



universität
wien

Masterarbeit / Master's Thesis

Titel der Masterarbeit / Title of the Master's Thesis

On the Influence of Earth's Rotation on the Propagation of Light

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2019 / Vienna, 2019

Studienkennzahl lt. Studienblatt /
degree programme code as it appears on
the student record sheet:

UA 066 876

Studienrichtung lt. Studienblatt /
degree programme as it appears on
the student record sheet:

Masterstudium Physik

Betreut von /
Supervisor:

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Abstract

In this thesis, the influence of Earth's rotation (both around its own axis and around the Sun) on the propagation of light in optical media is analysed. This is done using both geometrical optics and a perturbative calculation based on Maxwell's equations in rotating coordinates in flat spacetime. Considering light propagation in cylindrical step-index waveguides in particular, the first order correction to electromagnetic modes is computed. The calculation shows that Earth's rotation causes a weak mode coupling, giving rise to sidebands, whose amplitudes are computed as well. The correction to the dispersion relation derived here allows to assess the anisotropy of light propagation due to Earth's rotation. The linearisation of this result is found to agree numerically with a simple formula derived from geometrical optics.

Zusammenfassung

In dieser Arbeit wird der Einfluss der Erdrotation (sowohl um die eigene Achse, als auch um die Sonne) auf Lichtausbreitung in optischen Medien analysiert. Dies geschieht sowohl im Rahmen der geometrischen Optik, als auch anhand von perturbativen Rechnungen basierend auf den Maxwell-Gleichungen (Wellenoptik) in rotierenden Koordinaten in flacher Raumzeit. Insbesondere werden Korrekturen erster Ordnung zu den elektromagnetischen Moden in zylindrischen Stufenindex-Wellenleitern berechnet. Die Rechnung zeigt, dass die Erdrotation eine schwache Modenkopplung hervorruft und daher Seitenbänder anregt, deren Amplituden ebenfalls bestimmt wurden. Die hier hergeleitete Korrektur zur Dispersionsrelation erlaubt es, die Anisotropie der Lichtausbreitung aufgrund der Erdrotation zu quantifizieren. Die Linearisierung dieses Resultats stimmt numerisch mit einer einfachen Formel überein, die aus der geometrischen Optik abgeleitet wird.

Contents

I	Introduction	I
1.1	Outline of this Work	I
1.2	Model Assumptions	2
1.3	Conventions	2
2	Geometrical Optics	3
2.1	The Metric Tensor in Born Coordinates	3
2.2	The Optical Metric	4
2.3	The Eikonal Equation	4
2.4	Geometrical Optics Approximation	5
3	Maxwell's Equations in Rotating Coordinates	7
3.1	Field Equations	7
3.2	Decomposition of the Field Strength Tensors	7
3.3	Decomposition of the Field Equations	8
3.4	Electric-Magnetic Symmetry	9
3.5	The Wave Equation for the Electromagnetic Field	10
4	Maxwell's Equations in Cylindrical Waveguides	13
4.1	Outline of the Calculation	13
4.1.1	Unperturbed Case	13
4.1.2	First Order Corrections	14
4.2	Unperturbed Solution	14
4.2.1	Dispersion Relation	16
4.2.2	Generalisation	16
5	Main Corrections	19
5.1	Notation	19
5.2	The Wave Equation for the z Component	19
5.3	The Radial Equations	20
5.4	The Wave Equation in the Core	22
5.5	The Wave Equation in the Cladding	23
5.6	Collection of Intermediate Results	24
5.7	Compatibility of Approximations	25
5.8	The Transverse Components of d and b	26
5.9	The Field Components of e and h	28
5.10	Consistency of the Approximations	28
5.11	Continuity Conditions	30
5.12	Continuity of the Main Mode	33

5.13	Dispersion Relation	35
5.14	Continuity of the Sidebands	38
6	Corrections of Relative Order λ/R	41
6.1	The Wave Equations	41
6.2	The Radial Equations	42
6.3	The Remaining Field Components	43
6.4	Order of Magnitude	43
6.5	Continuity Conditions	43
7	Corrections of Relative Order ℓ/R	45
7.1	Corrections to the Velocity Vector Field	45
7.2	The Wave Equations	46
7.3	The Radial Equations	46
7.4	The Transverse Components of d and b	48
7.5	The Field Components of e and h	50
7.6	Consistency of the Approximations	50
7.7	Continuity Conditions	52
8	Numerical Examples	55
8.1	Corrections to the Dispersion Relation	55
8.2	Main Mode Amplitudes	57
8.3	Sideband Amplitudes	57
9	Conclusion	61
A	The Wave Equation Induced by the Optical Metric	63
B	Remarks on the Inhomogeneous Bessel Equation	65
B.1	General Definitions	65
B.2	General Properties of the Bessel Operator	65
B.3	The Ordinary Bessel Equation	66
B.4	The Modified Bessel Equation	66
B.5	Rescaled Arguments	67
B.6	Restrictions on Source Terms and Boundary Conditions	67
C	Some Integrals of Bessel Functions	69
D	Remarks on the Influence of Earth's Gravitational Field	71
	References	73

Chapter I

Introduction

The problem of determining the influence of Earth's rotation on optical phenomena was raised in the second half of the 19th century and was first discussed using the theory of a luminiferous aether, see e.g. [9, 10]. Using an argument based on Huygen's principle, Lorentz concluded that the effective refractive index n_{eff} is related to the ordinary refractive index n by

$$n_{\text{eff}} \approx n - \vec{m} \cdot \vec{v},$$

cf. [9, p. 126ff], where $\vec{v}$ is the velocity of aether and $\vec{m}$ is the unit vector in the direction of light propagation. This formula turns out to be accurate if one sets $\vec{v}$ to be the negative linear velocity due to Earth's rotation. However, at the time when this formula was published, it was still under debate whether the aether was set in motion by Earth's movement (aether drag hypothesis), which would invalidate this simple substitution.

The influence of Earth's rotation (about its own axis) on light was first demonstrated experimentally using a Sagnac interferometer [11], where a beam of light is split into two, which traverse a ring interferometer in opposite sense of rotation. If the apparatus rotates, the different light rays take unequal times to complete a full circle, leading to an interference pattern whose fringes are displaced from the positions they would have if the apparatus were an inertial system. Although Sagnac interpreted this effect as a proof for the existence of aether [12, 13], Laue gave an explanation in the framework of Special Relativity [15] and Langevin was able to explain the phenomenon in the language of General Relativity [7, 8]. Further development of such setups led to Ring Laser Gyroscopes, which, in 2011 reached the precision required to allow the measurement of the Chandler wobble [14].

In a 2017 paper [4], Hilweg et al. proposed an experiment to measure the effect of Earth's gravitational potential on single photons. This paper included an estimate for the influence of Earth's rotation, where the photon was modelled as a classical particle traversing a spooled waveguide. In a more recent paper [2], the shift of the wave vector due to Earth's gravitational field was calculated more accurately using Maxwell's equations in a post-Newtonian metric.

In this work, the effect of Earth's rotation is determined from first principles, using Maxwell's equations in a rotating frame in Minkowski spacetime.

1.1 Outline of this Work

Classical propagation of light can be described either by geometrical optics or wave optics. The latter method describes light by wave solutions of Maxwell's equations whereas geometrical optics is based on the eikonal equation, which can be understood as a limiting case of wave optics where the wavelength becomes infinitely short.

In Chapter 2, the influence of Earth's rotation on the propagation of light is estimated using geometrical optics. Chapter 3 is concerned with the derivation of a wave equation from Maxwell's equations in appropriate coordinates. A simplified version of the equation is solved in Chapter 5 and additional corrections are discussed in the subsequent chapters 6 and 7. Finally, in Chapter 8 numerical examples of a concrete setup are provided and the effects of Earth's rotation around its own axis and the rotation about the Sun are compared.

1.2 Model Assumptions

In order for wave optics to be applicable, we must specify the full geometry of the medium of interest. We consider cylindrical step index waveguides which consist of a cylindrical core of radius a and refractive index n_1 surrounded by a cladding of diameter $a' \gg a$ and refractive index $n_2 < n_1$. The dielectric will be assumed to be nonmagnetic and its field response will be assumed to be linear. The dependence of the fields $\vec{D}$ and $\vec{H}$ on $\vec{E}$ and $\vec{B}$ is described covariantly by means of the *optical metric*, introduced in [3] (not to be confused with the optical metric defined when studying null geodesics in stationary spacetimes).

We will neglect boundary effects which arise at the two ends of the waveguide and at the outer boundary of the cladding. In this sense, the waveguide will be treated as infinitely long and the cladding as infinitely thick.

To describe Earth's rotation we use rigidly rotating coordinates in flat spacetime. The metric tensor will be approximated under the assumption that the rotational velocity is sufficiently slow compared to the speed of light. Such a description applies both to Earth's rotation about its own axis and its orbit around the Sun (neglecting its orbital eccentricity).

1.3 Conventions

We use Heaviside-Lorentz units which, in our problem, coincide with Gaussian units, due to the absence of external charges and currents. Furthermore, unless explicitly specified otherwise, we use units where the speed of light in vacuum is equal to one. Compared to SI units, this means that

$$c = \mu_0 = \varepsilon_0 = 1.$$

The signature of the metric tensor g is

$$\text{sgn}(g) = (-, +, +, +).$$

The Einstein summation convention is used. Greek indices will range from 0 to 4 and Latin indices range from 1 to 3.

Chapter 2

Geometrical Optics

In this section we discuss the metric tensor used to describe rotating coordinate systems and introduce the optical metric to describe light propagation in rotating media. A first estimate on the correction to the dispersion relation is derived in the framework of geometrical optics.

2.1 The Metric Tensor in Born Coordinates

It is convenient to use coordinates in which the waveguide is at rest. For this we transform the metric tensor from inertial coordinates with origin at the centre of rotation, denoted by $\bar{x}^\mu$, to corotating coordinates x^μ :

$$t \equiv x^0 = \bar{x}^0, \quad \bar{x}^i = R^i_j(t)x^j, \quad (2.1)$$

where $R^i_j(t)$ denotes a matrix of uniform rotation along $\vec{\Omega}$ with angular velocity $\Omega := |\vec{\Omega}|$, thus

$$\frac{d}{dt}R^i_l(t) = \epsilon^i_{jk}\Omega^j R^k_l(t). \quad (2.2)$$

This gives the following form of the metric tensor in the new coordinates:

$$g = \eta_{\mu\nu}dx^\mu dx^\nu + 2\epsilon_{ijk}\Omega^j x^k dt dx^i + \mathcal{O}(\rho^2\Omega^2). \quad (2.3)$$

Equivalently, (2.3) may be obtained by expanding the Born metric, which describes rigid rotation in flat spacetime,

$$g = -(1 - \Omega^2 \rho^2)dt^2 + 2\Omega \rho^2 dt d\phi + d\rho^2 + \rho^2 d\phi^2 + dz^2$$

to first order in $\Omega\rho$. In the chosen coordinates, the inverse metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} + \mathcal{O}(\rho^2\Omega^2).$$

In what follows, the components of any tensor will refer to the corotating coordinate system x^μ . Furthermore, $\Omega|\vec{x}|/c$ will be assumed to be sufficiently small, so that all terms quadratic in $\rho\Omega$ can be neglected.

Note that wave equations (in particular those for the electromagnetic field) also depend on derivatives of the metric. By expanding the metric tensor to order $v = \Omega\rho$, the wave equation is not expanded in powers of $v \equiv |\vec{v}|$ alone, but also in powers of $L\Omega$, where L is any characteristic length scale of the problem: in our case these are the length ℓ and the radius a of the waveguide, as well as the wavelength λ . Thus, by neglecting terms of order $\mathcal{O}(v^2)$ in the components of the metric tensor, we also neglect terms of order $\mathcal{O}(a^2\Omega^2)$, $\mathcal{O}(\ell^2\Omega^2)$ and $\mathcal{O}(\lambda^2\Omega^2)$ in the field equations. Since the typical

length scales of the dielectric are much smaller than ρ (in our case, this is either Earth's radius or its distance to the Sun), these error terms are expected to be negligible in all practical applications. By slight abuse of notation, we will refer to all of these correction terms by $O(\Omega^2)$ and thus formally expand all equations in powers of the (dimensionful) angular velocity Ω .

2.2 The Optical Metric

The optical metric tensor γ of a linear dielectric with four-velocity vector field u is defined as

$$\begin{aligned}\gamma^{\mu\nu} &= g^{\mu\nu} + (1 - n^2)u^\mu u^\nu, \\ \gamma_{\mu\nu} &= g_{\mu\nu} + (1 - n^{-2})u_\mu u_\nu,\end{aligned}\tag{2.4}$$

where g is the spacetime metric and n is the refractive index. It is readily verified that $\gamma^{\mu\nu}$ and $\gamma_{\mu\nu}$ are inverse to each other.

In a pseudo-orthonormal basis (with respect to g), whose timelike vector coincides with u , the matrix representation of $\gamma_{\mu\nu}$ is given by

$$\text{diag}(-n^{-2}, 1, 1, 1),$$

which shows that γ is a Lorentzian metric tensor.

In the comoving coordinates the four-velocity of the co-rotating dielectric is given by

$$u = \frac{\partial}{\partial x^0}, \quad \Leftrightarrow \quad (u^\mu) = (1, 0, 0, 0).$$

Using the velocity vector field

$$\vec{v} = \vec{\Omega} \times \vec{x},$$

the components of γ (in the comoving coordinates) take the form

$$\gamma_{00} = -\frac{1}{n^2}, \quad \gamma_{0i} = \frac{v^i}{n^2}, \quad \gamma_{ij} = \delta_{ij},\tag{2.5}$$

and the components of the matrix inverse read

$$\gamma^{00} = -n^2, \quad \gamma^{0i} = v^i, \quad \gamma^{ij} = \delta^{ij},\tag{2.6}$$

where terms of order $O(v^2)$ have been neglected.

2.3 The Eikonal Equation

In geometrical optics, any component f of the electromagnetic field strength tensors is assumed to be of the form

$$f(x) = A(x)e^{i\psi(x)},$$

where the eikonal ψ is a rapidly varying function of space and time and the amplitude A varies slowly [5, p. 153f]. The wave covector k is defined as the differential of the eikonal:

$$k = d\psi.$$

In local inertial coordinates, k is typically decomposed as

$$d\psi = \omega dt - k_i dx^i,$$

to establish the analogy to plane waves, where $\psi = \omega t - k_i x^i$. In vacuum it holds that $\omega^2 = \vec{k}^2$, so k is lightlike with respect to the spacetime metric g . This generalises to light propagation in linear media, where k is null with respect to the optical metric γ ;

$$\gamma(d\psi, d\psi) = 0, \quad (2.7)$$

which is the general form of the eikonal equation.

2.4 Geometrical Optics Approximation

The eikonal equation may be used to obtain a first estimate on the influence of Earth's rotation on the propagation of light. One could try to solve (2.7) in a cylindrical waveguide, using appropriate conditions at the core-cladding interface, which we have not attempted to do. Instead, to simplify the problem, we look for plane wave solutions where the dispersion relation (which relates $\vec{k}^2$ and ω^2) *in the absence of rotation* reduces to $k = n\omega$, where n is the effective refractive index obtained by solving the full Maxwell's equations in the unperturbed case. Writing

$$\beta^2 = \vec{k}^2, \quad \vec{k} = \beta \vec{m},$$

where $\vec{m}$ is the unit vector along the constant vector $\vec{k}$, and using the explicit form (2.6) of the (inverse) optical metric, the eikonal equation (2.7) yields the quadratic equation

$$-n^2 \omega^2 + \beta^2 - 2\beta \omega \vec{m} \cdot \vec{v} = 0,$$

which has the positive solution

$$\beta = \omega \sqrt{n^2 + (\vec{m} \cdot \vec{v})^2} + \omega \vec{m} \cdot \vec{v}.$$

Expanding in powers of $v \equiv |\vec{v}|$ one obtains

$$\beta = n\omega + \omega \vec{m} \cdot \vec{v} + O(v^2),$$

from which one obtains the first order correction due to rotation

$$\frac{\delta\beta}{\beta} \approx \frac{\omega}{\beta} \vec{m} \cdot \vec{v}. \quad (2.8)$$

Using $\beta = n\omega + O(v)$ on the right hand side, this can be rewritten as

$$\frac{\delta\beta}{\beta} \approx \frac{1}{n} \vec{m} \cdot \vec{v}, \quad (2.9)$$

which approximates unexpectedly well one of the main results of our work below. As stated before, we require the dispersion relation to reproduce $\beta = n\omega$ in the unperturbed case, where n is the *effective* refractive index obtained from the exact solution of the unperturbed problem, which depends on the precise geometry of the waveguide.

In the next chapters, we will compute $\delta\beta$ using Maxwell's equations in the metric (2.3), which requires to specialise to a fixed geometry. We will consider an infinite cylindrical step index waveguide, for which an exact solution in the absence of rotation is known.

Chapter 3

Maxwell's Equations in Rotating Coordinates

In this chapter, we formulate Maxwell's equations in the linearised Born metric (2.3) and derive a wave equation for the electric and magnetic fields.

3.1 Field Equations

In manifestly covariant formulations of electrodynamics, the fields $\vec{E}$ and $\vec{B}$ are subsumed by a two-form F , and the fields $\vec{D}$ and $\vec{H}$ are combined to form the bivector $\bar{F}$. In the absence of electromagnetic charges or currents, Maxwell's equations can be written in the form

$$dF = 0, \quad \delta \bar{F} = 0,$$

where d denotes the exterior derivative and δ is the codifferential. In local coordinates, these equations take the form

$$\partial_{[\mu} F_{\nu\rho]} = 0, \quad \nabla_{\nu} \bar{F}^{\mu\nu} = 0,$$

where the square brackets indicates antisymmetrisation and ∇ denotes the Levi-Civita covariant derivative associated with the spacetime metric g . For linear media of permittivity ε and permeability μ , the optical metric γ allows to write the relation between F and $\bar{F}$ in the concise form

$$\mu \bar{F}^{\alpha\beta} = \gamma^{\alpha\rho} \gamma^{\beta\sigma} F_{\rho\sigma}, \quad (3.1)$$

which correctly reduces to $\bar{F}^{\alpha\beta} = F^{\alpha\beta}$ in vacuum.

We now discuss a “3+1 split” of these equations, which allows us to derive a wave equation for the electric and magnetic fields.

3.2 Decomposition of the Field Strength Tensors

Following [2, eq. (3.5)] we decompose the two-form F and the bivector $\bar{F}$ as

$$F = +e \wedge dx^0 + \epsilon_{ijk} b^i dx^j \wedge dx^k, \\ \bar{F} = -d \wedge \frac{\partial}{\partial x^0} + \epsilon^{ijk} h_i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k},$$

where e and h are spatial one-forms and b and d (not to be confused with the differential operator d) are spatial vectors. More explicitly, this means

$$\begin{aligned} e_i &= F_{i0} , \\ b^i &= \frac{1}{2} \epsilon^{ijk} F_{jk} , \\ d^i &= \bar{F}^{0i} , \\ h_i &= \frac{1}{2} \epsilon_{ijk} \bar{F}^{jk} , \end{aligned} \tag{3.2}$$

where, as already pointed out, all field components $F_{\mu\nu}$ and $\bar{F}^{\mu\nu}$ refer to the corotating coordinate system.

Using (3.1), we may express the d and h fields in terms of the e and b fields as

$$\begin{aligned} \mu d^i &= n^2 e_i - \epsilon^{ijk} v^j b^k , \\ \mu h_i &= b^i - \epsilon_{ijk} v^j e_k , \end{aligned} \tag{3.3}$$

where terms of order $O(v^2)$ have been neglected.

3.3 Decomposition of the Field Equations

From [2, eqns. (3.12) and (3.15)] we have the following decomposition of the field equations

$$\begin{aligned} \dot{b}^i + \epsilon^{ijk} \partial_j e_k &= 0 , \\ \dot{d}^i - \epsilon^{ijk} \partial_j h_k &= 0 , \\ \partial_i b^i &= 0 , \\ \partial_i d^i &= 0 . \end{aligned} \tag{3.4}$$

We need not make any distinction between upper and lower indices since they are manipulated with the spatial metric δ_{ij} . Since $\vec{b}$ and $\vec{d}$ have a vanishing spatial divergence, it is natural to work with these two fields and express $\vec{e}$ and $\vec{h}$ in terms of them. From (3.3) one finds to order $O(v)$

$$\begin{aligned} n^2 \vec{e} &= \mu \vec{d} + \vec{v} \times \vec{b} , \\ n^2 \vec{h} &= \epsilon \vec{b} - \vec{v} \times \vec{d} , \end{aligned} \tag{3.5}$$

where $n^2 = \epsilon\mu$ was used.

We note that for $n \neq 1$, we may set $\vec{u} := -\vec{v}/(n^2 - 1)$ to obtain

$$\begin{aligned} \vec{d} &= \epsilon \vec{e} + (n^2 - 1) \vec{u} \times \vec{h} , \\ \vec{b} &= \mu \vec{h} - (n^2 - 1) \vec{u} \times \vec{e} , \end{aligned}$$

which coincides with the relations between $\vec{e}$, $\vec{b}$ and $\vec{d}$, $\vec{h}$ in a medium which moves with velocity $\vec{u}$ relative to an inertial system [6, p. 329, eqns. (76,10) and (76,11)].

Substituting the expression for $\vec{e}$ into the first equation in (3.4), we obtain

$$\begin{aligned} n^2 \dot{b}^i + \epsilon^{ijk} \partial_j (n^2 e^k) &= n^2 \dot{b}^i + \mu \epsilon^{ijk} \partial_j d^k + \epsilon^{ijk} \partial_j (\epsilon^{klm} v^l b^m) \\ &= n^2 \dot{b}^i + \mu \epsilon^{ijk} \partial_j d^k + \delta_{lm}^{ij} (v^l \partial_j b^m + b^m \partial_j v^l) \\ &= n^2 \dot{b}^i + \mu \epsilon^{ijk} \partial_j d^k - v^j \partial_j b^i + b^j \partial_j v^i , \end{aligned}$$

where we have used the generalised Kronecker delta δ_{kl}^{ij} as well as the fact that $\vec{\nabla} \cdot \vec{b} = 0$ and $\vec{\nabla} \cdot \vec{v} = 0$. A similar argument holds for the second equation in (3.4).

Expressed entirely in terms of $\vec{d}$ and $\vec{b}$, the field equations thus take the form

$$\begin{aligned} n^2 \dot{\vec{b}} + \mu \vec{\nabla} \times \vec{d} - [\vec{v}, \vec{b}] &= 0, \\ n^2 \dot{\vec{d}} - \epsilon \vec{\nabla} \times \vec{b} - [\vec{v}, \vec{d}] &= 0, \\ \vec{\nabla} \cdot \vec{b} &= 0, \\ \vec{\nabla} \cdot \vec{d} &= 0, \end{aligned} \tag{3.6}$$

where $[\vec{v}, \vec{w}]$ denotes the commutator (Lie bracket) of the vector fields $\vec{v}$ and $\vec{w}$.

3.4 Electric-Magnetic Symmetry

Note that the field equations are invariant under the transformation J , which acts on $\vec{b}$ and $\vec{d}$ as

$$\vec{b} \mapsto +\frac{\mu}{n} \vec{d}, \quad \vec{d} \mapsto -\frac{\epsilon}{n} \vec{b}, \tag{3.7}$$

where $n^2 = \epsilon\mu$. This induces the following transformation of the fields $\vec{e}$ and $\vec{h}$:

$$\vec{h} \mapsto +\frac{\epsilon}{n} \vec{e}, \quad \vec{e} \mapsto -\frac{\mu}{n} \vec{h}. \tag{3.8}$$

Due to the linearity of the field equations the overall normalisation by the factor $1/n$ is arbitrary. This specific value was chosen such that J defines a complex structure on the fields, which means that $J^2 = -\text{id}$.

Defining the complex fields

$$\vec{z} = \vec{b} - iJ(\vec{b}), \quad \text{and} \quad \vec{\xi} = \vec{h} - iJ(\vec{h}),$$

such that

$$J(\vec{z}) = i\vec{z},$$

and similarly for $\vec{\xi}$, the equations (3.5) can be combined to yield

$$\mu \vec{\xi} = \vec{z} - \frac{i}{n} \vec{v} \times \vec{z},$$

and the four real equations (3.6) can be written as two complex equations

$$\begin{aligned} n^2 \dot{\vec{z}} + in \vec{\nabla} \times \vec{z} - [\vec{v}, \vec{z}] &= 0, \\ \vec{\nabla} \cdot \vec{z} &= 0. \end{aligned} \tag{3.9}$$

When solving Maxwell's equations, we may thus either solve the complex equations (3.9), or solve an equation for any of the two original fields ($\vec{d}$ or $\vec{b}$) and obtain a solution for the other field by applying the transformation (3.8). Since we use complex fields (the physical quantities are the real parts thereof), we choose to work with the latter method.

3.5 The Wave Equation for the Electromagnetic Field

In this section, we derive a wave equation for the electromagnetic field, taking all terms of order $O(\Omega)$ into account.

We start by writing (3.6) as

$$\begin{aligned} n^2 \dot{\vec{b}} + \mu \vec{\nabla} \times \vec{d} - \nabla_{\vec{v}} \vec{b} + \vec{\Omega} \times \vec{b} &= 0, \\ n^2 \dot{\vec{d}} - \epsilon \vec{\nabla} \times \vec{b} - \nabla_{\vec{v}} \vec{d} + \vec{\Omega} \times \vec{d} &= 0, \end{aligned} \quad (3.10)$$

which is equivalent to the original form since

$$[\vec{v}, \vec{w}]^i = v^j \partial_j w^i - w^j \partial_j v^i = v^j \partial_j w^i - w^j \epsilon^{jik} \Omega^k.$$

Differentiating the first equation in (3.10) with respect to time and multiplying by n^2 , one obtains

$$n^4 \ddot{\vec{b}} + \mu \vec{\nabla} \times (n^2 \dot{\vec{d}}) - n^2 \nabla_{\vec{v}} \dot{\vec{b}} + \vec{\Omega} \times (n^2 \dot{\vec{b}}) = 0.$$

Due to the second equation of (3.10) and $n^2 = \epsilon\mu$, the second term takes the form

$$\mu \vec{\nabla} \times (n^2 \dot{\vec{d}}) = n^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{b}) + \mu \vec{\nabla} \times (\nabla_{\vec{v}} \vec{d}) - \mu \vec{\nabla} \times (\vec{\Omega} \times \vec{d}).$$

Since $\vec{\nabla} \cdot \vec{b} = 0$, the double curl reduces to

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{b}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{b}) - \Delta \vec{b} = -\Delta \vec{b},$$

where $\Delta \vec{b}$ denotes the vector Laplacian of $\vec{b}$. Moreover, due to $\vec{\nabla} \cdot \vec{b} = 0$ and $\vec{v} = \vec{\Omega} \times \vec{r}$, we have

$$\vec{\nabla} \times (\nabla_{\vec{v}} \vec{d}) = \nabla_{\vec{v}} (\vec{\nabla} \times \vec{d}) - \vec{\nabla}(\vec{\Omega} \cdot \vec{d}).$$

Together with the intermediate result

$$\vec{\nabla} \times (\vec{\Omega} \times \vec{d}) = -(\vec{\Omega} \cdot \vec{\nabla}) \vec{d},$$

which is due to the fact that $\vec{\nabla} \cdot \vec{d} = 0$, we obtain

$$\begin{aligned} \mu \vec{\nabla} \times (n^2 \dot{\vec{d}}) &= -n^2 \Delta \vec{b} + \nabla_{\vec{v}} (\mu \vec{\nabla} \times \vec{d}) - \mu \vec{\nabla}(\vec{\Omega} \cdot \vec{d}) + \mu(\vec{\Omega} \cdot \vec{\nabla}) \vec{d} \\ &= -n^2 \Delta \vec{b} - n^2 \nabla_{\vec{v}} \dot{\vec{b}} - \mu[\vec{\nabla}(\vec{\Omega} \cdot \vec{d}) - (\vec{\Omega} \cdot \vec{\nabla}) \vec{d}] + O(\Omega^2). \end{aligned}$$

Combining this with

$$n^2 \dot{\vec{b}} \times \vec{\Omega} = \mu \vec{\Omega} \times (\vec{\nabla} \times \vec{d}) + O(\Omega^2) = \mu[\vec{\nabla}(\vec{\Omega} \cdot \vec{d}) - (\vec{\Omega} \cdot \vec{\nabla}) \vec{d}] + O(\Omega^2),$$

we arrive at the wave equation

$$n^2(n^2 \ddot{\vec{b}} - \Delta \vec{b}) = 2 \nabla_{\vec{v}}(n^2 \dot{\vec{b}}) - 2\mu[(\vec{\Omega} \cdot \vec{\nabla}) \vec{d} - \vec{\nabla}(\vec{\Omega} \cdot \vec{d})] + O(\Omega^2).$$

If we replace the time derivative of $\vec{b}$ with spatial derivatives of $\vec{d}$ using the second equation from (3.10), the wave equation takes the form

$$n^2(n^2 \ddot{\vec{b}} - \Delta \vec{b}) = -2\mu \nabla_{\vec{v}}(\vec{\nabla} \times \vec{d}) - 2\mu[(\vec{\Omega} \cdot \vec{\nabla}) \vec{d} - \vec{\nabla}(\vec{\Omega} \cdot \vec{d})] + O(\Omega^2).$$

Since the right hand side depends on $\vec{d}$ only, this form allows to compare the various source terms.

To get some insight into the relative magnitudes of the two source terms, let us estimate spatial derivatives of the fields by $1/\lambda$, where λ is the wavelength of the electromagnetic wave. Furthermore, $\vec{v}$ is roughly of the order of magnitude ΩR , where R is Earth's radius (or its distance to the Sun, depending on the problem), so we may estimate that the first order differential operator (whose terms involve $\vec{\Omega}$) is suppressed relative to the second order operator (which is directly proportional to $\vec{v}$) by the factor

$$\frac{\Omega\lambda^{-1}}{\Omega R\lambda^{-2}} = \frac{\lambda}{R},$$

which is clearly negligible. The validity of the estimate needs of course to be verified at the end of the calculation, which is done at the end of Chapter 6.

Furthermore, let us note that the directional derivative along $\vec{v}$ depends on the position in the dielectric. Decomposing the radius vector $\vec{x}$ as

$$\vec{x} = \vec{R} + \vec{r} \quad \Leftrightarrow \quad (x^1, x^2, x^3) = (R^1, R^2, R^3) + (x, y, z)$$

where $\vec{R}$ points from centre of rotation to one end of the dielectric. Choosing z to be the distance along the symmetry axis of the waveguide, the coordinate ranges inside the core are

$$x^2 + y^2 \leq a^2 \quad \text{and} \quad 0 \leq z \leq \ell.$$

Neglecting the dependence of $\vec{v}$ on x and y is expected to produce relative errors of order $O(a/R)$, while those resulting from neglecting its z dependence are expected to be

$$O(\ell/R) \gg O(a/R),$$

since the length of a waveguide is typically many orders of magnitude larger than its diameter. As a first approximation we will use $\vec{v} \approx \vec{V}$, where

$$\vec{V} = \vec{\Omega} \times \vec{R}.$$

This approximation will be removed in Chapter 7, leading to a result which is valid up to terms of order $O(a/R)$.

To summarise, we have found the following wave equation for the electromagnetic field:

$$\begin{aligned} \epsilon(n^2\ddot{\vec{b}} - \Delta\vec{b}) &= 2\epsilon\nabla_{\vec{v}}\dot{\vec{b}} - 2[(\vec{\Omega} \cdot \vec{\nabla})\vec{d} - \vec{\nabla}(\vec{\Omega} \cdot \vec{d})], \\ \mu(n^2\ddot{\vec{d}} - \Delta\vec{d}) &= 2\mu\nabla_{\vec{v}}\dot{\vec{d}} - 2[(\vec{\Omega} \cdot \vec{\nabla})\vec{b} - \vec{\nabla}(\vec{\Omega} \cdot \vec{b})]. \end{aligned} \quad (3.11)$$

The Ω terms on the right hand side are neglected, since they are estimated to be suppressed relative to the $\vec{v}$ terms by the factor λ/R . Furthermore, by approximating $\vec{v} \approx \vec{V}$, we expect to make errors of the order of ℓ/R and a/R . We thus consider the following approximate form of the wave equations:

$$\begin{aligned} n^2\ddot{\vec{b}} - \Delta\vec{b} &= 2\nabla_{\vec{V}}\dot{\vec{b}}, \\ n^2\ddot{\vec{d}} - \Delta\vec{d} &= 2\nabla_{\vec{V}}\dot{\vec{d}}. \end{aligned} \quad (3.12)$$

Chapter 4

Maxwell's Equations in Cylindrical Waveguides

In this chapter, we give an outline of the calculations in the framework of wave optics and review the solution to the unperturbed problem.

4.1 Outline of the Calculation

We seek solutions to Maxwell's equations inside and outside an infinite cylinder of radius a , where all field components depend on time t and z (distance along the symmetry axis) as $e^{i(\omega t - \beta z)}$. All calculations are performed in corotating cylindrical coordinates (t, r, θ, z) . We choose as “fundamental fields”

$$f = \begin{pmatrix} D^z \\ B^z \end{pmatrix},$$

which determine all other field components uniquely. Symbolically, we write $F = F(f; \beta, \omega)$, where F is an abbreviation for the collection of fields $(\vec{E}, \vec{D}, \vec{B}, \vec{H})$. Due to the linearity of the medium, F depends linearly on f , which satisfies a wave equation of the form

$$n^2 \ddot{f} - \Delta f = g[f],$$

where g is a linear differential operator, which vanishes in the unperturbed case.

At the core-cladding interface ($r = a$), the field components $D^r, E^\theta, E^z, B^r, H^\theta$ and H^z are required to be continuous. As only four of these conditions are linearly independent, it suffices to require the continuity of D^r, E^z, B^r and H^z , for which we symbolically write

$$\llbracket F(f; \beta, \omega) \rrbracket = 0,$$

where $\llbracket \cdot \rrbracket$ is a linear measure of the jump height.

The problem at hand thus reduces to a wave equation for f on two disjoint domains (core and cladding), subject to the linear continuity constraint at the interface $r = a$.

4.1.1 Unperturbed Case

In the unperturbed case, where g vanishes, we consider the excitation of a single mode, i.e.

$$f^{(0)}(t, r, \theta, z) = h(r; \beta, \omega) e^{i(\omega t - \beta z + m\theta)}.$$

The two-component radial function $h(r; \beta, \omega)$ has four (complex) degrees of freedom, since it satisfies a homogeneous ordinary differential equation (ODE) of second order on two disjoint domains (core

and cladding) and is required to be regular at the origin and to decay at large distances. We choose linear coordinates α on the solution space and write $h = h_\alpha$, such that the four linearly independent continuity conditions are equivalent to a matrix equation of the form

$$M^{(0)}(\beta, \omega) \alpha = 0.$$

To obtain a nontrivial solution ($\alpha \neq 0$) one is led to the equation

$$\det M^{(0)}(\beta, \omega) = 0,$$

which provides the dispersion relation linking β and ω . Finally, the coefficients α are chosen to be in the kernel of the $M^{(0)}$. An explicit calculation shows that this kernel is one-dimensional.

4.1.2 First Order Corrections

To describe linear corrections due to Earth's rotation, we write

$$f(t, r, \theta, z) = [h_\alpha(r; \beta, \omega) + f^{(1)}(r, \theta; \beta, \omega)] e^{i(\omega t - \beta z + m\theta)},$$

where $f^{(1)}$ also depends on θ due to the angular dependence of g . Decomposing $f^{(1)}$ into a Fourier series in the angular variable, every coefficient function $f_k^{(1)}$ satisfies an inhomogeneous ODE of second order, subject to the same constraints of above, and is thus an element of a four-dimensional (complex) solution space. Due to the linear independence of the angular modes, continuity conditions hold for all modes separately. Further analysis requires a distinction between the main mode (no further dependence on θ , so $k = 0$) and the sidebands ($k \neq 0$).

For the main mode, $f_0^{(1)}$ can be chosen to be any particular solution to the ODE, since a homogeneous part can be removed by redefining α . Thus, the radial function of the main mode depends linearly on α , so the continuity condition of this mode is again equivalent to a homogeneous matrix equation

$$(M^{(0)}(\beta, \omega) + M^{(1)}(\beta, \omega)) \alpha = 0,$$

which determines the first order correction to the dispersion relation.

For the sidebands, the radial functions $f_k^{(1)}$ satisfy inhomogeneous ODEs, whose inhomogeneities are linear in the coefficients α . Parameterising homogeneous solutions linearly with parameters χ_k , $f_k^{(1)}$ becomes a function linear in α and χ_k , so the continuity condition is equivalent to an inhomogeneous matrix equation of the form

$$M_k \chi_k = N_k \alpha,$$

whose solution χ_k turns out to be unique.

4.2 Unperturbed Solution

In this section, we review the solutions of the problem when $\Omega = 0$, using the notation of [2] with $\phi = 0$ (i.e. no gravitational potential) and $c = \mu_0 = \epsilon_0 = 1$.

Since the dielectric is assumed to be nonmagnetic, we have $\mu = 1$ and thus $n^2 = \epsilon$. The relation between $\vec{D}$, $\vec{H}$ and $\vec{E}$, $\vec{B}$ are thus

$$\begin{aligned} \vec{D} &= n^2 \vec{E}, \\ \vec{B} &= \vec{H}. \end{aligned} \tag{4.1}$$

The refractive index is assumed to be piecewise constant, namely

$$n(r) = \begin{cases} n_1, & r < a, \\ n_2, & r > a, \end{cases}$$

where a denotes the radius of the dielectric.

We separate the dependence of the fields on t, z and θ via

$$\begin{aligned} \vec{E}(t, \vec{r}) &= \vec{E}(r) e^{i(\omega t - \beta z + m\theta)}, \\ \vec{B}(t, \vec{r}) &= \vec{B}(r) e^{i(\omega t - \beta z + m\theta)}. \end{aligned}$$

In the absence of a gravitational potential ($\phi = 0$), the constants U and W defined in [2, eqns. (4.17, 4.18)] reduce to

$$\begin{aligned} U^2 &= a^2(n_1^2\omega^2 - \beta^2), \\ W^2 &= a^2(\beta^2 - n_2^2\omega^2), \end{aligned}$$

with U and W positive. The function

$$f(r) = \begin{cases} J_m(Ur/a)/J_m(U), & r < a, \\ K_m(Wr/a)/K_m(W), & r > a, \end{cases}$$

is the continuous solution of the equations

$$\begin{aligned} f''(r) + \frac{1}{r}f'(r) + \left(\frac{U^2}{a^2} - \frac{m^2}{r^2}\right)f(r) &= 0 \quad \text{for } r < a, \\ f''(r) + \frac{1}{r}f'(r) - \left(\frac{W^2}{a^2} + \frac{m^2}{r^2}\right)f(r) &= 0 \quad \text{for } r > a, \end{aligned}$$

which is bounded when $r \rightarrow 0$ and has finite total energy. The z components of the fields are then given by

$$\begin{aligned} E^z(r) &= Af(r), \\ H^z(r) &= Bf(r), \end{aligned}$$

where the constants A and B are to be determined from continuity conditions.

The transverse components of the fields are obtained from the z components by

$$\begin{aligned} i\zeta E^x &= \beta\partial_x E^z + \omega\partial_y H^z, \\ i\zeta E^y &= \beta\partial_y E^z - \omega\partial_x H^z, \\ i\zeta H^x &= \beta\partial_x H^z - \omega n^2\partial_y E^z, \\ i\zeta H^y &= \beta\partial_y H^z + \omega n^2\partial_x E^z, \end{aligned}$$

or, when referred to a cylindrical orthonormal frame

$$\begin{aligned} i\zeta E^r &= \beta\partial_r E^z + \omega\partial_\theta H^z/r, \\ i\zeta E^\theta &= \beta\partial_\theta E^z/r - \omega\partial_r H^z, \\ i\zeta H^r &= \beta\partial_r H^z - \omega n^2\partial_\theta E^z/r, \\ i\zeta H^\theta &= \beta\partial_\theta H^z/r + \omega n^2\partial_r E^z, \end{aligned} \tag{4.2}$$

where we have set

$$\zeta = \omega^2 n^2 - \beta^2, \tag{4.3}$$

Together with the relations (4.1), these equations constitute the map $F(f; \beta, \omega)$, which was introduced in the previous section. Due to the special form of (4.1), the continuous fields E^z and H^z satisfy the same wave equation as the fundamental fields D^z and B^z . Thus, their continuity can be implemented immediately and the nontrivial continuity conditions reduce to two linearly independent constraints, which can be written in the following form [2, eq. (4.41)]:

$$\begin{pmatrix} im\beta \left(\frac{1}{U^2} + \frac{1}{W^2} \right) & -\omega \left(\frac{1}{U} \frac{J'_m(U)}{J_m(U)} + \frac{1}{W} \frac{K'_m(W)}{K_m(W)} \right) \\ \omega \left(\frac{n_1^2}{U} \frac{J'_m(U)}{J_m(U)} + \frac{n_2^2}{W} \frac{K'_m(W)}{K_m(W)} \right) & im\beta \left(\frac{1}{U^2} + \frac{1}{W^2} \right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \quad (4.4)$$

This is the anticipated matrix formulation of the continuity condition $\llbracket F \rrbracket = 0$.

4.2.1 Dispersion Relation

The requirement of (4.4) to admit nontrivial solutions for A and B leads to the dispersion relation

$$m^2 \frac{\beta^2}{\omega^2} \left(\frac{1}{U^2} + \frac{1}{W^2} \right)^2 = \left(\frac{1}{U} \frac{J'_m(U)}{J_m(U)} + \frac{1}{W} \frac{K'_m(W)}{K_m(W)} \right) \left(\frac{n_1^2}{U} \frac{J'_m(U)}{J_m(U)} + \frac{n_2^2}{W} \frac{K'_m(W)}{K_m(W)} \right).$$

To solve this equation numerically, it is useful to introduce the following parameterisation. Since $U^2 + W^2 = \rho^2$, with $\rho^2 := a^2 \omega^2 (n_1^2 - n_2^2)$, we can write $U = \rho \cos \varphi$ and $W = \rho \sin \varphi$, where the angle φ is constrained by $0 \leq \varphi \leq \pi/2$. This leads to the following form of β :

$$\beta^2 / \omega^2 = n_1^2 \sin^2 \varphi + n_2^2 \cos^2 \varphi.$$

Multiplying the dispersion relation by $J_m(U)^2 K_m(W)^2$, we are led to the equivalent condition

$$\begin{aligned} \kappa_m(\varphi) &= 4m^2 \csc^2(2\varphi) J_m^2 K_m^2 (n_1^2 \sec^2 \varphi + n_2^2 \csc^2 \varphi) \\ &\quad - \rho^2 [\sec(\varphi) J'_m K_m + \csc(\varphi) J_m K'_m] [n_1^2 \sec(\varphi) J'_m K_m + n_2^2 \csc(\varphi) J_m K'_m] \stackrel{!}{=} 0, \end{aligned}$$

where the arguments of the Bessel functions are suppressed for brevity.

Figures 4.1a, 4.1b and 4.1c show plots of κ for $n_1 = 1.47$, $n_2 = 1.46$ and various values of $a\omega$ and m . Clearly, the value of $a\omega$ (and thus ρ) determines which values m can take, e.g. $m \geq 2$ is not allowed for $a\omega = 13.5$ (see figure 4.1a). The plots show that the roots move to higher values of φ as ρ increases, which leads to normal dispersion, as shown in figure 4.1d. Note however, that for sufficiently large values of ρ , *secondary* roots appear close to $\varphi = 0$, see e.g. Figure 4.1c.

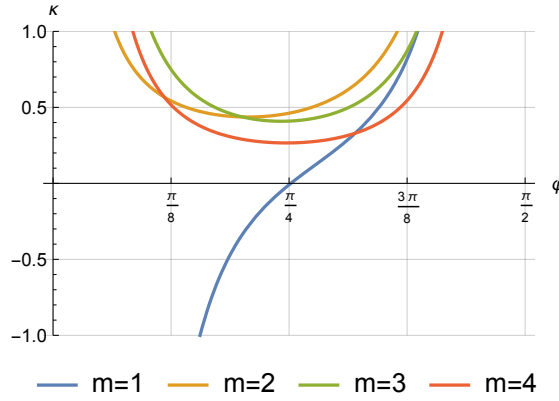
4.2.2 Generalisation

In light of the general formalism of the previous section, it will prove useful to generalise the expressions found above. To allow for discontinuous fundamental fields D^z and B^z , we define

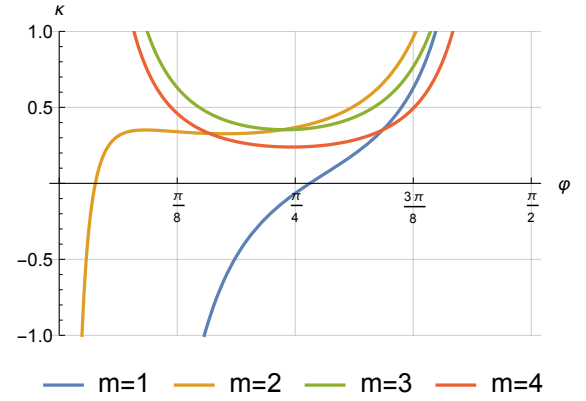
$$f_0(\alpha_1, \alpha_2; r) = \begin{cases} \alpha_1 J_m(Ur/a), & r < a, \\ \alpha_2 K_m(Wr/a), & r > a, \end{cases} \quad (4.5)$$

such that the solutions can be written as

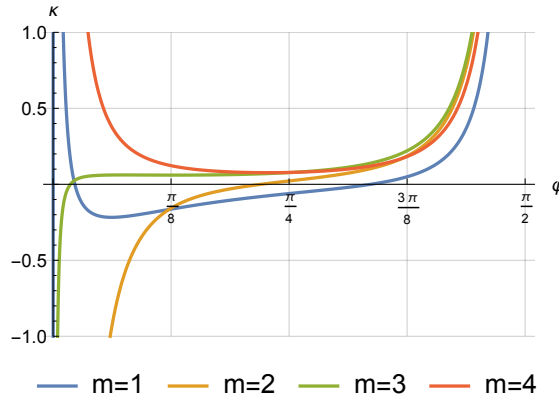
$$\begin{aligned} D^z(r) &= f_0(\alpha_1, \alpha_2; r), \\ B^r(r) &= f_0(\alpha_3, \alpha_4; r). \end{aligned} \quad (4.6)$$



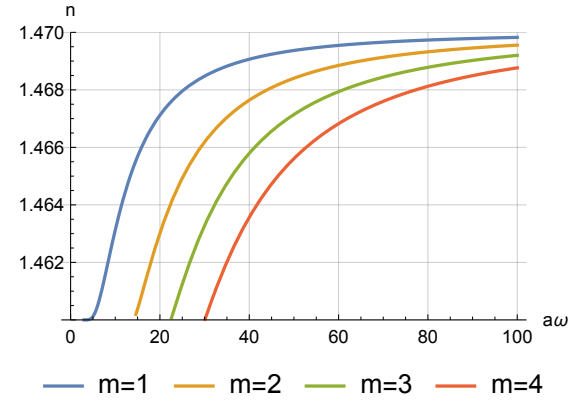
(a) For $a\omega = 13,5$, the root for $m = 1$ is clearly visible, while for $m > 1$ there are no solutions.



(b) For $a\omega = 14,5$, $m = 1$ and $m = 2$ admit different solutions.



(c) For $a\omega = 22,5$, the dispersion relation admits solutions for $m \leq 3$. Moreover, the curve for $m = 1$ features two roots.



(d) Frequency dependence of the effective refractive index on the frequency ω . Secondary roots are not shown.

Figure 4.1: The first three figures show the dependence of κ on the angle φ for various values of m and ρ . The functions have been multiplied by nonzero functions m for better visibility of the roots. The last figure shows the dispersion relation for different values of m .

The original expressions are thus obtained by setting

$$\alpha_1 = \frac{n_1^2 A}{J_m(U)}, \quad \alpha_2 = \frac{n_2^2 A}{K_m(W)}, \quad \alpha_3 = \frac{B}{J_m(U)}, \quad \alpha_4 = \frac{B}{K_m(W)}. \quad (4.7)$$

This form of the solution will be used in Chapter 5 to determine first order corrections due to Earth's rotation. In this calculation, it will prove useful to allow for parameters $\alpha = (\alpha_1, \dots, \alpha_4)$, which deviate from the unperturbed values given here.

Chapter 5

Main Corrections

In this chapter, we compute the first order corrections to the fields which arise from the approximate wave equation (3.12) and the modified relation between $\vec{e}$, $\vec{h}$ and $\vec{d}$, $\vec{b}$ given in (3.5).

5.1 Notation

To describe corrections to the unperturbed solution, we write

$$\begin{aligned}\vec{e} &= \vec{E} + \delta\vec{e}, \\ \vec{d} &= \vec{D} + \delta\vec{d}, \\ \vec{b} &= \vec{B} + \delta\vec{b}, \\ \vec{h} &= \vec{H} + \delta\vec{h},\end{aligned}\tag{5.1}$$

where $\vec{d}$, $\vec{b}$ etc. were defined in (3.2), $\vec{D}$, $\vec{B}$ etc. are the “unperturbed” fields given by (4.6) and $\delta\vec{d}$, $\delta\vec{b}$ etc. are the first corrections of order Ω . Note that $\vec{D}$, $\vec{H}$ are related to $\vec{E}$, $\vec{B}$ via $\vec{D} = n^2 \vec{E}$ and $\vec{B} = \vec{H}$, while $\vec{e}$, $\vec{h}$ are related to $\vec{d}$, $\vec{b}$ via the equations (3.5).

5.2 The Wave Equation for the z Component

In this section, we specialise the vectorial equations (3.12) to scalar wave equations for the z components of the fields $\vec{d}$ and $\vec{b}$.

From now on, we will use cylindrical coordinates (r, θ, z) and refer all tensorial objects to the orthonormal frame

$$\mathbf{e}_r = \frac{\partial}{\partial r}, \quad \mathbf{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \mathbf{e}_z = \frac{\partial}{\partial z},\tag{5.2}$$

whose coframe is given by

$$\mathbf{e}^r = dr, \quad \mathbf{e}^\theta = r d\theta, \quad \mathbf{e}^z = dz.\tag{5.3}$$

Since the wave equation (3.12) is independent of the spatial coordinate system, we may evaluate it in the coordinate system at hand. As stated in the outline, we are mainly interested in the wave equation for the z components of the fields for which the equation simplifies since the z component of the vector Laplacian coincides with the scalar Laplacian of the z component

$$(\Delta \vec{b})^z = \Delta(b^z),$$

and since the z component of the covariant derivative equals the partial derivative of the z component, i.e. $(\nabla_i \vec{b})^z = \partial_i b^z$. We thus obtain the scalar equations

$$\begin{aligned} n^2 \ddot{b}^z - \Delta b^z &= 2V(\dot{b}^z), \\ n^2 \ddot{d}^z - \Delta d^z &= 2V(\dot{d}^z), \end{aligned} \quad (5.4)$$

where $V(b^z)$ denotes the action of the vector field $\vec{V}$ on the scalar function b^z (i.e. the directional derivative). As shown in Appendix A, this is identical to the scalar wave equation for the functions d^z and b^z , where the wave operator is defined using the optical metric γ instead of the spacetime metric g , and where the approximation $\vec{v} \approx \vec{V}$ has been used.

Inserting the perturbative expansion (5.1), neglecting all terms of order $O(\Omega^2)$, and using the fact that D^z and B^z satisfy

$$\begin{aligned} n^2 \ddot{B}^z - \Delta B^z &= 0, \\ n^2 \ddot{D}^z - \Delta D^z &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} n^2 \delta \ddot{d}^z - \Delta \delta d^z &= 2V(\dot{D}^z), \\ n^2 \delta \ddot{b}^z - \Delta \delta b^z &= 2V(\dot{B}^z). \end{aligned} \quad (5.5)$$

5.3 The Radial Equations

Summarising, we need to analyse the following differential equations for the z components of $\delta \vec{d}$ and $\delta \vec{b}$:

$$\begin{aligned} n^2 \delta \ddot{d}^z - \Delta \delta d^z &= 2i\omega V(D^z), \\ n^2 \delta \ddot{b}^z - \Delta \delta b^z &= 2i\omega V(B^z), \end{aligned} \quad (5.6)$$

where the fields D^z and B^z are given by

$$\begin{aligned} D^z(t, r, \theta, z) &= f_0(\alpha_1, \alpha_2; r) e^{i(\omega t - \beta z + m\theta)}, \\ B^z(t, r, \theta, z) &= f_0(\alpha_3, \alpha_4; r) e^{i(\omega t - \beta z + m\theta)}. \end{aligned}$$

Here, the parameter β and the coefficients $\alpha_1, \dots, \alpha_4$ *do not necessarily* coincide with the unperturbed parameters, which would have been obtained when $\Omega = 0$. Note that in doing so, one modifies the right hand side of the wave equations by terms of order $O(\Omega^2)$, which we assume to be negligible. Note also that V on the right hand side depends on the angle θ via

$$\begin{aligned} V^r &= V^x \cos \theta + V^y \sin \theta, \\ V^\theta &= V^x \sin \theta - V^y \cos \theta, \end{aligned}$$

where V^x, V^y (and V^z) are constants. Thus, the θ -dependence of the source term is not given by $e^{im\theta}$ alone; instead, there are *sidebands* with an angular dependence of the form $e^{i(m\pm 1)\theta}$.

Defining the constants $V^\pm$ by

$$2V^\pm = iV^x \pm V^y$$

and decomposing all fields as a Fourier series in θ of the form

$$X = \sum_{n=-1}^{+1} \langle X \rangle_n e^{i(m+n)\theta}, \quad (5.7)$$

one has e.g.

$$\langle V^r \rangle_{\pm} = -iV^{\pm}, \quad \langle V^{\theta} \rangle_{\pm} = \pm V^{\pm}, \quad (5.8)$$

so one obtains for $w^i = D^z$ or $w^i = B^z$

$$\begin{aligned} \langle V(w^i) \rangle_0 &= -i\beta V^z w^i, \\ \langle V(w^i) \rangle_{\pm} &= -iV^{\pm} c_m^{\pm} w^i, \end{aligned}$$

where we have set

$$c_m^{\pm} = \partial_r \mp \frac{m}{r}. \quad (5.9)$$

For the perturbed fields $\delta d^z, \delta b^z$ we use the ansatz

$$\begin{aligned} \delta d^z(t, r, \theta, z) &= [\delta d_0^z(r) + \delta d_+^z(r)e^{+i\theta} + \delta d_-^z(r)e^{-i\theta}] e^{i(\omega t - \beta z + m\theta)}, \\ \delta b^z(t, r, \theta, z) &= [\delta b_0^z(r) + \delta b_+^z(r)e^{+i\theta} + \delta b_-^z(r)e^{-i\theta}] e^{i(\omega t - \beta z + m\theta)}, \end{aligned} \quad (5.10)$$

which leads to the following set of equations for the radial functions:

$$\begin{aligned} L_m^{\zeta} \delta b_0^z(r) &= -2\beta\omega V^z B^z(r), \\ L_m^{\zeta} \delta d_0^z(r) &= -2\beta\omega V^z D^z(r), \\ L_{m\pm 1}^{\zeta} \delta b_{\pm}^z(r) &= -2\omega V^{\pm} c_m^{\pm} B^z(r), \\ L_{m\pm 1}^{\zeta} \delta d_{\pm}^z(r) &= -2\omega V^{\pm} c_m^{\pm} D^z(r), \end{aligned} \quad (5.11)$$

where $\zeta = n^2\omega^2 - \beta^2$ as before and

$$L_{\nu}^{\zeta} f(r) := f''(r) + \frac{1}{r} f'(r) + \left(\zeta - \frac{\nu^2}{r^2} \right) f(r)$$

is either an ordinary or modified Bessel operator, depending on whether $\zeta > 0$ or $\zeta < 0$. Since the equations for $\vec{d}$ and $\vec{b}$ coincide, it suffices to consider those for $\vec{d}$. Using the expressions (4.6), we obtain equations of the form

$$\begin{aligned} L_m^{\zeta} p_0(\alpha_1, \alpha_2; r) &= a^{-2} f_0(\alpha_1, \alpha_2; r), \\ L_{m\pm 1}^{\zeta} p_{\pm}(\alpha_1, \alpha_2; r) &= a^{-1} c_m^{\pm} f_0(\alpha_1, \alpha_2; r), \end{aligned} \quad (5.12)$$

where the coefficients in front of the source terms have been dropped momentarily – they can be added later due to the linearity of the equations. The powers of a were chosen such that the functions $p_0(\alpha_i, \alpha_j; r)$ and $p_{\pm}(\alpha_i, \alpha_j; r)$ have the same dimension as $f_0(\alpha_i, \alpha_j; r)$, which, in turn, has the same dimension as the coefficients α_i, α_j . The homogeneous solutions of interest (we require the functions to be finite at the origin and to decay sufficiently fast at large distances) to the first equation are given by (4.5). The corresponding homogeneous solutions for the sidebands are then given by

$$f_{\pm}(\chi_1, \chi_2; r) = \begin{cases} \mp U \chi_1 J_{m\pm 1}(Ur/a), & r < a, \\ -W \chi_2 K_{m\pm 1}(Wr/a), & r > a. \end{cases} \quad (5.13)$$

The factors $\mp U$ and $-W$ were chosen for convenience.

5.4 The Wave Equation in the Core

In the core ($r < a$) we have $\zeta = U^2/a^2 > 0$, so L_ν^ζ is an ordinary Bessel operator (as opposed to the modified Bessel operator). According to (4.5) f_0 is given by

$$f_0(\alpha_1, \alpha_2; r < a) = \alpha_1 J_m(Ur/a).$$

Using the identity

$$c_m^\pm J_m(\lambda r) \equiv \partial_r J_m(\lambda r) \mp \frac{m}{r} J_m(\lambda r) = \mp \lambda J_{m\pm 1}(\lambda r), \quad (5.14)$$

cf. [1, p. 361, Equation 9.1.27], the radial equations take the form

$$\begin{aligned} L_m^{U^2/a^2} p_0(\alpha_1, \alpha_2; r < a) &= \frac{\alpha_1}{a^2} J_m(Ur/a), \\ L_{m\pm 1}^{U^2/a^2} p_\pm(\alpha_1, \alpha_2; r < a) &= \mp U \frac{\alpha_1}{a^2} J_{m\pm 1}(Ur/a). \end{aligned}$$

As shown in Appendix B (compare (B.4)), the general solution to $L_\nu^\zeta f(r) = g(r)$ is given by

$$\begin{aligned} f(r) &= J_\nu(Ur/a) \left[c_1 - \frac{\pi}{2} \int_{r_1}^r Y_\nu(Ur'/a) g(r') r' dr' \right] \\ &+ Y_\nu(Ur/a) \left[c_2 + \frac{\pi}{2} \int_{r_2}^r J_\nu(Ur'/a) g(r') r' dr' \right]. \end{aligned}$$

We choose $r_1 = a$ such that the first integral does not contribute at the core-cladding interface, where we will require continuity of some field components. Furthermore, choosing $r_2 = 0$, we set $c_2 = 0$ to obtain a solution free of singularities, provided that all other terms are sufficiently well behaved as $r \rightarrow 0$. The constant c_1 multiplies a homogeneous solution to the Bessel equation. Note that for $\nu = m$, this has the same form as the unperturbed field. Since we are only interested in the sum $\vec{d} = \vec{D} + \delta\vec{d}$ (and likewise for the magnetic field), we may set $c_1 = 0$ by allowing α_1 (α_3 for the magnetic field) to deviate from the unperturbed value specified in equation (4.7). For the other modes ($\nu = m \pm 1$) we choose to add a homogeneous solution separately and are thus free to set $c_1 = 0$ for those modes as well. We thus obtain the solution

$$\begin{aligned} p_0(\alpha_1, \alpha_2, r < a) &= \alpha_1 \frac{\pi}{2} [J_m(Ur/a) \Gamma_0(r/a) + Y_m(Ur/a) \Delta_0(r/a)], \\ p_\pm(\alpha_1, \alpha_2, r < a) &= \mp \alpha_1 U \frac{\pi}{2} [J_{m\pm 1}(Ur/a) \Gamma_\pm(r/a) + Y_{m\pm 1}(Ur/a) \Delta_\pm(r/a)], \end{aligned} \quad (5.15)$$

where we have introduced the dimensionless functions

$$\begin{aligned} \Gamma_n(z) &= \int_z^1 \xi Y_{m+n}(U\xi) J_{m+n}(U\xi) d\xi, \\ \Delta_n(z) &= \int_0^z \xi J_{m+n}(U\xi)^2 d\xi, \end{aligned} \quad (5.16)$$

and used the abbreviations $\Gamma_\pm = \Gamma_{\pm 1}$ etc. These integrals are given explicitly in Appendix C. Note that the derivatives of these particular solutions are given by

$$\begin{aligned} \partial_r p_0(\alpha_1, \alpha_2, r < a) &= U \frac{\alpha_1}{a} \frac{\pi}{2} [J'_m(Ur/a) \Gamma_0(r/a) + Y'_m(Ur/a) \Delta_0(r/a)], \\ \partial_r p_\pm(\alpha_1, \alpha_2, r < a) &= \mp U^2 \frac{\alpha_1}{a} \frac{\pi}{2} [J'_{m\pm 1}(Ur/a) \Gamma_\pm(r/a) + Y'_{m\pm 1}(Ur/a) \Delta_\pm(r/a)], \end{aligned}$$

where the terms containing derivatives of Δ and Γ cancel. For later use, we compute the limits of these functions and their first derivatives as r approaches a from below:

$$\begin{aligned} \lim_{r \nearrow a} p_0(\alpha_1, \alpha_2; r) &= \alpha_1 \frac{\pi}{2} Y_m(U) \Delta_0(1), \\ \lim_{r \nearrow a} p_{\pm}(\alpha_1, \alpha_2; r) &= \mp \alpha_1 U \frac{\pi}{2} Y_{m\pm 1}(U) \Delta_{\pm}(1), \\ \lim_{r \nearrow a} \partial_r p_0(\alpha_1, \alpha_2; r) &= \alpha_1 \frac{U}{a} \frac{\pi}{2} Y'_m(U) \Delta_0(1), \\ \lim_{r \nearrow a} \partial_r p_{\pm}(\alpha_1, \alpha_2; r) &= \mp \alpha_1 \frac{U^2}{a} \frac{\pi}{2} Y'_{m\pm 1}(U) \Delta_{\pm}(1). \end{aligned} \tag{5.17}$$

5.5 The Wave Equation in the Cladding

In the cladding ($r > a$) we have $\zeta = -W^2/a^2 < 0$ so $L_{\nu}^{\zeta} = \tilde{L}_{\nu}^{W^2/a^2}$ is a modified Bessel operator defined as

$$\tilde{L}_{\nu}^{\lambda^2} f(r) := f''(r) + \frac{1}{r} f'(r) - \left(\lambda^2 + \frac{\nu^2}{r^2} \right) f(r).$$

According to (4.6) the inhomogeneities assume the form

$$f_0(\alpha_1, \alpha_2; r > a) = \alpha_2 K_m(Wr/a).$$

By virtue of the identity

$$c_m^{\pm} K_m(\lambda r) \equiv \partial_r K_m(\lambda r) \mp \frac{m}{r} K_m(\lambda r) = -\lambda K_{m\pm 1}(\lambda r), \tag{5.18}$$

cf. [I, p. 376, Equation 9.6.26], one obtains the differential equations

$$\begin{aligned} \tilde{L}_m^{W^2/a^2} p_0(\alpha_1, \alpha_2; r > a) &= \frac{\alpha_2}{a^2} K_m(Wr/a), \\ \tilde{L}_{m\pm 1}^{W^2/a^2} p_{\pm}(\alpha_1, \alpha_2; r > a) &= -W \frac{\alpha_2}{a^2} K_{m\pm 1}(Wr/a). \end{aligned}$$

As shown in Appendix B, the general solution to $\tilde{L}_{\nu}^{W^2/a^2} f(r) = g(r)$ is given by

$$\begin{aligned} f(r) &= K_{\nu}(Wr/a) \left[c_1 - \int_{r_1}^r I_{\nu}(Wr'/a) g(r') r' dr' \right] \\ &\quad + I_{\nu}(Wr/a) \left[c_2 + \int_{r_2}^r K_{\nu}(Wr'/a) g(r') r' dr' \right]. \end{aligned}$$

As before, it is convenient to set $r_1 = a$. Furthermore, we choose $r_2 = \infty$ and $c_2 = 0$. Again, we choose $c_1 = 0$ for the m -mode by allowing α_2 and α_3 to differ from the unperturbed values given in equation (4.7) and also set $c_1 = 0$ for the sidebands by adding homogeneous solutions separately. We are thus led to the solutions

$$\begin{aligned} p_0(\alpha_1, \alpha_2; r > a) &= -\alpha_2 [K_m(Wr/a) \Sigma_0(r/a) + I_m(Wr/a) T_0(r/a)], \\ p_{\pm}(\alpha_1, \alpha_2; r > a) &= \alpha_2 W [K_{m\pm 1}(Wr/a) \Sigma_{\pm}(r/a) + I_{m\pm 1}(Wr/a) T_{\pm}(r/a)]. \end{aligned} \tag{5.19}$$

Here we have set

$$\begin{aligned} \Sigma_n(z) &= \int_1^z \xi I_{m+n}(W\xi) K_{m+n}(W\xi) d\xi, \\ T_n(z) &= \int_z^{\infty} \xi K_{m+n}(W\xi)^2 d\xi. \end{aligned} \tag{5.20}$$

Again, explicit forms of the integrals are given in Appendix C.

Similar to the discussion of the region $r < a$, we compute the limits of these functions and their first derivatives as r approaches a from above. The first derivatives of the solutions are given by

$$\begin{aligned}\partial_r p_0(\alpha_1, \alpha_2; r > a) &= -\alpha_2 a W [K'_m(Wr/a)\Sigma_0(r/a) + I'_m(Wr/a)T_0(r/a)] , \\ \partial_r p_{\pm}(\alpha_1, \alpha_2; r > a) &= \alpha_2 W^2 [K'_{m\pm 1}(Wr/a)\Sigma_{\pm}(r/a) + I'_{m\pm 1}(Wr/a)T_{\pm}(r/a)] ,\end{aligned}$$

where the terms involving derivatives of Σ and T cancel. The limits are then found to be given by

$$\begin{aligned}\lim_{r \searrow a} p_0(\alpha_1, \alpha_2; r) &= -\alpha_2 I_m(W)T_0(1), \\ \lim_{r \searrow a} p_{\pm}(\alpha_1, \alpha_2; r) &= +\alpha_2 W I_{m\pm 1}(W)T_{\pm}(1), \\ \lim_{r \searrow a} \partial_r p_0(\alpha_1, \alpha_2; r) &= -\alpha_2 \frac{W}{a} I'_m(W)T_0(1), \\ \lim_{r \searrow a} \partial_r p_{\pm}(\alpha_1, \alpha_2; r) &= +\alpha_2 \frac{W^2}{a} I'_{m\pm 1}(W)T_{\pm}(1).\end{aligned}\tag{5.21}$$

5.6 Collection of Intermediate Results

In this section, we summarise the main expressions for the z components of the fields $\vec{d} = \vec{D} + \delta\vec{d}$ and $\vec{b} = \vec{B} + \delta\vec{b}$.

The fields D^z and B^z are given by

$$\begin{aligned}D^z(t, r, \theta, z) &= f_0(\alpha_1, \alpha_2; r) e^{i(\omega t - \beta z + m\theta)}, \\ B^z(t, r, \theta, z) &= f_0(\alpha_3, \alpha_4; r) e^{i(\omega t - \beta z + m\theta)},\end{aligned}$$

cf. (4.6), where f_0 is defined in (4.5) and the coefficients $\alpha_1, \dots, \alpha_4, \beta$ may differ from their unperturbed values.

According to (5.10), the fields δd^z and δb^z have three Fourier components in the angular variable θ , whose radial parts are given by

$$\begin{aligned}\delta d_0^z(r) &= -2a^2 \beta \omega V^z p_0(\alpha_1, \alpha_2; r), \\ \delta b_0^z(r) &= -2a^2 \beta \omega V^z p_0(\alpha_3, \alpha_4; r), \\ \delta d_{\pm}^z(r) &= -2a \omega V^{\pm} p_{\pm}(\alpha_1, \alpha_2; r) + f_{\pm}(\chi_1^{\pm}, \chi_2^{\pm}; r), \\ \delta b_{\pm}^z(r) &= -2a \omega V^{\pm} p_{\pm}(\alpha_3, \alpha_4; r) + f_{\pm}(\chi_3^{\pm}, \chi_4^{\pm}; r).\end{aligned}\tag{5.22}$$

According to the recursion relation (5.14) and (5.18), the homogeneous solutions $f_{\pm}$ are related to f_0 by

$$f_{\pm}(\alpha_i, \alpha_j; r) = a c_m^{\pm} f_0(\alpha_i, \alpha_j; r),$$

which was the reason why the coefficients $\mp U$ and $-W$ were introduced in (5.13).

The particular solutions p_0 and $p_{\pm}$ are given by

$$\begin{aligned}p_0(\alpha_1, \alpha_2; r) &= \begin{cases} +\alpha_1 \frac{\pi}{2} [J_m(Ur/a)\Gamma_0(r/a) + Y_m(Ur/a)\Delta_0(r/a)] & r < a, \\ -\alpha_2 [K_m(Wr/a)\Sigma_0(r/a) + I_m(Wr/a)T_0(r/a)] & r > a, \end{cases} \\ p_+(\alpha_1, \alpha_2; r) &= \begin{cases} -\alpha_1 U \frac{\pi}{2} [J_{m+1}(Ur/a)\Gamma_+(r/a) + Y_{m+1}(Ur/a)\Delta_+(r/a)] & r < a, \\ +\alpha_2 W [K_{m+1}(Wr/a)\Sigma_+(r/a) + I_{m+1}(Wr/a)T_+(r/a)] & r > a, \end{cases} \\ p_-(\alpha_1, \alpha_2; r) &= \begin{cases} +\alpha_1 U \frac{\pi}{2} [J_{m-1}(Ur/a)\Gamma_-(r/a) + Y_{m-1}(Ur/a)\Delta_-(r/a)] & r < a, \\ +\alpha_2 W [K_{m-1}(Wr/a)\Sigma_-(r/a) + I_{m-1}(Wr/a)T_-(r/a)] & r > a, \end{cases}\end{aligned}\tag{5.23}$$

cf. (5.15) for the expressions with $r < a$ and (5.19) for those with $r > a$. The functions $\Gamma, \Delta, \Sigma, T$ are given explicitly in equation (C.1). The limiting behaviour of these particular solutions and their first derivatives as r approaches a are stated in (5.17) and (5.21).

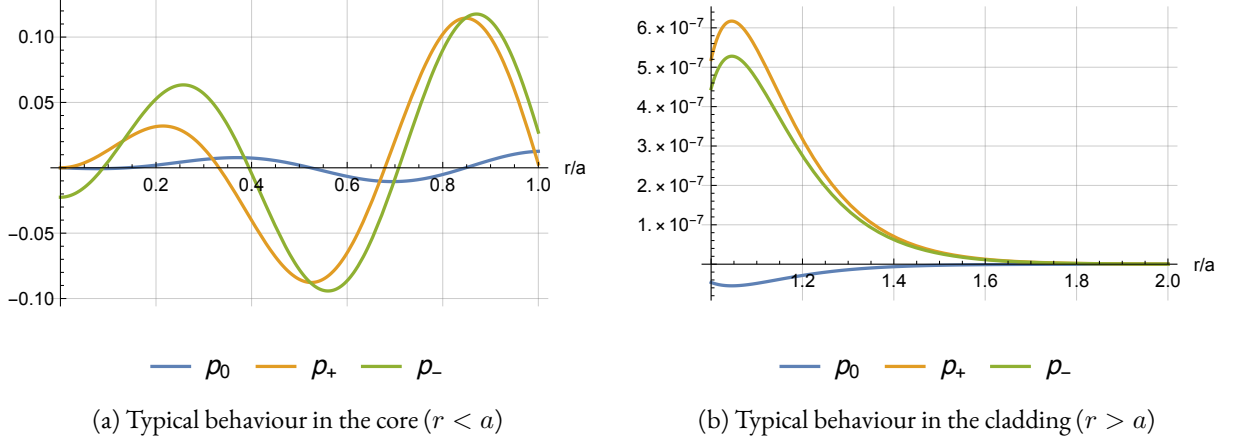


Figure 5.1: Plots of the functions p_0 and $p_{\pm}$ for $m = 1$ and $U = W = 10$.

The functions p_0 and $p_{\pm}$ are plotted in figure 5.1. Similar to the unperturbed solutions, the functions show oscillatory behaviour in the core and decay exponentially in the cladding. As expected, the functions are regular at the origin and decay rapidly in the cladding.

5.7 Compatibility of Approximations

Until now we were only concerned with the z components of the fields. In all calculations we have omitted terms quadratic in Ω and we have neglected some terms leading to relative errors of the order of the a/R or ℓ/R , where a denotes the radius of the dielectric, ℓ its length and R denotes Earth's radius.

In the next section we will compute the remaining components using similar approximations. There is however a subtlety: as we will now show, these approximations cannot be discussed term by term, since the resulting expressions must satisfy two equations, which relate three field components each.

We shall later require the continuity of six field components at $r = a$ (the core-cladding interface) due to the following conditions:

1. The non-existence of magnetic surface charges and surface currents is equivalent to the continuity of b^r , e^θ and e^z ; and
2. The absence of electric surface charges and surface currents is equivalent to continuity of d^r , h^θ and h^z .

Clearly, the continuity conditions must hold for all Fourier modes separately. Note, however, that these conditions constitute six linear equations in the four coefficients which parameterise the solutions in each mode. In order to obtain nontrivial solutions, two continuity conditions must therefore be expressible in terms of the remaining four equations. The existence of such a relation follows in fact

from the r -components of (3.4),

$$i\omega b^r + \frac{1}{r}\partial_\theta e^z - \partial_z e^\theta = 0, \quad (5.24)$$

$$i\omega d^r - \frac{1}{r}\partial_\theta h^z + \partial_z h^\theta = 0. \quad (5.25)$$

One sees that, for solutions as considered in this work, if (5.24) holds everywhere, and if any two of the fields appearing in (5.24) are continuous at the core-cladding interface, then the third one will also be continuous. A similarly statement holds for (5.25).

In order to obtain consistent equations, it is therefore necessary to make approximations which are compatible with these two equations.

The components e^z, h^z are determined by (3.5) whereas the components $b^r, d^r, h^\theta, e^\theta$ will be computed from (3.6). The approximations made in either set of equations thus determines the approximations in the other one.

5.8 The Transverse Components of d and b

We use the equations (3.4) to express the “transverse” r and θ components of the fields in terms of the “longitudinal” z component. Substituting (5.1), one obtains to first order in Ω

$$\begin{aligned} n^2 \delta \vec{b} + \nabla \times \delta \vec{d} - [\vec{v}, \vec{H}] &= 0, \\ \delta \vec{d} - \nabla \times \delta \vec{b} - [\vec{v}, \vec{E}] &= 0, \end{aligned} \quad (5.26)$$

where we have used $\mu = 1, \varepsilon = n^2, \vec{D} = n^2 \vec{E}, \vec{B} = \vec{H}$ and fact that n^2 is locally constant. If all field components depend on t and z via $\exp(i(\omega t - \beta z))$, the differential equations for the transverse components reduce to the following set of linear algebraic equations

$$\begin{aligned} i n^2 \omega \delta b^r + i \beta \delta d^\theta &= [\vec{v}, \vec{H}]^r - \partial_\theta \delta d^z / r, \\ i n^2 \omega \delta b^\theta - i \beta \delta d^r &= [\vec{v}, \vec{H}]^\theta + \partial_r \delta d^z, \\ i \omega \delta d^r - i \beta \delta b^\theta &= [\vec{v}, \vec{E}]^r + \partial_\theta \delta b^z / r, \\ i \omega \delta d^\theta + i \beta \delta b^r &= [\vec{v}, \vec{E}]^\theta - \partial_r \delta b^z, \end{aligned}$$

with the solution

$$\begin{aligned} i \zeta \delta d^r &= n^2 \omega ([\vec{v}, \vec{E}]^r + \partial_\theta \delta b^z / r) + \beta ([\vec{v}, \vec{H}]^\theta + \partial_r \delta d^z), \\ i \zeta \delta d^\theta &= n^2 \omega ([\vec{v}, \vec{E}]^\theta - \partial_r \delta b^z) - \beta ([\vec{v}, \vec{H}]^r - \partial_\theta \delta d^z / r), \\ i \zeta \delta b^r &= \omega ([\vec{v}, \vec{H}]^r - \partial_\theta \delta d^z / r) - \beta ([\vec{v}, \vec{E}]^\theta - \partial_r \delta b^z), \\ i \zeta \delta b^\theta &= \omega ([\vec{v}, \vec{H}]^\theta + \partial_r \delta d^z) + \beta ([\vec{v}, \vec{E}]^r + \partial_\theta \delta b^z / r), \end{aligned} \quad (5.27)$$

where $\zeta = n^2 \omega^2 - \beta^2$. Note that for any vector field $\vec{w}$ it holds that

$$[\vec{v}, \vec{w}] = \nabla_{\vec{v}} \vec{w} - \vec{\Omega} \times \vec{w},$$

where ∇ denotes the Levi-Civita covariant derivative with respect to the spatial metric. Indeed, in Cartesian coordinates one finds

$$[\vec{v}, \vec{w}]^i = v^j \partial_j w^i - w^l \partial_l (\epsilon^{ijk} \Omega^j x^k) = (\nabla_{\vec{v}} \vec{w} - \vec{\Omega} \times \vec{w})^i.$$

Due to the tensorial nature of this equation, the result holds in all coordinate systems.

As a first approximation, we write

$$[\vec{v}, \vec{w}] \approx \nabla_{\vec{v}} \vec{w}.$$

The approximation $\nabla_{\vec{v}} \vec{w} \approx \nabla_{\vec{v}} \vec{w}$ corresponds to $\vec{x} \approx \vec{R}$, where relative errors are expected to be of order a/R or ℓ/R . Moreover, neglecting $\vec{\Omega} \times \vec{w}$ compared to the first term is expected to produce relative errors of the order of λ/R , cf. the discussion in Section 3.5.

Defining the connection one-forms

$$\omega^a_b(X) = \mathbf{e}^a(\nabla_X \mathbf{e}_b),$$

we may decompose the covariant derivative as

$$(\nabla_X Y)^a = X(Y^a) + \omega^a_b(X)Y^b.$$

We find the only non-zero connection forms to be

$$\omega^{\theta}_r = +\frac{1}{r}\mathbf{e}^{\theta}, \quad \omega^r_{\theta} = -\frac{1}{r}\mathbf{e}^{\theta}.$$

Since the first order perturbation fields have three θ modes, e.g.

$$\delta d^z = e^{i(\omega t - \beta z + m\theta)} \sum_{n=-1}^{+1} \delta d_n(r) e^{in\theta},$$

we decompose all expressions in a Fourier series in θ using the notation introduced in (5.7). As before, we use the abbreviation $\langle X \rangle_{\pm} = \langle X \rangle_{\pm 1}$. Note that the cylindrical frame components of E^{θ} , E^r , B^{θ} and B^r of the unperturbed fields have vanishing $m \pm 1$ components by (4.2).

Using the notation (5.8) (and a similar notation for $\vec{\Omega}$), we obtain the following decomposition of the Lie bracket $[\vec{w}, \vec{v}]$, where $\vec{w}$ is any unperturbed field:

$$\begin{aligned} \langle [\vec{w}, \vec{v}]^r \rangle_0 &= i\beta V^z w^r, \\ \langle [\vec{w}, \vec{v}]^{\theta} \rangle_0 &= i\beta V^z w^{\theta}, \\ \langle [\vec{w}, \vec{v}]^z \rangle_0 &= i\beta V^z w^z, \\ \langle [\vec{w}, \vec{v}]^r \rangle_{\pm} &= iV^{\pm} c_m^{\pm} w^r \pm V^{\pm} w^{\theta}/r, \\ \langle [\vec{w}, \vec{v}]^{\theta} \rangle_{\pm} &= iV^{\pm} c_m^{\pm} w^{\theta} \mp V^{\pm} w^r/r, \end{aligned} \tag{5.28}$$

where we have used the operator $c_m^{\pm} = \partial_r \mp m/r$, defined in (5.9).

Combining (5.27) and (5.28), one obtains the following expressions for the m modes of the electric field $\vec{d}$ and the magnetic field $\vec{h}$

$$\begin{aligned} i\zeta \delta d_0^r &= \beta \partial_r \delta d_0^z + i\omega \frac{m}{r} n^2 \delta b_0^z - iV^z \beta (\beta H^{\theta} + \omega D^r), \\ i\zeta \delta d_0^{\theta} &= i\beta \frac{m}{r} \delta d_0^z - \omega n^2 \partial_r \delta b_0^z + iV^z \beta (\beta H^r - \omega D^{\theta}), \\ i\zeta \delta b_0^r &= \beta \partial_r \delta b_0^z - i\omega \frac{m}{r} \delta d_0^z + iV^z \beta (\beta E^{\theta} - \omega H^r), \\ i\zeta \delta b_0^{\theta} &= i\beta \frac{m}{r} \delta b_0^z + \omega \partial_r \delta d_0^z - iV^z \beta (\beta E^r + \omega H^{\theta}), \end{aligned} \tag{5.29}$$

and similarly for the $m \pm 1$ modes

$$\begin{aligned}
 i\zeta\delta d_{\pm}^r &= \beta\partial_r\delta d_{\pm}^z + i\omega\frac{m\pm 1}{r}n^2\delta b_{\pm}^z - iV^{\pm}c_m^{\pm}(\beta H^{\theta} + \omega D^r) \pm \frac{V^{\pm}}{r}(\beta H^r - \omega D^{\theta}), \\
 i\zeta\delta d_{\pm}^{\theta} &= i\beta\frac{m\pm 1}{r}\delta d_{\pm}^z - \omega n^2\partial_r\delta b_{\pm}^z + iV^{\pm}c_m^{\pm}(\beta H^r - \omega D^{\theta}) \pm \frac{V^{\pm}}{r}(\beta H^{\theta} + \omega D^r), \\
 i\zeta\delta b_{\pm}^r &= \beta\partial_r\delta b_{\pm}^z - i\omega\frac{m\pm 1}{r}\delta d_{\pm}^z + iV^{\pm}c_m^{\pm}(\beta E^{\theta} - \omega H^r) \mp \frac{V^{\pm}}{r}(\beta E^r + \omega H^{\theta}), \\
 i\zeta\delta b_{\pm}^{\theta} &= i\beta\frac{m\pm 1}{r}\delta b_{\pm}^z + \omega\partial_r\delta d_{\pm}^z - iV^{\pm}c_m^{\pm}(\beta E^r + \omega H^{\theta}) \mp \frac{V^{\pm}}{r}(\beta E^{\theta} - \omega H^r).
 \end{aligned} \tag{5.30}$$

5.9 The Field Components of e and h

We rewrite equation (3.5) with $\mu = 1$ and $\varepsilon = n^2$ in the form

$$\begin{aligned}
 n^2\vec{e} &= \vec{d} + \vec{v} \times \vec{b}, \\
 n^2\vec{h} &= n^2\vec{b} - \vec{v} \times \vec{d},
 \end{aligned}$$

where $\vec{v} = \vec{\Omega} \times \vec{x}$. Substituting the perturbative expansion (5.1) and using $\vec{D} = n^2\vec{E}$ as well as $\vec{B} = \vec{H}$, one obtains to first order in Ω

$$\begin{aligned}
 n^2\delta\vec{e} &= \delta\vec{d} + \vec{v} \times \vec{B}, \\
 \delta\vec{h} &= \delta\vec{b} - \vec{v} \times \vec{E}.
 \end{aligned} \tag{5.31}$$

We compute the θ and z components of these fields, which are required to be continuous. Using the approximation $\vec{v} \approx \vec{V}$ (in accordance with $\vec{x} = \vec{R} + \vec{r} \approx \vec{R}$), one finds for the m modes

$$\begin{aligned}
 n^2\delta e_0^{\theta} &= \delta d_0^{\theta} + V^z B^r, \\
 n^2\delta e_0^z &= \delta d_0^z, \\
 \delta h_0^{\theta} &= \delta b_0^{\theta} - V^z E^r, \\
 \delta h_0^z &= \delta b_0^z,
 \end{aligned} \tag{5.32}$$

and for the $m \pm 1$ modes

$$\begin{aligned}
 n^2\delta e_{\pm}^{\theta} &= \delta d_{\pm}^{\theta} + iV^{\pm}B^z, \\
 n^2\delta e_{\pm}^z &= \delta d_{\pm}^z - iV^{\pm}(B^{\theta} \mp iB^r), \\
 \delta h_{\pm}^{\theta} &= \delta b_{\pm}^{\theta} - iV^{\pm}E^z, \\
 \delta h_{\pm}^z &= \delta b_{\pm}^z + iV^{\pm}(E^{\theta} \mp iE^r).
 \end{aligned} \tag{5.33}$$

5.10 Consistency of the Approximations

In this section, we verify that the approximations made are consistent with the equations (5.24)-(5.25). Due to the electromagnetic symmetry, it suffices to verify the equation

$$i\omega d^r - \frac{1}{r}\partial_{\theta}h^z + \partial_z h^{\theta} = 0. \tag{5.34}$$

Let us start by showing that if all fields depend on z only through $e^{-i\beta z}$, and if the field components are related to each other as in the unperturbed case, i.e. the r and θ components are related to

the z components by (4.2) (or equivalently by (5.27) without the Lie bracket terms) and by $\vec{h} = \vec{b}$, $\vec{d} = n^2 \vec{e}$, then (5.24)-(5.25) hold.

Hence, we consider the following subset of the exact Maxwell's equations with $\Omega \equiv 0$:

$$\begin{aligned} i\zeta D^r &= n^2 \omega \frac{1}{r} \partial_\theta B^z + \beta \partial_r D^z, \\ i\zeta B^\theta &= \omega \partial_r D^z + \frac{1}{r} \beta \partial_\theta B^z, \\ H^\theta &= B^\theta, \\ H^z &= B^z, \end{aligned} \tag{5.35}$$

with ζ defined in (4.3), where all fields depend on z only as $e^{-i\beta z}$. We have

$$\begin{aligned} i\zeta(i\omega D^r + \partial_z H^\theta) &= i\zeta(i\omega D^r - i\beta H^\theta) \\ &= \frac{i}{r} n^2 \omega^2 \partial_\theta B^z + i\beta \omega \partial_r D^z - i\beta \omega \partial_r D^z - \frac{i}{r} \beta^2 \partial_\theta B^z = i\zeta \frac{1}{r} \partial_\theta B^z. \end{aligned} \tag{5.36}$$

Equation (5.34) with the $\vec{d}$ and $\vec{h}$ fields replaced by $\vec{D}$ and $\vec{H}$ now follows from the last equation in (5.35), namely $H^z = B^z$.

In fact, the calculation in (5.36) shows that if a set of equations with a structure as in (5.35) holds, then (5.34) will hold *regardless* of the precise form of the dependence upon r and θ of the z components of the fields.

Let us return to the problem at hand. As such, (5.27) can be written as

$$\begin{aligned} i\zeta d^r &= n^2 \omega \frac{1}{r} \partial_\theta b^z + \beta \partial_r d^z + \dots, \\ i\zeta b^\theta &= \omega \partial_r d^z + \frac{1}{r} \beta \partial_\theta b^z + \dots, \end{aligned} \tag{5.37}$$

while (5.31) reads

$$\begin{aligned} h^\theta &= b^\theta + \dots, \\ h^z &= b^z + \dots, \end{aligned} \tag{5.38}$$

The calculation in (5.36) applies to those terms in (5.37)-(5.38) which have been written out explicitly. Hence, to verify (5.34) it remains to check that an identity of the form (5.34) is satisfied by the terms, denoted by $\dots$ in (5.37)-(5.38), which have not been written in explicit form.

Equivalently, it remains to consider the deviations from the “unperturbed relations” when checking (5.34). Let us use the notation $x \rightsquigarrow y$ when x (e.g. δd^r) deviates from the “unperturbed expression” by y . For instance, for the m mode we have by equations (5.29) and (5.32), suppressing the common factor $-V^z$ temporarily

$$\begin{aligned} i\zeta \delta d_0^r &\rightsquigarrow i\beta(\beta H^\theta + \omega D^r), \\ i\zeta \delta b_0^\theta &\rightsquigarrow i\beta(\beta E^r + \omega H^\theta), \\ \partial_z \delta h_0^\theta + i\beta \delta b_0^\theta &\rightsquigarrow \partial_z E^r, \\ \delta h_0^z &\rightsquigarrow 0. \end{aligned}$$

Since $\zeta = n^2 \omega^2 - \beta^2$ it follows that

$$i\omega \delta d_0^r - i\beta \delta b_0^\theta \rightsquigarrow i\beta E^r,$$

and thus

$$i\omega\delta d_0^r + \partial_z\delta h_0^\theta = (i\omega\delta d_0^r - i\beta\delta b_0^\theta) + (\partial_z\delta h_0^\theta - i\beta\delta b_0^\theta) \rightsquigarrow i\beta E^r + \partial_z E^r = 0,$$

which establishes the consistency of the equations for the m mode.

For the sidebands, we have from (5.30) and (5.33), and after omitting the common factor $-V^\pm$,

$$\begin{aligned} i\zeta\delta d_\pm^r &\rightsquigarrow ic_m^\pm(\beta H^\theta + n^2\omega E^r) \mp \frac{1}{r}(\beta H^r - n^2\omega E^\theta), \\ i\zeta\delta b_\pm^\theta &\rightsquigarrow ic_m^\pm(\beta E^r + \omega H^\theta) \pm \frac{1}{r}(\beta E^\theta - \omega H^r), \\ \partial_z\delta h_\pm^\theta + i\beta\delta b_\pm^\theta &\rightsquigarrow i\partial_z E^r, \\ \delta h_\pm^z &\rightsquigarrow -iE^\theta \mp E^r. \end{aligned}$$

Using the definition $c_m^\pm = \partial_r \mp m/r$, one obtains

$$i\omega\delta d_\pm^r - i\beta\delta b_\pm^\theta \rightsquigarrow ic_m^\pm E^r \pm \frac{1}{r}E^\theta = i\partial_r E^r \mp \frac{i}{r}mE^r \pm \frac{1}{r}E^\theta.$$

Thus,

$$\begin{aligned} i\omega\delta d_\pm^r + \partial_z\delta h_\pm^\theta - i\frac{(m \pm 1)}{r}\delta h_\pm^z &= (i\omega\delta d_\pm^r - i\beta\delta b_\pm^\theta) + (\partial_z\delta h_\pm^\theta + i\beta\delta b_\pm^\theta) - i\frac{m \pm 1}{r}\delta h_\pm^z \\ &\rightsquigarrow i\partial_r E^r \mp \frac{i}{r}mE^r \pm \frac{1}{r}E^\theta + i\partial_z E^r + \frac{m \pm 1}{r}(\pm iE^r - E^\theta) \\ &= i\left(\partial_r E^r + \frac{1}{r}E^r + \frac{1}{r}\partial_\theta E^\theta + \partial_z E^z\right) = i(\vec{\nabla} \cdot \vec{E}) = 0. \end{aligned}$$

This ends the proof of the consistency of the approximate equations for the sidebands.

5.II Continuity Conditions

In this section, we discuss a general method to analyse the continuity conditions of the various field components.

According to the discussion in Section 5.7, the requirement that d^r, e^θ, e^z and b^r, h^θ, h^z be continuous at the core-cladding interface constitute only *four* linearly independent equations. It thus suffices to require d^r, e^z, b^r and h^z to be continuous, so we define a vector which summarises the relevant jump heights:

$$\llbracket F \rrbracket = \left(\llbracket d^r \rrbracket \quad \llbracket e^z \rrbracket \quad \llbracket b^r \rrbracket \quad \llbracket h^z \rrbracket \right)^\top,$$

where $x^\top$ denotes the transpose of x . Here, we have introduced the notation $\llbracket f \rrbracket$ for the jump height of a function f at $r = a$:

$$\llbracket f \rrbracket := \lim_{r \nearrow a} f(r) - \lim_{r \searrow a} f(r).$$

Discontinuity Matrices

Due to linearity of the system, $\llbracket F \rrbracket$ is a linear homogeneous function of the coefficients

$$\begin{aligned} \boldsymbol{\alpha} &= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}^\top, \\ \boldsymbol{\chi}^\pm &= \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 & \chi_4 \end{pmatrix}^\top, \end{aligned}$$

which parameterise the fields according to (4.6) and (5.22). Thus, there exist (complex) 4×4 matrices $M_1, M_2^\pm$ such that

$$\llbracket F \rrbracket = M_1 \alpha + M_2^\pm \chi^\pm.$$

We will refer to these matrices as *discontinuity matrices*. In fact, equations of this form hold for every Fourier mode of the fields separately.

To compute these matrices, we consider the jump height of a field component w as a linear map of the coefficients (e.g. α) to the field of complex numbers. We may thus identify $\llbracket w \rrbracket$ with a row vector of four complex entries, such that

$$\llbracket w \rrbracket = \llbracket w \rrbracket^i \alpha_i,$$

for which we write

$$\llbracket w \rrbracket \doteq \left(\llbracket w \rrbracket^1 \quad \llbracket w \rrbracket^2 \quad \llbracket w \rrbracket^3 \quad \llbracket w \rrbracket^4 \right)_\alpha.$$

The subscript α indicates that the form acts on the coefficients α , and not on the coefficients $\chi^\pm$. We will omit this subscript whenever there is no danger of confusion.

Using this notation, the rows of M_1 in the above equation are exactly the row vectors which represent $\llbracket d^r \rrbracket$, $\llbracket e^z \rrbracket$, $\llbracket b^r \rrbracket$ and $\llbracket h^z \rrbracket$, respectively. Using the expressions (4.5) and (4.6), we find jump heights of the unperturbed fields to be given by

$$\begin{aligned} \llbracket D^z \rrbracket &\doteq \begin{pmatrix} J_m(U) & -K_m(W) & 0 & 0 \end{pmatrix}, \\ \llbracket B^z \rrbracket &\doteq \begin{pmatrix} 0 & 0 & J_m(U) & -K_m(W) \end{pmatrix}, \\ \llbracket \partial_r D^z \rrbracket &= \begin{pmatrix} \frac{U}{a} J'_m(U) & -\frac{W}{a} K'_m(W) & 0 & 0 \end{pmatrix}, \\ \llbracket \partial_r B^z \rrbracket &= \begin{pmatrix} 0 & 0 & \frac{U}{a} J'_m(U) & -\frac{W}{a} K'_m(W) \end{pmatrix}. \end{aligned} \tag{5.39}$$

Since the unperturbed fields depend on $\alpha_1, \dots, \alpha_4$ only, it is clear that the suppressed subscript is α .

Multiplication of a field with a constant clearly multiplies the corresponding row vector by the same number. If a field component w , whose jump heights are given by

$$\llbracket w \rrbracket \doteq \begin{pmatrix} w^1 & w^2 & w^3 & w^4 \end{pmatrix},$$

is multiplied by a function f , whose left- and right-sided limits

$$f(a_-) := \lim_{r \nearrow a} f(r), \quad f(a_+) := \lim_{r \searrow a} f(r)$$

exist, then the jump height of fw is given by

$$\llbracket fw \rrbracket \doteq \begin{pmatrix} f(a_-)w^1 & f(a_+)w^2 & f(a_-)w^3 & f(a_+)w^4 \end{pmatrix}.$$

For example, if $f = n^2$ or $f = \zeta$, then

$$\begin{aligned} \llbracket n^2 w \rrbracket &\doteq \begin{pmatrix} n_1^2 w^1 & n_2^2 w^2 & n_1^2 w^3 & n_2^2 w^4 \end{pmatrix}, \\ \llbracket \zeta w \rrbracket &\doteq \begin{pmatrix} \frac{U^2}{a^2} w^1 & -\frac{W^2}{a^2} w^2 & \frac{U^2}{a^2} w^3 & -\frac{W^2}{a^2} w^4 \end{pmatrix}. \end{aligned}$$

Similar equations hold for arbitrary powers of n and ζ .

Symmetries of the discontinuity matrices

Due to the electromagnetic symmetry, all discontinuity matrices are of the form

$$\begin{pmatrix} a & b & n_1^2 e & n_2^2 f \\ c & d & g & h \\ -e & -f & a & b \\ -g & -h & n_1^2 c & n_2^2 d \end{pmatrix}. \quad (5.40)$$

Indeed, if the jump height of any component of the $\vec{d}$ field is given by

$$\llbracket d^i \rrbracket = \begin{pmatrix} \llbracket d^i \rrbracket^1 & \llbracket d^i \rrbracket^2 & \llbracket d^i \rrbracket^3 & \llbracket d^i \rrbracket^4 \end{pmatrix},$$

we may apply the symmetry transformation $\vec{d} \mapsto \vec{b}$ and $\vec{b} \mapsto -\vec{d}/n^2$ to obtain

$$\llbracket b^i \rrbracket = \begin{pmatrix} -\llbracket d^i \rrbracket^3/n_1^2 & -\llbracket d^i \rrbracket^4/n_2^2 & \llbracket d^i \rrbracket^1 & \llbracket d^i \rrbracket^2 \end{pmatrix}.$$

Similarly, if the jump height of e^i is given by

$$\llbracket e^i \rrbracket = \begin{pmatrix} \llbracket e^i \rrbracket^1 & \llbracket e^i \rrbracket^2 & \llbracket e^i \rrbracket^3 & \llbracket e^i \rrbracket^4 \end{pmatrix},$$

we may apply the same transformation as above, which entails $\vec{e} \mapsto \vec{h}/n^2$, and find

$$\llbracket h^i \rrbracket = \begin{pmatrix} -\llbracket e^i \rrbracket^3 & -\llbracket e^i \rrbracket^4 & n_1^2 \llbracket e^i \rrbracket^1 & n_2^2 \llbracket e^i \rrbracket^2 \end{pmatrix}.$$

It thus suffices to compute the jump heights of d^r and e^z , since the first two rows determine the entire matrix.

General Structure of Discontinuity matrices

In general, there are two contributions to these matrices. Some of the terms stem from the z components of the fields, which produce terms in all other field components; while other terms stem from the cross product in (3.5) which relates $\vec{d}$ and $\vec{b}$ to $\vec{e}$ and $\vec{h}$.

Let f be any function which enters the expressions for the z components of the fields as

$$\begin{aligned} d^z(r) &= f(\alpha_1, \alpha_2; r) + \dots, \\ b^z(r) &= f(\alpha_3, \alpha_4; r) + \dots. \end{aligned}$$

We denote by $D(f)$ the corresponding matrix defined by the condition that the such produced terms in $\llbracket F \rrbracket$ are

$$\llbracket F \rrbracket = D(f)\alpha + \dots$$

This method allows to describe all contributions to the discontinuity matrices which arise due to the z components of the fields alone. To cover the remaining terms which arise from the cross products in (3.5), we will introduce additional matrices D_1, D_2 etc. Since, at our level of approximation, the cross product terms are determined by the unperturbed fields $\vec{D}$ and $\vec{B}$, which are parameterised by α alone, these additional matrices will multiply α only.

5.12 Continuity of the Main Mode

According to (5.1), (4.6) and (5.22), the main Fourier modes of the z components of $\vec{d}$ and $\vec{b}$ are given by

$$\begin{aligned} d_0^z(r) &= D^z(r) + \delta d_0^z(r) = f_0(\alpha_1, \alpha_2; r) - 2a^2 \beta \omega V^z p_0(\alpha_1, \alpha_2; r), \\ b_0^z(r) &= B^z(r) + \delta b_0^z(r) = f_0(\alpha_3, \alpha_4; r) - 2a^2 \beta \omega V^z p_0(\alpha_3, \alpha_4; r). \end{aligned}$$

Correspondingly, we write the jump height of the fields in the form

$$\llbracket F_0 \rrbracket = D(f_0) \alpha - 2a^2 \beta \omega V^z (D(p_0) + D_1) \alpha,$$

where $D(f_0)$ describes the jump heights due to the functions f_0 alone, $D(p_0)$ those due to the presence of the function p_0 , and D_1 the terms which arise from the cross product terms in (3.5). The factor $2a^2 \beta \omega V^z$ was factored from D_1 for convenience.

We start by showing that the matrix $D(f_0)$ is given by

$$D(f_0) = \begin{pmatrix} -i \frac{a\beta}{U} J'_m & -i \frac{a\beta}{W} K'_m & mn_1^2 \frac{a\omega}{U^2} J_m & mn_2^2 \frac{a\omega}{W^2} K_m \\ \frac{1}{n_1^2} J_m & -\frac{1}{n_2^2} K_m & 0 & 0 \\ -m \frac{a\omega}{U^2} J_m & -m \frac{a\omega}{W^2} K_m & -i \frac{a\beta}{U} J'_m & -i \frac{a\beta}{W} K'_m \\ 0 & 0 & J_m & -K_m \end{pmatrix}, \quad (5.41)$$

where all functions whose arguments have been suppressed are evaluated at the core-cladding interface $r = a$. Thus, all *ordinary* Bessel functions are evaluated at U and all *modified* Bessel functions are evaluated at W . Similarly, the functions Δ_ν, T_ν with suppressed arguments will be evaluated at $r/a = 1$.

To compute this matrix, it suffices to consider the equations for the unperturbed fields, since the effects of all perturbations are described by $D(p_0)$ and D_1 . From (4.2) we have

$$i\zeta D^r = \beta \partial_r D^z + in^2 \omega \frac{m}{r} B^z.$$

Using (5.39), one obtains

$$\begin{aligned} \llbracket i\zeta D^r \rrbracket &= \beta \llbracket \partial_r D^z \rrbracket + i \frac{m\omega}{a} \llbracket n^2 B^z \rrbracket \\ &= \begin{pmatrix} \frac{\beta U}{a} J'_m & -\frac{\beta W}{a} K'_m & in_1^2 \frac{m\omega}{a} J_m & -in_2^2 \frac{m\omega}{a} K_m \end{pmatrix}, \end{aligned}$$

and thus

$$\llbracket D^r \rrbracket = \begin{pmatrix} -i \frac{a\beta}{U} J'_m & -i \frac{a\beta}{W} K'_m & n_1^2 \frac{m\omega}{U^2} J_m & n_2^2 \frac{m\omega}{W^2} K_m \end{pmatrix},$$

which is exactly the first row of the matrix given in (5.41). To compute the jump height of E^z , we use

$$E^z = \frac{1}{n^2} D^z.$$

By virtue of (5.39), we find

$$\llbracket E^z \rrbracket = \begin{pmatrix} \frac{1}{n_1^2} J_m & -\frac{1}{n_2^2} K_m & 0 & 0 \end{pmatrix},$$

which is the second row of $D(f_0)$. The third and fourth row are now determined by the electromagnetic symmetry, i.e. by the general form (5.40).

To compute $D(p_0)$, we note that $D(f)$ is uniquely determined by the limiting behaviour of f and its first derivative at $r = a$. Comparing the limiting behaviour of p_0 and its first derivatives, given explicitly in (5.17) and (5.21), with the limiting behaviour of f_0 , we find that the matrix $D(p_0)$ is obtained from $D(f_0)$ by the substitution

$$\begin{aligned} J_m(U) &\mapsto \frac{\pi}{2} Y_m(U) \Delta_0(1), \\ J'_m(U) &\mapsto \frac{\pi}{2} Y'_m(U) \Delta_0(1), \\ K_m(U) &\mapsto -I_m(U) T_0(1), \\ K'_m(U) &\mapsto -I'_m(U) T_0(1). \end{aligned}$$

We thus obtain

$$D(p_0) = \begin{pmatrix} -i \frac{a\beta}{U} \frac{\pi}{2} Y'_m \Delta_0 & i \frac{a\beta}{W} I'_m T_0 & n_1^2 \frac{ma\omega}{U^2} \frac{\pi}{2} Y_m \Delta_0 & -n_2^2 \frac{ma\omega}{W^2} I_m T_0 \\ \frac{1}{n_1^2} \frac{\pi}{2} Y_m \Delta_0 & \frac{1}{n_2^2} I_m T_0 & 0 & 0 \\ -\frac{ma\omega}{U^2} \frac{\pi}{2} Y_m \Delta_0 & \frac{ma\omega}{W^2} I_m T_0 & -i \frac{a\beta}{U} \frac{\pi}{2} Y'_m \Delta_0 & i \frac{a\beta}{W} I'_m T_0 \\ 0 & 0 & \frac{\pi}{2} Y_m \Delta_0 & I_m T_0 \end{pmatrix}. \quad (5.42)$$

Finally, we compute D_1 explicitly. According to (5.29), the relevant term for δd_0^z is

$$i\zeta \delta d_0^r = -iV^z \beta (\beta H^\theta + \omega D^r) + \dots$$

From (4.2) we obtain

$$i\zeta (\beta H^\theta + \omega D^r) = 2\beta\omega \partial_r D^z + i\bar{\zeta} \frac{m}{r} B^z, \quad (5.43)$$

where we have set

$$\bar{\zeta} = n^2 \omega^2 + \beta^2 \equiv \zeta + 2\beta^2 = \begin{cases} \bar{U}^2/a^2, & r < a, \\ \bar{W}^2/a^2, & r > a. \end{cases}$$

Here, we have introduced the constants $\bar{U}$ and $\bar{W}$ according to

$$\begin{aligned} \bar{U}^2/a^2 &= \beta^2 + n_1^2 \omega^2 \equiv 2\beta^2 + U^2/a^2, \\ \bar{W}^2/a^2 &= \beta^2 + n_2^2 \omega^2 \equiv 2\beta^2 - W^2/a^2. \end{aligned}$$

Having established that

$$\zeta^2 \delta d_0^r = iV^z \beta \left(2\beta\omega \partial_r D^z + i\bar{\zeta} \frac{m}{r} B^z \right) + \dots,$$

we may use (5.39) to obtain

$$\llbracket \zeta^2 \delta d_0^r \rrbracket = iV^z \beta \left(2\beta\omega \frac{U}{a} J'_m \quad -2\beta\omega \frac{W}{a} K'_m \quad i \frac{m\bar{U}^2}{a^3} J_m \quad -i \frac{m\bar{W}^2}{a^3} K_m \right)_\alpha + \dots,$$

from which it follows that

$$\llbracket \delta d_0^r \rrbracket = -2a^2 \beta\omega V^z \left(-i \frac{a\beta}{U^3} J'_m \quad +i \frac{a\beta}{W^3} K'_m \quad + \frac{m}{2a\omega} \frac{\bar{U}^2}{U^4} J_m \quad - \frac{m}{2a\omega} \frac{\bar{W}^2}{W^4} K_m \right)_\alpha + \dots$$

This determines the first row of D_1 . According to (5.32) we have $n^2 \delta e_0^z = \delta d_0^z$, so $\vec{V}$ does not contribute to δe_0^z . Accordingly, the second row of D_1 vanishes. The remaining rows are then fully

determined by the general form (5.40), so we arrive at the result

$$D_1 = \begin{pmatrix} -i \frac{a\beta}{U^3} J'_m & i \frac{a\beta}{W^3} K'_m & \frac{m}{2a\omega} \frac{\bar{U}^2}{U^4} J_m & -\frac{m}{2a\omega} \frac{\bar{W}^2}{W^4} K_m \\ 0 & 0 & 0 & 0 \\ -\frac{m}{2a\omega} \frac{1}{n_1^2} \frac{\bar{U}^2}{U^4} J_m & \frac{m}{2a\omega} \frac{1}{n_2^2} \frac{\bar{W}^2}{W^4} K_m & -i \frac{a\beta}{U^3} J'_m & i \frac{a\beta}{W^3} K'_m \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.44)$$

The continuity conditions of the main mode now take the concise form

$$\llbracket F_0 \rrbracket = D(f_0) \boldsymbol{\alpha} - 2a^2 \beta \omega V^z (D(p_0) + D_1) \boldsymbol{\alpha} \stackrel{!}{=} 0. \quad (5.45)$$

5.13 Dispersion Relation

We have established that the continuity condition of the main mode takes the form

$$(M^{(0)} + \delta M) \boldsymbol{\alpha} = 0, \quad (5.46)$$

where

$$\begin{aligned} M^{(0)} &= D(f_0), \\ \delta M &= -2a^2 \beta \omega V^z (D(p_0) + D_1). \end{aligned}$$

In the unperturbed case, where $\Omega = 0$, M reduces to $M^{(0)}$, so δM is the first order perturbation of M due to Earth's rotation.

Generically, the kernel of the matrix M is trivial, so we must arrange parameters such that its determinant vanishes, if we wish to obtain a nontrivial solution. Since the only remaining parameters are β and ω , one is determined by the other via the dispersion relation

$$\det M = \det(M^{(0)} + \delta M) \stackrel{!}{=} 0. \quad (5.47)$$

We choose to take the frequency ω as the independent variable, such that the dispersion relation determines $\beta = \beta(\omega)$. Equation (5.47) is of major importance since encodes how the wave vector β changes due to Earth's rotation. As in the unperturbed case, this equation can only be solved numerically. If a solution to the unperturbed dispersion relation is known, one may obtain the first order correction to β by expanding the dispersion relation (5.47) to first order in the velocity V .

Here, we see the importance of letting the parameters $\boldsymbol{\alpha}$ deviate from their unperturbed values $\boldsymbol{\alpha}^{(0)}$. Had we added homogeneous solutions to the wave equations with parameters $\boldsymbol{\alpha}^{(1)}$ and neglected the term $\delta M \boldsymbol{\alpha}^{(1)}$, which is quadratic in the velocity $\vec{V}$, instead of (5.46) we would have obtained

$$M^{(0)} \boldsymbol{\alpha}^{(1)} + \delta M \boldsymbol{\alpha}^{(0)} = 0,$$

since $M^{(0)} \boldsymbol{\alpha}^{(0)} = 0$. This equation already suggests that the dispersion relation must be modified since it cannot be solved for $\boldsymbol{\alpha}^{(1)}$ due to M being singular when β assumes its unperturbed value. Further analysis would have been necessary to obtain the dispersion relation (5.47).

Linearising the condition $\det M = 0$ in the vicinity of $V = 0$ using Jacobi's formula

$$\frac{\partial}{\partial A_{ij}} \det A = \text{adj}(A)^{ji},$$

where $\text{adj}(A)$ is the adjugate matrix of A (the transpose of the cofactor matrix), one obtains

$$\delta(\det M) = \underbrace{\left[\frac{\partial |M^{(0)}|}{\partial \beta} + \frac{\partial |M^{(0)}|}{\partial U} \frac{\partial U}{\partial \beta} + \frac{\partial |M^{(0)}|}{\partial W} \frac{\partial W}{\partial \beta} \right]}_{\equiv \frac{d}{d\beta} \det M^{(0)}} \delta\beta + \text{tr} [\text{adj}(M^{(0)}) \delta M] = 0,$$

which implies

$$\delta\beta = - \frac{\text{tr} [\text{adj}(M^{(0)}) \delta M]}{\frac{d}{d\beta} \det M^{(0)}}.$$

When solving the equation $\det M = 0$, we may equally replace M by $\tilde{M} = A_1 M A_2$, where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & i n_1^2 \frac{a\beta}{U} \frac{J'_m}{J_m} & 0 & -n_1^2 \frac{m a \omega}{U^2} \\ 0 & 0 & 0 & 1 \\ 0 & n_1^2 \frac{m a \omega}{U^2} & 1 & i \frac{a\beta}{U} \frac{J'_m}{J_m} \end{pmatrix}, \quad A_2 = \begin{pmatrix} n_1^2 & 0 & 0 & 0 \\ 0 & n_2^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since these matrices are nonsingular (in fact $\det A_1 = 1$ and $\det A_2 = n_1^2 n_2^2$) this does not change the roots of $\det M = 0$.

It proves useful to introduce the abbreviations

$$\Delta n^2 = n_1^2 - n_2^2, \\ \nu = \frac{a^2 \beta^2}{U^2 W^2} m \Delta n^2 \omega,$$

as well as the brackets

$$(J_m, K_m) = \frac{1}{U} J'_m(U) K_m(W) + \frac{1}{W} J_m(U) K'_m(W), \\ \{J_m, K_m\} = \frac{1}{U^3} J'_m(U) K_m(W) - \frac{1}{W^3} J_m(U) K'_m(W), \\ (J_m, K_m)_{n^2} = \frac{n_1^2}{U} J'_m(U) K_m(W) + \frac{n_2^2}{W} J_m(U) K'_m(W), \\ \{J_m, K_m\}_{n^2} = \frac{n_1^2}{U^3} J'_m(U) K_m(W) - \frac{n_2^2}{W^3} J_m(U) K'_m(W).$$

$\tilde{M}^{(0)}$ then takes the concise form

$$\tilde{M}^{(0)} = \begin{pmatrix} J_m & -K_m & 0 & 0 \\ 0 & -i \frac{a\beta}{J_m} (J_m, K_m)_{n^2} & 0 & a\nu K_m \\ 0 & 0 & J_m & -K_m \\ 0 & -a\nu K_m & 0 & -i \frac{a\beta}{J_m} (J_m, K_m) \end{pmatrix},$$

from which one can read off the determinant

$$\det \tilde{M}^{(0)} = a^2 \nu^2 J_m(U)^2 K_m(W)^2 - a^2 \beta^2 (J_m, K_m) (J_m, K_m)_{n^2}.$$

Since similar structures will appear again in the following calculations, we set

$$\Psi(f, g) := a^2 \nu^2 J_m(U)^2 f(W) g(W) - a^2 \beta^2 (J_m, f) (J_m, g)_{n^2},$$

such that the defining equation of the unperturbed value of β takes the form

$$\det \tilde{M}^{(0)} = \Psi(K_m, K_m) = 0. \quad (5.48)$$

Using the fact that

$$\frac{n_1^2}{U^2} + \frac{n_2^2}{W^2} = \frac{a^2 \beta^2}{U^2 W^2} \Delta n^2,$$

as well as the Wronskian

$$J_m(U)Y'_m(U) - J'_m(U)Y_m(U) = \frac{2}{\pi U},$$

the adjugate of $\tilde{M}^{(0)}$ and the perturbations $\delta \tilde{M}_1, \delta \tilde{M}_2$ take the form

$$\begin{aligned} \text{adj}(\tilde{M}^{(0)}) &= \begin{pmatrix} 0 & -ia\beta(J_m, K_m)K_m & 0 & -a\nu J_m K_m^2 \\ 0 & -ia\beta(J_m, K_m)J_m & 0 & -a\nu J_m^2 K_m \\ 0 & a\nu J_m K_m^2 & 0 & -ia\beta(J_m, K_m)n^2 K_m \\ 0 & a\nu J_m^2 K_m & 0 & -ia\beta(J_m, K_m)n^2 J_m \end{pmatrix}, \\ \delta \tilde{M}_1 &= -2a^2 \beta \omega V^z \begin{pmatrix} \frac{\pi}{2} Y_m \Delta_0 & I_m T_0 & 0 & 0 \\ -i \frac{n_1^2}{U^2} \frac{a\beta}{J_m} \Delta_0 & i \frac{a\beta}{J_m} (J_m, I_m) n^2 T_0 & 0 & -a\nu I_m T_0 \\ 0 & 0 & \frac{\pi}{2} Y_m \Delta_0 & I_m T_0 \\ 0 & a\nu I_m T_0 & -i \frac{1}{U^2} \frac{a\beta}{J_m} \Delta_0 & i \frac{a\beta}{J_m} (J_m, I_m) T_0 \end{pmatrix}, \\ \delta \tilde{M}_2 &= -2a^2 \beta \omega V^z \begin{pmatrix} 0 & 0 & 0 & 0 \\ -in_1^2 \frac{a\beta}{U^3} J'_m & in_2^2 \frac{a\beta}{W^3} K'_m & \frac{m}{2a\omega} \frac{\bar{U}^2}{U^4} J_m & -\frac{m}{2a\omega} \frac{\bar{W}^2}{W^4} K_m \\ 0 & 0 & 0 & 0 \\ -\frac{m}{2a\omega} \frac{\bar{U}^2}{U^4} J_m & \frac{m}{2a\omega} \frac{\bar{W}^2}{W^4} K_m & -i \frac{a\beta}{U^3} J'_m & i \frac{a\beta}{W^3} K'_m \end{pmatrix}. \end{aligned}$$

From these matrices, one obtains the following expressions for their traces with $\tilde{M}^{(0)}$:

$$\begin{aligned} \frac{\text{tr}[\tilde{M}^{(0)} \delta \tilde{M}_1]}{2a^2 \beta \omega V^z} &= \frac{a^2 \beta^2}{U^2} \frac{K}{J} (n_1^2(J, K) + (J, K)_{n^2}) \Delta_0 + (\Psi(K, I) + \Psi(I, K)) T_0, \\ \frac{\text{tr}[\tilde{M}^{(0)} \delta \tilde{M}_2]}{2a^2 \beta \omega V^z} &= a^2 \beta^2 ((J, K)\{J, K\}_{n^2} + (J, K)_{n^2}\{J, K\}) + \frac{m\nu}{a^2 \omega} J^2 K^2 \left(\frac{\bar{W}^2}{W^4} - \frac{\bar{U}^2}{U^4} \right). \end{aligned}$$

We thus arrive at the following result for the linear change in β :

$$\delta \beta = -\frac{2a^2 \beta \omega V^z}{\frac{d}{d\beta} \det \tilde{M}^{(0)}} \left[A + B + C + D \right], \quad (5.49)$$

where

$$\begin{aligned} A &= (\Psi(K_m, I_m) + \Psi(I_m, K_m)) T_0(1), \\ B &= \frac{a^2 \beta^2}{U^2} \frac{K_m(W)}{J_m(U)} (n_1^2(J_m, K_m) + (J_m, K_m)_{n^2}) \Delta_0(1), \\ C &= a^2 \beta^2 ((J_m, K_m)\{J_m, K_m\}_{n^2} + (J_m, K_m)_{n^2}\{J_m, K_m\}), \\ D &= a^2 \beta^2 m^2 \Delta n^2 \frac{J_m(U)^2 K_m(W)^2}{U^2 W^2} \left(\frac{\bar{W}^2}{W^4} - \frac{\bar{U}^2}{U^4} \right). \end{aligned}$$

This formula is the main result of this section: it determines the linearised deviation from the unperturbed dispersion relation (5.48). If β solves the unperturbed equation for given ω , (5.49) determines the first order correction to β for constant ω .

Once the perturbed value of β is known, the coefficients α (which parameterise the main mode) are obtained by finding the kernel of the matrix

$$M = M^{(0)} + \delta M ,$$

cf. (5.45) and (5.46). From the unperturbed problem it is known that the kernel of $M^{(0)}$ is one-dimensional. Since the dimension of the kernel is an upper semi-continuous function, the same will hold for M , provided that the velocity V is sufficiently small. Note that the determination of α to order $O(V)$ requires $M^{(0)}$ to be computed using the perturbed value of β . Since δM is already of order $O(V)$ one may use the unperturbed value of β in this matrix.

5.14 Continuity of the Sidebands

In this section, we compute the discontinuity matrices associated to the sidebands.

From (5.22) it is clear that the jump height of the sidebands takes the form

$$\llbracket F_{\pm} \rrbracket = D(f_{\pm})\chi^{\pm} - 2a\omega V^{\pm}D(p_{\pm})\alpha - iV^{\pm}D_2^{\pm}\alpha . \quad (5.50)$$

The coefficient in front of $D_2^{\pm}$, which contains the contributions from the usual cross product terms, was factored for convenience.

We may obtain $D(f_{\pm})$ from $D(f_0)$ by noting that

$$f_{\pm}(\alpha_1, \alpha_2; r) = \begin{cases} \mp U J_{m\pm 1}(Ur/a) & r < a , \\ -W K_{m\pm 1}(Wr/a) & r > a , \end{cases}$$

is related to $f_0(\alpha_1, \alpha_2; r)$ by the substitution

$$\begin{aligned} J_m &\mapsto \pm U J_{m\pm 1} , \\ K_m &\mapsto -W K_{m\pm 1} . \end{aligned}$$

Applying this to $D(f_0)$ and replacing all factors m (which are due to derivatives with respect to the angle θ) by $m \pm 1$, we find

$$D(f_{\pm}) = \begin{pmatrix} \pm ia\beta J'_{m\pm 1} & ia\beta K'_{m\pm 1} & \mp n_1^2 \frac{(m\pm 1)a\omega}{U} J_{m\pm 1} & -n_2^2 \frac{(m\pm 1)a\omega}{W} K_{m\pm 1} \\ \mp \frac{U}{n_1^2} J_{m\pm 1} & \frac{W}{n_2^2} K_{m\pm 1} & 0 & 0 \\ \pm \frac{(m\pm 1)a\omega}{U} J_{m\pm 1} & \frac{(m\pm 1)a\omega}{W} K_{m\pm 1} & \pm ia\beta J'_{m\pm 1} & ia\beta K'_{m\pm 1} \\ 0 & 0 & \mp U J_{m\pm 1} & W K_{m\pm 1} \end{pmatrix} .$$

Similarly, comparing p_0 with $p_{\pm}$ in (5.23), find that $D(p_{\pm})$ is related to $D(p_0)$ by the substitution

$$\begin{aligned} m &\mapsto m \pm 1 , \\ \Delta_0 &\mapsto \mp \Delta_{\pm} , \\ T_0 &\mapsto -T_{\pm} , \end{aligned}$$

so we find

$$D(p_{\pm}) = \begin{pmatrix} \pm ia\beta Y'_{m\pm 1} & -ia\beta I'_{m\pm 1} & \mp n_1^2 \frac{(m\pm 1)a\omega}{U} Y_{m\pm 1} & n_2^2 \frac{(m\pm 1)a\omega}{W} I_{m\pm 1} \\ \mp \frac{U}{n_1^2} Y_{m\pm 1} & -\frac{W}{n_2^2} I_{m\pm 1} & 0 & 0 \\ \pm \frac{(m\pm 1)a\omega}{U} Y_{m\pm 1} & -\frac{(m\pm 1)a\omega}{W} I_{m\pm 1} & \pm ia\beta Y'_{m\pm 1} & -ia\beta I'_{m\pm 1} \\ 0 & 0 & \mp U Y_{m\pm 1} & -W I_{m\pm 1} \end{pmatrix} \Lambda^{\pm}(\Delta, T),$$

where we have set

$$\Lambda^{\pm}(\Delta, T) = \text{diag} \left(\frac{\pi}{2} \Delta_{\pm}(1), T_{\pm}(1), \frac{\pi}{2} \Delta_{\pm}(1), T_{\pm}(1) \right).$$

Finally, we compute the matrix $D_2^{\pm}$. To this end, we note that the relevant terms in the equation for $\delta d_{\pm}^r$ are

$$i\zeta \delta d_{\pm}^r = -iV^{\pm} c_m^{\pm} (\beta H^{\theta} + \omega D^r) \pm \frac{V^{\pm}}{r} (\beta H^r - \omega D^{\theta}) + \dots$$

Using (5.43) and the analogous equation

$$i\zeta (\beta H^r - \omega D^{\theta}) = \bar{\zeta} \partial_r B^z - 2i\beta\omega \frac{m}{r} D^z, \quad (5.51)$$

we obtain

$$\zeta^2 \delta d_{\pm}^r = 2i\beta\omega V^{\pm} \left(c_m^{\pm} (\partial_r D^z) \pm \frac{m}{r^2} D^z \right) - \bar{\zeta} V^{\pm} \left(c_m^{\pm} \left(\frac{m}{r} B^z \right) \pm \frac{1}{r} \partial_r B^z \right).$$

Simplifying this expression using the identities

$$\begin{aligned} \partial_r (c_m^{\pm} f) &= c_m^{\pm} (\partial_r f) \pm \frac{m}{r^2} f, \\ \frac{m \pm 1}{r} c_m^{\pm} (f) &= c_m^{\pm} \left(\frac{m}{r} f \right) \pm \frac{1}{r} \partial_r f, \end{aligned}$$

one obtains

$$\zeta^2 \delta d_{\pm}^r = 2i\beta\omega V^{\pm} \partial_r (c_m^{\pm} D^z) - \bar{\zeta} V^{\pm} \frac{m \pm 1}{r} c_m^{\pm} B^z.$$

Using the explicit expressions for D^z and B^z , as well as the recursion relations (5.14), (5.18), we find

$$\begin{aligned} \llbracket \partial_r c_m^{\pm} D^z \rrbracket &= \left(\mp \frac{U^2}{a^2} J'_{m\pm 1} \quad \frac{W^2}{a^2} K'_{m\pm 1} \quad 0 \quad 0 \right), \\ \llbracket \bar{\zeta} c_m^{\pm} B^z \rrbracket &= \left(0 \quad 0 \quad \mp \frac{U \bar{U}^2}{a^3} J_{m\pm 1} \quad \frac{W \bar{W}^2}{a^3} K_{m\pm 1} \right), \end{aligned}$$

which implies

$$\llbracket \delta d_{\pm}^r \rrbracket = -iV^{\pm} \left(\pm \frac{2a^2\beta\omega}{U^2} J'_{m\pm 1} \quad -\frac{2a^2\beta\omega}{W^2} K'_{m\pm 1} \quad \pm i \frac{m\pm 1}{U} \frac{\bar{U}^2}{U^2} J_{m\pm 1} \quad -i \frac{m\pm 1}{W} \frac{\bar{W}^2}{W^2} K_{m\pm 1} \right).$$

This determines the first row of $D_2^{\pm}$.

Next, we compute the second row, which is (up to the factor $-iV^{\pm}$) given by the jump height of $\delta e_{\pm}^z$. From (5.33) we find the relevant term to be

$$\delta e_{\pm}^z = -iV^{\pm} (B^{\theta} \mp iB^r).$$

From (4.2) we have

$$i\zeta (B^{\theta} \mp iB^r) = c_m^{\pm} (\omega D^z \mp i\beta B^z).$$

Substituting this into the above expression for $\delta e_{\pm}^z$ and computing the jump height, we find

$$\llbracket n^2 \zeta \delta e_{\pm}^z \rrbracket = -V^{\pm} \left(\mp \frac{\omega U}{a} J_{m\pm 1} \quad \frac{\omega W}{a} K_{m\pm 1} \quad i \frac{\beta U}{a} J_{m\pm 1} \quad \mp i \frac{\beta W}{a} K_{m\pm 1} \right),$$

which implies

$$\llbracket \delta e_{\pm}^z \rrbracket = -iV^{\pm} \left(\pm \frac{i}{n_1^2} \frac{a\omega}{U} J_{m\pm 1} \quad \frac{i}{n_2^2} \frac{a\omega}{W} K_{m\pm 1} \quad \frac{1}{n_1^2} \frac{a\beta}{U} J_{m\pm 1} \quad \pm \frac{1}{n_2^2} \frac{a\beta}{W} K_{m\pm 1} \right).$$

Using the general form (5.40), we thus arrive at the result

$$D_2^{\pm} = \begin{pmatrix} \pm \frac{2a^2\beta\omega}{U^2} J'_{m\pm 1} & -\frac{2a^2\beta\omega}{W^2} K'_{m\pm 1} & \pm i \frac{m\pm 1}{U} \frac{\bar{U}^2}{U^2} J_{m\pm 1} & -i \frac{m\pm 1}{W} \frac{\bar{W}^2}{W^2} K_{m\pm 1} \\ \pm \frac{i}{n_1^2} \frac{a\omega}{U} J_{m\pm 1} & \frac{i}{n_2^2} \frac{a\omega}{W} K_{m\pm 1} & \frac{1}{n_1^2} \frac{a\beta}{U} J_{m\pm 1} & \pm \frac{1}{n_2^2} \frac{a\beta}{W} K_{m\pm 1} \\ \mp \frac{i}{n_1^2} \frac{m\pm 1}{U} \frac{\bar{U}^2}{U^2} J_{m\pm 1} & \frac{i}{n_2^2} \frac{m\pm 1}{W} \frac{\bar{W}^2}{W^2} K_{m\pm 1} & \pm \frac{2a^2\beta\omega}{U^2} J'_{m\pm 1} & -\frac{2a^2\beta\omega}{W^2} K'_{m\pm 1} \\ -\frac{1}{n_1^2} \frac{a\beta}{U} J_{m\pm 1} & \mp \frac{1}{n_2^2} \frac{a\beta}{W} K_{m\pm 1} & \pm i \frac{a\omega}{U} J_{m\pm 1} & i \frac{a\omega}{W} K_{m\pm 1} \end{pmatrix}.$$

Having computed all matrices which enter (5.50) and having computed the coefficients α of the main mode, the parameters $\chi^{\pm}$ are determined by

$$\llbracket F_{\pm} \rrbracket = D(f_{\pm}) \chi^{\pm} - 2a\omega V^{\pm} D(p_{\pm}) \alpha - iV^{\pm} D_2^{\pm} \alpha \stackrel{!}{=} 0. \quad (5.52)$$

Here and in the following sections, we shall assume that the matrices $D(f_{\pm})$ are invertible. Recall that the unperturbed dispersion relation is such that $\det D(f_0) = 0$. The special case where both $D(f_0)$ and either one of the two matrices $D(f_{\pm})$ are singular for the same value of β would require separate analysis.

Chapter 6

Corrections of Relative Order λ/R

In the previous chapter, we have solved the approximate wave equation (3.12). We now improve the approximation by solving the more accurate wave equation (3.11) and verify that the corrections are indeed of relative order $O(\lambda/R)$.

6.1 The Wave Equations

In the previous analysis, the source term proportional to Ω in (3.11) was neglected. In this section, we briefly discuss the influence of this term and thus consider the equations

$$\begin{aligned}\mu(n^2\delta\ddot{d}^z - \Delta\delta d^z) &= +2\Omega(B^z) + 2i\beta\vec{\Omega} \cdot \vec{B}, \\ \varepsilon(n^2\delta\ddot{b}^z - \Delta\delta b^z) &= -2\Omega(D^z) - 2i\beta\vec{\Omega} \cdot \vec{D}.\end{aligned}$$

Note first that the new source term does not modify the main mode, since its m -th Fourier component vanishes:

$$\langle \Omega(D^z) + i\beta\vec{\Omega} \cdot \vec{D} \rangle_0 = \Omega^z \partial_z D^z + i\beta\Omega^z D^z = 0.$$

Thus, this correction does not modify our result for the shift in β but merely modifies the coefficients $\chi_1^\pm, \dots, \chi_4^\pm$, which parameterise the sidebands according to (5.22). We thus consider the $m \pm 1$ -modes only, for which we rewrite the source term as a differential operator acting on D^z and B^z . Decomposing the angular velocity $\vec{\Omega}$ in the same way as the linear velocity $\vec{V}$ was decomposed in (5.8), we have

$$\langle \Omega(D^z) \rangle_\pm = -i\Omega^\pm c_m^\pm(D^z),$$

from which it follows that

$$\langle \Omega(D^z) + i\beta\vec{\Omega} \cdot \vec{D} \rangle_\pm = -i\Omega^\pm c_m^\pm(D^z) + \beta\Omega^\pm(D^r \pm iD^\theta).$$

Inserting the expressions for D^r and D^θ from (4.2) one obtains

$$i\zeta(D^r \pm iD^\theta) = c_m^\pm(\beta D^z \mp in^2\omega B^z),$$

so, using $\zeta = n^2\omega^2 - \beta^2$, we find

$$i\zeta \langle \Omega(D^z) + i\beta\vec{\Omega} \cdot \vec{D} \rangle_\pm = n^2\Omega^\pm c_m^\pm(\omega^2 D^z \mp i\beta\omega B^z).$$

In our case, where $\varepsilon = n^2$, the wave equation thus takes the form

$$(n^2\delta\ddot{b}_\pm^z - \Delta\delta b_\pm^z) = +\frac{2}{\zeta}\Omega^\pm c_m^\pm(i\omega^2 D^z \pm \beta\omega B^z). \quad (6.1)$$

The corresponding wave equation for δd^z is obtained by applying the electromagnetic symmetry transformation $\vec{b} \mapsto \vec{d}$ and $\vec{d} \mapsto -n^2 \vec{b}$:

$$(n^2 \ddot{\delta d}_{\pm}^z - \Delta \delta d_{\pm}^z) = -\frac{2}{\zeta} \Omega^{\pm} c_m^{\pm} (in^2 \omega^2 B^z \mp \beta \omega D^z). \quad (6.2)$$

6.2 The Radial Equations

Substituting

$$D^z(t, r, \theta, z) = f_0(\alpha_1, \alpha_2; r) e^{i(\omega t - \beta z + m\theta)},$$

$$B^z(t, r, \theta, z) = f_0(\alpha_3, \alpha_4; r) e^{i(\omega t - \beta z + m\theta)},$$

into (6.1) and (6.2) and using the usual ansatz for the two sidebands

$$d_{\pm}^r(t, r, \theta, z) = d_{\pm}^r(r) e^{i(\omega t - \beta z + (m \pm 1)\theta)},$$

one obtains the differential equations

$$\begin{aligned} L_{m \pm 1}^{\zeta} \delta d_{\pm}^z &= 2a^2 \Omega^{\pm} c_m^{\pm} f_0 \left(\frac{\mp \beta \omega \alpha_1 + in_1^2 \omega^2 \alpha_3}{U^2}, \frac{\pm \beta \omega \alpha_2 - in_2^2 \omega^2 \alpha_4}{W^2}; r \right), \\ L_{m \pm 1}^{\zeta} \delta b_{\pm}^z &= 2a^2 \Omega^{\pm} c_m^{\pm} f_0 \left(\frac{-i\omega^2 \alpha_1 \mp \beta \omega \alpha_3}{U^2}, \frac{i\omega^2 \alpha_2 \pm \beta \omega \alpha_4}{W^2}; r \right). \end{aligned} \quad (6.3)$$

We define a new set of parameters $\hat{\alpha}$, which are related to the old set α by

$$\hat{\alpha} = A\alpha,$$

where A is the matrix

$$A = \begin{pmatrix} \mp \frac{a^2 \beta \omega}{U^2} & 0 & in_1^2 \frac{a^2 \omega^2}{U^2} & 0 \\ 0 & \pm \frac{a^2 \beta \omega}{W^2} & 0 & -in_2^2 \frac{a^2 \omega^2}{W^2} \\ -i \frac{a^2 \omega^2}{U^2} & 0 & \mp \frac{a^2 \beta \omega}{U^2} & 0 \\ 0 & i \frac{a^2 \omega^2}{W^2} & 0 & \pm \frac{a^2 \beta \omega}{W^2} \end{pmatrix}, \quad (6.4)$$

which maps α to the arguments of f_0 in (6.3) (multiplied by a^2 such that the matrix is dimensionless). The differential equations then take the form

$$\begin{aligned} L_{m \pm 1}^{\zeta} \delta d_{\pm}^z &= 2\Omega^{\pm} c_m^{\pm} f_0(\hat{\alpha}_1, \hat{\alpha}_2; r), \\ L_{m \pm 1}^{\zeta} \delta b_{\pm}^z &= 2\Omega^{\pm} c_m^{\pm} f_0(\hat{\alpha}_3, \hat{\alpha}_4; r). \end{aligned}$$

Since the functions $p_{\pm}$, given explicitly in (5.23), satisfy

$$L_{m \pm 1}^{\zeta} p_{\pm}(\alpha_i, \alpha_j; r) = \frac{1}{a} c_m^{\pm} f_0(\alpha_i, \alpha_j; r),$$

cf. (5.12), particular solutions to these equations are given by

$$\begin{aligned} \delta d_{\pm}^z(r) &= 2a\Omega^{\pm} p_{\pm}(\hat{\alpha}_1, \hat{\alpha}_2; r), \\ \delta b_{\pm}^z(r) &= 2a\Omega^{\pm} p_{\pm}(\hat{\alpha}_3, \hat{\alpha}_4; r). \end{aligned}$$

Adding these solutions to the expressions given in (5.22), we obtain the following solution:

$$\begin{aligned} \delta d_{\pm}^z(r) &= f_{\pm}(\chi_1^{\pm}, \chi_2^{\pm}; r) - 2a\omega V^{\pm} p_{\pm}(\alpha_1, \alpha_2; r) + 2a\Omega^{\pm} p_{\pm}(\hat{\alpha}_1, \hat{\alpha}_2; r), \\ \delta b_{\pm}^z(r) &= f_{\pm}(\chi_3^{\pm}, \chi_4^{\pm}; r) - 2a\omega V^{\pm} p_{\pm}(\alpha_3, \alpha_4; r) + 2a\Omega^{\pm} p_{\pm}(\hat{\alpha}_3, \hat{\alpha}_4; r). \end{aligned} \quad (6.5)$$

6.3 The Remaining Field Components

Since the additional source term in the wave equation is the only correction of relative order $O(\lambda/R)$, the remaining field components are related to the d^z and b^z in the same way as before. Moreover, since the considered correction does not modify δd_0^z and δb_0^z , the main modes are unaltered. The transverse components of the fields $\delta \vec{d}_\pm$ and $\delta \vec{b}_\pm$ are then given by (5.30) and the fields $\delta e_\pm$ and $\delta h_\pm$ are given by (5.33).

No calculation is required to verify that the given corrections are consistent in the sense that (5.24) and (5.25) are satisfied: the argument in Section 5.10 applies without modification, since it does not rely on the explicit form of the z components of $\delta \vec{d}$ and $\delta \vec{b}$.

6.4 Order of Magnitude

We estimate the order of magnitude by comparing the coefficient of α_1 in the first term in (6.5) to the coefficient in the second term (which enters through $\hat{\alpha}_1 = \mp a^2 \beta \omega U^{-2} \alpha_1 + \dots$). Estimating $V^\pm / \Omega^\pm \simeq R$, where R is Earth's radius, we find that this correction is suppressed relative to the main contribution by

$$\frac{2a^2\omega^2}{2\omega U^2} \frac{\Omega^\pm}{V^\pm} \simeq \frac{a^2\omega}{U^2 R}.$$

Approximating the wave vector β as

$$\beta^2 \simeq \frac{1}{2}(n_1^2 + n_2^2)\omega^2,$$

one obtains

$$U^2/a^2 = n_1^2\omega^2 - \beta^2 \simeq \frac{\Delta n^2}{2}\omega^2,$$

where we have defined $\Delta n^2 = n_1^2 - n_2^2$. The relative amplitude is thus roughly

$$\frac{a^2\omega}{U^2 R} \simeq \frac{2}{\Delta n^2 \omega R} = \frac{1}{\pi \Delta n^2} \frac{\lambda}{R},$$

where we have defined λ by

$$\lambda\omega = 2\pi.$$

This results confirms our first estimate, according to which the relative error in neglecting this contribution is of the order of λ/R . Note, however, that this correction can be relevant if the alignment of the waveguide is such that V^+ or V^- is close or equal to zero, invalidating the approximation $\Omega^\pm / V^\pm \simeq 1/R$. Such a situation occurs, for example, at Earth's poles, where the velocity $\vec{V}$ vanishes identically. In this case, the main mode is unaltered (to first order) and the modification of the sidebands is the only influence of Earth's rotation. We also note that the above estimate suggests that this contribution is dominant in the weakly guiding case, where Δn^2 is close to zero.

6.5 Continuity Conditions

We close this section by giving the discontinuity matrix corresponding to the above correction. From (6.5) one can read off that (5.50) is replaced by

$$\llbracket F_{\pm} \rrbracket = D(f_{\pm})\chi^{\pm} - 2a\omega V^{\pm}D(p_{\pm})\alpha - iV^{\pm}D_2^{\pm}\alpha + D_{\lambda}^{\pm}\alpha \stackrel{!}{=} 0, \quad (6.6)$$

where

$$D_{\lambda}^{\pm} = 2a\Omega^{\pm}D(p_{\pm})A.$$

The matrix A , defined in (6.4), transforms the coefficients α to $\hat{\alpha}$, which parameterise $p_{\pm}$ in (6.5). Thus, the matrix product $D(p_{\pm})A$ accounts for the occurrence of $p_{\pm}(\hat{\alpha}_i, \hat{\alpha}_j; r)$ instead of $p_{\pm}(\alpha_i, \alpha_j; r)$.

Similar to the previous case (see e.g. (5.52)), equation (6.6) allows to compute the parameters $\chi^{\pm}$ once the coefficients α have been determined from the continuity conditions for the main mode (5.45).

Chapter 7

Corrections of Relative Order ℓ/R

Hitherto, the velocity vector field $\vec{v}$ was approximated by the velocity of the center of the waveguide $\vec{V}$. In this chapter, we compute the corrections to the fields which arise from the variation of $\vec{v}$ along the symmetry axis of the waveguide.

7.1 Corrections to the Velocity Vector Field

Hitherto, we have approximated

$$\vec{x} = \vec{R} + \vec{r} \approx \vec{R},$$

to obtain the simplified wave equation (3.12) and to compute the transverse field components, e.g. in (5.29) and (5.32). This approximation is expected to produce relative errors of the order of a/R or ℓ/R , where a is the radius of the dielectric, ℓ its length and R is Earth's radius. In this section, we discuss the effect of neglecting only a/R but not ℓ/R , which may be relevant for long waveguides. Instead of the above approximation, we thus write

$$\vec{x} = \vec{R} + \vec{r} \approx R + z\vec{e}_z.$$

The corresponding approximation for the velocity vector field $\vec{v}$ reads

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r} \approx \vec{V} + \vec{u}, \quad (7.1)$$

where the relative velocity $\vec{u} = \vec{\Omega} \times (z\vec{e}_r)$ has the following components when referred to a cylindrical orthonormal frame:

$$u^r = +\Omega^\theta z, \quad u^\theta = -\Omega^r z, \quad u^z = 0.$$

Note that the components of $\vec{u}$ depend on Ω^r and Ω^θ , but not on Ω^z . Hence, to first order in Ω , all terms involving u *do not contribute* to the main Fourier mode, for whenever u is multiplied by a component of an unperturbed field, w^i say, it produces only terms in the sidebands, e.g.

$$\begin{aligned} \langle u(w^i) \rangle_\pm &= \pm z \Omega^\pm c_m^\pm(w^i), \\ \langle (\nabla_{\vec{u}} \vec{w})^r \rangle_\pm &= \pm z \Omega^\pm c_m^\pm(w^r) - iz \Omega^\pm w^\theta / r, \\ \langle (\nabla_{\vec{u}} \vec{w})^\theta \rangle_\pm &= \pm z \Omega^\pm c_m^\pm(w^\theta) + iz \Omega^\pm w^r / r, \\ \langle (\vec{u} \times \vec{w})^r \rangle_\pm &= iz \Omega^\pm w^z, \\ \langle (\vec{u} \times \vec{w})^\theta \rangle_\pm &= \mp z \Omega^\pm w^z, \\ \langle (\vec{u} \times \vec{w})^z \rangle_\pm &= -z \Omega^\pm (iw^r \mp w^\theta). \end{aligned} \quad (7.2)$$

We thus conclude that the entire analysis of the continuity conditions for the main mode is unaltered by the terms introduced here.

7.2 The Wave Equations

In Section 3.5 we have derived the wave equations

$$\begin{aligned}\mu(n^2\ddot{d}^z - \Delta d^z) &= 2i\omega v(D^z), \\ \varepsilon(n^2\ddot{b}^z - \Delta b^z) &= 2i\omega v(B^z).\end{aligned}$$

While the previous analysis has used the approximation $\vec{v} \approx \vec{V}$, we now consider the improved approximation $\vec{v} \approx \vec{V} + \vec{u}$. By linearity we may use the old solution for the source terms proportional to $\vec{V}$ and simply add a solution for the equation where the only source term is proportional to $\vec{u}$. We thus consider the wave equations

$$\begin{aligned}\mu(n^2\delta\ddot{d}^z - \Delta\delta d^z) &= 2i\omega u(D^z), \\ \varepsilon(n^2\delta\ddot{b}^z - \Delta\delta b^z) &= 2i\omega u(B^z).\end{aligned}\tag{7.3}$$

7.3 The Radial Equations

Since the source terms in (7.3) depend on z as $ze^{-i\beta z}$, we consider wave equations of the form

$$(n^2\partial_t^2 - \Delta)f(t, r, \theta, z) = zg(r)e^{i(\omega t - \beta z + \nu\theta)}.$$

The ansatz

$$f(t, r, \theta, z) = \left(f_{(0)}(r) + \frac{z}{a}f_{(1)}(r)\right)e^{i(\omega t - \beta z + \nu\theta)}$$

leads to

$$\frac{z}{a}(L_\nu^\zeta f_{(1)} + ag(r)) + (L_\nu^\zeta f_{(0)} - 2ia^{-1}\beta f_{(1)}) = 0,$$

where (as always) $\zeta = n^2\omega^2 - \beta^2$. Setting the coefficients of z^0 and z^1 to zero, one obtains the following set of equations for $f_{(0)}$ and $f_{(1)}$:

$$\begin{aligned}L_m^\zeta f_1(r) &= -ag(r), \\ L_m^\zeta f_0(r) &= 2ia^{-1}\beta f_1(r).\end{aligned}\tag{7.4}$$

These equations can be solved using the techniques discussed in Appendix B.

Let us apply this to the problem at hand. Since the m -mode of the source term vanishes, we consider the sidebands only. Furthermore, due to the electromagnetic symmetry, we may restrict the discussion to the electric field. The corresponding equations for the magnetic field can then be obtained by applying the symmetry transformation (3.7). Using the decomposition of $u(w^i)$ from (7.2), we find the inhomogeneities to take the form

$$2i\omega \langle u(D^z) \rangle_\pm = \pm 2iz\omega\Omega^\pm c_m^\pm(D^z) = \pm 2iz\omega\Omega^\pm c_m^\pm f_0(\alpha_1, \alpha_2; r).$$

In line with the previous discussion of such equations, we seek solutions of the form

$$\delta d_\pm^r = \left(\delta d_{(0)\pm}^r + \frac{z}{a}\delta d_{(1)\pm}^r\right)e^{i(\omega t - \beta z + (m\pm 1)\theta)},\tag{7.5}$$

for which we obtain the following system of inhomogeneous Bessel equations:

$$\begin{aligned}L_{m\pm 1}^\zeta \delta d_{(1)\pm}^z &= \mp 2ia\omega\Omega^\pm c_m^\pm f_0(\alpha_1, \alpha_2; r), \\ L_{m\pm 1}^\zeta \delta d_{(0)\pm}^z &= 2ia^{-1}\beta \delta d_{(1)\pm}^z.\end{aligned}\tag{7.6}$$

The equation for $\delta d_{(1)\pm}^z$ is of the same form as those for the sidebands in (5.11), for which we have already found a solution. The general solution is thus

$$\delta d_{(1)\pm}^z = \mp 2ia^2\omega\Omega^\pm p_\pm(\alpha_1, \alpha_2; r) + f_\pm(\sigma_1^\pm, \sigma_2^\pm; r),$$

where $\sigma_1^\pm, \sigma_2^\pm$ are constants. Substituting this into the second equation of (7.6), one obtains

$$L_{m\pm 1}^\zeta \delta d_{(0)\pm}^z = \pm 4a\beta\omega\Omega^\pm p_\pm(\alpha_1, \alpha_2; r) + 2i\beta c_m^\pm f_0(\sigma_1^\pm, \sigma_2^\pm; r).$$

The source term proportional to $c_m^\pm f_0$ is of the same form as the one discussed before: a particular solution is given by

$$2ia\beta p_\pm(\sigma_1^\pm, \sigma_2^\pm; r).$$

It thus remains to find a function $\varpi_\pm$ which satisfies

$$L_{m\pm 1}^\zeta \varpi_\pm(\alpha_1, \alpha_2; r) = a^{-2} p_\pm(\alpha_1, \alpha_2; r).$$

Furthermore, $\varpi_\pm$ is required to remain finite as $r \rightarrow 0$ and to decay sufficiently rapidly as $r \rightarrow \infty$. The computation of $\varpi_\pm$ is completely analogous to the one of p_0 or $p_\pm$. We obtain

$$\varpi_\pm(\alpha_1, \alpha_2; r) = \begin{cases} +\alpha_1 \frac{\pi}{2} [J_{m\pm 1}(Ur/a)\Phi_\pm(r/a) + Y_{m\pm 1}(Ur/a)\Pi_\pm(r/a)] & r < a, \\ -\alpha_2 [K_{m\pm 1}(Wr/a)\Upsilon_\pm(r/a) + I_{m\pm 1}(Wr/a)\Xi_\pm(r/a)] & r > a, \end{cases}$$

where we have introduced the functions

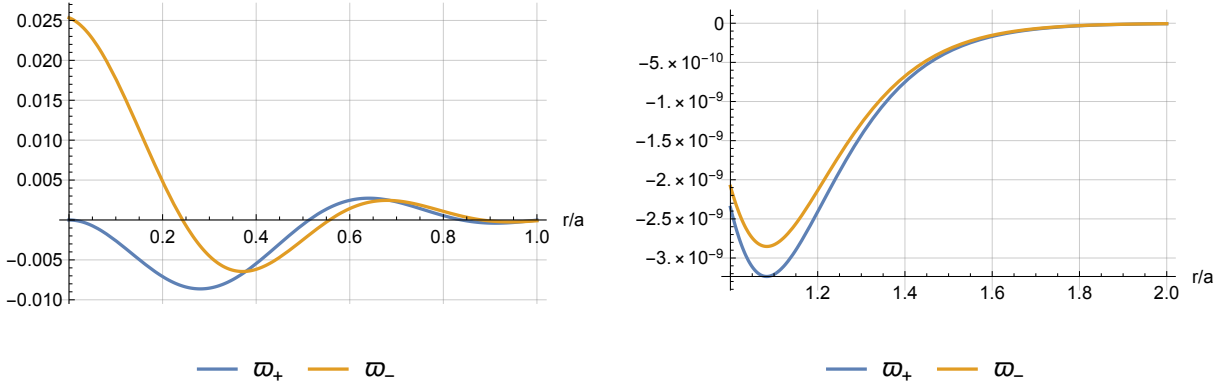
$$\begin{aligned} \Phi_\pm(z) &= \int_z^1 Y_{m\pm 1}(U\xi) p_\pm(1, 1; a\xi) \xi d\xi, \\ \Pi_\pm(z) &= \int_0^z J_{m\pm 1}(U\xi) p_\pm(1, 1; a\xi) \xi d\xi, \\ \Upsilon_\pm(z) &= \int_1^z I_{m\pm 1}(W\xi) p_\pm(1, 1; a\xi) \xi d\xi, \\ \Xi_\pm(z) &= \int_z^\infty K_{m\pm 1}(W\xi) p_\pm(1, 1; a\xi) \xi d\xi. \end{aligned} \tag{7.7}$$

Plots of the functions $\varpi_\pm$ are shown in figure 7.1. As for the functions $p_\pm$, they oscillate in the core and decay exponentially at large distances in the cladding. The functions clearly show the expected regularity at the origin and decay fast enough to ensure that the solution has finite energy.

We conclude that the perturbations of d^z and b^z due to the z dependence of the velocity vector field $\vec{v}$ are given by

$$\begin{aligned} \delta d_{(1)\pm}^z &= f_\pm(\sigma_1^\pm, \sigma_2^\pm; r) \mp 2ia^2\omega\Omega^\pm p_\pm(\alpha_1, \alpha_2; r), \\ \delta b_{(1)\pm}^z &= f_\pm(\sigma_3^\pm, \sigma_4^\pm; r) \mp 2ia^2\omega\Omega^\pm p_\pm(\alpha_3, \alpha_4; r), \\ \delta d_{(0)\pm}^z &= f_\pm(\tau_1^\pm, \tau_2^\pm; r) \pm 4a^3\beta\omega\Omega^\pm \varpi_\pm(\alpha_1, \alpha_2; r) + 2ia\beta p_\pm(\sigma_1^\pm, \sigma_2^\pm; r), \\ \delta b_{(0)\pm}^z &= f_\pm(\tau_3^\pm, \tau_4^\pm; r) \pm 4a^3\beta\omega\Omega^\pm \varpi_\pm(\alpha_3, \alpha_4; r) + 2ia\beta p_\pm(\sigma_3^\pm, \sigma_4^\pm; r). \end{aligned} \tag{7.8}$$

We thus obtain two sets of four parameters each $(\sigma_A^\pm, \tau_B^\pm)$ which must be chosen such that the fields proportional to z^0 and z^1 are continuous *individually*. Note that all coefficients $\alpha_I, \sigma_J^\pm$ and $\tau_K^\pm$ have the same dimension.


 Figure 7.1: Plots of the functions $\varpi_{\pm}$ for $m = 1$ and $U = W = 10$.

7.4 The Transverse Components of d and b

As in Section 5.8, we determine the r - and θ -components of the fields from (5.26). Keeping in mind that the z dependence is different from before, e.g.

$$\delta d^r = \left(\delta d_{(0)}^r + \frac{z}{a} \delta d_{(1)}^r \right) e^{i(\omega t - \beta z)}, \quad (7.9)$$

(the angular dependence has been suppressed) we separate the various powers of z in (5.26). The equations involving z^1 read

$$\begin{aligned} in^2 \omega \delta b_{(1)}^r + i\beta \delta d_{(1)}^\theta &= [\vec{v}, \vec{H}]_{(1)}^r - \partial_\theta \delta d_{(1)}^z / r, \\ in^2 \omega \delta b_{(1)}^\theta - i\beta \delta d_{(1)}^r &= [\vec{v}, \vec{H}]_{(1)}^\theta + \partial_r \delta d_{(1)}^z, \\ i\omega \delta d_{(1)}^r - i\beta \delta b_{(1)}^\theta &= [\vec{v}, \vec{E}]_{(1)}^r + \partial_\theta \delta b_{(1)}^z / r, \\ i\omega \delta d_{(1)}^\theta + i\beta \delta b_{(1)}^r &= [\vec{v}, \vec{E}]_{(1)}^\theta - \partial_r \delta b_{(1)}^z. \end{aligned}$$

while those for z^0 are

$$\begin{aligned} in^2 \omega \delta b_{(0)}^r + i\beta \delta d_{(0)}^\theta - \delta d_{(1)}^\theta / a &= [\vec{v}, \vec{H}]_{(0)}^r - \partial_\theta \delta d_{(0)}^z / r, \\ in^2 \omega \delta b_{(0)}^\theta - i\beta \delta d_{(0)}^r + \delta d_{(1)}^r / a &= [\vec{v}, \vec{H}]_{(0)}^\theta + \partial_r \delta d_{(0)}^z, \\ i\omega \delta d_{(0)}^r - i\beta \delta b_{(0)}^\theta + \delta b_{(1)}^\theta / a &= [\vec{v}, \vec{E}]_{(0)}^r + \partial_\theta \delta b_{(0)}^z / r, \\ i\omega \delta d_{(0)}^\theta + i\beta \delta b_{(0)}^r - \delta b_{(1)}^r / a &= [\vec{v}, \vec{E}]_{(0)}^\theta - \partial_r \delta b_{(0)}^z. \end{aligned}$$

The solution for the z^1 components is

$$\begin{aligned} i\zeta \delta d_{(1)}^r &= n^2 \omega ([\vec{v}, \vec{E}]^r + \partial_\theta \delta b^z / r)_{(1)} + \beta ([\vec{v}, \vec{H}]^\theta + \partial_r \delta d^z)_{(1)}, \\ i\zeta \delta d_{(1)}^\theta &= n^2 \omega ([\vec{v}, \vec{E}]^\theta - \partial_r \delta b^z)_{(1)} - \beta ([\vec{v}, \vec{H}]^r - \partial_\theta \delta d^z / r)_{(1)}, \\ i\zeta \delta b_{(1)}^r &= \omega ([\vec{v}, \vec{H}]^r - \partial_\theta \delta d^z / r)_{(1)} - \beta ([\vec{v}, \vec{E}]^\theta - \partial_r \delta b^z)_{(1)}, \\ i\zeta \delta b_{(1)}^\theta &= \omega ([\vec{v}, \vec{H}]^\theta + \partial_r \delta d^z)_{(1)} + \beta ([\vec{v}, \vec{E}]^r + \partial_\theta \delta b^z / r)_{(1)}, \end{aligned} \quad (7.10)$$

while the one for the z^0 components reads

$$\begin{aligned} i\zeta\delta d_{(0)}^r &= n^2\omega([\vec{v}, \vec{E}]^r + \partial_\theta\delta b^z/r)_{(0)} + \beta([\vec{v}, \vec{H}]^\theta + \partial_r\delta d^z)_{(0)} - n^2a^{-1}\omega\delta b_{(1)}^\theta - a^{-1}\beta\delta d_{(1)}^r, \\ i\zeta\delta d_{(0)}^\theta &= n^2\omega([\vec{v}, \vec{E}]^\theta - \partial_r\delta b^z)_{(0)} - \beta([\vec{v}, \vec{H}]^r - \partial_\theta\delta d^z/r)_{(0)} + n^2a^{-1}\omega\delta b_{(1)}^r - a^{-1}\beta\delta d_{(1)}^\theta, \\ i\zeta\delta b_{(0)}^r &= \omega([\vec{v}, \vec{H}]^r - \partial_\theta\delta d^z/r)_{(0)} - \beta([\vec{v}, \vec{E}]^\theta - \partial_r\delta b^z)_{(0)} + a^{-1}\omega\delta d_{(1)}^\theta - a^{-1}\beta\delta b_{(1)}^r, \\ i\zeta\delta b_{(0)}^\theta &= \omega([\vec{v}, \vec{H}]^\theta + \partial_r\delta d^z)_{(0)} + \beta([\vec{v}, \vec{E}]^r + \partial_\theta\delta b^z/r)_{(0)} - a^{-1}\omega\delta d_{(1)}^r - a^{-1}\beta\delta b_{(1)}^\theta. \end{aligned} \quad (7.11)$$

Hitherto, the Lie bracket of the velocity vector field $\vec{v}$ and any electric or magnetic field $\vec{w}$ was approximated as

$$[\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \vec{\Omega} \times \vec{w} \approx \nabla_{\vec{v}}\vec{w}.$$

Instead, we now write

$$[\vec{v}, \vec{w}] \approx \nabla_{\vec{v}}\vec{w} + \nabla_{\vec{u}}\vec{w} - \vec{\Omega}_\perp \times \vec{w},$$

where the transverse part of $\vec{\Omega}$ is defined as

$$\vec{\Omega}_\perp = \vec{\Omega} - \Omega^z \vec{e}_z.$$

The reason for keeping $\vec{\Omega}_\perp$ is twofold:

1. As we will see later, this is consistent with approximating $\vec{v} \times \vec{B} \approx \vec{V} \times \vec{B} + \vec{u} \times \vec{B}$ (and similarly for $\vec{D}$ instead of $\vec{B}$) in equation (3.5) in the sense that (5.34) is satisfied by the approximate fields.
2. The terms involving $\vec{\Omega}_\perp$ in the r and θ components of the Lie brackets (which are the only ones of interest here) stem from derivatives of those components of $\vec{v}$ which scale linearly with z . This is readily seen in Cartesian coordinates, e.g.

$$(\nabla_{\vec{w}}\vec{v})^x = \nabla_{\vec{w}}(\Omega^y z - \Omega^z x) \approx \nabla_{\vec{w}}(\Omega^y z) = \Omega^y w^z = (\vec{\Omega}_\perp \times \vec{w})^x.$$

We thus obtain

$$\begin{aligned} [\vec{v}, \vec{w}]_\pm^r &\approx \langle \nabla_{\vec{v}} w^r \rangle_\pm \pm z\Omega^\pm c_m^\pm(w^r) - iz\Omega^\pm w^\theta/r \mp \Omega^\pm w^z, \\ [\vec{v}, \vec{w}]_\pm^\theta &\approx \langle \nabla_{\vec{v}} w^\theta \rangle_\pm \pm z\Omega^\pm c_m^\pm(w^\theta) + iz\Omega^\pm w^r/r - i\Omega^\pm w^z. \end{aligned}$$

Separating the terms proportional to z^0 from those proportional to z^1 , we find

$$\begin{aligned} [\vec{v}, \vec{w}]_{(0)\pm}^r &\approx \langle \nabla_{\vec{v}} w^r \rangle_\pm \mp \Omega^\pm w^z, \\ [\vec{v}, \vec{w}]_{(0)\pm}^\theta &\approx \langle \nabla_{\vec{v}} w^\theta \rangle_\pm - i\Omega^\pm w^z, \\ [\vec{v}, \vec{w}]_{(1)\pm}^r &\approx \pm a\Omega^\pm c_m^\pm(w^r) - ia\Omega^\pm w^\theta/r, \\ [\vec{v}, \vec{w}]_{(1)\pm}^\theta &\approx \pm a\Omega^\pm c_m^\pm(w^\theta) + ia\Omega^\pm w^r/r. \end{aligned}$$

One thus obtains the following expression for the z^1 components:

$$\begin{aligned} i\zeta\delta d_{(1)\pm}^r &= \beta\partial_r\delta d_{(1)\pm}^z + in^2\omega\frac{m\pm 1}{r}\delta b_{(1)\pm}^z \pm a\Omega^\pm c_m^\pm(n^2\omega E^r + \beta H^\theta) - \frac{i}{r}a\Omega^\pm(n^2\omega E^\theta - \beta H^r), \\ i\zeta\delta d_{(1)\pm}^\theta &= i\beta\frac{m\pm 1}{r}\delta d_{(1)\pm}^z - n^2\omega\partial_r\delta b_{(1)\pm}^z \pm a\Omega^\pm c_m^\pm(n^2\omega E^\theta - \beta H^r) + \frac{i}{r}a\Omega^\pm(n^2\omega E^r + \beta H^\theta), \\ i\zeta\delta b_{(1)\pm}^r &= \beta\partial_r\delta b_{(1)\pm}^z - i\omega\frac{m\pm 1}{r}\delta d_{(1)\pm}^z \pm a\Omega^\pm c_m^\pm(\omega H^r - \beta E^\theta) - \frac{i}{r}a\Omega^\pm(\omega H^\theta + \beta E^r), \\ i\zeta\delta b_{(1)\pm}^\theta &= i\beta\frac{m\pm 1}{r}\delta b_{(1)\pm}^z + \omega\partial_r\delta d_{(1)\pm}^z \pm a\Omega^\pm c_m^\pm(\omega H^\theta + \beta E^r) + \frac{i}{r}a\Omega^\pm(\omega H^r - \beta E^\theta). \end{aligned} \quad (7.12)$$

For the z^0 components, we merely write the corrections to (5.30):

$$\begin{aligned}
 i\zeta\delta d_{(0)\pm}^r &= \Omega^\pm(\mp n^2\omega E^z - i\beta H^z) - n^2a^{-1}\omega\delta b_{(1)\pm}^\theta - a^{-1}\beta\delta d_{(1)\pm}^r + \dots, \\
 i\zeta\delta d_{(0)\pm}^\theta &= \Omega^\pm(\pm\beta H^z - in^2\omega E^z) + n^2a^{-1}\omega\delta b_{(1)\pm}^r - a^{-1}\beta\delta d_{(1)\pm}^\theta + \dots, \\
 i\zeta\delta b_{(0)\pm}^r &= \Omega^\pm(\mp\omega H^z + i\beta E^z) + a^{-1}\omega\delta d_{(1)\pm}^\theta - a^{-1}\beta\delta b_{(1)\pm}^r + \dots, \\
 i\zeta\delta b_{(0)\pm}^\theta &= \Omega^\pm(\mp\beta E^z - i\omega H^z) - a^{-1}\omega\delta d_{(1)\pm}^r - a^{-1}\beta\delta b_{(1)\pm}^\theta + \dots.
 \end{aligned} \tag{7.13}$$

Here, the suppressed terms (denoted by $\dots$) represent the terms given explicitly in (5.30).

7.5 The Field Components of e and h

We determine the θ - and z -components of the fields $\vec{e}$ and $\vec{h}$ from (3.5) using the perturbative expansion (5.1) and the approximation (7.1), from which we obtain

$$\begin{aligned}
 n^2\delta\vec{e} &= \delta\vec{d} + \vec{V} \times \vec{H} + \vec{u} \times \vec{H}, \\
 \delta\vec{h} &= \delta\vec{b} - \vec{V} \times \vec{E} - \vec{u} \times \vec{E}.
 \end{aligned}$$

From (7.2) we have the following decomposition of the cross product of $\vec{u}$ with any unperturbed field $\vec{w}$:

$$\begin{aligned}
 \langle(\vec{u} \times \vec{w})^\theta\rangle_\pm &= \mp z\Omega^\pm w^z, \\
 \langle(\vec{u} \times \vec{w})^z\rangle_\pm &= -z\Omega^\pm(iw^r \mp w^\theta).
 \end{aligned}$$

For the z^0 component, one obtains

$$\begin{aligned}
 n^2\delta e_{(0)\pm}^\theta &= \delta d_{(0)\pm}^\theta + iV^\pm H^z, \\
 n^2\delta e_{(0)\pm}^z &= \delta d_{(0)\pm}^z - V^\pm(iH^\theta \pm H^r), \\
 \delta h_{(0)\pm}^\theta &= \delta b_{(0)\pm}^\theta - iV^\pm E^z, \\
 \delta h_{(0)\pm}^z &= \delta b_{(0)\pm}^z + V^\pm(iE^\theta \pm E^r),
 \end{aligned} \tag{7.14}$$

as in (5.33); whereas for the z^1 components one finds

$$\begin{aligned}
 n^2\delta e_{(1)\pm}^\theta &= \delta d_{(1)\pm}^\theta \mp a\Omega^\pm H^z, \\
 n^2\delta e_{(1)\pm}^z &= \delta d_{(1)\pm}^z - a\Omega^\pm(iH^r \mp H^\theta), \\
 \delta h_{(1)\pm}^\theta &= \delta b_{(1)\pm}^\theta \pm a\Omega^\pm E^z, \\
 \delta h_{(1)\pm}^z &= \delta b_{(1)\pm}^z + a\Omega^\pm(iE^r \mp E^\theta).
 \end{aligned} \tag{7.15}$$

7.6 Consistency of the Approximations

We verify that the approximations made in the previous calculations are consistent with the equations (5.24) and (5.25). As in Section 7.6, we write $x \rightsquigarrow y$ if x differs from y by terms whose consistency has already been established. Due to the electromagnetic symmetry, it again suffices to show that

$$i\omega\delta d_\pm^r - i\frac{m \pm 1}{r}\delta h_\pm^z + \partial_z\delta h_\pm^\theta = 0.$$

We show the consistency of the approximations for the terms scaling as z^0 or z^1 separately.

z^1 terms:

From (7.12) we have

$$i\omega\delta d_{(1)\pm}^r - i\beta\delta b_{(1)\pm}^\theta \rightsquigarrow \pm a\Omega^\pm c_m^\pm(E^r) - i\frac{a}{r}\Omega^\pm E^\theta.$$

Using (7.15) and the fact that $(\partial_z f)_{(1)} = -i\beta f_{(1)}$, we find

$$(\partial_z \delta h^\theta + i\beta \delta b^\theta)_{(1)\pm} = \mp i a \beta \Omega^\pm E^z = \pm a \Omega^\pm \partial_z E^z.$$

From (7.15) it also follows that

$$\delta h_{(1)\pm}^z \rightsquigarrow a\Omega^\pm (iE^r \pm E^\theta).$$

We thus have

$$\begin{aligned} & i\omega\delta d_{(1)\pm}^r - i\frac{m \pm 1}{r}\delta h_{(1)\pm}^z + (\partial_z \delta h^\theta)_{(1)\pm} \\ &= (i\omega\delta d_{(1)\pm}^r - i\beta\delta b_{(1)\pm}^\theta) - i\frac{m \pm 1}{r}\delta h_{(1)\pm}^z + (\partial_z \delta h^\theta + i\beta\delta b^\theta)_{(1)\pm} \\ &\rightsquigarrow \pm a\Omega^\pm (\nabla \cdot \vec{E}) = 0. \end{aligned}$$

which establishes the consistency of the approximations for the z^1 terms.

z^0 terms:

From (7.14) we have

$$i\omega\delta d_{(0)\pm}^r - i\beta\delta b_{(0)\pm}^\theta \rightsquigarrow \mp \Omega^\pm E^z - a^{-1}\delta b_{(1)\pm}^\theta.$$

Using the fact that

$$(\partial_z \delta h_\pm^\theta)_{(0)} = -i\beta \partial_z \delta h_{(0)\pm}^\theta + a^{-1}\delta h_{(1)\pm}^\theta,$$

which follows from

$$\delta h_\pm^\theta = \delta h_{(0)\pm}^\theta + \frac{z}{a}\delta h_{(1)\pm}^\theta,$$

and using the relation between $\delta h_{(1)\pm}^\theta$ and $\delta b_{(1)\pm}^\theta$ from (7.15), we find

$$(\partial_z \delta h_\pm^\theta + i\beta \delta b_\pm^\theta)_{(0)\pm} \rightsquigarrow \pm \Omega^\pm E^z + a^{-1}\delta b_{(1)\pm}^\theta.$$

Moreover, from (7.14) it follows that

$$\delta h_{(0)\pm}^z \rightsquigarrow 0.$$

Using all these intermediate results, we arrive at

$$i\omega\delta d_{(0)\pm}^r + (\partial_z \delta h_\pm^\theta)_{(0)} = i\omega\delta d_{(0)\pm}^r - i\beta\delta b_{(0)\pm}^\theta + (\partial_z \delta h_\pm^\theta + i\beta\delta b_\pm^\theta)_{(0)} \rightsquigarrow 0,$$

which shows that the approximations are fully consistent.

7.7 Continuity Conditions

We discuss the modifications to the continuity conditions due to the corrections derived in the previous sections.

Corresponding to the decomposition (7.5), the jump heights $\llbracket F \rrbracket$ contain terms which – except for $e^{-i\beta z}$ – are independent of z , while all other terms scale linearly with z . Denoting these terms by $\llbracket F_{(0)} \rrbracket$ and $\llbracket F_{(1)} \rrbracket$, respectively, we have

$$\llbracket F \rrbracket = \llbracket F_{(0)} \rrbracket + \llbracket F_{(1)} \rrbracket z/a \stackrel{!}{=} 0.$$

Since the fields are required to be continuous for all values of z , it follows that $\llbracket F_{(0)} \rrbracket$ and $\llbracket F_{(1)} \rrbracket$ must both vanish.

Without the ℓ/R correction, $\llbracket F_{(1)} \rrbracket$ vanishes identically and $\llbracket F_{(0)} \rrbracket$ is given by (5.50). By linearity, the corrections of order ℓ/R will add additional terms. Henceforward, we will assume that the coefficients $\chi^\pm$ are chosen such that the expression given in (5.50) vanishes, so that we may write the corrections to $\llbracket F_{(0)} \rrbracket$ without copying the old terms.

We show that the jump height $\llbracket F_{(1)\pm} \rrbracket$ is given by

$$\llbracket F_{(1)\pm} \rrbracket = D(f_\pm) \sigma^\pm \mp 2ia^2 \omega \Omega^\pm D(p_\pm) \alpha \pm a \Omega^\pm D_2 \alpha, \quad (7.16)$$

where the first two terms are due to the functions $p_\pm$ and $f_\pm$ in the expressions for the z components of the fields, given in (7.8). The matrix D_2 is the same one that occurred already in (5.50) since the $\Omega^\pm$ terms in (7.12) and (7.15) are obtained from the $V^\pm$ terms in (5.30) and (5.33) by the substitution $V^\pm \mapsto \pm ia \Omega^\pm$. The coefficients $-iV^\pm$ and $\pm a \Omega^\pm$ of D_2 in (5.50) and (7.16) reflect this symmetry.

Next, we compute the jump height $\llbracket F_{(0)\pm} \rrbracket$. From (7.8) we can read off the following structure:

$$\llbracket F_{(0)\pm} \rrbracket = D(f_\pm) \tau^\pm \pm 4a^3 \beta \omega \Omega^\pm D(\varpi_\pm) \alpha + 2ia \beta D(p_\pm) \sigma^\pm + a \Omega^\pm D_3^\pm \alpha + \text{coupling}. \quad (7.17)$$

The first three terms are due to the form of the z components given in (7.8), $D_3^\pm$ stems from the $\Omega^\pm$ terms in (7.13) and the coupling terms contain all contributions which arise due to the presence of $\delta d_{(1)}^r, d_{(1)}^\theta, b_{(1)}^r$ and $b_{(1)}^\theta$ in (7.13).

By comparison of (5.23) with (7.7) we see that $D(\varpi_\pm)$ is obtained from $D(p_\pm)$ by the substitution

$$\Delta_\pm \mapsto \mp \frac{1}{U} \Pi_\pm, \quad T_\pm \mapsto -\frac{1}{W} \Xi_\pm,$$

so we obtain

$$D(\varpi_\pm) = \begin{pmatrix} -i \frac{a\beta}{U} Y'_{m\pm 1} & i \frac{a\beta}{W} I'_{m\pm 1} & n_1^2 \frac{(m\pm 1)a\omega}{U^2} Y_{m\pm 1} & -n_2^2 \frac{(m\pm 1)a\omega}{W^2} I_{m\pm 1} \\ \frac{1}{n_1^2} Y_{m\pm 1} & \frac{1}{n_2^2} I_{m\pm 1} & 0 & 0 \\ -\frac{(m\pm 1)a\omega}{U^2} Y_{m\pm 1} & \frac{(m\pm 1)a\omega}{W^2} I_{m\pm 1} & -i \frac{a\beta}{U} Y'_{m\pm 1} & i \frac{a\beta}{W} I'_{m\pm 1} \\ 0 & 0 & Y_{m\pm 1} & I_{m\pm 1} \end{pmatrix} \Lambda^\pm(\Pi, \Xi),$$

where $\Lambda^\pm(\Pi, \Xi)$ is defined as

$$\Lambda^\pm(\Pi, \Xi) = \text{diag} \left(\frac{\pi}{2} \Pi_\pm(1), \Xi_\pm(1), \frac{\pi}{2} \Pi_\pm(1), \Xi_\pm(1) \right).$$

Next, we show that the matrix $D_3^\pm$, which encodes the jump heights due to terms multiplied by Ω in (7.13) and (7.14), is given by

$$D_3^\pm = \begin{pmatrix} \pm i \frac{a\omega}{U^2} J_m & \pm i \frac{a\omega}{W^2} K_m & -\frac{a\beta}{U^2} J_m & -\frac{a\beta}{W^2} K_m \\ 0 & 0 & 0 & 0 \\ \frac{1}{n_1^2} \frac{a\beta}{U^2} J_m & \frac{1}{n_2^2} \frac{a\beta}{W^2} K_m & \pm i \frac{a\omega}{U^2} J_m & \pm i \frac{a\omega}{W^2} K_m \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Indeed, from (7.13) we have

$$i\zeta\delta d_{(0)\pm}^r = \Omega^\pm(\mp\omega D^z - i\beta B^z) + \dots,$$

which entails

$$\llbracket \delta d_{(0)\pm}^r \rrbracket = a\Omega^\pm \left(\pm i \frac{a\omega}{U^2} J_m \quad \pm i \frac{a\omega}{W^2} K_m \quad -\frac{a\beta}{U^2} J_m \quad -\frac{a\beta}{W^2} K_m \right) + \dots,$$

which determines the first row of $D_3^\pm$. The second row vanishes since the expression for $\delta e_{(0)\pm}^z$ in (7.14) has no term which is multiplied by $\Omega^\pm$. As usual, the last two rows are determined completely by the first two rows.

Finally, we consider the coupling terms introduced in (7.17). Since the fields $\delta d_{(1)}^r, \delta d_{(1)}^\theta$ etc. in (7.11) yield contributions both from the unperturbed fields, which are parameterised by α , and also contributions from the fields $\delta d_{(1)}^z, \delta b_{(1)}^z$, which – according to (7.8) – are parameterised by α and $\sigma^\pm$, we have

$$\text{Coupling terms} = ia\Omega^\pm C_1 \alpha + 2ia^2\omega\Omega^\pm C_2 \alpha + C_3 \sigma^\pm.$$

The factors were chosen for convenience. For the matrix C_1 , which contains all contributions from the unperturbed fields, we find

$$C_1 = \begin{pmatrix} \frac{a\omega}{U^2} \frac{\widehat{U}^2}{U^2} J'_{m\pm 1} & \pm \frac{a\omega}{W^2} \frac{\widehat{W}^2}{W^2} K'_{m\pm 1} & i \frac{m\pm 1}{U} \frac{a\beta}{U^2} \frac{\widetilde{U}^2}{U^2} J_{m\pm 1} & \pm i \frac{m\pm 1}{W} \frac{a\beta}{W^2} \frac{\widetilde{W}^2}{W^2} K_{m\pm 1} \\ 0 & 0 & 0 & 0 \\ -i \frac{m\pm 1}{n_1^2} \frac{a\beta}{U} \frac{\widetilde{U}^2}{U^2} J_{m\pm 1} & \mp i \frac{m\pm 1}{n_2^2} \frac{a\beta}{W} \frac{\widetilde{W}^2}{W^2} K_{m\pm 1} & \frac{a\omega}{U^2} \frac{\widehat{U}^2}{U^2} J'_{m\pm 1} & \pm \frac{a\omega}{W^2} \frac{\widehat{W}^2}{W^2} K'_{m\pm 1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where we have set

$$\begin{aligned} \widehat{U}^2 &= 2a^2\beta^2 + \bar{U}^2 \equiv n_1^2 a^2 \omega^2 + 3a^2 \beta^2, & \widetilde{U}^2 &= 2n_1^2 a^2 \omega^2 + \bar{U}^2 \equiv 3n_1^2 a^2 \omega^2 + a^2 \beta^2, \\ \widehat{W}^2 &= 2a^2\beta^2 + \bar{W}^2 \equiv n_2^2 a^2 \omega^2 + 3a^2 \beta^2, & \widetilde{W}^2 &= 2n_2^2 a^2 \omega^2 + \bar{W}^2 \equiv 3n_2^2 a^2 \omega^2 + a^2 \beta^2. \end{aligned}$$

C_2 , which contains all contributions due to the $p_\pm$ terms in (7.8), is given by

$$C_2 = \begin{pmatrix} \frac{\bar{U}^2}{U^2} Y'_{m\pm 1} & \pm \frac{\bar{W}^2}{W^2} I'_{m\pm 1} & in_1^2 \frac{m\pm 1}{U} \frac{2a^2\beta\omega}{U^2} Y_{m\pm 1} & \pm in_2^2 \frac{m\pm 1}{W} \frac{2a^2\beta\omega}{W^2} I_{m\pm 1} \\ 0 & 0 & 0 & 0 \\ -i \frac{m\pm 1}{U} \frac{2a^2\beta\omega}{U^2} Y_{m\pm 1} & \mp i \frac{m\pm 1}{W} \frac{2a^2\beta\omega}{W^2} I_{m\pm 1} & \frac{\bar{U}^2}{U^2} Y'_{m\pm 1} & \pm \frac{\bar{W}^2}{W^2} I'_{m\pm 1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Lambda,$$

where we have factored the matrix Λ , defined as

$$\Lambda = \Lambda^\pm(\Delta, T) = \text{diag} \left(\frac{\pi}{2} \Delta_\pm(1), T_\pm(1), \frac{\pi}{2} \Delta_\pm(1), T_\pm(1) \right),$$

for readability. Finally, C_3 , which arises due to the $f_\pm$ terms in (7.8) is found to be

$$C_3 = \begin{pmatrix} \mp \frac{\bar{U}^2}{U^2} J'_{m\pm 1} & \frac{\bar{W}^2}{W^2} K'_{m\pm 1} & \mp in_1^2 \frac{m\pm 1}{U} \frac{2a^2\beta\omega}{U^2} J_{m\pm 1} & in_2^2 \frac{m\pm 1}{W} \frac{2a^2\beta\omega}{W^2} K_{m\pm 1} \\ 0 & 0 & 0 & 0 \\ \pm i \frac{m\pm 1}{U} \frac{2a^2\beta\omega}{U^2} J_{m\pm 1} & -i \frac{m\pm 1}{W} \frac{2a^2\beta\omega}{W^2} K_{m\pm 1} & \mp \frac{\bar{U}^2}{U^2} J'_{m\pm 1} & \frac{\bar{W}^2}{W^2} K'_{m\pm 1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Given the parameters α of the main mode, the coefficients $\sigma^\pm$ are now determined by the equation

$$\llbracket F_{(1)\pm} \rrbracket = D(f_{\pm})\boldsymbol{\sigma}^{\pm} \mp 2ia^2\omega\Omega^{\pm}D(p_{\pm})\boldsymbol{\alpha} \pm a\Omega^{\pm}D_2\boldsymbol{\alpha} \stackrel{!}{=} 0. \quad (7.18)$$

The solution to this equation is then used to determine the parameters $\boldsymbol{\tau}^{\pm}$ via the continuity condition

$$\begin{aligned} \llbracket F_{(0)\pm} \rrbracket &= D(f_{\pm})\boldsymbol{\tau}^{\pm} + 2ia\beta D(p_{\pm})\boldsymbol{\sigma}^{\pm} + C_3\boldsymbol{\sigma}^{\pm} \\ &\pm 4a^3\beta\omega\Omega^{\pm}D(\varpi_{\pm})\boldsymbol{\alpha} + a\Omega^{\pm}D_3^{\pm}\boldsymbol{\alpha} + ia\Omega^{\pm}C_1\boldsymbol{\alpha} + 2ia^2\omega\Omega^{\pm}C_2\boldsymbol{\alpha} \stackrel{!}{=} 0. \end{aligned} \quad (7.19)$$

In general, these equations are to be solved numerically.

Chapter 8

Numerical Examples

We apply our formulae to the same setup as considered in [2]. Table 8.1 lists both the parameters of the waveguide (refractive indices and core diameter) and the parameters of the light wave (wavelength and angular mode number).

	Quantity	Magnitude
n_1	Refractive index (core)	1,4712
n_2	Refractive index (cladding)	1,4659
n	Effective refractive index	1,4682
a	Core diameter	4,1 μm
λ	Wavelength	1550 nm
m	Angular mode number	1

Table 8.1: Parameters used for numerical examples.

The effective refractive index n was found by solving (5.48) numerically, which yields

$$\beta \approx 1.4682 \omega .$$

This is well approximated by

$$\sqrt{(n_1^2 + n_2^2)/2} \approx 1.4685 .$$

which was used for the estimates in Section 6.4.

8.1 Corrections to the Dispersion Relation

The results derived in this work may be applied both to Earth's rotation about its own axis (spin) and its orbital rotation about the Sun, if the eccentricity of Earth's orbit is neglected.

In the first case, we use $R = R_E \approx 6371$ km and $\Omega = 2\pi / \text{d}$. Since Earth is slightly oblate, more accurate results could be obtained by setting R to be the local distance to Earth's axis. To illustrate the dependence of the effect on the latitude ψ , we compute $\delta\beta$ both at the equator ($\psi = \pi/2$) and in Vienna ($\psi = 48^\circ 12' 30''$). In both cases, we assume the waveguide to be aligned with the velocity vector due to the rotation, i.e. the z axis points east.

To estimate the effect of Earth's orbital motion, we set $\Omega = 2\pi / \text{yr}$ and $R = 1$ au. Again, we choose to align the waveguide such that the z axis points along the momentary direction of motion.

Note however, that if the alignment relative to Earth is kept fixed, then V^z varies over time due to Earth's spin, so $\delta\beta$ will oscillate with a period of one day.

	Spin (Equator)	Spin (Vienna)	Orbit
Geometrical Optics	$1,0526 \cdot 10^{-6}$	$7,8478 \cdot 10^{-7}$	$6,7669 \cdot 10^{-5}$
Linearisation	$1,0526 \cdot 10^{-6}$	$7,8478 \cdot 10^{-7}$	$6,7669 \cdot 10^{-5}$
Exact	$1,0527 \cdot 10^{-6}$	$7,8487 \cdot 10^{-7}$	$6,8408 \cdot 10^{-5}$

Table 8.2: Numerical results for $\delta\beta/\beta$ due to Earth's rotation. Here and in the following tables, “spin” refers to Earth's rotation about its own axis and “orbit” refers to the rotation about the Sun.

Table 8.2 summarises the numerical results for the relative shift $\delta\beta/\beta$. Here, the first estimate was obtained from (2.9), the “linearisation” result was obtained from (5.49) and the “exact” number was found by solving (5.47) numerically. The linearised result agrees numerically with the prediction from geometrical optics which approximates the “exact” result reasonably well.

While these numbers quantify the anisotropy in the propagation of light, one may also compare the wavelength of two parallel light rays propagating at different heights, e.g. in a Mach-Zehnder type interferometer whose two arms are displaced by a height difference Δh . In view of the high quality of the geometrical optics approximation, we discuss this setup using the formula (2.9). If ϑ denotes the angle between the direction of wave propagation and the local velocity vector field $\vec{v}$, the geometrical optics formula

$$\delta\beta = \omega v_{\parallel} = \omega v \cos \vartheta,$$

implies a difference in wavelengths in the two arms of

$$\Delta\beta = \omega\Omega\Delta h \cos \vartheta.$$

If ℓ denotes the length of the interferometer arms, the phase difference at the recombination point evaluates to

$$\Delta\Phi_{\text{rot.}} = \omega A \cos \vartheta,$$

where $A = \ell\Delta h$ is the enclosed area.

We compare this effect with the shift of β caused by Earth's gravitational field. In [2] it was found that a vertical displacement of the dielectric by a height δh results in the following first order change in the wavelength β :

$$\left. \frac{\delta\beta}{\beta} \right|_{\text{G}} \approx -\delta h \times 2,19 \cdot 10^{-16} \text{ m}^{-1}.$$

Since $-2g/c^2 \approx 2,18 \cdot 10^{-16} \text{ m}^{-1}$, this is in good agreement with the following formula, which we derive in Appendix D:

$$\left. \frac{\delta\beta}{\beta} \right|_{\text{G}} \approx -2g \delta h / c^2,$$

where g is the local gravitational acceleration. Even for vertical distances of the order of 100 m, the anisotropy effect due to Earth's spin in Vienna is seven orders of magnitude larger. This suggests that second order effects due to Earth's spin may still be comparable with the first order effect due to Earth's gravitational field. Furthermore, the third order effect due to Earth's orbital motion might still be larger than the first order gravitational effect. At such a level of accuracy, additional effects stemming from the eccentricity of the orbit might become relevant as well.

In the same Mach-Zehnder setup as described above, the phase shift due to the gravitational potential is found to be given by

$$\Delta\Phi_{\text{grav.}} = n\omega g A ,$$

see e.g. [4]. Comparing with the previous formula and restoring the correct powers of c , one finds obtains the ratio

$$\frac{\Delta\Phi_{\text{rot.}}}{\Delta\Phi_{\text{grav.}}} = \frac{\Omega c}{ng} \cos\vartheta .$$

In the considered setup, the maximal ration of the two phase shifts evaluates to $\Omega c/(ng) \approx 1,6 \cdot 10^3$, so the rotational effect is three orders of magnitude larger than the gravitational one.

Let us also note that Earth's spin causes a height difference of $\Delta h = 2R_E$ in the gravitational field of the Sun twice a day. Since the Sun has a local acceleration of roughly $g_S \approx 5,93 \cdot 10^{-3} \text{ m/s}^2$, which – by the above formula – corresponds to a relative shift in β of $-1,68 \cdot 10^{-12}$. Even though this effect is much larger than the gravitational effect discussed in [2], it is unlikely to be directly observable since the effect seems to effect light in all arms of an interferometer in the same way.

8.2 Main Mode Amplitudes

Having solved for β in terms of ω , the coefficients α must be in the kernel of the continuity matrix for the main mode, as stated in (5.45). The kernel was computed numerically and the spanning vector α was normalised to $\alpha_1 = 1$.

	Unperturbed	Spin (Equator)	Spin (Vienna)	Orbit
α_1	1	1	1	1
α_2	1,6799	1,6800	1,6800	1,6811
$i\alpha_3$	0,6769	0,6765	0,6766	0,6512
$i\alpha_4$	1,1454	1,1447	1,1449	1,1027

Table 8.3: Comparison of the main mode coefficients α in the unperturbed and perturbed case. The linearity of the equations was used to normalise the coefficients such that $\alpha_1 = 1$. “Spin” refers to Earth's intrinsic rotation and “Orbit” to the rotation around the Sun.

Table 8.3 shows the numerical results for the coefficients α (normalised to $\alpha_1 = 1$). As before, the alignment was chosen such that the direction of wave propagation is parallel to the local velocity vector $\vec{V}$. We note that the perturbation changes only the modulus of the coefficients, but not their phase.

Comparing, for example, the various results for α_4 , we find a relative change of 3.7% due to Earth's orbit around the Sun and a change of only 0.06% due to its intrinsic rotation at the equator. This ratio is due to the fact that Earth's orbital velocity is roughly 60 times larger than the tangential velocity at the equator.

8.3 Sideband Amplitudes

Having found to coefficients α , which determine the amplitudes of the main mode, one may compute the coefficients for the sidebands. If all corrections of relative order $O(\lambda/R)$ and $O(\ell/R)$ are

neglected, the sidebands are determined by the coefficients $\chi^\pm$ alone (cf. (5.50)). However, if these corrections are taken into account, one has to introduce the additional parameters $\sigma^\pm$ and $\tau^\pm$ (cf. (7.8)).

We first discuss the simpler case, where terms of order $O(\lambda/R)$ and $O(\ell/R)$ are neglected. In this case, the parameters $\chi^\pm$ are determined by (5.52). Clearly, the parameters depend linearly on $V^\pm$. For the orientation discussed before, where $\vec{V} = V^z \mathbf{e}_z$ the components $V^\pm$ vanish, so we thus consider a different orientation of the waveguide, where the z axis is parallel to the angular velocity vector $\vec{\Omega}$ (i.e. parallel to the axis of rotation) such that $\vec{\Omega} = \Omega^z \mathbf{e}_z$. When considering Earth's rotation about its own axis, this means that the guided waves propagate north.

	Spin (Equator)	Spin (Vienna)	Orbit
χ_1^-	$2,31 \cdot 10^{-5}$	$1,73 \cdot 10^{-5}$	$1,49 \cdot 10^{-3}$
χ_2^-	$1,33 \cdot 10^{-5}$	$9,90 \cdot 10^{-6}$	$8,53 \cdot 10^{-4}$
χ_3^-	$1,56 \cdot 10^{-5}$	$1,16 \cdot 10^{-5}$	$1,00 \cdot 10^{-3}$
χ_4^-	$9,05 \cdot 10^{-6}$	$6,75 \cdot 10^{-6}$	$5,82 \cdot 10^{-4}$
χ_1^+	$1,74 \cdot 10^{-5}$	$1,30 \cdot 10^{-5}$	$1,12 \cdot 10^{-3}$
χ_2^+	$4,83 \cdot 10^{-6}$	$3,60 \cdot 10^{-6}$	$3,11 \cdot 10^{-4}$
χ_3^+	$1,18 \cdot 10^{-5}$	$8,77 \cdot 10^{-6}$	$7,56 \cdot 10^{-4}$
χ_4^+	$3,30 \cdot 10^{-6}$	$2,46 \cdot 10^{-6}$	$2,21 \cdot 10^{-4}$

Table 8.4: Absolute values of the coefficients $\chi^\pm$ when the waveguide is aligned in parallel with the axis of rotation $\vec{\Omega}$. “Spin” refers to Earth's intrinsic rotation and “Orbit” to the rotation around the Sun.

In general, the various coefficients $\chi^\pm$ have different phases, but since we are mainly interested in the amplitudes of the sidebands relative to the main mode, we restrict our attention to their absolute values.

Table 8.4 shows the moduli of the sideband coefficients $\chi^\pm$ for a setup where the waveguide is aligned in parallel with the axis of rotation. Since the amplitudes scale linearly with $V^\pm$, the influence of Earth's orbital motion is two orders of magnitude larger than the one of Earth's spin. Note that the energy of the electromagnetic field scales with the square of the amplitudes, so the excitation of the sidebands due to Earth's orbit carry about six orders of magnitude less energy than the main mode, whereas those due to the intrinsic rotation carry roughly ten orders of magnitude less energy.

We now consider the more exact calculation, where corrections of relative order $O(\lambda/R)$ and $O(\ell/R)$ are taken into account. As can be seen from (6.3) and (7.6), both contributions scale with $\Omega^\pm$ and are independent of the velocity vector $\vec{V}$. Thus, these corrections vanish in the above configuration where the wave propagation is parallel to the axis of rotation (in which case we have $\vec{\Omega} = \Omega^z \mathbf{e}_z$ and consequently $\Omega^\pm = 0$). We thus return to the first alignment, where the waveguide is oriented such that $\vec{V} = V^z \mathbf{e}_z$ (i.e. the waves propagate along the rotational direction of motion of the laboratory), where the contribution to $\llbracket F_\pm \rrbracket$ from (5.50) vanishes. In this case, the coefficients $\chi^\pm$ are thus determined by (6.6) alone. The absolute values of $\chi^\pm$ for this setup are listed in table 8.5. For a generic orientation, $\llbracket F_\pm \rrbracket$ would be given by the sum of the terms written in (5.50) and (6.6).

Comparing the value of χ_1^- at the equator with the one given in table 8.4, we find that the correction which was estimated to be of relative order λ/R is suppressed by roughly $2,3 \cdot 10^{-13}$, which is indeed close to $\lambda/R_E \approx 2,4 \cdot 10^{-13}$. Even though this ratio is different for the other parameters, they are generally suppressed by more than eleven orders of magnitude. The λ/R corrections arising from

	Spin (Equator)	Spin (Vienna)	Orbit
χ_1^-	$5,25 \cdot 10^{-18}$	$5,25 \cdot 10^{-18}$	$1,42 \cdot 10^{-20}$
χ_2^-	$1,26 \cdot 10^{-16}$	$1,26 \cdot 10^{-16}$	$3,39 \cdot 10^{-19}$
χ_3^-	$2,88 \cdot 10^{-18}$	$2,88 \cdot 10^{-18}$	$7,79 \cdot 10^{-21}$
χ_4^-	$8,60 \cdot 10^{-17}$	$8,60 \cdot 10^{-17}$	$2,31 \cdot 10^{-19}$
χ_1^+	$4,99 \cdot 10^{-20}$	$6,04 \cdot 10^{-20}$	$6,61 \cdot 10^{-21}$
χ_2^+	$8,12 \cdot 10^{-20}$	$8,14 \cdot 10^{-20}$	$4,75 \cdot 10^{-23}$
χ_3^+	$2,12 \cdot 10^{-19}$	$2,05 \cdot 10^{-19}$	$5,16 \cdot 10^{-21}$
χ_4^+	$6,70 \cdot 10^{-20}$	$6,68 \cdot 10^{-20}$	$3,04 \cdot 10^{-22}$

Table 8.5: Absolute values of the sideband coefficients $\chi^\pm$ when the waveguide is aligned in parallel with the velocity vector $\vec{V}$. In the unperturbed case, all coefficients $\chi^\pm$ vanish. “Spin” refers to Earth’s intrinsic rotation and “Orbit” to the rotation around the Sun.

Earth’s orbital motion behave entirely analogously: the value of χ_1^- in table 8.5 is roughly $9,6 \cdot 10^{-18}$ times smaller than the one listed in 8.4, which is close to the ratio of the wavelength to the astronomical unit, which evaluates to $1,0 \cdot 10^{-7}$ in the considered case.

Since the terms of relative order $O(\lambda/R)$ scale with $\vec{\Omega}$ instead of $\vec{V}$, the influence of Earth’s rotation about its own axis exceeds the one of the rotation about the Sun by about two orders of magnitude. Note, however, that the various coefficients scale differently since the parameters α are not the same in the considered cases.

Next, we consider the coefficients $\sigma^\pm$, which determine the part of the fields which grow linearly along the length of the waveguide (see e.g (7.8)). These parameters are related to the coefficients α and $\chi^\pm$ by the matrix equation (7.18).

	Spin (Equator)	Spin (Vienna)	Orbit
σ_1^-	$6,58 \cdot 10^{-18}$	$6,58 \cdot 10^{-18}$	$1,80 \cdot 10^{-20}$
σ_2^-	$5,38 \cdot 10^{-18}$	$5,38 \cdot 10^{-18}$	$1,48 \cdot 10^{-20}$
σ_3^-	$4,47 \cdot 10^{-18}$	$4,47 \cdot 10^{-18}$	$1,18 \cdot 10^{-20}$
σ_4^-	$3,67 \cdot 10^{-18}$	$3,67 \cdot 10^{-18}$	$9,69 \cdot 10^{-21}$
σ_1^+	$1,12 \cdot 10^{-17}$	$1,12 \cdot 10^{-17}$	$3,06 \cdot 10^{-20}$
σ_2^+	$3,10 \cdot 10^{-18}$	$3,10 \cdot 10^{-18}$	$7,57 \cdot 10^{-21}$
σ_3^+	$7,56 \cdot 10^{-18}$	$7,56 \cdot 10^{-18}$	$2,00 \cdot 10^{-20}$
σ_4^+	$2,13 \cdot 10^{-18}$	$2,13 \cdot 10^{-18}$	$6,24 \cdot 10^{-21}$

Table 8.6: Absolute values of the sideband coefficients $\sigma^\pm$, all of which are zero in the unperturbed case. The alignment of the waveguide is identical to the one in table 8.5, i.e. the waveguides is parallel to the velocity vector $\vec{V}$. “Spin” refers to Earth’s intrinsic rotation and “Orbit” to the rotation around the Sun.

Table 8.6 shows the absolute values of the coefficients $\sigma^\pm$ for the same alignment as before (where $\vec{V} = V^z \mathbf{e}_z$). Note that all of these coefficients would vanish if the terms of relative order $O(\ell/R)$ were not taken into account. Comparing these numbers with the ones listed in table 8.5, we find that

the corrections of relative order $O(\lambda/R)$ and $O(\ell/R)$ are of the same order of magnitude. Since the fields parameterised $\sigma^\pm$ grow linearly with z , their ratio to the parameters $\chi^\pm$ determine the length at which the two contributions become equally strong (measured in units of the core radius).

	Spin (Equator)	Spin (Vienna)	Orbit
τ_1^-	$9,10 \cdot 10^{-17}$	$9,10 \cdot 10^{-17}$	$2,49 \cdot 10^{-19}$
τ_2^-	$3,59 \cdot 10^{-16}$	$3,59 \cdot 10^{-16}$	$9,84 \cdot 10^{-19}$
τ_3^-	$6,16 \cdot 10^{-17}$	$6,16 \cdot 10^{-17}$	$1,63 \cdot 10^{-19}$
τ_4^-	$2,45 \cdot 10^{-16}$	$2,46 \cdot 10^{-16}$	$6,48 \cdot 10^{-19}$
τ_1^+	$2,27 \cdot 10^{-15}$	$2,27 \cdot 10^{-15}$	$6,05 \cdot 10^{-18}$
τ_2^+	$8,47 \cdot 10^{-16}$	$8,47 \cdot 10^{-16}$	$2,26 \cdot 10^{-18}$
τ_3^+	$1,54 \cdot 10^{-15}$	$1,54 \cdot 10^{-15}$	$4,16 \cdot 10^{-18}$
τ_4^+	$5,77 \cdot 10^{-16}$	$5,77 \cdot 10^{-16}$	$1,57 \cdot 10^{-18}$

Table 8.7: Absolute values of the sideband coefficients $\tau^\pm$, which vanish in the unperturbed case. As in table 8.5, the waveguide was chosen to be aligned in parallel with the velocity vector $\vec{V}$. “Spin” refers to Earth’s intrinsic rotation and “Orbit” to the rotation around the Sun.

Finally, the coefficients $\tau^\pm$ may be calculated from the parameters α and $\sigma^\pm$ using (7.19). The absolute values of $\tau^\pm$ are listed in table 8.7. We note that these parameters are roughly two orders of magnitude larger than the various coefficients $\sigma^\pm$ (see table 8.6). Comparing (6.5) with (7.8), we note that both $\chi^\pm$ and $\tau^\pm$ parameterise the z components of the fields which *do not* scale linearly with z in the same way. The ratio of the coefficients $\chi^\pm$ to $\tau^\pm$ thus measures whether the $O(\lambda/R)$ or the $O(\ell/R)$ corrections dominate. Comparing the two tables 8.5 and 8.7, one finds that the terms of order $O(\ell/R)$ exceed those of order $O(\lambda/R)$ by one to two orders of magnitude.

In conclusion, we have found that the main corrections to the unperturbed case are the modified dispersion relation (table 8.2), the perturbed values of the main mode coefficients α (table 8.3) and the excitations of the sidebands of order $\vec{V}$ (table 8.4). The corrections of higher order were indeed found to be suppressed by the ratios λ/R and ℓ/R , respectively.

Chapter 9

Conclusion

We have computed the first order corrections to electromagnetic waves in cylindrical waveguides due to Earth's rotation by solving Maxwell's equations in the linearised Born metric. The correction to the dispersion relation was found to be in good agreement with the geometrical optics formula

$$\frac{\delta\beta}{\beta} = \frac{v_{\parallel}}{n},$$

where n is the effective refractive index and $v_{\parallel}$ is the projection of the velocity vector $\vec{v}$ on the direction of wave propagation. The relative change of the wave vector β due to Earth's rotation about the Sun was found to be roughly $7 \cdot 10^{-5}$, while the effect of Earth's rotation about its own axis causes relative changes of up to $1 \cdot 10^{-6}$ at the equator (the effect depends on the latitude and on the orientation of the waveguide in space). These corrections (relative to the unperturbed value) are many orders of magnitude larger than those due to Earth's gravitational field. Furthermore, considering a Mach-Zehnder type interferometer whose arms are placed at different heights, it was shown that the phase shift due to rotation is – depending on the orientation – up to three orders of magnitude larger than the one due to Earth's gravitational field.

It was found that Earth's rotation couples neighbouring modes of different angular dependence. The relative amplitude of the sidebands caused by Earth's intrinsic rotation was found to be of the order of 10^{-5} , whereas Earth's orbital rotation around the Sun induces relative amplitudes of the order of 10^{-3} .

We expect that higher order contributions to the dispersion relation cannot be computed using geometrical optics alone, since this method only approximates the result obtained here with an error which we estimate to be much larger than the second order correction.

We hope that the methods developed here will be useful for future calculations assessing other influences on light propagation in waveguides. In particular, we expect that the method of discontinuity matrices is applicable to similar problems and provides an efficient method to ensure continuous fields at interfaces.

Appendix A

The Wave Equation Induced by the Optical Metric

We consider the scalar wave operator $\square_\gamma$, which is defined with respect to the Levi-Civita connection of the optical metric and compute $\square_\gamma \varphi$ to order Ω .

According to (2.5), the components of the optical metric in corotating coordinates are given by

$$\gamma_{00} = -1/n^2, \quad \gamma_{0i} = \frac{1}{n^2} \epsilon^{ijk} \Omega^j x^k, \quad \gamma_{ij} = \delta_{ij}. \quad (\text{A.1})$$

We show that the associated Christoffel symbols are given by

$$\Gamma^0_{\rho\sigma} = 0 + O(\Omega^2), \quad \Gamma^i_{jk} = 0 + O(\Omega^2), \quad \Gamma^i_{0j} = -\frac{1}{n^2} \epsilon^{ijk} \Omega^k + O(\Omega^2). \quad (\text{A.2})$$

We consider the equations of motion associated to the ‘‘Lagrangian’’

$$L = \frac{n^2}{2} \gamma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{2} \dot{t}^2 + \dot{t} \dot{x}^i \epsilon^{ijk} \Omega^j x^k + \frac{1}{2} n^2 \delta_{ij} \dot{x}^i \dot{x}^j.$$

Using the derivatives

$$\begin{aligned} \frac{\partial L}{\partial t} &= 0, & \frac{\partial L}{\partial x^i} &= \epsilon^{ijk} \Omega^k \dot{x}^j, \\ \frac{\partial L}{\partial \dot{t}} &= -\dot{t} + \dot{x}^i \epsilon^{ijk} \Omega^j x^k, & \frac{\partial L}{\partial \dot{x}^i} &= n^2 \dot{x}^i + \dot{t} \epsilon^{ijk} \Omega^j x^k, \end{aligned}$$

the equations of motion are found to be

$$\begin{aligned} \ddot{t} - \epsilon^{ijk} \ddot{x}^i \Omega^j x^k &= 0, \\ n^2 \ddot{x}^i + \ddot{t} \epsilon^{ijk} \Omega^j x^k - 2 \epsilon^{ijk} \Omega^k \dot{t} \dot{x}^j &= 0. \end{aligned}$$

Since the two equations imply $\ddot{t} = O(\Omega)$ and $\ddot{x}^i = O(\Omega)$, we have the expansion

$$\begin{aligned} \ddot{t} &= 0 + O(\Omega^2), \\ n^2 \ddot{x}^i - 2 \epsilon^{ijk} \Omega^k \dot{t} \dot{x}^j &= 0 + O(\Omega^2), \end{aligned}$$

which is equivalent to the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0,$$

with the Christoffel symbols given in (A.2).

We now consider the scalar wave equation

$$\square_\gamma \varphi = 0$$

where the wave operator $\square_\gamma$ is defined with respect to the optical metric. We have

$$\nabla_\mu \nabla_\nu \varphi = \partial_\mu \partial_\nu \varphi + \Gamma_{\mu\nu}^i \partial_i \varphi + O(\Omega^2)$$

since all Christoffel symbols with the upper index equal to zero vanish to order Ω . Contracting with the inverse optical metric

$$\gamma^{00} = -n^2, \quad \gamma^{0i} = -v^i, \quad \gamma^{ij} = \delta^{ij},$$

we obtain

$$\square_\gamma \varphi = \gamma^{\mu\nu} \partial_\mu \partial_\nu \varphi - 2\gamma^{0j} \Gamma_{0j}^i \partial_i \varphi = \gamma^{\mu\nu} \partial_\mu \partial_\nu \varphi + O(\Omega^2).$$

The Christoffel symbols do not contribute at the required order, since both Γ_{0j}^i and γ^{0i} are $O(\Omega)$. Using once again the explicit form of the inverse optical metric, one obtains

$$\square_\gamma \varphi = -n^2 \ddot{\varphi} + \Delta \varphi + 2v(\dot{\varphi}) + O(\Omega^2).$$

Appendix B

Remarks on the Inhomogeneous Bessel Equation

B.1 General Definitions

We define the Bessel operator of order ν as

$$L_\nu f(r) := f''(r) + \frac{1}{r}f'(r) + \left(1 - \frac{\nu^2}{r^2}\right)f(r). \quad (\text{B.1})$$

Similarly, the modified Bessel operator of order ν is defined as

$$\tilde{L}_\nu f(r) := f''(r) + \frac{1}{r}f'(r) - \left(1 + \frac{\nu^2}{r^2}\right)f(r). \quad (\text{B.2})$$

We consider the following family of inner products

$$\langle f, g \rangle_{R_1}^{R_2} := \int_{R_1}^{R_2} f(r)g(r)r \, dr.$$

The Wronskian of two functions f, g is defined as

$$W(f, g; r) = f(r)g'(r) - f'(r)g(r).$$

B.2 General Properties of the Bessel Operator

We show the following identity

$$\langle f, L_\nu g \rangle_{R_1}^{R_2} - \langle L_\nu f, g \rangle_{R_1}^{R_2} = R_2 W(f, g; R_2) - R_1 W(f, g; R_1).$$

This is readily verified by noting that the first two terms of the Bessel operator may be written as $(rf')'/r$, from which it follows that

$$\begin{aligned} \langle f, (rf')'/r \rangle_{R_1}^{R_2} &= \int_{R_1}^{R_2} f(rg')' \, dr \\ &= [rf(r)g'(r)]_{R_1}^{R_2} - \int_{R_1}^{R_2} (rf')g' \, dr \\ &= [rf(r)g'(r) - rf'(r)g(r)]_{R_1}^{R_2} - \int_{R_1}^{R_2} (rf')'g \, dr. \end{aligned}$$

Since the rest of the Bessel operator is clearly self-adjoint with respect to $\langle \cdot, \cdot \rangle_{R_1}^{R_2}$, the result follows. Clearly, the same statement also holds for the modified Bessel operator.

We conclude that if the Bessel operator is restricted to functions defined on an interval (R_1, R_2) satisfying Dirichlet or von Neumann boundary conditions, L_ν and $\tilde{L}_\nu$ are formally self-adjoint.

B.3 The Ordinary Bessel Equation

We seek a solution of the equation

$$L_\nu f = g$$

on an interval (R_1, R_2) . A particular solution to this equation is given by

$$f(r) = \frac{\pi}{2} Y_\nu(r) \langle J_\nu, g \rangle_{R_1}^r - \frac{\pi}{2} J_\nu(r) \langle Y_\nu, g \rangle_{R_1}^r ,$$

provided the integrals exist, i.e. $\langle g, h \rangle_{R_1}^r$ is finite for all $h \in \ker B_\nu$ and $r \in [R_1, R_2]$. If these requirements are met, we have

$$f'(r) = \frac{\pi}{2} Y'_\nu(r) \langle J_\nu, g \rangle_{R_1}^r - \frac{\pi}{2} J'_\nu(r) \langle Y_\nu, g \rangle_{R_1}^r ,$$

(the derivative of the inner products vanish) and

$$f''(r) = \frac{\pi}{2} Y''_\nu(r) \langle J_\nu, g \rangle_{R_1}^r - \frac{\pi}{2} J''_\nu(r) \langle Y_\nu, g \rangle_{R_1}^r + \frac{\pi r}{2} W(J_\nu, Y_\nu; r) g(r) .$$

The result then follows from the fact that

$$W(J_\nu, Y_\nu; r) = \frac{2}{\pi r} ,$$

see e.g. [1, p. 360, Equation 9.1.16].

B.4 The Modified Bessel Equation

A solution to the equation

$$\tilde{L}_\nu f = g$$

on the interval (R_1, R_2) is given by

$$f(r) = I_\nu(r) \langle K_\nu, g \rangle_{R_1}^r - K_\nu(r) \langle I_\nu, g \rangle_{R_1}^r ,$$

provided the integrals exist. The proof is completely analogous to the one for the ordinary equation except for the Wronskian

$$W(K_\nu, I_\nu; r) = \frac{1}{r} ,$$

see e.g. [1, p. 375, eq. 9.6.15].

B.5 Rescaled Arguments

Consider the rescaled Bessel operator

$$L_\nu^{\lambda^2} f(r) := f''(r) + \frac{1}{r} f'(r) + \left(\lambda^2 - \frac{\nu^2}{r^2} \right) f(r). \quad (\text{B.3})$$

Writing $f(r) = \hat{f}(\rho)$, where $\rho = \lambda r$, one obtains

$$L_\nu^{\lambda^2} f(r) = \lambda^2 B_\nu \hat{f}(\rho),$$

so the inhomogeneous equation

$$L_\nu^{\lambda^2} f(r) = g(r)$$

is equivalent to

$$L_\nu \hat{f}(\rho) = \hat{g}(\rho),$$

where

$$\hat{g}(\rho) = \frac{1}{\lambda^2} g(r).$$

By the above formula, a particular solution to this equation is given by

$$\hat{f}(\rho) = \frac{\pi}{2} Y_\nu(\rho) \int_{\rho_0}^{\rho} J_\nu(\rho') \hat{g}(\rho') \rho' d\rho' - \frac{\pi}{2} J_\nu(\rho) \int_{\rho_0}^{\rho} Y_\nu(\rho') \hat{g}(\rho') \rho' d\rho'.$$

Substituting $\rho' = \lambda r'$, one obtains

$$f(r) = \frac{\pi}{2} Y_\nu(\lambda r) \int_{r_0}^r J_\nu(\lambda r') g(r') \rho' dr' - \frac{\pi}{2} J_\nu(\lambda r) \int_{r_0}^r Y_\nu(\lambda r') g(r') r' dr'. \quad (\text{B.4})$$

A similar argument applies to the modified Bessel operator: a particular solution to

$$\tilde{L}_\nu^{\lambda^2} f(r) = g(r)$$

is given by

$$f(r) = I_\nu(\lambda r) \int_{r_0}^r K_\nu(\lambda r') g(r') \rho' dr' - K_\nu(\lambda r) \int_{r_0}^r I_\nu(\lambda r') g(r') r' dr'. \quad (\text{B.5})$$

B.6 Restrictions on Source Terms and Boundary Conditions

Let $h \in \ker B_\nu$ and $g \in \text{im } B_\nu$, so $g = B_\nu f$ for some function f . By the above identity, we have

$$\langle h, g \rangle_{R_1}^{R_2} = R_2 W(h, f; R_2) - R_1 W(h, f; R_1).$$

Thus, g must be orthogonal (with respect to $\langle \cdot, \cdot \rangle_{R_1}^{R_2}$) to all $h \in \ker B_\nu$ for which the right hand side vanishes. The same statement also holds for the case where B_ν is replaced by $\tilde{B}_\nu$.

Example: Consider the equation

$$B_\nu f = J_\nu$$

on the interval $(0, R)$, where $R = j_{\nu,k}$ is the k -th zero of J_ν . Since $\langle J_\nu, J_\nu \rangle_0^R > 0$ the above equation implies that there is no solution f to this equation which satisfies $f(R) = 0$ and is sufficiently well behaved at the origin as to satisfy $rW(h, f; r) \rightarrow 0$ as $r \rightarrow 0$. However, no such restriction arises if R does not coincide with any root of J_ν .

Appendix C

Some Integrals of Bessel Functions

We give explicit formulae for some integrals of products of Bessel functions.

For integrals of Bessel functions with respect to the measure $r dr$, we find the following formulae

$$\begin{aligned}\int J_m(r)^2 r dr &= \frac{r^2}{2} (J_m(r)^2 - J_{m-1}(r)J_{m+1}(r)) , \\ \int K_m(r)^2 r dr &= \frac{r^2}{2} (K_m(r)^2 - K_{m-1}(r)K_{m+1}(r)) , \\ \int Y_m(r)J_m(r) r dr &= \frac{r^2}{2} (J_m(r)Y_m(r) - J_{m-1}(r)Y_{m+1}(r)) , \\ \int I_m(r)K_m(r) r dr &= \frac{r^2}{2} (K_m(r)I_m(r) + K_{m-1}(r)I_{m+1}(r)) ,\end{aligned}$$

which are readily verified using the recursion relations (5.14) and (5.18).

The functions Γ_n, Δ_n , which were defined in (5.16), and the functions Σ_n, T_n , defined in (5.20), can thus be written as

$$\begin{aligned}\Delta_n(z) &= \frac{z^2}{2} (J_{m+n}(Uz)^2 - J_{m+n-1}(Uz)J_{m+n+1}(Uz)) , \\ T_n(z) &= -\frac{z^2}{2} (K_{m+n}(Wz)^2 - K_{m+n-1}(Wz)K_{m+n+1}(Wz)) , \\ \Gamma_n(z) &= \left[\frac{\xi^2}{2} (J_{m+n}(U\xi)Y_{m+n}(U\xi) - J_{m+n-1}(U\xi)Y_{m+n+1}(U\xi)) \right]_{\xi=z}^{\xi=1} , \\ \Sigma_n(z) &= \left[\frac{\xi^2}{2} (K_{m+n}(W\xi)I_{m+n}(W\xi) + K_{m+n-1}(W\xi)I_{m+n+1}(W\xi)) \right]_{\xi=1}^{\xi=z} .\end{aligned}\tag{C.1}$$

Appendix D

Remarks on the Influence of Earth's Gravitational Field

We give a derivation of the first order change in the wave vector due to Earth's gravitational field using geometrical optics.

We start from the post-Newtonian metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\phi\delta_{\mu\nu} ,$$

where ϕ is Newton's potential. To first order in ϕ , the components of the inverse metric tensor read

$$g^{\mu\nu} = \eta^{\mu\nu} + 2\phi\delta^{\mu\nu} .$$

A medium at rest in the considered coordinates has four velocity $(u^\mu) = (u^0, \vec{0})$, where u^0 is determined from $g(u, u) = -1$, which yields

$$(u^0)^2 = -\frac{1}{g_{00}} \approx 1 - 2\phi .$$

Inserting this into (2.6), one obtains the components of the inverse optical metric

$$\begin{aligned} \gamma^{00} &= -1 + 2\phi + (1 - n^2)(1 - 2\phi) = -n^2(1 - 2\phi) , \\ \gamma^{ij} &= (1 + 2\phi)\delta^{ij} , \\ \gamma^{0i} &= \gamma^{i0} = 0 . \end{aligned}$$

The condition of $k = d\psi = \omega dt - \beta dz$ being null with respect to γ then yields the quadratic equation

$$(1 + 2\phi)\beta^2 - n^2(1 - 2\phi)\omega^2 = 0 ,$$

whose solution for positive β ($\omega > 0$) is

$$\beta = n\omega \sqrt{\frac{1 - 2\phi}{1 + 2\phi}} \approx n\omega(1 - 2\phi) .$$

The linearised change in β due to the gravitational field is thus

$$\frac{\delta\beta}{\beta} = -2n\frac{\omega}{\beta}\delta\phi ,$$

or, using $\beta = n\omega + O(\phi)$ and $\delta\phi = g\delta h$, where g is the local gravitational acceleration and δh is a difference in height

$$\frac{\delta\beta}{\beta} = -2g\delta h .$$

This is the same result as equation (1.1) in [2].

Bibliography

- [1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. U.S. Govt. Print. Off., Washington, 1964.
- [2] R. Beig, P. T. Chruściel, C. Hilweg, P. Kornreich, and P. Walther. Weakly gravitating isotropic waveguides. *Classical and Quantum Gravity*, 35(24):244001, November 2018.
- [3] W. Gordon. Zur Lichtfortpflanzung nach der Relativitätstheorie. *Annalen der Physik*, 377(22):421–456, 1923.
- [4] C. Hilweg, F. Massa, D. Martynov, J. Mavalvala, P. T. Chruściel, and P. Walther. Gravitationally induced phase shift on a single photon. *New Journal of Physics*, 19(3):033028, March 2017.
- [5] L. D. Landau and E. M. Lifschitz. *Klassische Feldtheorie*. Number 2 in Lehrbuch der theoretischen Physik. Akademie-Verlag Berlin, 8th edition, 1981.
- [6] L. D. Landau and E. M. Lifschitz. *Elektrodynamik der Kontinua*. Number 8 in Lehrbuch der theoretischen Physik. Akademie-Verlag Berlin, 4th edition, 1985.
- [7] P. Langevin. Sur la théorie de la relativité et l’expérience de M. Sagnac. *Comptes Rendus Hebdomadaires des Seances de l’Academie des Sciences*, 173:831–834, 1921.
- [8] P. Langevin. Sur l’expérience de Sagnac. *Comptes Rendus Hebdomadaires des Seances de l’Academie des Sciences*, 205:304–306, 1937.
- [9] H. A. Lorentz. De l’influence du mouvement de la terre sur les phénomènes lumineux. *Archives néerlandaises des sciences exactes et naturelles*, 21:103–176, 1887.
- [10] H. A. Lorentz. The Rotation of the Earth and its Influence on Optical Phenomena. *Nature*, 112(2803):103–104, 1923.
- [11] G. Pascoli. The Sagnac effect and its interpretation by Paul Langevin. *Comptes Rendus Physique*, 18(9-10):563–569, November 2017.
- [12] M. G. Sagnac. L’éther lumineux démontré par l’effet du vent relatif d’éther dans un interféromètre en rotation uniforme. *Comptes rendus hebdomadaires des séances de l’Académie des sciences*, 157:708–710, 1913.
- [13] M. G. Sagnac. Effet tourbillonnaire optique. La circulation de l’éther lumineux dans un interférographe tournant. *Journal de Physique Théorique et Appliquée*, 4(1):177–195, 1914.
- [14] K. U. Schreiber, T. Klügel, J.-P. R. Wells, R. B. Hurst, and A. Gebauer. How to Detect the Chandler and the Annual Wobble of the Earth with a Large Ring Laser Gyroscope. *Physical Review Letters*, 107(17), October 2011.
- [15] M. v. Laue. Zum Versuch von F. Harreß. *Annalen der Physik*, 367(13):448–463, 1920.

