)+\log(1+\rho t_1t_2))}\gamma_1(\rho,s;i\omega)d\omega\right)dt_2dt_1\right| \\
\leq &\sup_{t_1\in(-1,1),t_2\in(0,1)}\left|\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s)+\log(1+\rho t_1t_2))}\gamma_1(\rho,s;i\omega)d\omega\right|.
\end{aligned}$$

The order of integrations can be changed because $\rho\in(0,1)$ is fixed. Moreover, thanks to the cut-off $\chi(\rho\langle\omega\rangle)$, we have $|(1+\rho t_1t_2)^{-\frac{3}{2}}|\lesssim 1$ and $|\log(1+\rho t_1t_2)|\lesssim 1$ for all t_1,t_2 in the respective domains. So we can use Lemma 2.4.1 and get

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s)+\log(1+\rho t_1t_2))}\gamma_1(\rho,s;i\omega)d\omega\right| \\
&\lesssim \langle\tau+\log(1-s)+\log(1+\rho t_1t_2)\rangle^{-2}\lesssim\langle\tau+\log(1-s)\rangle^{-2}.
\end{aligned}$$

Plugging this back in we have

$$|I_1(\rho,s,\tau)|\lesssim 1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}, \tag{2.29}$$

so

$$\begin{aligned}
\|I_1(\cdot,s,\tau)\|_{L^5(\mathbb{B}^5)} &\lesssim \left(\int_0^1\left|1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\right|^5\rho^4d\rho\right)^{\frac{1}{5}} \\
&= s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\left(\int_0^s\rho^4d\rho\right)^{\frac{1}{5}} \\
&\cong s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}.
\end{aligned}$$

For I_2 we directly use that

$$\begin{aligned}
I_2(\rho,s,\tau)=1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}&\int_{\mathbb{R}}e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega) \\
&\times\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-3})\chi(\rho\langle\omega\rangle)d\omega,
\end{aligned}$$

and we look at the integral kernel,

$$\begin{aligned}
&\int_{\mathbb{R}}e^{i\omega(\tau+\log(1-s))}\chi(\rho\langle\omega\rangle)\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-3})\varphi_0(\rho;i\omega)d\omega \\
=&\int_{\mathbb{R}}e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\chi(\rho\langle\omega\rangle)\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-3}) \\
&\times\frac{1}{\rho^3}\left[(1+\rho)^{\frac{1}{2}}(2+\rho(-1+2i\omega))-e^{iw(\log(1+\rho)-\log(1-\rho))}(1-\rho)^{\frac{1}{2}}(2+\rho(1-2i\omega))\right]d\omega.
\end{aligned}$$

Since on the support of the cut-off $\chi(\rho\langle\omega\rangle)$, we have $e^{iw(\log(1+\rho)-\log(1-\rho))}=1+\mathcal{O}(\omega\rho)$, the function

$$g(\rho,\omega):=(1+\rho)^{\frac{1}{2}}(2+\rho(-1+2i\omega))-e^{iw(\log(1+\rho)-\log(1-\rho))}(1-\rho)^{\frac{1}{2}}(2+\rho(1-2i\omega))$$

is of symbol type. Furthermore, we have

$$g(0,\omega)=\partial_\rho g(\rho,\omega)|_{\rho=0}=\partial_\rho^2 g(\rho,\omega)|_{\rho=0}=0 \text{ for all } \omega\in\mathbb{R},$$

so by Cauchy remainder we have

$$g(\rho,\omega)=\frac{\rho^3}{2!}\int_0^1 \partial_1^3 g(\rho t,\omega)(1-t)^2dt,$$

where ∂_1 is the derivative with respect to the first variable. Since $|\partial_1^3 g(\rho t,\omega)|\lesssim\langle\omega\rangle^3$ on the support of $\chi(\rho\langle\omega\rangle)$, we obtain $|g(\rho,\omega)|\lesssim\rho^3\langle\omega\rangle^3$. Hence $g(\rho,\omega)=\mathcal{O}(\rho^3\langle\omega\rangle^3)$, and on the support of $\chi(\rho\langle\omega\rangle)$, we also have $g(\rho,\omega)=\mathcal{O}(\rho^{\frac{21}{10}}\langle\omega\rangle^{\frac{21}{10}})$. Thus Lemma 2.4.2 gives the estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-s))}\chi(\rho\langle\omega\rangle)\mathcal{O}(\rho^0(1-\rho)^0 s^0(1-s)^0\langle\omega\rangle^{-3})\varphi_0(\rho;i\omega)d\omega\right|\\
&=\left|\rho^{-\frac{9}{10}}\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\chi(\rho\langle\omega\rangle)\mathcal{O}(\rho^0\langle\omega\rangle^{-\frac{9}{10}})d\omega\right|\\
&\lesssim\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)-\log(1+\rho)\rangle^{-2}\\
&\le\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-2},
\end{aligned}$$

so

$$\begin{aligned}
&|I_2(\rho,s,\tau)|\\
&\lesssim 1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-2}.
\end{aligned}\tag{2.30}$$

Then we have

$$\begin{aligned}
&\|I_2(\cdot,s,\tau)\|_{L^5(\mathbb{B}^5)}\\
&\lesssim s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\left(\int_0^s\left(\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\right)^5\rho^4d\rho\right)^{\frac{1}{5}}\\
&\le s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\left(\int_0^s\rho^{-\frac{1}{2}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{2}}d\rho\right)^{\frac{1}{5}}.
\end{aligned}$$

We estimate

$$\begin{aligned}
\left|\int_0^1\rho^{-\frac{1}{2}}|x-\log(1+\rho)|^{-\frac{1}{2}}d\rho\right|&\le\int_0^{\frac{\log(2)}{x}}(e^{xy}-1)^{-\frac{1}{2}}|x-xy|^{-\frac{1}{2}}xe^{xy}dy\\
&=x|x|^{-\frac{1}{2}}\int_0^{\frac{\log(2)}{x}}(e^{xy}-1)^{-\frac{1}{2}}|1-y|^{-\frac{1}{2}}e^{xy}dy\\
&\le x|x|^{-\frac{1}{2}}\int_0^{\frac{1}{x}}(xy)^{-\frac{1}{2}}|1-y|^{-\frac{1}{2}}e^{xy}dy\\
&\lesssim\int_0^{\frac{1}{|x|}}|y|^{-\frac{1}{2}}|1-y|^{-\frac{1}{2}}dy\\
&\lesssim\Big|\log|x|\Big|+1\lesssim|x|^{-\frac{1}{2}}+1.
\end{aligned}$$

Since $s \leq 1$,

$$\left| \int_0^s \rho^{-\frac{1}{2}} |\tau + \log(1-s) - \log(1+\rho)|^{-\frac{1}{2}} d\rho \right| \lesssim |\tau + \log(1-s)|^{-\frac{1}{2}} + 1.$$

With this, we obtain the bound

$$\begin{aligned} \|I_2(\cdot, s, \tau)\|_{L^5(\mathbb{B}^5)} &\lesssim s^2(1-s)^{-\frac{1}{2}} \left(|\tau + \log(1-s)|^{-\frac{1}{2}} + 1 \right)^{\frac{1}{5}} \langle \tau + \log(1-s) \rangle^{-2} \\ &\lesssim s^2(1-s)^{-\frac{1}{2}} |\tau + \log(1-s)|^{-\frac{1}{10}} \langle \tau + \log(1-s) \rangle^{-1}. \end{aligned}$$

Next, for G_2 we have

$$\begin{aligned} &\int_{\mathbb{R}} e^{i\omega\tau} G_2(\rho, s; \omega) d\omega \\ &= 2 \cdot 1_{\mathbb{R}_+}(s-\rho) s(1-s)^{-\frac{1}{2}} \int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] \frac{e^{i\omega(\tau + \log(1-s))} \varphi_1(\rho; i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)} \gamma_2(\rho, s; i\omega) d\omega \\ &\quad - 1_{\mathbb{R}_+}(s-\rho) s^2(1-s)^{-\frac{1}{2}} \int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] \frac{e^{i\omega(\tau + \log(1-s))} \varphi_1(\rho; i\omega)}{(3-2i\omega)(1+2i\omega)} \gamma_2(\rho, s; i\omega) d\omega \\ &=: 2I_1(\rho, s, \tau) + I_2(\rho, s, \tau). \end{aligned}$$

In I_1 we look at

$$\begin{aligned} &\int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] \frac{e^{i\omega(\tau + \log(1-s))} \varphi_1(\rho; i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)} \gamma_2(\rho, s; i\omega) d\omega \\ &= \int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] e^{i\omega(\tau + \log(1-s))} \varphi_1(\rho; i\omega) \mathcal{O}(\rho^0 (1-\rho)^0 s^0 (1-s)^0 \langle \omega \rangle^{-4}) d\omega \\ &= (1+\rho)^{\frac{1}{2}} \rho^{-2} \int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] e^{i\omega(\tau + \log(1-s) - \log(1+\rho))} \mathcal{O}(\rho^0 (1-\rho)^0 s^0 (1-s)^0 \langle \omega \rangle^{-3}) d\omega \end{aligned}$$

Using Lemma 2.4.3 we get

$$\begin{aligned} &\rho^{-2} \int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] e^{i\omega(\tau + \log(1-s) - \log(1+\rho))} \mathcal{O}(\rho^0 (1-\rho)^0 s^0 (1-s)^0 \langle \omega \rangle^{-3}) d\omega \\ &= O(\langle \tau + \log(1-s) - \log(1+\rho) \rangle^{-2}). \end{aligned}$$

Then since $\log(1+\rho)$ is bounded for all $\rho \in [0,1]$, we get

$$\begin{aligned} |I_1(\rho, s, \tau)| &\lesssim 1_{\mathbb{R}_+}(s-\rho) s(1-s)^{-\frac{1}{2}} \langle \tau + \log(1-s) - \log(1+\rho) \rangle^{-2} \\ &\lesssim 1_{\mathbb{R}_+}(s-\rho) s(1-s)^{-\frac{1}{2}} \langle \tau + \log(1-s) \rangle^{-2}, \end{aligned}$$

which is the same as (2.29) and the rest follows the same way. In I_2 the expression becomes

$$\begin{aligned} I_2(\rho, s, \tau) = 1_{\mathbb{R}_+}(s-\rho) s^2(1-s)^{-\frac{1}{2}} (1+\rho)^{\frac{1}{2}} \rho^{-2} &\int_{\mathbb{R}} [1 - \chi(\rho \langle \omega \rangle)] e^{i\omega(\tau + \log(1-s) - \log(1+\rho))} \\ &\times \mathcal{O}(\rho^0 (1-\rho)^0 s^0 (1-s)^0 \langle \omega \rangle^{-2}) d\omega. \end{aligned}$$

Here we first use Lemma 2.4.3 to estimate

$$\rho^{-2}\left|\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\mathcal{O}\left(\langle\omega\rangle^{-2}\right)d\omega\right|$$
$$\lesssim\rho^{-1}\langle\tau+\log(1-s)-\log(1+\rho)\rangle^{-2}\lesssim\rho^{-1}\langle\tau+\log(1-s)\rangle^{-2}. \tag{2.31}$$

On the other hand, using Lemma 2.4.4 we have

$$\rho^{-2}\left|\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\mathcal{O}\left(\langle\omega\rangle^{-2}\right)d\omega\right|$$
$$\lesssim|\tau+\log(1-s)-\log(1+\rho)|^{-1}\langle\tau+\log(1-s)-\log(1+\rho)\rangle^{-2}$$
$$\lesssim|\tau+\log(1-s)-\log(1+\rho)|^{-1}\langle\tau+\log(1-s)\rangle^{-2}.$$

We can then interpolate the two bounds and get

$$\rho^{-2}\left|\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\mathcal{O}\left(\langle\omega\rangle^{-2}\right)d\omega\right|$$
$$\lesssim\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-2}.$$

Now

$$|I_2(\rho,s,\tau)|$$
$$\lesssim 1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-2}$$
$$\lesssim 1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}\rho^{-\frac{9}{10}}|\tau+\log(1-s)-\log(1+\rho)|^{-\frac{1}{10}}\langle\tau+\log(1-s)\rangle^{-2},$$

which is the same as (2.30), and the rest follows the same way.

Similar to G_2, in G_3 we have to bound the two terms

$$I_1(\rho,s,\tau)=1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}(1-\rho)^{\frac{1}{2}}\rho^{-2}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1-\rho))}$$
$$\times\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-3})d\omega$$

and

$$I_2(\rho,s,\tau)=1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}(1-\rho)^{\frac{1}{2}}\rho^{-2}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1-\rho))}$$
$$\times\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-2})d\omega.$$

For I_1, from Lemma 2.4.3 we have the bound

$$(1-\rho)^{\frac{1}{2}}\rho^{-2}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1-\rho))}\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-2})d\omega$$
$$=(1-\rho)^{\frac{1}{2}}O(\langle\tau+\log(1-s)-\log(1-\rho)\rangle^{-2}).$$

We then use

$$(1-\rho)^{\frac{1}{2}}\langle\tau+\log(1-s)-\log(1-\rho)\rangle^{-2}\lesssim\langle\tau+\log(1-s)\rangle^{-2} \tag{2.32}$$

and obtain the exact same estimate as for I_1 in the G_2 case. For I_2, similar to before, we interpolate the bounds from Lemma 2.4.3 and 2.4.4,

$$\begin{aligned}
&(1-\rho)^{\frac{1}{2}}\rho^{-2}\left|\int_{\mathbb{R}}\left[1-\chi(\rho\langle\omega\rangle)\right]e^{i\omega(\tau+\log(1-s)-\log(1-\rho))}\mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-2})d\omega\right| \\
&\lesssim \rho^{-\frac{9}{10}}\left|\tau+\log(1-s)-\log(1-\rho)\right|^{-\frac{1}{10}}(1-\rho)^{\frac{1}{2}}\left\langle\tau+\log(1-s)-\log(1-\rho)\right\rangle^{-2} \\
&\lesssim \rho^{-\frac{9}{10}}\left|\tau+\log(1-s)-\log(1-\rho)\right|^{-\frac{1}{10}}\left\langle\tau+\log(1-s)\right\rangle^{-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|I_2(\cdot,s,\tau)\|_{L^5(\mathbb{B}^5)} \\
&\lesssim s^2(1-s)^{-\frac{1}{2}}\left\langle\tau+\log(1-s)\right\rangle^{-2}\left(\int_0^s\rho^{-\frac{1}{2}}|\tau+\log(1-s)-\log(1-\rho)|^{-\frac{1}{2}}\right)^{\frac{1}{5}}.
\end{aligned}$$

We estimate

$$\begin{aligned}
&\left|\int_0^1\rho^{\frac{1}{2}}\left|x-\log(1-\rho)\right|^{-\frac{1}{2}}d\rho\right| \\
&\le |x|\int_0^\infty(1-e^{-|x|y})^{-\frac{1}{2}}\Big|x+|x|y\Big|^{-\frac{1}{2}}e^{-|x|y}dy \\
&\lesssim |x|^{\frac{1}{2}}\int_0^2(|x|y)^{-\frac{1}{2}}|1-y|^{-\frac{1}{2}}dy+|x|^{\frac{1}{2}}\int_2^\infty(|x|y)^{-1}|1-y|^{-\frac{1}{2}}dy \\
&\lesssim \int_0^2y^{-\frac{1}{2}}|1-y|^{-\frac{1}{2}}dy+|x|^{-\frac{1}{2}}\int_2^\infty y^{-\frac{3}{2}}dy \\
&\lesssim |x|^{-\frac{1}{2}}+1.
\end{aligned}$$

Using this we have

$$\begin{aligned}
\|I_2(\cdot,s,\tau)\|_{L^5(\mathbb{B}^5)} &\lesssim s^2(1-s)^{-\frac{1}{2}}\left(|\tau+\log(1-s)|^{-\frac{1}{2}}+1\right)^{\frac{1}{5}}\left\langle\tau+\log(1-s)\right\rangle^{-2} \\
&\lesssim s^2(1-s)^{-\frac{1}{2}}|\tau+\log(1-s)|^{-\frac{1}{10}}\left\langle\tau+\log(1-s)\right\rangle^{-1}.
\end{aligned}$$

For G_4 we have

$$\int_{\mathbb{R}}e^{i\omega\tau}G_4(\rho,s;\omega)d\omega=2I_1(\rho,s,\tau)+I_2(\rho,s,\tau),$$

with

$$\begin{aligned}
I_1(\rho,s,\tau) &:= 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1-s^2)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}\chi(s\langle\omega\rangle)e^{i\omega(\tau+\log(1-s^2)-\log(1+\rho))}\frac{\varphi_0(s;i\omega)\gamma_4(\rho,s;i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)}d\omega
\end{aligned}$$

and

$$\begin{aligned}
I_2(\rho,s,\tau) &:= 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^2}(1-s^2)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}\chi(s\langle\omega\rangle)e^{i\omega(\tau+\log(1-s^2)-\log(1+\rho))}\frac{\varphi_0(s;i\omega)\gamma_4(\rho,s;i\omega)}{(3-2i\omega)(-1-2i\omega)}d\omega.
\end{aligned}$$

In I_1 we use the expression (2.28) for $\varphi_0(s;i\omega)$, and write

$$
\begin{aligned}
I_1(\rho,s,\tau) \cong\ & 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1-s^2)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\int_{-1}^{1}\int_0^1 (1+st_1t_2)^{-\frac{3}{2}}t_1^2 \\
& \times \int_{\mathbb{R}} \chi(s\langle\omega\rangle)e^{i\omega(\tau+\log(1-s^2)-\log(1+\rho)-\log(1+st_1t_2))}\gamma_4(\rho,s;i\omega)d\omega dt_2 dt_1,
\end{aligned}
$$

where the order of integration in t_1, t_2 and ω can be changed because s is fixed. Similar to the G_1 case, on the support of the cut-off $\chi(s\langle\omega\rangle)$, we have $|(1+st_1t_2)^{-\frac{3}{2}}| \lesssim 1$ and $|\log(1+st_1t_2)| \lesssim 1$ for all $t_1 \in [-1,1]$ and $t_2 \in [0,1]$, so

$$
\begin{aligned}
&\Big|\int_{-1}^{1}\int_0^1 (1+st_1t_2)^{-\frac{3}{2}}t_1^2 \int_{\mathbb{R}} e^{i\omega(\tau+\log(1-s^2)-\log(1+\rho)-\log(1+st_1t_2))}\gamma_4(\rho,s;i\omega)d\omega dt_2 dt_1 \\
&\qquad \times \left[\mathcal{O}(\rho^0 s^0)\mathcal{O}_0(\langle\omega\rangle^{-1}) + \mathcal{O}((1-\rho)^0 s^0\rho^0(1-s)^0\langle\omega\rangle^{-2})\right]\Big| \\
&\lesssim \sup_{t_1\in(-1,1),t_2\in(0,1)} \left\langle \tau+\log(1-s^2)-\log(1+\rho)-\log(1+st_1t_2)\right\rangle^{-2} \\
&\lesssim \langle\tau+\log(1-s)\rangle^{-2}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
|I_1(\rho,s,\tau)| &\lesssim 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1-s^2)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\lesssim 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}. \qquad (2.33)
\end{aligned}
$$

The situation for I_2 is similar, we use (2.27) and write

$$
\begin{aligned}
I_2(\rho,s,\tau) =\ & 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s^2)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\int_{-1}^{1}(1+st)^{-\frac{1}{2}}t \\
& \times \int_{\mathbb{R}} \chi(s\langle\omega\rangle)e^{i\omega(\tau+\log(1-s^2)-\log(1+\rho)-\log(1+st))}\frac{1-2i\omega}{1+2i\omega}\gamma_4(\rho,s;i\omega)d\omega dt
\end{aligned}
$$

Since

$$
\frac{1-2i\omega}{1+2i\omega} = \frac{1-4i\omega-4\omega^2}{1-4\omega^2} = \mathcal{O}_e(\langle\omega\rangle^0) + \mathcal{O}_o(\langle\omega\rangle^{-1}) + \mathcal{O}(\langle\omega\rangle^{-2}),
$$

we see that $\frac{1-2i\omega}{1+2i\omega}\gamma_4(\rho,s;i\omega)$ has the same form as $\gamma_4(\rho,s;i\omega)$. Hence by the same logic as the I_1 case, we obtain the same bound

$$
|I_2(\rho,s,\tau)| \lesssim 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2},
$$

so for $n = 1,2$ we have

$$
\begin{aligned}
\|I_n(\cdot,s,\tau)\|_{L^5(\mathbb{B}^5)} &\lesssim s^3(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\left(\int_s^1 \rho^{-6}d\rho\right)^{\frac{1}{5}} \\
&= s^3(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}\left(-\frac{1}{5}+\frac{1}{5s^5}\right)^{\frac{1}{5}} \\
&\lesssim s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}.
\end{aligned}
$$

For G_5 we use that on the support of the cut-off $1-\chi(s\langle\omega\rangle)$ we have $s^{-1}\lesssim\langle\omega\rangle$, and write

$$\begin{aligned}
G_5(\rho,s;i\omega) &= 1_{\mathbb{R}_+}(\rho-s)\,[1-\chi(s\langle\omega\rangle)]\,s^4(1-s)^{-\frac{1}{2}+i\omega}\varphi_1(\rho;\lambda)s^{-2}\mathcal{O}(\langle\omega\rangle^{-3}) \\
&= 2\cdot 1_{\mathbb{R}_+}(\rho-s)\,[1-\chi(s\langle\omega\rangle)]\,\frac{s^4}{\rho^3}(1-s)^{-\frac{1}{2}+i\omega}(1+\rho)^{\frac{1}{2}-i\omega}s^{-2}\mathcal{O}(\langle\omega\rangle^{-3}) \\
&\quad + 1_{\mathbb{R}_+}(\rho-s)\,[1-\chi(s\langle\omega\rangle)]\,\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}+i\omega}(1+\rho)^{\frac{1}{2}-i\omega}s^{-1}\mathcal{O}(\langle\omega\rangle^{-2}).
\end{aligned}$$

So we need to bound the two expressions

$$\begin{aligned}
I_1(\rho,s,\tau) &= 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1-s)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times s^{-2}\int_{\mathbb{R}}[1-\chi(s\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\mathcal{O}(\langle\omega\rangle^{-3})d\omega,
\end{aligned}$$

and

$$\begin{aligned}
I_2(\rho,s,\tau) &= 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times s^{-1}\int_{\mathbb{R}}[1-\chi(s\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1-s)-\log(1+\rho))}\mathcal{O}(\langle\omega\rangle^{-2})d\omega.
\end{aligned}$$

Now we apply Lemma 2.4.3. Since $\log(1+\rho)$ is bounded, we obtain the bounds

$$\begin{aligned}
|I_1(\rho,s,\tau)| &\lesssim 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\leq 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2},
\end{aligned}$$

and

$$|I_2(\rho,s,\tau)|\lesssim 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}.$$

These bounds are the same as (2.33).

The last term, G_6, is similar to G_5. We look at

$$\begin{aligned}
I_1(\tau)f(\rho) &= 1_{\mathbb{R}_+}(\rho-s)\frac{s^4}{\rho^3}(1+s)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}_+}[1-\chi(s\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1+s)-\log(1+\rho))}s^{-2}\mathcal{O}(\langle\omega\rangle^{-3})d\omega
\end{aligned}$$

and

$$\begin{aligned}
I_2(\tau)f(\rho) &= 1_{\mathbb{R}_+}(\rho-s)\frac{s^3}{\rho^2}(1+s)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}_+}[1-\chi(s\langle\omega\rangle)]\,e^{i\omega(\tau+\log(1+s)-\log(1+\rho))}s^{-1}\mathcal{O}(\langle\omega\rangle^{-2})d\omega.
\end{aligned}$$

Using

$$(1+s)^{-\frac{1}{2}} \langle \tau + \log(1+s) - \log(1+\rho)\rangle^{-2} \lesssim \langle \tau \rangle^{-2} \lesssim (1-s)^{-\frac{1}{2}} \langle \tau + \log(1-s)\rangle^{-2},$$

we obtain the same bounds as G_5. $\square$

Using this, we obtain boundedness of the operators T_n.

Proposition 2.4.6. *Let $p \in [2,\infty]$ and $q \in [\frac{10}{3}, 5]$ such that $\frac{1}{p} + \frac{5}{q} = \frac{3}{2}$. Then we have the bound*

$$\|T_n(\cdot)f\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|f\|_{L^2(\mathbb{B}^5)}$$

for all $f \in C([0,1])$ and $n \in \{1,2,\dots,6\}$.

Proof. From Proposition 2.4.5 we have

$$\begin{aligned}
\|T_n(\tau)f\|_{L^5(\mathbb{B}^5)} &\lesssim \int_0^1 |f(s)| \left\| \int_{\mathbb{R}} e^{i\omega\tau} G_n(\rho,s;\omega) d\omega \right\|_{L^5_\rho(\mathbb{B}^5)} ds \\
&\lesssim \int_0^1 |f(s)| s^2 (1-s)^{-\frac{1}{2}} |\tau + \log(1-s)|^{-\frac{1}{10}} \langle \tau + \log(1-s)\rangle^{-1} ds \\
&\lesssim \int_0^\infty \left| f(1-e^{-y}) \right| (1-e^{-y})^2 e^{-\frac{1}{2}y} |\tau - y|^{-\frac{1}{10}} \langle \tau - y \rangle^{-1} dy,
\end{aligned}$$

by doing a change of variables $s = 1 - e^{-y}$. Applying Young's inequality gives

$$\begin{aligned}
\|T_n(\cdot)f\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} &\lesssim \left(\int_0^\infty \left| f(1-e^{-y}) \right|^2 (1-e^{-y})^4 e^{-y} dy \right)^{\frac{1}{2}} \int_{\mathbb{R}} |y|^{-\frac{1}{10}} \langle y \rangle^{-1} dy \\
&\lesssim \|f\|_{L^2(\mathbb{B}^5)}.
\end{aligned}$$

On the other hand, Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|T_n(\tau)f\|_{L^5(\mathbb{B}^5)} &\lesssim \|f\|_{L^2(\mathbb{B}^5)} \left\| |\tau - \cdot|^{-\frac{1}{10}} \langle \tau - \cdot \rangle^{-1} \right\|_{L^2(\mathbb{R})} \\
&\lesssim \|f\|_{L^2(\mathbb{B}^5)},
\end{aligned}$$

so $\|T_n(\cdot)f\|_{L^\infty(\mathbb{R}_+;L^{\frac{10}{3}}(\mathbb{B}^5))} \lesssim \|I_2\|_{L^\infty(\mathbb{R}_+;L^5(\mathbb{B}^5))} \lesssim \|f\|_{L^2(\mathbb{B}^5)}$. Therefore the claim follows by interpolation. $\square$

2.4.3 The term containing λ

We recall that the function F_λ contains a term $\lambda \tilde{f}_1$. For convenience we rewrite this term as

$$F_\lambda(s) = s\tilde{f}_1'(s) + \left(\lambda + \frac{5}{2}\right) \tilde{f}_1(s) + \tilde{f}_2(s) = s\tilde{f}_1'(s) + \left(\lambda + \frac{1}{2}\right) \tilde{f}_1(s) + 2\tilde{f}_1(s) + \tilde{f}_2(s)$$

and define for $\tau \geq 0$, $\rho \in (0,1)$, $f \in C^1([0,1])$, and $n \in \{1,2,\dots,6\}$,

$$\dot{T}_{n,\epsilon}(\tau)f(\rho) := \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\epsilon-iN}^{\epsilon+iN} \frac{1}{2}(1+2\lambda) e^{\lambda\tau} \int_0^1 G_n(\rho,s;\lambda) f(s) ds d\lambda.$$

In order to prove the Strichartz estimates, we also need to have bounds for the operators $\dot{T}_n$.

We first present two lemmas concerning norm equivalence.

Lemma 2.4.7. *We have*

$$\|(\cdot)^{-1}f\|_{L^2(\mathbb{B}^5)} = \|(\cdot)f\|_{L^2(0,1)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$$

for all $f \in C^1([0,1])$.

Proof. We compute

$$\begin{aligned}
\int_0^1 |f(s)|^2 s^2 ds &= \frac{1}{3}\int_0^1 |f(s)|^2 \partial_s(s^3) ds \\
&\lesssim |f(1)|^2 + \int_0^1 |f(s)||f'(s)|s^3 ds \\
&\lesssim \|f\|_{H^1(\mathbb{B}^5)} + \epsilon \int_0^1 |f(s)|^2 s^2 ds + \frac{1}{\epsilon}\int_0^1 |f'(s)|^2 s^4 ds,
\end{aligned}$$

where we have used Lemma 2.2.2. Since this is true for any $\epsilon > 0$, we obtain the desired conclusion. □

Lemma 2.4.8. *We have*

$$\|(\cdot)f\|_{L^5(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$$

for all $f \in C^1([0,1])$.

Proof. We compute

$$\begin{aligned}
\|(\cdot)f\|_{L^5(\mathbb{B}^5)} &= \left(\int_0^1 (|f(\rho)|\rho)^5 \rho^4 d\rho\right)^{\frac{1}{5}} \leq \left(\int_0^1 (|f(\rho)|\rho)^5 \rho^2 d\rho\right)^{\frac{1}{5}} \\
&= \|(\cdot)f\|_{L^5(\mathbb{B}^3)} \lesssim \|(\cdot)f\|_{L^6(\mathbb{B}^3)} \lesssim \|(\cdot)f\|_{H^1(\mathbb{B}^3)} \\
&= \left(\int_0^1 (|f(\rho)|\rho)^2 \rho^2 d\rho\right)^{\frac{1}{2}} + \left(\int_0^1 \left(\frac{d}{d\rho}(f(\rho)\rho)\right)^2 \rho^2 d\rho\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2(\mathbb{B}^5)} + \|f'\|_{L^2(\mathbb{B}^5)} + \|(\cdot)f\|_{L^2(0,1)} \\
&\lesssim \|f\|_{H^1(\mathbb{B}^5)},
\end{aligned}$$

where we used Sobolev embedding in 3 dimensions $H^1 \hookrightarrow L^6(\mathbb{B}^3)$, and Lemma 2.4.7. □

With the extra growth in λ, we no longer have absolute convergence in the λ integral. Hence we must first do an integration by parts in s to obtain the desired decay, and treat the boundary terms separately.

Proposition 2.4.9. *Let* $p \in [2,\infty]$ *and* $q \in [\frac{10}{3}, 5]$ *be such that* $\frac{1}{p} + \frac{5}{q} = \frac{3}{2}$. *For* $n \in \{1,2,\dots,6\}$, *setting* $\dot{T}_n(\tau)f(\rho) := \lim_{\epsilon\to 0+} \dot{T}_{n,\epsilon}(\tau)f(\rho)$, *we have the bound*

$$\|\dot{T}_n(\cdot)f\|_{L^p(\mathbb{R}_+;L^q(\mathbb{B}^5))} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$$

for all $f \in C^1([0,1])$.

Proof. Let $\lambda = \epsilon + i\omega$ where $\epsilon \in [0, \frac{1}{4}]$ and $\omega \in \mathbb{R}$.

For $n = \{1, 2, 3\}$, we have that each G_n is given by

$$G_n(\rho, s; \lambda) = g_n(\rho; \lambda) 1_{\mathbb{R}_+}(s-\rho) s(1-s)^{-\frac{1}{2}+\lambda}(2 + s(-1+2\lambda))\gamma_n(\rho, s; \lambda),$$

where

$$\begin{aligned} g_1(\rho; \lambda) &= \chi(\rho \langle\omega\rangle) \frac{\varphi_0(\rho; \lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)}, \\ g_2(\rho; \lambda) &= [1 - \chi(\rho \langle\omega\rangle)] \frac{\varphi_1(\rho; \lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)}, \\ g_3(\rho; \lambda) &= [1 - \chi(\rho \langle\omega\rangle)] \frac{\tilde{\varphi}_1(\rho; \lambda)}{(3-2\lambda)(1+2\lambda)(1-2\lambda)}. \end{aligned}$$

We note that $|g_n(\rho; \lambda)| \lesssim \rho^{-3} \langle\omega\rangle^{-2}$. Doing an integration by parts on the s-integral, we get

$$\begin{aligned} &\int_0^1 1_{\mathbb{R}_+}(s-\rho) s(1-s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda)) f(s) \gamma_n(\rho, s; \lambda) ds \\ &= -\frac{2}{1+2\lambda} \int_\rho^1 \partial_s (1-s)^{\frac{1}{2}+\lambda} s(2+s(-1+2\lambda)) f(s)\gamma_n(\rho, s; \lambda) ds \\ &= \frac{-2}{1+2\lambda}(1-\rho)^{\frac{1}{2}+\lambda} \rho(2+\rho(-1+2\lambda)) f(\rho)\gamma_n(\rho, \rho; \lambda) \\ &\quad + \frac{2}{1+2\lambda} \int_\rho^1 (1-s)^{\frac{1}{2}+\lambda} \partial_s \left[s(2+s(-1+2\lambda)) f(s) \gamma_n(\rho, s; \lambda) \right] ds. \end{aligned} \tag{2.34}$$

Inserting the integral terms back to $\dot{T}_{n,\epsilon}$ we see that the λ integral is now absolutely convergent, so limits exist. Now

$$\begin{aligned} &\partial_s \left[s(2+s(-1+2i\omega)) f(s)\gamma_n(\rho, s; i\omega) \right] \\ &= (2+s(-1+2i\omega)) s f(s) \partial_s \gamma_n(\rho, s; \lambda) + (2+2s(-1+2i\omega)) f(s)\gamma_n(\rho, s; i\omega) \\ &\quad + f'(s) s(2+s(-1+2i\omega))\gamma_n(\rho, s; i\omega), \end{aligned}$$

and by Lemma 2.3.11 we see that

$$s\partial_s \gamma_n(\rho, s; \lambda) = O(\rho^0 s^0) O_0(\langle\omega\rangle^{-1}) + O(\rho^0 (1-\rho)^0 s^0 (1-s)^0 \langle\omega\rangle^{-2}). \tag{2.35}$$

So we could follow what we did in Proposition 2.4.5. Using $\|(\cdot) f\|_{L^2(0,1)} + \|(\cdot)^2 f'\|_{L^2(0,1)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$, we obtain the desired bound. The boundary terms give rise to operators

$$B_{n,\epsilon}(\tau) f(\rho) := \frac{(1-\rho)^{\frac{1}{2}} \rho f(\rho)}{2\pi i} \lim_{N\to\infty} \int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}(1-\rho)^\lambda (2+\rho(-1+2\lambda))\gamma_n(\rho, \rho; \lambda) g_n(\rho; \lambda) d\lambda.$$

The integrands are bounded by $C\rho^{-3}\langle\omega\rangle^{-2}$, we can take the limits

$$B_n(\tau) f(\rho) := \lim_{\epsilon\to 0} B_{n,\epsilon}(\tau) f(\rho).$$

For B_1 we have

$$B_1(\tau)f(\rho) = \frac{2(1-\rho)^{\frac{1}{2}}\rho f(\rho)}{2\pi}\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-\rho))}\frac{\chi(\rho\langle\omega\rangle)\varphi_0(\rho;i\omega)}{(3-2i\omega)(1-2i\omega)(1+2i\omega)}\gamma_1(\rho,\rho;i\omega)d\omega$$
$$+\frac{2(1-\rho)^{\frac{1}{2}}\rho^2 f(\rho)}{2\pi}\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-\rho))}\frac{\chi(\rho\langle\omega\rangle)\varphi_0(\rho;i\omega)}{(3-2i\omega)(-1-2i\omega)}\gamma_1(\rho,\rho;i\omega)d\omega.$$

For the two terms, we use the two representations (2.28) and (2.27) respectively. Since $\frac{1-2i\omega}{1+2i\omega} = \mathcal{O}_e(\langle\omega\rangle^0) + \mathcal{O}(\langle\omega\rangle^{-1})$, by Lemma 2.4.1, we obtain

$$|B_1(\tau)f(\rho)| \lesssim \rho|f(\rho)|(1-\rho)^{\frac{1}{2}}\langle\tau+\log(1-\rho)\rangle^{-2} \lesssim \rho|f(\rho)|\langle\tau\rangle^{-2}.$$

For B_2 we have

$$B_2(\tau)f(\rho) = \frac{\rho f(\rho)(1-\rho)^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}}{2\pi}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle]e^{i\omega(\tau+\log(1-\rho)-\log(1+\rho))}\rho^{-1}\mathcal{O}(\langle\omega\rangle^{-2})d\omega,$$

and using Lemma 2.4.3 we have

$$|B_2(\tau)f(\rho)| \lesssim \rho|f(\rho)|(1-\rho)^{\frac{1}{2}}\langle\tau+\log(1-\rho)\rangle^{-2} \lesssim \rho|f(\rho)|\langle\tau\rangle^{-2}.$$

Similarly,

$$B_3(\tau)f(\rho) = \frac{\rho f(\rho)(1-\rho)(1+\rho)^{\frac{1}{2}}}{2\pi}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle]e^{i\omega\tau}\rho^{-1}\mathcal{O}(\langle\omega\rangle^{-2})d\omega,$$

so we also have $|B_3(\tau)f(\rho)| \lesssim \rho|f(\rho)|\langle\tau\rangle^{-2}$. Then by Lemma 2.4.8 we obtain the bounds $\|B_n(\tau)f\|_{L^5(\mathbb{B}^5)} \lesssim \langle\tau\rangle^{-2}\|f\|_{H^1(\mathbb{B}^5)}$ for $n = \{1,2,3\}$.

For $n = 4$, we decompose $\dot{T}_{4,\epsilon}$ into two parts according to $\varphi_1(\rho;\lambda)$. We have

$$\begin{aligned}\dot{T}_{4,\epsilon}(\tau)f(\rho) &= \frac{1}{4\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{e^{\lambda\tau}\varphi_1(\rho;\lambda)}{(3-2\lambda)(1-2\lambda)} \\ &\quad\times\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^4(1-s^2)^{-\frac{1}{2}+\lambda}\varphi_0(s;\lambda)\gamma_4(\rho,s;\lambda)f(s)dsd\lambda \\ &= I_{1,\epsilon}(\tau,\rho) + I_{2,\epsilon}(\tau,\rho),\end{aligned}$$

where

$$\begin{aligned}I_{1,\epsilon}(\tau,\rho) &= \frac{\rho^{-3}(1+\rho)^{\frac{1}{2}}}{4\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{(1+\rho)^{-\lambda}e^{\lambda\tau}}{(3-2\lambda)(1-2\lambda)} \\ &\quad\times\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^4(1-s^2)^{-\frac{1}{2}+\lambda}\varphi_0(s;\lambda)\gamma_4(\rho,s;\lambda)f(s)dsd\lambda, \\ I_{2,\epsilon}(\tau,\rho) &= \frac{\rho^{-2}(1+\rho)^{\frac{1}{2}}}{4\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{(1+\rho)^{-\lambda}e^{\lambda\tau}}{(3-2\lambda)} \\ &\quad\times\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^4(1-s^2)^{-\frac{1}{2}+\lambda}\varphi_0(s;\lambda)\gamma_4(\rho,s;\lambda)f(s)dsd\lambda.\end{aligned}$$

The integrals in I_1 are already absolutely convergent, so we can take the limit and use Fubini,

$$\begin{aligned}I_1(\tau,\rho) = \lim_{\epsilon\to 0+} I_{1,\epsilon}(\rho) = \rho^{-3}(1+\rho)^{\frac{1}{2}} \int_0^\rho s^4(1-s^2)^{-\frac{1}{2}} f(s) \\ \times \int_{\mathbb{R}} \frac{e^{i\omega(\tau-\log(1+\rho)+\log(1-s^2))}}{(3-2i\omega)(1-2i\omega)} \chi(s\langle\omega\rangle)\varphi_0(s;i\omega)\gamma_4(\rho,s;i\omega)d\omega ds.\end{aligned}$$

This is similar to the G_4 case in Proposition 2.4.5. We use the expression (2.27) for $\varphi_0(s;i\omega)$ and obtain

$$\|I_1(\tau,\cdot)\|_{L^5(\mathbb{B}^5)} \lesssim \|(\cdot)f\|_{L^2(0,1)} \langle\tau\rangle^{-2} \lesssim \|f\|_{H^1(\mathbb{B}^5)} \langle\tau\rangle^{-2}.$$

For $I_{2,\epsilon}$, the s-dependent part is given by

$$1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s\left[(1-s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda)) - (1+s)^{-\frac{1}{2}+\lambda}(2+s(1-2\lambda))\right]\gamma_4(\rho,s;\lambda).$$

Now we write

$$\begin{aligned}&(1-s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda)) - (1+s)^{-\frac{1}{2}+\lambda}(2+s(1-2\lambda)) \\ &= 2s^2\partial_s\left[\frac{(1+s)^{\frac{1}{2}+\lambda}-(1-s)^{\frac{1}{2}+\lambda}}{s}\right]\end{aligned}$$

and do an integration by parts in the s integral. We obtain

$$\begin{aligned}&\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)f(s)\gamma_4(\rho,s;\lambda) \\ &\quad\times s\left[(1-s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda)) - (1+s)^{-\frac{1}{2}+\lambda}(2+s(1-2\lambda))\right]ds \\ &= 2\int_0^\rho \chi(s\langle\omega\rangle)s^3 f(s)\gamma_4(\rho,s;\lambda)\partial_s\left[\frac{(1+s)^{\frac{1}{2}+\lambda}-(1-s)^{\frac{1}{2}+\lambda}}{s}\right]ds \\ &= \chi(\rho\langle\omega\rangle)\rho^2 f(\rho)\gamma_4(\rho,\rho;\lambda)\left((1+\rho)^{\frac{1}{2}+\lambda}-(1-\rho)^{\frac{1}{2}+\lambda}\right) \\ &\quad+\int_0^\rho \frac{(1+s)^{\frac{1}{2}+\lambda}-(1-s)^{\frac{1}{2}+\lambda}}{s}\partial_s\left[\chi(s\langle\omega\rangle)s^3 f(s)\gamma_4(\rho,s;\lambda)\right]ds.\end{aligned}$$

For the integral term, reinserting back into $I_{2,\epsilon}$, we see that the λ integrals are now absolutely convergent, so we can take the limits and use Fubini. When the derivative does not fall on the cut-off $\chi(s\langle\omega\rangle)$, notice that

$$\partial_s\left[s^3 f(s)\gamma_4(\rho,s;\lambda)\right] = 3s^2 f(s)\gamma_4(\rho,s;\lambda) + s^3 f'(s)\gamma_4(\rho,s;\lambda) + s^3 f(s)\partial_s\gamma_4(\rho,s;\lambda),$$

and again we use that $s\partial_s\gamma_4(\rho,s;\lambda)$ has the form as in (2.35). Moreover, we write

$$\frac{(1+s)^{\frac{1}{2}+i\omega}-(1-s)^{\frac{1}{2}+i\omega}}{s} = \frac{1}{2}(1+2i\omega)\int_{-1}^1 (1+st)^{-\frac{1}{2}+i\omega}dt. \tag{2.36}$$

Now $\frac{1+2i\omega}{3-2i\omega} = \mathcal{O}_e(\langle\omega\rangle^0) + \mathcal{O}(\langle\omega\rangle^{-1})$, so we can proceed as in the G_4 case in Proposition 2.4.5 to obtain the bound by $\|(\cdot)f\|_{L^2(0,1)} + \|(\cdot)^2 f'\|_{L^2(0,1)}$. When the derivative falls on the cut-off, we have to look at the term

$$\rho^{-2}(1+\rho)^{\frac{1}{2}} s^2 (1\pm s)^{\frac{1}{2}} f(s) \int_{\mathbb{R}} e^{i\omega(\tau-\log(1+\rho)+\log(1\pm s))} \chi'(s\langle\omega\rangle)\mathcal{O}(\langle\omega\rangle^{-1}) d\omega.$$

Since $\rho \geq s$, and on the cut-off we have also $s\langle\omega\rangle \simeq 1$, so

$$\begin{aligned}
&\rho^{-2}(1+\rho)^{\frac{1}{2}} s^2 (1\pm s)^{\frac{1}{2}} |f(s)| \left| \int_{\mathbb{R}} e^{i\omega(\tau-\log(1+\rho)+\log(1\pm s))} \chi'(s\langle\omega\rangle)\mathcal{O}(\langle\omega\rangle^{-1}) d\omega \right| \\
&\lesssim \frac{s^2}{\rho^2} |f(s)| \left| s^{-1} \int_{\mathbb{R}} e^{i\omega(\tau-\log(1+\rho)+\log(1\pm s))} \chi'(s\langle\omega\rangle)\mathcal{O}(\langle\omega\rangle^{-2}) d\omega \right| \\
&\lesssim \frac{s^2}{\rho^2} |f(s)| \langle \tau + \log(1-s)\rangle^{-2} \lesssim \frac{s^2}{\rho^2} (1-s)^{-\frac{1}{2}} |f(s)| \langle \tau + \log(1-s)\rangle^{-2}
\end{aligned}$$

using the same logic as in the proof of Lemma 2.4.3. Then again we obtain the bound $\|(\cdot)f\|_{L^2(0,1)}$ when taking the $L^2(\mathbb{R}_+; L^5(\mathbb{B}^5))$ norm. On the other hand, for the boundary term we can again take the limit

$$\begin{aligned}
&B_4(\tau)f(\rho) \\
&:= \frac{(1+\rho)^{\frac{1}{2}} f(\rho)}{4\pi} \lim_{\epsilon\to 0+} \lim_{N\to\infty} \int_{\epsilon-iN}^{\epsilon+iN} \frac{e^{\lambda(\tau+\log(1+\rho))}}{(3-2\lambda)} \\
&\qquad\qquad \times \chi(\rho\langle\omega\rangle)\gamma_4(\rho,\rho;\lambda) \left((1+\rho)^{\frac{1}{2}+\lambda} - (1-\rho)^{\frac{1}{2}+\lambda} \right) d\lambda \\
&= \frac{(1+\rho)^{\frac{1}{2}} f(\rho)}{4\pi} \int_{\mathbb{R}} \frac{e^{i\omega(\tau+\log(1+\rho))}}{(3-2i\omega)} \chi(\rho\langle\omega\rangle)\gamma_4(\rho,\rho;i\omega) \left((1+\rho)^{\frac{1}{2}+i\omega} - (1-\rho)^{\frac{1}{2}+i\omega} \right) d\omega.
\end{aligned}$$

By (2.36) and Lemma 2.4.1 we obtain $|B_4(\tau)f(\rho)| \lesssim \rho|f(\rho)|\langle\tau\rangle^{-2}$, and again Lemma 2.4.8 gives that $\|B_4(\tau)f\|_{L^5(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}\langle\tau\rangle^{-2}$.

For $n = \{5, 6\}$, the s dependent part is

$$1_{\mathbb{R}_+}(\rho - s)[1-\chi(s\langle\omega\rangle)] s(1\pm s)^{-\frac{1}{2}+\lambda}(2 \pm s(1-2\lambda)\gamma_n(\rho, s;\lambda),$$

while the s independent part splits into two terms,

$$\frac{e^{\lambda\tau}\varphi_1(\rho;\lambda)}{(3-2\lambda)(1-2\lambda)} = \frac{2e^{\lambda\tau}\rho^{-3}(1+\rho)^{\frac{1}{2}-\lambda}}{(3-2\lambda)(1-2\lambda)} - \frac{e^{\lambda\tau}\rho^{-3}(1+\rho)^{\frac{1}{2}-\lambda}}{(3-2\lambda)}.$$

Again the first part already has decay $\langle\omega\rangle^{-2}$, so we can follow what we did for G_5 and G_6 in Proposition 2.4.5 to obtain the bounds by $\|(\cdot)f\|_{L^2(0,1)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$. For the second part we do an integration by parts in s,

$$\begin{aligned}
&\int_0^1 1_{\mathbb{R}_+}(\rho - s)(1\pm s)^{-\frac{1}{2}+\lambda}[1-\chi(s\langle\omega\rangle)] s(2\pm s(1-2\lambda)) f(s)\gamma_n(\rho,s;\lambda) ds \\
&= \frac{\pm 2}{1+2\lambda} \int_0^\rho \partial_s (1\pm s)^{\frac{1}{2}+\lambda}[1-\chi(s\langle\omega\rangle)] s(2\pm s(1-2\lambda)) f(s)\gamma_n(\rho,s;\lambda) ds \\
&= \frac{\pm 2}{1+2\lambda} (1\pm\rho)^{\frac{1}{2}+\lambda}[1-\chi(\rho\langle\omega\rangle)]\rho(2\pm\rho(1-2\lambda)) f(\rho)\gamma_n(\rho,\rho;\lambda) \\
&\quad - \frac{\pm 2}{1+2\lambda} \int_0^\rho (1\pm s)^{\frac{1}{2}+\lambda} \partial_s \left[[1-\chi(s\langle\omega\rangle)] s(2\pm s(1-2\lambda)) f(s)\gamma_n(\rho,s;\lambda) \right] ds. \qquad (2.37)
\end{aligned}$$

Using that $s\partial_s \gamma_n(\rho, s; \lambda)$ has the form (2.35), the integral part is treated the same way as before and is bounded by $\|(\cdot)f\|_{L^2(0,1)} + \|(\cdot)^2 f'\|_{L^2(0,1)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$. The boundary term is given by

$$
\begin{aligned}
&B_n(\tau)f(\rho)\\
&= \frac{(1+\rho)^{\frac{1}{2}}(1\pm\rho)^{\frac{1}{2}} f(\rho)}{4\pi} \lim_{\epsilon\to 0+} \lim_{N\to\infty} \int_{\epsilon - iN}^{\epsilon+iN} e^{\lambda(\tau - \log(1+\rho)+\log(1\pm\rho))}[1-\chi(\rho\langle\omega\rangle)]\mathcal{O}(\rho^0 \langle\omega\rangle^{-2})d\lambda\\
&= \frac{(1+\rho)^{\frac{1}{2}}(1\pm\rho)^{\frac{1}{2}} f(\rho)}{4\pi} \int_{\mathbb{R}} e^{i\omega(\tau - \log(1+\rho)+\log(1\pm\rho))}[1-\chi(\rho\langle\omega\rangle)]\mathcal{O}(\rho^0 \langle\omega\rangle^{-2})d\omega.
\end{aligned}
$$

So Lemma 2.4.3 gives that

$$
|B_n(\tau)f(\rho)| \lesssim \rho|f(\rho)|(1\pm\rho)^{\frac{1}{2}} \langle \tau - \log(1+\rho) + \log(1\pm\rho)\rangle^{-2} \lesssim \rho|f(\rho)|\langle\tau\rangle^{-2},
$$

and as before we have $\|B_n(\tau)f\|_{L^5(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}\langle\tau\rangle^{-2}$. □

2.4.4 Strichartz estimates

Theorem 2.4.10. *Let $p \in [2,\infty]$ and $q \in [\frac{10}{3}, 5]$ such that $\frac{1}{p} + \frac{5}{q} = \frac{3}{2}$. Then we have the bound*

$$
\|[\mathbf{S}(\cdot)(\mathbf{I}-\mathbf{P})\mathbf{f}]_1\|_{L^p(\mathbb{R}_+, L^q(\mathbb{B}^5))} \lesssim \|(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}
$$

for all $\mathbf{f} \in \mathcal{H}$. And hence we also have

$$
\left\| \int_0^\tau [\mathbf{S}(\tau-\sigma)(\mathbf{I}-\mathbf{P})\mathbf{h}(\sigma,\cdot)]_1 \, d\sigma \right\|_{L^p_\tau(\mathbb{R}_+; L^q(\mathbb{B}^5))} \lesssim \|(\mathbf{I}-\mathbf{P})\mathbf{h}\|_{L^1(\mathbb{R}_+,\mathcal{H})}
$$

or all $\mathbf{h} \in C([0,\infty),\mathcal{H}) \cap L^1(\mathbb{R}_+,\mathcal{H})$

Proof. From Equation (2.26), we have

$$
[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1 = [\mathbf{S}_0(\tau)\tilde{\mathbf{f}}]_1(\rho) + \sum_{n=1}^{6} \left[T_n(\tau)\left(|\cdot|\tilde{f}_1' + 2\tilde{f}_1 + \tilde{f}_2\right) + \dot{T}_n(\tau)\tilde{f}_1\right].
$$

Then by using Proposition 2.2.4, 2.4.6, 2.4.9, the stated result follows by the same argument as [8, Theorem 4.1]. □

2.5 Improved energy estimates

We also need to take $\epsilon \to 0+$ in Lemma 2.2.9. But since the constant depends on ϵ, we cannot directly do this. Instead, we show explicitly in this section that the constant C_ϵ can be chosen independent of ϵ, allowing us to take $\epsilon = 0$.

2.5.1 Preliminaries

We first introduce the following result which allows us to interchange a limit and an integral, which is not absolutely convergent. This is in the same spirit as [8, Lemma 5.1].

Lemma 2.5.1. *Let f be a function of two variables $\epsilon \in [0, \frac{1}{4})$ and $\omega \in \mathbb{R}$ satisfying the properties that*

- *$f(\epsilon, \cdot)$ is an odd function for every fixed ϵ ,*
- *$f \in C^1([0, \frac{1}{4}) \times \mathbb{R})$,*
- *$|f(\epsilon, \omega)| \leq C_1 \langle \omega \rangle^{-1}$ and $|\partial_\omega f(\epsilon, \omega)| \leq C_2 \langle \omega \rangle^{-2}$ with C_1, C_2 constants independent of ϵ.*

Then

$$\lim_{\epsilon \to 0+} \int_{\mathbb{R}} e^{ia\omega} f(\epsilon, \omega) d\omega = \int_{\mathbb{R}} e^{ia\omega} f(0, \omega) d\omega$$

for all $a \in \mathbb{R} \backslash \{0\}$.

Proof. We do an integration by parts,

$$\int_{\mathbb{R}} e^{ia\omega} f(\epsilon, \omega) d\omega = -\frac{1}{ia} \int_{\mathbb{R}} e^{ia\omega} \partial_\omega f(\epsilon, \omega) d\omega.$$

This integral is now absolutely convergent, so we can take the limit in ϵ by dominated convergence, then undo the integration by parts,

$$\begin{aligned} \lim_{\epsilon \to 0+} \int_{\mathbb{R}} e^{ia\omega} f(\epsilon, \omega) d\omega &= \lim_{\epsilon \to 0+} -\frac{1}{ia} \int_{\mathbb{R}} e^{ia\omega} \partial_\omega f(\epsilon, \omega) d\omega \\ &= -\frac{1}{ia} \int_{\mathbb{R}} e^{ia\omega} \partial_\omega f(0, \omega) d\omega \\ &= \int_{\mathbb{R}} e^{ia\omega} f(0, \omega) d\omega, \end{aligned}$$

as stated. □

2.5.2 Decomposition

Let $\tilde{\mathbf{f}} \in \operatorname{rg}(\mathbf{I} - \mathbf{P}) \cap C^2 \times C^1([0, 1])$, we have

$$\partial_\rho [\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1(\rho) = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\epsilon - iN}^{\epsilon - iN} e^{\lambda\tau} \int_0^1 G'(\rho, s; \lambda) F_\lambda(s) ds,$$

where

$$G'(\rho, s; \lambda) = \frac{s^4 (1 - s^2)^{-\frac{1}{2} + \lambda}}{(3 - 2\lambda)(1 + 2\lambda)(1 - 2\lambda) w_0(\lambda)} \begin{cases} \partial_\rho u_0(\rho; \lambda) u_1(s; \lambda) \text{ if } \rho \leq s \\ \partial_\rho u_1(\rho; \lambda) u_0(s; \lambda) \text{ if } \rho \geq s \end{cases}$$

and $F_\lambda(s) = s\tilde{f}_1'(s) + \left(\lambda + \frac{5}{2}\right) \tilde{f}_1(s) + \tilde{f}_2(s)$. The free part is given by

$$G_0'(\rho, s; \lambda) = \frac{s^4 (1 - s^2)^{-\frac{1}{2} + \lambda}}{(3 - 2\lambda)(1 + 2\lambda)(1 - 2\lambda)} \begin{cases} \partial_\rho \varphi_0(\rho; \lambda) \varphi_1(s; \lambda) \text{ if } \rho \leq s \\ \partial_\rho \varphi_1(\rho; \lambda) \varphi_0(s; \lambda) \text{ if } \rho \geq s \end{cases} .$$

Lemma 2.5.2. *We have the decomposition*

$$G'(\rho,s;\lambda)=G_0'(\rho,s;\lambda)+\sum_{n=1}^{6}\rho^{-1}\tilde{G}_n(\rho,s;\lambda)+\sum_{n=1}^{6}G_n'(\rho,s;\lambda),$$

where $\tilde{G}_n$ *has the same form as in Lemma 2.3.11, and*

$$\begin{aligned}
G_1'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(s-\rho)\chi(\rho\langle\omega\rangle)\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}\left(g_1(\rho;\lambda)-\tilde{g}_1(\rho;\lambda)\right)\varphi_1(s;\lambda)\gamma_1'(\rho,s;\lambda),\\
G_2'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(s-\rho)[1-\chi(\rho\langle\omega\rangle)]\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}g_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_2'(\rho,s;\lambda),\\
G_3'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(s-\rho)[1-\chi(\rho\langle\omega\rangle)]\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}\tilde{g}_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_3'(\rho,s;\lambda),\\
G_4'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}g_1(\rho;\lambda)\varphi_0(s;\lambda)\gamma_4'(\rho,s;\lambda),\\
G_5'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(\rho-s)[1-\chi(s\langle\omega\rangle)]\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}g_1(\rho;\lambda)\varphi_1(s;\lambda)\gamma_5'(\rho,s;\lambda),\\
G_6'(\rho,s;\lambda)&=1_{\mathbb{R}_+}(\rho-s)[1-\chi(s\langle\omega\rangle)]\frac{s^4(1-s^2)^{-\frac{1}{2}+\lambda}}{1+2\lambda}g_1(\rho;\lambda)\tilde{\varphi}_1(s;\lambda)\gamma_6'(\rho,s;\lambda),
\end{aligned}$$

with

$$\begin{aligned}
g_1(\rho;\lambda)&=\rho^{-2}(1+\rho)^{-\frac{1}{2}-\lambda},\\
\tilde{g}_1(\rho;\lambda)&=\rho^{-2}(1-\rho)^{-\frac{1}{2}-\lambda},
\end{aligned}$$

and

$$\gamma_n'(\rho,s;\lambda)=\mathcal{O}(\rho^0s^0)\mathcal{O}_0(\langle\omega\rangle^{-1})+\mathcal{O}((1-\rho)^0s^0\langle\omega\rangle^{-2})+\mathcal{O}(\rho^0(1-s)\langle\omega\rangle^{-2}).$$

Proof. First of all, differentiating explicitly we have

$$\begin{aligned}
\partial_\rho\varphi_1(\rho;\lambda)&=-3\rho^{-1}\varphi_1(\rho;\lambda)-\frac{1}{2}(1-2\lambda)(3-2\lambda)\rho^{-2}(1+\rho)^{-\frac{1}{2}-\lambda},\\
\partial_\rho\tilde{\varphi}_1(\rho;\lambda)&=-3\rho^{-1}\tilde{\varphi}_1(\rho;\lambda)-\frac{1}{2}(1-2\lambda)(3-2\lambda)\rho^{-2}(1-\rho)^{-\frac{1}{2}-\lambda},\\
\partial_\rho\varphi_0(\rho;\lambda)&=\partial_\rho\varphi_1(\rho;\lambda)-\partial_\rho\tilde{\varphi}_1(\rho;\lambda)\\
&=-3\rho^{-1}\varphi_0(\rho;\lambda)-\frac{1}{2}(1-2\lambda)(3-2\lambda)\rho^{-2}\left((1+\rho)^{-\frac{1}{2}-\lambda}-(1-\rho)^{-\frac{1}{2}-\lambda}\right).
\end{aligned}$$

Now, using Equation (2.24) we have

$$
\begin{aligned}
&\partial_\rho u_1(\rho;\lambda)\\
&= \partial_\rho\varphi_1(\rho;\lambda)[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})]\\
&\quad+\varphi_1(\rho;\lambda)[\mathcal{O}(\rho^{-1})\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^{-1}(1-\rho)^0\langle\omega\rangle^{-2})]\\
&= \partial_\rho\varphi_1(\rho;\lambda)+\left(-3\rho^{-1}\varphi_1(\rho;\lambda)-\frac{1}{2}(1-2\lambda)(3-2\lambda)\rho^{-2}(1+\rho)^{-\frac{1}{2}-\lambda}\right)\\
&\quad\times[\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})]\\
&\quad+\rho^{-1}\varphi_1(\rho;\lambda)[\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})]\\
&= \partial_\rho\varphi_1(\rho;\lambda)+\rho^{-1}\varphi_1(\rho;\lambda)[\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})]\\
&\quad-(1-2\lambda)(3-2\lambda)\rho^{-2}(1+\rho)^{-\frac{1}{2}-\lambda}[\mathcal{O}(\rho^0)O_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})].
\end{aligned}
$$

From Lemma 2.3.4, we have

$$
\begin{aligned}
&\chi(\rho\langle\omega\rangle)\partial_\rho u_0(\rho;\lambda)\\
&= \chi(\rho\langle\omega\rangle)\partial_\rho\varphi_0(\rho;\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]+\chi(\rho\langle\omega\rangle)\varphi_0(\rho;\lambda)\mathcal{O}(\rho\langle\omega\rangle)\\
&= \chi(\rho\langle\omega\rangle)\partial_\rho\varphi_0(\rho;\lambda)+\chi(\rho\langle\omega\rangle)\rho^{-1}\varphi_0(\rho;\lambda)\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\\
&\quad-\chi(\rho\langle\omega\rangle)\frac{1}{2}(1-2\lambda)(3-2\lambda)\rho^{-2}\left((1+\rho)^{-\frac{1}{2}-\lambda}-(1-\rho)^{-\frac{1}{2}-\lambda}\right)\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\\
&= \chi(\rho\langle\omega\rangle)\partial_\rho\varphi_0(\rho;\lambda)+\chi(\rho\langle\omega\rangle)\rho^{-1}\varphi_0(\rho;\lambda)\mathcal{O}(\rho^0\langle\omega\rangle^{-2})\\
&\quad-\chi(\rho\langle\omega\rangle)(1-2\lambda)(3-2\lambda)\rho^{-2}\left((1+\rho)^{-\frac{1}{2}-\lambda}-(1-\rho)^{-\frac{1}{2}-\lambda}\right)\mathcal{O}(\rho^0\langle\omega\rangle^{-2}).
\end{aligned}
$$

From Lemma 2.3.8, we have

$$
\begin{aligned}
&[1-\chi(\rho\langle\omega\rangle)]\partial_\rho u_0(\rho;\lambda)\\
&= [1-\chi(\rho\langle\omega\rangle)]\partial_\rho\varphi_1(\rho;\lambda)[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})+\mathcal{O}(\langle\omega\rangle^{-2})]\\
&\quad+[1-\chi(\rho\langle\omega\rangle)]\varphi_1(\rho;\lambda)[\mathcal{O}(\rho^{-1})\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^{-1}(1-\rho)^0\langle\omega\rangle^{-2})]\\
&\quad-[1-\chi(\rho\langle\omega\rangle)]\partial_\rho\tilde{\varphi}_1(\rho;\lambda)[1+\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-2})+\mathcal{O}(\langle\omega\rangle^{-2})]\\
&\quad-[1-\chi(\rho\langle\omega\rangle)]\tilde{\varphi}_1(\rho;\lambda)[\mathcal{O}(\rho^{-1})\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^{-1}(1-\rho)^0\langle\omega\rangle^{-2})]\\
&= [1-\chi(\rho\langle\omega\rangle)]\partial_\rho\varphi_0(\rho;\lambda)\\
&\quad+[1-\chi(\rho\langle\omega\rangle)]\rho^{-1}\varphi_1(\rho;\lambda)[\mathcal{O}(\rho^0)O_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})]\\
&\quad-[1-\chi(\rho\langle\omega\rangle)]\rho^{-1}\tilde{\varphi}_1(\rho;\lambda)[\mathcal{O}(\rho^0)O_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})]\\
&\quad+[1-\chi(\rho\langle\omega\rangle)](1-2\lambda)(3-2\lambda)\rho^{-2}(1+\rho)^{-\frac{1}{2}-\lambda}[\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})]\\
&\quad-[1-\chi(\rho\langle\omega\rangle)](1-2\lambda)(3-2\lambda)\rho^{-2}(1-\rho)^{-\frac{1}{2}-\lambda}[\mathcal{O}(\rho^0)\mathcal{O}_o(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^0\langle\omega\rangle^{-2})].
\end{aligned}
$$

Combining with (2.24) and (2.25), we obtain the desired decomposition. □

Now for $n\in\{1,2,\dots,6\}$, $\tau>0$, $\rho\in(0,1)$, and $f\in C^1([0,1])$, we define operators

$$
\begin{aligned}
\tilde{S}_{n,\epsilon}(\tau)f(\rho) &:= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}e^{\lambda\tau}\int_0^1\rho^{-1}\tilde{G}_n(\rho,s;\lambda)f(s)dsd\lambda\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}e^{(\epsilon+i\omega)\tau}\int_0^1\rho^{-1}\tilde{G}_n(\rho,s;\epsilon+i\omega)f(s)dsd\omega.
\end{aligned}
$$

Then by dominated convergence and Fubini, we have

$$\tilde{S}_n(\tau)f(\rho) := \lim_{\epsilon\to 0+} \tilde{S}_{n,\epsilon}(\tau)f(\rho) = \frac{1}{2\pi}\int_0^1 f(s)\int_{\mathbb{R}} e^{i\omega\tau}\rho^{-1}\tilde{G}_n(\rho,s;i\omega)d\omega ds.$$

On the other hand, for $n = \{1,2,\cdots,6\}$, $\tau > 0$, $\rho \in (0,1)$, and $f \in C^1([0,1])$, we define

$$\begin{aligned}S'_{n,\epsilon}(\tau)f(\rho) &= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}\int_0^1 G'_n(\rho,s;\lambda)f(s)dsd\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\epsilon+i\omega)\tau}\int_0^1 G'_n(\rho,s;\epsilon+i\omega)f(s)dsd\omega,\end{aligned}$$

and we also want to take $\epsilon \to 0+$. Indeed, by the same argument as [8, Lemma 5.3], we have

$$\begin{aligned}S'_n(\tau)f(\rho) := \lim_{\epsilon\to 0+} S'_{n,\epsilon}(\tau)f(\rho) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega\tau}\int_0^1 G'_n(\rho,s;i\omega)f(s)dsd\omega \\ &= \frac{1}{2\pi}\int_0^1 f(s)\int_{\mathbb{R}} e^{i\omega\tau}G'_n(\rho,s;i\omega)d\omega ds.\end{aligned}$$

Our goal now is to obtain bounds for $\|\tilde{S}_n(\tau)f\|_{L^2(\mathbb{B}^5)}$ and $\|S'_n(\tau)f\|_{L^2(\mathbb{B}^5)}$.

2.5.3 Kernel estimates

We have bounds for the kernel of the operators $\tilde{S}_n$.

Lemma 2.5.3. *We have*

$$\left|\int_{\mathbb{R}} e^{i\omega\tau}\tilde{G}_n(\rho,s;i\omega)d\omega\right| \lesssim \rho^{-1}s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2},$$

for all $\tau > 0$, $\rho, s \in (0,1)$, and $n \in \{1,2,\ldots,6\}$.

Proof. Since the form of $\tilde{G}_n$ is the same as G_n, the treatment here is similar to Proposition 2.4.5. For $\tilde{G}_1$, we have

$$\int_{\mathbb{R}} e^{i\omega\tau}\tilde{G}_1(\rho,s;\omega)d\omega = 2I_1(\rho,s,\tau) + I_2(\rho,s\tau)$$

where

$$\begin{aligned}I_1(\rho,s,\tau) &= 1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(1+2i\omega)(1-2i\omega)}\tilde{\gamma}_1(\rho,s;i\omega)d\omega, \\ I_2(\rho,s,\tau) &= 1_{\mathbb{R}_+}(s-\rho)s^2(1-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(-1-2i\omega)}\tilde{\gamma}_1(\rho,s;i\omega)d\omega.\end{aligned}$$

From the proof of Proposition 2.4.5 we already see that

$$\begin{aligned}|I_1(\rho,s,\tau)| &\lesssim 1_{\mathbb{R}_+}(s-\rho)s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\ &\le \rho^{-1}s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}.\end{aligned}$$

For I_2, we use (2.27), and as before we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \chi(\rho\langle\omega\rangle)\frac{e^{i\omega(\tau+\log(1-s))}\varphi_0(\rho;i\omega)}{(3-2i\omega)(-1-2i\omega)}\tilde{\gamma}_1(\rho,s;i\omega)d\omega\right| \\
&\cong \left|\rho^{-1}\int_{-1}^{1}(1+\rho t)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s)-\log(1-\rho t))}\frac{1-2i\omega}{1+2i\omega}\tilde{\gamma}_1(\rho,s;i\omega)d\omega dt\right| \\
&\leq \rho^{-1}\langle\tau+\log(1-s)\rangle^{-2},
\end{aligned}$$

This gives the bound

$$|I_2(\rho,s,\tau)| \lesssim \rho^{-1}s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}$$

For $\tilde{G}_2$, we do the exact same thing in G_2 but we stop at (2.31), then we obtain the desired bound. Similarly, $\tilde{G}_3$ is exactly the same as G_3, where we use (2.32).

For $n=\{4,5,6\}$, by the same method as the case of G_n in Proposition 2.4.5, we have the bounds as in (2.33),

$$\begin{aligned}
\left|\int_{\mathbb{R}} e^{i\omega\tau}\tilde{G}_n(\rho,s;i\omega)d\omega\right| &\lesssim 1_{\mathbb{R}_+}(\rho-s)\rho^{-2}s^3(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\leq \rho^{-1}s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2}
\end{aligned}$$

as desired. □

Similarly, we have the following result for S_n'.

Lemma 2.5.4. *We have*

$$\left|\int_{\mathbb{R}} e^{i\omega\tau}G_n'(\rho,s;i\omega)d\omega\right| \lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}$$

for all $\tau>0$, $\rho,s\in(0,1)$, and $n\in\{1,2,\dots,6\}$.

Proof. The logic is still very similar to the proof of Proposition 2.4.5. For G_1', we have

$$\int_{\mathbb{R}} e^{i\omega\tau}G_1'(\rho,s;i\omega)d\omega = 2I_1(\rho,s,\tau)+I_2(\rho,s,\tau),$$

where

$$\begin{aligned}
I_1(\rho,s,\tau) &= 1_{\mathbb{R}+}(s-\rho)s(1-s)^{-\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s))}\frac{g_1(\rho;i\omega)-\tilde{g}_1(\rho;i\omega)}{1+2i\omega}\gamma_1'(\rho,s;i\omega)d\omega, \\
I_2(\rho,s,\tau) &= 1_{\mathbb{R}+}(s-\rho)s^2(1-s)^{-\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s))}\left(g_1(\rho;i\omega)-\tilde{g}_1(\rho;i\omega)\right)\frac{-1+2i\omega}{1+2i\omega}\gamma_1'(\rho,s;i\omega)d\omega.
\end{aligned}$$

For I_1, we use the expression

$$g_1(\rho;i\omega)-\tilde{g}_1(\rho;i\omega) = -\frac{1}{2\rho}(1+2i\omega)\int_{-1}^{1}(1+\rho t)^{-\frac{3}{2}-i\omega}dt. \tag{2.38}$$

Then, again thanks to the cut-off, we have

$$\begin{aligned}
&|I_1(\rho,s,\tau)| \\
&\le 1_{\mathbb{R}+}(s-\rho)\rho^{-1}s(1-s)^{-\frac{1}{2}} \\
&\quad\times \int_{-1}^{1}\left|(1+\rho t)^{-\frac{3}{2}}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau+\log(1-s)-\log(1+\rho t))}\frac{-1+2i\omega}{1+2i\omega}\gamma_1'(\rho,s;i\omega)d\omega\right|dt \\
&\lesssim 1_{\mathbb{R}+}(s-\rho)\rho^{-1}s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}
\end{aligned}$$

by Lemma 2.4.1. For I_2, we need to bound the expression

$$\begin{aligned}
I_{2,\pm}(\rho,s,\tau) &= 1_{\mathbb{R}+}(s-\rho)\rho^{-2}(1\pm\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau-\log(1\pm\rho)+\log(1-s))}\frac{-1+2i\omega}{1+2i\omega}\gamma_1'(\rho,s;i\omega)d\omega.
\end{aligned}$$

Again by Lemma 2.4.1 we have

$$|I_2(\rho,s,\tau)| \lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.$$

For G_2', the terms we need to bound are

$$\begin{aligned}
I_1(\rho,s,\tau) &= 1_{\mathbb{R}+}(s-\rho)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s(1-s)^{-\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1-s))}\frac{\gamma_2'(\rho,s;i\omega)}{1+2i\omega}d\omega
\end{aligned}$$

and

$$\begin{aligned}
I_2(\rho,s,\tau) &= 1_{\mathbb{R}+}(s-\rho)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}} \\
&\quad\times\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1-s))}\frac{-1+2i\omega}{1+2i\omega}\gamma_2'(\rho,s;i\omega)d\omega.
\end{aligned}$$

For I_1, notice that

$$\frac{\gamma_2'(\rho,s;i\omega)}{1+2i\omega} = \mathcal{O}(\rho^0(1-\rho)^0s^0(1-s)^0\langle\omega\rangle^{-2}),$$

we use Lemma 2.4.3 and obtain

$$\begin{aligned}
|I_1(\rho,s,\tau)| &\lesssim 1_{\mathbb{R}+}(s-\rho)\rho^{-1}(1+\rho)^{-\frac{1}{2}}s(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
\end{aligned}$$

For I_2, by Lemma 2.4.1 we have

$$\begin{aligned}
|I_2(\rho,s,\tau)| &\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau+\log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
\end{aligned}$$

Next, G_3' is the same as G_2' with $(1-\rho)$ instead of $(1+\rho)$, and we obtain the same bound.

For G_4', we first write

$$
\begin{aligned}
&s^4(1-s^2)^{-\frac{1}{2}+\lambda}\varphi_0(s;\lambda)\\
&= s\left((1-s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda)) - (1+s)^{-\frac{1}{2}+\lambda}(2+s(-1+2\lambda))\right)\\
&= s\left((1-s)^{-\frac{1}{2}+\lambda}(2(1-s)+s(1+2\lambda)) - (1+s)^{-\frac{1}{2}+\lambda}(2(1+s)-s(1+2\lambda))\right)\\
&= 2s\left((1-s)^{\frac{1}{2}+\lambda} - (1+s)^{\frac{1}{2}+\lambda}\right) + s^2(1+2\lambda)\left((1-s)^{-\frac{1}{2}+\lambda} + (1+s)^{-\frac{1}{2}+\lambda}\right). \qquad (2.39)
\end{aligned}
$$

Then, we write

$$
\int_{\mathbb{R}} e^{i\omega\tau} G_4'(\rho,s;i\omega)d\omega = 2I_1(\rho,s,\tau) + I_2(\rho,s,\tau),
$$

where

$$
\begin{aligned}
I_1(\rho,s,\tau) &= 1_{\mathbb{R}_+}(\rho-s)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s\\
&\quad\times \int_{\mathbb{R}} \chi(s\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho))}\frac{\left((1-s)^{\frac{1}{2}+i\omega} - (1+s)^{\frac{1}{2}+i\omega}\right)}{1+2i\omega}\gamma_4'(\rho,s;i\omega)d\omega,\\
I_2(\rho,s,\tau) &= 1_{\mathbb{R}_+}(\rho-s)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2\\
&\quad\times \int_{\mathbb{R}} \chi(s\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho))}\left((1-s)^{-\frac{1}{2}+i\omega} + (1+s)^{-\frac{1}{2}+i\omega}\right)\gamma_4'(\rho,s;i\omega)d\omega.
\end{aligned}
$$

In I_1, we use the expression (2.36), and because of the cut-off, we have

$$
\begin{aligned}
&|I_1(\rho,s,\tau)|\\
&\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s\int_{-1}^{1}\left|(1+st)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(s\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho)+\log(1+st))}\gamma_4'(\rho,s;i\omega)\right|dt\\
&\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s\langle\tau-\log(1+\rho)\rangle^{-2}\\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
\end{aligned}
$$

In I_2, we need to look at

$$
I_{2,\pm} = 1_{\mathbb{R}_+}(\rho-s)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1\pm s)^{-\frac{1}{2}}\int_{\mathbb{R}}\chi(s\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho)+\log(1\pm s))}\gamma_4'(\rho,s;i\omega)d\omega,
$$

and again we have

$$
|I_2(\rho,s,\tau)| \lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
$$

For G_5' we need to bound

$$
\begin{aligned}
I_1(\rho,s,\tau) &= 1_{\mathbb{R}+}(\rho-s)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s(1-s)^{-\frac{1}{2}}\\
&\quad\times\int_{\mathbb{R}}[1-\chi(s\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1-s))}\frac{\gamma_5'(\rho,s;i\omega)}{1+2i\omega}d\omega,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(\rho,s,\tau) &= 1_{\mathbb{R}+}(\rho-s)\rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\\
&\quad\times\int_{\mathbb{R}}[1-\chi(s\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1-s))}\frac{-1+2i\omega}{1+2i\omega}\gamma_5'(\rho,s;i\omega)d\omega.
\end{aligned}
$$

So for I_1 we use Lemma 2.4.3 and have

$$\begin{aligned}
&|I_1(\rho,s,\tau)| \\
&\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}} \\
&\quad \times \left| s^{-1}\int_{\mathbb{R}}[1-\chi(s\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1-s))}\mathcal{O}(\rho^0(1-\rho)^0 s^0(1-s)^0\langle\omega\rangle^{-2})d\omega\right| \\
&\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1+\rho)+\log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
\end{aligned}$$

And for I_2 we have

$$\begin{aligned}
|I_2(\rho,s,\tau)| &\lesssim \rho^{-2}(1+\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1+\rho)+\log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.
\end{aligned}$$

Finally, G_6' is similar to G_5', but we get $(1+s)$ instead of $(1-s)$. Then, we have

$$(1+s)^{-\frac{1}{2}}\langle\tau+\log(1+s)\rangle^{-2}\lesssim\langle\tau\rangle^{-2}\lesssim(1-\rho)^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}$$

and obtain the desired bound. □

Lemma 2.5.5. *We have*

$$\begin{aligned}
\|\tilde{S}_n(\tau)f\|_{L^2(\mathbb{B}^5)} &\lesssim \|f\|_{L^2(\mathbb{B}^5)}, \qquad n\in\{1,\cdots,6\}, \\
\|S_n'(\tau)f\|_{L^2(\mathbb{B}^5)} &\lesssim \|f\|_{L^2(\mathbb{B}^5)}, \qquad n\in\{1,\cdots,6\},
\end{aligned}$$

for $\tau>0$ *and* $f\in C^1([0,1])$.

Proof. We write

$$S_n'(\tau)f(\rho)=\frac{1}{2\pi}\int_0^1 f(s)K_n'(\rho,s;\tau)ds,$$

where

$$K_n'(\rho,s;\tau)=\int_{\mathbb{R}}e^{i\omega\tau}G_n'(\rho,s;i\omega)d\omega.$$

Then Lemma 2.5.4 gives the bounds

$$|K_n'(\rho,s;\tau)|\lesssim\rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}.$$

Doing a change of variables $\rho=1-e^{-x}$, $s=1-e^{-y}$, we get

$$\left|[S_n'(\tau)f](1-e^{-x})\right|\lesssim e^{\frac{x}{2}}(1-e^{-x})^{-2}\int_0^\infty\langle\tau-y+x\rangle^{-2}|f(1-e^{-y})|(1-e^{-y})^2e^{-\frac{y}{2}}dy,$$

so using Young's inequality, we have

$$\begin{aligned}
\|S'_n(\tau)f\|_{L^2(\mathbb{B}^5)} &\cong \left(\int_0^1 |S'_n(\tau)f(\rho)|^2\rho^4 d\rho\right)^{\frac{1}{2}} \\
&\cong \left(\int_0^\infty |[S'_n(\tau)]f(1-e^{-x})|^2(1-e^{-x})^4 e^{-x}dx\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^\infty \left|\int_0^\infty \langle \tau - y + x\rangle^{-2} |f(1-e^{-y})|(1-e^{-y})^2 e^{-\frac{y}{2}}dy\right|^2 dx\right)^{\frac{1}{2}} \\
&\lesssim \| \langle \tau - \cdot\rangle \|_{L^1(\mathbb{R})} \left(\int_0^1 |f(\rho)|^2\rho^4 d\rho\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2(\mathbb{B}^5)}.
\end{aligned}$$

For the operators $\tilde{S}_n$, from Lemma 2.5.3 we have

$$\begin{aligned}
\left|\rho^{-1}\int_{\mathbb{R}} e^{i\omega\tau}\tilde{G}_n(\rho,s;i\omega)d\omega\right| &\lesssim \rho^{-2}s^2(1-s)^{-\frac{1}{2}}\langle \tau + \log(1-s)\rangle^{-2} \\
&\lesssim \rho^{-2}(1-\rho)^{-\frac{1}{2}}s^2(1-s)^{-\frac{1}{2}}\langle \tau - \log(1-\rho) + \log(1-s)\rangle^{-2},
\end{aligned}$$

so the bounds for $\tilde{S}_n$ follows the same way. □

2.5.4 The term containing λ

As before we need to deal with the $\lambda\tilde{f}_1$ term in F_λ. Here we also need to deal with the operators

$$\begin{aligned}
\dot{\tilde{S}}_{n,\epsilon}(\tau)f(\rho) &:= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{1}{2}(1+2\lambda)e^{\lambda\tau}\int_0^1 \rho^{-1}\tilde{G}_n(\rho,s;\lambda)f(s)dsd\lambda, \\
\dot{S}'_{n,\epsilon}(\tau)f(\rho) &:= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{1}{2}(1+2\lambda)e^{\lambda\tau}\int_0^1 G'_n(\rho,s;\lambda)f(s)dsd\lambda.
\end{aligned}$$

Lemma 2.5.6. *Setting $\dot{\tilde{S}}_n(\tau)f(\rho) := \lim_{\epsilon\to 0+}\dot{\tilde{S}}_{n,\epsilon}(\tau)f(\rho)$, we have the bound*

$$\|\dot{\tilde{S}}_n(\tau)f\|_{L^2(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$$

for all $\tau > 0$, $f \in C^1([0,1])$, and $n \in \{1,2,\dots,6\}$.

Proof. We follow the logic in Proposition 2.4.9 by an integration by parts in s. Indeed, the form of $\dot{\tilde{S}}_{n,\epsilon}(\tau)$ and $\dot{T}_{n,\epsilon}(\tau)$ only differ in a factor of ρ^{-1}. After an integration by parts, the integral terms can be treated the same way as in Lemma 2.5.3 followed by Lemma 2.5.5, and the $L^2(\mathbb{B}^5)$ norms are bounded by $C\left(\|f'\|_{L^2(\mathbb{B}^5)} + \|(\cdot)^{-1}f\|_{L^2(\mathbb{B}^5)}\right) \lesssim \|f\|_{H^1(\mathbb{B}^5)}$ for all $\tau > 0$. On the other hand, the boundary terms are (pointwise) bounded by $C|f(\rho)|\langle\tau\rangle^{-2}$ (see proof of Proposition 2.4.9), so the $L^2(\mathbb{B}^5)$ norm is bounded by $\|f\|_{L^2(\mathbb{B}^5)}$ for all $\tau > 0$. □

Lemma 2.5.7. *Setting* $\dot{S}'_n(\tau)f(\rho) := \lim_{\epsilon\to0+}\dot{S}'_{n,\epsilon}(\tau)f(\rho)$*, we have the bound*

$$\|\dot{S}'_n(\tau)f\|_{L^2(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$$

for all $\tau > 0$*,* $f \in C^1([0,1])$*, and* $n \in \{1, 2, \dots, 6\}$ *.*

Proof. Again, we follow the same method as Proposition 2.4.9. The s dependent part of $\dot{S}'_n$ has the same form as $\dot{T}_n$ for each $n = \{1, \cdots, 6\}$.

For $n = \{1, 2, 3\}$, we do an integration by parts as in (2.34). The integral part is dealt with by the same way as Lemma 2.5.4 (for each fixed N, we use Fubini to interchange the ω and s integral, then we use dominated convergence for the s integral to take the limit $N \to \infty$). Then the method in Lemma 2.5.5 gives the same bound as in Lemma 2.5.6. For the boundary term, with the logic in Lemma 2.5.1 we can take the limit $\epsilon \to 0$. For $n = 1$, we have

$$\begin{aligned}
&B'_1(\tau)f(\rho)\\
&= \frac{2(1-\rho)^{\frac{1}{2}}\rho f(\rho)}{2\pi i}\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-\rho))}\frac{\chi(\rho\langle\omega\rangle)(g_1(\rho;i\omega)-\tilde{g}_1(\rho;i\omega))}{1+2i\omega}\gamma'_1(\rho,\rho;i\omega)d\omega\\
&\quad+\frac{(1-\rho)^{\frac{1}{2}}\rho^2 f(\rho)}{2\pi i}\int_{\mathbb{R}} e^{i\omega(\tau+\log(1-\rho))}\chi(\rho\langle\omega\rangle)(g_1(\rho;i\omega)-\tilde{g}_1(\rho;i\omega))\frac{-1+2i\omega}{1+2i\omega}\gamma'_1(\rho,\rho;i\omega)d\omega.
\end{aligned}$$

In the first term we use (2.38) and in the second term we use the expression of g_1 and $\tilde{g}_1$ directly, similar to proof of Lemma 2.5.4. So we get the bound $|B'_1(\tau)f(\rho)| \lesssim |f(\rho)|\langle\tau\rangle^{-2}$. For $n = \{2, 3\}$, we have

$$\begin{aligned}
&B'_n(\tau)f(\rho)\\
&= \frac{2(1-\rho)^{\frac{3}{2}}(1\pm\rho)^{-\frac{1}{2}}f(\rho)}{2\pi i}\rho^{-1}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau+\log(1-\rho)-\log(1\pm\rho))}\frac{\gamma'_n(\rho,\rho;i\omega)}{1+2i\omega}d\omega\\
&\quad+\frac{(1-\rho)^{\frac{1}{2}}(1\pm\rho)^{-\frac{1}{2}}f(\rho)}{2\pi i}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau+\log(1-\rho)-\log(1\pm\rho))}\gamma'_n(\rho,\rho;i\omega)d\omega.
\end{aligned}$$

The same treatment as in the proof of Lemma 2.5.4 gives $|B'_n(\tau)f(\rho)| \lesssim |f(\rho)|\langle\tau\rangle^{-2}$.

For $n = 4$, we decompose the operator into two parts according to (2.39). So we have

$$\begin{aligned}
&\dot{S}'_{4,\epsilon}(\tau)f(\rho)\\
&= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda\tau}g_1(\rho;\lambda)\\
&\quad\times\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s\left((1-s)^{\frac{1}{2}+\lambda}-(1+s)^{\frac{1}{2}+\lambda}\right)f(s)\gamma'_4(\rho,s;\lambda)dsd\lambda\\
&\quad+\frac{1}{2\pi i}\lim_{N\to\infty}\int_{\epsilon-iN}^{\epsilon+iN}\frac{1}{2}(1+2\lambda)e^{\lambda\tau}g_1(\rho;\lambda)\\
&\quad\times\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^2\left((1-s)^{-\frac{1}{2}+\lambda}-(1+s)^{-\frac{1}{2}+\lambda}\right)f(s)\gamma'_4(\rho,s;\lambda)dsd\lambda\\
&=: I_{1,\epsilon}(\tau,\rho)+I_{2,\epsilon}(\tau,\rho)
\end{aligned}$$

In I_1 we can take the limit in ϵ and change order of integration (same argument as in [8, Lemma 5.3]). Then we have $I_1(\tau,\rho) := \lim_{\epsilon\to 0+} I_{1,\epsilon}(\tau,\rho)$, and

$$I_1(\tau,\rho) = \frac{\rho^{-2}(1+\rho)^{-\frac{1}{2}}f(\rho)}{2\pi}\int_0^1 1_{\mathbb{R}_+}(\rho-s)s$$
$$\times \int_{\mathbb{R}} \chi(s\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho))}\left((1-s)^{\frac{1}{2}+\lambda}-(1+s)^{\frac{1}{2}+\lambda}\right)\gamma_4'(\rho,s;i\omega)d\omega ds.$$

Following the method in the proof of Lemma 2.5.4 (the I_2 term in G_4'), we get the bound $\|I_1(\tau)f\| \lesssim \|(\cdot)^{-1}f\|_{L^2(\mathbb{B}^5)} \lesssim \|f\|_{H^1(\mathbb{B}^5)}$ for all $\tau > 0$. In I_2 we do an integration by parts in s using

$$(1\pm s)^{-\frac{1}{2}+\lambda} = \frac{\pm 2}{1+2\lambda}\partial_s(1\pm s)^{\frac{1}{2}+\lambda},$$

so

$$\begin{aligned}
&\int_0^1 1_{\mathbb{R}_+}(\rho-s)\chi(s\langle\omega\rangle)s^2(1\pm s)^{-\frac{1}{2}+\lambda}f(s)\gamma_4'(\rho,s;\lambda)ds\\
&=\frac{\pm 2}{1+2\lambda}\int_0^\rho \partial_s(1\pm s)^{\frac{1}{2}+\lambda}\chi(s\langle\omega\rangle)s^2f(s)\gamma_4'(\rho,s;\lambda)ds\\
&=\frac{\pm 2}{1+2\lambda}(1\pm\rho)^{\frac{1}{2}+\lambda}\chi(\rho\langle\omega\rangle)\rho^2f(\rho)\gamma_4'(\rho,\rho;\lambda)\\
&\quad-\frac{\pm 2}{1+2\lambda}\int_0^\rho(1\pm s)^{\frac{1}{2}+\lambda}\partial_s\left[\chi(s\langle\omega\rangle)s^2f(s)\gamma_4'(\rho,s;\lambda)\right]ds.
\end{aligned}$$

The integral part is dealt with by the same argument as in Lemma 2.5.4 (the I_2 term in G_4'), and we again obtain a bound by $C\left(\|f'\|_{L^2(\mathbb{B}^5)} + \|(\cdot)^{-1}f\|_{L^2(\mathbb{B}^5)}\right) \lesssim \|f\|_{H^1(\mathbb{B}^5)}$ for all $\tau > 0$. In the boundary term we take the limit $\epsilon \to 0+$ by Lemma 2.5.1, so

$$B_4'(\tau)f(\rho) = \frac{\pm 2(1\pm\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}}f(\rho)}{2\pi i}\int_{\mathbb{R}}\chi(\rho\langle\omega\rangle)e^{i\omega(\tau-\log(1+\rho)+\log(1\pm\rho))}\gamma_4'(\rho,\rho;i\omega)d\omega,$$

and we get the bound $|B_4'(\tau)f(\rho)| \lesssim f(\rho)\langle\tau\rangle^{-2}$.

For $n = \{5,6\}$, as before we do an integration by parts in s as in (2.37), and the integral terms are handled by the same way as in Lemma 2.5.4 (the I_2 term in G_5' and G_6'). The boundary terms are

$$\begin{aligned}
&B_n'(\tau)f(\rho)\\
&=\frac{\pm(1\pm\rho)^{\frac{3}{2}}(1+\rho)^{-\frac{1}{2}}f(\rho)}{\pi i}\rho^{-1}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1\pm\rho))}\frac{\gamma_n'(\rho,\rho;i\omega)}{1+2i\omega}d\omega\\
&\quad\pm\frac{(1\pm\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}}f(\rho)}{2\pi i}\int_{\mathbb{R}}[1-\chi(\rho\langle\omega\rangle)]e^{i\omega(\tau-\log(1+\rho)+\log(1\pm\rho))}\gamma_n'(\rho,\rho;i\omega)d\omega.
\end{aligned}$$

So we also have $|B_n'(\tau)f(\rho)| \lesssim f(\rho)\langle\tau\rangle^{-2}$ for $n = \{5,6\}$. □

2.5.5 Improved energy bounds

Lemma 2.5.8. *The semigroup* $\mathbf{S}$ *admits the bound*

$$\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}} \lesssim \|(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}$$

for all $\tau \geq 0$ *and* $f \in \mathcal{H}$.

Proof. Here we have the representation of the first component

$$[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1 = [\mathbf{S}_0(\tau)\tilde{\mathbf{f}}]_1(\rho) + \sum_{n=1}^{6} \left[T_n(\tau)\left(|\cdot|\tilde{f}_1' + 2\tilde{f}_1 + \tilde{f}_2\right) + \dot{T}_n(\tau)\tilde{f}_1\right],$$

and the definition of the second component

$$[\mathbf{S}(\tau)\tilde{\mathbf{f}}]_2(\rho) = \left[\partial_\tau + \rho\partial_\rho + \frac{3}{2}\right][\mathbf{S}(\tau)\tilde{\mathbf{f}}]_1(\rho).$$

This is the same proof as [8, Lemma 5.7]. □

2.6 Proof of the theorem

We look at the nonlinear problem

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}(\tau-\sigma)\mathbf{N}(\Phi(\sigma))d\sigma,$$

where

$$\mathbf{N}(\mathbf{u})(\rho) := \begin{pmatrix} 0 \\ N(u_1(\rho)) \end{pmatrix}.$$

with

$$N(x) = |c_5 + x|^{\frac{4}{3}}(c_5 + x) - c_5^{\frac{7}{3}} - \frac{35}{4}x$$

Lemma 2.6.1. *We have*

$$\|\mathbf{N}(\mathbf{u})\|_{\mathcal{H}} \lesssim \|u_1\|^2_{L^5(\mathbb{B}^5)} + \|u_1\|^{\frac{7}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)}$$

and

$$\begin{aligned}\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\|_{\mathcal{H}} &\lesssim \|u_1 - v_1\|_{L^5(\mathbb{B}^5)}\left(\|u_1\|_{L^5(\mathbb{B}^5)} + \|v_1\|_{L^5(\mathbb{B}^5)}\right) \\ &\quad + \|u_1 - v_1\|_{L^{\frac{14}{3}}(\mathbb{B}^5)}\left(\|u_1\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)} + \|v_1\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)}\right)\end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$.

Proof. We compute

$$N'(x) = \frac{7}{3}|c_5 + x|^{\frac{4}{3}} - \frac{35}{4} = \frac{7}{3}\left|\left(\frac{15}{4}\right)^{\frac{3}{4}} + x\right|^{\frac{4}{3}} - \frac{35}{4},$$

and observe that $N(0) = N'(0) = 0$. Hence

$$|N(x)| \lesssim |x|^2 + |x|^{\frac{7}{3}}, \qquad |N'(x)| \lesssim |x| + |x|^{\frac{4}{3}},$$

for all $x \in \mathbb{R}$. Since N is monotone, we also have

$$|N(x) - N(y)| \lesssim |x-y|\left(|N'(x)| + |N'(y)|\right) \lesssim |x-y|\left(|x| + |x|^{\frac{4}{3}} + |y| + |y|^{\frac{4}{3}}\right).$$

Then by Hölder's inequality we get

$$\begin{aligned}
\|\mathbf{N}(\mathbf{u})\|_{\mathcal{H}} &= \|N(u_1)\|_{L^2(\mathbb{B}^5)} \lesssim \|u_1^2\|_{L^2(\mathbb{B}^5)} + \|u_1^{\frac{7}{3}}\|_{L^2(\mathbb{B}^5)} \\
&\lesssim \|u_1\|^2_{L^4(\mathbb{B}^5)} + \|u_1\|^{\frac{7}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)} \lesssim \|u_1\|^2_{L^5(\mathbb{B}^5)} + \|u_1\|^{\frac{7}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)},
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbf{N}(\mathbf{u}) - \mathbf{N}(\mathbf{v})\|_{\mathcal{H}} &\lesssim \|N(u_1) - N(v_1)\|_{L^2(\mathbb{B}^5)} \\
&\lesssim \|u_1 - v_1\|_{L^2(\mathbb{B}^5)} \left(\|u_1\|_{L^2(\mathbb{B}^5)} + \|u_1\|^{\frac{4}{3}}_{L^2(\mathbb{B}^5)} + \|v_1\|_{L^2(\mathbb{B}^5)} + \|v_1\|^{\frac{4}{3}}_{L^2(\mathbb{B}^5)} \right) \\
&\lesssim \|u_1 - v_1\|_{L^5(\mathbb{B}^5)} \left(\|u_1\|_{L^5(\mathbb{B}^5)} + \|v_1\|_{L^5(\mathbb{B}^5)} \right) \\
&\quad + \|u_1 - v_1\|_{L^{\frac{14}{3}}(\mathbb{B}^5)} \left(\|u_1\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)} + +\|v_1\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)} \right)
\end{aligned}$$

as required. □

2.6.1 The nonlinear problem

In the following, we first modify the equation to remove the unstable direction, then we show how to remove this modification.

For $\Psi(\tau)(\rho) = (\varphi_1(\tau, \rho), \varphi_2(\tau, \rho))$, we define

$$\|\Psi\|^2_{\mathcal{X}} := \|\Psi\|^2_{L^\infty(\mathbb{R}_+;\mathcal{H})} + \|\varphi_1\|^2_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))},$$

and introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0, \infty), \mathcal{H}) : \varphi_1 \in L^2(\mathbb{R}_+; L^5(\mathbb{B}^5)), \|\Phi\|_{\mathcal{X}} < \infty\}.$$

Moreover, we set

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\| \leq \delta\}.$$

For initial data $\mathbf{u} \in \mathcal{H}$, we define

$$\mathbf{K_u}(\Phi)(\tau) = \mathbf{S}(\tau) \left[\mathbf{u} - \mathbf{C}(\Phi, \mathbf{u})\right] + \int_0^\tau \mathbf{S}(\tau - \sigma)\mathbf{N}(\Phi(\mathbf{u}))d\sigma,$$

where

$$\mathbf{C}(\Phi, \mathbf{u}) := \mathbf{P} \left[\mathbf{u} + \int_0^\infty e^{-\sigma}\mathbf{N}(\Phi(\sigma))d\sigma\right].$$

Lemma 2.6.2. *There exist $c, \delta > 0$ such that if $\|\mathbf{u}\|_{\mathcal{H}} \leq \frac{\delta}{c}$ and $\Phi \in \mathcal{X}_\delta$, then $\mathbf{K_u}(\Phi) \in \mathcal{X}_\delta$.*

Proof. From Lemmas 2.5.8 and 2.6.1, we have

$$\begin{aligned}
\|(\mathbf{I} - \mathbf{P})\mathbf{K_u}(\Phi)(\tau)\|_{\mathcal{H}} &\lesssim \|\mathbf{u}\|_{\mathcal{H}} + \int_0^\tau \|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma \\
&\lesssim \frac{\delta}{c} + \int_0^\tau \left(\|\varphi_1(\sigma, \cdot)\|^2_{L^5(\mathbb{B}^5)} + \|\varphi_1(\sigma, \cdot)\|^{\frac{7}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)} \right) d\sigma \\
&\lesssim \frac{\delta}{c} + \delta^2 + \left(\|\varphi_1\|^\theta_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} \|\varphi_1\|^{1-\theta}_{L^\infty(\mathbb{R}_+;L^{\frac{10}{3}}(\mathbb{B}^5))} \right)^{\frac{7}{3}} \\
&\lesssim \frac{\delta}{c} + \delta^2 + \left(\|\varphi_1\|^\theta_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} \|\Phi\|^{1-\theta}_{L^\infty(\mathbb{R}_+;\mathcal{H})} \right)^{\frac{7}{3}} \\
&\lesssim \frac{\delta}{c} + \delta^2 + \delta^{\frac{7}{3}}.
\end{aligned}$$

Moreover, from the Strichartz estimates in Theorem 2.4.10, we have

$$\begin{aligned}
\|(\mathbf{I}-\mathbf{P})\mathbf{K_u}(\Phi)(\tau)\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} &\lesssim \|\mathbf{u}\|_{\mathcal{H}} + \int_0^\infty \|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma \\
&\lesssim \frac{\delta}{c} + \delta^2 + \delta^{\frac{7}{3}}.
\end{aligned}$$

Next, from Lemma 2.2.8 we have

$$\begin{aligned}
\mathbf{PK_u}(\Phi)(\tau) &= \mathbf{P}\left[\mathbf{S}(\tau)\left[\mathbf{u}-\mathbf{C}(\Phi,\mathbf{u})\right] + \int_0^\tau \mathbf{S}(\tau-\sigma)\mathbf{N}(\Phi(\mathbf{u}))d\sigma\right] \\
&= -\int e^{\tau-\sigma}\mathbf{PN}(\Phi(\sigma))d\sigma + \int_0^\tau e^{\tau-\sigma}\mathbf{PN}(\Phi(\sigma))d\sigma \\
&= -\int_\tau^\infty e^{\tau-\sigma}\mathbf{PN}(\Phi(\sigma))d\sigma.
\end{aligned}$$

Since we also have $\operatorname{rg}\mathbf{P} = \langle \mathbf{g} \rangle$, so by the Riesz representation theorem, there exists $\mathbf{g}^* \in \mathcal{H}$ with $\mathbf{Pf} = (\mathbf{f}|\mathbf{g}^*)_{\mathcal{H}}\mathbf{g}$ for all $\mathbf{f} \in \mathcal{H}$. Then

$$\begin{aligned}
\|\mathbf{PK_u}(\Phi)(\tau)\|_{\mathcal{H}} &\lesssim \int_\tau^\infty e^{\tau-\sigma}\left|(\mathbf{N}(\Phi(\sigma))|\mathbf{g}^*)_{\mathcal{H}}\right| d\sigma \\
&\lesssim \int_\tau^\infty \|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma \\
&\lesssim \delta^2 + \delta^{\frac{7}{3}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\mathbf{PK_u}(\Phi)(\tau)\|_{L^5(\mathbb{B}^5)} &\lesssim \int_\tau^\infty e^{\tau-\sigma}\|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma \\
&= 1_{[0,\infty)}(\tau)\int_{\mathbb{R}} 1_{[-\infty,0]}(\tau-\sigma)e^{\tau-\sigma}\|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma,
\end{aligned}$$

then Young's inequality gives

$$\begin{aligned}
\|\mathbf{PK_u}(\Phi)\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} &\lesssim \|1_{[-\infty,0]}e^{(\cdot)}\|_{L^2(\mathbb{R})}\int_0^\infty \|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}} d\sigma \\
&\lesssim \delta^2 + \delta^{\frac{7}{3}}.
\end{aligned}$$

Therefore, we have $\|\mathbf{K_u}(\Phi)\|_{\mathcal{X}} \lesssim \frac{\delta}{c} + \delta^2 + \delta^{\frac{7}{3}}$, which implies the claim. □

Lemma 2.6.3. *Let $\delta > 0$ be small and $\mathbf{u} \in \mathcal{H}$. Then*

$$\|\mathbf{K_u}(\Phi) - \mathbf{K_u}(\Psi)\|_{\mathcal{H}} \le \frac{1}{2}\|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$.

Proof. From Lemmas 2.5.8 and 2.6.1, we have

$$
\begin{aligned}
&\|(\mathbf{I}-\mathbf{P})\left[\mathbf{K}_{\mathbf{u}}(\Phi)(\tau)-\mathbf{K}_{\mathbf{u}}(\Psi)(\tau)\right]\|_{\mathcal{H}} \\
\lesssim& \int_0^\tau \|\mathbf{N}(\Phi(\sigma))-\mathbf{N}(\Psi(\sigma))\|_{\mathcal{H}} d\sigma \\
\lesssim& \int_0^\tau \|\varphi_1(\sigma)-\psi_1(\sigma)\|_{L^5(\mathbb{B}^5)}\left(\|\varphi_1(\sigma)\|_{L^5(\mathbb{B}^5)}+\|\psi_1(\sigma)\|_{L^5(\mathbb{B}^5)}\right) \\
&+\|\varphi_1(\sigma)-\psi_1(\sigma)\|_{L^{\frac{14}{3}}(\mathbb{B}^5)}\left(\|\varphi_1(\sigma)\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)}+\|\psi_1(\sigma)\|^{\frac{4}{3}}_{L^{\frac{14}{3}}(\mathbb{B}^5)}\right) d\sigma \\
\lesssim& \|\varphi_1-\psi_1\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))}\left(\|\varphi_1\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))}+\|\psi_1\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))}\right) \\
&+\|\varphi_1-\psi_1\|_{L^{\frac{7}{3}}(\mathbb{R}_+;L^{\frac{14}{3}}(\mathbb{B}^5))}\left(\|\varphi_1\|^{\frac{4}{3}}_{L^{\frac{7}{3}}(\mathbb{R}_+;L^{\frac{14}{3}}(\mathbb{B}^5))}+\|\psi_1\|^{\frac{4}{3}}_{L^{\frac{7}{3}}(\mathbb{R}_+;L^{\frac{14}{3}}(\mathbb{B}^5))}\right) \\
\lesssim& \delta\|\Phi-\Psi\|_{\mathcal{X}}.
\end{aligned}
$$

Furthermore, from Theorem 2.4.10 we have

$$
\begin{aligned}
\|(\mathbf{I}-\mathbf{P})\left[\mathbf{K}_{\mathbf{u}}(\Phi)(\tau)-\mathbf{K}_{\mathbf{u}}(\Psi)(\tau)\right]\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} &\lesssim \int_0^\infty \|\mathbf{N}(\Phi(\sigma))-\mathbf{N}(\Psi(\sigma))\|_{\mathcal{H}} d\sigma \\
&\lesssim \delta\|\Phi-\Psi\|_{\mathcal{X}}.
\end{aligned}
$$

Following the same logic as in the proof of Lemma 2.6.2, we have

$$
\begin{aligned}
\|\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Phi)(\tau)-\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Psi)(\tau)\|_{\mathcal{H}} &\lesssim \int_\tau^\infty e^{\tau-\sigma}\left|(\mathbf{N}(\Phi(\sigma))-\mathbf{N}(\Psi(\sigma))|\mathbf{g}^*)\right|_{\mathcal{H}} d\sigma \\
&\lesssim \delta\|\Phi-\Psi\|_{\mathcal{X}}.
\end{aligned}
$$

Lastly,

$$
\|\left[\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Phi)(\tau)-\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Psi)(\tau)\right]_1\|_{L^5(\mathbb{B}^5)} \lesssim \int_\tau^\infty e^{\tau-\sigma}\|\mathbf{N}(\Phi(\sigma))-\mathbf{N}(\Psi(\sigma))\|_{\mathcal{H}} d\sigma,
$$

and by Young's inequality we have

$$
\|\left[\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Phi)-\mathbf{P}\mathbf{K}_{\mathbf{u}}(\Psi)\right]_1\|_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} \lesssim \delta\|\Phi-\Psi\|_{\mathcal{X}}.
$$

Hence $\|\mathbf{K}_{\mathbf{u}}(\Phi)-\mathbf{K}_{\mathbf{u}}(\Psi)\|_{\mathcal{H}} \lesssim \delta\|\Phi-\Psi\|_{\mathcal{X}}$. By choosing δ sufficiently small, we get the inequality as claimed. □

Corollary 2.6.4. *There exist $c, \delta > 0$ such that if $\|\mathbf{u}\|_{\mathcal{H}} \le \frac{\delta}{c}$, then there exists a unique $\Phi \in \mathcal{X}_\delta$ satisfying $\Phi = \mathbf{K}_{\mathbf{u}}(\Phi)$.*

Proof. We obtain the conclusion from Lemmas 2.6.2 and 2.6.3, and the Banach fixed point theorem. □

Finally, we show that if we choose the blowup time correctly, the correction term $\mathbf{C}(\Phi, \mathbf{u})$ is in fact 0. Recall that the initial data $\Phi(0) = (\varphi_1(0), \varphi_2(0))$ are given by

$$
\begin{aligned}
\varphi_1(0) &= \psi_1(0,\rho) - c_5 = T^{\frac{3}{2}} f(T\rho) - c_5 \\
\varphi_2(0) &= \psi_2(0,\rho) - \frac{3}{2}c_5 = T^{\frac{5}{2}} g(T\rho) - \frac{3}{2}c_5.
\end{aligned}
$$

The ODE blowup solution u^1, transformed to similarity coordinates, is given by

$$\psi^1(\tau,\rho) = T^{\frac{3}{2}} e^{-\frac{3}{2}\tau} u^1(T - Te^{-\tau}, Te^{-\tau}\rho) = T^{\frac{3}{2}} e^{-\frac{3}{2}\tau} c_5 (1 - T + Te^{-\tau})^{-\frac{3}{2}}.$$

As in (2.5), we set $\psi_1^1 := \psi^1$, and

$$\psi_2^1(\tau,\rho) = \partial_\tau \psi_1^1 + \rho\partial_\rho \psi_1^1 + \frac{3}{2}\psi_1^1 = \frac{3}{2} c_5 T^{\frac{5}{2}} e^{-\frac{3}{2}\tau} (1 - T + Te^{-\tau})^{-\frac{5}{2}}.$$

Thus intial data for this solution is given by

$$\psi_1^1(0,\rho) = c_5 T^{\frac{3}{2}}, \psi_2^1(0,\rho) = \frac{3}{2} c_5 T^{\frac{5}{2}}.$$

Now we rewrite the initial condition for Φ as

$$\begin{aligned}\Phi(0)(\rho) &= \left(T^{\frac{3}{2}}\left(f(T\rho) - c_5\right) + c_5 T^{\frac{3}{2}} - c_5, T^{\frac{5}{2}}\left(g(T\rho) - \frac{3}{2}c_5\right) + \frac{3}{2}c_5 T^{\frac{5}{2}} - \frac{3}{2}c_5\right) \\ &= \mathbf{U}\left(T, \left(f - c_5, g - \frac{3}{2}c_5\right)\right)(\rho),\end{aligned}$$

where

$$\mathbf{U}(T,\mathbf{v})(\rho) := \left(T^{\frac{3}{2}} v_1(T\rho), T^{\frac{5}{2}} v_2(T\rho)\right) + \left(c_5 T^{\frac{3}{2}}, \frac{3}{2}T^{\frac{5}{2}}\right) - \left(c_5, \frac{3}{2}c_5\right).$$

We can see that, for $\delta > 0$ sufficently small and $\mathbf{v} \in H^1 \times L^2(\mathbb{B}^5_{1+\delta})$, the map

$$\mathbf{U}(\cdot,\mathbf{v}) : [1-\delta, 1+\delta] \to H^1 \times L^2(\mathbb{B}^5)$$

is continuous (see for example the proof of [12, Lemma 4.14]).

Lemma 2.6.5. *There exist $M \geq 1$ and $\delta > 0$ such that, given $\|\mathbf{v}\|_{H^1\times L^2(\mathbb{B}^5_{1+\delta})} < \frac{\delta}{M}$, there exist $T^* \in [1-\delta, 1+\delta]$ and $\|\Phi\| \in \mathcal{X}_\delta$ with $\Phi = \mathbf{K}_{\mathbf{U}(T^*,\mathbf{v})}(\Phi)$ and $\mathbf{C}(\Phi, \mathbf{U}(T^*,\mathbf{v})) = 0$.*

Proof. First, we have

$$\partial_T \begin{pmatrix} c_5 T^{\frac{3}{2}} \\ \frac{3}{2} c_5 T^{\frac{5}{2}} \end{pmatrix}\Bigg|_{T=1} = \begin{pmatrix} \frac{3}{2}c_5 \\ \frac{15}{4}c_5 \end{pmatrix} = \frac{3}{4} c_5 \mathbf{g}.$$

So we can write

$$\mathbf{U}(T,\mathbf{v})(\rho) = \left(T^{\frac{3}{2}} v_1(T\rho), T^{\frac{5}{2}} v_2(T\rho)\right) + \frac{3}{4}c_5 \mathbf{g}(T-1) + (T-1)^2 \mathbf{f}_T$$

where $\|\mathbf{f}_T\|_{\mathcal{H}} \lesssim 1$ on $T \in [\frac{1}{2}, \frac{3}{2}]$. Hence

$$(\mathbf{U}(T,\mathbf{v})|\mathbf{g}) = O\left(\frac{\delta}{M}T^0\right) + \frac{3}{4}c_5\|\mathbf{g}\|^2(T-1) + O(\delta^2 T^0)$$

for all $T \in [1-\delta, 1+\delta]$, $\delta \in [0, \frac{1}{2}]$, and $M \geq 1$.

Now notice that

$$\|\mathbf{U}(T,\mathbf{v})\|_{\mathcal{H}} \lesssim \|v\|_{H^1\times L^2(\mathbb{B}^5_{1+\delta})} + |T-1|,$$

so for every $T \in [1-\delta, 1+\delta]$, by Corollary 2.6.4, there exists $\Phi_T \in \mathcal{X}_\delta$ with $\Phi_T = \mathbf{K}_{\mathbf{U}(T,\mathbf{v})}(\Phi)$ provided that $\delta > 0$ is sufficiently small and $M \geq 1$ is sufficiently large.

Recall that

$$\mathbf{C}(\Phi, \mathbf{u}) := \mathbf{P}\left[\mathbf{u} + \int_0^\infty e^{-\sigma}\mathbf{N}(\Phi(\sigma))d\sigma\right].$$

and notice that

$$\int_0^\infty e^{-\sigma}\|\mathbf{N}(\Phi(\sigma))\|_{\mathcal{H}}d\sigma \lesssim \delta^2,$$

we have

$$(\mathbf{C}(\Phi, \mathbf{u})|\mathbf{g})_{\mathcal{H}} = O\left(\frac{\delta}{M}T^0\right) + \frac{3}{4}c_5\|\mathbf{g}\|^2(T-1) + O(\delta^2 T^0).$$

Since $\operatorname{rg}\mathbf{C}(\Phi, \mathbf{u}) \subset \langle \mathbf{g} \rangle$, we see that $\mathbf{C}(\Phi, \mathbf{u}) = 0$ is equivalent to $T - 1 = F(T)$, where F is a continuous function of T on $[1-\delta, 1+\delta]$ with $|F(T)| \lesssim \frac{\delta}{M} + \delta^2$. Choosing M sufficiently large, we have that $\operatorname{rg}(1+F)$ is a continuous function from $[1-\delta, 1+\delta]$ to itself, so $1+F$ has a fixed point T^*. □

2.6.2 Proof of the main theorem

Finally, we turn to our main result.

Proof of Theorem 2.1.1. Let $M \geq 1$ be sufficiently large and $\delta > 0$ sufficiently small. For (f, g) with

$$\|(f,g) - u^1[0]\|_{H^1 \times L^2(\mathbb{B}^5_{1+\delta})} \leq \frac{\delta}{M},$$

let $\mathbf{v} = (f,g) - u^1[0]$. Then we have the associated $\Phi \in \mathcal{X}_\delta$ and T given by Lemma 2.6.5. Then from (2.4), (2.5), and (2.7), we have

$$\begin{aligned}
\delta^2 &\geq \|\varphi_1\|^2_{L^2(\mathbb{R}_+;L^5(\mathbb{B}^5))} = \int_0^\infty \|\varphi_1(\tau,\cdot)\|^2_{L^5(\mathbb{B}^5)}d\tau \\
&= \int_0^\infty \|\psi_1(\tau,\cdot) - c_5\|^2_{L^5(\mathbb{B}^5)}d\tau \\
&= \int_0^T \left\|\psi_1\left(\log\frac{T}{T-t}, \cdot\right) - c_5\right\|^2_{L^5(\mathbb{B}^5)}\frac{dt}{T-t} \\
&= \int_0^T \left\|\psi_1\left(\log\frac{T}{T-t}, \frac{\cdot}{T-t}\right) - c_5\right\|^2_{L^5(\mathbb{B}^5_{T-t})}\frac{dt}{(T-t)^3} \\
&= \int_0^T (T-t)^3\left\|(T-t)^{-\frac{3}{2}}\psi_1\left(\log\frac{T}{T-t}, \frac{\cdot}{T-t}\right) - (T-t)^{-\frac{3}{2}}c_5\right\|^2_{L^5(\mathbb{B}^5_{T-t})}\frac{dt}{(T-t)^3} \\
&= \int_0^T \|u(t,\cdot) - u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^5_{T-t})}dt \\
&\simeq \int_0^T \frac{\|u - u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^3_{T-t})}}{\|u^T(t,\cdot)\|^2_{L^5(\mathbb{B}^3_{T-t})}}\frac{dt}{T-t}
\end{aligned}$$

as required. □

Appendix A

Properties of symbol type functions

Here we discuss some properties of functions of symbol type. From the Leibniz rule we have the following property.

Lemma A.0.1. *Let $f, g : (0,1) \to \mathbb{C}$, $a, b \in [0,1]$, and $\alpha, \beta \in \mathbb{R}$. If $f(x) = \mathcal{O}((x-a)^\alpha)$, $g(x) = \mathcal{O}((x-b)^\beta)$, then $f(x)g(x) = \mathcal{O}((x-a)^\alpha(x-b)^\beta)$.*

In particular, if $f(x) = \mathcal{O}((x-a)^\alpha)$, $g(x) = \mathcal{O}((x-a)^\beta)$, then $f(x)g(x) = \mathcal{O}((x-a)^{\alpha+\beta})$.

Proof. Fix $j \in \mathbb{N}_0$. By assumption, f and g satisfy

$$\left|\partial_x^k f(x)\right| \lesssim |x-a|^{\alpha-j} \text{ and } \left|\partial_x^k g(x)\right| \lesssim |x-b|^{\beta-j}$$

for all $x \in (0,1)$ and $k \in \mathbb{N}_0$. Then by the Leibniz rule we have

$$\begin{aligned}
\left|\partial_x^j \left(f(x)g(x)\right)\right| &\leq \sum_{k=0}^{j} \binom{j}{k} \left|\partial_x^{j-k} f(x)\right| \left|\partial_x^k g(x)\right| \\
&\lesssim \sum_{k=0}^{j} \binom{j}{k} |x-a|^{\alpha-(j-k)} |x-b|^{\beta-k} \\
&\lesssim |x-a|^{\alpha-j}|x-b|^{\beta-j},
\end{aligned}$$

since $|x-a| < 1$ and $|x-b| < 1$. $\square$

Symbol behaviour is also stable under integration in certain situations. An example is the following.

Lemma A.0.2. *Let $f : (0,1) \to \mathbb{C}$ and $\alpha > 1$ with $f(x) = \mathcal{O}(x^{-\alpha})$. Define a function $g : (0,c] \to \mathbb{C}$ with $c \in (0,1)$ by*

$$g(x) = \int_x^c f(y)dy.$$

Then $g(x) = \mathcal{O}(x^{-\alpha+1})$.

Proof. First of all, since $0 < x \le c < 1$ and $-\alpha + 1 < 0$, we have

$$|g(x)| \le \int_x^c |f(y)| dy \lesssim \int_x^c y^{-\alpha} dy \lesssim c^{-\alpha+1} + x^{-\alpha+1} \lesssim x^{-\alpha+1}.$$

Moreover, since

$$\partial_x \int_x^c f(y) dy = -f(x),$$

estimates on derivatives follow from the symbol behaviour of f. $\square$

Lemma A.0.3. *Let $f : I \subset \mathbb{R} \to \mathbb{C}$, $0 \in \overline{I}$ be such that $f \ne 0$ and $f(x) = 1 + \mathcal{O}(x)$. Then there exists $\delta > 0$ such that $1/f(x) = 1 + \mathcal{O}(x)$ on $(0, \delta)$.*

Proof. This follows from the Taylor expansion. $\square$

Next, we show some properties that are specifically used in the derivative estimates of the Volterra solution from Lemma 2.3.6.

Lemma A.0.4. *Let $x \in [0, \infty)$ and $j \in \mathbb{N}_0$. Then*

$$|\partial_x^j (e^{-x} \mathcal{O}(x^0))| \lesssim e^{-x} x^{-j} \langle x \rangle^j .$$

Proof. By the Leibniz rule, we have

$$|\partial_x^j (e^{-x} \mathcal{O}(x^0))| = \left| \sum_{l=0}^{j} \binom{j}{l} \partial_x^{j-l} e^{-x} \partial_x^l \mathcal{O}(x^0) \right| \lesssim e^{-x} x^{-j} \langle x \rangle^j .$$

$\square$

Lemma A.0.5. *Let $x, y \in [0, \infty)$ and $j \in \mathbb{N}_0$. Then*

$$\partial_x^j \mathcal{O}(x^0 (x+y)^0) = \mathcal{O}(x^{-j} (x+y)^0).$$

Proof. We have

$$|\partial_x^j \mathcal{O}(x^0 (x+y)^0)| \lesssim \sum_{l=0}^{j} x^{-(j-l)} (x+y)^{-l} \lesssim x^{-j},$$

since $x + y \ge x$, so $(x+y)^{-l} \le x^{-l}$. $\square$

Lemma A.0.6. *Let $x \in [0, \infty)$, $y, \omega \in \mathbb{R}$ such that y and ω have the same sign, and $k \in \mathbb{N}_0$. Then*

$$\partial_\omega^k \mathcal{O}\left(\left(\frac{y}{\omega} + x \right)^0 \right) = \mathcal{O}\left(\left(\frac{y}{\omega} + x \right)^0 \omega^{-k} \right).$$

Proof. First, the claim is true for $k = 0$. For $k \ge 1$, we show that

$$\partial_\omega^k \mathcal{O}\left(\left(\frac{y}{\omega} + x \right)^0 \right) = \sum_{m=1}^{k} \mathcal{O}\left(\left(\frac{y}{\omega} + x \right)^{-m} \right) y^m \omega^{-k-m}.$$

We do this by induction. When $k = 1$, we have

$$\partial_\omega \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0\right) = \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^{-1}\right) y\omega^{-2}$$

which is what the formula claimed. Next, suppose that this is true for all derivatives up to order $k - 1$. Then

$$\begin{aligned}
\partial_\omega^k \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0\right) &= \partial_\omega \sum_{m=1}^{k-1} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^{-m}\right) y^m \omega^{-(k-1)-m} \\
= \sum_{m=2}^{k} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^{-m}\right) y^m \omega^{-k-m} &+ \sum_{m=1}^{k-1} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^{-m}\right) y^m \omega^{-k-m} \\
= \sum_{m=1}^{k} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^{-m}\right) y^m \omega^{-k-m}.
\end{aligned}$$

So we can estimate

$$\begin{aligned}
\left|\partial_\omega^k \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0\right)\right| &\lesssim \sum_{m=1}^{k} \left|\left(\frac{y}{\omega} + x\right)^{-m} y^m \omega^{-k-m}\right| \\
&= \sum_{m=1}^{k} \left|\left(\frac{\omega}{y + \omega x}\right)^m y^m \omega^{-k-m}\right| \lesssim \left|\frac{y}{y + \omega x}\right|^m |\omega|^{-k} \leq |\omega|^{-k},
\end{aligned}$$

since $|y| \leq |y + \omega x|$ for ω and y with the same sign and $x \geq 0$. □

Lemma A.0.7. *Let x, y, ω, k be as in the previous lemma. Then*

$$\partial_\omega^k \left(e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0\right)\right) = \sum_{m=0}^{k} e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0 y^m \omega^{-k-m}\right).$$

Proof. The claim is true for $k = 0$. Now suppose the statement is true for up to $k - 1$ derivatives. Then

$$\begin{aligned}
\partial_\omega^k \left(e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0\right)\right) &= \partial_\omega \sum_{m=0}^{k-1} e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(-\frac{y}{\omega} + x\right)^0 y^m \omega^{-(k-1)-m}\right) \\
= \sum_{m=1}^{k} e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0 y^m \omega^{-k-m}\right) &+ \sum_{m=0}^{k-1} e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0 y^m \omega^{-k-m}\right) \\
= \sum_{m=0}^{k} e^{-\frac{y}{\omega}} \mathcal{O}\left(\left(\frac{y}{\omega} + x\right)^0 y^m \omega^{-k-m}\right),
\end{aligned}$$

as required. □

Appendix B

Derivatives of the Volterra solution

Here we show that the solution to the Volterra equation (2.22) is of symbol type. We introduce the variables $x = \varphi(\rho)$, $y = \varphi(s)$ with $\varphi(s) = \frac{1}{2}\log\frac{1+s}{1-s}$. Define $\tilde{H}(x;\lambda) := h_1(\varphi^{-1}(x);\lambda)$, by change of variables, (2.22) is transformed into a Volterra equation of $\tilde{H}$,

$$\tilde{H}(x;\lambda) = 1 + \int_x^\infty \tilde{K}(x,y;\lambda)\tilde{H}(y;\lambda)dy,$$

where $\tilde{K}(x,y;\lambda) := K(\varphi^{-1}(x),\varphi^{-1}(y);\lambda)(\varphi^{-1})'(y)$. Notice that we have $|\partial_y^j\varphi^{-1}(y)| \lesssim_j e^{-2y}$ for all $j \in \mathbb{N}$ and $y \geq 0$, and

$$\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}-\lambda}\left(\frac{1-s}{1+s}\right)^{-\frac{1}{2}+\lambda} = e^{(1-2\lambda)(y-x)} = e^{2i\omega(y-x)}e^{(1-2\epsilon)(y-x)}.$$

Since $\epsilon \in [0,\frac{1}{4}]$ and $x \leq y$, we also have $e^{(1-2\epsilon)(y-x)} \leq e^{(y-x)}$. Hence we can write

$$\tilde{K}(x,y;\lambda)(x,y;\lambda) = a(y;\lambda) + b(x,y;\lambda)e^{-2i\omega(y-x)},$$

where $a(y;\lambda) = e^{-2y}\mathcal{O}(y^0\langle\omega\rangle^{-1})$ and $b(x,y;\lambda) = e^{-x}e^{-y}\mathcal{O}(x^0y^0\langle\omega\rangle^{-1})$.

We rewrite this equation further, by setting

$$\tilde{H}(x;\lambda) = 1 + e^{-2x}H(x;\lambda),$$

so

$$H(x;\lambda) := e^{2x}(H(x;\lambda) - 1).$$

Then we obtain the Volterra equation for H, given by

$$H(x;\lambda) = g(x;\lambda) + \int_x^\infty \hat{K}(x,y;\lambda)H(y;\lambda)dy \tag{B.1}$$

where

$$g(x;\lambda) := e^{2x}\int_x^\infty \tilde{K}(x,y;\lambda)dy$$

and

$$\hat{K}(x,y;\lambda) := \tilde{K}(x,y;\lambda)e^{2x}e^{-2y} = \hat{a}(x,y;\lambda) + \hat{b}(x,y;\lambda)e^{-2i\omega(y-x)},$$

with $\hat{a}(x,y;\lambda) = e^{2x}e^{-4y}\mathcal{O}(y^0\langle\omega\rangle^{-1})$ and $\hat{b}(x,y;\lambda) = e^{x}e^{-3y}\mathcal{O}(x^0y^0\langle\omega\rangle^{-1})$.

Now we turn to (B.1). We would like to show that a solution H to (B.1) is of symbol type, then conclude that a solution to (2.22) is also of symbol type. The argument is broken down into a few steps.

Step 1: Firstly, from (2.23) we have $|H(x;\lambda)| \lesssim \langle\omega\rangle^{-1}$. This also follows directly from Volterra iterations of (B.1), since

$$|g(x;\lambda)| \lesssim e^{2x}\int_x^\infty e^{-2y} + e^{-x}e^{-y}dy\,\langle\omega\rangle^{-1} \lesssim \langle\omega\rangle^{-1}.$$

Hence $\|g(\cdot;\lambda)\|_{L^\infty} \lesssim \langle\omega\rangle^{-1}$. Moreover, the Kernel $\hat{K}$ satisfies

$$|\hat{K}(x,y;\lambda)| \lesssim \left(e^{2x}e^{-4y} + e^{x}e^{-3y}\right)\langle\omega\rangle^{-1},$$

so

$$\int_{\delta_1\langle\omega\rangle^{-1}}^\infty \sup_{x\in(\delta_1\langle\omega\rangle^{-1},y)} |\hat{K}(x,y;\lambda)|dy \lesssim \int_{\delta_1\langle\omega\rangle^{-1}}^\infty \left(e^{2x}e^{-4y} + e^{x}e^{-3y}\right)dy\,\langle\omega\rangle^{-1} \lesssim \langle\omega\rangle^{-1}.$$

Step 2: We use induction to show that $|\partial_x^j H(x;\lambda)| \lesssim x^{-j}\langle\omega\rangle^{-1}$ for all $j\in\mathbb{N}_0$. This is true for $j=0$ from step 1. Now suppose that $|\partial_x^l H(x;\lambda)| \lesssim x^{-l}\langle\omega\rangle^{-1}$ for all $l=0,\cdots,j-1$. Taking derivatives of (B.1) we have

$$\partial_x^j H(x;\lambda) = g_{j,0}(x;\lambda) + \int_x^\infty \hat{K}(x,y;\lambda)\partial_y^j h(y;\lambda)dy$$

where

$$g_{j,0}(x;\lambda) := \partial_x^j g(x;\lambda) + \sum_{l=0}^{j-1}\binom{j}{l}\int_0^\infty \partial_x^{j-l}\hat{K}(x,y+x;\lambda)\partial_x^l H(y+x;\lambda)dy.$$

We compute

$$\begin{aligned}\partial_x^j g(x;\lambda) &= \partial_x^j\left(e^{2x}\int_0^\infty \tilde{K}(x,y+x;\lambda)dy\right)\\ &= \int_0^\infty e^{-2y}\mathcal{O}((y+x)^{-j}\langle\omega\rangle^{-1}) + e^{-y}\mathcal{O}(x^{-j}(y+x)^0\langle\omega\rangle^{-1})e^{-2i\omega y}dy.\end{aligned}$$

Hence $|\partial_x^j g(x;\lambda)| \lesssim x^{-j}\langle\omega\rangle^{-1}$. Moreover,

$$\begin{aligned}&|\partial_x^{j-l}\hat{K}(x,y+x;\lambda)|\\ =\;& \left|e^{-4y}\partial_x^{j-l}\left(e^{-2x}\mathcal{O}((y+x)^0\langle\omega\rangle^{-1})\right)\right| + \left|e^{-3y}\partial_x^{j-l}\left(e^{-2x}\mathcal{O}(x^0(y+x)^0\langle\omega\rangle^{-1})\right)\right|\\ \lesssim\;& x^{-(j-l)}\langle x\rangle^{j-l}\left(e^{-4y}e^{-2x} + e^{-3y}e^{-2x}\right)\langle\omega\rangle^{-1}.\end{aligned}$$

We also have

$$|\partial_x^l H(y+x;\lambda)| \lesssim (y+x)^{-l}\langle\omega\rangle^{-1} \lesssim x^{-l}\langle\omega\rangle^{-1}$$

for $l=0,\cdots,j-1$ from the inductive hypothesis. Hence

$$\left|\sum_{l=0}^{j-1}\binom{j}{l}\int_0^\infty \partial_x^{j-l}\hat{K}(x,y+x;\lambda)\partial_x^l h(y+x;\lambda)dy\right| \lesssim e^{-2x}x^{-j}\langle x\rangle^j\langle\omega\rangle^{-1} \lesssim x^{-j}\langle\omega\rangle^{-1}.$$

Therefore $|g_{j,0}(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1}$.

Now define $\tilde{g}_{j,0}(x;\lambda) := x^j g_{j,0}(x;\lambda)$ and $H_{j,0}(x;\lambda) := x^j \partial_x^j H(x;\lambda)$. Then $H_{j,0}$ solves the Volterra equation

$$H_{j,0}(x;\lambda) = \tilde{g}_{j,0}(x;\lambda) + \int_x^\infty \hat{K}(x,y;\lambda) x^j y^{-j} H_{j,0}(y;\lambda) dy$$

with $\|\tilde{g}_{j,0}(\cdot;\lambda)\|_{L^\infty} \lesssim \langle\omega\rangle^{-1}$ and $|\hat{K}(x,y;\lambda)x^j y^{-j}| \leq |\hat{K}(x,y;\lambda)|$. Hence Volterra iterations give the bound $\|H_{j,0}(\cdot;\lambda)\|_{L^\infty} \lesssim \langle\omega\rangle^{-1}$, and therefore $|\partial_x^j H(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1}$.

Step 3: We show that $|\partial_x^j \partial_\omega^k H(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1-k}$ for all $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$, by induction on k. The case when $k = 0$ is done in step 2. Now suppose that we have $|\partial_x^j \partial_\omega^m H(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1-m}$ for all $j \in \mathbb{N}_0$ and $m = 0, 1, \cdots k-1$.

Differentiating (B.1) directly, we have

$$\partial_x^j \partial_\omega^k H(x;\lambda) = g_{j,k}(x;\lambda) + \int_x^\infty \hat{K}(x,y;\lambda) \partial_y^j \partial_\omega^k H(y;\lambda) dy$$

where

$$\begin{aligned} g_{j,k}(x;\lambda) &:= \partial_x^j \partial_\omega^k g(x;\lambda) \\ &\quad + \sum_{\substack{0\leq l\leq j, 0\leq m\leq k,\\ (l,m)\neq(j,k)}} \binom{j}{l}\binom{k}{m} \int_0^\infty \partial_x^{j-l}\partial_\omega^{k-m}\hat{K}(x,y+x;\lambda)\partial_x^l \partial_\omega^m h(y+x;\lambda)dy. \end{aligned}$$

Using the forms of g and $\hat{K}$ and the inductive hypothesis, we find that $|g_{j,k}(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1}$. Then following the same argument as in step 2, Volterra iterations give the bound $|\partial_x^j \partial_\omega^k H(x;\lambda)| \lesssim x^{-j} \langle\omega\rangle^{-1}$.

So now we can assume that $|\omega| \geq 1$ and use a scaling argument to obtain the decay in ω. Observe that both g and $\hat{K}$ have a symbol part and an oscillatory part. Since the symbol part already gives enough decay, we only need to be concerned with the oscillatory part. In $g(x;\lambda)$, we write

$$\begin{aligned} e^{2x}\int_x^\infty b(x,y;\lambda)e^{-2i\omega(y-x)}dy &= e^x \int_x^\infty e^{-y}\mathcal{O}(x^0 y^0 \langle\omega\rangle^{-1})e^{-2i\omega(y-x)}dy \\ &= \int_0^\infty e^{-\frac{y}{\omega}}\mathcal{O}\left(x^0\left(\frac{y}{\omega}+x\right)^0\omega^{-2}\right)e^{-2iy}dy. \end{aligned}$$

Then no derivative will fall on the oscillatory term. Using Lemmas A.0.6 and A.0.7, we have

$$\begin{aligned} &\partial_x^j \partial_\omega^k \int_0^\infty e^{-\frac{y}{\omega}}\mathcal{O}\left(x^0\left(\frac{y}{\omega}+x\right)^0\omega^{-2}\right)e^{-2iy}dy \\ &= \sum_{m=0}^k \int_0^\infty e^{-y}\mathcal{O}\left(x^{-j}(y+x)^0 y^m \omega^{-1-k}\right)e^{-2i\omega y}dy. \end{aligned}$$

Taking the absolute value we have

$$\begin{aligned} \left|\sum_{m=0}^k \int_0^\infty e^{-y}\mathcal{O}\left(x^{-j}(y+x)^0 y^m\omega^{-1-k}\right)e^{-2i\omega y}dy\right| &\lesssim \sum_{m=0}^k x^{-j}\omega^{-1-k}\int_0^\infty e^{-y}y^m dy \\ &\lesssim x^{-j}\omega^{-1-k}. \end{aligned}$$

Hence we indeed have $|\partial_x^j\partial_\omega^k g(x;\lambda)| \lesssim x^{-j}\omega^{-1-k}$. Similarly, the $\hat{a}$ part of $\hat{K}$ already gives enough decay, and we write

$$\int_x^\infty \hat{b}(x,y;\lambda)e^{-2i\omega(y-x)}H(y;\lambda)dy = \int_0^\infty \hat{b}\left(x, \frac{y}{\omega}+x;\lambda\right)e^{-2iy}H\left(\frac{y}{\omega}+x;\lambda\right)\frac{dy}{\omega}.$$

So we can have

$$\begin{aligned}&\partial_x^j\partial_\omega^k\int_0^\infty \hat{b}\left(x, \frac{y}{\omega}+x;\lambda\right)e^{-2iy}H\left(\frac{y}{\omega}+x;\lambda\right)\frac{dy}{\omega}\\ &= b_{j,k}(x;\lambda) + \int_x^\infty \hat{b}(x,y;\lambda)\partial_x^j\partial_m^k H(y;\lambda)e^{-2i\omega(y-x)}dy,\end{aligned}$$

where

$$\begin{aligned}&b_{j,k}(x;\lambda)\\ :=& \sum_{\substack{0\le l\le j, 0\le m\le k,\\ (l,m)\neq(j,k)}}\binom{j}{l}\binom{k}{m}\int_0^\infty e^{-2iy}\partial_x^{j-l}\partial_\omega^{k-m}\frac{\hat{b}\left(x,\frac{y}{\omega}+x;\lambda\right)}{\omega}\partial_x^l\partial_\omega^m H\left(\frac{y}{\omega}+x;\lambda\right)dy\\ &+\int_0^\infty e^{-2iy}\frac{\hat{b}\left(x,\frac{y}{\omega}+x;\lambda\right)}{\omega}\left(\partial_x^j\sum_{m=0}^{k-1}\binom{k}{m}\partial_1^{k-m}\partial_2^m H\left(\frac{y}{\omega}+x;\lambda\right)\right)dy.\end{aligned}$$

By inductive hypothesis, H behaves like a symbol when the second slot is differentiated up to $k-1$ times. So we have

$$\left|\partial_x^l\partial_\omega^m H\left(\frac{y}{\omega}+x;\lambda\right)\right| \lesssim \left(\frac{y}{\omega}+x\right)^{-l}\omega^{-1-m} \lesssim x^{-l}\omega^{-1-m},$$

for all $l\in\mathbb{N}_0$ and $m=1,\cdots,k-1$. Moreover,

$$\left|\partial_x^j\sum_{m=0}^{k-1}\binom{k}{m}\partial_1^{k-m}\partial_2^m H\left(\frac{y}{\omega}+x;\lambda\right)\right| \lesssim \left(\frac{y}{\omega}+x\right)^{-j}\omega^{-1-k}\lesssim x^{-j}\omega^{-1-k}.$$

Similarly,

$$\begin{aligned}\partial_x^{j-l}\partial_\omega^{k-m}\frac{\hat{b}\left(x,\frac{y}{\omega}+x;\lambda\right)}{\omega} &= \partial_x^{j-l}\partial_\omega^{k-m}\left(e^x e^{-3\left(\frac{y}{\omega}-x\right)}\mathcal{O}\left(x^0\left(\frac{y}{\omega}-x\right)^0\omega^{-2}\right)\right)\\ &= \partial_x^{j-l}\left(e^{-2x}\sum_{m'=0}^{k-m}e^{-3\frac{y}{\omega}}\mathcal{O}\left(x^0\left(\frac{y}{\omega}-x\right)^0 y^{m'}\omega^{-2-(k-m)-m'}\right)\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\partial_x^{j-l}\left(e^{-2x}\sum_{m'=0}^{k-m}\mathcal{O}\left(x^0\left(\frac{y}{\omega}-x\right)^0 y^{m'}\omega^{-(k-m)-m'}\right)\right)\right|\\ &\lesssim e^{-2x}x^{-(j-l)}\langle x\rangle^{j-l}\sum_{m'=0}^{k-m}y^{m'}\omega^{-2-(k-m)-m'}.\end{aligned}$$

Now we can estimate

$$
\begin{aligned}
|b_{j,k}(x;\lambda)| &\lesssim e^{-2x}x^{-j}\langle x\rangle^{j}\left(\sum_{m=0}^{k}\sum_{m'=0}^{k-m}\int_0^\infty e^{-3\frac{y}{\omega}}y^{m'}dy\omega^{-3-k-m'}+e^{-2x}\int_0^\infty e^{-3\frac{y}{\omega}}dy\omega^{-3-k}\right)\\
&\lesssim x^{-j}\left(\sum_{m'=0}^{k}\int_0^\infty e^{-3y}y^{m'}dy\omega^{-2-k}+\int_0^\infty e^{-3y}dy\omega^{-2-k}\right)\\
&\lesssim x^{-j}\omega^{-2-k}.
\end{aligned}
$$

Hence we conclude that

$$
|g_{j,k}(x;\lambda)|\lesssim x^{-j}\langle\omega\rangle^{-1-k},
$$

and solving the Volterra equation we obtain $|\partial_x^j\partial_\omega^k H(x;\lambda)|\lesssim x^{-j}\langle\omega\rangle^{-1-k}$.

Transforming back to h_1: we show that $|\partial_\rho^j\partial_\omega^k h_1(\rho;\lambda)|\lesssim \rho^{-j}(1-\rho)^{1-j}\langle\omega\rangle^{-1-k}$ for $j+k\geq 1$.

So far we have shown that $\tilde{H}(x;\lambda)=1+e^{-2x}\mathcal{O}(x^0\langle\omega\rangle^{-1})$. Hence for any $j+k\geq 1$, we have

$$
\left|\partial_x^j\partial_\omega^k\tilde{H}(x;\lambda)\right|\lesssim e^{-2x}x^{-j}\langle x\rangle^{j}\langle\omega\rangle^{-1-k}.
$$

Recall that $\varphi(\rho)=\frac{1}{2}\log\frac{1+\rho}{1-\rho}$, so we have

$$
\left|\partial_\rho^j\varphi(\rho)\right|\lesssim(1-\rho)^{-j}
$$

for $\rho\in(0,1)$ and $j\in\mathbb{N}$. Now

$$
\partial_\rho^j h_1(\rho;\lambda)=\partial_\rho^j\tilde{H}(\varphi(\rho);\lambda)=\sum_{l=1}^{j}\tilde{H}^{(j,0)}(\varphi(\rho);\lambda)\tilde{\varphi}_{j,l}(\rho)
$$

where $|\tilde{\varphi}_{j,l}(\rho)|\lesssim(1-\rho)^{-j}$. So

$$
\begin{aligned}
|\partial_\rho^j\partial_\omega^k h_1(\rho;\lambda)| &\lesssim\sum_{l=1}^{j}\left|\tilde{H}^{(j,k)}(\varphi(\rho);\lambda)\right|(1-\rho)^{-j}\\
&\lesssim\varphi(\rho)^{-j}\langle\varphi(\rho)\rangle^{j}e^{-2\varphi(\rho)}\langle\omega\rangle^{-1-k}(1-\rho)^{-j}\\
&\lesssim\rho^{-j}(1-\rho)^{1-j}\langle\omega\rangle^{-1-k}
\end{aligned}
$$

as desired.

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