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"Pathwise Otto calculus and the likelihood ratio process as a backwards martingale"

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Abstract

Fontbona and Jourdain have discussed in [FJ16] that the *likelihood ratio process* between two Markov measures can be viewed as a backwards martingale. We revisit and rigorously prove this fact in Chapter 1 of this thesis under the general assumption that one of the Markov measures is σ-finite and not necessarily a probability measure.

In Chapter 2 we follow [Léo14a, Léo14b] and introduce the concept of *reversible Brownian motion*. This is a path measure which corresponds to a Brownian motion whose random initial position is uniformly distributed. The fundamental feature of reversible Brownian motion is its invariance under time-reversal.

In the final Chapter 3 we place ourselves in the setting of [KST19] and provide proofs for two limiting assertions stated therein. Additionally, we give further details on other relevant subjects related to *Pathwise Otto calculus*.

Zusammenfassung

Fontbona und Jourdain haben in [FJ16] gezeigt, dass der *Dichteprozess* zwischen zwei Markov-Maßen als ein Rückwärtsmartingal aufgefasst werden kann. In Kapitel 1 dieser Arbeit diskutieren und beweisen wir diese Tatsache unter der allgemeinen Annahme, dass eines der Markov-Maße σ-endlich ist und kein Wahrscheinlichkeitsmaß zu sein braucht.

In Kapitel 2 folgen wir [Léo14a, Léo14b] und führen das Konzept der *reversiblen Brownschen Bewegung* ein. Diese ist ein Pfadmaß, welches einer Brownschen Bewegung mit gleichverteiltem Anfangswert entspricht. Die fundamentale Eigenschaft der reversiblen Brownschen Bewegung ist deren Invarianz unter Zeitumkehr.

Im letzten Kapitel 3 begeben wir uns in das Setting von [KST19] und beweisen zwei dort getroffene Grenzwertaussagen. Des Weiteren stellen wir zusätzliche Details bereit, welche der Leserschaft Hintegrundinformationen zum Thema *Pathwise Otto calculus* liefern.

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Contents

1 The likelihood ratio process as a backwards martingale

In the introduction of the seminal work [FJ16] by Fontbona and Jourdain it is remarked and explained that the *likelihood ratio process* $(\ell_t(X_t))_{t\geqslant 0}$ of a Markov measure $\mathbb{P}$ with respect to another Markov measure $\mathbb{Q}$ can be viewed as a backwards $\mathbb{Q}$-martingale, see Definition 1.18 and Theorem 1.19. The goal of this chapter is to understand and rigorously prove this insight in a certain general setting, which is relevant for many applications. The chapter at hand is organized as follows.

In Section 1.1 and especially in Lemma 1.1 we prove a preliminary result, which will be the key to our final upshot.

Section 1.2 establishes the general setting and provides the necessary notation for the remainder of this chapter. In Subsection 1.2.1 we state and explain our standard assumptions, which are taken for granted in what follows. The purpose of Subsection 1.2.2 is to define martingales on a σ-finite path space, and in Subsection 1.2.3 we describe how to define well the relative entropy of a probability measure with respect to a σ-finite measure, see [Léo14b, Section 2].

Subsequently, in Section 1.3 we apply the preliminary result of Section 1.1 in the general setting of Section 1.2 and state the main result of this chapter as Theorem 1.19.

In Section 1.4 we present an alternative short proof of Theorem 1.19, whose idea is due to Fontbona and Jourdain, see [FJ16].

Finally, in Section 1.5 we collect some immediate consequences of Theorem 1.19. In particular, in Subsection 1.5.1 we apply the convergence theorem for backwards martingales in order to show that the likelihood ratio process converges $\mathbb{Q}$-almost surely and in the norm of $L^1(\mathbb{Q})$, see Corollary 1.21. Following this, in Subsection 1.5.2 we introduce the *relative entropy process* $\big(\ell_t(X_t)\log \ell_t(X_t)\big)_{t\geqslant 0}$, see Definition 1.23, of $\mathbb{P}$ with respect to $\mathbb{Q}$, and prove that it is a backwards $\mathbb{Q}$-submartingale, see Corollary 1.25.

1.1 A preliminary result

Let $\mathcal{X}_0$ and $\mathcal{X}_1$ be two Polish spaces with corresponding Borel σ-algebras $\mathcal{B}(\mathcal{X}_0)$ and $\mathcal{B}(\mathcal{X}_1)$.

We equip the product space $\mathcal{X}_0 \times \mathcal{X}_1$ with the product σ-algebra $\mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathcal{X}_1)$. Since the underlying spaces are Polish, the product σ-algebra $\mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathcal{X}_1)$ coincides with the natural Borel σ-algebra $\mathcal{B}(\mathcal{X}_0 \times \mathcal{X}_1)$ on the product space $\mathcal{X}_0 \times \mathcal{X}_1$.

For $i \in \{0,1\}$, we define the canonical coordinate mappings $\chi_i \colon \mathcal{X}_0 \times \mathcal{X}_1 \to \mathcal{X}_i$ by $\chi_i(x_0,x_1) = x_i$ for a pair $(x_0,x_1) \in \mathcal{X}_0 \times \mathcal{X}_1$. Note that each χ_i is measurable with respect to the product σ-algebra $\mathcal{B}(\mathcal{X}_0) \otimes \mathcal{B}(\mathcal{X}_1)$ in the domain and $\mathcal{B}(\mathcal{X}_i)$ in the codomain.

Consider a Markovian transition kernel $P \colon \mathcal{X}_0 \times \mathcal{B}(\mathcal{X}_1) \to [0,1]$ from $\mathcal{X}_0$ into $\mathcal{X}_1$. This means that, for every point $x_0 \in \mathcal{X}_0$, the function $P(x_0,\,\cdot\,) \colon \mathcal{B}(\mathcal{X}_1) \to [0,1]$ given by $B_1 \mapsto P(x_0,B_1)$ is a probability measure on $\mathcal{X}_1$, and for every set $B_1 \in \mathcal{B}(\mathcal{X}_1)$, the map $P(\,\cdot\,,B_1) \colon \mathcal{X}_0 \to [0,1]$ defined via $x_0 \mapsto P(x_0,B_1)$ is $\mathcal{B}(\mathcal{X}_0)$-measurable.

Choose a probability measure μ_0 on $\mathcal{X}_0$ and a σ-finite measure ν_0 on $\mathcal{X}_0$. We define a probability measure P and a σ-finite measure Q on the product space $\mathcal{X}_0 \times \mathcal{X}_1$ by

$$\mathsf{P}[B_0 \times B_1] = \int_{B_0} \mu_0(\mathrm{d}x_0)\, P(x_0, B_1) \qquad \text{and} \qquad \mathsf{Q}[B_0 \times B_1] = \int_{B_0} \nu_0(\mathrm{d}x_0)\, P(x_0, B_1) \tag{1.1}$$

for $B_0 \in \mathcal{B}(\mathcal{X}_0)$ and $B_1 \in \mathcal{B}(\mathcal{X}_1)$.

The marginal measures $\mathsf{P}_0 = (\chi_0)_\#\mathsf{P}$ and $\mathsf{Q}_0 = (\chi_0)_\#\mathsf{Q}$ at time zero are equal to μ_0 and ν_0, respectively. At time one, the marginal measures $\mathsf{P}_1 = (\chi_1)_\#\mathsf{P}$ and $\mathsf{Q}_1 = (\chi_1)_\#\mathsf{Q}$ are given by

$$\mathsf{P}_1[B_1] = \int_{\mathcal{X}_0} \mu_0(\mathrm{d}x_0)\, P(x_0, B_1) \qquad \text{and} \qquad \mathsf{Q}_1[B_1] = \int_{\mathcal{X}_0} \nu_0(\mathrm{d}x_0)\, P(x_0, B_1) \tag{1.2}$$

for $B_1 \in \mathcal{B}(\mathcal{X}_1)$.

In the following, we assume that the Markovian transition kernel P from $\mathcal{X}_0$ into $\mathcal{X}_1$ and the σ-finite measure ν_0 on $\mathcal{X}_0$ are chosen such that the marginal measure Q_1 is a σ-finite measure on $\mathcal{X}_1$. This assumption ensures that conditional expectations of the form $\mathbb{E}_\mathsf{Q}[\,\cdot \,|\, \chi_1]$ are well-defined, see [Léo14b, Definition 1.8]. In this setting, we now formulate our first preliminary result.

Lemma 1.1. *Under the assumptions of* Section 1.1, *assume that* $\mathsf{P}_0 \ll \mathsf{Q}_0$ *and let* $l_0 = \frac{\mathrm{d}\mathsf{P}_0}{\mathrm{d}\mathsf{Q}_0}$ *be the Radon-Nikodym derivative of* P_0 *with respect to* Q_0*. Then also* $\mathsf{P}_1 \ll \mathsf{Q}_1$ *and the Radon-Nikodym derivative* $l_1 = \frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{Q}_1}$ *of* P_1 *with respect to* Q_1 *satisfies*

$$\mathbb{E}_\mathsf{Q}[l_0(\chi_0) \,|\, \chi_1] = l_1(\chi_1). \tag{1.3}$$

Proof. First, it is easy to see that the assumption $\mathsf{P}_0 \ll \mathsf{Q}_0$ implies $\mathsf{P} \ll \mathsf{Q}$. At the same time, the latter relation entails that also $\mathsf{P}_1 \ll \mathsf{Q}_1$.

Now we turn to the proof of (1.3). By the factorization lemma, see [Kle14, Corollary 1.97], there exists a measurable function $\varphi_1 \colon \mathcal{X}_1 \to \overline{\mathbb{R}}$ such that

$$\mathbb{E}_\mathsf{Q}[l_0(\chi_0) \,|\, \chi_1] = \varphi_1(\chi_1). \tag{1.4}$$

Since Q_1 is a σ-finite measure on $\mathcal{X}_1$, according to [Léo14b], there exists a regular conditional distribution $\mathsf{Q}[\chi_0 \in \cdot \,|\, \chi_1]$ of χ_0 given χ_1 with respect to the measure Q such that

$$\mathbb{E}_\mathsf{Q}[l_0(\chi_0) \,|\, \chi_1] = \int_{\mathcal{X}_0} \mathsf{Q}[\chi_0 \in \mathrm{d}x_0 \,|\, \chi_1]\, l_0(x_0) \tag{1.5}$$

and

$$\varphi_1(x_1) = \int_{\mathcal{X}_0} \mathsf{Q}[\chi_0 \in \mathrm{d}x_0 \,|\, \chi_1 = x_1]\, l_0(x_0) \tag{1.6}$$

for $x_1 \in \mathcal{X}_1$. Moreover, the measure Q can be represented as

$$\mathsf{Q}[\,\cdot\,] = \mathbb{E}_\mathsf{Q}[\mathsf{Q}[\,\cdot \,|\, \chi_1]] = \int_{\mathcal{X}_1} \mathsf{Q}_1(\mathrm{d}x_1)\, \mathsf{Q}[\;\cdot\; \,|\, \chi_1 = x_1]. \tag{1.7}$$

In order to prove the assertion of the lemma, we have to show that $\varphi_1 = l_1$, i.e. $\mathsf{P}_1 = \varphi_1 \odot \mathsf{Q}_1$. To see this, let $B_1 \in \mathcal{B}(\mathcal{X}_1)$. On the one hand, by (1.2) we have that

$$\mathsf{P}_1[B_1] = \int_{\mathcal{X}_0} \mu_0(\mathrm{d}x_0)\, P(x_0, B_1) \tag{1.8}$$

and because of $\mu_0 = l_0 \odot \nu_0$, we get

$$\mathsf{P}_1[B_1] = \int_{\mathcal{X}_0} \nu_0(\mathrm{d}x_0)\, l_0(x_0)\, P(x_0, B_1). \tag{1.9}$$

On the other hand, (1.6) yields

$$(\varphi_1 \odot \mathsf{Q}_1)[B_1] = \int_{B_1} \mathsf{Q}_1(\mathrm{d}x_1)\, \varphi_1(x_1) = \int_{\mathcal{X}_1} \mathsf{Q}_1(\mathrm{d}x_1)\, \mathbb{1}_{B_1}(x_1) \int_{\mathcal{X}_0} \mathsf{Q}[\chi_0 \in \mathrm{d}x_0 \,|\, \chi_1 = x_1]\, l_0(x_0). \tag{1.10}$$

As a consequence of (1.7), the last double integral is equal to

$$\int_{\mathcal{X}_0 \times \mathcal{X}_1} \mathsf{Q}(\mathrm{d}(x_0, x_1))\, l_0(x_0)\, \mathbb{1}_{B_1}(x_1) = \mathbb{E}_{\mathsf{Q}}\left[\ell_0(\chi_0)\, \mathbb{1}_{B_1}(\chi_1)\right]. \tag{1.11}$$

Finally, from the definition of the measure Q in (1.1), we conclude that

$$(\varphi_1 \odot \mathsf{Q}_1)[B_1] = \int_{\mathcal{X}_0} \nu_0(\mathrm{d}x_0)\, l_0(x_0)\, P(x_0, B_1). \tag{1.12}$$

Comparing with (1.9), we see that $\mathsf{P}_1 = \varphi_1 \odot \mathsf{Q}_1$ as required. □

1.2 Notation and general setting

Let $\mathcal{X}$ be a Polish space with Borel σ-algebra $\mathcal{B}(\mathcal{X})$. We define the path space $\Omega = \mathcal{C}(\mathbb{R}_+, \mathcal{X})$ as the space of all continuous functions from the time interval $\mathbb{R}_+ = [0, \infty)$ into the state space $\mathcal{X}$. For each $t \geqslant 0$, we define the canonical coordinate mappings $X_t \colon \Omega \to \mathcal{X}$ by $X_t(\omega) = \omega_t$ for a continuous path $\omega = (\omega_s)_{s \geqslant 0} \in \Omega$. Furthermore, we write $(\mathcal{F}_t)_{t \geqslant 0}$ for the natural filtration generated by the canonical coordinate process $(X_t)_{t \geqslant 0}$ and equip the path space Ω with the σ-algebra $\mathcal{F}_\infty = \sigma(X_t \colon t \geqslant 0)$.

Given some measurable space $(\Omega', \mathcal{F}')$, we shall denote by $\mathcal{M}(\Omega')$ the set of all measures on Ω' and by $\mathcal{M}^1(\Omega')$ the subset of all probability measures on Ω'. Any measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ on the path space Ω is called a *path measure*.

If $F \colon \Omega' \to \mathcal{X}$ is a measurable function and $\mathbb{Q} \in \mathcal{M}(\Omega')$, we denote the pushforward measure of $\mathbb{Q}$ under F by $F_\# \mathbb{Q} \in \mathcal{M}(\mathcal{X})$. In particular, if $\Omega' = \Omega$ and $F = X_t$ for some $t \geqslant 0$, we write $\mathbb{Q}_t = (X_t)_\# \mathbb{Q} \in \mathcal{M}(\mathcal{X})$ for the marginal measure of $\mathbb{Q}$ at time t.

For the vector space of all bounded and measurable functions on $\mathcal{X}$ with values in $\mathbb{R}$ we write $B(\mathcal{X}, \mathbb{R})$.

We fix a transition semigroup $(P_t)_{t \geqslant 0}$, which is a collection of Markovian transition kernels on $\mathcal{X}$ satisfying the following three properties:

(1) For every point $x \in \mathcal{X}$, we have that $P_0(x, \cdot\,) = \delta_x$.

(2) For all $s, t \geqslant 0$ and $B \in \mathcal{B}(\mathcal{X})$, we have that

$$P_{t+s}(x, B) = \int_{\mathcal{X}} P_t(x, \mathrm{d}y)\, P_s(y, B). \qquad \text{(Chapman-Kolmogorov identity)}$$

(3) For every set $B \in \mathcal{B}(\mathcal{X})$, the map $P_\cdot(\,\cdot\,, B) \colon \mathbb{R}_+ \times \mathcal{X} \to [0, 1]$ given by $(t, x) \mapsto P_t(x, B)$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{X})$ in the domain and $\mathcal{B}([0, 1])$ in the codomain.

If $f \in B(\mathcal{X}, \mathbb{R})$ and $t \geqslant 0$, we can define a function $P_t f \in B(\mathcal{X}, \mathbb{R})$ by

$$P_t f(x) = \int_{\mathcal{X}} P_t(x, \mathrm{d}y)\, f(y) \tag{1.13}$$

for $x \in \mathcal{X}$.

For each $x \in \mathcal{X}$, let $\mathbb{P}^x \in \mathcal{M}^1(\Omega)$ be the unique path measure under which the canonical coordinate process $(X_t)_{t\geqslant 0}$ is a Markov process (with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$) with transition semigroup $(P_t)_{t\geqslant 0}$, and the law of X_0 is δ_x, i.e. $(\mathbb{P}^x)_0 = \delta_x$. The canonical coordinate process $(X_t)_{t\geqslant 0}$ being a Markov process under $\mathbb{P}^x$ means that

$$\mathbb{E}_{\mathbb{P}^x}[f(X_{s+t}) \,|\, \mathcal{F}_s] = P_t f(X_s) \qquad \mathbb{P}^x\text{-almost surely} \tag{1.14}$$

and in particular

$$\mathbb{E}_{\mathbb{P}^x}[f(X_{s+t}) \,|\, \mathcal{F}_s] = \mathbb{E}_{\mathbb{P}^x}[f(X_{s+t}) \,|\, X_s] \qquad \mathbb{P}^x\text{-almost surely} \tag{1.15}$$

for every $f \in B(\mathcal{X}, \mathbb{R})$ and all $s, t \geqslant 0$.

For a short introduction to Markov processes we refer to [LG16, Chapter 6], from where we have borrowed most of the notation and terminology used above.

Choose a probability measure $\mu \in \mathcal{M}^1(\mathcal{X})$ and a σ-finite measure $\nu \in \mathcal{M}(\mathcal{X})$. We define a probability measure $\mathbb{P} \in \mathcal{M}^1(\Omega)$ and a measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ on the path space $(\Omega, \mathcal{F}_\infty)$ by

$$\mathbb{P}[A] = \int_{\mathcal{X}} \mu(\mathrm{d}x)\, \mathbb{P}^x[A] \qquad \text{and} \qquad \mathbb{Q}[A] = \int_{\mathcal{X}} \nu(\mathrm{d}x)\, \mathbb{P}^x[A] \tag{1.16}$$

for $A \in \mathcal{F}_\infty$. Then $\mathbb{P}$ is the unique probability measure on the path space $(\Omega, \mathcal{F}_\infty)$ under which the canonical coordinate process $(X_t)_{t\geqslant 0}$ is a Markov process (with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$) with transition semigroup $(P_t)_{t\geqslant 0}$, and the law of X_0 is μ, i.e. $\mathbb{P}_0 = \mu$. Similarly, $\mathbb{Q}$ is the unique measure on Ω under which the canonical coordinate process is a Markov process with semigroup $(P_t)_{t\geqslant 0}$ and $\mathbb{Q}_0 = \nu$. We call such measures $\mathbb{P}$ and $\mathbb{Q}$ *Markov measures.*

Remark 1.2. For the proper definition of conditional expectations with respect to a σ-finite measure with possibly infinite mass, such as $\mathbb{Q}$, we need conditions (i) and (ii) of Assumptions 1.3 below; see [Léo14b] and Remark 1.4 below. ◇

Let $t > 0$. We use the notation $\mathbb{P}_{0,t} = (X_0, X_t)_\# \mathbb{P}$ and analogously $\mathbb{Q}_{0,t} = (X_0, X_t)_\# \mathbb{Q}$. Note that we have

$$\mathbb{P}_{0,t}[B_0 \times B_t] = \int_{B_0} \mu(\mathrm{d}\omega_0)\, P_t(\omega_0, B_t) \tag{1.17}$$

and similarly

$$\mathbb{Q}_{0,t}[B_0 \times B_t] = \int_{B_0} \nu(\mathrm{d}\omega_0)\, P_t(\omega_0, B_t) \tag{1.18}$$

for $B_0, B_t \in \mathcal{B}(\mathcal{X})$.

At time $t \geqslant 0$, the marginal measures of $\mathbb{P}$ and $\mathbb{Q}$ are given by

$$\mathbb{P}_t[B_t] = \int_{\mathcal{X}} \mu(\mathrm{d}\omega_0)\, P_t(\omega_0, B_t) \qquad \text{and} \qquad \mathbb{Q}_t[B_t] = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0)\, P_t(\omega_0, B_t) \tag{1.19}$$

for $B_t \in \mathcal{B}(\mathcal{X})$. In particular, $\mathbb{P}_0 = \mu$ and $\mathbb{Q}_0 = \nu$.

1.2.1 Standard assumptions of Chapter 1

For the remainder of this chapter, we impose that the following three standard assumptions are satisfied.

Assumptions 1.3 (Standard assumptions of Chapter 1)**.**

(i) *σ-finiteness of the initial marginal measure* $\mathbb{Q}_0$*:* The initial marginal measure $\mathbb{Q}_0 = \nu$ is a σ-finite measure on $\mathcal{X}$. As a consequence, the path measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ is σ-finite, too. This can be seen as follows. Since the marginal measure $\mathbb{Q}_0 \in \mathcal{M}(\mathcal{X})$ is σ-finite by assumption, we can choose a sequence $(B_n)_{n\geqslant 1} \subseteq \mathcal{B}(\mathcal{X})$ which satisfies $\mathbb{Q}_0[B_n] < \infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n\geqslant 1} B_n = \mathcal{X}$. Now we consider the sets $A_n = \{X_0 \in B_n\} \in \mathcal{F}_\infty$ for $n \in \mathbb{N}$. Then $\mathbb{Q}[A_n] = \mathbb{Q}_0[B_n] < \infty$ for all $n \in \mathbb{N}$ and we have that $\bigcup_{n\geqslant 1} A_n = \Omega$, which shows the σ-finiteness of $\mathbb{Q}$.

(ii) *σ-finiteness of the marginal measures* $\mathbb{Q}_t$ *for all* $t > 0$*:* The transition semigroup $(P_t)_{t\geqslant 0}$ and the σ-finite measure ν on $\mathcal{X}$ are chosen such that the marginal measures $\mathbb{Q}_t$ are σ-finite measures on $\mathcal{X}$ for all $t > 0$.

(iii) *Absolute continuity of* $\mathbb{P}_0$ *with respect to* $\mathbb{Q}_0$*:* The initial marginal measures $\mathbb{P}_0$ and $\mathbb{Q}_0$ satisfy $\mathbb{P}_0 \ll \mathbb{Q}_0$. In other words, μ is absolutely continuous with respect to ν, which implies that $\mathbb{P} \ll \mathbb{Q}$. This follows directly from the definition of the path measures $\mathbb{P}$ and $\mathbb{Q}$ in (1.16). Additionally, the absolute continuity of $\mathbb{P}$ with respect to $\mathbb{Q}$ entails that $\mathbb{P}_t \ll \mathbb{Q}_t$ for all $t \geqslant 0$, which is an immediate consequence of the definition of the marginal measures $\mathbb{P}_t$ and $\mathbb{Q}_t$.

Remark 1.4. Conditions (i) and (ii) of Assumptions 1.3 imply that the marginal measures $\mathbb{Q}_t$ are σ-finite measures on $\mathcal{X}$ for all $t \geqslant 0$. This assumption ensures that $\mathbb{Q}$ is a *conditionable path measure*, see [Léo14b, Definition 1.8], which means that conditional expectations of the form $\mathbb{E}_\mathbb{Q}[\,\cdot\,|\,(X_i)_{i\in I}]$ are well-defined for any subset $I \subseteq \mathbb{R}_+$. $\diamond$

Definition 1.5. As a result of Assumptions 1.3, the Radon-Nikodym derivative of $\mathbb{P}_t$ with respect to $\mathbb{Q}_t$ is well-defined for all $t \geqslant 0$ and we denote it by $\ell_t = \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t}$. We refer to ℓ_t as the *likelihood ratio* of the Markov measure $\mathbb{P}$ with respect to the Markov measure $\mathbb{Q}$ at time $t \geqslant 0$. Note that $\ell_t \in L^1(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{Q}_t)$.

1.2.2 Martingales on a σ-finite path space

In this subsection we define martingales on a σ-finite path space and also discuss backwards martingales.

Definition 1.6. Let $(\mathcal{H}_t)_{t\geqslant 0}$ be a filtration on the σ-finite path space $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$. We assume that for each $t \geqslant 0$ there exists a subset $I \subseteq \mathbb{R}_+$ such that $\mathcal{H}_t = \sigma((X_i)_{i\in I})$. Together with conditions (i) and (ii) of Assumptions 1.3, this ensures that conditional expectations of the form $\mathbb{E}_\mathbb{Q}[\,\cdot\,|\,\mathcal{H}_t]$ are well-defined for every $t \geqslant 0$. Let $(M_t)_{t\geqslant 0}$ be a real-valued stochastic process which is adapted to the filtration $(\mathcal{H}_t)_{t\geqslant 0}$ and satisfies $M_t \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ for all $t \geqslant 0$. Then $(M_t)_{t\geqslant 0}$ is called a $\mathbb{Q}$-*martingale* with respect to the filtration $(\mathcal{H}_t)_{t\geqslant 0}$ if

$$\mathbb{E}_\mathbb{Q}[M_t\,|\,\mathcal{H}_s] = M_s \tag{1.20}$$

for all $s, t \geqslant 0$ with $s \leqslant t$. We define $\mathbb{Q}$-*submartingales* and $\mathbb{Q}$-*supermartingales* in the obvious ways.

Remark 1.7. Let $(M_t)_{t\geqslant 0}$ be a $\mathbb{Q}$-martingale with respect to some filtration $(\mathcal{H}_t)_{t\geqslant 0}$ on the path space $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$. Then Definition 1.6 tacitly requires the filtration $(\mathcal{H}_t)_{t\geqslant 0}$ to be of the special form described above. $\diamond$

Analogously, we define $\mathbb{Q}$-martingales when the path space $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ remains the same but the index set $\mathbb{R}_+$ is replaced by $\mathbb{R}_- = (-\infty, 0]$. In this setting, we define backwards $\mathbb{Q}$-martingales as follows.

Definition 1.8. Let $(\overline{\mathcal{H}}_t)_{t\leqslant 0}$ be a filtration on $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ and assume that for each $t \leqslant 0$ there is a subset $I \subseteq \mathbb{R}_+$ such that $\overline{\mathcal{H}}_t = \sigma((X_i)_{i\in I})$. Let $(\overline{M}_t)_{t\leqslant 0}$ be a real-valued stochastic process which is adapted to the filtration $(\overline{\mathcal{H}}_t)_{t\leqslant 0}$ and satisfies $\overline{M}_t \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ for all $t \leqslant 0$. Then $(\overline{M}_t)_{t\leqslant 0}$ is a $\mathbb{Q}$-martingale with respect to the filtration $(\overline{\mathcal{H}}_t)_{t\leqslant 0}$ if

$$\mathbb{E}_{\mathbb{Q}}[\overline{M}_t \,|\, \overline{\mathcal{H}}_s] = \overline{M}_s \tag{1.21}$$

for all $s, t \leqslant 0$ with $s \leqslant t$. Now for $t \geqslant 0$ set $\mathcal{A}_t = \overline{\mathcal{H}}_{-t}$. For $s, t \geqslant 0$ with $s \leqslant t$ we have that $\mathcal{A}_t \subseteq \mathcal{A}_s$ and hence we call $(\mathcal{A}_t)_{t\geqslant 0}$ a *backwards filtration* on $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$. The stochastic process $(\overline{M}_{-t})_{t\geqslant 0}$ is called a *backwards* $\mathbb{Q}$*-martingale* with respect to the backwards filtration $(\mathcal{A}_t)_{t\geqslant 0}$ and satisfies

$$\mathbb{E}_{\mathbb{Q}}[\overline{M}_{-s} \,|\, \mathcal{A}_t] = \overline{M}_{-t} \tag{1.22}$$

for all $s, t \geqslant 0$ with $s \leqslant t$. Analogously we define *backwards* $\mathbb{Q}$*-submartingales* and *backwards* $\mathbb{Q}$*-supermartingales.*

1.2.3 Relative entropy with respect to a σ-finite measure

First of all, in the definition below, we recall the notion of *relative entropy* between two probability measures.

Definition 1.9. Let $\mathcal{P}$ and $\mathcal{Q}$ be two probability measures on some measurable space $(\Omega', \mathcal{F}')$. The *relative entropy* or *Kullback-Leibler divergence* of $\mathcal{P}$ with respect to $\mathcal{Q}$ is defined by

$$H(\mathcal{P} \,|\, \mathcal{Q}) = \int_{\Omega'} \log\big(\tfrac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}\big)\,\mathrm{d}\mathcal{P} \tag{1.23}$$

if $\mathcal{P} \ll \mathcal{Q}$ and $H(\mathcal{P} \,|\, \mathcal{Q}) = \infty$ otherwise.

Remark 1.10. The function $f(p) = p\log p - p + 1$ is non-negative for $p > 0$ and we clearly have

$$H(\mathcal{P} \,|\, \mathcal{Q}) = \int_{\Omega'} f\big(\tfrac{\mathrm{d}\mathcal{P}}{\mathrm{d}\mathcal{Q}}\big)\,\mathrm{d}\mathcal{Q} \tag{1.24}$$

for $\mathcal{P}, \mathcal{Q} \in \mathcal{M}^1(\Omega')$ with $\mathcal{P} \ll \mathcal{Q}$. This implies that the relative entropy $H(\mathcal{P} \,|\, \mathcal{Q})$ between two probability measures is well-defined and takes values in the interval $[0, \infty]$. ◇

We are now interested in the more general situation where $\mathrm{Q} \in \mathcal{M}(\Omega')$ is a σ-finite measure. In this case we have to identify the collection of probability measures $\mathrm{P} \in \mathcal{M}^1(\Omega')$ for which the relative entropy $H(\mathrm{P} \,|\, \mathrm{Q})$ given in formula (1.23) is well-defined. To this end we pursue the following procedure, which is recorded in [Léo14b, Section 2].

Let $\mathrm{Q} \in \mathcal{M}(\Omega')$ be a σ-finite measure and let $\mathrm{P} \in \mathcal{M}^1(\Omega')$ be such that $\mathrm{P} \ll \mathrm{Q}$. Using the definition of σ-finiteness, it is easy to construct a measurable function $g\colon \Omega' \to (0,1)$ such that the expectation $\mathbb{E}_{\mathrm{Q}}[g]$ is finite. Then the strictly positive measurable function $h = -\log g$ satisfies $\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}] < \infty$. Next, we consider the probability measure

$$\mathcal{Q} = \tfrac{\mathrm{e}^{-h}}{\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]} \odot \mathrm{Q} \tag{1.25}$$

on Ω'. Obviously we have that

$$\tfrac{\mathrm{dP}}{\mathrm{dQ}} = \tfrac{\mathrm{e}^{-h}}{\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-g}]}\,\tfrac{\mathrm{dP}}{\mathrm{d}\mathcal{Q}} \tag{1.26}$$

and taking the logarithm on both sides yields

$$\log\big(\tfrac{\mathrm{dP}}{\mathrm{dQ}}\big) = \log\big(\tfrac{\mathrm{dP}}{\mathrm{d}\mathcal{Q}}\big) - h - \log\big(\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]\big). \tag{1.27}$$

In case that $\mathbb{E}_{\mathrm{P}}[h]$ is finite we can take the expectation with respect to P and obtain

$$\mathbb{E}_{\mathrm{P}}\big[\log\big(\tfrac{\mathrm{dP}}{\mathrm{dQ}}\big)\big] = \mathbb{E}_{\mathrm{P}}\big[\log\big(\tfrac{\mathrm{dP}}{\mathrm{d}\mathcal{Q}}\big)\big] - \mathbb{E}_{\mathrm{P}}[h] - \log\big(\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]\big), \tag{1.28}$$

which is a number in the interval $(-\infty, \infty]$. In light of (1.23), this means we have

$$H(\mathrm{P}\,|\,\mathrm{Q}) = H(\mathrm{P}\,|\,\mathcal{Q}) - \mathbb{E}_{\mathrm{P}}[h] - \log\big(\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]\big). \tag{1.29}$$

Summarizing, we have proven the following lemma.

Lemma 1.11 ([Léo14b])**.** *Let* $\mathrm{Q} \in \mathcal{M}(\Omega')$ *be a* σ*-finite measure and let* $\mathrm{P} \in \mathcal{M}^1(\Omega')$ *be a probability measure with* $\mathrm{P} \ll \mathrm{Q}$. *Choose a strictly positive measurable function* h *such that* $\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]$ *is finite. Then the relative entropy of* P *with respect to* Q *given in formula* (1.23) *is well-defined. More precisely,* $H(\mathrm{P}\,|\,\mathrm{Q})$ *takes values in the interval* $(-\infty, \infty]$ *and can be computed as*

$$H(\mathrm{P}\,|\,\mathrm{Q}) = H(\mathrm{P}\,|\,\mathcal{Q}) - \mathbb{E}_{\mathrm{P}}[h] - \log\big(\mathbb{E}_{\mathrm{Q}}[\mathrm{e}^{-h}]\big), \tag{1.30}$$

where the probability measure $\mathcal{Q}$ *is defined as in* (1.25)*.*

1.3 The Fontbona-Jourdain theorem

Throughout this section, we use the notation and setting of the previous Section 1.2. Recall that according to conditions (i) and (ii) of Assumptions 1.3, the marginal measures $\mathbb{Q}_t$ are assumed to be σ-finite for all $t \geqslant 0$. As already mentioned, condition (iii) of Assumptions 1.3 implies that $\mathbb{P}_t$ is absolutely continuous with respect to $\mathbb{Q}_t$ for all $t \geqslant 0$, and we write $\ell_t = \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t}$ for the Radon-Nikodym derivative of $\mathbb{P}_t$ with respect to $\mathbb{Q}_t$.

The following lemma shows how to compute the likelihood ratio ℓ_t as a conditional expectation under the measure $\mathbb{Q}$. It is a direct consequence of the preliminary result formulated in Lemma 1.1 of Section 1.1.

Lemma 1.12. *Under the* Assumptions 1.3*, for all* $t \geqslant 0$ *we have that*

$$\mathbb{E}_{\mathbb{Q}}[\ell_0(X_0)\,|\,X_t] = \ell_t(X_t). \tag{1.31}$$

Proof. As one would expect, the proof of Lemma 1.12 follows from an unsurprising application of Lemma 1.1 in the obvious way. Therefore, the reader who suffers from a lack of time or feels very comfortable with our preliminary result may safely skip the subsequent explanation.

Since the claim is trivial for $t = 0$, let us fix a time $t > 0$. In the setting of Section 1.1, consider the case where the two Polish spaces $\mathcal{X}_0$ and $\mathcal{X}_1$ are both equal to $\mathcal{X}$. Then the canonical coordinate mappings $\chi_0, \chi_1 \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are given by $\chi_0(\omega_s, \omega_t) = \omega_s$ and $\chi_1(\omega_s, \omega_t) = \omega_t$ for a pair $(\omega_s, \omega_t) \in \mathcal{X} \times \mathcal{X}$. We now take $P = P_t$ as Markovian transition kernel and use the measures $\mu_0 = \mu \in \mathcal{M}^1(\mathcal{X})$ and $\nu_0 = \nu \in \mathcal{M}(\mathcal{X})$. Then (1.1) turns into (1.17) and (1.18). In other words, $\mathsf{P} = \mathbb{P}_{0,t}$ and $\mathsf{Q} = \mathbb{Q}_{0,t}$. Moreover, we see that (1.2) becomes (1.19). To put it another way, $\mathsf{P}_1 = \mathbb{P}_t$ and $\mathsf{Q}_1 = \mathbb{Q}_t$. It is also clear that $\mathsf{P}_0 = \mathbb{P}_0$ and $\mathsf{Q}_0 = \mathbb{Q}_0$. In particular, $l_0 = \ell_0$. Applying Lemma 1.1 in this situation with $\mathbb{P}_t = \mathsf{P}_1 \ll \mathsf{Q}_1 = \mathbb{Q}_t$ shows that the Radon-Nikodym derivative $l_1 = \frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{Q}_1} = \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t} = \ell_t$ satisfies

$$\mathbb{E}_{\mathsf{Q}}\big[l_0(\chi_0)\,|\,\chi_1\big] = l_1(\chi_1) \tag{1.32}$$

or equivalently

$$\mathbb{E}_{\mathbb{Q}_{0,t}}\big[\ell_0(\chi_0)\,|\,\chi_1\big] = \ell_t(\chi_1). \tag{1.33}$$

Finally, note that the identity (1.33) holds on the measure space $(\mathcal{X} \times \mathcal{X}, \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X}), \mathbb{Q}_{0,t})$, but it clearly implies (1.31), which holds on the path space $(\Omega, \mathcal{F}_\infty, \mathbb{Q})$. $\square$

Remark 1.13. Note that conditions (i) and (ii) of Assumptions 1.3 ensure the conditional expectation in (1.31) to be well-defined for every $t \geqslant 0$. Furthermore, in order to apply Lemma 1.1 in the proof of Lemma 1.12, we also require conditions (i) and (ii) of Assumptions 1.3 because we need the σ-finiteness of the marginal measure $\mathsf{Q}_1 = \mathbb{Q}_t$ for all $t \geqslant 0$. ◇

Our next goal is to generalize Lemma 1.12 by showing that the initial time zero in (1.31) can be replaced by any time $s \geqslant 0$ with $s \leqslant t$. The following lemma makes this precise and its proof only uses the definitions of the involved objects.

Lemma 1.14. *Under the* Assumptions 1.3, *for any* $s, t \geqslant 0$ *with* $s \leqslant t$ *we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, X_t\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0) \,|\, X_t\big]. \tag{1.34}$$

Proof. Let $s, t \geqslant 0$ with $s \leqslant t$. In order to verify (1.34), by definition of conditional expectation, it suffices to show that

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\, \mathbb{1}_{B_t}(X_t)\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\, \mathbb{1}_{B_t}(X_t)\big] \tag{1.35}$$

for all $B_t \in \mathcal{B}(\mathcal{X})$. Fix a set $B_t \in \mathcal{B}(\mathcal{X})$. On the one hand, from the definition of the measure $\mathbb{Q}$, we conclude that

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\, \mathbb{1}_{B_t}(X_t)\big] = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0) \int_{\mathcal{X}} P_s(\omega_0, \mathrm{d}\omega_s)\, \ell_s(\omega_s) \int_{\mathcal{X}} P_{t-s}(\omega_s, \mathrm{d}\omega_t)\, \mathbb{1}_{B_t}(\omega_t) \tag{1.36}$$

and therefore

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\, \mathbb{1}_{B_t}(X_t)\big] = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0) \int_{\mathcal{X}} P_s(\omega_0, \mathrm{d}\omega_s)\, \ell_s(\omega_s)\, P_{t-s}(\omega_s, B_t). \tag{1.37}$$

The representations of the marginal measures of $\mathbb{P}$ and $\mathbb{Q}$ in (1.19) imply that

$$\int_{\mathcal{X}} \mathbb{P}_s(\mathrm{d}\omega_s)\, P_{t-s}(\omega_s, B_t) = \int_{\mathcal{X}} \mu(\mathrm{d}\omega_0) \int_{\mathcal{X}} P_s(\omega_0, \mathrm{d}\omega_s)\, P_{t-s}(\omega_s, B_t) \tag{1.38}$$

and similarly

$$\int_{\mathcal{X}} \mathbb{Q}_s(\omega_s)\, \ell_s(\omega_s)\, P_{t-s}(\omega_s, B_t) = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0) \int_{\mathcal{X}} P_s(\omega_0, \mathrm{d}\omega_s)\, \ell_s(\omega_s)\, P_{t-s}(\omega_s, B_t). \tag{1.39}$$

At the same time, the relation $\mathbb{P}_s = \ell_s \odot \mathbb{Q}_s$ entails

$$\int_{\mathcal{X}} \mathbb{P}_s(\mathrm{d}\omega_s)\, P_{t-s}(\omega_s, B_t) = \int_{\mathcal{X}} \mathbb{Q}_s(\mathrm{d}\omega_s)\, \ell_s(\omega_s)\, P_{t-s}(\omega_s, B_t). \tag{1.40}$$

As a consequence,

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\, \mathbb{1}_{B_t}(X_t)\big] = \int_{\mathcal{X}} \mu(\mathrm{d}\omega_0) \int_{\mathcal{X}} P_s(\omega_0, \mathrm{d}\omega_s)\, P_{t-s}(\omega_s, B_t) \tag{1.41}$$

and by the Chapman-Kolmogorov identity this is equal to

$$\int_{\mathcal{X}} \mu(\mathrm{d}\omega_0)\, P_t(\omega_0, B_t) = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0)\, \ell_0(\omega_0)\, P_t(\omega_0, B_t), \tag{1.42}$$

where we used that $\mu = \ell_0 \odot \nu$. On the other hand, again by definition of the measure $\mathbb{Q}$ we have that

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\, \mathbb{1}_{B_t}(X_t)\big] = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0)\, \ell_0(\omega_0) \int_{\mathcal{X}} P_t(\omega_0, \mathrm{d}\omega_t)\, \mathbb{1}_{B_t}(\omega_t) \tag{1.43}$$

and hence

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\, \mathbb{1}_{B_t}(X_t)\big] = \int_{\mathcal{X}} \nu(\mathrm{d}\omega_0)\, \ell_0(\omega_0)\, P_t(\omega_0, B_t) \tag{1.44}$$

as required. □

Summarizing Lemma 1.12 and Lemma 1.14, we obtain the following result.

Lemma 1.15. *Under the* Assumptions 1.3*, for any $s, t \geqslant 0$ with $s \leqslant t$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\,|\,X_t\big] = \ell_t(X_t). \tag{1.45}$$

Proof. Combining Lemma 1.12 and Lemma 1.14 yields the claim. □

We observe that the identity (1.45) suggests the stochastic process $(\ell_t(X_t))_{t\geqslant 0}$ to be a backwards $\mathbb{Q}$-martingale with respect to an appropriate backwards filtration. Indeed, we just have to replace the conditional expectation $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\,X_t]$ in (1.45) by $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\,\mathcal{G}_t]$, where the σ-algebra $\mathcal{G}_t$ is defined as $\mathcal{G}_t = \sigma(X_u \colon u \geqslant t)$ for $t \geqslant 0$. This is justified in the lemma below, which essentially shows that the Markov process $(X_t)_{t\geqslant 0}$ running backwards in time maintains the Markov property.

Lemma 1.16. *Under the* Assumptions 1.3*, let $s, t \geqslant 0$ with $s \leqslant t$ and $f \in L^1(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{Q}_s)$. Then $f(X_s) \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ and we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[f(X_s)\,|\,\mathcal{G}_t\big] = \mathbb{E}_{\mathbb{Q}}\big[f(X_s)\,|\,X_t\big]. \tag{1.46}$$

Proof. Apart from handling conditional expectations, the proof of Lemma 1.16 is a direct consequence of the fact that $(X_t)_{t\geqslant 0}$ is a Markov process under the path measure $\mathbb{Q}$.

Fix $s, t \geqslant 0$ with $s \leqslant t$. Using the change of variables formula for the pushforward measure, we notice that $f \in L^1(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{Q}_s)$ if and only if we have that $f(X_s) \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$. In order to prove (1.46), we set $Z = \mathbb{E}_{\mathbb{Q}}[f(X_s)\,|\,X_t]$. Then, by definition of conditional expectation, we have to show that

$$\mathbb{E}_{\mathbb{Q}}\big[f(X_s)\,\mathbb{1}_B\big] = \mathbb{E}_{\mathbb{Q}}\big[Z\,\mathbb{1}_B\big] \tag{1.47}$$

for all $B \in \mathcal{G}_t$. To see this, let us consider the path measures $\mathbb{Q}^{(1)} = f(X_s) \odot \mathbb{Q}$ as well as $\mathbb{Q}^{(2)} = Z \odot \mathbb{Q}$. Since $f(X_s) \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ and $Z \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$, both measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ are finite. According to (1.47), we have to verify that the measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ agree on the σ-algebra $\mathcal{G}_t$. Now the π-system

$$\Pi = \big\{X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k \colon\ k \in \mathbb{N},\, t \leqslant t_1 \leqslant \ldots \leqslant t_k,\, B_1, \ldots, B_k \in \mathcal{B}(\mathcal{X})\big\} \tag{1.48}$$

generates $\mathcal{G}_t$ and we clearly have $\mathbb{Q}^{(1)}[\Omega] = \mathbb{Q}^{(2)}[\Omega] < \infty$. Therefore, using a standard uniqueness result for finite measures, see e.g. [Kle14, Lemma 1.42], it suffices to verify that $\mathbb{Q}^{(1)}|_\Pi = \mathbb{Q}^{(2)}|_\Pi$. In other words, we have to check that (1.47) holds for all $B \in \Pi$. To this end, let $k \in \mathbb{N}$, choose times $t_1, \ldots, t_k$ with $t \leqslant t_1 \leqslant \ldots \leqslant t_k$, and sets $B_1, \ldots, B_k \in \mathcal{B}(\mathcal{X})$. Then the equation

$$\mathbb{E}_{\mathbb{Q}}\big[f(X_s)\,\mathbb{1}_{B_1}(X_{t_1}) \cdot \ldots \cdot \mathbb{1}_{B_k}(X_{t_k})\big] = \mathbb{E}_{\mathbb{Q}}\big[Z\,\mathbb{1}_{B_1}(X_{t_1}) \cdot \ldots \cdot \mathbb{1}_{B_k}(X_{t_k})\big] \tag{1.49}$$

must be verified. Defining the function

$$g(\omega_t) = \int_{B_1} P_{t_1 - t}(\omega_t, \mathrm{d}\omega_{t_1}) \int_{B_2} P_{t_2 - t_1}(\omega_{t_1}, \mathrm{d}\omega_{t_2}) \cdot \ldots \cdot \int_{B_k} P_{t_k - t_{k-1}}(\omega_{t_{k-1}}, \mathrm{d}\omega_{t_k}) \tag{1.50}$$

for $\omega_t \in \mathcal{X}$ and using the Chapman-Kolmogorov identity, (1.49) can be equivalently written as

$$\mathbb{E}_{\mathbb{Q}}\big[f(X_s)\,g(X_t)\big] = \mathbb{E}_{\mathbb{Q}}\big[Z\,g(X_t)\big]. \tag{1.51}$$

But the equation (1.51) follows immediately from the definition of Z as the conditional expectation $\mathbb{E}_{\mathbb{Q}}[f(X_s)\,|\,X_t]$. □

Combining Lemma 1.15 and Lemma 1.16, we arrive at the following conclusion.

Lemma 1.17. *Under the* Assumptions 1.3*, for any $s, t \geqslant 0$ with $s \leqslant t$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, \mathcal{G}_t\big] = \ell_t(X_t). \tag{1.52}$$

Proof. Let $s, t \geqslant 0$ with $s \leqslant t$. According to Lemma 1.15, we have that

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, X_t\big] = \ell_t(X_t). \tag{1.53}$$

As $\ell_s \in L^1(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathbb{Q}_s)$, we can apply Lemma 1.16 with $f = \ell_s$ and obtain

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, X_t\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, \mathcal{G}_t\big], \tag{1.54}$$

which yields the assertion. □

Definition 1.18. Under the Assumptions 1.3, the stochastic process $(\ell_t(X_t))_{t\geqslant 0}$ is called the *likelihood ratio process* of the Markov measure $\mathbb{P}$ with respect to the Markov measure $\mathbb{Q}$.

For a negative time $t \leqslant 0$, we define $\overline{X}_t = X_{-t}$ and write $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$ for the natural filtration generated by the time-reversed canonical coordinate process $(\overline{X}_t)_{t\leqslant 0}$, i.e. $\overline{\mathcal{F}}_t = \sigma(\overline{X}_s \colon s \leqslant t)$ for $t \leqslant 0$. Furthermore, we set $\overline{\ell}_t = \ell_{-t}$ for $t \leqslant 0$. With this notation, we can recapitulate our findings as follows.

Theorem 1.19 (Fontbona-Jourdain theorem [FJ16])**.** *Under the* Assumptions 1.3*, the likelihood ratio process $(\ell_t(X_t))_{t\geqslant 0}$ is a backwards $\mathbb{Q}$-martingale with respect to the backwards filtration $(\mathcal{G}_t)_{t\geqslant 0}$. That is, for any $s, t \geqslant 0$ with $s \leqslant t$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s) \,|\, \mathcal{G}_t\big] = \ell_t(X_t). \tag{1.55}$$

Equivalently, the time-reversed likelihood ratio process $(\overline{\ell}_t(\overline{X}_t))_{t\leqslant 0}$ is a $\mathbb{Q}$-martingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$. In other words, for any $s, t \leqslant 0$ with $s \leqslant t$ we have that

$$\mathbb{E}_{\mathbb{Q}}\big[\overline{\ell}_t(\overline{X}_t) \,|\, \overline{\mathcal{F}}_s\big] = \overline{\ell}_s(\overline{X}_s). \tag{1.56}$$

Proof. Equation (1.55) is nothing else than (1.52) from Lemma 1.17. Using the notation introduced above and observing that for $t \leqslant 0$ we have that $\overline{\mathcal{F}}_t = \sigma(X_u \colon u \geqslant -t) = \mathcal{G}_{-t}$, we see that (1.55) implies (1.56). □

In the following, we refer to Theorem 1.19 as the *Fontbona-Jourdain theorem*.

1.4 Another proof of the Fontbona-Jourdain theorem

We continue using the notation introduced in the previous sections. In the present section we display an alternative short proof of Theorem 1.19, whose idea is due to Fontbona and Jourdain, see [FJ16].

We have already remarked that condition (i) of Assumptions 1.3 implies that the path measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ is σ-finite, and condition (iii) of Assumptions 1.3 shows the absolute continuity of $\mathbb{P}$ with respect to $\mathbb{Q}$. As a consequence, the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ is well-defined. Furthermore, by its very definition, the *Radon-Nikodym density process* $(\overline{Z}_t)_{t\leqslant 0}$ of $\mathbb{P}$ with respect to $\mathbb{Q}$ generated by the filtration $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$, which is given by

$$\overline{Z}_t = \mathbb{E}_{\mathbb{Q}}\big[\tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} \,|\, \overline{\mathcal{F}}_t\big], \qquad t \leqslant 0, \tag{1.57}$$

is a $\mathbb{Q}$-martingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$ in the sense of Definition 1.8. Next, observe that

$$\mathbb{E}_{\mathbb{Q}}\big[\tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\,\big|\,\overline{\mathcal{F}}_t\big] = \frac{\mathrm{d}\mathbb{P}|_{\overline{\mathcal{F}}_t}}{\mathrm{d}\mathbb{Q}|_{\overline{\mathcal{F}}_t}} = \tfrac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\big|_{\overline{\mathcal{F}}_t}, \qquad t\leqslant 0, \tag{1.58}$$

which easily follows from the definition of conditional expectation and the Radon-Nikodym derivative. In order to conclude that the time-reversed likelihood ratio process $(\overline{\ell}_t(\overline{X}_t))_{t\leqslant 0}$ is a $\mathbb{Q}$-martingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$, it suffices to show that

$$\frac{\mathrm{d}\mathbb{P}|_{\overline{\mathcal{F}}_t}}{\mathrm{d}\mathbb{Q}|_{\overline{\mathcal{F}}_t}} = \overline{\ell}_t(\overline{X}_t), \qquad t\leqslant 0. \tag{1.59}$$

Recall that by Assumptions 1.3 the Radon-Nikodym derivative or likelihood ratio $\ell_t = \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t}$ is well-defined for all $t\geqslant 0$.

We are now prepared to give another proof of the *Fontbona-Jourdain theorem.*

Alternative proof of Theorem 1.19: According to the considerations above, we have to verify (1.59). To this end, let us fix a time $t\geqslant 0$ and recall that $\overline{\mathcal{F}}_{-t} = \sigma(X_u\colon u\geqslant t) = \mathcal{G}_t$. Hence the equation (1.59) is equivalent with

$$\mathbb{P}|_{\mathcal{G}_t} = \ell_t(X_t)\odot\mathbb{Q}|_{\mathcal{G}_t}, \qquad t\geqslant 0. \tag{1.60}$$

In other words, we have to show that the probability measures $\mathbb{P}$ and $\ell_t(X_t)\odot\mathbb{Q}$ agree on the σ-algebra $\mathcal{G}_t$. This can be done using a similar strategy as in the proof of Lemma 1.16. In fact, it suffices to check that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_B\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_t(X_t)\,\mathbb{1}_B\big] \tag{1.61}$$

for all sets B in the π-system

$$\Pi = \big\{X_{t_1}\in B_1,\dots,X_{t_k}\in B_k\colon\ k\in\mathbb{N},\, t\leqslant t_1\leqslant\ldots\leqslant t_k,\, B_1,\dots,B_k\in\mathcal{B}(\mathcal{X})\big\}, \tag{1.62}$$

which generates the σ-algebra $\mathcal{G}_t$. Now let $k\in\mathbb{N}$, choose times $t_1,\dots,t_k$ with $t\leqslant t_1\leqslant\ldots\leqslant t_k$, and sets $B_1,\dots,B_k\in\mathcal{B}(\mathcal{X})$. Then the equation

$$\mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_{B_1}(X_{t_1})\cdot\ldots\cdot\mathbb{1}_{B_k}(X_{t_k})\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_t(X_t)\,\mathbb{1}_{B_1}(X_{t_1})\cdot\ldots\cdot\mathbb{1}_{B_k}(X_{t_k})\big] \tag{1.63}$$

must be verified. Defining the function

$$g(\omega_t) = \int_{B_1} P_{t_1-t}(\omega_t,\mathrm{d}\omega_{t_1})\int_{B_2} P_{t_2-t_1}(\omega_{t_1},\mathrm{d}\omega_{t_2})\cdot\ldots\cdot\int_{B_k} P_{t_k-t_{k-1}}(\omega_{t_{k-1}},\mathrm{d}\omega_{t_k}) \tag{1.64}$$

for $\omega_t\in\mathcal{X}$ and using the Chapman-Kolmogorov identity, (1.63) can be equivalently written as

$$\mathbb{E}_{\mathbb{P}}\big[g(X_t)\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_t(X_t)\,g(X_t)\big]. \tag{1.65}$$

But since ℓ_t is the Radon-Nikodym derivative of $\mathbb{P}_t$ with respect to $\mathbb{Q}_t$, we see that both sides of equation (1.65) are equal to

$$\int_{\mathcal{X}} \mathbb{P}_t(\mathrm{d}\omega_t)\,g(\omega_t), \tag{1.66}$$

which finishes the proof. □

1.5 Consequences of the Fontbona-Jourdain theorem

In this final section of Chapter 1 we make use of the *Fontbona-Jourdain theorem* and show some of its consequences. As customary, we use the terminology introduced in the preceding sections.

1.5.1 Convergence of the likelihood ratio process

In Theorem 1.19 we have seen that the likelihood ratio process $(\ell_t(X_t))_{t\geqslant 0}$ is a backwards martingale and as such it is uniformly integrable. We state this fact in the corollary below.

Corollary 1.20. *Under the* Assumptions 1.3*, the likelihood ratio process* $(\ell_t(X_t))_{t\geqslant 0}$ *is uniformly* $\mathbb{Q}$*-integrable.*

Proof. By Theorem 1.19 we have that

$$\ell_t(X_t) = \mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\,|\,\mathcal{G}_t\big] \tag{1.67}$$

for any $t \geqslant 0$. As $\ell_0(X_0) \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$, it follows that the likelihood ratio process $(\ell_t(X_t))_{t\geqslant 0}$ is uniformly $\mathbb{Q}$-integrable, see e.g. [Kle14, Corollary 8.22]. □

Exclusively in the following corollary we assume that $\mathbb{Q}$ is a probability measure in order to apply the convergence theorem for backwards martingales. As a result, we will see that the likelihood ratio process $(\ell_t(X_t))_{t\geqslant 0}$ converges $\mathbb{Q}$-almost surely as well as in the norm of $L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$.

Corollary 1.21. *Under the* Assumptions 1.3*, assume in addition that* $\mathbb{Q}$ *is a probability measure. Then the likelihood ratio process* $(\ell_t(X_t))_{t\geqslant 0}$ *converges* $\mathbb{Q}$*-almost surely as well as in the norm of* $L^1(\Omega, \mathcal{F}_\infty, \mathbb{Q})$ *to a random variable* $\ell_\infty(X_\infty)$*. More precisely, we have*

$$\ell_\infty(X_\infty) = \mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\,|\,\mathcal{G}_\infty\big], \tag{1.68}$$

where $\mathcal{G}_\infty = \bigcap_{t\geqslant 0} \mathcal{G}_t$*.*

Proof. The assertion of the corollary follows from the backwards martingale convergence theorem, see e.g. [Kle14, Theorem 12.14]. □

Remark 1.22. The σ-algebra $\mathcal{G}_\infty$ is equal to $\overline{\mathcal{F}}_{-\infty} = \bigcap_{t\leqslant 0} \overline{\mathcal{F}}_t$. ◇

1.5.2 The relative entropy process as a backwards submartingale

In this subsection we introduce the *relative entropy process* and show that it is a backwards $\mathbb{Q}$-submartingale.

Definition 1.23. The stochastic process $\big(\ell_t(X_t)\log \ell_t(X_t)\big)_{t\geqslant 0}$ is called the *relative entropy process* of the Markov measure $\mathbb{P}$ with respect to the Markov measure $\mathbb{Q}$.

Remark 1.24. The reason for calling the stochastic process $\big(\ell_t(X_t)\log \ell_t(X_t)\big)_{t\geqslant 0}$ relative entropy process of $\mathbb{P}$ with respect to $\mathbb{Q}$ is that, in case of existence, the relative entropy $H(\mathbb{P}_t\,|\,\mathbb{Q}_t)$ of $\mathbb{P}_t$ with respect to $\mathbb{Q}_t$ at time $t \geqslant 0$ is equal to the $\mathbb{Q}$-expectation of $\ell_t(X_t)\log \ell_t(X_t)$, i.e.

$$H(\mathbb{P}_t\,|\,\mathbb{Q}_t) = \mathbb{E}_{\mathbb{Q}_t}\big[\ell_t \log \ell_t\big] = \mathbb{E}_{\mathbb{Q}}\big[\ell_t(X_t)\log \ell_t(X_t)\big]. \tag{1.69}$$

Of course we can write the relative entropy also as a $\mathbb{P}$-expectation, namely

$$H(\mathbb{P}_t\,|\,\mathbb{Q}_t) = \mathbb{E}_{\mathbb{P}_t}\big[\log \ell_t\big] = \mathbb{E}_{\mathbb{P}}\big[\log \ell_t(X_t)\big]. \tag{1.70}$$

The advantage of the representation in (1.70) is that it involves the probability measure $\mathbb{P}$ and not the σ-finite measure $\mathbb{Q}$. ◇

The following corollary is an immediate consequence of the *Fontbona-Jourdain theorem* and Jensen's inequality for conditional expectations. It shows that the relative entropy process $\big(\ell_t(X_t)\log\ell_t(X_t)\big)_{t\geqslant 0}$ of $\mathbb{P}$ with respect to $\mathbb{Q}$ is a backwards $\mathbb{Q}$-submartingale.

Corollary 1.25. *Under the* Assumptions 1.3*, suppose that the relative entropy $H(\mathbb{P}_0\,|\,\mathbb{Q}_0)$ between the initial marginal measures $\mathbb{P}_0=\mu$ and $\mathbb{Q}_0=\nu$ exists in the sense of* Subsection 1.2.3 *and is finite. Then the relative entropy $H(\mathbb{P}_t\,|\,\mathbb{Q}_t)$ exists and is finite for all $t\geqslant 0$. Furthermore, the relative entropy process*

$$\Big(\ell_t(X_t)\log\ell_t(X_t)\Big)_{t\geqslant 0} \tag{1.71}$$

is a backwards $\mathbb{Q}$-submartingale with respect to the backwards filtration $(\mathcal{G}_t)_{t\geqslant 0}$. That is, for any $s,t\geqslant 0$ with $s\leqslant t$ we have that

$$\mathbb{E}_{\mathbb{Q}}\Big[\ell_s(X_s)\log\ell_s(X_s)\;\big|\;\mathcal{G}_t\Big]\geqslant \ell_t(X_t)\log\ell_t(X_t). \tag{1.72}$$

Equivalently, the time-reversed relative entropy process

$$\Big(\overline{\ell}_t(\overline{X}_t)\log\overline{\ell}_t(\overline{X}_t)\Big)_{t\leqslant 0} \tag{1.73}$$

is a $\mathbb{Q}$-submartingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{t\leqslant 0}$. In other words, for any $s,t\leqslant 0$ with $s\leqslant t$ we have that

$$\mathbb{E}_{\mathbb{Q}}\Big[\overline{\ell}_t(\overline{X}_t)\log\overline{\ell}_t(\overline{X}_t)\;\big|\;\overline{\mathcal{F}}_s\Big]\geqslant \overline{\ell}_s(\overline{X}_s)\log\overline{\ell}_s(\overline{X}_s). \tag{1.74}$$

Proof. Let $s,t\geqslant 0$ with $s\leqslant t$ and consider the convex function $\varphi(p)=p\log p$ for $p\in[0,\infty)$. By Jensen's inequality, which is also valid for conditional expectations with respect to the σ-finite measure $\mathbb{Q}$, we have

$$\varphi\Big(\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\,|\,\mathcal{G}_t\big]\Big)\leqslant \mathbb{E}_{\mathbb{Q}}\big[\varphi(\ell_s(X_s))\,|\,\mathcal{G}_t\big], \tag{1.75}$$

where $\varphi(\ell_s(X_s))^-\in L^1(\Omega,\mathcal{F}_\infty,\mathbb{Q})$ and hence the expectation on the right-hand side of the inequality (1.75) exists with values in $(-\infty,\infty]$. In particular, the relative entropy $H(\mathbb{P}_s\,|\,\mathbb{Q}_s)$ exists with values in $(-\infty,\infty]$. According to (1.75), we have

$$\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\,|\,\mathcal{G}_t\big]\Big(\log\mathbb{E}_{\mathbb{Q}}\big[\ell_s(X_s)\,|\,\mathcal{G}_t\big]\Big)\leqslant \mathbb{E}_{\mathbb{Q}}\Big[\ell_s(X_s)\log\ell_s(X_s)|\mathcal{G}_t\Big] \tag{1.76}$$

and using equation (1.55) from Theorem 1.19 yields

$$\ell_t(X_t)\log\ell_t(X_t)\leqslant \mathbb{E}_{\mathbb{Q}}\Big[\ell_s(X_s)\log\ell_s(X_s)\;\big|\;\mathcal{G}_t\Big], \tag{1.77}$$

which is (1.72). Taking expectations with respect to the σ-finite measure $\mathbb{Q}$ and setting $s=0$ in this inequality leads to

$$H(\mathbb{P}_t\,|\,\mathbb{Q}_t)=\mathbb{E}_{\mathbb{Q}}\big[\ell_t(X_t)\log\ell_t(X_t)\big]\leqslant \mathbb{E}_{\mathbb{Q}}\big[\ell_0(X_0)\log\ell_0(X_0)\big]=H(\mathbb{P}_0\,|\,\mathbb{Q}_0), \tag{1.78}$$

which by assumption implies that the relative entropy $H(\mathbb{P}_t\,|\,\mathbb{Q}_t)$ is finite. Finally, it is clear that (1.72) implies (1.74). $\square$

1.6 The Fontbona-Jourdain theorem for a finite time interval

We use the notation and setting introduced in the previous sections of Chapter 1. In the section at hand we formulate the *Fontbona-Jourdain theorem* for a finite time interval.

We replace the unbounded index set $\mathbb{R}_+$ by the finite time interval $[0,T]$, where $T > 0$ is some fixed terminal time. Accordingly, we consider the path space $\Omega = \mathcal{C}([0,T],\mathcal{X})$ of continuous functions from $[0,T]$ into the state space $\mathcal{X}$. For each time $t \in [0,T]$, the canonical coordinate mappings $X_t\colon \Omega \to \mathcal{X}$ are given by $X_t(\omega) = \omega_t$ for a continuous path $\omega = (\omega_s)_{0\leqslant s\leqslant T} \in \Omega$. We denote by $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ the natural filtration generated by the canonical coordinate process $(X_t)_{0\leqslant t\leqslant T}$ and equip the path space Ω with the σ-algebra $\mathcal{F}_T$. For $t \in [0,T]$, we set $\overline{X}_t = X_{T-t}$ and let $(\overline{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ be the natural filtration generated by the time-reversed canonical coordinate process $(\overline{X}_t)_{0\leqslant t\leqslant T}$. In other words, we have

$$\overline{\mathcal{F}}_t = \sigma\big(\overline{X}_s\colon\ 0 \leqslant s \leqslant t\big) = \sigma\big(X_s\colon\ T-t \leqslant s \leqslant T\big) \tag{1.79}$$

for all $t \in [0,T]$.

Under the Assumptions 1.3, recall that we denote the likelihood ratio by $\ell_t = \frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{Q}_t}$ for $t \in [0,T]$. Furthermore, we set $\overline{\ell}_t = \ell_{T-t}$ for $t \in [0,T]$.

The formulation of the *Fontbona-Jourdain theorem* for the finite time interval $[0,T]$ reads as follows.

Theorem 1.26 (Fontbona-Jourdain theorem [FJ16]). *Under the* Assumptions 1.3*, the time-reversed likelihood ratio process $(\overline{\ell}_t(\overline{X}_t))_{0\leqslant t\leqslant T}$ is a $\mathbb{Q}$-martingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$. In other words, for any times s,t with $0 \leqslant s \leqslant t \leqslant T$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big[\overline{\ell}_t(\overline{X}_t)\,|\,\overline{\mathcal{F}}_s\big] = \overline{\ell}_s(\overline{X}_s). \tag{1.80}$$

Proof. Theorem 1.26 is the equivalent formulation of Theorem 1.19 for the index set $[0,T]$ instead of $\mathbb{R}_+$. □

Similarly, we can translate the statement of Corollary 1.25 into the new setting as follows.

Corollary 1.27. *Under the* Assumptions 1.3*, suppose that the relative entropy $H(\mathbb{P}_0\,|\,\mathbb{Q}_0)$ between the initial marginal measures $\mathbb{P}_0 = \mu$ and $\mathbb{Q}_0 = \nu$ exists in the sense of* Subsection 1.2.3 *and is finite. Then the relative entropy $H(\mathbb{P}_t\,|\,\mathbb{Q}_t)$ exists and is finite for all $t \in [0,T]$. Furthermore, the time-reversed relative entropy process*

$$\Big(\overline{\ell}_t(\overline{X}_t)\log\overline{\ell}_t(\overline{X}_t)\Big)_{0\leqslant t\leqslant T} \tag{1.81}$$

is a $\mathbb{Q}$-submartingale with respect to the filtration $(\overline{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$. In other words, for any times s,t with $0 \leqslant s \leqslant t \leqslant T$ we have that

$$\mathbb{E}_{\mathbb{Q}}\Big[\overline{\ell}_t(\overline{X}_t)\log\overline{\ell}_t(\overline{X}_t)\ \big|\ \overline{\mathcal{F}}_s\Big] \geqslant \overline{\ell}_s(\overline{X}_s)\log\overline{\ell}_s(\overline{X}_s). \tag{1.82}$$

Proof. Corollary 1.27 corresponds to the analogue of Corollary 1.25 for the finite time interval $[0,T]$ instead of $\mathbb{R}_+$. □

2 Reversible Brownian motion

In this chapter we use some of the notation introduced in Section 1.2 of Chapter 1 and introduce the notion of *reversible Brownian motion*, see e.g. [Léo14a] and [Léo14b]. Roughly speaking, this is a Brownian motion whose random initial position is uniformly distributed. The eponymous feature of reversible Brownian motion is its invariance under time-reversal, see Lemma 2.2.

We fix $T > 0$ and consider the path space $\Omega = \mathcal{C}([0,T], \mathbb{R}^d)$ of continuous functions from $[0,T]$ into $\mathbb{R}^d$. For $t \in [0,T]$, the canonical coordinate mappings $X_t \colon \Omega \to \mathbb{R}^d$ are given by $X_t(\omega) = \omega_t$ for $\omega = (\omega_s)_{0 \leqslant s \leqslant T} \in \Omega$. For each $x \in \mathbb{R}^d$, we let $\mathbb{W}^x \in \mathcal{M}^1(\Omega)$ be the Wiener measure starting from x. We denote by $\mathbb{W} = \mathbb{W}^0$ the standard Wiener measure on $\mathbb{R}^d$. Under this probability measure $\mathbb{W}$, the canonical coordinate process $(X_t)_{0 \leqslant t \leqslant T}$ is a d-dimensional Brownian motion starting from 0.

Now we look at the time-reversed canonical coordinate process $(\overline{X}_t)_{0 \leqslant t \leqslant T}$, which is given by $\overline{X}_t = X_{T-t}$ for $t \in [0,T]$. Evidently, this process cannot be again a $\mathbb{W}$-Brownian motion. In other words, the path measure $\mathbb{W}$ is not invariant under time-reversal. Motivated by this, we want to find a path measure $\mathbb{Q}$ on Ω which is related to $\mathbb{W}$ and is invariant under time-reversal. This can be accomplished by the so-called *reversible Brownian motion* on $\mathbb{R}^d$, see e.g. [Léo14a] and [Léo14b]. Reversible Brownian motion is the path measure $\mathbb{Q} \in \mathcal{M}(\Omega)$ under which the canonical coordinate process $(X_t)_{0 \leqslant t \leqslant T}$ is a Markov process with transition semigroup $(P_t)_{0 \leqslant t \leqslant T}$ being the usual heat semigroup of Brownian motion and the initial marginal measure $\mathbb{Q}_0$ is the σ-finite Lebesgue measure λ on $\mathbb{R}^d$, i.e. the random initial position X_0 is uniformly distributed on $\mathbb{R}^d$. In other words, reversible Brownian motion on $\mathbb{R}^d$ is the σ-finite path measure

$$\mathbb{Q}[\,\cdot\,] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x[\,\cdot\,]. \tag{2.1}$$

Remark 2.1. Recall that the heat semigroup $(P_t)_{0 \leqslant t \leqslant T}$ of Brownian motion is given by

$$P_t(x, \mathrm{d}y) = \varphi_t(y - x)\, \lambda(\mathrm{d}y), \qquad x, y \in \mathbb{R}^d, \tag{2.2}$$

for $t \in (0,T]$, where the function φ_t given by

$$\varphi_t(z) = \tfrac{1}{(2\pi t)^{d/2}} \exp\big(-\tfrac{|z|^2}{2t}\big), \qquad z \in \mathbb{R}^d, \tag{2.3}$$

is the probability density of a Gaussian vector in $\mathbb{R}^d$ with covariance matrix $t\,\mathrm{Id}_d$. ◇

In the following lemma we will see that the reversible Brownian motion actually deserves its name, i.e., it is invariant under time-reversal.

Lemma 2.2 ([Léo14a])**.** *The reversible Brownian motion $\mathbb{Q} \in \mathcal{M}(\Omega)$ is a reversible path measure with the Lebesgue measure $\lambda \in \mathcal{M}(\mathbb{R}^d)$ as its reversing measure in the following sense: The initial marginal measure $\mathbb{Q}_0$ is equal to the Lebesgue measure λ and $\mathbb{Q}$ is invariant with respect to time-reversal, i.e.,*

$$\overline{X}_{\#}\mathbb{Q} = X_{\#}\mathbb{Q}. \tag{2.4}$$

Furthermore, not only at the initial time 0 but also at every time $t \in (0,T]$ the marginal measure $\mathbb{Q}_t$ is equal to the Lebesgue measure λ.

Proof. At time $t \in [0,T]$, the marginal measure of $\mathbb{Q}$ is given by

$$\mathbb{Q}_t[\,\cdot\,] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}\omega_0)\, P_t(\omega_0, \,\cdot\,). \tag{2.5}$$

Because of $P_0(\omega_0, \,\cdot\,) = \delta_{\omega_0}$, we clearly have $\mathbb{Q}_0 = \lambda$. Now let $t \in (0,T]$ and $B_t \in \mathcal{B}(\mathbb{R}^d)$. Using (2.2), we compute

$$\mathbb{Q}_t[B_t] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}\omega_0) \int_{B_t} \lambda(\mathrm{d}\omega_t)\, \varphi_t(\omega_t - \omega_0) = \int_{B_t} \lambda(\mathrm{d}\omega_t) \int_{\mathbb{R}^d} \lambda(\mathrm{d}\omega_0)\, \varphi_t(\omega_t - \omega_0) = \lambda[B_t], \tag{2.6}$$

which shows that $\mathbb{Q}_t = \lambda$. In order to prove (2.4), we let $k \in \mathbb{N}$, choose times $t_1, \ldots, t_k$ with $0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant T$, and sets $B_0, B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}^d)$. On the one hand, we have to calculate

$$(X_{\#}\mathbb{Q})[B_0 \times B_1 \times \ldots \times B_k] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x[X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k]. \tag{2.7}$$

For the sake of better readability, we set $t_0 = 0$, $\Delta t_i = t_i - t_{i-1}$ and $\Delta\omega_i = \omega_i - \omega_{i-1}$ for $i \in \{1, \ldots, k\}$. Then, using the heat semigroup $(P_t)_{0 \leqslant t \leqslant T}$ defined as in (2.2), we can express (2.7) as

$$\int_{B_0} \lambda(\mathrm{d}\omega_0) \int_{B_1} \lambda(\mathrm{d}\omega_1)\, \varphi_{\Delta t_1}(\Delta\omega_1) \ldots \int_{B_k} \lambda(\mathrm{d}\omega_k)\, \varphi_{\Delta t_k}(\Delta\omega_k). \tag{2.8}$$

On the other hand, we should get the same result for

$$(\overline{X}_{\#}\mathbb{Q})[B_0 \times B_1 \times \ldots \times B_k] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x[X_{T-t_k} \in B_k, \ldots, X_{T-t_1} \in B_1, X_T \in B_0]. \tag{2.9}$$

We introduce a new variable of integration ω_{k+1} and set $\Delta\omega_{k+1} = \omega_{k+1} - \omega_k$, then (2.9) equals

$$\int_{\mathbb{R}^d} \lambda(\mathrm{d}\omega_{k+1}) \int_{B_k} \lambda(\mathrm{d}\omega_k)\, \varphi_{T-t_k}(\Delta\omega_{k+1}) \int_{B_{k-1}} \lambda(\mathrm{d}\omega_{k-1})\, \varphi_{\Delta t_k}(\Delta\omega_k) \ldots \int_{B_0} \lambda(\mathrm{d}\omega_0)\, \varphi_{\Delta t_1}(\Delta\omega_1). \tag{2.10}$$

Finally, this expression can be simplified to

$$\int_{B_k} \lambda(\mathrm{d}\omega_k) \int_{B_{k-1}} \lambda(\mathrm{d}\omega_{k-1})\, \varphi_{\Delta t_k}(\Delta\omega_k) \ldots \int_{B_1} \lambda(\mathrm{d}\omega_1)\, \varphi_{\Delta t_2}(\Delta\omega_2) \int_{B_0} \lambda(\mathrm{d}\omega_0)\, \varphi_{\Delta t_1}(\Delta\omega_1), \tag{2.11}$$

which is indeed the same as (2.8). □

The identity (2.4) from Lemma 2.2 can also be interpreted as follows.

Corollary 2.3. *For each $x \in \mathbb{R}^d$, let $\overline{\mathbb{W}}^x \in \mathcal{M}^1(\Omega)$ be the probability measure under which the time-reversed canonical coordinate process $(\overline{X}_t)_{0 \leqslant t \leqslant T}$ is a d-dimensional Brownian motion starting from x. Then the path measure*

$$\overline{\mathbb{Q}}[\,\cdot\,] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \overline{\mathbb{W}}^x[\,\cdot\,] \tag{2.12}$$

is equal to the reversible Brownian motion $\mathbb{Q}$.

Proof. We let $k \in \mathbb{N}$ and choose times $t_0, t_1, \ldots, t_k$ with $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant T$. Moreover, let $B_0, B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}^d)$. Then we have to verify the identity

$$\int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x[X_{t_0} \in B_0, \ldots, X_{t_k} \in B_k] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \overline{\mathbb{W}}^x[X_{t_0} \in B_0, \ldots, X_{t_k} \in B_k]. \tag{2.13}$$

The right-hand side of (2.13) can also be expressed as

$$\int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x\big[\overline{X}_{t_0} \in B_0, \ldots, \overline{X}_{t_k} \in B_k\big] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x\big[X_{T-t_0} \in B_0, \ldots, X_{T-t_k} \in B_k\big]. \quad (2.14)$$

However, from the proof of Lemma 2.2 we know that

$$\int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x\big[X_{T-t_0} \in B_0, \ldots, X_{T-t_k} \in B_k\big] = \int_{\mathbb{R}^d} \lambda(\mathrm{d}x)\, \mathbb{W}^x\big[X_{t_0} \in B_0, \ldots, X_{t_k} \in B_k\big], \quad (2.15)$$

which shows (2.13). □

3 Pathwise Otto calculus

In this chapter we place ourselves in the setting of [KST19] and provide proofs for two limiting assertions stated therein. In [KST19], these proofs were left as exercises for the diligent reader. We also give further details on other relevant subjects coming up in [KST19].

3.1 The proof of the assertion (3.39) of [KST19]

Under the Assumptions 1.2 of [KST19], we let $t_0 \geqslant 0$. The limiting identity (3.39) of [KST19] asserts that we have

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0} \, \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] = \lim_{t \downarrow t_0} \frac{1}{t-t_0} \, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell\big(u, X(u)\big)^2} \, \mathrm{d}u \right], \tag{3.1}$$

as long as one of the limits exists.

Proof of the limiting assertion (3.1)*:* Let $t_0 \geqslant 0$. We show that the limit

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0} \, \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \tag{3.2}$$

exists if and only if the limit

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0} \, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \tag{3.3}$$

exists. Furthermore, we prove that

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0} \left(\mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \right) = 0, \tag{3.4}$$

as long as one of the limits (3.2), (3.3) exists.

In order to do this, we first assume that the limit (3.2) exists and estimate

$$\left| \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \right| \tag{3.5}$$

$$\leqslant \left| \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \right| \tag{3.6}$$

$$+ \left| \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta\big(u, X(u)\big)^2} \, \mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell\big(u, X(u)\big)^2} \, \mathrm{d}u \right] \right|. \tag{3.7}$$

After an application of the Fubini-Tonelli theorem, we see that the expression of (3.6) is equal to

$$\left| \int_{t_0}^{t} \mathbb{E}_{\mathbb{P}^\beta} \left[\frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta(u, X(u))^2} \cdot \mathbb{E}_{\mathbb{P}^\beta} \Big[\Big(1 - Z^{-1}(u) \Big) \;\Big|\; X(u) \Big] \right] \mathrm{d}u \right|, \tag{3.8}$$

where the density process $(Z(t))_{t_0 \leqslant t \leqslant T}$ is given by

$$Z(t) = \frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}} \bigg|_{\mathcal{F}(t)} = \exp\bigg(- \int_{t_0}^{t} \Big\langle \beta(X(u)) \, , \, \mathrm{d}W(u) \Big\rangle_{\mathbb{R}^n} - \tfrac{1}{2} \int_{t_0}^{t} \big|\beta(X(u))\big|^2 \, \mathrm{d}u \bigg) \tag{3.9}$$

for $t_0 \leqslant t \leqslant T$, see Lemma 4.11 of [KST19]. By analogy with (4.50) of [KST19] we have

$$(Y^\beta)^{-1}(t, X(t)) = \mathbb{E}_{\mathbb{P}^\beta}[Z^{-1}(t) \,|\, X(t)], \qquad t_0 \leqslant t \leqslant T, \tag{3.10}$$

where the "perturbed-to-unperturbed" ratio is defined as

$$Y^\beta(t, x) = \frac{\ell^\beta(t, x)}{\ell(t, x)} = \frac{p^\beta(t, x)}{p(t, x)}, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n. \tag{3.11}$$

Therefore the quantity in (3.8) is not greater than

$$\int_{t_0}^{t} \mathbb{E}_{\mathbb{P}^\beta} \left[\frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta(u, X(u))^2} \cdot \Big| 1 - (Y^\beta)^{-1}(u, X(u)) \Big| \right] \mathrm{d}u. \tag{3.12}$$

Lemma 4.12 of [KST19] implies that we have

$$\Big| 1 - (Y^\beta)^{-1}(u, X(u)) \Big| \leqslant C\,(t - t_0), \qquad t_0 \leqslant u \leqslant t. \tag{3.13}$$

Applying this inequality and using once again the Fubini-Tonelli theorem, we conclude that (3.12) is less than or equal to

$$C\,(t - t_0)\, \mathbb{E}_{\mathbb{P}^\beta} \left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta(u, X(u))^2} \, \mathrm{d}u \right]. \tag{3.14}$$

We now see that

$$\lim_{t \downarrow t_0} \frac{C\,(t - t_0)}{t - t_0} \, \mathbb{E}_{\mathbb{P}^\beta} \left[\int_{t_0}^{t} \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta(u, X(u))^2} \, \mathrm{d}u \right] = 0, \tag{3.15}$$

since the limit (3.2) exists by assumption.

The expression of (3.7) is less than or equal to

$$\mathbb{E}_{\mathbb{P}} \left[\int_{t_0}^{t} \left| \frac{\big|\nabla \ell^\beta(u, X(u))\big|^2}{\ell^\beta(u, X(u))^2} - \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell(u, X(u))^2} \right| \mathrm{d}u \right] \tag{3.16}$$

$$\leqslant \mathbb{E}_{\mathbb{P}} \left[\int_{t_0}^{t} \left| \frac{\nabla \ell^\beta(u, X(u))}{\ell^\beta(u, X(u))} - \frac{\nabla \ell(u, X(u))}{\ell(u, X(u))} \right|^2 \mathrm{d}u \right] \tag{3.17}$$

$$+ 2\, \mathbb{E}_{\mathbb{P}} \left[\int_{t_0}^{t} \left| \frac{\nabla \ell^\beta(u, X(u))}{\ell^\beta(u, X(u))} - \frac{\nabla \ell(u, X(u))}{\ell(u, X(u))} \right| \cdot \left| \frac{\nabla \ell^\beta(u, X(u))}{\ell^\beta(u, X(u))} \right| \mathrm{d}u \right]. \tag{3.18}$$

According to Lemma 4.12 of [KST19], there is a constant $C > 0$ such that

$$\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right] \leqslant C\,(t-t_0)^2 \tag{3.19}$$

holds for all $t_0 \leqslant t \leqslant T$. This implies that

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0}\,\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right] = 0. \tag{3.20}$$

It remains to show that

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0}\,2\,\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right| \cdot \left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right| \mathrm{d}u\right] = 0, \tag{3.21}$$

under the assumption that the limit (3.2) exists. Applying two times the Cauchy-Schwarz inequality, we find

$$\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right| \cdot \left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right| \mathrm{d}u\right] \tag{3.22}$$

$$\leqslant \mathbb{E}_{\mathbb{P}}\left[\left(\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right)^{1/2} \cdot \left(\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right|^2 \mathrm{d}u\right)^{1/2}\right] \tag{3.23}$$

$$\leqslant \left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2} \cdot \left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2} \tag{3.24}$$

$$= \left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2} \cdot \left(\mathbb{E}_{\mathbb{P}^{\beta}}\left[\frac{1}{Z(t)}\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2} \tag{3.25}$$

$$\leqslant C\left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2} \cdot \left(\mathbb{E}_{\mathbb{P}^{\beta}}\left[\int_{t_0}^{t}\left|\frac{\nabla \ell^{\beta}(u,X(u))}{\ell^{\beta}(u,X(u))}\right|^2 \mathrm{d}u\right]\right)^{1/2}, \tag{3.26}$$

where we have used in the last step that the density process $(Z^{-1}(t))_{t_0 \leqslant t \leqslant T}$ is uniformly bounded, see Lemma 4.11 of [KST19]. Finally, dividing the last line by $\sqrt{t-t_0} \cdot \sqrt{t-t_0}$ and using (3.19) as well as the fact that the limit (3.2) exists, we obtain (3.21).

Summing up, under the assumption that the limit (3.2) exists, we have shown the limiting assertion (3.4). This also implies that the limit (3.3) exists.

Secondly, we assume that the limit (3.3) exists and estimate

$$\left| \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^t \frac{\left|\nabla \ell^\beta(u,X(u))\right|^2}{\ell^\beta(u,X(u))^2} \,\mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \frac{\left|\nabla \ell(u,X(u))\right|^2}{\ell(u,X(u))^2} \,\mathrm{d}u \right] \right| \tag{3.27}$$

$$\leqslant \left| \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^t \frac{\left|\nabla \ell^\beta(u,X(u))\right|^2}{\ell^\beta(u,X(u))^2} \,\mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^t \frac{\left|\nabla \ell(u,X(u))\right|^2}{\ell(u,X(u))^2} \,\mathrm{d}u \right] \right| \tag{3.28}$$

$$+ \left| \mathbb{E}_{\mathbb{P}^\beta}\left[\int_{t_0}^t \frac{\left|\nabla \ell(u,X(u))\right|^2}{\ell(u,X(u))^2} \,\mathrm{d}u \right] - \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \frac{\left|\nabla \ell(u,X(u))\right|^2}{\ell(u,X(u))^2} \,\mathrm{d}u \right] \right|. \tag{3.29}$$

The expression of (3.28) is less than or equal to

$$\mathbb{E}_{\mathbb{P}}\left[Z(t) \int_{t_0}^t \left| \frac{\left|\nabla \ell^\beta(u,X(u))\right|^2}{\ell^\beta(u,X(u))^2} - \frac{\left|\nabla \ell(u,X(u))\right|^2}{\ell(u,X(u))^2} \right| \mathrm{d}u \right] \tag{3.30}$$

$$\leqslant C\, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right] \tag{3.31}$$

$$+ 2\, C\, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \cdot \left| \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \mathrm{d}u \right], \tag{3.32}$$

where we have used that the density process $(Z(t))_{t_0 \leqslant t \leqslant T}$ is uniformly bounded, see Lemma 4.11 of [KST19]. Furthermore, from (3.19) we conclude that

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0}\, C\, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right] = 0. \tag{3.33}$$

It remains to show that

$$\lim_{t \downarrow t_0} \frac{1}{t-t_0}\, 2\, C\, \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \cdot \left| \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \mathrm{d}u \right] = 0, \tag{3.34}$$

under the assumption that the limit (3.3) exists. Once again, applying two times the Cauchy-Schwarz inequality, we find

$$\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \cdot \left| \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right| \mathrm{d}u \right] \tag{3.35}$$

$$\leqslant \mathbb{E}_{\mathbb{P}}\left[\left(\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right)^{1/2} \cdot \left(\int_{t_0}^t \left| \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right)^{1/2} \right] \tag{3.36}$$

$$\leqslant \left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell^\beta(u,X(u))}{\ell^\beta(u,X(u))} - \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right] \right)^{1/2} \cdot \left(\mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^t \left| \frac{\nabla \ell(u,X(u))}{\ell(u,X(u))} \right|^2 \mathrm{d}u \right] \right)^{1/2}. \tag{3.37}$$

Finally, dividing the last line by $\sqrt{t-t_0} \cdot \sqrt{t-t_0}$ and using (3.19) as well as the fact that the limit (3.3) exists, we get (3.34).

Applying the Fubini-Tonelli theorem, we see that the quantity in (3.29) is equal to

$$\left| \int_{t_0}^{t} \mathbb{E}_{\mathbb{P}} \left[\frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell(u, X(u))^2} \cdot \mathbb{E}_{\mathbb{P}} \Big[\big(Z(u) - 1 \big) \mid X(u) \Big] \right] \mathrm{d}u \right|. \tag{3.38}$$

According to (4.50) of [KST19] we have

$$Y^{\beta}(t, X(t)) = \mathbb{E}_{\mathbb{P}} \big[Z(t) \,|\, X(t) \big], \qquad t_0 \leqslant t \leqslant T, \tag{3.39}$$

and hence the expression in (3.38) is less than or equal to

$$\int_{t_0}^{t} \mathbb{E}_{\mathbb{P}} \left[\frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell(u, X(u))^2} \cdot \big| Y^{\beta}(u, X(u)) - 1 \big| \right] \mathrm{d}u. \tag{3.40}$$

By Lemma 4.12 of [KST19] we have

$$\big| Y^{\beta}(u, X(u)) - 1 \big| \leqslant C\,(t - t_0), \qquad t_0 \leqslant u \leqslant t. \tag{3.41}$$

Applying this inequality and using once again the Fubini-Tonelli theorem, we deduce that (3.40) is not greater than

$$C\,(t - t_0)\, \mathbb{E}_{\mathbb{P}} \left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell(u, X(u))^2} \,\mathrm{d}u \right]. \tag{3.42}$$

We now conclude that

$$\lim_{t \downarrow t_0} \frac{C\,(t - t_0)}{t - t_0}\, \mathbb{E}_{\mathbb{P}} \left[\int_{t_0}^{t} \frac{\big|\nabla \ell(u, X(u))\big|^2}{\ell(u, X(u))^2} \,\mathrm{d}u \right] = 0, \tag{3.43}$$

as the limit (3.3) exists by assumption.

Summing up, under the assumption that the limit (3.3) exists, we have shown the limiting assertion (3.4), which also implies the existence of the limit (3.2).

Altogether, this completes the proof of the limiting identity (3.4), as long as one of the limits (3.2), (3.3) exists. □

3.2 The proof of the assertion (3.54) in Proposition 3.14 of [KST19]

Under the Assumptions 1.2 of [KST19], we let $t_0 \geqslant 0$ be such that the limit

$$\lim_{t \downarrow t_0} \frac{H(P(t) \,|\, \mathrm{Q}) - H(P(t_0) \,|\, \mathrm{Q})}{t - t_0} = -\tfrac{1}{2}\, \mathbb{E}_{\mathbb{P}} \left[\frac{\big|\nabla \ell(t_0, X(t_0))\big|^2}{\ell(t_0, X(t_0))^2} \right] \tag{3.44}$$

exists. The limiting identity (3.54) in Proposition 3.14 of [KST19] asserts that the relative entropy process

$$\log \ell(t, X(t)) = \log \left(\frac{p(t, X(t))}{q(X(t))} \right) = \log p(t, X(t)) + 2\,\Psi(X(t))\,, \qquad 0 \leqslant t \leqslant T, \tag{3.45}$$

satisfies, with $T > t_0$, the trajectorial relation

$$\lim_{s \uparrow T-t_0} \frac{\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta(t_0, X(t_0)) \;\big|\; \mathcal{G}(T-s) \Big] - \log \ell^\beta(T-s, X(T-s))}{T-t_0-s}$$
$$= \frac{1}{2} \frac{\big|\nabla \ell(t_0, X(t_0))\big|^2}{\ell(t_0, X(t_0))^2} - \operatorname{div} \beta(X(t_0)) - \Big\langle \beta(X(t_0)) \,,\, \nabla \log p(t_0, X(t_0)) \Big\rangle_{\mathbb{R}^n}, \tag{3.46}$$

where the limit (3.46) exists in the norms of both $L^1(\mathbb{P})$ and $L^1(\mathbb{P}^\beta)$.

Proof of the limiting assertion (3.46)*:* Let $t_0 \geqslant 0$ be such that the limiting identity (3.44) is valid. Select $T > t_0$. The time-reversed perturbed likelihood ratio process

$$\ell^\beta(T-s, X(T-s)) = \frac{p^\beta(T-s, X(T-s))}{q(X(T-s))}, \qquad 0 \leqslant s \leqslant T-t_0, \tag{3.47}$$

and its logarithm satisfy respectively the stochastic differential equations

$$\frac{\mathrm{d}\ell^\beta(T-s, X(T-s))}{\ell^\beta(T-s, X(T-s))} = \Big(\langle \beta \,,\, 2\nabla\Psi \rangle_{\mathbb{R}^n} - \operatorname{div}\beta \Big)(X(T-s))\, \mathrm{d}s$$
$$+ \frac{\big|\nabla \ell^\beta(T-s, X(T-s))\big|^2}{\ell^\beta(T-s, X(T-s))^2}\, \mathrm{d}s + \left\langle \frac{\nabla \ell^\beta(T-s, X(T-s))}{\ell^\beta(T-s, X(T-s))} \,,\, \mathrm{d}\overline{W}^{\mathbb{P}^\beta}(T-s) \right\rangle_{\mathbb{R}^n} \tag{3.48}$$

and

$$\mathrm{d}\log \ell^\beta(T-s, X(T-s)) = \Big(\langle \beta \,,\, 2\nabla\Psi \rangle_{\mathbb{R}^n} - \operatorname{div}\beta \Big)(X(T-s))\, \mathrm{d}s$$
$$+ \frac{1}{2}\frac{\big|\nabla \ell^\beta(T-s, X(T-s))\big|^2}{\ell^\beta(T-s, X(T-s))^2}\, \mathrm{d}s + \left\langle \frac{\nabla \ell^\beta(T-s, X(T-s))}{\ell^\beta(T-s, X(T-s))} \,,\, \mathrm{d}\overline{W}^{\mathbb{P}^\beta}(T-s) \right\rangle_{\mathbb{R}^n}, \tag{3.49}$$

for $0 \leqslant s \leqslant T-t_0$, with respect to the backwards filtration $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$. By means of the stochastic equation (3.49), the difference

$$\log \ell^\beta(t_0, X(t_0)) - \log \ell^\beta(T-s, X(T-s)) \tag{3.50}$$

is equal to the sum of the three integral terms

$$\mathcal{I}_1(s) := \int_s^{T-t_0} \Big(\langle \beta \,,\, 2\nabla\Psi \rangle_{\mathbb{R}^n} - \operatorname{div}\beta \Big)(X(T-u))\, \mathrm{d}u, \tag{3.51}$$

$$\mathcal{I}_2(s) := \int_s^{T-t_0} \frac{1}{2}\frac{\big|\nabla \ell^\beta(T-u, X(T-u))\big|^2}{\ell^\beta(T-u, X(T-u))^2}\, \mathrm{d}u, \tag{3.52}$$

$$\mathcal{I}_3(s) := \int_s^{T-t_0} \left\langle \frac{\nabla \ell^\beta(T-u, X(T-u))}{\ell^\beta(T-u, X(T-u))} \,,\, \mathrm{d}\overline{W}^{\mathbb{P}^\beta}(T-u) \right\rangle_{\mathbb{R}^n}, \tag{3.53}$$

for $0 \leqslant s \leqslant T-t_0$. In order to pass to the $\mathbb{P}$-Brownian motion $\big(\overline{W}^{\mathbb{P}}(T-s)\big)_{0 \leqslant s \leqslant T-t_0}$ from the $\mathbb{P}^\beta$-Brownian motion $\big(\overline{W}^{\mathbb{P}^\beta}(T-s)\big)_{0 \leqslant s \leqslant T-t_0}$, we use the relation

$$\mathrm{d}(\overline{W}^{\mathbb{P}} - \overline{W}^{\mathbb{P}^\beta})(T-u) = \left(\beta(X(T-u)) + \frac{\nabla Y^\beta(T-u, X(T-u))}{Y^\beta(T-u, X(T-u))} \right) \mathrm{d}u, \qquad 0 \leqslant u \leqslant T-t_0, \tag{3.54}$$

where the "perturbed-to-unperturbed" ratio $Y^\beta(\,\cdot\,,\,\cdot\,)$ is given by

$$Y^\beta(t,x) = \frac{\ell^\beta(t,x)}{\ell(t,x)} = \frac{p^\beta(t,x)}{p(t,x)}, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n. \tag{3.55}$$

With the help of (3.54) we can express the integral term $\mathcal{I}_3(s)$ of (3.53) as the sum of the quantities

$$\mathcal{J}_1(s) := \int_s^{T-t_0} \left\langle \frac{\nabla \ell^\beta(T-u, X(T-u))}{\ell^\beta(T-u, X(T-u))}\,,\, \mathrm{d}\overline{W}^{\mathbb{P}}(T-u) \right\rangle_{\mathbb{R}^n}, \tag{3.56}$$

$$\mathcal{J}_2(s) := -\int_s^{T-t_0} \left\langle \frac{\nabla \ell^\beta(T-u, X(T-u))}{\ell^\beta(T-u, X(T-u))}\,,\, \beta(X(T-u)) \right\rangle_{\mathbb{R}^n} \mathrm{d}u, \tag{3.57}$$

$$\mathcal{J}_3(s) := -\int_s^{T-t_0} \left\langle \frac{\nabla \ell^\beta(T-u, X(T-u))}{\ell^\beta(T-u, X(T-u))}\,,\, \frac{\nabla Y^\beta(T-u, X(T-u))}{Y^\beta(T-u, X(T-u))} \right\rangle_{\mathbb{R}^n} \mathrm{d}u. \tag{3.58}$$

Recapitulating, we have

$$\log \ell^\beta(t_0, X(t_0)) - \log \ell^\beta(T-s, X(T-s)) = (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3)(s). \tag{3.59}$$

Next, we take expectations with respect to the probability measure $\mathbb{P}$ in this equation and argue why the expectation of the stochastic integral in (3.56) vanishes. From (4.41) of [KST19] we know that the quantity

$$\mathbb{E}_{\mathbb{P}^\beta}\left[\int_0^{T-t_0} \frac{1}{2} \frac{\left|\nabla \ell^\beta(T-u, X(T-u))\right|^2}{\ell^\beta(T-u, X(T-u))^2} \,\mathrm{d}u \right] \tag{3.60}$$

is finite. According to Lemma 4.11 of [KST19], the density process $(Z(t))_{t_0 \leqslant t \leqslant T}$, given by

$$Z(t) = \frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}(t)} = \exp\left(-\int_{t_0}^t \big\langle \beta(X(u))\,,\, \mathrm{d}W(u) \big\rangle_{\mathbb{R}^n} - \tfrac{1}{2}\int_{t_0}^t \left|\beta(X(u))\right|^2 \mathrm{d}u \right) \tag{3.61}$$

for $t_0 \leqslant t \leqslant T$, is uniformly bounded. Hence also the expression

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^{T-t_0} \frac{1}{2} \frac{\left|\nabla \ell^\beta(T-u, X(T-u))\right|^2}{\ell^\beta(T-u, X(T-u))^2} \,\mathrm{d}u \right] \tag{3.62}$$

is finite. Thus the stochastic integral of (3.56) is a square-integrable $\mathbb{P}$-martingale. In particular, $\mathbb{E}_{\mathbb{P}}[\mathcal{J}_1(s)] = 0$. Therefore taking expectations in (3.59) yields

$$\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta(t_0, X(t_0)) - \log \ell^\beta(T-s, X(T-s)) \Big] = \mathbb{E}_{\mathbb{P}}\Big[(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{J}_2 + \mathcal{J}_3)(s) \Big]. \tag{3.63}$$

Since $\nabla \log \ell^\beta(t,x) = \nabla \log Y^\beta(t,x) + \nabla \log \ell(t,x)$, we can express the integral term $\mathcal{I}_2(s)$ of (3.52) as

$$\mathcal{K}_1(s) := \int_s^{T-t_0} \frac{1}{2} \left| \frac{\nabla Y^\beta(T-u, X(T-u))}{Y^\beta(T-u, X(T-u))} + \frac{\nabla \ell(T-u, X(T-u))}{\ell(T-u, X(T-u))} \right|^2 \mathrm{d}u. \tag{3.64}$$

Similarly, the quantity $\mathcal{J}_2(s)$ of (3.57) equals the sum of the expressions

$$\mathcal{K}_2(s) := -\int_s^{T-t_0} \left\langle \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)} \,,\, \beta\big(X(T-u)\big) \right\rangle_{\mathbb{R}^n} \mathrm{d}u, \tag{3.65}$$

$$\mathcal{K}_3(s) := -\int_s^{T-t_0} \left\langle \frac{\nabla \ell\big(T-u, X(T-u)\big)}{\ell\big(T-u, X(T-u)\big)} \,,\, \beta\big(X(T-u)\big) \right\rangle_{\mathbb{R}^n} \mathrm{d}u; \tag{3.66}$$

and the integral term $\mathcal{J}_3(s)$ of (3.58) is equal to the sum of the integrals

$$\mathcal{K}_4(s) := -\int_s^{T-t_0} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2} \,\mathrm{d}u, \tag{3.67}$$

$$\mathcal{K}_5(s) := -\int_s^{T-t_0} \left\langle \frac{\nabla \ell\big(T-u, X(T-u)\big)}{\ell\big(T-u, X(T-u)\big)} \,,\, \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)} \right\rangle_{\mathbb{R}^n} \mathrm{d}u. \tag{3.68}$$

With that said, we have

$$\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta\big(t_0, X(t_0)\big) - \log \ell^\beta\big(T-s, X(T-s)\big)\Big] = \mathbb{E}_{\mathbb{P}}\Big[\big(\mathcal{I}_1 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4 + \mathcal{K}_5\big)(s)\Big]. \tag{3.69}$$

Squaring out the absolute value of (3.64), we obtain

$$\big(\mathcal{I}_1 + \mathcal{K}_1 + \mathcal{K}_3 + \mathcal{K}_4 + \mathcal{K}_5\big)(s) = \big(\mathcal{L}_1 + \mathcal{L}_1 + \mathcal{L}_3\big)(s), \tag{3.70}$$

with

$$\mathcal{L}_1(s) := \int_s^{T-t_0} \frac{1}{2} \frac{\big|\nabla \ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2} \,\mathrm{d}u, \tag{3.71}$$

$$\mathcal{L}_2(s) := \int_s^{T-t_0} \Big(-\operatorname{div} \beta\big(X(T-u)\big) - \Big\langle \beta\big(X(T-u)\big) \,,\, \nabla \log p\big(T-u, X(T-u)\big) \Big\rangle_{\mathbb{R}^n} \Big) \,\mathrm{d}u, \tag{3.72}$$

$$\mathcal{L}_3(s) := -\int_s^{T-t_0} \frac{1}{2} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2} \,\mathrm{d}u. \tag{3.73}$$

Hence we have

$$\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta\big(t_0, X(t_0)\big) - \log \ell^\beta\big(T-s, X(T-s)\big)\Big] = \mathbb{E}_{\mathbb{P}}\Big[\big(\mathcal{K}_2 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3\big)(s)\Big]. \tag{3.74}$$

We divide now this equation by $T - t_0 - s$ and study the behavior of the resulting ratios as $s \uparrow T - t_0$. The function $(t,x) \mapsto \nabla \log Y^\beta(t,x)$ is continuous on $(0,\infty) \times \mathbb{R}^n$, thus the first expression $\mathcal{K}_2(s)$ of (3.65) is uniformly bounded on $[0, T-t_0] \times \operatorname{supp} \beta$. Since $\log Y^\beta(t_0, \,\cdot\,) = 0$, we can use continuity and uniform boundedness to conclude that

$$-\frac{1}{T-t_0-s} \int_s^{T-t_0} \left\langle \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)} \,,\, \beta\big(X(T-u)\big) \right\rangle_{\mathbb{R}^n} \mathrm{d}u \tag{3.75}$$

converges $\mathbb{P}$-almost surely as well as in $L^1(\mathbb{P})$ to zero. Regarding the second integral term $\mathcal{L}_1(s)$ of (3.71), we recall that $t_0 \geqslant 0$ is chosen such that the limiting assertion (3.44) is valid. In other words, we have that

$$\lim_{s \uparrow T-t_0} \frac{1}{T-t_0-s} \, \mathbb{E}_{\mathbb{P}} \left[\int_s^{T-t_0} \frac{1}{2} \frac{\big|\nabla \ell(T-u, X(T-u))\big|^2}{\ell(T-u, X(T-u))^2} \, \mathrm{d}u \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{1}{2} \frac{\big|\nabla \ell(t_0, X(t_0))\big|^2}{\ell(t_0, X(t_0))^2} \right]. \tag{3.76}$$

In order to handle the third expression $\mathcal{L}_2(s)$ of (3.72), we use continuity and uniform boundedness once again to see that

$$\frac{1}{T-t_0-s} \int_s^{T-t_0} \Big(-\operatorname{div} \beta(X(T-u)) - \Big\langle \beta(X(T-u)) \, , \, \nabla \log p(T-u, X(T-u)) \Big\rangle_{\mathbb{R}^n} \Big) \, \mathrm{d}u \tag{3.77}$$

converges $\mathbb{P}$-almost surely as well as in $L^1(\mathbb{P})$ to

$$-\operatorname{div} \beta(X(t_0)) - \Big\langle \beta(X(t_0)) \, , \, \nabla \log p(t_0, X(t_0)) \Big\rangle_{\mathbb{R}^n}. \tag{3.78}$$

Finally, we turn to the fourth and last integral term $\mathcal{L}_3(s)$ of (3.73). Owing to Lemma 4.12 of [KST19], there is a constant $C > 0$ such that

$$\frac{1}{T-t_0-s} \, \mathbb{E}_{\mathbb{P}} \left[\int_s^{T-t_0} \frac{\big|\nabla Y^\beta(T-u, X(T-u))\big|^2}{Y^\beta(T-u, X(T-u))^2} \, \mathrm{d}u \right] \leqslant C \, (T-t_0-s) \tag{3.79}$$

holds for all $0 \leqslant s \leqslant T - t_0$. This implies that

$$\lim_{s \uparrow T-t_0} -\frac{1}{T-t_0-s} \, \mathbb{E}_{\mathbb{P}} \left[\int_s^{T-t_0} \frac{1}{2} \frac{\big|\nabla Y^\beta(T-u, X(T-u))\big|^2}{Y^\beta(T-u, X(T-u))^2} \, \mathrm{d}u \right] = 0. \tag{3.80}$$

Summing up, we have shown that

$$\begin{aligned} &\lim_{s \uparrow T-t_0} \frac{\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta(t_0, X(t_0)) - \log \ell^\beta(T-s, X(T-s)) \Big]}{T-t_0-s} \\ &\quad = \mathbb{E}_{\mathbb{P}} \left[\frac{1}{2} \frac{\big|\nabla \ell(t_0, X(t_0))\big|^2}{\ell(t_0, X(t_0))^2} - \operatorname{div} \beta(X(t_0)) - \Big\langle \beta(X(t_0)) \, , \, \nabla \log p(t_0, X(t_0)) \Big\rangle_{\mathbb{R}^n} \right]. \end{aligned} \tag{3.81}$$

Now Proposition D.2 in Appendix D of [KST19] implies that we also have the limiting identity

$$\begin{aligned} &\lim_{s \uparrow T-t_0} \frac{\mathbb{E}_{\mathbb{P}}\Big[\log \ell^\beta(t_0, X(t_0)) - \log \ell^\beta(T-s, X(T-s)) \;\big|\; \mathcal{G}(T-s) \Big]}{T-t_0-s} \\ &\quad = \frac{1}{2} \frac{\big|\nabla \ell(t_0, X(t_0))\big|^2}{\ell(t_0, X(t_0))^2} - \operatorname{div} \beta(X(t_0)) - \Big\langle \beta(X(t_0)) \, , \, \nabla \log p(t_0, X(t_0)) \Big\rangle_{\mathbb{R}^n}, \end{aligned} \tag{3.82}$$

where the limit exists in the norm of $L^1(\mathbb{P})$. This proves the limiting assertion (3.46) with respect to the probability measure $\mathbb{P}$. According to Lemma 4.11 of [KST19], the probability measures $\mathbb{P}$ and $\mathbb{P}^\beta$ are equivalent, and the mutual Radon-Nikodým derivatives $\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}}$ and $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^\beta}$ are bounded on the σ-algebra $\mathcal{F}(T) = \mathcal{G}(0)$. Hence, convergence in $L^1(\mathbb{P})$ is equivalent to convergence in $L^1(\mathbb{P}^\beta)$. This establishes the $L^1(\mathbb{P}^\beta)$-convergence of (3.82), which completes the proof of the limiting assertion (3.46). $\square$

3.3 The backward dynamics of the probability density process

The probability density function $(t,x) \mapsto p(t,x)$ and its perturbed version $(t,x) \mapsto p^{\beta}(t,x)$ satisfy the Fokker-Planck equations

$$\partial_t p(t,x) = \operatorname{div}\left(\nabla\Psi(x)\, p(t,x)\right) + \tfrac{1}{2}\Delta p(t,x), \qquad t > 0 \tag{3.83}$$

and

$$\partial_t p^{\beta}(t,x) = \operatorname{div}\Big(\big(\nabla\Psi(x) + \beta(x)\big)\, p^{\beta}(t,x)\Big) + \tfrac{1}{2}\Delta p^{\beta}(t,x), \qquad t \geqslant t_0, \tag{3.84}$$

respectively. Under the probability measure $\mathbb{P}$, the time-reversed canonical coordinate process $(X(T-s))_{0 \leqslant s \leqslant T}$ satisfies the stochastic differential equation

$$\mathrm{d}X(T-s) = \Big(\nabla \log p\big(T-s, X(T-s)\big) + \nabla\Psi\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbb{P}}(T-s), \tag{3.85}$$

for $0 \leqslant s \leqslant T$, with respect to the backwards filtration $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T}$. In contrast, under the probability measure $\mathbb{P}^{\beta}$, the semimartingale decomposition of the time-reversed canonical coordinate process $(X(T-s))_{0 \leqslant s \leqslant T-t_0}$ is given by

$$\mathrm{d}X(T-s) = \Big(\nabla \log p^{\beta}\big(T-s, X(T-s)\big) + (\nabla\Psi + \beta)\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbb{P}^{\beta}}(T-s), \tag{3.86}$$

for $0 \leqslant s \leqslant T - t_0$, with respect to the backwards filtration $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$.

With these preparations, we obtain the following stochastic differentials in the backwards direction of time.

Lemma 3.1. *Under the* Assumptions 1.2 *of* [KST19]*, let $T > 0$. The time-reversed probability density process $\big(p(T-s, X(T-s))\big)_{0 \leqslant s \leqslant T}$ and its logarithm satisfy respectively the stochastic differential equations*

$$\begin{aligned} \frac{\mathrm{d}p(T-s,X(T-s))}{p(T-s,X(T-s))} &= \left(\frac{\big|\nabla p(T-s,X(T-s))\big|^2}{p\big(T-s,X(T-s)\big)^2} - \Delta\Psi\big(X(T-s)\big) \right) \mathrm{d}s \\ &\qquad + \left\langle \frac{\nabla p(T-s,X(T-s))}{p(T-s,X(T-s))}\,,\, \mathrm{d}\overline{W}^{\mathbb{P}}(T-s) \right\rangle_{\mathbb{R}^n} \end{aligned} \tag{3.87}$$

and

$$\begin{aligned} \mathrm{d}\log p\big(T-s,X(T-s)\big) &= \left(\frac{1}{2}\frac{\big|\nabla p(T-s,X(T-s))\big|^2}{p\big(T-s,X(T-s)\big)^2} - \Delta\Psi\big(X(T-s)\big) \right) \mathrm{d}s \\ &\qquad + \left\langle \frac{\nabla p(T-s,X(T-s))}{p(T-s,X(T-s))}\,,\, \mathrm{d}\overline{W}^{\mathbb{P}}(T-s) \right\rangle_{\mathbb{R}^n}, \end{aligned} \tag{3.88}$$

for $0 \leqslant s \leqslant T$, with respect to the backwards filtration $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T}$.

Proof. Applying Itô's formula and using (3.85) as well as (3.83), one obtains the stochastic equation (3.87). Another application of Itô's formula leads to (3.88). $\square$

Lemma 3.2. *Under the* Assumptions 1.2 *of* [KST19], *let* $t_0 \geqslant 0$ *and* $T > t_0$. *The time-reversed perturbed density process* $\big(p^\beta(T-s, X(T-s))\big)_{0 \leqslant s \leqslant T-t_0}$ *and its logarithm satisfy respectively the stochastic differential equations*

$$\begin{aligned}\frac{\mathrm{d}p^\beta(T-s,X(T-s))}{p^\beta(T-s,X(T-s))} &= -\Big(\operatorname{div}\beta\big(X(T-s)\big) + \Delta\Psi\big(X(T-s)\big)\Big)\,\mathrm{d}s \\ &\quad + \frac{\big|\nabla p^\beta(T-s,X(T-s))\big|^2}{p^\beta(T-s,X(T-s))^2}\,\mathrm{d}s + \left\langle \frac{\nabla p^\beta(T-s,X(T-s))}{p^\beta(T-s,X(T-s))}, \mathrm{d}\overline{W}^{\mathbb{P}^\beta}(T-s)\right\rangle_{\mathbb{R}^n}\end{aligned} \tag{3.89}$$

and

$$\begin{aligned}\mathrm{d}\log p^\beta(T-s,X(T-s)) &= -\Big(\operatorname{div}\beta\big(X(T-s)\big) + \Delta\Psi\big(X(T-s)\big)\Big)\,\mathrm{d}s \\ &\quad + \frac{1}{2}\frac{\big|\nabla p^\beta(T-s,X(T-s))\big|^2}{p^\beta(T-s,X(T-s))^2}\,\mathrm{d}s + \left\langle \frac{\nabla p^\beta(T-s,X(T-s))}{p^\beta(T-s,X(T-s))}, \mathrm{d}\overline{W}^{\mathbb{P}^\beta}(T-s)\right\rangle_{\mathbb{R}^n},\end{aligned} \tag{3.90}$$

for $0 \leqslant s \leqslant T-t_0$, *with respect to the backwards filtration* $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T-t_0}$.

Proof. The stochastic equations (3.89) and (3.90) follow from Itô's formula in conjunction with (3.86) and (3.84). □

3.4 The forward dynamics of the "perturbed-to-unperturbed" ratio

The probability density function $(t,x) \mapsto p(t,x)$ and its perturbed version $(t,x) \mapsto p^\beta(t,x)$ induce the respective likelihood ratios

$$\ell(t,x) = \frac{p(t,x)}{q(x)}, \qquad \ell^\beta(t,x) = \frac{p^\beta(t,x)}{q(x)}, \tag{3.91}$$

which satisfy the backward Kolmogorov-type equations

$$\partial_t \ell(t,x) = \tfrac{1}{2}\Delta\ell(t,x) - \langle \nabla\ell(t,x), \nabla\Psi(x)\rangle_{\mathbb{R}^n}, \qquad t > 0 \tag{3.92}$$

and

$$\begin{aligned}\partial_t \ell^\beta(t,x) &= \tfrac{1}{2}\Delta\ell^\beta(t,x) + \langle \nabla\ell^\beta(t,x), \beta(x) - \nabla\Psi(x)\rangle_{\mathbb{R}^n} \\ &\quad + \ell^\beta(t,x)\Big(\operatorname{div}\beta(x) - 2\,\langle\beta(x),\nabla\Psi(x)\rangle_{\mathbb{R}^n}\Big), \qquad t > t_0,\end{aligned} \tag{3.93}$$

respectively. The analogous partial differential equation for the "perturbed-to-unperturbed" ratio

$$Y^\beta(t,x) = \frac{\ell^\beta(t,x)}{\ell(t,x)} = \frac{p^\beta(t,x)}{p(t,x)}, \qquad (t,x) \in [t_0,\infty)\times\mathbb{R}^n \tag{3.94}$$

can be derived as follows. Substituting the product $\ell^\beta(t,x) = Y^\beta(t,x)\,\ell(t,x)$ and its consequences

$$\nabla\ell^\beta(t,x) = \ell(t,x)\,\nabla Y^\beta(t,x) + Y^\beta(t,x)\,\nabla\ell(t,x), \tag{3.95}$$

$$\Delta\ell^\beta(t,x) = \ell(t,x)\,\Delta Y^\beta(t,x) + Y^\beta(t,x)\,\Delta\ell(t,x) + 2\,\langle\nabla\ell(t,x),\nabla Y^\beta(t,x)\rangle_{\mathbb{R}^n}, \tag{3.96}$$

into the backward Kolmogorov equation (3.93) for $\ell^{\beta}(t,x)$, and summoning the backward Kolmogorov equation (3.92) for $\ell(t,x)$, we arrive at the partial differential equation

$$\begin{aligned}\partial_t Y^{\beta}(t,x) = \tfrac{1}{2}\Delta Y^{\beta}(t,x) + \big\langle \nabla Y^{\beta}(t,x), \beta(x) + \nabla \log p(t,x) + \nabla\Psi(x)\big\rangle_{\mathbb{R}^n} \\ + Y^{\beta}(t,x)\Big(\operatorname{div}\beta(x) + \big\langle \beta(x), \nabla \log p(t,x)\big\rangle_{\mathbb{R}^n}\Big), \qquad t > t_0,\end{aligned} \tag{3.97}$$

with $Y^{\beta}(t_0,\,\cdot\,) = 1$, for the ratio in (3.94). In conjunction with

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t), \qquad t \geqslant 0, \tag{3.98}$$

the equation (3.97) leads to the following forward dynamics.

Lemma 3.3. *Under the* Assumptions 1.2 *of* [KST19]*, let* $t_0 \geqslant 0$*. The "perturbed-to-unperturbed" ratio process* $\big(Y^{\beta}(t,X(t))\big)_{t\geqslant t_0}$ *and its logarithm satisfy respectively the stochastic differential equations*

$$\begin{aligned}\mathrm{d}Y^{\beta}(t,X(t)) = \Big(\Delta Y^{\beta}(t,X(t)) + \Big\langle \nabla Y^{\beta}(t,X(t))\,,\, \beta(X(t)) + \nabla \log p(t,X(t))\Big\rangle_{\mathbb{R}^n}\Big)\,\mathrm{d}t \\ + Y^{\beta}(t,X(t))\Big(\operatorname{div}\beta(X(t)) + \Big\langle \beta(X(t))\,,\, \nabla\log p(t,X(t))\Big\rangle_{\mathbb{R}^n}\Big)\,\mathrm{d}t + \Big\langle \nabla Y^{\beta}(t,X(t))\,,\, \mathrm{d}W(t)\Big\rangle_{\mathbb{R}^n}\end{aligned} \tag{3.99}$$

and

$$\begin{aligned}\mathrm{d}\log Y^{\beta}(t,X(t)) = \left(\frac{\Delta Y^{\beta}(t,X(t))}{Y^{\beta}(t,X(t))} + \left\langle \frac{\nabla Y^{\beta}(t,X(t))}{Y^{\beta}(t,X(t))}\,,\, \beta(X(t)) + \nabla\log p\big(t,X(t)\big)\right\rangle_{\mathbb{R}^n}\right)\mathrm{d}t \\ + \left(\operatorname{div}\beta(X(t)) + \Big\langle \beta(X(t))\,,\, \nabla\log p(t,X(t))\Big\rangle_{\mathbb{R}^n} - \frac{1}{2}\frac{\big|\nabla Y^{\beta}(t,X(t))\big|^2}{Y^{\beta}\big(t,X(t)\big)^2}\right)\mathrm{d}t \\ + \left\langle \frac{\nabla Y^{\beta}(t,X(t))}{Y^{\beta}(t,X(t))}\,,\, \mathrm{d}W(t)\right\rangle_{\mathbb{R}^n},\end{aligned} \tag{3.100}$$

for $t \geqslant t_0$*, with respect to the forward filtration* $(\mathcal{F}(t))_{t\geqslant t_0}$.

Bibliography

[FJ16] J. Fontbona and B. Jourdain. *A trajectorial interpretation of the dissipations of entropy and Fisher information for stochastic differential equations.* The Annals of Probability, 44(1):131–170, 2016.

[Kle14] A. Klenke. *Probability Theory: A Comprehensive Course.* Universitext. Springer, London, second edition, 2014.

[KST19] I. Karatzas, W. Schachermayer, and B. Tschiderer. *Pathwise Otto calculus.* arXiv:1811.08686v2, 2019.

[Léo14a] C. Léonard. *A survey of the Schrödinger problem and some of its connections with optimal transport.* Discrete and Continuous Dynamical Systems - Series A, 34(4):1533–1574, 2014.

[Léo14b] C. Léonard. *Some Properties of Path Measures.* In Séminaire de Probabilités XLVI, volume 2123 of Lecture Notes in Mathematics, pages 207–230. Springer, 2014.

[LG16] J.-F. Le Gall. *Brownian Motion, Martingales, and Stochastic Calculus*, volume 274 of Graduate Texts in Mathematics. Springer, 2016.