DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

„Post-Lie algebra structures on classes of Lie algebras“

verfasst von / submitted by

Christof Ender, BSc MSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2019 / Vienna, 2019

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Prof. Dr. Dietrich Burde

Abstract

In this thesis we study *post-Lie algebra structures* (vector spaces V with three bilinear operations $[,], \{,\}, \cdot$, such that $(V, [,])$ is a Lie algebra $\mathfrak{g}$, $(V, \{,\})$ a Lie algebra $\mathfrak{n}$, $(V, \cdot)$ is a non-associative algebra and certain compatibility conditions between $[,], \{,\}$ and $\cdot$ are satisfied) on pairs of complex Lie algebras $(\mathfrak{g}, \mathfrak{n})$. Post-Lie algebra structures come up (among other areas) in geometry in the study of crystallographic groups and a question of John Milnor; in Chapter 1 we explain this geometric background in detail.
After giving an overview on definitions and important, already known, theorems on Lie algebras and post-Lie algebra structures in Chapter 2, we study the existence of post-Lie algebra structures on pairs of Lie algebras (Chapter 3). We prove existence and non-existence of post-Lie algebra structures on pairs of Lie algebras subject to algebraic properties of the Lie algebras; afterwards, we prove the non-existence of post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where $\dim(\mathfrak{g}) = \dim(\mathfrak{n}) < 45$ and $\mathfrak{g}$ is semisimple (non-simple) and $\mathfrak{n}$ simple. Then we investigate post-Lie algebra structures on Lie algebras satisfying $[x, y] = a(\{x, y\})$ for a scalar $a \in \mathbb{C}^*$ (as a generalization of the important class of commutative post-Lie algebra structures) and post-Lie algebras structures in dimension 3.
In Chapter 4, we study post-Lie algebra structures on complete Lie algebras. First, we classify all complete Lie algebras up to dimension 7; then we study post-Lie algebras on pairs $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is complete such that $[x, y] = R(\{x, y\})$ holds for a linear map R. In the case where $R = a\,\mathrm{Id}, a \in \mathbb{C}$, we can refine our results from Chapter 3; in the more general case, we find a characterization of those post-Lie algebra structures in terms of certain inner derivations of $\mathfrak{n}$.
Post-Lie algebra structures (and in particular, commutative ones) on nilpotent Lie algebras are studied in Chapter 5. We find correspondences of commutative post-Lie algebra structures on 2-step nilpotent Lie algebras to pre-Lie algebra structures and LR-structures (as a corollary, we obtain results on the completeness of those structures on Heisenberg Lie algebras). Afterwards, we investigate commutative post-Lie algebra structures on filiform Lie algebras and prove that they can be (if non-metabelian) essentially classified. We conclude Chapter 5 by proving connections between post-Lie algebra structures on nilpotent Lie algebras and Poisson-admissible Lie algebras.
Afterwards, we state open problems (Chapter 6) and give examples and classifications of post-Lie algebra structures in small dimensions (Appendix B).

Acknowledgments

Many people helped me – knowingly or not – during my Ph.D. studies. I would like to take a moment to express my gratitude, even though it is not an easy task to thank them appropriately with only a few words.

First and foremost, I want to thank my supervisor, Prof. Dietrich Burde, for his excellent guidance. He always provided support when I needed it – thank you especially for providing many new ideas and directions, carefully reading drafts of this thesis, valuable comments and giving me freedom to pursue my research interests! It was a great pleasure to work and discuss with you!

I would also like to thank Prof. Karel Dekimpe (KU Leuven) and Prof. Pasha Zusmanovich (University of Ostrava) for refereeing my thesis and serving as committee members for my defense.

I am very grateful to my friends and colleagues, who helped me to mature both as a person and as a mathematician. Thank you, in particular, to Alexandre, Andreas, Andrei, Arindam, Aya, Carina, Christian, Christina, Christopher, Christopher, Claudia, David, Dominik, Federico, Gabriel, Giancarlo, Hana, Irmgard, Jelena, Kasimir, Lorenz, Magdalena, Markus, Markus, Melanie, Monika, Patrick, Sabine, Theresa, Thomas, Vsevolod, Wolfgang and Wolfgang for all the interesting discussions and great times we had!

I also want to thank the Vienna Doctoral School of Mathematics for bringing together Ph.D. students from different areas of mathematics and for organizing a lot of scientific and fun activities (providing teaching opportunities, organizing summer schools, retreats, colloquia, ...). I am very happy to be part of the VDS Mathematics.

Moreover, I am especially indebted to my parents, Silvia and Hermann Ender, for their unconditional love, support and encouragement during my whole life, without which my studies could not have been possible.

My very special thanks go to the most important person, Noëma Nicolussi. Thank you for always cheering me up during stressful times, making me laugh and always being on my side no matter what! You are incredible!

Funding acknowledgments

I gratefully acknowledge support by the Austrian Science Fund (FWF), project P 28079-N35: Nil-affine Crystallographic Groups and Algebraic Structures, and by an "Abschlussstipendium" of the VDS Mathematics.

Contents

Outline

This thesis is organized as follows:

In Chapter 1, we explain how post-Lie algebra structures arise in differential geometry and algebra. After explaining Bieberbach's theory of *Euclidean crystallographic groups* (and their geometry), we introduce *affine crystallographic groups* and *Auslander's* and *Milnor's* famous questions which motivate the algebraic notion of *pre-Lie algebra structures*. As an important generalization of affine crystallographic groups, we define *nil-affine crystallographic groups* – on an algebraic level, they correspond to *post-Lie algebra structures*. Afterwards, we mention briefly other areas in which post-Lie algebra structures come up.

Chapter 2 has two parts: In Section 2.1, we state the background on Lie algebras (definitions and important theorems) which we need in the sequel. Afterwards, in Section 2.2, we define *post-Lie algebra structures* and related structures (LR-structures and pre-Lie algebra structures) and in particular *commutative post-Lie algebra structures* (which will be abbreviated in the following as *CPA-structures*), state the notions we want to use and summarize some already known theorems on post-Lie algebra structures.

In Chapter 3, we study *general questions on the (non-)existence of post-Lie algebra structures* (over $\mathbb{C}$). First (Section 3.1), we ask: Given two algebraic properties P, Q of Lie algebras (e.g. being nilpotent or complete), is there a pair $(\mathfrak{g}, \mathfrak{n})$ of Lie algebras – with $\mathfrak{g}$ and $\mathfrak{n}$ having properties P and Q, respectively – such that $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure? Some questions of this kind have already been answered in the literature, we summarize them and also state new results (especially for *complete* Lie algebras). These results are summarized in Table 1 below. A "✓" means that there is some pair $(\mathfrak{g}, \mathfrak{n})$ with the corresponding properties, a "✗" means that there is no such pair $(\mathfrak{g}, \mathfrak{n})$ at all. (So if if there is a "✓", not necessarily all combinations of such Lie algebras admit post-Lie algebra structures.) The properties in Table 1 (except "complete") should be read mutually exclusively – that is, e.g. "nilpotent" means "nilpotent, not abelian". (See Section 3.1 for more details.)
Next, in Section 3.2, we are interested in a special case of the above question: in the non-existence of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is *semisimple and non-simple*, $\mathfrak{n}$ is *simple* – we give a proof (based on Dynkin's paper of subalgebras of semisimple Lie algebras) that post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ cannot exist if one of the following conditions holds (Theorem 3.32): (i) $\mathfrak{g}$ is simple and $\mathfrak{g} \not\cong \mathfrak{n}$, (ii) $\mathfrak{g}$ has exactly two simple factors, (iii) $\mathfrak{n}$ is an exceptional simple Lie algebra, (iv) $\dim(\mathfrak{n}) < 45$.
In Section 3.3, we study post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ where

$\mathfrak{g}$'s Lie bracket is a *scalar multiple* of $\mathfrak{n}$'s Lie bracket (as a generalization of [28, Chapter 3]). Our result is that the existence of a post-Lie algebra structure on such a pair $(\mathfrak{g}, \mathfrak{n})$ already implies that $\mathfrak{g}$ and $\mathfrak{n}$ are solvable (Corollary 3.47) unless the scalar is $0, -1$ or 1.
Finally, in Section 3.4 we investigate post-Lie algebra structures in *small dimensions* – we state which pairs $(\mathfrak{g}, \mathfrak{n})$ of Lie algebras in dimensions 2 and 3 admit post-Lie algebra structures. (The proof is deferred to Appendix B.1).
Also Appendix B.2 deals with 3-dimensional Lie algebras: here, we explicitly classify all CPA-structures on 3-dimensional Lie algebras.

Table 1: Existence of post-Lie algebra structures over $\mathbb{C}$. (See Table 5 for more details.)

$\mathfrak{g}$ \ $\mathfrak{n}$	abelian	nilpotent	solvable	simple	semisimple	reductive	complete
abelian	✓	✓	✓	✗	✗	✗	✓
nilpotent	✓	✓	✓	✗	✗	✗	✓
solvable	✓	✓	✓	✓	✓	✓	✓
simple	✗	✗	✗	✓	✗	✗	✗
semisimple	✗	✗	✗	✗	✓	?	✗
reductive	✓	?	?	?	?	✓	✓
complete	✓	✓	✓	?	?	✓	✓

In Chapter 4, we are interested in post-Lie algebra structures on *complete* Lie algebras (over $\mathbb{C}$). More precisely, our main motivating question is: Given a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ (where $\mathfrak{n}$ is a complete Lie algebra), such that $[x, y] = R(\{x, y\})$ with a linear transformation R, can we classify the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$? (For the subcase of $\mathfrak{n}$ being *semisimple*, this question has been answered in [23].)
After stating some general structure results on post-Lie algebra structures on complete Lie algebras which will be useful in the sequel (Section 4.1), we *classify*, in Section 4.2, all *complete complex Lie algebras up to dimension* 7 to get interesting, concrete examples of complete Lie algebras. We do this by applying the theory of *rigid* Lie algebras. We introduce useful tools and state already known results on complete Lie algebras.
In Section 4.3, we restrict our motivating question to scalars $a \in \mathbb{C}^*$ (that is, $R = a\,\mathrm{Id}, a \in \mathbb{C}^*$). Since we already considered this situation before for *non-solvable* Lie algebras (Section 3.3) and found that always $a \in \{-1, 1\}$, we now restrict ourselves to *solvable, but complete* Lie algebras and show that in this case also $a \in \{-1, 1\}$ holds (Theorem 4.40).
Finally, in Section 4.4, we deal with the question more generally. We restrict ourselves to Lie algebras $\mathfrak{n}$ which satisfy certain conditions (being generated by special weight

spaces, solvable-complete and satisfying $H^0(T, \mathfrak{n}) = T$) to use Leger's and Luks' theory on quasiderivations of Lie algebras. We state examples of Lie algebras satisfying these conditions and prove that a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ (such that there is a linear map R as above) can be written as $x \cdot y = \{\varphi(x), y\}$ or $x \cdot y = \{\varphi(x) - x, y\}$, where $\varphi = \mathrm{ad}(z)$ is an *inner derivation* of $\mathfrak{n}$ (Theorem 4.63). From that, we can derive corollaries on the element z (Lemma 4.67 and Proposition 4.71), the structure of the Lie algebra $\mathfrak{g}$ (Proposition 4.73) and the operator R (Lemma 4.74). The results also extend work on CPA-structures on complete Lie algebras done in [33].
Appendix B.3 is also relevant here: There, we explicitly describe the possibilities for post-Lie algebra structures in the above situation (i.e., $\mathfrak{n}$ is complete and there exists a linear map $R \in \mathrm{End}(V)$ such that $[x, y] = R(\{x, y\})$ for all $x, y \in \mathfrak{n}$).

In Chapter 5, we are interested in post-Lie algebra structures on *nilpotent* Lie algebras (over $\mathbb{C}$).
First, in Section 5.1, we study one generalization of CPA-structures – namely post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{n}_3(\mathbb{C})$, the three-dimensional Heisenberg algebra over $\mathbb{C}$. We list the *complete classification* of these structures (Proposition 5.1) and draw corollaries out of it.
In Section 5.2, we study post-Lie algebra structures, left-symmetric structures and pre-Lie algebra structures on 2-*step nilpotent* Lie algebras and find that under suitable conditions, they correspond to each other. In the case where $\mathfrak{g}$ and $\mathfrak{n}$ are Heisenberg algebras of dimension ≥ 5, the correspondence is particularly nice and we find corollaries regarding the *completeness* of LR-structures and pre-Lie algebra structures (Corollaries 5.17 and 5.23).
Section 5.3 studies CPA-structures on another important Lie algebras: the family of Lie algebras of *strictly upper triangular matrices*. It turns out that all of those CPA-structures have a very "nice" form – in particular, they satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ (Proposition 5.27). Using this form, we can derive the CPA-structures on the (non-nilpotent) Lie algebra of (non-strictly) *upper triangular matrices* (Proposition 5.34).
In Section 5.4, we study CPA-structures on the well-known class of *filiform* Lie algebras – we prove Theorem 5.44 which characterizes CPA-structures on filiform Lie algebras $(\mathfrak{g}, [\,,])$: All CPA-structures on a filiform Lie algebra $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ if and only if $\mathfrak{g}$ has solvability class greater than 2 and all CPA-structures on the filiform Lie algebra $\mathfrak{g}$ satisfy $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ if and only if $\mathfrak{g}$ is not isomorphic to the standard graded filiform Lie algebra L_n. For the proof of Theorem 5.44, we divide the class of CPA-structures on filiform Lie algebras into four subclasses and deal with each of those classes separately. As the methods are similar, but involve more and more details, the proof for three of these classes is deferred to Appendix A.
Afterwards Section 5.4.5 is about post-Lie algebra structures on *infinite-dimensional filiform* Lie algebras – we note that the results we proved on finite-dimensional filiform Lie algebras extends to the infinite-dimensional case (Corollary 5.63). This is the only time where infinite-dimensional Lie algebras play a role in this thesis.
Next, in Section 5.5, we apply our knowledge about CPA-structures on general filiform Lie algebras (from Theorem 5.44) to explicitly compute all CPA-structures on *certain*

important families of filiform Lie algebras (L_n, Q_n, R_n, W_n and a family of characteristically nilpotent Lie algebras).
For several classes of nilpotent Lie algebras $(\mathfrak{g}, [,])$ studied in this chapter (for example, the non-metabelian filiform Lie algebras or the Lie algebras of strictly upper triangular matrices of dimension n), we obtain (among other things) that every CPA-structure on $\mathfrak{g}$ satisfies $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$. This condition allows us to view $(\mathfrak{g}, \cdot)$ as an *associative algebra* – we do this in Section 5.6. Another condition which CPA-structures can have is the one of being *central* – we prove that CPA-structures are central if and only if they are *Poisson algebras*. We present connections between *Poisson admissible algebras* and LR-structures – as a corollary, we find that associative LR-structures on two-step nilpotent stem Lie algebras are always complete (Corollary 5.103).

Chapter 6 concludes this thesis; there we state some *open questions* for future work on post-Lie algebra structures.

While many results presented here hold for Lie algebras over any algebraically closed field of characteristic zero, all Lie algebras we study here are assumed to be defined over the complex numbers.

1 Motivation

This section describes one possible motivation for coming up with post-Lie algebra structures. While our techniques and results will be algebraic, the motivation comes from differential geometry. We will not give proofs in this chapter; they can be found in the referenced literature. Mainly we will follow the surveys [1], [20], [44, Chapter 1] and [48, Chapter 1].

In the center of the geometric motivation stands a famous question of John Milnor:

Question 1.1 (Milnor's question, 1977, [77])**.** *Does every solvable Lie group G admit a complete affinely flat structure which is invariant under left translation, or equivalently, does the universal covering group $\widetilde{G}$ operate simply transitively by affine transformations of $\mathbb{R}^k$?*

1.1 Euclidean crystallographic groups

Post-Lie algebra structures arise in the study of *nil-affine crystallographic actions.* To motivate this notion, we will first introduce *crystallographic groups* and present the three famous Bieberbach theorems.

Definition 1.2. The *affine group of the Euclidean space* $\mathbb{R}^n$, denoted by $\operatorname{Aff}(\mathbb{R}^n)$, is given by the semidirect product

$$\operatorname{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R}),$$

where $\mathrm{GL}_n(\mathbb{R})$ is the general linear group over $\mathbb{R}$ of degree n. $\operatorname{Aff}(\mathbb{R}^n)$ forms a group via the multiplication $(a, A) \cdot (b, B) := (Ba + b, BA)$.
An element $x \in \operatorname{Aff}(\mathbb{R}^n)$ can be represented in a unique way as $x = v + A, v \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})$. However, it will sometimes be more convenient to think of x as a block matrix $x = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}, A \in \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n$. In this way, $\operatorname{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n \right\}$ is a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$ with multiplication

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & Aw + v \\ 0 & 1 \end{pmatrix}.$$

The group $\operatorname{Aff}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$ by

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + v \\ 1 \end{pmatrix}$$

for all $x \in \mathbb{R}^n$.

Definition 1.3. The *isometry group of the Euclidean space $\mathbb{R}^n$,* denoted by $\mathrm{Iso}(\mathbb{R}^n)$, is a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$; it is given by

$$\mathrm{Iso}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O_n(\mathbb{R}),$$

where $O_n := \{A \in \mathrm{Mat}_n(\mathbb{R})\colon \langle Aa, Ab\rangle = \langle a, b\rangle \text{ for all } a, b \in \mathbb{R}^n\}$ is the *orthogonal group of* $\mathbb{R}^n$.

Definition 1.4. Let Γ be a group acting on $\mathbb{R}^n$ via a homomorphism $\rho\colon \Gamma \to \mathrm{Iso}(\mathbb{R}^n)$ such that

(i) the action is *properly discontinuous*, i.e. for each compact subset $K \subseteq \mathbb{R}^n$, the set $\{\gamma \in \Gamma\colon \rho(\gamma)K \cap K \neq \emptyset\}$ is finite and

(ii) the action is *cocompact*, that is, the orbit space $\mathbb{R}^n/\rho(\Gamma)$ is compact.

Then, the action is called a *(Euclidean) crystallographic action* and the image $\rho(\Gamma)$ is called a *(Euclidean) crystallographic group.*

Example 1.5. *Let $\Gamma = \mathbb{Z}^n$ and let Γ act on $\mathbb{R}^n$ via translations, i.e. consider the action $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n, (z, r) \mapsto (z + r)$. This is a crystallographic action.*

Example 1.6. *The fundamental group of the Klein bottle, $\langle a, b | aba^{-1} = b\rangle$ (also known as the Baumslag-Solitar group $BS(1, -1)$) admits a crystallographic action on $\mathbb{R}^2 = \{(x, y, 1)\colon x, y \in \mathbb{R}\}$ via*

$$a \mapsto \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.7 (See e.g. [90]). Let Γ be a group acting on $\mathbb{R}^n$ via $\rho\colon \Gamma \to \mathrm{Iso}(\mathbb{R}^n)$.

(i) The action is properly discontinuous if and only if Γ is a discrete group.

(ii) Let Γ be discrete. Then the action is *free* (meaning whenever for an $x \in \mathbb{R}^n$ and a $\gamma \in \Gamma$, we have $\rho(\gamma)(x) = x$, then $\gamma = 1$) if and only if Γ is torsion-free.

(iii) If the action is properly discontinuous and free, then the quotient space $\mathbb{R}^n/\rho(\Gamma)$ is a manifold.

1.2 Crystallographic groups in nature

The name "crystallographic group" is a very colorful one; thus we want to say a few words about crystallographic groups occurring in nature.

Two-dimensional crystallographic groups are known as *wallpaper groups*; they correspond to tessellations of the plane $\mathbb{R}^2$ via periodic tilings with two linearly independent directions of translation. In 1891, Federov proved that there are (up to isomorphism)

exactly 17 wallpaper groups. Most, if not all of them, were found as ornaments in the Alhambra fortress in Grenada, Spain (see also Marcus du Sautoy's popular scientific book [49] where he describes a visit to the Alhambra).
In tilings of ancient Egypt, twelve of these wallpaper groups were found. Also many paintings by the famous Dutch artist M. C. Escher feature wallpaper groups.

In three dimensions, crystallographic groups are precisely the symmetry groups of crystals occurring in nature – the first correct classification of the 219 three-dimensional crystallographic groups was given by Federov and Schönflies in 1892.

1.3 The Bieberbach theorems

A part of Hilbert's 18th problem was the question (see Hilbert's paper [60]):

> *"Is there in n-dimensional Euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region?"*

In different terms, the question was, whether or not, given any $n \in \mathbb{N}$, there exist (up to isomorphism) only finitely many n-dimensional crystallographic groups. This question received a lot of attention; among many other people, Ludwig Bieberbach studied their structure and proved the famous three *Bieberbach theorems* in [14] and [15] we will state now.

Theorem 1.8 (Bieberbach's first theorem). *Every crystallographic group is virtually abelian. More precisely, if Γ is an n-dimensional crystallographic group, then Γ contains $\mathbb{Z}^n$ as a normal subgroup and the quotient $\Gamma/\mathbb{Z}^n$ is finite. (In particular, Γ is finitely generated.)*

Here, we have used the term *virtually abelian.* The phrase "virtually" is commonly used in group theory; it is defined as follows:

Definition 1.9. Let P be a property of groups (such as being abelian, nilpotent, free etc.). We say that a group G is *virtually P* or *P-by-finite*, if G contains a subgroup of finite index with the property P.

Thus, "virtually abelian" means "has an abelian subgroup of finite index".
The second Bieberbach theorem characterizes isomorphisms between two crystallographic groups: An isomorphism exists precisely between two "affinely equivalent" groups:

Theorem 1.10 (Bieberbach's second theorem). *Two crystallographic groups $\Gamma, \Gamma' \leq \mathrm{Iso}(\mathbb{R}^n)$ are isomorphic as abstract groups via the map $\varphi \colon \Gamma \to \Gamma'$ if and only if there is an affine transformation $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ such that $\varphi(\gamma) = \alpha^{-1}\gamma\alpha$ for all $\gamma \in \Gamma$.*

Finally, Bieberbach's third theorem gives the answer to Hilbert's question:

Theorem 1.11 (Bieberbach's third theorem). *For any $n \in \mathbb{N}$, there exist (up to isomorphism) only finitely many different n-dimensional crystallographic groups.*

Table 2: Number of crystallographic and Bieberbach groups in dimensions ≤ 6 ([40])

Dimension	2	3	4	5	6
Number of crystallographic groups	17	219	4783	222018	28927922
Number of Bieberbach groups	2	10	74	1060	38746

Remark 1.12. The precise numbers of crystallographic groups are known up to dimension 6 – see Table 2.

Among the crystallographic groups, those which do also have the property of being torsion-free, are of special interest (as we will see soon).

Definition 1.13. A torsion-free crystallographic group is called a *Bieberbach group*.

Remark 1.14. In two dimensions, there are only two Bieberbach groups (up to isomorphism), namely the fundamental groups of the torus and the Klein bottle. The number of Bieberbach groups is known up to dimension 6 – see Table 2.

There is an important consequence of the Bieberbach theorems: The Bieberbach theorems imply (see [102] and [100, Theorem 3.2.9]) that the crystallographic groups are precisely the (infinite) finitely-generated virtually abelian groups.

1.4 Bieberbach theorems geometrically

From a geometrical point of view, Bieberbach groups are also of great interest: Bieberbach groups are precisely the fundamental groups of connected, compact, flat Riemannian manifolds (this is the Killing–Hopf theorem, see [100, Corollary 2.4.10]). This means that the three Bieberbach theorems also have a geometrical interpretation which we will state in this section. A good reference for the geometric world of Bieberbach groups is [39].

Theorem 1.15 (Bieberbach's first theorem)**.** *Any compact, connected flat Riemannian manifold is finitely covered by a flat torus.*

Theorem 1.16 (Bieberbach's second theorem)**.** *Let Γ_1 and Γ_2 be the fundamental groups of the compact, connected flat Riemannian manifolds M_1 and M_2, respectively. Then $\Gamma_1 \cong \Gamma_2$ if and only if M_1 and M_2 are affinely equivalent.*

Theorem 1.17 (Bieberbach's third theorem)**.** *For every $n \in \mathbb{N}$, there are only finitely many n-dimensional connected, compact, flat Riemannian manifolds (up to affine equivalence).*

And we can characterize the fundamental groups of these Riemannian manifolds:

Theorem 1.18 ([100, Theorem 3.3.2])**.** *The fundamental groups of the connected compact flat Riemannian manifolds are precisely the torsion-free virtually abelian finitely-generated groups.*

For more information on the geometric and algebraic structure of Bieberbach groups, see Charlap's book [39].

1.5 Affine crystallographic groups

Thanks to the Bieberbach theorems, the structure of Euclidean crystallographic groups is well-understood. A natural generalization are *affine crystallographic groups*:

Definition 1.19. Let Γ be a group acting on $\mathbb{R}^n$ via a homomorphism $\rho\colon \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ such that

(i) the action is *properly discontinuous* and

(ii) the orbit space $\mathbb{R}^n/\rho(\Gamma)$ is compact.

Then, the action is called an *affine crystallographic action* and the image $\rho(\Gamma)$ is called an *affine crystallographic group*.

Example 1.20. *The group* $\left\{ \begin{pmatrix} 1 & 0 & a \\ a & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ *is an affine crystallographic group acting on* $\mathbb{R}^2 = \{(x, y, 1)\colon x, y \in \mathbb{R}\}$.

Note that this definition is the same as the one of Euclidean crystallographic actions (Definition 1.4) except that we now require ρ to map into the (larger) group $\mathrm{Aff}(\mathbb{R}^n)$ (instead of $\mathrm{Iso}(\mathbb{R}^n)$). So it indeed makes sense to call this notion a generalization of the one of Euclidean crystallographic groups.

However, none of the Bieberbach theorems does hold if we replace "crystallographic groups" by "affine crystallographic groups" (see [4] for counterexamples). So a natural question arises: Can we find theorems analogous to the Bieberbach theorems for affine crystallographic groups?

Before we deal with this question, let us describe the geometric aspects behind affine crystallographic groups.

1.6 Affinely flat manifolds

Theorem 1.18 completely characterizes the fundamental group of the connected compact flat Riemannian manifolds. So one could ask what the geometric interpretation of affine crystallographic groups is – we will explain this in the next section. To this end, we define the class of *affinely flat manifolds*:

Definition 1.21. A manifold M is called *affinely flat*, if it has an *affinely flat* atlas $\{\varphi_\alpha\colon U_\alpha \to \varphi(U_\alpha)\}$, that is, whenever $U_\alpha \cap U_\beta \neq \emptyset$, every coordinate change homeomorphism

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a restriction of a map in $\mathrm{Aff}(\mathbb{R}^n)$. We say that M has an *affinely flat structure* or just an *affine structure.*

Example 1.22 ([44]). *Special subclasses of affinely flat manifolds are the previously considered* Riemannian-flat manifolds, *where the coordinate change homeomorphisms extend to isometries in* $\mathrm{Iso}(\mathbb{R}^n)$*, that is, affine transformations* $x \mapsto Ax + b$*, where* $A \in O_n(\mathbb{R})$*. Another special subclass is the one of* Lorentz-flat manifolds*, where the coordinate change homeomorphisms extend to affine transformations* $x \mapsto Ax + b$*, where* A *belongs to the* Lorentz group $O(n-1, 1)$.

Benzecri ([13]) characterized closed surfaces admitting an affinely flat structure:

Theorem 1.23 (Benzecri, 1959). *A closed surface admits an affinely flat structure if and only if its Euler characteristic vanishes.*

In particular, two-dimensional closed surfaces different from the 2-torus or the Klein bottle do not admit an affinely flat structure.

Concerning affinely flat manifolds, one can define *geodesics*:

Definition 1.24 ([44]). A *geodesic* in an affinely flat manifold M is a curve $\gamma\colon [a, b] \to M$ (where $[a, b]$ denotes an interval in $\mathbb{R}$) such that for every chart φ of the affinely flat atlas, the composition $\varphi \circ \gamma\colon [a, b] \to \mathbb{R}^n$ is the restriction of an affine map.
If we replace in this definition the interval $[a, b]$ by the whole $\mathbb{R}$, we speak of a *complete geodesic* $\gamma\colon \mathbb{R} \to M$.

Definition 1.25. An affinely flat manifold M is *complete* if every geodesic $\gamma\colon [a, b] \to M$ can be extended to a complete geodesic $\gamma\colon \mathbb{R} \to M$.

1.7 Auslander's conjecture and Milnor's question

In [7], Auslander and Markus characterized connected, complete affinely flat manifolds:

Theorem 1.26 (Auslander–Markus). *The connected, complete affinely flat manifolds are exactly the quotients* $\mathbb{R}^n/\Gamma$*, where* $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ *acts properly discontinuously and freely on* $\mathbb{R}^n$.

This theorem implies that a similar correspondence as in the Euclidean case does hold for affine crystallographic groups: Torsion-free affine crystallographic groups are precisely the fundamental groups of connected, complete, compact affinely flat manifolds (see also [100, Corollary 1.9.6]).

Remark 1.27. In the situation of Riemannian manifolds, we could omit the word "complete" as compact Riemannian manifolds are always complete (Hopf–Rinow theorem, see e.g. [39, Theorem 4.1]).

For Bieberbach-like theorems in the setting of *affine-crystallographic groups*, a reasonable generalization of the term "virtually abelian" is "virtually polycyclic". So we define polycyclicity:

Definition 1.28. Let G be a group. We say G is *polycyclic,* if G admits a subnormal series

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \ldots \triangleright G_{n-1} \triangleright G_n = 1,$$

such that for all $i \in \{0, \ldots, n-1\}$, the group G_{i+1} is a normal subgroup of G_i and the quotient G_i/G_{i+1} is a cyclic group.

Examples for polycyclic groups include finite solvable groups and finitely generated nilpotent groups. In particular, every finitely generated abelian group is polycyclic.

By [61, Chapter 14.2], we have:

Theorem 1.29. *Every discrete solvable linear subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is polycyclic. In particular, for discrete linear groups, the terms "polycyclic" and "solvable" are equivalent.*

In [77, Chapter 4], John Milnor proved that every torsion-free, virtually polycyclic group is the fundamental group of some connected, complete affinely flat manifold, that is, admits a properly discontinuous affine action on some $\mathbb{R}^n$.
It was already known that every torsion-free, virtually polycyclic group is the fundamental group of a connected, compact manifold (see [77]). Now Milnor asked whether one could combine those two results:

Question 1.30 (Milnor's question, 1977, [77])**.** *Let* Γ *be a torsion-free, virtually polycyclic group. Is* Γ *the fundamental group of a connected, compact, complete affinely flat manifold?*

In fact, in [77, Chapter 4], Milnor asks Question 1.30, and, "as a first step", the corresponding question for Lie groups (which is Question 1.1). If, in Question 1.1, G admits such a structure, then, for every discrete subgroup Γ, G/Γ is a complete affinely flat manifold and Γ can often be chosen such that G/Γ is compact (see [77]). (Affine crystallographic actions of finitely generated, torsion-free nilpotent groups can be identified with simply transitive affine actions of their *Mal'cev completition*, which is a connected and simply-connected nilpotent Lie group; for details, see [44, 47].)

Both questions received a lot of attention. (Milnor's questions are now known to have a negative answer; we will come to this in Section 1.9.)
The so-called Auslander conjecture asks for the converse: In 1964, Louis Auslander published a theorem ([5]) that the fundamental group of any connected, complete affinely flat manifold is virtually polycyclic. Unfortunately, his proof turned out to be incorrect – moreover, Margulis ([71], see also [1, Chapter 8]) even constructed examples of connected, complete affinely flat manifolds with fundamental groups not being virtually polycyclic. Thus, not only the proof, but also the statement of Auslander's theorem was incorrect. However, the statement is still open for the case of *compact* manifolds and known as Auslander's conjecture:

Conjecture 1.31 (Auslander's conjecture)**.** *The fundamental group of a connected, compact, complete affinely flat manifold is virtually polycyclic.*

Comparing Auslander's conjecture and Milnor's question with Theorem 1.18, one finds that they indeed would be – if true – generalizations of Theorem 1.18 to affinely flat manifolds.
For manifolds of dimension up to 5, the answer to Auslander's conjecture is known to be affirmative (see Section 1.9). However, the conjecture is widely believed to be true in general.

1.8 Milnor's and Auslander's questions on the level of discrete groups

Via the correspondence between affinely flat manifolds and affine crystallographic groups, we may translate Milnor's question and Auslander's conjecture to the level of discrete groups:

Question 1.32 (Milnor)**.** *Is every torsion-free, virtually polycyclic group an affine crystallographic group?*

Question 1.33 (Auslander)**.** *Is every affine crystallographic group virtually polycyclic?*

Remark 1.34. Sometimes, e.g. in [1] or [53], these two questions are stated in terms of virtual solvability (instead of virtual polycyclicity). However, by Theorem 1.29, these two formulations are equivalent.

Remark 1.35. A positive answer to Milnor's question and to Auslander's conjecture would generalize Bieberbach's theorems to affine crystallographic actions and would give a geometric characterization of the class of virtually polycyclic groups.

Another way to look at the two questions is a famous theorem by Tits:

Theorem 1.36 (Tits alternative, [95], 1971)**.** *Every subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is either virtually solvable or contains a free, non-abelian subgroup.*

As described in Section 1.2, we can regard $\mathrm{Aff}(\mathbb{R}^n)$ as a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$. So we may read the questions of Milnor and Auslander as asking in which of those two types of groups in the Tits alternative our affine crystallographic groups belong.

1.9 Results on Auslander's and Milnor's conjectures

There is positive evidence for Milnor's question 1.30: The question has an affirmative answer for 2-step and 3-step nilpotent groups ([87]), for certain classes of solvable Lie groups ([101]) and for Lie groups with a Lie algebra admitting an invertible derivation ([86]).

However, the answer to Milnor's question eventually turned out to be negative. Those results were obtained on the level of Lie algebras (see Section 1.10).
Yves Benoist published a counterexample to Milnor's question in 1995 (see [12]): Benoist

constructed an 11-dimensional filiform Lie algebra which does not admit an affine structure. Dietrich Burde and Fritz Grunewald generalized this counterexample in 1995 ([30]) and in 1996, Burde presented a family of counterexamples and determined exactly which 10-dimensional filiform Lie algebras admit affine structures and proved that filiform Lie algebras of dimension smaller than 10 always admit affine structures (see [18, 32]).

Thus, we have the following theorem:

Theorem 1.37 (Benoist, Burde–Grunewald, Burde)**.** *Milnor's question has a negative answer: There exist finitely-generated torsion-free nilpotent groups which are not affine crystallographic groups.*

On the other hand, Auslander's conjecture is still open. The affirmative answer is know up to dimension 5 ([96]) (and perhaps up to dimension 6, see [2]).

As Milnor's question turned out to have a negative answer, one cannot characterize the affine crystallographic groups to be the virtually polycyclic groups. However, there were several attempts to generalize the Bieberbach theorems. We want to mention a generalization of Bieberbach's second theorem. To present this theorem, we have to introduce the set of polynomial automorphisms $P(\mathbb{R}^n)$:

Definition 1.38. A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a *polynomial map* if it can be expressed as $f = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$ for some polynomials $F_1, \dots, F_m \in \mathbb{R}[x_1, \dots, x_n]$.
A *polynomial automorphism of* $\mathbb{R}^n$ is a bijective map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that f and f^{-1} are polynomial maps.
The set of all polynomial automorphisms of $\mathbb{R}^n$ forms a group which will be denoted by $P(\mathbb{R}^n)$.

The group $P(\mathbb{R}^n)$ provides a generalization of the second Bieberbach theorem:

Theorem 1.39 (Fried–Goldman, [53, Theorem 1.20])**.** *Two virtually polycyclic affine crystallographic groups* Γ, Γ' *are isomorphic via the map* $\varphi\colon \Gamma \to \Gamma'$ *if and only if there is a polynomial automorphism* $\alpha \in P(\mathbb{R}^n)$ *such that* $\varphi(\gamma) = \alpha^{-1}\gamma\alpha$ *for all* $\gamma \in \Gamma$.

1.10 Milnor's question on the level of Lie algebras

In this section, we want to describe how Milnor's question 1.1 can be translated to the level of Lie algebras. This approach provided the framework for the counterexamples to Milnor's question (cf. Theorem 1.37). We will follow [22, Chapter 1.4] here (see also [47, Chapter 1]) for more details).

Let us explain the term "left-invariant" in Milnor's question 1.1:

Definition 1.40. An affine structure on a Lie group G over $\mathbb{R}$ is called *left-invariant*, if for all $g \in G$, the left-multiplication operator $L(g)\colon G \to G$ is an affine diffeomorphism.

Left-invariant affine structures can be reformulated via pre-Lie algebra structures:

Theorem 1.41 ([22, Proposition 1.4.6 and Proposition 1.4.8]). *Let G be a connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then there is a 1-1-correspondence between left-invariant affine structures on G and pre-Lie algebra structures on $\mathfrak{g}$ (up to suitable equivalences). Moreover, there are 1-1-correspondences (up to suitable equivalences) between*

(i) Complete left-invariant affine structures on G.

(ii) Simply transitive actions of G by affine transformations.

(iii) Complete pre-Lie algebra structures on $\mathfrak{g}$.

Simply transitive actions by affine transformations are defined as follows:

Definition 1.42. Let G be an n-dimensional Lie group.

(i) An action via a homomorphism $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^n)$ is called an *affine transformation of G.*

(ii) If for all $x_1, x_2 \in \mathbb{R}^n$ there is a unique $g \in G$ such that $\rho(g)(x_1) = x_2$, we say that ρ is a *simply transitive action by affine transformations.*

Definition 1.43 (See e.g. [88]). Let $\mathfrak{g}$ be a Lie algebra over the field k. We say that a k-bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto x \cdot y$, is a *pre-Lie algebra structure* or *affine structure* on $\mathfrak{g}$, if $x \cdot y$ satisfies the two identities

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y] \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.
A pre-Lie algebra structure on $\mathfrak{g}$ is said to be *complete*, if for all $x \in \mathfrak{g}$, the right-multiplication operator $R(x)\colon \mathfrak{g} \to \mathfrak{g}$, defined by $R(x)(y) \coloneqq y \cdot x$, is nilpotent.

Remark 1.44.

(i) Pre-Lie algebras arise in different areas of mathematics and physics and there exists a vast amount of literature on them. For a detailed survey, see [20].

(ii) Sometimes, the definition of a pre-Lie algebra is stated as a vector space V together with a bilinear map $V \times V \to V, (x, y) \mapsto x \cdot y$, such that

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z)$$

holds for all $x, y, z \in V$. This definition is seen to be equivalent to Definition 1.43 if we introduce a Lie bracket $[,]$ by $[x, y] \coloneqq x \cdot y - y \cdot x$ (and note that this indeed defines a Lie bracket).
This definition seemingly relates to the *associator* of $(V, \cdot)$, defined by $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ and can be rewritten as $(x, y, z) = (y, x, z)$. Thus, pre-Lie algebra structures are also know as *left-symmetric algebras.*

If G admits a left-invariant affine structure, then there is an affine connection ∇ on G which is left-invariant, torsion-free and flat. Then, for two left-invariant vector fields $X, Y \in \mathfrak{g}$, one can check that $X \cdot Y = \nabla_X(Y)$ defines a pre-Lie algebra structure on $\mathfrak{g}$. (For details, see [22, 47]).

Example 1.45. *The Lie algebra $\mathfrak{gl}_n(k)$ of $n \times n$-matrices with Lie bracket $[x, y] = x \cdot y - y \cdot x$, where $x \cdot y$ denotes the matrix multiplication of the matrices x and y, is a pre-Lie algebra structure.*

So an algebraic reformulation (on the level of Lie algebras) of Milnor's question 1.1 is:

Question 1.46 (Milnor)**.** *Does every solvable Lie algebra admit a complete affine structure?*

Counterexamples could now be given via the following connection to representation theory of Lie algebras ([22, Proposition 1.5.1, Lemma 3.1.18]):

Proposition 1.47. *If $\mathfrak{g}$ is an n-dimensional Lie algebra over a field of characteristic zero which admits an affine structure, then there is a faithful $\mathfrak{g}$-module of dimension $n+1$.*

Now, on the level of Lie algebras, in [22, Chapter 5], one can find examples of filiform Lie algebras of dimensions $n = 10, 11, 12$ not admitting faithful modules of dimension $n+1$ implying that Milnor's questions (both 1.1 and 1.30) have a negative answer. This proves Theorem 1.37.

Let us mention that the converse to Milnor's question 1.1 is true:

Theorem 1.48 (Auslander, [6])**.** *Let G be a Lie group admitting a simply transitive affine action. Then G is solvable.*

1.11 Nil-affine crystallographic actions and post-Lie algebra structures

Since Milnor's question fails for affine crystallographic groups, people tried to find a suitable generalization of affine crystallographic groups such that the generalized setting is still of geometric meaning and the corresponding statements of Milnor's and Auslander's conjectures are true.

It is proved in [46] that virtually polycyclic groups Γ admit a *polynomial structure*, that is, a properly discontinuous homomorphism with compact orbit space into $P(\mathbb{R}^n)$ (see Definition 1.38). However, it turned out that the *nil-affine* setting was much better suited for geometry than $P(\mathbb{R}^n)$ (see [45]):

In the nil-affine setting, we let our group act on *any* nilpotent Lie group instead of $\mathbb{R}^n$. To be more precise, let N be a nilpotent Lie group and let $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ be the group of affine transformations of N.

The group $\mathrm{Aff}(N)$ acts on N via *nil-affine transformations*, that is, via ${}^{(m,\alpha)}n = m \cdot \alpha(n)$

for $(m,\alpha) \in \mathrm{Aff}(N), n \in N$. The Lie algebra of the Lie group $\mathrm{Aff}(N)$ will be denoted by $\mathfrak{aff}(\mathfrak{n}) = \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ with Lie bracket

$$[(n_1, D_1), (n_2, D_2)] = ([n_1, n_2] + D_1(n_2) - D_2(n_1), [D_1, D_2]).$$

Then, we can define nil-affine crystallographic groups very similar to affine crystallographic groups:

Definition 1.49. Let Γ be a group acting on a nilpotent Lie group N via a homomorphism $\rho\colon \Gamma \to \mathrm{Aff}(N)$. We call this action a *nil-affine crystallographic action,* if

(i) Γ acts properly discontinuously on N, i.e. for each compact subset $K \subseteq N$, the set $\{\gamma \in K\colon \rho(\gamma)K \cap K \neq \emptyset\}$ is finite and

(ii) the orbit space $N/\rho(\Gamma)$ is compact.

Then, the action is called a *nil-affine crystallographic action* and we call $\rho(\Gamma)$ a *nil-affine crystallographic group.*

Remark 1.50. Note that the nil-affine crystallographic setting is indeed a generalization of the affine crystallographic setting (see Definition 1.19): If we replace N by $\mathbb{R}^n$, we obtain the notion of affine crystallographic groups.

As before, we can view this situation geometrically: If $\Gamma \subseteq \mathrm{Aff}(N)$ acts freely and properly discontinuously on N, we call the quotient space N/Γ a *nil-affine flat manifold* with fundamental group Γ; if Γ is a nil-affine crystallographic group, then N/Γ is a compact nil-affine flat manifold. Via the left-invariant affine connection from N, one may define geodesics and thus completeness (any partial geodesic can be extended to the real line; see [45] for details).
Now, what makes the nil-affine situation very interesting is that the analogue of Milnor's question does hold for nil-affine crystallographic actions:

Theorem 1.51 (Dekimpe, [45], Baues, [10])**.** *Every virtually polycyclic group admits a nil-affine crystallographic action.*
In other (geometric) words, every torsion-free, virtually polycyclic group is the fundamental group of some compact, complete, connected nil-affinely flat manifold.

The generalized Auslander conjecture becomes in this setting:

Conjecture 1.52 (Generalized Auslander conjecture, nil-affine setting)**.** *Every group admitting a nil-affine crystallographic action is virtually polycyclic.*

Regarding this Generalized Auslander conjecture, we want to mention two results from [25]: Conjecture 1.52 is proven for all nilpotent Lie groups up to dimension 5; moreover, for all 2-step nilpotent Lie groups, the generalized Auslander conjecture is equivalent to the classical Auslander conjecture.

We want to derive a analogous correspondence to Theorem 1.41 for nil-affine crystallographic actions.

Definition 1.53. A group G acts *simply transitively* on the Lie group N *by nil-affine transformations*, if there is a homomorphism $\rho\colon G \to \operatorname{Aff}(N)$ letting G act on N such that for all $x_1, x_2 \in N$ there is a unique $g \in G$ with $\rho(g)(x_1) = x_2$.

And the nil-affine analogue to Theorem 1.41 is as follows:

Theorem 1.54 (Burde–Dekimpe–Vercammen, [28])**.** *Let G and N be connected and simply-connected nilpotent Lie groups. Then there exists a simply transitive action by nil-affine transformations of G on N if and only if the corresponding pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ admits a* left-nil post-Lie algebra structure*.*

(Left-nil) post-Lie algebra structures are defined as follows:

Definition 1.55. Let $(\mathfrak{g}, [,]), (\mathfrak{n}, \{,\})$ be two Lie algebras with underlying k-vector space V. A *post-Lie algebra structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a k-bilinear map $(x, y) \mapsto x \cdot y$ satisfying the three identities

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y] - \{x, y\} \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ x \cdot \{y, z\} &= \{x \cdot y, z\} + \{y, x \cdot z\} \end{aligned}$$

for all $x, y, z \in V$.
We say that a post-Lie algebra structure is *left-nil* if all left-multiplication operators $L(x)\colon V \to V, L(x)(y) \coloneqq x \cdot y$, are nilpotent.

Remark 1.56. In the case of pre-Lie algebra structures, the condition of completeness (all right-multiplication operators are nilpotent) is equivalent to the condition of being left-nil (all left-multiplication operators are nilpotent), see [65, Theorem 2.1 and Theorem 2.2].

This motivates us to study post-Lie algebra structures in general, i.e. without assumptions on nilpotency of $\mathfrak{g}$ and $\mathfrak{n}$ or being left-nil. However, we do get interesting results if we assume $\mathfrak{g}$ or $\mathfrak{n}$ (or both) to be nilpotent (see Chapter 5).
The following remark puts post-Lie algebra structures and pre-Lie algebras and LR-structures into perspective:

Remark 1.57.

(i) If $\mathfrak{n}$ is an *abelian* Lie algebra (i.e. $\{\mathfrak{n}, \mathfrak{n}\} = 0$), then a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is a *pre-Lie algebra structure* on $\mathfrak{g}$, compare Definition 1.43.
This is no surprise, but follows from our general theory of affine transformations: If $\mathfrak{n}$ is abelian, then a nil-affine transformation of G on N is just an affine transformation of G (on $\mathbb{R}^n$). Thus one here recovers the definition of a pre-Lie algebra structure on $\mathfrak{g}$.

(ii) On the other hand, if not $\mathfrak{n}$ but $\mathfrak{g}$ is an abelian Lie algebra, then a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ corresponds to what is called an *LR-structure* on $\mathfrak{n}$ (see Definition 2.37).

Thus one can view post-Lie algebra structures as a common generalization of pre-Lie algebra structures and LR-structures. Both are well-studied notions; there exists a vast amount of literature on pre-Lie algebra structures and LR-structures (see e.g. [20, 26, 27]).

1.12 Post-Lie algebra structures in other contexts

We presented a geometric-algebraic approach based on Milnor's question to come up with post-Lie algebra structures; however, post-Lie algebra structures have also been discovered by completely different means.

Post-Lie algebra structures were originally introduced by Bruno Vallette in 2006 ([98]) in operad theory:
There, the *poset of pointed partitions* is defined as the set of partitions of $\{1, \ldots, n\}$ into blocks $\{B_1, \ldots, B_k\}$ such that in each block B_i exactly one element is chosen which is "pointed". This forms a poset via the following definition of $\leq$: Given two pointed partitions λ, μ, we say $\lambda \leq \mu$ if μ is a refinement of λ (as a partition) and every pointed element in λ is a pointed element in μ. Vallette shows that a Π_{Perm}, the operad corresponding to a Perm-algebra (a special kind of associative algebras) is isomorphic to the poset of pointed partitions; the Koszul dual of Π_{Perm} is $PreLie$, the operad corresponding to pre-Lie algebra structures. This fact is used to compute the homology of the poset of pointed partitions ([98, Theorem 13]).

In the same way, [98] also defines the *poset of multi-pointed partitions*: the set of partitions of $\{1, \ldots, n\}$ into blocks $\{B_1, \ldots, B_k\}$ such that in each block at least one element is pointed. We say $\lambda \leq \mu$ if μ is a refinement of λ (as a partition) and for every block B_i of μ, every pointed element in B is pointed in λ or every pointed element in B is unpointed in λ.
This poset is isomorphic to the operadic poset $\Pi_{ComTrias}$, which is the operad corresponding to commutative trialgebras, a subclass of Perm-algebras. And $PostLie$, the operad corresponding to post-Lie algebras, is proven to be the Koszul dual of $ComTrias$, which is again used to compute the homology of the poset of multi-pointed partitions ([98, Theorem 17]).

Post-Lie algebras are defined in [98] as follows:

Definition 1.58. A *post-Lie algebra* $(L, \circ, [,])$ is a k-vector space with two binary operations, $\circ$ and $[,]$, such that $(L, [,])$ is a Lie algebra and the two identities

$$\begin{aligned} x \circ [y, z] &= (x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y), \\ [x, y] \circ z &= [x \circ z, y] + [x, y \circ z] \end{aligned}$$

hold for all $x, y, z \in L$.

This definition is the one of a right-post-Lie algebra, while Definition 1.55 uses multiplication from the left. It is equivalent to Definition 1.55 via:

Lemma 1.59. *If* $(L, \circ, [,]_V)$ *is a post-Lie algebra as in Definition 1.58, then* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{g}$ *has Lie bracket* $[,]$ *and* $\mathfrak{n}$ *has Lie bracket* $\{,\}$ *(and* $\mathfrak{g}$ *and* $\mathfrak{n}$ *have* L*'s underlying vector space), defined by*

$$\begin{aligned} x \cdot y &:= y \circ x \\ \{x, y\} &:= -[x, y]_V \\ [x, y] &:= -[x, y]_V + y \circ x - x \circ y \end{aligned}$$

is a post-Lie algebra structure $(\mathfrak{g}, \mathfrak{n}, \cdot)$ *as in Definition 1.55. (In particular,* L *admits another Lie bracket (namely the Lie bracket of* $\mathfrak{g}$*).)*

We will work with Definition 1.55 in the sequel.

Post-Lie algebra structures have also been studied in connection with the classical Yang-Baxter equation, see [80] and in connection with Lie algebroids in [16].

In [51, 79], the authors study post-Lie algebras in the context of Lie-Butcher Series, geometric integration, differential equation on homogeneous spaces and moving frame theory, see [42] for an overview about post-Lie algebra structures in geometric integration theory. The considerations there are geometrical; we, on the other hand, will pursue algebraic approaches to the theory of post-Lie algebra structures. In [52], post-Lie algebra structures arising in Control Theory are studied.

There are also strong connections between post-Lie algebra structures and Rota-Baxter operators, see e.g. [31], where post-Lie algebra structures are studied via Rota-Baxter theory.

2 Background on Lie algebras and post-Lie algebra structures

The goal of this chapter is twofold: To remind the reader of some basic notions in the realm of Lie algebras and to state already known theorems on post-Lie algebra structures.

2.1 Lie algebras

We want to recall some fundamental definitions and results of Lie algebras and clarify our notation. Proofs are usually omitted; details can be found in any introductory textbook on the topic (such as [62, 64, 94]).

2.1.1 Basic definitions

Definition 2.1. A *Lie algebra* L over a field k is a k-vector space together with a k-bilinear map $[,]\colon L \times L \to L$, called the *Lie bracket*, which is *anti-commutative* and satisfies the *Jacobi identity*, that is

(i) $[x, x] = 0$ for all $x \in L$,

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Remark 2.2. Instead of capital letters like L, Lie algebras are often denoted by small gothic letters such as $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}$. We use both conventions to emphasize what we are currently interested in: When considering post-Lie algebra (or related) structures, we will refer to Lie algebras by gothic letters; if we consider Lie algebras in general (without referring to post-Lie algebra structures), we will instead refer to them by capital letters.

Remark 2.3. The condition $[x, x] = 0$ implies $[x, y] = -[y, x]$ for all $x, y \in L$ (thus the name "*anti-commutativity*"), as

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

As with other algebraic structures, we can define subalgebras, ideals and homomorphisms of Lie algebras:

Definition 2.4. Let $(L, [,])$ be a Lie algebra over k.

(i) A subspace M of L is called a *subalgebra* of L, if $[x, y] \in M$ for all $x, y \in M$.

(ii) If M_1, M_2 are subalgebras of L, we denote by $[M_1, M_2]$ the subalgebra spanned by all $[x, y]$ with $x \in M_1, y \in M_2$. (In general, this is different from the set $\{[x, y], x \in M_1, y \in M_2\}$.)

(iii) If M is a subalgebra of L such that $[M, L] \subseteq M$, we call M a *(Lie) ideal* of L and write $M \triangleleft L$.

(iv) If $(M, \{,\})$ is another Lie algebra over k, we call a map $\varphi\colon L \to M$ a *homomorphism of Lie algebras*, if $\varphi([x, y]) = \{\varphi(x), \varphi(y)\}$ holds for all $x, y \in L$. If, in addition, φ is bijective, we call φ an *isomorphism of Lie algebras* and write $L \cong M$.

(v) A Lie algebra isomorphism $\varphi\colon L \to L$ is called an *automorphism of* L; the set of automorphisms of L forms the *automorphism group of* L, denoted by $\mathrm{Aut}(L)$.

Example 2.5. *The generic example for a Lie algebra is the* general linear Lie algebra $\mathfrak{gl}(V)$*, the set of endomorphism of a vector space V together with the commutator as Lie bracket, that is $[\varphi, \psi] := \varphi \circ \psi - \psi \circ \varphi$. For the general linear Lie algebra over the n-dimensional k-vector space, we write $\mathfrak{gl}_n(k)$.*

Every Lie algebra L has the two *trivial ideals* 0 and L.

Definition 2.6. A non-abelian Lie algebra without non-trivial ideals is called a *simple* Lie algebra.

Example 2.7. *The simple Lie algebra over $\mathbb{C}$ of smallest dimension is known by the name $\mathfrak{sl}_2(\mathbb{C})$ (the* special linear Lie algebra*). It can be defined as the vector space of traceless 2×2-matrices with the commutator as Lie bracket.*
More generally, $\mathfrak{sl}_n(\mathbb{C}) := \{X \in \mathfrak{gl}_n(\mathbb{C})\colon \mathrm{tr}(X) = 0\}$ defines a simple Lie algebra over $\mathbb{C}$ in dimension $n^2 - 1$.

Definition 2.8. Let $(L, [,])$ be a Lie algebra over k.

(i) The *commutator algebra* of L is the ideal $[L, L] \triangleleft L$. If $L = [L, L]$, we call L *perfect*.

(ii) The *center* of L is the ideal $\mathrm{Z}(L) := \{x \in L\colon [x, y] = 0 \text{ for all } y \in L\}$.

(iii) If $\varphi\colon L \to M$ is a Lie algebra homomorphism, its *kernel* $\ker \varphi := \{x \in L\colon \varphi(x) = 0\}$ is an ideal of L; its *image* $\mathrm{Im}(\varphi) := \{y \in M\colon y = \varphi(y) \text{ for an } x \in L\}$ is a subspace of M.

The *derivation algebra* of a Lie algebra is very important; we call a linear map $D\colon L \to L$ a *derivation* of L if it satisfies the Leibniz rule with respect to the Lie bracket, i.e.

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in L$. The set of all derivations forms a Lie algebra, denoted by $\mathrm{Der}(L)$, with Lie bracket $[D, D'] := D \circ D' - D' \circ D$.
The most important kind of derivations are *inner derivations*; if $x \in L$, we define the *adjoint map* $\mathrm{ad}(x)\colon L \to L$ by $\mathrm{ad}(x)(y) := [x, y]$. Every adjoint map is a derivation of L; derivations of this form are called *inner*, non-inner derivations are called *outer derivations*.
The Lie algebra $\mathrm{ad}(L) := \{\mathrm{ad}(x)\colon x \in L\}$ forms an ideal in $\mathrm{Der}(L)$, because we have

$\mathrm{ad}(D(x)) = [D, \mathrm{ad}(x)]$ for every $D \in \mathrm{Der}(L), x \in L$.

Finally, a *(Lie algebra) representation* of a k-Lie algebra is a k-vector space V together with a homomorphism $\rho\colon L \to \mathfrak{gl}(V)$; a representation is called *faithful*, if ρ is injective. Equivalently, a representation of L is a vector space together with a bilinear map $\cdot\colon L \times V \to V$ satisfying $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$ for all $x, y \in L, z \in V$.

Example 2.9. *The* adjoint representation *of L is given by* $\mathrm{ad}\colon L \to \mathfrak{gl}(L), x \mapsto \mathrm{ad}(x)$.

2.1.2 Abelian, nilpotent and solvable Lie algebras

Definition 2.10. Let $(L, [,])$ be a Lie algebra.

(i) The *lower central series of* L is defined by

$$L^0 = L, L^1 = [L, L], \dots, L^{i+1} = [L, L^i], \dots$$

Note that every L^i is an ideal of L.

(ii) The *upper central series of* L is defined by

$$\mathrm{Z}_0 = 0, \mathrm{Z}_1 = \mathrm{Z}(L), \dots, \mathrm{Z}_{i+1} = \{x \in L\colon [x, y] \in \mathrm{Z}_i \text{ for all } y \in L\}, \dots$$

Note that every Z_i is an ideal of L.

(iii) The *derived series of* L is defined by

$$L^{(0)} = L, L^{(1)} = [L, L], \dots, L^{(i+1)} = [L^{(i)}, L^{(i)}], \dots$$

Note that every $L^{(i)}$ is an ideal of L.

(iv) The lower central series eventually terminates (i.e. $L^k = 0$ for some k) if and only if the upper central series eventually terminates (i.e. $\mathrm{Z}_k = L$ for some k). If they do terminate, they have the same length and we call L *nilpotent*. If $L^k = 0$ and $L^{k-1} \neq 0$, we say that L has *nilindex* k or that L is *k-step nilpotent*.

(v) If the derived series eventually terminates, i.e. $L^{(k)} = 0$ for some k, we call L *solvable*. We refer to the lowest k such that $L^{(k)} = 0$ as the *solvability length* of L. If $k \leq 2$, we say L is *metabelian*.

(vi) The *nilradical* of L, denoted by $\mathrm{nil}(L)$, is the (unique) largest nilpotent ideal of L.

(vii) The *radical* of L, denoted by $\mathrm{rad}(L)$, is the (unique) largest solvable ideal of L.

Example 2.11. *The 3-dimensional* Heisenberg Lie algebra $\mathfrak{n}_3$ *over $\mathbb{C}$, which has the basis $\{e_1, e_2, e_3\}$ and non-zero Lie bracket $[e_1, e_2] = e_3$, is 2-step nilpotent.*

Example 2.12. *The three-dimension complex Lie algebras are listed in Table 3.*

An important subclass of nilpotent Lie algebras is the class of abelian Lie algebras:

Table 3: Lie algebras in dimension 3

$\mathfrak{g}$	non-zero Lie brackets
$\mathbb{C}^3$	–
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3$ with $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ if and only if $\mu = \frac{1}{\lambda}$ or $\mu = \lambda$
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$

Definition 2.13. A Lie algebra L is called *abelian*, if $[x, y] = 0$ for all $x, y \in L$.

Lemma 2.14. *Every nilpotent Lie algebra is solvable; every abelian Lie algebra is nilpotent.*

Proof. Indeed, from $L^{(k)} \subseteq L^k$ follows the first claim; the latter statement holds since $L^1 = 0$ if L is abelian. $\square$

Definition 2.15. Let $M \triangleleft L$ be an ideal of a Lie algebra L. We say M is a *characteristic ideal*, if M is stable under every derivation of L (meaning $D(M) \subseteq L$ for every $D \in \mathrm{Der}(L)$).

Example 2.16. *Let L be a finite-dimensional Lie algebra over a field of characteristic* 0*. Then* $\mathrm{rad}(L)$ *and* $\mathrm{nil}(L)$ *are characteristic ideals.*

Lemma 2.17. *Every term L^k of the lower central series and every term $L^{(k)}$ of the derived series is a* characteristic ideal *of L.*

Proof. If L_1 and L_2 are characteristic ideals of L, then so is $[L_1, L_2]$, as $D([L_1, L_2]) = [D(L_1), L_2] + [L_1, D(L_2)] \subseteq [L_1, L_2]$, for $D \in \mathrm{Der}(L)$. The lemma follows by the simple observation that L is a characteristic ideal of L. $\square$

Remark 2.18. Sometimes the lower central series is denoted by $\gamma_i = \gamma_i(L) := L^{i-1}, i \geq 1$.

Nilpotent Lie algebras can be characterized by Engel's famous theorem:

Theorem 2.19 (Engel)**.** *A finite-dimensional Lie algebra is nilpotent if and only if all adjoint operators are nilpotent.*

Finite-dimensional Lie algebras over fields of characteristic zero are solvable if and only if their commutator algebra is nilpotent.

2.1.3 Direct and semidirect products

If $L_i, i \in I$ are Lie algebras, their vector space direct sum $L := \bigoplus_{i\in I} L_i$, together with the map $[(x_i)_{i\in I}, (y_i)_{i\in I}] := ([x_i, y_i]_{i\in I})$ forms again a Lie algebra, the *direct sum of the L_i*.

Let L_1 and L_2 be Lie algebras over k and $\varphi\colon L_1 \to \operatorname{Der}(L_2)$ a Lie algebra homomorphism. Then the *semidirect product* of L_1 and L_2 (with respect to φ), denoted by $L_1 \ltimes_\varphi L_2$ or simpler $L_1 \ltimes L_2$, is defined as the vector space $L_1 \times L_2$ with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2], [y_1, y_2] + \varphi(x_1)(y_2) - \varphi(x_2)(y_1)).$$

In this case, L_1 is a subalgebra and L_2 an ideal of $L_1 \ltimes L_2$.
If $\varphi \equiv 0$, we recover the notion of the direct sum of L_1 and L_2.

The direct sum $L \oplus L \oplus \ldots \oplus L$ with n equal (or isomorphic) factors will be written as L^n.

2.1.4 Semisimple Lie algebras

Definition 2.20. A non-abelian Lie algebra which is the direct sum of simple Lie algebras is called *semisimple*.

Complex semisimple Lie algebras have some useful properties:

Theorem 2.21. *Let L be a semisimple Lie algebra over $\mathbb{C}$. Then:*

(i) The Lie algebra L is perfect*, that is, $[L, L] = L$.*

(ii) All derivations of L are inner.

(iii) Ideals and images of L under Lie algebra homomorphisms are semisimple.

(iv) The center of L is trivial.

(v) The radical of L is trivial.

Other important theorems on semisimple Lie algebras will be presented in Section 2.1.5.

Simple (and thus also semisimple) finite-dimensional, complex Lie algebras can be classified completely, see Table 4.
The Lie algebras in Table 4 are defined by

$$\begin{aligned}
\mathfrak{sl}_m(\mathbb{C}) &:= \{X \in \mathfrak{gl}_m(\mathbb{C})\colon \operatorname{tr}(X) = 0\},\\
\mathfrak{so}_m(\mathbb{C}) &:= \{X \in \mathfrak{gl}_m(\mathbb{C})\colon X + X^t = 0\},\\
\mathfrak{sp}_{2n}(v) &:= \{X \in \mathfrak{gl}_{2n}(\mathbb{C})\colon JX + X^t J = 0\},
\end{aligned}$$

where $J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_m & 0 \end{pmatrix}$. In all cases the Lie brackets are given by the commutator of the matrix product.

Table 4: Classification of finite-dimensional simple complex Lie algebras

L	Type	dimension
$\mathfrak{sl}_{n+1}(\mathbb{C}), n \geq 1$	A_n	$n(n+2)$
$\mathfrak{so}_{2n+1}(\mathbb{C}), n \geq 2$	B_n	$n(2n+1)$
$\mathfrak{sp}_{2n}(\mathbb{C}), n \geq 3$	C_n	$n(2n+1)$
$\mathfrak{so}_{2n}(\mathbb{C}), n \geq 4$	D_n	$n(2n-1)$
$\mathfrak{g}_2(\mathbb{C})$	G_2	14
$\mathfrak{f}_4(\mathbb{C})$	F_4	52
$\mathfrak{e}_6(\mathbb{C})$	E_6	78
$\mathfrak{e}_7(\mathbb{C})$	E_7	133
$\mathfrak{e}_8(\mathbb{C})$	E_8	248

The Lie algebras of type A_n, B_n, C_n, D_n are called the *classical Lie algebras*. The name of the "types" comes from *Dynkin diagrams*, which are used in the proof of this classification.

Remark 2.22. There are isomorphisms for simple Lie algebras of small rank:

$$A_1 \cong B_1 \cong C_1, B_2 \cong C_2, D_2 \cong A_1 \oplus A_1, D_3 \cong A_3.$$

We will refer to the simple Lie algebras both by their name and by their type.

As semisimple Lie algebras are direct sums of simple Lie algebras, we obtain as a corollary also a classification of semisimple Lie algebras over $\mathbb{C}$.
Later (in Chapter 4), we will analyze a generalization of semisimple Lie algebras, namely *complete Lie algebras*:

Definition 2.23. A Lie algebra is called *complete*, if it has trivial center and only inner derivations.

Note that by Theorem 2.21, every semisimple Lie algebra is complete. Chapter 4 is entirely devoted to complete Lie algebras.
We want to introduce one more class of Lie algebras:

Definition 2.24.

(i) A Lie algebra is called *reductive*, if its adjoint representation is semisimple.

(ii) A Lie subalgebra M of L is called *reductive in* L, if the adjoint representation $\mathrm{ad}\colon M \to \mathfrak{gl}(L)$ is semisimple.

Proposition 2.25. *A Lie algebra L is reductive if and only if it can be written as $L = A \oplus S$, where A is an abelian and S a semisimple ideal of L.*

2.1.5 Classical structure theorems for Lie algebras

In this section, we shall state important classical structure results on Lie algebras.

Theorem 2.26 (Levi decomposition). *Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L can be decomposed as a semidirect sum*

$$L = \mathfrak{s} \ltimes \operatorname{rad}(L),$$

where $\mathfrak{s}$ is a semisimple subalgebra of L. In this case, $\mathfrak{s}$ is called a Levi complement *or* Levi subalgebra *of L. A Levi subalgebra is a maximal semisimple subalgebra of L.*

As already mentioned, the "most natural" Lie algebras to consider are matrix algebras, i.e., subalgebras of $\mathfrak{gl}_n(k)$ for some $n \in \mathbb{N}$.
The following famous theorem states that in some regard, these are also the most important Lie algebras to consider:

Theorem 2.27 (Ado). *Let L be a finite-dimensional Lie algebra over a field k of characteristic zero. Then L has a finite-dimensional faithful representation, i.e. is isomorphic to a subalgebra of some $\mathfrak{gl}_n(k)$ (where the Lie bracket is given by the matrix commutator, $[A, B] = AB - BA$).*

Theorem 2.28 (Cartan). *Let L be a finite-dimensional Lie algebra over a field k of characteristic zero and $\kappa\colon L \times L \to k, \kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$ the* Killing form *of L.*

(1) The following two conditions are equivalent:
   - *(i) The Lie algebra L is semisimple.*
   - *(ii) The Killing form of L is non-degenerate.*

(2) The following two conditions are equivalent:
   - *(i) The Lie algebra L is solvable.*
   - *(ii) We have $\kappa(L, [L, L]) = 0$.*

Theorem 2.29 (Weyl). *Let L be a semisimple Lie algebra over a algebraically closed field of characteristic zero. Then every finite-dimensional representation of L is semisimple.*

2.1.6 Graded and filtered Lie algebras

Definition 2.30. A Lie algebra L is called *A-graded* for an abelian group A if L can be written as a direct sum of vector spaces, $L = \bigoplus_{i \in A} L_i$ such that $[L_i, L_j] \subseteq L_{i+j}$.

Definition 2.31. A Lie algebra L is called *filtered* if there are subspaces $L_0 \subseteq L_1 \subseteq \ldots \subseteq L$ such that $L = \bigcup_{i \in \mathbb{N}} L_i$ and $[L_i, L_j] \subseteq L_{i+j}$.

2.2 Post-Lie algebra structures, pre-Lie algebra structures and LR-structures

We repeat the definition of a post-Lie algebra structure we gave in the introductory part.

Definition 2.32. Let V be a vector space over a field k and $\mathfrak{g} = (V, [,]), \mathfrak{n} = (V, \{,\})$ be two Lie algebras with underlying vector space V. A *post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ is a k-bilinear map $V \times V \to V, (x, y) \mapsto x \cdot y$ satisfying the following three identities:

$$x \cdot y - y \cdot x = [x, y] - \{x, y\} \tag{PA1}$$
$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \tag{PA2}$$
$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \tag{PA3}$$

for all $x, y, z \in V$.

Note that as vector spaces, $V = \mathfrak{g} = \mathfrak{n}$. Therefore, the statements "$x \in V$", "$x \in \mathfrak{g}$" and "$x \in \mathfrak{n}$" will be used interchangeably.

The special case where $\mathfrak{g} = \mathfrak{n}$, or, equivalently, $x \cdot y$ is commutative, is known as a *CPA-structure*:

Definition 2.33. Let V be a vector space over a field k and $\mathfrak{g} = (V, [,])$ a Lie algebra with underlying vector space V. A *commutative post-Lie algebra structure* or a *CPA-structure* on $\mathfrak{g}$ is a k-bilinear map $V \times V \to V, (x, y) \mapsto x \cdot y$ satisfying the following three identities:

$$x \cdot y = y \cdot x \tag{CPA1}$$
$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \tag{CPA2}$$
$$x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z] \tag{CPA3}$$

for all $x, y, z \in V$.

As the topic of this thesis are post-Lie algebra structures, we want to fix some notation for the remainder of the thesis:

Convention 2.34.

(i) Even though many results may be extended to other fields, we will always consider Lie algebras over the ground field $k = \mathbb{C}$.

(ii) When considering post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, we will always implicitly assume that $\mathfrak{g}$ has a Lie bracket denoted by $[,]$ and $\mathfrak{n}$ has a Lie bracket denoted by $\{,\}$. Usually, post-Lie algebra structures are denoted by $x \cdot y$. When talking about post-Lie algebra structures, V is always meant to be the underlying vector space for $\mathfrak{g}$ and $\mathfrak{n}$.

(iii) If not said otherwise, V will be a finite-dimensional vector space. We will say otherwise only once, namely in Section 5.4.5.

(iv) Commutative post-Lie algebra structures will usually be considered over a Lie algebra called $\mathfrak{g}$ with a Lie bracket denoted by $[,]$.

(v) Given a (commutative) post-Lie algebra structure $x \cdot y$ on $(\mathfrak{g}, \mathfrak{n})$ and $x \in V$, we denote the *left-multiplication operator* $L(x)$ by $L(x) := V \to V, L(x) := x \cdot y$.

Lemma 2.35. *By (PA2), the map* $L\colon \mathfrak{g} \to \operatorname{End}(V), x \mapsto L(x)$ *is a representation of* $\mathfrak{g}$.

Lemma 2.36. *By (PA3), for every* $x \in \mathfrak{n}$*, the left-multiplication operator* $L(x)$ *is a derivation of* $\mathfrak{n}$.

Post-Lie algebra structures can be seen as generalizations of pre-Lie algebra structures and LR-structures; pre-Lie algebra structures and LR-structures correspond to the "extreme" case where one of $\mathfrak{g}$ and $\mathfrak{n}$ is abelian:

Definition 2.37 ([26])**.** A Lie algebra $(\mathfrak{g}, [,])$ over k with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto x \cdot y$ is called an *LR-structure*, if

$$
\begin{aligned}
[x, y] &= x \cdot y - y \cdot x \\
x \cdot (y \cdot z) &= y \cdot (x \cdot z) \\
(x \cdot y) \cdot z &= (x \cdot z) \cdot y
\end{aligned}
$$

for all $x, y, z \in \mathfrak{g}$.

Definition 2.38 ([20])**.** A Lie algebra $(\mathfrak{g}, [,])$ over k with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto x \cdot y$ is called a *pre-Lie algebra structure*, if

$$
\begin{aligned}
[x, y] &= x \cdot y - y \cdot x \\
x \cdot (y \cdot z) - (x \cdot y) \cdot z &= y \cdot (x \cdot z) - (y \cdot x) \cdot z
\end{aligned}
$$

for all $x, y, z \in \mathfrak{g}$.

Definition 2.39. Let $x \cdot y$ define a post-Lie algebra structure, an LR-structure, or a left-symmetric structure on $(\mathfrak{g}, \mathfrak{n})$ or $\mathfrak{g}$ (as applicable). Then the structure is called *complete*, if all right-multiplication operators $R(x)$, defined by $R(x)(y) := y \cdot x$, are nilpotent.

Lemma 2.40.

(i) If $\mathfrak{g}$ *is abelian and* $x \cdot y$ *a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, then* $x \circ y$ *with* $x \circ y := -x \cdot y$ *is an LR-structure on* $\mathfrak{n}$ *(and vice versa).*

(ii) If $\mathfrak{n}$ *is abelian and* $x \cdot y$ *a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, then* $x \cdot y$ *is also a pre-Lie algebra structure on* $\mathfrak{g}$ *(and vice versa).*

In Section 5.2 we will establish further relations between CPA-structures and pre-Lie algebra structures/LR-structures on two-step nilpotent Lie algebras.

As we will often list explicit Lie algebras in the sequel, we make the following (usual and intuitive) convention for listing their Lie brackets:

Convention 2.41. If we define a Lie algebra L via a basis and list the Lie brackets of pairs of basis elements, it is understood that we are talking about the Lie algebra with the given relations and the relations obtained by antisymmetry – all other Lie brackets between basis elements are defined to be 0. Then, the Lie bracket is extended bilinearly to all of L. (We used this convention before in this section, see e.g. Examples 2.11 and 2.12.)

2.2.1 Review of results on post-Lie algebra structures

In this section, we want to present some of the results on post-Lie algebra structures which already exist in the literature. Some of those will be used or generalized in the sequel.

Example 2.42. *Given any Lie algebra $\mathfrak{g}$, there is a (commutative) post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{g})$, defined by $x \cdot y = 0$ for all $x, y \in \mathfrak{g}$. This CPA-structure will be referred to as the* trivial *(commutative) post-Lie algebra structure.*
Moreover, there is a post-Lie algebra structure on $(\mathfrak{g}, -\mathfrak{g})$ (where the Lie bracket $\{,\}$ of $-\mathfrak{g}$ is given by $\{x, y\} = -[x, y]$ for all $x, y \in -\mathfrak{g}$).

In Section 3.1, we will need the following result for post-Lie algebra structures on complete Lie algebras, which will be proven in Section 4.1:

Proposition 2.43 ([28, Lemma 2.9, Proposition 2.10])**.** *Let $x \cdot y$ define a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is complete. Then there is a unique linear map $\varphi\colon V \to V$ with $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in V$ (in other words: $L(x) = \operatorname{ad}(\varphi(x))$, where* ad *denotes the adjoint operator with respect to $\{,\}$). The map φ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{n}$ and satisfies*

$$\{\varphi(x), y\} + \{x, \varphi(y)\} = [x, y] - \{x, y\}$$

for all $x, y \in V$.

Example 2.44 ([28, Proposition 4.7])**.** *If the pair $(\mathfrak{g}, \mathfrak{sl}_2(\mathbb{C}))$ admits a post-Lie algebra structure, then $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{g} \cong \mathfrak{r}_{3,\lambda}(\mathbb{C})$ for a $\lambda \neq -1$.*

Given a Lie algebra $(L, [,])$, the semidirect product $L \rtimes \operatorname{Der}(L)$ has a Lie bracket

$$[(x, D), (x', D')] := ([x, x'] + D(x') - D'(x), [D, D']),$$

$x \in L, D \in \operatorname{Der}(L)$. (As usual, for $D, D \in \operatorname{Der}(L), [D, D'] := D \circ D' - D' \circ D$ is the commutator of derivations.)

Proposition 2.45 ([28, Proposition 2.11])**.** *There is a 1-1 correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$ for which the projection $p\colon \mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n}) \to \mathfrak{n}$ onto the first factor induces a Lie algebra isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.*

There is a specialization of Proposition 2.45 to the case where $\mathfrak{n}$ is semisimple:

Proposition 2.46 ([28, Proposition 2.14]). *Let $\mathfrak{n}$ be a semisimple Lie algebra. Then there is a 1-1 correspondence between the post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and the subalgebras $\mathfrak{h} \leq \mathfrak{n} \oplus \mathfrak{n}$ for which the map $p\colon \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}, (x, y) \mapsto x - y$ induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.*

Proposition 2.47 ([28, Proposition 3.1]). *A bilinear map defined by $x \cdot y = \lambda[x, y]$, where $\lambda \in \mathbb{C}\backslash\{0, 1\}$, defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if $\mathfrak{g}$ are nilpotent with nilindex 1 or 2 and $\{x, y\} = (1 - 2\lambda)[x, y]$.*

A Lie algebra is called *unimodular*, if all adjoint operators have trace zero.

Proposition 2.48 ([23, Theorem 3.3]). *If $\mathfrak{n}$ is a semisimple Lie algebra and $\mathfrak{g}$ is solvable and unimodular, then $(\mathfrak{g}, \mathfrak{n})$ does not admit a post-Lie algebra structure.*

Proposition 2.49 ([24, Proposition 5.4]). *CPA-structures on semisimple Lie algebras are trivial.*

This even extends to *perfect* Lie algebras (see Definition 2.8):

Proposition 2.50 ([33, Theorem 3.3]). *CPA-structures on perfect Lie algebras are trivial.*

Proposition 2.51 ([24, Corollary 5.5]). *CPA-structures on a Lie algebra $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{rad}(\mathfrak{g})$.*

A complete Lie algebra can always be decomposed uniquely into *simply-complete* ideals, that is, complete ideals which can not be decomposed further (see Proposition 4.16). For simply-complete, non-metabelian Lie algebras there exists the following classification of post-Lie algebra structures:

Proposition 2.52 ([33, Theorem 4.10]). *Let $\mathfrak{g}$ be simply-complete and non-metabelian and let $\mathfrak{g}$ satisfy $\mathrm{nil}(\mathfrak{g}) = [\mathfrak{g}, \mathrm{nil}(\mathfrak{g})]$. Then there is a 1-1 correspondence between CPA-structures on $\mathfrak{g}$ and elements $z \in \mathrm{Z}([\mathfrak{g}, \mathfrak{g}])$, by $x \cdot z = [[z, x], y]$.*

In Sections 4.3 and 4.4, we present some generalizations of this result.

Definition 2.53. A Lie algebra is called *stem*, if $\mathrm{Z}(L) \subseteq [L, L]$.

Proposition 2.54 ([34, Theorem 3.6]). *CPA-structures on stem nilpotent Lie algebras are complete.*

There exist also many other structure results in the literature of the following form: Given a pair $(\mathfrak{g}, \mathfrak{n})$ admitting a post-Lie algebra structure, where $\mathfrak{g}$ (or $\mathfrak{n}$) has a certain algebraic property P (like being nilpotent or semisimple), then $\mathfrak{n}$ (or $\mathfrak{g}$) must have a certain property Q. To avoid redundancy, these statements will be presented in Section 3.1.

3 General existence questions on post-Lie algebra structures

3.1 Existence of post-Lie algebra structures

We want to start this chapter with a general question on the existence of post-Lie algebra structures. More precisely, what we are interested in is:

Question 3.1. *Given two "algebraic" properties P and Q of Lie algebras (like being nilpotent, abelian etc.), does there exist a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ admitting a post-Lie algebra structure such that $\mathfrak{g}$ has property P and $\mathfrak{n}$ has property Q?*

The results we prove here are summarized in Table 5 – let us explain its notation:

(a) Consider the cell in the row with property P and column with property Q. A check mark (✓) in this cell means that there is a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ with a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ having property P and $\mathfrak{n}$ having property Q.
A cross (✗) means that there is no pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ with a post-Lie algebra structure where $\mathfrak{g}$ has property P and $\mathfrak{n}$ has property Q.
Question marks (?) indicate that the answer is – to the best of our knowledge – not known.

(b) The first row and the first column contain the same properties, however, due to space constraints, the contents of the first row had to be abbreviated. Therefore, e.g. the entries "sol." and "ssim." have to be read as "solvable, non-nilpotent" and "semisimple, non-simple", respectively (and the other entries analogously).

(c) All listed properties except for the last (complete, non-semisimple) are chosen such that they are mutually exclusive. However, there is an intersection between "complete, non-semisimple" Lie algebras and "solvable, non-nilpotent" Lie algebras (see also Chapter 4).

(d) The diagonal consists only of check marks for if $\mathfrak{g} \cong \mathfrak{n}$, then there is always the *trivial* CPA-structure, see Example 2.42.

For some cells, we also want to give further remarks/ closer specializations which we prove in this section:

(e) There exist no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is abelian and $\mathfrak{n}$ is not metabelian.

Table 5: Existence of post-Lie algebra structures over $\mathbb{C}$: A "✓" ("✗") means that there is a (no) pair of $\mathbb{C}$-Lie algebras where $\mathfrak{g}$ and $\mathfrak{n}$ have the corresponding algebraic properties. If it is unknown, the cell contains a "?". For the cells with superscripts, we give a remark below.

$\mathfrak{g}$ \ $\mathfrak{n}$	ab.	nil.	sol.	sim.	ssim.	red.	com.
abelian	✓	✓(e)	✓(e)	✗	✗	✗	✓(e)
nilpotent, non-abelian	✓	✓	✓(f)	✗	✗	✗	✓(g)
solvable, non-nilpotent	✓	✓	✓	✓	✓	✓	✓
simple	✗	✗	✗	✓(h)	✗	✗	✗
semisimple, non-simple	✗	✗	✗	✗(i)	✓	?	✗
reductive, non-semisimple, non-abelian	✓	?	?	?	?	✓	✓(j)
complete, non-semisimple	✓	✓	✓	?	?	✓	✓(j)

(f) For every solvable non-metabelian Lie algebra $\mathfrak{n}$ there is a nilpotent, non-abelian Lie algebra $\mathfrak{g}$ with a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

(g) If $\mathfrak{n}$ is complete, then post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ – where $\mathfrak{g}$ is nilpotent, non-abelian and $\mathfrak{n}$ is complete, non-semisimple – are precisely possible if $\mathfrak{n}$ is solvable and non-metabelian.

(h) Post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are both simple, do exist precisely if $\mathfrak{g} \cong \mathfrak{n}$ – and then, only the two "trivial" structures (see Example 2.42) are possible.

(i) We will give a proof in Section 3.2 that there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is simple and $\mathfrak{g}$ is semisimple (but not simple) are not possible if $\dim(\mathfrak{g}) < 45$ (and under certain other conditions). However, it has been proven recently that there are no post-Lie algebra structures at all (see [31, Theorem 3.1]).

(j) If $\mathfrak{n}$ is complete and solvable, then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ unless $\mathfrak{g}$ is also solvable.

Remark 3.2. A "✓" does not mean that *all* pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ has property P and $\mathfrak{n}$ has property Q, admit post-Lie algebra structures. E.g. there exist post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is nilpotent and non-abelian and $\mathfrak{n}$ is solvable and non-nilpotent – but, no 3-dimensional pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C}), \mathfrak{n} \cong \mathfrak{r}_{3,-1}(\mathbb{C})$ does admit post-Lie algebra structures – see Section 3.4.

Remark 3.3. Table 5 shows that the definition of post-Lie algebra structures is not symmetric in $\mathfrak{g}$ and $\mathfrak{n}$.

The goal of this chapter is to "prove Table 5". That is, we prove (or give references in the literature) the existence of post-Lie algebra structures (mostly by giving concrete examples) indicated with a check mark and prove (or reference) the non-existence of post-Lie algebra structures indicated with a cross.

$\mathfrak{g}$ abelian Let us start with the case where $\mathfrak{g}$ is abelian. Note that these structures correspond to LR-structures (see Lemma 2.40).

Theorem 3.4. *Let $\mathfrak{g}$ be an abelian Lie algebra.*

(i) There do exist post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is abelian, nilpotent or solvable. (There also do exist examples where $\mathfrak{n}$ is solvable, but non-nilpotent and nilpotent, but non-abelian).

(ii) There do not exist post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is reductive and non-abelian (including $\mathfrak{n}$ being simple or semisimple).

(iii) If $\mathfrak{n}$ is complete, then there exist post-Lie structures on $(\mathfrak{g}, \mathfrak{n})$ if and only if $\mathfrak{n}$ is metabelian.

We prove this theorem via the following lemmas:

Lemma 3.5.

(i) There exist post-Lie algebra structures on $(\mathbb{C}^n, \mathbb{C}^n)$ for every $n \in \mathbb{N}$.

(ii) There exist post-Lie algebra structures on $(\mathbb{C}^3, \mathfrak{n}_3(\mathbb{C}))$.

(iii) There exist post-Lie algebra structures on $(\mathbb{C}^2, \mathfrak{r}_2(\mathbb{C}))$.

Proof.

(i) The simplest one (of many examples) is the trivial structure $x \cdot y = 0$ for all $x, y \in \mathbb{C}^n$.

(ii) In [26, Proposition 3.1], all LR-structures on $\mathfrak{n}_3(\mathbb{C})$ are classified (more precisely, they are given as LR-structures on $\mathfrak{n}_3(\mathbb{R})$ but, in fact, are equal to the LR-structures on $\mathfrak{n}_3(\mathbb{C})$). They stand in 1-1 correspondence to post-Lie algebra structures on $(\mathbb{C}^3, \mathfrak{n}_3(\mathbb{C}))$.

(iii) LR-structures on $\mathfrak{r}_2(\mathbb{C})$ exist and are classified in [26, Example 1.6].

□

Remark 3.6. In [26], there are more examples of LR-structures on nilpotent and solvable Lie algebras. There are results on classes of Lie algebras always admitting (complete) LR-structures (e.g. 2-step nilpotent Lie algebras, free 3-step nilpotent Lie algebras, 2-step solvable filiform Lie algebras).

However, if $\mathfrak{n}$ is not metabelian, there are no post-Lie algebra structures on $(\mathbb{C}^{\dim(\mathfrak{n})}, \mathfrak{n})$ (which implies that there do not exist post-Lie algebra structures on reductive, non-abelian Lie algebras):

Lemma 3.7. *If there is an LR-structure on $\mathfrak{n}$ (and hence, a post-Lie algebra structure on $(\mathbb{C}^n, \mathfrak{n})$ for some $n \in \mathbb{N}$), then $\mathfrak{n}$ is metabelian.*

Proof. See [26, Proposition 2.1]. □

Lemma 3.8. *Let $(\mathfrak{g}_1, \mathfrak{n}_1)$ admit a post-Lie algebra structure $x \circ y$ and $(\mathfrak{g}_2, \mathfrak{n}_2)$ admit a post-Lie algebra structure $x * y$. Then the "direct sum" of $\circ$ and $*$, i.e. the bilinear map defined by*

$$x \cdot y = \begin{cases} x \circ y & \text{if } x, y \in \mathfrak{g}_1 \\ x * y & \text{if } x, y \in \mathfrak{g}_2 \\ 0 & \text{else} \end{cases}$$

(and extended bilinearly) is a post-Lie algebra structure on the pair $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{n}_1 \oplus \mathfrak{n}_2)$.

Lemma 3.9 ([89])**.** *Every complete complex metabelian Lie algebra is isomorphic to $(\mathfrak{r}_2(\mathbb{C}))^n$ for some $n \in \mathbb{N}$.*

Lemma 3.10. *(Semi)simple and reductive (non-abelian) Lie algebras do not admit LR-structures; complete Lie algebras if and only if they are metabelian.*

Proof. This follows from Lemma 3.7: (semi)simple, reductive (non-abelian) and complete, non-metabelian Lie algebras are never metabelian, thus admit no LR-structures. To show that complete metabelian Lie algebras always admit post-Lie algebra structures, note that they are all of the form $(\mathfrak{r}_2(\mathbb{C}))^n$ for an $n \in \mathbb{N}$ by Lemma 3.9. Post-Lie algebra structures on $(\mathbb{C}^2, \mathfrak{r}_2(\mathbb{C}))$ exist, see Lemma 3.5, and their direct sums is a post-Lie algebra structure on $(\mathbb{C}^{2n}, \mathfrak{r}_2(\mathbb{C})^n)$. □

$\mathfrak{g}$ nilpotent and non-abelian Let $\mathfrak{g}$ be nilpotent and non-abelian. Let us first note that $(\mathfrak{g}, \mathfrak{n})$ does not admit post-Lie algebra structures unless $\mathfrak{n}$ is solvable ([28, Proposition 4.3]).
We get the following classification:

Theorem 3.11.

(i) There exist pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is nilpotent and non-abelian and $\mathfrak{n}$ is abelian or nilpotent (and non-abelian) or solvable (and non-nilpotent) with post-Lie algebra structures.

(ii) If $\mathfrak{n}$ is reductive (and non-abelian) including the cases where $\mathfrak{n}$ is (semi)simple, then there exist no post-Lie algebra structures on any pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ nilpotent.

(iii) Let $\mathfrak{n}$ be solvable, complete and non-metabelian. Then there exists a nilpotent (non-abelian) Lie algebra $\mathfrak{g}$ admitting post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ – conversely, if there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is nilpotent and non-abelian and $\mathfrak{n}$ is complete, then $\mathfrak{n}$ is also solvable and non-metabelian.

We again split the proof up into several lemmas:

Lemma 3.12.

(i) There exist pairs $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{g}$ *nilpotent (and non-abelian) and* $\mathfrak{n}$ *abelian admitting a post-Lie algebra structure.*

(ii) Post-Lie algebra structures exist on every pair $(\mathfrak{n}, \mathfrak{n})$ *with* $\mathfrak{n}$ *nilpotent.*

(iii) If $(\mathfrak{g}, \mathfrak{n})$ *admits a post-Lie algebra structure and* $\mathfrak{g}$ *is nilpotent, then* $\mathfrak{n}$ *is solvable.*

Proof.

(i) See [20, Theorem 2.33].

(ii) Again, take the trivial structure defined by $x \cdot y = 0$.

(iii) See [28, Proposition 4.3].

□

This takes care of the cases where $\mathfrak{n}$ is abelian, nilpotent, (semi)simple or reductive. It remains to consider the solvable and the complete case.

Lemma 3.13. *Let* $\mathfrak{n}$ *be a solvable complete Lie algebra. Then there is a nilpotent Lie algebra* $\mathfrak{g}$ *such that there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$.

Proof. Any solvable complete Lie algebra $\mathfrak{n}$ can be decomposed as a direct sum of vector spaces $T \oplus N$, where T is an abelian Lie algebra and N its nilradical (see Proposition 4.21). So $\mathfrak{n}$ admits a decomposition into two nilpotent Lie algebras. But if $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$ is such a decomposition, by [23, Proposition 2.13], the (nilpotent!) Lie algebra $\mathfrak{g}$ with underlying vector space $\mathfrak{a} \oplus \mathfrak{b}$ and Lie brackets

$$[a + b, a' + b'] = \{a, a'\} - \{b, b'\}$$

$(a \in \mathfrak{a}, b \in \mathfrak{b})$ admits a post-Lie algebra structure via

$$(a + b) \cdot (a' + b') = -\{b, a' + b'\}.$$

□

For the "$\mathfrak{g}$ nilpotent, non-abelian, $\mathfrak{n}$ complete and metabelian"-case, we will need Itô's Theorem:

Lemma 3.14 (Itô's Theorem for Lie algebras)**.** *Let* L *be a Lie algebra which is the vector space sum* $L = \mathfrak{a} + \mathfrak{b}$ *of two abelian Lie algebras. Then* L *is metabelian.*

The analogous result for groups was proven by Itô in [63]. The proof for Lie algebras is also analogous – as the proof is rather short, we want to give it here. This is the same proof as the one in [66], where the statement is proven for Lie rings – but the very same argument also works for Lie algebras.

Proof. Since $[\mathfrak{a},\mathfrak{a}]=[\mathfrak{b},\mathfrak{b}]=0$, it is enough to show that $[[a',b'],[a,b]]=0$ for $a,a'\in\mathfrak{a}$ and $b,b'\in\mathfrak{b}$.
By the Jacobi identity and $\mathfrak{a},\mathfrak{b}$ being abelian, this can be written as

$$\begin{aligned}[[a',b'],[a,b]] &= -[[b',[a,b]],a']-[[[a,b],a'],b'] \\ &= [[a,[b,b']],a']+[[b,[b',a]],a']+[[[b,a'],a],b']+[[[a',a],b],b'] \\ &= [a',[[b',a],b]+[[a,[a',b]],b'].\end{aligned}$$

Let us introduce the notation $[b',a]=a''+b'''$, $[a',b]=a'''+b''$, where $a'',a'''\in\mathfrak{a}$, $b'',b'''\in\mathfrak{b}$. Then we can continue the above calculation by

$$\begin{aligned}[[a',b'],[a,b]] &= [a',[a''+b''',b]+[[a,a'''+b''],b'] = [a',[a'',b]]+[[a,b''],b'] \\ &= -[a'',[b,a']]-[b,[a',a'']]-[[b'',b'],a]-[[b',a],b''] \\ &= [[b,a'],a'']+[[a,b'],b''] = -[a'''+b'',a'']-[a''+b''',b''] \\ &= [a'',b'']-[a'',b'']=0.\end{aligned}$$

This is what was to be shown. ☐

Lemma 3.15. *Let $\mathfrak{n}$ be complete and metabelian. Then there is no nilpotent, non-abelian Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure.*

Proof. If $\mathfrak{n}$ is complete and metabelian, then $\mathfrak{n}\cong(\mathfrak{r}_2(\mathbb{C}))^n$ by Lemma 3.9 (with basis $\{e_1,\dots,e_n\}$ and Lie brackets $\{e_{2i+1},e_{2i+2}\}=e_{2i+2}, 0\le i\le n-1)$; as it is complete, $\mathrm{Der}(\mathfrak{r}_2(\mathbb{C})^n)\cong\mathrm{ad}((\mathfrak{r}_2(\mathbb{C}))^n)$. So we can write $\mathfrak{n}\rtimes\mathrm{Der}((\mathfrak{r}_2(\mathbb{C}))^n)$ with a basis $\{e_1,\dots,e_{2n}\}\cup\{A_1,\dots,A_{2n}\}$ and non-zero Lie brackets $[A_{2i+1},A_{2i+2}]=A_{2i+2}$, $[e_{2i+1},e_{2i+2}]=e_{2i+2},[A_{2i+1},e_{2i+2}]=e_{2i+2},[A_{2i+2},e_{2i+1}]=e_{2i+2}, 0\le i\le n-1$.
Via the basis change $f_{4j+1}=e_{2j+1}, f_{4j+2}=e_{2j+2}, f_{4j+3}=-e_{2j+1}+A_{2j+1}, f_{4j+4}=e_{2j+2}+A_{2j+2}, 0\le j\le\frac{n}{2}-1$, one gets the basis $\{f_1,f_2,\dots,f_{2n}\}$ with $[f_{2i+1},f_{2i+2}]=f_{2i+2}, 0\le i\le 2n-1$, meaning that $\mathfrak{n}\rtimes\mathrm{Der}(\mathfrak{r}_2(\mathbb{C})^n)\cong(\mathfrak{r}_2(\mathbb{C}))^{2n}$.
But any subalgebra of $(\mathfrak{r}_2(\mathbb{C}))^m$ either contains $\mathfrak{r}_2(\mathbb{C})$ (and thus is non-nilpotent) or is abelian.
So there is no nilpotent, but non-abelian $\mathfrak{g}$ such that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure – otherwise, by Proposition 2.45, $\mathfrak{g}$ would be isomorphic to a subalgebra of $(\mathfrak{r}_2(\mathbb{C}))^{2n}$ which is a contradiction. ☐

Corollary 3.16. *Let $\mathfrak{n}$ be complete. Then there exists a nilpotent (and non-abelian) Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g},\mathfrak{n})$ admits a post-Lie algebra structure precisely if $\mathfrak{n}$ is solvable and non-metabelian.*

Proof. For the first direction, let $\mathfrak{n}$ be complete and non-solvable or metabelian.

(i) If $\mathfrak{n}$ is metabelian, no pair $(\mathfrak{g},\mathfrak{n})$ (where $\mathfrak{g}$ is nilpotent and non-abelian) admits a post-Lie algebra structure by Lemma 3.15.

(ii) If $\mathfrak{n}$ is not solvable, then by Lemma 3.12, (iii), there is no post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$.

On the other hand, let $\mathfrak{n}$ be complete, solvable and non-metabelian. We can find a nilpotent Lie algebra $\mathfrak{g}$ with a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ by Lemma 3.13. It remains to show that this $\mathfrak{g}$ is not abelian: But if it was, then the construction in Lemma 3.13 would imply that $\mathfrak{n}$ had a vector space decomposition into two abelian Lie algebras – by Lemma 3.14, $\mathfrak{n}$ was metabelian then, a contradiction. □

By the above construction, we can give an example of a pair where $\mathfrak{g}$ is nilpotent (and non-abelian) and $\mathfrak{n}$ is complete (and non-semisimple):

Example 3.17. *Let $\mathfrak{n} = T_2 \ltimes \mathfrak{n}_3$ be the five-dimensional complete Lie algebra with basis $\{e_1, e_2, e_3, t_1, t_2\}$ and non-zero brackets*

$$\{e_1, e_2\} = e_3, \{t_1, e_1\} = e_1, \{t_1, e_3\} = e_3, \{t_2, e_2\} = e_2, \{t_2, e_3\} = e_3.$$

Define $\mathfrak{a}$ to be the vector space spanned by $\{e_1, e_2, e_3\}$, $\mathfrak{b}$ to be the vector space spanned by $\{t_1, t_2\}$; then $\mathfrak{a} \oplus \mathfrak{b} \cong \mathfrak{g} \cong \mathfrak{n}_3 \oplus \mathbb{C}^2$ with non-zero bracket

$$[e_1, e_2] = e_3$$

is a non-abelian, nilpotent Lie algebra and

$$(e_i + t_i) \cdot (e_j + t_j) = -\{t_i, e_j + t_j\}$$

is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

$\mathfrak{g}$ solvable (and non-nilpotent). If $\mathfrak{g}$ is solvable and non-nilpotent, then all cases we consider are possible:

Theorem 3.18. *There are examples of $(\mathfrak{g}, \mathfrak{n})$ admitting post-Lie algebra structures where $\mathfrak{g}$ is solvable (and non-nilpotent) and $\mathfrak{n}$ is (i) abelian, (ii) nilpotent and non-abelian, (iii) solvable and non-nilpotent, (iv) simple, (v) semisimple and non-simple, (vi) reductive, non-semisimple and non-abelian, (vii) complete and non-semisimple.*

Proof.

(i) See [20, Example 3.34].

(ii) There exist post-Lie algebra structures on the 3-dimensional pair $(\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}, \mathfrak{n}_3(\mathbb{C}))$, see Example B.3.

(iii) The trivial post-Lie algebra structure $x \cdot y = 0$ on any pair $(\mathfrak{g}, \mathfrak{g})$.

(iv) An example for a post-Lie algebra structure on $(\mathfrak{sl}_2(\mathbb{C}), \mathfrak{r}_{3,\lambda}(\mathbb{C}))$ is given in [23, Proposition 4.7].

(v) [23, Proposition 3.1] constructs for any semisimple Lie algebra $\mathfrak{n}$ a solvable, non-nilpotent Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure.

(vi) See Corollary 3.19.

(vii) See the appendix, Proposition B.17 for an example in dimension 5. (In Section B.3, one can also find more examples.)

□

Corollary 3.19. *Let $\mathfrak{n}$ be reductive, $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{s}$ with $\mathfrak{a}$ abelian and $\mathfrak{s}$ semisimple. If there exists a solvable Lie algebra $\mathfrak{g}'$ such that $(\mathfrak{g}', \mathfrak{a})$ admits a post-Lie algebra structure, then there also is a solvable Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. (In particular, by Theorem 3.18, part (i), there are pairs $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ solvable and $\mathfrak{n}$ reductive, admitting a post-Lie algebra structure.)*

Proof. By [23, Proposition 3.1], there exists a solvable Lie algebra $\mathfrak{g}''$ such that $(\mathfrak{g}'', \mathfrak{s})$ admits a post-Lie algebra structure. So by Lemma 3.8, we find a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, with $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$. □

$\mathfrak{g}$ simple. The situation where $\mathfrak{g}$ is simple is just the opposite: Post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ do exist only if $\mathfrak{g} \cong \mathfrak{n}$ and then are just given by the two trivial ones.

Theorem 3.20. *Let $(\mathfrak{g}, \mathfrak{n})$ admit a post-Lie algebra structure where $\mathfrak{g}$ is simple. Then $\mathfrak{g} \cong \mathfrak{n}$. Moreover, the only two post-Lie algebra structures possible are the trivial ones: $x \cdot y = 0$ where $[x, y] = \{x, y\}$ and $x \cdot y = [x, y] = -\{x, y\}$.*

Proof. By [24, Theorem 3.1], $\mathfrak{n} \cong \mathfrak{g}$; by [28, Proposition 4.6], only the trivial structures are possible. □

$\mathfrak{g}$ semisimple and non-simple. In the case where $\mathfrak{g}$ is semisimple and non-simple, we also have non-existence results:

Theorem 3.21. *Let $\mathfrak{g}$ be semisimple and non-simple. There do not exist any pairs $(\mathfrak{g}, \mathfrak{n})$ admitting a post-Lie algebra structure where $\mathfrak{n}$ is solvable (including $\mathfrak{n}$ being abelian, nilpotent or solvable-complete).*
There also do not exist post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with simple or complete $\mathfrak{n}$, but there exist post-Lie algebra structures where $\mathfrak{n}$ is semisimple.

Proof. For the assertion regarding $\mathfrak{n}$ being solvable, see [23, Theorem 4.2]. The assertion with $\mathfrak{n}$ being simple will be proven in Section 3.2. If $\mathfrak{n}$ is complete, then by [31, Proposition 3.8], we do not have post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$. □

Remark 3.22. There do not only exist the trivial structures on pairs $(\mathfrak{g}, \mathfrak{g})$ where $\mathfrak{g}$ is semisimple (as in the case where $\mathfrak{g}$ is simple). See [23, Example 2.11] for a non-trivial post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{g})$ where $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

The next proposition states an obstruction which can be used to show that certain pairs of Lie algebras admit no post-Lie algebra structure:

Proposition 3.23. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{n}$ a complete Lie algebra, let $\mathfrak{n} = \mathfrak{s} \ltimes \mathfrak{r}$ be the Levi decomposition of $\mathfrak{n}$. If there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, then both $\mathfrak{s}$ and $\mathfrak{r}$ are – with the Lie bracket $[,]$ – subalgebras of $\mathfrak{g}$.*

Proof. Suppose there is a post-Lie algebra structure $x \cdot y$ on $(\mathfrak{g}, \mathfrak{n})$; then by Proposition 2.43, there is a unique homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{n}$ satisfying $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in V$.
As $\mathfrak{s}$ is a semisimple Lie algebra, its image under the homomorphism φ is (since $\varphi(\mathfrak{g}) \cong \mathfrak{g}/\ker(\varphi)$) semisimple too, so $\varphi(\mathfrak{g}) \leq \mathfrak{s}$. We obtain

$$\mathfrak{g} \cdot \mathfrak{s} \subseteq \{\varphi(\mathfrak{g}), \mathfrak{s}\} \subseteq \{\mathfrak{s}, \mathfrak{s}\} \subseteq \mathfrak{s},$$
$$\mathfrak{g} \cdot \mathfrak{r} \subseteq \{\varphi(\mathfrak{g}), \mathfrak{r}\} \subseteq \{\mathfrak{s}, \mathfrak{r}\} \subseteq \mathfrak{r}.$$

Furthermore,

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s} \cdot \mathfrak{s} - \mathfrak{s} \cdot \mathfrak{s} + \{\mathfrak{s}, \mathfrak{s}\} \subseteq \mathfrak{s},$$
$$[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} \cdot \mathfrak{r} - \mathfrak{r} \cdot \mathfrak{r} + \{\mathfrak{r}, \mathfrak{r}\} \subseteq \mathfrak{r},$$

what we claimed. □

$\mathfrak{g}$ reductive. Here our result is:

Theorem 3.24. *There do exist examples of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is reductive and $\mathfrak{n}$ is (i) abelian, (ii) reductive, non-semisimple and non-abelian, (iii) complete and non-semisimple. If $\mathfrak{n}$ is solvable and complete, then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. For (ii), we can again take the trivial post-Lie algebra structure. For (i), the ordinary matrix multiplication gives a pre-Lie algebra structure (that is, a post-Lie algebra structure with $\mathfrak{n}$ abelian) on $(\mathfrak{gl}_n(\mathbb{C}), \mathbb{C}^{n^2})$. An example for (iii) is given in Proposition B.29. □

Note that the Lie algebra $L_{7,4}$, which is used in Proposition B.29, is complete, but not solvable; indeed we cannot replace it by a solvable-complete Lie algebra:

Lemma 3.25. *If $\mathfrak{n}$ is a complete and solvable Lie algebra and $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure, then $\mathfrak{g}$ is also solvable.*

Proof. Denote by $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$, and $\mathfrak{n}^{(0)} = \mathfrak{n}, \mathfrak{n}^{(i+1)} = \{\mathfrak{n}^{(i)}, \mathfrak{n}^{(i)}\}$ the derived series of V with respect to $[,]$ and $\{,\}$, respectively. We show $\mathfrak{g}^{(i)} \subseteq \mathfrak{n}^{(i)}$ for all $i \in \mathbb{N}$:
We have $\mathfrak{g}^{(0)} = V = \mathfrak{n}^{(0)}$ as sets.
Now, for the induction step $i \to i+1$, let $z \in \mathfrak{g}^{(i)}$. Then z is a linear combination of elements of the form $[x, y]$ with $x, y \in \mathfrak{g}^{(i)} \subseteq \mathfrak{n}^{(i)}$. As $\mathfrak{n}$ is complete, we have a unique linear map $\varphi\colon V \to V$ with $x \cdot y = \{\varphi(x), y\}$ (see Proposition 2.43) – so we can write

$$[x, y] = \{\varphi(x), y\} + \{x, \varphi(y)\} + \{x, y\}$$

and since $\mathfrak{n}^{(i)}$ is an ideal of $\mathfrak{n}$, we obtain $[x, y] \in \mathfrak{n}^{(i)}$. So it follows $\mathfrak{g}^{(i)} \subseteq \mathfrak{n}^{(i)}$.
So if we choose i large enough such that $\mathfrak{n}^{(i)} = 0$, then also $\mathfrak{g}^{(i)} = 0$, so $\mathfrak{g}$ is solvable too. □

Corollary 3.26. *If* $\mathfrak{n}$ *is solvable and complete and* $\mathfrak{g}$ *is reductive (and non-abelian), then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$.

Proof. By Lemma 3.25, $\mathfrak{g}$ is solvable too, so $\mathfrak{g}$ cannot have a simple ideal. □

$\mathfrak{g}$ complete and non-semisimple.

Theorem 3.27. *There are examples of post-Lie algebras on* $(\mathfrak{g}, \mathfrak{n})$ *where* $\mathfrak{g}$ *is complete and non-semisimple and* $\mathfrak{n}$ *is (i) abelian, (ii) nilpotent and non-abelian, (iii) solvable and non-nilpotent, (iv) reductive, non-semisimple and non-abelian, (v) complete and non-semisimple.*

Proof. Example 3.28 below is an example for (i), Corollary 3.30 gives examples for (ii). For (iii) and (v), one may take a solvable complete Lie algebra $\mathfrak{g}$ and the trivial post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{g})$ and (iv) is proven in Corollary 3.31. □

Example 3.28. *This is an example for* $\mathfrak{n}$ *abelian and* $\mathfrak{g}$ *complete and non-semisimple: Take* $\mathfrak{n} = \mathbb{C}^5, \mathfrak{g} = T_2 \ltimes \mathfrak{n}_3$ *(as in Table 12 in Section 4.2). Then* $\mathfrak{g}$ *has a basis* $\{e_1, e_2, e_3, t_1, t_2\}$ *with non-zero Lie brackets* $[e_1, e_2] = e_3, [t_1, e_1] = e_1, [t_1, e_3] = e_3, [t_2, e_2] = e_2, [t_2, e_3] = e_3$ *and the bilinear map given by*

$$\begin{aligned} e_1 \cdot e_2 = e_3, e_1 \cdot t_1 = -e_1, t_1 \cdot e_3 = e_3 \\ t_1 \cdot t_1 = -t_1, t_2 \cdot e_2 = e_2, t_2 \cdot e_3 = e_3 \end{aligned}$$

and extended bilinearly, is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

For the case "$\mathfrak{g}$ complete, $\mathfrak{n}$ nilpotent", we first prove a lemma on post-Lie algebra structures on semidirect products:

Lemma 3.29. *Let* $\mathfrak{g}$ *be a semidirect product* $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$. *Then there is a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$*, given by*

$$x \cdot y = \begin{cases} 0 & \textit{if } x, y \in \mathfrak{a} \\ 0 & \textit{if } x \in \mathfrak{b} \\ [x, y] & \textit{if } x \in \mathfrak{a}, y \in \mathfrak{b} \end{cases}$$

and extended bilinearly.

Proof. We check that the identities (PA1), (PA2) and (PA3) do hold for elements x, y, z in $\mathfrak{a} \cup \mathfrak{b}$. Note that if $x, y \in \mathfrak{a}$ or $x, y \in \mathfrak{b}$, then $[x, y] = \{x, y\}$. Identity (PA1) says $x \cdot y - y \cdot x = [x, y] - \{x, y\}$.

(i) If $x, y \in \mathfrak{a}$ or $x, y \in \mathfrak{b}$, then $x \cdot y = y \cdot x = 0$ and also $[x, y] = \{x, y\}$.

(ii) If $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ or vice versa, then $y \cdot x = 0, \{x, y\} = 0$, but $x \cdot y = [x, y]$.

Concerning identity (PA2), which says $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$:

(i) If $y \in \mathfrak{b}$ (or, by symmetry, $x \in \mathfrak{b}$), then $[x, y] \cdot z = x \cdot (y \cdot z) = y \cdot (x \cdot z) = 0$ (since $\mathfrak{b} \cdot \mathfrak{g} = 0$).

(ii) If $x, y, z \in \mathfrak{a}$, then $[x, y] \in \mathfrak{a}$, so $[x, y] \cdot z = x \cdot (y \cdot z) = y \cdot (x \cdot z) = 0$.

(iii) The case $x, y \in \mathfrak{a}, z \in \mathfrak{b}$ remains, but as $[x, y] \cdot z = [[x, y], z], y \cdot (x \cdot z) = [y, [x, z]]$, $x \cdot (y \cdot z) = [x, [y, z]]$, it reduces to the Jacobi identity.

A similar case distinction works for (PA3), which says $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$:

(i) If $x \in \mathfrak{b}$, then $x \cdot \{y, z\} = \{x \cdot y, z\} = \{y, x \cdot z\} = 0$.

(ii) If $x, y \in \mathfrak{a}$, then also $x \cdot \{y, z\} = \{x \cdot y, z\} = 0, \{y, x \cdot z\} = \{y, [x, z]\} = 0$.

(iii) For $x, z \in \mathfrak{a}, y \in \mathfrak{b}$, we also find that all three terms evaluate to zero.

(iv) The only case left is $x \in \mathfrak{a}, y, z \in \mathfrak{b}$. But then $x \cdot \{y, z\} = x \cdot [y, z] = [x, [y, z]]$, $\{x \cdot y, z\} = \{[x, y], z\} = [[x, y], z], \{y, x \cdot z\} = \{y, [x, z]\} = [y, [x, z]]$, so identity (PA3) simplifies to the Jacobi identity.

Using bilinearity, one finds that the structure is indeed a post-Lie algebra structure. □

Solvable complete Lie algebras L can be written as semidirect products of an abelian Lie algebra T and its nilradical $\mathfrak{m}$ (which is abelian by Ito's Theorem 3.14 if and only if L is metabelian). Therefore, we obtain:

Corollary 3.30. *Let $\mathfrak{g}$ be solvable and complete, $\mathfrak{g} = T \ltimes \mathfrak{m}, T$ abelian, $\mathfrak{m}$ nilpotent. Let* $\dim(T) = s$. *Then there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n} \cong \mathbb{C}^s \oplus \mathfrak{m}$, given by*

$$x \cdot y = \begin{cases} 0 & \text{if } x \in \mathfrak{m} \\ [x, y] & \text{if } x \in T \end{cases} .$$

Proof. In Lemma 3.29, choose $\mathfrak{a} = T, \mathfrak{b} = \mathfrak{m}$. Then one obtains the statement. □

Note that $\mathfrak{n} = \mathbb{C}^{\dim(T)} \oplus \mathfrak{m}$ is nilpotent (and, if $\mathfrak{g}$ is not metabelian, $\mathfrak{n}$ is non-abelian) – thus we have examples of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is complete (and non-semisimple) and $\mathfrak{n}$ is nilpotent (and non-abelian).

We also obtain the case where $\mathfrak{g}$ is complete (non-semisimple) and $\mathfrak{n}$ is reductive (non-abelian, non-semisimple):

Corollary 3.31. *There is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is the complete Lie algebra $L_{7,4}$ and $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4$.*

For the definition of $L_{7,4}$, see Table 14 in Section 4.2.

Proof. The Lie algebra $L_{7,4}$ can be seen as a semidirect product $L_{7,4} = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \ltimes_\varphi \mathbb{C}^3$ with respect to

$$\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} \to \mathrm{Der}(\mathbb{C}^3), \varphi(ae_1 + be_2 + ce_3 + de_4) = \begin{pmatrix} 2c+d & 2a & 0 \\ b & d & a \\ 0 & 2b & -2c+d \end{pmatrix}$$

Therefore, Lemma 3.29 gives us a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. □

See Proposition B.29 for an explicit structure on $(L_{7,4}, \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4)$.

3.2 Non-existence of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ semisimple, $\mathfrak{n}$ simple

In this subsection we shall investigate post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{n}$ is a simple Lie algebra.

The theorem we want to prove in this section is the following:

Theorem 3.32. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{n}$ a simple Lie algebra. Suppose that one of the following conditions holds:*

(i) The Lie algebra $\mathfrak{g}$ is simple and not isomorphic to $\mathfrak{n}$.

(ii) The Lie algebra $\mathfrak{g}$ is a direct sum of (exactly) two simple factors $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

(iii) The Lie algebra $\mathfrak{n}$ is an exceptional simple Lie algebra.

(iv) The Lie algebra $\mathfrak{n}$ satisfies $\dim(\mathfrak{n}) < 45$.

Then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.

Remark 3.33. Recently it was proved that no post-Lie algebra structures exist *at all* on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is simple unless $\mathfrak{g} \cong \mathfrak{n}$ (see [31, Theorem 3.1]).

Nevertheless, as our methods are different, we want to give our proof of Theorem 3.32 here.

Note that we already know that (i) implies the non-existence of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, see Theorem 3.20. (But this will also appear as a corollary in this section, cf. Remark 3.40.)
In this section, our two main ingredients are Proposition 2.46 and Dynkin's paper on semisimple subalgebras of semisimple Lie algebras: Our proof of Theorem 3.32 will rely on a theorem of Dynkin which classifies maximal subalgebras of semisimple Lie algebras. The theorem is given in [50, Theorem 15.1]. For the reader's convenience, we shall repeat it and its proof:

Proposition 3.34 (Dynkin). *Let L be a semisimple Lie algebra and*

$$L = L_1 \oplus L_2 \oplus \ldots \oplus L_s$$

its decomposition as a direct sum of simple ideals. Then all maximal subalgebras of L are given by the formulas

$$R(i, \widetilde{L}) := \bigoplus_{\substack{k=1 \\ k \neq i}}^{s} L_k \oplus \widetilde{L} \tag{1}$$

and

$$R(i, j, \psi) := \bigoplus_{\substack{k=1 \\ k \neq i,j}}^{s} L_k \oplus \{L_i, L_j, \psi\}, \tag{2}$$

where $\tilde{L}$ is a maximal subalgebra of $L_i, \psi \colon L_i \to L_j$ is an isomorphism between L_i and L_j and $\{L_i, L_j, \psi\}$ is defined as the set of pairs $\{(x, \psi(x)) \colon x \in L_i\}$.

Proof. Let $p_i \colon L \to L_i$ be the projection homomorphism onto L_i; i.e., if $x \in L$ has the unique decomposition $x = \sum_{i=1}^{s} x_i, x_i \in L_i$, then $p_i(x) = x_i$.
Let R be a maximal proper subalgebra of L.
First we assume that there is an $i \in \{1, \ldots, s\}$ with $p_i(R) \neq L_i$. Let $\widetilde{L}$ be a maximal subalgebra of L_i. Since $p_i(R) \neq L_i$ is a subalgebra of L_i, we have $p_i(R) \subseteq \widetilde{L}$. But then, $R \subseteq R(i, \widetilde{L}) \neq L$ and by the maximality of R it follows $R = R(i, \widetilde{L})$.

Now suppose $p_i(R) = L_i$ for all $i \in \{1, \ldots, s\}$. First we shall show that for every i, the set $L_i \cap R$ is an ideal in L_i.
Let $y \in L_i \cap R$ and $z \in L_i$. Because of $p_i(R) = L_i$, there is an $x \in R$ with $p_i(x) = z$. Because R is a subalgebra of L and L_i is an ideal of L, we get $[y, x] \in R \cap L_i$ and thus $p_i([y, x]) = [y, x]$. Since $y \in R \cap L_i$, also $p_i(y) = y$.
In particular,

$$[y, z] = [y, p_i(x)] = [p_i(y), p_i(x)] = p_i([y, x]) = [y, x] \in R \cap L_i.$$

So $R \cap L_i$ is an ideal of L_i – but as L_i is simple, there are only two possibilities:

$$R \cap L_i = L_i \text{ (i.e., } L_i \subseteq R) \text{ or } L_i \cap R = 0.$$

Now if $L_i \subseteq R$ for all $i \in \{1, \ldots, s\}$, then $R = L$. So there is at least one i with $L_i \cap R = 0$. But there is also another one: Suppose $L_k \subseteq R$ for all $k \in \{1, \ldots, s\} \setminus \{i\}$. But then, if $z \in L_i$ and $p_i(x) = z$ for an $x \in R$ (such an x exists because of $p_i(R) = L_i$) and $x = \sum_{k=1}^{s} x_k, x_k \in L_k$, then also $x_i = x - \sum_{\substack{k=1 \\ k \neq i}}^{s} x_k \in R$, so $L_i \subseteq R$, a contradiction.

Thus there are at least two (different) indices $i, j \in \{1, \dots, s\}$ with $L_i \cap R = 0 = L_j \cap R$. Now set

$$L^* := R \cap (L_i \oplus L_j) \text{ and } R^* := L^* \oplus \bigoplus_{\substack{k=1 \\ k \neq i,j}}^{s} L_k.$$

Then R^* contains R and is not L, thus, by maximality of R we have $R^* = R$.
Claim: For every $x \in L_i$, there is exactly one $y \in L_j$ with $x + y \in L^*$.
For $x \in L_i$ take $z \in R$ with $p_i(z) = x$; then $x + p_j(z) \in L^*$, so we may choose $y := p_j(z)$.
Now suppose, this y was not unique: Let $x + y \in L^*$ and $x + z \in L^*$ for some $y, z \in L_j$. But then,

$$y - z = x + y - (x + z) \in L^* \Rightarrow y - z \in L_j \cap R = 0,$$

so $y = z$.
Thus we can define $\psi\colon L_i \to L_j$ by setting $\psi(x)$ to be the $y \in L_j$ with $x + y \in L^*$.
It is clear that ψ is bijective; we show that ψ is an isomorphism:
If $x_1 + y_1, x_2 + y_2 \in L^*$, $x_1, x_2 \in L_i, y_1, y_2 \in L_j$, then $[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2]$ by the simplicity of L_i, L_j. Now $[x_1 + y_1, x_2 + y_2] \in L^*$, so $[y_1, y_2]$ is the element such that $\psi([x_1, x_2]) = [y_1, y_2]$. Thus $\psi([x_1, x_2]) = [y_1, y_2] = [\psi(x_1), \psi(x_2)]$.
By definition of L^* we see

$$L^* = \{(x, \psi(x))\colon x \in L_i\} =: \{L_i, L_j, \psi\}$$

and therefore, we have proven the proposition. □

Now, we can apply Proposition 3.34 to find semisimple subalgebras of a simple Lie algebra $\mathfrak{n}$:

Lemma 3.35. *Let $\mathfrak{n}$ be a simple Lie algebra. Then every semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ is isomorphic to one of the form*

$$\mathfrak{h}' = \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c}, \tag{3}$$

where $\mathfrak{a} \oplus \mathfrak{b} \leq \mathfrak{n}, \mathfrak{b}' \oplus \mathfrak{c} \leq \mathfrak{n}$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}', \mathfrak{c}$ are semisimple.
Moreover, if $p|_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{g}, p((x_1, x_2)) = x_1 - x_2$ is an isomorphism, then $p|_{\mathfrak{h}'}$ is an isomorphism onto $\mathfrak{g}$ too.

Proof. Let $\mathfrak{h}$ be a semisimple subalgebra of $\mathfrak{n} \oplus \mathfrak{n}$. Then there is a (finite) chain of semisimple subalgebras

$$\mathfrak{h} = \mathfrak{h}_n \subseteq \mathfrak{h}_{n-1} \subseteq \ldots \subseteq \mathfrak{h}_0 = \mathfrak{n} \oplus \mathfrak{n},$$

where $\mathfrak{h}_{i+1}$ is a maximal proper semisimple subalgebra of $\mathfrak{h}_i$.
Each step $\mathfrak{h}_{i-1} \to \mathfrak{h}_i$ is done by either passing from $\mathfrak{h}_{i-1}$ to a subalgebra $\widetilde{\mathfrak{h}}$ of type (1) and finding $\mathfrak{h}_i$ as a Levi subalgebra of $\widetilde{\mathfrak{h}}$ or by passing to a subalgebra of type (2) (note that a subalgebra of type (2) is automatically semisimple).

We will prove the lemma inductively. First, any maximal proper semisimple subalgebra of $\mathfrak{n} \oplus \mathfrak{n}$ is of form (3) by Proposition 3.34.
Now, let $\mathfrak{h}_{i-1} = \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c}$ and $\mathfrak{h}_i$ a maximal semisimple subalgebra of $\mathfrak{h}_{i-1}$ – we shall show that $\mathfrak{h}_i$ is isomorphic to a Lie algebra of form (3) too. Let the decompositions of $\mathfrak{a}, \{\mathfrak{b}, \mathfrak{b}', \psi\}, \mathfrak{c}$ into simple ideals read $\mathfrak{a} = \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_k, \{\mathfrak{b}, \mathfrak{b}', \psi\} = \{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\} \oplus \ldots \oplus \{\mathfrak{b}_\ell, \mathfrak{b}_\ell', \psi_\ell\}, \mathfrak{c} = \mathfrak{c}_1 \oplus \ldots \oplus \mathfrak{c}_m$, respectively.
If $\mathfrak{h}_i$ arises from $\mathfrak{h}_{i-1}$ by type (1), then $\mathfrak{h}_i$ clearly is of the desired form (3); so we suppose that $\mathfrak{h}_i$ is of type (2) with respect to $\mathfrak{h}_{i-1}$.
When passing to a subalgebra of type (2), two simple isomorphic ideals of $\mathfrak{h}_i$ get combined. Therefore, without loss of generality, we have four possible cases:

(i) $\mathfrak{a}_1, \mathfrak{a}_2$ play the role of L_i, L_j in (2).

(ii) $\mathfrak{a}_1, \{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}$ play the role of L_i, L_j in (2).

(iii) $\{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}, \{\mathfrak{b}_2, \mathfrak{b}_2', \psi_2\}$ play the role of L_i, L_j in (2).

(iv) $\mathfrak{a}_1, \mathfrak{c}_1$ play the role of L_i, L_j in (2).

In each of those cases, we have to show that $\mathfrak{h}_i$ is isomorphic to a Lie algebra $\mathfrak{h}_i'$ of form (3) and that if $p|_{\mathfrak{h}_i}$ is an isomorphism onto $\mathfrak{g}$, then also $p|_{\mathfrak{h}_i'}$.

ad (i). Let $\mathfrak{a}_1$ be isomorphic to $\mathfrak{a}_2$ via ψ_0. Then $\mathfrak{h}_i = \{\mathfrak{a}_1, \mathfrak{a}_2, \psi_0\} \oplus \mathfrak{a}_3 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c}$ is isomorphic to

$$\mathfrak{h}_i' := \{a + \psi_0(a), a \in \mathfrak{a}_1\} \oplus \mathfrak{a}_3 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c},$$

which, in fact, is of the desired form (3).
We show that $\mathfrak{h}_i \cong \mathfrak{h}_i'$: Let w.l.o.g. $\mathfrak{h}_i = \{\mathfrak{a}_1, \mathfrak{a}_2, \psi_0\}$. Then $\mathfrak{h}_i' = \{a + \psi_0(a), a \in \mathfrak{a}_1\}$.
Let $\varphi\colon \mathfrak{h}_i \to \mathfrak{h}_i', (a, \psi_0(a)) \mapsto a + \psi_0(a)$.
Since $\mathfrak{a}_1, \mathfrak{a}_2$ are semisimple, $a + \psi_0(a) = 0$ implies $a = 0 = \psi_0(a)$, thus φ is injective; surjectivity is clear since given $a + \psi_0(a) \in \mathfrak{h}_i'$, the element $(a, \psi_0(a))$ maps to $a + \psi_0(a)$.
And φ respects the Lie brackets:

$$\begin{aligned}
\varphi([(a, \psi_0(a)), (a', \psi_0(a'))]) &= \varphi(([a, a'], \psi_0([a, a']))) = [a, a'] + \psi_0([a, a']) \\
&= [a, a'] + \underbrace{[\psi_0(a), a']}_{=0} + \underbrace{[a, \psi_0(a')]}_{=0} + [\psi_0(a), \psi_0(a')] = [a + \psi_0(a), a' + \psi_0(a')] \\
&= [\varphi(a, \psi_0(a)), \varphi(a', \psi_0(a'))].
\end{aligned}$$

So $\mathfrak{h}_i$ and $\mathfrak{h}_i'$ are isomorphic; we also have to show that if $p|_{\mathfrak{h}_i}$ is an isomorphism onto $\mathfrak{g}$, then also $p|_{\mathfrak{h}_i'}$. But this is clear since $p|_{\mathfrak{h}_i} \equiv p|_{\mathfrak{h}_i'}$.

ad (ii). Let $\mathfrak{a}_1$ be isomorphic to $\{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}$ via ψ_0. If $\psi_0(a) = (b_1, b_1')$, let $\psi_0'(a) = b_1, \psi_0''(a) = b_1'$. Then

$$\mathfrak{h}_i = \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{\mathfrak{a}_1, \{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}, \psi_0\} \oplus \{\mathfrak{b}_2, \mathfrak{b}_2'\} \oplus \ldots \oplus \{\mathfrak{b}_\ell, \mathfrak{b}_\ell'\} \oplus \mathfrak{c},$$

which, in turn, is isomorphic to

$$\mathfrak{h}_i' \coloneqq \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{(a + \psi_0'(a), \psi_0''(a)), a \in \mathfrak{a}_1\} \oplus \{\mathfrak{b}_2, \mathfrak{b}_2', \psi_2\} \oplus \ldots \oplus \{\mathfrak{b}_\ell, \mathfrak{b}_\ell', \psi_\ell\} \oplus \mathfrak{c}$$

of form (3).

ad (iii). Let $\{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}$ be isomorphic to $\{\mathfrak{b}_2, \mathfrak{b}_2', \psi_2\}$. Then $\mathfrak{b}_1$ is isomorphic to $\mathfrak{b}_2$ via ψ_0, say. Then

$$\mathfrak{h}_i = \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{\{\mathfrak{b}_1, \mathfrak{b}_1', \psi_1\}, \{\mathfrak{b}_2, \mathfrak{b}_2', \psi_2\}, \psi_0\} \oplus \{\mathfrak{b}_3, \mathfrak{b}_3', \psi_3\} \oplus \ldots \oplus \{\mathfrak{b}_\ell, \mathfrak{b}_\ell', \psi_\ell\} \oplus \mathfrak{c}$$

is isomorphic to

$$\begin{aligned} \mathfrak{h}_i' \coloneqq \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{(b + \psi_0(b), \psi_1(b) + \psi_2(\psi_0(b))), b \in \mathfrak{b}_1\} \\ \oplus \{\mathfrak{b}_3, \mathfrak{b}_3', \psi_3\} \oplus \ldots \oplus \{\mathfrak{b}_\ell, \mathfrak{b}_\ell', \psi_\ell\} \oplus \mathfrak{c}, \end{aligned}$$

a subalgebra of type (3).

ad (iv). Let $\mathfrak{a}_1$ be isomorphic to $\mathfrak{c}_1$ via ψ_0. Then

$$\mathfrak{h}_i' \coloneqq \mathfrak{h}_i = \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_k \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \{\mathfrak{a}_1, \mathfrak{c}_1, \psi_0\} \oplus \mathfrak{c}_2 \oplus \ldots \oplus \mathfrak{c}_m,$$

and this is also of form (3).

In (ii) and (iii), it is analogous to (i) to see that $\mathfrak{h}_i \equiv \mathfrak{h}_i'$ and that p stays an isomorphism. □

Concerning decompositions of Lie algebras, we will make use of a theorem of Koszul:

Theorem 3.36 (Koszul, [67]). *Let L be a reductive Lie algebra over a field of characteristic 0. Suppose that L is the direct vector space sum of two subalgebras*

$$L = A_1 \oplus A_2 \text{ (direct sum of vector spaces)},$$

which are reductive in L. Then L is the direct Lie algebra sum of two ideals

$$L = B_1 \oplus B_2 \text{ (direct sum of Lie algebras)},$$

such that A_1 is naturally isomorphic to B_1 and A_2 is naturally isomorphic to B_2.

Remembering that if $A \leq L$ is a (semi)simple Lie algebra, then A is reductive in L (Theorem 2.29), we obtain the following corollary:

Corollary 3.37. *Let L be a simple Lie algebra over a field of characteristic 0. Then L can decomposed as a direct vector space sum of semisimple Lie algebras only in a trivial way, i.e. if $L = A_1 \oplus A_2$ is a direct vector space sum with A_1, A_2 semisimple subalgebras of L, then either $A_1 = 0$ or $A_2 = 0$.*

Indeed, if there was another decomposition, then L could also be decomposed into two ideals which is not possible since L is simple.

Corollary 3.38. *Let $\mathfrak{h}$ be a semisimple subalgebra of $\mathfrak{n} \oplus \mathfrak{n}$ ($\mathfrak{n}$ simple) such that the restriction of $p\colon \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{g}$ to $\mathfrak{h}$ is an isomorphism. Let $\mathfrak{h}$ be of the form described in Lemma 3.35, i.e.*

$$\mathfrak{h} = \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c},$$

where $\mathfrak{a} \oplus \mathfrak{b} \leq \mathfrak{n}, \mathfrak{b}' \oplus \mathfrak{c} \leq \mathfrak{n}$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}', \mathfrak{c}$ are semisimple. Then none of the factors $\mathfrak{a}, \{\mathfrak{b}, \mathfrak{b}', \psi\}, \mathfrak{c}$ is 0. In particular,

(i) $\mathfrak{g}$ has at least three simple ideals,

(ii) $\mathfrak{a} \oplus \mathfrak{b}$ and $\mathfrak{b}' \oplus \mathfrak{c}$ are not simple (i.e., $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}', \mathfrak{c} \neq 0$).

Proof. The case $\mathfrak{c} = 0$ is analogous to $\mathfrak{a} = 0$; so we deal with the cases $\mathfrak{a} = 0$ and $\mathfrak{b} = 0$. We shall show that in either case, $\mathfrak{h} = \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c}$ is isomorphic to $\mathfrak{n}$ – which is a contradiction, since $\mathfrak{n}$ is simple, but $\mathfrak{h} \cong \mathfrak{g}$ is not.

(a) If $\mathfrak{a} = 0$, then, as $\dim(\mathfrak{h}) = \dim(\mathfrak{n})$, $\dim(\mathfrak{n}) = \dim(\mathfrak{b}') + \dim(\mathfrak{c})$ and $\mathfrak{b}' \oplus \mathfrak{c}$ is a semisimple subalgebra of $\mathfrak{n}$. So either $\mathfrak{b}' = 0$ and thus $\mathfrak{c} \cong \mathfrak{n}$ or $\mathfrak{c} = 0$ and thus $\mathfrak{b}' \cong \mathfrak{n}$. So in either case, $\mathfrak{h} \cong \mathfrak{n}$.

(b) Now suppose $\mathfrak{b} = 0$. Note that $\mathfrak{a} \cap \mathfrak{c} = 0$ (for if $x \in \mathfrak{a} \cap \mathfrak{c}$, then $(x, x) \in \mathfrak{h}$ which gets mapped to 0 under p – but since p is an isomorphism, $\ker(p) = 0$). So then, $\mathfrak{a} \oplus \mathfrak{c} = \mathfrak{n}$ as a direct vector space sum. But then, by Corollary 3.37, $\mathfrak{a} \cong \mathfrak{n}$ or $\mathfrak{c} \cong \mathfrak{n}$, so $\mathfrak{h} \cong \mathfrak{n}$.

□

Corollary 3.38, in turn, yields that condition (ii) of Theorem 3.32 implies the non-existence of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$:

Corollary 3.39. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of exactly two simple Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and $\mathfrak{n}$ is simple, then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. If there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, then, by Proposition 2.46, $\mathfrak{n} \oplus \mathfrak{n}$ has a subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{g}$ via $p\colon \mathfrak{h} \leq \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{g}, (x, y) \mapsto x - y$. But by Lemma 3.35, $\mathfrak{h} \cong \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c}$ with semisimple factors which are all nonzero by Corollary 3.38, part (i). So $\mathfrak{h}$ and thus also $\mathfrak{g}$ has at least three simple factors. □

Remark 3.40. Note that the proof of Corollary 3.38 also implies that if $\mathfrak{g}$ and $\mathfrak{n}$ are both simple and the restriction of p to $\mathfrak{h}$ is an isomorphism, then $\mathfrak{g} \cong \mathfrak{n}$. (So we get back the statement of Theorem 3.20 via Corollary 3.39.)

In [50] and [69] (see also [78]), one finds tables listing the maximal semisimple subalgebras of the five exceptional complex Lie algebras and of the classical complex Lie algebras up to rank 6. The results we will need are summarized in Table 6. Let $\mathfrak{n}$ be simple; we find all semisimple subalgebras of $\mathfrak{n}$ by iterative application of the results in Table 6; i.e., we find the maximal semisimple subalgebras of $\mathfrak{n}$ with Table 6, then the maximal subalgebras among them and so forth.

Table 6: Maximal semisimple subalgebras of simple Lie algebras $\mathfrak{n}$, according to [50] and [69].

$\mathfrak{n}$	$\dim(\mathfrak{n})$	max. semisimple subalgebras of $\mathfrak{n}$	resp. dimensions
A_1	3	–	–
A_2	8	A_1	3
A_3	15	A_1^2, B_2	$6, 10$
A_4	24	$A_3, A_1 \oplus A_2$	$15, 11$
A_5	35	$A_4, A_1 \oplus A_3, A_2^2, C_3$	$24, 18, 16, 21$
A_6	48	$A_5, A_1 \oplus A_4, A_2 \oplus A_3, G_2, B_3$	$35, 27, 23, 14, 21$
B_2	10	A_1^2	6
B_3	21	B_2, A_1^3, A_3, G_2	$10, 9, 15, 14$
B_4	36	$B_3, A_1^2 \oplus B_2, A_3 \oplus A_1, D_4$	$21, 16, 18, 28$
B_5	55	$B_4, A_1^2 \oplus B_3, A_3 \oplus B_2, D_4 \oplus A_1, D_5$	$36, 27, 25, 31, 45$
B_6	78	$B_5, A_1^2 \oplus B_4, A_3 \oplus B_3, D_4 \oplus B_2, D_5 \oplus A_1, D_6$	$55, 42, 36, 38, 48, 66$
C_3	21	$A_2, A_1 \oplus B_2$	$8, 13$
C_4	36	$A_3, A_1 \oplus C_3, B_2^2, A_1^3$	$15, 24, 20, 9$
C_5	55	$A_4, A_1 \oplus C_4, B_2 \oplus C_3$	$24, 39, 31$
C_6	78	$A_5, A_1 \oplus C_5, B_2 \oplus C_4, C_3^2$	$35, 58, 46, 42$
D_4	28	$A_3, A_1^4, A_3 \oplus A_1, G_2, B_3$	$15, 12, 18, 14, 21$
D_5	45	$D_4, A_4, A_1^2 \oplus A_3, D_4 \oplus A_1, B_4, B_2^2$	$28, 24, 21, 31, 36, 20$
D_6	66	$D_5, A_5, A_1^2 \oplus D_4, A_3^2, D_5 \oplus A_1, B_5,$ $G_2 \oplus A_1, B_3 \oplus B_2, C_3 \oplus A_1$	$45, 35, 34, 30, 48, 55,$ $17, 31, 24$
G_2	14	A_1^2, A_2	$6, 8$
F_4	52	$A_1 \oplus C_3, A_2^2, A_3 \oplus A_1, B_4, G_2 \oplus A_1$	$24, 16, 18, 36, 17$
E_6	78	$D_5, A_5 \oplus A_1, A_2^3, C_4, G_2 \oplus A_2, F_4$	$45, 38, 24, 36, 22, 52$
E_7	133	$E_6, D_6 \oplus A_1, A_5 \oplus A_2, A_3^2 \oplus A_1, A_7,$ $G_2 \oplus C_3, F_4 \oplus A_1, G_2 \oplus A_1$	$78, 69, 43, 33, 63,$ $35, 55, 17$
E_8	248	$A_1 \oplus E_7, A_2 \oplus E_6, A_3 \oplus D_5, A_4^2,$ $A_5 \oplus A_2 \oplus A_1, A_7 \oplus A_1, D_8, A_8, G_2 \oplus F_4$	$136, 86, 60, 48,$ $46, 66, 120, 80, 66$

Now we can apply Corollary 3.38 to our problem – remember that we have a semisimple Lie algebra $\mathfrak{g}$ and a simple Lie algebra $\mathfrak{n}$. If $\mathfrak{n}$ is exceptional, then there exists no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ (which is condition (iii) of Theorem 3.32):

Corollary 3.41. *Let $\mathfrak{n} \in \{G_2, E_6, E_7, E_8, F_4\}$ be an exceptional Lie algebra and $\mathfrak{g}$ be semisimple and non-simple. Then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. If there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, then there is a semisimple subalgebra $\mathfrak{h} \leq \mathfrak{n} \oplus \mathfrak{n}$ (by Proposition 2.46) with

$$\mathfrak{h} = \mathfrak{a} \oplus \{\mathfrak{b}, \mathfrak{b}', \psi\} \oplus \mathfrak{c},$$

where $\mathfrak{a}\oplus\mathfrak{b}\leq\mathfrak{n}$, $\mathfrak{b}'\oplus\mathfrak{c}\leq\mathfrak{n}$ and $\mathfrak{a},\mathfrak{b},\mathfrak{b}',\mathfrak{c}$ are semisimple and the map ψ is an isomorphism. If $\max(\dim(\mathfrak{a}\oplus\mathfrak{b}),\dim(\mathfrak{b}'\oplus\mathfrak{c}))<\frac{\dim(\mathfrak{n})}{2}$, then $\dim(\mathfrak{a}\oplus\mathfrak{b}\oplus\mathfrak{b}'\oplus\mathfrak{c})<\dim(\mathfrak{n})$, so we can assume $\dim(\mathfrak{a}\oplus\mathfrak{b})\geq\frac{\dim(\mathfrak{n})}{2}$.
We thus shall find with the help of Table 6

(a) all semisimple subalgebras $\mathfrak{a}\oplus\mathfrak{b}$ of $\mathfrak{n}$ of dimension $\geq\frac{\dim(\mathfrak{n})}{2}$.

(b) for each such semisimple subalgebra $\mathfrak{a}\oplus\mathfrak{b}$ of dimension m, each semisimple subalgebra $\mathfrak{b}'\oplus\mathfrak{c}$ of $\mathfrak{n}$, which has dimension k, such that $m+k\geq\dim(\mathfrak{n})$.

Then, if there is a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, we must have $\mathfrak{g}\cong\mathfrak{a}\oplus\mathfrak{b}\oplus\mathfrak{c}$.
By Corollary 3.38, none of $\mathfrak{a}\oplus\mathfrak{b}$ and $\mathfrak{b}'\oplus\mathfrak{c}$ can be simple. This excludes a lot of possibilities.
Then we can compute the dimension of $\mathfrak{b}$ as

$$\dim(\mathfrak{b})=\dim(\mathfrak{a}\oplus\mathfrak{b}\oplus\mathfrak{b}'\oplus\mathfrak{c})-\dim(\mathfrak{n})$$

(since $\mathfrak{b}\cong\mathfrak{b}'$, so $\dim(\mathfrak{b})=\dim(\mathfrak{b}')$) and

$$\dim(\mathfrak{a})=\dim(\mathfrak{a}\oplus\mathfrak{b})-\dim(\mathfrak{b}),\dim(\mathfrak{c})=\dim(\mathfrak{b}'\oplus\mathfrak{c})-\dim(\mathfrak{b}).$$

If there is a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, then $\mathfrak{a}\oplus\mathfrak{b}$ allows a decomposition into a direct sum of a Lie algebra of dimension $\dim(\mathfrak{a})$ and one of dimension $\dim(\mathfrak{b})$. Analogously, $\mathfrak{b}'\oplus\mathfrak{c}$ allows a decomposition into a direct sum of a Lie algebra of dimension $\dim(\mathfrak{b})$ and one of dimension $\dim(\mathfrak{c})$. This will often lead to a contradiction.
We illustrate this method with the exceptional Lie algebras:

(i) $\mathfrak{n}=G_2$. Then $\dim(\mathfrak{n})=14$; we thus have to find all semisimple subalgebras $\mathfrak{a}\oplus\mathfrak{b}\leq\mathfrak{n}$ of dimension ≥ 7. But there is only one, namely A_2. But since A_2 is simple, there is no post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$.

(ii) $\mathfrak{n}=F_4$. Then $\dim(\mathfrak{n})=52$; we thus have to find all semisimple subalgebras $\mathfrak{a}\oplus\mathfrak{b}\leq\mathfrak{n}$ of dimension ≥ 26. However, again there are only simple ones (namely D_4 and B_4). So again, there can be no post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$.

(iii) $\mathfrak{n}=E_6$. Then $\dim(\mathfrak{n})=78$, we thus have to find all semisimple subalgebras $\mathfrak{a}\oplus\mathfrak{b}\leq\mathfrak{n}$ of dimension ≥ 39. However, again they are only given by simple ones (namely D_5 and F_4).

(iv) $\mathfrak{n}=E_7$. Then $\dim(\mathfrak{n})=133$; we thus have to find all semisimple subalgebras $\mathfrak{a}\oplus\mathfrak{b}\leq\mathfrak{n}$ of dimension ≥ 67. These are given by E_6 (which is simple) and $D_6\oplus A_1$ (of dimension 69). So $\dim(\mathfrak{a}\oplus\mathfrak{b})=69$ and thus $\dim(\mathfrak{b}'\oplus\mathfrak{c})\geq 64$. All semisimple subalgebras of dimension ≥ 64 are again given by E_6 and $D_6\oplus A_1$. We can exclude E_6 (since it is simple) from our considerations; so $\mathfrak{a}\oplus\mathfrak{b}\cong\mathfrak{b}'\oplus\mathfrak{c}\cong D_6\oplus A_1$. Now we have

$$\dim(\mathfrak{b})=\dim(\mathfrak{a}\oplus\mathfrak{b}\oplus\mathfrak{b}'\oplus\mathfrak{c})-\dim(\mathfrak{n})=69+69-133=5.$$

But this means that $D_6 \oplus A_1$ contains a semisimple subalgebra $\mathfrak{b}$ of dimension 5. This is a contradiction (since there does not exist any semisimple Lie algebra of dimension 5).

(v) $\mathfrak{n} = E_8$. Then $\dim(\mathfrak{n}) = 248$. As before, we shall find all semisimple subalgebras of dimension ≥ 124 – and (besides the simple one, E_7) there is only one, $A_1 \oplus E_7$. It is of dimension 136, which means $\dim(\mathfrak{b}' \oplus \mathfrak{c}) \geq 248 - 136 = 112$ and so $A_1 \oplus E_7$ also is given by $A_1 \oplus E_7$ (we again exclude the possibility of the simple subalgebra E_7).
Then,

$$\dim(\mathfrak{b}) = \dim(\mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{b}' \oplus \mathfrak{c}) - \dim(\mathfrak{n}) = 136 + 136 - 248 = 24.$$

But the Lie algebra $A_1 \oplus E_7$ can not be decomposed into a semisimple subalgebra of dimension 24 and one of dimension $136 - 24 = 112$, a contradiction.

□

With the same method, one may show that some classical simple Lie algebras $\mathfrak{n}$ admit no post-Lie algebra structures with semisimple, non-simple $\mathfrak{g}$:

Corollary 3.42. *Let $\mathfrak{n} \in \{A_i, B_i, C_i, D_i, A_5, A_6, C_5\}, i \leq 4$ be a classical simple Lie algebra and $\mathfrak{g}$ be semisimple and non-simple. Then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. The proof works the same way as the one of Corollary 3.41. We shall therefore be brief:

$\mathfrak{n} = A_1, A_2$: All semisimple subalgebras of $\mathfrak{n}$ are of dimension $< \frac{\dim(\mathfrak{n})}{2}$ (in fact, A_1 has none and A_2 only A_1). So there is no post-Lie algebra structure with a semisimple $\mathfrak{g}$.

$\mathfrak{n} = A_3, A_4, A_5$: All "large enough" semisimple subalgebras are simple (namely B_2 in A_3, A_3 in A_4, A_4 and C_4 in A_5). So there is again no post-Lie algebra structure with a semisimple $\mathfrak{g}$.

$\mathfrak{n} = A_6$: The semisimple subalgebras of $\mathfrak{n}$ of dimension $\geq \frac{\dim(\mathfrak{n})}{2}$ are the simple A_5 and A_4 and $A_1 \oplus A_4$. If $\mathfrak{a} \oplus \mathfrak{b} \cong A_1 \oplus A_4$, then, because $\mathfrak{b}' \oplus \mathfrak{c}$ is not simple, it is either $A_1 \oplus A_4$ or $A_2 \oplus A_3$. But this leads to $\dim(\mathfrak{b}) \in \{4, 2\}$ and there are no semisimple Lie algebras of these dimensions. So again, there is no semisimple $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure.

The cases $\mathfrak{n} = B_2, B_3, B_4, C_3, C_4, C_5, D_4$ are very similar.

□

Remark 3.43. Concerning B_5, B_6 and D_5, the "problematic" cases, i.e. those pairs $(\mathfrak{g}, \mathfrak{n})$ where this method does not exclude post-Lie algebra structures, are

$$(D_4 \oplus A_1 \oplus A_1 \oplus B_3, B_5), (B_4 \oplus A_1 \oplus A_1 \oplus B_4, B_6), (D_4 \oplus A_1 \oplus G_2, D_5),$$

respectively.

In other words, Corollary 3.42 gives us the case of condition (iv) (of Theorem 3.32).

Corollary 3.44. *Let $\mathfrak{n}$ be a simple Lie algebra of dimension* $\dim(\mathfrak{n}) \leq 44$ *over* $\mathbb{C}$ *and* $\mathfrak{g}$ *be semisimple and non-simple. Then there are no post-Lie algebra structures on* $(\mathfrak{g},\mathfrak{n})$.

Summing up all together, we obtain the statement of Theorem 3.32:

Proof of Theorem 3.32. If (i) holds, we have no post-Lie algebra structures on $(\mathfrak{g},\mathfrak{n})$ by Theorem 3.20 (or, by Remark 3.40), (ii) is the case of Corollary 3.39, (iii) is Corollary 3.41 and condition (iv) is the case of Corollary 3.44. □

3.3 Post-Lie algebra structures with $[x,y]=a(\{x,y\})$

In this section, we want to study special kinds of post-Lie algebra structures, namely those where one Lie bracket is a scalar multiple of the other, i.e. we have $[x,y]=a(\{x,y\})$ for some $a\in\mathbb{C}$ and all $x,y\in V$ (see also [28, Chapter 3]).
Note that if $a\neq 0$, $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic while if $a=0$, then $(\mathfrak{g},[,])$ is abelian (that is, this case corresponds to the case of LR-structures (cf. Definition 2.37)).
If $a\neq 0$, the post-Lie algebra structure axioms (PA1), (PA2), (PA3) become

$$x\cdot y-y\cdot x=(a-1)\{x,y\} \tag{4}$$
$$[x,y]\cdot z=x\cdot(y\cdot z)-y\cdot(x\cdot z) \tag{5}$$
$$x\cdot[y,z]=[x\cdot y,z]+[y,x\cdot z] \tag{6}$$

for all $x,y,z\in V$.
The scalar a being 1 is equivalent to considering a *commutative post-Lie algebra structure* (cf. Definition 2.33).
Since $[x,y]=a\{x,y\}$, any ideal in $\mathfrak{g}$ is also an ideal in $\mathfrak{n}$ and vice versa.
There is a correspondence between those structures for $a=1$ and $a=-1$:

Lemma 3.45. *If $x\circ y$ is a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}$'s Lie bracket satisfies $[x,y]=-\{x,y\}$ (that is, $a=-1$) for all $x,y\in\mathfrak{g}$, then $x\cdot y$ given by $x\cdot y\coloneqq x\circ y-[x,y]$ is a CPA-structure on $\mathfrak{g}$ with respect to $[,]$ (that is, $a=1$) and vice versa.*

Proof. We only show one direction. Let $x\circ y$ be a post-Lie algebra structure as above. The bilinear map $\circ$ satisfies the three properties

$$x\circ y-y\circ x=2[x,y] \tag{7}$$
$$[x,y]\circ z=x\circ(y\circ z)-y\circ(x\circ z) \tag{8}$$
$$x\circ[y,z]=[x\circ y,z]+[y,x\circ z] \tag{9}$$

for all $x,y,z\in\mathfrak{g}$.
We set $x\cdot y\coloneqq x\circ y-[x,y]$ and show that $x\cdot y$ defines a commutative post-Lie algebra structure on $\mathfrak{g}$ with respect to $[,]$. As a difference of two bilinear functions, $(x,y)\mapsto x\cdot y$

itself is bilinear.
Identity (4): We have

$$x \cdot y - y \cdot x = x \circ y - [x,y] - y \circ x + [y,x] \overset{(7)}{=} 2[x,y] - [x,y] - [x,y] = 0.$$

Identity (6): Here, by (9) we get

$$\begin{aligned} x \cdot [y,z] - [x \cdot y, z] - [y, x \cdot z] &= x \circ [y,z] - [x,[y,z]] - [x \circ y, z] \\ &\quad + [[x,y],z] - [y, x \circ z] + [y,[x,z]] \\ &\overset{(9)}{=} -[x,[y,z]] + [[x,y],z] + [y,[x,z]] \end{aligned}$$

and this equals 0 due to the Jacobi identity.
Identity (5): Here we will need all of (7), (8) and (9). We have

$$\begin{aligned} &[x,y] \cdot z - x \cdot (y \cdot z) + y \cdot (x \cdot z) \\ &\quad = [x,y] \circ z - [[x,y],z] - x \cdot (y \circ z - [y,z]) + y \cdot (x \circ z - [x,z]) \\ &\quad = [x,y] \circ z - [[x,y],z] - x \circ (y \circ z) + [x, y \circ z] + x \circ [y,z] \\ &\quad - [x,[y,z]] + y \circ (x \circ z) - [y, x \circ z] - y \circ [x,z] + [y,[x,z]] \\ &\quad \overset{(8)}{=} [z,[x,y]] + [x, y \circ z] + x \circ [y,z] - [x,[y,z]] - [y, x \circ z] - y \circ [x,z] + [y,[x,z]] \\ &\quad = [z,[x,y]] - [x,[y,z]] - [y,[z,x]] + [x, y \circ z] + x \circ [y,z] - [y, x \circ z] - y \circ [x,z] \\ &\quad \overset{\text{Jacobi}}{=} 2[z,[x,y]] + [x, y \circ z] + x \circ [y,z] - [y, x \circ z] - y \circ [x,z] \\ &\quad \overset{(9)}{=} [z, 2[x,y]] + [x, y \circ z] + [x \circ y, z] + [y, x \circ z] - [y, x \circ z] - [y \circ x, z] - [x, y \circ z] \\ &\quad = [z, 2[x,y]] + [x \circ y, z] - [y \circ x, z] \\ &\quad \overset{(7)}{=} [z, x \circ y - y \circ x] + [x \circ y, z] - [y \circ x, z] \\ &\quad = -[x \circ y, z] + [y \circ x, z] + [x \circ y, z] - [y \circ x, z] = 0. \end{aligned}$$

□

Given any Lie algebra $\mathfrak{g}$, we can always define the trivial commutative post-Lie algebra structure $x \cdot y \coloneqq 0$ on $\mathfrak{g}$. The corresponding structure for $a = -1$ by Lemma 3.45 is $x \cdot y = -[x,y]$ for all $x, y \in \mathfrak{g}$ (cf. Example 2.42). So for $[x,y] = a\{x,y\}, a \in \{\pm 1\}$, there are always post-Lie algebra structures.
Consequently, we are interested in the existence of post-Lie algebra structures subject to $[x,y] = a\{x,y\}$, where a is not in $\{-1,0,1\}$.

Lemma 3.46. *Suppose there is a post-Lie algebra structure on* $(\mathfrak{g},\mathfrak{n})$ *where there is an* $a \in \mathbb{C}\backslash\{0\}$ *with* $[x,y] = a\{x,y\}$ *for all* $x, y \in \mathfrak{n}$.
Then either $\mathfrak{n} \cdot \mathfrak{n} \subseteq \operatorname{rad}(\mathfrak{n})$ *or* $a = -1$ *(i.e.,* $\{x,y\} = -[x,y]$*).*

Proof. This is a generalization of [24, Corollary 5.5]. Let $\mathfrak{n} = \mathfrak{s} \ltimes \mathfrak{r}$ be the Levi decomposition of $\mathfrak{n}$, with $\mathfrak{s}$ semisimple and $\mathfrak{r} = \operatorname{rad}(\mathfrak{n}) = \operatorname{rad}(\mathfrak{g})$.

The left-multiplication operators $L(x)$ are derivations of $\mathfrak{n}$ for all $x \in \mathfrak{n}$. We thus have $L(x)(\mathfrak{r}) \subseteq \mathfrak{r}$ for all $x \in \mathfrak{n}$ and therefore $\mathfrak{n} \cdot \mathfrak{r} \subseteq \mathfrak{r}$ (remember that by Example 2.16, $\mathfrak{r}$ is a characteristic ideal of $\mathfrak{n}$).
On the other hand, by (4),

$$\mathfrak{r} \cdot \mathfrak{n} \subseteq (a-1)\{\mathfrak{r}, \mathfrak{n}\} + \mathfrak{n} \cdot \mathfrak{r} \subseteq \mathfrak{r} + \mathfrak{r} = \mathfrak{r}.$$

So we can define a bilinear map $(x + \mathfrak{r}) \circ (y + \mathfrak{r})$ on the semisimple quotient $\mathfrak{s} \cong \mathfrak{n}/\mathfrak{r}$ by

$$\circ : (x + \mathfrak{r}, y + \mathfrak{r}) \mapsto (x + \mathfrak{r}) \circ (y + \mathfrak{r}) = x \cdot y + \mathfrak{r}.$$

This is a post-Lie algebra structure on $(\mathfrak{s}, \mathfrak{s})$ with respect to the induced Lie brackets $[x+\mathfrak{r}, y+\mathfrak{r}] = [x, y]+\mathfrak{r}$ and $\{x+\mathfrak{r}, y+\mathfrak{r}\} = \{x, y\}+\mathfrak{r}$ satisfying $[x+\mathfrak{r}, y+\mathfrak{r}] = a\{x+\mathfrak{r}, y+\mathfrak{r}\}$. Now, by [23, Theorem 6.4 and Proposition 6.5], the map $(x + \mathfrak{r}) \circ (y + \mathfrak{r})$ is either given by $(x+\mathfrak{r}) \circ (y+\mathfrak{r}) = \mathfrak{r}$ or by $(x+\mathfrak{r}) \circ (y+\mathfrak{r}) = [x, y]+\mathfrak{r}$ or by $(x+\mathfrak{r}) \circ (y+\mathfrak{r}) = [x, y]+\mathfrak{r}$ (and $[x + \mathfrak{r}, y + \mathfrak{r}] = -\{x + \mathfrak{r}, y + \mathfrak{r}\}$).
In the former case, we infer $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathfrak{r}$.
So if $\mathfrak{n} \cdot \mathfrak{n} \nsubseteq \mathfrak{r}$, the latter case must occur and we have $a = -1$. □

Lemma 3.46 implies that such structures with $a \notin \{-1, 0, 1\}$ can only appear on solvable Lie algebras:

Corollary 3.47. *Let $\mathfrak{n}$ be a Lie algebra and $x \cdot y$ a post-Lie algebra structure on $\mathfrak{n}$ with respect to $[x, y] = a\{x, y\}$ for all $x, y \in \mathfrak{n}$, where $a \in \mathbb{C}\backslash\{0, 1, -1\}$. Then $\mathfrak{n}$ is solvable.*

Proof. Suppose $\mathfrak{n}$ is not solvable and let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{n}$. Let $s_1, s_2 \in \mathfrak{s}$ with $\{s_1, s_2\} \neq 0$.
The post-Lie algebra structure $x \cdot y$ satisfies

$$s_1 \cdot s_2 - s_2 \cdot s_1 = (a-1)\{s_1, s_2\}.$$

But since $s_1, s_2 \in \mathfrak{s}$ and $a \neq 1$, the right-hand side is contained in $\mathfrak{s}\backslash\{0\}$ (as $\{s_1, s_2\} \neq 0$). On the other hand, since $a \neq -1$, $\mathfrak{n} \cdot \mathfrak{n} \subseteq \operatorname{rad}(\mathfrak{n})$ by Lemma 3.46. This is a contradiction. □

Solvable Lie algebras can also admit scalars other than -1 and 1:

Example 3.48. *The three-dimensional Lie algebra $\mathfrak{r}_{3,-1}(\mathbb{C})$ with non-zero Lie brackets*

$$\{e_1, e_2\} = e_2, \{e_1, e_3\} = -e_3.$$

It has for every $a \in \mathbb{C}^*$ a post-Lie algebra structure with

$$[e_1, e_2] = ae_2, [e_1, e_3] = -ae_3,$$

for example by non-zero relations

$$e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = -ae_3, e_2 \cdot e_1 = e_2, e_3 \cdot e_1 = -e_3.$$

Table 7: Lie algebras in dimension 3

$\mathfrak{g}$	non-zero Lie brackets
$\mathbb{C}^3$	–
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3$ with $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ if and only if $\mu = \frac{1}{\lambda}$ or $\mu = \lambda$
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$

While general solvable Lie algebras admit other scalars than 1 and -1, solvable *complete* Lie algebras (where the nilradical and the nilpotent ideal coincide) do not. We will study this situation in Section 4.3.

Remark 3.49. We can refine Corollary 3.47 to the case where $D(\{x, y\}) = [x, y]$ for a $D \in \operatorname{Der}(\mathfrak{n})$. Then, we have to assume that $\operatorname{rad}(\mathfrak{n})$ is a Lie ideal in $\mathfrak{g}$ (to make sense of the quotient $\mathfrak{g}/\operatorname{rad}(\mathfrak{n})$) and that $\mathfrak{n}$ is not a direct sum of copies of $\mathfrak{sl}_2(\mathbb{C})$ (this assumption we do need because otherwise, by [23, Proposition 6.5], there are more possibilities for post-Lie algebra structures on the quotient $\mathfrak{g}/\operatorname{rad}(\mathfrak{n})$). Under these two assumptions, we also get the result that $\mathfrak{n}$ is solvable.

3.4 Existence of post-Lie algebra structures in low dimensions

In this section, we answer the following question:

Question 3.50. *Let $n \leq 3$ and let μ, μ' be two n-dimensional Lie algebra laws. Is there a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ has law μ and $\mathfrak{n}$ has law μ', that admits a post-Lie algebra structure?*

In dimension 1, there is only one Lie algebra (the abelian one, $\mathbb{C}$) and in dimension 2, there are two non-isomorphic Lie algebras ($\mathbb{C}^2$ (the abelian Lie algebra) and $\mathfrak{r}_2(\mathbb{C})$ (with basis $\{e_1, e_2\}$ and Lie brackets $[e_1, e_2] = e_1$)). In [28], all post-Lie algebra structures on two-dimensional Lie algebras are classified. In particular, the classification implies that all possible pairs (namely $(\mathbb{C}, \mathbb{C}), (\mathbb{C}^2, \mathbb{C}^2), (\mathbb{C}^2, \mathfrak{r}_2(\mathbb{C})), (\mathfrak{r}_2(\mathbb{C}), \mathbb{C}^2)$ and $(\mathfrak{r}_2(\mathbb{C}), \mathfrak{r}_2(\mathbb{C}))$ admit post-Lie algebra structures (with a suitable choice of bases).

In dimension 3, not all pairs of Lie algebras (the 3-dimensional Lie algebras are, up to isomorphism, listed in Table 7) admit post-Lie algebra structures. We get the following result, which is summarized in Table 8:

Proposition 3.51. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of 3-dimensional Lie algebras.*

(i) If $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{n} \ncong \mathfrak{sl}_2(\mathbb{C})$, then there are no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.

Table 8: Existence of post-Lie algebra structures in dimension 3

$\mathfrak{g}$ \ $\mathfrak{n}$	$\mathbb{C}^3$	$\mathfrak{n}_3(\mathbb{C})$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{r}_3(\mathbb{C})$	$\mathfrak{r}_{3,1}(\mathbb{C})$	$\mathfrak{r}_{3,\mu}(\mathbb{C}), \mu \neq 1$	$\mathfrak{sl}_2(\mathbb{C})$
$\mathbb{C}^3$	✓	✓	✓	✓	✓	✓	✗
$\mathfrak{n}_3(\mathbb{C})$	✓	✓	✓	✓	✓	✗	✗
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	✓	✓	✓	✓	✓	✓	✗
$\mathfrak{r}_3(\mathbb{C})$	✓	✓	✓	✓	✓	✗	✗
$\mathfrak{r}_{3,1}(\mathbb{C})$	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \neq 1$	✓	✓	✓	✓	✓	✓	✓(iff $\lambda \neq -1$)
$\mathfrak{sl}_2(\mathbb{C})$	✗	✗	✗	✗	✗	✗	✓

(ii) *If* $\mathfrak{n} \cong \mathfrak{sl}_2(\mathbb{C})$ *and* $\mathfrak{g}$ *is neither isomorphic to* $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ *(for any* λ*) nor isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*, then there are no post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$*.*
Moreover, there are also no post-Lie algebra structures on the pair $(\mathfrak{r}_{3,-1}(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C}))$*.*

(iii) *If* $\mathfrak{n} \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$*, where* $\mu \neq 1$ *and* $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C})$ *or* $\mathfrak{g} \cong \mathfrak{r}_3(\mathbb{C})$*, then there are no post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$*.*

(iv) *For any pair* $(\mathfrak{g}, \mathfrak{n})$ *of* 3*-dimensional Lie algebras not listed in (i), (ii), (iii), there exists a pair* $(\mathfrak{g}', \mathfrak{n}')$ *with a post-Lie algebra structure where* $\mathfrak{g} \cong \mathfrak{g}'$ *and* $\mathfrak{n} \cong \mathfrak{n}'$*.*

For the proof, the reader is referred to the appendix, Section B.1.

4 Post-Lie algebra structures on complete Lie algebras

In this chapter, we consider post-Lie algebra structures on *complete* Lie algebras. Since a general classification in this case seems rather hopeless, we try to answer a related question, motivated by the results in [23, Chapter 6] on semisimple Lie algebras:

Question 4.1. *Can we classify post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{n}$ *is complete and there exists a linear map* $R \in \operatorname{End}(V)$ *such that* $[x, y] = R(\{x, y\})$ *for all* $x, y \in V$*?*

4.1 General results

Definition 4.2. A Lie algebra L is called *complete* if $\mathrm{Z}(L) = 0$ and $\operatorname{Der}(L) = \operatorname{ad}(L)$, that is, all derivations of L are inner.

Remark 4.3. The class of complete Lie algebras is a natural generalization of the (important) class of *semisimple* Lie algebras: Every semisimple Lie algebra has trivial center and only inner derivations (see Theorem 2.21)!

Given a Lie algebra L, we have $H^0(L, L) = \mathrm{Z}(L)$ and $H^1(L, L) = \operatorname{Der}(L)/\operatorname{ad}(L)$. Therefore, using the language of Lie algebra cohomology, a Lie algebra L is complete if and only if $H^0(L, L) = H^1(L, L) = 0$. Compare this to Remark 4.12 (there, the condition $H^2(L, L) = 0$ defines *cohomological rigidity*).

Two classes of complete Lie algebras are especially important (see e.g. [94]):

Definition 4.4. Let L be a Lie algebra. A maximal (with respect to inclusion) solvable subalgebra of L is called a *Borel subalgebra*. A subalgebra of L containing a Borel subalgebra is called a *parabolic subalgebra*.

Proposition 4.5. *If* L *is a semisimple Lie algebra, then its Borel and parabolic subalgebras are all complete.*

Proof. See [74, Theorem 4.7]. □

Concerning post-Lie algebra structures, the Lie algebra $\mathfrak{n}$ being complete allows us to view post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ (for any $\mathfrak{g}$) differently:

Proposition 4.6 ([28]). *Let* $x \cdot y$ *define a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *where* $\mathfrak{n}$ *is complete. Then there is a unique linear map* $\varphi \colon V \to V$ *with* $x \cdot y = \{\varphi(x), y\}$ *for all* $x, y \in V$ *(in other words:* $L(x) = \operatorname{ad}(\varphi(x))$*, where* ad *denotes the adjoint operator with respect to* $\{,\}$*).*

As the proof (in [28]) is rather short, we give it here. Note that the proof exactly uses the properties of $\mathfrak{n}$ having a trivial center and all of $\mathfrak{n}$'s derivations being inner:

Proof. Consider the homomorphism $\psi\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{n}), \psi(x) = \mathrm{ad}(x)$. As $\mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n})$, the map ψ is surjective, as $\mathrm{Z}(\mathfrak{n}) = 0$, $\ker\psi = \mathrm{Z}(\mathfrak{n}) = 0$, thus ψ is bijective.
So if $x \in \mathfrak{n}$, there is a unique element $\varphi(x) \in \mathfrak{n}$ with $L(x) = \mathrm{ad}(\varphi(x))$. So we defined the operator φ; we show that φ is linear: For $x, y', y \in \mathfrak{n}, \lambda \in \mathbb{C}$,

$$\begin{aligned}\{\varphi(x+\lambda x'), y\} &= (x+\lambda x')\cdot y\\ &= x\cdot y + \lambda x'\cdot y\\ &= \{\varphi(x)+\lambda\varphi(x'), y\}\end{aligned}$$

and since $\mathrm{Z}(\mathfrak{n}) = 0, \varphi(x+\lambda x') = \varphi(x) + \lambda\varphi(x')$. So φ is linear. □

Via this endomorphism $\varphi\colon V \to V$, the post-Lie algebra axioms may be translated (see also [28]):

Proposition 4.7. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras where $\mathfrak{n}$ is complete. Then, writing $x \cdot y = \{\varphi(x), y\}$, the post-Lie algebra axioms translate to the two identities*

$$\begin{aligned}\{\varphi(x), y\} + \{x, \varphi(y)\} &= [x, y] - \{x, y\}\\ \varphi([x, y]) &= \{\varphi(x), \varphi(y)\}.\end{aligned}$$

This makes it possible to study post-Lie algebra structures on complete Lie algebras via Rota-Baxter theory:

Definition 4.8. Let $(L, [,])$ be a Lie algebra over k and $\lambda \in k$. A linear operator $R\colon L \to L$ is called a *Rota-Baxter operator* on L of weight λ if

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y])$$

for all $x, y \in L$.

Now, from Proposition 4.7 it follows that φ is a Rota-Baxter operator on $\mathfrak{n}$ of weight 1.

Actually, also the other direction is true: For a complete Lie algebra $\mathfrak{n}$ and every Rota-Baxter operator on $\mathfrak{n}$ of weight 1, there is a Lie algebra $\mathfrak{g}$ and a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ (cf. [31]).

4.2 Classification of complete Lie algebras up to dimension 7

Let us start with a classification of complete (complex) Lie algebras in small dimensions.

More specifically, in this chapter, we shall classify all complete Lie algebras up to dimension 7 over $\mathbb{C}$. Up to dimension 7, complete Lie algebras always have the property to be *rigid* (see [37]), i.e. their Lie algebra law has open orbit (with respect to basis

change) in $L_n(\mathbb{C})$ (the variety of Lie algebras in dimension n over $\mathbb{C}$) with respect to the Zariski topology. Moreover, for solvable Lie algebras up to dimension 7, the properties "complete" and "rigid" turn out to be equivalent (see [37, Corollaire 2.1(3)]). Carles and Diakité also give classifications of rigid Lie algebras up to dimension 7 in [37] – for the solvable and the non-solvable case.
Therefore, for our classification problem, we can exploit this property – we shall adopt the tables in [37] and find out for each non-solvable rigid Lie algebra if they are complete or not.

We want to point out that Zhu and Meng ([103]) also classified the complete Lie algebras over $\mathbb{C}$ up to dimension 7 by completely different means. In Section 4.2.3 we compare our classification with the one of Zhu and Meng.

4.2.1 Rigid Lie algebras

To define "rigid", we will mainly follow [55]:

Let $\mathfrak{g}$ be an n-dimensional Lie algebra over $\mathbb{C}$. One can view $\mathfrak{g}$ as a pair $(\mathbb{C}^n, \mu)$, where $\mu\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie algebra law of $\mathfrak{g}$.
We denote by L^n the set of all Lie algebra laws on n-dimensional complex Lie algebras; L^n can be seen as a subset of the set of all alternating bilinear maps on $\mathbb{C}^n$. The linear algebraic group $\mathrm{GL}_n(\mathbb{C})$ now acts on L^n by

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C}) \times L^n &\to L^n, \\ (g, \mu) &\mapsto g * \mu, \end{aligned}$$

where $(g * \mu)(X, Y) := g^{-1}(\mu(g(X), g(Y)))$ for all $X, Y \in \mathbb{C}^n$.
The orbit $\mathcal{O}(\mu)$ of μ with respect to this action equals the set of the Lie algebra laws which are isomorphic to μ, i.e., the n-dimensional Lie algebra laws μ' such that $(\mathbb{C}^n, \mu) \cong (\mathbb{C}^n, \mu')$ as Lie algebras.

One can now fix a basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$. Given an n-dimensional Lie algebra law μ, one can identify μ with its *structure constants*, that is, the complex numbers C_{ij}^k such that

$$\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k$$

(which, of course, depend on the chosen basis). By the anticommutativity and the Jacobi identity, the structure constants satisfy

$$C_{ij}^k = -C_{ji}^k \text{ for } 1 \le i < j \le n, 1 \le k \le n$$

and

$$\sum_{m=1}^{n} (C_{ij}^m C_{mk}^\ell + C_{jk}^m C_{mi}^\ell + C_{ki}^m C_{mj}^\ell) = 0 \text{ for } 1 \le i < j < k \le n, 1 \le \ell \le n.$$

Now, we can identify the Lie algebra with law μ with its *structure vector* $(C_{ij}^k)_{i,j,k} \in \mathbb{C}^{n^3}$. In this way, L^n becomes an affine variety (as a subvariety of $\mathbb{C}^{n^3}$) as in the following definition:

Definition 4.9. Let k be an algebraically closed field. A subset $\mathbb{V} \subseteq k^m$ is called an *algebraic set* over k, if there is a set S of polynomials in $k[X_1, \dots, X_m]$ such that

$$\mathbb{V} = \{x \in k^m \colon f(x) = 0 \text{ for all } f \in S\}.$$

If $\mathbb{V}$ is *irreducible*, i.e. $\mathbb{V} \neq \emptyset$ and $\mathbb{V}$ is not the union of two non-empty algebraic sets, then $\mathbb{V}$ is called an *(algebraic) affine variety* over k.

On algebraic varieties, the natural topology is the Zariski topology:

Definition 4.10. Let $\mathbb{V}$ be an affine variety. Then one can define a topology on $\mathbb{V}$, called the *Zariski topology*, by calling zero loci of polynomials closed, that is,

$$V \subset \mathbb{V} \text{ is Zariski-closed} :\Longleftrightarrow V \text{ is an algebraic set.}$$

So via the correspondence $\{$Lie algebra laws in $\mathbb{C}^n\} \leftrightarrow \{$structure vectors in $\mathbb{C}^{n^3}\}$, we may transport the Zariski topology to L^n and finally, we can define *rigid* Lie algebras:

Definition 4.11. The Lie algebra $\mathfrak{g} = (\mathbb{C}^n, \mu)$ is said to be *rigid* if the orbit $\mathcal{O}(\mu)$ as discussed above is open in L^n with respect to the Zariski topology.

The orbit $\mathcal{O}(\mu)$ being open means that all Lie algebra laws near to μ (in the Zariski topology) are isomorphic to μ.

Remark 4.12. There is also a stronger condition than rigidity, namely cohomological rigidity. We say that a Lie algebra L is *cohomologically rigid* if its second cohomology group $H^2(L, L)$ vanishes. Every cohomologically rigid Lie algebra is rigid ([81]); however, there are examples of rigid Lie algebras with $H^2(L, L) \neq 0$ (see e.g. [84], where for every odd $n > 5$ a rigid Lie algebra L_n of dimension $2n + 4$ is given which satisfies $H^2(L_n, L_n) \neq 0$). Recall that completeness of L means that $H^0(L, L)$ and $H^1(L, L)$ vanish.

Semisimple Lie algebras (over a field of characteristic zero) are rigid. This follows from Whitehead's lemma ($H^2(L, L) = 0$ for semisimple Lie algebras L) and cohomological rigidness implying rigidness. However, up to dimension 7, even more is true – the structure theorem we will use often is the following:

Theorem 4.13. *Let $\mathfrak{g}$ be a Lie algebra of dimension ≤ 7 over $\mathbb{C}$. Then:*

(i) If $\mathfrak{g}$ is complete, $\mathfrak{g}$ is also rigid.

(ii) If $\mathfrak{g}$ is solvable, then $\mathfrak{g}$ is complete if and only if $\mathfrak{g}$ is rigid.

Proof. See [37]. ∎

Remark 4.14. If $\mu \in L^n$ is rigid, then $\overline{\mathcal{O}(\mu)}$, the closure of the orbit $\mathcal{O}(\mu)$, is an irreducible component of the variety L^n. Since there are only finitely many irreducible components, there are also only finitely many non-isomorphic rigid Lie algebras in dimension n.

So Theorem 4.13 suggests the following strategy to find the complete Lie algebras: For a given dimension ≤ 7, we shall look at lists of rigid Lie algebras: For solvable ones, we know by Theorem 4.13, (ii), to find the non-solvable rigid Lie algebras, we only have to consider the non-solvable rigid Lie algebras.

Two results on the structure of complete Lie algebras will be very helpful in the sequel: The first one is a decomposition of complete Lie algebras into *simply-complete* ideals, similar to a decomposition of semisimple Lie algebras into their simple ideals:

Definition 4.15. Let L be a complete Lie algebra. We say that L is *simply-complete*, if no non-trivial ideal of L is complete.

Proposition 4.16.

(i) Let $L_1, \dots, L_n$ be simply-complete Lie algebras. Then $L := L_1 \oplus \dots \oplus L_n$ is complete.

(ii) If L is a complete Lie algebra, then there exist simply-complete ideals $L_1, \dots, L_n$ with $L = L_1 \oplus \dots \oplus L_n$. Moreover, such a decomposition is unique (up to reordering of the ideals).

Proof. See [74, Theorem 3.7]. $\square$

Remark 4.17. If L is semisimple, Proposition 4.16 gives us back the decomposition of L into simple ideals.

Remark 4.18. One can also refine Proposition 4.5 to simple Lie algebras: If L is a simple Lie algebra, then its Borel and parabolic subalgebras are all *simply-complete* ([74, Theorem 4.7]).

The other important structure result is a decomposition of rigid Lie algebras into a semidirect sum of a maximal torus and the nilradical:

Definition 4.19 (See e.g. [85]). Let L be a Lie algebra. A maximal abelian subalgebra T of L consisting of semisimple elements is called a *maximal torus* of L. The dimension of a maximal torus is an invariant of L; we define $\operatorname{rank}(L) := \dim(T)$ and call it the *rank* of L.

Proposition 4.20.

(i) Every rigid Lie algebra is algebraic*, that is, the Lie algebra of a (linear) algebraic group.*

(ii) Let L be an algebraic Lie algebra. Then $\operatorname{rad}(L)$ has a decomposition $\operatorname{rad}(L) = T \ltimes \operatorname{nil}(L)$, where $\operatorname{nil}(L)$ is the nilradical of L and T is a maximal (non-zero) torus of $\operatorname{rad}(L)$.

Table 9: Rigid Lie algebras in dimension 2

Lie algebra	non-zero Lie brackets	solvable?	complete?
$\mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_1$	✓	✓

Proof. See [36, Proposition 1.5 and Proposition 4.1]. □

In particular, solvable rigid Lie algebras L have a decomposition $T \ltimes \mathrm{nil}(L)$ as a semidirect product of a maximal torus and their nilradical. We shall henceforth write solvable Lie algebras in this form – the terminology for the nilradical corresponds with those given in [70]. As Carles and Diakité do in [37], we denote the basis of $\mathfrak{g}$ by $\{t_1, \dots, t_i, e_i, \dots, e_j\}$, where $\{t_1, \dots, t_i\}$ forms a basis for the maximal torus T.

It will also be important later that solvable complete Lie algebras can be decomposed in the same way:

Proposition 4.21. *Let L be a solvable complete Lie algebra. Then L has a decomposition $L = T \ltimes \mathrm{nil}(L)$, where $\mathrm{nil}(L)$ is the nilradical of L and T is maximal (non-zero) torus of L.*

Proof. See [75, Theorem 1]. □

For Lie algebras of dimension ≤ 7, one also obtains Proposition 4.21 by combining Theorem 4.13 and Proposition 4.20.

4.2.2 The classification

Whenever the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ appears in the sequel, we denote a basis by $\{e_1, e_2, e_3\}$ with relations $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_1$ and $[e_2, e_3] = 2e_2$ (if we do not say otherwise).

Dimension 1 In dimension 1 there is only one Lie algebra, namely the abelian $\mathbb{C}$. It is neither rigid nor complete.

Proposition 4.22. *In dimension* 1 *there is neither a rigid nor a complete Lie algebra.*

Dimension 2 In dimension 2, there exist only two Lie algebras – the abelian algebra $\mathbb{C}^2$ and the Lie algebra $\mathfrak{r}_2(\mathbb{C})$ with $[e_1, e_2] = e_1$. It is straightforward to show that only the latter one is complete (and therefore also rigid).

Proposition 4.23. *In dimension* 2, *$\mathfrak{r}_2(\mathbb{C})$ is the only complete Lie algebra. It is also the only rigid Lie algebra in dimension* 2.

Table 10: Rigid Lie algebras in dimension 3

Lie algebra	non-zero Lie brackets	solvable?	complete?
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$	✗	✓

Table 11: Rigid Lie algebras in dimension 4

Lie algebra	non-zero Lie brackets	solvable?	complete?
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_1, [e_3, e_4] = e_3$	✓	✓
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$	✗	✗

Dimension 3 In dimension 3, one finds that exactly one is complete, namely $\mathfrak{sl}_2(\mathbb{C})$. Its completeness follows from the fact that (semi)simple Lie algebras always are complete. According to [37], it also is the only rigid Lie algebra in dimension 3.

Proposition 4.24. *In dimension* 3*,* $\mathfrak{sl}_2(\mathbb{C})$ *is the only complete Lie algebra. It is also the only rigid Lie algebra in dimension* 3*.*

Dimension 4 Using a classification of 4-dimensional Lie algebras as the one given in [35], one finds out that there is also only one complete Lie algebra: the algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$. To avoid showing directly that this Lie algebra is complete, we make use of Proposition 4.16 – as a direct sum of two complete Lie algebras, $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ is also complete.

[37] gives two rigid Lie algebras in dimension 4, namely $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$. (Since $\mathrm{Z}(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \cong \mathbb{C} \neq 0$, the latter one is not complete.)

Proposition 4.25. *In dimension* 4*, there are two rigid Lie algebras:* $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ *and* $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$*. The first one is complete, the latter one not.*

Dimension 5 Before we start with dimension 5, we make a remark about notation:

It is well-known that in each dimension n there is exactly one n-dimensional irreducible representation φ_n of $\mathfrak{sl}_2(\mathbb{C})$. We will write $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(n)$ for $\mathfrak{sl}_2(\mathbb{C}) \ltimes_{\varphi_n} \mathbb{C}^n$.

Lemma 4.26. *If* $\mathfrak{g}$ *is a Lie algebra of dimension* 5 *and* $\mathfrak{g}$ *is not solvable then there are exactly three possibilities: Either* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ *or* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$ *or* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$*.*

Proof. By Levi's Theorem, there exists a decomposition $\mathfrak{g} \cong \mathfrak{s} \ltimes \mathrm{rad}(\mathfrak{g})$ with $\mathfrak{s}$ semisimple. Since $\mathfrak{g}$ is assumed to be non-solvable, $\mathfrak{s}$ cannot be zero. But $\dim(\mathfrak{g}) = \dim(\mathfrak{s}) + \dim(\mathrm{rad}(\mathfrak{g}))$ and the only semisimple Lie algebra up to dimension 5 is $\mathfrak{sl}_2(\mathbb{C})$.
So $\mathrm{rad}(\mathfrak{g})$ has to be two-dimensional; so $\mathrm{rad}(\mathfrak{g})$ is either $\mathfrak{r}_2(\mathbb{C})$ or $\mathbb{C}^2$.
If $\mathrm{rad}(\mathfrak{g}) = \mathfrak{r}_2(\mathbb{C})$, we have (since $\mathfrak{r}_2(\mathbb{C})$ is complete) $\mathrm{Der}(\mathfrak{r}_2(\mathbb{C})) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, a, b \in \mathbb{C} \right\} \cong \mathfrak{r}_2(\mathbb{C})$.
Now, let $\varphi = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix}$ be any homomorphism $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{r}_2(\mathbb{C}) \cong \mathrm{Der}(\mathfrak{r}_2(\mathbb{C}))$.

Table 12: Rigid Lie algebras in dimension 5

Lie algebra	non-zero Lie brackets	solv.?	compl.?
$T_2 \ltimes \mathfrak{n}_3$	$[e_1,e_2]=e_3, [t_1,e_1]=e_1, [t_1,e_3]=e_3,$ $[t_2,e_2]=e_2, [t_2,e_3]=e_3$	✓	✓
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1,e_2]=e_3, [e_1,e_3]=-2e_1,$ $[e_2,e_3]=2e_2, [e_4,e_5]=e_5$	✗	✓
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$	$[e_1,e_2]=e_3, [e_1,e_3]=-2e_1, [e_1,e_5]=e_4,$ $[e_2,e_3]=2e_2, [e_2,e_4]=e_5,$ $[e_3,e_4]=e_4, [e_3,e_5]=-e_5$	✗	✗

Then

$$\begin{aligned}
\begin{pmatrix}\alpha_3\\ \alpha_6\end{pmatrix} &= \varphi([e_1,e_2]) = [\varphi(e_1),\varphi(e_2)] = \left[\begin{pmatrix}\alpha_1\\ \alpha_4\end{pmatrix},\begin{pmatrix}\alpha_2\\ \alpha_5\end{pmatrix}\right] = \begin{pmatrix}\alpha_1\alpha_5-\alpha_2\alpha_4\\ 0\end{pmatrix},\\
-2\begin{pmatrix}\alpha_1\\ \alpha_4\end{pmatrix} &= \varphi([e_1,e_3]) = [\varphi(e_1),\varphi(e_3)] = \left[\begin{pmatrix}\alpha_1\\ \alpha_4\end{pmatrix},\begin{pmatrix}\alpha_3\\ \alpha_6\end{pmatrix}\right] = \begin{pmatrix}\alpha_1\alpha_6-\alpha_3\alpha_4\\ 0\end{pmatrix},\\
2\begin{pmatrix}\alpha_2\\ \alpha_5\end{pmatrix} &= \varphi([e_2,e_3]) = [\varphi(e_2),\varphi(e_3)] = \left[\begin{pmatrix}\alpha_2\\ \alpha_5\end{pmatrix},\begin{pmatrix}\alpha_3\\ \alpha_6\end{pmatrix}\right] = \begin{pmatrix}\alpha_2\alpha_6-\alpha_3\alpha_5\\ 0\end{pmatrix}.
\end{aligned}$$

Comparing the second rows yields $\alpha_4=\alpha_5=\alpha_6=0$, comparing the first rows afterwards yields $\alpha_1=\alpha_2=\alpha_3=0$.
So any homomorphism $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathfrak{r}_2(\mathbb{C}))$ has to be identically 0, which corresponds to $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{r}_2(\mathbb{C})$.
On the other hand, let $\mathrm{rad}(\mathfrak{g})=\mathbb{C}^2$ and φ be a homomorphism $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathbb{C}^2)=\mathfrak{gl}_2(\mathbb{C})$. But then, φ is a representation of $\mathfrak{sl}_2(\mathbb{C})$. So we can assume that $\varphi=0$ or φ is the irreducible 2-dimensional representation. This proves the theorem. □

The algebra $\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}^2$ has non-trivial center (and, additionally, is not rigid). The other two non-solvable Lie algebras are rigid; while $\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{r}_2(\mathbb{C})$ is complete as a direct sum of two complete Lie algebras, $\mathfrak{sl}_2(\mathbb{C})\ltimes V(2)$ is not complete – $\mathrm{diag}(0,0,0,1,1)$ is an outer derivation.
So we have identified which non-solvable Lie algebras in dimension 5 are complete. Remember that every solvable rigid Lie algebra is complete – and there is one solvable rigid Lie algebra in [37], namely $T_2\ltimes\mathfrak{n}_3$ (with the Lie brackets indicated in Table 12). We therefore obtain the following proposition:

Proposition 4.27. *The three rigid and two complete Lie algebras in dimension 5 are the ones given in Table 12.*

Dimension 6

Lemma 4.28. *There are exactly three non-isomorphic Lie algebras of dimension 6 over $\mathbb{C}$ with non-trivial Levi decomposition (that is, Lie algebras $\mathfrak{s}\ltimes_\varphi\mathfrak{h}$ with $\mathfrak{s}$ semisimple, $\mathfrak{h}$*

solvable, $\mathfrak{s}, \mathfrak{h} \neq 0$) which are, additionally, not direct sums of ideals.
They are given by

- $L_{6,1} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$ *(i.e. $\mathfrak{sl}_2(\mathbb{C})$ acting on $\mathbb{C}^3$ via its irreducible 3-dimensional representation); that is, the explicit Lie brackets are given by* $[e_1, e_3] = -2e_1, [e_1, e_2] = e_3, [e_1, e_5] = 2e_4, [e_1, e_6] = e_5, [e_2, e_3] = 2e_2, [e_2, e_4] = e_5, [e_2, e_5] = 2e_6, [e_3, e_4] = 2e_4, [e_3, e_6] = -2e_6$*;*

- $L_{6,2} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\varphi \mathfrak{n}_3$*, where $\mathfrak{n}_3$ denotes the 3-dimensional Heisenberg algebra (with basis $\{e_4, e_5, e_6\}$, where $[e_4, e_5] = e_6$) over $\mathbb{C}$, subject to $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathfrak{n}_3)$,*

$$\varphi(ae_1 + be_2 + ce_3) = \begin{pmatrix} c & a & 0 \\ b & -c & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

- $L_{6,3} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{r}_{3,1}(\mathbb{C})$ *with Lie brackets* $[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_1, e_5] = e_4, [e_2, e_3] = 2e_2, [e_2, e_4] = e_5, [e_3, e_4] = e_4, [e_3, e_5] = -e_5, [e_4, e_6] = e_4, [e_5, e_6] = e_5$.

Proof. In [97], Turkowski lists all real Lie algebras of dimension $5, 6, 7$ and 8 with non-trivial Levi decomposition.
Taking into account that $\mathfrak{so}_3$ and $\mathfrak{sl}_2$ agree over $\mathbb{C}$ (and the other do not), one obtains the lemma. □

Remark 4.29. The explicit Lie brackets for $L_{6,2}$ can be found in Table 13 below.

Again, using Carles' and Diakité's classifications, one finds the solvable and non-solvable rigid Lie algebras and has to check completeness for the non-solvable ones.
Out of the Lie algebras with non-trivial Levi decomposition, $L_{6,2}$ is rigid; one also has to take the semisimple Lie algebras and direct sums into account:
The non-solvable rigid Lie algebras in dimension 6, are, according to Carles and Diakité, given by

- $L_{6,2}$ as stated above
- $L_{6,4} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$;
- $L_{6,5} = \mathfrak{aff}(\mathbb{C}^2) = \mathfrak{gl}_2(\mathbb{C}) \ltimes_\varphi \mathbb{C}^2$ with respect to $\varphi\colon \mathfrak{gl}_2(\mathbb{C}) \to \mathrm{Der}(\mathbb{C}^2) = \mathfrak{gl}_2(\mathbb{C}), \varphi = \mathrm{id}$.

The explicit Lie brackets can be found in Table 13 below.
In fact, two of the mentioned Lie algebras are isomorphic:

Lemma 4.30. *There exists an isomorphism between $L_{6,5} = \mathfrak{aff}(\mathbb{C}^2)$ and $L_{6,3} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{r}_{3,1}(\mathbb{C})$ (where $L_{6,3}$ is as described above).*

Proof. Let us denote the standard basis of $\mathfrak{gl}_2(\mathbb{C})$ by $\{a_1, a_2, a_3, a_4\}$,

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and write $a_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a_6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Then $L_{6,5} = \mathfrak{aff}(\mathbb{C}^2)$ has the following non-zero Lie brackets:

$$[a_1, a_2] = a_2, [a_1, a_3] = -a_3, [a_1, a_5] = a_5, [a_2, a_3] = a_1 - a_4,$$
$$[a_2, a_4] = a_2, [a_2, a_6] = a_5, [a_3, a_4] = -a_3, [a_3, a_5] = a_6, [a_4, a_6] = a_6.$$

Consider $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \ltimes_\varphi \mathbb{C}^2$ with basis $\{b_1, \ldots, b_6\}$. The set $\{b_1, b_2, b_3\}$ shall be the usual basis of $\mathfrak{sl}_2(\mathbb{C})$ and φ the map

$$\varphi\colon (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C}), \varphi(\alpha b_1 + \beta b_2 + \gamma b_3 + \delta b_4) = \begin{pmatrix} \delta + \gamma & \alpha \\ \beta & \delta - \gamma \end{pmatrix}.$$

That is, the non-zero Lie brackets in $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \ltimes_\varphi \mathbb{C}^2$ are given by

$$[b_1, b_2] = b_3, [b_1, b_3] = -2b_1, [b_1, b_6] = b_5, [b_2, b_3] = 2b_2,$$
$$[b_2, b_5] = b_6, [b_3, b_5] = b_5, [b_3, b_6] = -b_6, [b_4, b_5] = b_5, [b_4, b_6] = b_6.$$

One now can construct an isomorphism between $\mathfrak{aff}(\mathbb{C}^2) = \mathfrak{gl}_2(\mathbb{C}) \ltimes_{\mathrm{id}} \mathbb{C}^2$ and $(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}) \ltimes_\varphi \mathbb{C}^2$ by the correspondence $b_1 \leftrightarrow a_2, b_2 \leftrightarrow a_3, b_3 \leftrightarrow a_1 - a_4$, $b_4 \leftrightarrow a_1 + a_4, b_5 \leftrightarrow a_5, b_6 \leftrightarrow a_6$.
Then, by another transformation,

$$b_1 \leftrightarrow e_1, b_2 \leftrightarrow e_2, b_3 \leftrightarrow e_3, b_4 \leftrightarrow -e_6, b_5 \leftrightarrow e_4, b_6 \leftrightarrow e_5,$$

one obtains the Lie algebra $L_{6,3}$. □

Lemma 4.31. *The Lie algebras $L_{6,4}$ and $L_{6,5}$ are complete, $L_{6,2}$ is not complete.*

Proof. As a semisimple Lie algebra, $L_{6,4}$ is complete.
One can show that, with respect to the basis $\{a_1, \ldots, a_6\}$ as in the proof of Lemma 4.30, every derivation of $L_{6,5}$ is of the form

$$\begin{pmatrix} 0 & -c & b & 0 & 0 & 0 \\ -b & a & 0 & b & 0 & 0 \\ c & 0 & -a & -c & 0 & 0 \\ 0 & c & -b & 0 & 0 & 0 \\ e & f & 0 & 0 & a+d & b \\ 0 & 0 & e & f & c & d \end{pmatrix}, a, b, c, d, e, f \in \mathbb{C},$$

meaning that every derivation of $L_{6,5}$ is inner. Since also $\mathrm{Z}(L_{6,5}) = 0$ holds, $L_{6,5}$ is complete.
On the other hand, $L_{6,2}$ is not complete; denoting by $\{e_4, e_5, e_6\}$ the basis of its nilradical $\mathfrak{n}_3$ satisfying $[e_4, e_5] = e_6$, we get $e_6 \in \mathrm{Z}(L_{6,2})$, so $L_{6,2}$ is not complete because of having a nontrivial center. □

So we can list the six-dimensional rigid Lie algebras and state whether or not they are solvable/complete (see Proposition 4.32 and Table 13).

Proposition 4.32. *The six rigid and five complete Lie algebras in dimension 6 are given in Table 13.*

Table 13: Rigid Lie algebras in dimension 6

Lie algebra	non-zero Lie brackets	solv.?	compl.?
$T_2 \ltimes \mathfrak{g}_4$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [t_1, e_1] = e_1, [t_1, e_3] = e_3,$ $[t_1, e_4] = 2e_4, [t_2, e_2] = e_2, [t_2, e_3] = e_3, [t_2, e_4] = e_4$	✓	✓
$T_1 \ltimes \mathfrak{g}_{5,6}$	$[e_1, e_i] = e_{i+1}, 2 \leq i \leq 4, [e_2, e_3] = e_5,$ $[t_1, e_i] = ie_i, 1 \leq i \leq 5$	✓	✓
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_1, [e_3, e_4] = e_3, [e_5, e_6] = e_5$	✓	✓
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2,$ $[e_4, e_5] = e_6, [e_4, e_6] = -2e_4, [e_5, e_6] = 2e_5$	✗	✓
$\mathfrak{aff}(\mathbb{C}^2)$	$[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_1, e_5] = e_5,$ $[e_2, e_3] = e_1 - e_4, [e_2, e_4] = e_2, [e_2, e_6] = e_5,$ $[e_3, e_4] = -e_3, [e_3, e_5] = e_6, [e_4, e_6] = e_6$	✗	✓
$L_{6,2} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_1, e_5] = e_4, [e_2, e_3] = 2e_2,$ $[e_2, e_4] = e_5, [e_3, e_4] = e_4, [e_3, e_5] = -e_5, [e_4, e_5] = e_6$	✗	✗

Dimension 7

Lemma 4.33. *There are exactly five non-isomorphic families of Lie algebras in dimension 7 over $\mathbb{C}$ with non-trivial Levi decomposition.*
They are given by

- $L_{7,1} = \mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2) \times V(2)) = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\varphi \mathbb{C}^4$, *where* $\varphi(e_1) = \begin{pmatrix} \psi(e_1) & 0 \\ 0 & \psi(e_1) \end{pmatrix}, \varphi(e_2) = \begin{pmatrix} \psi(e_2) & 0 \\ 0 & \psi(e_2) \end{pmatrix}, \varphi(e_3) = \begin{pmatrix} \psi(e_3) & 0 \\ 0 & \psi(e_3) \end{pmatrix}$ *(4×4-block matrices) with ψ the 2-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ (i.e.,* $\psi(e_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \psi(e_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \psi(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$*);*
- $L_{7,2} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(4)$*;*
- *the family* $L^p_{7,3}$ *given by* $[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_1, e_5] = e_4, [e_2, e_3] = 2e_2, [e_2, e_4] = e_5, [e_3, e_5] = -e_5, [e_4, e_7] = e_4, [e_5, e_7] = e_5, [e_6, e_7] = pe_6$ *with* $p \in \mathbb{C}, p \neq 0$*;*
- $L_{7,4} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\varphi \mathfrak{g}_6$ *(here, $\mathfrak{g}_6$ denotes the 4-dimensional Lie algebra with basis* $\{e_4, e_5, e_6, e_7\}$ *and relations* $[e_4, e_5] = e_5, [e_4, e_6] = e_6, [e_4, e_7] = e_7$*) with*
$$\varphi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathfrak{g}_6), \varphi(ae_1 + be_2 + ce_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2c & 2a & 0 \\ 0 & b & 0 & a \\ 0 & 0 & 2b & -2c \end{pmatrix};$$
- $L_{7,5} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\varphi \mathfrak{g}_8$ *(here, $\mathfrak{g}_8$ denotes the 4-dimensional Lie algebra with basis* $\{e_4, e_5, e_6, e_7\}$ *and relations* $[e_4, e_5] = e_5, [e_4, e_6] = e_6, [e_4, e_7] = 2e_7, [e_5, e_6] = e_7$*)*

with

$$\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Der}(\mathfrak{g}_8), \varphi(ae_1+be_2+ce_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c & a & 0 \\ 0 & b & -b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 4.34. The explicit Lie brackets for $L_{7,1}, L_{7,2}, L_{7,4}$ and $L_{7,5}$ can be found in Table 14 below.

Proof. As in dimension 6. □

To classify the complete Lie algebras in dimension 7, we again only have to check the non-solvable rigid Lie algebras of Carles' and Diakité's list. They are given by

- $L_{7,1}$;
- $L_{7,2}$;
- $L_{7,4}$;
- $L_{7,5}$;
- $L_{7,6} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$;
- $L_{7,7} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$.

As before, the explicit Lie brackets can be found in Table 14.

Lemma 4.35. *The Lie algebras $L_{7,4}, L_{7,5}, L_{7,6}$ are complete, $L_{7,1}, L_{7,2}, L_{7,7}$ are not.*

Proof. The Lie algebra $L_{7,6}$ is complete as a direct sum of complete Lie algebras. One can explicitly compute the derivation algebras and finds

$$\mathrm{Der}(L_{7,4}) = \left\{ \begin{pmatrix} 2c & 0 & -2a & 0 & 0 & 0 & 0 \\ 0 & -2c & 2b & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2f & 0 & -2e & -e & 2c+d & 2a & 0 \\ -g & -e & 0 & -f & b & d & a \\ 0 & -2f & 2g & -g & 0 & 2b & d-2c \end{pmatrix} : a,b,c,d,e,f,g \in \mathbb{C} \right\},$$

$$\mathrm{Der}(L_{7,5}) = \left\{ \begin{pmatrix} 2c & 0 & -2a & 0 & 0 & 0 & 0 \\ 0 & -2c & 2b & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f & 0 & -e & -e & c+d & a & 0 \\ 0 & -e & f & -f & b & d-c & 0 \\ 0 & 0 & 0 & -2g & -f & e & 2d \end{pmatrix} : a,b,c,d,e,f,g \in \mathbb{C} \right\},$$

meaning that all derivations are inner. In addition, both $L_{7,4}$ and $L_{7,5}$ have trivial center. This shows that both are complete.
On the other hand, $\mathrm{diag}(0,0,0,1,1,1,1)$ is an outer derivation of both $L_{7,1}$ and $L_{7,2}$. And $L_{7,7}$ has nontrivial center, i.e., $L_{7,1}, L_{7,2}$ and $L_{7,7}$ are not complete. □

If one also takes the solvable Lie algebras into account, one obtains Proposition 4.36:

Proposition 4.36. *The fourteen rigid and eleven complete Lie algebras in dimension 7 are the ones given in Table 14.*

4.2.3 Comparison with Zhu's and Meng's list

In [103], Zhu and Meng also classify the complex complete Lie algebras up to dimension 7 by different means. (They additionally classify all solvable Lie algebras with nilradical of dimension ≤ 6.)

Concerning non-solvable complete Lie algebras, our lists only differ by a minor issue: Zhu's and Meng's Table 6 (in which they list the non-solvable complete Lie algebras) lacks the Lie algebra which they call L_7^2 – however, they construct it later in the proof of their Theorem 4.5. There also is one minor issue in this Lie algebra L_7^2 – one Lie bracket is incorrect and has to read $[h, e_3] = -2e_3$ instead of $[h, e_3] = 2e_3$ (otherwise, L_7^2 would not be a Lie algebra due to not satisfying the Jacobi identity).
With this correction, one sees quickly that their Lie algebras $\mathfrak{sl}_2, \mathfrak{sl}_2 \oplus L_2, L_6^1, L_7^1, L_7^2, L_7^3$ correspond to "our" $\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), L_{7,6}, L_{7,4}, L_{7,5}$, respectively – since their Lie algebra L_6^2 satisfies $\dim[L_6^2, L_6^2] = 5$, it equals "our" $\mathfrak{aff}(\mathbb{C}^2)$ (as, e.g. by checking Turkowski's list in [97], there is (up to isomorphism) only one six-dimensional Lie algebra with five-dimensional commutator algebra).

Concerning solvable complete Lie algebras, all of the Lie algebras we listed but one appear in [103] – in Table 15, we compare our and their notation.
"Our" seven-dimensional complete Lie algebra $T_1 \ltimes \mathcal{G}_{6,12}$ does not appear in Zhu's and Meng's list – in their notation, it would be $G_6^1 = T_1 \ltimes \mathcal{G}_{6,12}$. Zhu and Meng prove in [103, Proposition 3.8] that $T_1 \ltimes \mathcal{G}_{6,12}$ is not complete by constructing an outer derivation ϕ for this Lie algebra – however, this ϕ does not seem to be a derivation of $T_1 \ltimes \mathcal{G}_{6,12}$ since it does not satisfy $\phi([e_1, e_5]) = [\phi(e_1), e_5] + [e_1, \phi(e_5)]$ (with respect to Zhu's and Meng's basis).
On the other hand, all solvable complete Lie algebras up to dimension 7 that Zhu and Meng list (that is, in Tables 4 and 5 and Proposition 3.10) also can be found in our Tables 9 – 14.
Moreover, we want to point out that the (eight-dimensional) Lie algebra listed in Zhu's and Meng's Table 5 by the name of G_5^1 seems to be erroneous as it does not satisfy the Jacobi identity for e_3, e_4, t_3 – if one takes the matrix $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ instead of

Table 14: Rigid Lie algebras in dimension 7

Lie algebra	non-zero Lie brackets	s.?	c.?
$(T_2 \ltimes \mathfrak{n}_3) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1,e_2]=e_3,[t_1,e_1]=e_1,[t_1,e_3]=e_3,$ $[t_2,e_2]=e_2,[t_2,e_3]=e_3,[e_6,e_7]=e_7$	✓	✓
$T_2 \ltimes \mathfrak{g}_{5,3}$	$[e_1,e_2]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5,[t_1,e_1]=e_1,$ $[t_1,e_3]=2e_3,[t_1,e_4]=e_4,[t_1,e_5]=2e_5,$ $[t_2,e_2]=e_2,[t_2,e_4]=e_4,[t_2,e_5]=e_5$	✓	✓
$T_2 \ltimes \mathfrak{g}_{5,4}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_2,e_3]=e_5,[t_1,e_1]=e_1,$ $[t_1,e_3]=e_3,[t_1,e_4]=2e_4,[t_1,e_5]=e_5,[t_2,e_2]=e_2,$ $[t_2,e_3]=e_3,[t_2,e_4]=e_4,[t_2,e_5]=2e_5$	✓	✓
$T_2 \ltimes \mathfrak{g}_{5,5}$	$[e_1,e_i]=e_{i+1},2\le i\le 4,[t_1,e_1]=e_1,[t_1,e_3]=e_3,$ $[t_1,e_4]=2e_4,[t_1,e_5]=3e_5,[t_2,e_2]=e_2,$ $[t_2,e_3]=e_3,[t_2,e_4]=e_4,[t_2,e_5]=e_5$	✓	✓
$T_1 \ltimes \mathcal{G}_{6,12}$	$[e_1,e_5]=[e_2,e_3]=[e_2,e_4]=e_6,[e_1,e_2]=e_4,$ $[e_1,e_4]=e_5,[t_1,e_1]=e_1,[t_1,e_2]=2e_2,$ $[t_1,e_3]=3e_3,[t_1,e_4]=3e_4,[t_1,e_5]=4e_5,$ $[t_1,e_6]=5e_6$	✓	✓
$T_1 \ltimes \mathcal{G}_{6,17}$	$[e_1,e_i]=e_{i+1},2\le i\le 5,[e_2,e_3]=e_6,[t_1,e_1]=e_1,$ $[t_1,e_i]=(i+1)e_i,2\le i\le 6$	✓	✓
$T_1 \ltimes \mathcal{G}_{6,19}$	$[e_1,e_i]=e_{i+1},2\le i\le 5,[e_2,e_3]=e_5,[e_2,e_4]=e_6,$ $[t_1,e_i]=ie_i,1\le i\le 6$	✓	✓
$T_1 \ltimes \mathcal{G}_{6,20}$	$[e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=[e_2,e_3]=e_5,$ $[e_2,e_5]=[e_4,e_3]=e_6,[t_1,e_i]=ie_i,1\le i\le 5,$ $[t_1,e_6]=7e_6$	✓	✓
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{g}_6$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_1,e_6]=2e_5,$ $[e_1,e_7]=e_6,[e_2,e_3]=2e_2,[e_2,e_5]=e_6,$ $[e_2,e_6]=2e_7,[e_3,e_5]=2e_5,[e_3,e_7]=-2e_7,$ $[e_4,e_5]=e_5,[e_4,e_6]=e_6,[e_4,e_7]=e_7$	✗	✓
$\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{g}_8$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_1,e_6]=e_5,$ $[e_2,e_3]=2e_2,[e_2,e_5]=e_6,[e_3,e_5]=e_5,$ $[e_3,e_6]=-e_6,[e_4,e_5]=e_5,[e_4,e_6]=e_6,$ $[e_4,e_7]=2e_7,[e_5,e_6]=e_7$	✗	✓
$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{r}_2(\mathbb{C}))^2$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_2,e_3]=2e_2,$ $[e_4,e_5]=e_4,[e_6,e_7]=e_7$	✗	✓
$\mathfrak{sl}_2(\mathbb{C}) \ltimes (V(2))^2$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_1,e_5]=e_4,$ $[e_1,e_7]=e_6,[e_2,e_3]=2e_2,[e_2,e_4]=e_5,$ $[e_2,e_6]=e_7,[e_3,e_4]=e_4,[e_3,e_5]=-e_5,$ $[e_3,e_6]=e_6,[e_3,e_7]=-e_7$	✗	✗
$\mathfrak{sl}_2(\mathbb{C}) \ltimes V(4)$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_1,e_5]=e_4,$ $[e_1,e_6]=2e_5,[e_1,e_7]=3e_6,[e_2,e_3]=2e_2,$ $[e_2,e_4]=3e_5,[e_2,e_5]=2e_6,[e_2,e_6]=e_7,$ $[e_3,e_4]=3e_4,[e_3,e_5]=e_5,$ $[e_3,e_6]=-e_6,[e_3,e_7]=-3e_7$	✗	✗
$(\mathfrak{sl}_2(\mathbb{C})(\mathbb{C}))^2 \oplus \mathbb{C}$	$[e_1,e_2]=e_3,[e_1,e_3]=-2e_1,[e_2,e_3]=2e_2,$ $[e_4,e_5]=e_6,[e_4,e_6]=-2e_4,[e_5,e_6]=2e_5$	✗	✗

Table 15: Complete Lie algebras – notations here and in [103]

dimension	notation here	notation in [103]
2	$\mathfrak{r}_2(\mathbb{C})$	G_1
3	$\mathfrak{sl}_2(\mathbb{C})$	$\mathfrak{sl}_2$
4	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$(G_1)^2$
5	$T_2 \ltimes \mathfrak{n}_3$	G_3
5	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	G_3
6	$T_2 \ltimes \mathfrak{n}_4$	G_4^1
6	$T_1 \ltimes \mathfrak{n}_{5,6}$	G_5^6
6	$\mathfrak{r}_2(\mathbb{C})^3$	$(G_1)^3$
6	$(\mathfrak{sl}_2(\mathbb{C}))^2$	L_6^1
6	$\mathfrak{aff}(\mathbb{C}^2)$	L_6^2
7	$(T_2 \ltimes \mathfrak{n}_3) \oplus \mathfrak{r}_2(\mathbb{C})$	$G_3 \oplus G_1$
7	$T_2 \ltimes \mathfrak{n}_{5,3}$	G_5^3
7	$T_2 \ltimes \mathfrak{n}_{5,4}$	G_5^4
7	$T_2 \ltimes \mathfrak{n}_{5,5}$	G_5^7
7	$T_1 \ltimes \mathfrak{n}_{6,12}$	missing
7	$T_1 \ltimes \mathfrak{n}_{6,17}$	G_6^7
7	$T_1 \ltimes \mathfrak{n}_{6,19}$	G_6^5
7	$T_1 \ltimes \mathfrak{n}_{6,20}$	G_6^6
7	$L_{7,3}$	L_7^2
7	$L_{7,4}$	L_7^3
7	$\mathfrak{sl}_2(\mathbb{C}) \oplus (\mathfrak{r}_2(\mathbb{C}))^2$	L_7^1

the matrix $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, one indeed obtains a complete Lie algebra. Another 8-dimensional Lie algebra there given as G_6^8 and its nilradical $\mathfrak{g}_6^8$ do also not satisfy the Jacobi identity and thus are no Lie algebras.

4.3 Post-Lie algebra structures related by a scalar for complete Lie algebras

In Section 3.3, we studied the case of post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$'s Lie bracket is a scalar (non-zero) multiple of $\mathfrak{n}$'s Lie bracket, $[x, y] = a\{x, y\}$ for all $x, y \in V$ for an $a \in \mathbb{C}^*$. We found that if $\mathfrak{n}$ is not solvable, then $a \in \{1, -1\}$ (Corollary 3.47).
For the case that $\mathfrak{n}$ is *complete*, we will find the same result. This is the goal of this section.
Because we know that the scalar a must be 1 or -1 for non-solvable Lie algebras by Corollary 3.47, we will assume that our complete Lie algebra $\mathfrak{n}$ is *solvable*.

Recall that every solvable complete Lie algebra L has the form $L = T \ltimes \mathfrak{m}$ with nilradical $\mathfrak{m}$ and T a maximal torus of semisimple derivations of $\mathfrak{m}$ (Proposition 4.21). For the nilradical $\mathfrak{m}$, the lower central series is given by

$$\mathfrak{m} = \mathfrak{m}^0 \supseteq \mathfrak{m}^1 \supseteq \ldots \supseteq \mathfrak{m}^i = 0,$$

where $\mathfrak{m}^0 = \mathfrak{m}$ and $\mathfrak{m}^j = [\mathfrak{m}, \mathfrak{m}^{j-1}]$. Let $\mathfrak{s}^0$ be the vector space $\mathfrak{m}/\mathfrak{m}^1$ (which we occasionally identify with its image in $\mathfrak{m}$ under the canonical map $\mathfrak{m}/\mathfrak{m}^1 \to \mathfrak{m}$).
We can find a (vector space) basis X of $\mathfrak{m}$ consisting of eigenvectors of the $t_i \in T$, where $\{t_1, \ldots, t_n\}$ is a basis of T. This means that for every basis element $x \in X$ and every $t_i \in T$, we have $[t_i, x] = \alpha_{t_i,x} x$ for some $\alpha_{t_i,x} \in \mathbb{C}$.
Now, consider $Y := X \backslash \mathfrak{m}^1$ (which generates $\mathfrak{s}^0$ as a vector space). Let $B \subseteq Y$ be a (vector space) basis of $\mathfrak{s}^0$. This (vector space) basis of $\mathfrak{s}^0$ still consists of eigenvectors of T. We call B a *minimal system of generators* of L, as $B \cup \{t_1, \ldots, t_n\}$ generates L as a Lie algebra.
By [85, Lemma 2.8], it always holds $\dim(T) \leq \dim(\mathfrak{m}/\mathfrak{m}^1)$.

One question on complete Lie algebras which, surprisingly, is open, is whether or not the *nilradical* and the *nilpotent radical* coincide (cf. [33, Remark 4.11]). More precisely, given a Lie algebra L, we can define, as always, the *nilradical* $\mathrm{nil}(L)$ as the maximal (with respect to inclusion) nilpotent ideal of L and the *nilpotent radical* $\mathrm{n}(L)$ as the intersection of all kernels of finite-dimensional irreducible representations of L. It is known that (see e.g. Bourbaki [17, §5])

$$\mathrm{n}(L) = [L, L] \cap \mathrm{rad}(L) = [L, \mathrm{rad}(L)] \subseteq \mathrm{nil}(L).$$

However, what is unknown is the answer to the question if $\mathrm{n}(L) = \mathrm{nil}(L)$ holds for complete Lie algebras L. (For an arbitrary Lie algebra, the statement is not true.)
[41, Proposition 1, (iii)] says that $\mathrm{n}(L) = \mathrm{nil}(L)$ for complete Lie algebras – however, for the proof, the reader is referred to their reference no. 4, where we could not find it.

Remark 4.37. For parabolic subalgebras of (complex) semisimple Lie algebras (those are complete, cf. Proposition 4.5), it is known that the nilradical and the nilpotent radical coincide (see [9]).

Like it is also done in [33], we will impose in the section the assumption that the Lie algebra $\mathfrak{n}$ (or, equivalently, $\mathfrak{g}$, since $[x, y] = a\{x, y\}$) satisfies $\mathrm{n}(\mathfrak{n}) = \mathrm{nil}(\mathfrak{n})$ (which may or may not be true automatically). Note that if $\mathfrak{n}$ is solvable, $\mathrm{n}(\mathfrak{n}) = \mathrm{nil}(\mathfrak{n})$ is equivalent to $\{\mathfrak{n}, \mathfrak{n}\} = \mathrm{nil}(\mathfrak{n})$.

If $\mathfrak{n}$ is complete, each post-Lie algebra structure is given by $x \cdot y = \{\varphi(x), y\}$, where $\varphi \colon \mathfrak{n} \to \mathfrak{n}$ is a linear map (see Proposition 4.6) and the axioms in Proposition 4.7 hold.

Lemma 4.38. *Let L be a solvable complete Lie algebra with $[L, L] = \mathrm{nil}(L) =: \mathfrak{m}$. Then there are elements $x_1, x_2 \in \mathfrak{m}/\mathfrak{m}^1$ with $[x_1, x_2] \neq 0$ if and only if L is not metabelian.*

Proof. We have:

$$\begin{aligned} L \text{ is metabelian } &\Leftrightarrow [[L,L],[L,L]] = 0 \\ &\Leftrightarrow [\mathfrak{m},\mathfrak{m}] = 0 \\ &\Leftrightarrow \mathfrak{m}^1 = 0 \\ &\Leftrightarrow \text{ there are no } x_1, x_2 \in \mathfrak{m}/\mathfrak{m}^1 \text{ with } [x_1,x_2] \neq 0. \end{aligned}$$

□

Given a solvable complete Lie algebra L, we can find a "good" minimal system of generators:

Lemma 4.39. *Let L be a solvable complete Lie algebra with nilradical $\mathfrak{m}$ and maximal torus T. Let $\dim(T) = n$ and $\dim(\mathfrak{m}/\mathfrak{m}^1) = m$. Then there are a basis $\{t_1,\dots,t_n\}$ of T and n linearly independent elements $x_1,\dots,x_n$ in $\mathfrak{m}/\mathfrak{m}^1$ such that $[t_i,x_j] = \delta_{ij}x_j$ (where δ_{ij} denotes the Kronecker delta). (In particular, $\{x_1,\dots,x_n\}$ is a minimal system of generators consisting of eigenvectors of T.)*

Proof. Let $\{t'_1,\dots,t'_n\}$ be any basis of T and $\{y_1,\dots,y_m\}$ a basis of the vector space $\mathfrak{m}/\mathfrak{m}^1$ such that $[t'_i,y_j] = \alpha_i^j y_j$.
Consider the $n \times m$-matrix

$$A = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_1^m \\ \vdots & \ddots & \vdots \\ \alpha_n^1 & \dots & \alpha_n^m \end{pmatrix};$$

we show that A has rank n: Suppose $\sum\limits_{i=1}^n c_i(\alpha_i^1,\dots,\alpha_i^m) = 0$ for some scalars $c_i \in \mathbb{C}$. But this means for any element $g \in \mathfrak{m}/\mathfrak{m}^1, g = a_1y_1 + \ldots + a_my_m + \mathfrak{m}^1$ and $t := \sum\limits_{i=1}^n c_it'_i$, that,

$$[t,g] = \sum_{i=1}^n c_i(a_1\alpha_i^1 y_1 + \ldots + a_m\alpha_i^m y_m + \mathfrak{m}^1) = 0.$$

Thus $[t,\mathfrak{m}/\mathfrak{m}^1] = 0$ and consequently, as $\mathfrak{m}/\mathfrak{m}^1$ (more precisely: its image under the embedding $\mathfrak{m}/\mathfrak{m}^1 \to \mathfrak{m}$) generates the nilradical $\mathfrak{m}$ as a Lie algebra, $[t,\mathfrak{m}] = 0$. Since T is abelian, we also have $[t,T] = 0$ and conclude $t \in \mathrm{Z}(L)$. But $\mathrm{Z}(L) = 0$, meaning the c_i are all 0 and so the rows of A are linearly independent.

So A has rank n and thus its reduced row echelon form has the form $\begin{pmatrix} E_n & B \end{pmatrix}$ (after a permutation of the columns). But taking linear combinations of the rows just means to take linear combinations of the t'_i – permuting columns means a renumbering of the x_j. Thus we end up with a basis $\{t_1,\dots,t_n\}$ of T and n linearly independent elements $x_1,\dots,x_n$ such that $[t_i,x_j] = \delta_{ij}x_j$. □

After this preparation, let us study post-Lie algebra structures with $[x, y] = a\{x, y\}$ for a scalar $a \neq 0$ on complete Lie algebras. We begin with non-metabelian Lie algebras:

Theorem 4.40. *Let $\mathfrak{n}$ be a complete non-metabelian Lie algebra satisfying $\{\mathfrak{n}, \mathrm{rad}(\mathfrak{n})\} = \mathrm{nil}(\mathfrak{n}){=:}\mathfrak{m}$. Then every post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $[x, y] = a\{x, y\}, a \in \mathbb{C}\backslash\{0\}$ satisfies $a \in \{\pm 1\}$.*

Proof. By Corollary 3.47, we may assume that $\mathfrak{n}$ is solvable, thus it has the form $\mathfrak{n} = T \ltimes \mathfrak{m}$ with maximal torus T and nilradical $\mathfrak{m}$.

Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $[x, y] = a\{x, y\}$ and

$$\mathfrak{m} = \mathfrak{m}^0 \supseteq \mathfrak{m}^1 \supseteq \ldots \supseteq \mathfrak{m}^{n-1} \supseteq \mathfrak{m}^n = 0$$

the lower central series of the nilradical $\mathfrak{m}$. We define the vector spaces $\mathfrak{s}^0, \mathfrak{s}^1, \ldots, \mathfrak{s}^{n-1}$ as before by

$$\mathfrak{s}^j := \mathfrak{m}^j / \mathfrak{m}^{j+1}$$

and note that $\mathfrak{m} = \bigoplus_{i=0}^{n-1} \mathfrak{s}^i$ (if we identify each $\mathfrak{s}^i$ with its image in $\mathfrak{m}$).
Let $B = \{b_1, \ldots, b_m\}$ be a basis for $\mathfrak{s}^0$ as in Lemma 4.39, that is, if $\{t_1, \ldots, t_n\}$ is a basis of T (which then satisfies $n \leq m$), then $\{t_i, b_j\} = \delta_{ij} b_j$ for $1 \leq j \leq n$.
Since $\mathfrak{n}$ is complete, there is a unique endomorphism $\varphi\colon \mathfrak{n} \to \mathfrak{n}$ with

$$x \cdot y = \{\varphi(x), y\}$$

for all $x, y \in \mathfrak{n}$ (cf. Proposition 4.6). Moreover, φ satisfies the two conditions

$$\begin{aligned} \{\varphi(x), y\} + \{x, \varphi(y)\} &= [x, y] - \{x, y\} = (a-1)\{x, y\}, \\ a\varphi(\{x, y\}) &= \varphi([x, y]) = \{\varphi(x), \varphi(y)\} \end{aligned}$$

for all $x, y \in \mathfrak{n}$ by Proposition 4.7.
Note that if $y \in \mathfrak{s}^0\backslash\{0\}$, then there exists a $t \in T$ with $\{t, y\} \neq 0$ (otherwise $\{\mathfrak{n}, \mathrm{rad}(\mathfrak{n})\} \neq \mathrm{nil}(\mathfrak{n})$, as $y \in \mathfrak{s}^0 \subseteq \mathfrak{m} = \mathrm{nil}(\mathfrak{n})$, but $y \notin \{\mathfrak{n}, \mathrm{rad}(\mathfrak{n})\}$).

First step: The endomorphism φ maps $\mathfrak{m}$ into itself: It is enough to show that $\varphi(b) \in \mathfrak{m}$ for $b \in B$. Choose, for given $b \in B$, a $t \in T$ with $\{t, b\} = \lambda b \neq 0$. Then

$$\varphi(b) = \frac{a\lambda}{a\lambda}\varphi(b) = \frac{a}{a\lambda}\varphi(\{t, b\}) = \frac{1}{a\lambda}\{\varphi(t), \varphi(b)\} \in \{\mathfrak{n}, \mathfrak{n}\} \subseteq \mathfrak{m}.$$

Second step: The endomorphism φ maps $b \in B$ to the vector space $\mathrm{span}(b) + \mathfrak{m}^1$: Let $b \in B$ and $\{b, b_1, \ldots, b_{m-1}\} = B$.
We write $\varphi(b) = \alpha b + \alpha_1 b_1 + \ldots + \alpha_{m-1} b_{m-1} + \mathfrak{m}^1$ (by the first step) and show that

$\alpha_1 = \ldots = \alpha_{m-1} = 0$. Suppose the α_i are not all zero, then we can choose a $t \in T$ with $\{t, \sum_{i=1}^{m-1} \alpha_i b_i\} \neq 0$. Writing $\varphi = \psi + \varphi'$ with $\psi(t) \in \mathfrak{m}, \varphi'(t) \in T$, we get

$$\underbrace{(a-1)\{t,b\}}_{\in \operatorname{span}(b)} = \{\varphi(t), b\} + \{t, \varphi(b)\} = \underbrace{\{\psi(t), b\}}_{\in \mathfrak{m}^1} + \underbrace{\{\varphi'(t), b\}}_{\in \operatorname{span}(b)} + \{t, \varphi(b)\}$$

and so $\{t, \varphi(b)\} \in \operatorname{span}(b) + \mathfrak{m}^1$.
But

$$\begin{aligned}
\underbrace{\{t, \varphi(b)\}}_{\in \operatorname{span}(b)+\mathfrak{m}^1} &= \{t, \alpha b + \alpha_1 b_1 + \ldots + \alpha_{m-1} b_{m-1} + \mathfrak{m}^1\} \\
&= \underbrace{\{t, \alpha b\}}_{\in \operatorname{span}(b)} + \underbrace{\{t, \alpha_1 b_1 + \ldots + \alpha_{m-1} b_{m-1}\}}_{\in \mathfrak{s}^0 / \operatorname{span}(b)} + \underbrace{\{t, \mathfrak{m}^1\}}_{\in \mathfrak{m}^1}
\end{aligned}$$

and so $\{t, \alpha_1 b_1 + \ldots + \alpha_{m-1} b_{m-1}\} = 0$, a contradiction.

Third step: Every basis element $t_i \in T$ satisfies $\varphi(t_i) = \alpha t_i + \mathfrak{m}$ for some $\alpha \in \mathbb{C}$:
Let $\varphi(t_i) = \alpha_1 t_1 + \ldots + \alpha_n t_n + \mathfrak{m}$ – we show that $\alpha_j = 0$ for $j \neq i$. Let $j \neq i$ and choose a $b \in B$ with $\{t_j, b\} = b \neq 0$ and $\{t_k, b\} = 0$ for $k \neq j$ (which exists by Lemma 4.39). By the second step, $\varphi(b) = \beta b + \mathfrak{m}^1$ for a $\beta \in \mathbb{C}$. Thus we have

$$\begin{aligned}
\underbrace{(a-1)\{t_i, b\}}_{=0} &= \{\varphi(t_i), b\} + \{t_i, \varphi(b)\} \\
&\in \{\alpha_1 t_1 + \ldots + \alpha_n t_n, b\} + \underbrace{\{\mathfrak{m}, b\}}_{\subseteq \mathfrak{m}^1} + \underbrace{\{t_i, \beta b\}}_{=0} + \underbrace{\{t_i, \mathfrak{m}^1\}}_{\subseteq \mathfrak{m}^1} \\
&\subseteq \alpha_j b + \mathfrak{m}^1
\end{aligned}$$

and thus, as $b \notin \mathfrak{m}^1$, we get $\alpha_j = 0$.

Fourth step: For $b \in B, \varphi(b) = \beta b + \mathfrak{m}^1$ (which we can write by the second step), we either have $\beta = 0$ or $\beta = -1$.
To show this, choose a basis element $t \in T$ satisfying $\{t, b\} = \tau b \neq 0$; let $\varphi(t) = \mathfrak{m} + \lambda t$. From $a\varphi(\{t, b\}) = \{\varphi(t), \varphi(b)\}$ we infer

$$\begin{aligned}
a\tau\varphi(b) &\in \{\lambda t + \mathfrak{m}, \beta b + \mathfrak{m}^1\} \\
&\subseteq \lambda\beta\{t, b\} + \{\lambda t, \mathfrak{m}^1\} + \{\mathfrak{m}, \beta b\} + \{\mathfrak{m}, \mathfrak{m}^1\} \\
&\subseteq \lambda\beta\tau b + \mathfrak{m}^1 + \mathfrak{m}^1 + \mathfrak{m}^2.
\end{aligned}$$

Since $a\tau\varphi(b) = a\tau\beta b + a\tau\mathfrak{m}^1$ (second step), we get

$$a\tau\beta = \lambda\tau\beta$$

and so ($\tau \neq 0$) either $\beta = 0$ or $\lambda = a$.
If $\lambda = a$, by the properties of φ,

$$\begin{aligned}(a-1)\tau b = (a-1)\{t,b\} &= \{\varphi(t),b\} + \{t,\varphi(b)\}\\ &\in \{\lambda t + \mathfrak{m}, b\} + \{t, \beta b + \mathfrak{m}^1\}\\ &\subseteq \lambda\{t,b\} + \{\mathfrak{m},b\} + \beta\{t,b\} + \{t,\mathfrak{m}^1\}\\ &\subseteq \lambda\tau b + \mathfrak{m}^1 + \beta\tau b + \mathfrak{m}^1\end{aligned}$$

and so $(a-1)\tau = \lambda\tau + \beta\tau$ and since we assumed $\lambda = a$, we get $\beta = -1$.

Last step: We finally show $a \in \{\pm 1\}$.
By Lemma 4.38, we may choose (since $\mathfrak{n}$ is non-metabelian), two elements $b_1, b_2 \in B$ with $\{b_1, b_2\} =: z \in \mathfrak{s}^1, z \neq 0$.
By the fourth step, let us write $\varphi(b_1) = \beta_1 b_1 + \mathfrak{m}^1, \varphi(b_2) = \beta_2 b_2 + \mathfrak{m}^1$ (with $\beta_1, \beta_2 \in \{-1, 0\}$).
We get

$$\begin{aligned}(a-1)z = (a-1)\{b_1,b_2\} &= \{\varphi(b_1),b_2\} + \{b_1,\varphi(b_2)\}\\ &\in \{\beta_1 b_1 + \mathfrak{m}^1, b_2\} + \{b_1, \beta_2 b_2 + \mathfrak{m}^1\}\\ &\subseteq \beta_1 z + \{\mathfrak{m}^1, b_2\} + \beta_2 z + \{b_1, \mathfrak{m}^1\}\\ &\subseteq (\beta_1 + \beta_2)z + \mathfrak{m}^2 + \mathfrak{m}^2\end{aligned}$$

and therefore $a - 1 = \beta_1 + \beta_2$. This means $a - 1 \in \{-2, -1, 0\}$ and thus $a \in \{-1, 0, 1\}$. Since we have excluded $a = 0$, we get $a \in \{\pm 1\}$, as desired. □

In the metabelian case, there is not much to do since there is only one simply-complete metabelian Lie algebra, namely $(\mathfrak{r}_2(\mathbb{C}), \{,\})$ (with basis elements t_1, e_1 and non-zero Lie bracket $\{t_1, e_1\} = e_1$). All complete metabelian Lie algebras are of the form $(\mathfrak{r}_2(\mathbb{C}))^n$ for an $n \in \mathbb{N}$, see Lemma 3.9.

Remark 4.41. The result of Theorem 4.40 indeed fails for $\mathfrak{r}_2(\mathbb{C})$. Here, given any $a \in \mathbb{C}\backslash\{0\}$, all post-Lie algebra structures with $[x,y] = a\{x,y\}$ are given by one of the endomorphisms

$$\varphi_1 = \begin{pmatrix} a & 0 \\ \beta & -1 \end{pmatrix} \text{ and } \varphi_2 = \begin{pmatrix} a-1 & 0 \\ \beta & 0 \end{pmatrix}$$

(with $\beta \in \mathbb{C}$ arbitrary) and $x \cdot y = \{\varphi_i(x), y\}$. (In particular, post-Lie algebra structures do exist not only for $a \in \{-1, 1\}$, but for any $a \in \mathbb{C}^*$.)

The proof of Theorem 4.40 also shows:

Remark 4.42. As a rigid solvable Lie algebra $\mathfrak{n}$ also has the form $\mathfrak{n} = T \ltimes \mathfrak{m}$, T a maximal torus, $\mathfrak{m}$ the nilradical (cf. Proposition 4.20), "inner" post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ (that is, post-Lie algebra structures $x \cdot y$ with a linear map φ satisfying $\{\varphi(x), y\} = x \cdot y$) with $[x,y] = a\{x,y\}$ also force a to be 1 or -1 (provided that $\{\mathfrak{n}, \mathfrak{n}\} = \mathrm{nil}(\mathfrak{n})$ as before).

4.4 Post-Lie algebra structures on Lie algebra double pairs

In this section, we are interested in post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$'s Lie bracket is the image of $\mathfrak{n}$'s Lie bracket under a linear map, that is, there is an $R \in \operatorname{End}(V)$ such that $[x,y] = R(\{x,y\})$ for all $x, y \in \mathfrak{n}$. This is motivated by the the study of this situation in the case where $\mathfrak{n}$ is semisimple in [23, Chapter 6] (remember that the class of complete Lie algebra contains the class of semisimple Lie algebras). We will study this case for complete $\mathfrak{n}$.

We want to remind the reader about our convention on the notation of Lie algebras: If we state results on post-Lie algebra structures, we will call the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$; if we state results on Lie algebras, we will call the Lie algebra L. In the following, we will state some results on a complete Lie algebra L which we will later apply to the Lie algebra $\mathfrak{n}$ when studying post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.

Definition 4.43. Let $(\mathfrak{g}, [,]), (\mathfrak{n}, \{,\})$ be two Lie algebras with the same underlying vector space V. If there is a linear transformation $R \in \operatorname{End}(V)$ with $[x,y] = R(\{x,y\})$ for all $x, y \in \mathfrak{n}$, we call $(\mathfrak{g}, \mathfrak{n}, R)$ a *Lie algebra double pair*.

In this section, we will give a (partial) answer to the following question:

Question 4.44. *Let $\mathfrak{n}$ be a complete Lie algebra. For which $\mathfrak{g}$ and R is the triple $(\mathfrak{g}, \mathfrak{n}, R)$ a Lie algebra double pair which admits a post-Lie algebra structure?*

Surely the case $[x,y] = a(\{x,y\}), a \in \mathbb{C}$ and, in particular, the case $R = \operatorname{Id}$ leading to *commutative post-Lie algebra structures*, see [33], is a special case of this setting (we addressed this case in Section 4.3).
We will answer the question for solvable complete Lie algebras $\mathfrak{n}$ generated by special weight spaces satisfying $H^0(T, \mathfrak{n}) = T$ (the definitions will follow later).
Remember that since $\mathfrak{n}$ is complete, every post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n}, R)$ is given by $x \cdot y = \{\varphi(x), y\}$, where $\varphi \colon \mathfrak{n} \to \mathfrak{n}$ is a linear map (Proposition 4.6). Then the post-Lie algebra axioms read as follows (see Proposition 4.7):

$$R(\{x,y\}) - \{x,y\} = \{\varphi(x), y\} + \{x, \varphi(y)\} \tag{10}$$

$$\{\varphi(x), \varphi(y)\} = \varphi([x,y]) = \varphi(R(\{x,y\})) \tag{11}$$

Identity (10) has some resemblance to the definition of a derivation. Indeed, we can and will make use of this fact (and the theory of *quasiderivations*), as we will see later.

Remark 4.45. The case "$\mathfrak{n}$ simple" has been addressed in [23, Proposition 6.1 and Theorem 6.4], with the following result:
If $\mathfrak{n} \ncong \mathfrak{sl}_2(\mathbb{C})$ is simple, then either $x \cdot y = 0$ and $[x,y] = \{x,y\}$ or $x \cdot y = -\{x,y\} = [x,y]$.
If $\mathfrak{n} \cong \mathfrak{sl}_2(\mathbb{C})$, then there are a lot of different post-Lie algebra structures on Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}, R)$ which seem very hard to classify.

Recall (Proposition 4.21) that every solvable complete Lie algebra L is of the form $L = T \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is the nilradical of L and T is a maximal torus of L, that is, an abelian subalgebra of $\mathrm{Der}(L)$ whose elements are all semisimple. This structure will be used often in the sequel.

Moreover, also recall Proposition 4.16: the decomposition of a complete Lie algebra into simply-complete ideals: This decomposition theorem is useful for our purposes – post-Lie algebra structures on Lie algebra double pairs may be reduced to those on the simply-complete factors:

Lemma 4.46. *Let $\mathfrak{n}$ be a complete Lie algebra with decomposition into simply-complete ideals $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_n$ and V_i the underlying vector space of $\mathfrak{n}_i$. Suppose there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $[x,y] = R(\{x,y\})$ for all $x, y \in \mathfrak{n}$. Then*

(i) Each $\mathfrak{n}_i$ is closed with respect to $[,]$ and thus the set $\mathfrak{n}_i$ with respect to the Lie bracket $[,]$ is again a Lie algebra, which we will denote by $\mathfrak{g}_i$.

(ii) All post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ are given by

$$x \cdot y = \begin{cases} f_i(x,y) & \text{if } x, y \in V_i \\ 0 & \text{otherwise} \end{cases}$$

extended bilinearly. Here, $f_i \colon V_i \times V_i \to V_i$ is any post-Lie algebra structure on $(\mathfrak{g}_i, \mathfrak{n}_i)$.

Proof.

(i) Let $x, y \in \mathfrak{n}_i$. Then

$$[x,y] = \{x,y\} + \{\varphi(x), y\} + \{x, \varphi(y)\}$$

and, since $\mathfrak{n}_i$ is an ideal with respect to $\{,\}$, the claim follows.

(ii) This result is a generalization of [33, Proposition 4.3] (where it is given for commutative post-Lie algebra structures). By Lemma 3.8, a structure defined in this way is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ – for the other direction, we need to show that $V_i \cdot V_j \subseteq V_i \cap V_j = 0$.
Note that $x_i \cdot x_j = x_j \cdot x_i$ for $x_i \in V_i, x_j \in V_j, i \neq j$, since by (10),

$$x_i \cdot x_j - x_j \cdot x_i = [x_i, x_j] - \{x_i, x_j\} = R(0) - 0 = 0.$$

We have, since all derivations of $\mathfrak{n}$ are inner (the direct sum of complete Lie algebras is again complete, cf. Proposition 4.16),

$$V_i \cdot V_j \subseteq L(V_i)V_j \subseteq \mathrm{ad}(\mathfrak{n})V_j \subseteq \{\mathfrak{n}, V_j\} \subseteq V_j$$

and because of $V_i \cdot V_j = V_j \cdot V_i$ also $V_i \cdot V_j \subseteq V_i$. Thus $V_i \cdot V_j = 0$.
So we may reduce post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ to post-Lie algebra structures on the factors $(\mathfrak{g}_i, \mathfrak{n}_i)$.

□

Remark 4.47. In particular, using the results of [23, Chapter 6] one obtains the following result if $\mathfrak{n}$ is semisimple:
Let $\mathfrak{n}$ be semisimple and $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_n$ be its decomposition into simple ideals. Suppose $[x, y] = R(\{x, y\})$ for a linear transformation $R \in \mathrm{End}(V)$. As described in Lemma 4.46, we can reduce these structures to the ones on the simple factors $\mathfrak{n}_i$:
If no factor $\mathfrak{n}_i$ is $\mathfrak{sl}_2(\mathbb{C})$, then every post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ is given by

$$x_i \cdot y_i = 0 \text{ or } x_i \cdot y_i = -\{x_i, y_i\}, x_i, y_i \in \mathfrak{n}_i.$$

If $\mathfrak{n}_i = \mathfrak{sl}_2(\mathbb{C})$ for some i, then there are more (and complicated) possibilities. The reason for that lies in the theory of quasiderivations (which we will introduce later): The space of quasiderivations of $\mathfrak{sl}_2(\mathbb{C})$ is the whole $\mathrm{End}(\mathfrak{sl}_2(\mathbb{C}))$, whereas for all other complex simple Lie algebras L, the space of quasiderivations is given by $\mathrm{ad}(L) \oplus \mathbb{C} \cdot \mathrm{Id}$ (see [23, Theorem 5.6 and Proposition 5.7]).

Remark 4.48. If one drops the condition $R(\{x, y\}) = [x, y]$, Lemma 4.46 no longer holds true (not even for semisimple Lie algebras). For a counterexample, consider e.g. [23, Example 2.11]: There, a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where both $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, is given. This post-Lie algebra structure cannot be decomposed into post-Lie algebra structures on $\mathfrak{sl}_2(\mathbb{C})$.

So via Lemma 4.46, we can reduce the study of post-Lie algebra structures on Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ complete to the study of post-Lie algebra structures on Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}), \mathfrak{n}$ simply-complete.

We want to use Leger's and Luks' theory of quasiderivations and thus shall adopt several notations from [68]:

Let $(L, [,])$ be a Lie algebra containing a (maximal torus T (for the definition of a maximal torus, see Definition 4.21).
We say $\alpha \in T^*$ (where T^* denotes the dual space of T) is a *weight* of T on L if

$$L_\alpha \coloneqq \{x \in L | [t, x] = \alpha(t) \cdot x \text{ for all } t \in T\} \neq 0.$$

Given a weight α, the set L_α is called the *weight space* with respect to α. Note that 0 is also a weight and $L_0 = H^0(T, L) = \{x \in L \colon [T, x] = 0\}$ (the zero-th cohomology group) and, as T is abelian, $T \subseteq L_0$. We write $L_{(\alpha)} \coloneqq \bigoplus_{c \in \mathbb{C}} L_{c\alpha}$ and say α is *special* if $L_{(\alpha)}$ does not contain a nontrivial ideal of L.
Leger and Luks ([68]) study (among other things) the *space of quasiderivations* of a Lie algebra L (which itself forms a Lie algebra with respect to the commutator of functions), defined by

$$\mathrm{QDer}(L) \coloneqq \{f \in \mathrm{End}(L, L) : [f(x), y] + [x, f(y)] = f'[x, y] \text{ for some } f' \in \mathrm{End}(L, L)\}.$$

We say that L is *generated by special weight spaces*, if there exist weight spaces $(L_{\alpha_i})_{i \in I}$ of T on L such that $\bigcup_{i \in I} L_{\alpha_i}$ generates L as a Lie algebra and each α_i is special.

For the remainder of this chapter, let us fix the following notation:

Let $(\mathfrak{n}, \{,\}) = (L, \{,\})$ be a simply-complete solvable non-metabelian Lie algebra which is generated by special weight spaces. We write $L = T \ltimes \mathfrak{m}$, where T is a maximal torus of L and $\mathfrak{m}$ the nilradical of L. Moreover, for every $x \in L$ we assume that there is an $t \in T$ with $\{t, x\} \neq 0$ (in other words, $T = L_0$ or $H^0(T, L) = T$).

Remark 4.49. Given the decomposition $L = T \ltimes \mathfrak{m}$ into a maximal torus T and nilradical $\mathfrak{m}$, the statement $H^0(T, L) = T$ is equivalent to T being a *Cartan subalgebra* of L (meaning T is nilpotent and self-normalizing).

Furthermore, we denote the lower central series of $\mathfrak{m}$ by

$$\mathfrak{m}^0 = \mathfrak{m}, \mathfrak{m}^1 = \{\mathfrak{m}^0, \mathfrak{m}^0\}, \mathfrak{m}^2 = \{\mathfrak{m}^0, \mathfrak{m}^1\}, \ldots, \mathfrak{m}^s = \{\mathfrak{m}^0, \mathfrak{m}^{s-1}\} = 0.$$

We know that there is a basis of $\mathfrak{m}$ consisting of eigenvectors for T. Let us denote this basis by B and as before, consider the vector spaces $\mathfrak{s}^i := \mathfrak{m}^i/\mathfrak{m}^{i+1}$ with $\bigoplus_{i=0}^{s-1} \mathfrak{s}^i = \mathfrak{m}$.
We write $\operatorname{rank}(L) := \dim(T)$ (as before) and $\operatorname{type}(L) := \dim H^1(\mathfrak{m}, \mathbb{C})$ (in other words, $\operatorname{type}(L) = \dim(\mathfrak{m}/\mathfrak{m}^1)$) and refer to the *rank* and *type* of L, respectively (as in [103, Definition 2.5] or [85, Chapter 2]). Let $B^j := B \cap \mathfrak{s}^j$ (which is a basis of $\mathfrak{s}^j$).

The condition $H^0(T, L) = T$ can be seen as a replacement for the condition $\mathrm{n}(L) = \mathrm{nil}(L)$ (which we considered in Section 4.3):

Lemma 4.50. *Let L be solvable and complete. The condition $H^0(T, L) = T$ implies the condition $\mathrm{n}(L) = \mathrm{nil}(L)$.*

Proof. We know that $\mathrm{n}(L) = [L, \mathrm{rad}(L)]$ and $\mathrm{n}(L) \subseteq \mathrm{nil}(L)$ always hold.
As L is solvable, $\mathrm{rad}(L) = L$; as $H^0(T, L) = T$, the only elements x of L satisfying $[T, x] = 0$ are already in T.
Choose a basis $\{x_1, \ldots, x_n\}$ of $\mathrm{nil}(L)$ which consists of eigenvectors of T – then for each i there is a $t \in T$ with $[t, x_i] = \alpha_i x_i$ for some $\alpha_i \neq 0$. However, this means $\mathrm{nil}(L) = \mathrm{span}\{\alpha_1 x_1, \ldots, \alpha_n x_n\} \subseteq [L, L] = \mathrm{n}(L)$. ☐

Up to dimension 9, all solvable complete Lie algebras satisfy $H^0(T, L) = T$ (cf. Remark 4.60).

Lemma 4.51. *Let $L = T \ltimes \mathfrak{m}$ be generated by special weight spaces. For the dimension of the maximal torus T it holds $\dim(T) \geq 2$.*

Proof. See [68]. If α was the only special weight of L, we had $\mathfrak{m} \subseteq L_{(\alpha)}$ contradicting α being special. ☐

Any Lie algebra L with $\operatorname{type}(L) = \operatorname{rank}(L)$ is generated by special weight spaces. Before we can prove that, we need another lemma:

Lemma 4.52. *If $L = T \ltimes \mathfrak{m}$ is a solvable simply-complete Lie algebra (T a maximal torus, $\mathfrak{m}$ the nilradical) then $\mathfrak{m}$ can not be decomposed into a direct sum of two ideals.*

Proof. Suppose $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. By [103, Proposition 3.4], $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are completable with maximal tori T_1 and T_2, respectively. But then, by the proof of that proposition, $T_1 \oplus T_2$ is a maximal torus on $\mathfrak{m}$. Finally, [103, Lemma 3.2] assures T and $T_1 \oplus T_2$ to be isomorphic. So we get the decomposition $L = (T_1 \ltimes \mathfrak{m}_1) \oplus (T_2 \ltimes \mathfrak{m}_2)$, a contradiction to L being simply-complete. □

The following easy lemma will be helpful for calculations:

Lemma 4.53. *Let $L = T \ltimes \mathfrak{m}$ be solvable complete, $x, y \in \mathfrak{m}, t \in T$ with $[x, y] = z, [t, x] = \alpha x, [t, y] = \beta y, \alpha, \beta \in \mathbb{C}$. Then $[t, z] = (\alpha + \beta)z$.*

Proof. This follows from the Jacobi identity:

$$[t, z] = [t, [x, y]] = -[x, [y, t]] - [y, [t, x]] = [x, \beta y] + [\alpha x, y] = (\alpha + \beta)z.$$

□

The conditions we impose on $\mathfrak{n}$ (namely, being generated by special weight spaces and satisfying $H^0(T, \mathfrak{n}) = T$) are satisfied for instance by all solvable simply-complete Lie algebras with $\operatorname{type}(\mathfrak{n}) = \operatorname{rank}(\mathfrak{n})$. The nilradical is then said to be of *maximal rank*, see e.g. [85] and [103]:

Definition 4.54. Let $L = T \ltimes \mathfrak{m}$ be a solvable complete Lie algebra. If $\operatorname{type}(L) = \operatorname{rank}(L)$, we say that the nilradical $\mathfrak{m}$ is of *maximal rank*.

Lemma 4.55. *Let $L = T \ltimes \mathfrak{m}$ be a solvable simply-complete non-metabelian Lie algebra and $\mathfrak{m}$ of maximal rank $m = \dim(T) = \dim H^1(\mathfrak{m}, \mathbb{C})$. Then L is generated by special weight spaces and $H^0(T, L) = T$.*

Proof. Let $\{e_1, \ldots, e_m\}$ be a basis of $\mathfrak{s}^0$.
By [103, Proposition 3.6] (or, alternatively, Lemma 4.39), we can find a basis $\{t_1, \ldots, t_m\}$ of T with $[t_i, e_j] = \delta_{ij} e_j$.
Let $\alpha_i \in T^*$ be given by $\alpha_i(t_j) = \delta_{ij}$. Then the space $L_{(\alpha_i)}$ is given as

$$L_{(\alpha_i)} = \{x \in L \colon [t_i, x] = \beta x, [t_j, x] = 0 \text{ for all } t_j \in \{t_1, \ldots, t_m\} \backslash \{t_i\} \text{ and a } \beta \in \mathbb{C}\}.$$

We shall show that the weights

$$0, \alpha_1, \alpha_2, \ldots, \alpha_m$$

are special.
To show this, we show $L_{(\alpha_i)} = \{a e_i \colon a \in \mathbb{C}\}$ for every $i \in \{1, \ldots, m\}$. The inclusion "$\supseteq$" holds clearly, so we show "$\subseteq$": Let $x \in L, x \notin \{a e_i \colon a \in \mathbb{C}\}$ (it follows $x \neq 0$).
Suppose $x \in L_{(\alpha_i)}$.
First we assume $x \in \mathfrak{s}^0$. Then $x = \beta_1 e_1 + \ldots + \beta_m e_m$ for some $\beta_1, \ldots, \beta_m \in \mathbb{C}$. Then $[t_j, x] = \beta_j e_j$ and this expression is zero for all $t \in \{t_1, \ldots, t_m\} \backslash \{t_i\}$ only if $\beta_j = 0$ for $j \neq i$. So x is a scalar multiple of e_i.

Now we show that elements $x \in \mathfrak{m}^1$ can not be in $L_{(\alpha_i)}$. We show this for basis elements $x \in B^\ell$ (and we can assume x to be of the form $x = [a, b], a, b \in \bigcup_{k=0}^{\ell-1} B^k$) inductively, then it follows for all $x \in \mathfrak{m}^1$. More precisely, we show: If $x \in B^\ell, \ell > 0$, then

- for all $t \in T, [t, x] = \alpha_t x$, where $\alpha_t \geq 0$ and
- there are $j, k = 1, \ldots, m, j \neq k$ with $[e_j, x] = \alpha x, [e_k, x] = \beta x$ and $\alpha, \beta > 0$.

This holds for $x \in B^1$: Write $x = [e_j, e_k]$, $e_j, e_k \in \mathfrak{s}^0$. Then $[t_j, e_j] = e_j, [t_j, e_k] = 0$, thus (Lemma 4.53) $[t_j, x] = x$ and also $[t_k, e_k] = e_k, [t_k, e_j] = 0$, thus $[t_k, x] = x$. And also by Lemma 4.53, $[t, x] = x$ or $[t, x] = 0$ for all $t \in T$, thus $[t, x] = \alpha_t x$ with $\alpha_t \geq 0$ for all $t \in T$.

Now suppose we have shown this for $x \in B^{\ell-1}$, let us show the two conditions for $x \in B^\ell$: As before, write $x = [a, b], a, b \in \bigcup_{k=0}^{\ell-1} B^k$. By induction hypothesis, there are $t_j, t_k \in T$ with $[t_j, a] = \alpha_1 a, [t_k, b] = \alpha_2 a, \alpha_1, \alpha_2 > 0$. Write (by induction hypothesis) $[t_j, b] = \beta_1 b, [t_k, b] = \beta_2 b, \beta_1, \beta_2 \geq 0$, then by Lemma 4.53, $[t_j, x] = (\alpha_1 + \beta_1)x, [t_k, x] = (\alpha_2 + \beta_2)x$ and $\alpha_1 + \beta_1, \alpha_2 + \beta_2 > 0$. And for $t \in T$ with $[t, a] = \alpha a, [t, b] = \beta b, \alpha, \beta \geq 0$, we find $[t, x] = (\alpha + \beta)x$ with $\alpha + \beta \geq 0$.

So we have shown that for every element in $x \in \mathfrak{s}^0$, $x \in L_{(\alpha_i)}$ precisely if x is a multiple of e_i and for $x \in \mathfrak{m}^1$, there are two elements $t_j, t_k \in T$ with $[t_j, x], [t_k, x] \neq 0$, meaning $x \notin L_{(\alpha_i)}$.
So indeed, $L_{(\alpha_i)} = \{ae_i \colon a \in \mathbb{C}\}$.

Now, if $L_{(\alpha_i)}$ would contain a non-trivial ideal of L, then, in particular, $L_{(\alpha_i)}$ was an ideal of $\mathfrak{m}$ and $M \coloneqq \bigoplus_{i=1}^m L_{(\alpha_i)}$ too. But $\mathfrak{n} = M \oplus \mathfrak{m}^1$. But this would contradict Lemma 4.52 about $\mathfrak{m}$ being directly indecomposable.
As $e_1, \ldots, e_m, t_1, \ldots, t_m$ generate L, the Lie algebra L is indeed generated by special weight spaces via the weights $0, \alpha_1, \ldots, \alpha_m$.

From the above induction we also infer that if $[t, x] = 0$ for every $t \in T$ and an $x \in L$, then $x \in T$. But this exactly means $H^0(T, L) = T$. □

So, thanks to Lemma 4.55 we have a class of Lie algebras (namely those of maximal rank) satisfying the conditions we impose on $\mathfrak{n}$. For a Lie algebra with known maximal torus and nilradical, it is easy to check if it has maximal rank.

Let us give two examples of important families of Lie algebras which have maximal rank. We say that a nilpotent Lie algebra N is *completable of maximal rank*, if there is a complete Lie algebra L such that $\mathrm{nil}(L) = N$ and L has maximal rank. Then, we refer to L as the *completition* of N.

Example 4.56. *The free-nilpotent Lie algebra $F_{n,c}$ with nilpotency class c on n generators is completable of maximal rank; indeed given the a minimal system of generators $e_1, \ldots, e_n$ for $F_{n,c}$, we can add an n-dimensional torus T by $[t_i, e_j] = \delta_{ij} e_j$ for $1 \leq j \leq n$.*

The action of T to the other elements of $F_{n,c}$ follows by the Jacobi identity. By [75, Theorem 3], $T \ltimes F_{n,c}$ is complete. So $F_{n,c}$ is of maximal rank (as $\dim(T) = n$ and a minimal system of generators has n elements) and thus, by Lemma 4.55, is generated by special weight spaces and satisfies $H^0(T, F_{n,c}) = T$.

Example 4.57. *The standard graded filiform Lie algebra L_n ($n \geq 3$) with basis $\{e_1, \dots, e_n\}$ and Lie brackets $[e_1, e_i] = e_{i+1}, i = 2, \dots, n-1$ is completable of maximal rank (see [3]); its completition (with torus $T = \operatorname{span}(\{t_1, t_2\})$ and relations $[t_1, e_1] = e_1, [t_1, e_i] = (i-2)e_i, i = 2, \dots, n; [t_2, e_i] = e_i, i = 2, \dots, n$) therefore is generated by special weight spaces and $H^0(T, L_n) = T$. (The elements e_1, e_2 are a minimal system of generators, so again, the rank is maximal.)*
In particular, the standard graded filiform Lie algebras of dimensions $3, 4, 5$ thus have a completition of dimension $5, 6, 7$, respectively – in Tables 12, 13, 14 they can be found under the names $T_2 \ltimes \mathfrak{n}_3, T_2 \ltimes \mathfrak{g}_4, T_2 \ltimes \mathfrak{g}_{5,5}$.

Remark 4.58. By [57, Corollaire 4], any filiform Lie algebra has rank $0, 1$ or 2. Out of those, precisely the Lie algebras of rank 2 are completable of maximal rank and are given by L_n and Q_n (see [3]).
A basis for the Lie algebra Q_n (which is defined only for even $n \geq 4$) can be found in Definition 5.69, which is, however, not a basis of eigenvectors for a maximal torus on Q_n – one can find a maximal torus and a basis of eigenvectors in [56, Chapter 3.1.1].
Thus, by Lemma 4.55, also Q_n is generated by special weight spaces and satisfies $H^0(T, Q_n) = T$.

Remark 4.59. We will study filiform Lie algebras in Section 5.4 in the context of CPA-structures.

Not all complete Lie algebras have a nilradical of maximal rank. The non-metabelian ones with one-dimensional maximal torus do not (and they are also not generated by special weight spaces, see Lemma 4.51). There are also other, e.g. the 7-dimensional Lie algebra $T_2 \ltimes \mathfrak{g}_{5,3}$ (see Table 14 for its definition).
While not all complete Lie algebras have maximal rank, we can at least say the following:

Remark 4.60. Up to dimension 9, all solvable complete Lie algebras L with $\operatorname{rank}(L) \geq 2$, are generated by special weight spaces and satisfy $H^0(T, L) = T$. (Here, we used the classifications given in Section 4.2 and in [103, Tables 4 and 5].)

Now that we gave some examples for Lie algebras L generated by special weight spaces with $H^0(T, L) = T$, we explain the reason we are interested in them: The reason we study Lie algebras generated by special weight spaces is the following theorem ([68, Theorem 4.12]):

Theorem 4.61 (Leger, Luks). *Let L be a directly indecomposable Lie algebra, $T \subseteq L$ a maximal torus with $H^0(T, L) = T$ and $\mathrm{Z}(L) = 0$. Suppose L is generated by special weight spaces. Then*

$$\operatorname{QDer}(L) = \operatorname{Der}(L) \oplus (\mathrm{Id}).$$

This theorem helps us to find out the forms of φ and R since, fortunately, simply-complete Lie algebras are directly indecomposable:

Lemma 4.62. *A complete Lie algebra is simply-complete if and only if it is directly indecomposable.*

Proof. See [74, Theorem 3.2]. $\square$

We can now state our result on post-Lie algebra structures on Lie algebra double pairs:

Theorem 4.63. *Let $\mathfrak{n} = T \ltimes \mathfrak{m}$ be simply-complete, non-metabelian and generated by special weight spaces with $H^0(T, \mathfrak{n}) = T$. Let $\operatorname{ad}(z)$ denote the adjoint operator with respect to $\mathfrak{n}$.*
Then for every Lie algebra double pair $(\mathfrak{g}, \mathfrak{n}, R)$ with a post-Lie algebra structure $x \cdot y = \{\varphi(x), y\}$, there is an element $z \in \mathfrak{n}$ such that the post-Lie algebra structure is given by one of the following two cases:

(i) $\varphi = \operatorname{ad}(z), R|_{\mathfrak{m}} = \operatorname{Id} + \operatorname{ad}(z)$ *and $R|_T$ an arbitrary linear map $T \to \mathfrak{n}$ or*
(ii) $\varphi = \operatorname{ad}(z) - \operatorname{Id}, R|_{\mathfrak{m}} = -\operatorname{Id} + \operatorname{ad}(z)$ *and $R|_T$ an arbitrary linear map $T \to \mathfrak{n}$.*

Proof. By (10),

$$\{\varphi(x), y\} + \{x, \varphi(y)\} = (R - \operatorname{Id})(\{x, y\}),$$

so φ is a quasiderivation of $\mathfrak{n}$.
By Theorem 4.61, we have

$$\operatorname{QDer}(\mathfrak{n}) = \operatorname{Der}(\mathfrak{n}) \oplus (\operatorname{Id}),$$

which means (since $\mathfrak{n}$ is complete) that there are $z \in \mathfrak{n}, a \in \mathbb{C}$ with $\varphi(x) = ax + \{z, x\}$.

So, by (10), we have

$$\begin{aligned} R(\{x, y\}) &= \{\varphi(x), y\} + \{x, \varphi(y)\} + \{x, y\} \\ &= \{ax + \{z, x\}, y\} + \{x, ay + \{z, y\}\} + \{x, y\} \\ &= (2a + 1)\{x, y\} + \{\{z, x\}, y\} + \{x, \{z, y\}\} \\ &= (2a + 1)\{x, y\} + \{z, \{x, y\}\}. \end{aligned}$$

And in particular, given $x \in B^j$, we can find a $t \in T$ with $\{t, x\} = \lambda x, \lambda \neq 0$ and thus

$$\begin{aligned} R(x) &= \frac{1}{\lambda} R(\lambda x) = \frac{1}{\lambda} R(\{t, x\}) \\ &= \frac{1}{\lambda}(2a + 1)\{t, x\} + \frac{1}{\lambda}\{z, \{t, x\}\} \\ &= (2a + 1)x + \{z, x\}. \end{aligned}$$

So we can write

$$\varphi(x) = d_x x + \mathfrak{m}^{j+1} \text{ and } R(x) = \mu_x x + \mathfrak{m}^{j+1}$$

for $x \in B^j$ and some $d_x, \mu_x \in \mathbb{C}$.
However, we can not immediately conclude $d_x = a$ and $\mu_x = 2a+1$, since $\{z,x\} \subseteq \operatorname{span}(x) + \mathfrak{m}^{j+1}$.
But since for any $t \in T, \{z,t\} \subseteq \mathfrak{m}$, we have $\varphi(t) = at + \mathfrak{m}$ for all $t \in T$.

Now fix $x \in B^j$ and take $t \in T$ with $\{t,x\} = \lambda x \neq 0$. Equation (10) states

$$\begin{aligned}
\{\varphi(t), x\} + \{t, \varphi(x)\} &= R(\{t,x\}) - \{t,x\} \\
\{at + \mathfrak{m}, x\} + \{t, d_x x + \mathfrak{m}^{j+1}\} &= R(\lambda x) - \lambda x \\
\lambda a x + \mathfrak{m}^{j+1} + \lambda d_x x + \mathfrak{m}^{j+1} &= \lambda \mu_x x + \mathfrak{m}^{j+1} - \lambda x.
\end{aligned}$$

So comparing the x-terms, we get (as $\lambda \neq 0$)

$$a + d_x = \mu_x - 1.$$

By identity (11),

$$\begin{aligned}
\{\varphi(t), \varphi(x)\} &= \varphi(R(\{t,x\})) \\
\{at + \mathfrak{m}, d_x x + \mathfrak{m}^{j+1}\} &= \lambda \varphi(\mu_x x + \mathfrak{m}^{j+1}) \\
\lambda d_x a x + \mathfrak{m}^{j+1} + \mathfrak{m}^{j+1} + \mathfrak{m}^{j+2} &= \lambda d_x \mu_x x + \mathfrak{m}^{j+1}
\end{aligned}$$

and again comparing the x-terms, we get

$$\lambda d_x a = \lambda d_x \mu_x.$$

Now using $\lambda \neq 0$ and $\mu_x = 1 + a + d_x$, we get

$$d_x a = d_x(1 + a + d_x)$$

and hence $d_x = 0$ or $d_x = -1$.
It remains to show that $a \in \{0, -1\}$. Take $x_1, x_2 \in B^0$ with $y \coloneqq \{x_1, x_2\} \in \mathfrak{s}^1, y \neq 0$ (which exists since $\mathfrak{n}$ is not metabelian).
Let y be expanded into the basis B as $y = \alpha_1 y_1 + \ldots + \alpha_m y_m + \mathfrak{m}^2$, where $y_1, \ldots, y_m \in B^1$ and w.l.o.g. $\alpha_1 \neq 0$. Write $d_1 \coloneqq d_{x_1}, d_2 \coloneqq d_{x_2}$.

We have

$$\begin{aligned}
\{\varphi(x_1), x_2\} + \{x_1, \varphi(x_2)\} &= R(y) - y \\
\{d_1 x_1 + \mathfrak{m}^1, x_2\} + \{x_1, d_2 x_2 + \mathfrak{m}^1\} &= \sum_{i=1}^m \alpha_i \mu_{y_i} y_i + \mathfrak{m}^2 - \sum_{i=1}^m \alpha_i y_i + \mathfrak{m}^2 \\
d_1(\alpha_1 y_1 + \ldots + \alpha_m y_m) + \mathfrak{m}^2 + d_2(\alpha_1 y_1 + \ldots + \alpha_m y_m) + \mathfrak{m}^2 &= \sum_{i=1}^m \alpha_i \mu_{y_i} y_i + \mathfrak{m}^2 - \sum_{i=1}^m \alpha_i y_i + \mathfrak{m}^2
\end{aligned}$$

and if we compare the y_1-terms, we get

$$d_1\alpha_1 + d_2\alpha_1 = \alpha_1\mu_{y_1} - \alpha_1,$$

so $d_1 + d_2 = \mu_{y_1} - 1$. Since $\mu_{y_1} = d_{y_1} + 1 + a$, we get $d_1 + d_2 - d_{y_1} = a$.
Since $d_1, d_2, d_{y_1} \in \{0, -1\}$, we get $a \in \{0, -1\}$ unless $(d_1, d_2, d_{y_1}) \in \{(0, 0, -1), (-1, -1, 0)\}$, however, we show that those cases are contradictory:

With notation as above, by identity (11) we get

$$\begin{aligned}\{\varphi(x_1), \varphi(x_2)\} &= \varphi(R(\{x_1, x_2\}))\\ \{d_1x_1 + \mathfrak{m}^1, d_2x_2 + \mathfrak{m}^1\} &= \varphi(R(y)) = \varphi(\alpha_1\mu_{y_1}y_1 + \ldots + \alpha_m\mu_{y_m}y_m + \mathfrak{m}^2)\\ d_1d_2(\alpha_1y_1 + \ldots + \alpha_my_m) + \mathfrak{m}^2 &= d_{y_1}\alpha_1\mu_{y_1}y_1 + \ldots + d_{y_m}\alpha_1\mu_{y_m}y_m + \mathfrak{m}^2.\end{aligned}$$

Comparing again the y_1-terms gives

$$d_1d_2 = d_{y_1}\mu_{y_1} = d_{y_1}(d_{y_1} + 1 + a).$$

As $a = d_1 + d_2 - d_{y_1}$, the two cases we want to exclude are $(d_1, d_2, d_{y_1}, a) = (0, 0, -1, 1)$ and $(d_1, d_2, d_{y_1}, a) = (-1, -1, 0, -2)$. But inserting those cases into $d_1d_2 = d_{y_1}(d_{y_1}+1+a)$ (which is the equation we just obtained) gives a contradiction.

So we know $a \in \{-1, 0\}$ – by inserting this into the formulas

$$\varphi(x) = ax + \{z, x\}, R(x_0) = (2a + 1)(x_0) + \{z, x_0\} \text{ for } x \in \mathfrak{n}, x_0 \in \mathfrak{m}$$

we obtain the statement.

Finally, if $\{t_1, \ldots, t_n\}$ is a basis of T, equations (10) and (11) are satisfied for every choice of the $R(t_i)$.
This proves Theorem 4.63. □

Recall that a maximal torus of $\mathfrak{n}$ has to be at least two-dimensional (Lemma 4.51) for Theorem 4.63 to hold. Indeed, the statement of Theorem 4.63 does not carry over to Lie algebras with one-dimensional maximal torus, for there are counterexamples in dimension 7:

Example 4.64. *Consider the (complete, see Table 14) Lie algebra $T_1 \ltimes \mathcal{G}_{6,12}$ (with non-zero brackets $\{e_1, e_5\} = \{e_2, e_3\} = \{e_2, e_4\} = e_6, \{e_1, e_2\} = e_4, \{e_1, e_4\} = e_5, \{t, e_i\} = ie_i, 1 \le i \le 3, \{t, e_i\} = (i-1)e_i, 4 \le i \le 6$). All post-Lie algebra structures on any Lie algebra double pair $(\mathfrak{g}, T_1 \ltimes \mathcal{G}_{6,12}, R)$ with respect to this basis are given by*

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 - 2c_3\\ 0 & 0 & 0 & 0 & 0 & 0 & -c_1\\ 0 & 0 & 0 & 0 & 0 & 0 & -c_2\\ 0 & 0 & 0 & 0 & 0 & 0 & c_4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & r_1\\ 0 & 1 & 0 & 0 & 0 & 0 & r_2\\ 0 & 0 & 1 & 0 & 0 & 0 & r_3\\ 0 & 0 & 0 & 1 & 0 & 0 & r_4\\ c_1 & 0 & 0 & 0 & 1 & 0 & r_5\\ c_2 & c_3 & 0 & 0 & 0 & 1 & r_6\\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 \end{pmatrix},$$

$c_1, \dots, c_4, r_1, \dots, r_7 \in \mathbb{C}$ and by $\tilde{\varphi} := -\operatorname{Id} - \varphi, \tilde{R} := -R$.
Choose e.g. $c_1 = c_2 = 1$, then φ is not a derivation of $\mathfrak{n}$, as $\{\varphi(t), e_1\} + \{t, \varphi(e_1)\} = e_5 + e_6$, but $\varphi(\{t, e_1\}) = 0$. Also we have $\varphi(e_1) = \varphi(e_2) = 0$ – thus there is no $z \in \mathfrak{n}$ such that φ has the form $-\operatorname{Id} + \operatorname{ad}(z)$ or $\varphi = \operatorname{ad}(z)$ (that is, Theorem 4.63 does not hold for $T_1 \ltimes \mathcal{G}_{6,12}$).

Remark 4.65. We can recover a special case of Proposition 2.52 by $R = \operatorname{Id}$, which means to study commutative post-Lie algebra structures. Then by

$$R(x) = x + \{z, x\} = \operatorname{Id}(x) \text{ or } R(x) = -x + \{z, x\} = \operatorname{Id}(x),$$

we get $\{z, x\} = 0$ or $\{z, x\} = 2x$ for all $x \in \mathfrak{m}$.
It follows $z \in \mathrm{Z}(\mathfrak{m})$ or $\{z, x\} = 2x$ for all $x \in \mathfrak{m}$. But the latter is impossible if $\mathfrak{n}$ is non-metabelian (since if $\{x_1, x_2\} = x_3, \{z, x_1\} = 2x_1, \{z, x_2\} = 2x_2$, then $\{z, x_3\} = 4x_3$ (Lemma 4.53)).
So under these conditions ($\mathfrak{n}$ is solvable, complete, non-metabelian and generated by special weight spaces, $H^0(T, \mathfrak{n}) = T$), every *commutative post-Lie algebra structure* on $(\mathfrak{n}, \mathfrak{n})$ is given by $x \cdot y = \{6\{z, x\}, y\}$, where $z \in \mathrm{Z}(\mathfrak{m})$ (cf. Proposition 2.52).

Lemma 4.66. *If a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n}, R)$ is given by $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in \mathfrak{n}$, then $x \circ y = \{(-\operatorname{Id} - \varphi)(x), y\}$ defines a post-Lie algebra structure on $(\mathfrak{g}_1, \mathfrak{n})$, where $\mathfrak{g}_1$ is equipped with the Lie bracket $[,]_1$ given by $[x, y]_1 := -R\{x, y\} = -[x, y]$.*

Proof. The linear maps $-\operatorname{Id} - \varphi$ and $-R$ indeed satisfy identities (10) and (11):
We have

$$\begin{aligned} \{(-\operatorname{Id} - \varphi)(x), y\} &+ \{x, (-\operatorname{Id} - \varphi)(y)\} + \{x, y\} \\ &= -\{x, y\} - \{\varphi(x), y\} - \{x, y\} - \{x, \varphi(y)\} + \{x, y\} \\ &= -R\{x, y\}. \end{aligned}$$

Concerning identity (11), we have

$$\begin{aligned} \{(-\operatorname{Id} - \varphi)(x), (-\operatorname{Id} - \varphi)(y)\} &= \{(-\operatorname{Id} - \varphi)(x), (-\operatorname{Id} - \varphi)(y)\} \\ &= \{x, y\} + \{\varphi(x), y\} + \{x, \varphi(y)\} + \{\varphi(x), \varphi(y)\} \\ &= \{x, y\} + \{\varphi(x), y\} + \{x, \varphi(y)\} + \varphi(R\{x, y\}) \\ &= R\{x, y\} + \varphi(R\{x, y\}) = (-\operatorname{Id} - \varphi)(-R\{x, y\}). \end{aligned}$$

□

This means that we can "identify" the structures given by $\varphi = \operatorname{ad}(z), R|_{\mathfrak{m}} = \operatorname{Id} + \operatorname{ad}(z)$ and $\widetilde{\varphi} = \operatorname{ad}(z) - \operatorname{Id}, \widetilde{R}|_{\mathfrak{m}} = -\operatorname{Id} + \operatorname{ad}(z)$. In the following, we will mostly investigate the first structure where φ is an inner derivation.

Let $\mathfrak{n}$ be a Lie algebra as before. We have just shown that there is an injective map

$$(\{\text{post-Lie algebra structures on Lie algebra double pairs } (\mathfrak{g}, \mathfrak{n}, R)\} / \sim) \to \mathfrak{n},$$

where the equivalence relation $\sim$ identifies the post-Lie algebra structures φ and $\widetilde{\varphi} = -\varphi - \mathrm{Id}$.

However, this map is not surjective, that is, not every element $z \in \mathfrak{n}$ defines a post-Lie algebra structure via $\varphi = \mathrm{ad}(z)$. So, we investigate which $z \in \mathfrak{n}$ do:

Lemma 4.67. *Let $\mathfrak{n} = T \ltimes \mathfrak{m}$ be a solvable complete Lie algebra and $z \in \mathfrak{n}$. Then $\varphi := \mathrm{ad}(z), R|_{\mathfrak{m}} := \mathrm{Id} + \mathrm{ad}(z)$ (or $\widetilde{\varphi} := \mathrm{ad}(z) - \mathrm{Id}, R|_{\mathfrak{m}} := -\mathrm{Id} + \mathrm{ad}(z)$) and $R|_T$ arbitrary defines a post-Lie algebra structure if and only if z satisfies*

$$\{\{z, x\}, \{z, y\}\} = \{z, \{x, y\}\} + \{z, \{z, \{x, y\}\}\} \tag{12}$$

for all $x, y \in \mathfrak{n}$.

Proof. Suppose $\varphi := \mathrm{ad}(z), R|_{\mathfrak{m}} := \mathrm{Id} + \mathrm{ad}(z)$ defines a post-Lie algebra structure. Then, by (10) we have

$$R(\{x, y\}) = \{x, y\} + \{\{z, x\}, y\} + \{x, \{z, y\}\} = \{x, y\} + \{z, \{x, y\}\}$$

and thus, by (11),

$$\begin{aligned}\{\{z, x\}, \{z, y\}\} &= \varphi(R\{x, y\}) = \varphi(\{x, y\}) + \varphi(\{z, \{x, y\}\}) \\ &= \{z, \{x, y\}\} + \{z, \{z, \{x, y\}\}.\end{aligned}$$

Similarly, one shows that (10) and (11) hold for $\varphi := \mathrm{ad}(z)$ and $R|_{\mathfrak{m}} := \mathrm{Id} + \mathrm{ad}(z)$ if z satisfies (12). □

Remark 4.68. Equation (12) is a form of the modified Yang-Baxter equation studied e.g. in [21] (and the references therein),

$$[z, [z, [x, y]]] = [[z, x], [z, y]] + \lambda[x, y],$$

see [21, Corollary 3.3].

Remark 4.69. Condition (12) is satisfied if $z \in \mathrm{Z}(\{\mathfrak{n}, \mathfrak{n}\})$. However, the next example shows that, in general, there are elements of $\mathfrak{n}$ satisfying (12) despite not being contained in $z \in \mathrm{Z}(\{\mathfrak{n}, \mathfrak{n}\})$.

Example 4.70. *Consider the seven-dimensional Lie algebra $\mathfrak{n} := T_2 \ltimes \mathfrak{g}_{5,3}$ with basis $\{t_1, t_2, e_1, e_2, e_3, e_4, e_5\}$ and non-zero brackets*

$$\begin{aligned}&\{e_1, e_2\} = e_4, \{e_1, e_4\} = e_5, \{e_2, e_3\} = e_5, \{t_1, e_1\} = e_1, \{t_1, e_3\} = 2e_3, \\ &\{t_1, e_4\} = e_4, \{t_1, e_5\} = 2e_5, \{t_2, e_2\} = e_2, \{t_2, e_4\} = e_4, \{t_2, e_5\} = e_5.\end{aligned}$$

Then the element $z := t_2 + e_2$ defines a post-Lie algebra structure by $\varphi(x) = \mathrm{ad}(t_2 + e_2), R|_{\mathfrak{g}_{5,3}} = \mathrm{Id} + \mathrm{ad}(z), R(t)$ arbitrary for $t \in T$, even though $z \notin \mathrm{Z}(\{\mathfrak{n}, \mathfrak{n}\}) = \langle e_5 \rangle$.

Explicitly, φ and R are given by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 1 & 0 & 0 & \alpha_3 & \beta_3 \\ -1 & 0 & 0 & 0 & 0 & \alpha_4 & \beta_4 \\ 0 & 0 & 1 & 0 & 0 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & 0 & \alpha_6 & \beta_6 \\ 0 & 0 & 0 & 0 & 0 & \alpha_7 & \beta_7 \end{pmatrix}$$

with $\alpha_i, \beta_i \in \mathbb{C}, i = 1, \ldots, 7$.

So we can not only choose $z \in \mathrm{Z}(\{\mathfrak{n}, \mathfrak{n}\})$, but which $z \in \mathfrak{n}$ satisfy the above condition? Looking at the Jordan decomposition of $\mathrm{ad}(z)$, we get the following result (here, we do not impose any conditions on $\mathfrak{n}$ other than completeness):

Proposition 4.71. *Let $\mathfrak{n}$ be a complete Lie algebra. If $z \in \mathfrak{n}$ defines a post-Lie algebra structure on the Lie algebra double pair $(\mathfrak{g}, \mathfrak{n}, R)$ by $x \cdot y = \{\mathrm{ad}(z)(x), y\}$, then the operator $\mathrm{ad}(z)$ has the Jordan decomposition*

$$J = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & J_{n-1} & \\ & & & & J_n \end{pmatrix},$$

where each J_i is a Jordan block of the form

$$J_i = \begin{pmatrix} 0 \end{pmatrix}, J_i = \begin{pmatrix} -1 \end{pmatrix} \text{ or } J_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Moreover, let $\mathrm{ad}(z)$ have the above Jordan decomposition. Then there is a basis $\{e_1, \ldots, e_n\}$ of V consisting of generalized eigenvectors of $\mathrm{ad}(z)$ to the eigenvalues 0 and -1. The fact that $\mathrm{ad}(z)$ defines a post-Lie algebra structure by $x \cdot y = \{\{z, x\}, y\}$ is equivalent to the following condition:
If x_1, x_2, y, a_1, a_2 are generalized eigenvectors of $\mathrm{ad}(z)$ with

$$\{z, x_1\} = -x_1, \{z, x_2\} = -x_2, \{z, y\} = 0, \{z, a_1\} = b_1, \{z, a_2\} = b_2,$$

then

$$\{x_1, x_2\} = \{y, b_1\} = \{a_1, b_2\} + \{b_1, a_2\} + \{b_1, b_2\} = 0. \tag{13}$$

In particular, $\mathrm{ad}(z)$ has only eigenvalues from $\{0, -1\}$ and -1 has the same algebraic and geometric multiplicity as an eigenvalue.

Proof. Let z define a post-Lie algebra structure on $\mathfrak{n}$. As seen before,

$$\{\{z,x\},\{z,y\}\} = \{z,\{z,\{x,y\}\}\} + \{z,\{x,y\}\} \text{ for all } x,y \in \mathfrak{n}. \tag{14}$$

In particular, if $x = z$, we get

$$0 = \mathrm{ad}(z)^2 + \mathrm{ad}(z)^3.$$

So the minimal polynomial of $\mathrm{ad}(z)$ divides $X^2 \cdot (1+X) \in \mathbb{C}[X]$. Thus $\mathrm{ad}(z)$ can only have eigenvalues 0 and -1; 0 can have algebraic multiplicity 1 or 2; -1 can only have algebraic multiplicity 0.

We now want to show why we need condition (13). In particular, we show: A map $\mathrm{ad}(z)$ with a Jordan decomposition of the above form satisfies (14) for all $x,y \in \mathfrak{n}$ if and only if (13) holds.

We first show: if x_1,x_2,y_1,y_2,a_1,a_2 are generalized eigenvectors of $\mathrm{ad}(z)$ with

$$\{z,x_i\} = -x_i, \{z,y_i\} = 0, \{z,a_i\} = b_i, \text{ where } \{z,b_i\} = 0,$$

then

$$\begin{aligned}
&\{z,\{x_1,x_2\}\} = \{x_1,\{z,x_2\}\} + \{\{z,x_1\},x_2\} = \{x_1,-x_2\} + \{-x_1,x_2\} = -2\{x_1,x_2\};\\
&\{z,\{x_1,y_1\}\} = \{x_1,\{z,y_1\}\} + \{\{z,x_1\},y_1\} = 0 + \{-x_1,y_1\} = -\{x_1,y_1\};\\
&\{z,\{x_1,a_1\}\} = \{x_1,\{z,a_1\}\} + \{\{z,x_1\},a_1\} = \{x_1,b_1\} + \{-x_1,a_1\} = \{x_1,b_1\} - \{x_1,a_1\};\\
&\{z,\{y_1,y_2\}\} = \{y_1,\{z,y_2\}\} + \{\{z,y_1\},y_2\} = 0 + 0 = 0;\\
&\{z,\{y_1,a_1\}\} = \{y_1,\{z,a_1\}\} + \{\{z,y_1\},a_1\} = \{y_1,b_1\} + 0 = \{y_1,b_1\};\\
&\{z,\{a_1,a_2\}\} = \{a_1,\{z,a_2\}\} + \{\{z,a_1\},a_2\} = \{a_1,b_2\} + \{b_1,a_2\}.
\end{aligned}$$

With these facts, we show that equation (14) is satisfied if and only if (13) holds:

That $\mathrm{ad}(z)$ has the above Jordan form is equivalent to the fact that there exists a basis of $\mathfrak{n}$ consisting of generalized eigenvectors for $\mathrm{ad}(z)$ of the forms x_i, y_i, a_i as above. Inserting any pair of them into (14) gives one of the following six equations:

- $\{x_1,x_2\} = \{\{z,x_1\},\{z,x_2\}\} = \{z,\{z,\{x_1,x_2\}\}\} + \{z,\{x_1,x_2\}\} = 4\{x_1,x_2\} - 2\{x_1,x_2\} = 2\{x_1,x_2\}$.
- $0 = \{\{z,x_1\},\{z,y_1\}\} = \{z,\{z,\{x_1,y_1\}\}\} + \{z,\{x_1,y_1\}\} = \{x_1,y_1\} - \{x_1,y_1\} = 0$.
- $-\{x_1,b_1\} = \{\{z,x_1\},\{z,a_1\}\} = \{z,\{z,\{x_1,a_1\}\}\} + \{z,\{x_1,a_1\}\} = \{z,\{x_1,b_1\}\} - \{z,\{x_1,a_1\}\} + \{x_1,b_1\} - \{x_1,a_1\} = -\{x_1,\{b_1,z\}\} - \{b_1,\{z,x_1\}\} + \{x_1,\{a_1,z\}\} + \{a_1,\{z,x_1\}\} + \{x_1,b_1\} - \{x_1,a_1\} = 0 + \{b_1,x_1\} - \{x_1,b_1\} - \{a_1,x_1\} + \{x_1,b_1\} - \{x_1,a_1\} = -\{x_1,b_1\}$.
- $0 = \{\{z,y_1\},\{z,y_2\}\} = \{z,\{z,\{y_1,y_2\}\}\} + \{z,\{y_1,y_2\}\} = 0 + 0 = 0$.
- $0 = \{\{z,y_1\},\{z,a_1\}\} = \{z,\{z,\{y_1,a_1\}\}\} + \{z,\{y_1,a_1\}\} = \{z,\{y_1,b_1\}\} + \{y_1,b_1\} = -\{y_1,\{b_1,z\}\} - \{b_1,\{z,y_1\}\} + \{y_1,b_1\} = \{y_1,b_1\}$.

- $\{b_1,b_2\} = \{\{z,a_1\},\{z,a_2\}\} = \{z,\{z,\{a_1,a_2\}\}\} + \{z,\{a_1,a_2\}\} = \{z,\{a_1,b_2\}\} + \{z,\{b_1,a_2\}\} + \{a_1,b_2\} + \{b_1,a_2\} = -\{a_1,\{b_2,z\}\} - \{b_2,\{z,a_1\}\} - \{b_1,\{a_2,z\}\} - \{a_2,\{z,b_1\}\} + \{a_1,b_2\} + \{b_1,a_2\} = \{b_1,b_2\} + \{b_1,b_2\} + \{a_1,b_2\} + \{b_1,a_2\}$.

So, given the Jordan form, (13) holds if and only if z satisfies (14).

For the sufficiency, let $z \in \mathfrak{n}$ and let $\mathrm{ad}(z)$ have a Jordan decomposition as described above. Then all generalized eigenvectors v will satisfy $\{z,v\} \in \{-v,0,w\}$, where $\{z,w\}=0$. But we have already checked that for such generalized eigenvectors, the second condition makes sure that (14) is satisfied. Thus z indeed defines a post-Lie algebra structure on $\mathfrak{n}$. ☐

Remark 4.72. If φ defines a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n},R)$ with respect to $[,]$, then $-\mathrm{Id}-\varphi$ does too (with respect to $-[,]$). So the map $\widetilde{\varphi} = -\mathrm{Id}-\mathrm{ad}(z)$ defines a post-Lie algebra structure on some Lie algebra double pair $(\mathfrak{g},\mathfrak{n},R)$ if and only $\mathrm{ad}(z)$ satisfies the two conditions stated in Proposition 4.71.

In the following, we continue to study non-metabelian, simply-complete solvable Lie algebras $\mathfrak{n}$ generated by special weight spaces with $H^0(T,\mathfrak{n}) = T$ and ask what can be said about $\mathfrak{g}$ provided that there is a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n},R)$.
Remember that by Lemma 3.25, $\mathfrak{g}$ is also solvable.

Proposition 4.73. *Let $(\mathfrak{g},\mathfrak{n},R)$ be a Lie algebra double pair, $\mathfrak{n}$ solvable, complete and let $\varphi = \mathrm{ad}(z)$ be a post-Lie algebra structure as before. If $\mathrm{Z}(\mathfrak{g})=0$, then $\mathfrak{g}\cong\mathfrak{n}$.*

Proof. By Theorem 4.63, we know that there is an element $z\in\mathfrak{n}$ such that $\varphi = \mathrm{ad}(z), R|_{\mathfrak{m}} = \mathrm{Id}+\mathrm{ad}(z)$ (where $\mathfrak{m}$, as always, is the nilradical of $\mathfrak{n}$) and $R(t)$ is arbitrary for $t\in T$. Note that we may assume, without changing $\mathfrak{g}$, that $R|_T = \mathrm{Id}$.
We shall investigate the element z. Let us write $z = t + z_0$ with $t\in T, z_0\in\mathfrak{m}$.
First let us suppose $t=0$.
Then we have $R(x) = x + \{z_0,x\}$ for $x\in\mathfrak{m}$. In particular, $R-\mathrm{Id}$ is nilpotent, hence R is unipotent, thus invertible, so $\mathfrak{g}\cong\mathfrak{n}$.

Now suppose $t\neq 0$. We will construct a non-zero element in $\mathrm{Z}(\mathfrak{g})$. Remember our notation $\mathfrak{s}^i := \mathfrak{m}^i/\mathfrak{m}^{i+1}$ and B a basis of $\mathfrak{n}$ which consists of eigenvectors for T.
Let v be an eigenvector for t, i.e. $\{t,v\} = \lambda v$. Then $\lambda = 0$ or $\lambda=-1$: This follows form Lemma 4.67 if we set $x = z = t+z_0, y = v$.

All $x_i\in B$ are eigenvectors for t; since $\{t,x_i\} = 0$ for all $x_i\in B$ is impossible (this would imply $t\in\mathrm{Z}(\mathfrak{n})=0$), we can find an $x_i\in B$ with $\{t,x_i\} = -x_i$.

Claim: There is an element $y\in\mathfrak{m}$ satisfying the following three conditions:

(i) $\{y,x_i\} = 0$ for all $x_i\in B$

(ii) $\{t,y\} = -y$

(iii) y is an eigenvector for all $u \in T$, i.e. for all $u \in T$ there is some $\alpha_u \in \mathbb{C}$ with $\{u, y\} = \alpha_u y$.

Proof of Claim: We know that there is an $y_0 \in B$ with $\{t, y_0\} = -y_0$. Let k be such that $y_0 \in \mathfrak{s}^{k-1}$. So y_0 satisfies (ii) and (iii).
If for all $x_i \in B, \{y_0, x_i\} = 0$, we are done.
Otherwise there is an $x \in B$ with $\{y_0, x\} \neq 0$.
If $\{t, x\} = -x$, then

$$\{t, \{y_0, x\}\} = -2\{y_0, x\}$$

by Lemma 4.53. But then $\{y_0, x\}$ was an eigenvector for T with eigenvalue -2, which is impossible. So $\{t, x\} = 0$ and it follows $\{t, \{y_0, x\}\} = -\{y_0, x\}$ by Lemma 4.53.
Thus we have found an element $y_1 \coloneqq \{y_0, x\} \in \mathfrak{s}^k$ with $\{t, y_1\} = -y_1$.
Since y_0 and x are eigenvectors for all $u \in T$, so is y_1.
We can repeat this procedure; since $\mathfrak{m}$ is nilpotent, the procedure eventually terminates in an element y with $\{y, x_i\} = 0$ for all $x_i \in B$ which satisfies conditions (i), (ii) and (iii).
This proves the claim.

As $\{y, x_i\} = 0$ for all basis elements $x_i \in B$, we have $\{y, \mathfrak{m}\} = 0$. And $y \in \ker(R)$:

$$R(y) = y + \{z, y\} = y + \{t, y\} + \{z_0, y\} = y - y + 0 = 0$$

and we conclude that $y \in \mathrm{Z}(\mathfrak{g})$: Let $x \in \mathfrak{m}$. Then $[x, y] = R(\{x, y\}) = R(0) = 0$. If $u \in T$, then $[u, y] = R(\{u, y\}) = R(\alpha_u y) = \alpha_u R(y) = 0$.
This shows $y \in \mathrm{Z}(\mathfrak{g})$ and thus $\mathrm{Z}(\mathfrak{g}) \neq 0$, a contradiction to the assumption. □

However, there are Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}, R)$ with post-Lie algebra structures where $\mathfrak{n}$ is complete, but $\mathfrak{g}$ is not isomorphic to $\mathfrak{n}$ (and hence $\mathfrak{g}$ has nontrivial center). See the appendix, Section B.3, for examples.
If the element $z \in \mathfrak{n}$ lies in the nilradical of $\mathfrak{n}$, then it must even be in the center of the nilradical:

Lemma 4.74. *Consider a post-Lie algebra structure on a Lie algebra double pair $(\mathfrak{g}, \mathfrak{n}, R)$ with $\mathfrak{n}$ solvable complete, $H^0(T, \mathfrak{n}) = T$ and $\varphi = \mathrm{ad}(z)$. If z is in the nilradical $\mathfrak{m} = \mathrm{nil}(\mathfrak{n})$, then $z \in \mathrm{Z}(\mathfrak{m})$. So in this case, $R|_{\mathfrak{m}} = \pm \mathrm{Id}$.*

Proof. Suppose $z \in \mathfrak{m}$, but $z \notin \mathrm{Z}(\mathfrak{m})$. Then there is an $y \in B$ satisfying $\{z, y\} \neq 0$.
As $H^0(T, \mathfrak{n}) = T$, we can choose a $t \in T$ with $\{t, y\} = \alpha y \neq 0, \alpha \in \mathbb{C}$.
By Lemma 4.67, we have

$$\{z, \{z, \{t, y\}\}\} + \{z, \{t, y\}\} = \{\{z, t\}, \{z, y\}\},$$

meaning

$$\alpha\{z, \{z, y\}\} + \alpha\{z, y\} = \{\{z, t\}, \{z, y\}\}.$$

But if k is maximal such that $\{z, y\} \in \mathfrak{m}^k$, then $\{z, \{z, k\}\}, \{\{z, t\}, \{z, y\}\} \in \mathfrak{m}^{k+1}$, a contradiction.
So, $z \in \mathrm{Z}(\mathfrak{m})$.
Therefore, for $x \in \mathfrak{m}, R(x) = \pm\,\mathrm{Id} + \mathrm{ad}(z) = \pm\,\mathrm{Id}$. This proves the lemma. □

Remark 4.75.

(i) Lemma 4.74 does assume $H^0(T, \mathfrak{n}) = T$; but not $\mathfrak{n}$ being generated by special weight spaces. It does hold whenever the post-Lie algebra structure is of the form $\varphi = \mathrm{ad}(z)$.

(ii) By the proof of Proposition 4.73, if $\mathrm{Z}(\mathfrak{g}) = 0$ and $z = t + z_0$, then $t = 0$. So if $\mathrm{Z}(\mathfrak{g}) = 0$, then $z \in \mathfrak{m}$ and thus $z \in \mathrm{Z}(\mathfrak{m})$. (In particular, in the case of CPA-structures on complete Lie algebras, we have $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{n}) = 0$, so $x \cdot y = \{\{z, x\}, y\}$ for a $z \in \mathrm{Z}(\mathfrak{m})$, cf. Proposition 2.52.)

So far, we always excluded metabelian Lie algebras in our considerations. Therefore, it remains to study the case $\mathfrak{n}$ metabelian. This is not too complicated as there is only one simply-complete metabelian Lie algebra, namely $\mathfrak{r}_2(\mathbb{C})$ (see Lemma 3.9).

Fix for $\mathfrak{r}_2(\mathbb{C})$ a basis with the basis elements t_1, e_1 and relation $\{t_1, e_1\} = e_1$ (cf. also Remark 4.41).

Lemma 4.76. *All post-Lie algebra structures on any Lie algebra double pair $(\mathfrak{g}, \mathfrak{n}, R)$, where $\mathfrak{n} \cong \mathfrak{r}_2(\mathbb{C})$, are given by*

(i) $\varphi = \begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix}, R = \begin{pmatrix} \delta & 0 \\ \epsilon & 1+\alpha \end{pmatrix}$

(ii) $\varphi = \begin{pmatrix} -\alpha & 0 \\ \gamma & -1 \end{pmatrix}, R = \begin{pmatrix} \delta & 0 \\ \epsilon & 1+\alpha \end{pmatrix}$

(iii) $\varphi = \begin{pmatrix} \alpha & \beta \\ -\frac{\alpha(1+\alpha)}{\beta} & -1-\alpha \end{pmatrix}, R = \begin{pmatrix} \delta & 0 \\ \epsilon & 0 \end{pmatrix}$ *(here, $\beta \neq 0$),*

(with respect to the basis above) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ arbitrary.
In case (iii) and in the cases (i) and (ii) for $\alpha = 0, \mathfrak{g}$ is the two-dimensional abelian Lie algebra, in the cases (i) and (ii) for $\alpha \neq 0$ we have $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C})$.

Proof. This follows from the ansatz $\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \zeta \end{pmatrix}, R = \begin{pmatrix} \delta & \eta \\ \epsilon & \theta \end{pmatrix}$ when considering equations (10) and (11) for $x = t_1, y = e_1$. □

Remark 4.77. In the appendix, Section B.3, we list all possible post-Lie algebra structures on Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}, R)$, where $\mathfrak{n}$ is simply-complete and solvable (with the help of the classification of complete Lie algebras given in Section 4.2).

5 Post-Lie algebra structures on nilpotent Lie algebras

5.1 Post-Lie structures on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}\cong\mathfrak{n}\cong\mathfrak{n}_3(\mathbb{C})$

In this section, we want to list all post-Lie algebra structures on $(\mathfrak{g},\mathfrak{n})$, where both $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic to the three-dimensional Heisenberg algebra. These results have been published in [29].

Let us fix a basis $\{e_1,e_2,e_3\}$ of $\mathfrak{g}$ with Lie brackets $[e_1,e_2]=e_3$. We can express the Lie brackets of $\mathfrak{n}$ in terms of the basis $\{e_1,e_2,e_3\}$ as

$$\begin{aligned}\{e_1,e_2\}&=r_1e_1+r_2e_2+r_3e_3\\ \{e_1,e_3\}&=r_4e_1+r_5e_2+r_6e_3\\ \{e_2,e_3\}&=r_7e_1+r_8e_2+r_9e_3\end{aligned}$$

and define the structure vector $r\coloneqq(r_1,\ldots,r_9)$. Let $\operatorname{Ad}(x)$ be the adjoint operator of $x\in\mathfrak{n}$ with respect to $\{,\}$. Then, we obtain by using the software Mathematica the following classification:

Proposition 5.1. *Every post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}\cong\mathfrak{n}\cong\mathfrak{n}_3(\mathbb{C})$ is one of the following list. In all cases, we have $L(e_3)=-\frac{1}{2}\operatorname{Ad}(e_3)$ (and thus do not list $L(e_3)$ here).*

(1) The structure vector of $\mathfrak{n}$ is $r=(r_1,r_2,r_3,-\frac{r_1r_2}{r_3},-\frac{r_2^2}{r_3},-r_2,\frac{r_1^2}{r_3},\frac{r_1r_2}{r_3},r_1)$, $r_2,r_3\neq 0$ and

$$L(e_1)=\begin{pmatrix}\frac{r_1\alpha}{r_2} & -\frac{r_1(2r_1\alpha+r_2^2)}{2r_2^2} & \frac{r_1r_2}{2r_3}\\ \alpha & -\frac{2r_1\alpha+r_2^2}{2r_2} & \frac{r_2^2}{2r_3}\\ \beta & -\frac{2r_1\beta+r_2r_3}{2r_2} & \frac{r_2}{2}\end{pmatrix},\ L(e_2)=\begin{pmatrix}\frac{r_1(r_2^2-2r_1\alpha)}{2r_2^2} & \frac{r_1^3\alpha}{r_2^3} & -\frac{r_1^2}{2r_3}\\ \frac{r_2^2-2r_1\alpha}{2r_2} & \frac{r_1^2\alpha}{r_2^2} & -\frac{r_1r_2}{2r_3}\\ \frac{r_2(r_3-2)-2r_1\beta}{2r_2} & \frac{r_1(r_1\beta+r_2)}{r_2^2} & -\frac{r_1}{2}\end{pmatrix}.$$

(2) The structure vector of $\mathfrak{n}$ is $r=(r_1,0,r_3,0,0,0,\frac{r_1^2}{r_3},0,r_1)$, $r_3\neq 0$ and

$$L(e_1)=\begin{pmatrix}0 & -\frac{r_1}{2} & 0\\ 0 & 0 & 0\\ 0 & \frac{2-r_3}{2} & 0\end{pmatrix},\ L(e_2)=\begin{pmatrix}\frac{r_1}{2} & \alpha & -\frac{r_1^2}{2r_3}\\ 0 & 0 & 0\\ \frac{r_3}{2} & \beta & -\frac{r_1}{2}\end{pmatrix}.$$

(3) The structure vector of $\mathfrak{n}$ *is* $r = (0,0,r_3,0,0,0,0,0,0), r_3 \neq 0$ *and*

$$L(e_1) = \begin{pmatrix} \alpha & -\frac{\alpha^2}{\beta} & 0 \\ \beta & -\alpha & 0 \\ \gamma & \delta & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} -\frac{\alpha^2}{\beta} & \frac{\alpha^3}{\beta^2} & 0 \\ -\alpha & \frac{\alpha^2}{\beta} & 0 \\ r_3 - 1 + \delta & \frac{\alpha(\beta(1-r_3)-\alpha\gamma-2\beta\delta)}{\beta^2} & 0 \end{pmatrix}.$$

(4) The structure vector of $\mathfrak{n}$ *is* $r = (0,0,r_3,0,0,0,0,0,0), r_3 \neq 0$ *and*

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ r_3 - 1 + \beta & \delta & 0 \end{pmatrix}$$

where $\alpha\gamma = 0$.

(5) The structure vector of $\mathfrak{n}$ *is* $r = (0,0,0,r_4,r_5,0,-\frac{r_4^2}{r_5},-r_4,0), r_5 \neq 0$ *and*

$$L(e_1) = \begin{pmatrix} \frac{r_4\alpha}{r_5} & -\frac{r_4^2\alpha}{r_5^2} & -\frac{r_4}{2} \\ \alpha & -\frac{r_4\alpha}{r_5} & -\frac{r_5}{2} \\ \beta & -\frac{r_4\beta}{r_5} & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} -\frac{r_4^2\alpha}{r_5^2} & \frac{r_4^3\alpha}{r_5^3} & \frac{r_4^2}{2r_5} \\ -\frac{r_4\alpha}{r_5} & \frac{r_4^2\alpha}{r_5^2} & \frac{r_4}{2} \\ -\frac{r_4\beta+r_5}{r_5} & \frac{r_4(r_4\beta+r_5)}{r_5^2} & 0 \end{pmatrix}.$$

(6) The structure vector of $\mathfrak{n}$ *is* $r = (0,0,0,0,0,0,r_7,0,0), r_7 \neq 0$ *and*

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & \alpha & -\frac{r_7}{2} \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix}.$$

Apart from the stated conditions, all variables can be chosen arbitrarily in $\mathbb{C}$.

From Proposition 5.1, we obtain certain structure results:

Corollary 5.2. *Let* $x \cdot y$ *define a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$, *where* $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{n}_3(\mathbb{C})$. *Then all left-multiplication operators are nilpotent and we obtain the following identities:*

$$\begin{aligned} x \cdot \{y, z\} &= 0, \\ [x, y] \cdot z &= z \cdot [y, x] \\ [x, y \cdot z] + [x, z \cdot y] &= [y, x \cdot z] + [y, z \cdot x] \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$. *The bilinear map* $x \circ y$, *defined by the anti-commutator*

$$x \circ y := \frac{1}{2}(x \cdot y + y \cdot x)$$

is a CPA-structure on $\mathfrak{g}$.

Proof. Follows from the explicit classification in Proposition 5.1. □

For other Heisenberg algebras than the 3-dimensional one (see Section 5.2.2 for their definition), all of these identities fail and the anti-commutator does no longer define a CPA-structure in general.

It would be interesting to find ways to obtain the identities in Corollary 5.2 and also the surprising identity $L(e_3) = -\frac{1}{2}\operatorname{Ad}(e_3)$ without using the explicit classification given here.

5.2 CPA-structures on 2-step nilpotent stem Lie algebras

In this section, we associate CPA-structures to post-Lie algebra structures, LR-structures and pre-Lie algebra structures on 2-step nilpotent Lie algebras. These results have been published in [29].
Since we want to use Proposition 2.54, we assume the two-step nilpotent Lie algebras L in this section to be *stem*, that is, $\mathrm{Z}(L) \subseteq [L, L]$ (see Definition 2.53).

5.2.1 $\mathfrak{g}, \mathfrak{n}$ both 2-step nilpotent stem

Proposition 5.3. *Let $x \cdot y$ define a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are 2-step nilpotent and $\mathrm{Z} := \mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{n})$. Further suppose $\mathrm{Z} \cdot \mathfrak{g} = 0$. Then we can associate a CPA-structure to $x \cdot y$ by*

$$x \circ y := x \cdot y + \frac{1}{2}(\{x, y\} - [x, y])$$

if and only if $[x \cdot y, z] = [x \cdot z, y]$ for all $x, y, z \in \mathfrak{g}$.

Proof. Let us make two observations we will need later:

(i) Since $\mathfrak{g} \cdot \mathrm{Z} = \mathrm{Z} \cdot \mathfrak{g} + [\mathfrak{g}, \mathrm{Z}] - \{\mathfrak{g}, \mathrm{Z}\} = 0 + 0 + 0 = 0$, we find that all terms of the forms $a \cdot \{b, c\}, a \cdot [b, c], \{b, c\} \cdot a, [b, c] \cdot a, \{a, [b, c]\}, \{a, \{b, c\}\}, [a, \{b, c\}], [a, [b, c]], a, b, c \in \mathfrak{g}$ are zero.

(ii) We have $\{a \cdot b, c\} = \{b \cdot a, c\}$ and $[a \cdot b, c] = [b \cdot a, c]$ for all $a, b, c \in \mathfrak{g}$:
Indeed, axiom (PA1) gives us $\{a \cdot b, c\} = \{b \cdot a - [a, b] + \{a, b\}, c\} = \{b \cdot a, c\} + \{\mathrm{Z}, c\} = \{b \cdot a, c\}$. The same proof also works for the bracket $[,]$.

Define $a \circ b := a \cdot b + \frac{1}{2}(\{a, b\} - [a, b])$. We shall show that $x \circ y$ satisfies axioms (CPA1), (CPA2), (CPA3) if and only if $[x \cdot y, z] = [x \cdot z, y]$ for all $x, y, z \in \mathfrak{g}$.
Axiom (CPA1) is clear by definition.
Axiom (CPA3): We have to show

$$a \circ [b, c] - [a \circ b, c] - [b, a \circ c] = 0.$$

Using the definition of $x \circ y$, the left-hand side reads

$$\begin{aligned} a \circ [b,c] - [a \circ b, c] - [b, a \circ c] &= a \cdot [b,c] - \frac{1}{2}[a,[b,c]] + \frac{1}{2}\{a,[b,c]\} \\ &- [a \cdot b, c] + \frac{1}{2}[[a,b],c] - \frac{1}{2}[\{a,b\},c] - [b, a \cdot c] + \frac{1}{2}[b,[a,c]] - \frac{1}{2}[b,\{a,c\}] \\ &\overset{\text{(i)}}{=} -[a \cdot b, c] - [b, a \cdot c] = -([a \cdot b, c] + [b, a \cdot c]). \end{aligned}$$

Axiom (CPA2): We have to show

$$[a,b] \circ c - a \circ (b \circ c) + b \circ (a \circ c) = 0.$$

Using the definition of $x \circ y$, the left-hand side reads

$$\begin{aligned} [a,b] \circ c - a \circ (b \circ c) + b \circ (a \circ c) &= [a,b] \cdot c - \frac{1}{2}[[a,b],c] + \frac{1}{2}\{[a,b],c\} \\ &- a \cdot (b \cdot c) + \frac{1}{2} a \cdot [b,c] - \frac{1}{2} a \cdot \{b,c\} + \frac{1}{2}[a, b \cdot c] - \frac{1}{4}[a,[b,c]] + \frac{1}{4}[a,\{b,c\}] \\ &- \frac{1}{2}\{a, b \cdot c\} + \frac{1}{4}\{a,[b,c]\} - \frac{1}{4}\{a,\{b,c\}\} + b \cdot (a \cdot c) - \frac{1}{2} b \cdot [a,c] + \frac{1}{2} b \cdot \{a,c\} \\ &- \frac{1}{2}[b, a \cdot c] + \frac{1}{4}[b,[a,c]] - \frac{1}{4}[b,\{a,c\}] + \frac{1}{2}\{b, a \cdot c\} - \frac{1}{4}\{b,[a,c]\} + \frac{1}{4}\{b,\{a,c\}\} \\ &\overset{\text{(i)}}{=} [a,b] \cdot c - a \cdot (b \cdot c) + \frac{1}{2}[a, b \cdot c] - \frac{1}{2}\{a, b \cdot c\} + b \cdot (a \cdot c) - \frac{1}{2}[b, a \cdot c] + \frac{1}{2}\{b, a \cdot c\} \\ &= \frac{1}{2}[a, b \cdot c] - \frac{1}{2}\{a, b \cdot c\} - \frac{1}{2}[b, a \cdot c] + \frac{1}{2}\{b, a \cdot c\} \\ &= \frac{1}{2}([a, b \cdot c] - [b, a \cdot c] + \{b, a \cdot c\} - \{a, b \cdot c\}) \\ &= \frac{1}{2}([a, b \cdot c] - [b, a \cdot c] + \{b, a \cdot c\} + \{b \cdot c, a\}) \\ &\overset{\text{(ii)}}{=} \frac{1}{2}([a, b \cdot c] - [b, a \cdot c] + \{b, c \cdot a\} + \{c \cdot b, a\}) \\ &= \frac{1}{2}([a, b \cdot c] - [b, a \cdot c] + c \cdot \{b, a\}) \\ &\overset{\text{(i)}}{=} \frac{1}{2}([a, b \cdot c] - [b, a \cdot c]) = \frac{1}{2}([a \cdot c, b] + [a, b \cdot c]) \overset{\text{(ii)}}{=} \frac{1}{2}([c \cdot a, b] + [a, c \cdot b]). \end{aligned}$$

So both axiom (CPA3) and axiom (CPA2) are equivalent to the identity $[x \cdot y, z] = [x \cdot z, y]$ for all $x, y, z \in \mathfrak{g}$. ☐

Remark 5.4. Similarly, if $\mathfrak{g}$ is any Lie algebra, $\mathfrak{n}$ two-step nilpotent and $x \cdot y$ a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, then $x \circ y := x \cdot y + \frac{1}{2}\{x, y\}$ defines a pre-Lie algebra structure on $\mathfrak{g}$. In particular, if there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ two-step nilpotent, then $\mathfrak{g}$ cannot be semisimple (as semisimple Lie algebras do not admit pre-Lie algebra structures, cf. Table 5). This has also been proven in [28, Proposition 4.2].

Remark 5.5. The identity $[x \cdot y, z] = [x \cdot z, y]$ for all $x, y, z \in \mathfrak{g}$ means that all left-multiplication operators $L(x), x \in \mathfrak{g}$, are $(0,1,1)$-derivations of $\mathfrak{g}$ in the sense of [82]. In the sense of Leger and Luks ([68]), the $L(x)$ are quasiderivations of $\mathfrak{g}$.

Remember that a CPA-structure (or LR-structure or pre-Lie algebra structure) is *complete* if all right-multiplication operators are nilpotent (see Definition 2.39). When working with nilpotent Lie algebras, we may, however, also consider left-multiplication operators instead:

Lemma 5.6. *Given a CPA-structure, an LR-structure or a pre-Lie algebra structure* $x \cdot y$ *on a nilpotent Lie algebra* $\mathfrak{g}$. *Then all right-multiplication operators* $R(x)\colon \mathfrak{g} \to \mathfrak{g}, y \mapsto y \cdot x$ *are nilpotent if and only if all left-multiplication operators* $L(x)\colon \mathfrak{g} \to \mathfrak{g}, y \mapsto x \cdot y$ *are nilpotent.*

Proof. For CPA-structures, this is clear since $L(x) = R(x)$. For LR-structures, see [27, Proposition 2.2] and for pre-Lie algebra structures, see [65, Theorem 2.1 and Theorem 2.2]. $\square$

Corollary 5.7. *Let* $\mathfrak{g}, \mathfrak{n}$ *be* 2*-step nilpotent and stem,* $x \cdot y$ *a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$. *If* $\mathrm{Z} = \mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{n}), \mathrm{Z} \cdot \mathfrak{g} = 0$ *and* $[x \cdot y, z] = [x \cdot z, y]$ *for all* $x, y, z \in \mathfrak{g}$, *then the post-Lie algebra structure* $x \cdot y$ *is complete.*

Proof. Let $\widetilde{L}(x)$ be the left-multiplication operator with respect to $x \cdot y$. By Proposition 5.3, $x \circ y := x \cdot y + \frac{1}{2}(\{x, y\} - [x, y])$ is a CPA-structure; let $\widetilde{L}(x)$ be the left-multiplication operators with respect to $x \circ y$. We have $L(x) = \widetilde{L}(x) - \frac{1}{2}\operatorname{ad}(x) + \frac{1}{2}\operatorname{Ad}(x)$. (Here, $\operatorname{ad}(x)$ and $\operatorname{Ad}(x)$ define the adjoint operators with respect to $\mathfrak{g}$ and $\mathfrak{n}$, respectively.) As $\widetilde{L}(x)$ is associated to a CPA-structure, it is nilpotent (Proposition 2.54), say $(\widetilde{L}(x))^s = 0$. Expanding $(L(x))^{s+1}$ gives

$$(L(x))^{s+1} = \sum A_1 \circ \ldots \circ A_{s+1},$$

where each A_i is one of the operators $-\frac{1}{2}\operatorname{ad}(x), \frac{1}{2}\operatorname{Ad}(x)$ and $\widetilde{L}(x)$.
Now if $A_i, i \neq 1$ is $\operatorname{ad}(x)$ or $\operatorname{Ad}(x)$, we have $A_i(y) \in \mathrm{Z}(\mathfrak{g})$ for all $y \in \mathfrak{g}$, thus $A_{i-1} \circ A_i(y) \in A_{i-1}(\mathrm{Z}(\mathfrak{g})) = 0$. So from the sum, only the terms $A_1 \circ (\widetilde{L}(x))^s$ remain; however, they are 0 by the nilpotency of $\widetilde{L}(x)$. $\square$

5.2.2 $\mathfrak{g}, \mathfrak{n}$ both Heisenberg algebras

Let us define the n-dimensional Heisenberg algebra ($n \geq 3, n$ odd) to have a basis $\{p_1, \ldots, p_{\frac{n-1}{2}}, q_1, \ldots, q_{\frac{n-1}{2}}, z\}$ and Lie brackets $[p_i, q_i] = z, i \in \{1, \ldots, \frac{n-1}{2}\}$. It is 2-step nilpotent and stem.
We show that the condition "$\mathrm{Z} \cdot \mathfrak{g} = 0$" of Corollary 5.7 is fulfilled if $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic to a Heisenberg algebra of dimension ≥ 5:

Proposition 5.8. *Let* $x \cdot y$ *define a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$, *where* $\mathfrak{g}$ *and* $\mathfrak{n}$ *are both isomorphic to a Heisenberg Lie algebra of dimension* ≥ 5 *and* $\mathrm{Z} := \mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{n})$. *Then* $\mathrm{Z} \cdot \mathfrak{g} = 0$. *(In particular, every CPA-structure on* $\mathfrak{g}$ *satisfies* $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ *if* $(\mathfrak{g}, [,])$ *is a Heisenberg algebra of dimension* ≥ 5.*)*

Proof. Note that if x is a basis element of $\mathfrak{n}$, we can find elements $p, q \in \mathfrak{n}$ such that $\{p, q\} = z, \{p, x\} = 0 = \{q, x\}$. Let us make two observations:

(i) By (PA3), we have

$$\{b,c\} = 0 \Rightarrow \{a\cdot b, c\} = \{a\cdot c, b\} \text{ for all } a \in \mathfrak{n}.$$

Indeed, by axiom (PA3), $0 = a\cdot\{b,c\} = \{a\cdot b, c\} + \{b, a\cdot c\}$ implying $\{a\cdot b, c\} = \{a\cdot c, b\}$.

(ii) We have $\{a\cdot b, c\} = \{b\cdot a, c\}$ for all $a,b,c \in \mathfrak{g}$ (by the same argument as in Proposition 5.3).

Now let $x \notin \mathrm{Z}$ be a basis element of $\mathfrak{n}$ and let $z \in \mathrm{Z}$. Choose $p,q \in \mathfrak{n}$ such that $\{p,q\} = z, \{p,x\} = 0 = \{q,x\}$. Then

$$\{p, x\cdot q\} = -\{x\cdot q, p\} \overset{\text{(ii)}}{=} -\{q\cdot x, p\} \overset{\text{(i)}}{=} -\{q\cdot p, x\} \overset{\text{(ii)}}{=} -\{p\cdot q, x\} \overset{\text{(i)}}{=} -\{p\cdot x, q\} \overset{\text{(ii)}}{=} -\{x\cdot p, q\}.$$

Now by axiom (PA3),

$$x\cdot z = x\cdot\{p,q\} = \{x\cdot p, q\} + \{p, x\cdot q\} = \{x\cdot p, q\} - \{x\cdot p, q\} = 0,$$

so $\mathfrak{n}\cdot \mathrm{Z} = 0$ and thus also $\mathrm{Z}\cdot\mathfrak{n} = \mathfrak{n}\cdot\mathrm{Z} - [\mathrm{Z},\mathfrak{n}] + \{\mathrm{Z},\mathfrak{n}\} = 0$.
To show $z\cdot z = 0$, take again two elements p,q with $\{p,q\} = z$:

$$z\cdot z = z\cdot\{p,q\} \overset{\text{(PA3)}}{=} \{z\cdot p, q\} + \{p, z\cdot q\} \overset{\text{(ii)}}{=} \{p\cdot z, q\} + \{p, q\cdot z\} = 0 + 0 = 0.$$

□

From Corollary 5.7 and Proposition 5.8, we get:

Corollary 5.9. *Let $x\cdot y$ define a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are both isomorphic to a Heisenberg Lie algebra of dimension ≥ 5 and $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}(\mathfrak{n})$. Then we can associate a CPA-structure to $x\cdot y$ by*

$$x\circ y := x\cdot y - \frac{1}{2}[x,y] + \frac{1}{2}\{x,y\}$$

if and only if $[x\cdot y, z] = [x\cdot z, y]$ for all $x,y,z \in \mathfrak{g}$. In particular, all post-Lie algebra structures on $(\mathfrak{g},\mathfrak{n})$ with $[x\cdot y, z] = [x\cdot z, y]$ for all $x,y,z \in \mathfrak{g}$ are complete.

5.2.3 $\mathfrak{n}$ a 2-step nilpotent stem Lie algebra, $\mathfrak{g}$ an abelian Lie algebra

Now let $\mathfrak{g}$ be an abelian Lie algebra and $\mathfrak{n}$ a 2-step nilpotent stem Lie algebra of dimension ≥ 5.
We can state the analogous statement to Proposition 5.3:

Proposition 5.10. *Let $x\cdot y$ define a post-Lie algebra structure on $(\mathfrak{g},\mathfrak{n})$, where $\mathfrak{g}$ is abelian and $\mathfrak{n}$ is a 2-step nilpotent stem Lie algebra. If $\mathrm{Z}(\mathfrak{n})\cdot\mathfrak{n} = 0$, then the bilinear map defined by $x\circ y := x\cdot y + \frac{1}{2}\{x,y\}$ is a CPA-structure both on $(\mathfrak{g},[,])$ and on $(\mathfrak{n},\{,\})$.*

Proof. Exactly as the proof of Proposition 5.3. □

Corollary 5.11. *Let $\mathfrak{n}$ be a two-step stem nilpotent Lie algebra. Then there is a bijection*

$$\{LR\text{-}structures\ on\ \mathfrak{n}\ with\ \mathrm{Z}(\mathfrak{n})\cdot\mathfrak{n}=0\} \longleftrightarrow \{CPA\text{-}structures\ on\ \mathfrak{n}\ with\ \mathrm{Z}(\mathfrak{n})\cdot\mathfrak{n}=0\},$$
$$x\cdot y \longmapsto -x\cdot y+\frac{1}{2}\{x,y\},$$
$$-(x\circ y-\frac{1}{2}\{x,y\}) \longleftarrow x\circ y.$$

Proof. Note that the transformation preserves the property $\mathrm{Z}(\mathfrak{n})\cdot\mathfrak{n}=0$.
If $x\cdot y$ is an LR-structure, then $-x\cdot y$ is a post-Lie algebra structure on $(\mathbb{C}^n,\mathfrak{n})$ (where $\mathbb{C}^n$ is the abelian Lie algebra with dimension $n\coloneqq\dim(\mathfrak{n})$) and by Proposition 5.10, $-x\cdot y+\frac{1}{2}\{x,y\}$ is a CPA-structure on $\mathfrak{n}$.
Let $x\circ y$ define a CPA-structure on $\mathfrak{n}$ with $\mathrm{Z}(\mathfrak{n})\circ\mathfrak{n}=0$, we show that $x\cdot y\coloneqq x\circ y-\frac{1}{2}\{x,y\}$ satisfies the following two axioms from Definition 2.37:

$$x\cdot(y\cdot z)=y\cdot(x\cdot z) \tag{LR1}$$
$$(x\cdot y)\cdot z=(x\cdot z)\cdot y \tag{LR2}$$

for all $x,y,z\in\mathfrak{n}$.
Axiom (LR1):

$$\begin{aligned}
x\cdot(y\cdot z)-y\cdot(x\cdot z) &= x\cdot(y\circ z)-\frac{1}{2}x\cdot\{y,z\}-y\cdot(x\circ z)+\frac{1}{2}y\cdot\{x,z\}\\
&= x\circ(y\circ z)-\frac{1}{2}\{x,y\circ z\}-\frac{1}{2}x\circ\{y,z\}+\frac{1}{4}\{x,\{y,z\}\}\\
&\quad -y\circ(x\circ z)+\frac{1}{2}\{y,x\circ z\}+\frac{1}{2}y\circ\{x,z\}-\frac{1}{4}\{y,\{x,z\}\}\\
&= x\circ(y\circ z)-y\circ(x\circ z)+\frac{1}{2}\left(\{y,x\circ z\}-\{x,y\circ z\}\right)\\
&= \{x,y\}\circ z+\frac{1}{2}\left(\{y,x\circ z\}+\{y\circ z,x\}\right)\\
&= \{x,y\}\circ z+\frac{1}{2}\left(\{y,z\circ x\}+\{z\circ y,x\}\right)\\
&= \frac{1}{2}\left(z\circ\{y,x\}\right)=0.
\end{aligned}$$

Axiom (LR2) is a similar computation:

$$\begin{aligned}
(x\cdot y)\cdot z-(x\cdot z)\cdot y &= (x\circ y)\cdot z-\frac{1}{2}\{x,y\}\cdot z-(x\circ z)\cdot y+\frac{1}{2}\{x,z\}\cdot y\\
&= (x\circ y)\circ z-\frac{1}{2}\{x\circ y,z\}-\frac{1}{2}\{x,y\}\circ z+\frac{1}{4}\{\{x,y\},z\}\\
&\quad -(x\circ z)\circ y+\frac{1}{2}\{x\circ z,y\}+\frac{1}{2}\{x,z\}\circ y-\frac{1}{4}\{\{x,z\},y\}\\
&= (x\circ y)\circ z-(x\circ z)\circ y+\frac{1}{2}(\{z,x\circ y\}+\{x\circ z,y\})\\
&= z\circ(y\circ x)-y\circ(z\circ x)+\frac{1}{2}(x\circ\{z,y\})\\
&= \{z,y\}\circ x=0.
\end{aligned}$$

So $x \cdot y$ is an LR-structure on $\mathfrak{n}$. $\square$

Similarly as in Corollary 5.7, this means:

Corollary 5.12. *If $\mathfrak{n}$ is two-step nilpotent and stem, then all LR-structures on $\mathfrak{n}$ satisfying $\mathrm{Z} \cdot \mathfrak{n} = 0$ are complete.*

Proof. Let $x \cdot y$ be a LR-structure on $\mathfrak{n}$ and $L(x)$ be the left-multiplication operator of x with respect to $x \cdot y$. By Corollary 5.11, we know there is a CPA-structure $x \circ y$ on $\mathfrak{n}$ with $L(x) = -\widetilde{L}(x) + \frac{1}{2}\operatorname{ad}(x)$, where $\widetilde{L}(x)$ denotes the left-multiplication operator of x with respect to $x \circ y$. We know that "$\circ$" is nilpotent, say $(\widetilde{L}(x))^s = 0$. Expanding $(L(x))^{s+1}$ gives

$$(L(x))^{s+1} = \sum A_1 \circ \ldots \circ A_{s+1},$$

where each A_i is one of the operators $\frac{1}{2}\operatorname{ad}(x)$ and $-\widetilde{L}(x)$.
Now if $A_i, i \neq 1$ is the ad-operator, we have $A_i(y) \in \mathrm{Z}(\mathfrak{n})$ for all $y \in \mathfrak{n}$, thus $A_{i-1} \circ A_i(y) \in A_{i-1}(\mathrm{Z}(\mathfrak{n})) = 0$. So from the sum, only the terms $A_1 \circ (\widetilde{L}(x))^s$ remain; however, they are 0 by the nilpotency of $\widetilde{L}(x)$. $\square$

Remark 5.13. There are indeed two-step nilpotent Lie algebras $\mathfrak{n}$ with LR-structures *not* being complete: Take e.g. the five-dimensional two-step nilpotent Lie algebra $\mathfrak{n} = L_{5,8}$ from [43] with Lie brackets $\{e_1, e_2\} = e_4, \{e_1, e_3\} = e_5$. Then a non-complete LR-structure on $\mathfrak{n}$ is given by

$$e_1 \cdot e_1 = e_1, e_1 \cdot e_4 = e_4, e_1 \cdot e_5 = e_5, e_2 \cdot e_1 = -e_4, e_3 \cdot e_1 = -e_5, e_4 \cdot e_1 = e_4, e_5 \cdot e_1 = e_5.$$

One cannot associate a CPA-structure to this LR-structure in the way described above.

Remark 5.14. Another LR-structure on the same Lie algebra $\mathfrak{n} = L_{5,8}$ shows that not all complete LR-structures satisfy $\mathrm{Z}(\mathfrak{n}) \cdot \mathfrak{n} = 0$: An LR-structure on $\mathfrak{n}$ (with respect to the basis given above) is given by

$$e_2 \cdot e_1 = -e_4, e_3 \cdot e_1 = -e_5, e_3 \cdot e_3 = e_2, e_3 \cdot e_5 = e_4, e_5 \cdot e_3 = e_4.$$

All left-multiplication operators are three-step nilpotent; however, $\mathrm{Z}(\mathfrak{n}) \cdot \mathfrak{n} \neq 0$.
Again, one cannot associate a CPA-structure in the way described above to this LR-structure.

5.2.4 $\mathfrak{g}$ abelian, $\mathfrak{n}$ Heisenberg

Let us specialize Corollary 5.12 to the situation where $\mathfrak{n}$ is a Heisenberg Lie algebra of dimension ≥ 5. We find, similar to Proposition 5.8:

Proposition 5.15. *Let $x \cdot y$ define a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is abelian and $\mathfrak{n}$ is isomorphic to a Heisenberg Lie algebra of dimension ≥ 5. Then $\mathrm{Z}(\mathfrak{n}) \cdot \mathfrak{n} = \mathfrak{n} \cdot \mathrm{Z}(\mathfrak{n}) = 0$.*

Proof. The proof is the same as the one of Proposition 5.8. $\square$

So in the "Heisenberg of dimension ≥ 5"-case, the assumption $\mathrm{Z}(\mathfrak{n}) \cdot \mathfrak{n} = 0$ of Corollaries 5.11 and 5.12 becomes superfluous:

Corollary 5.16. *Let $\mathfrak{n}$ be a Heisenberg algebra of dimension ≥ 5. Then there is a bijection*

$$\begin{aligned} \{\textit{LR-structures on } \mathfrak{n}\} &\longleftrightarrow \{\textit{commutative post-Lie algebra structures on } \mathfrak{n}\}, \\ x \cdot y &\longmapsto -x \cdot y + \frac{1}{2}\{x, y\}, \\ -(x \circ y - \frac{1}{2}\{x, y\}) &\longleftarrow\!\shortmid\, x \circ y. \end{aligned}$$

Corollary 5.17. *All LR-structures on $\mathfrak{n}$, where $\mathfrak{n}$ is a Heisenberg algebra of dimension ≥ 5, are complete.*

What about the 3-dimensional Heisenberg algebra $\mathfrak{n}_3 = \mathfrak{n}_3(\mathbb{C})$? By inspecting the classification in [26, Proposition 3.1], one finds:

Remark 5.18. There is an LR-structure on $\mathfrak{n}_3$ which does not satisfy $\mathrm{Z}(\mathfrak{n}_3) \cdot \mathfrak{n}_3 = 0$; namely the structure A_4 from [26, Proposition 3.1]:

$$e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = e_2, e_2 \cdot e_3 = e_3, e_3 \cdot e_2 = e_3$$

(where $\{e_1, e_2\} = e_3$). Here, $e_2 \cdot \{e_1, e_2\} = e_2 \cdot e_3 = e_3 \neq 0$.

However, all *complete* LR-structures on the 3-dimensional Heisenberg algebra satisfy $\mathfrak{n}_3 \cdot \{\mathfrak{n}_3, \mathfrak{n}_3\} = 0$. Thus there is a bijection

$$\begin{aligned} \{\text{complete LR-structures on } \mathfrak{n}_3\} &\longleftrightarrow \{\text{commutative post-Lie algebra structures on } \mathfrak{n}_3\}, \\ x \cdot y &\longmapsto -x \cdot y + \frac{1}{2}\{x, y\}, \\ -(x \circ y - \frac{1}{2}\{x, y\}) &\longleftarrow\!\shortmid\, x \circ y. \end{aligned}$$

Proof. We only have to check that if $x \circ y$ is a CPA-structure, then $x \circ y - \frac{1}{2}\{x, y\}$ is a *complete* LR-structure. But this follows as in Corollary 5.12. $\square$

5.2.5 $\mathfrak{g}$ two-step nilpotent stem, $\mathfrak{n}$ abelian

Now we ask the opposite question: Let $x \cdot y$ define a *pre-Lie algebra structure* on the 2-step nilpotent Lie algebra $\mathfrak{g}$. When does $x \circ y \coloneqq x \cdot y - \frac{1}{2}[x, y]$ define a CPA-structure on $\mathfrak{g}$?

We remind the reader about the definition of a pre-Lie algebra structure (see Definition 2.38): A *pre-Lie algebra structure on* $\mathfrak{g}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathbb{C}^n)$, where $\mathbb{C}^n$ is the abelian Lie algebra of dimension $n \coloneqq \dim(\mathfrak{g})$.

Here the important condition for the structure $x \cdot y$ is to make all left-multiplication operators into derivations:

Proposition 5.19. *Let $x \cdot y$ define a pre-Lie algebra structure on the two-step nilpotent stem Lie algebra $\mathfrak{g}$. The bilinear map defined by $x \circ y := x \cdot y - \frac{1}{2}[x, y]$ is a CPA-structure on $\mathfrak{g}$ if and only if all left-multiplication operators $L(x)$ (with respect to the pre-Lie algebra structure) are derivations of $\mathfrak{g}$.*

Proof. The structure $x \circ y$ is clearly commutative; it remains to check that it defines a post-Lie algebra structure if and only if the left-multiplication operators are derivations. Axiom (CPA3) becomes

$$\begin{aligned} & a \circ [b, c] - [a \circ b, c] - [b, a \circ c] \\ & \quad = a \cdot [b, c] - \frac{1}{2}[a, [b, c]] - [a \cdot b, c] + \frac{1}{2}[[a, b], c] - [b, a \cdot c] + \frac{1}{2}[b, [a, c]] \\ & \quad = a \cdot [b, c] - [a \cdot b, c] - [b, a \cdot c] \end{aligned}$$

which is zero precisely if all left-multiplication operators (with respect to $x \cdot y$) are derivations of $\mathfrak{g}$. Suppose that they are; then axiom (CPA2) reads

$$\begin{aligned} & [a, b] \circ c - a \circ (b \circ c) + b \circ (a \circ c) \\ & \quad = [a, b] \cdot c - \frac{1}{2}[[a, b], c] - a \circ (b \cdot c) + \frac{1}{2} a \circ [b, c] + b \circ (a \cdot c) - \frac{1}{2} b \circ [a, c] \\ & \quad = [a, b] \cdot c - \frac{1}{2}[[a, b], c] - a \cdot (b \cdot c) + \frac{1}{2}[a, b \cdot c] + \frac{1}{2} a \cdot [b, c] \\ & \quad - \frac{1}{4}[a, [b, c]] + b \cdot (a \cdot c) - \frac{1}{2}[b, a \cdot c] - \frac{1}{2} b \cdot [a, c] + \frac{1}{4}[b, [a, c]] \\ & \quad = [a, b] \cdot c - a \cdot (b \cdot c) + \frac{1}{2}[a, b \cdot c] + \frac{1}{2} a \cdot [b, c] + b \cdot (a \cdot c) - \frac{1}{2}[b, a \cdot c] - \frac{1}{2} b \cdot [a, c] \\ & \quad = [a, b] \cdot c - a \cdot (b \cdot c) + b \cdot (a \cdot c) + \frac{1}{2}([a, b \cdot c] + a \cdot [b, c] - [b, a \cdot c] - b \cdot [a, c]) \\ & \quad = 0 + \frac{1}{2}([a, b \cdot c] + [a \cdot b, c] + [b, a \cdot c] - [b, a \cdot c] - [b \cdot a, c] - [a, b \cdot c]) \\ & \quad = \frac{1}{2}([a \cdot b, c] - [b \cdot a, c]) = \frac{1}{2}([a \cdot b, c] - [a \cdot b + [b, a], c]) = \frac{1}{2}([a \cdot b, c] - [a \cdot b, c]) = 0. \end{aligned}$$

□

Remark 5.20. On the 3-dimensional Heisenberg Lie algebra $\mathfrak{n}_3(\mathbb{C})$, there are pre-Lie algebra structures where not all left-multiplication operators are derivations, e.g. the structure defined by

$$L(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Neither $L(e_1)$ nor $L(e_2)$ is a derivation of $\mathfrak{n}_3(\mathbb{C})$.

Corollary 5.21. *Let $\mathfrak{g}$ be a two-step nilpotent stem Lie algebra. Then there is a bijection*

$$\left\{\begin{array}{c}\text{pre-Lie algebra structures on } \mathfrak{g} \text{ where all}\\ \text{left-multiplication operators are derivations}\end{array}\right\} \longleftrightarrow \{\text{CPA-structures on } \mathfrak{g}\}$$

$$\begin{aligned} x \cdot y &\longmapsto x \cdot y - \frac{1}{2}[x,y] \\ x \circ y + \frac{1}{2}[x,y] &\longleftarrow x \circ y. \end{aligned}$$

Remark 5.22. Pre-Lie algebra structures where all left-multiplication operators are derivations have been studied by Medina in [72, 73].

Proof. We have shown so far that for each pre-Lie algebra structure $x \cdot y$ (with corresponding left-multiplication operator $L(x)$) on $\mathfrak{g}$ such that all $L(x)$ are derivations, one can associate a CPA-structure $x \circ y$ on $\mathfrak{g}$. In particular, all $\widetilde{L}(x)$ (where $\widetilde{L}(x)$ is the left-multiplication operator with respect to the map $x \circ y$) are derivations of $\mathfrak{g}$ by (CPA3); so $L(x) = \widetilde{L}(x) + \frac{1}{2}\operatorname{ad}(x)$ is (as a sum of derivations) too. Similar as in Proposition 5.3, one checks that $x \cdot y$ indeed defines a pre-Lie algebra structure on $\mathfrak{g}$. □

The bijection gives us, as in Corollary 5.12:

Corollary 5.23. *A pre-Lie algebra structure on a two-step nilpotent stem Lie algebra, where all left-multiplication operators are derivations, is complete.*

So we have established two bijections onto the set of CPA-structures. If we put them together (in case of the Heisenberg algebra), we obtain:

Corollary 5.24. *Let $(\mathfrak{g}, [,])$ be a Heisenberg algebra of dimension ≥ 5. Then there is a bijection*

$$\{\text{LR-structures on } \mathfrak{g}\} \longleftrightarrow \left\{\begin{array}{c}\text{pre-Lie algebra structures on } \mathfrak{g} \text{ where all}\\ \text{left-multiplication operators are derivations}\end{array}\right\}$$

$$\begin{aligned} x \cdot y &\longmapsto -x \cdot y + [x,y], \\ -(x \circ y - [x,y]) &\longleftarrow x \circ y. \end{aligned}$$

One can also take $\mathfrak{g}$ to be a Heisenberg algebra of dimension 3 – then the term "LR-structures" has to be replaced with "complete LR-structures".

5.3 CPA-structures on the Lie algebra of strictly upper triangular matrices

Given a matrix $A = (A_{j,k})_{j,k}$, we call the collection of entries at positions $A_{j,k}, k - j = i$ the *i-th diagonal*. The i-th diagonal is above (below) the main diagonal if $i > 0$ ($i < 0$) and equals the main diagonal if $i = 0$. The 1-th diagonal is the *superdiagonal*, the -1-th diagonal the *subdiagonal* of A.

We are now interested in CPA-structures on the Lie algebra $\mathfrak{g}$ of strictly upper triangular matrices of size n (which is m-step nilpotent, where $m \coloneqq n-1$). It has a natural grading $\mathfrak{g} = \bigoplus_{i=1}^{n-1} \mathfrak{g}_i$, where $\mathfrak{g}_i = \operatorname{span}\{E_{j,k}, 1 \leq j < k \leq n \colon k-j = i\}$ is the set of strictly upper triangular matrices with zeros outside of the i-th diagonal. We have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$; the non-zero Lie brackets are given as $[E_{j,k}, E_{k,\ell}] = E_{j,\ell}$.

Lemma 5.25. *Let $\mathfrak{g}$ be the Lie algebra of strictly upper triangular matrices of size n and $m \coloneqq n-1$. If $[a, [\mathfrak{g}, \mathfrak{g}]] = 0$ for an $a \in \mathfrak{g}$, then $a \in \mathfrak{g}^{m-3}$.*

Proof. Suppose $[a, [\mathfrak{g}, \mathfrak{g}]] = 0$ and write $a = \sum_{1 \leq i < j \leq n} \alpha_{(i,j)} E_{i,j}$ as linear combination of the elementary matrices $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}, 1 \leq i < j \leq n$. The matrices $E_{1,3}, E_{2,4}, E_{3,5}, \ldots,$ $E_{n-2,n}$ are in $[\mathfrak{g}, \mathfrak{g}]$; we compute $[a, b]$, where b is one of these matrices:

- $[a, E_{1,3}] = -\sum_{j=4}^{n} \alpha_{(3,j)} E_{1,j}$
- $[a, E_{2,4}] = \sum_{i=1}^{1} \alpha_{(i,2)} E_{i,4} - \sum_{j=5}^{n} \alpha_{(4,j)} E_{2,j}$
- $[a, E_{3,5}] = \sum_{i=1}^{2} \alpha_{(i,3)} E_{i,5} - \sum_{j=6}^{n} \alpha_{(5,j)} E_{3,j}$
- ...
- $[a, E_{n-2,n}] = \sum_{i=1}^{n-3} \alpha_{(i,n-2)} E_{i,n}$.

So as $[a, [\mathfrak{g}, \mathfrak{g}]] = 0$, all of these expressions vanish, thus what remains is $a = \alpha_{(1,n-1)} E_{1,n-1} + \alpha_{(2,n-1)} E_{2,n-1} + \alpha_{(1,n)} E_{1,n} + \alpha_{(2,n)} E_{2,n}$. But this means $a \in \mathfrak{g}^{m-3}$. $\square$

From [83, Theorem 3.2], we infer the form of $\mathfrak{g}$'s nilpotent derivations:

Lemma 5.26 ([83])**.** *Every nilpotent derivation $D \in \operatorname{Der}(\mathfrak{g})$ has the form $D = \operatorname{ad}(x) + \psi, x \in \mathfrak{g}, \psi \in \operatorname{Der}(\mathfrak{g})$, where ψ satisfies $\psi(\mathfrak{g}) \subseteq \mathfrak{g}^{m-2}, \psi([\mathfrak{g}, \mathfrak{g}]) = 0$.*

Proposition 5.27. *Let $m = n-1$ and let $\mathfrak{g}$ be the (m-step nilpotent) Lie algebra of strictly upper triangular matrices of size $n \times n$, where $n \geq 5$. Then all CPA-structures on $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{m-2}$ and $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.*

Proof. For $n = 5, 6$, we compute the CPA-structures directly. So let us assume $n \geq 7$ – we want to show the result inductively. Let $x \cdot y$ be a CPA-structure on $\mathfrak{g}$. By Proposition 2.54, we know that all left-multiplication operators with respect to $x \cdot y$ are nilpotent derivations.
By Lemma 5.26, we may assume that for every $g \in \mathfrak{g}$, the left-multiplication operator $L(g)$ is of the form $L(g) = \operatorname{ad}(x_g) + \psi_g$ for some $x_g \in \mathfrak{g}, \psi_g \in \operatorname{Der}(\mathfrak{g})$ with $\psi_g([\mathfrak{g}, \mathfrak{g}]) = 0, \psi_g(\mathfrak{g}) \subseteq \mathfrak{g}^{m-2}$.

If $E_{i,j}$ denotes the matrix with 1 at position (i, j) and 0 else, the Lie brackets between basis elements of $\mathfrak{g}$ are given as

$$\begin{aligned}
[E_{i,j}, E_{k,\ell}] &= E_{i,\ell} \text{ if } j = k \\
[E_{i,j}, E_{k,\ell}] &= -E_{k,i} \text{ if } i = \ell \\
[E_{i,j}, E_{k,\ell}] &= 0 \text{ otherwise.}
\end{aligned}$$

This implies that the first row, $\mathfrak{a} \coloneqq \operatorname{span}\{E_{1,i}, i = 2, \ldots, n\}$ is an Lie ideal in $\mathfrak{g}$. Also, the last column, $\mathfrak{b} \coloneqq \operatorname{span}\{E_{i,n}, i = 1, \ldots, n-1\}$ is an Lie ideal in $\mathfrak{g}$.
Let us also consider the ideal $\mathfrak{g}^{m-2} = \operatorname{span}\{E_{1,n-1}, E_{2,n}, E_{1,n}\}$. We write $\mathfrak{a}' \coloneqq \mathfrak{a} + \mathfrak{g}^{m-2}$ and observe that

$$\mathfrak{g}/\mathfrak{a}' \cong \mathfrak{h}/\operatorname{Z}(\mathfrak{h}),$$

where $\mathfrak{h}$ is the Lie algebra of strictly upper triangular $(n-1) \times (n-1)$-matrices.
By induction, we know that every CPA-structure on $\mathfrak{h}$ satisfies $\mathfrak{h} \cdot \mathfrak{h} \in \mathfrak{h}^{m-3}, \mathfrak{h} \cdot [\mathfrak{h}, \mathfrak{h}] = 0$.
Now, the CPA-structure satisfies $\mathfrak{g} \cdot \mathfrak{a}' \in \mathfrak{a}'$: Take any element $g \in \mathfrak{g}$. Then

$$g \cdot \mathfrak{a}' = L(g)(\mathfrak{a}') = (\operatorname{ad}(x_g) + \psi_g)(\mathfrak{a}') \subseteq [x_g, \mathfrak{a}'] + \psi_g(\mathfrak{a}') \subseteq \mathfrak{a}' + \mathfrak{g}^{m-2} \subseteq \mathfrak{a}',$$

as $\mathfrak{g}^{m-2} \subseteq \mathfrak{a}'$.

So if we have a CPA-structure on $\mathfrak{g}$, we may consider the quotient CPA-structure on $\mathfrak{g}/\mathfrak{a}'$, which, by $\mathfrak{g}/\mathfrak{a}' \cong \mathfrak{h}/\operatorname{Z}(\mathfrak{h})$, satisfies $\mathfrak{g}/\mathfrak{a}' \cdot \mathfrak{g}/\mathfrak{a}' \in \operatorname{Z}(\mathfrak{g}/\mathfrak{a}'), \mathfrak{g}/\mathfrak{a}' \cdot [\mathfrak{g}/\mathfrak{a}', \mathfrak{g}/\mathfrak{a}'] = 0$.
This means that

$$\mathfrak{g} \cdot \mathfrak{g} \in \operatorname{Z}(\mathfrak{g}/\mathfrak{a}') + \mathfrak{a}', \qquad \mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \in \mathfrak{a}'$$

for the CPA-structure on $\mathfrak{g}$.
We also observe that, writing $\mathfrak{b}' \coloneqq \mathfrak{b} + \mathfrak{g}^{m-2}$, we get

$$\mathfrak{g}/\mathfrak{b}' \cong \mathfrak{h}/\operatorname{Z}(\mathfrak{h})$$

and

$$\mathfrak{g} \cdot \mathfrak{g} \in \operatorname{Z}(\mathfrak{g}/\mathfrak{b}') + \mathfrak{b}', \qquad \mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \in \mathfrak{b}'$$

for the CPA-structure on $\mathfrak{g}$.
In particular,

$$\mathfrak{g} \cdot \mathfrak{g} \in (\operatorname{Z}(\mathfrak{g}/\mathfrak{a}') + \mathfrak{a}') \cap (\operatorname{Z}(\mathfrak{g}/\mathfrak{b}') + \mathfrak{b}') = \mathfrak{g}^{m-3}$$

and

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}' \cap \mathfrak{b}' = \mathfrak{g}^{m-2}.$$

We have $\mathfrak{g}\cdot\mathfrak{g}^1 = \mathfrak{g}\cdot[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}^{m-2}$, thus by axiom (CPA3)

$$\begin{aligned}\mathfrak{g}\cdot\mathfrak{g}^2 = \mathfrak{g}\cdot[\mathfrak{g},\mathfrak{g}^1] &\subseteq [\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}^1] + [\mathfrak{g},\mathfrak{g}\cdot\mathfrak{g}^1]\\ &\subseteq [\mathfrak{g}\cdot\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] + [\mathfrak{g},\mathfrak{g}^{m-2}]\\ &\subseteq [\mathfrak{g},[\mathfrak{g},\mathfrak{g}\cdot\mathfrak{g}]] + [\mathfrak{g},[\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}]] + \mathfrak{g}^{m-1}\\ &\subseteq [\mathfrak{g},[\mathfrak{g},\mathfrak{g}^{m-3}]] + [\mathfrak{g},[\mathfrak{g}^{m-3},\mathfrak{g}]] + \mathfrak{g}^{m-1} \subseteq \mathfrak{g}^{m-1} + \mathfrak{g}^{m-1} + \mathfrak{g}^{m-1} \subseteq \mathfrak{g}^{m-1}.\end{aligned}$$

And this in turn implies again by axiom (CPA3)

$$\begin{aligned}\mathfrak{g}\cdot\mathfrak{g}^3 &\subseteq [\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}^2] + [\mathfrak{g},\mathfrak{g}\cdot\mathfrak{g}^2]\\ &\subseteq [\mathfrak{g}\cdot\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] + [\mathfrak{g},\mathfrak{g}^{m-1}]\\ &\subseteq [\mathfrak{g},[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}\cdot\mathfrak{g}]] + [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}]] + \mathfrak{g}^m\\ &\subseteq [\mathfrak{g},[[\mathfrak{g},\mathfrak{g}\cdot\mathfrak{g}],\mathfrak{g}]] + [\mathfrak{g},[[\mathfrak{g}\cdot\mathfrak{g},\mathfrak{g}],\mathfrak{g}]] + [[\mathfrak{g},\mathfrak{g}],\mathfrak{g}^{m-2}] + \mathfrak{g}^m\\ &\subseteq [\mathfrak{g},[\mathfrak{g}^{m-2},\mathfrak{g}]] + [\mathfrak{g},[\mathfrak{g}^{m-2},\mathfrak{g}]] + [[\mathfrak{g},\mathfrak{g}^{m-2}],\mathfrak{g}] + [[\mathfrak{g}^{m-2},\mathfrak{g}],\mathfrak{g}] + \mathfrak{g}^m\\ &\subseteq \mathfrak{g}^m + \mathfrak{g}^m + \mathfrak{g}^m + \mathfrak{g}^m + \mathfrak{g}^m = \mathfrak{g}^m = 0.\end{aligned}$$

Now as $m \geq 6$, $\mathfrak{g}^3 \supseteq \mathfrak{g}^{m-3}$, thus $\mathfrak{g}\cdot\mathfrak{g}^{m-3} = 0$.
We conclude by axiom (CPA2),

$$[\mathfrak{g},\mathfrak{g}]\cdot\mathfrak{g} \subseteq \mathfrak{g}\cdot(\mathfrak{g}\cdot\mathfrak{g}) - \mathfrak{g}\cdot(\mathfrak{g}\cdot\mathfrak{g}) \subseteq \mathfrak{g}\cdot(\mathfrak{g}\cdot\mathfrak{g}) \subseteq \mathfrak{g}\cdot\mathfrak{g}^{m-3} = 0.$$

It remains to prove $\mathfrak{g}\cdot\mathfrak{g} \subseteq \mathfrak{g}^{m-2}$.
But suppose $L(g)(\mathfrak{g}) \not\subseteq \mathfrak{g}^{m-2}$ for some element $g \in \mathfrak{g}$, then $\operatorname{ad}(x_g)(\mathfrak{g}) + \psi_g(\mathfrak{g}) \not\subseteq \mathfrak{g}^{m-2}$. Since $\psi_g(\mathfrak{g}) \subseteq \mathfrak{g}^{m-2}$, this implies $[x_g,\mathfrak{g}] \not\subseteq \mathfrak{g}^{m-2}$, meaning $x_g \notin \mathfrak{g}^{m-3}$. This, however, means by Lemma 5.25, that $[x_g,[\mathfrak{g},\mathfrak{g}]] \neq 0$ – but then $L(g)([\mathfrak{g},\mathfrak{g}]) \neq 0$ (since $\psi_g([\mathfrak{g},\mathfrak{g}]) = 0$ by Lemma 5.26). However, this in turn means $g\cdot[\mathfrak{g},\mathfrak{g}] \neq 0$, a contradiction to $\mathfrak{g}\cdot[\mathfrak{g},\mathfrak{g}] = 0$ what we already have established.
So we have $L(\mathfrak{g})(\mathfrak{g}) \subseteq \mathfrak{g}^{m-2}$ and this is what we wanted to show. □

Remark 5.28. From the structure of $\mathfrak{g}$'s derivations given in [83], we also infer that $L(E_{i,j}) \in \mathrm{Z}(\mathfrak{g})$ for $1 \leq i < j \leq n$ if $(i,j) \notin \{(1,2),(n-1,n)\}$.

Proposition 5.27 is not true for $n \leq 4$. Indeed, for $n = 2$, we have the one-dimensional Lie algebra where any bilinear map is a CPA-structure, for $n = 3$ we obtain the 3-dimensional Heisenberg algebra $\mathfrak{n}_3(\mathbb{C})$ (the CPA-structures on $\mathfrak{n}_3(\mathbb{C})$ are listed in the appendix, Proposition B.10), for $n = 4$, we get one additional type of CPA-structures:

Example 5.29. *Let $\mathfrak{g}$ be the Lie algebra of strictly upper triangular matrices of size 4×4 with basis $\{e_1 = E_{1,2}, e_2 = E_{2,3}, e_3 = E_{3,4}, e_4 = E_{1,3}, e_5 = E_{2,4}, e_6 = E_{1,4}\}$ and Lie brackets $[e_1,e_2] = e_4, [e_1,e_5] = e_6, [e_2,e_3] = e_5, [e_3,e_4] = -e_6$.*
The CPA-structures on $\mathfrak{g}$ are given by the following three types:

(i) $e_1\cdot e_1 = \alpha_1 e_4 + \alpha_2 e_5 + \alpha_3 e_6, e_2\cdot e_1 = e_1\cdot e_2 = \alpha_4 e_6, e_3\cdot e_1 = e_1\cdot e_3 = \alpha_5 e_4 - \alpha_1 e_5 + \alpha_6 e_6, e_2\cdot e_2 = \alpha_7 e_6, e_3\cdot e_2 = e_2\cdot e_3 = \alpha_8 e_6, e_3\cdot e_3 = \alpha_9 e_4 - \alpha_5 e_5 + \alpha_{10} e_6$ *with* $\alpha_1,\dots,\alpha_{10} \in \mathbb{C}$.

(ii) $e_1 \cdot e_1 = \alpha_1 e_4 + \alpha_2 e_5 + \alpha_3 e_6, e_2 \cdot e_1 = e_1 \cdot e_2 = e_4 + \alpha_4 e_6, e_3 \cdot e_1 = e_1 \cdot e_3 = \alpha_5 e_4 - \alpha_1 e_5 + \alpha_6 e_6, e_5 \cdot e_1 = e_1 \cdot e_5 = e_6, e_2 \cdot e_2 = \alpha_7 e_6, e_3 \cdot e_2 = e_2 \cdot e_3 = -e_5 + \alpha_8 e_6, e_3 \cdot e_3 = \alpha_9 e_4 - \alpha_5 e_5 + \alpha_{10} e_6, e_4 \cdot e_3 = e_3 \cdot e_4 = -e_6$ *with* $\alpha_1, \ldots, \alpha_{10} \in \mathbb{C}$.

(iii) $e_1 \cdot e_1 = \alpha_1 e_4 + \alpha_2 e_5 + \alpha_3 e_6, e_2 \cdot e_1 = e_1 \cdot e_2 = \frac{1}{2} e_4 + \alpha_4 e_5 + \alpha_5 e_6, e_3 \cdot e_1 = e_1 \cdot e_3 = -\frac{\alpha_2}{4\alpha_4^2} e_4 - \alpha_1 e_5 + \alpha_6 e_6, e_4 \cdot e_1 = e_1 \cdot e_4 = \alpha_4 e_6, e_5 \cdot e_1 = e_1 \cdot e_5 = \frac{1}{2} e_6, e_2 \cdot e_2 = \alpha_7 e_6, e_3 \cdot e_2 = e_2 \cdot e_3 = -\frac{1}{4\alpha_4} e_4 - \frac{1}{2} e_5 + \alpha_8 e_6, e_3 \cdot e_3 = \frac{\alpha_1}{4\alpha_4^2} e_4 + \frac{\alpha_2}{4\alpha_4^2} e_5 + \alpha_9 e_6, e_4 \cdot e_3 = e_3 \cdot e_4 = -\frac{1}{2} e_6, e_5 \cdot e_3 = e_3 \cdot e_5 = -\frac{1}{4\alpha_4} e_6$ *with* $\alpha_1, \ldots, \alpha_9 \in \mathbb{C}, \alpha_4 \neq 0$.

While $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{m-2} = \mathfrak{g}^1 = \operatorname{span}\{e_4, e_5, e_6\} = [\mathfrak{g}, \mathfrak{g}]$ *holds for all three types,* $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ *does not hold for types (ii) and (iii).*

Example 5.30. *For* $n = 5$*, the explicit CPA-structures on* $\mathfrak{g}$ *are given by the non-zero products between basis elements*

$$
\begin{aligned}
e_1 \cdot e_1 &= -\alpha_1 e_8 + \alpha_2 e_9 + \alpha_3 e_{10} \\
e_1 \cdot e_2 &= \alpha_4 e_{10} \\
e_1 \cdot e_3 &= \alpha_5 e_{10} \\
e_1 \cdot e_4 &= -\alpha_6 e_8 + \alpha_1 e_9 + \alpha_7 e_{10} \\
e_2 \cdot e_2 &= \alpha_8 e_{10} \\
e_2 \cdot e_3 &= \alpha_9 e_{10} \\
e_2 \cdot e_4 &= \alpha_{10} e_{10} \\
e_3 \cdot e_3 &= \alpha_{11} e_{10} \\
e_3 \cdot e_4 &= \alpha_{12} e_{10} \\
e_4 \cdot e_4 &= \alpha_{13} e_8 + \alpha_6 e_9 + \alpha_{14} e_{10}
\end{aligned}
$$

(and the relations obtained by commutativity), $\alpha_1, \ldots, \alpha_{14} \in \mathbb{C}$*. (Here,* $e_1 = E_{1,2}, \ldots, e_4 = E_{4,5}, e_5 = E_{1,3}, \ldots, e_7 = E_{3,5}, e_8 = E_{1,4}, e_9 = E_{2,5}, e_{10} = E_{3,4}$*.) So we really do have* $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{m-2}$ *(and not* $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathfrak{g}^{m-1} = \mathrm{Z}(\mathfrak{g})$*, that is, the CPA-structure is not central as per [34, Definition 3.9]).*

5.3.1 CPA-structures on the Lie algebra of upper triangular matrices

Now we are interested in the CPA-structures on the Lie algebra $\mathfrak{g}$ of (non-strict) upper triangular square matrices (of size n). This is not a nilpotent Lie algebra, but rather a solvable one.
As $\mathfrak{g}$ is an algebraic Lie algebra (of the algebraic group of invertible upper triangular matrices), we can decompose $\mathfrak{g}$ into torus and nilradical (Proposition 4.20). By $E_{i,j}$, we denote the matrix with a 1 at position (i, j) and 0 everywhere else (as before):

Lemma 5.31. *The Lie algebra* $\mathfrak{g}$ *of upper triangular matrices (of size* $n \geq 2$*) has the form* $\mathfrak{g} = T \ltimes \mathfrak{m}$*, where* T *is a torus of derivations of* $\mathfrak{g}$ *and* $\mathfrak{m}$ *the nilradical of* $\mathfrak{g}$*. In particular, we have* $[T, \mathfrak{m}] \subseteq \mathfrak{m}, [T, \mathrm{Z}(\mathfrak{m})] \subseteq \mathrm{Z}(\mathfrak{m})$ *and* $[T, [\mathfrak{m}, \mathfrak{m}]] \subseteq [\mathfrak{m}, \mathfrak{m}]$.
Note that for every elementary matrix $m = E_{i,j} \in \mathfrak{m}$*, where* $i \neq j$*, there is a* $t \in T$ *(e.g.* $t = E_{i,i}$*) with* $[m, t] = \alpha m, \alpha \neq 0$.

Remark 5.32. It is $T = \mathrm{span}\{E_{i,i} \colon i = 1, \ldots, n\}$ and $\mathfrak{n} = \mathrm{span}\{E_{i,j} \colon 1 \leq i < j \leq n\}$.

Remark 5.33. The Lie algebra $\mathfrak{g}$ has a non-trivial center which is given by $\{a \cdot (E_{1,1} + \ldots + E_{n,n}) \colon a \in \mathbb{C}\}$.

Proposition 5.34. *The CPA-structures on the Lie algebra $\mathfrak{g}$ of upper triangular matrices of size n, where $n \geq 3$, all satisfy $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}([\mathfrak{g}, \mathfrak{g}])$ and $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.*

Proof. For $n = 3, 4$, we compute the CPA-structures directly; for $n \geq 5$, we may use Proposition 5.27. Let $n \geq 5$ and let us write $\mathfrak{g} = T \ltimes \mathfrak{m}$, where T is a torus and $\mathfrak{m}$ the nilradical of $\mathfrak{g}$. As $\mathfrak{m}$ is isomorphic to the Lie algebra of strictly upper triangular matrices of size n, we know that every CPA-structure on $\mathfrak{m}$ satisfies $\mathfrak{m} \cdot \mathfrak{m} \subseteq \mathfrak{m}^{n-3} \subseteq [\mathfrak{m}, \mathfrak{m}]$.
Let $x \cdot y$ be a CPA-structure on $\mathfrak{g}$.
Step 0: We show that $\mathfrak{g} \cdot \mathfrak{m} \subseteq \mathfrak{m}$. But this holds since $\mathfrak{m}$ is a characteristic ideal, thus stable under $L(\mathfrak{g})$. This implies that the restriction of $x \cdot y$ to $\mathfrak{m}$ is a CPA-structure on $\mathfrak{m}$. So we have $\mathfrak{m} \cdot \mathfrak{m} \subseteq \mathfrak{m}^{n-2} \subseteq [\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}] = 0$.
Step 1: We have $\mathfrak{m} \cdot T \subseteq [\mathfrak{m}, \mathfrak{m}]$, because by axiom (CPA3)

$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m} \cdot T] \subseteq \mathfrak{m} \cdot \underbrace{[[\mathfrak{m}, \mathfrak{m}], T]}_{\subseteq [\mathfrak{m}, \mathfrak{m}]} - [\underbrace{\mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}]}_{=0}, T] \subseteq \mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}] + 0 = 0.$$

Thus by Lemma 5.25, $\mathfrak{m} \cdot T \subseteq \mathfrak{m}^{n-4} \subseteq [\mathfrak{m}, \mathfrak{m}]$.
Step 2: We have $T \cdot [\mathfrak{m}, \mathfrak{m}] = 0$, because by axiom (CPA2)

$$[\mathfrak{m}, \mathfrak{m}] \cdot T \subseteq \mathfrak{m} \cdot \underbrace{(\mathfrak{m} \cdot T)}_{\subseteq [\mathfrak{m}, \mathfrak{m}]} - \mathfrak{m} \cdot \underbrace{(\mathfrak{m} \cdot T)}_{\subseteq [\mathfrak{m}, \mathfrak{m}]} \subseteq \mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}] = 0.$$

Step 3: We have $\mathfrak{m} \cdot \mathfrak{m} = 0$: Let $m_1, m_2 \in \mathfrak{m}$ and choose $t \in T$ with $[m_1, t] = \alpha m_1$ with $\alpha \neq 0$. Then, by axiom (CPA2),

$$m_1 \cdot m_2 = \frac{1}{\alpha}([m_1, t] \cdot m_2) = \frac{1}{\alpha}(m_1 \cdot \underbrace{(t \cdot m_2)}_{\in [\mathfrak{m}, \mathfrak{m}]} - t \cdot \underbrace{(m_1 \cdot m_2)}_{\in [\mathfrak{m}, \mathfrak{m}]}) \subseteq \mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}] + T \cdot [\mathfrak{m}, \mathfrak{m}] = 0.$$

Step 4: We have $\mathfrak{m} \cdot T \subseteq \mathrm{Z}(\mathfrak{m})$, because by axiom (CPA3)

$$[\mathfrak{m} \cdot T, \mathfrak{m}] \subseteq \mathfrak{m} \cdot \underbrace{[T, \mathfrak{m}]}_{\subseteq \mathfrak{m}} - [T, \underbrace{\mathfrak{m} \cdot \mathfrak{m}}_{=0}] \subseteq \mathfrak{m} \cdot \mathfrak{m} + 0 = 0$$

and it follows $\mathfrak{m} \cdot T \subseteq \mathrm{Z}(\mathfrak{m})$.
Step 5: We have $T \cdot T \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}_2(\mathfrak{m})$, since by axiom (CPA3),

$$[T \cdot T, \mathfrak{m}] \subseteq T \cdot \underbrace{[T, \mathfrak{m}]}_{\subseteq \mathfrak{m}} - [T, \underbrace{T \cdot \mathfrak{m}}_{\subseteq \mathrm{Z}(\mathfrak{m})}] \subseteq T \cdot \mathfrak{m} + \mathrm{Z}(\mathfrak{m}) \subseteq \mathrm{Z}(\mathfrak{m}).$$

This however means, that, writing $t_1 \cdot t_2 = t + m$, where $t \in T, m \in \mathfrak{m}$, we have $[t + m, \mathfrak{m}] \subseteq \mathrm{Z}(\mathfrak{m})$. However, this is only possible if $t \in \mathrm{Z}(\mathfrak{g})$ and $m \in \mathrm{Z}_2(\mathfrak{m})$.
Step 6: We have $\mathfrak{m} \cdot \mathrm{Z}(\mathfrak{g}) = 0$, because by axiom (CPA3),

$$[\mathfrak{m} \cdot \mathrm{Z}(\mathfrak{g}), T] \subseteq \mathfrak{m} \cdot [\mathrm{Z}(\mathfrak{g}), T] - [\mathrm{Z}(\mathfrak{g}), \mathfrak{m} \cdot T] = 0.$$

Thus $\mathfrak{m} \cdot \mathrm{Z}(\mathfrak{g}) = 0$ (otherwise, we could find a $t \in T$ with $[\mathfrak{m} \cdot \mathrm{Z}(\mathfrak{g}), T] \neq 0$).
Step 7: We have $T \cdot \mathfrak{m} = 0$ by axiom (CPA2). Indeed, let $m \in \mathfrak{m}, t \in T$ and let $t_m \in T$ be such that $[t_m, m] = \alpha m$ with $\alpha \neq 0$. Then

$$m \cdot t = \frac{1}{\alpha}[t_m, m] \cdot t = \frac{1}{\alpha}(t_m \cdot \underbrace{(m \cdot t)}_{\in \mathrm{Z}(\mathfrak{m})} - m \cdot \underbrace{(t_m \cdot t)}_{\in \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}_2(\mathfrak{m})}) \subseteq T \cdot [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m} \cdot \mathrm{Z}(\mathfrak{g}) + \mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}]$$
$$\subseteq 0 + 0 + 0 = 0.$$

Step 8: We have $T \cdot T \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}(\mathfrak{m})$, since, by axiom (CPA3),

$$[T \cdot T, \mathfrak{m}] \subseteq T \cdot \underbrace{[T, \mathfrak{m}]}_{\subseteq \mathfrak{m}} - [T, \underbrace{T \cdot \mathfrak{m}}_{=0}] \subseteq T \cdot \mathfrak{m} + 0 = 0$$

and thus, as before, $T \cdot T \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}(\mathfrak{m})$.
So, in total, we have

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \cdot \mathfrak{m} = 0$$

by Step 3 and Step 7, and

$$\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}(\mathfrak{m}) = \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}([\mathfrak{g}, \mathfrak{g}])$$

by Step 3, Step 7 and Step 8. This is what we have claimed. □

Remark 5.35. Suppose we have a (non-necessarily nilpotent) Lie algebra $\mathfrak{m}$ such that all CPA-structures on $\mathfrak{m}$ satisfy $\mathfrak{m} \cdot [\mathfrak{m}, \mathfrak{m}] = 0$ and $\mathfrak{m} \cdot \mathfrak{m} \subseteq [\mathfrak{m}, \mathfrak{m}]$. (For example, $\mathfrak{m}$ a non-metabelian filiform Lie algebra, cf. Chapter 5.4.)
If $\mathfrak{g} = T \ltimes \mathfrak{m}$ is a Lie algebra with a torus T (see Proposition 4.20), then, under "certain technical assumptions", we can copy the proof of Proposition 5.34 and also find $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{Z}(\mathfrak{g}) + \mathrm{Z}([\mathfrak{g}, \mathfrak{g}])$ and $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.
The "certain technical assumptions" are about the interaction between T and $\mathfrak{m}$ and have to make sure that the conclusions in Steps 1, 3, 5, 6, 7 and 8 are still valid.
One could, for example, repeat the proof for certain solvable algebraic Lie algebras (see Proposition 4.20). (For complete Lie algebras however, the result of Proposition 5.34 is already implied by Proposition 2.52.)

Remark 5.36. Let $\mathfrak{g}$ now be the Lie algebra of all *traceless* upper triangular matrices. As $\mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, we also get the result that all CPA-structures on $\mathfrak{g}$ satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ – this follows from Proposition 4.5 and Proposition 2.52. So, we recover [33, Corollary 4.16].

5.4 Classification of CPA-structures on filiform Lie algebras

In this section, we are interested in CPA-structures on *filiform* Lie algebras. (Finite-dimensional) filiform Lie algebras can be thought of as the "least nilpotent" Lie algebras, that is, the lower central series is as long as possible, but still terminates. In other words, if $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}], i \geq 1$, is the lower central series of the n-dimensional Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is filiform per definitionem precisely if $\dim(\mathfrak{g}^0/\mathfrak{g}^1) = 2$ and $\dim(\mathfrak{g}^{i+1}/\mathfrak{g}^i) = 1$ for $i \leq n-1$. We do assume that $\dim(\mathfrak{g}) \geq 3$. It is well-known that every filiform Lie algebra admits an *adapted basis*:

Proposition 5.37 (Vergne, [99])**.** *Every n-dimensional filiform Lie algebra has an* adapted basis*, that is, a vector space basis $\{e_1, \dots, e_n\}$ such that*

$$\begin{aligned} [e_1, e_i] &= e_{i+1} \text{ for } 2 \leq i \leq n-1, \\ [e_i, e_j] &\in \operatorname{span}(\{e_{i+j}, \dots, e_n\}) \text{ for } i, j \geq 2 \text{ with } i + j \leq n, \\ [e_i, e_{n+1-i}] &= (-1)^{i+1} \tau e_n \text{ for } 2 \leq i \leq n-1 \end{aligned}$$

with $\tau \in \mathbb{C}$ where $\tau = 0$ always holds if n is odd.
Moreover, the brackets $[e_i, e_j]$ with $i, j \geq 2$ are completely determined if one knows the brackets

$$[e_k, e_{k+1}] = \sum_{s=2k+1}^{n} \alpha_{k,s} e_s \text{ for } 2 \leq k \leq \lfloor n/2 \rfloor$$

(cf. also [22]).

(However, one does not automatically obtain a Lie algebra by choosing the above brackets arbitrarily; the Jacobi identity is equivalent to certain polynomial equations in the coefficients $\alpha_{k,s}$.)

Example 5.38. *Consider the Lie algebra given by the basis $\{e_1, \dots, e_n\}$ and non-zero Lie brackets $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n-1$. This is the* standard graded filiform *Lie algebra and will be denoted by the name L_n.*

In the following, let $\mathfrak{g}$ be a filiform Lie algebra with an adapted basis $\{e_1, \dots, e_n\}$ as above, $x \cdot y$ a CPA-structure on $\mathfrak{g}$ and $L_i \coloneqq L(e_i)$ the left-multiplication operator with respect to the structure $x \cdot y$. Further define for $i \in \mathbb{N}$ the characteristic ideals

$$\delta_i = \operatorname{span}(\{e_i, \dots, e_n\})$$

with the convention $\delta_i = 0$ if $i \geq n+1$. As $\delta_1 = \mathfrak{g}, \delta_i = \mathfrak{g}^{i-2}$ for $i \geq 3$ and $\delta_i \subseteq \delta_j$ if $i \geq j$, $(\delta_1, \delta_2, \dots)$ is a refinement of the lower central series. Note that the filtration given by the δ_i has also been used in the paper [8], where δ_i is called $V_{(i)}$.
Let us collect some observations:

Lemma 5.39. *Let $\mathfrak{g}$ be a filiform Lie algebra, $x \cdot y$ a CPA-structure on $\mathfrak{g}$ and $L_i \coloneqq L(e_i)$ the left-multiplication operator with respect to an adapted basis $\{e_1, \dots, e_n\}$. Then:*

(i) The Lie algebra $\mathfrak{g}$ is metabelian precisely if $[\delta_3, \delta_3] = 0$.

(ii) $L_i(\delta_j) \subseteq \delta_{j+1}$ for all $i, j = 1, \dots, n$ (that is, all L_i are lower triangular with respect to the adapted basis).

(iii) *We have $\mathfrak{g} \cdot \mathfrak{g} \subseteq \delta_3$ if and only if $e_1 \cdot e_1 \in \delta_3$ and $\mathfrak{g} \cdot \delta_2 \subseteq \delta_4$ if and only if $e_1 \cdot e_2 \in \delta_4$ and $e_2 \cdot e_2 \in \delta_4$. (Later we will see that $e_2 \cdot e_2 \in \delta_4$ holds even if $\mathfrak{g} \cdot \delta_2 \not\subseteq \delta_4$.)*

Proof.

(i) This holds since $\delta_3 = [\mathfrak{g}, \mathfrak{g}]$.

(ii) Note that the L_i are all derivations by Lemma 2.36.
All δ_j are characteristic ideals, thus $L_i(\delta_j) \subseteq \delta_j$. But since $x \cdot y$ is a CPA-structure on a nilpotent Lie algebra, L_i is nilpotent (by Proposition 2.54) – thus $L_i(\delta_j)$ must really lie in δ_{j+1}.

(iii) This follows from (ii): We have $L_i(\delta_2) \subseteq \delta_3$ for all $1 \le i \le n$, thus $\mathfrak{g} \cdot \delta_2 \subseteq \delta_3$. So if $e_1 \cdot e_1 \in \delta_3$, then $\mathfrak{g} \cdot \mathfrak{g} \subseteq \delta_3$. Also, $\mathfrak{g} \cdot \delta_3 \subseteq \delta_4$. Thus if $e_1 \cdot e_2 \in \delta_4$ and $e_2 \cdot e_2 \in \delta_4$, then $\mathfrak{g} \cdot \delta_2 \subseteq \delta_4$.

□

Definition 5.40. Let $\mathfrak{g}$ be a Lie algebra. We say that a CPA-structure $x \cdot y$ on $\mathfrak{g}$ is *associative*, if $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

Remark 5.41. We call CPA-structures satisfying $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ associative, because $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ if and only if the algebra $(A, \cdot)$ is associative – see Proposition 5.90.

As filiform Lie algebras satisfy $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = 2$, associative CPA-structures on filiform Lie algebras only have three products between basis elements which are possibly non-zero: $e_1 \cdot e_1, e_1 \cdot e_2 = e_2 \cdot e_1$ and $e_2 \cdot e_2$.

Let $\mathfrak{g}$ be a filiform Lie algebra of dimension $n \ge 7$ with a CPA-structure $x \cdot y$. By Lemma 5.39, (ii), we always have $\mathfrak{g} \cdot \mathfrak{g} = \delta_1 \cdot \delta_1 \subseteq \delta_2$ and $\mathfrak{g} \cdot \delta_2 = \delta_1 \cdot \delta_2 \subseteq \delta_3$.

Definition 5.42. If $\mathfrak{g} \cdot \mathfrak{g} \subseteq \delta_3$, we say $x \cdot y$ is in *Class A*.
Class A is the disjoint union of Class A_1 and Class A_2: If $x \cdot y$ is in Class A and $\mathfrak{g} \cdot \delta_2 \subseteq \delta_4$, we say $x \cdot y$ is in *Class A_1*; if $x \cdot y$ is in Class A and $\mathfrak{g} \cdot \delta_2 \not\subseteq \delta_4$, we say $x \cdot y$ is in *Class A_2*.
If $\mathfrak{g} \cdot \mathfrak{g} \not\subseteq \delta_3$, we say $x \cdot y$ is in *Class B*.
Class B is the disjoint union of Class B_1 and Class B_2: If $x \cdot y$ is in Class B and $\mathfrak{g} \cdot \delta_2 \subseteq \delta_4$, we say $x \cdot y$ is in *Class B_1*; if $x \cdot y$ is in Class B and $\mathfrak{g} \cdot \delta_2 \not\subseteq \delta_4$, we say $x \cdot y$ is in *Class B_2*.

Note that every CPA-structure on $\mathfrak{g}$ is in exactly one of Classes A and B and in exactly one of Classes A_1, A_2, B_1, B_2.
Writing $e_1 \cdot e_1 = \alpha e_2 + \delta_3, e_1 \cdot e_2 = \beta e_3 + \delta_4$ (with $\alpha, \beta \in \mathbb{C}$), by Lemma 5.39, (iii), $x \cdot y$ is in Class A if and only if $\alpha = 0$ and in Class A_1 or B_1 if and only if $\beta = 0$ and $e_2 \cdot e_2 \in \delta_4$. We will later see that $\beta \in \{0, 1\}$ and that $e_2 \cdot e_2 \in \delta_4$ always holds.

Remark 5.43. Given a CPA-structure on $\mathfrak{g}$, its class is invariant under a change of adapted bases of $\mathfrak{g}$.

We shall prove the following:
CPA-structures in Class A_1 are associative. CPA-structures in Class A_2 satisfy the weaker condition $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ and only exist on metabelian Lie algebras. CPA-structures in Class B only appear on $\mathfrak{g} \cong L_n$, the standard graded filiform Lie algebra (which is in particular metabelian). All together, we will obtain:

Theorem 5.44. *Let $\mathfrak{g}$ be a filiform Lie algebra of dimension* $\dim(\mathfrak{g}) \geq 6$.

(i) Every CPA-structure on $\mathfrak{g}$ is associative if and only if $\mathfrak{g}$ is non-metabelian.

(ii) Every CPA-structure on $\mathfrak{g}$ satisfies $[\mathfrak{g},\mathfrak{g}]\cdot[\mathfrak{g},\mathfrak{g}] = 0$ if and only if $\mathfrak{g}$ is not isomorphic to L_n.

(iii) The only remaining filiform Lie algebra, L_n, admits two types of structures, see Proposition 5.66.

Remark 5.45. For dimensions n smaller than 6, there are no filiform Lie algebras if $n = 1, 2$ and only one (namely L_n) if $n = 3, 4$. For $n = 4$, the CPA-structures are given in Remark 5.67, for $n = 3$ in Proposition B.10. For $n = 5$, there are two non-isomorphic Lie algebras, L_5 and R_5; see Section 5.5 for their post-Lie algebra structures. While R_5 is metabelian, not all post-Lie algebra structures on R_5 satisfy the condition in (ii) (that is, Theorem 5.44 does not hold in dimension 5).

We will assume in the following that the dimension of the filiform Lie algebra $\mathfrak{g}$ is at least 7. However, one may investigate the (five, up to isomorphism; see [38] or [54]) filiform Lie algebras in dimension 6 and finds that Theorem 5.44 still holds.

Let us distinguish two cases: As said before, every n-dimensional filiform Lie algebra has an adapted basis $\{e_1, \dots, e_n\}$ such that the non-zero brackets are given by

$$\begin{aligned}
[e_1, e_i] &= e_{i+1}, i = 2, \dots, n-1 \\
[e_i, e_j] &\in \operatorname{span}\{e_{i+j}, \dots, e_n\} = \delta_{i+j}, i + j \leq n, \\
[e_i, e_{n-i+1}] &= (-1)^{i+1}\tau e_n, 2 \leq i \leq n-1
\end{aligned}$$

where the scalar $\tau \in \mathbb{C}$ is always zero if n is odd. If n is even, we have $\tau = 0$ if and only if $\mathfrak{g}^{(n-4)/2}$ is abelian (cf. [22, Lemma 2.5.9]).
If $\dim(\mathfrak{g}){=:}n$ is odd, the Lie algebra $\mathfrak{g}^{\lceil(n-4)/2\rceil}$ is always abelian.

To prove Theorem 5.44, we will first assume that $\mathfrak{g}^{\lceil(n-4)/2\rceil}$ is abelian (that is, $\tau = 0$). Then we have $[e_i, e_j] \in \delta_{i+j}$ for all $i, j \in \{1, \dots, n\}$. We will later (in Section 5.4.3) consider the case where $\mathfrak{g}^{\lceil(n-4)/2\rceil}$ is not abelian (i.e. $\tau \neq 0$).

We will in this section only give the proof that CPA-structures in Class A_1 are associative. As the proofs for the corresponding statements for CPA-structures in Classes A_2, B_1 and B_2 are similar, but require to take care of many more details. The proof is given in Appendix A.

In the following, $\mathfrak{g}$ will always be a filiform Lie algebra of dimension ≥ 7 and $x \cdot y$ a CPA-structure on $\mathfrak{g}$.

5.4.1 CPA-structures in Class A

In this section we investigate the case where $\mathfrak{g}\cdot\mathfrak{g}$ is contained in the characteristic ideal δ_3, that is, $x \cdot y$ is in Class A.

We start by introducing a condition which implies that $x \cdot y$ is in Class A (Lemma 5.46 and Proposition 5.47): If our Lie algebra has an adapted basis with $[e_2, e_3] \in \alpha e_5 + \delta_6$, where $\alpha \neq 0$, then every CPA-structure on $\mathfrak{g}$ is in Class A.

Lemma 5.46. *If the filiform Lie algebra $\mathfrak{g}$ has an adapted basis $\{e_1, \ldots, e_n\}$ with $[e_2, e_3] = \alpha e_5 + \delta_6, \alpha \neq 0$, then there is an adapted basis $\{f_1, \ldots, f_n\}$ of $\mathfrak{g}$ with $[f_2, f_3] = f_5 + \delta_6$.*

Proof. Let us set

$$f_1 \coloneqq e_1, f_i \coloneqq \frac{1}{\alpha} e_i, i \geq 2.$$

Then $\{f_1, f_2, \ldots, f_n\}$ is still an adapted basis for $\mathfrak{g}$ and

$$[f_2, f_3] = \frac{1}{\alpha} \cdot \frac{1}{\alpha}[e_2, e_3] \in \frac{1}{\alpha^2}(\alpha e_5 + \delta_6) \subseteq \frac{1}{\alpha} e_5 + \delta_6 = f_5 + \delta_6.$$

□

Proposition 5.47. *Let $\mathfrak{g}$ be filiform and suppose $[e_2, e_3] \in e_5 + \delta_6$. Then all CPA-structures on $\mathfrak{g}$ are in Class A.*

Proof. Let us first note that $[e_2, e_4] \in e_6 + \delta_7$ by the Jacobi identity: We have

$$[e_2, e_4] = -[e_2, [e_3, e_1]] = [e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] \in [e_1, e_5 + \delta_6] + [e_3, e_3] \subseteq e_6 + \delta_7.$$

Let us now consider the left-multiplication operator $L_1 \coloneqq L(e_1)$ and write $L_1(e_1) \in \alpha e_2 + \delta_3, L_1(e_2) \in \beta e_3 + \delta_4$ with some $\alpha, \beta \in \mathbb{C}$. Our goal is to show that $\alpha = 0$. Since L_1 is a derivation, we find

$$\begin{aligned}
L_1(e_3) &= [L_1(e_1), e_2] + [e_1, L_1(e_2)] \in [\alpha e_2 + \delta_3, e_2] + [e_1, \beta e_3 + \delta_4] \subseteq \beta e_4 + \delta_5, \\
L_1(e_4) &= [L_1(e_1), e_3] + [e_1, L_1(e_3)] \in [\alpha e_2 + \delta_3, e_3] + [e_1, \beta e_4 + \delta_5] \subseteq (\alpha + \beta) e_5 + \delta_6, \\
L_1(e_5) &= [L_1(e_1), e_4] + [e_1, L_1(e_4)] \in [\alpha e_2 + \delta_3, e_4] + [e_1, (\alpha + \beta) e_5 + \delta_6] \subseteq (2\alpha + \beta) e_6 + \delta_7.
\end{aligned}$$

Now we use that L_1 is a derivation on the element $[e_2, e_3]$ and find:

$$L_1([e_2, e_3]) = [L_1(e_2), e_3] + [e_2, L_1(e_3)] \in [\beta e_3 + \delta_4, e_3] + [e_2, \beta e_4 + \delta_5] \subseteq \beta e_6 + \delta_7.$$

But on the other hand, $L_1([e_2, e_3]) \in L_1(e_5) + L_1(\delta_6) \subseteq (2\alpha + \beta) e_6 + \delta_7$ and in particular, $2\alpha + \beta = \beta$ which means $\alpha = 0$. So indeed $e_1 \cdot e_1 \in \delta_3$. □

Remark 5.48. There is another interpretation of $[e_2, e_3] \in e_5 + \delta_6$ (see [22, Lemma 2.5.9]): If a filiform Lie algebra does not contain a one-codimensional subspace $U \supseteq \mathfrak{g}^1$ such that $[U, \mathfrak{g}^1] \subseteq \mathfrak{g}^4$ (where $\mathfrak{g}^0 = \mathfrak{g}, \ldots, \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ denotes the lower central series of $\mathfrak{g}$), then $\mathfrak{g}$ has an adapted basis $\{e_1, \ldots, e_n\}$ with $[e_2, e_3] \in e_5 + \delta_6$.

In this and the next section, we assume that $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian – we study filiform Lie algebras where this is not the case in Section 5.4.3.

Recall that the *subdiagonal* of a quadratic matrix lies directly under the main diagonal.

Proposition 5.49. *Let $\mathfrak{g}$ be filiform with $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ abelian and $x \cdot y$ a CPA-structure on $\mathfrak{g}$; suppose that $x \cdot y$ is in Class A, that is, $e_1 \cdot e_1 \in \delta_3$.*

(i) We either have $e_1 \cdot e_2 \in \delta_4$ ($x \cdot y$ is in Class A_1) or $e_1 \cdot e_2 \in e_3 + \delta_4$ ($x \cdot y$ is in Class A_2). In particular, if $x \cdot y$ is in Class A_2 and $e_1 \cdot e_2 \in \beta e_3 + \delta_4$, then $\beta = 1$.

(ii) If $x \cdot y$ is in Class A_1, we have $e_1 \cdot e_j \in \delta_{j+2}$ for all $j \in \{1, \dots, n\}$; if $x \cdot y$ is in Class A_2, we have $e_1 \cdot e_j \in e_{j+1} + \delta_{j+2}$ for all $j \in \{2, \dots, n-1\}$.

(iii) We have $e_2 \cdot e_j \in \delta_{2+j}$ for all $j \in \{1, \dots, n\}$.

(iv) If $x \cdot y$ is in Class A_1, we have $e_i \cdot e_j \in \delta_{i+j}$ for all $i, j \in \{1, \dots, n\}$. (We will show later that this also holds for Class A_2.)

Proof. Let $L_i := L(e_i)$ with respect to the adapted basis. Let $L_i = \begin{pmatrix} a^i_{11} & a^i_{12} & \dots & a^i_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a^i_{n1} & a^i_{n2} & \dots & a^i_{nn} \end{pmatrix}$.
By Lemma 5.39, every L_i is lower triangular (with respect to the adapted basis). Let us investigate the subdiagonal of L_1 and L_2. By assumption, $e_1 \cdot e_1 \in \delta_3$. Now we use axiom (CPA3) for $2 \le i \le n-2$ and get

$$\underbrace{\underbrace{e_1 \cdot [e_1, e_i]}_{e_1 \cdot e_{i+1}}}_{\in a^1_{i+2,i+1} e_{i+2} + \delta_{i+3}} = \underbrace{[\underbrace{e_1 \cdot e_1}_{\in \delta_3}, e_i]}_{\subseteq \delta_{i+3}} + \underbrace{[e_1, \underbrace{e_1 \cdot e_i}_{\in a^1_{i+1,i} e_{i+1} + \delta_{i+2}}]}_{\subseteq a^1_{i+1,i} e_{i+2} + \delta_{i+3}}.$$

In particular, $\alpha := a^1_{i+1,i} = a^1_{i+2,i+1} = a^1_{i+3,i+2} = \dots = a^1_{n,n-1}$. That is, the subdiagonal of L_1 consists of equal entries. So L_1 looks like

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ * & \alpha & 0 & \dots & 0 & 0 \\ * & * & \alpha & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ * & * & * & \dots & \alpha & 0 \end{pmatrix}.$$

Exactly the same argument gives that L_2 has the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ * & \beta & 0 & \dots & 0 & 0 \\ * & * & \beta & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ * & * & * & \dots & \beta & 0 \end{pmatrix}.$$

We show that $\beta = 0$ via axiom (CPA2): We have

$$\underbrace{\underbrace{[e_1, e_2] \cdot e_2}_{e_2 \cdot e_3}}_{\in \beta e_4 + \delta_5} = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_2)}_{\in \beta e_3 + \delta_4}}_{\subseteq \alpha\beta e_4 + \delta_5} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_2)}_{\in \alpha e_3 + \delta_4}}_{\subseteq \alpha\beta e_4 + \delta_5}$$

and we get $\beta = 0$, which proves part (iii).
If we use axiom (CPA2) again, we find

$$\underbrace{[e_1, e_2] \cdot e_1}_{\underbrace{e_1 \cdot e_3}_{\in \alpha e_4 + \delta_5}} = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_1)}_{\in \alpha e_3 + \delta_4}}_{\subseteq \alpha^2 e_4 + \delta_5} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_3}}_{\subseteq \delta_5},$$

thus $\alpha = \alpha^2$ and so, $\alpha \in \{0, 1\}$. This proves (i) and (ii).
Now we prove part (iv): The additional assumption that $x \cdot y$ is in Class A_1 just means $\alpha = 0$. In particular, both L_1 and L_2 do not have a subdiagonal. Thus $e_1 \cdot e_i \in \delta_{i+2}, e_2 \cdot e_i \in \delta_{i+2}$.
It remains to show $e_i \cdot e_j \in \delta_{i+j}$ for $i, j \geq 3$.
Let $i \geq 3$ be fixed. Induction start:

$$e_3 \cdot e_i = [e_1, e_2] \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_i)}_{\in \delta_{i+2}}}_{\subseteq \delta_{i+3}} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_{i+3}} \in \delta_{i+3}$$

and inductively ($j \to j+1$):

$$e_{j+1} \cdot e_i = [e_1, e_j] \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_i)}_{\in \delta_{i+j}}}_{\subseteq \delta_{i+j+1}} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_{i+j+1}} \in \delta_{i+j+1}.$$

So we have proven (iv): $e_i \cdot e_j \in \delta_{i+j}$ for all $i, j \in \{1, \ldots, n\}$. □

Note that, by Proposition 5.49, the assumptions of the following lemma are satisfied for CPA-structures in Class A_1 on a filiform Lie algebra $\mathfrak{g}$ with $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ abelian for $\ell = 0$:

Lemma 5.50. *Let $\mathfrak{g}$ be a filiform Lie algebra with a CPA-structure in Class A_1 (i.e. $e_1 \cdot e_1 \in \delta_3, e_1 \cdot e_2 \in \delta_4$). If for some integer $\ell \geq 0$,*

- $e_2 \cdot e_2 \in \delta_4$,
- $e_1 \cdot e_j \in \delta_{j+\ell+2}$ *for all* $j \neq 1, 2$,
- $e_i \cdot e_j \in \delta_{i+j+\ell}$ *for all pairs* $(i, j) \notin \{(1,1), (1,2), (2,1), (2,2)\}$,

then also

- $e_2 \cdot e_2 \in \delta_4$,
- $e_1 \cdot e_j \in \delta_{j+\ell+3}$ *for all* $j \neq 1, 2$,
- $e_i \cdot e_j \in \delta_{i+j+\ell+1}$ *for all other pairs* $(i, j) \notin \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Here, we do not assume that $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ *is abelian.*

Proof. First we show that $e_2 \cdot e_j \in \delta_{3+j+\ell}, j \geq 3$:

$$e_2 \cdot e_3 = e_3 \cdot e_2 = [e_1, e_2] \cdot e_2 = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_2)}_{\in \delta_4}}_{\subseteq \delta_{4+2+\ell}} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_2)}_{\in \delta_4}}_{\subseteq \delta_{4+2+\ell}} \in \delta_{6+\ell}.$$

This was the induction start, now the induction step $j \to j + 1$:

$$e_2 \cdot e_{j+1} = e_{j+1} \cdot e_2 = [e_1, e_j] \cdot e_2 = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_2)}_{\in \delta_{3+j+\ell}}}_{\subseteq \delta_{3+j+\ell+2+\ell}} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_2)}_{\in \delta_4}}_{\subseteq \delta_{j+4+\ell}} \in \delta_{4+j+\ell}.$$

Now we show it for e_3, i.e. $e_3 \cdot e_j \in \delta_{4+j+\ell}, j \geq 3$:

$$e_3 \cdot e_3 = [e_1, e_2] \cdot e_3 = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_3)}_{\in \delta_{6+\ell}}}_{\subseteq \delta_{8+2\ell}} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_3)}_{\in \delta_{5+\ell}}}_{\subseteq \delta_{8+2\ell}} \in \delta_{7+\ell}.$$

This was the induction start, now the induction step $j \to j + 1$:

$$e_3 \cdot e_{j+1} = e_{j+1} \cdot e_3 = [e_1, e_j] \cdot e_3 = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_3)}_{\in \delta_{4+j+\ell}}}_{\subseteq \delta_{4+j+\ell+2+\ell}} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_3)}_{\in \delta_{5+\ell}}}_{\subseteq \delta_{5+\ell+j+\ell}} \in \delta_{5+j+\ell}.$$

We can also do this for $e_4, \ldots, e_n$. Thus we have proven so far that $e_i \cdot e_j \in \delta_{i+j+\ell+1}$ if $(i, j) \notin \{(1, j), (i, 1), (2, 2)\}$.
So it remains to show $e_1 \cdot e_j \in \delta_{j+\ell+3}$ for $j \neq 2$.
This is also an application of axiom (CPA2): As an induction start, we have

$$e_1 \cdot e_3 = e_3 \cdot e_1 = [e_1, e_2] \cdot e_1 = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_1)}_{\in \delta_4}}_{\subseteq \delta_{6+\ell}} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_3}}_{\subseteq \delta_{6+\ell}} \in \delta_{6+\ell}$$

and the induction step $j \to j + 1$ reads

$$e_1 \cdot e_{j+1} = e_{j+1} \cdot e_1 = [e_1, e_j] \cdot e_1 = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_1)}_{\in \delta_{j+\ell+3}}}_{\subseteq \delta_{j+\ell+4}} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_3}}_{\subseteq \delta_{4+j+\ell}} \in \delta_{j+\ell+4}.$$

Here we have used that by assumption $e_1 \cdot e_k \in \delta_{k+2+\ell}$ (for $k \geq 3$) and that $e_i \cdot e_j \in \delta_{i+j+\ell+1}$ (which we proved before). So we have proven the induction step. $\square$

Corollary 5.51. *Let $\mathfrak{g}$ be a filiform Lie algebra with $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ abelian. Then every CPA-structure in Class A_1 is associative.*

Proof. Proposition 5.49 implies the case $\ell = 0$ in Lemma 5.50. So, we know that $e_i \cdot e_j \in \delta_{n+1} = 0$ for all pairs (i, j) except $(1, 1), (1, 2), (2, 1), (2, 2)$; however, this exactly means that $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$, thus $x \cdot y$ is associative. □

We have dealt now with Class A_1. CPA-structures in Class A_2 and Class B will be investigated by similar means – however, they are much more work (as the conditions $\mathfrak{g} \cdot \mathfrak{g} \subseteq \delta_3$ and $\mathfrak{g} \cdot \delta_2 \subseteq \delta_4$ simplify the situation for Class A_1 tremendously).
Thus, the proofs for the following theorems can be found in Appendix A.

Proposition 5.52. *Suppose we have a CPA-structure on the filiform Lie algebra $\mathfrak{g}$ in Class A_2. Then $\mathfrak{g}$ is metabelian and $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.*

Proof. See Proposition A.2. □

5.4.2 CPA-structures in Class B

The remaining case is the one where $e_1 \cdot e_1 \notin \delta_3$ (meaning $x \cdot y$ is in Class B).

Lemma 5.53. *Let $\mathfrak{g}$ be filiform and $x \cdot y$ a CPA-structure on $\mathfrak{g}$ in Class B. Then $e_1 \cdot e_2 \in \beta e_2 + \delta_3$, where $\beta \in \{0, 1\}$.*

Proof. See Lemma A.3. □

Let $x \cdot y$ be a CPA-structure with $\mathfrak{g} \cdot \mathfrak{g} \not\subseteq \delta_3$, write $e_1 \cdot e_2 \in \beta e_3 + \delta_4$. We now know that $x \cdot y$ is in Class B_1 if and only if $\beta = 0$ and in Class B_2 if and only if $\beta = 1$.
Again, we deal with both cases (Class B_1 and Class B_2) separately.

Proposition 5.54. *Let $\mathfrak{g}$ be filiform (and n-dimensional), $x \cdot y$ a CPA-structure on $\mathfrak{g}$ in Class B_1 (meaning $e_1 \cdot e_1 \notin \delta_3$ and $e_1 \cdot e_2 \in \delta_4$). Then $\mathfrak{g} \cong L_n$ (where L_n denotes the standard graded filiform Lie algebra).*

Proof. See Proposition A.4. □

Proposition 5.55. *If $x \cdot y$ is a CPA-structure in Class B_2 (meaning $e_1 \cdot e_1 \notin \delta_3$ and $e_1 \cdot e_2 \notin \delta_4$) on $\mathfrak{g}$ (where $\mathfrak{g}$ is filiform and n-dimensional), then $\mathfrak{g} \cong L_n$.*

Proof. See Proposition A.6. □

Corollary 5.56. *If $\mathfrak{g}$ is n-dimensional filiform, $x \cdot y$ a CPA-structure on $\mathfrak{g}$ in Class B, then $\mathfrak{g} \cong L_n$.*

Proof. This statement is the combination of Proposition 5.54 and Proposition 5.55. □

5.4.3 The case $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ not abelian

In Section 5.4.1, we assumed that $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian. Now, the case remains where $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is not abelian. Let $\mathfrak{g}$ be filiform of dimension ≥ 7 and let $\mathfrak{g}$ have an adapted basis with $[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, [e_i, e_j] \in \delta_{i+j}, 1 \leq i, j \leq n, [e_{i+1}, e_{n-i}] = (-1)^i \tau e_n, 1 \leq i < n-1$ where $\tau \neq 0$.

We will first show (Lemma 5.57) that every CPA-structure on $\mathfrak{g}$ is in Class A. Afterwards, we prove in Proposition 5.58 that the assumptions of Lemma 5.50 still hold, meaning that we can as before conclude that every CPA-structure on $\mathfrak{g}$ is associative.

Lemma 5.57. *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra and $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ not abelian. Then every CPA-structure on $\mathfrak{g}$ is in Class A.*

Proof. This is one simple application of axiom (CPA3). We have $[e_2, e_{n-1}] = -\tau e_n, \tau \neq 0$ and $e_1 \cdot e_1 \in \alpha e_2 + \delta_3$. By axiom (CPA3) we obtain

$$\begin{aligned} 0 = e_1 \cdot e_n = [e_1 \cdot e_1, e_{n-1}] + [e_1, e_1 \cdot e_{n-1}] &\in [\alpha e_2 + \delta_3, e_{n-1}] + [e_1, \delta_n] \\ &\subseteq \alpha [e_2, e_{n-1}] + [\delta_3, e_{n-1}] + 0 \subseteq -\alpha \tau e_n + 0. \end{aligned}$$

So since $\tau \neq 0$, we have $\alpha = 0$, meaning that $x \cdot y$ is in Class A. □

Let $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ be non-abelian. Note that Lemma 5.46 and Proposition 5.47 still hold for $\mathfrak{g}$ (as they do not assume $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ to be abelian). We want to prove for $\mathfrak{g}$ a result similar to Proposition 5.49:

Proposition 5.58. *Let $\mathfrak{g}$ be n-dimensional filiform, $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ not abelian, $x \cdot y$ a CPA-structure on $\mathfrak{g}$ (which is in Class A by Lemma 5.57). Then*

(i) $x \cdot y$ is in Class A_1.

(ii) $e_1 \cdot e_j \in \delta_{j+2}$ for all $j \neq 1, 2$.

(iii) $e_i \cdot e_j \in \delta_{i+j}$ if neither of i and j is 1.

Proof. The proof is very similar to the one of Proposition 5.49. Write, as in the proof of Proposition 5.49, $L_i = (a^i_{k\ell})_{1 \leq k, \ell \leq n}$.
As in Proposition 5.49, we use (CPA3) for $2 \leq i \leq n-3$ and get

$$\underbrace{\underbrace{e_1 \cdot [e_1, e_i]}_{e_1 \cdot e_{i+1}}}_{\in a^1_{i+2,i+1} e_{i+2} + \delta_{i+3}} = \underbrace{[\underbrace{e_1 \cdot e_1}_{\in \delta_3}, e_i]}_{\subseteq \delta_{i+3}} + \underbrace{[e_1, \underbrace{e_1 \cdot e_i}_{\in a^1_{i+1,i} e_{i+1} + \delta_{i+2}}]}_{\subseteq a^1_{i+1,i} e_{i+2} + \delta_{i+3}}.$$

(This time, however, we cannot do this step for $i = n-2$.)
Doing the same for L_2, we obtain

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ * & \alpha & 0 & \dots & 0 & 0 & 0 \\ * & * & \alpha & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ * & * & * & \dots & \alpha & 0 & 0 \\ * & * & * & \dots & * & * & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ * & \beta & 0 & \dots & 0 & 0 & 0 \\ * & * & \beta & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ * & * & * & \dots & \beta & 0 & 0 \\ * & * & * & \dots & * & * & 0 \end{pmatrix}.$$

As before, $[e_1, e_2] \cdot e_2 = e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2)$ implies $\beta = 0$ and $[e_1, e_2] \cdot e_1 = e_1 \cdot (e_2 \cdot e_1) - e_2 \cdot (e_1 \cdot e_1)$ implies $\alpha \in \{0, 1\}$. (Later we will show that $\alpha = 1$ is impossible, meaning that $x \cdot y$ is in Class A_1.)
Let us now show that for $i, j \geq 3$, we have $e_i \cdot e_j \in \delta_{i+j}$ if $i + j \neq n + 1$ and $e_i \cdot e_j \in \delta_n$ if $i + j = n + 1$. We do this inductively: We fix i and do an induction on $j \geq 3$. For the induction base, we distinguish three cases: (a) $i + 3 = n + 1$, (b) $i + 3 = n + 2$, (c) else.
Case (a): Suppose $i + 3 = n + 1$, show $e_3 \cdot e_i \in \delta_n$: By (CPA2),

$$e_3 \cdot e_i = e_3 \cdot e_{n-2} = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_{n-2})}_{\in \delta_n}}_{=0} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_{n-2})}_{\in \delta_{n-1}}}_{\subseteq \delta_n} \in \delta_n.$$

Case (b): Suppose $i + 3 = n + 2$, show $e_3 \cdot e_i \in \delta_{i+3}$: Again, by (CPA2),

$$e_3 \cdot e_i = e_3 \cdot e_{n-1} = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_{n-1})}_{\in \delta_n}}_{=0} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_{n-1})}_{\in \delta_n}}_{=0} = 0 = \delta_{i+3}.$$

Case (c): Suppose $i + 3 \neq n + 2, i + 3 \neq n + 1$, show $e_3 \cdot e_i \in \delta_{i+3}$: Again, by (CPA2),

$$e_3 \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_2 \cdot e_i)}_{\in \delta_{i+2}}}_{\subseteq \delta_{i+3}} - \underbrace{e_2 \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_{i+3}} \in \delta_{i+3}.$$

Now the induction step $j \to j+1$: We again distinguish three cases: (a) $j + (i+1) = n+1$, (b) $i + j = n + 1$, (c) else.
Case (a): Suppose $i + j = n$, show $e_{j+1} \cdot e_i \in \delta_n$: By (CPA2),

$$e_{j+1} \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_i)}_{\in \delta_{i+j}}}_{\subseteq \delta_{i+j+1} = 0} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_n} \in \delta_n$$

(as $e_j \cdot \delta_{i+1} \subseteq \delta_n$ by induction hypothesis).
Case (b): Suppose $i+j=n+1$, show $e_{j+1} \cdot e_i \in \delta_{i+j+1}$: Again, by (CPA2),

$$e_{j+1} \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_i)}_{\in \delta_n}}_{=0} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_{i+j+1}=0} = 0 = \delta_{i+j+1}.$$

Case (c): Suppose $i+j \neq n, i+j \neq n+1$, show $e_{j+1} \cdot e_i \in \delta_{i+j+1}$: Again, by (CPA2),

$$e_{j+1} \cdot e_i = \underbrace{e_1 \cdot \underbrace{(e_j \cdot e_i)}_{\in \delta_{i+j}}}_{\subseteq \delta_{i+j+1}} - \underbrace{e_j \cdot \underbrace{(e_1 \cdot e_i)}_{\in \delta_{i+1}}}_{\subseteq \delta_{i+j+1}} \in \delta_{i+j+1}.$$

This proves our claim.
Now we are ready to prove that $x \cdot y$ is in Class A_1: Suppose not, then we have $\alpha = 1$. We want to show that then $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian (this would be a contradiction to our assumption). For that, it is enough to show that $[e_{n/2}, e_{n/2+1}] = 0$.
By the above, we can write $e_{n/2} \cdot e_{n/2} = \alpha_1 e_n, e_{n/2} \cdot e_{n/2+1} = \alpha_2 e_n$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. This implies by axiom (CPA2)

$$\begin{aligned} 0 &= [e_1, e_{n/2}] \cdot e_{n/2} - \underbrace{e_1 \cdot (e_{n/2} \cdot e_{n/2})}_{=0} + e_{n/2} \cdot (e_1 \cdot e_{n/2}) \\ &\in e_{n/2+1} \cdot e_{n/2} + e_{n/2} \cdot (e_{n/2+1} + \delta_{n/2+2}) \\ &\subseteq \alpha_2 e_n + \alpha_2 e_n + 0, \end{aligned}$$

meaning $\alpha_2 = 0$. So by axiom (CPA3),

$$\begin{aligned} 0 = e_{n/2} \cdot e_{n/2+1} = e_{n/2} \cdot [e_1, e_{n/2}] &= [e_{n/2} \cdot e_1, e_{n/2}] + [e_1, e_{n/2} \cdot e_{n/2}] \\ &\in [e_{n/2+1} + \delta_{n/2+2}, e_{n/2}] + [e_1, \alpha_1 e_n] \subseteq [e_{n/2+1}, e_{n/2}] + 0 \\ &\subseteq -[e_{n/2}, e_{n/2+1}] + 0, \end{aligned}$$

thus $[e_{n/2}, e_{n/2+1}] = 0$.
So we have a contradiction and obtain that $x \cdot y$ is in Class A_1. This shows (i).
Let us show now (iii). We have proven (iii) so far for $i = 2, j \neq n-1$ (since $\beta = 0$) and for $i, j \geq 3$ with $i + j \neq n+1$. Now let $i \geq 2, j \geq 3, i+j = n+1$. For this, we write $e_i \cdot e_{j-1} = \alpha_1 e_n, e_i \cdot e_j = \alpha_2 e_n$. Then

$$\begin{aligned} 0 &= [e_1, e_{j-1}] \cdot e_i - e_1 \cdot (e_{j-1} \cdot e_i) + e_{j-1} \cdot (e_1 \cdot e_i) \\ &\in e_j \cdot e_i - \underbrace{e_1 \cdot (\alpha_1 e_n)}_{=0} + \underbrace{e_{j-1} \cdot \delta_{i+2}}_{=0} \subseteq \alpha_2 e_n + 0 \end{aligned}$$

meaning $\alpha_2 = 0$ which, in turn, implies $e_i \cdot e_j = 0$, that is, $\delta_i \cdot \delta_j \in \delta_{i+j}$ also holds for $i + j = n + 1$.
This proves (iii). However, if we repeat this last step and set $i = 1, j = n-1$ (and use that we have just proven $\delta_3 \cdot \delta_{n-2} = 0$), we also obtain (ii). $\square$

But the statements of Proposition 5.58 mean that we are in the situation of Lemma 5.50 for $\ell = 0$. Lemma 5.50 does not assume $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ to be abelian and thus it can be applied. Therefore:

Corollary 5.59. *If $\mathfrak{g}$ is filiform and $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ not abelian then all CPA-structures on $\mathfrak{g}$ are associative.*

Proof. Proposition 5.58 implies the assumptions of Lemma 5.50; by Lemma 5.50 we find, inductively as before, that $e_i \cdot e_j \in \delta_{n+1}$ if (i, j) is not one of $(1, 1), (1, 2), (2, 1), (2, 2)$ – so $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$. □

Putting everything we proved so far on filiform Lie algebras together, we can prove the "classification theorem", Theorem 5.44.

Proof of Theorem 5.44. If $\mathfrak{g}$ is filiform and $x \cdot y$ is a CPA-structure on $\mathfrak{g}$, then $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian or not.

(a) Let $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ be not abelian. Then, by Corollary 5.59, the CPA-structure is associative. Note that as $\dim(\mathfrak{g}) \geq 6$, we have $[e_3, e_{n-2}] = e_n$, so $\mathfrak{g}$ is non-metabelian.

(b) Let $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ be abelian.

   (i) If $\mathfrak{g}$ is non-metabelian, then $x \cdot y$ cannot be in Class B (Corollary 5.56) and not in Class A_2 (Proposition 5.52). So it is in Class A_1 and thus associative by Corollary 5.51.

   (ii) If $\mathfrak{g}$ is metabelian and $\mathfrak{g} \ncong L_n$, then $x \cdot y$ cannot be in Class B (Corollary 5.56), so it is in Class A_1 or A_2, meaning $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ or $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ by Corollary 5.51 or Proposition 5.52. In both cases, $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

This proves the "if"-directions of (i) and (ii). The "only if"-directions will be proven by presenting a non-associative CPA-structure on metabelian Lie algebras and by presenting a CPA-structure on L_n not satisfying $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$, see Proposition 5.60 and Proposition 5.66, respectively.

□

5.4.4 CPA-structures on metabelian filiform Lie algebras

We have just proven that every CPA-structure on a filiform non-metabelian Lie algebra satisfies $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$. For metabelian filiform Lie algebras, there are indeed CPA-structures not satisfying this condition:

Proposition 5.60. *If $\mathfrak{g}$ is filiform and metabelian of dimension $\dim(\mathfrak{g}) \geq 4$, then there is a non-associative CPA-structure on $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}$ be filiform and metabelian; let $\{e_1, e_2, \dots, e_n\}$ be an adapted basis of $\mathfrak{g}$. Write $[e_2, e_3] = \alpha_{2,5}e_5 + \alpha_{2,6}e_6 + \ldots + \alpha_{2,n}e_n$. Then we define a bilinear map by

$$\begin{aligned} e_i \cdot e_1 &= e_1 \cdot e_i = [e_1, e_i] \text{ for all } i \geq 1, \\ e_j \cdot e_2 &= e_2 \cdot e_j = [e_2, e_j] \text{ for all } j \geq 3, \\ e_2 \cdot e_2 &= 2\alpha_{2,5}e_4 + 2\alpha_{2,6}e_5 + \ldots + 2\alpha_{2,n}e_{n-1}. \end{aligned}$$

(If $\dim(\mathfrak{g}) = 4$, set $e_2 \cdot e_2 = 0$.)
The commutativity (axiom (CPA1)) of $x \cdot y$ is clear, we show that axioms (CPA2) and (CPA3) hold:
First note that we have:

- $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.
- If $x \in [\mathfrak{g}, \mathfrak{g}]$, then for all $y \in \mathfrak{g}$ we have $x \cdot y = y \cdot x = [y, x]$.
- $e_1 \cdot (e_2 \cdot e_2) = [e_1, e_2 \cdot e_2] = 2[e_2, e_3] = 2e_2 \cdot e_3$.

It is enough to show axioms (CPA2) and (CPA3) for the basis vectors $e_1, \dots, e_n$.
Axiom (CPA3):
We have to show $x \cdot [y, z] - [x \cdot y, z] - [y, x \cdot z] = 0$.
If $x = e_1$, this equality is satisfied since $L(e_1) = \operatorname{ad}(e_1)$ is a derivation of $\mathfrak{g}$.
If $x \in [\mathfrak{g}, \mathfrak{g}]$, we have

$$x \cdot [y, z] - [x \cdot y, z] - [y, x \cdot z] = 0 - [[y, x], z] - [y, [z, x]] = [[z, y], x]$$

which is 0 since $x \in [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}$ is metabelian.
The case $x = e_2$ remains; since $L(e_2)|_{[\mathfrak{g},\mathfrak{g}]} = \operatorname{ad}(e_2)|_{[\mathfrak{g},\mathfrak{g}]}$, we may assume that not both of y and z are in $[\mathfrak{g}, \mathfrak{g}]$. So we have four cases left:
Case 1: If $y = z = e_1$ or $y = z = e_2$, then axiom (CPA3) is trivially satisfied.
Case 2: If $y = e_1$ and $z = e_2$ (or, by symmetry, vice versa), then we have

$$\begin{aligned} e_2 \cdot [e_1, e_2] - [e_2 \cdot e_1, e_2] - [e_1, e_2 \cdot e_2] &= e_2 \cdot e_3 - [e_3, e_2] - [e_1, e_2 \cdot e_2] \\ &= 2[e_2, e_3] - [e_1, e_2 \cdot e_2] = 0. \end{aligned}$$

Case 3: If $y = e_1$ and $z \in [\mathfrak{g}, \mathfrak{g}]$ (or vice versa), then

$$e_2 \cdot [e_1, z] - [e_1 \cdot e_2, z] - [e_1, e_2 \cdot z] = [e_2, [e_1, z]] - [[e_1, e_2], z] - [e_1, [e_2, z]] = 2[z, [e_1, e_2]] = 0$$

again by the fact that $\mathfrak{g}$ is metabelian.
Case 4: If $y = e_2$ and $z \in [\mathfrak{g}, \mathfrak{g}]$ (or vice versa), then

$$e_2 \cdot [e_2, z] - [\underbrace{e_2 \cdot e_2}_{\in [\mathfrak{g},\mathfrak{g}]}, z] - [e_2, e_2 \cdot z] = [e_2, [e_2, z]] - 0 - [e_2, [e_2, z]] = 0.$$

So axiom (CPA3) holds for all combinations of basis vectors and thus for all elements of $\mathfrak{g}$.

Axiom (CPA2) is a similar check:
If $z \in [\mathfrak{g}, \mathfrak{g}]$, then

$$[x,y] \cdot z - x \cdot (y \cdot z) + y \cdot (x \cdot z) = 0 - [x,[y,z]] + [y,[x,z]] = [z,[x,y]] = 0$$

since $\mathfrak{g}$ is metabelian.
If $z = e_1$, we have

$$[x,y] \cdot e_1 - x \cdot (y \cdot e_1) + y \cdot (x \cdot e_1) = [e_1,[x,y]] - [x,[e_1,y]] + [y,[e_1,x]] = 0$$

by the Jacobi identity. So let $z = e_2$. If $x = y$ or $x, y \in [\mathfrak{g}, \mathfrak{g}]$, axiom (CPA2) is trivially satisfied. It remains to check two cases:
Case 1: $x = e_1, y = e_2$ (or vice versa): Here, we have

$$\begin{aligned}[e_1,e_2] \cdot e_2 - e_1 \cdot (e_2 \cdot e_2) + e_2 \cdot (e_1 \cdot e_2) &= e_2 \cdot e_3 - e_1 \cdot (e_2 \cdot e_2) + e_2 \cdot e_3 \\ &= 2e_2 \cdot e_3 - e_1 \cdot (e_2 \cdot e_2) = 0.\end{aligned}$$

Case 2: $x = e_1$ or $x = e_2, y \in [\mathfrak{g}, \mathfrak{g}]$ (or vice versa): Here, we have

$$[x,y] \cdot e_2 - x \cdot (y \cdot e_2) + y \cdot \underbrace{(x \cdot e_2)}_{\in [\mathfrak{g},\mathfrak{g}]} = [e_2,[x,y]] - [x,[e_2,y]] + 0 = [y,[x,e_2]] = 0$$

since $y \in [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}$ is metabelian.
Finally, as $e_1 \cdot e_3 = e_4$, the CPA-structure is non-associative. □

Remark 5.61.

(i) For the metabelian $\mathfrak{n}_3(\mathbb{C})$ (which is the only filiform Lie algebra in dimension 3), all CPA-structures are on $\mathfrak{n}_3(\mathbb{C})$ are associative (that is, Proposition 5.60 does not hold in dimension 3), see Proposition B.10.

(ii) Proposition 5.60 proves the "only if"-direction in Theorem 5.44, (i).

5.4.5 CPA-structures on infinite-dimensional filiform Lie algebras

In this section, we shall consider post-Lie algebra structures on infinite-dimensional filiform Lie algebras. An infinite-dimensional filiform Lie algebra $\mathfrak{g}$ is not nilpotent, but rather *residually nilpotent*, meaning $\bigcap_{i=0}^{\infty} \mathfrak{g}^i = 0$ (where $\mathfrak{g}^0, \mathfrak{g}^1, \ldots$ denotes the lower central series of $\mathfrak{g}$). The definition is as follows:

Definition 5.62 ([76]). Let $\mathfrak{g}$ be a finitely-generated, infinite-dimensional Lie algebra and $\mathfrak{g}^0, \mathfrak{g}^1, \mathfrak{g}^2, \ldots$ its lower central series. If $\dim(\mathfrak{g}^0/\mathfrak{g}^1) = 2$ and $\dim(\mathfrak{g}^i/\mathfrak{g}^{i+1}) = 1$ for all $i \geq 1$, then $\mathfrak{g}$ is called a *filiform Lie algebra*.

So when passing from the ideal $\mathfrak{g}^i$ to the ideal $\mathfrak{g}^{i+1}$ (with $i \geq 1$), the dimension decreases exactly by 1. Note that this is a reasonable generalization of the term "filiform" to infinite-dimensional Lie algebras.
Infinite-dimensional filiform Lie algebras can be seen as direct limits of finite-dimensional filiform Lie algebras (where each one of those is a central extension of the previous Lie algebra) and thus also have an adapted basis, that is, a basis $\{e_1, e_2, e_3, \ldots, \}$ such that

$$[e_1, e_i] = e_{i+1} \text{ for all } i \geq 2,$$
$$[e_i, e_j] \in \operatorname{span}(\{e_{i+j}, e_{i+j+1}, \ldots\})$$

(cf. [76]).

Also CPA-structures on infinite-dimensional filiform Lie algebras are direct limits of CPA-structures on filiform Lie algebras: Given a CPA-structure on $\mathfrak{g}$, define $\mathfrak{g}_n := \mathfrak{g}/\mathfrak{g}^{n+1}$ and let $\overline{a}$ be the image of a in $\mathfrak{g}_n$. Then $\mathfrak{g}_n$ is finite-dimensional filiform and a CPA-structure on $\mathfrak{g}_n$ is given by $\overline{a} \cdot \overline{b} := \overline{(ab)}$. Then $\mathfrak{g}$ is the direct limit of the $\mathfrak{g}_n$ and the CPA-structure on $\mathfrak{g}$ is the direct limit of the CPA-structures on the $\mathfrak{g}_n$. This implies that Theorem 5.44 extends to infinite-dimensional filiform Lie algebras:

Corollary 5.63. *Let $\mathfrak{g}$ be an infinite-dimensional filiform Lie algebra with a CPA-structure $x \cdot y$.*

(i) We have $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ for every CPA-structure on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is non-metabelian.

(ii) We have $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ for every CPA-structure on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to the direct limit of the standard graded filiform Lie algebras L_n.

Proof. For the "only if"-directions, see Proposition 5.64 and Remark 5.68, respectively. We show the "if"-directions:

(i) Let $\mathfrak{g}$ be non-metabelian. Then there is an n such that for all $\mathfrak{g}_m, m \geq n, \mathfrak{g}_m$ is non-metabelian.
Thus to prove $a \cdot [b, c] = 0$ for all $a, b, c \in \mathfrak{g}$, choose n such that $\mathfrak{g}_n$ is non-metabelian. Then in all $\mathfrak{g}_m, m \geq n$, we have $\overline{a} \cdot [\overline{b}, \overline{c}] = 0$, thus $a \cdot [b, c] = 0$ in $\mathfrak{g}$.

(ii) If $\mathfrak{g}$ is metabelian, then all $\mathfrak{g}_m$ are metabelian and there is an n such that for all $\mathfrak{g}_m, \geq n, \mathfrak{g}_m \not\cong L_m$. To prove $[a, b] \cdot [c, d] = 0$ for $a, b, c, d \in \mathfrak{g}$, note that in all $\mathfrak{g}_m$, $[\overline{a}, \overline{b}] \cdot [\overline{c}, \overline{d}] = 0$, thus $[a, b] \cdot [c, d] = 0$ also in $\mathfrak{g}$.

□

The CPA-structure given in Proposition 5.60 can be extended to infinite-dimensional filiform Lie algebras:

Proposition 5.64. *If $\mathfrak{g}$ is infinite-dimensional, filiform and metabelian, then there exists a non-associative CPA-structure on $\mathfrak{g}$.*

Proof. Let $\{e_1, e_2, \ldots\}$ be an adapted basis of $\mathfrak{g}$ and let $[e_2, e_3] = \sum_{i=5}^{m} \alpha_{2,i} e_i$ for some $m \in \mathbb{N}$. Analogously to the proof of Proposition 5.60, define a bilinear map by

$$\begin{aligned} e_i \cdot e_1 &= e_1 \cdot e_i = [e_1, e_i] \text{ for all } i \geq 1, \\ e_j \cdot e_2 &= e_2 \cdot e_2 = [e_2, e_j] \text{ for all } j \geq 2, \\ e_2 \cdot e_2 &= 2 \sum_{i=4}^{m-1} \alpha_{2,i+1} e_i \end{aligned}$$

and copy the proof of Proposition 5.60. □

Post-Lie algebra structures on infinite-dimensional Lie algebras were also studied by other authors ([91, 92]).

5.5 Explicit CPA-structures on certain filiform Lie algebras

With these results, we want to derive explicitly the CPA-structures on certain families of filiform Lie algebras. [58, Chapter 4, page 111] states that "among all these [filiform] algebras, only four play an important role". These families of Lie algebras are known by the names L_n, Q_n, R_n and W_n – we will derive all CPA-structures on them. Afterwards, we will calculate the CPA-structures on a family of characteristic nilpotent filiform Lie algebras.
Our strategy is to use Theorem 5.44 and then to derive the concrete CPA-structures with the help of the derivation algebras of those families.

5.5.1 The Lie algebra L_n

Lemma 5.65 ([58, Chapter 4, Proposition 1]). *Let L_n be the n-dimensional filiform Lie algebra ($n \geq 4$) defined by*

$$[e_1, e_i] = e_{i+1}, i = 2, \ldots, n-1.$$

Let $t_1, t_2, t_3, h_2, h_3, \ldots, h_{n-2}$ be the endomorphisms

$$\begin{aligned} & t_1(e_i) = e_i, 2 \leq i \leq n, t_1(e_1) = 0, \\ & t_2(e_1) = e_1, t_2(e_i) = (i-1)e_i, 2 \leq i \leq n, \\ & t_3(e_1) = e_2, t_3(e_i) = 0, 2 \leq i \leq n, \\ & h_s(e_i) = e_{i+s}, 2 \leq i \leq n-s, h_s(e_1) = 0, s = 2, \ldots, n-2, h_s(e_j) = 0, n-s < j \leq n. \end{aligned}$$

Then the endomorphisms $\operatorname{ad} e_i, i = 1, \ldots, n-1, t_1, t_2, t_3, h_2, h_3, \ldots, h_{n-2}$ *form a basis of* $\operatorname{Der}(L_n)$. *In particular,* $\dim(\operatorname{Der}(L_n)) = 2n - 1$.

Proposition 5.66. *Let $n \geq 5$. Then all CPA-structures on L_n are given by one of the two following types (with respect to the basis above):*

Type (i): $e_1 \cdot e_1 = \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_n e_n, e_1 \cdot e_2 = e_2 \cdot e_1 = \beta e_{n-1} + \gamma e_n, e_1 \cdot e_3 = e_3 \cdot e_1 = \beta e_n, e_2 \cdot e_2 = \delta e_n$; *all other products between basis elements are zero and the condition* $-\delta\alpha_2 = \beta$ *holds.*

Type (ii): $e_1 \cdot e_1 = \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_n e_n, e_1 \cdot e_2 = e_2 \cdot e_1 = e_3 + \beta e_{n-1} + \gamma e_n, e_1 \cdot e_3 = e_3 \cdot e_1 = e_4 + \beta e_n, e_1 \cdot e_k = e_k \cdot e_1 = e_{k+1}$ $(4 \leq k \leq n-1)$, $e_2 \cdot e_2 = \delta e_n$; *all other products between basis elements are zero and the condition* $\delta\alpha_2 = \beta$ *holds.*

In both cases, $\alpha_2, \ldots, \alpha_n, \beta, \gamma, \delta \in \mathbb{C}$ *arbitrarily.*

Proof. One has to check that the structures given in the proposition are indeed CPA-structures on L_n. On the other hand, let $x \cdot y$ be any CPA-structure on L_n, we show that it is of the form (i) or (ii) by induction on the dimension n. For $n = 5$, one calculates that every CPA-structure has one of these two forms, now consider the induction step $n-1 \to n$:
We have $L_n / \mathrm{Z}(L_n) \cong L_{n-1}$ and, as the center of L_n is one-dimensional, $\mathfrak{g} \cdot \mathrm{Z}(L_n) = 0$ ([34, Corollary 3.4]).
So $x \cdot y$ induces a CPA-structure $x \circ y$ on L_{n-1}, which, by induction hypothesis, is of type (i) or (ii).

Case 1: Let us first assume that $x \circ y$ is of type (i). So we can lift it back to L_n and see that $x \cdot y$ has the form

$$\begin{aligned} e_1 \cdot e_1 &\in \alpha_2 e_2 + \ldots + \alpha_{n-1} e_{n-1} + \mathrm{Z}(L_n) \\ e_1 \cdot e_2 = e_2 \cdot e_1 &\in \beta' e_{n-2} + \gamma' e_{n-1} + \mathrm{Z}(L_n) \\ e_1 \cdot e_3 = e_3 \cdot e_1 &\in \beta' e_{n-1} + \mathrm{Z}(L_n) \\ e_2 \cdot e_2 &\in \delta' e_{n-1} + \mathrm{Z}(L_n) \end{aligned}$$

subject to $-\alpha_2 \delta' = \beta'$ and all other products of basis elements lie in $\mathrm{Z}(L_n)$.
As all left-multiplication operators lie in $\mathrm{Der}(L_n)$, by Lemma 5.65 we obtain $[L_n, L_n] \cdot [L_n, L_n] = 0, \delta_2 \cdot \delta_4 = L_n \cdot \delta_5 = 0$ (where we write $\delta_i \coloneqq \mathrm{span}\{e_j, j \geq i\}$). So we have (the variables' names resemble the form we want in the end)

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_2 e_2 + \ldots + \alpha_n e_n \\ e_1 \cdot e_2 = e_2 \cdot e_1 &= \beta' e_{n-2} + \gamma' e_{n-1} + \gamma e_n \\ e_1 \cdot e_3 = e_3 \cdot e_1 &= \beta' e_{n-1} + \beta e_n \\ e_1 \cdot e_4 = e_4 \cdot e_1 &= \varepsilon e_n \\ e_2 \cdot e_2 &= \delta' e_{n-1} + \delta e_n \\ e_2 \cdot e_3 = e_3 \cdot e_2 &= \zeta e_n \end{aligned}$$

for some $\alpha_i, \beta, \beta', \gamma, \gamma', \delta, \delta', \varepsilon, \zeta \in \mathbb{C}$. By Lemma 5.65, we further obtain $\beta = \gamma', \varepsilon = \beta'$ and $\zeta = \delta'$.
Now axiom (CPA2) gives us

$$\begin{aligned} \zeta e_n = e_2 \cdot e_3 = [e_1, e_2] \cdot e_2 &= e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2) \\ &= e_1 \cdot (\delta' e_{n-1} + \delta e_n) - e_2 \cdot (\beta' e_{n-2} + \beta e_{n-1} + \gamma e_n) = 0, \end{aligned}$$

meaning $\zeta = \delta' = 0$.
And axiom (CPA2) also implies

$$\begin{aligned}\beta' e_{n-1} + \beta e_n = e_3 \cdot e_1 = [e_1, e_2] \cdot e_1 &= e_1 \cdot (e_2 \cdot e_1) - e_2 \cdot (e_1 \cdot e_1) \\ &= e_1 \cdot (\beta' e_{n-2} + \beta e_{n-1} + \gamma e_n) - e_2 \cdot (\alpha_2 e_2 + \ldots + \alpha_n e_n) \\ &= -\alpha_2 \delta e_n.\end{aligned}$$

So $\beta' = 0$ and $\beta = -\alpha_2\delta$, meaning we have proven that $x \cdot y$ is a CPA-structure of type (i).
Case 2: Now we assume that $x \circ y$, the CPA-structure on the quotient $L_n / \operatorname{Z}(L_n) \cong L_{n-1}$, is of type (ii). So, lifting $x \circ y$ to L_n, we see that $x \cdot y$ satisfies

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_2 e_2 + \ldots + \alpha_{n-1} e_{n-1} + \alpha_n e_n \\ e_1 \cdot e_2 = e_2 \cdot e_1 &= e_3 + \beta' e_{n-2} + \gamma' e_{n-1} + \gamma e_n \\ e_1 \cdot e_3 = e_3 \cdot e_1 &= e_4 + \beta' e_{n-1} + \beta e_n \\ e_1 \cdot e_k = e_k \cdot e_1 &= e_{k+1}, 4 \le k \le n-2 \\ e_1 \cdot e_{n-1} = e_{n-1} \cdot e_1 &= \varepsilon e_n \\ e_2 \cdot e_2 &= \delta' e_{n-1} + \delta e_n \\ e_2 \cdot e_3 = e_3 \cdot e_2 &= \zeta e_n \end{aligned}$$

with $\delta' \alpha_2 = \beta'$ and the other constants arbitrarily (after using the form of L_n's derivations). Moreover, the fact that $L(e_1)$ and $L(e_2)$ are derivations also implies

$$\varepsilon = 1, \beta = \gamma' \text{ and } \zeta = \delta'.$$

And as in Case 1, axiom (CPA2), used on the triples (e_1, e_2, e_2) and (e_1, e_2, e_1), gives $\delta' = 0$ and $\beta' = 0, \beta = \alpha_2 \delta$, respectively. So we obtain that $x \cdot y$ is a CPA-structure of type (ii) and are done. $\square$

Remark 5.67. What if $n \le 4$? The Lie algebras L_1 and L_2 are abelian, L_3 is the 3-dimensional Heisenberg algebra. The Lie algebra L_4 has only one type of CPA-structures, namely:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \beta & 0 & 0 \\ \alpha_4 & \gamma & \beta & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{pmatrix}, L(e_4) = 0$$

with arbitrary elements $\alpha_2, \alpha_3, \alpha_4, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha_2\delta + \beta - \beta^2 = 0$.

Remark 5.68. Setting the variables β, γ, δ in Proposition 5.66 to zero, one can check that the analogously defined bilinear maps

$$e_1 \cdot e_1 = \sum_{i=2}^{\infty} \alpha_i e_i, e_j \cdot e_k = 0 \text{ if } j \text{ or } k \text{ is not } 1$$

and

$$e_1 \cdot e_1 = \sum_{i=2}^{\infty} \alpha_i e_i, e_1 \cdot e_j = e_j \cdot e_1 = e_{j+1}, e_j \cdot e_k = 0 \text{ if } j \text{ and } k \text{ are not } 1$$

(and extended bilinearly) also define CPA-structures on the *infinite-dimensional* standard graded filiform Lie algebra (with non-zero brackets $[e_1, e_j] = e_{j+1}$).

5.5.2 The Lie algebra Q_n

Lemma 5.69 ([58, Chapter 4, Proposition 3]). *Let Q_n be the n-dimensional filiform Lie algebra ($n \geq 6$ even) defined by*

$$[e_1, e_i] = e_{i+1}, i = 2, \ldots, n-1,$$
$$[e_i, e_{n-i+1}] = (-1)^{i+1} e_n, i = 2, \ldots, n/2.$$

Let $t_0, t_1, t_2, h_3, h_5, h_7, \ldots, h_{n-3}$ be the endomorphisms

$$\begin{aligned}
&t_0(e_1) = e_n, t_0(e_i) = 0, i \neq 1, \\
&t_1(e_1) = -e_2, t_1(e_i) = e_i, 2 \leq i \leq n-1, t_1(e_n) = 2e_n, \\
&t_2(e_1) = e_1 + e_2, t_2(e_i) = (i-2)e_i, 2 \leq i \leq n-1, t_2(e_n) = (n-3)e_n, \\
&h_s(e_i) = e_{i+s}, 2 \leq i \leq n-s, h_s(e_1) = h_s(e_j) = 0, n-s < j \leq n.
\end{aligned}$$

Then the endomorphisms $\operatorname{ad} e_i, i = 1, \ldots, n-1, t_0, t_1, t_2, h_3, h_5, h_7, \ldots, h_{n-3}$ *form a basis of* $\operatorname{Der}(Q_n)$. *In particular,* $\dim(\operatorname{Der}(Q_n)) = \frac{3n}{2}$.

Remark 5.70. Some remarks are in order:

(i) In [58], the endomorphism t_0 is missing.

(ii) For $n = 4$, we have $Q_4 \cong L_4$.

(iii) If $n \geq 6$, then all derivations of Q_n are (with respect to the basis given above) lower triangular.

Proposition 5.71. *Let $n \geq 6$. Then all CPA-structures on Q_n are (with respect to the basis given above) given by*

$$e_1 \cdot e_1 = \alpha e_{n-1} + \beta e_n, e_1 \cdot e_2 = e_2 \cdot e_1 = -\alpha e_{n-1} + \gamma e_n, e_2 \cdot e_2 = \alpha e_{n-1} + \delta e_n, \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

and all other products are zero.

Proof. For one direction of the proposition, one can check the given structures are indeed CPA-structures on Q_n.

For the other direction, let again $L_i := L(e_i)$ be the left-multiplication operator.

Note that $Q_n^{\lceil (n-4)/2 \rceil}$ is not abelian – so by Corollary 5.59, we have $Q_n \cdot [Q_n, Q_n] = 0$. To deduce the precise form of the CPA-structure, expand L_1 and L_2 into the basis of derivations,

$$L_i = \sum_{k=1}^{n-1} \alpha_{i,k} \operatorname{ad} e_k + \sum_{k=1}^{n/2-2} \beta_{i,k} h_{2k+1} + \gamma_i t_0 + \gamma_{i,1} t_1 + \gamma_{i,2} t_2.$$

The fact $L_i(\delta_3) = 0, i = 1, 2$ (since $Q_n \cdot [Q_n, Q_n] = 0$), implies that most coefficients vanish. In fact, $L_i = 0$ for $i \geq 3$ and for $i = 1, 2$ we have

$$L_i = \alpha_{i,n-2} \operatorname{ad} e_{n-2} + \alpha_{i,n-1} \operatorname{ad} e_{n-1} + \beta_{i,n-2} h_{2n-3} + \gamma_i t_0.$$

To make notation simpler, let us write

$$\begin{aligned} L_1 &= \alpha_1 \operatorname{ad} e_{n-2} + \gamma \operatorname{ad} e_{n-1} + \alpha_2 h_{2n-3} + (\gamma + \beta) t_0, \\ L_2 &= \alpha_3 \operatorname{ad} e_{n-2} + \delta \operatorname{ad} e_{n-1} + \alpha_4 h_{2n-3} + (\epsilon + \delta) t_0. \end{aligned}$$

This means $L_1(e_1), L_1(e_2), L_2(e_1), L_2(e_2) \in \operatorname{span}\{e_{n-1}, e_n\}$.

To conclude the proof, let us compute $L_i(e_j), i, j = 1, 2$:

$$\begin{aligned} L_1(e_1) &= -\alpha_1 e_{n-1} - \gamma e_n + (\gamma + \beta) e_n, \\ L_1(e_2) &= \gamma e_n + \alpha_2 e_{n-1}, \\ L_2(e_1) &= -\alpha_3 e_{n-1} - \delta e_n + (\epsilon + \delta) e_n, \\ L_2(e_2) &= \delta e_n + \alpha_4 e_{n-1}. \end{aligned}$$

Since $L_1(e_2) = L_2(e_1)$, $\alpha_2 = -\alpha_3$ and $\epsilon = \gamma$.
We know $e_1 \cdot e_3 = 0 = e_2 \cdot e_3$, but

$$\begin{aligned} L_1(e_3) &= -\alpha_1 e_n + \alpha_2 e_n \\ L_2(e_3) &= -\alpha_3 e_n + \alpha_4 e_n, \end{aligned}$$

so $\alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4$, set $\alpha \coloneqq -\alpha_1$.
Thus finally

$$\begin{aligned} e_1 \cdot e_1 &= \alpha e_{n-1} + \beta e_n, \\ e_1 \cdot e_2 &= -\alpha e_{n-1} + \gamma e_n, \\ e_2 \cdot e_2 &= \alpha e_{n-1} + \delta e_n. \end{aligned}$$

and all other products are zero, as desired. □

Remark 5.72. For the CPA-structures on Q_4, see Remark 5.67 (as $Q_4 \cong L_4$).

5.5.3 The Lie algebra R_n

Lemma 5.73 ([58, Chapter 4, Proposition 4]). *Let R_n be the n-dimensional filiform Lie algebra ($n \geq 5$) defined by*

$$[e_1, e_i] = e_{i+1}, i = 2, \ldots, n-1,$$
$$[e_2, e_i] = e_{i+2}, i = 3, \ldots, n-2.$$

Let $t, h_2, \ldots, h_{n-2}$ be the endomorphisms

$$t(e_i) = ie_i, 1 \leq i \leq n,$$
$$h_k(e_i) = e_{i+k}, 2 \leq i \leq n-2, h_k(e_i) = 0 \text{ else.}$$

Then the endomorphisms $\operatorname{ad} e_i, i = 1, \ldots, n-1, t, h_2, \ldots, h_{n-2}$ form a basis of $\operatorname{Der}(R_n)$. In particular, $\dim(\operatorname{Der}(R_n)) = 2n-3$.

Proposition 5.74. *Let $n \geq 6$. Then all CPA-structures on R_n are given (with respect to the basis given above) by*

(a) $e_1 \cdot e_1 = \alpha_3 e_3 + \alpha_4 e_4 + \ldots + \alpha_{n-1} e_{n-1} + \alpha_n e_n, e_1 \cdot e_2 = \alpha_3 e_4 + \alpha_4 e_5 + \ldots + \alpha_{n-2} e_{n-1} + \beta e_n, e_2 \cdot e_2 = \alpha_3 e_5 + \alpha_4 e_6 + \ldots + \alpha_{n-3} e_{n-1} + \gamma e_n$

(b) $e_1 \circ e_i = e_1 \cdot e_i + [e_1, e_i], e_2 \circ e_2 = e_2 \cdot e_2 + 2e_4, e_2 \circ e_i = [e_2, e_i], i \geq 3$, *where $x \cdot y$ is a CPA-structure as in case (a).*

The variables are arbitrary complex numbers.

Proof. One may check that the given maps are indeed CPA-structures on R_n. Now let $x \cdot y$ be any CPA-structure on R_n.
As R_n is not isomorphic to L_n, we have two cases: $x \cdot y$ being in Class A_1 or in Class A_2. In the first case, by Corollary 5.51, the product is associative, i.e. $R_n \cdot [R_n, R_n] = 0$. It remains to calculate the products $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2$.
If we expand the left-multiplication operators $L_i := L(e_i)$ in terms of the basis of $\operatorname{Der}(R_n)$ given in Lemma 5.73,

$$L_i = \sum_{k=1}^{n-1} \alpha_{i,k} \operatorname{ad} e_k + \sum_{k=2}^{n-2} \beta_{i,k} h_k + \gamma_i t,$$

we have $\gamma_1 = \gamma_2 = \alpha_{1,1} = \alpha_{2,1} = \beta_{1,i} = \beta_{2,i} = 0, 3 \leq i \leq n-2$ since $L_1(e_3) = L_2(e_3) = 0$ and $\beta_{1,2} = -\alpha_{1,2}, \beta_{2,2} = -\alpha_{2,2}$.
So the remaining products are given as

$$e_1 \cdot e_1 = L_1(e_1) = -\alpha_{1,2} e_3 - \alpha_{1,3} e_4 - \ldots - \alpha_{1,n-1} e_n,$$
$$e_1 \cdot e_2 = L_1(e_2) = -\alpha_{1,3} e_5 - \alpha_{1,4} e_6 - \ldots - \alpha_{1,n-2} e_n + \beta_{1,2} e_4 + \beta_{1,n-2} e_n,$$
$$e_2 \cdot e_1 = L_2(e_1) = -\alpha_{2,2} e_3 - \alpha_{2,3} e_4 - \ldots - \alpha_{2,n-1} e_n,$$
$$e_2 \cdot e_2 = L_2(e_2) = -\alpha_{2,3} e_5 - \alpha_{2,4} e_6 - \ldots - \alpha_{2,n-2} e_{n-2} + \beta_{2,2} e_n + \beta_{2,n-2} e_n.$$

Remember that $e_2 \cdot e_1 = e_1 \cdot e_2$. Now to obtain Case (a), we just set

$$\alpha_3 = -\alpha_{1,2} = \beta_{1,2} = -\alpha_{2,3}, \alpha_4 = -\alpha_{1,3} = -\alpha_{2,4}, \dots, \alpha_n = -\alpha_{1,n-1},$$
$$\beta = -\alpha_{1,n-2} + \beta_{1,n-2}, \gamma = \beta_{2,2} + \beta_{2,n-2}.$$

The second case (where the CPA-structure is in Class A_2) is more involved:
We again write

$$L_i = \sum_{k=1}^{n-1} \alpha_{i,k} \operatorname{ad} e_k + \sum_{k=2}^{n-2} \beta_{i,k} h_k$$

for the left-multiplication operator $L_i = L(e_i)$. By Proposition 5.52, we know that $L_3(e_3) = 0$ and $L_1(e_3) = L_3(e_1) \in e_4 + \mathrm{Z}(R_n)$. This makes the form of L_3 easier:

$$0 = L_3(e_3) = \alpha_{3,1}e_4 + \alpha_{3,2}e_5 + \beta_{3,2}e_5 + \beta_{3,3}e_6 + \ldots + \beta_{3,n-3}e_n,$$

which implies $\alpha_{3,1} = \beta_{3,3} = \ldots = \beta_{3,n-3} = 0$ and $\beta_{3,2} = -\alpha_{3,2}$.
So we can write $L_3 = \sum\limits_{k=2}^{n-1} \alpha_{3,k} \operatorname{ad}(e_k) - \alpha_{3,2}h_2 + \beta_{3,n-2}h_{n-2}$.
But since $L_3(e_1) \in e_4 + \mathrm{Z}(R_n)$, we can further conclude $\alpha_{3,2} = \alpha_{3,4} = \ldots = \alpha_{3,n-2} = 0$ and $\alpha_{3,3} = -1$ meaning $L_3 = -\operatorname{ad}(e_3) + \alpha_{3,n-1} \operatorname{ad}(e_{n-1}) + \beta_{3,n-2}h_{n-2}$.
Moreover, $L_1(e_3) \in e_4 + \mathrm{Z}(R_n)$ implies $\alpha_{1,1} = 1, \beta_{1,2} = -\alpha_{1,2}, \beta_{1,3} = \ldots = \beta_{1,n-4} = 0$ and allows us to write L_1 as

$$L_1 = \operatorname{ad}(e_1) + \sum_{k=2}^{n-1} \operatorname{ad}(e_k) - \alpha_{1,2}h_2 + \beta_{1,n-3}h_{n-3} + \beta_{1,n-2}h_{n-2}.$$

In particular, $L_3(e_2) = e_5 + \beta_{3,n-2}e_n$. This of course implies $L_2(e_3) = e_5 + \beta_{3,n-2}e_n$, so

$$e_5 + \beta_{3,n-2}e_n = L_2(e_3) = \alpha_{2,1}e_4 + (\alpha_{2,2} + \beta_{2,2})e_5 + \beta_{2,3}e_5 + \beta_{2,4}e_7 + \ldots + \beta_{2,n-4}e_{n-1} + \beta_{2,n-3}e_n$$

means $\alpha_{2,1} = \beta_{2,3} = \ldots = \beta_{2,n-4} = 0$ and $\beta_{2,2} = 1 - \alpha_{2,2}$. If we introduce the notation

$$\alpha_3 \coloneqq -\alpha_{1,2}, \alpha_4 \coloneqq -\alpha_{1,3}, \dots, \alpha_n \coloneqq -\alpha_{1,n-1},$$

we obtain

$$\begin{aligned}
&e_1 \cdot e_1 = a_3e_3 + a_4e_4 + \ldots + a_{n-1}e_{n-1} + a_ne_n,\\
&e_1 \cdot e_2 = e_3 + a_3e_4 + a_4e_5 + \ldots + a_{n-3}e_{n-2} + (a_{n-2} + \beta_{1,n-3})e_{n-1} + (a_{n-1} + \beta_{1,n-2})e_n,\\
&e_2 \cdot e_1 = -\alpha_{2,2}e_3 - \alpha_{2,3}e_4 - \alpha_{2,4}e_5 - \ldots - \alpha_{2,n-2}e_{n-1} - \alpha_{2,n-1}e_n,\\
&e_2 \cdot e_2 = (1 - \alpha_{2,2})e_4 - \alpha_{2,3}e_5 - \ldots - \alpha_{2,n-4}e_{n-2} + (-\alpha_{2,n-3} + \beta_{2,n-3})e_{n-1} + (-\alpha_{2,n-2} + \beta_{2,n-2})e_n.
\end{aligned}$$

Since $e_3 \cdot e_2 = e_5 + \beta_{3,n-2}e_n$ and $e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2) = e_5$, by axiom (CPA2),

$$e_5 + \beta_{3,n-2}e_n = e_3 \cdot e_2 = [e_1, e_2] \cdot e_2 = e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2) = e_5,$$

meaning $\beta_{3,n-2} = 0$. Thus $L_3(e_2) = L_2(e_3) = e_5$, so the coefficient $\beta_{2,n-3}$ is also zero. By another application of axiom (CPA2), we find that $e_3 \cdot e_1 = e_1 \cdot (e_2 \cdot e_1) - e_2 \cdot (e_1 \cdot e_1)$ implies $\alpha_{2,n-2} = a_{n-2}$. So the products $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_1, e_2 \cdot e_2$ read

$$
\begin{aligned}
&e_1 \cdot e_1 = \alpha_3 e_3 + \alpha_4 e_4 + \ldots + \alpha_{n-1} e_{n-1} + \alpha_n e_n,\\
&e_1 \cdot e_2 = e_3 + \alpha_3 e_4 + \alpha_4 e_5 + \ldots + \alpha_{n-3} e_{n-2} + (\alpha_{n-2} + \beta_{1,n-3}) e_{n-1} + (\alpha_{n-1} + \beta_{1,n-2}) e_n,\\
&e_2 \cdot e_1 = -\alpha_{2,2} e_3 - \alpha_{2,3} e_4 - \alpha_{2,4} e_5 - \ldots - \alpha_{2,n-3} e_{n-2} + \alpha_{n-2} e_{n-1} - \alpha_{2,n-1} e_n,\\
&e_2 \cdot e_2 = (1 - \alpha_{2,2}) e_4 - \alpha_{2,3} e_5 - \ldots - \alpha_{2,n-4} e_{n-2} - \alpha_{2,n-3} e_{n-1} + (-\alpha_{2,n-2} + \beta_{2,n-2}) e_n.
\end{aligned}
$$

Now we compare $e_1 \cdot e_2$ with $e_2 \cdot e_1$; we find $\alpha_{2,2} = -1, \beta_{1,n-3} = 0$ and $\alpha_{2,3} = -\alpha_3, \ldots,$ $\alpha_{2,n-3} = -\alpha_{n-3}$. Setting $\beta \coloneqq \alpha_{n-1} + \beta_{1,n-2}$ and $\gamma \coloneqq -\alpha_{n-2} + \beta_{2,n-2}$, we obtain

$$
\begin{aligned}
e_1 \cdot e_1 &= \alpha_3 e_3 + \alpha_4 e_4 + \ldots + \alpha_{n-1} e_{n-1} + \alpha_n e_n,\\
e_1 \cdot e_2 = e_2 \cdot e_1 &= e_3 + \alpha_3 e_4 + \alpha_4 e_5 + \ldots + \alpha_{n-3} e_{n-2} + \alpha_{n-2} e_{n-1} + \beta e_n,\\
e_2 \cdot e_2 &= 2 e_4 + \alpha_3 e_5 + \alpha_4 e_6 - \ldots + \alpha_{n-4} e_{n-2} + \alpha_{n-3} e_{n-1} + \gamma e_n,
\end{aligned}
$$

the desired form for the products $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2$. Inductively, using $e_1 \cdot e_{j+1} = [e_1 \cdot e_1, e_j] + [e_1, e_1 \cdot e_j]$ and $e_2 \cdot e_{j+1} = [e_1 \cdot e_2, e_j] + [e_2, e_1 \cdot e_j]$, one obtains that the other products are also given as claimed. □

Remark 5.75. On R_5 there are three types of CPA-structures:

(i) $e_1 \cdot e_1 = \alpha e_3 + \beta e_4 + \gamma e_5, e_1 \cdot e_2 = \alpha e_4 + \delta e_5, e_2 \cdot e_2 = \epsilon e_5$

(ii) $e_1 \cdot e_1 = \alpha e_3 + \beta e_4 + \gamma e_5, e_1 \cdot e_2 = e_3 + \alpha e_4 + \delta e_5, e_1 \cdot e_3 = e_4, e_1 \cdot e_4 = e_5, e_2 \cdot e_2 = 2e_4 + \epsilon e_5, e_2 \cdot e_3 = e_5$

(iii) $e_1 \cdot e_1 = -1/2 e_2 + \alpha e_3 + \beta e_4 + \gamma e_5, e_1 \cdot e_2 = 1/2 e_3 + \delta e_4 + \epsilon e_5, e_1 \cdot e_3 = 1/2 e_4 + (\delta - \alpha) e_5, e_2 \cdot e_2 = 1/2 e_4 + (\delta - \alpha) e_5$

with all variables in $\mathbb{C}$, extended bilinearly and commutatively.
In particular, Theorem 5.44 does not hold in dimension 5 – type (iii) would be a contradiction.

5.5.4 The Witt algebra W_n

Definition 5.76. The n-dimensional Witt algebra W_n has basis vectors $e_1, \ldots, e_n$ and relations

$$[e_i, e_j] = (j - i) e_{i+j} \text{ for } i + j \leq n.$$

Remark 5.77. In [58], the authors give two different definitions of W_n, in Chapter 2 and Chapter 4. However, both appear to be incorrect (i.e., they do not satisfy the Jacobi identity).

Remark 5.78. The basis of W_n given above is not adapted; we could define an adapted basis $\{f_1, \ldots, f_n\}$ of W_n by setting $a_1 = 1, a_2 = 6, a_i = (i-2)a_{i-1}, i \geq 3, f_i := a_i e_i$. This basis is adapted, since

$$[f_1, f_i] = [e_1, (i-2)a_{i-1}e_i] = (i-2)a_{i-1}[e_1, e_i] = (i-1)(i-2)a_{i-1}e_{i+1} = (i-1)a_i e_{i+1} = f_{i+1}$$

and $[f_i, f_j] \in \operatorname{span}(e_{i+j}, \ldots, e_n)$ still holds.

Lemma 5.79. *Let $n \geq 7$. Then the endomorphisms $\operatorname{ad} e_1, \ldots, \operatorname{ad} e_{n-1}, t, g_1, g_2, g_3$ form a basis of $\operatorname{Der}(W_n)$, where t, g_1, g_2, g_3 are given as*

$$\begin{aligned}
&t(e_i) = ie_i, 1 \leq i \leq n,\\
&g_1(e_2) = e_{n-2}, g_1(e_3) = (n-3)e_{n-1}, g_1(e_4) = \binom{n-2}{2} e_n, g_1(e_i) = 0 \textit{ if } i \neq 2, 3, 4,\\
&g_2(e_2) = e_{n-1}, g_2(e_3) = (n-2)e_n, g_2(e_i) = 0 \textit{ if } i \neq 2, 3,\\
&g_3(e_2) = e_n, g_3(e_i) = 0 \textit{ if } i \neq 2.
\end{aligned}$$

In particular, $\dim(\operatorname{Der}(W_n)) = n + 3$.

As we were unable to find this statement (or a proof) in the literature, we give a proof here:

Proof. One may check that these endomorphisms are linearly independent and derivations of W_n. Now let D be any derivation of W_n; let us write $D = (a_{ij})_{1 \leq i,j \leq n}$ as a matrix with respect to the basis given above. As each $\delta_j = \operatorname{span}\{e_j, \ldots, e_n\}$ is a characteristic ideal, we find $a_{ij} = 0$ for $i < j, i = 3, \ldots, n$. Comparing the coefficient of e_4 in $D[e_2, e_3] = [De_2, e_3] + [e_2, De_3]$, we find $0 = 2a_{1,2}$, so $a_{1,2} = 0$ and D is lower triangular. For $i = 2, \ldots, n-1$, comparing the e_{i+1}-coefficient in $D[e_1, e_i] = [De_1, e_i] + [e_1, De_i]$ reads

$$\begin{aligned}
a_{3,3} &= a_{1,1} + a_{2,2},\\
a_{4,4} &= a_{1,1} + a_{3,3} = 2a_{1,1} + a_{2,2},\\
&\vdots\\
a_{n,n} &= a_{1,1} + a_{n-1,n-1} = (n-2)a_{1,1} + a_{2,2}
\end{aligned}$$

and by $[De_2, e_3] + [e_2, De_3] - D[e_2, e_3] = 0$,

$$0 \in a_{2,2}e_5 + \delta_6 + a_{1,1}e_5 + a_{2,2}e_5 + \delta_6 - a_{5,5}e_5 + \delta_6,$$

so $a_{2,2} = 2a_{1,1}$.
Thus only the derivation t contributes to the diagonal of D; we may thus assume that its contribution is 0 and consider $D - \alpha t$ such that this matrix's diagonal vanishes.
Now we investigate the subdiagonal: Again, by $[De_1, e_i] + [e_1, De_i] - D[e_1, e_i] = 0$, we have $(i = 2, \ldots, n-2)$

$$0 \in (i-2)a_{2,1}e_{i+2} + \delta_{i+3} + ia_{i+1,i}e_{i+2} + \delta_{i+3} - (i-1)a_{i+2,i+1}e_{i+2} + \delta_{i+3},$$

i.e.

$$\begin{aligned} a_{4,3} &= 2a_{3,2}, \\ 2a_{5,4} &= a_{2,1} + 3a_{4,3}, \\ 3a_{6,5} &= 2a_{2,1} + 4a_{5,4}, \\ &\vdots \\ (n-3)a_{n,n-1} &= (n-4)a_{2,1} + (n-2)a_{n-1,n-2} \end{aligned}$$

and $[De_2, e_3] + [e_2, De_3] - D[e_2, e_3] = 0$ gives

$$0 \in \delta_7 + 2a_{4,3}e_6 + \delta_7 - a_{6,5}e_6 + \delta_7.$$

These relations imply that the subdiagonal is a multiple of $\mathrm{ad}(e_1)$. We can again assume that this multiple is 0.

Inductively, if everything above the $-i$-th diagonal is zero, analogously to the previous step, one uses the relations $D[e_1, e_i] = [De_1, e_i] + [e_1, De_i]$ and $D[e_2, e_3] = [De_2, e_3] + [e_2, De_3]$ and finds that the $-i$-th diagonal is a multiple of $\mathrm{ad}(e_i)$. This step involves comparing the coefficient of e_{i+5}; thus the very same procedure works until the $-(n-5)$-th diagonal.
So the i-th step looks like this:
Everything above the $-i$-th diagonal is 0. So for every j with $2 \leq j, j+i+1 \leq n$, we obtain

$$\begin{aligned} 0 &= [D(e_1), e_j] + [e_1, D(e_j)] - D([e_1, e_j]) \\ &\in [a_{i+1,1}e_{i+1} + \delta_{i+2}, e_j] + [e_1, a_{i+j,j}e_{i+j} + \delta_{i+j+1}] - (j-1)a_{j+1+i,j+1}e_{j+i+1} + \delta_{j+i+2} \\ &\subseteq (j-(i+1))a_{i+1,1}e_{i+j+1} + \delta_{i+j+2} + (j+i-1)a_{i+j,j}e_{i+j+1} + \delta_{j+i+2} \\ &- (j-1)a_{j+1+i,j+1}e_{j+i+1} + \delta_{j+i+2}. \end{aligned}$$

In particular, for $j = 2, 3, 4$, we obtain

$$\begin{aligned} a_{3+i,3} &= (1-i)a_{i+1,1} + (1+i)a_{i+2,2}, \\ 2a_{4+i,4} &= (2-i)a_{i+1,1} + (2+i)a_{i+3,3}, \\ 3a_{5+i,5} &= (3-i)a_{i+1,1} + (3+i)a_{i+4,4}. \end{aligned}$$

To these three linear equations we add another one. Namely, from $[D(e_2), e_3] + [e_2, D(e_3)] - D(e_5) = 0$ we get

$$\begin{aligned} 0 &\in [a_{2+i,2}e_{2+i} + \delta_{3+i}, e_3] + [e_2, a_{3+i,3}e_{3+i} + \delta_{4+i}] - a_{5+i,5}e_{6+i} + \delta_{7+i} \\ &\subseteq (1-i)a_{2+i,2}e_{5+i} + (1+i)a_{3+i,3}e_{5+i} + \delta_{6+i} - a_{5+i,5}e_{6+i} + \delta_{7+i}. \end{aligned}$$

We write $a_{i+1,1} = (1-i) \cdot k$ for a $k \in \mathbb{C}$. Now, we have a system of four linearly independent linear equations with four variables – the solution is given as

$$a_{2+i,2} = (2-i) \cdot k, a_{3+i,3} = (3-i) \cdot k, a_{4+i,4} = (4-i)k, a_{5+i,5} = (5-i)k.$$

And for general j, we thus find recursively that $a_{j+i,j} = (j-i)\cdot k$ as long as $j+i+1 \le n$. But this just means that the $-i$-th diagonal is $k \cdot \operatorname{ad}(e_i)$.

For the $-(n-4)$-th diagonal, we get the relations

$$\begin{aligned} 0 \in D[e_1, e_2] &= [De_1, e_2] + [e_1, De_2] - a_{n-1,3}e_{n-1} + \delta_n \\ &\subseteq -(n-5)a_{n-3,1}e_{n-1} + \delta_n + (n-3)a_{n-2,2}e_{n-1} + \delta_n - a_{n-1,3}e_{n-1} + \delta_n \end{aligned}$$

and

$$2a_{n,1}e_n = D[e_1, e_3] = [De_1, e_3] + [e_1, De_3] = -(n-6)a_{n-3,1}e_n + (n-2)a_{n-1,3}e_n.$$

So

$$\begin{aligned} a_{n-1,3} &= -(n-5)a_{n-3,1} + (n-3)a_{n-2,2}, \\ a_{n,1} &= -\frac{n-6}{2}a_{n-3,1} + \frac{n-2}{2}a_{n-1,3}. \end{aligned}$$

We set $\alpha \coloneqq a_{n-3,1}, \beta \coloneqq a_{n-2,2} - \frac{n-6}{n-5}\alpha$. Then

$$\begin{aligned} a_{n-1,3} &= -(n-5)a_{n-3,1} + (n-3)a_{n-2,2} \\ &= -(n-5)\alpha + (n-3)\beta + \alpha\frac{(n-6)(n-3)}{n-5} \\ &= (n-3)\beta + \frac{n-7}{n-5}\alpha \end{aligned}$$

and

$$\begin{aligned} a_{n,1} &= -\frac{n-6}{2}a_{n-3,1} + \frac{n-2}{2}a_{n-1,3} \\ &= -\frac{n-6}{2}\alpha + \frac{(n-2)(n-3)}{2}\beta + \frac{(n-7)(n-2)}{2(n-5)}\alpha \\ &= \binom{n-2}{2}\beta + \frac{n-8}{n-5}\alpha. \end{aligned}$$

But this exactly means that the $-(n-4)$-th diagonal can be expressed as $\beta g_1 + \alpha(n-5)\operatorname{ad}(n-4)$.We again can assume that $\alpha = \beta = 0$.
Now we investigate the $-(n-3)$-th diagonal and have only one relation left:

$$\begin{aligned} a_{n,3}e_n = D[e_1, e_2] &= [De_1, e_2] + [e_1, De_2] \\ &= -(n-4)a_{n-2,1}e_n + (n-2)a_{n-1,2}e_n. \end{aligned}$$

So writing $\alpha \coloneqq a_{n-2,1}, \beta \coloneqq a_{n-1,2} - \frac{n-5}{n-4}\alpha$, we get

$$\begin{aligned} a_{n,3} &= -(n-4)\alpha + (n-2)\beta + (n-2)\frac{n-5}{n-4}\alpha \\ &= (n-2)\beta + \frac{n-6}{n-4}\alpha, \end{aligned}$$

which means that the $-(n-3)$-th diagonal is $\beta g_2 + \alpha(n-4)\operatorname{ad}(n-3)$.
So we set the $-(n-3)$-th diagonal to be 0.
Now we are left with a matrix having only three non-zero entries; namely $a_{n-1,1}, a_{n,1}, a_{n,2}$ which can be chosen arbitrarily. So the remaining derivation is a linear combination of $g_3, \operatorname{ad}(n-2)$ and $\operatorname{ad}(n-1)$.
In total, we have proven that

$$D = \sum_{k=1}^{n-1} \alpha_k \operatorname{ad}(e_k) + \beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 + \gamma t,$$

with $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \beta_2, \beta_3, \gamma \in \mathbb{C}$. $\square$

Remark 5.80. For $n = 3,4,5,6$, the space $\operatorname{Der}(W_n)$ is different; however, W_3, W_4, W_5, W_6 are isomorphic to L_3, L_4, R_5, R_6, respectively.

Proposition 5.81. *Let $n \geq 7$. Then the CPA-structures on W_n are given (with respect to the basis given above) by*

$$\begin{aligned} e_1 \cdot e_1 &= \left(\frac{n-2}{n-4}\right) \alpha_1 e_{n-2} + \alpha_2 e_{n-1} + \alpha_3 e_n, \\ e_1 \cdot e_2 &= \alpha_1 e_{n-1} + \alpha_4 e_n, \\ e_2 \cdot e_2 &= \alpha_5 e_n, \end{aligned}$$

$\alpha_1, \ldots, \alpha_5 \in \mathbb{C}$ arbitrary.

Proof. The given structures are evidently CPA-structures; we show that every CPA-structure on W_n is of this form. Since $W_n, n \geq 7$ is non-metabelian, $W_n \cdot [W_n, W_n] = 0$ by Theorem 5.44 – we only have to deduce the precise form of the products $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2$. Let us expand the left-multiplication operators in terms of the basis given in Lemma 5.79 as $(i = 1,2)$

$$L_i := L(e_i) = \sum_{k=1}^{n-1} \alpha_{i,k} \operatorname{ad}(e_k) + \sum_{k=1}^{3} \beta_{i,k} g_k$$

(note that by the nilpotency of the CPA-structure, there is no contribution of the endomorphism t in L_i).

The only open thing is the precise form of the products $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2$.
By using $e_1 \cdot e_3 = 0 = e_2 \cdot e_3$, we find the e_n-coordinate of $L_i(e_3), i = 1,2$ to be

$$\begin{aligned} -(n-6)\alpha_{1,n-3} e_n + (n-2)\beta_{1,2} e_n = 0, \\ -(n-6)\alpha_{2,n-3} e_n + (n-2)\beta_{2,2} e_n = 0, \end{aligned}$$

thus $\beta_{1,2} = \left(\frac{n-6}{n-2}\right) \alpha_{1,n-3}, \beta_{2,2} = \left(\frac{n-6}{n-2}\right) \alpha_{2,n-3}$. A lot of coefficients of L_1, L_2 do vanish since $L_i([W_n, W_n]) = 0$; what remains is (we use the coefficients $a_1, \ldots, a_4, b_1, \ldots, b_4$ here instead of the α_i and $\beta_{i,k}$)

$$L_1 = a_1 \operatorname{ad} e_{n-3} + a_2 \operatorname{ad} e_{n-2} + a_3 \operatorname{ad} e_{n-1} + \left(\frac{n-6}{n-2}\right) a_1 g_2 + a_4 g_3,$$
$$L_2 = b_1 \operatorname{ad} e_{n-3} + b_2 \operatorname{ad} e_{n-2} + b_3 \operatorname{ad} e_{n-1} + \left(\frac{n-6}{n-2}\right) b_1 g_2 + b_4 g_3$$

This means that

$$\begin{aligned}
e_1 \cdot e_1 &= -(n-4)a_1 e_{n-2} - (n-3)a_2 e_{n-1} - (n-2)a_3 e_n, \\
e_1 \cdot e_2 &= -(n-5)a_1 e_{n-1} + \left(\frac{n-6}{n-2}\right) a_1 e_{n-1} - (n-4)a_2 e_n + a_4 e_n, \\
e_2 \cdot e_1 &= -(n-4)b_1 e_{n-2} - (n-3)b_2 e_{n-1} - (n-2)b_3 e_n, \\
e_2 \cdot e_2 &= -(n-5)b_1 e_{n-1} + \left(\frac{n-6}{n-2}\right) b_1 e_{n-1} - (n-4)b_2 e_n + b_4 e_n.
\end{aligned}$$

Now using $e_1 \cdot e_2 = e_2 \cdot e_1$ gives the desired result: The coefficient a_1 can be derived from b_2 by calculating

$$a_1 = -(n-3)b_2 \Big/ \left(-(n-5) + \frac{n-6}{n-2}\right) = \frac{(n-3)(n-2)b_2}{(n-4)^2},$$

so the ratio of the e_{n-2}-coefficient in $e_1 \cdot e_1$ and the e_{n-1}-coefficient in $e_2 \cdot e_1$ is

$$\frac{-(n-4)a_1}{-(n-3)b_2} = \frac{n-2}{n-4}.$$

Note that by comparing $e_1 \cdot e_2$ and $e_2 \cdot e_1$, the coefficient b_1 is zero; to get the form in the statement, we now simply set

$$\begin{aligned}
\alpha_1 \coloneqq -(n-3)b_2, \alpha_2 \coloneqq -(n-3)a_2, \alpha_3 \coloneqq -(n-2)a_3, \\
\alpha_4 \coloneqq a_4 - (n-4)a_2, \alpha_5 \coloneqq b_4 - (n-4)b_2.
\end{aligned}$$

□

Remark 5.82. With respect to the adapted basis $\{f_1, \ldots, f_n\}$ of $W_n, n \geq 7$ given in Remark 5.78, all CPA-structures on W_n are given by

$$\begin{aligned}
f_1 \cdot f_1 &= \beta_1 + \beta_2 f_{n-1} + \beta_3 e_n, \\
f_2 \cdot f_1 = f_1 \cdot f_2 &= \frac{6(n-4)}{(n-2)(n-3)} \beta_1 f_{n-1} + \beta_4 f_n, \\
f_2 \cdot f_2 &= \beta_5 f_n
\end{aligned}$$

(extended bilinearly) for arbitrary constants $\beta_1, \ldots, \beta_5 \in \mathbb{C}$.

5.5.5 A family of strongly-nilpotent filiform Lie algebras

Lemma 5.83. *Let $\mathfrak{g}_n, n \geq 8$ be the n-dimensional filiform Lie algebra from [19] with basis $\{e_1, \ldots, e_n\}$ and Lie brackets $[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, [e_2, e_3] = e_{n-1}, [e_2, e_4] = e_n, [e_2, e_5] = -e_n, [e_3, e_4] = e_n$. Then the endomorphisms $\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_{n-1})$, $h_3, \ldots, h_{n-2}, t$ form a basis of $\mathrm{Der}(\mathfrak{g}_n)$, where the h_i and t are defined as*

$$h_i(e_j) = e_{i+j} \text{ if } j \neq 1 \text{ and } i+j \leq n, h_i(e_j) = 0 \text{ otherwise,}$$
$$t(e_1) = e_2, t(e_4) = e_{n-1}, t(e_5) = 2e_n, t(e_6) = -e_n, t(e_j) = 0 \text{ otherwise.}$$

(In particular, $\dim(\mathrm{Der}(\mathfrak{g}_n)) = 2n-4$.)

Remark 5.84. This is a *strongly-nilpotent* Lie algebra, i.e. it has only nilpotent prederivations. (A *prederivation* of a Lie algebra $\mathfrak{g}$ is a linear map $P\colon \mathfrak{g} \to \mathfrak{g}$ satisfying $P([x,[y,z]]) = [P(x),[y,z]] + [x,[P(y),z]] + [x,[y,P(z)]]$ for all $x,y,z \in \mathfrak{g}$ (cf. [19]) – clearly, every derivation is a prederivation.) In particular, it is a *characteristically nilpotent* Lie algebra, meaning that all derivations are nilpotent.

Remark 5.85. One can also define the Lie algebra $\mathfrak{g}_7$ in the same way; however, then one has to make changes in the given basis for $\mathrm{Der}(\mathfrak{g}_7)$: The endomorphism t is not a derivation and has to be replaced by t', where $t'(e_1) = 2e_2, t'(e_2) = e_4, t'(e_3) = e_5, t'(e_4) = 3e_6, t'(e_5) = 5e_7, t'(e_6) = -2e_7, t'(e_7) = 0$.
The analogous defined Lie algebra $\mathfrak{g}_6$ has a non-nilpotent derivation: $e_i \mapsto ie_i, i \leq 5, e_6 \mapsto 7e_6$.

Proposition 5.86. *Let $\mathfrak{g}_n$ be the n-dimensional filiform strongly-nilpotent Lie algebra defined above and $n \geq 8$. Then all CPA-structures on $\mathfrak{g}_n$ are given (with respect to the basis above) by*

$$e_1 \cdot e_1 = \alpha e_5 + \alpha_6 e_6 + \ldots + \alpha_n e_n,$$
$$e_1 \cdot e_2 = e_2 \cdot e_1 = -\alpha e_{n-1} + \beta e_n,$$
$$e_2 \cdot e_2 = \gamma e_n,$$

where the variables are arbitrary complex numbers and all other products between basis vectors are zero.

Proof. As $\mathfrak{g}_n$ is non-metabelian, by Theorem 5.44, we have $\mathfrak{g}_n \cdot [\mathfrak{g}_n, \mathfrak{g}_n] = 0$. Let $L_i = L(e_i)$ be the left-multiplication operator with respect to e_i; as always, we write

$$L_i = \alpha_i t + \sum_{j=3}^{n-2} \beta_{i,j} h_j + \sum_{j=1}^{n-1} \gamma_{i,j}\, \mathrm{ad}(e_j)$$

for some $\alpha_i, \beta_{i,j}, \gamma_{i,j} \in \mathbb{C}$. As $\mathfrak{g}_n \cdot [\mathfrak{g}_n, \mathfrak{g}_n] = 0$, we are interested in $e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2$. These products are so far given by

$$L_1(e_1) = -\gamma_{1,4} e_5 - \gamma_{1,5} e_6 - \ldots - \gamma_{1,n-1} e_n,$$
$$L_1(e_2) = -\gamma_{1,4} e_{n-1} + (\beta_{1,n-2} - \gamma_{1,4} + \gamma_{1,5}) e_n,$$
$$L_2(e_1) = -\gamma_{2,4} e_5 - \gamma_{2,5} e_6 - \ldots - \gamma_{2,n-1} e_n,$$
$$L_2(e_2) = -\gamma_{2,4} e_{n-1} + (\beta_{2,n-2} - \gamma_{2,4} + \gamma_{2,5}) e_n.$$

and one obtains the proposition by noting $L_1(e_2) = L_2(e_1)$ and setting $\alpha := -\gamma_{1,4}, \alpha_i := -\gamma_{1,i-1}, i = 6, \ldots, n, \beta := \beta_{1,n-2} - \gamma_{1,4} + \gamma_{1,5} = -\gamma_{2,n-1}$ and $\gamma := \beta_{2,n-2} - \gamma_{2,4} + \gamma_{2,5}$. $\square$

Remark 5.87. Proposition 5.86 also holds for $n = 7$ (that is, the CPA-structures on $\mathfrak{g}_7$ have the same form as the ones of higher dimension). In those cases, as the basis of $\mathrm{Der}(\mathfrak{g}_7)$ is different, one would have to adapt the proof slightly (or determine the CPA-structures directly).

5.6 Post-Lie algebra structures and algebras

In this section, we want to present connections between CPA-structures/LR-structures and other algebras.

By an *algebra* $(A, \cdot)$, we mean a vector space (over some field) with a bilinear map $\cdot\colon A \times A \to A$.

Definition 5.88.

(i) An *associative algebra* $(A, \cdot)$ is an algebra satisfying

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in A$.

(ii) An algebra $(A, \cdot)$ is called a *Jordan algebra*, if the two identities

$$x \cdot y = y \cdot x,$$
$$(x \cdot y) \cdot (x \cdot x) = x \cdot (y \cdot (x \cdot x))$$

hold for all $x, y \in A$.

(iii) A *Poisson algebra* is a triple $(\mathfrak{g}, [,], \circ)$, where $\mathfrak{g}$ is a Lie algebra and $x \circ y$ commutative, associative and for all $x, y, z \in \mathfrak{g}$ the identity

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \tag{15}$$

holds.

(iv) (See [11, 59].) An algebra $(A, \cdot)$ is called *Poisson admissible*, if A endowed with the Lie bracket $[,]$ and the bilinear map $x \circ y$, defined by

$$[x, y] := x \cdot y - y \cdot x, \qquad x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$$

is a Poisson algebra.

Remark 5.89. Every commutative associative algebra $(A, \cdot)$ is a Jordan algebra.

Proposition 5.90. *Let $(\mathfrak{g}, [,], \cdot)$ be a CPA-structure. The following are equivalent:*

(i) $(\mathfrak{g}, \cdot)$ *is an associative algebra.*

(ii) $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

(iii) $(\mathfrak{g}, \cdot)$ *is Poisson admissible. (In this case, the associated Poisson algebra has trivial Lie bracket.)*

Remark 5.91. This proposition explains why we called CPA-structures with $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ associative in Section 5.4.

Proof.

(i)⇔(ii) Using the commutativity and (CPA2), we get

$$\begin{aligned}(x \cdot y) \cdot z - x \cdot (y \cdot z) &= z \cdot (x \cdot y) - x \cdot (y \cdot z) \\ &= x \cdot (z \cdot y) + [z, x] \cdot y - x \cdot (y \cdot z) \\ &= x \cdot (y \cdot z) - x \cdot (y \cdot z) + [z, x] \cdot y = y \cdot [z, x],\end{aligned}$$

so associativity is equivalent to $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.

(i)⇔(iii) Since $x \cdot y$ is commutative, for the Lie bracket $\{,\}$ associated to the Poisson algebra, we obtain $\{x, y\} \coloneqq x \cdot y - y \cdot x = 0$. So (15) is trivially satisfied and $(A, \{,\}, \circ)$ being a Poisson algebra (where $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x) = x \cdot y$) is equivalent to $x \cdot y$ being associative.

□

Remark 5.92. Similarly, a CPA-structure on $\mathfrak{g}$ is a Jordan algebra if and only if $[x, y] \cdot (x \cdot x) = [x \cdot x, y] \cdot x$ for all $x, y \in \mathfrak{g}$.

There are several classes of Lie algebra where we proved that every CPA-structure satisfies $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$:

Corollary 5.93. *Let* $(\mathfrak{g}, [,], \cdot)$ *be a CPA-structure on (i) a Heisenberg algebra, (ii) a non-metabelian filiform Lie algebra, (iii) a complete, non-metabelian Lie algebra* $\mathfrak{g}$ *with* $\operatorname{nil}(\mathfrak{g}) = [\mathfrak{g}, \operatorname{nil}(\mathfrak{g})]$*, (iv) the Lie algebra of strictly upper triangular matrices or (v) the Lie algebra of upper triangular matrices (of size* $n \geq 5$*). Then* $(\mathfrak{g}, \cdot)$ *is an associative algebra. On the Lie algebra of upper triangular matrices of size* 4*, all CPA-structures are Jordan algebras, but not necessarily associative algebras.*

Proof. The condition $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ holds in the following cases: (i) a Heisenberg algebra of dimension ≥ 5 (by Proposition 5.8), (ii) a non-metabelian filiform Lie algebra (by Theorem 5.44), (iii) a complete, non-metabelian Lie algebra $\mathfrak{g}$ with $\operatorname{nil}(\mathfrak{g}) = [\mathfrak{g}, \operatorname{nil}(\mathfrak{g})]$ (note that $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ follows from Proposition 2.52) (iv) a Lie algebra of strictly upper triangular matrices of size $n \geq 5$ (by Proposition 5.27), (v) a Lie algebra of strictly upper triangular matrices of size $n \geq 3$ (by Proposition 5.34).
For Heisenberg algebras of dimension 3, one can prove the associativity of all CPA-structures (which are classified in Proposition B.10); for the Lie algebra of upper triangular matrices of size 4, one proves that $(x \cdot y) \cdot (x \cdot x) = x \cdot (y \cdot (x \cdot x))$ holds by checking this

equation for all CPA-structures (they are classified in Example 5.29); CPA-structures of types (ii) and (iii) in Example 5.29 are not associative. □

For the Lie algebra of upper triangular matrices of size 3×3 (which is isomorphic to $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$), Corollary 5.93 does not hold:

Example 5.94. *With respect to the basis $\{e_1, e_2, e_3\}$ and relations $[e_1, e_2] = e_1$, the CPA-structure on $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ (cf. Proposition B.11, where all CPA-structures on $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ are classified) given by*

$$L(e_1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not a Jordan algebra (and thus also not associative).

Not even all CPA-structures on a *nilpotent* Lie algebra are associative algebras:

Example 5.95. *On the standard graded filiform Lie algebra $L_n, n \geq 5$, the CPA-structure given by $e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = -e_{n-1}, e_1 \cdot e_3 = e_3 \cdot e_1 = -e_n, e_2 \cdot e_2 = e_n$ (cf. Proposition 5.66) is not a Jordan algebra (and thus also not an associative algebra).*

Definition 5.96 ([34])**.** A CPA-structure $(\mathfrak{g}, [,], \cdot)$ is called *central*, if $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{Z}(\mathfrak{g})$.

Remark 5.97. If $x \cdot y$ is a central CPA-structure, then the quotient CPA-structure on $\overline{\mathfrak{g}} \coloneqq \mathfrak{g}/\mathrm{Z}(\mathfrak{g})$ satisfies $\overline{\mathfrak{g}} \cdot \overline{\mathfrak{g}} = 0$.

Being central implies the condition $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ we had before:

Lemma 5.98 ([34, Lemma 3.10])**.** *Let $(\mathfrak{g}, \cdot)$ be a central CPA-structure. Then $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$; if $\mathfrak{g}$ is stem, then also $\mathfrak{g} \cdot \mathrm{Z}(\mathfrak{g}) = 0$.*

Proof. We have

$$\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g} \cdot \mathfrak{g}, \mathfrak{g}] + [\mathfrak{g}, \mathfrak{g} \cdot \mathfrak{g}] \subseteq [\mathrm{Z}(\mathfrak{g}), \mathfrak{g}] + [\mathfrak{g}, \mathrm{Z}(\mathfrak{g})] = 0$$

by axiom (CPA3). If $\mathrm{Z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$, we also have $\mathfrak{g} \cdot \mathrm{Z}(\mathfrak{g}) = 0$. □

Remark 5.99. Among other things, [34] studies CPA-structures on free-nilpotent Lie algebras. There, it is conjectured (and there are several affirmative results in this direction, see [34, Conjecture 4.9]) that all CPA-structures on $\mathfrak{g} = F_{g,c}, g \geq 2, c \geq 3$ (the free-nilpotent Lie algebra with g generators and nilpotency class c) are central.

Lemma 5.100. *Let $(\mathfrak{g}, [,], \cdot)$ be a CPA-structure. Then $(\mathfrak{g}, [,], \cdot)$ is a Poisson algebra if and only if $x \cdot y$ is central.*

Proof. If $x \cdot y$ is central, then,

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = x \cdot (z \cdot y) - z \cdot (x \cdot y) = [x, z] \cdot y = 0$$

by Lemma 5.98. This proves associativity; $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ and $[\mathfrak{g}, \mathfrak{g} \cdot \mathfrak{g}] \subseteq [\mathfrak{g}, \mathrm{Z}(\mathfrak{g})] = 0$ imply (15).
On the other hand, if $x \cdot y$ is a CPA-structure and a Poisson algebra, then, by Proposition 5.90, $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$. Thus, by (15), we obtain $[x, y \cdot z] = 0$ for all $x, y, z \in \mathfrak{g}$ meaning $\mathfrak{g} \cdot \mathfrak{g} \subseteq \mathrm{Z}(\mathfrak{g})$. □

Let us now consider LR-structures (viewed as post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ abelian) and their connection to Poisson admissible algebras:

Proposition 5.101. *Let $(\mathfrak{n}, \{,\}, \cdot)$ be an LR-structure. The following are equivalent:*

(i) $(\mathfrak{n}, \cdot)$ *is Poisson-admissible.*

(ii) $(\mathfrak{n}, \cdot)$ *is associative.*

(iii) $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{Z}(\mathfrak{n})$.

In this case, the associated Poisson algebra $(\mathfrak{n}, [,], \circ)$ satisfies $x \circ [y, z] = 0$ for all $x, y, z \in \mathfrak{n}$.

Proof.

(i)⇔(ii) This is part of [11, Proposition 3.2].

(ii)⇔(iii) The associator is given by

$$\begin{aligned} x \cdot (y \cdot z) - (x \cdot y) \cdot z &= x \cdot (y \cdot z) - z \cdot (x \cdot y) + \{x \cdot y, z\} \\ &= x \cdot (y \cdot z) - x \cdot (z \cdot y) + \{x \cdot y, z\} \\ &= x \cdot (y \cdot z) - x \cdot (y \cdot z) + x \cdot \{z, y\} + \{x \cdot y, z\} \\ &= \{x \cdot z, y\} + \{z, x \cdot y\} + \{x \cdot y, z\} = \{x \cdot z, y\} \end{aligned}$$

which is zero precisely if $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{Z}(\mathfrak{n})$.

□

Corollary 5.102. *Let $(\mathfrak{n}, \{,\}, \cdot)$ be an LR-structure. If $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{Z}(\mathfrak{n})$ (or equivalently, $(\mathfrak{n}, \cdot)$ is associative), then $\mathfrak{n}$ is two-step nilpotent.*

Proof. By [11, Corollary 3.1], an associative algebra is Poisson admissible if and only if the associated Lie algebra is 2-step nilpotent. Thus, the claim follows by Proposition 5.101. □

Corollary 5.103. *Let $x \cdot y$ be an associative LR-structure on the stem Lie algebra $(\mathfrak{n}, \{,\})$. Then $(\mathfrak{n} \cdot \mathfrak{n}) \cdot \mathfrak{n} = \mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) = 0$ (in particular, the LR-structure is complete).*

Proof. As $x \cdot y$ is associative, $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{Z}(\mathfrak{n})$ by Proposition 5.101. Now since $\mathfrak{n}$ is stem, note that as in Lemma 5.98, $\mathfrak{n} \cdot \mathfrak{n} \subseteq \mathrm{Z}(\mathfrak{n})$ implies $\mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) = 0$:

$$\mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) \subseteq \mathfrak{n} \cdot \mathrm{Z}(\mathfrak{n}) \subseteq \mathfrak{n} \cdot \{\mathfrak{n}, \mathfrak{n}\} \subseteq \{\mathfrak{n} \cdot \mathfrak{n}, \mathfrak{n}\} + \{\mathfrak{n}, \mathfrak{n} \cdot \mathfrak{n}\} \subseteq \{\mathrm{Z}(\mathfrak{n}), \mathfrak{n}\} = 0$$

and also

$$(\mathfrak{n} \cdot \mathfrak{n}) \cdot \mathfrak{n} \subseteq \mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) + \{\mathfrak{n}, \mathfrak{n} \cdot \mathfrak{n}\} \subseteq \mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) + \{\mathfrak{n}, \mathrm{Z}(\mathfrak{n})\} = 0.$$

□

Remark 5.104. The following examples from [26] are defined over $\mathbb{R}$, however, nothing changes if we replace the field $\mathbb{R}$ by $\mathbb{C}$:

(i) The assumption "stem" in Corollary 5.103 is necessary; take for example the LR-structure $A_4(\alpha)$ from [26, Proposition 3.3] on the Lie algebra $\mathfrak{n} = \mathfrak{n}_3(\mathbb{R}) \oplus \mathbb{R}$ (which has a basis $\{e_1, \ldots, e_4\}$ and non-zero Lie bracket $[e_1, e_2] = e_3$ and is therefore not stem). The LR-structure $A_4(\alpha), \alpha \in \{0, 1\}$, given by

$$e_1 \cdot e_1 = e_4, e_1 \cdot e_4 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = \alpha e_3, e_4 \cdot e_1 = e_3$$

is associative, but $e_1 \cdot (e_1 \cdot e_1) = e_3 \neq 0$.

(ii) The assumption "associative" in Corollary 5.103 is necessary; take for example the LR-structure A_4 from [26, Proposition 3.1] on the Heisenberg Lie algebra $\mathfrak{n} = \mathfrak{n}_3(\mathbb{R})$ (which has a basis $\{e_1, e_2, e_3\}$ and non-zero Lie bracket $[e_1, e_2] = e_3$). The LR-structure A_4 is given by

$$e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = e_2, e_2 \cdot e_3 = e_3, e_3 \cdot e_2 = e_3$$

and is not associative ($e_2 \cdot (e_1 \cdot e_2) - (e_2 \cdot e_1) \cdot e_2 = e_3$); the LR-structure does not satisfy $\mathfrak{n} \cdot (\mathfrak{n} \cdot \mathfrak{n}) = 0$ and is not even complete (as $e_2 \cdot e_2 = e_2$; cf. Remark 5.18).

6 Open questions and future work

6.1 General questions

The existence table in Section 3.1 (Table 5) contains some question marks. It would be interesting to solve the missing cases, for example to find out what properties $\mathfrak{n}$ can have if $\mathfrak{g}$ is reductive or complete and $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure. Also other questions about the relation between $\mathfrak{g}$ and $\mathfrak{n}$ (where $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie algebra structure) would be worth studying – e.g. whether or not the nilpotency class/solvability class of $\mathfrak{g}$ has implications on the nilpotency class/solvability class of $\mathfrak{n}$ or vice versa (cf. Lemma 3.7).

There seems to be little hope that post-Lie algebra structures can be classified nicely in general – even in dimension 3, a complete description of post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ or $\mathfrak{n}$ is solvable, is very complicated.
Therefore, it would be nice to find interesting classes of Lie algebras with "easy" post-Lie algebra structures (like the ones presented in Chapters 4 and 5).
In [24, 28], there are results on post-Lie algebra structures on *perfect* (which is, besides "complete", another generalization of semisimple, see Definition 2.8) Lie algebras. Are there results on post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ or $\mathfrak{n}$ is perfect?

Question 6.1.

(i) In Table 5, for every cell which now contains a "?", should it contain a "✓" (i.e. there is a pair of Lie algebras with the corresponding properties admitting a post-Lie algebra structure) or a "–" (i.e. there is no pair of Lie algebras with the corresponding properties admitting a post-Lie algebra structure)?

(ii) Which results of Table 5 can be refined?

Another possible extension of the work presented here is the extension to different fields. Our considerations were all done over the field of complex numbers: this assumption was often crucial since important theorems we used need not hold over fields of positive characteristic or over non-algebraically closed fields (the Levi decomposition theorem does not hold in positive characteristic, some theorems on decompositions of Lie algebras over $\mathbb{C}$ do not even extend to Lie algebras over the real numbers, ...).
It would be interesting to study post-Lie algebra structures on Lie algebras over other fields, especially over the real numbers (since the geometric motivation we presented leads to *real* Lie algebras) or to find ways to transfer results on post-Lie algebra structures on Lie algebras over $\mathbb{C}$ to results on post-Lie algebra structures on Lie algebras over $\mathbb{R}$.

For fields of positive characteristics, we expect that many results presented here do not extend to them.

6.2 Questions for complete Lie algebras

In Sections 4.3 and 4.4, we studied post-Lie algebra structures on post-Lie algebra double pairs where $\mathfrak{n}$ is a (solvable) complete Lie algebra. However, we had to impose some assumptions on $\mathfrak{n}$ (namely $\mathrm{nil}(\mathfrak{n}) = \mathrm{n}(\mathfrak{n})$ in Section 4.3 and the two assumptions $H^0(T, \mathfrak{n}) = T$ and $\mathfrak{n}$ generated by special weight spaces in Section 4.4). As we have seen that not all solvable complete Lie algebras are generated by special weight spaces (and for those who are not, Theorem 4.63 does not hold, see Example 4.64), we should ask whether the other two conditions are vacuous or even necessary for our purposes:

Question 6.2.

(i) Are the conditions $\mathrm{nil}(\mathfrak{n}) = \mathrm{n}(\mathfrak{n})$ *and* $H^0(T, \mathfrak{n}) = T$ *satisfied for every solvable complete Lie algebra* $\mathfrak{n}$*?*

(ii) If they are not satisfied for some solvable complete Lie algebra $\mathfrak{n}$*, do the results of Sections 4.3 and 4.4 still hold – that is, do post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n}, R)$ *have the same simple structures as in Theorems 4.40 and 4.63?*

Another way to generalize the result would be to consider rigid Lie algebras instead of complete ones:

Question 6.3. *Can one classify post-Lie algebra structures on Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$*, where* $\mathfrak{n}$ *is rigid?*

6.3 Questions for nilpotent Lie algebras

In Section 5.1 we listed all post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where both $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic to the three-dimensional Heisenberg Lie algebra $\mathfrak{n}_3$. From this explicit classification, we drew some surprising consequences (Corollary 5.2) like that the anti-commutator always defines a CPA-structure. It would be interesting to know if one could derive this result without the explicit classification:

Question 6.4. *How can one prove (without using a complete classification of all post-Lie algebra structures) that, given a post-Lie algebra structure* $x \cdot y$ *on* $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{n}_3$*, the bilinear map* $x \circ y := \frac{1}{2}(x \cdot y + y \cdot x)$ *defines a CPA-structure on* $\mathfrak{g}$*?*

That one can associate a CPA-structure in this way is particularly nice – it would be interesting to know for which pairs of Lie algebras it is possible to do so. (It is, for example, not true for pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are both isomorphic to a Heisenberg Lie algebra of dimension ≥ 5.)

Question 6.5. *Given a pair* $(\mathfrak{g}, \mathfrak{n})$ *with a post-Lie algebra structure* $x \cdot y$*, which conditions on* $\mathfrak{g}$ *and* $\mathfrak{n}$ *ensure that*

$$x \circ y := \frac{1}{2}(x \cdot y + y \cdot x)$$

defines a CPA-structure on $\mathfrak{g}$ *or* $\mathfrak{n}$*?*

Question 6.6. *Given a pair* $(\mathfrak{g}, \mathfrak{n})$ *with a post-Lie algebra structure* $x \cdot y$*, is there another way to associate a CPA-structure, an LR-structure or a pre-Lie algebra structure as via the anti-commutator? If so, which properties (like e.g. completeness of the respective structures) does this correspondence preserve?*

Given Theorem 5.44 which classifies CPA-structures on filiform Lie algebras in terms of their solvability class, one may ask if there are other families of Lie algebras where the form of the CPA-structures is connected to their solvability class (or nilpotency class):

Question 6.7. *Are there other "large" and "interesting" families of Lie algebras where CPA-structures can be classified similarly as in Theorem 5.44?*

Computational examples suggest that CPA-structures on *quasi-filiform* stem Lie algebras (an n-dimensional Lie algebra is quasi-filiform, if it is $(n-2)$-step nilpotent) also always satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ if $\mathfrak{g}$ is non-metabelian (like CPA-structures on filiform Lie algebras do, by Theorem 5.44). (Remember that $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ is equivalent to $(\mathfrak{g}, \cdot)$ being an associative algebra by Proposition 5.90.) This suggests to try to prove a similar result as Theorem 5.44 for quasi-filiform Lie algebras:

Question 6.8. *Can we classify CPA-structures on quasi-filiform stem Lie algebras* $\mathfrak{g}$ *in terms of their solvability class? Do they all satisfy* $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$*?*

The conjecture that $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ for all CPA-structures on non-metabelian Lie algebras with "high" nilpotency class is not true:

Example 6.9. *The* 9*-dimensional stem nilpotent Lie algebra* $\mathfrak{g}$ *(with solvability class* 3 *and nilpotency class* 5*), given by*

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_6] = e_8, [e_2, e_3] = e_7, [e_2, e_5] = -e_9, [e_3, e_4] = e_9$$

admits CPA-structures with $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] \neq 0$*, e.g. the one given by*

$$e_1 \cdot e_1 = e_1 \cdot e_2 = e_2 \cdot e_2 = e_6, e_1 \cdot e_3 = e_1 \cdot e_6 = e_2 \cdot e_3 = e_8.$$

As the classification of all post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{n}_3$, yielded interesting results (Corollary 5.2), it would be worthwhile to study this situation for the "opposite" case, namely filiform Lie algebras:

Question 6.10. *Given two filiform non-metabelian isomorphic Lie algebras* $\mathfrak{g} \cong \mathfrak{n}$ *and a (non-necessarily commutative) post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*. Is it still true (as in the case for CPA-structures), that* $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ *or* $\mathfrak{n} \cdot \{\mathfrak{n}, \mathfrak{n}\} = 0$*? Can one impose certain conditions (e.g. terms of the lower central series of* $\mathfrak{g}$ *and* $\mathfrak{n}$ *being "sufficiently similar") on the pair* $(\mathfrak{g}, \mathfrak{n})$ *which imply one of these statements?*

Remark 6.11. One might also state an analogous question to Question 6.10 for a pair of *non-isomorphic* filiform Lie algebras $(\mathfrak{g}, \mathfrak{n})$; then it is necessary that both $\mathfrak{g}$ and $\mathfrak{n}$ are metabelian, as counterexamples on $(\mathfrak{g}, \mathfrak{n}) = (L_8, R_8)$ and $(\mathfrak{g}, \mathfrak{n}) = (R_8, L_8)$ show (in both cases, it is possible to find post-Lie algebra structures which do neither satisfy $\mathfrak{g} \cdot [\mathfrak{g}, \mathfrak{g}] = 0$ nor $\mathfrak{n} \cdot \{\mathfrak{n}, \mathfrak{n}\} = 0$).

Another open question is related to completeness: All CPA-structures on nilpotent Lie algebras are complete (Proposition 2.54) and thus the left-multiplication operators and right-multiplication operators are nilpotent. Can this result be extended to non-commutative post-Lie algebra structures?

Question 6.12. *Given a pair of nilpotent stem Lie algebras* $(\mathfrak{g}, \mathfrak{n})$*. Is it true that all left-multiplication operators* $L(x)$ *are nilpotent? Is it true if we assume* $\mathfrak{g} \cong \mathfrak{n}$*?*

Again, examples suggest that the answer is yes if $\mathfrak{g}$ and $\mathfrak{n}$ are stem (and is no, if we let $\mathfrak{g}$ or $\mathfrak{n}$ have an abelian ideal).
The analogous question for the right-multiplication operators (that is, completeness) is wrong:

Example 6.13 ([29, Remark 5.5])**.** *Consider* $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{n}_3(\mathbb{C})$*, the* 3*-dimensional Heisenberg algebra. If a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *is given as in Proposition 5.1, type (6), then* $R(e_2)$ *is not nilpotent (even though* $\mathfrak{g} \cong \mathfrak{n}$ *are nilpotent).*

Another important class of nilpotent Lie algebra (which did not come up in this thesis) is the one of *free-nilpotent Lie algebras.* Similar to the case of a free group, given a set of generators X, the *free Lie algebra* L *on* X is the (unique) Lie algebra L with a map $i\colon X \to L$ such that for every Lie algebra L' and map $f\colon X \to L'$, there is a unique homomorphism of Lie algebras $\varphi\colon L \to L'$ such that $f = \varphi \circ i$. Free-nilpotent Lie algebras are quotients of L by an ideal of the lower central series:

Definition 6.14. The *free-nilpotent Lie algebra* $F_{g,c}$ *with* g *generators and nilpotency class* c is given by $F_{g,c} = L_g / L_g^{c+1}$. Here, L_g denotes the free Lie algebra on g generators.

The article [34] conjectures that for $c \geq 3, g \geq 2$, all CPA-structures on $F_{g,c}$ are central (see Definition 5.96). This problem is connected to linear equations in $F_{g,c}$: Consider the linear system of equations in $F_{g,c}$

$$[u_{i,j}, x_k] + [x_j, u_{i,k}] = 0$$

(in the variables $u_{i,j}$ with $u_{i,j} = u_{j,i}$ for all i, j) for all $1 \leq i, j, k \leq g$. It is proven in [34] that if $g \geq 3$, all solutions $u_{i,j}$ which lie in the commutator $[F_{g,c}, F_{g,c}]$ are contained in $\mathrm{Z}(F_{g,c})$. This result is used in proving that, given $g \geq 3$, if all CPA-structures on $F_{g,3}$ are central, then the same is true for $F_{g,c}$ for any $c \geq 3$.
However, for $g = 2$ an analogous result is not yet proven (for $g = 2$, it is proven in [34] that $F_{2,3}$ admits only central CPA-structures – thus such a result would imply $F_{2,c}$ having only central CPA-structures for every $c \geq 3$) and for $g \geq 3$, the base case (namely that $F_{g,3}$ only admits central CPA-structures) is still open.
The system of equations for $F_{2,3}$ can be simplified:

Question 6.15 ([34]).

(i) *Given the linear system of equations in* $F_{2,3}$,

$$\begin{aligned} [u_{1,1}, x_2] + [x_1, u_{1,2}] &= 0, \\ [u_{2,2}, x_1] + [x_2, u_{1,2}] &= 0, \end{aligned}$$

where x_1, x_2 *are generators of* $F_{2,3}$. *Are all solutions* $(u_{1,1}, u_{1,2}, u_{2,2})$, *which are contained in the commutator of* $F_{2,3}$, *contained in the center of* $F_{2,3}$*?*

(ii) *Are all CPA-structures on* $F_{g,3}$, $g \geq 3$, *central?*

A Proof of Theorem 5.44

Here, we want to give the missing proofs from Sections 5.4.1 and 5.4.2 which are necessary in proving Theorem 5.44. We consider, as before, a filiform Lie algebra $\mathfrak{g}$ with an adapted basis $\{e_1, \dots, e_n\}$ of dimension $n \geq 7$. As we are considering CPA-structures in Classes A$_2$ and B, we know that $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian by Proposition 5.58.

As before, we denote by δ_i the vector space $\delta_i \coloneqq \operatorname{span}\{e_j : j \geq i\}$.

A.1 CPA-structures in Class A$_2$

We shall consider CPA-structures in Class A$_2$, i.e. CPA-structures on a filiform Lie algebra $\mathfrak{g}$ (where $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian) with $e_1 \cdot e_1 \in \delta_3$ and $e_1 \cdot e_2 \notin \delta_4$. By Proposition 5.49, this implies $e_1 \cdot e_2 = e_3 + \delta_4$.

Lemma A.1. *Let $x \cdot y$ be a CPA-structure in Class A_2. Then $e_1 \cdot e_1 \in \delta_3, e_1 \cdot e_i = e_{i+1} + \delta_{i+2}$ (for $2 \leq i \leq n-1$) and $e_i \cdot e_j \in \delta_{i+j}$ for $2 \leq i, j \leq n$.*

Proof. We know that $e_1 \cdot e_1 \in \delta_3, e_1 \cdot e_i \in e_{i+1} + \delta_{i+2}$ (for $2 \leq i \leq n-1$) and $e_2 \cdot e_i \in \delta_{2+i}$ by Proposition 5.49.

Now we show that $e_i \cdot e_j \in \delta_{i+j}$ for $i, j \geq 3$:

We have $e_3 \cdot e_3 \in \delta_6$:

$$e_3 \cdot e_3 = e_3 \cdot [e_1, e_2] = [e_3 \cdot e_1, e_2] + [e_1, e_3 \cdot e_2] \in [\delta_4, e_2] + [e_1, \delta_5] \subseteq \delta_6 + \delta_6.$$

and find

$$e_3 \cdot e_4 = [e_3 \cdot e_1, e_3] + [e_1, e_3 \cdot e_3] \in [\delta_4, e_3] + [e_1, \delta_6] \subseteq \delta_7$$

and inductively $e_3 \cdot e_j \in \delta_{3+j}$.

To show $e_i \cdot e_j \in \delta_{i+j}$ is just another induction: If $e_i \cdot e_{j-1} \in \delta_{i+j-1}$, then

$$e_i \cdot e_j = [e_i \cdot e_1, e_{j-1}] + [e_1, e_i \cdot e_{j-1}] \in [\delta_{i+1}, e_{j-1}] + [e_1, \delta_{i+j-1}] \subseteq \delta_{i+1+j-1} + \delta_{i+j} \subseteq \delta_{i+j}.$$

This proves Lemma A.1. $\square$

So we can prove Proposition 5.52, which says:

Proposition A.2. *Suppose we have a CPA-structure on the filiform Lie algebra $\mathfrak{g}$ in Class A_2. Then $\mathfrak{g}$ is metabelian and $[\mathfrak{g}, \mathfrak{g}] \cdot [\mathfrak{g}, \mathfrak{g}] = 0$.*

Proof. Keep in mind that by Lemma A.1, the CPA-structure satisfies $e_1 \cdot e_1 \in \delta_3, e_1 \cdot e_i = e_{i+1} + \delta_{i+2}$ for $i \neq 1, n$ (in particular, $e_1 \cdot e_{n-1} = e_n$), $e_i \cdot e_j \in \delta_{i+j}$ for all $i, j = 1, \ldots, n$ and $e_1 \cdot e_n = 0$.
Now we claim the following: Suppose for all $j, k \in \{3, \ldots, n\}$ and for an $\ell \in \{0, \ldots, n-7\}$, the following three conditions do hold:

(i) $[e_j, e_k] \in \delta_{j+k+\ell}$

(ii) $e_j \cdot e_k \in \delta_{j+k+\ell}$

(iii) $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+2} + \delta_n$ for $j \notin \{n-1, n\}$.

Then these three conditions also hold for $\ell + 1$, that is:

(i)' $[e_j, e_k] \in \delta_{j+k+\ell+1}$

(ii)' $e_j \cdot e_k \in \delta_{j+k+\ell+1}$

(iii)' $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+3} + \delta_n$ for $j \notin \{n-1, n\}$.

Note that we know that the three conditions are true for $\ell = 0$. So if we have proven this claim, we know inductively that they also hold for $\ell = n - 6$.

The first thing we shall prove is that $[e_j, e_{j+1}] \in \delta_{2j+\ell+2}$ for $j, k \geq 3$. Note that by (i), we have $[e_j, e_{j+1}] \in \delta_{2j+\ell+1}$. Further note that we may assume $2j+\ell+1 \leq n$ – otherwise $\delta_{2j+\ell+1} = 0 = \delta_{2j+\ell+2}$ and there is nothing to prove.

To prove the claim, we first need to do a distinction between $j = 3$ and $j > 3$. (The reason is that if $j = 3$, we cannot use assumption (ii) for the products $e_2 \cdot e_3$ and $e_2 \cdot e_4$ as it is only valid from $j \geq 3$ on).

Case 1: Let us suppose $j > 3$. By (ii), we can write

$$\begin{aligned} e_j \cdot e_j &\in \alpha_1 e_{2j+\ell} + \delta_{2j+\ell+1}, \\ e_j \cdot e_{j+1} &\in \alpha_2 e_{2j+\ell+1} + \delta_{2j+\ell+2}, \\ e_{j-1} \cdot e_j &\in \alpha_3 e_{2j+\ell-1} + \delta_{2j+\ell}, \\ e_{j-1} \cdot e_{j+1} &\in \alpha_4 e_{2j+\ell} + \delta_{2j+\ell+1} \end{aligned}$$

with coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$.
We will now apply axioms (CPA2) and (CPA3) and shall conclude $\alpha_1 = 0 = \alpha_2$.

Step 1: Assume for a moment that $2j + \ell + 1 < n$. Let us use axiom (CPA2) on the triple (e_1, e_j, e_j). We obtain

$$\begin{aligned} 0 &= [e_1, e_j] \cdot e_j - e_1 \cdot (e_j \cdot e_j) + e_j \cdot (e_1 \cdot e_j) \\ &\in e_j \cdot e_{j+1} - e_1 \cdot (\alpha_1 e_{2j+\ell} + \delta_{2j+\ell+1}) + e_j \cdot (e_{j+1} + \delta_{j+\ell+2} + \delta_n) \\ &\subseteq \alpha_2 e_{2j+\ell+1} + \delta_{2j+\ell+2} - \alpha_1 e_{2j+\ell+1} + \delta_{2j+\ell+2} + \delta_n + \alpha_2 e_{2j+\ell+1} + \delta_{2j+\ell+2} \\ &\subseteq \alpha_2 e_{2j+\ell+1} - \alpha_1 e_{2j+\ell+1} + \alpha_2 e_{2j+\ell+1} + \delta_{2j+\ell+2} + \delta_n \end{aligned}$$

and get $2\alpha_2 - \alpha_1 = 0$. Note that we have $2j+\ell+1 < n$. Thus there is no multiple of $e_{2j+\ell+1}$ contained in δ_n.
If instead $2j+\ell+1 = n$ (note that we ruled out the case $2j+\ell+1 > n$), then we instead have

$$\begin{aligned} 0 &\in \alpha_2 e_n - e_1 \cdot (\alpha_1 e_{n-1} + \delta_n) + e_j \cdot (e_{j+1} + \delta_{j+\ell+2} + \delta_n) \\ &\subseteq \alpha_2 e_n - \alpha_1 e_n + \alpha_2 e_n, \end{aligned}$$

so we also find $\alpha_1 = 2\alpha_2$ (note here that $e_1 \cdot e_{n-1} = e_n$).

Step 2: We use axiom (CPA2) again; this time on the triple (e_1, e_{j-1}, e_j). We have

$$\begin{aligned} 0 &= [e_1, e_{j-1}] \cdot e_j - e_1 \cdot (e_{j-1} \cdot e_j) + e_{j-1} \cdot (e_1 \cdot e_j) \\ &\in e_j \cdot e_j - e_1 \cdot (\alpha_3 e_{2j+\ell-1} + \delta_{2j+\ell}) + e_{j-1} \cdot (e_{j+1} + \delta_{j+2} + \delta_n) \\ &\subseteq \alpha_1 e_{2j+\ell} + \delta_{2j+\ell+1} - \alpha_3 e_{2j+\ell} + \delta_{2j+\ell+1} + \delta_n + \alpha_4 e_{2j+\ell} + \delta_{2j+\ell+1} \\ &\subseteq \alpha_1 e_{2j+\ell} - \alpha_3 e_{2j+\ell} + \alpha_4 e_{2j+\ell} + \delta_{2j+\ell+1} + \delta_n \end{aligned}$$

and conclude (by comparing the $e_{2j+\ell}$-coefficient) that $\alpha_1 = \alpha_3 - \alpha_4$.

Step 3: Now we apply axiom (CPA3) on the triple (e_{j-1}, e_1, e_j):

$$\begin{aligned} 0 &= e_{j-1} \cdot [e_1, e_j] - [e_{j-1} \cdot e_1, e_j] - [e_1, e_{j-1} \cdot e_j] \\ &\in e_{j-1} \cdot e_{j+1} - [e_j + \delta_{j+1} + \delta_n, e_j] - [e_1, \alpha_3 e_{2j+\ell-1} + \delta_{2j+\ell}] \\ &\subseteq \alpha_4 e_{2j+\ell} + \delta_{2j+\ell+1} - 0 + \delta_{j+1+j+\ell} - 0 - \alpha_3 e_{2j+\ell} + \delta_{2j+\ell+1} \\ &\subseteq \alpha_4 e_{2j+\ell} - \alpha_3 e_{2j+\ell} + \delta_{2j+\ell+1}. \end{aligned}$$

Here we have used assumption (i); we conclude that $\alpha_4 = \alpha_3$ and since $2\alpha_2 = \alpha_1 = \alpha_3 - \alpha_4$, we find $\alpha_1 = 0 = \alpha_2$.
Step 4: Our last step is again an application of axiom (CPA3) (on the triple (e_j, e_1, e_j)) – we show that $[e_j, e_{j+1}] \in \delta_{2j+\ell+2}$: We have

$$\begin{aligned} 0 &= e_j \cdot [e_1, e_j] - [e_j \cdot e_1, e_j] - [e_1, e_j \cdot e_j] \\ &\in e_j \cdot e_{j+1} - [e_{j+1}, e_j] - [\delta_{j+2} + \delta_n, e_j] - [e_1, \alpha_1 e_{2j+\ell} + \delta_{2j+\ell+1}] \\ &\subseteq \alpha_2 e_{2j+\ell+1} + \delta_{2j+\ell+2} - [e_{j+1}, e_j] + \delta_{2j+\ell+2} - \alpha_1 e_{2j+\ell+1} + \delta_{2j+\ell+2} \\ &\subseteq \alpha_2 e_{2j+\ell+1} + [e_j, e_{j+1}] - \alpha_1 e_{2j+\ell+1} + \delta_{2j+\ell+2}. \end{aligned}$$

But since $\alpha_1 = 0 = \alpha_2$, we find $[e_j, e_{j+1}] \in \delta_{2j+\ell+2}$. This completes Case 1.

Case 2: Before we can go on, we need to prove the same thing (namely $[e_j, e_{j+1}] \in \delta_{2j+\ell+2}$) for $j = 3$. In this case, the products $e_2 \cdot e_3, e_2 \cdot e_4$ read more complicated as before – we can write

$$\begin{aligned}
e_3 \cdot e_3 &\in \alpha_1 e_{6+\ell} + \delta_{7+\ell},\\
e_3 \cdot e_4 &\in \alpha_2 e_{7+\ell} + \delta_{8+\ell},\\
e_2 \cdot e_3 &\in \sum_{i=5}^{4+\ell} \beta_i e_i + \alpha_3 e_{5+\ell} + \delta_{6+\ell},\\
e_2 \cdot e_4 &\in \sum_{i=6}^{5+\ell} \gamma_i e_i + \alpha_4 e_{6+\ell} + \delta_{7+\ell}.
\end{aligned}$$

However, we will show that the additional terms do not matter (thanks to assumption (iii)). We show again that $\alpha_1 = 0 = \alpha_2$.

This involves four steps similar to the ones before:

Step 1 can be copied (with $j = 3$) completely from above (as it involves only the products $e_3 \cdot e_3$ and $e_3 \cdot e_4$ having the same form as before). So we again have $\alpha_1 = 2\alpha_2$.

Step 2 is again an application of axiom (CPA2) on (e_1, e_2, e_3):

$$\begin{aligned}
0 &= e_3 \cdot e_3 - e_1 \cdot (e_2 \cdot e_3) + e_2 \cdot (e_1 \cdot e_3)\\
&\in \alpha_1 e_{6+\ell} + \delta_{7+\ell} - e_1 \cdot \left(\sum_{i=5}^{4+\ell} \beta_i e_i + \alpha_3 e_{5+\ell} + \delta_{6+\ell} \right) + e_2 \cdot (e_4 + \delta_{5+\ell} + \delta_n)\\
&\subseteq \alpha_1 e_{6+\ell} + \delta_{7+\ell} - \sum_{i=5}^{4+\ell} \beta_i e_{i+1} - \underbrace{\sum_{i=5}^{4+\ell} \delta_{i+\ell+2}}_{\subseteq \delta_{7+\ell}} - \alpha_3 e_{6+\ell} + \delta_n + \delta_{7+\ell} + \sum_{i=6}^{5+\ell} \gamma_i e_i + \alpha_4 e_{6+\ell} + \delta_{7+\ell}\\
&\subseteq \alpha_1 e_{6+\ell} - \alpha_3 e_{6+\ell} + \alpha_4 e_{6+\ell} - \sum_{i=5}^{4+\ell} \beta_i e_{i+1} + \sum_{i=6}^{5+\ell} \gamma_i e_i + \delta_{7+\ell} + \delta_n
\end{aligned}$$

and comparing the $e_{6+\ell}$-coefficient gives again $\alpha_1 = \alpha_3 - \alpha_4$.

Here we needed that $6 + \ell < n$ – otherwise we could not compare the corresponding coefficients. (But this is satisfied since $\ell \leq n - 7$ by assumption.)

Step 3 is an application of axiom (CPA3) to (e_2, e_1, e_3):

$$\begin{aligned}
0 &= e_2 \cdot [e_1, e_3] - [e_2 \cdot e_1, e_3] - [e_1, e_2 \cdot e_3] \\
&\in e_2 \cdot e_4 - [e_3 + \delta_4, e_3] - [e_1, \sum_{i=5}^{4+\ell} \beta_i e_i + \alpha_3 e_{5+\ell} + \delta_{6+\ell}] \\
&\subseteq \sum_{i=6}^{5+\ell} \gamma_i e_i + \alpha_4 e_{6+\ell} + \delta_{7+\ell} + \delta_{7+\ell} - \sum_{i=5}^{4+\ell} \beta_i e_{i+1} - \underbrace{\sum_{i=5}^{4+\ell} \delta_{i+\ell+2}}_{\in \delta_{7+\ell}} - \alpha_3 e_{6+\ell} + \delta_{7+\ell} \\
&\subseteq \alpha_4 e_{6+\ell} - \alpha_3 e_{6+\ell} + \sum_{i=6}^{5+\ell} \gamma_i e_i - \sum_{i=5}^{4+\ell} \beta_i e_{i+1} + \delta_{7+\ell}
\end{aligned}$$

meaning $\alpha_4 = \alpha_3$ and thus $\alpha_1 = 0 = \alpha_2$ as before. (We again needed $6 + \ell < n$.)
Step 4, completely as before, gives $[e_3, e_4] \in \delta_{8+\ell}$.

So we have for all $j \geq 3$ that $[e_j, e_{j+1}] \in \delta_{2j+\ell+2}$.
This implies inductively that $[e_j, e_{j+i}] \in \delta_{2j+i+\ell+1}$ for all $j \geq 3$: By the Jacobi identity,

$$\begin{aligned}
[e_j, e_{j+i}] &= [e_j, [e_1, e_{j+i-1}]] \\
&= -[e_1, \underbrace{[e_{j+i-1}, e_j]}_{\in \delta_{2j+\ell+i} \text{ (induction)}}] + \underbrace{[e_{j+i-1}, e_{j+1}]}_{\in \delta_{2j+i+\ell+1} \text{ (induction)}} \in \delta_{2j+i+\ell+1}.
\end{aligned}$$

This proves that (i)' holds.

Let us prove (ii)': We have $e_j \cdot e_j \in \delta_{2j+\ell+1}$ (since $\alpha_1 = 0$). Inductively, by axiom (CPA3),

$$\begin{aligned}
e_j \cdot e_{j+i} = e_j \cdot [e_1, e_{j+i-1}] &= [\underbrace{e_j \cdot e_1}_{\in \delta_{j+1}}, e_{j+i-1}] + [e_1, \underbrace{e_j \cdot e_{j+i-1}}_{\in \delta_{j+j+i-1+\ell+1} \text{ (induction)}}] \\
&\in \delta_{2j+1+\ell+i}.
\end{aligned}$$

However, this step does no longer work if $\ell = n - 6$, because then we do not have $e_j \cdot e_j \in \delta_{2j+\ell+1}$ for $j = 3$.
This proves (ii)'; we conclude by proving (iii)':

Let us distinguish three cases:
Case 1: $j + \ell + 3 \geq n + 1$. Then, as $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+2} + \delta_n = e_{j+1} + \delta_n$ by assumption (iii), we automatically have $e_1 \cdot e_j \in e_{j+1} + \underbrace{\delta_{j+\ell+3}}_{=0} + \delta_n$.

Case 2: $j + \ell + 3 < n$. Then write the products $e_1 \cdot e_j$ and $e_1 \cdot e_{j+1}$ as

$$\begin{aligned}
e_1 \cdot e_j &\in e_{j+1} + \beta_1 e_{j+\ell+2} + \delta_{j+\ell+3}, \\
e_1 \cdot e_{j+1} &\in e_{j+2} + \beta_2 e_{j+\ell+3} + \delta_{j+\ell+4}.
\end{aligned}$$

Then, by axiom (CPA2),

$$\begin{aligned}
0 &= e_{j+1} \cdot e_1 - e_1 \cdot (e_j \cdot e_1) + e_j \cdot (e_1 \cdot e_1)\\
&\in e_{j+2} + \beta_2 e_{j+\ell+3} + \delta_{j+\ell+4} - e_1 \cdot (e_{j+1} + \beta_1 e_{j+\ell+2} + \delta_{j+\ell+3}) + e_j \cdot \delta_3\\
&\subseteq e_{j+2} + \beta_2 e_{j+\ell+3} + \delta_{j+\ell+4} - e_{j+2} - \beta_2 e_{j+\ell+3} + \delta_{j+\ell+4} - \beta_1 e_{j+\ell+3} + \delta_{j+\ell+4} + \delta_{j+\ell+4}\\
&\subseteq -\beta_1 e_{j+\ell+3} + \delta_{j+\ell+4}
\end{aligned}$$

(note that $e_j \cdot \delta_3 \in \delta_{j+\ell+4}$ by (ii)' which we have just proven).

Hence, as the $e_{j+\ell+3}$-coefficient "exists", $\beta_1 = 0$. But this exactly means what we claimed, i.e. $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+3}$.

Case 3: $j + \ell + 3 = n$. This case is very similar to Case 2.
We write

$$\begin{aligned}
e_1 \cdot e_j &\in e_{j+1} + \beta_1 e_{j+\ell+2} + \delta_{j+\ell+3} + \delta_n = e_{j+1} + \beta_1 e_{n-1} + \delta_n,\\
e_1 \cdot e_{j+1} &= e_{j+2} + \beta_2 e_n.
\end{aligned}$$

And as before, by axiom (CPA2),

$$\begin{aligned}
0 &= [e_1, e_j] \cdot e_1 - e_1 \cdot (e_j \cdot e_1) + e_j \cdot (e_1 \cdot e_1)\\
&= e_{j+1} \cdot e_1 - e_1 \cdot (e_j \cdot e_1) + e_j \cdot (e_1 \cdot e_1)\\
&\in e_{j+2} + \beta_2 e_n - e_1 \cdot (e_{j+1} + \beta_1 e_{n-1} + \delta_n) + e_j \cdot \delta_3\\
&\subseteq e_{j+2} + \beta_2 e_n - e_{j+2} - \beta_2 e_n - \beta_1 e_n + 0,
\end{aligned}$$

so again $\beta_1 = 0$.
This shows (iii)'.

Now since (i), (ii) and (iii) hold for $\ell = 0$, they do also hold for $\ell = n - 6$. So in particular, for $j, k \geq 3$,

$$[e_j, e_k] \in \delta_{j+k+n-6} \text{ and } e_j \cdot e_k \in \delta_{j+k+n-6}.$$

But if $j \geq 3, k \geq 4$, then this means $[e_j, e_k] \in \delta_{n+1} = 0, e_j \cdot e_k \in \delta_{n+1} = 0$. Thus $[\delta_3, \delta_3] = 0$ and $\mathfrak{g}$ is metabelian.
However, we do "only" have $e_3 \cdot e_3 \in \delta_n$ (and not $e_3 \cdot e_3 = 0$).
So let us show that indeed $e_3 \cdot e_3 = 0$. By (iii)', we have $e_1 \cdot e_i = e_{i+1} + \delta_n, 3 \leq i \leq n-1$; let us show that for $4 \leq i \leq n-1$, we indeed have $e_1 \cdot e_i = e_{i+1}$:

$$e_1 \cdot e_i = e_1 \cdot [e_1, e_{i-1}] = [e_1 \cdot e_1, e_{i-1}] + [e_1, e_1 \cdot e_{i-1}] \in [\delta_3, e_{i-1}] + [e_1, e_i + \delta_n] \subseteq 0 + e_{i+1} + 0$$

using the fact that $\mathfrak{g}$ is metabelian.
This, however, implies that $[e_1, e_2 \cdot e_3] = e_1 \cdot (e_2 \cdot e_3)$, which, by (CPA3) implies that $e_2 \cdot e_4 = e_1 \cdot (e_2 \cdot e_3)$:

$$e_2 \cdot e_4 = \underbrace{[e_2 \cdot e_1, e_3]}_{=0} + [e_1, e_2 \cdot e_3] = e_1 \cdot (e_2 \cdot e_3).$$

And now, we can show that $e_3 \cdot e_3 = 0$ (CPA2):

$$0 = [e_1, e_3] \cdot e_2 - e_1 \cdot (e_3 \cdot e_2) + e_3 \cdot (e_1 \cdot e_2) = e_2 \cdot e_4 - e_1 \cdot (e_2 \cdot e_3) + e_3 \cdot (e_1 \cdot e_2) \in e_3 \cdot (e_3 + \delta_4).$$

Since $e_3 \cdot \delta_4 = 0$ by the above, we now know that $e_3 \cdot e_3 = 0$, which we wanted to show. □

A.2 CPA-structures in Class B

We again assume that $\mathfrak{g}$ is filiform of dimension ≥ 7; the remaining case is the one where $e_1 \cdot e_1 \notin \delta_3$ (meaning $x \cdot y$ is in Class B). By Lemma 5.57, $\mathfrak{g}^{\lceil (n-4)/2 \rceil}$ is abelian.

Lemma A.3. *Let $\mathfrak{g}$ be filiform and $x \cdot y$ a CPA-structure on $\mathfrak{g}$ in Class B. Then*

(i) $e_1 \cdot e_2 \in \beta e_3 + \delta_4$, *where* $\beta \in \{0, 1\}$.

(ii) $e_2 \cdot e_2 \in \delta_5, e_2 \cdot e_3 \in \delta_6$.

Proof. As $x \cdot y$ is in Class B, we write $e_1 \cdot e_1 \in \alpha e_2 + \delta_3, e_1 \cdot e_2 \in \beta e_3 + \delta_4$ and $e_2 \cdot e_2 \in \gamma e_4 + \delta_5$ with $\alpha \neq 0$. We have to show $\gamma = 0$ and $\beta \in \{0, 1\}$.
By axiom (CPA3), we have

$$e_2 \cdot e_3 = [e_2 \cdot e_1, e_2] + [e_1, e_2 \cdot e_2] \in [\delta_3, e_2] + [e_1, \gamma e_4 + \delta_5] \subseteq \delta_6 + \gamma e_5 + \delta_6.$$

Here we used that $[e_2, \delta_3] \in \delta_6$ (which holds by Lemma 5.46 and Proposition 5.47).
In the same way we get $e_2 \cdot e_4 \in \gamma e_6 + \delta_7$ and $e_1 \cdot e_3 \in \beta e_4 + \delta_5, e_1 \cdot e_4 \in \beta e_5 + \delta_6$. But on the other hand,

$$\begin{aligned}0 &= e_3 \cdot e_2 - e_1 \cdot (e_2 \cdot e_2) + e_2 \cdot (e_1 \cdot e_2) \\ &\in \gamma e_5 + \delta_6 - e_1 \cdot (\gamma e_4 + \delta_5) + e_2 \cdot (\beta e_3 + \delta_4) \subseteq \gamma e_5 - \beta\gamma e_5 + \delta_6 + \beta\gamma e_5 + \delta_6 \subseteq \gamma e_5 + \delta_6.\end{aligned}$$

Thus $\gamma = 0$, meaning $e_2 \cdot e_2 \in \delta_5$. This however implies

$$\begin{aligned}0 &= e_3 \cdot e_1 - e_1 \cdot (e_2 \cdot e_1) + e_2 \cdot (e_1 \cdot e_1) \\ &\in \beta e_4 + \delta_5 - e_1 \cdot (\beta e_3 + \delta_4) + e_2 \cdot \delta_2 \subseteq \beta e_4 + \delta_5 - \beta^2 e_4 + \delta_5,\end{aligned}$$

so $\beta \in \{0, 1\}$. □

So if $x \cdot y$ is in Class B, $x \cdot y$ is in Class B$_1$ if and only if $e_1 \cdot e_2 \in \delta_4$ and in Class B$_2$ if and only if $e_1 \cdot e_2 \in e_3 + \delta_4$.

We have to deal with both cases (Class B$_1$ and Class B$_2$) separately.

Proposition A.4. *Let $\mathfrak{g}$ be filiform (and n-dimensional), $x \cdot y$ a CPA-structure on $\mathfrak{g}$ in Class B$_1$ (meaning $e_1 \cdot e_1 \notin \delta_3$ and $e_1 \cdot e_2 \in \delta_4$). Then $\mathfrak{g} \cong L_n$.*

Proof. We will prove the result in five steps:

(1) We show that $e_2 \cdot e_j \in \delta_{j+3}$ for all $j \geq 2$. We need this for step (2) and (3). From this, it also follows that $e_i \cdot e_j \in \delta_{i+j}$ for $i, j \geq 2$.

(2) We show $e_1 \cdot e_j \in \delta_{j+2}$ for all $j \geq 2$. We need this statement quite a few times in step (3).

(3) Next we do an induction to show that $e_i \cdot e_j = 0$ if

$$(i,j) \notin \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}.$$

(4) Using step (3), we show (inductively) that $[e_i, e_j] = 0$ if $i, j \neq 1, (i,j) \notin \{(2,3),(3,2)\}$.

(5) We also show $[e_2, e_3] = 0$. This finally means that $\mathfrak{g} \cong L_n$.

Step (1): We already know $e_2 \cdot e_2 \in \delta_5$ (by Lemma A.3). We show $e_2 \cdot e_j \in \delta_{j+3}$ by induction on j – the induction basis is $e_2 \cdot e_2 \in \delta_5$. The induction step $j \to j+1$ is, via (CPA3)

$$e_2 \cdot e_{j+1} = e_2 \cdot [e_1, e_j] = [\underbrace{e_2 \cdot e_1}_{\in \delta_4}, e_j] + [e_1, \underbrace{e_2 \cdot e_j}_{\in \delta_{j+3}}] \in \delta_{j+4}.$$

Step (2): We already know $e_1 \cdot e_2 \in \delta_4$ (by Lemma A.3 and our assumption $\beta = 0$). We show $e_1 \cdot e_j \in \delta_{j+2}$ by induction on j – the equation $e_1 \cdot e_2 \in \delta_4$ is the induction basis. The induction step $j \to j+1$ reads, via axiom (CPA2)

$$e_1 \cdot e_{j+1} = e_{j+1} \cdot e_1 = [e_1, e_j] \cdot e_1 = e_1 \cdot \underbrace{(e_j \cdot e_1)}_{\in \delta_{j+2}} - e_j \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_2} \in \delta_{j+3} - \delta_{j+3} \subseteq \delta_{j+3}.$$

Step (3): The statement we want to show here is: If for an $\ell \geq 0$ the inclusions

- $e_1 \cdot e_k \in \delta_{k+\ell+1}$ and
- $e_i \cdot e_j \in \delta_{i+j+\ell}$

hold for all $i \geq 2, j \geq 3, k \geq 4$, then they do also hold for $\ell + 1$, that is, the inclusions

- $e_1 \cdot e_k \in \delta_{k+\ell+2}$ and
- $e_i \cdot e_j \in \delta_{i+j+\ell+1}$

hold for all $i \geq 2, j \geq 3, k \geq 4$.

We start by showing $e_3 \cdot e_j \in \delta_{j+\ell+4}, j \geq 3$: We have

$$e_3 \cdot e_j = e_1 \cdot \underbrace{(e_2 \cdot e_j)}_{\in \delta_{j+2+\ell}} - e_2 \cdot \underbrace{(e_1 \cdot e_j)}_{\in \delta_{j+2}} \in \delta_{j+4+\ell} - \delta_{j+4+\ell} \subseteq \delta_{j+4+\ell}.$$

Now we show that if $e_i \cdot e_j \in \delta_{i+j+\ell+1}$, then also $e_{i+1} \cdot e_j \in \delta_{i+1+j+\ell+1}$:

$$e_{i+1} \cdot e_j = e_1 \cdot \underbrace{(e_i \cdot e_j)}_{\in \delta_{i+j+\ell+1}} - e_i \cdot \underbrace{(e_1 \cdot e_j)}_{\in \delta_{j+2}} \in \delta_{i+j+\ell+2} - \delta_{i+j+\ell+2} \subseteq \delta_{i+j+\ell+2}.$$

So the second inclusion holds if $i \geq 3$ and $j \geq 3$. Now we also show this inclusion for $i = 2$: Let $j \geq 3$, then

$$e_2 \cdot e_j = e_j \cdot e_2 = e_1 \cdot \underbrace{(e_{j-1} \cdot e_2)}_{\in \delta_{j+2}} - e_{j-1} \cdot \underbrace{(e_1 \cdot e_2)}_{\in \delta_4} \in \delta_{3+j+\ell}.$$

Now we show the remaining part – namely $e_1 \cdot e_k \in \delta_{k+\ell+2}, k \geq 4$. Similarly to above, we have

$$e_1 \cdot e_k = e_k \cdot e_1 = [e_1, e_{k-1}] \cdot e_1 = e_1 \cdot \underbrace{(e_{k-1} \cdot e_1)}_{\in \delta_{k+1}} - e_{k-1} \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_2} \in \delta_{k+\ell+2}.$$

For the second summand, we used here what we just have proven – namely $e_{k-1} \cdot e_2 \in \delta_{k-1+2+\ell+1}$.
Hence we have proven the claim. Now note that we know that the statement is true for $\ell = 0$, so if we choose ℓ large enough, we have $e_1 \cdot e_k = e_i \cdot e_j = 0$ for $i \geq 2, j \geq 3, k \geq 4$. In other words, the only (potentially) non-zero products are $e_1 \cdot e_1, e_1 \cdot e_2 = e_2 \cdot e_1, e_1 \cdot e_3 = e_3 \cdot e_1$ and $e_2 \cdot e_2$.

Step (4): We do another induction: Claim: If for an $\ell \geq 0$, the inclusion $[e_i, e_j] \in \delta_{i+j+\ell}$ holds for all $i \geq 2, j \geq 4$, then this inclusion holds also for $\ell + 1$, that is $[e_i, e_j] \in \delta_{i+j+\ell+1}$ holds for all $i \geq 2, j \geq 4$.
To prove this, let us do an induction on i. Remember that we write $e_1 \cdot e_1 = \alpha e_2 + \delta_3$ and assume $\alpha \neq 0$. First suppose $i = 2$. Note that since $j \geq 4$, we have $e_1 \cdot e_{j+1} = e_1 \cdot e_j = 0$. Now, by axiom (CPA3),

$$0 = e_1 \cdot e_{j+1} - [e_1 \cdot e_1, e_j] - [e_1, \underbrace{e_1 \cdot e_j}_{=0}] \in - \underbrace{\alpha}_{\neq 0} [e_2, e_j] - \underbrace{[\delta_3, e_j]}_{\subseteq \delta_{j+\ell+3}}$$

implying $[e_2, e_j] \in \delta_{j+\ell+3}$. So our claim holds for $i = 2$.
We do the induction step $i \to i + 1$ with help of the Jacobi identity: We have

$$[e_{i+1}, e_j] = [[e_1, e_i], e_j] = -[\underbrace{[e_i, e_j]}_{\in \delta_{i+j+\ell+1}}, e_1] - [\underbrace{[e_j, e_1]}_{\in \delta_{j+1}}, e_i] \in \delta_{i+j+\ell+2}.$$

Since in the beginning, we are in the case $\ell = 0$, we may conclude as before (by choosing ℓ large enough) that $[\delta_2, \delta_4] = 0$. However, we cannot do this procedure for $[e_2, e_j]$ where $j = 3$ – in this case, we do not have the equality $e_1 \cdot e_j = 0$ (and we used this equality). So to identify $\mathfrak{g}$'s true form as L_n, we still have to prove $[e_2, e_3] = 0$.
Step (5). Let us show $[e_2, e_3] = 0$.
First note that due to the Jacobi identity,

$$[e_1, [e_2, e_3]] = -[e_2, \underbrace{[e_3, e_1]}_{=-e_4}] - [e_3, [e_1, e_2]] = [e_2, e_4] - [e_3, e_3] = 0 - 0 = 0,$$

implying $[e_2, e_3] \in \delta_n$. Let us apply axioms (CPA3) and (CPA2) a few times:

$$0 = e_1 \cdot e_4 = \underbrace{[\underbrace{e_1 \cdot e_1}_{\in \delta_2}, e_3]}_{\in \delta_n} + [e_1, e_1 \cdot e_3]$$

implying $[e_1, e_1 \cdot e_3] \in \delta_n$ and thus $e_1 \cdot e_3 \in \delta_{n-1}$.
Another application is

$$\underbrace{e_1 \cdot [e_1, e_2]}_{\in \delta_{n-1}} = \underbrace{[e_1 \cdot e_1, e_2]}_{\in \delta_n} + [e_1, e_1 \cdot e_2]$$

implying $[e_1, e_1 \cdot e_2] \in \delta_{n-1}$ and thus $e_1 \cdot e_2 \in \delta_{n-2}$. We use this axiom again and find

$$\underbrace{e_2 \cdot [e_1, e_2]}_{=0} = \underbrace{[e_2 \cdot e_1, e_2]}_{=0} + [e_1, e_2 \cdot e_2],$$

so $e_2 \cdot e_2 \in \delta_n$. Now by axiom (CPA2),

$$e_3 \cdot e_1 = [e_1, e_2] \cdot e_1 = e_1 \cdot \underbrace{(e_2 \cdot e_1)}_{\in \delta_{n-2}} - e_2 \cdot \underbrace{(e_1 \cdot e_1)}_{\in \delta_2} \in 0 - \delta_n = \delta_n.$$

And now, finally,

$$0 = e_1 \cdot [e_1, e_3] = [e_1 \cdot e_1, e_3] + [e_1, \underbrace{e_1 \cdot e_3}_{\in \delta_n}] \in [\alpha e_2 + \delta_3, e_3] + 0 = \underbrace{\alpha}_{\neq 0} [e_2, e_3]$$

which means $[e_2, e_3] = 0$. So $\mathfrak{g} = L_n$. $\square$

So we completed now the case of Class B_1 and found out that $\mathfrak{g} \cong L_n$. There is one case left, namely CPA-structures in Class B_2. They satisfy $e_1 \cdot e_1 = \alpha e_2 + \delta_3, e_1 \cdot e_2 = e_3 + \delta_4, \alpha \neq 0$. We will show that if $x \cdot y$ is a CPA-structure in Class B_2 on $\mathfrak{g}$, then also $\mathfrak{g} \cong L_n$.

Lemma A.5. *Let $\mathfrak{g}$ be n-dimensional filiform, $x \cdot y$ a CPA-structure on $\mathfrak{g}$ of Class B_2 with $e_1 \cdot e_1 \notin \delta_3$ and $e_1 \cdot e_2 \notin \delta_4$ (by Lemma A.3, this implies $e_1 \cdot e_2 = e_3 + \delta_4$). Then we have (a) $e_1 \cdot e_i = e_{i+1} + \delta_{i+2}$ for all $i \notin \{1, n\}$, (b) $e_2 \cdot e_j \in \delta_{3+j}$ and (c) $[e_2, e_j] \in \delta_{3+j}$ for all $j \geq 2$.*

Proof. By induction on i, we will show

(i) $e_1 \cdot e_i \in e_{i+1} + \delta_{i+2}$

(ii) $e_{i-1} \cdot e_i \in \delta_{2i}$

(iii) $[e_{i-1}, e_i] \in \delta_{2i}$.

for all $i \in \{3, \ldots, n-1\}$. Note that by Lemma A.3, the statements are true for $i = 3$ (here note that by axiom (CPA3), if $e_1 \cdot e_2 \in e_3 + \delta_4$, then also $e_1 \cdot e_3 \in e_4 + \delta_5$). We assume it holds for i and show it for $i+1$:
Axiom (CPA3) on (e_{i-1}, e_1, e_i):

$$e_{i-1} \cdot e_{i+1} = [\underbrace{e_{i-1} \cdot e_1}_{\in \delta_i}, e_i] + [e_1, \underbrace{e_{i-1} \cdot e_i}_{\in \delta_{2i}}] \in \delta_{2i+1}.$$

Axiom (CPA2) on (e_1, e_{i-1}, e_i):

$$e_i \cdot e_i = e_1 \cdot (\underbrace{e_{i-1} \cdot e_i}_{\in \delta_{2i}}) - e_{i-1} \cdot (\underbrace{e_1 \cdot e_i}_{\in \delta_{i+1}}) \in \delta_{2i+1}.$$

Axiom (CPA2) on (e_1, e_i, e_i): Let us write $e_i \cdot e_{i+1} = \beta_1 e_{2i+1} + \delta_{2i+2}$.

$$\begin{aligned} \underbrace{e_{i+1} \cdot e_i}_{\beta_1 e_{2i+1} + \delta_{2i+2}} &= e_1 \cdot (e_i \cdot e_i) - e_i \cdot (e_1 \cdot e_i) \\ &\in e_1 \cdot \delta_{2i+1} - e_i \cdot (e_{i+1} + \delta_{i+2}) \subseteq \delta_{2i+2} - \beta_1 e_{2i+1} + \delta_{2i+2}, \end{aligned}$$

thus $\beta_1 = 0$ and so $e_i \cdot e_{i+1} \in \delta_{i+2}$.
Axiom (CPA3) on (e_i, e_1, e_i):

$$\underbrace{e_i \cdot e_{i+1}}_{\in \delta_{2i+2}} = \underbrace{[e_i \cdot e_1, e_i]}_{[e_{i+1}, e_i] + \delta_{2i+2}} + [e_1, \underbrace{e_i \cdot e_i}_{\in \delta_{2i+1}}],$$

meaning $[e_i, e_{i+1}] \in \delta_{2i+2}$.
This implies inductively (by the Jacobi identity) $[e_{i-1}, e_{i+1}] \in \delta_{2i+1}, [e_{i-2}, e_{i+1}] \in \delta_{2i}$, ..., up to $[e_2, e_{i+1}] \in \delta_{i+4}$ implying for $i < n-1$

$$e_1 \cdot e_{i+1} = [e_1 \cdot e_1, e_i] + [e_1, e_1 \cdot e_i] \in \delta_{i+3} + e_{i+2} + \delta_{i+3} \subseteq e_{i+2} + \delta_{i+3}.$$

This shows that (i), (ii) and (iii) also hold for $i+1$.

So (a) and (c) are proven; (b) is another induction using axiom (CPA3). □

Proposition A.6. *If $x \cdot y$ is a CPA-structure of Class B_2 on $\mathfrak{g}$ (where $\mathfrak{g}$ is filiform and n-dimensional), then $\mathfrak{g} \cong L_n$.*

Proof. We will show the following: Suppose for all $j, k \in \{3, \ldots, n\}$ and for an $\ell \in \{0, \ldots, n-5\}$, the following five conditions do hold:

(i) $[e_j, e_k] \in \delta_{j+k+\ell} + \delta_n$

(ii) $e_j \cdot e_k \in \delta_{j+k+\ell}$

(iii) $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+2} + \delta_n$ for $j \neq n-1, n$ (and $e_1 \cdot e_{n-1} = e_n$).

(iv) $e_2 \cdot e_j \in \delta_{3+j+\ell} + \delta_n$ for $j \geq 2$.

(v) $[e_2, e_j] \in \delta_{3+j+\ell}$ for $j \geq 3$.

Then these five conditions also hold for $\ell+1$, that is:

(i)' $[e_j, e_k] \in \delta_{j+k+\ell+1}$

(ii)' $e_j \cdot e_k \in \delta_{j+k+\ell+1}$

(iii)' $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+3} + \delta_n$ for $j \neq n-1, n$ (and $e_1 \cdot e_{n-1} = e_n$).

(iv)' $e_2 \cdot e_j \in \delta_{4+j+\ell}$ for $j \geq 2$.

(v)' $[e_2, e_j] \in \delta_{4+j+\ell}$ for $j \geq 3$.

(Again note that by Lemma A.5, these conditions are satisfied for $\ell = 0$).

We would like to do the proof similar to the one of Proposition A.2. However, there is a slight problem: When we proved (iii), we used that $e_1 \cdot e_1 \in \delta_3$. Since we now assume $x \cdot y$ to be in Class B, this is no longer the case. We therefore have to refine the proof.

With the same proof as of Proposition A.2 we can prove that (i)' and (ii)' hold. We now show that (iii)' does hold, i.e. $e_1 \cdot e_j \in e_{j+1} + \delta_{j+\ell+3} + \delta_n$ for $j \neq n$.
By assumptions (iii) and (iv) we can write

$$\begin{aligned} e_1 \cdot e_2 &\in e_3 + \beta_1 e_{4+\ell} + \delta_{\ell+5}, \\ e_2 \cdot e_2 &\in \gamma_1 e_{5+\ell} + \delta_{6+\ell}, \\ e_1 \cdot e_3 &\in e_4 + \beta_2 e_{5+\ell} + \delta_{\ell+6}, \\ e_2 \cdot e_3 &\in \gamma_2 e_{6+\ell} + \delta_{7+\ell} \end{aligned}$$

and so on, up to

$$\begin{aligned} e_1 \cdot e_{n-\ell-3} &\in e_{n-\ell-2} + \beta_{n-\ell-4} e_{n-1} + \delta_n, \\ e_2 \cdot e_{n-\ell-3} &\in \gamma_{n-\ell-4} e_n. \end{aligned}$$

Using axiom (CPA2) on the triple (e_1, e_2, e_1), we get

$$\begin{aligned} 0 &= e_3 \cdot e_1 - e_1 \cdot (e_2 \cdot e_1) + e_2 \cdot (e_1 \cdot e_1) \\ &\in e_4 + \beta_2 e_{5+\ell} + \delta_{6+\ell} - e_1 \cdot (e_3 + \beta_1 e_{4+\ell} + \delta_{5+\ell}) + e_2 \cdot (\alpha e_2 + \delta_3) \\ &\subseteq e_4 + \beta_2 e_{5+\ell} + \delta_{6+\ell} - e_4 - \beta_2 e_{5+\ell} - \beta_1 e_{5+\ell} + \delta_{6+\ell} + \alpha\gamma_1 e_{5+\ell} + \delta_{6+\ell} + \delta_n \\ &\subseteq -\beta_1 e_{5+\ell} + \alpha\gamma_1 e_{5+\ell} + \delta_{6+\ell} + \delta_n. \end{aligned}$$

So we have $\beta_1 = \alpha\gamma_1$ and similarly (by using (CPA2) for the triple (e_1, e_{s+1}, e_1)) that $\beta_s = \alpha\gamma_s$ for $1 \leq s \leq n-\ell-4$.
Axiom (CPA3) on the triple (e_1, e_1, e_2) yields $\gamma_2 = \gamma_1$ (more precisely, $\beta_1 = \beta_2$, but $\beta_i = \alpha\gamma_i$ and $\alpha \neq 0$), but axiom (CPA2) on the triple (e_2, e_1, e_2) yields $\gamma_1 = 2\gamma_2$. So $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$ meaning $e_1 \cdot e_2 \in e_3 + \delta_{\ell+5}, e_1 \cdot e_3 \in e_4 + \delta_{\ell+6}, e_2 \cdot e_2 \in \delta_{6+\ell}, e_2 \cdot e_3 \in$

$\delta_{7+\ell}$. However, this has consequences on the Lie bracket: By axiom (CPA3) on the triple (e_2, e_1, e_2), we have

$$\underbrace{e_2 \cdot e_3}_{\in \delta_{7+\ell}} = \underbrace{[e_2 \cdot e_1, e_2]}_{\in [e_3+\delta_{\ell+5}, e_2]} + \underbrace{[e_1, e_2 \cdot e_2]}_{\in \delta_{7+\ell}}$$

implying $[e_2, e_3] \in \delta_{\ell+7}$. The Jacobi identity then implies $[e_2, e_4] \in \delta_{\ell+8}$.

We need three more applications of the CPA-axioms. First, by axiom (CPA2) on (e_2, e_1, e_2) and (CPA3) on (e_1, e_1, e_4), we also find $\beta_3 = \beta_4 = 0$, then, by axiom (CPA3) on (e_2, e_1, e_4), we have $[e_3, e_4] \in \delta_{9+\ell}$. The Jacobi identity both implies $[e_2, e_j] \in \delta_{4+j+\ell}$ and $[e_3, e_j] \in \delta_{5+j+\ell}$ for all $j \geq 3$. Now we may finally prove that all of the β_i's and γ_i's are zero – this is an application of axiom (CPA3) to (e_1, e_1, e_{s+1}). This proves (iii)', (iv)' and (v)'.

This procedure works until $\ell = n-5$. This means, in the last step, we get the non-zero products

$$e_1 \cdot e_1 \in \delta_2, e_1 \cdot e_2 \in e_3 + \delta_{n-2}, e_1 \cdot e_3 \in e_4 + \delta_{n-1},$$
$$e_1 \cdot e_j \in e_{j+1} + \delta_n, j \geq 4; , e_2 \cdot e_2 \in \delta_{n-1}, e_2 \cdot e_3 \in \delta_n$$

and have that $[\delta_3, \delta_3] = [\delta_2, \delta_4] = 0$. To show that $\mathfrak{g} \cong L_n$, it remains to show $[e_2, e_3] = 0$ (up to now, we only have $[e_2, e_3] \in \delta_n$). This is similar to the end of Proposition A.4's proof; a sequence of applications of axioms (CPA2) and (CPA3) gives the result. $\square$

B Explicit post-Lie algebra structures

In this appendix, we want to state some proofs and classifications which have been obtained with help of the software Mathematica.

B.1 Existence of post-Lie algebra structures in dimension 3

Here, we want to prove Proposition 3.51 from Section 3.4, namely, which pairs of 3-dimensional Lie algebras admit a post-Lie algebra structure.

The situation in dimension 3 is manageable, mainly for two reasons: There are only "a few" Lie algebras in dimension 3 and moreover, there are many useful invariants of 3-dimensional Lie algebras like the one from [93], which can be computed via the Killing form of a given basis. This invariant helps to distinguish non-nilpotent Lie algebras in dimension 3 and is thus very useful for concrete calculations.

To prove Proposition 3.51, we give an example for each "existence" case. For our examples, if not said otherwise, we assume $\mathfrak{g}$ to have the basis $\{e_1, e_2, e_3\}$ with Lie brackets given in Table 16 and indicate the post-Lie algebra structure by stating the Lie brackets of $\mathfrak{n}$ and a possible post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ in terms of the left-multiplication operators $L(e_1), L(e_2), L(e_3)$.

Example B.1. *Let $\mathfrak{g} \cong \mathbb{C}^3$. We can find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ isomorphic to (i) $\mathbb{C}^3$, (ii) $\mathfrak{n}_3(\mathbb{C})$, (iii) $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, (iv) $\mathfrak{r}_3(\mathbb{C})$, (v) $\mathfrak{r}_{3,1}(\mathbb{C})$, (vi) $\mathfrak{r}_{3,\lambda}(\mathbb{C})$.*

Proof.

(i) The trivial zero-product: $\operatorname{ad}(e_1) = \operatorname{ad}(e_2) = \operatorname{ad}(e_3) = L(e_1) = L(e_2) = L(e_3) = 0$.

(ii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_3$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_1$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

Table 16: Lie algebras in dimension 3

$\mathfrak{g}$	non-zero Lie brackets
$\mathbb{C}^3$	–
$\mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3$ with $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \cong \mathfrak{r}_{3,\mu}(\mathbb{C})$ if and only if $\mu = \frac{1}{\lambda}$ or $\mu = \lambda$
$\mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_3$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(vi) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mu \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

$\square$

Example B.2. *Let $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C})$. We can find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ isomorphic to (i) $\mathbb{C}^3$, (ii) $\mathfrak{n}_3(\mathbb{C})$, (iii) $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, (iv) $\mathfrak{r}_3(\mathbb{C})$, (v) $\mathfrak{r}_{3,1}(\mathbb{C})$.*

Proof.

(i) $\mathfrak{n}$ the abelian Lie algebra, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L(e_2) = L(e_3) = 0.$$

(ii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_3$, post-Lie algebra structure: the trivial zero-product: $L(e_1) = L(e_2) = L(e_3) = 0$.

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L(e_3) = 0$$

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$, Lie brackets of $\mathfrak{g}$: $[e_1, e_3] = e_2$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_3$, Lie brackets of $\mathfrak{g}$: $[e_1, e_3] = e_2$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

Example B.3. *Let $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$. We can find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ isomorphic to (i) $\mathbb{C}^3$, (ii) $\mathfrak{n}_3(\mathbb{C})$, (iii) $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, (iv) $\mathfrak{r}_3(\mathbb{C})$, (v) $\mathfrak{r}_{3,\mu}(\mathbb{C})$ (μ may or may not be 1).*

Proof.

(i) $\mathfrak{n}$ the abelian Lie algebra, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

(ii) Lie brackets of $\mathfrak{n}$: $\{e_2, e_3\} = e_1$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_1$, post-Lie algebra structure: the trivial zero-product: $L(e_1) = L(e_2) = L(e_3) = 0$.

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_3\} = e_1 + e_2, \{e_2, e_3\} = e_2$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = e_2$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = 0, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ \mu & 0 & -1 \end{pmatrix}$$

□

Example B.4. *Let* $\mathfrak{g} \cong \mathfrak{r}_3(\mathbb{C})$. *We can find post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{n}$ *isomorphic to (i)* $\mathbb{C}^3$, *(ii)* $\mathfrak{n}_3(\mathbb{C})$, *(iii)* $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, *(iv)* $\mathfrak{r}_3(\mathbb{C})$, *(v)* $\mathfrak{r}_{3,1}(\mathbb{C})$.

Proof.

(i) $\mathfrak{n}$ the abelian Lie algebra, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = 0, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) Lie brackets of $\mathfrak{n}$: $\{e_2, e_3\} = e_1$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_1$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = -e_1 + e_3, [e_2, e_3] = e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_3) = 0.$$

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$, post-Lie algebra structure: the trivial zero-product: $L(e_1) = L(e_2) = L(e_3) = 0$.

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

□

Example B.5. *Let* $\mathfrak{g} \cong \mathfrak{r}_{3,1}(\mathbb{C})$. *We can find post-Lie algebra structures on* $(\mathfrak{g}, \mathfrak{n})$ *with* $\mathfrak{n}$ *isomorphic to (i)* $\mathbb{C}^3$, *(ii)* $\mathfrak{n}_3(\mathbb{C})$, *(iii)* $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, *(iv)* $\mathfrak{r}_3(\mathbb{C})$, *(v)* $\mathfrak{r}_{3,1}(\mathbb{C})$, *(vi)* $\mathfrak{r}_{3,\mu}, \mu \neq 1$, *(vii)* $\mathfrak{sl}_2(\mathbb{C})$.

Proof.

(i) $\mathfrak{n}$ the abelian Lie algebra, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

(ii) Lie brackets of $\mathfrak{n}$: $\{e_2, e_3\} = e_1$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = 0, L(e_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_1$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = -e_1, [e_2, e_3] = e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_3) = 0.$$

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_3$, post-Lie algebra structure: the trivial zero-product: $L(e_1) = L(e_2) = L(e_3) = 0$.

(vi) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = 0, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}.$$

(vii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_3, \{e_1, e_3\} = -2e_1, \{e_2, e_3\} = 2e_2$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = e_1, [e_1, e_3] = -e_1, [e_2, e_3] = -e_2 - e_3$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$\square$

Example B.6. *Let $\mathfrak{g} \cong \mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \neq 1$. We can find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ isomorphic to (i) $\mathbb{C}^3$, (ii) $\mathfrak{n}_3(\mathbb{C})$, (iii) $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, (iv) $\mathfrak{r}_3(\mathbb{C})$, (v) $\mathfrak{r}_{3,\mu}$ (where μ may or may not be 1), (vi) $\mathfrak{sl}_2(\mathbb{C})$. Case (vii) is not possible if $\mu = -1$.*

Proof.

(i) $\mathfrak{n}$ the abelian Lie algebra, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, L(e_2) = 0, L(e_3) = 0.$$

(ii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_3\} = e_2$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1/2 \\ 0 & 0 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1-\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 1-\lambda & 0 & 0 \end{pmatrix}$$

(iii) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_1$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = -\frac{1}{\lambda} e_1, [e_2, e_3] = e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L(e_3) = 0.$$

(iv) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = e_2 + e_3$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = (-1 + \frac{1}{1-\lambda}) e_2, [e_1, e_3] = (\frac{1}{1-\lambda}) e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda/(1-\lambda) & -1 \\ 0 & 0 & \lambda/(1-\lambda) \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = 0.$$

(v) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$, left-multiplication operators:

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, L(e_2) = 0, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}$$

(vi) Lie brackets of $\mathfrak{n}$: $\{e_1, e_2\} = e_3, \{e_1, e_3\} = -2e_1, \{e_2, e_3\} = 2e_2$, Lie brackets of $\mathfrak{g}$: $[e_1, e_2] = -\lambda e_1, [e_1, e_3] = -\frac{2\lambda}{\lambda+1} e_1, [e_2, e_3] = \frac{1-\lambda^2}{2} e_1 - \frac{2}{\lambda+1} e_2 + e_3$, left-multiplication operators:

$$L(e_1) = 0, L(e_2) = \begin{pmatrix} \lambda & 0 & \frac{1-\lambda^2}{2} \\ 0 & -\lambda & -2 \\ 1 & \frac{\lambda^2-1}{4} & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} -\frac{2}{\lambda+1} & 0 & 2 \\ 0 & \frac{2}{\lambda+1} & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

This does work if and only if $\lambda \neq -1$.

□

Example B.7. *Let $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$. We can find post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ if $\mathfrak{n}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. An example is the trivial post-Lie algebra structure, where $\mathfrak{n}$'s Lie brackets are $\{e_1, e_2\} = e_3, \{e_1, e_3\} = -2e_1, \{e_2, e_3\} = 2e_3$ and the left-multiplication operators are $L(e_1) = L(e_2) = L(e_3) = 0$. □

Remark B.8. This structure and the other trivial post-Lie algebra structure are the only ones on $(\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C}))$ (cf. Example 2.42).

All together, these examples prove (iv) of Proposition 3.51.
It remains to prove the non-existence of post-Lie algebra structures on certain combinations of Lie algebras, namely cases (i), (ii), (iii) of Proposition 3.51:

Proof of Proposition 3.51, cases (i), (ii), (iii).

(i) This holds because of Theorem 3.20: If there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is simple, then $\mathfrak{n} \cong \mathfrak{g}$.

(ii) This is exactly the statement of Example 2.44. (Alternatively, it also follows from Proposition 2.48.)

(iii) If $\mathfrak{n} \cong \mathfrak{r}_{3,\mu}(\mathbb{C}), \mu \neq 1$ and $\mathfrak{g} \cong \mathfrak{n}_3(\mathbb{C})$, we can assume $\mathfrak{n}$ to have a basis $\{e_1, e_2, e_3\}$ with $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$. As $\mathfrak{g}$ is nilpotent, all its adjoint operators are too and thus their trace vanishes; we may write

$$\operatorname{ad}(e_1) = \begin{pmatrix} 0 & r_1 & r_4 \\ 0 & r_2 & r_5 \\ 0 & r_3 & -r_2 \end{pmatrix}, \operatorname{ad}(e_2) = \begin{pmatrix} -r_1 & 0 & r_6 \\ -r_2 & 0 & -r_4 \\ -r_3 & 0 & r_1 \end{pmatrix}, \operatorname{ad}(e_3) = \begin{pmatrix} -r_4 & -r_6 & 0 \\ -r_5 & r_4 & 0 \\ r_2 & -r_1 & 0 \end{pmatrix}$$

for the adjoint operators with respect to $[,]$ and e_1, e_2, e_3.
After applying axioms (PA1) and (PA3) we find (using $\mu \neq 1$)

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_5 & \alpha_1 \\ \alpha_6 & 0 & \alpha_7 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_{10} & 0 \\ \alpha_4 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_3 & 0 & 0 \\ \alpha_8 & 0 & \alpha_9 \end{pmatrix}$$

and $r_1 = r_4 = r_6 = 0, r_2 = \alpha_1 - \alpha_2 + 1, r_3 = -\alpha_4, r_5 = -\alpha_3$ for some $\alpha_1, \ldots, \alpha_{10} \in \mathbb{C}$ subject to the condition $\mu = -1 - \alpha_1 + \alpha_2 - \alpha_7 + \alpha_8$. From the fact that $(\operatorname{ad}(e_1))^2 = 0$ combined with axiom (PA2) one obtains that either $\mathfrak{g}$ is abelian or $\mu = 1$, both is a contradiction to our assumption.
The case where $\mathfrak{n} \cong \mathfrak{r}_{3,\mu}(\mathbb{C}), \mu \neq 1$ and $\mathfrak{g} \cong \mathfrak{r}_3(\mathbb{C})$ is similar; when one fixes $\mathfrak{n}$'s basis $\{e_1, e_2, e_3\}$ again to satisfying $\{e_1, e_2\} = e_2, \{e_1, e_3\} = \mu e_3$, one arrives (relatively quickly, by making use of Tasaki's and Umehara's invariant χ (see [93]), for which one knows $\chi(\mathfrak{r}_3(\mathbb{C})) = 4$) at a contradiction (meaning either $\mathfrak{g} \cong \mathfrak{r}_{3,1}(\mathbb{C})$ or $\mu = 1$ or $\dim([\mathfrak{g}, \mathfrak{g}]) < 2$).

□

Remark B.9. These calculations are fairly manageable in dimension 3. In dimension 4, however, due to many factors (e.g. the larger number of Lie algebra families and less good invariants to distinguish Lie algebras efficiently in calculations), they become quite hard and complicated.

B.2 Explicit description of all CPA-structures in dimension 3

In this section, we want to state explicitly all commutative post-Lie algebra structures in dimension 3 for non-abelian Lie algebras.

We will write down the post-Lie algebra structures by indicating the left-multiplications operators $L(e_1), L(e_2), L(e_3)$ with respect to the basis given in Table 16.

Proposition B.10. *The Lie algebra $\mathfrak{n}_3(\mathbb{C})$ has the following CPA-structures:*

$$\text{(i)}\ \ L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ a_1 & a_2 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & a_5 & 0 \\ 0 & 0 & 0 \\ a_2 & a_3 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$a_1a_5 = a_4a_5 = a_3a_4 = 0.$$

$$\text{(ii)}\ \ L(e_1) = \begin{pmatrix} a_2^2/a_1 & -a_2 & 0 \\ a_2^3/a_1^2 & -a_2^2/a_1 & 0 \\ -\frac{a_2(2a_1a_3+a_2a_4)}{a_1^2} & a_3 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} -a_2 & a_1 & 0 \\ -a_2^2/a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \end{pmatrix},$$
$$L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_1 \neq 0.$$

Apart from the conditions $a_1a_5 = a_4a_5 = a_3a_4 = 0$ and $a_1 \neq 0$ in the first and in the second case, respectively, all variables can be chosen arbitrarily in $\mathbb{C}$.

Proposition B.11. *The Lie algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ has the following CPA-structures:*

$$\text{(i)}\ \ L(e_1) = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & a_3 & a_4 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_4 & a_5 \end{pmatrix},$$

where the variables can be chosen arbitrarily in $\mathbb{C}$ subject to the conditions $a_1(a_1+1) = 0$ and $a_4^2 = a_3 \cdot a_5$.

Proposition B.12. *The Lie algebra $\mathfrak{r}_3(\mathbb{C})$ has the following CPA-structures:*

$$\text{(i)}\ \ L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_3 & a_3 \\ a_2 & 0 & a_3 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix}$$

with $a_1, a_2 \in \mathbb{C}, a_3 \in \{0, 1\}$.

Proposition B.13. *The Lie algebra $\mathfrak{r}_{3,\lambda}(\mathbb{C}), \lambda \neq 1$ has the following CPA-structures:*

(i) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_2 & 0 \\ a_3 & 0 & a_4 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & 0 & 0 \end{pmatrix},$$

where $a_2(a_2-1)=0$ and $a_4(a_4-\lambda)=0$. Apart from that condition, the variables can be chosen to be any complex numbers.

Proposition B.14. *The Lie algebra $\mathfrak{r}_{3,1}(\mathbb{C})$ has the following CPA-structures:*

(i) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_3 & 0 \\ a_2 & 0 & a_3 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, a_3 \in \{0,1\}.$$

(ii) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1a_2 & 0 & a_2 \\ a_1 & 0 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_2a_3 & -a_2^2a_3 \\ 0 & a_3 & -a_2a_3 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & -a_2^2a_3 & a_2^3a_3 \\ 1 & -a_2a_3 & a_2^2a_3 \end{pmatrix}.$$

(iii) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & 1 & a_3 \\ a_2 & 0 & 0 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & 0 & a_4 \\ 0 & 0 & 0 \end{pmatrix}, a_2a_4 = 0.$$

(iv) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_3 & \frac{a_3-a_3^2}{a_4} \\ a_2 & a_4 & 1-a_3 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{a_3-a_3^2}{a_4} & 0 & 0 \\ 1-a_3 & 0 & 0 \end{pmatrix},$$

$a_4 \neq 0$.

(v) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & 0 & a_4 \\ a_2 & a_3 & 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & a_5 & 0 \end{pmatrix}, L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$a_1a_5 = a_3a_4 = a_4a_5 = 0$.

(vi) $$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ -a_1a_2/a_3 & a_4 & a_2(a_4-1)/a_3 \\ a_1 & -a_3a_4/a_2 & 1-a_4 \end{pmatrix}, L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ a_4 & -a_2 & -a_2^2/a_3 \\ -a_3a_4/a_2 & a_3 & a_2 \end{pmatrix},$$

$$L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ a_2(a_4-1)/a_3 & -a_2^2/a_3 & -a_2^3/a_3^2 \\ 1-a_4 & a_2 & a_2^2/a_3 \end{pmatrix}, a_2a_3 \neq 0.$$

Apart from the stated conditions, the variables can be chosen arbitrarily in $\mathbb{C}$.

Proposition B.15. *The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has only the following CPA-structure:*

(i) $$L(e_1) = L(e_2) = L(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark B.16. That $\mathfrak{sl}_2(\mathbb{C})$ only has the trivial CPA-structure is not surprising as $\mathfrak{sl}_2(\mathbb{C})$ is simple (cf. Theorem 3.20).

B.3 Post-Lie algebra structures on Lie algebra double pairs with $\mathfrak{n}$ complete

In Section 4.4, we studied post-Lie algebra structures on Lie algebra double pairs with a complete Lie algebra $\mathfrak{n}$, i.e., post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is complete and there is a linear map $R\colon V \to V$ satisfying $[x, y] = R(\{x, y\})$ for all $x, y \in \mathfrak{g}$. Now, we want to list explicitly all such structures for simply-complete Lie algebras $\mathfrak{n}$ up to dimension 7.
Note that since $\mathfrak{n}$ is complete, there is a linear map $\varphi\colon \mathfrak{n} \to \mathfrak{n}$ with $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in \mathfrak{n}$. Therefore, we will list the post-Lie algebra structures by listing the possible pairs (φ, R).

We use the classification of complete Lie algebras given in Section 4.2 and study all simply-complete, non-semisimple, non-metabelian Lie algebras $\mathfrak{n}$ up to dimension 7. (For the metabelian simply-complete Lie algebra, see Lemma 4.76, for semisimple Lie algebras, see [23].) Post-Lie algebra structures on Lie algebra double pairs where $\mathfrak{n}$ is not simply-complete can be reduced to the ones on Lie algebra double pairs where $\mathfrak{n}$ is simply-complete (cf. Proposition 4.16 and Lemma 4.46).
The definition of these Lie algebras can be found in Tables 12 (dimension 5), 13 (dimension 6) and 14 (dimension 7) on pages 62, 65 and 68, respectively.
Note that for dimensions less than 5, there are only three complete Lie algebras: the simple $\mathfrak{sl}_2(\mathbb{C})$ and the two metabelian Lie algebras $\mathfrak{r}_2(\mathbb{C})$ and $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$ (so we do not include them here).

5-dimensional Lie algebras

Proposition B.17. *If $\mathfrak{n} = T_2 \ltimes \mathfrak{n}_3$, then all Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}, R)$ with a post-Lie algebra structure are given (with respect to the basis $\{e_1, e_2, e_3, t_1, t_2\}$) by the following:*

$$\text{(i)}\ \ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & r_1 & r_6 \\ 0 & 1 & 0 & r_2 & r_7 \\ 0 & 0 & 1 & r_3 & r_8 \\ 0 & 0 & 0 & r_4 & r_9 \\ 0 & 0 & 0 & r_5 & r_{10} \end{pmatrix}$$

$$\text{(ii)}\ \ \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 0 & r_1 & r_6 \\ 0 & -1 & 0 & r_2 & r_7 \\ 0 & 0 & 0 & r_3 & r_8 \\ 0 & 0 & 0 & r_4 & r_9 \\ 0 & 0 & 0 & r_5 & r_{10} \end{pmatrix}$$

(iii) $\varphi_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & c & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 & 0 & r_1 & r_6 \\ 0 & 0 & 0 & r_2 & r_7 \\ 0 & 0 & 0 & r_3 & r_8 \\ 0 & 0 & 0 & r_4 & r_9 \\ 0 & 0 & 0 & r_5 & r_{10} \end{pmatrix}$

(iv) $\varphi_4 = -\varphi_3 - \mathrm{Id}, R_4 = -R_3$

(v) $\varphi_5 = -\varphi_2 - \mathrm{Id}, R_5 = -R_2$

(vi) $\varphi_6 = -\varphi_1 - \mathrm{Id}, R_6 = -R_1$

All variables are arbitrary scalars.
In cases (i) and (vi), $\mathfrak{g} \cong T_2 \ltimes \mathfrak{n}_3 = \mathfrak{n}$*, in the other cases,* $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^3$.

6-dimensional Lie algebras

Proposition B.18. *If* $\mathfrak{n} = T_2 \ltimes \mathfrak{g}_4$*, then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, t_1, t_2\}$*) by the following:*

(i) $\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & r_1 & r_7 \\ 0 & 1 & 0 & 0 & r_2 & r_8 \\ 0 & 0 & 1 & 0 & r_3 & r_9 \\ 0 & 0 & 0 & 1 & r_4 & r_{10} \\ 0 & 0 & 0 & 0 & r_5 & r_{11} \\ 0 & 0 & 0 & 0 & r_6 & r_{12} \end{pmatrix}$

(ii) $\varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & c_2 \\ c_2 & 0 & -1 & 0 & c_3 & c_3 \\ c_3 & 0 & 0 & -1 & 2c_1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & r_1 & r_7 \\ 0 & 0 & 0 & 0 & r_2 & r_8 \\ c_2 & 0 & 0 & 0 & r_3 & r_9 \\ c_3 & 0 & 0 & 0 & r_4 & r_{10} \\ 0 & 0 & 0 & 0 & r_5 & r_{11} \\ 0 & 0 & 0 & 0 & r_6 & r_{12} \end{pmatrix}$

(iii) $\varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$

(iv) $\varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$

All variables are arbitrary scalars.
In cases (i) and (iv), $\mathfrak{g} \cong T_2 \ltimes \mathfrak{g}_4 = \mathfrak{n}$*, in the other cases,* $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^4$.

Proposition B.19. *If* $\mathfrak{n} = T_1 \ltimes \mathfrak{g}_{5,6}$*, then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, t_1\}$*) by the following:*

$$\text{(i)}\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2c_1 \\ 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 1 & 0 & 0 & r_3 \\ 2c_1 & 0 & 0 & 1 & 0 & r_4 \\ c_2 & c_1 & 0 & 0 & 1 & r_5 \\ 0 & 0 & 0 & 0 & 0 & r_6 \end{pmatrix}$$

$$\text{(ii)}\ \varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_1 \ltimes \mathfrak{g}_{5,6} = \mathfrak{n}$.

And there is one non-solvable, non-semisimple complete Lie algebra in dimension 6:

Proposition B.20. *If* $\mathfrak{n} = \mathfrak{aff}(\mathbb{C}^2)$*, then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6\}$*) by the following:*

$$\text{(i)}\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} c_1 & 0 & 0 & c_1 - 1 & 0 & 0 \\ c_2 & 1 & 0 & c_2 & 0 & 0 \\ c_3 & 0 & 1 & c_3 & 0 & 0 \\ c_4 & 0 & 0 & c_4 + 1 & 0 & 0 \\ c_5 & 0 & 0 & c_5 & 1 & 0 \\ c_6 & 0 & 0 & c_6 & 0 & 1 \end{pmatrix}$$

$$\text{(ii)}\ \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c_5 - c_7 & c_8 & 0 & 0 & -1 & 0 \\ 0 & 0 & c_5 - c_7 & c_8 & 0 & -1 \end{pmatrix}, R_2 = \begin{pmatrix} c_1 & 0 & 0 & c_1 - 1 & 0 & 0 \\ c_2 & 1 & 0 & c_2 & 0 & 0 \\ c_3 & 0 & 1 & c_3 & 0 & 0 \\ c_4 & 0 & 0 & c_4 + 1 & 0 & 0 \\ c_5 & c_8 & 0 & c_7 & 0 & 0 \\ c_6 & 0 & c_5 - c_7 & c_6 + c_8 & 0 & 0 \end{pmatrix}$$

$$\text{(iii)}\ \varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$$

$$\text{(vi)}\ \varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$$

All variables are arbitrary scalars.
In cases (i) and (iv), $\mathfrak{g} \cong \mathfrak{aff}(\mathbb{C}^2) = \mathfrak{n}$.
In the other cases, $\mathfrak{g}$ *is isomorphic to a Lie algebra of the form* $\mathfrak{gl}_2(\mathbb{C}) \ltimes \mathbb{C}^2$.

Remark B.21. The fact that these structures look different compared to the other ones is due to $\mathfrak{aff}(\mathbb{C}^2)$ being expressed in terms of a basis which does not highlight the torus of $\mathfrak{aff}(\mathbb{C}^2)$'s radical. (Remember that by Proposition 4.20, we can write every rigid Lie algebra L as $L = \mathfrak{s} \ltimes (T \ltimes \mathfrak{m})$, where $\mathfrak{s}$ is a Levi subalgebra, T a torus of the radical of L and $\mathfrak{m}$ the nilradical of L.) In fact, this decomposition appears in the proof of Lemma 4.30 – a basis of $\mathfrak{aff}(\mathbb{C}^2)$ is given by $\{b_1, \dots, b_6\}$ with Lie brackets

$[b_1, b_2] = b_3, [b_1, b_3] = -2b_1, [b_1, b_6] = b_5, [b_2, b_3] = 2b_2, [b_2, b_5] = b_6, [b_3, b_5] = b_5, [b_3, b_6] = -b_6, [b_4, b_5] = b_5, [b_4, b_6] = b_6$. Thus, one sees that

$$\operatorname{span}\{b_1, b_2, b_3\} = \mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C}), \operatorname{span}\{b_4\} = T \cong \mathbb{C}, \operatorname{span}\{b_5, b_6\} = \mathfrak{m} \cong \mathbb{C}^2.$$

With respect to this basis, the four different types of post-Lie algebra structures are:

$$\text{(i)}\ \ \varphi_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, R_1 = \begin{pmatrix} 1&0&0&r_1&0&0\\ 0&1&0&r_2&0&0\\ 0&0&1&r_3&0&0\\ 0&0&0&r_4&0&0\\ 0&0&0&r_5&1&0\\ 0&0&0&r_6&0&1 \end{pmatrix}$$

$$\text{(ii)}\ \ \varphi_2 = \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ c_1&0&c_2&c_2&0&0\\ 0&c_2&c_1&-c_1&0&0 \end{pmatrix}, R_2 = \begin{pmatrix} -1&0&0&r_1&0&0\\ 0&-1&0&r_2&0&0\\ 0&0&-1&r_3&0&0\\ 0&0&0&r_4&0&0\\ -c_1&0&c_2&r_5&0&0\\ 0&c_2&c_1&r_6&0&0 \end{pmatrix}$$

(iii) $\varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$

(vi) $\varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$,

where all variables are arbitrary scalars. (Compare these structures to those of the 7-dimensional Lie algebras $L_{7,4}$ and $L_{7,5}$ in Proposition B.29 and Proposition B.30, respectively.)

7-dimensional Lie algebras

Proposition B.22. *If $\mathfrak{n} = T_2 \ltimes \mathfrak{g}_{5,3}$, then all Lie algebra double pairs $(\mathfrak{g}, \mathfrak{n}, R)$ with a post-Lie algebra structure are given (with respect to the basis $\{e_1, e_2, e_3, e_4, e_5, t_1, t_2\}$) by the following:*

$$\textit{(i)}\ \ \varphi_1 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&2c_1&c_1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, R_1 = \begin{pmatrix} 1&0&0&0&0&r_1&r_8\\ 0&1&0&0&0&r_2&r_9\\ 0&0&1&0&0&r_3&r_{10}\\ 0&0&0&1&0&r_4&r_{11}\\ 0&0&0&0&1&r_5&r_{12}\\ 0&0&0&0&0&r_6&r_{13}\\ 0&0&0&0&0&r_7&r_{14} \end{pmatrix}$$

$$\textit{(ii)}\ \ \varphi_2 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&-c_2\\ 0&0&0&0&0&0&0\\ -c_2&0&0&-1&0&c_3&c_3\\ c_3&0&c_2&0&-1&2c_1&c_1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, R_2 = \begin{pmatrix} 1&0&0&0&0&r_1&r_8\\ 0&0&0&0&0&r_2&r_9\\ 0&0&1&0&0&r_3&r_{10}\\ -c_2&0&0&0&0&r_4&r_{11}\\ c_3&0&c_2&0&0&r_5&r_{12}\\ 0&0&0&0&0&r_6&r_{13}\\ 0&0&0&0&0&r_7&r_{14} \end{pmatrix}$$

(iii) $\varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$

(iv) $\varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$

All variables are arbitrary scalars.
In cases (i) and (iv), $\mathfrak{g} \cong T_2 \ltimes \mathfrak{g}_{5,3} = \mathfrak{n}$, *in the other cases,* $\mathfrak{g}$ *is isomorphic to a Lie algebra of the form* $(\mathfrak{r}_{3,\frac{1}{2}}(\mathbb{C}) \oplus \mathbb{C}^2) \ltimes \mathbb{C}^2$.

Proposition B.23. *If* $\mathfrak{n} = T_2 \ltimes \mathfrak{g}_{5,4}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, t_1, t_2\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 2c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & r_1 & r_8 \\ 0 & 1 & 0 & 0 & 0 & r_2 & r_9 \\ 0 & 0 & 1 & 0 & 0 & r_3 & r_{10} \\ 0 & 0 & 0 & 1 & 0 & r_4 & r_{11} \\ 0 & 0 & 0 & 0 & 1 & r_5 & r_{12} \\ 0 & 0 & 0 & 0 & 0 & r_6 & r_{13} \\ 0 & 0 & 0 & 0 & 0 & r_7 & r_{14} \end{pmatrix}$$

(ii) $\varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_2 \ltimes \mathfrak{g}_{5,4} = \mathfrak{n}$.

Proposition B.24. *If* $\mathfrak{n} = T_2 \ltimes \mathfrak{g}_{5,5}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, t_1, t_2\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3c_1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & r_1 & r_8 \\ 0 & 1 & 0 & 0 & 0 & r_2 & r_9 \\ 0 & 0 & 1 & 0 & 0 & r_3 & r_{10} \\ 0 & 0 & 0 & 1 & 0 & r_4 & r_{11} \\ 0 & 0 & 0 & 0 & 1 & r_5 & r_{12} \\ 0 & 0 & 0 & 0 & 0 & r_6 & r_{13} \\ 0 & 0 & 0 & 0 & 0 & r_7 & r_{14} \end{pmatrix}$$

$$(ii)\ \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & c_2 \\ c_2 & 0 & -1 & 0 & 0 & c_3 & c_3 \\ c_3 & 0 & 0 & -1 & 0 & 2c_4 & c_4 \\ c_4 & 0 & 0 & 0 & -1 & 3c_1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & r_1 & r_8 \\ 0 & 0 & 0 & 0 & 0 & r_3 & r_9 \\ c_2 & 0 & 0 & 0 & 0 & r_3 & r_{10} \\ c_3 & 0 & 0 & 0 & 0 & r_4 & r_{11} \\ c_4 & 0 & 0 & 0 & 0 & r_5 & r_{12} \\ 0 & 0 & 0 & 0 & 0 & r_6 & r_{13} \\ 0 & 0 & 0 & 0 & 0 & r_7 & r_{14} \end{pmatrix}$$

(iii) $\varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$

(iv) $\varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$

All variables are arbitrary scalars.
In cases (i) and (iv), $\mathfrak{g} \cong T_2 \ltimes \mathfrak{g}_{5,5} = \mathfrak{n}$, *in the other cases,* $\mathfrak{g} \cong \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^5$.

Proposition B.25. *If* $\mathfrak{n} = T_1 \ltimes \mathcal{G}_{6,12}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, t_1\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 - 2c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & r_4 \\ c_1 & 0 & 0 & 0 & 1 & 0 & r_5 \\ c_2 & c_3 & 0 & 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 \end{pmatrix}$$

$$(ii)\ \varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_1 \ltimes \mathcal{G}_{6,12} = \mathfrak{n}$.

Proposition B.26. *If* $\mathfrak{n} = T_1 \ltimes \mathcal{G}_{6,17}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, t_1\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & r_3 \\ 3c_1 & 0 & 0 & 1 & 0 & 0 & r_4 \\ -c_2 & 0 & 0 & 0 & 1 & 0 & r_5 \\ -c_3 & c_1 & 0 & 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 \end{pmatrix}$$

$$(ii)\ \varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_1 \ltimes \mathcal{G}_{6,17} = \mathfrak{n}$.

Proposition B.27. *If* $\mathfrak{n} = T_1 \ltimes \mathcal{G}_{6,19}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, t_1\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & r_3 \\ 2c_1 & 0 & 0 & 1 & 0 & 0 & r_4 \\ 2c_2 & c_1 & 0 & 0 & 1 & 0 & r_5 \\ -c_3 & c_2 & 0 & 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 \end{pmatrix}$$

(ii) $\varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_1 \ltimes \mathcal{G}_{6,19} = \mathfrak{n}$.

Proposition B.28. *If* $\mathfrak{n} = T_1 \ltimes \mathcal{G}_{6,20}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, t_1\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & r_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & r_5 \\ 0 & c_1 & 0 & 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 \end{pmatrix}$$

(ii) $\varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong T_1 \ltimes \mathcal{G}_{6,20} = \mathfrak{n}$.

In dimension 7, we additionally have two non-solvable (non-semisimple) complete Lie algebras, namely $L_{7,4}$ and $L_{7,5}$:

The Lie algebra $L_{7,4}$ can be decomposed as $L_{7,4} = \mathfrak{s} \ltimes (T \ltimes \mathfrak{m})$ (with $\mathfrak{s}$ semisimple, T a torus of the radical, $\mathfrak{m}$ the nilradical), where $\mathfrak{s} = \mathrm{span}\{e_1, e_2, e_3\} \cong \mathfrak{sl}_2(\mathbb{C}), T = \mathrm{span}\{e_4\} \cong \mathbb{C}, \mathfrak{m} = \mathrm{span}\{e_5, e_6, e_7\} \cong \mathbb{C}^3$.

Proposition B.29. *If* $\mathfrak{n} = L_{7,4}$, *then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$*) by the following:*

$$(i)\ \varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & r_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & r_6 & 0 & 1 & 0 \\ 0 & 0 & 0 & r_7 & 0 & 0 & 1 \end{pmatrix}$$

$$(ii)\ \varphi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2c_3 & 0 & 2c_1 & c_1 & -1 & 0 & 0 \\ c_2 & c_1 & 0 & c_3 & 0 & -1 & 0 \\ 0 & 2c_3 & -2c_2 & c_2 & 0 & 0 & -1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 & r_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 & 0 & 0 \\ 2c_3 & 0 & 2c_1 & r_5 & 0 & 0 & 0 \\ c_2 & c_1 & 0 & r_6 & 0 & 0 & 0 \\ 0 & 2c_3 & -2c_2 & r_7 & 0 & 0 & 0 \end{pmatrix}$$

(iii) $\varphi_3 = -\varphi_2 - \mathrm{Id}, R_3 = -R_2$

(iv) $\varphi_4 = -\varphi_1 - \mathrm{Id}, R_4 = -R_1$

All variables are arbitrary scalars.
In cases (i) and (iv), $\mathfrak{g} \cong L_{7,4} = \mathfrak{n}$*, in the other cases,* $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^4$.

The Lie algebra $L_{7,5}$ can be decomposed as $L_{7,5} = \mathfrak{s} \ltimes (T \ltimes \mathfrak{m})$ (with $\mathfrak{s}$ semisimple, T a torus of the radical, $\mathfrak{m}$ the nilradical), where $\mathfrak{s} = \mathrm{span}\{e_1, e_2, e_3\} \cong \mathfrak{sl}_2(\mathbb{C}), T = \mathrm{span}\{e_4\} \cong \mathbb{C}, \mathfrak{m} = \mathrm{span}\{e_5, e_6, e_7\} \cong \mathfrak{n}_3$ (the Heisenberg Lie algebra in dimension 3).

Proposition B.30. *If* $\mathfrak{n} = L_{7,5}$*, then all Lie algebra double pairs* $(\mathfrak{g}, \mathfrak{n}, R)$ *with a post-Lie algebra structure are given (with respect to the basis* $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$*) by the following:*

(i) $$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 & r_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & r_6 & 0 & 1 & 0 \\ 0 & 0 & 0 & r_7 & 0 & 0 & 1 \end{pmatrix}$$

(ii) $\varphi_2 = -\varphi_1 - \mathrm{Id}, R_2 = -R_1$

All variables are arbitrary scalars.
In both cases, $\mathfrak{g} \cong L_{7,5} = \mathfrak{n}$.

Bibliography

[1] Herbert Abels, *Properly Discontinuous Groups of Affine Transformations: A Survey*, Geometriae Dedicata **87** (2001), no. 1-3, 309–333.

[2] Herbert Abels, Gregory A. Margulis, and Gregory A. Soifer, *The Auslander conjecture for dimension less than 7*, arXiv: 1211.2525v4 [math.GR] (2013).

[3] José M. Ancochea Bermúdez and Rutwig Campoamor Stursberg, *Completable filiform Lie algebras*, Linear Algebra and Its Applications **367** (2003), 185–191.

[4] Louis Auslander, *Examples of Locally Affine Spaces*, Annals of Mathematics, Second Series **64** (1956), no. 2, 255–259.

[5] Louis Auslander, *The structure of compact locally affine manifolds*, Topology **3** (1964), 131–139.

[6] Louis Auslander, *Simply Transitive Groups of Affine Motions*, American Journal of Mathematics **99** (1977), no. 4, 809–826.

[7] Louis Auslander and Lawrence Markus, *Holonomy of Flat Affinely Connected Manifolds*, Annals of Mathematics **62** (1955), no. 1, 139–151.

[8] Yuri Bahturin, Michel Goze, and Elisabeth Remm, *Group Gradings on Filiform Lie algebras*, Communications in Algebra **44** (2016), no. 1, 40–62.

[9] Ignacio Bajo, *Maximal isotropy groups of Lie groups related to nilradicals of parabolic subalgebras*, Tohoku Mathematical Journal **52** (2000), no. 1, 19–29.

[10] Oliver Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology **43** (2004), no. 4, 903–924.

[11] Saïd Benayadi and Mohamed Boucetta, *Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures*, Differential Geometry and its Applications **36** (2014), 66–89.

[12] Yves Benoist, *Une nilvariété non affine*, Journal of Differential Geometry **41** (1995), no. 1, 21–52 (in French).

[13] Jean-Paul Benzécri, *Variétés localement affines*, Séminaire Ehresmann. Topologie et géométrie différentielle **2** (1959), no. 7, 42 pp (in French).

[14] Ludwig Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume. (Erste Abhandlung)*, Mathematische Annalen **70** (1911), 297–336 (in German).

[15] Ludwig Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume. (Zweite Abhandlung.) Die Gruppen mit einem endlichen Fundamentalbereich*, Mathematische Annalen **72** (1912), 400–412 (in German).

[16] Anthony D. Blaom, *The Lie algebroid associated with a hypersurface*, arXiv:1702.03452v2 [math.DG] (2017).

[17] Nicolas Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie*, Hermann Paris, 1960 (in French).

[18] Dietrich Burde, *Affine structures on nilmanifolds*, International Journal of Mathematics **70** (1996), no. 5, 599–616.

[19] Dietrich Burde, *Lie algebra prederivations and strongly nilpotent Lie algebras*, Communications in Algebra **30** (2002), no. 7, 3157–3175.

[20] Dietrich Burde, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*, Central European Journal of Mathematics **4** (2006), no. 3, 325–357.

[21] Dietrich Burde, *Derivation double Lie algebras*, Journal of Algebra and Its Applications **15** (2016), no. 6, 17 pp.

[22] Dietrich Burde, *Left-invariant affine structures on nilpotent Lie groups, Habilitationsschrift*, Düsseldorf, 1999.

[23] Dietrich Burde and Karel Dekimpe, *Post-Lie algebra structures and generalized derivations of semisimple Lie algebras*, Moscow Mathematical Journal **13** (2013), no. 1, 1–18.

[24] Dietrich Burde and Karel Dekimpe, *Post-Lie algebra structures on pairs of Lie algebras*, Journal of Algebra **464** (2016), 226–245.

[25] Dietrich Burde, Karel Dekimpe, and Sandra Deschamps, *The Auslander conjecture for NIL-affine crystallographic groups*, Mathematische Annalen **332** (2005), no. 1, 161–176.

[26] Dietrich Burde, Karel Dekimpe, and Sandra Deschamps, *LR-algebras*, Contemporary Mathematics **491** (2009), 125–140.

[27] Dietrich Burde, Karel Dekimpe, and Kim Vercammen, *Complete LR-structures on solvable Lie algebras*, Journal of Group Theory **13** (2010), no. 5, 703–719.

[28] Dietrich Burde, Karel Dekimpe, and Kim Vercammen, *Affine actions on Lie groups and post-Lie algebra structures*, Linear Algebra and Its Applications **437** (2012), no. 5, 1250–1263.

[29] Dietrich Burde, Christof Ender, and Wolfgang A. Moens, *Post-Lie algebra structures for nilpotent Lie algebras*, International Journal of Algebra and Computation **28** (2018), no. 5, 915–933.

[30] Dietrich Burde and Fritz Grunewald, *Modules for certain Lie algebras of maximal class*, Journal of Pure and Applied Algebra **99** (1995), no. 3, 239–254.

[31] Dietrich Burde and Vsevolod Gubarev, *Rota–Baxter operators and post-Lie algebra structures on semisimple Lie algebras*, To appear in Communications of Algebra (2019), arXiv:1805.05104 [math.RA].

[32] Dietrich Burde and Wolfgang A. Moens, *Faithful Lie algebra modules and quotients of the universal enveloping algebra*, Journal of Algebra **325** (2011), no. 1, 440–460.

[33] Dietrich Burde and Wolfgang A. Moens, *Commutative post-Lie algebra structures on Lie algebras*, Journal of Algebra **467** (2016), 183–201.

[34] Dietrich Burde, Wolfgang A. Moens, and Karel Dekimpe, *Commutative post-Lie algebra structures and linear equations for nilpotent Lie algebras*, To appear in Journal of Algebra (2019), arXiv:1711.01964 [math.RA].

[35] Dietrich Burde and Christine Steinhoff, *Classification of orbit closures of* 4*-dimensional complex Lie algebras*, Journal of Algebra **214** (1999), 729–739.

[36] Roger Carles, *Sur la structure des algèbres de Lie rigides*, Annales de l'institut Fourier **34** (1984), no. 3, 65–82 (in French).

[37] Roger Carles and Yoro Diakité, *Sur les variétés d'Algèbres de Lie de dimension* ≤ 7, Journal of Algebra **91** (1984), 53–63 (in French).

[38] Manuel Ceballos, Juan Nuñez, and Ángel F. Tenorio, *New Results in the Classification of Filiform Lie Algebras*, Bulletin of the Malaysian Mathematical Sciences Society **40** (2017), no. 1, 409–437.

[39] Leonard S. Charlap, *Bieberbach Groups and Flat Manifolds*, Springer-Verlag, 1986.

[40] Carlos Cid and Tilman Schulz, *Computation of Five- and Six-Dimensional Bieberbach Groups*, Experimental Mathematics **10** (2001), no. 1, 109–115.

[41] Jiang Cuibo and Daoji Meng, *The realization and structure of complete Lie algebras whose nilpotent radicals are Heisenberg algebras*, Chinese Science Bulletin **43** (1998), no. 5, 383–385.

[42] Charles Curry, Kurusch Ebrahimi-Fard, and Hans Munthe-Kaas, *What is a post-Lie algebra and why is it useful in geometric integration*, arXiv:1712.09415v1 (2017).

[43] Willem A. de Graaf, *Classification of* 6*-dimensional nilpotent Lie algebras over fields of characteristic not* 2, Journal of Algebra **309** (2007), 640–653.

[44] Karel Dekimpe, *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Springer-Verlag, 1996.

[45] Karel Dekimpe, *Any virtually polycyclic group admits a NIL-affine crystallographic action*, Topology **42** (2003), no. 4, 821–832.

[46] Karel Dekimpe and Paul Igodt, *Polycyclic-by-finite groups admit a bounded-degree polynomial structure*, Inventiones mathematicae **129** (1997), no. 1, 121–140.

[47] Karel Dekimpe and Wim Malfait, *Affine structures on a class of virtually nilpotent groups*, Topology and its Applications **73** (1996), 97–119.

[48] Sandra Deschamps, *An algebraic approach to simply transitive and crystallographic NIL-affine actions and the generalized Auslander conjecture*, PhD thesis (2008), Katholieke Universiteit Leuven.

[49] Marcus du Sautoy, *Finding Moonshine: A Mathematician's Journey Through Symmetry*, Harpercollins Publishers, 2008.

[50] Evgeniĭ B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Matematicheskii Sbornik **30** (1952), no. 72, 349–462, English translation in American Mathematical Society Translation, Series 2, Vol. 6 (1957), 111–244.

[51] Kurusch Ebrahimi-Fard, Alexander Lundervold, Igor Mencattini, and Hans Z. Munthe-Kaas, *Post-Lie Algebras and Isospectral Flows*, SIGMA **11** (2015), no. 093, 16 pp.

[52] Loïc Foissy, *Extension of the product of a post-Lie algebra and application to the SISO feedback transformation group*, arXiv: 1705.08132v1 [math. RA] (2017).

[53] David Fried and William M. Goldman, *Three-Dimensional Affine Crystallographic Groups*, Advances in Mathematics **47** (1983), 1–49.

[54] José R. Gómez, Antonio Jimenéz-Merchán, and Yusupdjan Khakimdjanov, *Low-dimensional filiform Lie algebras*, Journal of Pure and Applied Algebra **130** (1998), 133–158.

[55] Michel Goze, *Lie algebras: Classification, Deformations and Rigidity*, arXiv: math/0611793 [math.RA] (2006).

[56] Michel Goze and Jose M. Ancochea Bermudez, *On the classification of rigid Lie algebras*, Journal of Algebra **245** (2001), no. 1, 68–91.

[57] Michel Goze and You Hakimjanov, *Sur les algèbres de Lie nilpotentes admettant un tore de dérivations*, Manuscripta math. **84** (1994), 115–124 (in French).

[58] Michel Goze and Yusupdjan Khakimdjanov, *Nilpotent Lie Algebras*, vol. 361 of Mathematics and Its Applications, Kluwer Academic Publishers, 1996.

[59] Michel Goze and Elisabeth Remm, *Poisson algebras in terms of non-associative algebras*, Journal of Algebra **320** (2008), 294–317.

[60] David Hilbert, *Mathematical problems*, Bulletin of the American Mathematical Society **87** (1902), no. 10, 437–479.

[61] Joachim Hilgert and Karl-Hermann Neeb, *Structure and Geometry of Lie Groups*, Springer Monographs in Mathematics, 2012.

[62] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1972.

[63] Noboru Itô, *Über das Produkt von zwei abelschen Gruppen*, Mathematische Zeitschrift **63** (1955), 400–401 (in German).

[64] Nathan Jacobson, *Lie Algebras and Algebraic Groups*, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, 1962.

[65] Hyuk Kim, *Complete left-invariant affine structures on nilpotent Lie groups*, Journal of Differential Geometry **24** (1986), 373–394.

[66] Bernard Kolman, *Semi-modular Lie Algebras*, Journal of Science of the Hiroshima University, Series A-I (Mathematics) **29** (1965), 149–163.

[67] Jean L. Koszul, *Variante d'un théoremè de H. Ozeki*, Osaka Journal of Mathematics **15** (1978), 547–551 (in French).

[68] George F. Leger and Eugene M. Luks, *Generalized Derivations of Lie Algebras*, Journal of Algebra **228** (2000), 165–203.

[69] María Lorente and Bruno Gruber, *Classification of Semisimple Subalgebras of Simple Lie Algebras*, Journal of Mathematical Physics **13** (1972), no. 10, 1639–1663.

[70] Louis Magnin, *Sur les algèbres de Lie nilpotentes de dimension* ≤ 7, Journal of Geometry and Physics **3** (1986), 119–144 (in French).

[71] Gregory Margulis, *Complete affine locally flat manifolds with a free fundamental group*, Journal of Soviet Mathematics **36** (1987), no. 1, 129–139.

[72] Alberto Medina, *Sur quelques algèbres symétriques à gauche dont l'algèbre de Lie sous-jacente est résoluble*, Comptes Rendus de l'Académie des Sciences, Séries A-B **286** (1978), no. 3, A173–176 (in French).

[73] Alberto Medina Perea, *Flat left-invariant connections adapted to the automorphism structure of a Lie group*, Journal of Differential Geometry **16** (1981), no. 3, 445–474.

[74] Dao Ji Meng, *Some results on complete Lie algebras*, Communications in Algebra **22** (1994), no. 13, 5457–5507.

[75] Dao Ji Meng and Lin Sheng Zhu, *Solvable complete Lie algebras I*, Communications in Algebra **24** (1996), no. 13, 4181–4197.

[76] Dmitry Millionshchikov, *The variety of Lie algebras of maximal class*, Proceedings of the Steklov Institute of Mathematics **266** (2009), 177–194.

[77] John Milnor, *On fundamental groups of complete affinely flat manifolds*, Advances in Mathematics **25** (1977), no. 2, 178–187.

[78] Andrei N. Minchenko, *The semisimple subalgebras of exceptional Lie algebras*, Transactions of the Moscow Mathematical Society **67** (2006), 225–259.

[79] Hans Munthe-Kaas and Alexander Lundervold, *On post-Lie algebras, Lie-Butcher series and moving frames*, Foundations of Computational Mathematics **13** (2013), 583–613.

[80] Xiang Ni, Chengming Bai, and Li Guo, *Nonabelian Generalized Lax Pairs, the Classical Yang-Baxter Equation and PostLie Algebras*, Communications in Mathematical Physics **297** (2010), no. 2, 553–596.

[81] Albert Nijenhuis and Roger W. Richardson, Jr., *Cohomology and deformations in graded Lie algebras*, Bulletin of the American Mathematical Society **72** (1966), 1–29.

[82] Petr Novotný and Jiří Hrivnák, *On (α, β, γ)-derivations of Lie algebras and corresponding invariant functions*, Journal of Geometry and Physics **58** (2008), no. 2, 208–217.

[83] Shikun Ou, Dengyin Wang, and Ruiping Yao, *Derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring*, Linear Algebra and Its Applications **424** (2007), 378–383.

[84] Roger W. Richardson, Jr., *On the Rigidity of semi-direct Products of Lie Algebras*, Pacific Journal of Mathematics **22** (1967), no. 2, 339–344.

[85] Louis J. Santharoubane, *Kac-Moody Lie algebras and the classification of nilpotent algebras of maximal rank*, Canadian Journal of Mathematics **34** (1982), 1215–1239.

[86] John Scheuneman, *Examples of compact locally affine spaces*, Bulletin of the American Mathematical Society **77** (1971), no. 4, 589–592.

[87] John Scheuneman, *Affine Structures on Three-Step Nilpotent Lie Algebras*, Proceedings of the Americal Mathematical Society **46** (1974), no. 3, 451–454.

[88] Dan Segal, *The structure of complete left-symmetric algebras*, Mathematische Annalen **293** (1992), 569–578.

[89] Ernest L. Stitzinger, *On Lie algebras with Only Inner Derivations*, Journal of Algebra **105** (1987), 341–343.

[90] Andrzej Szczepański, *Geometry of crystallographic groups*, World Scientific Publishing Co., 2012.

[91] Xiaomin Tang, *Biderivations, linear commuting maps and commutative post-Lie algebra structures on W-algebras*, Communications in Algebra **45** (2017), no. 12, 5252–5261.

[92] Xiaomin Tang, *Post-Lie algebra structures on the Witt algebra*, arXiv:1611.04082v2 [math.RA] (2017).

[93] Hiroyuki Tasaki and Masaaki Umehara, *An invariant on* 3*-dimensional Lie algebras*, Proceedings of the Americal Mathematical Society **115** (1992), no. 2, 293–294.

[94] Patrice Tauvel and Rupert W. T. Yu, *Lie Algebras and Algebraic Groups*, Springer, 2005.

[95] Jacques Tits, *Free subgroups in linear groups*, Journal of Algebra **20** (1972), no. 2, 250–270.

[96] George Tomanov, *Properly discontinuous group actions on affine homogeneous spaces*, Proceedings of the Steklov Institute of Mathematics **292** (2016), no. 1, 260–271.

[97] Paweł Turkowski, *Low-dimensional real Lie algebras*, Journal of Mathematical Physics **29** (1988), 2139–2144.

[98] Bruno Vallette, *Homology of generalized partition posets*, Journal of Pure and Applied Algebra **208** (2007), no. 2, 699–725.

[99] Michèle Vergne, *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*, Bulletin de la Société Mathématique de France **98** (1970), 81–116 (in French).

[100] Joseph A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, 1967.

[101] Satoru Yamaguchi, *Affine structure on some solvable Lie group*, Memoirs of the Faculty of Science, Kyushu University, Series A **34** (1980), no. 2, 339–345.

[102] Hans Zassenhaus, *Über einen Algorithmus zur Bestimmung der Raumgruppen*, Commentarii mathematici Helvetici **21** (1948), 117–141 (in German).

[103] Linsheng Zhu and Daoji Meng, *The Classification of Complete Lie Algebras with Low Dimensions*, Algebra Colloquium **4** (1997), no. 1, 95–109.

Zusammenfassung

In dieser Dissertation studieren wir *Post-Lie-Algebra-Strukturen* (Vektorräume V mit drei bilinearen Operationen $[,], \{,\}, \cdot$, so dass $(V, [,])$ eine Lie-Algebra $\mathfrak{g}$, $(V, \{,\})$ eine Lie-Algebra $\mathfrak{n}$ und $(V, \cdot)$ eine nicht-assoziative Algebra ist ist sowie gewisse Verträglichkeitsbedingungen zwischen den drei Operationen erfüllt sind) auf Paaren komplexer Lie-Algebren $(\mathfrak{g}, \mathfrak{n})$. Post-Lie-Algebra-Strukturen tauchen (unter anderem) in der Geometrie in der Untersuchung kriststallographischer Gruppen und einer Frage von John Milnor auf; in Kapitel 1 erklären wir den geometrischen Hintergrund im Detail.
Nachdem wir einen Überblick über Definitionen und wichtige, bereits bekannte, Sätze über Lie-Algebren und Post-Lie-Algebra-Strukturen gegeben haben (Kapitel 2), studieren wir die Existenz von Post-Lie-Algebra-Strukturen in Bezug auf algebraische Eigenschaften der Lie-Algebren $\mathfrak{g}$ und $\mathfrak{n}$ (Kapitel 3). Danach beweisen wir die Nichtexistenz von Post-Lie-Algebra-Strukturen auf Paaren $(\mathfrak{g}, \mathfrak{n})$ mit $\dim(\mathfrak{g}) = \dim(\mathfrak{n}) < 45$, wobei $\mathfrak{g}$ halbeinfach (nicht einfach) und $\mathfrak{n}$ einfach ist. Danach untersuchen wir Post-Lie-Algebra-Strukturen auf Lie-Algebren, die $[x, y] = a(\{x, y\})$ für einen Skalar $a \in \mathbb{C}^*$ erfüllen (als Verallgemeinerung der wichtigen Klasse kommutativer Post-Lie-Algebra-Stukturen) und Post-Lie-Algebra-Strukturen in Dimension 3.
In Kapitel 4 studieren wir für Post-Lie-Algebra-Strukturen auf vollständigen Lie-Algebren. Zuerst klassifizieren wir alle vollständigen Lie-Algebren über $\mathbb{C}$ bis Dimension 7; danach studieren wir Post-Lie-Algebra-Strukturen auf Paaren $(\mathfrak{g}, \mathfrak{n})$, wobei $\mathfrak{n}$ vollständig ist und $[x, y] = R(\{x, y\})$ für eine lineare Abbildung R gilt. In dem Fall $R = a\,\mathrm{Id}, a \in \mathbb{C}$, können wir unsere Resultate aus Kapitel 3 verbessern; im allgemeineren Fall können wir die entstehenden Post-Lie-Algebra-Strukturen durch bestimmte innere Derivationen von $\mathfrak{n}$ charakterisieren.
Post-Lie-Algebra-Strukturen (insbesondere kommutative) auf nilpotenten Lie-Algebren werden in Kapitel 5 untersucht. Wir beweisen Korrespondenzen zwischen kommutativen Post-Lie-Algebra-Strukturen auf 2-stufig nilpotenten Lie-Algebren und Pre-Lie-Algebra-Strukturen bzw. LR-Strukturen (als ein Korollar erhalten wir Resultate über die Vollständigkeit dieser Strukturen auf Heisenberg-Lie-Algebren). Danach untersuchen wir kommutative Post-Lie-Algebra-Strukturen auf filiformen Lie-Algebren und beweisen, dass diese für nicht-metabelsche Lie-Algebren im Wesentlichen klassifiziert werden können. Wir schließen Kapitel 5 mit Beweisen über Zusammenhänge zwischen Post-Lie-Algebra-Strukturen auf nilpotenten Lie-Algebren und Poisson-zulässigen Lie-Algebren ab.
Danach geben wir offene Fragen an (Kapitel 6) und Beispiele bzw. Klassifikationen von Post-Lie-Algebra-Stukturen in niedrigen Dimensionen (Appendix B).