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Contents

Declararion of Authorship

I hereby certify that this master thesis I am submitting is entirely my own, original work, except where otherwise indicated. I am aware of the university's examination regulations concerning plagiarism. Throughout the course of my studies, I have neither submitted an identical or comparable thesis before, nor have I failed it definitely.

Vienna, September 25, 2019

Robert Biermann

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Abstract [German]

Kontingenztafeln werden vielfach genutzt, um diskrete multikategorische Variablen auf Abhängigkeit zu testen. Komplikationen entstehen, wenn die dahinterliegenden Variablen autokorreliert sind. Dieser Fall kann in beinahe allen Wissenschaften eintreten, insbesondere in der medizinischen Forschung, Meteorologie und Ökonomie. Findet hier keine Berichtigung an einem typischen Pearson χ^2-Test statt, erhöht sich der Umfang des Tests stark. Diese Arbeit behandelt eine Reihe an Tests, die diese Abweichung zu korrigieren versuchen. Die zwei betrachteten Hauptmethoden sind ein dynamisch angepasstes reduziertes Rangregressionsmodel, welches die kanonischen Korrelationen zwischen den Variablen testet und eine Methode, welche die Form der χ^2-Verteilung mit Hilfe der Eigenwerte der Übergangsmatrizen der Variablen korrigiert. Monte Carlo Simulationen beschreiben die Vor- und Nachteile der beiden Herangehensweisen. Die Tests wurden in einem empirisches Datensatz zu Preisveränderungen zwischen Produktkategorien angewandt, um die praktische Relevanz der Tests zu illustrieren.[1]

Schlagwörter: Kontingenztafeln, Autokorrelation, Abhängigkeitstests, Monte Carlo Simulationen, Markovketten

[1]Sollten Sie die Programmierergebnisse der Arbeit reproduzieren wollen, kontaktieren Sie bitte robert.biermann@web.de

Abstract

Contingency tables are widely used to test dependence among discrete multi-category variables. A complication arises, however, if the underlying variables are autocorrelated. Such situations can occur in almost all sciences, especially medical research, meteorology and economics. Here, if no adjustment is made to a typical Pearson χ^2 test, its size gets inflated. This thesis explores a number of tests attempting to correct for such cases. The two main approaches here are a dynamically augmented reduced rank regression model which tests the canonical correlations between the variables and a method to correct the shape of the χ^2 distribution using the eigenvalues of the variables' transition matrices. Monte Carlo simulations of the tests are used to describe the trade-offs between the test approaches. The tests are applied on an empirical data set on price changes between and within product categories to illustrate their practical relevance.[2]

Key words: Contingency Tables, Autocorrelation, Dependency Tests, Monte Carlo Simulations, Markov Chains

[2]If you wish to reproduce this thesis' programming results, kindly contact robert.biermann@web.de

1 Introduction

Discrete multi category variables are found in a plethora of scientific contexts. Often it is of interest to test for dependence between such variables. In the gathering of the category values there may have been a significant time delay, giving rise to the possibility of auto (or synonymously serial-) correlation within each variable. Such cases arise frequently in medical studies, meteorology, economics and finance (cf. introduction in Pesaran and Timmermann (2009)). In above cases the possibility of a strong time-dependence is often intuitively clear. Econometric literature, for instance, has a long tradition in modelling time series processes in macroeconomics. In meteorology several models are based on differential equations. While those models usually describe continuous processes (either in time or in the covariates), there may be cases where only categorical data are available or where it is sensible to categorise continuous values. Another field of growing relevance is online surveys. Here, there is often an initial pool of participants, each of which is asked to forward the survey to acquaintances. As some preferences between colleagues or friends are likely to be correlated, this would then give rise to autocorrelation within a subset of answers.

Typical tests for independence between categorical variables are the Fisher test, as well as Pearson's χ^2 test. If the variables are autocorrelated, however, the test statistics get inflated, leading to implications of dependence well beyond the set significance level. An early adjustment of the Pearson χ^2 statistic can be found in Tavaré (1983). He shows, that for reversible Markov chains in the bivariate case the Pearson χ^2 statistic is asymptotically distributed as the sum of weighted χ^2 statistics. The weights here use eigenvalues of the variables' transition matrices. This approach was generalised by Porteous (1987) for the multivariate case. Elsinger (2017, 2019) further refines the method by exploring alternative test statistics and loosens the requirement of reversibility.

Pesaran and Timmermann (2009) find a different way to filter the autocorrelation by employing a dynamically augmented reduced rank regression model. The model can be extended to the multivariate case, as well, and allows a high degree of serial correlation in the variables without becoming oversized, given enough observations.

Both ways of modelling yield positive and less convincing results when subjected to simulation under variation of observation size, degree of autocorrelation, number of lags in the serial process, et cetera.
As an empirical demonstration the tests have been applied on a data set provided by the University of Chicago Booth School of Business on product scanner data of finer foods store Dominick's. As argued in Pesendorfer (2002) retail prices often behave in a way that categorisation of price levels or price changes makes sense. We check, whether there is dependence in price levels between various product categories. Due to the significant time span of the data set (weekly data from 1989 to 1994) autocorrelation exists and for certain product groups the standard χ^2 leads to different conclusions than the adjusted tests.

2 Statistical Tests

This section outlines the mathematical background of the contingency tests discussed in this thesis. The aim is to provide a general understanding of the construction methods used in each test. We begin by reviewing common definitions required for describing the tests.

2.1 Review of Common Definitions

The following definitions can be found in virtually any literature on stochastic processes and mathematical statistics, such as Klenke (2006) and Rüschendorf (2014) and have been adopted in notation for the purpose of this thesis. The definitions are kept in a straightforward manner to avoid distracting meticulousness such as a detour into measure theory. This applies especially for the simplified definitions on statistical tests employed here.

Definition 1. *A* ***stochastic process*** *is a set of random variables* $\{X(t,\omega) : t = 1,...,T\}$*, where* ω *is an element of the sample space* Ω *and* $X(\cdot,\omega)$ *maps into the process' state space* S*.*

In the context of categorical variables, this means that for $1,...,T$ observations of variable X with m_x categories, $\Omega = \{1,...,m_x\}^T$ and $X(\cdot,\omega) \longrightarrow \{1,...m_x\}$.

Definition 2. *For $X_t \in \mathbb{R}$, an **autoregressive process of order p [AR(p)]** is defined as*

$$X_t = c + \sum_{i=1}^{p} \phi_p X_{t-p} + \varepsilon_t,$$

with $c, \phi_i \in \mathbb{R}$ and ε white noise.

Definition 3. *A stochastic process is **stationary** if its unconditional joint probability distribution does not change with time shifts.*

Definition 4. *A (discrete-time) l-order **Markov chain** is a sequence of random variables $X_1, X_2, ...,$ such that the probability of moving to the next state is dependent only on the past states up to order l and not previous ones and $X_1, X_2, ...$ map into a countable state space S.*

Definition 5. *For a first order Markov chain, let $q_{ij} = \mathbb{P}(X_{t+1,j} = 1 | X_{t,i} = 1)$ be the probability to reach state j from state i and $\boldsymbol{Q} := (q_{ij})_{ij}$ be its corresponding **transition matrix**.*

Definition 6. *For a first order Markov chain, let $p_{ij} = \mathbb{P}(X_{t+1,j} = 1, X_{t,i} = 1)$ be the **joint probability** to reach state j from state i and $\boldsymbol{P} := (p_{ij})_{ij}$ be its summarising matrix, with $\boldsymbol{D} := diag(p_1, ..., p_{|S|})$ and thus $\boldsymbol{Q} = \boldsymbol{D}^{-1}\boldsymbol{P}$.*

Definition 7. *A Markov chain is **irreducible** if it is possible to reach any state from any state within $n_{ij} \in \mathbb{N}$ steps, e.g. if $\mathbb{P}(X_{n_{ij}} = j | X_0 = i) > 0$.*

Definition 8. *The period of a state i in a Markov chain is defined as*

$$k = \gcd\{n > 0 : \mathbb{P}(X_n = i | X_0 = i) > 0\}.$$

*If $k = 1$, the state is aperiodic. A Markov chain is **aperiodic** if all of its states are aperiodic.*

Definition 9. *A Markov chain is **ergodic** if it is irreducible and aperiodic.*

Definition 10. *A Markov chain is **reversible** if there are probability distributions π_i, π_j, such that $\pi_i q_{ij} = \pi_j q_{ji}$, for all $i, j \in S$.*

Let the order of a Markov chain X be l_X and consider $l_X > 1$. As shown in Cox and Miller (1965, pp. 132-133) such a chain can be transformed into a first order chain.

For example, if $l_X = 2, m_x = 2$, then the transition matrix is extended to

$$
\tilde{\boldsymbol{Q}} = \begin{array}{ccc} & & X_{t-1} \\ X_{t-2} & X_{t-1} & X_t \\ 1 & 1 & \\ 1 & 2 & \\ 2 & 1 & \\ 2 & 2 & \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{array} \\ \begin{pmatrix} p_{111} & p_{112} & 0 & 0 \\ 0 & 0 & p_{121} & p_{122} \\ p_{211} & p_{212} & 0 & 0 \\ 0 & 0 & p_{221} & p_{222} \end{pmatrix} \end{array}
$$

due to the expansion of the state space S_X from $S_X|_{l=1} = \{\{1\},\{2\}\}$ to $S_X|_{l=2} = \{\{1,1\},\{1,2\},\{2,1\},\{2,2\}\}$. Thus, in general $|S| = m^l$, leading to an exponential increase in the size of our transition matrix with each lag added and hence requires $(1-m)m^l$ independent parameters to be estimated. In above case $\tilde{\boldsymbol{Q}}$ is still irreducible as can quickly be checked. It is, however, also aperiodic as case (2,1) can be reached at steps $\{2,3,4,...\}$ by the paths $(2,1) \to (1,2) \to (2,1)$ for even steps and $(2,1) \to (1,1) \to (1,2) \to (2,1)$ for odd steps, thus allowing a combination of both to reach (2,1) again for every natural number greater than 1. Analogously with (2,1) and (1,1), (2,2) communicate directly. This neat result can easily be extended to the general case of $l > 1$, which will be useful later for generalising the tests for higher lags:

Lemma 1. *For a Markov chain X with lag l, let $\tilde{X}$ be the equivalent process with extended state space $S_{\tilde{X}} = S_X^l$. If X is ergodic, then $\tilde{X}$ is also ergodic.*

Proof. For ergodic X, $p_X = (p_1, ..., p_{m_X}) > 0$. Thus, also $p_{\tilde{X}} > 0$ and $\tilde{X}$ is irreducible. A well-known theorem states that if in an irreducible Markov chain one state is aperiodic, then all states are aperiodic. As $(\underbrace{1,1,...,1}_{l-times}) \to (\underbrace{1,1,...,1}_{l-times})$ always holds, the first state of $\tilde{X}$ is aperiodic and thus $\tilde{X}$ aperiodic and ergodic. $\square$

In below notation let $\{X_i\}_t$, $i = 1,...,d$ be d category variables of m_i categories

(or states) which occur as time series of $t = 1, ..., T$ observations. Let $\boldsymbol{x}_{i,T\times 1}$ be the vector of numeric category realisations $1, ..., m_i$ at each $t = 1, ..., T$. In matrix representation let $\boldsymbol{X}_{i,T\times m_i}$ be so that $x_{i,tj} = 1 \iff \boldsymbol{x}_{i,t} = j$ and otherwise $x_{i,tj} = 0$. Let $\boldsymbol{X}_{i,-1}$ be the sequence's matrix representation shifted one row backwards, e.g. row t of $\boldsymbol{X}_{i,-1}$ is row $t+1$ in $\boldsymbol{X}_i$ (and $t = T$ in $\boldsymbol{X}_{i,-1}$ maps to $t = 1$ in $\boldsymbol{X}_i$).
In case of $l_i > 1$ let $\tilde{\boldsymbol{x}}_{i,T\times 1}$ be the vector of extended enumerated category realisations for $S_i|_{l_{X_i}}$ and construct analogously $\tilde{\boldsymbol{X}}_{i,T\times m_i^{l_i}}$.

Consider also the following conventions for state probability vectors. Let $p_{j_1,...,j_i,...,j_d}$ be the unconditional probability that $X_{1,j_1,t} = 1 \wedge ... \wedge X_{i,j_i,t} = 1 \wedge ... \wedge X_{d,j_d,t} = 1$. Then the $\prod_i m_i \times 1$ vector of unconditional state probabilities is

$$\boldsymbol{p} = (p_{(1,1,...,1)}, ..., p_{(1,1,...,m_d)}, ..., p_{(m_1,...,m_d)})'.$$

The vector of marginal probabilities of $\{X_i\}_t$ is thus

$$\boldsymbol{p}_{x_i} = \left(\sum p_{(...,j_i=1,...)}, ..., \sum p_{(...,j_i=m_i,...)}\right) = \left(\boldsymbol{1}'_{\prod_{i+1}^{d+1} m_i} \otimes \boldsymbol{I}_{m_i} \otimes \boldsymbol{1}'_{\prod_{i=0}^{i-1} m_i}\right) \boldsymbol{p},$$

with $m_0, m_{d+1} := 1$.

Definition 11. *A* ***statistical test*** *is a decision rule to decide between accepting a null hypothesis H_0 or rejecting it in favour of an alternative hypothesis H_1.*
A test statistic leads to rejecting H_0 at significance level α, if its value is larger than the α-quantile limit of the specified test's distribution.

Definition 12. *The* ***test size****, or type I error, is the probability of falsely rejecting the null hypothesis at significance level α.*

Definition 13. *The* ***test power****, β or type II error is the probability of failing to correctly reject the null hypothesis at significance level α.*

In the following we consider H_0 to be X, Y independent, with H_1 to be X, Y dependent and $\alpha = 0.05$. Dependency here is expressed as cross-correlation r_{xy}.

2.2 Fisher, Pearson-Chi-Squared for Contingency Tables

In the case of $d = 2$, $X := X_1, Y := X_2$ and X, Y not being serially correlated we may consider the canonical Pearson-Chi-Squared test. For $m_x, m_y \in \mathbb{N}$ we get below contingency table structure,

Y, X	1	2	$\cdots$	m_x	
1	n_{11}	n_{12}	$\cdots$	n_{1m_x}	n_{1+}
2	n_{21}	n_{22}	$\cdots$	n_{2m_x}	n_{2+}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_y	n_{m_y1}	n_{m_y2}	$\cdots$	$n_{m_ym_x}$	n_{m_y+}
	n_{+1}	n_{+2}	$\cdots$	n_{+m_x}	$n = T$

for which the Pearson-Chi-Squared statistic is

$$C_n = \sum_{i,j} \frac{n(n_{ij} - n_{i+}n_{+j}/n)^2}{n_{i+}n_{+j}} = n\left(\sum_{i,j} \frac{n_{ij}^2}{n_{i+}n_{+j}} - 1\right)$$

and as $n \to \infty$ the convergence $C_n \sim \chi^2_{(m_x-1)(m_y-1)}$ holds asymptotically.
If $m_x = m_y = 2$ we may also employ Fisher's Exact Test, which follows a hypergeometric distribution. Here,

$$p_{\text{Fisher}} = \frac{\binom{n_{1+}}{n_{11}}\binom{n_{2+}}{n_{21}}}{\binom{n}{n_{+1}}} = \mathbb{P}(H_0|H_1).$$

Note that for large N Fisher's and Pearson's tests converge.

2.3 Tavaré (1983), for bivariate case of reversible Markov chains

For X, Y being reversible, ergodic Markov chains, Tavaré (1983) shows that serial correlation inflates or deflates the Pearson-Chi-Squared test statistic, such that a correction can be achieved by

$$C_n \sim \sum_{i=1}^{m_x-1} \sum_{j=1}^{m_y-1} \underbrace{\left(\frac{1+\lambda_i\mu_j}{1-\lambda_i\mu_j}\right)}_{\rho_k} Z_{ij}^2,$$

where λ_i, μ_j are non-unit eigenvalues of the transition matrices of X, Y respectively, and Z_{ij} are i.i.d. $N(0,1)$. The reversibility condition ensures λ_i, μ_j are real. For

$m_X > 2$ (or $m_Y > 2$ analogously) the critical value becomes non-trivial as the weighted sum of χ^2_1 distributions has no closed solution. It can, however, be estimated quickly by Monte-Carlo simulation for any ρ_k, $k = 1, ..., (m_x-1)(m_y-1)$.

2.4 Porteous (1987), Elsinger (2017), generalised Tavaré (1983) test for multivariate cases, relaxed reversibility

Consider $X_1, ..., X_d$ variables with m_i categories each.
Porteous (1987) generalises Tavaré's (1983) approach by showing that for d variables the equivalent Pearson statistic given by

$$X^2_{I(d)} = T \sum_j \frac{\{\hat{p}(j) - \hat{p}(j_1)...\hat{p}(j_d)\}^2}{\hat{p}(j_1)...\hat{p}(j_d)}$$

for $T \to \infty$ asymptotically follows

$$X^2_{I(d)} \sim \sum_{k=1}^{\nu} \frac{1+\gamma_k}{1-\gamma_k} Z^2_1, \tag{1}$$

where $\nu = r_1...r_d - (r_1 + ... + r_d) - (d+1)$.
An alternative representation of the test statistic is given in Elsinger (2017, 2019) by

$$X^2_{A(d)} = T(\hat{\boldsymbol{p}} - \bigotimes_{i=1}^{d} \hat{\boldsymbol{p}}_{X_i})' \bigotimes_{i=1}^{d} (\boldsymbol{D}^{-1}_{X_i} - \mathbf{1}_{m_i}\mathbf{1}'_{m_i})(\hat{\boldsymbol{p}} - \bigotimes_{i=1}^{d} \hat{\boldsymbol{p}}_{X_i}),$$

$\boldsymbol{D}_{X_i} = \boldsymbol{X}'_i \boldsymbol{X}_i / T$. Here,

$$X^2_{A(d)} \sim \sum_{k=1}^{\tilde{d}} \frac{1+\gamma_k}{1-\gamma_k} Z^2_1,$$

with $\tilde{d} = (m_1 - 1)...(m_d - 1)$. This simplifies the weight structure compared to $X^2_{I(d)}$, as now each γ_l is one of the $\tilde{d}$ permutations of the products of non-unit eigenvalues $\{\lambda_{1,h_1}...\lambda_{d,h_d}\}$, $h_i = 1, ..., m_i$ of the transition matrices $Q_1, ..., Q_d$.
Elsinger (2017, 2019) further shows, if the reversibility assumption is violated, the sum of the weights ρ_k is still given by $\sum_k \frac{1+\gamma_k}{1-\gamma_k}$. We may then use approximations of the distribution such as $a\chi^2_b, c\chi^2_{\tilde{d}}$, with $c = \bar{\boldsymbol{\rho}} = \frac{\sum \rho_j}{\tilde{d}}$, $\quad a = \frac{\sum \rho_j^2}{\sum \rho_j}$, $\quad b = \frac{(\sum \rho_j)^2}{\sum \rho_j}$.

Both options yield similar results and in applications we shall here use $c\chi^2_{\tilde{d}}$.

2.5 Pesaran and Timmermann (2009), dynamically augmented trace statistic for multivariate cases

Let $d = 2$, $X := X_1, Y := X_2$. In the two chain case, first consider X, Y to be serial processes of lag order 1. The general idea of the test is to regress y on x and their lags as in

$$\theta' y_t = c(1-\varphi) + \gamma' x_t - \varphi\gamma' x_{t-1} + \varphi\theta' y_{t-1} + \varepsilon_t,$$

and then test if $\gamma = 0$ holds (θ is a nuisance parameter, ε an independent martingale difference process).

Put into matrix notation, one may write

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\Pi}' + \boldsymbol{W}\boldsymbol{B} + \boldsymbol{E} \tag{2}$$

and test for $\boldsymbol{\Pi} = 0$, where $\boldsymbol{W} = (\tau, \boldsymbol{X}_{-1}, \boldsymbol{Y}_{-1})$ and $\boldsymbol{E}$ a matrix of serially uncorrelated errors. Pesaran and Timmermann (2009) then show that this test is equivalent to testing the maximum or average of the canonical correlation coefficients of $\boldsymbol{Y}$ and $\boldsymbol{X}$ by computing

$$\boldsymbol{S} = \boldsymbol{S}^{-1}_{yy,w}\boldsymbol{S}_{yx,w}\boldsymbol{S}^{-1}_{xx,w}\boldsymbol{S}_{xy,w}. \tag{3}$$

As the eigenvalues of S_w capture the distorting effect of serial correlation and with $tr(S_w)$ being the sum of its eigenvalues, Pesaran and Timmermann's (2009) trace test statistic reads $T \cdot tr(S_w) \overset{a}{\sim} \chi^2_{(m_y-1)(m_x-1)}$.
To capture the cross-relations of X, Y and their lags in S_w, the transformations

$$\boldsymbol{S}_{yy,w} = T^{-1}\boldsymbol{Y}'\boldsymbol{M}_w\boldsymbol{Y}, \quad \boldsymbol{S}_{xx,w} = T^{-1}\boldsymbol{X}'\boldsymbol{M}_w\boldsymbol{X},$$
$$\boldsymbol{S}_{yx,w} = T^{-1}\boldsymbol{Y}'\boldsymbol{M}_w\boldsymbol{X}, \quad \boldsymbol{M}_w = \boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'$$

have been used.
Note. In above notation, non-singularity of all matrices is ensured, by reducing

the dimensions of X and Y by 1 to $m_x - 1$ and $m_y - 1$ respectively. This also requires $\tau = \mathbf{1}_{T\times 1}$ in $\boldsymbol{W}$. This complication can be eliminated via a contribution by Elsinger (2017, 2019) in substituting the matrix inverse with the Moore-Penrose pseudo inverse, showing that all results from Pesaran and Timmermann (2009) still hold. This simplification is especially relevant in numerical implementations of Pesaran and Timmermann's (2009) tests.

Pesaran and Timmermann (2009) also develop an F test based on the maximum eigenvalue of $\boldsymbol{S}$ in equation (3). For both these tests the initial foundation is a static version, which follows the same regression principle but does not include the lagged values in $\boldsymbol{W}$. We include them in the simulation output to show the results align with those in Pesaran and Timmermann (2009).

2.5.1 Higher Order of Auto Regression

Let the order of the auto regressive process of X or Y be l_X, l_Y respectively and consider the case where, w.l.o.g. $l_X > 1$ or $l_X, l_Y > 1$. Pesaran and Timmermann (2009) remark in their proof that by transforming $\boldsymbol{W}$ to include the respective lags in equation (3), the test asymptotics still hold, as a quick verification of the thus modified variance-covariance matrix of $\boldsymbol{W}$, arguably a key component of the proof, shows: Consider the relation $p\lim_{T\to\infty} T^{-1}\boldsymbol{W}'\boldsymbol{W} = \boldsymbol{\Sigma}_{ww} = Var(\varepsilon_t)$, with

$$\boldsymbol{\Sigma} ww = \begin{pmatrix} 1 & \boldsymbol{p}_y & \boldsymbol{p}_x \\ \boldsymbol{p}_y & \boldsymbol{D}_y & \mathbf{0} \\ \boldsymbol{p}_y & \mathbf{0} & \boldsymbol{D}_x \end{pmatrix}$$

with $\boldsymbol{p}_x = (p_{x,1}, ..., p_{x,m_x})$ and $\boldsymbol{D}_x = diag(\boldsymbol{p}_x)$. Non-singularity of $\boldsymbol{\Sigma}_{ww}$ is ensured by the Markov chains in $\boldsymbol{W}$ being ergodic. The transformation of $\boldsymbol{W}$ to $\tilde{\boldsymbol{W}}$ hence includes $p_{x,m_x^{l_x}}$ marginal probabilities needed due to the higher lag and $\boldsymbol{\Sigma}\tilde{w}\tilde{w}$ is still non-singular, as for ergodic X, $\tilde{X}$ is also ergodic as shown in lemma 1. Note, however, that we then also have to transform X, Y to $\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}$ to keep matrix dimensions consistent in equation (3).

2.5.2 Higher Number of Categorical Variables

For $d = 3$, $Z := X_3$ Pesaran and Timmermann (2009) produce a test of joint independence of X, Z from Y, in similar notation as above, such that

$$T \cdot tr(S_{yy,w}^{-1} S_{yq,w} S_{qq,w}^{-1} S_{qy,w}) \overset{a}{\sim} \chi^2_{(my-1)(m_x+m_z-2)}, \tag{4}$$

with

$$\begin{gathered} \boldsymbol{S}_{yq,w} = T^{-1} Y' \boldsymbol{M}_w \boldsymbol{Q}, \quad \boldsymbol{S}_{qq,w} = T^{-1} \boldsymbol{Q}' \boldsymbol{M}_w \boldsymbol{Q}, \quad \boldsymbol{Q} = (\boldsymbol{X}, \boldsymbol{Z}), \\ \boldsymbol{M}_w = \boldsymbol{I} - \boldsymbol{W}'(\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}', \quad \boldsymbol{W} = (\boldsymbol{\tau}, \boldsymbol{X}_{-1}, \boldsymbol{Y}_{-1}, \boldsymbol{Z}_{-1}). \end{gathered}$$

Elsinger (2017, 2019) shows that above equation tests for simultaneous pairwise independence of $(\{X\}, \{Z\})$ from $\{Y\}$ rather than joint independence and suggests to redefine $\boldsymbol{Q}$ as $\tilde{\boldsymbol{Q}} := \boldsymbol{X} \otimes \boldsymbol{Z}$, with the resulting test statistic asymptotically following a $\chi^2_{(m_x m_z - 1)(m_y - 1)}$ distribution.

2.5.3 Generalised Pesaran Timmermann (2009) Trace Test

With above extensions to equation (3), we can formulate a general Pesaran-Timmermann trace statistic for joint independence of $X_2, ..., X_d$ from $X_1 =: Y$ under arbitrary lags in each X_i as

$$T \cdot tr(\boldsymbol{S}^+_{\tilde{y}\tilde{y},\tilde{w}} \boldsymbol{S}_{\tilde{y}\tilde{q},\tilde{w}} \boldsymbol{S}^+_{\tilde{q}\tilde{q},\tilde{w}} \boldsymbol{S}_{\tilde{q}\tilde{y},\tilde{w}}) \overset{a}{\sim} \chi^2_{(m_y-1)(\sum_{i \neq 1} m_i - d + 1)}, \tag{5}$$

with

$$\begin{gathered} \boldsymbol{S}_{\tilde{y}\tilde{q},\tilde{w}} = T^{-1} \tilde{\boldsymbol{Y}}' \boldsymbol{M}_{\tilde{w}} \tilde{\boldsymbol{Q}}, \quad \boldsymbol{S}_{\tilde{q}\tilde{q},\tilde{w}} = T^{-1} \tilde{\boldsymbol{Q}}' \boldsymbol{M}_{\tilde{w}} \tilde{\boldsymbol{Q}}, \quad \tilde{\boldsymbol{Q}} = \left(\bigotimes_{i \neq 1} \tilde{\boldsymbol{X}}_{\boldsymbol{i}} \right), \\ \boldsymbol{M}_{\tilde{w}} = \boldsymbol{I} - \tilde{\boldsymbol{W}}'(\tilde{\boldsymbol{W}}'\tilde{\boldsymbol{W}})^+ \tilde{\boldsymbol{W}}', \quad \tilde{\boldsymbol{W}} = (\tilde{\boldsymbol{X}}_{2,-1}, \tilde{\boldsymbol{Y}}_{-1}, ..., \tilde{\boldsymbol{X}}_{d,-1}). \end{gathered}$$

3 Monte-Carlo Simulation

This section details the construction and results of the Monte-Carlo Simulations of the tests to gauge their convergence patterns.

3.1 Fisher Test under Autocorrelation

Consider X following an AR(1) process with $\phi \in [0, 1)$, $m_X = 2$, categorised with equal probability and an i.i.d. process Y. To quickly see why the Fisher test on independence of X and Y is inappropriate here, consider figure 1, whose graph shows the simulated test size for above scenario for series of length $T = 1000$. Clearly, as $\phi > 0.5$ the test rejects the null hypothesis of independence far more

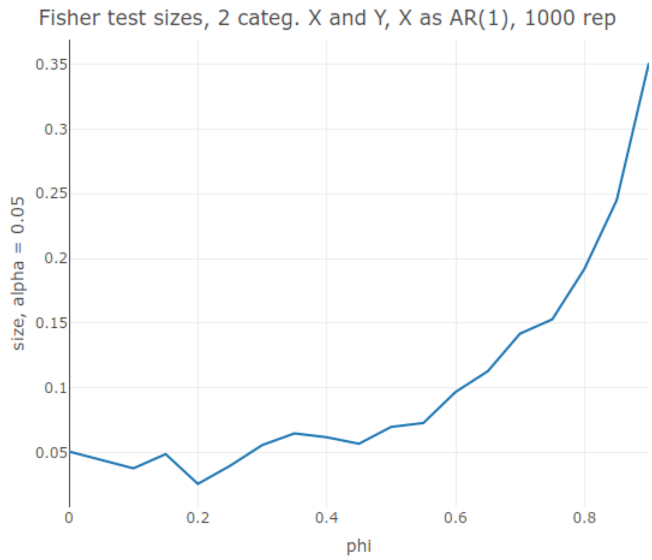

Figure 1: Fisher test sizes, 1000 realisations at each point

often than specified under the significance level.

3.2 Pesaran Timmermann (2009), Tavaré (1983) Test Simulation

Following Pesaran and Timmermann (2009) [PT (2009)], we simulate Markov chains X, Y via AR(1) processes with $\phi \in \{0, 0.5, 0.8, 0.95\}$, whose realisations are categorised into equally probable bins with $m_x = m_y$ and $m_x, m_y \in \{2, 3, 4, 5\}$ over sample sizes of $T \in \{20, 50, 100, 500, 1000\}$.

On these chains Pesaran and Timmermann's (2009) Trace Statistic and Maximum Correlation Statistic are applied, as well as Tavaré's (1983) Test for comparison, to determine the observed test size. The tests were carried out for significance level $\alpha = 0.05$. A similar exercise follows for the tests' power under cross-correlations between X, Y, $r_{xy} \in \{0, 0.2, 0.8\}$.

In Pesaran and Timmermann (2009), the processes' order is determined by AIC, limiting the search to $1 \leq \hat{l} \leq 4$. It is unclear here, if they apply the information criterion on the underlying time series or on the resulting Markov chain. Furthermore, it is not documented how the test statistic is supposed to be computed if for instance $T \in \{50, 100\}$, $m = 3$, and $\hat{l} = 3$, as this would require an estimation of $(m-1)m^{\hat{l}} = 54$ parameters for the first order chain transformed matrices in equation (3). This simulation attempts to replicate the results by estimating the AR process's order by AIC via a common algorithm implemented in R and using the simplification $\boldsymbol{W} = (\boldsymbol{X}_{-l_X}, \boldsymbol{Y}_{-l_Y})$. For the Tavaré (1983) test we simplify by always using the straight test for lags of order 1. Furthermore, Pesaran and Timmermann (2009) use simulated critical values for their F-statistic, weighted χ^2 values for the trace test. Under these simplifying specifications we find our simulation results resemble those of Pesaran and Timmermann (2009) closely in both size and power for $T \geq 500$. For lower T the simulation results appear slightly oversized by about 0.01 to 0.02 compared to the tables in Pesaran and Timmermann (2006) and Pesaran and Timmermann (2009). This may be due to the fact the our bootstrapped F-statistic critical values differ slightly from those in Pesaran and Timmermann (2006). Further disturbances may come from slightly different convergence speeds in using the Moore-Person inverse instead of a truncated matrix with standard inverse. Furthermore note that we are basing the results on 10^4 repetitions instead of 2000 in Pesaran and Timmermann (2006). Finally the lag implementation is not entirely clear and will be explored further. Tables 6 to 10 attempt to recreate the power tables shown in Pesaran and Timmermann (2009). The same tests as in the size section for tables 2 and 3 were used, and the results resemble those of Pesaran and Timmermann (2009). Once $T \geq 100$ all tests rapidly approach 1 for a cross-correlation of 0.8.

In terms of size even if granting Tavaré's (1983) test the true lag value of 1, it performs worse than the dynamic trace statistic under autocorrelations above 0.8

in the cases of $T \in \{500, 1000\}$. Even though the dynamic trace test's size deviates slightly under $\phi = 0.95$, we can see a convergence towards $\alpha = 0.05$ with increasing T. The Tavaré (1983) test becomes instead spurious for high ϕ, even if we increase the number of observations. This behaviour may stem from the fact that the estimation of eigenvalues in transition matrices becomes numerically more difficult the closer they approach unity, as the underlying AR process edges closer to becoming non-stationary. Note, however, that for medium auto correlation the Tavaré test is very robust to an increase in the number of categories, even for low T. Here, the dynamic trace statistic becomes oversized quickly with increasing m. We thus also have to consider the case of using the truly expanded model with $\tilde{\boldsymbol{X}}$. For 10,000 time series realisations we get about 73% lag 1 estimates. Under these estimations we find in Tables 4 and 5 that the test size mostly holds up. The increase in parameters to be estimated for the remaining 27% of lag 2 estimates does not cause large enough inference issues to significantly disturb the trace statistic. In fact, the Pesaran and Timmermann (2009) dynamic trace test performs in many cases better than in the previous simplified version. Note, however, that for $m = 5$ we can see the exponential increase in parameters to affect the size.

To separate the issue of parameter increase cleanly, we investigate the convergence pattern of the Pesaran and Timmermann (2009) dynamic trace test and Tavaré (2008) test for high m and high T under perfect knowledge of $l = 1$, as after all a higher lagged value is transformed into an equivalent lag 1 test with higher parameter space.

In figure 2 we find high T slightly reducing the Pesaran and Timmermann (2009) dynamic trace test size, if we consider the value for $T = 8000$ an outlier. The Tavaré (1983) test continues to grow in size to slightly over 0.09 for $T = 7000$.

A different picture emerges in case of an escalating number of categories m. Here, the Pesaran and Timmermann (2009) dynamic trace test requires a very high T to keep the test size from increasing dramatically, as figure 3 shows. Tavaré's (1983) test, however, strives further towards $\alpha = 0.05$ in size with higher m, with relatively similar sizes for both $T = 1000$ and $T = 5000$.

In the Pesaran and Timmermann (2009) trace statistic matrices $\boldsymbol{S}_{(\cdot,\cdot)}$ are of dimension $m_{(\cdot)} \times m_{(\cdot)}$. Hence, the inference burden rises quadratically with an increase in $m_{(\cdot)}$, accentuating the disturbing effects of 'unlucky' category realisations which

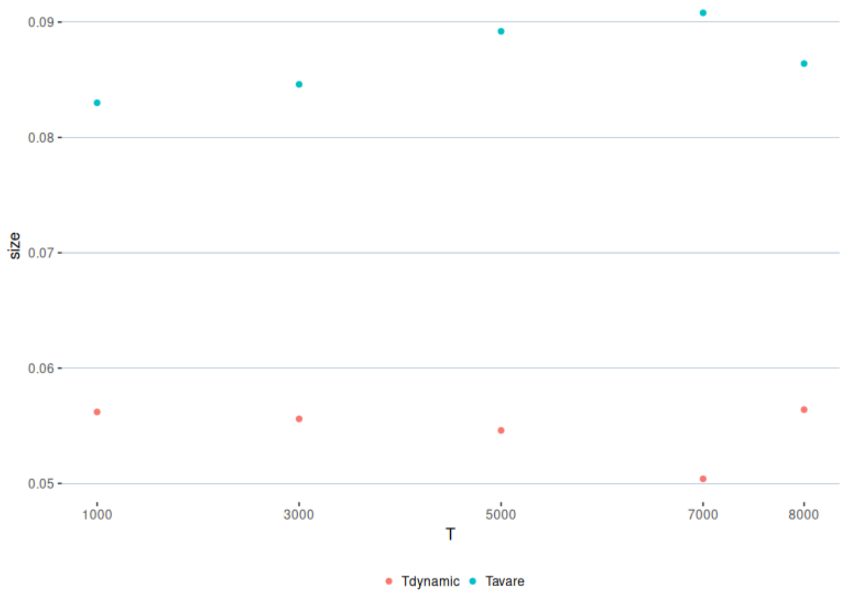

Figure 2: Test sizes PT (2009) Dynamic Trace, Tavaré (1983) for $\phi = 0.8$, $m = 3$, $l = 1$ under high T, 5000 simulations each step.

mislead from the true nature of the time series due to relatively small T being unable to 'correct' later. The quadratic nature of size development under escalating m is especially clear for the case PT (2009), $T = 1000$ in figure 3.
The Tavaré (1983) test's opposite behaviour may instead be explained by an increase in the number of weights γ_k leading to slightly higher critical values, thus fewer rejections of the null hypothesis and an overall more stable looking convergence pattern towards α for increasing m. Whether the effect is due to aforementioned potential eigenvalue estimation issues requires extensive analysis of the underlying theoretical properties of the test beyond the scope of this project.

Figure 3: left: Test sizes PT (2009) Dynamic Trace, Tavaré (1983) for $\phi = 0.8$, $l = 1$ under high T and high m, 5000 simulations each step. Dashed lines by polynomial smoothing for illustrative purpose.

3.3 Porteous (1987) Elsinger (2017) Test, Pesaran Timmermann (2009) Test with Generalised d

If we relax the number of variables to $d > 2$, we may compare the generalised Pesaran Timmermann test for joint independence and the Porteous (1987) Elsinger (2017) test. We consider the case where $m_i = 2$, $l = 1$ and $T = 1000$. Figure 4 shows the test size develops with some similarity to the bivariate case of Pesaran and Timmermann's (2009) trace statistic and the Tavaré (1983) test. Whereas the Porteous (1987) Elsinger (2017) test becomes oversized rapidly for $\phi > 0.8$ and $d < 5$, PT (2009) maintains a steady size around 0.05 up to very high T and d. Note, that the Pesaran and Timmermann (2009) test benefits in the multivariate case from a different number of degrees of freedom in the asymptotic χ^2 statistic compared to a bivariate case of the same total number of categories. Nevertheless, an increase in m_i would eventually lead to results akin to those in figure 3, as similar inference issues exist.

The Porteous (1987) Elsinger (2017) test not only inherits the characteristics of

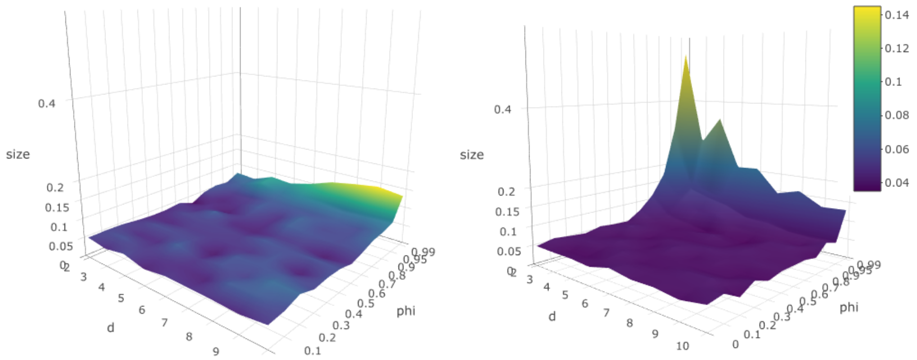

Figure 4: left: (PT 2009) mutual independence, right: Porteous (1987) Elsinger (2017), for $m_i = 2$, $l = 1$ and $T = 1000$, 10^3 simulations each step

the similar Tavaré (1983) test, but also shows to jump in size from relatively high size at even d and relatively low size for odd d. The reason for this is not entirely clear, given the symmetric formulas of test statistic and critical value and would, again, require analytical investigation beyond the scope of this thesis.

4 Empirical Application

In consumer products many economic questions occur, such as the behaviour of prices between different product groups. Such relationships may map interdependences between industries or consumer preferences. In this example consider super market scanner data from *Dominick's Finer Foods*, a food chain, in Chicago from the years 1989 to 1994. We are interested in comparing general price trends between different product groups and testing for dependence between members of each group and between members across product groups. As shown in Pesendorfer (2002) super market prices of a product over time can often be sensibly categorised, as there are usually a number of small fluctuations until a bigger step in pricing is undertaken. Here we consider categorising into

1. Price Change: 3 categories; *Up/Down* if deviation greater than half a standard deviation prices, otherwise *Steady*

2. Price Level: 3 categories; *High, Medium, Low* by quantiles [0, 1/3), [1/3,

2/3], (2/3, 1].

4.1 The Data

The whole data set comprises about 100 stores. We select one store (№128) with high movement of goods and two, by intuition, unrelated product types, namely beer and shampoo. Within those, we again choose two brands each with high purchasing frequency, which are *Becks* and *Corona* for beer and *Head & Shoulders* and *Suave* for shampoo.

From figures 5 and 6 we may glance that our categorisation approach is justified. To exemplify categorisation method 2, consider Becks prices divided by red dotted lines accordingly in figure 7. Missing values were imputed simply by selecting the previous value and the number of weeks is homogenised to $T = 272$. Due to the data's overall time span of five years we can naturally expect a high degree of autocorrelation in prices.

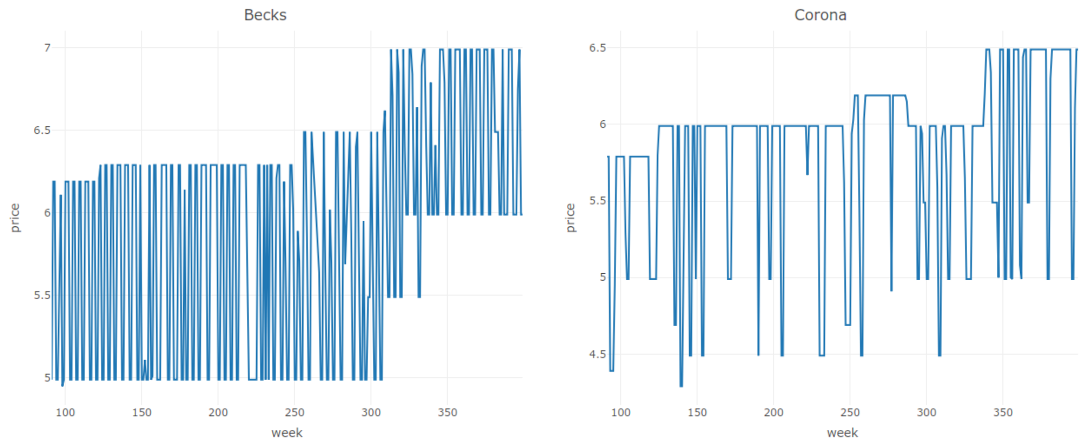

Figure 5: Weekly prices Becks, Corona

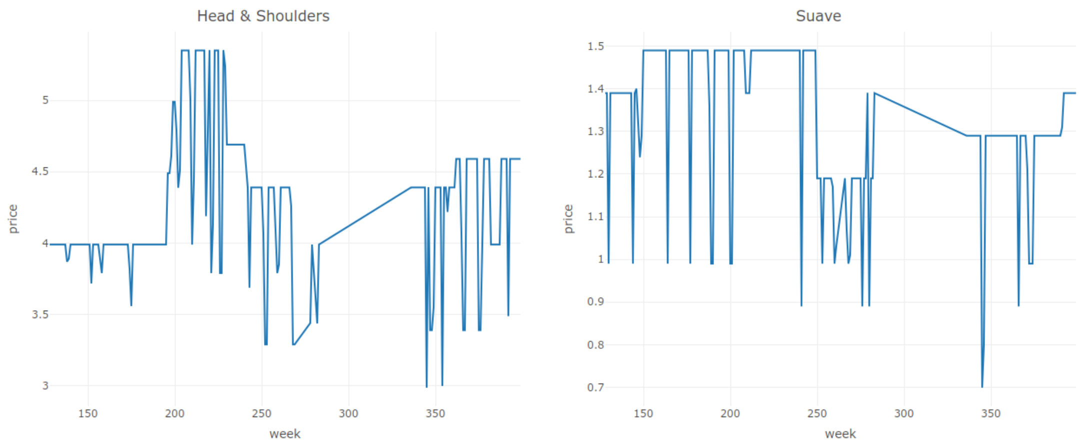

Figure 6: Weekly prices Head & Shoulders, Suave

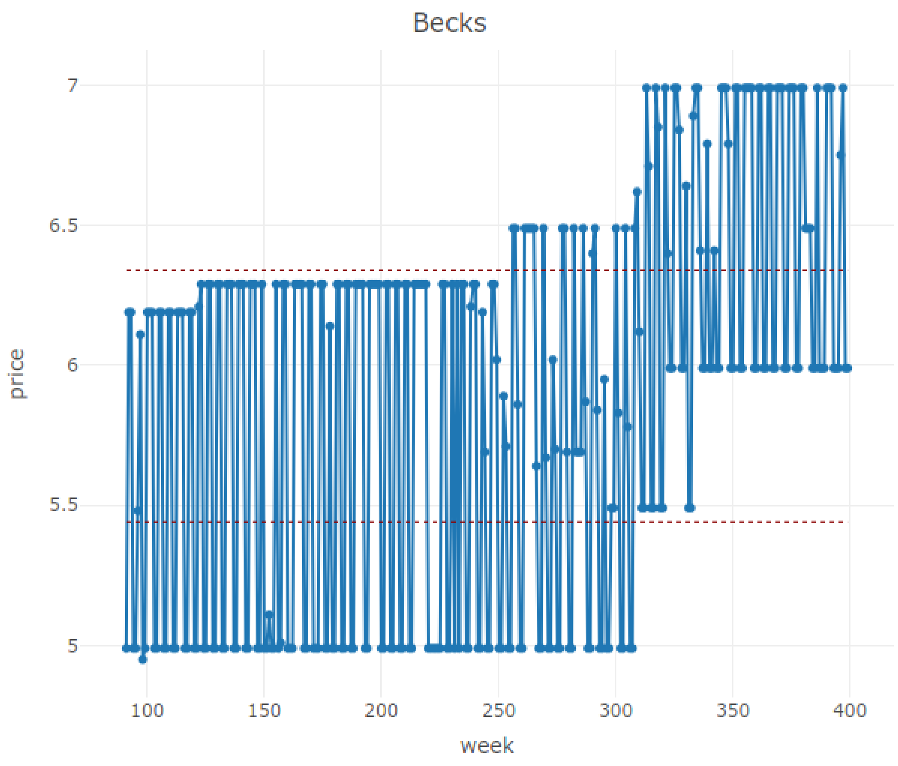

Figure 7: Price discretisation example (H/M/L), as separated by red dotted lines

4.2 Results

Dicky-Fuller unit-root tests on above price time series reject the null hypothesis of 'unit-root' in favour of 'stationarity' as the alternative. For lag estimates by AIC we receive values between 3 for Corona and $9 - 11$ for the other products. Due to dimensionality issues discussed above, a practical decision was made to a) consider all lags equal 3 and b) all lags equal one, as the highest autocorrelation estimate was present in the first lag. There are arguments of AIC leading to a higher estimation of lags than, for instance, the Bayesian Information Criterion (BIC), such as described in Raftery (1985) and argued even more extensively for in Kass and Raftery (1995) that an over-complication of the model (by relying on AIC) may not lead to more valid outcomes. Incidentally, the Pesaran and Timmermann (2009) test leads to identical results in both cases. For Tavaré's (1982) test, only the lag one case could be considered, as the contingency table for higher lags contained zeroes, violating the ergodicity assumption for the observed chain and making the computation of the tests statistic impossible. Even under the lag one assumption we lose out on two combinations here. Note, that applying Pesaran and Timmermann's (2009) test remains numerically possible due to the use of the Moore-Penrose-Inverse as shown in Elsinger (2017, 2019) and can arguably justified theoretically by postulating that the general Markov chain is ergodic after all and only the finite sample we observe suffers this defect.

Table 1 clearly demonstrates the difference in (naive) Fisher vs. adjusted test outcomes in case of L-M-H categorisation. Whereas the Fisher test picks up the underlying autocorrelation and implies dependence across product categories, Pesaran and Timmermann's (2009) dynamic trace test and, where numerically possible[3], Tavaré's (1983) test support the much more intuitive supposition that beer and shampoo prices are uncorrelated, yet within these product types brand prices are correlated. In terms of price changes the degree of serial correlation appears not to be strong enough to affect the simple Fisher test. With the exception of pairing (Becks, H&S), above hypothesis is supported. This exception may be ex-

[3]For (Becks, Suave), (Corona, Suave) the case (High, High) does not occur, which in consequence makes the calculation of the test statistic impossible due to division of zero as mentioned above. Clearly it would be desirable to derive a more robust test statistic formula to address such practical issues.

	Price Changes			L-M-H Prices		
	Fisher	PT	Tavaré	Fisher	PT	Tavaré
Becks, Corona	✗	✗	✗	✗	✗	✗
H&S, Suave	✗	✗	✗	✗	✗	✗
Becks, H&S	✗	✗	✗	✓	✓	✓
Corona, H&S	✓	✓	✓	✗	✓	✓
Becks, Suave	✓	✓	✓	✗	✓	N.A.
Corona, Suave	✓	✓	✓	✗	✓	N.A.

Table 1: Test results; ✗implies no independence, ✓implies independence

plained by both products being imports from the same country.
Despite the simplicity of the tests' hypothesis statement, they may be used to cross-validate or at least support more complex theories. Eichenbaum, Jaimovich, and Rebelo (2011) find that products "with a high probability of reference have a high probability of reference price changes" in their work on a retail data set, together with Dominick's as a robustness check. If we assume similar cost structures for similar products as by above considerations, the test results support the paper's implications.
Categorising by price changes appears not to have left a sufficiently strong serial correlation, as all tests have the same indication. This, however, also speaks for the power of the serial correction methods not suffering significantly due to the modifications to accommodate dynamic structures.

5 Conclusion

The thesis explores two major test approaches in dealing with serial correlation in contingency tables. We formulate generalised versions of the tests in both lags of the underlying data and number of variables. Both Pesaran and Timmermann's (2009) trace test and Tavaré's (1983) test, which is extended smoothly by Porteous (1987) and Elsinger (2017, 2019), show certain advantages and disadvantages in keeping the test size at the specified significance level under an escalation of the number of categories and order of lags. These results were obtained by extensive Monte-Carlo simulations which not only widely match the results in Pesaran and

Timmermann (2009) but also extend them. These simulations can be of further use as they integrate innovations by Elsinger (2017, 2019) to improve the ease of numerical implementation of the models. This should allow easier access to the trace test for future empirical work as the arbitrary deletion of one category column is spared.

Simulating with an increase in the number of variables supports the previous findings in Pesaran and Timmermann (2009) as the mathematical complexity of the model is mostly determined by the overall number of categories to be included in the transition and lagged matrices for a given number of observations. It may be of further interest to consider simulation results under an explicit increase in lags greater one. Essentially, however, these test ought to morph into equivalent lag one test with an extended state space, whose effects on the tests' size and power were shown here.

A brief empirical application on super market prices underlines the general importance of accounting for serial correlation in contingency tables if the time horizon of gathering the data is relatively long. The variable dependency structure indicated by the trace test finds stronger intuitive justification than that of a classical Fisher test. Even though the complex autocorrelation structure indicated by the AR-fit could not fully be accounted for, this example clearly shows how powerful the consideration of only one lag can already be.

Of further interest may be whether the positive characteristics of Pesaran and Timmermann's (2009) trace statistic and the Tavaré (1983)/Porteous (1987)/Elsinger (2017, 2019) approach could be combined without inheriting the respective drawbacks. One such path might lead to implementing a Mixture Transition Model (MTD) by Raftery (1985) or one of its abstractions mentioned in Berchtold, Raftery, et al. (2002). This would temper the inference issues of an exploding state space, as in a MTD an increase in lags leads only to a linear increase in parameters to be estimated in the transition matrix. The efficiency gains as well as fitting techniques have been shown in various papers such as Raftery and Tavaré (1994). It would be non-trivial to include this approach in Pesaran and Timmermann's (2009) trace test as the equation structure does not directly involve transition matrices. On the other hand, a much more straightforward implementation into the Tavaré (1983) test may be of little use as an increase in categories

here hardly seems to have a poor effect on test size.
The current 'discrete-time-discrete-variables' dependency tests may also be of help in devising tests on the natural abstractions of dependence between continuous variables gathered over discrete time or, conversely, discrete Markov processes in continuous time.

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6 Appendix

6.1 Program Description

This section details the programming steps taken to implement the test equations, hopefully efficiently, by focusing on vectorising and using linear algebra routines as much as possible. The scripts embedding these functions to produce test size and power tables, graphs and empirical results can be requested from the email address submitted in a footnote to the abstract which cannot be repeated here for legal reasons.

Construction of $\tilde{X}$: Here, let $\boldsymbol{x}$ be the vector of numerical realisations of X with values in $\{0, 1, ..., m_X - 1\}$. For general $l > 1$ we need to restructure $\boldsymbol{x}$ to $\tilde{\boldsymbol{X}}_{T \times m^l}$ to use equation (3): Let first $\boldsymbol{\Xi}$ be the matrix that has $\boldsymbol{x}$ up to shifted $\boldsymbol{x}_{l-1}$ as columns, such that

$$\boldsymbol{\Xi} = (\boldsymbol{x}, \boldsymbol{x}_{-1}, \cdots, \boldsymbol{x}_{-(l-1)})_{T \times l}. \tag{6}$$

Then with $\boldsymbol{m} := \left(m^0, m^1, ..., m^{l-1}\right)'$,

$$\tilde{\boldsymbol{x}} = \boldsymbol{\Xi m} \tag{7}$$

and $\tilde{\boldsymbol{X}}$ can be constructed directly from $\tilde{\boldsymbol{x}}$.

6.1.1 Test Statistic Tavaré

The function `TavareStat.cpp` has been implemented for X, Y category variables with T realisation each. The implementation requires matrices $\boldsymbol{X}, \boldsymbol{Y}$ of dimensions $T \times m_X, T \times m_Y$. From that the function constructs the contingency table

$$M1 = \boldsymbol{X'Y}$$

and the matrix of $(n_{i+} \cdot n_{+j})_{ij}$ products

$$M2 = \boldsymbol{X' \tau \tau' Y},$$

where $\boldsymbol{\tau}$ is a $1 \times T$ vector of ones. Then simply

$$C_T = \sum_i \sum_j \frac{(M1_{ij} - M2_{ij}/T)^2}{M2_{ij}}$$

6.1.2 Critical Values Tavaré

The function `TavareCrit.cpp` has been implemented for X, Y category variables with T realisation each. The implementation requires matrices $\boldsymbol{X}, \boldsymbol{Y}$ of dimensions $T \times m_X, T \times m_Y$. From that the algorithm computes

$$\begin{aligned} D_X &= X'X/T \\ P_X &= X'X_{-1}/T \\ Q_X &= D_X^{-1} P_X \end{aligned}$$

and the analogous Y matrices. Then, $\boldsymbol{\lambda_x}, \boldsymbol{\lambda_y}$ are the respective vectors of non-unit eigenvalues of Q_x, Q_y. The weights

$$\boldsymbol{\rho} = \left(\frac{1 + \lambda_{xi}\lambda_{xj}}{1 - \lambda_{xi}\lambda_{xj}} \right)_{\substack{i=1,\dots,m_x-1 \\ j=1,\dots,m_y-1}} \tag{8}$$

are the used to simulate a distribution $\sum_k \rho_k Z^2$ by constructing a matrix $\boldsymbol{Z}$ of $10^4 \times (m_X - 1)(m_Y - 1)$ random $\{N(0,1)\}^2$ realisations, such that

$$\boldsymbol{\zeta} = \boldsymbol{Z\rho}$$

can be used as a critical value estimate by finding the α-quantile value of $\boldsymbol{\zeta}$. Note, that `Tavarecrit.cpp` returns $\boldsymbol{\zeta}$, such that the respective quantile value has to be found at another step. If $m_X = m_Y = 2$, the function instead returns $\rho = \frac{1+\lambda_x\lambda_y}{1-\lambda_x\lambda_y}$.

6.1.3 Test Statistic Porteous, Elsinger

The function `X2Adstat` has been implemented for the general case of $X_1, \dots, X_d$ category variables with T realisations each. The implementation requires a matrix $\boldsymbol{X}$ of dimension $T \times d$, where each column i represents the category realisations of X_i, $i = 1, \dots, d$, where the possible outcomes of each X_i are mapped numerically from 0 to $m_{X_i} - 1$ and a vector

$$\boldsymbol{m} := (m_{X_1}, ..., m_{X_d})',$$

eg the vector of possible states of each X_i.
The function then computes

$$X^2_{A(d)} = T(\hat{\boldsymbol{p}} - \bigotimes_{i=1}^{d} \hat{\boldsymbol{p}}_{X_i})' \bigotimes_{i=1}^{d} (\boldsymbol{D}_{X_i}^{-1} - \boldsymbol{1}_{m_i}\boldsymbol{1}'_{m_i})(\hat{\boldsymbol{p}} - \bigotimes_{i=1}^{d} \hat{\boldsymbol{p}}_{X_i}).$$

Construction of $\hat{\boldsymbol{p}}, \hat{\boldsymbol{p}}_{X_i}, \boldsymbol{D}_{X_i}$:
For $\hat{\boldsymbol{p}}$ consider the mapping $u : \boldsymbol{X}[t] \to \hat{\boldsymbol{p}}_j$, showing which permutation $j \in \{1, 2, ..., \prod_i m_i\}$ was hit at time t as given in row t of $\boldsymbol{X}$. One can find that

$$u(t) = \sum_{i=1}^{d} \left(\prod_{i+1}^{d+1} m_i \right) \boldsymbol{X}_{t,i},$$

with $m_{d+1} := 1$. In an equivalent vector notation this means

$$\boldsymbol{u} = \boldsymbol{X}\boldsymbol{m}_{\text{cumprod}},$$

where $\boldsymbol{m}_{\text{cumprod}} := (\prod_{i=2}^{d} m_i, \prod_{i=3}^{d} m_i, ..., m_d, 1)$.
Now, simply generate $\boldsymbol{U_f}$ as the $T \times \prod_i m_i$ matrix that is 1 at $(t, \boldsymbol{u}_t + 1)$, for $t = 1, ..., T$ and otherwise 0 and find that

$$\hat{\boldsymbol{p}} = diag(\boldsymbol{U_f}'\boldsymbol{U_f})/T.$$

The marginal probabilities of X_i can be expressed as

$$\hat{\boldsymbol{p}}_{X_i} = \left(\boldsymbol{1}'_{\prod_{i+1}^{d+1} m_i} \otimes \boldsymbol{I}_{m_i} \otimes \boldsymbol{1}'_{\prod_{i=0}^{i-1} m_i} \right) \hat{\boldsymbol{p}},$$

with $m_0, m_{d+1} := 1$.
Finally, let $\boldsymbol{X}_{fi}$ be the $T \times m_i$ matrix that is 1 at $(t, X_t + 1)$ and otherwise 0. Then,

$$\boldsymbol{D}_{X_i} = \boldsymbol{X}'_{fi}\boldsymbol{X}_{fi}/T.$$

6.1.4 Critical Values Porteous, Elsinger

The function `critdbyEV` has been implemented for the general case of $X_1, ..., X_d$ category variables with T realisations each. The implementation requires a matrix $\boldsymbol{X}$ of dimension $T \times d$, where each column i represents the category realisations of X_i, $i = 1, ..., d$, where the possible outcomes of each X_i are mapped numerically from 0 to $m_i - 1$ and a vector

$$\boldsymbol{m} := (m_1, ..., m_d)',$$

e.g. the vector of possible states of each X_i. The function then returns a vector (a, b, c) for the test statistics $c\chi^2_{\tilde{d}}, a\chi^2_b$, where $\tilde{d} = \prod_{i=1}^d (m_i - 1)$.
Let the weights to be computed be ρ_j, $j = 1, ..., \tilde{d}$ and

$$\boldsymbol{\gamma} := \left(\left\{ \prod_{i=1}^{d} \lambda_{i,(\cdot)} \right\}_{\sigma(\boldsymbol{m})} \right),$$

the vector of the $\tilde{d}$ permutations $\sigma(\boldsymbol{m})$ of the product of non-unit eigenvalues $\lambda_{i,(\cdot)}$ of the matrices $\boldsymbol{Q}_1, ..., \boldsymbol{Q}_i, ..., \boldsymbol{Q}_d$. Then,

$$\boldsymbol{\rho} = \left(\frac{1 + \gamma_j}{1 - \gamma_j} \right)_{j=1,...,\tilde{d}}.$$

Computation of $\boldsymbol{\gamma}$: For each X_i in above order, the corresponding permutation matrix for the vector $\boldsymbol{\lambda}_i = (\lambda_{i,1}, ..., \lambda_{i,m_i-1})'$ is

$$\boldsymbol{\Lambda}_i = \mathbf{1}'_{\prod_{i+1}^{d+1}(m_i-1)} \otimes \boldsymbol{I}_{m_i-1} \otimes \mathbf{1}'_{\prod_{i=0}^{i-1}(m_i-1)}.$$

This means,

$$\boldsymbol{\tau} = \bigodot_{i=1}^{d} (\boldsymbol{\Lambda}_i \boldsymbol{\lambda}_i),$$

with $\bigodot$ indicating element-wise multiplication.
Finally, use

$$c = \bar{\boldsymbol{\rho}} = \frac{\sum \rho_j}{\tilde{d}}, \quad a = \frac{\sum \rho_j^2}{\sum \rho_j}, \quad b = \frac{(\sum \rho_j)^2}{\sum \rho_j}.$$

7 Tables Test Size and Power

T	Phi	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
20	0	2	0.05470000	0.05470000	0.07100000	0.07100000	0.05460000
		3	0.05190000	0.05140000	0.16140000	0.16980000	0.04379196
		4	0.05380000	0.05060000	0.38400000	0.41380000	0.03038079
		5	0.04780000	0.05230000	0.75460000	0.77640000	0.03126751
	0.5	2	0.07520000	0.07530000	0.07550000	0.07550000	0.05735736
		3	0.06740000	0.06660000	0.15030000	0.15580000	0.03928194
		4	0.06100000	0.05830000	0.34410000	0.34670000	0.03488226
		5	0.05080000	0.04970000	0.59680000	0.55870000	0.03517241
	0.8	2	0.13680000	0.13680000	0.08430000	0.08430000	0.05850948
		3	0.11980000	0.11390000	0.11990000	0.11450000	0.04381085
		4	0.09480000	0.07490000	0.18380000	0.14530000	0.04182087
		5	0.06740000	0.04080000	0.26050000	0.17160000	0.03678606
	0.95	2	0.08730000	0.08730000	0.04370000	0.04370000	0.07338820
		3	0.08360000	0.06890000	0.04360000	0.03330000	0.06021898
		4	0.07190000	0.03110000	0.04530000	0.02900000	0.06250000
		5	0.05700000	0.01110000	0.06010000	0.01060000	0.00000000
50	0	2	0.05000000	0.05000000	0.05950000	0.05950000	0.05620000
		3	0.05260000	0.05230000	0.07910000	0.07940000	0.04840000
		4	0.05060000	0.04580000	0.10950000	0.11140000	0.04220000
		5	0.05360000	0.04890000	0.16210000	0.18400000	0.04110411
	0.5	2	0.07810000	0.07810000	0.05920000	0.05920000	0.05480000
		3	0.07980000	0.08010000	0.08150000	0.08300000	0.04680000
		4	0.07430000	0.07290000	0.11390000	0.11970000	0.04373997
		5	0.06760000	0.06740000	0.16910000	0.19140000	0.03886352
	0.8	2	0.19600000	0.19600000	0.06700000	0.06700000	0.07206486
		3	0.24050000	0.24170000	0.09370000	0.09290000	0.05418566
		4	0.23220000	0.22200000	0.11850000	0.12010000	0.04134763
		5	0.20810000	0.18910000	0.15990000	0.15330000	0.03505399
	0.95	2	0.27810000	0.27810000	0.07320000	0.07320000	0.09967497
		3	0.35660000	0.33840000	0.08640000	0.07630000	0.07879377
		4	0.36560000	0.30240000	0.09120000	0.06610000	0.04916847
		5	0.35290000	0.24690000	0.09970000	0.06150000	0.03246753
100	0	2	0.04520000	0.04540000	0.05110000	0.05110000	0.05220000
		3	0.05200000	0.05180000	0.06130000	0.06380000	0.05170000
		4	0.04990000	0.04830000	0.07540000	0.07580000	0.04830000
		5	0.05440000	0.05360000	0.09080000	0.09820000	0.04770000
	0.5	2	0.08260000	0.08270000	0.05490000	0.05490000	0.05970000
		3	0.07980000	0.08040000	0.06400000	0.06560000	0.04870000
		4	0.07530000	0.07300000	0.07410000	0.06970000	0.04560000
		5	0.06900000	0.06710000	0.08680000	0.09530000	0.04180418
	0.8	2	0.21380000	0.21380000	0.05330000	0.05330000	0.08220000
		3	0.28760000	0.29650000	0.06150000	0.06380000	0.06764058
		4	0.28880000	0.29220000	0.08230000	0.08250000	0.05458757
		5	0.28600000	0.28640000	0.10480000	0.10920000	0.04558671
	0.95	2	0.44090000	0.44090000	0.07200000	0.07200000	0.14523372
		3	0.60240000	0.60300000	0.10230000	0.10020000	0.13823456
		4	0.64640000	0.62330000	0.10780000	0.09390000	0.09904153
		5	0.64980000	0.59470000	0.11580000	0.08740000	0.05949730

Table 2: Size Table PT, T = 20, 50, 100; 10^4 simulations each cell

T	Phi	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
500	0	2	0.0491000	0.0491000	0.0489000	0.0489000	0.0513000
		3	0.0487000	0.0514000	0.0503000	0.0524000	0.0485000
		4	0.0483000	0.0463000	0.0502000	0.0487000	0.0503000
		5	0.0508000	0.0526000	0.0589000	0.0590000	0.0530000
	0.5	2	0.0851000	0.0851000	0.0540000	0.0540000	0.0582000
		3	0.0826000	0.0867000	0.0485000	0.0525000	0.0500000
		4	0.0738000	0.0716000	0.0540000	0.0477000	0.0478000
		5	0.0733000	0.0692000	0.0561000	0.0501000	0.0456000
	0.8	2	0.2273000	0.2273000	0.0518000	0.0518000	0.0829000
		3	0.3153000	0.3249000	0.0589000	0.0588000	0.0860000
		4	0.3366000	0.3388000	0.0543000	0.0517000	0.0714000
		5	0.3317000	0.3462000	0.0541000	0.0547000	0.0650000
	0.95	2	0.5361000	0.5361000	0.0651000	0.0651000	0.2038000
		3	0.8340000	0.8463000	0.0803000	0.0833000	0.2522000
		4	0.9223000	0.9334000	0.0882000	0.0806000	0.2401922
		5	0.9492000	0.9632000	0.0936000	0.0942000	0.2162814
1000	0	2	0.0518000	0.0518000	0.0522000	0.0522000	0.0527000
		3	0.0517000	0.0538000	0.0539000	0.0541000	0.0520000
		4	0.0407000	0.0428000	0.0430000	0.0456000	0.0449000
		5	0.0599000	0.0522000	0.0619000	0.0560000	0.0517000
	0.5	2	0.0826000	0.0826000	0.0501000	0.0501000	0.0517000
		3	0.0850000	0.0876000	0.0525000	0.0531000	0.0532000
		4	0.0731000	0.0755000	0.0450000	0.0481000	0.0506000
		5	0.0791000	0.0773000	0.0572000	0.0536000	0.0483000
	0.8	2	0.2247000	0.2247000	0.0566000	0.0566000	0.0814000
		3	0.3251000	0.3327000	0.0602000	0.0643000	0.0886000
		4	0.3281000	0.3507000	0.0465000	0.0497000	0.0750000
		5	0.3512000	0.3605000	0.0575000	0.0553000	0.0699000
	0.95	2	0.5392000	0.5392000	0.0663000	0.0663000	0.2003000
		3	0.8426000	0.8520000	0.0707000	0.0721000	0.2707000
		4	0.9271000	0.9428000	0.0709000	0.0759000	0.2740000
		5	0.9660000	0.9765000	0.0838000	0.0827000	0.2592000

Table 3: Size Table PT, T = 500, 1000; 10^4 simulations each cell

T	Phi	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
20	0	2	0.08960000	0.09680000	0.05120000	0.06500000	0.05044301
		3	0.11370000	0.19860000	0.02900000	0.10830000	0.03858375
		4	0.12130000	0.22010000	0.00000000	0.26010000	0.03116006
		5	0.12340000	0.21970000	0.00000000	0.57330000	0.02797393
	0.5	2	0.13080000	0.14230000	0.04890000	0.06190000	0.05356602
		3	0.15510000	0.26010000	0.02890000	0.09980000	0.03797299
		4	0.14330000	0.27820000	0.00000000	0.19370000	0.03220368
		5	0.13940000	0.28070000	0.00000000	0.34480000	0.03607262
	0.8	2	0.18210000	0.19310000	0.05740000	0.06890000	0.05542597
		3	0.17480000	0.24090000	0.03800000	0.07780000	0.04448743
		4	0.13660000	0.23260000	0.00000000	0.08600000	0.03107198
		5	0.13190000	0.22920000	0.00000000	0.09310000	0.04353562
	0.95	2	0.10990000	0.11360000	0.03230000	0.03610000	0.07851434
		3	0.09910000	0.10120000	0.01760000	0.02750000	0.07565012
		4	0.08150000	0.06820000	0.00000000	0.02080000	0.00000000
		5	0.06530000	0.04570000	0.00000000	0.00770000	0.00000000
50	0	2	0.08790000	0.09050000	0.04730000	0.05080000	0.05300000
		3	0.16610000	0.19830000	0.02810000	0.05550000	0.04664048
		4	0.20120000	0.21070000	0.00900000	0.08580000	0.04318281
		5	0.17210000	0.20740000	0.00040000	0.12760000	0.04515130
	0.5	2	0.15610000	0.16250000	0.03950000	0.04360000	0.05480000
		3	0.25990000	0.32090000	0.02610000	0.05550000	0.04129956
		4	0.30890000	0.34570000	0.00550000	0.06960000	0.04312967
		5	0.25300000	0.33190000	0.00040000	0.11760000	0.03669464
	0.8	2	0.27850000	0.28500000	0.05220000	0.05610000	0.07214429
		3	0.37280000	0.42490000	0.03360000	0.06040000	0.05918654
		4	0.37610000	0.43540000	0.01360000	0.07270000	0.04226279
		5	0.32370000	0.41950000	0.00550000	0.09820000	0.03423800
	0.95	2	0.31660000	0.31920000	0.05590000	0.05900000	0.10258927
		3	0.41050000	0.41680000	0.04200000	0.05590000	0.08659397
		4	0.40410000	0.38010000	0.02470000	0.04990000	0.05797101
		5	0.37570000	0.34820000	0.01220000	0.04290000	0.04848485
100	0	2	0.08610000	0.08850000	0.04230000	0.04480000	0.05170000
		3	0.16000000	0.18940000	0.02840000	0.04960000	0.04770477
		4	0.19850000	0.20120000	0.00870000	0.05700000	0.04434835
		5	0.20100000	0.20270000	0.00120000	0.07150000	0.04147171
	0.5	2	0.15750000	0.16150000	0.03990000	0.04190000	0.05660000
		3	0.27730000	0.32780000	0.02320000	0.04270000	0.05196387
		4	0.33440000	0.34310000	0.00800000	0.05030000	0.04812438
		5	0.33740000	0.34170000	0.00130000	0.06130000	0.04528986
	0.8	2	0.28940000	0.29260000	0.04250000	0.04450000	0.08230000
		3	0.43320000	0.47530000	0.02960000	0.04740000	0.07762739
		4	0.47650000	0.50180000	0.01240000	0.05690000	0.05578366
		5	0.47840000	0.49300000	0.00210000	0.07100000	0.04928342
	0.95	2	0.48320000	0.48520000	0.06090000	0.06330000	0.14100470
		3	0.65340000	0.66540000	0.05670000	0.07130000	0.13879709
		4	0.71070000	0.70850000	0.03480000	0.07250000	0.10909993
		5	0.70430000	0.68150000	0.01680000	0.06130000	0.06622517

Table 4: Size Table PT, fully expanded lags, T = 20, 50, 100; 10^4 simulations each cell

T	Phi	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
500	0	2	0.08800000	0.08870000	0.04310000	0.04350000	0.05110000
		3	0.15560000	0.18730000	0.02990000	0.04470000	0.04960000
		4	0.19840000	0.19970000	0.00900000	0.04770000	0.05440000
		5	0.20060000	0.20020000	0.00200000	0.04670000	0.04860000
	0.5	2	0.16480000	0.16830000	0.03910000	0.03930000	0.05560000
		3	0.28290000	0.32860000	0.02790000	0.04140000	0.05160000
		4	0.34990000	0.35010000	0.00780000	0.03810000	0.05206030
		5	0.35060000	0.34740000	0.00090000	0.04450000	0.05066426
	0.8	2	0.30170000	0.30520000	0.04020000	0.04040000	0.08760000
		3	0.48350000	0.52100000	0.02750000	0.04190000	0.08285651
		4	0.52480000	0.53510000	0.00900000	0.03960000	0.07196065
		5	0.53130000	0.53950000	0.00110000	0.04100000	0.07219402
	0.95	2	0.58880000	0.58930000	0.04900000	0.04900000	0.19320000
		3	0.86360000	0.87750000	0.04000000	0.05740000	0.24709463
		4	0.94370000	0.95620000	0.01840000	0.06420000	0.25971878
		5	0.96570000	0.97530000	0.00630000	0.06690000	0.21892081
1000	0	2	0.09140000	0.09230000	0.04410000	0.04430000	0.05380000
		3	0.16410000	0.18980000	0.02700000	0.04220000	0.05200000
		4	0.18450000	0.18810000	0.01010000	0.04730000	0.04670000
		5	0.20020000	0.19440000	0.00130000	0.04530000	0.04720000
	0.5	2	0.15880000	0.16110000	0.03880000	0.03920000	0.05450000
		3	0.27620000	0.31950000	0.02510000	0.03830000	0.05280000
		4	0.33460000	0.34400000	0.00920000	0.03890000	0.05231046
		5	0.34310000	0.33970000	0.00170000	0.04010000	0.04768742
	0.8	2	0.30110000	0.30380000	0.03890000	0.03890000	0.08440000
		3	0.47670000	0.50890000	0.02720000	0.04060000	0.07995941
		4	0.51680000	0.53650000	0.01010000	0.04020000	0.07630633
		5	0.54920000	0.55400000	0.00120000	0.03790000	0.07144882
	0.95	2	0.58490000	0.58560000	0.05010000	0.05010000	0.19410000
		3	0.87710000	0.88720000	0.03790000	0.05430000	0.26909142
		4	0.94830000	0.96080000	0.01460000	0.05350000	0.27942410
		5	0.97310000	0.98250000	0.00280000	0.05630000	0.25377517

Table 5: Size Table PT, fully expanded lags, T = 500, 1000; 10^4 simulations each cell

Phi	rxy	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
0	0.2	2	0.08550000	0.08550000	0.10680000	0.10680000	0.08530000
		3	0.06820000	0.07230000	0.17950000	0.19290000	0.06350796
		4	0.06650000	0.06960000	0.40150000	0.43600000	0.04765328
		5	0.05970000	0.06450000	0.76470000	0.78180000	0.03726500
	0.8	2	0.78680000	0.78680000	0.76730000	0.76730000	0.78460000
		3	0.80480000	0.79910000	0.81830000	0.82040000	0.76996293
		4	0.74830000	0.73010000	0.87130000	0.88150000	0.65233012
		5	0.61840000	0.61410000	0.92760000	0.92200000	0.50776797
0.5	0.2	2	0.11240000	0.11250000	0.10030000	0.10030000	0.08566276
		3	0.09120000	0.09370000	0.18540000	0.19160000	0.06405656
		4	0.07520000	0.07550000	0.36360000	0.37100000	0.04655870
		5	0.06880000	0.06200000	0.61490000	0.56880000	0.04628043
	0.8	2	0.72850000	0.72850000	0.67910000	0.67910000	0.68591155
		3	0.72950000	0.72790000	0.74950000	0.75460000	0.67067004
		4	0.67380000	0.65290000	0.79250000	0.77140000	0.59623380
		5	0.55620000	0.53130000	0.82850000	0.76970000	0.52043949
0.8	0.2	2	0.14730000	0.14730000	0.09240000	0.09240000	0.06674626
		3	0.13750000	0.13280000	0.14410000	0.13460000	0.05314515
		4	0.11420000	0.08610000	0.19980000	0.15840000	0.04312268
		5	0.08380000	0.04660000	0.28370000	0.18880000	0.03523035
	0.8	2	0.55990000	0.56000000	0.43330000	0.43330000	0.43255259
		3	0.54530000	0.53300000	0.48960000	0.47320000	0.40776388
		4	0.50930000	0.44000000	0.53950000	0.46430000	0.40178571
		5	0.41930000	0.31960000	0.54560000	0.41800000	0.36156764
0.95	0.2	2	0.09230000	0.09230000	0.04910000	0.04910000	0.07901907
		3	0.09930000	0.08160000	0.05340000	0.04270000	0.05677656
		4	0.07570000	0.03480000	0.05420000	0.03310000	0.12500000
		5	0.06220000	0.01360000	0.05680000	0.01280000	0.00000000
	0.8	2	0.20810000	0.20810000	0.12140000	0.12140000	0.24274406
		3	0.25970000	0.23040000	0.15720000	0.13590000	0.33559783
		4	0.26120000	0.17220000	0.17430000	0.12240000	0.26956522
		5	0.23540000	0.10540000	0.18160000	0.07650000	0.30000000

Table 6: Power Table PT, T = 20 , 10^4 simulations each cell

Phi	rxy	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
0	0.2	2	0.14260000	0.14200000	0.15600000	0.15600000	0.15360000
		3	0.12460000	0.12480000	0.15920000	0.16180000	0.11770000
		4	0.09820000	0.10090000	0.17630000	0.18520000	0.09420000
		5	0.08890000	0.09000000	0.21480000	0.24340000	0.07780778
	0.8	2	0.99270000	0.99270000	0.99130000	0.99130000	0.99370000
		3	0.99920000	0.99940000	0.99860000	0.99880000	0.99940000
		4	0.99920000	0.99890000	0.99920000	0.99850000	0.99880000
		5	0.99850000	0.99630000	0.99810000	0.99690000	0.99529953
0.5	0.2	2	0.17480000	0.17460000	0.13510000	0.13510000	0.13500000
		3	0.16340000	0.16750000	0.14810000	0.15090000	0.11312262
		4	0.13390000	0.13090000	0.17040000	0.18020000	0.08522158
		5	0.12010000	0.11950000	0.21860000	0.26080000	0.08110862
	0.8	2	0.98750000	0.98750000	0.97400000	0.97400000	0.98280000
		3	0.99600000	0.99490000	0.99310000	0.99330000	0.99179836
		4	0.99610000	0.99420000	0.99670000	0.99530000	0.98976828
		5	0.99370000	0.98820000	0.99520000	0.99400000	0.98382009
0.8	0.2	2	0.26760000	0.26750000	0.10610000	0.10610000	0.12855427
		3	0.30740000	0.30830000	0.13570000	0.13920000	0.08923395
		4	0.28380000	0.27520000	0.15980000	0.16090000	0.06936075
		5	0.26010000	0.22930000	0.20360000	0.20780000	0.05892104
	0.8	2	0.92330000	0.92330000	0.81270000	0.81270000	0.85195195
		3	0.95810000	0.95580000	0.90260000	0.90240000	0.87925504
		4	0.95570000	0.94750000	0.93500000	0.92700000	0.88057454
		5	0.94360000	0.92230000	0.95320000	0.92930000	0.85659683
0.95	0.2	2	0.30210000	0.30210000	0.08210000	0.08210000	0.11250448
		3	0.38500000	0.36620000	0.09710000	0.08790000	0.09037002
		4	0.39380000	0.33370000	0.10830000	0.08230000	0.07619048
		5	0.38130000	0.27120000	0.11330000	0.07080000	0.04487179
	0.8	2	0.60730000	0.60720000	0.32690000	0.32690000	0.45363615
		3	0.72390000	0.70790000	0.41040000	0.39180000	0.52731377
		4	0.75960000	0.70460000	0.45170000	0.39180000	0.47729730
		5	0.76900000	0.66480000	0.50410000	0.41800000	0.39089692

Table 7: Power Table PT, T = 50 , 10^4 simulations each cell

Phi	rxy	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
0	0.2	2	0.23240000	0.23240000	0.24100000	0.24100000	0.25250000
		3	0.20780000	0.21040000	0.22310000	0.22500000	0.20970000
		4	0.16790000	0.16550000	0.20540000	0.20860000	0.16370000
		5	0.14570000	0.14850000	0.20380000	0.22170000	0.13930000
	0.8	2	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
		3	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
		4	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
		5	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.5	0.2	2	0.26430000	0.26440000	0.19680000	0.19680000	0.21330000
		3	0.25360000	0.25550000	0.19490000	0.20100000	0.19360000
		4	0.21670000	0.21180000	0.19460000	0.19740000	0.15750000
		5	0.18930000	0.18850000	0.19670000	0.21340000	0.13641364
	0.8	2	1.00000000	1.00000000	0.99990000	0.99990000	1.00000000
		3	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
		4	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
		5	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.8	0.2	2	0.33640000	0.33640000	0.13170000	0.13170000	0.18000000
		3	0.41720000	0.42270000	0.15170000	0.15310000	0.15213043
		4	0.41500000	0.41500000	0.16220000	0.16880000	0.12854556
		5	0.39360000	0.39120000	0.18230000	0.19260000	0.11641698
	0.8	2	0.99740000	0.99740000	0.97920000	0.97920000	0.99100000
		3	0.99940000	0.99930000	0.99740000	0.99700000	0.99709942
		4	0.99920000	0.99930000	0.99990000	0.99990000	0.99688003
		5	0.99930000	0.99930000	0.99990000	0.99980000	0.99616620
0.95	0.2	2	0.47570000	0.47570000	0.09910000	0.09910000	0.18523039
		3	0.62890000	0.62880000	0.13400000	0.13000000	0.17321362
		4	0.67200000	0.65000000	0.13710000	0.11940000	0.13328341
		5	0.67880000	0.62910000	0.14100000	0.11390000	0.09453376
	0.8	2	0.87890000	0.87890000	0.57390000	0.57390000	0.74537696
		3	0.94230000	0.94010000	0.69010000	0.68540000	0.81091309
		4	0.96260000	0.95430000	0.79550000	0.77090000	0.80854376
		5	0.96890000	0.95650000	0.85920000	0.81850000	0.76007950

Table 8: Power Table PT, T = 100 , 10^4 simulations each cell

Phi	rxy	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
0	0.2	2	0.8054000	0.8054000	0.8033000	0.8033000	0.8108000
		3	0.8379000	0.8374000	0.8371000	0.8367000	0.8320000
		4	0.8060000	0.7771000	0.8076000	0.7797000	0.7871000
		5	0.7582000	0.7099000	0.7630000	0.7163000	0.7100000
	0.8	2	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		3	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.5	0.2	2	0.7879000	0.7879000	0.7065000	0.7065000	0.7266000
		3	0.8250000	0.8251000	0.7773000	0.7798000	0.7640000
		4	0.7921000	0.7649000	0.7628000	0.7299000	0.7167000
		5	0.7505000	0.7176000	0.7377000	0.6966000	0.6639000
	0.8	2	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		3	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.8	0.2	2	0.7136000	0.7136000	0.4379000	0.4379000	0.5220000
		3	0.7818000	0.7858000	0.5124000	0.5191000	0.5329000
		4	0.7794000	0.7779000	0.5495000	0.5250000	0.4931000
		5	0.7753000	0.7705000	0.5715000	0.5400000	0.4684000
	0.8	2	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		3	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.95	0.2	2	0.6699000	0.6699000	0.1710000	0.1710000	0.3587000
		3	0.8883000	0.8977000	0.2158000	0.2221000	0.4208000
		4	0.9532000	0.9609000	0.2369000	0.2359000	0.4117647
		5	0.9638000	0.9765000	0.2684000	0.2692000	0.3704707
	0.8	2	1.0000000	1.0000000	0.9967000	0.9967000	0.9997000
		3	1.0000000	1.0000000	0.9999000	0.9999000	1.0000000
		4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
		5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 9: Power Table PT, T = 500 , 10^4 simulations each cell

Phi	rxy	m | Test	Fstatic	Tstatic	Fdynamic	Tdynamic	Tavare
0	0.2	2	0.9789	0.9789	0.9790	0.9790	0.9792
		3	0.9917	0.9915	0.9919	0.9919	0.9911
		4	0.9908	0.9873	0.9904	0.9871	0.9876
		5	0.9882	0.9767	0.9883	0.9775	0.9760
	0.8	2	1.0000	1.0000	1.0000	1.0000	1.0000
		3	1.0000	1.0000	1.0000	1.0000	1.0000
		4	1.0000	1.0000	1.0000	1.0000	1.0000
		5	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.2	2	0.9683	0.9683	0.9420	0.9420	0.9515
		3	0.9824	0.9816	0.9809	0.9799	0.9709
		4	0.9800	0.9745	0.9812	0.9782	0.9624
		5	0.9748	0.9627	0.9803	0.9684	0.9489
	0.8	2	1.0000	1.0000	1.0000	1.0000	1.0000
		3	1.0000	1.0000	1.0000	1.0000	1.0000
		4	1.0000	1.0000	1.0000	1.0000	1.0000
		5	1.0000	1.0000	1.0000	1.0000	1.0000
0.8	0.2	2	0.8957	0.8957	0.7116	0.7116	0.7765
		3	0.9391	0.9398	0.8445	0.8433	0.8066
		4	0.9424	0.9424	0.8904	0.8818	0.7891
		5	0.9389	0.9332	0.9233	0.8934	0.7587
	0.8	2	1.0000	1.0000	1.0000	1.0000	1.0000
		3	1.0000	1.0000	1.0000	1.0000	1.0000
		4	1.0000	1.0000	1.0000	1.0000	1.0000
		5	1.0000	1.0000	1.0000	1.0000	1.0000
0.95	0.2	2	0.7740	0.7740	0.2891	0.2891	0.4922
		3	0.9309	0.9355	0.3482	0.3534	0.5577
		4	0.9735	0.9778	0.3933	0.4021	0.5616
		5	0.9843	0.9904	0.4930	0.4758	0.5434
	0.8	2	1.0000	1.0000	1.0000	1.0000	1.0000
		3	1.0000	1.0000	1.0000	1.0000	1.0000
		4	1.0000	1.0000	1.0000	1.0000	1.0000
		5	1.0000	1.0000	1.0000	1.0000	1.0000

Table 10: Power Table PT, T = 1000 , 10^4 simulations each cell