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Vilson Hoti

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Thermodynamic identities in black hole spacetimes

Contents

1. Introduction

In their seminal paper [4] in January 24 1973 (´´The Four Laws of Black Hole Mechanics´´), J. M. Bardeen, B. Carter and S. W. Hawking published the derivation of the first law of black hole mechanics (and, moreover, formulated other three laws of black hole mechanics, namely, the zeroth, second and the third law). This law establishes the relationship between the infinitesimal changes in mass-energy of a stationary, axisymmetric black hole to infinitesimal changes in its canonical angular momentum, horizon surface area and a matter (e.g. a stationary perfect fluid) surrounding the horizon of the black hole. The basic setting of their derivation consists in, firstly, identifying the existence of a unique stationary Killing field ξ^a and a unique rotational (axial) Killing field φ^a at spatial infinity in a stationary axisymmetric asymptotically black hole spacetime. The stationary Killing field ξ^a is taken to have a unit norm and the axial Killing field φ^a has a closed orbit with length 2π. Furthermore, δM is defined to be the difference between the masses of the stationary axisymmetric black hole solutions, which are infinitesimally close to each other; then the first law of black hole mechanics (or the differential mass formula, as they call it) is given by the identity [4]

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_B \delta J_B + \int_\Sigma \left(\bar{\mu}\delta N_{def} + \bar{T}\delta S_{def} + \Omega \delta J_{def} \right). \tag{I}$$

where M represents the ADM (Arnowitt-Deser-Misner) mass, κ is the surface gravity, A is the horizon surface area, Ω_B measures the angular velocity of the horizon, J_B represents the angular momentum measured at the horizon, $\bar{\mu}$ is the redshifted chemical potential, $\bar{T}$ is the redshifted temperature and Ω is the angular velocity of the perfect fluid (generally non-constant). A particle number density *n,* entropy per particle *s,* fluid 4-velocity u^a and the spacetime metric g_{ab} are going to be the physical fields which describe the fluid. The three form N_{def} is the particle current 3-form, J_{def} defines the angular momentum current 3-form and S_{def} represents the entropy current 3-form, and ε_{adef} is the volume element on spacetime. Σ is taken to be a spacelike surface that extends from the black hole bifurcation surface to spatial infinity.

In this thesis we generalize (I) (by following some developed ideas by Iyer in [2]) by relaxing all symmetry assumptions on matter fields. The generalized expression is going to be identical (structurally) to (I) with the main difference that δ is defined as an arbitrary perturbation (without necessity of stationary or axisymmetric perturbations) of the background spacetime. Furthermore, we compare it with the first law of thermodynamics for perfect fluid stars (derived explicitly in [5])

$$\delta M = \int_\Sigma \left(\bar{\mu}\delta N_{def} + \bar{T}\delta S_{def} + \Omega\, \delta J_{def} \right). \tag{II}$$

and we clearly notice that the difference between them lies in the additional terms defined at the horizon of the black hole absent in the perfect fluid stars. The remainder of this thesis is organized as follows: In section 2 we review some preliminary constructions on the Lagrangian diffeomorphism covariant theories and derive the *perturbative identity* (PI). In section 3 we discuss the physical interpretation of PI distinguishing two possible cases. In section 4 we review some relevant results on perfect fluids and perfect fluid stars necessary for the derivation of the first law. In section 5 we finally derive the first law of black hole mechanics involving perfect fluids and compare it with the first law of thermodynamics for perfect fluid stars in dynamic equilibrium. In section 6 we give another version of the first law, called the ´´physical process version´´ where we start with a stationary black hole and perturb it by some physical process (e.g. a rocket falling into the black hole). The underlying assumption is that the black hole is not destroyed by this process and that it settles down to a new stationary final state [12].

Our notation and conventions will mainly follow [2]. All spacetimes, tensor fields, and surfaces in this thesis will be assumed to be smooth (C^{∞}). A boldface letters will mean that the indices of differential forms have been omitted, e.g. $\boldsymbol{\theta} = \theta_{abc}$. And we are, essentially, going to proceed in the spirit of [2] in this thesis.

2. A Perturbative identity (PI) in the context of diffeomorphism covariant theories

2.1. Preliminaries

In this section, we will provide basic and necessary conceptual ingredients for Lagrangian diffeomorphism covariant theories needed for further calculations in other sections. More detailed analysis of what is discussed in this section (2.1 and 2.2) and section 3 are to be found in references [1-3, 5] and references therein.

We begin by considering a diffeomorphism covariant Lagrangian 4-form $\boldsymbol{L} = L_{abcd}$ on an 4-dimensional compact oriented manifold M, which depends on the spacetime metric, g_{ab}, and matter fields, denoted by ψ. We indicate all dynamical fields as $\phi = (g_{ab}, \psi)$. So, the Lagrangian is given as (see -[1] for its proof)

$$\boldsymbol{L}=\boldsymbol{L}\,(g_{ab}, R_{abcd}, \nabla_{e_1}R_{abcd}, \ldots, \nabla_{(e_1\ldots e_m)}R_{abcd}, \psi, \nabla_{e_1}\psi, \nabla_{(e_1\ldots e_m)}\psi), \tag{1}$$

where ∇ is a derivative operator (or connection) associated to (or compatible with the) g_{ab} and R_{abcd} indicates curvature of g_{ab}. It is normally assumed that multiple derivatives $\nabla_{(e_1\ldots e_m)}$ are symmetrized (refer to-[1] for a detailed discussion). By a diffeomorphism covariant Lagrangian we mean a Lagrangian that satisfies the condition

$$\boldsymbol{L}(f^*(\phi)) = f^*\boldsymbol{L}(\phi), \tag{2}$$

where f^* is the action on the fields induced by a diffeomorphism f: M→M.

It is also possible for the Lagrangian (1) to be (non-uniquely) broken up into two components (all important quantities expressed in terms of decomposition ´´vacuum-matter´´ below can be mainly found in [2])

$$\boldsymbol{L} = \boldsymbol{L_g} + \boldsymbol{L_m} \tag{3}$$

with $\boldsymbol{L_g} = \boldsymbol{L_g}(g_{ab}, R_{abcd}, \nabla_{e_1}R_{abcd}, \ldots, \nabla_{(e_1\ldots e_m)}R_{abcd})$ called ´´vacuum´´ Lagrangian which has (as obvious) only a functional dependence on a stationary and axisymmetric metric, g_{ab}, and other tensor fields, and $\boldsymbol{L_m} = \boldsymbol{L_m}(g_{ab}, R_{abcd}, \nabla_{f_1}R_{abcd}, \ldots, \nabla_{(f_1\ldots f_l)}R_{abcd}, \psi, \nabla_{f_1}\psi, \nabla_{(f_1\ldots f_k)}\psi)$ called ´´matter´´ Lagrangian which depends on g_{ab}, and matter fields ψ. The assumption that multiple derivatives are symmetrized holds also here.

We allow $\boldsymbol{L}_g$ and $\boldsymbol{L}_m$ to be diffeomorphism covariant in the sense of condition (2) and are, as remarked in [2], characterized by the following ambiguities: $\boldsymbol{L}_g \to \boldsymbol{L}_g + \lambda$ and $\boldsymbol{L}_m \to \boldsymbol{L}_m - \lambda$ with $\lambda = \lambda\,(g_{ab}, R_{abcd}, \nabla_{h_1}R_{abcd}, \ldots, \nabla_{(h_1\ldots h_s)}R_{abcd})$. And finally, one can impose the condition that no restrictions on the matter fields are imposed (see-[2] for a more complete account).

The (first order) variation of the Lagrangian (1) can be brought to the following compact form

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\boldsymbol{\theta}, \tag{4}$$

where $\mathbf{E}$=0 are the equations of motion, uniquely defined by (4) and locally constructed out of the dynamical fields, ϕ, $\mathbf{E}$= $\mathbf{E}(\phi)$, and $\boldsymbol{\theta}$ is the *symplectic potential 3-form,* locally constructed out of the fields, ϕ, and their variations, $\delta\phi$, $\boldsymbol{\theta} = \boldsymbol{\theta}(\phi, \delta\phi)$; $\delta\phi$ can be understood as a vector in the tangent space and ´´d´´ denotes an exterior derivative acting on forms.

It is proved that $\boldsymbol{\theta}$ can always be chosen to be covariant (see-[1-Lemma 3.1]) and we will assume that we restrict ourselves to this choice. It is important to notice that the Lagrangian is defined up to the addition of an exact 4-form (which means that the equations of motion don´t change even though one alters the Lagrangian for some quantity $d\boldsymbol{\mu}$)

$$\mathbf{L} \rightarrow \mathbf{L} + d\boldsymbol{\mu}, \tag{5}$$

which implies that the symplectic potential is defined (or it is ambiguous) up to the addition of two terms, $\delta\boldsymbol{\mu}$ and $d\mathbf{Y}(\phi, \delta\phi)$

$$\boldsymbol{\theta} \rightarrow \boldsymbol{\theta} + \delta\boldsymbol{\mu} + d\mathbf{Y}, \tag{6}$$

where $\boldsymbol{\mu}$ and $\mathbf{Y}$ are covariant forms. (see - [1] for a more complete account of ambiguities).

The (first order) variation of the Lagrangian (3) gives

$$\delta\mathbf{L} = \delta\boldsymbol{L_g} + \delta\boldsymbol{L_m} = \boldsymbol{E}_{\mathrm{g}}^{ab}\delta g_{ab} + \boldsymbol{E}_m\delta\psi + d\boldsymbol{\theta}, \tag{7}$$

with $\mathbf{E}\delta\phi = \boldsymbol{E}_{\mathrm{g}}^{ab}\delta g_{ab} + \boldsymbol{E}_m\delta\psi$, where $\boldsymbol{E}_{\mathrm{g}}^{ab} = \mathbf{0}$ are the equations of motion for the spacetime metric and $\boldsymbol{E}_m = 0$ are the equations of motion for the matter fields.

Furthermore, $\delta\boldsymbol{L_g}$ and $\delta\boldsymbol{L_m}$ are given by the following identities:

$$\delta\boldsymbol{L_g} = \boldsymbol{E}_{\mathrm{g}}^{*ab}\delta g_{ab} + d\boldsymbol{\theta_g}$$

$$\delta\boldsymbol{L_m} = \boldsymbol{E}_m\delta\psi + d\boldsymbol{\theta_m} + \tfrac{1}{2}\boldsymbol{\varepsilon}T^{ab}\delta g_{ab}, \tag{8}$$

with $\boldsymbol{\theta_g} = \boldsymbol{\theta_g}(g,\delta g)$ and $\boldsymbol{\theta_m} = \boldsymbol{\theta_m}(\phi, \delta\phi)$ and T^{ab} representing stress-energy tensor. (Refer to- [5-(84)] for a detailed calculation of $\delta\boldsymbol{L_m}$ in (8) which justifies the form of the identity).

After substituting (8) into the right side of (7)

$$\boldsymbol{E}_{\mathrm{g}}^{ab}\delta g_{ab} + \boldsymbol{E}_m\delta\psi + d\boldsymbol{\theta} = (\boldsymbol{E}_{\mathrm{g}}^{*ab} + \tfrac{1}{2}\boldsymbol{\varepsilon}T^{ab})\delta g_{ab} + \boldsymbol{E}_m\delta\psi + d(\boldsymbol{\theta_g} + \boldsymbol{\theta_m}), \tag{9}$$

it yields:

$$\boldsymbol{E}_{\mathrm{g}}^{ab} = \boldsymbol{E}_{\mathrm{g}}^{*ab} + \tfrac{1}{2}\boldsymbol{\varepsilon}T^{ab}, \tag{10}$$

and:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_g + \boldsymbol{\theta}_m. \tag{11}$$

The *symplectic current 3-form*, $\boldsymbol{\omega}$ **,** is defined by

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_2\boldsymbol{\theta}(\phi, \delta_1\phi) - \delta_1\boldsymbol{\theta}(\phi, \delta_2\phi), \tag{12}$$

and it is bilinear and antisymmetric (skew) in ($\delta_1\phi$, $\delta_2\phi$). Taking the exterior derivative of (12) gives the following

$$
\begin{aligned}
d\boldsymbol{\omega} &= \delta_2\,(d\boldsymbol{\theta}) - \delta_1(d\boldsymbol{\theta}) \\
&= \delta_2\,(\mathbf{E}\,\delta_1\phi - \delta_1\mathbf{L}) - \delta_1(\mathbf{E}\,\delta_2\phi - \delta_2\mathbf{L}) \\
&= \delta_1\,\mathbf{E}\cdot\delta_2\,\phi - \delta_2\,\mathbf{E}\cdot\delta_1\phi,
\end{aligned}
\tag{13}
$$

where we used the identity (4) and $[\delta_1, \delta_2] = 0$ in the second line.

Eq. (13) implies that $\boldsymbol{\omega}$ is closed ($d\boldsymbol{\omega}=0$) if $\delta_1\phi$ and $\delta_2\phi$ satisfy the linearized equations of motion $\delta_1\,\mathbf{E}=\delta_2\,\mathbf{E}=0$.

The *Noether current 3- form*, $\mathbf{J}[\xi]$, for any fixed smooth vector field ξ^a on the spacetime is defined by

$$\mathbf{J}[\xi]= \boldsymbol{\theta}(\phi, \pounds_\xi\phi) - \xi\cdot\boldsymbol{L}, \tag{14}$$

where "$\cdot$" denotes contraction of the vector field ξ^a into the first index of a differential 4-form $\boldsymbol{L}$ (in some literature you find the contraction denoted by: $\mathbf{i}_\xi\,\boldsymbol{L}$, e. g. [5]), and $\pounds$ denotes the spacetime Lie derivative. Applying an exterior derivative to $\mathbf{J}[\xi]$ reads

$$
\begin{aligned}
d\mathbf{J}[\xi] &= d(\boldsymbol{\theta} - \xi\cdot\boldsymbol{L}) \\
&= \delta\mathbf{L} - \mathbf{E}\delta\phi - d(\xi\cdot\boldsymbol{L}) \\
&= -\mathbf{E}\,\pounds_\xi\phi.
\end{aligned}
\tag{15}
$$

where in the second line we used (4).

As shown in [3] there exists a $\mathbf{Q}[\xi]$, called the *Noether charge 2-form*, which is locally constructed out of ϕ, ξ^a and their derivatives, such that

$$\mathbf{J}[\xi]=d\mathbf{Q}[\xi] + \xi^a\mathbf{C}_a, \tag{16}$$

where $\mathbf{C}_a$ is a covariant 3- form, locally constructed out of the dynamical fields, ϕ, and it has a property that whenever the field equations are satisfied $\mathbf{E}$=0, the *constraint equations* $\mathbf{C}_a = 0$ follow.

The Noether current $\mathbf{J}[\xi]$ can also be given as [2]

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] - (\boldsymbol{\varepsilon}\cdot E\cdot\phi\cdot\xi), \tag{17}$$

where for convenience we set $\mathbf{E} = \boldsymbol{\varepsilon}\,\mathrm{E}$ and the second term (3-form) on the right side of Eq.(17) is defined as follows [2]:

$$
\begin{aligned}
(\boldsymbol{\varepsilon}\cdot E\cdot\phi\cdot\xi)_{abc} = \boldsymbol{\varepsilon}_{eabc}\sum_i E_{\phi_i\, b_1\ldots b_{u_i}}{}^{a_1\ldots a_{d_i}}\big(-\phi_i{}^{e\ldots b_{u_i}}{}_{a_1\ldots a_{d_i}}\delta_s^{b_1}\ldots - \phi_i{}^{b_1\ldots e}{}_{a_1\ldots a_{d_i}}\delta_s^{b_{u_i}} \\
+\, \phi_i{}^{b_1\ldots b_{u_i}}{}_{s\ldots a_{d_i}}\delta^e_{a_1} + \ldots + \phi_i{}^{b_1\ldots b_{u_i}}{}_{a_1\ldots s}\delta^e_{a_{d_i}}\big)\xi^s,
\end{aligned}
\tag{18}
$$

Proof of Eq.(17) is given in [2-Lemma 1], where particular use of Eq.(15) is made. The ambiguity in $\boldsymbol{\theta}$ implies the ambiguity in $\mathbf{J}[\xi]$

$$\mathbf{J}[\xi] \to \mathbf{J}[\xi] + d\mathbf{Y}(\phi, \pounds_\xi\phi) + d(\xi\cdot\boldsymbol{\mu}), \tag{19}$$

and therefore the ambiguity in $\mathbf{Q}[\xi]$

$$\mathbf{Q}[\xi] \rightarrow \mathbf{Q}[\xi] + \mathbf{Y}(\phi, £_\xi \phi) + \xi \cdot \boldsymbol{\mu} + d\mathbf{Z}, \tag{20}$$

where $\mathbf{Z}$ is covariant 1-form. These ambiguities are inconsequential with respect to the results in the following sections (we refer the reader to [1] for a wider analysis).

Expressed in terms of $\boldsymbol{L_g}$ and $\boldsymbol{L_m}$, the Noether currents are defined by (see (14))

$$\mathbf{J}_g[\xi]= \boldsymbol{\theta}_g(g, £_\xi g) - \xi \cdot \boldsymbol{L_g}$$

$$\mathbf{J}_m[\xi]= \boldsymbol{\theta}_m(\phi, £_\xi \phi) - \xi \cdot \boldsymbol{L_m}, \tag{21}$$

and therefore

$$\mathbf{J}_g[\xi]+ \mathbf{J}_m[\xi] = (\boldsymbol{\theta}_g + \boldsymbol{\theta}_m) - \xi \cdot (\boldsymbol{L_g} + \boldsymbol{L_m}) = \boldsymbol{\theta} - \xi \cdot \mathbf{L}$$

$$\rightarrow \mathbf{J}[\xi]= \mathbf{J}_g[\xi]+ \mathbf{J}_m[\xi]. \tag{22}$$

The Noether current $\mathbf{J}[\xi]$ for $\boldsymbol{L_g}$ and $\boldsymbol{L_m}$ can also be computed using Eq.(17):

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] - \boldsymbol{\varepsilon}\cdot(E_g + E_m)\cdot \phi\cdot \xi = d\mathbf{Q}[\xi] - 2\boldsymbol{\varepsilon} \cdot E_g \cdot \xi - \boldsymbol{\varepsilon}\cdot E_m \cdot \psi \cdot \xi, \tag{23}$$

where we conveniently set $\boldsymbol{E}_{\mathrm{g}}^{ab} = \boldsymbol{\varepsilon}\, E_{\mathrm{g}}^{ab}$ and $\boldsymbol{E}_m = \boldsymbol{\varepsilon}\, E_m$ and by pure convention one sets the factor of two in the second term of the right side of (23) for simplicity considerations.

The Noether charges $\mathbf{Q}_g$ and $\mathbf{Q}_m$ are defined by [2]

$$\mathbf{J}_g[\xi] = d\mathbf{Q}_g[\xi] - 2\boldsymbol{\varepsilon}\cdot E_{\mathrm{g}}^{*ab}\cdot \xi,$$

$$\mathbf{J}_m[\xi] = d\mathbf{Q}_m[\xi] - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi - \boldsymbol{\varepsilon} \cdot T \cdot \xi. \tag{24}$$

where we imposed (17).

After inserting (23) and (24) into (22) and after using (10) in the second line below

$$d\mathbf{Q}[\xi] - 2\boldsymbol{\varepsilon} \cdot E_g \cdot \xi - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi = d\mathbf{Q}_g[\xi] - 2\boldsymbol{\varepsilon}\cdot E_{\mathrm{g}}^{*ab}\cdot\xi - d\mathbf{Q}_m[\xi] - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi - \boldsymbol{\varepsilon}\cdot T \cdot \xi, \ \rightarrow$$

$$d\mathbf{Q}[\xi] - 2\boldsymbol{\varepsilon}\cdot \boldsymbol{E}_{\mathrm{g}}^{*ab} - \boldsymbol{\varepsilon}\cdot T \cdot \xi - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi = d\mathbf{Q}_g[\xi] - 2\boldsymbol{\varepsilon}\cdot E_{\mathrm{g}}^{*ab}\cdot\xi - d\mathbf{Q}_m[\xi] - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi - \boldsymbol{\varepsilon}\cdot T \cdot \xi \tag{25}$$

we obtain the identity

$$d\mathbf{Q}[\xi] = d\mathbf{Q}_g[\xi] + d\mathbf{Q}_m[\xi]$$

$$\rightarrow \mathbf{Q}[\xi] = \mathbf{Q}_g[\xi] + \mathbf{Q}_m[\xi] + d\mathbf{H}, \tag{26}$$

where $\mathbf{H}$ is an arbitrary covariant 1-form. The identity (26) will play an important role in the calculations in the next section concerning the derivation of the *perturbative identity* (PI).

Theorem 1: Let $\delta_1\phi=£_\xi\phi$ and let $\delta_2\phi=\delta\phi$ be a variation to an infinitesimally close solution, let ξ^a be a fixed smooth vector field, then the symplectic current 3-form $\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$ is exact:

$$\boldsymbol{\omega}(\phi, £_\xi\phi, \delta\phi) = d[\delta\mathbf{Q}\,[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi)]. \tag{27}$$

Proof:

We let, following Wald and Iyer [1] closely,

- ϕ satisfy **E**=0 and
- $\delta\phi$ satisfy δ**E**=0.

Then, by varying Eq.(14), we have

$$\begin{aligned} \delta\mathbf{J}[\xi] &= \delta\boldsymbol{\theta}(\phi, £_\xi\phi) - \xi\cdot\,\delta\boldsymbol{L} \\ &= \delta\boldsymbol{\theta}(\phi, £_\xi\phi) - \xi\cdot\,(\mathbf{E}\delta\phi + d\boldsymbol{\theta}(\phi,\delta\phi)) \\ &= \delta\boldsymbol{\theta}(\phi, £_\xi\phi) - £_\xi\boldsymbol{\theta}(\phi,\delta\phi) + d(\xi\cdot\boldsymbol{\theta}(\phi,\delta\,\phi)), \end{aligned} \tag{28}$$

where in the first line we used the fact that ξ^a is fixed ($\delta\xi^a$=0), in the second line we used the identity (4) and in the third line we applied **E**=0 together with Cartan´s formula on Lie derivatives: $£_\xi\boldsymbol{\lambda} = \xi\cdot\, d\boldsymbol{\lambda} + d(\xi\cdot\boldsymbol{\lambda})$. Using definition (12) we further identify in (28)

$$\boldsymbol{\omega}(\phi, £_\xi\phi, \delta\phi) = \delta\boldsymbol{\theta}(\phi, £_\xi\phi) - £_\xi\boldsymbol{\theta}(\phi,\delta\phi), \tag{29}$$

due to the covariant nature of $\boldsymbol{\theta}$. Inserting it into (28) we get

$$\begin{aligned} \delta\mathbf{J}[\xi] &= \boldsymbol{\omega}(\phi, £_\xi\phi, \delta\phi) + d(\xi\cdot\boldsymbol{\theta}(\phi,\delta\,\phi)) \\ \rightarrow\quad \boldsymbol{\omega}(\phi, £_\xi\phi, \delta\phi) &= \delta\mathbf{J}[\xi] - d(\xi\cdot\boldsymbol{\theta}(\phi,\delta\phi)). \end{aligned} \tag{30}$$

Since the equations of motion are satisfied, **E**=0, using Eq.(16) with $\mathbf{C}_a = 0$, and using $[\delta, \mathrm{d}] = 0$ we finally obtain

$$\boldsymbol{\omega}(\phi, £_\xi\phi, \delta\phi) = d[\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi)]. \tag{31}$$

as desired to show. ☐

After establishing the conceptual structure above, we are now ready to take the black hole spacetime into the consideration in the context of a theory specified by the Lagrangian (1) (see- [2,5]):

1). Consider a black hole spacetime with an asymptotically flat, stationary metric ($£_\xi\phi = 0$) with ξ^a representing the stationary Killing field at spatial infinity and axisymmetric metric ($£_\varphi\phi = 0$) with φ^a denoting the axial Killing field at spatial infinity. ξ^a is taken to have a unit norm and φ^a has a closed orbit with length 2π.

2). Allow the black hole possess an asymptotically flat bifurcate Killing horizon, with bifurcation surface B.

3). Assume Σ to be a spacelike surface that extends from the black hole bifurcation surface B to spatial infinity S_∞.

4). Let ϕ be a dynamical field satisfying the equations of motion, **E**=0, and set $\delta\phi$ be an arbitrary perturbation of dynamical fields which satisfy the linearized equations of motion, δ**E**=0, in a neighborhood of infinity, but not necessarily throughout the spacetime.

We refer to this set as ***Assumptions 1)-4).***

If, under ***Assumptions 1)-4)***, we integrate (27) over Σ, and use Stokes´ theorem, we obtain the following identity:

$$\int_{\Sigma} \boldsymbol{\omega}(\phi, \pounds_{\xi}\phi, \delta\phi) = \int_{S_{\infty}} \delta\mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi) - \int_{B} \delta\mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi), \tag{32}$$

which under some reasonable and expected conditions reveals the first law of black hole mechanics. But, before we see that, we need to take care of the terms involved in (32), seen from the light of decomposition ´´vacuum-matter´´ (which formalism was established above), leading explicitly to PI.

2.2. Evaluation of the identity (32)

Now, we are going to apply some results of section 2.1 to derive (following [2] very closely) the PI by evaluating (32). This identity plays crucial role for the calculations in section 5, concerning the derivation of the first law of black hole mechanics involving perfect fluids.

Let us start by considering the left side of (32). We take into account the ***Assumptions 1)-4)*** with additional information that horizon Killing field of a bifurcate Killing horizon is defined (by rigidity theorem) by

$$\chi^a = \xi^a + \Omega_B \varphi^a, \tag{33}$$

with constant angular velocity Ω_B. The horizon Killing field of a bifurcate Killing horizon is defined such that on the bifurcation surface B gives vanishing result ($\chi^a(B) = 0$).

To treat the left side of (32) effectively, further assumptions and definition, respectively, are in order [2,5]:

- assume spacelike surface Σ to be axisymmetric. This means that φ^a is tangent to Σ, such that the pull back to Σ vanishes
- assume that the matter fields ψ satisfy $\boldsymbol{E}_m = 0$, and that $\delta\psi$ satisfy $\delta\boldsymbol{E}_m = 0$ and also assume $\boldsymbol{E}_g = 0$ and $\delta\boldsymbol{E}_g$=0
- we define δ as a perturbation to an arbitrary nearby solution with fixed ξ^a and φ^a, $\delta\xi^a = \delta\varphi^a = 0$.

Then we reason as follows: the stationarity and axisymmetry of the metric, and the assumption of no restrictions on matter fields establishes $\boldsymbol{\omega}(\phi, \pounds_{\xi}\phi, \delta\phi) \neq 0$. Then the left side of (32) (written out in the form (29)) reads

$$\begin{aligned}
\boldsymbol{\omega}\big(\phi, \pounds_{\xi}\phi, \delta\phi\big) &= \delta\boldsymbol{\theta}(\phi, \pounds_{\xi}\phi) - \pounds_{\xi}\boldsymbol{\theta}(\phi, \delta\phi) \\
&= \delta(\boldsymbol{\theta}_{\boldsymbol{g}}(g, \pounds_{\xi}g) + \boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \pounds_{\xi}\phi)) - \pounds_{\xi}(\boldsymbol{\theta}_{\boldsymbol{g}}(g, \pounds_{\xi}g) + \boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \pounds_{\xi}\phi)) \qquad \text{(by (11))} \\
&= \delta\boldsymbol{\theta}_{\boldsymbol{g}}(g, \pounds_{\xi}g) - \pounds_{\xi}\boldsymbol{\theta}_{\boldsymbol{g}}(g, \delta g) + \delta\boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \pounds_{\xi}\phi) - \pounds_{\xi}\boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \delta\phi) \\
&= \delta\boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \pounds_{\xi}\phi) - \pounds_{\xi}\boldsymbol{\theta}_{\boldsymbol{m}}(\phi, \delta\phi). \qquad \text{(by stationarity of } g_{ab}\text{)}
\end{aligned} \tag{34}$$

Combining second equation of (21) and second equation of (24) to find $\boldsymbol{\theta}_m(\phi, \pounds_{\xi}\phi)$

$$\boldsymbol{\theta}_m(\phi, \pounds_{\xi}\phi) = d\mathbf{Q}_m[\xi] - \boldsymbol{\varepsilon} \cdot T \cdot \xi + \xi \cdot \boldsymbol{L}_{\boldsymbol{m}} - \boldsymbol{\varepsilon} \cdot E_m \cdot \psi \cdot \xi, \tag{35}$$

inserting it into (34) and recalling the assumption $\boldsymbol{E}_m = 0$ and Cartan´s formula for Lie derivatives $£_\xi \boldsymbol{\lambda} = \xi \cdot d\boldsymbol{\lambda} + \mathrm{d}(\xi \cdot \boldsymbol{\lambda})$ we obtain

$$\boldsymbol{\omega}\big(\phi, £_\xi\phi, \delta\phi\big) = \delta(d\mathbf{Q}_m[\xi] - \boldsymbol{\varepsilon}\cdot T \cdot \xi + \xi \cdot \boldsymbol{L_m}) - \xi \cdot d\boldsymbol{\theta_m}(\phi, \delta\phi) - \mathrm{d}(\xi \cdot \boldsymbol{\theta_m}(\phi, \delta\phi))$$

$$= d(\delta\mathbf{Q}_m[\xi] - \xi \cdot \boldsymbol{\theta_m}(\phi, \delta\phi)) - \delta(\boldsymbol{\varepsilon}\cdot T \cdot \xi) + \xi \cdot \delta\boldsymbol{L_m} - \xi \cdot d\boldsymbol{\theta_m}(\phi, \delta\phi). \tag{36}$$

where $[\delta, \mathrm{d}] = 0$ was used in the second line.

Using second equation of (8) we finally get

$$\boldsymbol{\omega}\big(\phi, £_\xi\phi, \delta\phi\big) = d(\delta\mathbf{Q}_m[\xi] - \xi\cdot \boldsymbol{\theta_m}(\phi, \delta\phi)) - \delta(\boldsymbol{\varepsilon}\cdot T\cdot\xi) + \xi \cdot (d\boldsymbol{\theta_m} + \tfrac{1}{2}\boldsymbol{\varepsilon}T^{ab}\delta g_{ab}) - \xi \cdot d\boldsymbol{\theta_m}(\phi, \delta\phi)$$

$$= d(\delta\mathbf{Q}_m[\xi] - \xi \cdot \boldsymbol{\theta_m}(\phi, \delta\phi)) - \delta(\boldsymbol{\varepsilon}\cdot T \cdot \xi) + \frac{1}{2}\xi \cdot \boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab}, \tag{37}$$

Moreover, substituting (11), (26) and (37) into (32), integrating over Σ and using Stokes´ theorem

$$\int_\Sigma d(\delta\mathbf{Q}_m[\xi] - \xi\cdot\boldsymbol{\theta_m}) - \delta(\varepsilon\cdot T\cdot\xi) + \frac{1}{2}\xi\cdot\boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} = \int_{S_\infty} \delta\big(\mathbf{Q}_g[\xi] + \mathbf{Q}_m[\xi] + d\mathbf{H}\big) - \xi\cdot\big(\boldsymbol{\theta}_g + \boldsymbol{\theta}_m\big)$$

$$- \int_{\mathrm{B}} \delta\big(\mathbf{Q}_g[\xi] + \mathbf{Q}_m[\xi] + d\mathbf{H}\big) - \xi\cdot(\boldsymbol{\theta}_g + \boldsymbol{\theta}_m)$$

$$= \int_\Sigma d(\delta\mathbf{Q}_m[\xi] - \xi\cdot\boldsymbol{\theta_m})$$

$$+ \int_{S_\infty} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g) - \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g), \tag{38}$$

and after making use of the property of exterior derivative $d^2 = 0$ we get

$$\int_\Sigma \frac{1}{2}\xi \cdot \boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} - \delta(\boldsymbol{\varepsilon} \cdot T \cdot \xi) = \int_{S_\infty} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g) - \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g). \tag{39}$$

If we use (33) to write

$$\xi^a = \chi^a - \Omega_B\varphi^a, \tag{40}$$

and insert it into the second term on the right side of (39)

$$\int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g) = \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\chi^a - \Omega_B\varphi^a] - (\chi^a - \Omega_B\varphi^a)\cdot\boldsymbol{\theta_g}(g, \delta g)$$

$$= \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\chi] - \Omega_B \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\varphi] - \int_B(\chi\cdot\boldsymbol{\theta_g}(g, \delta g) - \Omega_B\,\varphi\cdot\boldsymbol{\theta_g}(g, \delta g)), \tag{41}$$

and bearing in mind that χ^a vanishes at the bifurcation surface B, which yields $\chi \cdot \boldsymbol{\theta_g} = 0$, and φ^a vanishes because it is tangent to B, which yields, $\varphi \cdot \boldsymbol{\theta_g} = 0$, we finally obtain:

Theorem 2: Consider a spacetime with two Killing vectors ξ and φ. Let B be the bifurcation surface associated with the Killing vector (40). Then, assuming that all integrals converge, we have:

$$\int_{S_\infty} \delta\,\mathbf{Q}_g[\xi] - \xi\cdot\boldsymbol{\theta_g}(g, \delta g) = \int_\Sigma \frac{1}{2}\xi \cdot \boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} - \delta(\boldsymbol{\varepsilon} \cdot T \cdot \xi) + \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\chi] - \Omega_B \int_{\mathrm{B}} \delta\,\mathbf{Q}_g[\varphi]. \tag{42}$$

We refer to this equation as the *perturbative identity* (PI) [2]. As we are going to show in section 5, the first law (with perfect fluids included) represents a special case of this identity. Before we show it, it is reasonable to reveal the physical meaning of the terms involved in (42).

3. The physical interpretation of the PI

With regard to a possibility of meaningful physical interpretation of the surface integrals involved in the identity (42) we distinguish two possible cases [2]:

Case (a): Vanishing set of matter fields ψ

Case (b): Non-vanishing set of matter fields ψ

Let us analyze each of these two cases individually:

Case (a): In this case it is obvious that: $\boldsymbol{L_m} = 0 \Rightarrow T^{ab} = 0 \Rightarrow \boldsymbol{L} = \boldsymbol{L}_g$, $\boldsymbol{\theta} = \boldsymbol{\theta}_g$, and $\mathbf{Q} = \mathbf{Q}_g$. At spatial infinity, one usually assumes that the following holds [1]

$$\lim_{S\to S_\infty} \int_S \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi) = \lim_{S\to S_\infty} \int_S \xi \cdot \delta\boldsymbol{B}\,(\phi), \tag{43}$$

for some 3-form (not necessarily covariant) $\boldsymbol{B}$ depending on dynamical fields φ. Under the condition of **case (a)** it is straightforward to define familiar quantities [1]:

• The *canonical energy* $\mathcal{E}$ of an asymptotically flat region is defined as

$$\mathcal{E} = \int_{S_\infty} \mathbf{Q}[\xi] - \xi \cdot \boldsymbol{B}. \tag{44}$$

It can be shown [1] that in this particular case (vacuum general relativity) the definition (44) reduces to the standard, ADM, definition of mass-energy.

• The J_B is *canonical angular momentum* of the system measured at the black hole horizon and is defined by

$$J_B = -\int_{\mathrm{B}} \boldsymbol{Q}[\varphi]. \tag{45}$$

(As relevantly remarked in [1], the difference in signs between (44) and (45) can be traced back to the Lorentzian signature, which also appears in the special relativity when we define the energy ($E = -p_a t^a$) and the angular momentum ($J = +p_a\varphi^a$) of a particle with momentum p_a).

• The *entropy*, S_B, of the black hole horizon is defined by

$$S_B = 2\pi \int_B X^{cd}\, \boldsymbol{\varepsilon}_{cd}, \tag{46}$$

where X^{cd} is covariant quantity locally constructed out of the dynamical fields and their derivatives and is given by [1]

$$(X^{cd}) = -E^{abcd}\boldsymbol{\varepsilon}_{ab}\,, \tag{47}$$

where $\boldsymbol{\varepsilon}_{ab}$ denotes the binormal to B. The quantity E^{abcd} is defined as [1-(31)]

$$\boldsymbol{\varepsilon}E^{abcd} = \boldsymbol{\varepsilon}\left(\frac{\partial \boldsymbol{L}}{\partial R_{abcd}} - \nabla_{e_1}\frac{\partial \boldsymbol{L}}{\partial \nabla_{e_1} R_{abcd}} + \cdots + (-1)^m \nabla_{(e_1 \ldots e_m)} \frac{\partial \boldsymbol{L}}{\partial \nabla_{(e_1 \ldots e_m)} R_{abcd}}\right). \tag{48}$$

Let us now prove that the following equation is true:

$$\frac{\kappa}{2\pi}\delta S_B = \delta \int_{\mathrm{B}} \boldsymbol{Q}[\chi]\,. \tag{49}$$

(In order to be convinced about the consistency of definition of entropy (46), see - [1-section 7], where some calculations are performed for some specific Lagrangians, including also the case of our particluar used Lagrangian, (88), leading to familiar expressions).

Proof of (49):

We follow the proof in Wald and Iyer [1]. As proven in there (Proposition 4.1) the Noether charge can be non-uniquely decomposed as:

$$\boldsymbol{Q} = \boldsymbol{W}^c(\phi)\xi_c + \boldsymbol{X}^{cd}(\phi)\nabla_{[c}\xi_{d]} + \boldsymbol{Y}\big(\phi, £_\xi\phi\big) + d\boldsymbol{Z}(\phi, \xi), \tag{50}$$

where $\boldsymbol{W}^c, \boldsymbol{X}^{cd}, \boldsymbol{Y}$, and $\boldsymbol{Z}$ are covariant quantities dependent on the fields ϕ and their derivatives $\delta\phi$, where in addition $\boldsymbol{Y}$ depends linearly on $£_\xi\phi$, and $\boldsymbol{Z}$ depends linearly on ξ.

Insert Eq.(50) into (49) ($\frac{\kappa}{2\pi}\delta S_B = \delta\int_{\mathrm{B}}[\boldsymbol{W}^c(\phi)\xi_c + \boldsymbol{X}^{cd}(\phi)\nabla_{[c}\xi_{d]} + \boldsymbol{Y}\big(\phi, £_\xi\phi\big) + d\boldsymbol{Z}(\phi, \xi)]$) and observe that, due to (by definition) vanishing ξ on B the first term gives vanishing contribution. Also the term $\boldsymbol{Y}\big(\phi, £_\xi\phi\big)$ vanishes in the stationary background due to its linear dependence on $£_\xi\phi = 0$. We need also to prove that its variation does not contribute, which can be clearly seen from the simple calculation:

$$\delta\boldsymbol{Y}\big(\phi, £_\xi\phi\big) = \boldsymbol{Y}\big(\phi, £_\xi\delta\phi\big) = £_\xi\boldsymbol{Y}(\phi, \delta\phi) = \xi\cdot d\boldsymbol{Y} + d(\xi\cdot\boldsymbol{Y}), \tag{51}$$

where in the last line we used $£_\xi\boldsymbol{\lambda} = \xi\cdot d\boldsymbol{\lambda} + d(\xi\cdot\boldsymbol{\lambda})$. Also $d\boldsymbol{Z}$ clearly vanishes, so we are finally left with

$$\delta\int_{\mathrm{B}}\boldsymbol{Q}[\chi] = \delta\int_B \boldsymbol{X}^{cd}(\phi)\nabla_{[c}\xi_{d]}. \tag{52}$$

In order to evaluate the integral on the right side of (52) we appeal to the fact that for stationary background the following expression is true [1- (98)]

$$\nabla_c\xi_d = \kappa\,\boldsymbol{\varepsilon}_{cd}, \tag{53}$$

otherwise, for some w_{cd}, (see -[1-(100)]) one can write

$$\mathrm{w}_{cd} = \nabla_{[c}\xi_{d]} - \kappa\,\boldsymbol{\varepsilon}_{cd}. \tag{54}$$

The calculation of $\delta\mathrm{w}_{cd}$ yields

$$\delta\mathrm{w}_{cd} = -\kappa\, g_{a[d}\delta\boldsymbol{\varepsilon}^a_{c]}, \tag{55}$$

with only a normal-tangential component ([1- (101)]).

Inserting (54) into (52) and using definition (46), $\int_B X^{cd}\,\boldsymbol{\varepsilon}_{cd} = \frac{S_B}{2\pi}$, we finally obtain

$$\delta\int_B \boldsymbol{X}^{cd}(\phi)(\kappa\,\boldsymbol{\varepsilon}_{cd} + \mathrm{w}_{cd}) = \frac{\kappa}{2\pi}\delta S_B + \int_B \boldsymbol{X}^{cd}\delta\mathrm{w}_{cd}. \tag{56}$$

To finally prove (49) we need the expression (56) to have vanishing second term on the right side, which is provided by the following argument:

First, we will lay down what we know about $\boldsymbol{X}^{cd}$ and $\delta\mathrm{w}_{cd}$: as established in calculation (55) $\delta\mathrm{w}_{cd}$ possesses only a normal-tangential component, and if, in addition, one uses lemma 2.3 in [11], then $\delta\mathrm{w}_{cd}$ must reverse its sign under the map of the tangent space to B. Furthermore, $£_\xi\boldsymbol{X}^{cd} = 0$ due to $£_\xi\phi = 0$, which implies the invariance of $\boldsymbol{X}^{cd}$ under the map of the tangent space to B. Finally, we consider the pull-back of $\boldsymbol{X}^{cd}\delta\mathrm{w}_{cd}$ to B, which has only tangential component and therefore is invariant under the map, and therefore gives vanishing contribution, $\int_B \boldsymbol{X}^{cd}\delta\mathrm{w}_{cd} = 0$, yielding finally to

$$\delta \int_{\mathrm{B}} \boldsymbol{Q}[\chi] = \frac{\kappa}{2\pi}\delta S_B, \tag{57}$$

as desired to show. ☐

Thus we show that (42) is reduced to:

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\delta S_B + \Omega_B \delta J_B\,. \tag{58}$$

One can use the formalism and results above to also show the equivalence between the canonical angular momentum of the black hole, J_B, and the canonical angular momentum, J, at spatial infinity

$$J = -\int_{S_\infty} \boldsymbol{Q}[\varphi]. \tag{59}$$

in a setting of axisymmetric configurations. - One begins by observing the expression

$$\mathbf{J}[\varphi] = \boldsymbol{\theta}(\phi, \pounds_\varphi \phi) - \varphi \cdot \boldsymbol{L}\,, \tag{60}$$

where one replaces ξ^a with φ^a, with φ^a being an axial Killing field. Under the assumption that the metric is axisymmetric ($\pounds_\varphi \phi = 0$) and observing that pull-back of $\varphi \cdot \boldsymbol{L}$ gives no contribution ($\varphi \cdot \boldsymbol{L} = \mathbf{0}$) to the surface at spatial infinity (because we have already specified the surface at infinity to be tangent to φ^a (see section 2.2)) we obtain $\mathbf{J}[\varphi]$=0. Then by integrating Eq.(16) with $\mathbf{C}_a = 0$ over Σ, $\int_\Sigma d\mathbf{Q}[\varphi] = 0$, and using Stokes´ theorem, we get

$$\int_{\mathrm{B}} \boldsymbol{Q}[\varphi] = \int_{S_\infty} \boldsymbol{Q}[\varphi]\,. \tag{61}$$

as wanted. (See-[2] for a more complete analysis of other subtleties of angular momentum).

Case (b): This case is much more subtle, because one cannot neglect the incorporation of the ambiguities $\boldsymbol{L}_g \rightarrow \boldsymbol{L}_g + \lambda$ and $\boldsymbol{L}_m \rightarrow \boldsymbol{L}_m - \lambda$, which prevent us to straightforwardly apply the lesson of the **case (a),** namely, that the surface integrals in (42) are to be physically identified as the (variations of) mass-energy, entropy and angular momentum. In what way are these ambiguities relevant? - Their relevance consists in the fact that [2] every time one chooses a different $\boldsymbol{L_g}$ one gets a different choice of $\mathbf{Q}_g$, and as we already saw, our quantities at spatial infinity and at the horizon are, mainly, defined in terms of $\mathbf{Q}$.

One way to solve this problem is [2]:

c) to choose some $\boldsymbol{L_g}$, fix it and treat it as it specifies an independent theory and,

d) impose the condition that the stress–energy tensor T^{ab} is compact (falls off sufficiently rapidly at spatial infinity).

If the condition **d)** holds such that [2] the metrics of two theories ($\boldsymbol{L}$-theory and $\boldsymbol{L_g}$-theory) approach each other, and make sure that $\mathrm{M_g}$ defined as (see (44))

$$\mathrm{M_g} = \int_{S_\infty} \boldsymbol{Q}_g[\xi] - \xi \cdot \mathbf{B}_g. \tag{62}$$

gives the same result for both metrics, then it is conceivable to identify the left side of (42) as the variation of mass-energy of the system. Here, again, is assumed that at spatial infinity one has

$$\lim_{S \to S_\infty} \int_S \xi \cdot \boldsymbol{\theta_g}(\phi, \delta\phi) = \lim_{S \to S_\infty} \int_S \xi \cdot \delta \mathbf{B}_g\,(\phi). \tag{63}$$

for some 3-form $\mathbf{B}_g$ (not necessarily covariant) dependent on dynamical fields φ.

Two other surface terms are to be physically interpreted as the angular momentum measured at the black hole horizon J_{g_B} defined by (see (45))

$$J_{g_B} = -\int_{\mathrm{B}} \boldsymbol{Q}[\varphi], \tag{64}$$

and the entropy of black hole horizon, S_{g} defined by (see (46))

$$S_B = 2\pi \int_B X^{cd}\, \boldsymbol{\varepsilon}_{cd}\,, \tag{65}$$

such that (as proved in **case (a)**)

$$\frac{\kappa}{2\pi}\delta S_{\mathrm{g}} \;=\; \delta \int_{\mathrm{B}} \boldsymbol{Q}_g[\chi], \tag{66}$$

where $\boldsymbol{Q}_g$ is the Noether charge 2-form of the $\boldsymbol{L_g}$-theory, $\boldsymbol{\varepsilon}_{ab}$ is the binormal to the bifurcation surface and κ is the surface gravity of the black hole horizon.

Substituting (62), (64) and (66) into (42) we obtain:

$$\delta \mathrm{M}_{\mathrm{g}} = \frac{\kappa}{2\pi}\delta S_g + \Omega_B \delta J_{g_B} + \int_\Sigma \frac{1}{2}\xi \cdot \boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} - \delta(\boldsymbol{\varepsilon}\cdot T\cdot \xi), \tag{67}$$

where δ is an arbitrary perturbation to an infinitesimally close solution.

One can also appeal to the last argument of the **case (a)** and apply it consequently to this case, which in addition, provides us with an alternative version of (67). This can be done by a slightly simple argument [2]: one wants to find an explicit relationship between J_{g_B} and J_{g_∞}, respectively to express J_{g_B} in terms of J_{g_∞}, and insert it into (67). To do that, one needs to consider the identity (32) when one has replaced ξ^a with φ^a:

$$\int_\Sigma \boldsymbol{\omega}(\phi, \pounds_\varphi \phi, \delta\phi) = \int_{S_\infty} \delta \mathbf{Q}[\varphi] - \varphi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_B \delta\mathbf{Q}[\varphi] - \varphi\cdot\boldsymbol{\theta}(\phi,\delta\phi), \tag{68}$$

One performs the calculation of the left side of (68), using standard argument that φ^a is tangent to the bifurcation surface B and S_∞, and the pullback of $\varphi\cdot\boldsymbol{\theta}$ vanishes, then Eq. (68) (expressed as (42) with replacement $\xi^a = \varphi^a$) reduces straightforwardly to

$$\int_{\mathrm{B}} \delta\, \mathbf{Q}_g[\varphi] = \int_\Sigma \delta(\boldsymbol{\varepsilon}\cdot T\cdot\varphi) + \int_{S_\infty} \delta\, \mathbf{Q}_g[\varphi], \tag{69}$$

respectively,

$$J_{g_B} = J_{g_\infty} - \int_\Sigma \delta(\boldsymbol{\varepsilon}\cdot T\cdot\varphi), \tag{70}$$

with $J_{g_\infty} = -\int_{S_\infty} \mathbf{Q}_g[\varphi]$, as the angular momentum of the system measured at spatial infinity. Substitution of (70) into (67) finally yields:

$$\delta \mathrm{M}_{\mathrm{g}} = \frac{\kappa}{2\pi}\delta S_g + \Omega_B \delta J_{g_\infty} - \Omega_B \int_\Sigma \delta(\boldsymbol{\varepsilon}\cdot T\cdot\varphi) + \int_\Sigma \frac{1}{2}\xi\cdot\boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} - \delta(\boldsymbol{\varepsilon}\cdot T\cdot\xi). \tag{71}$$

4. Perfect fluids and perfect fluid stars

In this section, we gather some relevant results we need, namely, basic definitions and properties of the self-gravitating perfect fluids and perfect fluid stars, for the calculations in the following section. More complete and detailed account of what we discuss here can be found in [5]. For a curious reader we refer to [6] where you find broader treatment of perfect fluids.

All properties of the perfect fluids (defined such that the heat conduction, viscosity and shear stresses are excluded) are characterized by an energy density ρ=$\rho(n, s)$ which depends on the particle number density denoted by n, and an entropy per particle denoted by *s.* A pair of quantities (n, s) are assumed to define the local state of the perfect fluids. So, the particle number density n, entropy per particle s, fluid four-velocity u^a and spacetime metric g_{ab} represent, as stated in the Introduction, physical fields of the fluid. A stress–energy tensor of the perfect fluid is given by

$$T^{ab} = (p + \rho)u^a u^b + pg^{ab}, \tag{72}$$

with normalization condition $u^a u_a = -1$.

The integrated Gibbs-Duhem relation which defines the pressure of the perfect fluid p is given by [10,16]

$$p = -\rho + \mu n + \mathrm{T}ns\,, \tag{73}$$

(We will give slightly straightforward derivation of Eq.(73) [16]:

One begins with the first law of thermodynamics:

$$\mathrm{dE=TdS} - p\mathrm{dV} + \mu\mathrm{dN}, \tag{73.1}$$

and defines total energy E, entropy S and particle number N in terms of densities:

$$\mathrm{E}= \rho V,\ \mathrm{S=nsV \ and \ N=nV}\,, \tag{73.2}$$

and substitutes into (73.1)

$$\mathrm{d}(\rho V)\mathrm{=Td(nsV)} - p\mathrm{dV} + \mu\mathrm{d(nV)}, \tag{73.3}$$

and expands it

$$\rho\mathrm{dV} + \mathrm{Vd}\rho = \mathrm{TnsdV + TVd(ns)} - p\mathrm{dV} + \mu\mathrm{Vdn} + \mu\mathrm{ndV}. \tag{73.4}$$

One can apply (73.3) for the case V=1 which yields

$$\mathrm{d}\rho\mathrm{=Td(ns)} + \mu\mathrm{dn}. \tag{73.5}$$

Substituting (73.5) into (73.4)

$$\rho\mathrm{dV + VTd(ns)} + \mu\mathrm{Vdn = TnsdV + TVd(ns)} - p\mathrm{dV} + \mu\mathrm{Vdn} + \mu\mathrm{ndV}, \tag{73.6}$$

one obtains

$$\rho = \mathrm{Tns} - p + \mu\mathrm{n} \tag{73.7}$$

respectively,

$$p = -\rho + \mu n + \mathrm{T}ns \tag{73.8}$$

as desired).

If we differentiate the function $\rho = \rho\ (n, s)$

$$\mathrm{d}\rho = \frac{\partial\rho}{\partial n}\mathrm{dn} + \frac{\partial\rho}{\partial s}\mathrm{ds}, \tag{74}$$

and insert definitions of the temperature, T, and chemical potential, μ

$$T=\frac{1}{n}\frac{\partial\rho}{\partial s} \tag{75}$$

$$\mu=\frac{\partial\rho}{\partial n}-Ts\,, \tag{76}$$

into (74) we have

$$\begin{aligned} d\rho &= (\mu + Ts)dn + nTds \\ &= \mu dn + Tsdn + nTds \\ &= \mu dn + Td(ns), \end{aligned} \tag{77}$$

which represents the local first law of thermodynamics.

If we differentiate (73) and use (77) we obtain an alternative version of the local first law of thermodynamics:

$$\begin{aligned} dp &= -\, d\rho + \mu dn + nd\mu + nsdT + Td(ns) \\ &= -\, \mu dn - Td(ns) + \mu dn + nd\mu + nsdT + Td(ns) \\ &= nd\mu + nsdT. \end{aligned} \tag{78}$$

The following equations of motion hold for perfect fluids:

$$\nabla_a T^{ab} = 0, \text{ and } \nabla_a (nu^a) = 0. \text{ and} \tag{79}$$

$$u^a \nabla_a s = 0. \tag{80}$$

where ∇_a denotes a covariant derivative.

Total particle number, N, and total entropy, S, of the relativistic perfect fluid star are defined by integrating the particle number current (nu^a) and entropy current (nsu^a) over Σ

$$N= \int_\Sigma nu^a \nu_a \tag{81}$$

$$S= \int_\Sigma nsu^a \nu_a, \tag{82}$$

where Σ is a spacelike surface that stretches from the black hole bifurcation surface B to spatial infinity S_∞ and ν_a is the future directed normal on Σ.

We define a *perfect fluid star* [5] as a globally hyperbolic solution of the Einstein–fluid equations with compactly supported particle number density n.

A condition that the perfect fluid star has to satisfy to be in *dynamic equilibrium* [5] is that it has to be stationary and axisymmetric solution of the Einstein-fluid equations, together with a specification that u^a satisfies a (known as) circular flow condition (introduced, mainly, for technical convenience)

$$u^a = \frac{\xi^a + \Omega\varphi^a}{|v|}, \tag{83}$$

where Ω is the angular velocity of the perfect fluid (which is generally different from Ω_B), ξ^a is the stationary Killing field, φ^a is the axial Killing field, and $|v|$ is defined such that the following holds:

$$|v|^2 = -(\xi^a + \Omega\varphi^a)(\xi_a + \Omega\varphi_a). \tag{84}$$

We finally define three further (conserved) quantities necessary for the calculations in section 5:

- the fluid number density (or the particle current 3-form), N_{def}, by

$$N_{def} = nu^a \varepsilon_{adef}, \tag{85}$$

- the angular momentum density (or the angular momentum current 3-form), J_{def}, by

$$J_{def} = \varphi^a T_a^b \varepsilon_{bdef}, \tag{86}$$

- the entropy density (or the entropy current 3-form), S_{def}, by

$$S_{def} = nsu^a \varepsilon_{adef}, \tag{87}$$

where ε_{adef} is the usual volume element on spacetime.

5. Incorporating perfect fluids into the first law of black hole mechanics

Now we have all necessary ingredients at our disposal to evaluate the identity (67). If we impose the ***Assumptions 1)-4)*** again, together with an inclusion of the horizon Killing field of a bifurcate Killing horizon (which is defined such that $\chi^a(B)$ vanishes on the bifurcation surface B) defined by (33), and let the Einstein-Hilbert Lagrangian given by

$$\mathbf{L_g} = \frac{1}{16\pi} \mathrm{R}\, \boldsymbol{\varepsilon}, \tag{88}$$

and set $\mathbf{L_m}$ to be a perfect fluid Lagrangian (e.g. $\mathbf{L_m} = -\rho(n, s)\boldsymbol{\varepsilon}$); then under these particular circumstances it is already proven that the following is true [1]:

• $\mathrm{M_g} = \mathrm{M}$, representing standard ADM mass-energy (see (44))

• $J_{g_B} = J_B = -\int_{\mathrm{B}} \boldsymbol{Q}[\varphi]$ for black hole angular momentum

• $S_{\mathrm{g}} = \frac{A_B}{4}$, with A_B defining the horizon surface area

and therefore (67) becomes

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_B \delta J_B + \int_\Sigma \frac{1}{2}\xi \cdot \boldsymbol{\varepsilon}\, T^{ab}\delta g_{ab} - \delta(\boldsymbol{\varepsilon} \cdot T \cdot \xi) \tag{89}$$

Furthermore, we adopt following assumptions [5], for the purpose of effective calculations of the volume integral in the right side of (89):

- we assume the stress-energy tensor T^{ab} together with δT^{ab} has a compact support,
- we assume that the perturbed spacetime is such that $\delta\xi^a = \delta\varphi^a = 0$
- we choose spacelike surface Σ to be axisymmetric (φ^a is tangent to Σ) such that the pull back of $\varphi^a \varepsilon_{abcd}$ to Σ vanishes and,
- finally we assume that the four-velocity u^a satisfies (83).

Bearing all the above in mind, we evaluate the volume integral by following Wald, Green and Schiffrin [5]:

Recalling (83)

$$\xi^a = u^a|v| - \Omega\varphi^a, \tag{90}$$

we get

$$\begin{aligned}
&\int_\Sigma \xi^a \left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right] \\
&= \int_\Sigma (u^a|v| - \Omega\varphi^a)\left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right] \\
&= \int_\Sigma \left\{u^a|v|\left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right] + \Omega\delta\big(\varphi^a T_a^b\varepsilon_{bdef}\big)\right\},
\end{aligned} \tag{91}$$

where in the second line we used $\varphi^a\varepsilon_{adef} = 0$.

By recalling definitions (85)-(87), the expression (91) reduces to

$$\begin{aligned}
&\int_\Sigma \left\{u^a|v|\left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right] + \delta\big(\varphi^a T_a^b\varepsilon_{bdef}\big)\right\} \\
&= \int_\Sigma \left\{u^a|v|\left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right] + \Omega\,\delta J_{def}\right\} \\
&= \int_\Sigma \{D|v| + \Omega\delta J_{def}\},
\end{aligned} \tag{92}$$

with D = $u^a\left[\tfrac{1}{2}T^{bc}\delta g_{bc}\varepsilon_{adef} - \delta\big(T_a^b\varepsilon_{bdef}\big)\right]$.

Using the stress-energy tensor of the form (72), the relations (73) and (78) and the normalization condition $u^a u_a = -1$ together with the identities

$$u_a\delta u^a = -\tfrac{1}{2}u^b u^c\delta g_{bc} \tag{93}$$

$$\delta\varepsilon_{adef} = \tfrac{1}{2}g^{bc}\delta g_{bc}\varepsilon_{adef} \tag{94}$$

we finally evaluate the first integrand of (92):

$$\begin{aligned}
\mathrm{D} &= \tfrac{1}{2}u^a T^{bc}\delta g_{bc}\varepsilon_{adef} - u^a\delta\big(T_a^b\varepsilon_{bdef}\big) \\
&= \tfrac{1}{2}u^a[(p+\rho)u^b u^c + pg^{bc}]\,\delta g_{bc}\varepsilon_{adef} - u^a\delta[(p+\rho)u^b u_a\varepsilon_{bdef} + p\varepsilon_{adef}] \\
&= \tfrac{1}{2}(p+\rho)u^a u^b u^c\delta g_{bc}\varepsilon_{adef} + \tfrac{1}{2}u^a pg^{bc}\,\delta g_{bc}\varepsilon_{adef} - u^a\delta[(\mu + \mathrm{Ts})nu^b u_a\varepsilon_{bdef} + p\varepsilon_{adef}] \\
&= \tfrac{1}{2}(p+\rho)u^a u^b u^c\delta g_{bc}\varepsilon_{adef} + \tfrac{1}{2}u^a pg^{bc}\,\delta g_{bc}\varepsilon_{adef} - u^a\delta[u_a\mu N_{def} + u_a\mathrm{T}\,S_{def} + p\varepsilon_{adef}] \\
&= \mu\delta N_{def} + \mathrm{T}\delta S_{def} + \tfrac{1}{2}(p+\rho)u^a u^c u^b\delta g_{bc}\varepsilon_{adef} + \tfrac{1}{2}pu^a g^{bc}\,\delta g_{bc}\varepsilon_{adef} - \tfrac{1}{2}\mu nu^a u^b u^c\,\delta g_{bc}\varepsilon_{adef} \\
&\quad + nu^a\varepsilon_{adef}\delta\mu - \tfrac{1}{2}\,\mathrm{Ts}nu^a u^b u^c\,\delta g_{bc}\varepsilon_{adef} + nsu^a\varepsilon_{adef}\delta\mathrm{T} - u^a\varepsilon_{adef}\,\delta p - \tfrac{1}{2}pu^a g^{bc}\,\delta g_{bc}\varepsilon_{adef} \\
&= \mu\delta N_{def} + \mathrm{T}\delta S_{def} + \tfrac{1}{2}(p+\rho)u^a u^c u^b\delta g_{bc}\varepsilon_{adef} - \tfrac{1}{2}(\mu n + \mathrm{Ts}n)u^a u^c u^b\delta g_{bc}\varepsilon_{adef} \\
&\quad + u^a\varepsilon_{adef}(n\delta\mu + ns\delta\mathrm{T}) - u^a\varepsilon_{adef}\,\delta p = \mu\delta N_{def} + \mathrm{T}\delta S_{def}.
\end{aligned} \tag{95}$$

Inserting (95) into (92) and all together into (89) we obtain

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_B\delta J_B + \int_\Sigma\big(\mu|v|\delta N_{def} + T|v|\delta S_{def} + \Omega\,\delta J_{def}\big). \tag{96}$$

Recalling that redshifted temperature, $\overline{T}$, and chemical potential, $\overline{\mu}$, in GR are defined by

$$\overline{\mathrm{T}} = \mathrm{T}|v|, \qquad \bar{\mu} = \mu|v|. \tag{97}$$

then (96) becomes

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_B \delta J_B + \int_\Sigma \left(\bar{\mu}\delta N_{def} + \overline{\mathrm{T}}\delta S_{def} + \Omega\, \delta J_{def} \right) \tag{98}$$

which is the desired form of the first law of black hole mechanics with perfect fluids, and, in comparison to the identity (I) given in the Introduction, is true for arbitrary perturbations δ of the background spacetime.

If only a perfect fluid star were present, there wouldn't be a contribution from black hole horizon and therefore (98) reduces to

$$\delta M = \int_\Sigma \left(\bar{\mu}\delta N_{def} + \overline{\mathrm{T}}\delta S_{def} + \Omega\, \delta J_{def} \right) \tag{99}$$

which represents the first law of thermodynamics for relativistic perfect fluid stars, holding for arbitrary perturbations of perfect fluid stars in dynamic equilibrium (details of (99) can be found in [5]).

6. Proof of the physical process version of the first law of black hole mechanics revisited

The first law version derived in the previous section is also known as an ´´equilibrium state version´´, where one takes two infinitesimally close black hole solutions, in the phase space of solutions, and one compares their intensive and extensive parameters using (58) (or for that matter (98)) and it satisfies it. But there is another more exciting and logically independent version of the first law called the ´´physical process version´´, where one considers a quasi-static process in which a flux of matter (dropped in) perturbs the horizon of the black hole, initially in stationary configuration, and lets it dynamically evolve to a slightly new configuration with different parameters. And it turns out that it also satisfies the first law. As one will notice in calculations below, the ´´physical process version´´ has a local character and does not make any requirement to asymptotic structure of spacetime in comparison to equilibrium version. Proof of the physical process version for uncharged black holes was given in [13], but we will follow here (and work in the spirit of) [12] closely, to establish its validity for the same type of black hole. The generalized form of the first law involving charged black holes in Einstein-Maxwell theory is proved in a lengthy detail in [12, 17]. A detailed analysis of the first law in Einstein-Yang–Mills theory is given in [17].

As usual, we begin by laying down our guiding assumptions [2,5]:

-We assume that the dynamical field ϕ satisfies $\mathbf{E}$=0, and let $\delta\phi$ be any variation of the dynamical field satisfying $\delta\boldsymbol{E} = 0$, in the neighborhood of infinity but not necessarily throughout the spacetime.

- We let ξ^a be any fixed smooth vector field on the spacetime and assume that it is an asymptotic symmetry ($\pounds_\xi \phi = 0$).

We vary again Eq.(14) with only difference that we include the term $\mathbf{C}_a$ from Eq.(16):

$$\delta\mathbf{J}[\xi] = \delta\boldsymbol{\theta}(\phi, \pounds_\xi\phi) - \xi\cdot \delta\boldsymbol{L}$$

$$= \delta\boldsymbol{\theta}(\phi, \pounds_\xi\phi) - \xi\cdot d\boldsymbol{\theta}(\phi,\delta\phi)) \qquad \text{(by (4) and } \mathbf{E}\text{=0)}$$

$$= \delta\boldsymbol{\theta}(\phi, \pounds_\xi\phi) - \pounds_\xi\boldsymbol{\theta}(\phi,\delta\phi) + \mathrm{d}(\xi\cdot\boldsymbol{\theta}(\phi, \delta\,\phi) \qquad \text{(by Cartan´s formula)}$$

$$= \boldsymbol{\omega}(\phi,\pounds_\xi\phi,\delta\phi) + \mathrm{d}(\xi\cdot\boldsymbol{\theta}(\phi, \delta\,\phi). \qquad \text{(by definition (12))} \qquad (100)$$

Respectively,

$$\boldsymbol{\omega}(\phi,\pounds_\xi\phi,\delta\phi\,)= \delta(\xi^a\mathbf{C}_a) + d[\delta\mathbf{Q}\,[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\,\phi)]. \qquad \text{(by (16))} \qquad (101)$$

Since $\boldsymbol{\omega}(\phi,\pounds_\xi\phi,\delta\phi)$ vanishes (by $\pounds_\xi\phi = 0$) we have

$$d[\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) = -\,\xi^a\delta\mathbf{C}_a. \qquad (102)$$

Integrating (102) over a spacelike surface, Σ, and using Stokes´ theorem we get:

$$\int_{S_\infty}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_{\partial\Sigma}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) = -\int_\Sigma\xi^a\delta\mathbf{C}_a, \qquad (103)$$

where $\partial\Sigma$ represents the boundary of Σ.

Recall that the first term on the left side of (103) is defined as [14]

$$\delta\mathrm{H}_\xi = \int_{S_\infty}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi). \qquad (104)$$

where H_ξ denotes a conserved quantity (see-[14] for detailed discussion on conserved quantities).

Thus, we rewrite (103) as:

$$\delta\mathrm{H}_\xi = \int_{\partial\Sigma}\delta\mathbf{Q}[\xi] - \xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_\Sigma\xi^a\delta\mathbf{C}_a\,. \qquad (105)$$

If ϕ satisfies the equations of motion and ξ^a is an asymptotic symmetry ($\pounds_\xi\phi = 0$), one can define the current α as [12]

$$\alpha^a = -\frac{1}{3!}\,\varepsilon^{abcd}\,\delta\mathrm{C}_{bcde}\xi^e. \qquad (106)$$

which satisfies conservation equation $\nabla_a\alpha^a = 0$, where ε^{abcd} is the invariant volume element and $\mathrm{C}_{bcd} = \mathrm{C}_{bcde}\xi^e$ is a constraint 3- form, locally constructed out of the dynamical fields, ϕ.

After setting the stage, we will now proceed by considering (for simplicity of demonstration) a special case, in which Lagrangian si given by (88) (which corresponds to the case **a)** of section 4), and calculate the variation of ADM mass δM, angular momentum δJ, and the variation of horizon area δA, and prove the validity of the first law of black hole mechanics (case **a)**, section 3, Eq.(58)).

For this Lagrangian the constraints C_{bcda} are given by

$$\mathrm{C}_{bcda} = \varepsilon_{ebcd}\;T^e_a\,. \qquad (107)$$

We want to establish the validity of Eq.(107) by means of the following calculation (see [15] for more calculations of this type for other given Lagrangians):

Let us write out the Lagrangian (88) in the form, $\mathrm{L}=\frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}R_{\mu\nu}$, and perform the first variation on it:

$$\delta\boldsymbol{L} = \frac{1}{16\pi}(\delta\sqrt{|g|}g^{\mu\nu}R_{\mu\nu} + \sqrt{|g|}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu})\,, \qquad (108)$$

with

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}, \tag{109}$$

$$\delta R_{\mu\nu} = \nabla_a\left(\delta\Gamma^a_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^a_{\mu a}\right), \tag{110}$$

where $\Gamma^a_{\mu\nu}$ are called Christoffel symbols and the explicit form of their variation is given by

$$\delta\Gamma^a_{\mu\nu} = \frac{1}{2}g^{ab}(\nabla_\mu\delta g_{b\nu} + \nabla_\nu\delta g_{b\mu} - \nabla_b\delta g_{\mu\nu}). \tag{111}$$

The identity (110) is given in a coordinate system in which $\Gamma^a_{\mu\nu} = 0$ called `´Riemann normal coordinates´´ [9].

With Eq.(109) and (110), Eq.(108) becomes

$$\delta\boldsymbol{L} = \frac{1}{16\pi}[\sqrt{|g|}(\mathrm{R}g^{ab} - R^{ab})\,\delta g_{ab} + \sqrt{|g|}g^{\mu\nu}(\nabla_a\delta\Gamma^a_{\mu\nu} - \nabla_\nu\delta\Gamma^a_{\mu a})]. \tag{112}$$

Comparing it with $\delta\mathbf{L} = \boldsymbol{E}^{ab}\delta g_{ab} + \nabla_a\boldsymbol{\theta}^a$ we clearly recognize

$$\boldsymbol{E}^{ab} = \frac{1}{16\pi}[\sqrt{|g|}(\mathrm{R}g^{ab} - R^{ab})] = \frac{1}{16\pi}\sqrt{|g|}G^{ab},$$

$$\boldsymbol{\theta}^a = \frac{1}{16\pi}\sqrt{|g|}\left(g^{\mu\nu}\delta\Gamma^a_{\mu\nu} - g^{\mu a}\delta\Gamma^\nu_{\mu\nu}\right). \tag{113}$$

By using (111) one can rewrite $\boldsymbol{\theta}^a$ as

$$\begin{aligned}\boldsymbol{\theta}^a &= \frac{1}{16\pi}\,[\frac{1}{2}\sqrt{|g|}g^{\mu\nu}g^{ab}(\nabla_\mu\delta g_{b\nu} + \nabla_\nu\delta g_{b\mu} - \nabla_b\delta g_{\mu\nu})\\ &\quad -\frac{1}{2}\sqrt{|g|}g^{\mu a}g^{\nu b}(\nabla_\mu\delta g_{b\nu} + \nabla_\nu\delta g_{b\mu} - \nabla_b\delta g_{\mu\nu})]\\ &= \frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}g^{ab}(\nabla_\nu\delta g_{b\mu} - \nabla_b\delta g_{\mu\nu})\,.\end{aligned} \tag{114}$$

The symplectic potential $\boldsymbol{\theta}^a$ written as a 3-form becomes

$$\theta_{\mu\nu b} = \frac{1}{16\pi}\varepsilon_{a\mu\nu b}g^{a\lambda}g^{\rho\sigma}\left(\nabla_\rho\delta g_{\lambda\sigma} - \nabla_\lambda\delta g_{\rho\sigma}\right). \tag{115}$$

We further calculate the Noether current (see definition (14)) written in a form notation

$$\begin{aligned}\mathrm{J}_{\mu\nu b} &= \theta_{\mu\nu b} - \xi^a L_{a\mu\nu b}\\ &= \left[g^{a\lambda}g^{\rho\sigma}\left(\nabla_\rho\delta g_{\lambda\sigma} - \nabla_\lambda\delta g_{\rho\sigma}\right) - R\xi^a\right]\varepsilon_{a\mu\nu b}.\end{aligned} \tag{116}$$

Using the definition of Einstein tensor: $G_{a\lambda} = R_{a\lambda} - \frac{1}{2}Rg_{a\lambda}$ and transforming it as

$$R\xi^a = 2(R^a_\lambda\xi^\lambda - G^a_\lambda\xi^\lambda), \tag{117}$$

and using $\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$, then the Noether current (116) becomes

$$\mathrm{J}_{\mu\nu b} = \frac{1}{8\pi}\left(\nabla_\lambda\left(\nabla^{[\lambda}\xi^{a]}\right) + G^a_\lambda\xi^\lambda\right)\varepsilon_{a\mu\nu b.} \tag{118}$$

The constraint 3-form ($C_\xi = \xi^a\mathrm{C}_a$) is obtained by taking $\nabla_\lambda\left(\nabla^{[\lambda}\xi^{a]}\right) = 0$

$$\left(C_\xi\right)_{\mu\nu b} = \frac{1}{8\pi}G^a_\lambda\xi^\lambda\varepsilon_{a\mu\nu b}. \tag{119}$$

After relabeling some indices we write

$$C_{abc} = \frac{1}{8\pi}\xi^d G^e_d\varepsilon_{eabc}. \tag{120}$$

Using Einstein equations

$$G_{ab} = 8\pi\, T_{ab,} \tag{121}$$

and inserting it into (119) we finally obtain (after relabelling the indices again)

$$\mathrm{C}_{bcda} = \varepsilon_{ebcd}\; T^{e}_{\;a}\;. \tag{122}$$

as desired to show.

The variation of (107) yields

$$\delta\mathrm{C}_{bcda} = \varepsilon_{ebcd}\;\delta T^{e}_{\;a} \tag{123}$$

where one usually assumes for g_{ab} to be a solution of Einstein equations and δg_{ab} are solutions of linearized Einstein equations with δT_{ab} as a source.

Inserting (123) into (105) we get:

$$\delta \mathrm{H}_{\xi} = \int_{\partial\Sigma} \delta\mathbf{Q}[\xi] - \;\xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_{\Sigma} \varepsilon_{ebcd}\,\xi^{a}\;\delta T^{e}_{\;a}\,. \tag{124}$$

As discussed in [14], if we choose ξ^{a} to be a stationary Killing field (with unit norm) at spatial infinity we get the variation of ADM mass

$$\delta M = \int_{\partial\Sigma} \delta\mathbf{Q}[\xi] - \;\xi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_{\Sigma} \varepsilon_{ebcd}\,\xi^{a}\;\delta T^{e}_{\;a}\,. \tag{125}$$

and if we choose ξ^{a} to be the axial Killing field (with closed orbits) φ^{a} then we get the variation of angular momentum

$$\delta J = \int_{\partial\Sigma} \delta\mathbf{Q}[\varphi] - \;\varphi\cdot\boldsymbol{\theta}(\phi,\delta\phi) - \int_{\Sigma} \varepsilon_{ebcd}\,\varphi^{a}\;\delta T^{e}_{\;a}\,. \tag{126}$$

Let us now imagine we throw some (uncharged) matter into the black hole, which is initially in a stationary configuration, and impose the following assumptions [12]:

I) assume that the black hole is not destroyed during this process
II) assume all of the matter falls into the black hole and that it settles down to a stationary final state
III) assume that δT_{ab} vanishes at spatial infinity and
IV) δg_{ab} vanishes in a neighborhood of horizon B and finally
V) let the horizon Killing field of this black hole be defined by (33) and Σ be a spacelike surface that terminates on the event horizon B of the black hole.

By using the assumption IV) and constructing the difference (125) – Ω_B(126) (which is plausible, because we are already familiar with the form of the first law, which we want to obtain) we have

$$\delta\mathrm{M} - \Omega_B\delta\, J = -\int_{\Sigma} \varepsilon_{ebcd}\,\xi^{a}\;\delta T^{e}_{\;a}\,, \tag{127}$$

which can be rewritten as

$$\delta\mathrm{M} - \Omega_B\delta\, J = \int_{\Sigma} \alpha^{e}\;\mathrm{n}_e\;\bar{\varepsilon}_{abc}\,, \tag{128}$$

where one identifies the current $\alpha^{a} = \xi^{a}\;\delta T^{e}_{\;a}$, n_e is future-directed normal to B and one choses the orientation such that $\bar{\varepsilon}_{abc} = n^{e}\varepsilon_{eabc}$.

Using the assumption II), and taking into account $\nabla_a\alpha^{a} = 0$ we have from (128)

$$\delta\mathrm{M} - \Omega_B\delta\, J = \int_{\mathrm{B}} \alpha^{e}\;\mathrm{k}_e\;\bar{\varepsilon}_{abc} \tag{129}$$

respectively,

$$\delta \mathrm{M} - \Omega_B \delta\, J = \int_{\mathrm{B}} \xi^e\, \delta T_e^d\, \mathrm{k}_d. \tag{130}$$

where k_d is tangent to null geodesic generator of the event horizon B of a stationary black hole and the substitution of the relation $\alpha^a = \xi^a\, \delta T_a^e$ in (130) is obvious.

In order to calculate the right side of (130) we will use the (perturbed) Raychaudhuri equation given by

$$\frac{d}{dV}(\delta\theta) = \delta\,(-\,\tfrac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b)\,, \tag{131}$$

with V defined as affine parameter associated to k^a, θ is the expansion (which measures the local change of cross-sectional area and is usually defined as $\theta = \nabla_a k^a$), and σ_{ab} represents the shear (which roughly measures the tendency of a matter to become distorted from one shape to the other). The expansion θ is also given by

$$\theta = \frac{1}{A}\frac{dA}{d\lambda}, \tag{132}$$

where λ denotes an affine parametrization of the null geodesic generators. For our case of a stationary background, Eq.(131) reduces to

$$\frac{d}{dV}(\delta\theta) = -8\pi\, \delta(T_{ab}k^a k^b)\,. \tag{133}$$

justified by the fact that the expansion, θ, and shear, σ_{ab}, give vanishing contribution in the stationary background, and the last term are simply Einstein equations: $R_{ab}k^a k^b = 8\pi\, T_{ab}k^a k^b$.

Furthermore, we need to take care of the term δk^a present in (133). To treat it effectively, we choose a reasonable gauge [12] such that the null generators of the event horizon of the perturbed and unperturbed black hole are identified, which leads to $\delta k^a \approx k^a$, and therefore (133) becomes

$$\frac{d}{dV}(\delta\theta) = -8\pi\, \delta T_{ab}k^a k^b. \tag{134}$$

A detailed calculation of (134) can be found in [13], but the idea is basically as follows:

One expresses k^a in the following form:

$$\begin{aligned} k^a &= \left(\frac{\partial}{\partial V}\right)^a \\ &= \frac{1}{\kappa \mathrm{V}}\chi^a \\ &= \frac{1}{\kappa \mathrm{V}}(\xi^a + \Omega_B \varphi^a), \end{aligned} \tag{135}$$

where in the last line Eq.(33) was used.

If we multiply both sides of (134) by κV and integrate over B we get

$$\kappa \int_0^\infty dV \int V\, \frac{d(\delta\theta)}{dV} d^2A = -8\pi\,(\delta \mathrm{M} - \Omega_B \delta\, J) \tag{136}$$

For the left side of Eq.(136) one performs an integration by parts, where the resulting first term $\int d^2A(\delta\theta\, V)$ vanishes, via the substitution of the upper and lower limit, and the second term gives $\int_0^\infty \delta\theta dV \int d^2A \;= -\delta \mathrm{A}$ after substituting Eq.(132). Therefore Eq. (136) becomes

$$- \kappa\, \delta \mathrm{A} = -8\pi\, (\delta \mathrm{M} - \Omega_B \delta\, J)$$

$$\rightarrow \quad \delta \mathrm{M} - \Omega_B \delta\, J = \frac{\kappa}{8\pi}\, \delta \mathrm{A} \tag{137}$$

If we insert (137) into (130) we obtain

$$\kappa\, \delta A = 8\pi \int_{\mathrm{B}} \xi^e\, \delta T_e^d\, \mathrm{k}_d, \tag{138}$$

which clearly proves the validity of the first law of black hole mechanics (58).

7. Conclusion and overview

In this thesis, using necessary mathematical background arising from the method of Lagrangian diffeomorphism covariant theories, we have derived the first law of black hole mechanics which, in addition, incorporates perfect fluids. This is, as shown, a special case of the perturbative identity (PI) (42), with $\mathbf{L_g}$ being the Einstein-Hilbert Langrangian for general relativity and $\mathbf{L_m}$ any Lagrangian for a perfect fluid. The assumption was that the background solution is an asymptotically flat, stationary, axisymmetric black hole with bifurcate Killing horizon. We demanded that metric g_{ab} is stationary and axisymmetric in the background spacetime, satisfying the field and constraint equations, and we imposed no restrictions on matter fields. In that case symplectic current 3-form in the left side of (32) was non-vanishing, leading us to the expression of the first law of black hole mechanics. Furthermore, we considered the first law of thermodynamics for perfect fluid stars (defined as [5] globally hyperbolic, asymptotically flat solution to Einstein-fluid equations) and compared it to the first law of black hole mechanics. The conclusion drawn is that the only difference between them lies in the terms involving contributions measured at the black hole horizon, namely, the entropy and angular momentum of black hole horizon. We also discussed another version of the first law, namely, the ´´physical process version´´ which simply involves the process of dropping in some matter into the initially stationary black hole under the assumption that it settles down to a stationary final state. We then computed δM, $\delta\, J$ and δA for the Lagrangian (88), and verified that the first law (58) holds. We finally conclude by giving a remark in Gao and Wald [12]: ``if the ´´physical process version´´ of the first law (58) were to fail, this would give rise to an inconsistency with the assumption that the black hole settles down to a final stationary state, and would thereby provide strong evidence against cosmic censorship. Conversely, a proof of the `´physical process´´ version of the first law would provide support for cosmic censorship.´´

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Abstract

We derive the first law of black hole mechanics involving perfect fluids in the context of Lagrangian diffeomorphism covariant theories. This is a generalized form of the formula derived by Bardeen, Carter and Hawking [4]. This is also known as equilibrium state version, because it compares the areas of two infinitesimally close stationary black hole solutions to Einstein equations. After defining some important quantities as preliminaries, we derive the equation to which we refer as the perturbative identity (PI). After identifying two possible meaningful ways of interpretation of surface terms appearing in this identity and gathering some definitions and relevant properties of the perfect fluids and the perfect fluid stars the first law follows naturally out of the PI. Furthermore, we analyze another version of the first law called the physical process version, which involves a dropping of a bit of matter into the horizon of initially stationary black hole and we assume that it settles down to a stationary final state. The fact that this law can be proved is, as rightly remarked in [12], a strong support for the ´´cosmic censorship´´.

Zusammenfassung

In dieser Arbeit leiten wir das erste Gesetz der Schwarzlochmechanik mit perfekten Flüssigkeiten im Kontext der Lagrange'schen Diffeomorphismus-Kovarianten-Theorien ab. Dies ist eine verallgemeinerte Form der Formel, die von Bardeen, Carter und Hawking [4] abgeleitet wurde. Dies wird auch als Gleichgewichtszustandsversion bezeichnet, da sie die Bereiche zweier infinitesimaler stationärer Schwarzlochlösungen mit Einstein-Gleichungen vergleicht. Nachdem wir einige wichtige Größen als Vorbedingungen definiert haben, leiten wir die Gleichung ab, auf die wir uns als Perturbative (störende) Identität (PI) beziehen. Nachdem zwei mögliche sinnvolle Interpretationsweisen von Oberflächenbegriffen, die in dieser Identität vorkommen, identifiziert wurden und einige Definitionen und relevante Eigenschaften von perfekten Flüssigkeiten und perfekten Fluidsternen gesammelt wurden, folgt das erste Gesetz natürlich aus dem PI. Darüber hinaus analysieren wir eine andere Version des ersten Gesetzes, die als physikalische Prozessversion bezeichnet wird, die ein Absinken eines Teils der Materie in den Horizont des zunächst stationären Schwarzen Lochs beinhaltet und wir gehen davon aus, dass es sich in einen stationären Endzustand überführt. Die Tatsache, dass dieses Gesetz nachgewiesen werden kann, ist, wie zu Recht in [12] erwähnt, eine starke Unterstützung für die ´´Kosmische Zensur (cosmic censorship)´´.