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„Extended TQFTs valued in the Landau-Ginzburg bicategory“

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Abstract

This thesis determines framed and oriented extended two-dimensional topological quantum field theories (TQFTs) valued in the bicategory of Landau-Ginzburg models $\mathcal{LG}$. This bicategory is important, for example, in knot theory, string theory, and homological mirror symmetry. More specifically, the present study is inspired by work on homological (or categorified) knot invariants.

First, we recall $\mathcal{LG}$ as a bicategory with adjoints. Roughly, objects of $\mathcal{LG}$ are polynomials with an isolated singularity at the origin. The category of morphisms between two such polynomials is a homotopy category of matrix factorizations of their difference. We detail the definition of $\mathcal{LG}$ including some subtleties not mentioned in the literature.

Then we construct an explicit symmetric monoidal structure on $\mathcal{LG}$. The monoidal product of two objects is basically the sum of polynomials. As for the morphisms in the symmetric monoidal structure the unit 1-morphisms in $\mathcal{LG}$ and their unitors are vital. Also functors of restriction of scalars for matrix factorizations along ring isomorphisms feature prominently.

In the third part of the body of the thesis we define the dual of an object of $\mathcal{LG}$, essentially minus the polynomial. Building on the unit 1-morphisms of $\mathcal{LG}$, we single out coevaluation and evaluation morphisms for these duals. It follows that every object of $\mathcal{LG}$ is fully dualizable.

Before turning to the last investigations of this thesis we recall the bicategorical cobordism hypothesis and its analogue for oriented bordisms. The former cobordism hypothesis combines with the conclusion of the preceding chapter to establish that every object determines a framed extended TQFT valued in $\mathcal{LG}$. Next, we prove that precisely for those objects of $\mathcal{LG}$ which are given by polynomials in an even number of variables the corresponding Serre automorphism is trivializable. This implies that these objects determine oriented extended TQFTs valued in $\mathcal{LG}$. Finally, we introduce a bicategory LG very similar to $\mathcal{LG}$. In particular, it has the same objects and 1-morphisms. As opposed to $\mathcal{LG}$, in LG there are 2-morphisms of both even and odd degree (each of these morphisms being identified with minus itself via a $\mathbb{Z}_2$-action). This enables that every one of its objects determines an oriented extended TQFT valued in LG. We discuss an example of such a TQFT related to our knot-theoretic inspiration.

Our proofs are greatly simplified by some coherence results. These generalize known coherence theorems to include the effects of functors of restriction of scalars along ring isomorphisms.

Contents

1. Introduction

This thesis is motivated by the desire to better understand one class of two-dimensional topological quantum field theories (2d TQFTs). Namely, we study the bicategory $\mathcal{LG}$ of Landau-Ginzburg models. Roughly, its objects are polynomials with an isolated singularity at the origin. 1-morphisms are matrix factorizations. This bicategory relates e.g. to conformal field theory and is thus of interest to string theory. Moreover, it features prominently in the categorified knot invariants defined by Mikhail Khovanov and Lev Rozansky in [KR1, KR2]. In this thesis, we determine extended 2d TQFTs valued in $\mathcal{LG}$.

We devote this introduction to an account of some context focusing mainly on the mathematical description of TQFTs. In particular we comment on extended and on defect TQFTs. The latter allow us to smoothly transition to a few words on $\mathcal{LG}$. Finally, we provide a short summary of our results.

TQFTs are a subject of mathematical physics and pure mathematics which is rooted in theoretical physics, cf. Edward Witten's [W]. As this thesis is dedicated to mathematical investigations related to TQFTs we do however not go into their origins in physics beyond the following remark.

In theoretical physics TQFTs are a special kind of quantum field theories. The latter are commonly described in terms of path integrals, which are, however, often ill-defined. These path integrals are used to compute certain numbers which for theories modeling phenomena that can be probed experimentally, can be compared with measured data. In TQFTs these numbers do not depend on the metric of space-time, thus the adjective *topological*.

Their eponymous property makes TQFTs particularly accessible to the approach to formalizing quantum field theories in mathematically rigorous terms originally proposed by Graeme Segal in [Se] for conformal field theories (in which not only the topology but also a conformal structure on space-time is relevant to the quantities of physical interest). In line with [Se] Michael Atiyah suggested an axiomatization of TQFTs, cf. [A]. For a pedagogical motivation of Atiyah's axioms we recommend [CR4]. Here we concentrate on the mathematical description of TQFTs inferred from [A], cf. eg. [K].

We denote by $\mathbb{N}$ the natural numbers without 0. Let $n \in \mathbb{N}$. The standard way of paraphrasing Atiyah's original formalization of an n-dimensional TQFT is that it is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\Bbbk}. \qquad (1.0.1)$$

Here $\mathrm{Vect}_{\mathrm{k}}$ is the category of vector spaces over a field k. Together with the tensor product of vector spaces, its canonical associators, the natural isomorphisms $V \otimes W \cong W \otimes V$, $V \in \mathrm{Vect}_{\mathrm{k}}$, $W \in \mathrm{Vect}_{\mathrm{k}}$ for a braiding, the field k as a unit object and its canonical unitors this is a symmetric monoidal category.

Maybe less popularly, $\mathrm{Bord}_{n,n-1}$ is a category whose objects are $(n-1)$-dimensional closed manifolds. Morphisms in $\mathrm{Bord}_{n,n-1}$ are certain diffeomorphism classes of n-bordisms, i.e. n-dimensional bordisms. Their precise definition takes into account some subtleties faced in realizing the idea that morphisms between two $(n-1)$-manifolds are n-manifolds whose boundary is the former's disjoint union. Bordisms are composed by gluing them along their common boundaries. Unit morphisms are represented by cylinders. $\mathrm{Bord}_{n,n-1}$ allows for a symmetric monoidal structure whose product is disjoint union, with "twist bordisms" (a special case of the so-called cylinder construction, i.e. their underlying n-manifolds are the disjoint union of two cylinders) as symmetry and the empty manifold as unit object.

Note that a closed n-manifold can be regarded as a bordism from the empty set to the empty set. The functor Z of (1.0.1) assigns a number to such a manifold which corresponds to the numbers which are computed in theoretical phyiscs, cf. above. Thereby TQFTs yield invariants of manifolds. This has sparked considerable interest in mathematicians.

What makes these invariants particularly computable is that since Z is a functor it goes well along with gluing of manifolds. This allows to calculate its value on a given manifold by splitting the latter into smaller pieces on which Z may be easier to evaluate, applying Z to these and composing the results. This strategy underpins the known classification results for TQFTs as in (1.0.1).

In the seminal work [BD] John Baez and James Dolan envisioned a program taking the logic of decomposing a manifold into smaller bits even more seriously. This is about *extended* TQFTs as higher functors between higher categories. More accurately, they consider what in later terminology is referred to as *fully extended* n-dimensional TQFTs which are symmetric monoidal n-functors valued in arbitrary symmetric monoidal n-categories. Their source n-category is a bordism n-category "extended" to the point.

For example, one can think of a bordism n-category $\mathrm{Bord}_n^{\mathrm{fr}}$ whose objects are disjoint unions of framed points, 1-morphisms are framed 1-bordisms, 2-morphisms are framed 2-bordisms etc. up to n-morphisms which are equivalence classes of framed n-bordisms. Then Baez and Dolan conjectured that fully extended TQFTs with source $\mathrm{Bord}_n^{\mathrm{fr}}$ correspond to 'objects with duals' in their target n-category. This is the *cobordism hypothesis*.

There is however a caveat: all higher categories and higher functors are assumed to be weak. Yet there is no complete description of such weak higher algebraic structures for $n > 4$.

This obstacle was overcome by Jacob Lurie in [Lu] using sophisticated techniques of (∞, n)-categories. These allowed him to turn the ideas of [BD] into clearer statements whose proof he sketched extensively. In particular he replaced 'objects with duals' by *fully dualizable* objects. Moreover he suggested how to generalize the cobordism hypothesis for bordism categories with tangential structures other than a framing.

In parallel to Lurie's work, Christopher Schommer-Pries developed a more explicit approach to extended 2d TQFTs in [SP1].[1] Indeed, in two dimensions there is a well-understood theory of weak 2-categories, i.e. bicategories, which are the appropriate stage on which to place extended 2d TQFTs. Schommer-Pries classified oriented and unoriented extended 2d TQFTs by giving a generators and relations presentation of the respective bordism bicategories. Hence every such TQFT is determined by what it assigns to a finite set of data.

Using similar techniques Piotr Pstragowski gave a presentation of the framed 2d bordism bicategory $\mathrm{Bord}_2^{\mathrm{fr}}$ in [P] thereby classifying framed extended 2d TQFTs with values in an arbitrary symmetric monoidal bicategory $\mathcal{B}$, i.e. symmetric monoidal 2-functors $Z : \mathrm{Bord}_2^{\mathrm{fr}} \to \mathcal{B}$. Moreover [P] employs this presentation to prove the bicategorical cobordism hypothesis as inspired by [Lu]. We recall some more details of this work in Section 5.1.1.

In [H] Jan Hesse also proves the bicategorical version of Lurie's cobordism hypothesis for oriented bordisms. This is reviewed in Section 5.1.2 below.

We now turn to another way in which higher categories feature in studying TQFTs, so-called *defect* TQFTs, cf. [DKR] for the two-dimensional case $n = 2$ and [CRS] for $n \in \mathbb{N}$. Here one equips the bordisms with embedded submanifolds. These are additionally labeled by some chosen data $\mathbb{D}$. Physically, the labels for top-dimensional manifolds can be interpreted as "theories". Lower dimensional submanifolds are thought of as "defects". Equipping both the objects and morphisms of $\mathrm{Bord}_{n,n-1}$ with such labeled submanifolds in a compatible way results in a symmetric monoidal category $\mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D})$. An n-dimensional defect TQFT is then defined to be a symmetric monoidal functor

$$Z : \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \to \mathrm{Vect}_{\mathrm{k}}.$$

It is expected that one can extract a weak n-category from such a functor. Its objects are the theories, 1-morphisms are composed of the labels of defects, etc. Finally, the set of n-morphisms results from applying Z to certain labeled $(n-1)$-spheres. For $n = 2$ this is worked out in [DKR], cf. the lecture notes [C2]. For $n = 3$ see [CMS].

An example of a bicategory which is believed to arise in this way from a 2d defect TQFT is the bicategory $\mathcal{LG}$ of Landau-Ginzburg models. For physicists Landau-Ginzburg models are given by specific action functionals whose infrared properties are governed by a single holomorphic function. This limit of the theory is supposed to be a conformal field theory and therefore relevant to string theory.

[1] In dimension two every extended TQFT is automatically fully extended.

Objects of the bicategory $\mathcal{LG}$ comprise polynomials with an isolated singularity at the origin. These represent the holomorphic functions which themselves are the essential ingredient of the physicists' theories. 1-morphisms in $\mathcal{LG}$ are matrix factorizations of the difference of their target and source objects. In accordance with the general way of constructing higher categories from defect TQFTs, these matrix factorizations are thought of as defects between their source and target theories. Chapter 2 is dedicated to introduce $\mathcal{LG}$ more thoroughly.

Besides being relevant to string theory another example where this bicategory appears in theoretical physics is the paper [KapR] on some conjectured three-dimensional TQFTs. Here $\mathcal{LG}$ is supposed to describe surface, line and point defects. Moreover, since $\mathcal{LG}$ embeds matrix factorizations into a higher algebraic context, it is relevant to the areas where the latter feature. These in particular include homological mirror symmetry, cf. [HKK+, ABC+], and the theories of categorified knot invariants defined by Mikhail Khovanov and Lev Rozansky in [KR1, KR2].

Indeed, the topic of this thesis springs from the groundbreaking paper [KR1] where Khovanov and Rozansky define knot homologies categorifying $SU(n)$-Witten-Reshetikhin-Turaev invariants using matrix factorizations. In [KR1, §9] it is claimed that there is an oriented "TQFT with corners" involved in the construction of these homologies. They "leave details to the reader". We assemble some of these details from the literature accumulated since the publication of [KR1] and the results of this thesis.

From another point of view, the question whether one can define extended 2d TQFTs valued in $\mathcal{LG}$ is just a special instance of the broader task of finding out to which degree defect and extended TQFTs are related.

Concluding this introduction we outline the further content of the present thesis. First, in Chapter 2 we introduce in detail the definition of $\mathcal{LG}$ which we work with. Then, in Chapter 3 we define the required data turning $\mathcal{LG}$ into a symmetric monoidal bicategory. The monoidal product of objects is basically the sum of polynomials and that of 1-morphisms is essentially the external tensor product of matrix factorizations. This results in Theorem 3.3.12 which in particular entails

Theorem A. There exists a symmetric monoidal structure for the bicategory $\mathcal{LG}$.

In Chapter 4 we equip every object of $\mathcal{LG}$ with a dual object, roughly minus the polynomial, and define the associated (co-)evaluation morphisms. This allows us to state Corollary 4.7 which we quote as

Proposition B. Every object of $\mathcal{LG}$ is fully dualizable.

Finally, we begin Chapter 5 by providing some background on the bicategorical versions of the framed and oriented cobordism hypothesis. This sets the stage for us to subsequently use Theorem 3.3.12 and Corollary 4.7 to prove Corollary 5.2.1 and Corollary 5.2.4. We summarize these results next.

Theorem C. Every object of $\mathcal{LG}$ gives rise to a framed extended 2d TQFT valued in $\mathcal{LG}$. Every object of $\mathcal{LG}$ with an even number of variables is the value of an oriented extended 2d TQFT valued in $\mathcal{LG}$ in the positively oriented point.

Then we show Lemma 5.3.5, Proposition 5.3.6 and Corollary 5.3.8 which taken together say

Proposition D. There is a symmetric monoidal bicategory LG such that every object of $\mathcal{LG}$ gives rise to an oriented extended 2d TQFT valued in LG.

As an illustrative example of such a TQFT we consider the assignments of Khovanov and Rozansky in [KR1, §9] in Section 5.3.2.

Much of what we show in this thesis is unsurprising to experts. It is our contribution to prove it.

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Para Victor Manuel Ruiz Montiel

2. The bicategory $\mathcal{LG}$

In this chapter we define the bicategory $\mathcal{LG}$ of Landau-Ginzburg models. We intend to present $\mathcal{LG}$ rather pedagogically trying to clearly motivate the precise choices of objects and morphisms in $\mathcal{LG}$. To this end we begin with a preview of the definition of $\mathcal{LG}$. This helps us to justify why we consider the various structures we present below in preparing the definition of $\mathcal{LG}$. For the definition of a bicategory we refer the reader to Definition A.1.1.

Preview 1. Objects of $\mathcal{LG}$ are pairs $(\boldsymbol{x}, W)$, where $\boldsymbol{x}$ is an ordered set of variables and $W \in \mathrm{k}[\boldsymbol{x}]$ for a field k of characteristic zero is a potential as in Definition 2.1.1. 1-morphisms in $\mathcal{LG}$ from $(\boldsymbol{x}, V)$ to $(\boldsymbol{y}, W)$ are matrix factorizations of $(\mathrm{k}[\boldsymbol{x}, \boldsymbol{y}], W(\boldsymbol{y}) - V(\boldsymbol{x}))$ according to Definition 2.2.1.

The most authoritative exposition of the bicategory $\mathcal{LG}$ which we are aware of is in [CM2]. For a very concise summary of the essentials required to compute in this bicategory we recommend [CM3].

2.1. Potentials

We begin by defining the central notion when it comes to the objects of $\mathcal{LG}$.

Let k be a field of characteristic zero. Throughout this thesis, we use bold type as in $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, $n \in \mathbb{N}$, to refer to ordered sets of variables. Denote by $(\boldsymbol{x})^2 \subset \mathrm{k}[\boldsymbol{x}]$ the ideal whose elements have no constant or linear terms.

Definition 2.1.1. Let $W \in \mathrm{k}[\boldsymbol{x}]$.

1. The *Jacobi ring* of W is the ring

$$\mathrm{Jac}_W := \mathrm{k}[x]/(\partial_{x_1} W, \ldots, \partial_{x_n} W).$$

2. The element $W \in \mathrm{k}[\boldsymbol{x}]$ is called a *potential* if $W \in (\boldsymbol{x})^2$ and

$$\dim_{\mathrm{k}}(\mathrm{Jac}_W) < \infty.$$

Remark 2.1.2. The polynomial function $\mathrm{k}^n \to \mathrm{k}$ corresponding to W has an isolated singularity at the origin if and only if W is a potential.

Examples. One family of potentials is $\{x^d \in \mathrm{k}[x],\ d \in \mathbb{N}_{\geq 2}\}$. Another family of potentials is $\{x^d + xy^2 \in \mathrm{k}[x, y],\ d \in \mathbb{N}_{\geq 3}\}$.

2.2. Matrix factorizations

In this section we collect the definitions of matrix factorizations – particular ones of which are 1-morphisms in $\mathcal{LG}$ – and their morphisms.

Let $\mathcal{R}$ be a ring. Below we use the following conventions. By "$\mathcal{R}$-module" we mean a left $\mathcal{R}$-module. Let M be an $\mathcal{R}$-module. We denote by $r.m$ the action of $r \in \mathcal{R}$ on $m \in M$. Also, we decompose a $\mathbb{Z}_2$-graded $\mathcal{R}$-module M as $M = M^0 \oplus M^1$ displaying the $\mathbb{Z}_2$-degree by superscripts. Moreover, we write $\mathrm{Mod}_{\mathcal{R}}$ for the category of $\mathcal{R}$-modules. For its objects we write $M \in \mathrm{Mod}_{\mathcal{R}}$ and the set of morphisms from $M \in \mathrm{Mod}_{\mathcal{R}}$ to $N \in \mathrm{Mod}_{\mathcal{R}}$ is denoted by $\mathrm{Mod}_{\mathcal{R}}(M, N)$. In general, for $\mathcal{C}$ a category and A as well as B objects of $\mathcal{C}$ we write $A \in \mathcal{C}$, $B \in \mathcal{C}$ and denote the collection of morphisms from A to B in $\mathcal{C}$ by $\mathcal{C}(A, B)$.

The following definition rephrases the one in [E, §5].

Definition 2.2.1. Let $\mathcal{R}$ be a commutative ring and $W \in \mathcal{R}$. A *matrix factorization* $X \equiv (X_m, d_X)$ of $(\mathcal{R}, W)$ consists of the following.

1. X_m is a $\mathbb{Z}_2$-graded free $\mathcal{R}$-module.
2. d_X is a $\mathbb{Z}_2$-odd $\mathcal{R}$-module endomorphism of X_m such that $d_X^2 = W.\,\mathrm{id}_{X_m}$.

We refer to d_X as the *(twisted) differential* of X and to X_m as the *underlying module* of X.

Observe that given two $\mathbb{Z}_2$-graded $\mathcal{R}$-modules M, N the $\mathcal{R}$-module $\mathrm{Mod}_{\mathcal{R}}(M, N)$ is $\mathbb{Z}_2$-graded with

$$\mathrm{Mod}_{\mathcal{R}}(M, N)^i \cong \{(\varphi^0, \varphi^1) |\ \varphi^j \in \mathrm{Mod}_{\mathcal{R}}(M^j, N^{(j+i) \bmod 2}), j \in \{0, 1\}\},\ i \in \{0, 1\}. \tag{2.2.1}$$

For a homogeneous element $\varphi \in \mathrm{Mod}_{\mathcal{R}}(M, N)^i$, we indicate its $\mathbb{Z}_2$-degree as $|\varphi| = i$.

Definition 2.2.2. The category $\mathrm{MF}_{\mathcal{R},W}$ has matrix factorizations of $(\mathcal{R}, W)$ as objects and for every pair (X, Y) of matrix factorizations of $(\mathcal{R}, W)$ the $\mathbb{Z}_2$-graded $\mathcal{R}$-module $\mathrm{MF}_{\mathcal{R},W}(X, Y) := \mathrm{Mod}_{\mathcal{R}}(X_m, Y_m)$ as morphisms. Composition and units in $\mathrm{MF}_{\mathcal{R},W}$ are those of $\mathrm{Mod}_{\mathcal{R}}$.

Remark 2.2.3. Morphisms in $\mathrm{MF}_{\mathcal{R},W}(X, Y)$ are defined independently of d_X and d_Y. Still, the latter twisted differentials can be used to endow $\mathrm{MF}_{\mathcal{R},W}$ with a non-trivial differential graded (dg) structure, cf. [Ke], that is not present on $\mathrm{Mod}_{\mathcal{R}}$. To wit, as we show in Lemma 2.6.1 below, the twisted differentials of X and Y combine into an honest differential on $\mathrm{MF}_{\mathcal{R},W}(X, Y)$. This differential is an $\mathcal{R}$-linear map of degree 1 with respect to the $\mathbb{Z}_2$-grading on $\mathrm{MF}_{\mathcal{R},W}(X, Y) = \mathrm{Mod}_{\mathcal{R}}(X_m, Y_m)$ displayed in (2.2.1). Thus, Definition 2.2.2 together with Lemma 2.6.1 specify a differential $\mathbb{Z}_2$-graded category of matrix factorizations of $(\mathcal{R}, W)$. Many aspects of this dg structure are elaborated in [Dyc].

Sometimes, cf. [Y, Definition 7.1], the $\mathbb{Z}_2$-degree zero cycles in the chain complex of morphisms from X to Y are referred to as morphisms of matrix factorizations. This notion of morphisms reduces to morphisms of 2-periodic chain complexes if one considers matrix factorizations of $(\mathcal{R}, 0)$.

Occasionally, we write MF instead of $\mathrm{MF}_{\mathcal{R},W}$. This convention is used for the categories of matrix factorizations which we introduce below, too. Similarly we write Mod for $\mathrm{Mod}_{\mathcal{R}}$.

Remark 2.2.4. Definition 2.2.1 as well as Definition 2.2.2 do not require $W \in \mathcal{R}$ to be a potential. In fact, potentials only re-enter our discussion in Section 2.8.

2.3. Extension and restriction of scalars

In line with Preview 1, the categories of 1- and 2-morphisms in $\mathcal{LG}$ are closely related to $\mathrm{MF}_{\mathcal{R},W}$ for specific pairs $(\mathcal{R}, W)$. As in every bicategory, cf. Definition A.1.1, the categories of 1- and 2-morphisms in $\mathcal{LG}$ are interconnected via functors which yield the horizontal composites of morphisms. Preparing the ground for introducing these functors we comment on extension and restriction of scalars for matrix factorizations which are mentioned e.g. in [CM1, Appendix A.1] or [DM, §12].

Let $\mathcal{R}$, $\mathcal{S}$ be two rings and $\varphi : \mathcal{R} \to \mathcal{S}$ a ring homomorphism. Then there is the functor

$$\varphi_* : \mathrm{Mod}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{R}} \tag{2.3.1}$$

of *restriction of scalars* along φ which turns $\mathcal{S}$-modules into $\mathcal{R}$-modules by letting $r \in \mathcal{R}$ act as $\varphi(r)$, and reinterprets $\mathcal{S}$-linear maps as $\mathcal{R}$-linear maps.

Moreover, there is a functor

$$\varphi^* : \mathrm{Mod}_{\mathcal{R}} \to \mathrm{Mod}_{\mathcal{S}}$$

of *extension of scalars* along φ which acts as $M \mapsto \tilde{\varphi}_*(\mathcal{S}) \otimes_{\mathcal{R}} M$, where $\tilde{\varphi}_*$ is restriction of scalars for right modules and $\tilde{\varphi}_*(\mathcal{S}) \otimes_{\mathcal{R}} M$ is considered as a left $\mathcal{S}$-module via the multiplication in $\mathcal{S}$. φ^* sends a morphism $\psi \in \mathrm{Mod}_{\mathcal{R}}(M, N)$ to $\mathrm{id}_{\tilde{\varphi}_*(\mathcal{S})} \otimes_{\mathcal{R}} \psi$. The following is in line with [DM].

Lemma 2.3.1. Let $\mathcal{R}$, $\mathcal{S}$ be two commutative rings, $W \in \mathcal{R}$ and $\varphi : \mathcal{R} \to \mathcal{S}$ a ring homomorphism such that $\varphi_*(\mathcal{S})$ is a free $\mathcal{R}$-module. There is a functor

$$\mathrm{MF}_{\mathcal{S},\varphi(W)} \to \mathrm{MF}_{\mathcal{R},W}, \; (X_m, d_X) \mapsto (\varphi_*(X_m), \varphi_*(d_X)), \; \psi \mapsto \varphi_*(\psi) \tag{2.3.2}$$

for $\psi \in \mathrm{MF}_{\mathcal{S},\varphi(W)}(X, Y)$.

Proof. $\varphi_*(X_m)$ is an $\mathcal{R}$-module with $\mathbb{Z}_2$-grading induced from X_m. Since we assume that $\varphi_*(\mathcal{S})$ is a free $\mathcal{R}$-module and X_m is a free $\mathcal{S}$-module, $\varphi_*(X_m)$ is a free $\mathcal{R}$-module. Furthermore, $\varphi_*(d_X)$ is a $\mathbb{Z}_2$-odd $\mathcal{R}$-module endomorphism of $\varphi_*(X_m)$ satisfying

$$(\varphi_*(d_X))^2 = \varphi_*(d_X^2) = \varphi_*(\varphi(W).\, \mathrm{id}_{X_m}) = W.\varphi_*(\mathrm{id}_{X_m}) = W.\, \mathrm{id}_{\varphi_*(X_m)}$$

Thus, $(\varphi_*(X_m), \varphi_*(d_X))$ is an object of $\mathrm{MF}_{\mathcal{R},W}$. As φ_* is a functor $\mathrm{Mod}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{R}}$ and MF has the same morphism sets as Mod, the prescription (2.3.2) is well-defined on morphisms, too, and yields a functor. □

We use the following conventions which in light of (2.3.1) introduce some ambiguity. This is resolved in each instance by the context.

Definition 2.3.2. The functor in Lemma 2.3.1 is *restriction of scalars* (for matrix factorizations), $\varphi_* : \mathrm{MF}_{\mathcal{S},\varphi(W)} \to \mathrm{MF}_{\mathcal{R},W}$.

Remark 2.3.3. The functor in Lemma 2.3.1 is referred to as "pushforward" in [CM1] and [DM].

When working with restriction of scalars as we do below, it is useful to observe that such functors only change the ring actions in modules, but leave their elements invariant.

Since restriction of scalars along isomorphisms of rings features particularly prominently later on we ponder on this special case separately.

Remark 2.3.4. Note that, as the proof of Lemma 2.3.1 illustrates, restriction of scalars for matrix factorizations only works under the conditions of Lemma 2.3.1. The ring morphism $\varphi : \mathcal{R} \to \mathcal{S}$ must be such that $\varphi_*(\mathcal{S}_m)$ is a free $\mathcal{R}$-module and the source category can only comprise matrix factorizations of pairs $(\mathcal{S}, \widetilde{W})$ such that $\widetilde{W} \in \mathrm{Im}(\varphi)$. Both requirements are met automatically if φ is an isomorphism of rings.

Corollary 2.3.5. Let $\mathcal{R}$ and $\mathcal{S}$ be commutative rings, $W \in \mathcal{S}$, and $\varphi : \mathcal{R} \to \mathcal{S}$ a ring isomorphism. Then $\varphi_* : \mathrm{MF}_{\mathcal{S},W} \to \mathrm{MF}_{\mathcal{R},\varphi^{-1}(W)}$ is an equivalence.

Proof. By Remark 2.3.4 we can apply Lemma 2.3.1 to conclude that φ_* is a functor. Similarly, $(\varphi^{-1})_*$ is a functor $\mathrm{MF}_{\mathcal{R},\varphi^{-1}(W)} \to \mathrm{MF}_{\mathcal{S},W}$. We assert that $(\varphi^{-1})_*$ is inverse to φ_*. φ_* turns an $\mathcal{S}$-module X_m underlying a matrix factorization X into an $\mathcal{R}$-module by letting $r \in \mathcal{R}$ act as $\varphi(r)$. Applying $(\varphi^{-1})_*$ to the resulting $\mathcal{R}$-module gives an $\mathcal{S}$-module on which $s \in \mathcal{S}$ acts as $\varphi^{-1}(s)$ which in turn acts as $\varphi(\varphi^{-1}(s)) = s$. That is, $((\varphi^{-1})_* \circ \varphi_*)(X_m) = X_m$. Conversely, $(\varphi_* \circ (\varphi^{-1})_*)(X_m) = X_m$. Since restriction of scalars leaves morphisms of modules – including the differentials of matrix factorizations – unaffected, $(\varphi^{-1})_*$ and φ_* are inverse to each other on this level, too. □

Remarks 2.3.6. Corollary 2.3.5 is a version for matrix factorizations of the fact that isomorphic rings are Morita equivalent.

Our proof shows that φ_* is an isomorphism of categories.

Extension of scalars for matrix factorizations, which we turn to now, features also in [KR2, (2.17)].

Lemma 2.3.7. Let $\mathcal{R}$ and $\mathcal{S}$ be commutative rings, $W \in \mathcal{R}$ and $\varphi : \mathcal{R} \to \mathcal{S}$ a ring homomorphism. There is a functor

$$\mathrm{MF}_{\mathcal{R},W} \to \mathrm{MF}_{\mathcal{S},\varphi(W)}, \ (X_m, d_X) \mapsto (\varphi^*(X_m), \varphi^*(d_X)), \ \psi \mapsto \varphi^*(\psi) \tag{2.3.3}$$

for $\psi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)$.

Proof. $\varphi^*(X_m)$ is a free $\mathcal{S}$-module since for an $\mathcal{R}$-basis $\{e_i\}_{i\in I}$ of X_m, $s \in \mathcal{S}$ and $e \in X_m$, i.e. $e = \sum_{i\in I} r_i.e_i$ for some $r_i \in \mathcal{R}$, $i \in I$, we have

$$s \otimes_{\mathcal{R}} e = s \otimes_{\mathcal{R}} \sum_{i\in I} r_i.e_i = \sum_{i\in I} s \otimes_{\mathcal{R}} r_i.e_i = \sum_{i\in I} s\cdot\varphi(r_i) \otimes_{\mathcal{R}} e_i = \sum_{i\in I} (s\cdot\varphi(r_i)).(1 \otimes_{\mathcal{R}} e_i)$$

which shows that $\{1 \otimes_{\mathcal{R}} e_i\}_{i\in I}$ is an $\mathcal{S}$-basis of $\varphi^*(X_m)$. This module inherits a $\mathbb{Z}_2$-grading

$$\varphi^*(X_m)^i = \tilde{\varphi}_*(\mathcal{S}) \otimes_{\mathcal{R}} X_m^i, \ i \in \{0,1\}.$$

Furthermore, $\varphi^*(d_X)$ is a $\mathbb{Z}_2$-odd $\mathcal{S}$-module endomorphism of $\varphi^*(X_m)$ for which

$$\begin{aligned}(\varphi^*(d_X))^2 &= (1 \otimes_{\mathcal{R}} d_X)^2 = 1 \otimes_{\mathcal{R}} d_X^2 = 1 \otimes_{\mathcal{R}} (W.\,\mathrm{id}_{X_m}) = \varphi(W) \otimes_{\mathcal{R}} \mathrm{id}_{X_m} \\ &= \varphi(W).(1 \otimes_{\mathcal{R}} \mathrm{id}_{X_m}) = \varphi(W).\,\mathrm{id}_{\varphi^*(X_m)}\,.\end{aligned}$$

Therefore, $(\varphi^*(X_m), \varphi^*(d_X)) \in \mathrm{MF}_{\mathcal{S},\varphi(W)}$. As in the proof of Lemma 2.3.1 above it follows directly from φ^* being a functor on categories of modules that the assignment (2.3.3) is well-defined on morphisms and gives a functor. □

Following up Definition 2.3.2 we introduce

Definition 2.3.8. The functor in Lemma 2.3.7 is *extension of scalars* (for matrix factorizations), $\varphi^* : \mathrm{MF}_{\mathcal{R},W} \to \mathrm{MF}_{\mathcal{S},\varphi(W)}$.

Remark 2.3.9. Note that extension of scalars for matrix factorizations is defined for arbitrary homomorphisms of commutative rings.

2.4. Tensor products of matrix factorizations

In this section we introduce functors which we use later to define the horizontal composition in $\mathcal{LG}$. In doing so we follow [DM, §12]. Other references for tensor products of matrix factorizations are e.g. [KR1] and [CR1].

From now on tensor products are assumed to be $\mathbb{Z}_2$-graded whenever this makes sense unless specified differently. For example, let ϕ, ψ, $\widetilde{\phi}$ and $\widetilde{\psi}$ be morphisms of

$\mathbb{Z}_2$-graded modules such that the following tensor products are defined. Assume furthermore that ψ and $\widetilde{\phi}$ are homogeneous. Then

$$(\phi \otimes \psi) \circ (\widetilde{\phi} \otimes \widetilde{\psi}) = (-1)^{|\psi||\widetilde{\phi}|}(\phi \circ \widetilde{\phi}) \otimes (\psi \circ \widetilde{\psi}). \tag{2.4.1}$$

The functors which we aim at are composed of extension and restriction of scalars and the functors which we introduce in Lemma 2.4.2 below. Before turning to these functors we explain the notation which we use for their source categories. This is a special instance of a construction that is well-known in certain mathematical circles, cf. Remark 2.4.3 below.

Lemma 2.4.1. Let k be a commutative ring, $\boldsymbol{x}$ and $\boldsymbol{y}$ ordered sets of variables and $V \in k[\boldsymbol{x}]$, $W \in k[\boldsymbol{y}]$. There is a category $\mathrm{MF}_{k[\boldsymbol{y}],W} \otimes \mathrm{MF}_{k[\boldsymbol{x}],V}$ defined by

$$(\mathrm{MF}_{k[\boldsymbol{y}],W} \otimes \mathrm{MF}_{k[\boldsymbol{x}],V})_{\mathrm{O}} := (\mathrm{MF}_{k[\boldsymbol{y}],W} \times \mathrm{MF}_{k[\boldsymbol{x}],V})_{\mathrm{O}},$$
$$(\mathrm{MF}_{k[\boldsymbol{y}],W} \otimes \mathrm{MF}_{k[\boldsymbol{x}],V})((Y_1, X_1), (Y_2, X_2)) := \mathrm{MF}_{k[\boldsymbol{y}],W}(Y_1, Y_2) \otimes_k \mathrm{MF}_{k[\boldsymbol{x}],V}(X_1, X_2)$$

with composition dictated by the rule (2.4.1). The identity morphism on an object $(Y, X) \in (\mathrm{MF}_{k[\boldsymbol{y}],W} \otimes \mathrm{MF}_{k[\boldsymbol{x}],V})$ is $\mathrm{id}_Y \otimes_k \mathrm{id}_X$.

Proof. The morphism $\mathrm{id}_Y \otimes_k \mathrm{id}_X$ serves as an identity morphism for (Y, X) as $|\mathrm{id}_X| = 0 = |\mathrm{id}_Y|$. The composition is associative since

$$\begin{aligned}
&(\psi_3 \otimes_k \phi_3) \circ ((\psi_2 \otimes_k \phi_2) \circ (\psi_1 \otimes_k \phi_1)) \\
=&(\psi_3 \otimes_k \phi_3) \circ ((-1)^{|\psi_1||\phi_2|}(\psi_2 \circ \psi_1) \otimes_k (\phi_2 \circ \phi_1)) \\
=&(-1)^{|\psi_1||\phi_2|+|\phi_3||\psi_2 \circ \psi_1|}(\psi_3 \circ \psi_2 \circ \psi_1) \otimes_k (\phi_3 \circ \phi_2 \circ \phi_1) \\
=&(-1)^{|\psi_1||\phi_2|+|\phi_3||\psi_2|+|\phi_3||\psi_1|}(\psi_3 \circ \psi_2 \circ \psi_1) \otimes_k (\phi_3 \circ \phi_2 \circ \phi_1), \\
&((\psi_3 \otimes_k \phi_3) \circ (\psi_2 \otimes_k \phi_2)) \circ (\psi_1 \otimes_k \phi_1) \\
=&(-1)^{|\phi_3||\psi_2|}((\psi_3 \circ \psi_2) \otimes_k (\phi_3 \circ \phi_2)) \circ (\psi_1 \otimes_k \phi_1) \\
=&(-1)^{|\phi_3||\psi_2|+|\phi_3 \circ \phi_2||\psi_1|}(\psi_3 \circ \psi_2 \circ \psi_1) \otimes_k (\phi_3 \circ \phi_2 \circ \phi_1) \\
=&(-1)^{|\phi_3||\psi_2|+|\phi_3||\psi_1|+|\phi_2||\psi_1|}(\psi_3 \circ \psi_2 \circ \psi_1) \otimes_k (\phi_3 \circ \phi_2 \circ \phi_1).
\end{aligned}$$

□

The functors introduced next provide us with a first means of composing categories of matrix factorizations.

Lemma 2.4.2. Let k be a commutative ring, $\boldsymbol{x}$ an ordered set of variables, set $\mathcal{R} := k[\boldsymbol{x}]$ and let $V \in \mathcal{R}$, $W \in \mathcal{R}$. There is a functor

$$\begin{aligned}
\otimes_{\mathcal{R}} : &\mathrm{MF}_{\mathcal{R},W} \otimes \mathrm{MF}_{\mathcal{R},V} \to \mathrm{MF}_{\mathcal{R},W+V}, \\
&(Y, X) \mapsto (Y_m \otimes_{\mathcal{R}} X_m, d_Y \otimes_{\mathcal{R}} 1 + 1 \otimes_{\mathcal{R}} d_X),\ \phi \otimes_k \varphi \mapsto \phi \otimes_{\mathcal{R}} \varphi
\end{aligned} \tag{2.4.2}$$

for $\phi \in \mathrm{MF}_{\mathcal{R},W}(Y, Y')$, $\varphi \in \mathrm{MF}_{\mathcal{R},V}(X, X')$.

Proof. $Y_m \otimes_{\mathcal{R}} X_m$ is a free $\mathcal{R}$-module with $\mathbb{Z}_2$-grading

$$(Y_m \otimes_{\mathcal{R}} X_m)^0 = ((Y_m)^0 \otimes_{\mathcal{R}} (X_m)^0) \oplus ((Y_m)^1 \otimes_{\mathcal{R}} (X_m)^1),$$
$$(Y_m \otimes_{\mathcal{R}} X_m)^1 = ((Y_m)^0 \otimes_{\mathcal{R}} (X_m)^1) \oplus ((Y_m)^1 \otimes_{\mathcal{R}} (X_m)^0).$$

Also, $d_Y \otimes_{\mathcal{R}} 1 + 1 \otimes_R d_X$ is odd with respect to this grading. Moreover,

$$\begin{aligned}(d_Y \otimes_{\mathcal{R}} 1 + 1 \otimes_{\mathcal{R}} d_X)^2 &= d_Y^2 \otimes_{\mathcal{R}} 1 + 1 \otimes_{\mathcal{R}} d_X^2 = W.\, \mathrm{id}_{Y_m} \otimes_{\mathcal{R}} 1 + 1 \otimes_{\mathcal{R}} V.\, \mathrm{id}_{X_m} \\ &= (W+V).\, \mathrm{id}_{Y_m \otimes_{\mathcal{R}} X_m}, \qquad (2.4.3)\end{aligned}$$

where we use

$$(d_Y \otimes_{\mathcal{R}} 1) \circ (1 \otimes_{\mathcal{R}} d_X) = d_Y \otimes_{\mathcal{R}} d_X = -(1 \otimes_{\mathcal{R}} d_X) \circ (d_Y \otimes_{\mathcal{R}} 1),$$

cf. (2.4.1), in the first equality. Therefore the prescription (2.4.2) is well-defined on objects. Furthermore, it preserves identities as it maps $\mathrm{id} \otimes_k \mathrm{id} \mapsto \mathrm{id} \otimes_{\mathcal{R}} \mathrm{id}$. Composition of morphisms is respected as it is defined by the rule (2.4.1) in both the source and the target. □

Remark 2.4.3. The computation (2.4.3) indicates that it yields consistent structures to work with $\mathbb{Z}_2$-graded tensor products in the context of matrix factorizations. Indeed, these tensor products are natural from the point of view of the differential $\mathbb{Z}_2$-graded structure on MF mentioned in Remark 2.2.3. Also the category introduced in Lemma 2.4.1 is the standard way of defining the tensor product of two differential graded categories as a differential graded category on its own.[1] Likewise, it is well established that this tensor product extends to dg functors, cf. [Ke, T, Drin]. This underlies Lemma 2.4.4 below.

It is crucial that the source category of $\otimes_{\mathcal{R}}$ in Lemma 2.4.2 is $\mathrm{MF}_{\mathcal{R},W} \otimes \mathrm{MF}_{\mathcal{R},V}$ rather than $\mathrm{MF}_{\mathcal{R},W} \times \mathrm{MF}_{\mathcal{R},V}$. Taking the latter category in (2.4.2) and setting (ϕ, φ)" $\mapsto$ "$\phi \otimes_{\mathcal{R}} \varphi$ instead, does not yield a functor. To wit, the resulting prescription is not compatible with composition:

$$(\phi_2, \varphi_2) \circ (\phi_1, \varphi_1) = (\phi_2 \circ \phi_1, \varphi_2 \circ \varphi_1)\text{“} \mapsto \text{”}(\phi_2 \circ \phi_1) \otimes_{\mathcal{R}} (\varphi_2 \circ \varphi_1), \qquad (2.4.4)$$
$$(\phi_2 \otimes_{\mathcal{R}} \varphi_2) \circ (\phi_1 \otimes_{\mathcal{R}} \varphi_1) = (-1)^{|\varphi_2||\phi_1|}(\phi_2 \circ \phi_1) \otimes_{\mathcal{R}} (\varphi_2 \circ \varphi_1),$$

where we assume that ϕ_1 and φ_2 are morphisms of homogeneous $\mathbb{Z}_2$-degree and apply (2.4.1). In Section 5.3.1 we encounter an alternative way of mitigating this obstacle without referring to $\mathrm{MF} \otimes \mathrm{MF}$. There we consider equivalence classes of morphisms of matrix factorizations modulo signs, following [KR1].

[1] That we use the tensor product of dg categories is inspired by the definition of a monoidal dg category in [Mo, Definition 2.1.1]. In fact, using this tensor product as we do, one can establish that certain categories of matrix factorizations are monoidal dg categories (as special categories of morphisms in a variant of a bicategory), cf. Remark 5.3.2.

The usual reference for certain functors that are very similar to those in Lemma 2.4.2 is [Y2]. In [Y2], however, only $\mathbb{Z}_2$-even morphisms of matrix factorizations are considered. In this case one can replace MF $\otimes$ MF in Lemma 2.4.2 by the cartesian product of the categories. We do so in most of this thesis. Moreover, if one restricts to $\mathbb{Z}_2$-even morphisms of matrix factorizations the ring $\mathcal{R}$ need not be a polynomial ring. Rather we can allow it to be an arbitrary commutative ring.

We resume working towards the functors which serve as prototypes for the horizontal composition of $\mathcal{LG}$ with Lemma 2.4.4. Let $F : \mathrm{MF}_{k[\boldsymbol{x}],V} \to \mathrm{MF}_{k[\boldsymbol{y}],W}$ and $G : \mathrm{MF}_{k[\widetilde{\boldsymbol{x}}],\widetilde{V}} \to \mathrm{MF}_{k[\widetilde{\boldsymbol{y}}],\widetilde{W}}$ be two functors such that the corresponding functions between sets of morphisms are maps of $\mathbb{Z}_2$-graded k-modules. Then we can construct a new functor from them as follows.

Lemma 2.4.4. Given two functors F, G as above, the following defines a functor $F \otimes G : \mathrm{MF}_{k[\boldsymbol{x}],V} \otimes \mathrm{MF}_{k[\widetilde{\boldsymbol{x}}],\widetilde{V}} \to \mathrm{MF}_{k[\boldsymbol{y}],W} \otimes \mathrm{MF}_{k[\widetilde{\boldsymbol{y}}],\widetilde{W}}$:

1. $(F \otimes G)_{\mathrm{O}} := F_{\mathrm{O}} \times G_{\mathrm{O}}$
2. $(F \otimes G)_{(X,\widetilde{X}),(Y,\widetilde{Y})} := F_{(X,Y)} \otimes_k G_{(\widetilde{X},\widetilde{Y})}$.

Proof. Since F and G preserve identity morphisms the same is true for $F \otimes G$. We show that the function on morphisms respects the composition of morphisms:

$$\begin{aligned}(\widetilde{\psi} \otimes_k \widetilde{\phi}) \circ (\psi \otimes_k \phi) &= (-1)^{|\widetilde{\phi}||\psi|}(\widetilde{\psi} \circ \psi) \otimes_k (\widetilde{\phi} \circ \phi)\\ &\mapsto (-1)^{|\widetilde{\phi}||\psi|}F(\widetilde{\psi} \circ \psi) \otimes_k G(\widetilde{\phi} \circ \phi)\\ &= (-1)^{|\widetilde{\phi}||\psi|}(F(\widetilde{\psi}) \circ F(\psi)) \otimes_k (G(\widetilde{\phi}) \circ G(\phi))\\ &= (F(\widetilde{\psi}) \otimes_k G(\widetilde{\phi})) \circ (F(\psi) \otimes_k G(\phi)).\end{aligned}$$

Note that the last equality hinges on the condition that neither F nor G change the $\mathbb{Z}_2$-degree of morphisms. □

Particular examples of functors to which Lemma 2.4.4 applies are the functors of extension of scalars for matrix factorizations from Lemma 2.3.7.

Next we turn to the main definition of this section. First, however, we introduce

Notation 2.4.5. For k a commutative ring, $V \in k[\boldsymbol{x}]$, $W \in k[\boldsymbol{y}]$ and $\iota_{\boldsymbol{x}} : k[\boldsymbol{x}] \hookrightarrow k[\boldsymbol{x},\boldsymbol{y}]$, $\iota_{\boldsymbol{y}} : k[\boldsymbol{y}] \hookrightarrow k[\boldsymbol{x},\boldsymbol{y}]$ the canonical inclusions of rings we write $V(\boldsymbol{x})$ for $\iota_{\boldsymbol{x}}(V)$ and $W(\boldsymbol{y})$ for $\iota_{\boldsymbol{y}}(W)$.

Definition 2.4.6. Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be ordered sets of variables, k a commutative ring, $U \in k[\boldsymbol{x}]$, $V \in k[\boldsymbol{y}]$, $W \in k[\boldsymbol{z}]$ and $\iota_{\boldsymbol{x},\boldsymbol{y}} : k[\boldsymbol{x},\boldsymbol{y}] \hookrightarrow k[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$, $\iota_{\boldsymbol{y},\boldsymbol{z}} : k[\boldsymbol{y},\boldsymbol{z}] \hookrightarrow k[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$, $\iota_{\boldsymbol{x},\boldsymbol{z}} : k[\boldsymbol{x},\boldsymbol{z}] \hookrightarrow k[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ the canonical inclusions of rings. The functor

$$\check{\otimes}_{k[\boldsymbol{y}]} : \mathrm{MF}_{k[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})} \otimes \mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})} \to \mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$$

is the composite

$$(\iota_{\boldsymbol{x},\boldsymbol{z}})_* \circ \otimes_{k[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \circ ((\iota_{\boldsymbol{y},\boldsymbol{z}})^* \otimes (\iota_{\boldsymbol{x},\boldsymbol{y}})^*).$$

Note that restriction of scalars along $\iota_{\boldsymbol{x},\boldsymbol{z}}$ as it features in Definition 2.4.6 is possible since $W(\boldsymbol{z}) - U(\boldsymbol{x}) \in k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ lies in the image of $\iota_{\boldsymbol{x},\boldsymbol{z}}$ and assuming $\boldsymbol{y} \equiv (y_1, \dots, y_m)$, $(\iota_{\boldsymbol{x},\boldsymbol{z}})_*(k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}])$ is a free $k[\boldsymbol{x}, \boldsymbol{z}]$-module with a basis given, for example, by $\{y_1^{i_1} \dots y_m^{i_m} \mid i_j \in \mathbb{N}_0,\ 1 \leq j \leq m\}$.

We write $Y \check{\otimes} X$ for $\check{\otimes}(Y, X)$ and for $e \in X_m$, $f \in Y_m$ we write $e \check{\otimes}_{k[\boldsymbol{y}]} f$ for the corresponding element of $(Y \check{\otimes}_{k[\boldsymbol{y}]} X)_m$.

We point out that the functor considered in [Y2] corresponds to $\check{\otimes}_k$, i.e. $\boldsymbol{y} = \emptyset$, $V = 0$, if one only considers $\mathbb{Z}_2$-even morphisms of matrix factorizations. As such it is a fundamental ingredient of the monoidal product of $\mathcal{LG}$ which we introduce in Chapter 3.

We comment in Remark 2.10.8 towards the end of this chapter on the tensor products defined in [CR1] which from some perspective are more natural to consider than the ones in Definition 2.4.6. The point is, though, that the two products yield equivalent structures and the bicategory obtained using Definition 2.4.6 appears to be the easier one to endow with a monoidal structure, cf. Remark 3.2.1.

2.5. Unit matrix factorizations and unitors

The horizontal composition of $\mathcal{LG}$ is modeled on the functor in Definition 2.4.6. As we recollect in Definition A.1.1, a bicategory needs, on top of composition functors, also unit 1-morphisms and unitor isomorphisms. These are considered next.

The differentials of the unit matrix factorizations are defined using the following in which we apply a version of Notation 2.4.5.

Definition 2.5.1. Given a commutative ring k, two ordered sets of variables $\boldsymbol{y} \equiv (y_1, \dots, y_n)$, $\boldsymbol{z} \equiv (z_1, \dots, z_n)$ and a polynomial $W \in k[x_1, \dots, x_n]$, we denote by $\partial_{[i]}^{\boldsymbol{z},\boldsymbol{y}} W$ the polynomial

$$\frac{W(y_1, \dots, y_{i-1}, z_i, \dots, z_n) - W(y_1, \dots, y_i, z_{i+1}, \dots, z_n)}{z_i - y_i} \in k[\boldsymbol{y}, \boldsymbol{z}],\ i \in \mathbb{N}_{\leq n}. \tag{2.5.1}$$

To see that Definition 2.5.1 is well-defined consider the numerator of (2.5.1) as a polynomial in $k[z_i]$. Its corresponding polynomial function $k \to k$ is zero at $z_i = y_i$.

In defining unit matrix factorizations, we employ the

Notation 2.5.2. Let $\mathcal{R}$ be a commutative ring and X a free $\mathcal{R}$-module with a basis $\{e_i\}_{i \in I}$. We denote by $\bigwedge_{\mathcal{R}} X$ the exterior algebra of X over $\mathcal{R}$. We sometimes write $e_i \dots e_j := e_i \wedge \dots \wedge e_j$ and $e_i \dots \widehat{e_j} \dots e_k := e_i \dots e_{j-1} e_{j+1} \dots e_k$.

Another notational device which we use is the following. Suppose we are given an ordered set $\boldsymbol{x}$ of n variables, a field k of characteristic zero and a potential $W \in \mathrm{k}[\boldsymbol{x}]$. According to Preview 1 these data define an object $(\boldsymbol{x}, W) \in \mathcal{LG}$. To describe endomorphisms of this object in $\mathcal{LG}$ we need to refine Preview 1. Namely, we decorate either the set of variables pertaining to the source object or that belonging to the target object with a prime such that an endomorphism of $(\boldsymbol{x}, W)$ in $\mathcal{LG}$ is a matrix factorization of $(\mathrm{k}[\boldsymbol{x}', \boldsymbol{x}], W(\boldsymbol{x}) - W(\boldsymbol{x}'))$ or of $(\mathrm{k}[\boldsymbol{x}, \boldsymbol{x}'], W(\boldsymbol{x}') - W(\boldsymbol{x}))$.

Lemma 2.5.3. Let k be a commutative ring and assume throughout this lemma that unspecified tensor products are over k. Set $\mathcal{R} := k[\boldsymbol{x}] \equiv k[x_1, \ldots, x_n]$, $\mathcal{R}^{\mathrm{e}} := \mathcal{R} \otimes \mathcal{R}$, let $W \in \mathcal{R}$ and write φ for the ring isomorphism

$$k[\boldsymbol{x}', \boldsymbol{x}] \to \mathcal{R}^{\mathrm{e}},\ x_i \mapsto x_i \otimes 1,\ x'_i \mapsto 1 \otimes x_i \ \forall i \in \mathbb{N}_{\leq n}. \tag{2.5.2}$$

The following specifies a matrix factorization of $(k[\boldsymbol{x}', \boldsymbol{x}], W(\boldsymbol{x}) - W(\boldsymbol{x}'))$.

1. Its $k[\boldsymbol{x}', \boldsymbol{x}]$-module is
$$\varphi_*\Big(\bigwedge_{\mathcal{R}^{\mathrm{e}}} \big(\bigoplus_{i=1}^{n} (\mathcal{R}^{\mathrm{e}}\theta_i)\big)\ \Big), \tag{2.5.3}$$
where $\{\theta_i \mid i \in \mathbb{N}_{\leq n}\}$ is a chosen basis of $(\mathcal{R}^{\mathrm{e}})^{\oplus n}$. The $\mathbb{Z}_2$-degree of a basis element of (2.5.3) is defined to be the parity of the number of θ it involves.

2. The differential is
$$\sum_{i=1}^{n} \big((x_i - x'_i).\theta_i^* + \partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'} W\ .\ \theta_i \wedge (-) \big), \tag{2.5.4}$$
where by definition for $j_l \in \{0,1\}$ $\forall l \in \mathbb{N}_{\leq n}$
$$\theta_i^*(\theta_1^{j_1} \ldots \theta_n^{j_n}) = (-1)^{\sum_{k=1}^{i-1} j_k} \delta_{j_i 1}\ \theta_1^{j_1} \ldots \hat{\theta}_i \ldots \theta_n^{j_n},$$
extended $k[\boldsymbol{x}', \boldsymbol{x}]$-linearly.

Proof. According to its definition the module in (2.5.3) is a free $k[\boldsymbol{x}', \boldsymbol{x}]$-module on the basis

$$\{\theta_1^{i_1} \ldots \theta_n^{i_n} \mid i_j \in \{0,1\}\ \forall j \in \mathbb{N}_{\leq n}\}. \tag{2.5.5}$$

We have defined its $\mathbb{Z}_2$-grading. Furthermore, we have specified a module endomorphism which is odd with respect to this grading as differential. Moreover, for

e an element of (2.5.3), writing d for (2.5.4) we have

$$\begin{aligned} d^2(e) =& \sum_{j=1}^{n}(x_j - x'_j).\theta_j^*\Big(\sum_{i=1}^{n}((x_i - x'_i).\theta_i^*(e) + \partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_i \wedge e)\Big) \\ &+ \sum_{j=1}^{n}\partial_{[j]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_j \wedge \Big(\sum_{i=1}^{n}((x_i - x'_i).\theta_i^*(e) + \partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_i \wedge e)\Big) \\ =& \sum_{j=1}^{n}\sum_{i=1}^{n}(x_j - x'_j)(x_i - x'_i).\theta_j^*(\theta_i^*(e)) + \sum_{j=1}^{n}\sum_{i=1}^{n}(x_j - x'_j)\partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_j^*(\theta_i \wedge e) \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{n}\partial_{[j]}^{\boldsymbol{x},\boldsymbol{x}'}(W)(x_i - x'_i).\theta_j \wedge (\theta_i^*(e)) \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{n}\partial_{[j]}^{\boldsymbol{x},\boldsymbol{x}'}(W)\partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_j \wedge (\theta_i \wedge e). \end{aligned} \tag{2.5.6}$$

The first and the last summand in (2.5.6) are zero since the polynomial coefficients are symmetric in i and j whereas the operators acting on e are antisymmetric in i and j. Similarly the summands with $i \neq j$ in the second summand cancel with those in the third summand. This leaves us with

$$\begin{aligned} d^2(e) =& \sum_{i=1}^{n}(x_i - x'_i)\partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W).\theta_i^*(\theta_i \wedge e) + \sum_{i=1}^{n}\partial_{[i]}^{\boldsymbol{x},\boldsymbol{x}'}(W)(x_i - x'_i).\theta_i \wedge (\theta_i^*(e)) \\ =& \sum_{i=1}^{n}(W(x'_1, \dots, x'_{i-1}, x_i, \dots, x_n) - W(x'_1, \dots, x'_i, x_{i+1}, \dots, x_n)) \\ & .(\theta_i^*(\theta_i \wedge e) + \theta_i \wedge (\theta_i^*(e))) \\ =& \sum_{i=1}^{n}(W(x'_1, \dots, x'_{i-1}, x_i, \dots, x_n) - W(x'_1, \dots, x'_i, x_{i+1}, \dots, x_n)).e \\ =& (W(\boldsymbol{x}) - W(\boldsymbol{x}')).e. \end{aligned}$$

□

Definition 2.5.4. The matrix factorization described in Lemma 2.5.3 is denoted $I_{\mathcal{R},W}$ and called the *unit matrix factorization of* $(\mathcal{R}, W)$.

Remark 2.5.5. The description of $I_{\mathcal{R},W}$ in Lemma 2.5.3 is well-suited for the computations we carry out in this thesis. The unit matrix factorization can however also be specified without explicitly presenting its underlying module and differential. It is the "stabilized diagonal", cf. [Dyc]. Similarly, the unitors which we introduce in Lemma 2.5.6 below can be specified through a universal property, cf. [CM2, §4.2].

As unit 1-morphism in a bicategory the matrix factorization of Lemma 2.5.3 comes together with unitor 2-morphisms which we define using the following notation. Let k be a commutative ring, $\boldsymbol{x}$, $\boldsymbol{y}$ two ordered sets of variables and $\iota_{\boldsymbol{x}} : k[\boldsymbol{x}] \hookrightarrow k[\boldsymbol{x}, \boldsymbol{y}]$, $\iota_{\boldsymbol{y}} : k[\boldsymbol{y}] \hookrightarrow k[\boldsymbol{x}, \boldsymbol{y}]$ be as in Notation 2.4.5. For $g \in k[\boldsymbol{y}]$, $f \in k[\boldsymbol{x}, \boldsymbol{y}]$ we write $g \cdot f := \iota_{\boldsymbol{y}}(g) \cdot f \in k[\boldsymbol{x}, \boldsymbol{y}]$ and similarly, if $g \in k[\boldsymbol{x}, \boldsymbol{y}]$, $f \in k[\boldsymbol{x}]$ then $g \cdot f := g \cdot \iota_{\boldsymbol{x}}(f) \in k[\boldsymbol{x}, \boldsymbol{y}]$.

Lemma 2.5.6. Let $X \in \mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$ for $\boldsymbol{x} \equiv (x_1, \ldots, x_m)$ and $\boldsymbol{y} \equiv (y_1, \ldots, y_n)$. Let φ be the ring morphism $k[\boldsymbol{x}', \boldsymbol{x}] \to k[\boldsymbol{x}],\ x_i \mapsto x_i,\ x_i' \mapsto x_i\ \forall i \in \mathbb{N}_{\leq m}$ and let ψ be the ring morphism $k[\boldsymbol{y}', \boldsymbol{y}] \to k[\boldsymbol{y}],\ y_i \mapsto y_i,\ y_i' \mapsto y_i\ \forall i \in \mathbb{N}_{\leq n}$. The following define morphisms of matrix factorizations

$$\begin{aligned} X \check{\otimes}_{k[\boldsymbol{x}]} I_{k[\boldsymbol{x}],V} &\to X & (2.5.7)\\ (g.x) \check{\otimes}_{k[\boldsymbol{x}]} (f.\theta_1^{i_1} \ldots \theta_m^{i_m}) &\mapsto (g \cdot \varphi(f)).x\ \delta_{i_1,0} \cdots \delta_{i_m,0} \\ I_{k[\boldsymbol{y}],W} \check{\otimes}_{k[\boldsymbol{y}]} X &\to X & (2.5.8)\\ (\tilde{g}.\theta_1^{i_1} \ldots \theta_n^{i_n}) \check{\otimes}_{k[\boldsymbol{y}]} (\tilde{f}.x) &\mapsto (\psi(\tilde{g}) \cdot \tilde{f}).x\ \delta_{i_1,0} \cdots \delta_{i_n,0} \end{aligned}$$

for polynomials $f \in k[\boldsymbol{x}', \boldsymbol{x}]$, $g \in k[\boldsymbol{x}, \boldsymbol{y}]$, $\tilde{g} \in k[\boldsymbol{y}', \boldsymbol{y}]$, $\tilde{f} \in k[\boldsymbol{x}, \boldsymbol{y}]$.

Proof. The prescriptions in (2.5.7) and (2.5.8) yield morphisms in $\mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$ since they are $k[\boldsymbol{x}, \boldsymbol{y}]$-linear maps of underlying modules: $y_j,\ j \in \mathbb{N}_{\leq n}$ acts on the left hand sides by multiplying g respectively $\tilde{g}$ while x_i, $i \in \mathbb{N}_{\leq m}$ multiplies f respectively $\tilde{f}$ and

$$(y_j \cdot g) \cdot \varphi(x_i \cdot f) = (y_j x_i) \cdot (g \cdot \varphi(f)),\ \psi(y_j \cdot \tilde{g}) \cdot (x_i \cdot \tilde{f}) = (y_j x_i) \cdot (\psi(\tilde{g}) \cdot \tilde{f}).$$

The claim follows since (2.5.7) and (2.5.8) are k-linear as multiplying in $k[\boldsymbol{x}, \boldsymbol{y}]$ is bilinear and φ as well as ψ are ring homomorphisms. □

Definition 2.5.7. The map (2.5.8) is $\lambda_X^{(\boldsymbol{x},V),(\boldsymbol{y},W)} \equiv \lambda_X$, the *left unitor (isomorphism)* for X. The map (2.5.7) is $\rho_X^{(\boldsymbol{x},V),(\boldsymbol{y},W)} \equiv \rho_X$, the *right unitor (isomorphism)* for X.

In words, we describe the actions of the unitor isomorphisms for a matrix factorization X as follows. They project $I_{\mathcal{R},V}$ (or $I_{\mathcal{S},W}$ etc.) to its θ-degree zero part, which we also call the 1-*component*. Moreover, they identify the "middle-variables" with the variables (acting) on the right in the case of ρ respectively on the left in the case of λ. Finally, they multiply the resulting coefficients of the 1-component of I and of basis elements of X in the polynomial ring with the "outer" variables.

In defining right inverses of λ and ρ we use

Definition 2.5.8. Let $X \in \mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$ have a basis $\{e_j\}_{j \in J}$ for some index set J. Denote by ι the ring homomorphism $k[\boldsymbol{x}', \boldsymbol{x}, \boldsymbol{y}] \to k[\boldsymbol{x}, \boldsymbol{y}]$ which maps $x'_i \mapsto x_i$, $x_i \mapsto x_i$ and $y_l \mapsto y_l$ for all i, l. Thus, $\{e_j\}_{j\in J}$ is a $k[\boldsymbol{x}',\boldsymbol{x},\boldsymbol{y}]$-basis of $\iota_*(X_m)$. Then $\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_r]} d_X$ is the following $k[\boldsymbol{x}',\boldsymbol{x},\boldsymbol{y}]$-linear operator on $\iota_*(X_m)$. Take the matrix representing the $k[\boldsymbol{x},\boldsymbol{y}]$-linear map d_X in the basis $\{e_j\}_{j\in J}$. Its entries are elements of $k[\boldsymbol{x},\boldsymbol{y}]$. View these as polynomials in the $\boldsymbol{x}$-variables only. Finally, apply Definition 2.5.1 to the latter. Analogously, $\partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i_r]} d_X$ is the $k[\boldsymbol{x},\boldsymbol{y}',\boldsymbol{y}]$-linear operator obtained by applying Definition 2.5.1 to the entries of the matrix representing d_X viewed as polynomials in the $\boldsymbol{y}$-variables only.

Definition 2.5.9. In the situation of Definition 2.5.8 the right *inverse (of the) right unitor* $(\rho_X^{(\boldsymbol{x},V),(\boldsymbol{y},W)})^{-1} \equiv \rho_X^{-1}$ is the map in $\mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}(X, X\check{\otimes}_{k[\boldsymbol{x}]} I_{k[\boldsymbol{x}],V})$ specified by

$$e_i \mapsto \sum_{l\geq 0}\sum_{i_1<\cdots<i_l}\sum_{j\in J}(-1)^{\binom{l}{2}+l|e_i|} e_j \check{\otimes}_{k[\boldsymbol{x}]}\{\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_1]} d_X \ldots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_l]} d_X\}_{ji}.\theta_{i_1}\ldots\theta_{i_l}. \quad (2.5.9)$$

Here $\binom{l}{2}$ is a binomial coefficient and $\{-\}_{ij}$ denotes the (ij)-entry of the matrix representing the $k[\boldsymbol{x}',\boldsymbol{x},\boldsymbol{y}]$-linear operator "$-$" in the basis $\{e_j\}_{j\in J}$. Using the same notation the right *inverse (of the) left unitor* $(\lambda_X^{(\boldsymbol{x},V),(\boldsymbol{y},W)})^{-1} \equiv \lambda_X^{-1}$ is the morphism in $\mathrm{MF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}(X, I_{k[\boldsymbol{y}],W}\check{\otimes}_{k[\boldsymbol{y}]}X)$ determined by

$$e_i \mapsto \sum_{l\geq 0}\sum_{i_1<\cdots<i_l}\sum_{j\in J}\{\partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i_l]} d_X \ldots \partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i_1]} d_X\}_{ji}.\theta_{i_1}\ldots\theta_{i_l}\check{\otimes}_{k[\boldsymbol{y}]} e_j. \quad (2.5.10)$$

We refer to the factor of $I_{\mathcal{R},W}$ in the tensor product of matrix factorizations in the target of λ^{-1} or ρ^{-1} as the unit matrix factorization *created* by λ^{-1} or ρ^{-1}.

We mention one consequence of Definition 2.5.7 and Definition 2.5.9 which is vital for our proof of Lemma 3.1.10 below. The latter, in turn, implies that all diagrams commute which have to commute for our definitions in the following chapters to indeed define the structure we are looking for.

Observation 2.5.10. Assume that ρ^{-1} is applied to some matrix factorization and that subsequently the resulting factor of $I_{R,W}$ is projected onto θ-degree zero. This means that in the formula (2.5.9) only the summand for $l=0$ is left. Therefore the sum over $i_1<\cdots<i_l$ is turned into an empty sum. Furthermore, $\{\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_1]} d_X \ldots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_l]} d_X\}_{ji}$ becomes $\{1\}_{ji} = \delta_{ji}$. Thus, solely e_i remains from the sum over $j\in J$. From this it follows e.g. that $\rho\circ\rho^{-1}$ is the identity map. A thoroughly analogous reasoning applies to λ^{-1} and $\lambda\circ\lambda^{-1}$.

Still, neither $\lambda^{-1}\circ\lambda$ nor $\rho^{-1}\circ\rho$ are the identity morphism in MF. Yet they are identity maps as 2-morphisms in $\mathcal{LG}$ as we see in our proof of Proposition 2.10.3.

2.6. A homotopy category of matrix factorizations

The unitors of $\mathcal{LG}$ as displayed in Lemma 2.5.6 are not isomorphisms in MF. This leads us to refine MF to a category whose morphisms are equivalence classes of morphisms in MF with respect to an equivalence relation which is such that $\lambda^{-1} \circ \lambda$ and $\rho^{-1} \circ \rho$ are equivalent to identity morphisms. As a prerequisite for this we introduce a map $\delta_{X,Y}$ for each pair (X, Y) of matrix factorizations as follows.

Lemma 2.6.1. Let X and Y be two matrix factorizations of $W \in \mathcal{R}$. Extending the following prescription $\mathcal{R}$-linearly yields an (honest) differential on $\mathrm{MF}_{\mathcal{R},W}(X, Y)$:

$$\begin{aligned} \delta_{X,Y} : \mathrm{MF}_{\mathcal{R},W}(X,Y)^i &\to \mathrm{MF}_{\mathcal{R},W}(X,Y)^{(i+1) \bmod 2}, \ i \in \{0,1\}. \\ \varphi &\mapsto d_Y \circ \varphi - (-1)^i\ \varphi \circ d_X \end{aligned} \tag{2.6.1}$$

Proof. For $\varphi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)^i$ we compute

$$\begin{aligned} \delta^2_{X,Y}(\varphi) &= \delta_{X,Y}(d_Y \circ \varphi - (-1)^i\ \varphi \circ d_X) \\ &= d_Y(d_Y \circ \varphi - (-1)^i\ \varphi \circ d_X) - (-1)^{i+1}(d_Y \circ \varphi - (-1)^i\ \varphi \circ d_X)d_X \\ &= d_Y^2 \circ \varphi - (-1)^i d_Y \circ \varphi \circ d_X - (-1)^{i+1}(d_Y \circ \varphi \circ d_X - (-1)^i\ \varphi \circ d_X^2) \\ &= (W.\,\mathrm{id}_{Y_m}) \circ \varphi - (-1)^i d_Y \circ \varphi \circ d_X - (-1)^{i+1} d_Y \circ \varphi \circ d_X - \varphi \circ (W.\,\mathrm{id}_{X_m}) \\ &= 0, \end{aligned}$$

using that φ is $\mathcal{R}$-linear in the last step. □

We say that $\delta_{X,Y}$-closed maps of matrix factorizations are *compatible with the differentials* (of X and Y).

Definition 2.6.2. The category $\mathrm{HMF}_{\mathcal{R},W}$ has the same objects as $\mathrm{MF}_{\mathcal{R},W}$ but

$$\begin{aligned} \mathrm{HMF}_{\mathcal{R},W}(X,Y) &:= \{\varphi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)^0 \mid \delta_{X,Y}(\varphi) = 0\}/\sim, \text{ where} \\ \varphi_1 \sim \varphi_2 &\Leftrightarrow \varphi_2 - \varphi_1 \in \mathrm{Im}(\delta_{X,Y}). \end{aligned}$$

Representatives for the composite of two morphisms in $\mathrm{HMF}_{\mathcal{R},W}$ are given by the composite in $\mathrm{MF}_{\mathcal{R},W}$ of representatives. $\mathrm{HMF}_{\mathcal{R},W}$ is called the *homotopy category of matrix factorizations* of $(\mathcal{R}, W)$.

That the composition of morphisms in $\mathrm{MF}_{\mathcal{R},W}$ respects their equivalence classes in $\mathrm{HMF}_{\mathcal{R},W}$ is a special case of Lemma 5.3.1 which we proof below.

Remark 2.6.3. In words, Definition 2.6.2 says that morphisms in $\mathrm{HMF}_{\mathcal{R},W}(X,Y)$ are $\mathbb{Z}_2$-even module maps from X_m to Y_m which are $\delta_{X,Y}$-closed modulo those which are $\delta_{X,Y}$-exact. Put differently, $\mathrm{HMF}_{\mathcal{R},W}(X,Y)$ is the zeroth $\delta_{X,Y}$-cohomology of $\mathrm{MF}_{\mathcal{R},W}(X,Y)$ which we write as $\mathrm{H}^0(\mathrm{MF}_{\mathcal{R},W}(X,Y))$.

Remark 2.6.4. We show in Remark 2.7.4 below that for all X, Y in $\mathrm{MF}_{\mathcal{R},W}$ there is an $\widetilde{X} \in \mathrm{MF}_{\mathcal{R},W}$ such that $\mathrm{H}^0(\mathrm{MF}_{\mathcal{R},W}(\widetilde{X},Y)) \cong \mathrm{H}^1(\mathrm{MF}_{\mathcal{R},W}(X,Y))$. In this sense we do not "lose" any structure by taking only the zeroth cohomology. Moreover, restricting to zeroth cohomology saves us from having to accommodate to various signs resulting from the rule (2.4.1).

In order to use homotopy categories for $\mathcal{LG}$ we translate Definition 2.4.6 to the level of HMF. We start with

Lemma 2.6.5. Extension and restriction of scalars for matrix factorizations descend to the level of homotopy categories of matrix factorizations.

Proof. Since homotopy categories HMF have the same objects as MF for matching indices, we only have to show that the functors restrict to the equivalence classes of morphisms in MF which are morphisms in HMF. In this vein, note first that restriction and extension of scalars for matrix factorizations leave the $\mathbb{Z}_2$-degree of homogeneous morphisms invariant. The rest follows from extension and restriction of scalars for matrix factorizations being functors and respecting sums. For example, let $\varphi : \mathcal{R} \to \mathcal{S}$ be a homomorphism of commutative rings, ψ, ψ_1 and ψ_2 morphisms in $\mathrm{MF}_{\mathcal{R},W}(X,Y)$, then

$$\begin{aligned} d_Y \circ \psi = \psi \circ d_X &\Rightarrow \varphi^*(d_Y \circ \psi) = \varphi^*(\psi \circ d_X) \Rightarrow \varphi^*(d_Y) \circ \varphi^*(\psi) = \varphi^*(\psi) \circ \varphi^*(d_X) \\ &\Rightarrow d_{\varphi^*(Y)} \circ \varphi^*(\psi) = \varphi^*(\psi) \circ d_{\varphi^*(X)}, \end{aligned}$$

$$\begin{aligned} \psi_1 = \psi_2 + h \circ d_X + d_Y \circ h \Rightarrow \varphi^*(\psi_1) &= \varphi^*(\psi_2 + h \circ d_X + d_Y \circ h) \\ &= 1 \otimes_{\mathcal{R}} (\psi_2 + h \circ d_X + d_Y \circ h) \\ &= 1 \otimes_{\mathcal{R}} \psi_2 + 1 \otimes_{\mathcal{R}} (h \circ d_X) + 1 \otimes_{\mathcal{R}} (d_Y \circ h) \\ &= \varphi^*(\psi_2) + \varphi^*(h \circ d_X) + \varphi^*(d_Y \circ h) \\ &= \varphi^*(\psi_2) + \varphi^*(h) \circ \varphi^*(d_X) + \varphi^*(d_Y) \circ \varphi^*(h) \\ &= \varphi^*(\psi_2) + \varphi^*(h) \circ d_{\varphi^*(X)} + d_{\varphi^*(Y)} \circ \varphi^*(h). \end{aligned}$$

□

In particular, we note in passing

Corollary 2.6.6. The isomorphisms of categories from Corollary 2.3.5 give isomorphisms of homotopy categories.

Proof. Let $\varphi : \mathcal{R} \to \mathcal{S}$ be an isomorphism of commutative rings. According to Lemma 2.6.5 this induces a functor $\mathrm{HMF}_{\mathcal{S},W} \to \mathrm{HMF}_{\mathcal{R},\varphi^{-1}(W)}$ which acts as φ_* both on objects and on representatives of morphisms. The same holds true for φ^{-1}. Since Corollary 2.3.5 states that φ_* and $(\varphi^{-1})_*$ are each other's inverses also the induced functors are inverse to one another. □

Causing some further ambiguity to be resolved by the context we use

Definition 2.6.7. Let $\varphi : \mathcal{R} \to \mathcal{S}$ be a homomorphism of commutative rings. The functors in Lemma 2.6.5 are $\varphi^* : \mathrm{HMF}_{\mathcal{R},W} \to \mathrm{HMF}_{\mathcal{S},\varphi(W)}$, *extension of scalars* (for HMF), and $\varphi_* : \mathrm{HMF}_{\mathcal{S},\varphi(W)} \to \mathrm{HMF}_{\mathcal{R},W}$, *restriction of scalars* (for HMF).

Definition 2.4.6 which we are about to mimic for HMF instead of MF relies on extension and restriction of scalars as well as on the functor $\otimes_{\mathcal{R}}$ from Lemma 2.4.2. We turn the latter into a functor on HMF next. As anticipated below Remark 2.4.3 we can employ the cartesian product of categories rather than HMF $\otimes$ HMF since morphisms in HMF are $\mathbb{Z}_2$-even module maps.

Lemma 2.6.8. The functor from Lemma 2.4.2 induces a functor

$$\mathrm{HMF}_{\mathcal{R},W} \times \mathrm{HMF}_{\mathcal{R},V} \to \mathrm{HMF}_{\mathcal{R},W+V}.$$

Proof. Since objects of HMF coincide with those of MF we only need to show that the functor from Lemma 2.4.2 sends morphisms in HMF to morphisms. For this let $\widetilde{\psi} \in \mathrm{HMF}_{\mathcal{R},W}(Y,\tilde{Y})$ have a representative $\psi \in \mathrm{MF}_{\mathcal{R},W}(Y,\tilde{Y})$ and similarly for $\widetilde{\phi} \in \mathrm{HMF}_{\mathcal{R},V}(X,\tilde{X})$ and $\phi \in \mathrm{MF}_{\mathcal{R},V}(X,\tilde{X})$. Then, writing $\otimes$ for $\otimes_{\mathcal{R}}$ and keeping in mind $|\psi| = 0 = |\phi|$ and (2.4.1), we have

$$d_{\tilde{Y}\otimes\tilde{X}}\circ(\psi\otimes\phi) = (d_{\tilde{Y}}\circ\psi)\otimes\phi+\psi\otimes(d_{\tilde{X}}\circ\phi) = (\psi\circ d_Y)\otimes\phi+\psi\otimes(\phi\circ d_X) = (\psi\otimes\phi)\circ d_{Y\otimes X}$$

which shows that $\psi \otimes \phi$ is $\delta_{Y\otimes X,\tilde{Y}\otimes\tilde{X}}$-closed. The following shows that the tensor product is well-defined on equivalence classes in the first argument.

$$\begin{aligned}
\psi \otimes \phi \sim &(\psi + h d_Y + d_{\tilde{Y}} h) \otimes \phi \\
=&\psi \otimes \phi + (h \otimes \phi) \circ (d_Y \otimes 1) + (d_{\tilde{Y}} \otimes 1) \circ (h \otimes \phi) \\
=&\psi \otimes \phi + (h \otimes \phi) \circ d_{Y\otimes X} - (h \otimes \phi) \circ (1 \otimes d_X) + d_{\tilde{Y}\otimes\tilde{X}} \circ (h \otimes \phi) \\
&- (1 \otimes d_{\tilde{X}}) \circ (h \otimes \phi) \\
=&\psi \otimes \phi + \delta_{Y\otimes X,\tilde{Y}\otimes\tilde{X}}(h \otimes \phi) - (h \otimes \phi) \circ (1 \otimes d_X) - (1 \otimes d_{\tilde{X}}) \circ (h \otimes \phi) \\
=&\psi \otimes \phi + \delta_{Y\otimes X,\tilde{Y}\otimes\tilde{X}}(h \otimes \phi) - h \otimes (\phi \circ d_X) + h \otimes (d_{\tilde{X}} \circ \phi) \\
=&\psi \otimes \phi + \delta_{Y\otimes X,\tilde{Y}\otimes\tilde{X}}(h \otimes \phi) \\
\sim&\psi \otimes \phi.
\end{aligned}$$

It can be seen similarly that the tensor product is well-defined on equivalence classes in the second argument. □

Just as Definition 2.6.7 does not distinguish between MF and HMF we use

Notation 2.6.9. We write $\otimes_{\mathcal{R}}$ for the functor described in Lemma 2.6.8.

Definition 2.6.10 which comes next introduces functors which upon restriction to a subcategory of HMF are the functors for horizontal composition in $\mathcal{LG}$. To stress that it is nontrivial that this restriction is possible – it is enabled only by [DM, Theorem 12.4] which we quote as Theorem 2.9.3 below – we include a tilde on the following functor, reserving untilded notation for the functors for horizontal composition in $\mathcal{LG}$.

Definition 2.6.10. In the situation of Definition 2.4.6 the functor

$$\widetilde{\otimes}_{k[\boldsymbol{y}]} : \mathrm{HMF}_{k[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})} \times \mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})} \to \mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$$

is the composite

$$(\iota_{\boldsymbol{x},\boldsymbol{z}})_* \circ \otimes_{k[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \circ ((\iota_{\boldsymbol{y},\boldsymbol{z}})^* \times (\iota_{\boldsymbol{x},\boldsymbol{y}})^*).$$

For $Y \in \mathrm{HMF}_{k[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})}$, $X \in \mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}$, $y \in Y_m$, $x \in X_m$ we write

$$Y \widetilde{\otimes}_{k[\boldsymbol{y}]} X := \widetilde{\otimes}_{k[\boldsymbol{y}]}(Y, X),\ y \widetilde{\otimes}_{k[\boldsymbol{y}]} x \in (Y \widetilde{\otimes}_{k[\boldsymbol{y}]} X)_m.$$

Sometimes we also omit the indices on $\widetilde{\otimes}$.

As we argue that we pass to homotopy categories of matrix factorizations to ensure that the maps in Lemma 2.5.6 become isomorphisms, it is appropriate to point out

Lemma 2.6.11. The maps in Lemma 2.5.6 are compatible with the differentials.

Proof. Let $X \in \mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$ and consider $\lambda_X : I_{k[\boldsymbol{y}],W} \widetilde{\otimes} X \to X$. We argue that

$$d_X \circ \lambda_X = \lambda_X \circ d_{I_{k[\boldsymbol{y}],W} \widetilde{\otimes} X} \tag{2.6.2}$$

as maps of modules. Note that $d_{I_{k[\boldsymbol{y}],W} \widetilde{\otimes} X} = d_{I_{k[\boldsymbol{y}],W}} \widetilde{\otimes} 1 + 1 \widetilde{\otimes} d_X$. The prescription in Lemma 2.5.6 says that λ_X leaves the right factor untouched apart from multiplying it with a scalar. As d_X is linear it follows that the equality (2.6.2) holds if $\lambda_X \circ (d_{I_{k[\boldsymbol{y}],W}} \widetilde{\otimes} 1) = 0$. This is true. λ_X projects $I_{k[\boldsymbol{y}],W}$ to its 1-component. Therefore only the summands $(y_i - y_i').\theta_i^*$ in (2.5.4) can contribute to $\lambda_X \circ (d_{I_{k[\boldsymbol{y}],W}} \widetilde{\otimes} 1)$. But since λ_X sets $y_i = y_i'$ also this contribution vanishes.

One can prove analogously that ρ_X is compatible with the differentials. □

Remark 2.6.12. At this point we are in a position to use the unitors of Definition 2.5.7 to define a bicategory. Its objects are pairs $(\boldsymbol{y}, W)$ where $\boldsymbol{y}$ is an ordered set of variables and $W \in k[\boldsymbol{y}]$ for some commutative ring k. The category of morphisms from $(\boldsymbol{x}, V)$ to $(\boldsymbol{y}, W)$ is $\mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$. Indeed, the proof of [CM2, §4] that the maps in Definition 2.5.9 are inverse to the unitors of Definition 2.5.7 works in this generality. Moreover, there are canonical associators which essentially act by moving brackets, cf. Lemma 2.10.1 below which generalizes to the present setting. That these satisfy the pentagon axiom and together with the unitors also the triangle axiom follows as in $\mathcal{LG}$, cf. Proposition 2.10.3. This concludes the description of a bicategory. However, for our applications this bicategory is not sufficient, cf. Section 2.8.

2.7. A triangulated structure

As $\mathcal{LG}$ is related to TQFTs, it has more structure than that of a bicategory. In particular there are adjoints for 1-morphisms. We define them using an ingredient of a triangulated structure on HMF which we present next. For the definition of a triangulated category we refer to [Nee, Definition 1.3.13]. That HMF is triangulated features in our proof of Corollary 2.9.2 below.

Lemma 2.7.1. The following defines a functor $[1] : \mathrm{MF}_{\mathcal{R},W} \to \mathrm{MF}_{\mathcal{R},W}$.

$$\begin{aligned}
&X \mapsto X[1],\ (X[1])^i_m := X_m^{(i+1)\bmod 2},\ d^i_{X[1]} := -d_X^{(i+1)\bmod 2},\ i \in \{0,1\},\\
&\text{for } \varphi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)^i\\
&\varphi \mapsto \varphi[1] \in \mathrm{MF}_{\mathcal{R},W}(X[1],Y[1])^i,\ \varphi[1]^j := (-1)^i\ \varphi^{(j+1)\bmod 2},
\end{aligned} \tag{2.7.1}$$

where we employ the notation of (2.2.1) and (2.7.1) is to be extended $\mathcal{R}$-linearly to $\mathrm{MF}_{\mathcal{R},W}(X,Y)$.

Proof. By definition, $(X[1])_m$ is a $\mathbb{Z}_2$-graded free $\mathcal{R}$-module and $d_{X[1]}$ a $\mathbb{Z}_2$-odd endomorphism thereof. Also, $d_X^2 = W.\,\mathrm{id}_{X_m}$ implies $d^2_{X[1]} = W.\,\mathrm{id}_{X[1]_m}$.

For φ a homogeneous morphism $\varphi[1]$ is defined as a pair of module homomorphisms which makes it a homogeneous morphism of the same degree as φ.

Finally, for $(i,j) \in \{0,1\}^2$ let $\phi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)^i$ and $\psi \in \mathrm{MF}_{\mathcal{R},W}(Y,Z)^j$. Then $\psi \circ \phi \in \mathrm{MF}_{\mathcal{R},W}(X,Z)^{(i+j)\mathrm{mod}2}$ and for $s \in \{0,1\}$ we compute

$$\begin{aligned}
(\psi \circ \phi)[1]^s &= (-1)^{(i+j)\mathrm{mod}2}(\psi\circ\phi)^{(s+1)\mathrm{mod}2}\\
&= (-1)^{i+j}\begin{cases}\psi^s \circ \phi^{(s+1)\mathrm{mod}2} & \text{for } i=1\\ \psi^{(s+1)\mathrm{mod}2}\circ\phi^{(s+1)\mathrm{mod}2} & \text{for } i=0\end{cases}
\end{aligned}$$

$$(\psi[1]\circ\phi[1])^s = \begin{cases}\psi[1]^{(s+1)\mathrm{mod}2}\circ\phi[1]^s = (-1)^j\psi^s \circ (-1)^i\phi^{(s+1)\mathrm{mod}2} & \text{for } i=1\\ \psi[1]^s\circ\psi[1]^s = (-1)^j\psi^{(s+1)\mathrm{mod}2}\circ(-1)^i\phi^{(s+1)\mathrm{mod}2} & \text{for } i=0.\end{cases}$$

This verifies that [1] respects the composition of morphisms. Furthermore [1] respects identity morphisms since

$$\mathrm{id}_X \in \mathrm{MF}_{\mathcal{R},W}(X,X)^0 \Rightarrow \mathrm{id}_X[1]^i = (-1)^0\mathrm{id}_X^{(i+1)\mathrm{mod}2} = \mathrm{id}^i_{X[1]}.$$

□

The functor in Lemma 2.7.1 is called the *shift functor*.

Lemma 2.7.1 is analogous to the case of "untwisted" (periodic) chain complexes. Likewise, as with "untwisted" complexes $\mathrm{MF}_{\mathcal{R},W}$ itself is not triangulated. But as in that situation $\mathrm{HMF}_{\mathcal{R},W}$ becomes a triangulated category using

Corollary 2.7.2. The functor from Lemma 2.7.1 induces an endofunctor on $\mathrm{HMF}_{\mathcal{R},W}$.

Proof. Observe that according to Lemma 2.7.1 $d_{X[1]} = d_X[1]$. Moreover, since the sum of two homogeneous morphisms in $\mathrm{MF}_{\mathcal{R},W}$ is the sum of their components we have $(\phi + \varphi)[1] = \phi[1] + \varphi[1]$ for $\phi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)$, $\varphi \in \mathrm{MF}_{\mathcal{R},W}(X,Y)$. From this it follows that $\delta_{X,Y}$-closed morphisms in $\mathrm{MF}_{\mathcal{R},W}(X,Y)^0$ are mapped to closed morphisms and homotopic ones to homotopic ones as

$$d_{Y[1]}\phi[1] = d_Y[1]\phi[1] = (d_Y\phi)[1] = (\phi d_X)[1] = \phi[1]d_X[1] = \phi[1]d_{X[1]},$$
$$\varphi = \phi + d_Y\psi + \psi d_X \Rightarrow \varphi[1] = \phi[1] + d_{Y[1]}\psi[1] + \psi[1]d_{X[1]}.$$

□

Paralleling the theory of categories of actual complexes [KR1] states

Proposition 2.7.3. $\mathrm{HMF}_{\mathcal{R},W}$ together with its inherited shift functor and the usual cones is a triangulated category.

At this place, we return to Remark 2.6.4.

Remark 2.7.4. As mentioned e.g. in [C1] given two matrix factorizations X and Y of $(\mathcal{R}, W)$ there is an isomorphism between the zeroth $\delta_{X,Y}$-cohomology of $\mathrm{MF}_{\mathcal{R},W}(X,Y)$ and the first $\delta_{X[1],Y}$-cohomology of $\mathrm{MF}_{\mathcal{R},W}(X[1],Y)$ which in the notation of (2.2.1) sends a morphism (φ_0, φ_1) to (φ_1, φ_0).

2.8. Adjoints for finite-rank matrix factorizations

Using the shift functors of the previous section we associate matrix factorizations $X^\dagger$ and ${}^\dagger X$ to matrix factorizations X whose underlying module has finite rank. As we recapitulate in Definition A.2.1 for these to qualify as adjoints to X there moreover have to be (co-)evaluation morphisms. We are not aware that the latter have been defined more generally than on a subcategory of HMF which we introduce below. For details we refer to [CM2], cf. [CR2] for earlier work in this direction.

Lemma 2.8.1. Let $X \in \mathrm{MF}_{\mathcal{R},W}$ and view $\mathcal{R}$ as a $\mathbb{Z}_2$-graded module concentrated in degree zero. Then $\mathrm{Mod}_{\mathcal{R}}(X_m, \mathcal{R})$ equipped with its $\mathcal{R}$-module endomorphism $f \mapsto -(-1)^{|f|} f \circ d_X$ is a matrix factorization of $(\mathcal{R}, -W)$.

Proof. Since X_m is a free $\mathcal{R}$-module so is $\mathrm{Mod}_{\mathcal{R}}(X_m, \mathcal{R})$. Moreover, $\mathrm{Mod}_{\mathcal{R}}(X_m, \mathcal{R})$ has the $\mathbb{Z}_2$-grading from (2.2.1). Furthermore, we have

$$(-1)^{|f \circ d_X|}((-1)^{|f|} f \circ d_X) \circ d_X = (-1)^{|f|+1+|f|} f \circ d_X^2 = -f \circ (W.\,\mathrm{id}_{X_m}) = -W.f$$

which verifies that the endomorphism in the lemma – which is $\mathbb{Z}_2$-odd as d_X is – squares to $-W.\,\mathrm{id}$. □

Definition 2.8.2. The matrix factorization described in Lemma 2.8.1 is denoted $X^\vee$ and referred to as the dual matrix factorization to X.

Next, we introduce the matrix factorizations which are adjoints in $\mathcal{LG}$.

Definition 2.8.3. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be ordered sets of $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ variables, respectively, k a field of characteristic zero, $V \in \mathrm{k}[\boldsymbol{x}]$, $W \in \mathrm{k}[\boldsymbol{y}]$ potentials and X a matrix factorization of $(\mathrm{k}[\boldsymbol{x}, \boldsymbol{y}], W(\boldsymbol{y}) - V(\boldsymbol{x}))$ such that X_m is a free module of finite rank. Writing ϕ for the canonical ring homomorphism $\mathrm{k}[\boldsymbol{y}, \boldsymbol{x}] \to \mathrm{k}[\boldsymbol{x}, \boldsymbol{y}]$ we set

$$X^\dagger := \phi_*(X^\vee[m]),\ {}^\dagger X := \phi_*(X^\vee[n]).$$

Note that this definition allows for $\boldsymbol{x} = \emptyset$ and $V = 0$ or $\boldsymbol{y} = \emptyset$ and $W = 0$.

In [CM2, §5, §6] it is shown that there are evaluation and coevaluation maps exhibiting $X^\dagger$ and ${}^\dagger X$ as right respectively left adjoints to X in $\mathcal{LG}$. The coevaluation maps are based on the elements $\sum_{i=1}^{r}(-1)^{|e_i|}e_i^*\widetilde{\otimes}e_i \in (X^\vee\widetilde{\otimes}X)_m$ and $\sum_{i=1}^{r}e_i\widetilde{\otimes}e_i^* \in (X\widetilde{\otimes}X^\vee)_m$ for a basis $\{e_1, \dots, e_r\}$ of X_m with dual basis $\{e_1^*, \dots, e_r^*\}$. Therefore it is indispensable for these adjunctions that X be of finite rank. Hence we define the subcategory of HMF for which these adjoints can be constructed.

Definition 2.8.4. A matrix factorization has *finite rank* if its underlying module has finite rank. $\mathrm{hmf}_{\mathcal{R},W}$ is the full subcategory of $\mathrm{HMF}_{\mathcal{R},W}$ whose objects are matrix factorizations of finite rank.

We mention as an aside

Lemma 2.8.5. The isomorphisms of Corollary 2.6.6 carry over to isomorphisms of homotopy categories of finite-rank matrix factorizations.

Proof. For two isomorphic rings restriction of scalars along their isomorphism turns a basis into a basis. In particular it preserves the rank. As restriction of scalars for matrix factorizations acts as restriction of scalars of the corresponding modules on the level of underlying modules and the rank of a matrix factorization is defined to be the rank of its underlying module this implies the claim. □

Refining what we wrote in the paragraph preceding Definition 2.8.4, the coevaluation maps of [CM2, §5] rely on the elements $\sum_{i=1}^{r}(-1)^{|e_i|}e_i^* \otimes e_i \in (X^\dagger \otimes X)_m$ and $\sum_{i=1}^{r}e_i \otimes e_i^* \in (X \otimes {}^\dagger X)_m$. Indeed, a basis of $X^\vee$ is also a basis of $X^\vee[1]$. Only the $\mathbb{Z}_2$-degree of each basis element changes when passing from $X^\vee$ to $X^\vee[1]$.

Remark 2.8.6. As stated in [O, Proposition 3.3] $\mathrm{hmf}_{\mathcal{R},W}$ can be equipped with the structure of a triangulated category whose shift functor is induced from the functor in Corollary 2.7.2. This is proven explicitly in [L, Theorem 5.1.1].

We end this section by introducing the (co-)evaluation maps, cf. Definition A.2.1, of [CM2, §5] for matrix factorizations of finite rank.

Proposition 2.8.7. Let $\boldsymbol{x} \equiv (x_1, \dots, x_m)$, $\boldsymbol{y} \equiv (y_1, \dots, y_n)$ be ordered sets of variables, $W \in \mathrm{k}[\boldsymbol{y}]$, $V \in \mathrm{k}[\boldsymbol{x}]$ be potentials and $X \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$. Let $\{e_i\}_{i\in I}$, I some index set, be a basis of X_m with dual basis $\{e_i^*\}_{i\in I}$. Then the following defines a coevaluation morphism:

$$\widetilde{\mathrm{coev}}_X \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{x}',\boldsymbol{x}],V(\boldsymbol{x})-V(\boldsymbol{x}')}(I_{\mathrm{k}[\boldsymbol{x}],V}, X^\dagger \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X),$$
$$\gamma \mapsto \sum_{(i,j)\in I^2} (-1)^{(l+1)|e_j|+s} \{\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[b_l]} d_X \dots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[b_1]} d_X\}_{ji}.e_i^* \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} e_j,$$

where $b_1 < \dots < b_l$ and s are such that $\gamma \wedge \theta_{b_1} \dots \theta_{b_l} = (-1)^s \theta_1 \dots \theta_m$. Similarly, there is a coevaluation map

$$\mathrm{coev}_X \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{y}',\boldsymbol{y}],W(\boldsymbol{y})-W(\boldsymbol{y}')}(I_{\mathrm{k}[\boldsymbol{y}],W}, X \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{x}]} {}^\dagger X),$$
$$\gamma \mapsto \sum_{(i,j)\in I^2} (-1)^{\binom{l+1}{2}+s+n+nl} \{\partial^{\boldsymbol{y},\boldsymbol{y}'}_{[b_1]} d_X \dots \partial^{\boldsymbol{y},\boldsymbol{y}'}_{[b_l]} d_X\}_{ij}.e_i \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{x}]} e_j^*,$$

where $b_1 < \dots < b_l$ and s fulfill $\gamma \wedge \theta_1 \dots \theta_{b_l} = (-1)^s \theta_1 \dots \theta_n$. The following formulas define evaluation maps:

$$\widetilde{\mathrm{ev}}_X \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{y}',\boldsymbol{y}],W(\boldsymbol{y})-W(\boldsymbol{y}')}(X \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{x}]} X^\dagger, I_{\mathrm{k}[\boldsymbol{y}],W}),$$
$$g.e_j \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{x}]} e_i^* \mapsto \sum_{l\geq 0} \sum_{i_1<\dots<i_l} (-1)^{l+(m+1)|e_j|} \theta_{i_1} \dots \theta_{i_l} \operatorname{Res}_{\mathrm{k}[\boldsymbol{x}]/\mathrm{k}} \left[\frac{\{\partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i_l]} d_X \dots \partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i_1]} d_X \Lambda^{(x)}\}_{ij} g \,\mathrm{d}\, x}{\partial_{x_1} V, \dots, \partial_{x_m} V} \right],$$

$$\mathrm{ev}_X \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{x}',\boldsymbol{x}],V(\boldsymbol{x})-V(\boldsymbol{x}')}({}^\dagger X \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X, I_{\mathrm{k}[\boldsymbol{x}],V}),$$
$$e_i^* \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} h.e_j \mapsto \sum_{l\geq 0} \sum_{i_1<\dots<i_l} (-1)^{\binom{l}{2}+l|e_j|+n} \theta_{i_1} \dots \theta_{i_l} \operatorname{Res}_{\mathrm{k}[\boldsymbol{y}]/\mathrm{k}} \left[\frac{\{\Lambda^{(y)} \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_1]} d_X \dots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i_l]} d_X\}_{ij} h \,\mathrm{d}\, y}{\partial_{y_1} W, \dots \partial_{y_n} W} \right]$$

for $g \in \mathrm{k}[\boldsymbol{x}]$, $h \in \mathrm{k}[\boldsymbol{y}]$, where $\Lambda^{(x)} := (-1)^m \partial_{x_1} d_X \dots \partial_{x_m} d_X$, $\mathrm{d}\, x := \mathrm{d}\, x_1 \dots \mathrm{d}\, x_m$ and $\Lambda^{(y)} := \partial_{y_1} d_X \dots \partial_{y_n} d_X$, $\mathrm{d}\, y := \mathrm{d}\, y_1 \dots \mathrm{d}\, y_n$. The symbol $\operatorname{Res}_{\mathrm{k}[\boldsymbol{x}]/\mathrm{k}} \left[\frac{-\mathrm{d}\, x}{\dots} \right]$ is a residue as in [CM2, §2.4.].

We only refer to [CM2, §2.4] for a general description of residues. We encounter particularly simple cases in Section 5.3.2 below. Furthermore, we note that the assignments in Proposition 2.8.7 define ev_X and $\widetilde{\mathrm{ev}}_X$ on a basis. Indeed, according to our observation preceding Remark 2.8.6, if $\{e_i\}_{i\in I}$ is a basis of X then $\{e_i^*\}_{i\in I}$ is a basis of $X^\dagger$ or of ${}^\dagger X$.

2.9. Morphism categories for $\mathcal{LG}$

The horizontal composition of $\mathcal{LG}$ is essentially given by the functor in Definition 2.6.10. However, according to the previous section for us to be able to define adjoints for 1-morphisms in $\mathcal{LG}$ we have to restrict the latter to a subcategory of HMF. Yet the functor in Definition 2.6.10 does not close on hmf. To wit, for $Y \in \mathrm{HMF}_{k[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})}$, $X \in \mathrm{HMF}_{k[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}$, whenever X_m and Y_m are not the trivial module $\{0\}$ and the set of variables $\boldsymbol{y}$ is not empty $(Y \widetilde{\otimes}_{k[\boldsymbol{y}]} X)_m$ is not a finite-rank $k[\boldsymbol{x},\boldsymbol{z}]$-module. This is in particular true even if both $X \in$ hmf and $Y \in$ hmf. Generically the horizontal composition in $\mathcal{LG}$ is based on $Y \widetilde{\otimes}_{k[\boldsymbol{y}]} X$ for $\boldsymbol{y} \neq \emptyset$, $Y \neq \{0\} \neq X$.

At this point [DM, §12] comes to our rescue. Since [DM] relies on HMF being idempotent complete, cf. [Bor, I.6.5] for a discussion of this property, we present one way to argue that HMF is idempotent complete. This in turn employs the notion of coproducts which is defined e.g. in [McL, §III.3]. First, we show

Lemma 2.9.1. $\mathrm{HMF}_{\mathcal{R},W}$ has countable coproducts.

Proof. Note that it is implicit in Proposition 2.7.3 that finitary coproducts exist in $\mathrm{HMF}_{\mathcal{R},W}$. This is equivalent to saying that binary coproducts exist in $\mathrm{HMF}_{\mathcal{R},W}$. Indeed, for $X \in \mathrm{HMF}_{\mathcal{R},W}$, $Y \in \mathrm{HMF}_{\mathcal{R},W}$ define the matrix factorization $X \oplus Y$ by

$$\begin{aligned} (X \oplus Y)^i_m &:= X^i_m \oplus Y^i_m,\ i \in \{0,1\}, \\ d_{X\oplus Y}(x+y) &:= d_X(x) + d_Y(y)\ \forall (x,y) \in X_m \times Y_m, \end{aligned} \tag{2.9.1}$$

such that $d_{X\oplus Y}$ is a $\mathbb{Z}_2$-odd module endomorphism of the free $\mathbb{Z}_2$-graded $\mathcal{R}$-module $(X \oplus Y)_m$ which squares to $W.\,\mathrm{id}$. Then $X \oplus Y \in \mathrm{HMF}_{\mathcal{R},W}$. Also, it follows from the definition of $d_{X\oplus Y}$ that the inclusion maps from X_m respectively Y_m to their coproduct $X_m \oplus Y_m$ in $\mathrm{Mod}_{\mathcal{R}}$ are compatible with the differentials. Since they are $\mathbb{Z}_2$-even they are morphisms in $\mathrm{HMF}_{\mathcal{R},W}$. Moreover, for $Z \in \mathrm{HMF}_{\mathcal{R},W}$, $\phi \in \mathrm{HMF}_{\mathcal{R},W}(X,Z)$, $\psi \in \mathrm{HMF}_{\mathcal{R},W}(Y,Z)$ we define their sum $\phi + \psi$ by letting ϕ act non-trivially only on the summand corresponding to X_m and similarly for ψ. This yields a morphism in $\mathrm{HMF}_{\mathcal{R},W}(X \oplus Y, Z)$ and it is unique with the property that it makes the defining diagram of a coproduct of X and Y commute.

Representing the coproduct of matrix factorizations as in (2.9.1) also $\bigoplus_{i\in\mathbb{N}} X_i \in \mathrm{HMF}_{\mathcal{R},W}$ for all matrix factorizations $X_i \in \mathrm{HMF}_{\mathcal{R},W}$, $i \in \mathbb{N}$. To wit, $\bigoplus_{i\in\mathbb{N}} (X_i)_m$ is a free $\mathcal{R}$-module with the $\mathbb{Z}_2$-grading $(\bigoplus_{i\in\mathbb{N}} (X_i)_m)^j = \bigoplus_{i\in\mathbb{N}} (X_i)^j_m$, $j \in \{0,1\}$. Similarly, iterating (2.9.1) the differentials d_{X_i}, $i \in \mathbb{N}$, assemble into a differential for this $\mathbb{Z}_2$-graded module. The tuple of this differential and module hence constitutes a matrix factorization in $\mathrm{HMF}_{\mathcal{R},W}$. This can be exhibited to satisfy the condition on a coproduct as in the case of binary coproducts. □

Corollary 2.9.2. $\mathrm{HMF}_{\mathcal{R},W}$ is idempotent complete.

Proof. According to [Nee, Proposition 1.6.8] this follows from Proposition 2.7.3 together with Lemma 2.9.1. □

We move on to quote the result of [DM] as Theorem 2.9.3. To set the latter's stage, let $\boldsymbol{y} \equiv (y_1, \dots, y_n)$ be an ordered set of variables, k a field of characteristic zero and $V \in \mathrm{k}[\boldsymbol{y}]$ a potential. Recall the Jacobi ring $\mathrm{Jac}_V := \mathrm{k}[\boldsymbol{y}]/(\partial_{y_1} V, \dots, \partial_{y_n} V)$ from Definition 2.1.1. Consider the matrix factorization J_V of $(\mathrm{k}[\boldsymbol{y}], 0)$ with underlying $\mathrm{k}[\boldsymbol{y}]$-module $(\mathrm{J}_V)^0_m := \mathrm{Jac}_V$, $(\mathrm{J}_V)^1_m := \{0\}$ and differential 0. Given two further ordered sets of variables $\boldsymbol{x}$, $\boldsymbol{z}$, write

$$\iota_{\boldsymbol{x},\boldsymbol{y}} : \mathrm{k}[\boldsymbol{x},\boldsymbol{y}] \hookrightarrow \mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],\ \iota_{\boldsymbol{y}} : \mathrm{k}[\boldsymbol{y}] \hookrightarrow \mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],$$
$$\iota_{\boldsymbol{y},\boldsymbol{z}} : \mathrm{k}[\boldsymbol{y},\boldsymbol{z}] \hookrightarrow \mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],\ \iota_{\boldsymbol{x},\boldsymbol{z}} : \mathrm{k}[\boldsymbol{x},\boldsymbol{z}] \hookrightarrow \mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$$

for the canonical inclusions of rings. For

$$Y \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})},\ X \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}$$

denote by $Y \otimes_{\mathrm{k}[\boldsymbol{y}]} \mathrm{J}_V \otimes_{\mathrm{k}[\boldsymbol{y}]} X$ the matrix factorization

$$(\iota_{\boldsymbol{x},\boldsymbol{z}})_*(\iota^*_{\boldsymbol{y},\boldsymbol{z}}(Y) \otimes_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} (\iota^*_{\boldsymbol{y}}(\mathrm{J}_V) \otimes_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \iota^*_{\boldsymbol{x},\boldsymbol{y}}(X)))$$

of $(\mathrm{k}[\boldsymbol{x},\boldsymbol{z}], W(\boldsymbol{z}) - U(\boldsymbol{x}))$, where we use the notation of Notation 2.6.9.

Recall that an endomorphism $e \in \mathcal{C}(Z,Z)$ of an object Z in some category $\mathcal{C}$ is an idempotent if $e \circ e = e$. An idempotent $e \in \mathcal{C}(Z,Z)$ is said to split if there exists an object $\widetilde{Z} \in \mathcal{C}$ together with morphisms $f \in \mathcal{C}(Z,\widetilde{Z})$ and $g \in \mathcal{C}(\widetilde{Z},Z)$ such that $g \circ f = e$ and $f \circ g = 1_{\widetilde{Z}}$.

Theorem 2.9.3. Setting $Z := (Y \otimes_{\mathrm{k}[\boldsymbol{y}]} \mathrm{J}_V \otimes_{\mathrm{k}[\boldsymbol{y}]} X)[n]$ there is an idempotent $e \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}(Z,Z)$ which splits to $Y \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X$ in $\mathrm{HMF}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$.

Remark 2.9.4. In [DM] it is not only proven that an idempotent e as in Theorem 2.9.3 exists but also an explicit formula for e is displayed. This enables e.g. [CM1] to compute the link homology defined in [KR1] for various links. In [KR1] *graded matrix factorizations* of pairs $(\mathcal{R}, W)$ are used, where $\mathcal{R}$ is a $\mathbb{Z}$-graded ring and $W \in \mathcal{R}$ a homogeneous element of even degree. These are matrix factorizations as in Definition 2.2.1 whose underlying modules are additionally $\mathbb{Z}$-graded $\mathcal{R}$-modules and whose differential has half the $\mathbb{Z}$-degree of W. Such graded matrix factorizations whose underlying module has finite rank are the objects of a category $\mathrm{hmf}^{\mathrm{gr}}_{\mathcal{R},W}$ analogous to $\mathrm{hmf}_{\mathcal{R},W}$. As opposed to hmf which is not idempotent complete $\mathrm{hmf}^{\mathrm{gr}}$ is, cf. [KST, Lemma 2.11]. In [CM1] a computer implementation of an algorithm splitting idempotents in the latter category is provided. Since the link homology of [KR1] is defined in terms of tensor products of graded matrix factorizations the code of [CM1] can be used to compute this homology.

In order to use Theorem 2.9.3 in our exposition of $\mathcal{LG}$ we introduce, mimicking [M],

Definition 2.9.5. $\mathrm{hmf}^{\oplus}_{\mathcal{R},W}$ is the full subcategory of $\mathrm{HMF}_{\mathcal{R},W}$ whose objects are direct summands in $\mathrm{HMF}_{\mathcal{R},W}$ of finite-rank matrix factorizations.

Remarks 2.9.6. Note that the notion of "direct sum" underlying Definition 2.9.5 is that of a categorical biproduct. The latter is defined only up to isomorphism, cf. [McL, §VIII.2]. Consequently, a direct summand in $\mathrm{HMF}_{\mathcal{R},W}$ of an object $Z \in \mathrm{hmf}_{\mathcal{R},W}$ is defined to be an object $X \in \mathrm{HMF}_{\mathcal{R},W}$ such that there exists an object $Y \in \mathrm{HMF}_{\mathcal{R},W}$ satisfying $Z \cong X \oplus Y$ in $\mathrm{HMF}_{\mathcal{R},W}$ for some way of representing the biproduct $X \oplus Y$ of X and Y. Since all such ways of representing the biproduct are isomorphic this implies $Z \cong X \oplus Y$ in $\mathrm{HMF}_{\mathcal{R},W}$ for all possible ways of representing $X \oplus Y$. Thus this yields a well-defined notion of direct summand.

A remarkable aspect of this definition is that while the direct sum as defined by (2.9.1) of two matrix factorizations one of which has infinite rank is again of infinite rank, it can be isomorphic in $\mathrm{HMF}_{\mathcal{R},W}$ to a finite-rank matrix factorization. This is possible since isomorphisms in $\mathrm{HMF}_{\mathcal{R},W}$ need not be isomorphisms of underlying modules as morphisms in $\mathrm{HMF}_{\mathcal{R},W}$ are homotopy classes of morphisms in $\mathrm{MF}_{\mathcal{R},W}$. Therefore, a matrix factorization in $\mathrm{HMF}_{\mathcal{R},W}$ of infinite rank can be a direct summand in $\mathrm{HMF}_{\mathcal{R},W}$ of a finite-rank matrix factorization.

Also, we point out that $\mathrm{hmf} \neq \mathrm{hmf}^{\oplus}$. For example, [KMvB, Example A.5] is cited in the literature as demonstrating that hmf is not idempotent complete. By definition, $\mathrm{hmf}^{\oplus}$ is idempotent complete and this property is preserved by equivalences of categories. Therefore the two categories cannot be equivalent.

We repeatedly use restriction of scalars along ring isomorphisms for morphism categories of $\mathcal{LG}$ below. This is enabled by

Lemma 2.9.7. Let $\mathcal{R}$ and $\mathcal{S}$ be commutative rings, $W \in \mathcal{S}$ and $\varphi : \mathcal{R} \to \mathcal{S}$ a ring isomorphism. Then $\varphi_* : \mathrm{hmf}^{\oplus}_{\mathcal{S},W} \to \mathrm{hmf}^{\oplus}_{\mathcal{R},\varphi^{-1}(W)}$ is an equivalence.

Proof. Applying restriction of scalars to the inclusion and projection morphisms for direct summands of finite-rank matrix factorizations yields inclusion and projection morphisms for the matrix factorizations obtained via restriction of scalars. Indeed, according to Corollary 2.6.6 restriction of scalars is a functor on HMF whereby it preserves the compositions of the inclusions and projections which are morphisms in HMF. Moreover, restriction of scalars sends the 0-morphism which multiplies with zero to the 0-morphism. Next, recall that Lemma 2.8.5 tells us that the matrix factorizations obtained from finite-rank matrix factorizations by restriction of scalars along ring isomorphisms are again finite-rank matrix factorizations. Thus, applying restriction of scalars along a ring isomorphism to a direct summand of a matrix factorization of finite rank yields a direct summand of a finite-rank matrix factorization. This proves the claim on objects which suffices as $\mathrm{hmf}^{\oplus}$ is a full subcategory of HMF. □

Since Theorem 2.9.3 tells us that

$$Z \cong \mathrm{Im}(e) \oplus \mathrm{Im}(1-e) \cong (Y \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X) \oplus \mathrm{Im}(1-e)$$

we have that $Y \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X \in \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$. From this it follows that $\tilde{\otimes}$ closes on $\mathrm{hmf}^{\oplus}$.

Corollary 2.9.8. Let $V \in \mathrm{k}[\boldsymbol{y}]$ be a potential. The functor described in Definition 2.6.10 descends to a functor

$$\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})} \times \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})} \to \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}.$$

Proof. Let $Z \in \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}$, $\tilde{Z} \in \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})}$. Then there are

$$X \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}, \tilde{X} \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})},$$
$$Y \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],V(\boldsymbol{y})-U(\boldsymbol{x})}, \tilde{Y} \in \mathrm{HMF}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W(\boldsymbol{z})-V(\boldsymbol{y})}$$

such that $X \cong Z \oplus Y$ and $\tilde{X} \cong \tilde{Z} \oplus \tilde{Y}$. This means that there are projection and inclusion maps between X and Z respectively between X and Y and similarly for their tilded versions. Taking these morphisms' tensor products we get maps showing that

$$\tilde{X} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X \cong (\tilde{Z} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Z) \oplus (\tilde{Z} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Y) \oplus (\tilde{Y} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Z) \oplus (\tilde{Y} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Y).$$

Since Theorem 2.9.3 tells us that $\tilde{X} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} X$ is a direct summand of a finite-rank matrix factorization $\widehat{X} \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$ this implies that also $\tilde{Z} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Z$ is a direct summand of $\widehat{X}$, i.e. $\tilde{Z} \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} Z \in \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{z}],W(\boldsymbol{z})-U(\boldsymbol{x})}$. □

Notation 2.9.9. We write $\otimes_{\mathrm{k}[\boldsymbol{y}]}$ for the functor in Corollary 2.9.8.

We use $Y \otimes_{\mathrm{k}[\boldsymbol{y}]} X := \otimes_{\mathrm{k}[\boldsymbol{y}]}(Y, X)$ and $y \otimes_{\mathrm{k}[\boldsymbol{y}]} x$ for the element of $(Y \otimes_{\mathrm{k}[\boldsymbol{y}]} X)_m$ corresponding to $x \in X_m$ and $y \in Y_m$. Sometimes we omit the index on $\otimes$.

Our motivation of the definition of hmf in Definition 2.8.4 above is that its objects are precisely those matrix factorizations to which Definition 2.8.3 applies. This defines matrix factorizations that [CM2] endows with the structure of adjoint 1-morphisms in $\mathcal{LG}$. Yet horizontal composition of matrix factorizations closes on $\mathrm{hmf}^{\oplus}$ but not on hmf. Still, Theorem 2.10.6 below tells us that adjunction morphisms compatible with those for matrix factorizations in hmf can be defined for all objects of $\mathrm{hmf}^{\oplus}$.

We move on to define $\mathcal{LG}$ in the next section. Prior to this we include a table with the categories of matrix factorizations we encounter above in order to help the reader get an overview.

Category	Objects	Morphisms $X \to Y$
$\mathrm{MF}_{\mathcal{R},W}$	all matrix factorizations of $(\mathcal{R}, W)$	$\mathrm{Mod}_{\mathcal{R}}(X_m, Y_m)$
$\mathrm{HMF}_{\mathcal{R},W}$	all matrix factorizations of $(\mathcal{R}, W)$	$\mathrm{H}^0_{\delta_{X,Y}}(\mathrm{MF}_{\mathcal{R},W}(X,Y))$
$\mathrm{hmf}_{\mathcal{R},W}$	finite-rank matrix factorizations of $(\mathcal{R}, W)$	$\mathrm{H}^0_{\delta_{X,Y}}(\mathrm{MF}_{\mathcal{R},W}(X,Y))$
$\mathrm{hmf}^{\oplus}_{\mathcal{R},W}$	direct summands of finite-rank matrix factorizations of $(\mathcal{R}, W)$ in $\mathrm{HMF}_{\mathcal{R},W}$	$\mathrm{H}^0_{\delta_{X,Y}}(\mathrm{MF}_{\mathcal{R},W}(X,Y))$

Table 2.1.: Some categories of matrix factorizations

2.10. The bicategory $\mathcal{LG}$

In this section we define the bicategory $\mathcal{LG}$ combining the structures that we present above.

There is one last prerequisite for $\mathcal{LG}$ which we introduce separately before turning to the definition of the bicategory. Indeed, there is one structure of a bicategory which we do not mention so far. Every bicategory is equipped with natural isomorphisms witnessing that its horizontal composition is associative. We now deal with these in the case of $\mathcal{LG}$.

The setup of the next Lemma 2.10.1 is the following. For every triple (A, B, C) of categories we denote by $\mathfrak{a}$ the functor $(A \times B) \times C \to A \times (B \times C)$ which acts on objects as $((a, b)c) \mapsto (a, (b, c))$ and acts analogously on morphisms. Moreover, we refine the notation of Notation 2.9.9 as follows. Given four ordered sets of variables $\boldsymbol{x}_i$, $i \in \{1, 2, 3, 4\}$ together with potentials $W_i \in \mathrm{k}[\boldsymbol{x}_i]$ $\forall i$, we write $\otimes_{\mathrm{k}[\boldsymbol{x}_r],\mathrm{k}[\boldsymbol{x}_s],\mathrm{k}[\boldsymbol{x}_t]}$, $(r, s, t) \in \{1, 2, 3, 4\}^3$ for the functor

$$\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x}_s,\boldsymbol{x}_t],W_t(\boldsymbol{x}_t)-W_s(\boldsymbol{x}_s)} \times \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x}_r,\boldsymbol{x}_s],W_s(\boldsymbol{x}_s)-W_r(\boldsymbol{x}_r)} \to \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x}_r,\boldsymbol{x}_t],W_t(\boldsymbol{x}_t)-W_r(\boldsymbol{x}_r)}$$

described in Corollary 2.9.8.

Lemma 2.10.1. There is a natural isomorphism

$$\otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_4]} \circ (\otimes_{\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_3],\mathrm{k}[\boldsymbol{x}_4]} \times 1) \to \otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_3],\mathrm{k}[\boldsymbol{x}_4]} \circ (1 \times \otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_3]}) \circ \mathfrak{a}$$

with $((Z, Y), X)$-component

$$\begin{aligned}(Z \otimes_{\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_3],\mathrm{k}[\boldsymbol{x}_4]} Y) \otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_4]} X &\to Z \otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_3],\mathrm{k}[\boldsymbol{x}_4]} (Y \otimes_{\mathrm{k}[\boldsymbol{x}_1],\mathrm{k}[\boldsymbol{x}_2],\mathrm{k}[\boldsymbol{x}_3]} X),\\ (\tilde{g} \otimes_{\mathrm{k}[\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4]} \tilde{f}) \otimes_{\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_4]} \tilde{e} &\mapsto \widehat{g} \otimes_{\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{x}_3,\boldsymbol{x}_4]} (\widehat{f} \otimes_{\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3]} \widehat{e})\end{aligned} \tag{2.10.1}$$

where e.g. $g \in Z_m$ and $\tilde{g} := 1 \otimes_{\mathrm{k}[\boldsymbol{x}_3,\boldsymbol{x}_4]} g$ is an element of the module obtained from Z_m by extension of scalars along $\mathrm{k}[\boldsymbol{x}_3, \boldsymbol{x}_4] \hookrightarrow \mathrm{k}[\boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4]$ while $\widehat{g} := 1 \otimes_{\mathrm{k}[\boldsymbol{x}_3,\boldsymbol{x}_4]} g$ is an element of the module obtained from Z_m by extension of scalars along $\mathrm{k}[\boldsymbol{x}_3, \boldsymbol{x}_4] \hookrightarrow \mathrm{k}[\boldsymbol{x}_1, \boldsymbol{x}_3, \boldsymbol{x}_4]$. Similarly, for some $f \in Y_m$, $e \in X_m$ we have the elements $\tilde{f}$, $\tilde{e}$, $\widehat{f}$ and $\widehat{e}$ of the modules obtained by extension of scalars according to the upper line in (2.10.1).

Proof. The prescription (2.10.1) is a map in the homotopy category. Indeed, the differentials of the matrix factorizations on both sides of (2.10.1) act non-trivially only as d_X on x, as d_Y on y and as d_Z on z. Since (2.10.1) sends x to x, y to y and z to z, it follows that it commutes with the differentials. Moreover, it is an invertible map of modules. Also, its inverse is compatible with the differentials for the same reason as (2.10.1). Therefore, (2.10.1) is invertible as a morphism in $\mathrm{hmf}^{\oplus}$. Finally, it is natural in all three arguments. □

Notation 2.10.2. We write $\alpha \equiv \alpha_{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4}$ for the natural isomorphism in Lemma 2.10.1.

The next proposition is essentially [CM2, Proposition 2.7].

Proposition 2.10.3. For k a field of characteristic zero the following defines a bicategory.

1. Objects are pairs $(\boldsymbol{x}, W)$ of an ordered set of variables $\boldsymbol{x} \equiv \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ and a potential $W \in \mathrm{k}[\boldsymbol{x}]$.
2. The category of morphisms from $(\boldsymbol{x}, V)$ to $(\boldsymbol{y}, W)$ is $\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$.
3. The horizontal composition is $\otimes_{\mathrm{k}[\boldsymbol{y}]}$ from Corollary 2.9.8.
4. The associators are α from Lemma 2.10.1.
5. Unit 1-morphisms are $I_{(\boldsymbol{x},W)} := I_{\mathrm{k}[\boldsymbol{x}],W}$, cf. Lemma 2.5.3 .
6. Unitors are the maps λ, ρ from Lemma 2.5.6.

Proof. Let $\boldsymbol{x} \equiv (x_1, \dots, x_n)$. Then $I_{(\boldsymbol{x},W)}$ is a matrix factorization of finite rank 2^n, cf. (2.5.5), and thus in particular $I_{(\boldsymbol{x},W)} \in \mathrm{hmf}_{\mathrm{k}[\boldsymbol{x}',\boldsymbol{x}],W(\boldsymbol{x})-W(\boldsymbol{x}')}$ is a 1-morphism in $\mathcal{LG}$.

Lemma 2.6.11 shows that the unitors λ and ρ are compatible with the differentials. Moreover they are $\mathbb{Z}_2$-even maps. They are non-zero only on tensor products of the 1-component of I with an arbitrary element of the other matrix factorization. The $\mathbb{Z}_2$-degree of such a tensor product is the degree of the unspecified element. The result of applying λ respectively ρ to it is a scalar multiple of this element and therefore has the same $\mathbb{Z}_2$-degree. Thus, λ and ρ preserve the $\mathbb{Z}_2$-degree when they are non-zero. Since 0 can be viewed as an element in either $\mathbb{Z}_2$-degree they also preserve the $\mathbb{Z}_2$-degree in the other case. Therefore, both λ and ρ are 2-morphisms in $\mathcal{LG}$.

Since the unitors λ_X, ρ_X only alter the factor of X in their source by multiplying it with a scalar they are natural with respect to maps $\phi : X \to Y$ which in particular are linear maps. That they are invertible in the homotopy category is [CM2, Lemma 4.8].

We turn to the axioms on associators and unitors of a bicategory. The associators are defined as module maps which only rebracket tensor products of elements and compatibly change the rings over which these tensor products are taken. They satisfy the pentagon axiom (A.1.1) since both concatenations of associators compared in this diagram yield the same bracketing of the same tensor products. Furthermore, that the triangle axiom (A.1.2) holds for the maps α, λ and ρ can be seen as follows. Either of the two 2-morphisms featuring in this diagram projects the unit matrix factorization in the horizontal composite of 1-morphisms constituting its source to its 1-component. Moreover, scalars of this matrix factorization become scalars in the ring over which the resulting tensor product is taken. Thereby it gives the same result if these scalars are multiplied with one or the other factor of the final tensor product. □

We use the following, which is not completely in line with the literature, cf. Remark 2.10.5.

Definition 2.10.4. The bicategory in Proposition 2.10.3 is the *bicategory* $\mathcal{LG} \equiv \mathcal{LG}_{\mathrm{k}}$ *of Landau-Ginzburg models.*

Remark 2.10.5. There are two reasons why we write "essentially" in the line preceding Proposition 2.10.3. On the one hand, [CM2] uses a more general notion of "potential", where k is allowed to be some commutative ring, cf. [CM2, Definition 2.4]. The setup we use is the one of [DM, Theorem 12.4]. We do not see any obstruction to lift the original work presented in this thesis to the framework of [CM2]. However, we do not know of any publication to refer to showing explicitly that the result of [DM] can be lifted to the more general level of [CM2] and prefer not to provide the details here.

On the other hand, instead of taking $\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})}$ as the category of morphisms from $(\boldsymbol{x}, V)$ to $(\boldsymbol{y}, W)$ [CM2] uses $\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{y}]\otimes\mathrm{k}[\boldsymbol{x}],W\otimes 1-1\otimes V}$, where tensor products are over k.[2] The horizontal composition for these categories is the one induced from the functors $\otimes_{\mathrm{k}[\boldsymbol{y}]}$ in Corollary 2.9.8 via the equivalence $\mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y}],W(\boldsymbol{y})-V(\boldsymbol{x})} \cong \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{y}]\otimes\mathrm{k}[\boldsymbol{x}],W\otimes 1-1\otimes V}$ which we get by applying Lemma 2.9.7 to the canonical isomorphism of rings $\mathrm{k}[\boldsymbol{x},\boldsymbol{y}] \cong \mathrm{k}[\boldsymbol{y}] \otimes \mathrm{k}[\boldsymbol{x}]$. The unit matrix factorization in [CM2] is the one obtained from $I_{\mathrm{k}[\boldsymbol{x}],W}$ as in Lemma 2.5.3 by this same restriction of scalars. Altogether, horizontal composition, unit 1-morphisms, unitors and associators in the bicategory of [CM2] are induced from those in Proposition 2.10.3 by the same isomorphisms.[3] Therefore, there is a strict 2-functor from $\mathcal{LG}$ to the bicategory of [CM2] which is the identity on objects (and thereby in particular biessentially surjective) and is locally given by isomorphisms from

[2]By minor abuse of terminology [CM2] refers to $\mathrm{hmf}^{\oplus}$ as idempotent closure hmf^{ω} of hmf in HMF. In fact, $\mathrm{hmf}^{\oplus}$ and hmf^{ω} are equivalent categories, cf. [M, (2.3)].

[3]Note, that natural isomorphisms induced by equivalences of categories (not necessarily isomorphisms) are again natural isomorphisms. Indeed, they are given by composing (combinations of identity natural transformations and) the original natural isomorphism with (combinations of identity natural transformations and) the natural isomorphisms witnessing the equivalence of categories.

restriction of scalars (in particular it is locally an equivalence). In other words, the bicategory of [CM2] is biequivalent to $\mathcal{LG}$. We work with $\mathcal{LG}$ since it is the bicategory we can more easily endow with a symmetric monoidal structure.

Other than the minimal difference of replacing $\mathrm{k}[\boldsymbol{x}]^{\mathrm{e}}$ in (2.5.3) by $\mathrm{k}[\boldsymbol{x}', \boldsymbol{x}]$ the bicategory $\mathcal{LG}$ is the one described in [M, §2.2]. That (2.5.3) is based on $\mathrm{k}[\boldsymbol{x}]^{\mathrm{e}}$ manifests that we have defined the unit matrix factorization as the one obtained from that in the bicategory of [CM2] via restriction of scalars along the isomorphism of rings $\mathrm{k}[\boldsymbol{x}]^{\mathrm{e}} \cong \mathrm{k}[\boldsymbol{x}', \boldsymbol{x}]$ in (2.5.2).

Continuing the considerations of Remark 2.10.5 shows that [CM2, Theorem 6.11] "essentially" is

Theorem 2.10.6. The bicategory $\mathcal{LG}$ has adjoints which coincide with the matrix factorizations of Definition 2.8.3 for matrix factorizations of finite rank. In the latter case the (co-)evaluation maps are those presented in Proposition 2.8.7.

Remark 2.10.7. Using these adjoints [CM2, Proposition 7.2] shows that $\mathcal{LG}$ is *graded pivotal*. A bicategory with adjoints in the sense of Definition A.2.2 in which for every 1-morphism its left and right adjoints coincide is called *pivotal* if the chosen adjunctions satisfy additional equations which are depicted e.g. in [CR3, ((2.12)]. $\mathcal{LG}$ cannot be pivotal since there are 1-morphisms X in $\mathcal{LG}$ for which $X^\dagger = \phi_*(X^\vee[m]) \ncong \phi_*(X^\vee[n]) = {}^\dagger X$. An example of such a 1-morphism is $X \in \mathcal{LG}((\emptyset, 0), (x, x^d))$, $d \in \mathbb{N}$, given by $X_m := \mathrm{k}[x] \oplus \mathrm{k}[x]\theta$, where $\{1, \theta\}$ is a chosen basis of $\mathrm{k}[x]^2$ as in $I_{\mathcal{R},W}$, cf. Lemma 2.5.3, and $d_X := x.\theta^* + x^{d-1}.\theta \wedge (-)$. In this case $X^\dagger \ncong {}^\dagger X$ can be verified by explicit computations.

As presented in [CM2, §7] it is however possible to keep track of the shifts obstructing $\mathcal{LG}$ from being pivotal in a way that allows to calculate in $\mathcal{LG}$ as if it was pivotal. $\mathcal{LG}$ being "pivotal up to shifts" makes it an instance of a graded pivotal bicategory.

Since $X[2] = X$ for all 1-morphisms X the above obstruction is not present in the subbicategory of $\mathcal{LG}$ whose objects depend on an even number of variables. Indeed, as mentioned in [CM2, Remark 7.3] this subbicategory of $\mathcal{LG}$ is pivotal. This is not true for the subbicategory whose objects depend on an odd number of variables as the canonical isomorphisms $Y[1] \otimes X \cong (Y \otimes X)[1] \cong Y \otimes X[1]$, cf. [CM2, (2.40),(2.41)] imply $Y^\dagger \otimes X^\dagger \cong Y^\vee \otimes X^\vee \ncong (Y^\vee \otimes X^\vee)[1] \cong (X \otimes Y)^\vee[1] = (X \otimes Y)^\dagger$ for 1-morphisms X and Y in this subbicategory whence it cannot be pivotal.

Remark 2.10.8. In [CR1] so-called *matrix bi-factorizations* with underlying bimodules rather than modules are defined. This has the conceptual advantage that bimodules can naturally be regarded as 1-morphisms in the bicategory Bimod whose objects are rings and where 2-morphisms are intertwiners. Therefore, matrix bi-factorizations can serve to motivate the horizontal composition of $\mathcal{LG}$.

In some more detail, for k a field, $\mathcal{R}$, $\mathcal{S}$ commutative k-algebras, $W \in \mathcal{S}$ and $V \in \mathcal{R}$ [CR1] defines a matrix bi-factorization of $(\mathcal{R}, V)$ and $(\mathcal{S}, W)$ as a free

$\mathbb{Z}_2$-graded $\mathcal{S}$-$\mathcal{R}$-bimodule X_b together with a $\mathbb{Z}_2$-odd bimodule-endomorphism d_X such that $d_X^2(x) = W.x - x.V \ \forall x \in X_b$. Here an $\mathcal{S}$-$\mathcal{R}$-bimodule is called free if the corresponding $\mathcal{S} \otimes_{\mathrm{k}} \mathcal{R}^{\mathrm{op}}$-module is free. Indeed, the relation between $\mathcal{S} \otimes_{\mathrm{k}} \mathcal{R}^{\mathrm{op}}$-modules and $\mathcal{S}$-$\mathcal{R}$-bimodules extends to a functor from a category of matrix bi-factorizations of $(\mathcal{R}, V)$ and $(\mathcal{S}, W)$ where morphisms are homomorphisms of underlying bimodules to $\mathrm{MF}_{\mathcal{S}\otimes_{\mathrm{k}}\mathcal{R}, W\otimes_{\mathrm{k}}1-1\otimes_{\mathrm{k}}V}$. This is an equivalence. In fact, one can proceed along the same lines as the above to construct a category of matrix bi-factorizations corresponding to each entry in Table 2.1. In each case the aforementioned equivalence carries over to the respective categories of matrix factorizations and matrix bi-factorizations.

One can compose a matrix bi-factorization (X_b, d_X) of $(\mathrm{k}[\boldsymbol{y}], V)$ and $(\mathrm{k}[\boldsymbol{x}], U)$ with a matrix bi-factorization (Y_b, d_Y) of $(\mathrm{k}[\boldsymbol{z}], W)$ and $(\mathrm{k}[\boldsymbol{y}], V)$ to get $(Y_b \otimes_{\mathrm{k}[\boldsymbol{y}]} X_b, d_Y \otimes_{\mathrm{k}[\boldsymbol{y}]} 1 + 1 \otimes_{\mathrm{k}[\boldsymbol{y}]} d_X)$, where the tensor products are those in Bimod. In particular there is no extension or restriction of scalars involved. Moreover, this induces the horizontal composition of $\mathcal{LG}$ via last paragraph's equivalences of categories. Similarly, the associators of Lemma 2.10.1 correspond to the associators of the tensor product of bimodules. Hence, taking as unit 1-morphisms and unitors those resulting from these equivalences applied to the respective morphisms in $\mathcal{LG}$, it follows as in Remark 2.10.5 that there is a bicategory equivalent to $\mathcal{LG}$ whose morphism categories consist of matrix bi-factorizations and their morphisms. This is the bicategory described in [RC]. In [CR2] adjoints for those 1-morphisms of this bicategory corresponding to finite-rank matrix factorizations are described via duals for free bimodules.

3. A symmetric monoidal structure on $\mathcal{LG}$

In this chapter we endow $\mathcal{LG}$ with further structure which turns it into a symmetric monoidal bicategory. The definition of the latter is included in Definition A.4.1.

Much of the symmetric monoidal structure on $\mathcal{LG}$ presented below is defined in terms of unit matrix factorizations and the unitors. The major technicalities involved in proving that our definitions satisfy the necessary conditions are condensed into the coherence results presented in Section 3.1.2.

For earlier work on a symmetric monoidal bicategory of Landau-Ginzburg models using a different setting and a less explicit approach we refer to [McN]. Other references anticipating our result are e.g. [C2, §2.4.4], [CM2, §9] and [CR2, §4]. It is also implicit in [KR1].[1]

3.1. The monoidal product

In this section we equip $\mathcal{LG}$ with a 2-functor $\mathcal{LG} \times \mathcal{LG} \to \mathcal{LG}$.

Remark 3.1.1. By Remark 2.10.8 the horizontal composition in $\mathcal{LG}$ can be viewed as being inspired from the horizontal composition of Bimod. Similarly we can let ourselves be guided by the following bicategory in defining a monoidal product for $\mathcal{LG}$. For k a commutative ring there is a bicategory whose objects are k-algebras, 1-morphisms are bimodules and 2-morphisms are intertwiners. This is denoted $\mathrm{Alg}(\mathrm{Mod}_k)$ in [GPS, 8.9]. By the latter reference's results this is a monoidal bicategory with $\otimes_k$ as monoidal product (cf. also [SP1, §3.8.4]). It is the aim of this section to mimic this result for $\mathcal{LG}$.

3.1.1. Two ingredients of the monoidal product

In this section we introduce two items which as we prove in Section 3.1.3 below are part of a 2-functor $\mathcal{LG} \times \mathcal{LG} \to \mathcal{LG}$.

[1] The figures [KR1, Figure 12, Figure 13] are string diagrams in the homotopy category h$\mathcal{LG}$ of $\mathcal{LG}$. The category h$\mathcal{LG}$ inherits a monoidal structure from the monoidal structure on the bicategory $\mathcal{LG}$ which we define. This monoidal structure of h$\mathcal{LG}$ is the one depicted in the figures.

The following is intended to make our exposition more readable. First, from now on, k is a field of characteristic zero and for indices $i \in \mathbb{N}$ we link potentials and variables according to the following table, such that e.g. $W \in \mathrm{k}[\boldsymbol{z}]$,

ordered set of variables	$\boldsymbol{z}$	$\boldsymbol{z}_i$	$\boldsymbol{y}$	$\boldsymbol{y}_i$	$\boldsymbol{x}$	$\boldsymbol{x}_i$	$\boldsymbol{w}$	$\boldsymbol{w}_i$	$\emptyset$
number of variables	n	n_i	m	m_i	l	l_i	k	k_i	0
potential	W	W_i	V	V_i	U	U_i	T	T_i	0

Table 3.1.: Conventions for variables and potentials

Sticking to the conventions of Table 3.1 we generally abbreviate e.g. $(\boldsymbol{z}, W)$ as W. Moreover, we write tuples of ordered sets of variables to refer to the ordered set whose first variables are those of the first entry in their order and so on, e.g. $(\boldsymbol{y}, \boldsymbol{z})$ is the ordered set whose first m entries are those of $\boldsymbol{y}$ in their order and whose next n variables are those of $\boldsymbol{z}$ in their order. For potentials which are sums of potentials[2] the order of the summands mirrors the order of the variables, e.g.

$$V + W \equiv ((\boldsymbol{y}, \boldsymbol{z}), V + W). \tag{3.1.1}$$

For sums of potentials in which the same potential features more than once as a summand we use a notation in line with that introduced in the paragraph preceding Lemma 2.5.3. That is, we distinguish the variables corresponding to the summands by adorning them with primes. For example $W + W$ is to be read as either $W(\boldsymbol{z}) + W(\boldsymbol{z}') \equiv ((\boldsymbol{z}, \boldsymbol{z}'), W(\boldsymbol{z}) + W(\boldsymbol{z}'))$ or $W(\boldsymbol{z}') + W(\boldsymbol{z}) \equiv ((\boldsymbol{z}', \boldsymbol{z}), W(\boldsymbol{z}') + W(\boldsymbol{z}))$. In reading the individual potentials in a sum in which some potentials feature more than once as objects in $\mathcal{LG}$, we remove the primes again. When we refrain from writing the polynomial ring in the index of hmf, $\mathrm{hmf}^{\oplus}$ or HMF we mean the polynomial ring which only contains the variables associated to the potential. In case the latter is a sum the variables are supposed to be ordered according to the order of the summands of the potential, e.g. $\mathrm{hmf}_{-V+W} \equiv \mathrm{hmf}_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}],W-V}$. Moreover, we use

Notation 3.1.2. Suppressing variables according to Table 3.1 and (3.1.1) and assuming $\boldsymbol{z}_i \cap \boldsymbol{z}_j = \emptyset$ for $i \neq j$, let $W = \sum_{i=1}^{N} W_i$ be an object of $\mathcal{LG}$. Let $\sigma \in S_N$ be a permutation and $\boldsymbol{z}_\sigma$ the list of variables obtained by reordering the blocks $\boldsymbol{z}_i$, $i \in \mathbb{N}_{\leq N}$, of entries of $\boldsymbol{z}$ according to σ. Denote by $\varphi : \mathrm{k}[\boldsymbol{z}_\sigma] \to \mathrm{k}[\boldsymbol{z}]$ the isomorphism of rings acting as the identity on elements. We write

$$\iota_{W,\varphi^{-1}(W)} : \mathrm{hmf}^{\oplus}_{W} \to \mathrm{hmf}^{\oplus}_{\varphi^{-1}(W)}$$

for the functor of restriction of scalars along φ described in Lemma 2.9.7. Here we assume that the order of the summands W_i of $\varphi^{-1}(W)$ reflects that of the

[2] That the sum of two potentials is a potential is proven as part 1 of Proposition 3.1.12 below.

sets $\boldsymbol{z}_i$ in $\boldsymbol{z}_\sigma$. Moreover, we suppress the rings in the indices of hmf as in the line above Notation 3.1.2. An example is $\iota_{U+V,V+U} : \mathrm{hmf}^\oplus_{U+V} \to \mathrm{hmf}^\oplus_{V+U}$.

The first ingredient of the 2-functor of Section 3.1.3 comes in Definition 3.1.3. Preparing the latter we note

$$\begin{aligned}(\mathcal{LG}\times\mathcal{LG})\big((V_1,V_2),(W_1,W_2)\big) &= \mathcal{LG}(V_1,W_1)\times\mathcal{LG}(V_2,W_2)\\ &= \mathrm{hmf}^\oplus_{-V_1+W_1} \;\times\; \mathrm{hmf}^\oplus_{-V_2+W_2}\\ &= \mathrm{hmf}^\oplus_{-0+(-V_1+W_1)}\times\mathrm{hmf}^\oplus_{-(V_2-W_2)+0},\\ \mathcal{LG}(V_1+V_2,W_1+W_2) &= \mathrm{hmf}^\oplus_{-V_1-V_2+W_1+W_2},\end{aligned}$$

and that $\otimes_\mathrm{k}$, cf. Notation 2.9.9, is a functor

$$\mathrm{hmf}^\oplus_{-0+(-V_1+W_1)}\times\mathrm{hmf}^\oplus_{-(V_2-W_2)+0}\to\mathrm{hmf}^\oplus_{-(V_2-W_2)+(-V_1+W_1)}.$$

Definition 3.1.3. The functor $\square_{(V_1,V_2),(W_1,W_2)}$ is the composite

$$\begin{aligned}(\mathcal{LG}\times\mathcal{LG})\big((V_1,V_2),(W_1,W_2)\big) &\xrightarrow{\otimes_\mathrm{k}} \mathrm{hmf}^\oplus_{-(V_2-W_2)+(-V_1+W_1)}\\ &\xrightarrow{F} \mathcal{LG}(V_1+V_2,W_1+W_2),\end{aligned}$$

where $F := \iota_{-V_2+W_2-V_1+W_1,-V_1-V_2+W_1+W_2}$.

Often we write $\square$ without index for the functor of Definition 3.1.3. Also, given two 1-morphisms X, Y in $\mathcal{LG}$ we use $Y\square X := \square(Y,X)$.

The second gadget which we display in this section is

Lemma 3.1.4. Let $r_s\in\{0,1\}$ $\forall s\in\mathbb{N}_{\le n_1+n_2}$. Extending

$$\theta_1^{r_1}\dots\theta_{n_1+n_2}^{r_{n_1+n_2}}\mapsto\theta_1^{r_1}\dots\theta_{n_1}^{r_{n_1}}\otimes_\mathrm{k}\;\theta_1^{r_{n_1+1}}\dots\theta_{n_2}^{r_{n_1+n_2}} \tag{3.1.2}$$

$\mathrm{k}[\boldsymbol{z}_1',\boldsymbol{z}_2',\boldsymbol{z}_1,\boldsymbol{z}_2]$-linearly yields an isomorphism $I_{W_1+W_2}\to I_{W_1}\square I_{W_2}$.

We postpone the proof in favor of a preparatory lemma.

Lemma 3.1.5. Let $\mathcal{R}$ be a commutative ring, $W\in\mathcal{R}$, X and Y matrix factorizations of $(\mathcal{R},W)$ and $\phi\in\mathrm{Mod}_\mathcal{R}(X_m,Y_m)$ an isomorphism such that ϕ represents a morphism $\widetilde{\phi}\in\mathrm{HMF}_{\mathcal{R},W}(X,Y)$. Then $\phi^{-1}\in\mathrm{Mod}_\mathcal{R}(Y_m,X_m)$ represents a morphism $\widetilde{\phi^{-1}}\in\mathrm{HMF}_{\mathcal{R},W}(Y,X)$.

Proof. Since ϕ has $\mathbb{Z}_2$-degree zero so does ϕ^{-1}. We need to show that ϕ^{-1} is $\delta_{Y,X}$-closed. For this, act on $\phi\circ d_X = d_Y\circ\phi$ with ϕ^{-1} from both sides. $\square$

Note that $\mathrm{hmf}_{\mathcal{R},W}$ and $\mathrm{hmf}^\oplus_{\mathcal{R},W}$ are full subcategories of $\mathrm{HMF}_{\mathcal{R},W}$ such that the preceding lemma applies also to these categories.

Proof of Lemma 3.1.4. We show that Lemma 3.1.4 defines a morphism in $\mathcal{LG}$. Once this is established it follows from Lemma 3.1.5 that it is an isomorphism: (3.1.2) is one-to-one on basis elements and thus defines an isomorphism of underlying modules. Recall our conventions from Table 3.1 and suppose that $U = V + W$ such that $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z})$. First, we rewrite the differential of $I_U = I_{V+W}$, cf. (2.5.4), as

$$
\begin{aligned}
d_{I_U} =& \sum_{r=1}^{l} \big((x_r - x_r').\theta_r^* + \partial_{[r]}^{\boldsymbol{x},\boldsymbol{x}'}(U) \ . \ \theta_r \wedge (-) \big) \\
=& \sum_{r=1}^{m} \big((y_r - y_r').\theta_r^* + \partial_{[r]}^{\boldsymbol{y},\boldsymbol{y}'}(V) \ . \ \theta_r \wedge (-) \big) \\
&+ \sum_{r=1}^{n} \big((z_r - z_r').\theta_{m+r}^* + \partial_{[r]}^{\boldsymbol{z},\boldsymbol{z}'}(W) \ . \ \theta_{m+r} \wedge (-) \big).
\end{aligned}
$$

Acting with this on a basis element $\theta_1^{s_1} \dots \theta_l^{s_l}$, cf. (2.5.5), yields using Notation 2.5.2

$$
\begin{aligned}
&\sum_{r=1}^{m} \Big((y_r - y_r').(-1)^{\sum_{t=1}^{r-1} s_t} \delta_{s_r 1} \ \theta_1^{s_1} \dots \theta_{r-1}^{s_{r-1}} \ \widehat{\theta_r} \ \theta_{r+1}^{s_{r+1}} \dots \theta_l^{s_l} \\
&+ \partial_{[r]}^{\boldsymbol{y},\boldsymbol{y}'}(V) \ . \ (-1)^{\sum_{t=1}^{r-1} s_t} \delta_{s_r 0} \ \theta_1^{s_1} \dots \theta_{r-1}^{s_{r-1}} \ \theta_r \ \theta_{r+1}^{s_{r+1}} \dots \theta_l^{s_l} \Big) \\
&+ \sum_{r=1}^{n} \Big((z_r - z_r').(-1)^{\sum_{t=1}^{m+r-1} s_t} \delta_{s_{m+r} 1} \ \theta_1^{s_1} \dots \theta_{m+r-1}^{s_{m+r-1}} \ \widehat{\theta_{m+r}} \ \theta_{m+r+1}^{s_{m+r+1}} \dots \theta_l^{j_l} \\
&+ \partial_{[r]}^{\boldsymbol{z},\boldsymbol{z}'}(W) \ . \ (-1)^{\sum_{t=1}^{m+r-1} s_t} \delta_{s_{m+r} 0} \ \theta_1^{s_1} \dots \theta_{m+r-1}^{s_{m+r-1}} \ \theta_{m+r} \ \theta_{m+r+1}^{s_{m+r+1}} \dots \theta_l^{s_l} \Big).
\end{aligned}
$$

Suppressing restriction of scalars this gets mapped by (3.1.2) to

$$
\begin{aligned}
&\sum_{r=1}^{m} \Big((y_r - y_r').(-1)^{\sum_{t=1}^{r-1} s_t} \delta_{s_r 1} \ \theta_1^{s_1} \dots \theta_{r-1}^{s_{r-1}} \ \widehat{\theta_r} \ \theta_{r+1}^{s_{r+1}} \dots \theta_m^{s_m} \otimes_{\mathrm{k}} \theta_1^{s_{m+1}} \dots \theta_n^{s_l} \\
&+ \partial_{[r]}^{\boldsymbol{y},\boldsymbol{y}'}(V) \ . \ (-1)^{\sum_{t=1}^{r-1} s_t} \delta_{s_r 0} \ \theta_1^{s_1} \dots \theta_{r-1}^{s_{r-1}} \ \theta_r \ \theta_{r+1}^{s_{r+1}} \dots \theta_m^{s_m} \otimes_{\mathrm{k}} \theta_1^{s_{m+1}} \dots \theta_n^{s_l} \Big) \\
&+ \sum_{r=1}^{n} \Big((z_r - z_r').(-1)^{\sum_{t=1}^{m+r-1} s_t} \delta_{s_{m+r} 1} \ \theta_1^{s_1} \dots \theta_m^{s_m} \otimes_{\mathrm{k}} \theta_1^{s_{m+1}} \dots \theta_{r-1}^{s_{m+r-1}} \ \widehat{\theta_r} \ \theta_{r+1}^{s_{m+r+1}} \dots \theta_n^{s_l} \\
&+ \partial_{[r]}^{\boldsymbol{z},\boldsymbol{z}'}(W) \ . \ (-1)^{\sum_{t=1}^{m+r-1} s_t} \delta_{s_{m+r} 0} \ \theta_1^{s_1} \dots \theta_m^{s_m} \otimes_{\mathrm{k}} \theta_1^{s_{m+1}} \dots \ \theta_{r-1}^{s_{m+r-1}} \ \theta_r \ \theta_{r+1}^{s_{m+r+1}} \dots \theta_n^{s_l} \Big)
\end{aligned}
$$

$$= d_{I_V}(\theta_1^{s_1}\dots\theta_m^{s_m})\otimes_{\mathrm{k}}\theta_1^{s_{m+1}}\dots\theta_n^{s_l} + (-1)^{\sum_{t=1}^{m} s_t}\theta_1^{s_1}\dots\theta_m^{s_m}\otimes_{\mathrm{k}} d_{I_W}(\theta_1^{s_{m+1}}\dots\theta_n^{s_l})$$
$$= (d_{I_V}\otimes_k 1)(\theta_1^{s_1}\dots\theta_m^{s_m}\otimes_{\mathrm{k}}\theta_1^{s_{m+1}}\dots\theta_n^{s_l}) + (1\otimes_k d_{I_W})(\theta_1^{s_1}\dots\theta_m^{s_m}\otimes_{\mathrm{k}}\theta_1^{s_{m+1}}\dots\theta_n^{s_l})$$
$$= (d_{I_V\Box I_W}\circ\Box_{(V,W)})(\theta_1^{s_1}\dots\theta_l^{s_l}),$$

which proves that (3.1.2) is compatible with the differentials. Here, the first equality holds since by definition $\mathrm{k}[\boldsymbol{y}',\boldsymbol{y}]$ acts on the first factor in $I_V\Box I_W$ and $\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]$ acts on the second factor. □

3.1.2. Some coherence results

In this section we collect some lemmas which are essential for many proofs in the following.

We start with Lemma 3.1.6 which makes a precise statement out of the idea that a map between tensor products of matrix factorizations which merely permutes the factors while multiplying with a minus sign for each pair of $\mathbb{Z}_2$-odd elements whose order is exchanged is an isomorphism in $\mathcal{LG}$.

To set the stage for Lemma 3.1.6 we note first, that if $\varphi_*(X)$ is a module obtained via restriction of scalars along a ring isomorphism φ from a free module X then a basis of X is one of $\varphi_*(X)$. Now let $n\in\mathbb{N}$ and, neglecting brackets, consider a functor

$$F:\mathrm{hmf}^{\oplus}_{V_1}\times\cdots\times\mathrm{hmf}^{\oplus}_{V_n}\to\mathrm{hmf}^{\oplus}_{W}$$

which is a composite of cartesian products of identities, functors of restriction of scalars along ring isomorphisms, $\Box$ and horizontal composition in $\mathcal{LG}$. For $i\in\mathbb{N}_{\le n}$ let $X_i\in\mathrm{hmf}^{\oplus}_{V_i}$ have bases $\{e_{i,j}\}_{j\in J_i}$ for some index sets J_i. Then $F(X_1,\dots,X_n)=: X$ has a basis

$$\{e_{1,j_1}\otimes\cdots\otimes e_{n,j_n}|j_i\in J_i\ \forall i\in\mathbb{N}_{\le n}\}$$

where the brackets and tensor products are those dictated by F. For $\sigma\in S_n$ a permutation denote by σ also the induced functor

$$\mathrm{hmf}^{\oplus}_{V_1}\times\cdots\times\mathrm{hmf}^{\oplus}_{V_n}\to\mathrm{hmf}^{\oplus}_{V_{\sigma^{-1}(1)}}\times\cdots\times\mathrm{hmf}^{\oplus}_{V_{\sigma^{-1}(n)}}.$$

Let

$$G:\mathrm{hmf}^{\oplus}_{V_{\sigma^{-1}(1)}}\times\cdots\times\mathrm{hmf}^{\oplus}_{V_{\sigma^{-1}(n)}}\to\mathrm{hmf}^{\oplus}_{W}$$

be a second functor composed of cartesian products of identities, functors of restriction of scalars along ring isomorphisms, $\Box$ and horizontal composition in $\mathcal{LG}$. A basis of $(G\circ\sigma)(X_1,\dots,X_n)=: X_\sigma$ is given by

$$\{e_{\sigma^{-1}(1),j_{\sigma^{-1}(1)}}\otimes\cdots\otimes e_{\sigma^{-1}(n),j_{\sigma^{-1}(n)}}|j_{\sigma^{-1}(i)}\in J_{\sigma^{-1}(i)}\ \forall i\in\mathbb{N}_{\le n}\},$$

where the brackets and tensor products are determined by G.

Lemma 3.1.6. Let F, G and σ be functors as above such that

$$e_{1,j_1} \otimes \cdots \otimes e_{n,j_n} \mapsto (-1)^s e_{\sigma^{-1}(1),j_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{\sigma^{-1}(n),j_{\sigma^{-1}(n)}}, \tag{3.1.3}$$

where $s = \sum_{(k,l)|\ k<l,\ \sigma(l)<\sigma(k)} |e_{k,j_k}||e_{l,j_l}|$, defines a map $\eta : X_m \to (X_\sigma)_m$ of modules. Then η is a representative of the $(X_1, \ldots, X_n)$-component of a natural isomorphism $F \to G \circ \sigma$.

Proof. The map η is an invertible module morphism of $\mathbb{Z}_2$-degree zero.

We show that η is natural in all X_i, $i \in \mathbb{N}_{\leq n}$. To this end let $\psi_i \in \mathrm{hmf}^{\oplus}_{V_i}(X_i, Y_i)$ for $i \in \mathbb{N}_{\leq n}$. Since functors of restriction of scalars do not change the functions underlying module maps we have

$$F(\psi_1, \ldots, \psi_n) = \psi_1 \otimes \cdots \otimes \psi_n,$$

where we assume the brackets and tensor products dictated by F. We observe that under η

$$\psi_1(e_1) \otimes \cdots \otimes \psi_n(e_n) \mapsto (-1)^s \psi_{\sigma^{-1}(1)}(e_{\sigma^{-1}(1)}) \otimes \cdots \otimes \psi_{\sigma^{-1}(n)}(e_{\sigma^{-1}(n)}),$$

where s is the one in (3.1.3) since $|\psi_i| = 0$ such that $|\psi_i(e_i)| = |e_i|$ for all $i \in \mathbb{N}_{\leq n}$. For all $e_i \in X_i$, $i \in \mathbb{N}_{\leq n}$ this is the same as applying (3.1.3) first and then acting with

$$G(\psi_{\sigma^{-1}(1)}, \ldots, \psi_{\sigma^{-1}(n)}).$$

We move on to demonstrate that η is compatible with the differentials. On the one hand, we have

$$\begin{aligned} &d_{X_\sigma}((-1)^s e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes e_{\sigma^{-1}(n)}) \\ =&(-1)^s d_{X_{\sigma^{-1}(1)}}(e_{\sigma^{-1}(1)}) \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes e_{\sigma^{-1}(n)} \\ &+ (-1)^{s+|e_{\sigma^{-1}(1)}|} e_{\sigma^{-1}(1)} \otimes d_{X_{\sigma^{-1}(2)}}(e_{\sigma^{-1}(2)}) \otimes e_{\sigma^{-1}(3)} \otimes \cdots \otimes e_{\sigma^{-1}(n)} + \ldots \\ &+ (-1)^{s+|e_{\sigma^{-1}(1)}|+\cdots+|e_{\sigma^{-1}(n-1)}|} e_{\sigma^{-1}(1)} \otimes \cdots \otimes e_{\sigma^{-1}(n-1)} \otimes d_{X_{\sigma^{-1}(n)}}(e_{\sigma^{-1}(n)}), \end{aligned} \tag{3.1.4}$$

since the actions of $d_{\iota_*(Y)}$, ι some ring isomorphism, and d_Y on elements agree for all matrix factorizations Y, and thus d_{X_σ} is the sum over all $i \in \mathbb{N}_{\leq n}$ of the appropriate tensor product of d_{X_i} in the $\sigma(i)^{\text{th}}$ factor times the identity on all others. On the other hand

$$\begin{aligned} d_X(e_1 \otimes e_2 \otimes \cdots \otimes e_n) =& d_{X_1}(e_1) \otimes e_2 \otimes \cdots \otimes e_n \\ &+ (-1)^{|e_1|} e_1 \otimes d_{X_2}(e_2) \otimes e_3 \otimes \cdots \otimes e_n + \ldots \\ &+ (-1)^{|e_1|+\cdots+|e_{n-1}|} e_1 \otimes \cdots \otimes e_{n-1} \otimes d_{X_n}(e_n) \end{aligned}$$

which η sends to

$$\begin{aligned}&(-1)^{s_1} e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes d_{X_1}(e_1) \otimes \cdots \otimes e_{\sigma^{-1}(n)} \\ &+ (-1)^{s_2+|e_1|} e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes d_{X_2}(e_2) \otimes \cdots \otimes e_{\sigma^{-1}(n)} + \dots \\ &+ (-1)^{s_n+|e_1|+\cdots+|e_{n-1}|} e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes d_{X_n}(e_n) \otimes \cdots \otimes e_{\sigma^{-1}(n)},\end{aligned} \tag{3.1.5}$$

where $s_j = s + \sum_{(i<j)\wedge(\sigma(j)<\sigma(i))} |e_i| + \sum_{(j<i)\wedge(\sigma(i)<\sigma(j))} |e_i|$, $j \in \mathbb{N}_{\leq n}$. Indeed, for each $j \in \mathbb{N}_{\leq n}$ this is precisely the change in s which results from replacing $|e_j|$ by $|d_{X_j}(e_j)| = |e_j| + 1$.

We compare the powers of (-1) in (3.1.4) and (3.1.5). In (3.1.4) the summand with d_{X_1} has

$$s + \sum_{\sigma(i)<\sigma(1)} |e_i|$$

as the exponent of (-1) in its prefactor. This is exactly s_1. The summand of (3.1.4) which has d_{X_2} as one of its factors has

$$s + \sum_{(2<i)\wedge(\sigma(i)<\sigma(2))} |e_i| + \sum_{(i<2)\wedge(\sigma(2)<\sigma(i))} |e_i| + |e_1|$$

as the exponent of its prefactor. This gives $(-1)^{s_2+|e_1|}$. This argument iterates to show that (3.1.4) and (3.1.5) coincide. This holds for all $e_i \in X_i$, $i \in \mathbb{N}_{\leq n}$.

The preceding shows that η is a representative of a morphism in $\mathrm{hmf}_W^{\oplus}$. Moreover, since it is an isomorphism of underlying modules it follows from Lemma 3.1.5 that it is a representative of an isomorphism in $\mathrm{hmf}_W^{\oplus}$. □

We exemplify a situation which does not meet the condition that (3.1.3) defines a module map. The functor F is allowed to be $\square \circ (\otimes_{\mathrm{k}[\boldsymbol{x}]} \times \otimes_{\mathrm{k}[\boldsymbol{x}]})$:

$$(\mathcal{LG}(U, U_2) \times \mathcal{LG}(U_1, U)) \times (\mathcal{LG}(U, U_2) \times \mathcal{LG}(U_3, U)) \to \mathcal{LG}(U_1 + U_3, U_2 + U_2),$$

i.e.

$$(\mathrm{hmf}^{\oplus}_{-U+U_2} \times \mathrm{hmf}^{\oplus}_{-U_1+U}) \times (\mathrm{hmf}^{\oplus}_{-U+U_2} \times \mathrm{hmf}^{\oplus}_{-U_3+U}) \to \mathrm{hmf}^{\oplus}_{-U_1-U_3+U_2+U_2},$$

such that

$$X := F(X_1, X_2, X_3, X_4) = (X_1 \otimes_{\mathrm{k}[\boldsymbol{x}]} X_2) \square (X_3 \otimes_{\mathrm{k}[\boldsymbol{x}]} X_4).$$

As permutation σ we can take $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$, $4 \mapsto 4$. Then we are free to choose $G = F$. Hence we have

$$X_\sigma := (G \circ \sigma)(X_1, X_2, X_3, X_4) = (X_3 \otimes_{\mathrm{k}[\boldsymbol{x}]} X_2) \square (X_1 \otimes_{\mathrm{k}[\boldsymbol{x}]} X_4).$$

At this point we cannot define a module map $X \to X_\sigma$ by

$$(e_{1,i} \otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j}) \otimes_{\mathrm{k}} (e_{3,k} \otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l}) \text{``} \mapsto \text{''} (e_{3,k} \otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j}) \otimes_{\mathrm{k}} (e_{1,i} \otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l})$$

as for example below

$$\begin{aligned}(x.e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j})\otimes_{\mathrm{k}}(e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l}) &= (e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} x.e_{2,j})\otimes_{\mathrm{k}}(e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l})\\ &\text{“}\mapsto\text{”}(e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} x.e_{2,j})\otimes_{\mathrm{k}}(e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l})\\ &\neq (e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j})\otimes_{\mathrm{k}}(x.e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l}),\\ (x.e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j})\otimes_{\mathrm{k}}(e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l}) &\text{“}\mapsto\text{”}(e_{3,k}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{2,j})\otimes_{\mathrm{k}}(x.e_{1,i}\otimes_{\mathrm{k}[\boldsymbol{x}]} e_{4,l}).\end{aligned}$$

We say that in such a situation (3.1.3) does not *respect the tensor products*.

The next Lemma 3.1.7 is a coherence result for the maps described in Lemma 3.1.6.

Lemma 3.1.7. Two concatenations of isomorphisms of the form described in Lemma 3.1.6 with the same source and target are equal.

Proof. Note that composing morphisms of the kind described in Lemma 3.1.6 results in another morphism of that sort. First, composing the module maps yields a morphism of modules. Secondly, the functions underlying the isomorphisms in Lemma 3.1.6 have the following effects: they move brackets, they change tensor products and they permute factors at the expense of a prefactor of (-1) each time the order of two $\mathbb{Z}_2$-odd elements is reversed. Composing two functions which merely act in these ways gives another function of the same kind. Therefore it suffices to show that two morphisms of the form described in Lemma 3.1.6 with equal source and target are the same.

Since we assume that the source and target of the two morphisms we compare are equal they necessarily result in the same bracketing and the same tensor products. Furthermore, as the order of factors agrees in both the domain and the target the two maps as in (3.1.3) yield the same sign prefactor. □

In Lemma 3.1.10 we extend the result of Lemma 3.1.7. We show that two morphisms with equal source and target composed not only of morphisms as in Lemma 3.1.6 but also some specified other maps necessarily coincide. Some of the other morphisms which we can allow are the following which reappear in Section 3.2.2.

Lemma 3.1.8. Let X be a 1-morphism in $\mathcal{LG}$. There are canonical 2-isomorphisms $\kappa_X : X \to I_0 \square X$, $e \mapsto 1 \otimes_{\mathrm{k}} e$ and $\sigma_X : X \to X \square I_0$, $e \mapsto e \otimes_{\mathrm{k}} 1$.

Proof. According to Lemma 2.5.3 $(I_{(\emptyset,0)})_m = \mathrm{k}$ and $d_{I_{(\emptyset,0)}} = 0$. Combining this with $\iota_{-V_2+W_2-0+0,-0-V_2+0+W_2} = \iota_{-V_2+W_2,-V_2+W_2} = 1$ we have $d_{I_0 \square X} = 1 \otimes_{\mathrm{k}} d_X$ and $d_{X \square I_0} = d_X \otimes_{\mathrm{k}} 1$ for all 1-morphisms X in $\mathcal{LG}$ by Definition 3.1.3. Thus, the maps κ_X and σ_X are compatible with the differentials $d_{I_0 \square X}$ respectively $d_{X \square I_0}$ and d_X. Since they are isomorphisms of underlying modules they are 2-isomorphisms in $\mathcal{LG}$ by Lemma 3.1.5. □

Special cases of Lemma 3.1.8 are $\kappa_{I_W} = \square_{(0,W)}$ and $\sigma_{I_W} = \square_{(W,0)}$. Analogously to how we refer to the factor of I in the target of an inverse unitor we say that κ and σ *create* I_0.

In our proof of Lemma 3.1.10 we use the following fact.

Lemma 3.1.9. Let $\mathfrak{C}$ be a category and $a \in \mathfrak{C}(A, B)$ as well as $c \in \mathfrak{C}(C, D)$ be two isomorphisms. Two morphisms $b, b' \in \mathfrak{C}(B, C)$ are equal if and only if $c \circ b \circ a = c \circ b' \circ a$.

Proof. $b = b' \Leftrightarrow c^{-1} \circ c \circ b \circ a \circ a^{-1} = b' \Leftrightarrow c \circ b \circ a = c \circ b' \circ a$. □

Now we turn to our main coherence result.

Lemma 3.1.10. Let there be two 2-morphisms in $\mathcal{LG}$ composed via horizontal and vertical composition in $\mathcal{LG}$ or $\square$ of morphisms as in Lemma 3.1.6, as in Lemma 3.1.4 or as in Lemma 3.1.8, unitors of $\mathcal{LG}$, inverses of either of the foregoing or functors of restriction of scalars as in Notation 3.1.2 applied to the aforementioned maps. If they share the same source and target they are equal.

Throughout the proof we do not explicitly say so when we refer not only to the maps introduced in some lemma above but also to their inverses.

Proof. Call the two 2-morphisms to be compared φ_1 and φ_2. Note that they have source and target of the form $X := F(X_1, \dots, X_m) \in \mathcal{LG}(W_1, W_2)$ respectively $Y := \widetilde{F}(Y_1, \dots, Y_n)$, where

$$F : \mathrm{hmf}^{\oplus}_{U_1} \times \cdots \times \mathrm{hmf}^{\oplus}_{U_m} \to \mathrm{hmf}^{\oplus}_{-W_1+W_2},$$
$$\widetilde{F} : \mathrm{hmf}^{\oplus}_{V_1} \times \cdots \times \mathrm{hmf}^{\oplus}_{V_n} \to \mathrm{hmf}^{\oplus}_{-W_1+W_2}$$

are composites of cartesian products of identities, functors of restriction of scalars along ring isomorphisms, $\square$ and horizontal composition in $\mathcal{LG}$. Indeed, every one of the morphisms of which φ_1 and φ_2 can be composed has source and target of such a form. Therefore also the sources and targets of φ_1 and φ_2 can be described as X and Y above.

We distinguish two cases. First, we assume that there is at least one $b \in \mathbb{N}_{\leq n}$ such that $Y_b \neq I$. This implies that there exists an $a \in \mathbb{N}_{\leq m}$ such that $X_a \neq I$. Indeed none of the maps in Lemma 3.1.6, those in Lemma 3.1.4, the ones in Lemma 3.1.8 or unitors “create” matrix factorizations other than unit matrix factorizations. Neither do the maps obtained from these via restriction of scalars. Thus, if $Y_b \neq I$ then there has to be an $a \in \mathbb{N}_{\leq m}$ with $X_a = Y_b$.

By Lemma 3.1.9 $\varphi_1 = \varphi_2$ if and only if for some isomorphisms γ, ν we have $\gamma \circ \varphi_1 \circ \nu = \gamma \circ \varphi_2 \circ \nu$. Since we assume $Y_b \neq I$, we can compose such an isomorphism γ by concatenating and using $\square$ or $\otimes_{\Bbbk[-]}$ (i.e. $\otimes_{\Bbbk[\boldsymbol{x}]}$ for various sets of variables $\boldsymbol{x}$) out of maps as in Lemma 3.1.4, as in (3.1.3), as in Lemma 3.1.8, unitors or functors as in Notation 3.1.2 applied to either of those morphisms such that all Y_j which are unit matrix factorizations are "removed" after projecting to their 1-component. Similarly, we can create all X_i which are unit matrix factorizations from the other X_k by some combination of inverse unitors, maps as in Lemma 3.1.4, as in (3.1.3), as in Lemma 3.1.8 or functors of restriction of scalars applied to such maps. Let ν be an isomorphism of this kind. We argue that this reduces $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ to 2-morphisms of the form described in Lemma 3.1.6.

The maps $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ are of the form of those described in Lemma 3.1.6 if their underlying functions are of the form (3.1.3). As the functions underlying maps of matrix factorizations are not affected by functors of restriction of scalars we do not consider such functors in showing that the functions underlying $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ are of the form (3.1.3).

Eventually, all factors of I featuring in intermediate stages of $\gamma \circ \varphi_1 \circ \nu$ or $\gamma \circ \varphi_2 \circ \nu$ are projected to their 1-component. Therefore, the maps from Lemma 3.1.4 only contribute through $1 \leftrightarrow 1 \otimes_{\Bbbk} 1$. In particular, no signs can arise from permuting factors of I with other matrix factorizations via the maps in Lemma 3.1.6. Also, according to Observation 2.5.10 once all factors of I_W, $W \neq 0$ are projected to their 1-components, only the elements present before inverse unitors are applied remain in the other factors. The same holds true for those I_0 created and "removed" by the maps in Lemma 3.1.8. Therefore, the overall effect of both $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ has to be of the form (3.1.3).

The above reduces $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ to maps as in Lemma 3.1.6 such that it follows from Lemma 3.1.7 that they are equal. By Lemma 3.1.9 this entails $\varphi_1 = \varphi_2$.

Next, assume that all X_i, $i \in \mathbb{N}_{\leq m}$ are unit matrix factorizations. Analogously to the preceding case this implies that all Y_j, $j \in \mathbb{N}_{\leq n}$, are unit matrix factorizations, too. Again we argue that we can compose φ_1 and φ_2 with isomorphisms such that the functions underlying the resulting morphisms are equal. As before this justifies that we do not consider functors of restriction of scalars for the rest of the proof.

We show that there are isomorphisms γ and ν such that $\gamma \circ \varphi_1 \circ \nu = \gamma \circ \varphi_2 \circ \nu$. We choose $\nu = \lambda_X^{-1}$. Since all Y_j, $j \in \mathbb{N}_{\leq n}$, are unit matrix factorizations we can use unitors, maps as in Lemma 3.1.4, as in Lemma 3.1.8 and as in Lemma 3.1.6 (not permuting factors but changing functors of restriction of scalars) or functors as in Notation 3.1.2 applied to such morphisms to build an isomorphism $\tau : Y \to \iota(I)$, where ι is some functor as in Notation 3.1.2. Then there is an isomorphism ψ as in Lemma 3.1.6 (not permuting factors) such that $\gamma = \iota'(\rho) \circ \psi \circ (1 \otimes \tau)$, ι' a functor as in Notation 3.1.2, is well-defined. Consequently, all Y_j, $j \in \mathbb{N}_{\leq n}$ are projected to their 1-components. This entails that all X_i, $i \in \mathbb{N}_{\leq m}$ are projected to their 1-components, too. Moreover, it follows that $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ do not feature any signs from maps as in Lemma 3.1.6 permuting factors.

Recall the formula (2.5.10) defining λ_X^{-1}:

$$e_i \mapsto \sum_{l\geq 0} \sum_{i_1<\cdots<i_l} \sum_{j\in J} \{\partial_{[i_l]}^{\boldsymbol{z}_2,\boldsymbol{z}_2'} d_X \dots \partial_{[i_1]}^{\boldsymbol{z}_2,\boldsymbol{z}_2'} d_X\}_{ji}.\theta_{i_1}\dots\theta_{i_l} \otimes_{\mathrm{k}[\boldsymbol{z}_2]} e_j.$$

By the preceding both $\gamma \circ \varphi_1 \circ \nu$ and $\gamma \circ \varphi_2 \circ \nu$ act as

$$e_i \mapsto \sum_{l\geq 0} \sum_{i_1<\cdots<i_l} \{\partial_{[i_l]}^{\boldsymbol{z}_2,\boldsymbol{z}_2'} d_X \dots \partial_{[i_1]}^{\boldsymbol{z}_2,\boldsymbol{z}_2'} d_X\}_{1i}.\theta_{i_1}\dots\theta_{i_l}.$$

This shows $\gamma \circ \varphi_1 \circ \nu = \gamma \circ \varphi_2 \circ \nu$ from which it follows with Lemma 3.1.9 that $\varphi_1 = \varphi_2$. □

Lemma 3.1.10 applies to all 2-morphisms which we define below. Thus, all diagrams which have to commute for these maps to endow $\mathcal{LG}$ with the structure we aim at commute by Lemma 3.1.10.

Remarks 3.1.11. As we show in Proposition 3.1.12 below the maps in Lemma 3.1.4 are constraint 2-morphisms of a 2-functor. We can prove this independently of this section's lemmas. Then it follows from coherence for 2-functors, cf. [G, Remark 3.1.6], that all diagrams containing only unitors and associators of $\mathcal{LG}$ or their inverses as well as maps as in Lemma 3.1.4 or inverses thereof commute.

Using that the structure morphisms of $\mathcal{LG}$ are natural, many of the diagrams appearing in the proofs that our definitions below endow $\mathcal{LG}$ with the desired structure can be rearranged into subdiagrams which commute by this coherence theorem. This already significantly simplifies the proofs. Yet only using coherence for 2-functors one still has to prove that one can accommodate to functors of restriction of scalars as in Notation 3.1.2 case-by-case.

Vice versa, as a special case, Lemma 3.1.10 verifies the coherence theorem as applied to the two-functor which we introduce momentarily in Proposition 3.1.12.

3.1.3. The monoidal product

We get back to the purpose of this section: defining a 2-functor $\mathcal{LG} \times \mathcal{LG} \to \mathcal{LG}$. The notion of a 2-functor is recalled in Definition A.1.2.

Proposition 3.1.12. Let k be a field of characteristic zero. The following defines a 2-functor $\square : \mathcal{LG} \times \mathcal{LG} \to \mathcal{LG}$.

1. Its action on objects is $\square_{\mathrm{O}} : (\mathcal{LG} \times \mathcal{LG})_{\mathrm{O}} \to \mathcal{LG}_{\mathrm{O}},\ (V, W) \mapsto V + W$.
2. Its functors on morphism categories are

   $$\square_{(V_1,V_2),(W_1,W_2)} : (\mathcal{LG} \times \mathcal{LG})((V_1, V_2), (W_1, W_2)) \to \mathcal{LG}(V_1 + V_2, W_1 + W_2)$$

   of Definition 3.1.3.

3. Its natural isomorphisms $\square_{(U_1,U_2),(V_1,V_2),(W_1,W_2)}$ expressing compatibility with horizontal composition are

$$\otimes^{\mathcal{LG}}_{U_1+U_2,V_1+V_2,W_1+W_2} \circ (\square \times \square) \to \square \circ \otimes^{\mathcal{LG}\times\mathcal{LG}}_{(U_1,U_2),(V_1,V_2),(W_1,W_2)}$$

given by the $((Y_1,Y_2),(X_1,X_2))$-component

$$\begin{aligned} &((Y_1 \square Y_2) \otimes_{\mathrm{k}[\boldsymbol{y}_1,\boldsymbol{y}_2]} (X_1 \square X_2))_m \to ((Y_1 \otimes_{\mathrm{k}[\boldsymbol{y}_1]} X_1) \square (Y_2 \otimes_{\mathrm{k}[\boldsymbol{y}_2]} X_2))_m \quad (3.1.6)\\ &(f_1 \otimes_{\mathrm{k}} f_2) \otimes_{\mathrm{k}[\boldsymbol{y}_1,\boldsymbol{y}_2]} (e_1 \otimes_{\mathrm{k}} e_2) \mapsto (-1)^{|f_2|\,|e_1|} (f_1 \otimes_{\mathrm{k}[\boldsymbol{y}_1]} e_1) \otimes_{\mathrm{k}} (f_2 \otimes_{\mathrm{k}[\boldsymbol{y}_2]} e_2). \end{aligned}$$

4. Its isomorphisms $\square_{(W_1,W_2)}$ on units are the morphisms $I_{W_1+W_2} \to I_{W_1} \square I_{W_2}$ of Lemma 3.1.4.

In part 3 of Proposition 3.1.12 the assignment (3.1.6) is intended to define a module map by extending it linearly. We leave this comment implicit in the following.

Definition 3.1.13. The 2-functor $\square$ described in Proposition 3.1.12 is the *monoidal product* for $\mathcal{LG}$. The 2-morphisms $\square_{(U_1,U_2),(V_1,V_2),(W_1,W_2)}$ in part 3 of Proposition 3.1.12 are the *tensorators*.

Where the context allows to do so unambiguously, we do not write indices on the components of $\square$. Note that according to this convention both the 2-morphisms specified by (3.1.6) and the maps $\square_{(W_1,W_2)}$ defined in part 4 of Proposition 3.1.12 are written as $\square$.

Proof of Proposition 3.1.12. First, we prove that $\square$ is well-defined on objects. Let $(V,W) \in \mathcal{LG}^2$. Since $(\boldsymbol{y})^2_{\mathrm{k}[\boldsymbol{y}]} \subset (\boldsymbol{y},\boldsymbol{z})^2_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]}$ and $(\boldsymbol{z})^2_{\mathrm{k}[\boldsymbol{z}]} \subset (\boldsymbol{y},\boldsymbol{z})^2_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]}$ we have

$$V \in (\boldsymbol{y})^2_{\mathrm{k}[\boldsymbol{y}]},\ W \in (\boldsymbol{z})^2_{\mathrm{k}[\boldsymbol{z}]} \Rightarrow V + W \in (\boldsymbol{y},\boldsymbol{z})^2_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]}.$$

Moreover, the canonical ring isomorphism $\mathrm{k}[\boldsymbol{y}] \otimes_{\mathrm{k}} \mathrm{k}[\boldsymbol{z}] \cong \mathrm{k}[\boldsymbol{y},\boldsymbol{z}]$ induces a k-linear ring morphism

$$\begin{aligned} &\mathrm{k}[\boldsymbol{y}]/(\partial_{y_1}V,\dots,\partial_{y_m}V) \otimes_{\mathrm{k}} \mathrm{k}[\boldsymbol{z}]/(\partial_{z_1}W,\dots,\partial_{z_n}W) \quad (3.1.7)\\ \to &\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]/(\partial_{y_1}V,\dots,\partial_{y_m}V,\partial_{z_1}W,\dots,\partial_{z_n}W) \quad (3.1.8) \end{aligned}$$

which sends a basis of the k-vector space (3.1.7) to a basis of the k-vector space (3.1.8), whereby (3.1.7) and (3.1.8) are isomorphic k-vector spaces and

$$\begin{aligned} &\dim_{\mathrm{k}}(\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]/(\partial_{y_1}V,\dots,\partial_{y_m}V,\partial_{z_1}W,\dots,\partial_{z_n}W))\\ =&\dim_{\mathrm{k}}(\mathrm{k}[\boldsymbol{y}]/(\partial_{y_1}V,\dots,\partial_{y_m}V) \otimes_{\mathrm{k}} \mathrm{k}[\boldsymbol{z}]/(\partial_{z_1}W,\dots,\partial_{z_n}W))\\ =&\dim_{\mathrm{k}}(\mathrm{k}[\boldsymbol{y}]/(\partial_{y_1}V,\dots,\partial_{y_m}V)) \cdot \dim_{\mathrm{k}}(\mathrm{k}[\boldsymbol{z}]/(\partial_{z_1}W,\dots,\partial_{z_n}W)) < \infty. \end{aligned}$$

Therefore, $V + W \in \mathcal{LG}$.

That $\Box_{(U_1,U_2),(V_1,V_2),(W_1,W_2)}$ according to part 3 of Proposition 3.1.12 is a natural isomorphism is a special case of Lemma 3.1.6. Indeed, (3.1.6) defines a $\mathrm{k}[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{z}_1, \boldsymbol{z}_2]$-module map, where on both of its sides $\mathrm{k}[\boldsymbol{x}_i] \subset \mathrm{k}[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{z}_1, \boldsymbol{z}_2]$ acts on X_i and $\mathrm{k}[\boldsymbol{z}_i] \subset \mathrm{k}[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{z}_1, \boldsymbol{z}_2]$ acts on Y_i for $i \in \{1, 2\}$. (3.1.6) respects the tensor products since for $i \in \{1, 2\}$ these identify prefactors in $\mathrm{k}[\boldsymbol{y}_i]$ in front of f_i and e_i, respectively, on both of its sides.

It is left to verify that the structure we have defined makes the required diagrams, cf. Definition A.1.2, commute. We begin with the diagrams involving the unit 1-morphisms, cf. (A.1.4). To assure that $\Box$ is compatible with the left unitors we assert that the following two 2-morphisms coincide:

$$\begin{aligned} I_{V+W} \otimes_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]} (X \Box Y) &\xrightarrow{\lambda_{X\Box Y}} X \Box Y \text{ and} \\ I_{V+W} \otimes_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]} (X \Box Y) &\xrightarrow{\Box\otimes 1} (I_V \Box I_W) \otimes_{\mathrm{k}[\boldsymbol{y},\boldsymbol{z}]} (X \Box Y) \\ &\xrightarrow{\Box} (I_V \otimes_{\mathrm{k}[\boldsymbol{y}]} X) \Box (I_W \otimes_{\mathrm{k}[\boldsymbol{z}]} Y) \\ &\xrightarrow{\lambda_X \Box \lambda_Y} X \Box Y. \end{aligned}$$

This is a special case of Lemma 3.1.10. Analogously, it is a consequence of Lemma 3.1.10 that $\Box$ is compatible with right unitors.

That $\Box$ is compatible with associators, cf. (A.1.3), means that the following compositions of 2-morphisms in which we omit indices on $\otimes$ are identical for suitably composable 1-morphisms X_r, Y_r, Z_r, $r \in \{0, 1\}$ in $\mathcal{LG}$:

$$\begin{aligned} ((Z_1 \Box Z_2) \otimes (Y_1 \Box Y_2)) \otimes (X_1 \Box X_2) &\xrightarrow{\alpha} (Z_1 \Box Z_2) \otimes ((Y_1 \Box Y_2) \otimes (X_1 \Box X_2)) \\ &\xrightarrow{1\otimes\Box} (Z_1 \Box Z_2) \otimes ((Y_1 \otimes X_1) \Box (Y_2 \otimes X_2)) \\ &\xrightarrow{\Box} (Z_1 \otimes (Y_1 \otimes X_1)) \Box (Z_2 \otimes (Y_2 \otimes X_2)) \\ ((Z_1 \Box Z_2) \otimes (Y_1 \Box Y_2)) \otimes (X_1 \Box X_2) &\xrightarrow{\Box\otimes 1} ((Z_1 \otimes Y_1) \Box (Z_2 \otimes Y_2)) \otimes (X_1 \Box X_2) \\ &\xrightarrow{\Box} ((Z_1 \otimes Y_1) \otimes X_1) \Box ((Z_2 \otimes Y_2) \otimes X_2) \\ &\xrightarrow{\alpha\Box\alpha} (Z_1 \otimes (Y_1 \otimes X_1)) \Box (Z_2 \otimes (Y_2 \otimes X_2)). \end{aligned} \tag{3.1.9}$$

Note that according to their definition in (2.10.1) the associators α are morphisms of the kind described in Lemma 3.1.6 and the same is true for the tensorators. Thus, all maps in (3.1.9) are of that form. Hence it follows from Lemma 3.1.7 that the two concatenations of 2-morphisms depicted above coincide, showing that $\Box$ is compatible with associators. This finishes our proof that $\Box$ is a 2-functor. $\Box$

3.2. The remaining monoidal structure

In what follows we define the further data turning $\mathcal{LG}$ into a monoidal bicategory, cf. Definition A.3.1.

3.2.1. Associativity

Denote by $\mathfrak{A}$ the strict 2-functor $(\mathcal{LG} \times \mathcal{LG}) \times \mathcal{LG} \to \mathcal{LG} \times (\mathcal{LG} \times \mathcal{LG})$ which acts on the level of objects, 1-morphisms and 2-morphisms by rebracketing as e.g. $((U,V),W) \mapsto (U,(V,W))$. Before defining a pseudonatural transformation $a : \square \circ (\square \times 1) \to \square \circ (1 \times \square) \circ \mathfrak{A}$ we consider the action of the two 2-functors between which it interpolates on objects:

$$\begin{aligned}(\square \circ (\square \times 1))((U,V),W) &= \square((U+V),W) = (U+V+W,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})), \quad (3.2.1)\\ (\square \circ (1 \times \square) \circ \mathfrak{A})((U,V),W) &= \square(U,(V+W)) = (U+V+W,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})).\end{aligned}$$

This shows that on objects both 2-functors agree. Therefore we have a distinguished choice for the component of a on objects: the unit 1-morphism. Moreover, we can use the unitors to define the components of a on 1-morphisms.

Remark 3.2.1. That $(\square \circ (\square \times 1))_O = (\square \circ (1 \times \square) \circ \mathfrak{A})_O$ is a first instance where we benefit from working with $\mathcal{LG}$ rather than the equivalent bicategory of [CM2] which we mention in Remark 2.10.5. To wit, the equality of both lines in (3.2.1) hinges on the strict associativity of addition and of building nested ordered sets. Had we in contrast e.g. allowed for tensor products in objects of $\mathcal{LG}$ and defined $\square(V,W) := (\mathrm{k}[\boldsymbol{y}] \otimes_{\mathrm{k}} \mathrm{k}[\boldsymbol{z}], V \otimes_{\mathrm{k}} 1 + 1 \otimes_{\mathrm{k}} W)$ as suggested e.g. in [CR2, §4] it would be more involved to define a since $\otimes_{\mathrm{k}}$ is not strictly associative. Our approach has been hinted at in [C2, §2.4.4].

Due to Lemma 3.1.6 the following ingredient of a is well-defined.

Definition 3.2.2. Let X, Y and Z be 1-morphisms in $\mathcal{LG}$. $\mathscr{A}_{X,Y,Z}$ is the following 2-isomorphism in $\mathcal{LG}$ which is natural in X, Y and Z:

$$X\square(Y\square Z) \to (X\square Y)\square Z,\ e \otimes_{\mathrm{k}} (f \otimes_{\mathrm{k}} g) \mapsto (e \otimes_{\mathrm{k}} f) \otimes_{\mathrm{k}} g. \qquad (3.2.2)$$

Indeed, let $X \in \mathcal{LG}(U_1,U_2)$, $Y \in \mathcal{LG}(V_1,V_2)$ and $Z \in \mathcal{LG}(W_1,W_2)$. Then (3.2.2) defines a $\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{y}_1,\boldsymbol{z}_1,\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$-module morphism: scalars in $\mathrm{k}[\boldsymbol{x}_i] \subset \mathrm{k}[\boldsymbol{x}_1,\boldsymbol{y}_1,\boldsymbol{z}_1,\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$, $i \in \{1,2\}$, act on the first factor of both the source and the target, scalars in $\mathrm{k}[\boldsymbol{y}_i]$, $i \in \{1,2\}$, act on the second factor of both source and target and those in $\mathrm{k}[\boldsymbol{z}_i]$, $i \in \{1,2\}$, act on the third factor.

We use this to state, recall Definition A.1.4 of a pseudonatural transformation,

Lemma 3.2.3. The following defines a pseudonatural transformation

$$a : \square \circ (\square \times 1) \to \square \circ (1 \times \square) \circ \mathfrak{A}.$$

1. Its 1-morphism component is $a_{((U,V),W)} := I_{U+V+W}$.
2. Its 2-isomorphism component $a_{((X,Y),Z)}$ is $\lambda^{-1}_{(X\square Y)\square Z} \circ \mathscr{A}_{X,Y,Z} \circ \rho_{X\square(Y\square Z)}$.

Definition 3.2.4. The pseudonatural transformation a described in Lemma 3.2.3 is the *associator* for the monoidal product on $\mathcal{LG}$.

In the proof of Lemma 3.2.3 as well as in the following we frequently omit indices on $\otimes$, I and morphisms of matrix factorizations.

Proof of Lemma 3.2.3. It follows from (3.2.1) that the component of a in part 1 of Lemma 3.2.3 has the correct source and target objects. Furthermore, the component of a in part 2 of Lemma 3.2.3 is defined as a 2-isomorphism which is natural in all three arguments. To wit, λ and ρ are natural isomorphisms and $\square\circ(\square\times 1)$ as well as $\square\circ(1\times\square)\circ\mathfrak{A}$ are functorial in all three arguments. We show that the necessary diagrams, cf. (A.1.5) and (A.1.6), commute, too. a respects compositions since the following two concatenations of maps coincide by Lemma 3.1.10.

$$
\begin{aligned}
&((X_1\square(Y_1\square Z_1))\otimes(X\square(Y\square Z)))\otimes I\\
&\xrightarrow{\alpha}(X_1\square(Y_1\square Z_1))\otimes((X\square(Y\square Z))\otimes I)\\
&\xrightarrow{1\otimes\rho}(X_1\square(Y_1\square Z_1))\otimes(X\square(Y\square Z))\\
&\xrightarrow{1\otimes\mathscr{A}}(X_1\square(Y_1\square Z_1))\otimes((X\square Y)\square Z)\\
&\xrightarrow{1\otimes\lambda^{-1}}(X_1\square(Y_1\square Z_1))\otimes(I\otimes((X\square Y)\square Z))\\
&\xrightarrow{\alpha^{-1}}((X_1\square(Y_1\square Z_1))\otimes I)\otimes((X\square Y)\square Z)\\
&\xrightarrow{\rho\otimes 1}(X_1\square(Y_1\square Z_1))\otimes((X\square Y)\square Z)\\
&\xrightarrow{\mathscr{A}\otimes 1}((X_1\square Y_1)\square Z_1)\otimes((X\square Y)\square Z)\\
&\xrightarrow{\lambda^{-1}\otimes 1}(I\otimes((X_1\square Y_1)\square Z_1))\otimes((X\square Y)\square Z)\\
&\xrightarrow{\alpha}I\otimes(((X_1\square Y_1)\square Z_1)\otimes((X\square Y)\square Z))\\
&\xrightarrow{1\otimes(\square\circ(\square\times 1))}I\otimes(((X_1\otimes X)\square(Y_1\otimes Y))\square(Z_1\otimes Z)),\\
&((X_1\square(Y_1\square Z_1))\otimes(X\square(Y\square Z)))\otimes I\\
&\xrightarrow{(\square\circ(1\times\square)\circ\mathfrak{A})\otimes 1}((X_1\otimes X)\square((Y_1\otimes Y)\square(Z_1\otimes Z)))\otimes I\\
&\xrightarrow{\rho}(X_1\otimes X)\square((Y_1\otimes Y)\square(Z_1\otimes Z))\\
&\xrightarrow{\mathscr{A}}((X_1\otimes X)\square(Y_1\otimes Y))\square(Z_1\otimes Z)\\
&\xrightarrow{\lambda^{-1}}I\otimes(((X_1\otimes X)\square(Y_1\otimes Y))\square(Z_1\otimes Z)).
\end{aligned}
$$

Finally, we assert that a is compatible with units. This amounts to the sequences of maps below being equal

$$
\begin{aligned}
&I\otimes I\xrightarrow{(\square\circ(1\times\square)\circ\mathfrak{A})\otimes 1}(I\square(I\square I))\otimes I\xrightarrow{\rho}I\square(I\square I)\xrightarrow{\mathscr{A}}(I\square I)\square I\xrightarrow{\lambda^{-1}}I\otimes((I\square I)\square I),\\
&I\otimes I\xrightarrow{\lambda}I\xrightarrow{\rho^{-1}}I\otimes I\xrightarrow{1\otimes(\square\circ(\square\times 1))}I\otimes((I\square I)\square I).
\end{aligned}
$$

Again, this is an instance of Lemma 3.1.10. □

According to Definition A.3.1 associators for the monoidal product on a bicategory come as an adjoint equivalence. Therefore we need

Lemma 3.2.5. There is a pseudonatural transformation $a^- : \square \circ (1 \times \square) \circ \mathfrak{A} \to \square \circ (\square \times 1)$, given by the following.

1. Its 1-morphism component $a^-_{((U,V),W)}$ is I_{U+V+W}.
2. Its 2-isomorphism component $a^-_{((X,Y),Z)}$ is $\lambda^{-1}_{X\square(Y\square Z)} \circ \mathscr{A}^{-1}_{X,Y,Z} \circ \rho_{(X\square Y)\square Z}$.

Proof. This is completely analogous to our proof of Lemma 3.2.3 above. □

Before we show that a and a^- form an adjoint equivalence we inspect the composite $a^- \circ a$, cf. Definition A.1.7. This has the 1-morphism component

$$(a^- \circ a)_{((U,V),W)} = (a^-)_{((U,V),W)} \otimes a_{((U,V),W)} = I_{U+V+W} \otimes_{\mathrm{k}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} I_{U+V+W}.$$

Its associated 2-morphism is

$$\begin{aligned} ((X\square Y)\square Z) \otimes I \otimes I &\xrightarrow{(\lambda^{-1}\circ\mathscr{A}^{-1}\circ\rho)\otimes 1} I \otimes (X\square(Y\square Z)) \otimes I \\ &\xrightarrow{1\otimes(\lambda^{-1}\circ\mathscr{A}\circ\rho)} I \otimes I \otimes ((X\square Y)\square Z). \end{aligned} \tag{3.2.3}$$

This consists exclusively of morphisms to which Lemma 3.1.10 applies.

The following lemma uses the notion of a modification, cf. Definition A.1.5.

Lemma 3.2.6. There are modifications

$$\begin{aligned} &\epsilon : a^- \circ a \to 1_{\square\circ(\square\times 1)},\ \epsilon_{((U,V),W)} := \lambda_I, \quad \eta : 1_{\square\circ(1\times\square)\circ\mathfrak{A}} \to a \circ a^-,\ \eta_{((U,V),W)} = \lambda_I^{-1} \\ &\tilde\epsilon : a \circ a^- \to 1_{\square\circ(1\times\square)\circ\mathfrak{A}},\ \tilde\epsilon_{((U,V),W)} = \lambda_I, \quad \tilde\eta : 1_{\square\circ(\square\times 1)} \to a^- \circ a,\ \tilde\eta_{((U,V),W)} = \lambda_I^{-1} \end{aligned}$$

witnessing an adjoint equivalence of a and a^-.

Proof. First we note that $\epsilon_{((U,V),W)} := \lambda_I$ is a 2-isomorphism from $(a^-\circ a)_{((U,V),W)} = I \otimes I$ to $(1_{\square\circ(\square\times 1)})_{((U,V),W)} = I$. It makes the square defining a modification, cf. (A.1.7), commute since as displayed in (3.2.3) $(a^- \circ a)_{((X,Y),Z)}$ consists only of morphisms to which Lemma 3.1.10 applies and thereby the whole square features but such morphisms and commutes.

We can argue analogously that $\tilde\epsilon_{((U,V),W)} := \lambda_I$, $\eta_{((U,V),W)} := \lambda_I^{-1}$ and $\tilde\eta_{((U,V),W)} := \lambda_I^{-1}$ yield modifications.

ϵ and η as well as their tilded versions are adjunction maps, cf. Definition A.2.1. We have

$$((1 \otimes \epsilon) \circ (\eta \otimes 1))_{((U,V),W)} : I \otimes I \xrightarrow{\lambda^{-1}\otimes 1} I \otimes I \otimes I \xrightarrow{1\otimes\lambda} I \otimes I$$

which by Lemma 3.1.10 is the same as the identity 2-morphism. For $\tilde\eta$ and $\tilde\epsilon$ we can iterate this reasoning. Therefore a and a^- form an adjoint equivalence. □

Note that the components of the modifications in Lemma 3.2.6 can be replaced e.g by ρ_I and its inverses according to Lemma 3.1.10.

Next we introduce a modification showing that the associators a satisfy the pentagon axiom up to 2-isomorphisms.

Lemma 3.2.7. There is an invertible modification $\pi : (1\Box a)\circ a\circ(a\Box 1)\to a\circ a$ with $(((T,U),V),W)$-component

$$\begin{aligned}
&(1_T\Box a_{((U,V),W)})\otimes a_{((T,U+V),W)}\otimes(a_{((T,U),V)}\Box 1_W)\\
&=(I_T\Box I_{U+V+W})\otimes I_{T+U+V+W}\otimes(I_{T+U+V}\Box I_W)\\
&\xrightarrow{\Box\otimes 1\otimes\Box} I_{T+U+V+W}\otimes I_{T+U+V+W}\otimes I_{T+U+V+W}\\
&\xrightarrow{\lambda} I_{T+U+V+W}\otimes I_{T+U+V+W}=a_{((T,U),V+W)}\otimes a_{((T+U,V),W)}.
\end{aligned}$$

Definition 3.2.8. The modification π of Lemma 3.2.7 is the *pentagonator.*

In proving Lemma 3.2.7 and later we use the following notation. Let X, Y be matrix factorizations and ϕ, φ be morphisms of matrix factorizations. Sometimes we write $XY := X\Box Y$ and $\phi\varphi := \phi\Box\varphi$. Moreover, we do not display bracketings for $\otimes$ where they are not essential for our arguments.

Proof of Lemma 3.2.7. By definition the prescription of Lemma 3.2.7 yields invertible 2-morphisms. We argue that they combine into a modification. Let X_i, $i\in\mathbb{N}_{\le 4}$ be matrix factorizations. The upper left corner of the square whose commutativity qualifies π as modification, cf. (A.1.7), is $(X_1(X_2(X_3X_4)))\otimes(Ia)\otimes a\otimes(aI)$. In the square's lower right corner we have $a\otimes a\otimes(((X_1X_2)X_3)X_4)$. The two morphisms in between these matrix factorizations which label the edges of the diagram are based on 2-morphism components of a and π. It follows that Lemma 3.1.10 applies to show that π is a modification. □

Note that Lemma 3.1.10 also applies to all maps obtained from π by taking adjoints of its source or target 1-morphisms. Indeed, it applies to the morphisms in Lemma 3.2.6 which are the (co-)evaluation maps introduced by taking these adjoints.

The following is the only one of the axioms of a monoidal bicategory which involves but the structure maps with which we have equipped $\mathcal{LG}$ so far, cf. Definition A.3.1.

Lemma 3.2.9. $\Box$, a and π satisfy the associahedron equation, i.e. the identity of pasting diagrams (A.3.1)=(A.3.2).

Proof. The 2-morphisms depicted in (A.3.1) and (A.3.2) are composed solely of morphisms to which Lemma 3.1.10 applies. □

3.2.2. Units

It remains to define the monoidal unit object and to prove that it satisfies the necessary conditions. For this, we denote by **1** the bicategory consisting of a single object $*$ and only identity morphisms. Our following definition is anticipated in [CR2, §4].

Definition 3.2.10. $\mathbf{I} : \mathbf{1} \to \mathcal{LG}$ is the 2-functor with $\mathbf{I}_{\mathrm{O}}(*) := (\emptyset, 0)$. The *monoidal unit object* of $\mathcal{LG}$ is $(\emptyset, 0)$.

Note that Definition 3.2.10 is well-defined as $0 \in \mathrm{k} \equiv \mathrm{k}[\emptyset]$ is indeed a potential according to Definition 2.1.1 since $(\emptyset) = \{0\}$ and $0 \in \{0\} = (\emptyset)^2$, $\dim_{\mathrm{k}}(\mathrm{k}/\{0\}) = \dim_{\mathrm{k}}(\mathrm{k}) = 1 < \infty$.

Next, we spell out the 2-functor $\Box \circ (\mathbf{I} \times 1) \equiv \mathbf{I}\Box 1$, which is the source of the pseudonatural transformation that we introduce in Lemma 3.2.11 below.

$$\begin{aligned}
\mathbf{I}\Box 1 :&\mathbf{1} \times \mathcal{LG} \to \mathcal{LG}, \\
(\mathbf{I}\Box 1)_{\mathrm{O}} :&(*, (\boldsymbol{z}, W)) \mapsto (\boldsymbol{z}, W), \\
(\mathbf{I}\Box 1)_{(*,V),(*,W)} :&(\mathbf{1} \times \mathcal{LG})((*, V), (*, W)) \to \mathcal{LG}(V, W), \\
&(1_*, X) \mapsto I_0 \Box X, \ (1_{1_*}, \phi) \mapsto 1_{I_0} \Box \phi \\
(\mathbf{I}\Box 1)_{(*,U),(*,V),(*,W)} :&(I_0 \Box Y) \otimes (I_0 \Box X) \overset{(\lambda_{I_0} \times 1)\circ\Box}{\longmapsto} I_0 \Box (Y \otimes X) \\
(\mathbf{I}\Box 1)_{(*,W)} :&I_W \overset{\Box_{(0,W)}}{\longrightarrow} I_0 \Box I_W.
\end{aligned}$$

Denote by $\Lambda : \mathbf{1} \times \mathcal{LG} \to \mathcal{LG}$ the 2-functor which projects onto the second component and recall the notation of Lemma 3.1.8.

Lemma 3.2.11. There is a pseudonatural transformation $l : \mathbf{I}\Box 1 \to \Lambda$ with $l_{(*,W)} := I_W$ and $l_{(1_*,X)} := (1 \otimes \kappa_X) \circ \lambda_X^{-1} \circ \rho_X$.

Definition 3.2.12. The pseudonatural transformation $l : \mathbf{I}\Box 1 \to \Lambda$ in Lemma 3.2.11 is the *left unitor* for $\Box$.

Proof of Lemma 3.2.11. By definition, the 2-morphism components of l are isomorphisms. Moreover, Lemma 3.1.10 can be applied to these 2-morphisms. Suppressing associators α the two 2-morphisms which have to be equal for l to be compatible with horizontal composition of 1-morphisms, cf. (A.1.5), are

$$\begin{aligned}
(Y \otimes X) \otimes I_U &\xrightarrow{1\otimes l_{(1_*,X)}} Y \otimes (I_V \otimes (I_0 \Box X)) \\
&\xrightarrow{l_{(1_*,Y)}\otimes 1} (I_W \otimes (I_0 \Box Y)) \otimes (I_0 \Box X) \\
&\xrightarrow{1\otimes((\lambda_{I_0}\Box 1)\circ\Box)} I_W \otimes (I_0 \Box (Y \otimes X)), \\
(Y \otimes X) \otimes I_U &\xrightarrow{l_{(1_*,Y\otimes X)}} I_W \otimes (I_0 \Box (Y \otimes X)).
\end{aligned}$$

These two morphisms are the same by Lemma 3.1.10.

The 2-morphism components of l are compatible with units, cf. (A.1.6), since the following two 2-morphisms coincide by Lemma 3.1.10:

$$I_W \otimes I_W \xrightarrow{\lambda} I_W \xrightarrow{\rho^{-1}} I_W \otimes I_W \xrightarrow{1\otimes\Box_{(0,W)}} I_W \otimes (I_0 \Box I_W),$$
$$I_W \otimes I_W \xrightarrow{l_{(1_*,I)}} I_W \otimes (I_0 \Box I_W).$$

□

One may proceed completely analogously as to prove

Lemma 3.2.13. There is a pseudonatural transformation $l^- : \Lambda \to \mathbf{I}\Box 1$ with $l^-_{(*,W)} := I_W$ and $l^-_{(1_*,X)} := \lambda_X^{-1} \circ \rho_X \circ (\kappa_X^{-1} \otimes 1)$.

Indeed, we have the following adjunction:

Lemma 3.2.14. There are modifications

$$\epsilon : l^- \circ l \to 1_{\Box\circ(\mathbf{I}\times 1)},\ \epsilon_{(*,W)} := \lambda_{I_W}, \quad \eta : 1_\Lambda \to l \circ l^-,\ \eta_{(*,W)} := \lambda_{I_W}^{-1}$$
$$\tilde{\epsilon} : l \circ l^- \to 1_\Lambda,\ \tilde{\epsilon}_{(*,W)} := \lambda_{I_W}, \qquad \tilde{\eta} : 1_{\Box\circ(\mathbf{I}\times 1)} \to l^- \circ l,\ \tilde{\eta}_{(*,W)} := \lambda_{I_W}^{-1}$$

witnessing that l and l^- are adjoint equivalent.

Proof. The diagrams which have to commute in order to assert Lemma 3.2.14, cf. (A.1.7), comprise only κ, κ^{-1}, α, α^{-1}, λ, λ^{-1}, ρ and ρ^{-1}. Therefore they commute by Lemma 3.1.10. □

In introducing the next gadget, we write $P : \mathcal{LG} \times \mathbf{1} \to \mathcal{LG}$ for the projection to the first component. The following is proven along the same lines as the foregoing.

Lemma 3.2.15. There are pseudonatural transformations $r : \Box \circ (1 \times \mathbf{I}) \to P$ and $r^- : P \to \Box \circ (1 \times \mathbf{I})$ with

$$r_{(W,*)} := I_W,\ r_{(X,1_*)} := (1 \otimes \sigma_X) \circ \lambda_X^{-1} \circ \rho_X$$
$$r^-_{(W,*)} := I_W,\ r^-_{(X,1_*)} := \lambda_X^{-1} \circ \rho_X \circ (\sigma_X^{-1} \otimes 1).$$

These form an adjoint equivalence whose (co-)evaluations are all given in terms of λ_{I_W} and $\lambda_{I_W}^{-1}$.

Definition 3.2.16. The pseudonatural transformation $r : \Box \circ (1 \times \mathbf{I}) \to P$ of Lemma 3.2.15 is the *right unitor* for $\Box$.

We equip $\mathcal{LG}$ with the necessary modifications for l and r to qualify as unitors. We use the standard symbols for these modifications, cf. [G, S, SP1, Sta]. This conflicts with our use of the standard symbols for the structure isomorphisms of a bicategory. However, the context allows to distinguish the maps. In particular, while the components of the modifications are indexed by tuples of objects of $\mathcal{LG}$ and $*$ the structure morphisms are indexed by 1-morphisms in $\mathcal{LG}$.

Lemma 3.2.17. The following prescriptions describe invertible modifications

$$\begin{aligned}
&\lambda : 1_\Box \otimes (l \times 1) \to (l \otimes 1_{(1\times\Box)\circ\mathfrak{A}}) \circ (a \otimes 1_{(\mathbf{I}\times 1)\times 1}),\\
&\quad \lambda_{((*,V),W)} := \lambda^{-1}_{I_{V+W}} \circ \Box^{-1}_{(V,W)},\\
&\rho : r \otimes 1_{\Box\times 1} \to (1_\Box \otimes (1 \times r) \otimes 1_{\mathfrak{A}}) \circ (a \otimes 1_{(1\times 1)\times \mathbf{I}}),\\
&\quad \rho_{((V,W),*)} := (\Box_{(V,W)} \otimes 1_{I_{V+W}}) \circ \lambda^{-1}_{I_{V+W}},\\
&\mu : 1_\Box \otimes (r \times 1) \to (1_\Box \otimes (1 \times l) \otimes 1_{\mathfrak{A}}) \circ (a \otimes 1_{(1\times \mathbf{I})\times 1}),\\
&\quad \mu_{((V,*),W)} := \rho^{-1}_{I_V \Box I_W}.
\end{aligned}$$

Proof. The 2-morphisms displayed in defining λ, ρ and μ are invertible. In all three cases the diagrams which show that these 2-morphisms are components of modifications, cf. (A.1.7), commute by Lemma 3.1.10. □

Theorem 3.2.18. $\mathcal{LG}$ together with $\Box$, a, π, $(0, \emptyset)$, l, λ, r, ρ, μ as specified above forms a monoidal bicategory.

Proof. It remains to show that λ, μ and ρ satisfy the necessary equations depicted in (A.3.3) and (A.3.4). In both cases this follows from Lemma 3.1.10. □

3.3. The braiding, the syllepsis, and more

We begin by defining a braiding, cf. Definition A.4.1. Then we introduce its syllepsis and the two other modifications needed to define a symmetric monoidal bicategory, cf. Definition A.4.1. Finally, we prove that these data satisfy the conditions to make $\mathcal{LG}$ a symmetric monoidal bicategory.

3.3.1. The braiding

Before introducing a braiding for the monoidal product $\Box$ on $\mathcal{LG}$ we point out three special instances of Lemma 3.1.6. Recall our conventions from Table 3.1 and Notation 3.1.2.

Definition 3.3.1. Let U_1, U_2, V, W_1, W_2 be potentials, $U := U_1 + U_2$, $W := W_1 + W_2$, $X \in \mathcal{LG}(U, V)$ and $Y \in \mathcal{LG}(V, W)$. Then there are the following 2-isomorphisms in $\mathcal{LG}$ which are natural in X and Y.

$$\begin{aligned}
&\chi_l : \iota_{-U+W,-U+W_2+W_1}(Y \otimes_{\mathrm{k}[\boldsymbol{y}]} X) \to \iota_{-V+W,-V+W_2+W_1}(Y) \otimes_{\mathrm{k}[\boldsymbol{y}]} X,\\
&\chi_r : \iota_{-U+W,-U_2-U_1+W}(Y \otimes_{\mathrm{k}[\boldsymbol{y}]} X) \cong Y \otimes_{\mathrm{k}[\boldsymbol{y}]} \iota_{-U+V,-U_2-U_1+V}(X),
\end{aligned}$$

both specified by

$$f \otimes_{\mathrm{k}[\boldsymbol{y}]} e \mapsto f \otimes_{\mathrm{k}[\boldsymbol{y}]} e. \tag{3.3.1}$$

Indeed, the prescription $f \otimes_{\mathrm{k}[\boldsymbol{y}]} e \mapsto f \otimes_{\mathrm{k}[\boldsymbol{y}]} e$ respects the tensor products in the sense specified preceding Lemma 3.1.7. Thus, Lemma 3.1.6 applies.

Definition 3.3.2. Let $\widetilde{\boldsymbol{y}}$ be some permutation of $\boldsymbol{y}$, $X \in \mathcal{LG}(U,(\boldsymbol{y},V))$ and $Y \in \mathcal{LG}((\widetilde{\boldsymbol{y}},V),W)$. Then $\iota_{\widetilde{\boldsymbol{y}},\boldsymbol{y}}$, or $\tilde{\iota}$ if the context specifies the index, is the following canonical 2-isomorphism in $\mathcal{LG}$ which is natural in X and Y:

$$Y \otimes_{\mathrm{k}[\widetilde{\boldsymbol{y}}]} \iota_{((\boldsymbol{x},\boldsymbol{y}),-U+V),((\boldsymbol{x},\widetilde{\boldsymbol{y}}),-U+V)}(X) \to \iota_{((\widetilde{\boldsymbol{y}},\boldsymbol{z}),-V+W),((\boldsymbol{y},\boldsymbol{z})-V+W)}(Y) \otimes_{\mathrm{k}[\boldsymbol{y}]} X$$
$$f \otimes_{\mathrm{k}[\widetilde{\boldsymbol{y}}]} e \mapsto f \otimes_{\mathrm{k}[\boldsymbol{y}]} e. \tag{3.3.2}$$

The functors of restriction of scalars are precisely such that (3.3.2) respects the tensor products. Furthermore, the rings act in the same way on the modules underlying both sides of (3.3.2).

Remark 3.3.3. Note that there are no factors of (-1) in the formulas (3.3.1), (3.3.2) specifying the actions of the isomorphisms of matrix factorizations in Definition 3.3.1 and Definition 3.3.2 on elements of the underlying modules. Indeed, according to the formula (3.1.3) the isomorphisms of matrix factorizations in Lemma 3.1.6 multiply elements of the underlying modules with signs only if the order of factors in tensor products of matrix factorizations is changed. But neither in Definition 3.3.1 nor in Definition 3.3.2 are the two factors in the respective tensor products of matrix factorizations permuted.

Definition 3.3.4. For $X \in \mathcal{LG}(V_1,W_1)$, $Y \in \mathcal{LG}(V_2,W_2)$, $V := V_1 + V_2$, $W := W_2 + W_1$ the 2-isomorphism $\beta_{(X,Y)}$ which is natural in X and Y is

$$\iota_{-V_2-V_1+W,-V+W}(Y \Box X) \to \iota_{-V+W_1+W_2,-V+W}(X \Box Y), \tag{3.3.3}$$
$$y \otimes_{\mathrm{k}} x \mapsto (-1)^{|y||x|} x \otimes_{\mathrm{k}} y.$$

That $\beta_{(X,Y)}$ in Definition 3.3.4 is well-defined is ensured by the functors of restriction of scalars. They are such that both sides of (3.3.3) are modules over the same ring and the map (3.3.3) is compatible with the action of the latter on either side.

We move on to introduce the braiding in Lemma 3.3.5. Here we denote by $\tau : \mathcal{LG} \times \mathcal{LG} \to \mathcal{LG} \times \mathcal{LG}$ the strict 2-functor which permutes the entries of the tuples which are objects, 1- and 2-morphisms of $\mathcal{LG} \times \mathcal{LG}$, e.g. $(V,W) \mapsto (W,V)$.

Lemma 3.3.5. There is a pseudonatural transformation $b : \Box \to \Box \circ \tau$ given by the following:

1. For $U := V + W$ set $b_{(V,W)} := \iota_{-U+U,-U+W+V}(I_{V+W})$.

2. In the situation of Definition 3.3.4 $b_{(X,Y)}$ is defined as

$$\begin{aligned}(Y \Box X) \otimes b_{(V_1,V_2)} &\xrightarrow{\tilde{\iota}} \iota_{-V_2-V_1+W,-V+W}(Y \Box X) \otimes I_V \\ &\xrightarrow{\rho} \iota_{-V_2-V_1+W,-V+W}(Y \Box X) \\ &\xrightarrow{\beta_{(X,Y)}} \iota_{-V+W_1+W_2,-V+W}(X \Box Y) \\ &\xrightarrow{\iota(\lambda^{-1})} \iota_{-V+W_1+W_2,-V+W}(I_{W_1+W_2} \otimes (X \Box Y)) \\ &\xrightarrow{\chi_l} \iota_{-W_1-W_2+W_1+W_2,-W_1-W_2+W}(I_{W_1+W_2}) \otimes (X \Box Y)\end{aligned} \tag{3.3.4}$$

$$= b_{(W_1,W_2)} \otimes (X\square Y).$$

Definition 3.3.6. The pseudonatural transformation b in Lemma 3.3.5 is the *braiding* for $\square$.

Explicitly, our definition says that $(b_{(V,W)})_m$ is a free $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{z}, \boldsymbol{y}]$-module with the same underlying set as $(I_{V+W})_m$ and the $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{z}, \boldsymbol{y}]$-action given by precomposing the action of $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{y}, \boldsymbol{z}]$ on $(I_{V+W})_m$ with the canonical ring isomorphism $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{z}, \boldsymbol{y}] \cong \mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{y}, \boldsymbol{z}]$. In particular every basis of $(I_{V+W})_m$, e.g. the one in (2.5.5), is a basis of $(b_{(V,W)})_m$. Moreover, $d_{b_{(V,W)}}$ is $d_{I_{V+W}}$ interpreted as a map of $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{z}, \boldsymbol{y}]$-modules rather than as map of $\mathrm{k}[\boldsymbol{y}', \boldsymbol{z}', \boldsymbol{y}, \boldsymbol{z}]$-modules.

We spell out the action of the 2-morphism $b_{(X,Y)}$ on a general basis element of $((Y\square X)\otimes b_{(V_1,V_2)})_m$ explicitly. For this let $\{e_a\}_{a\in I}$ and $\{f_b\}_{b\in J}$, I, J some index sets, be bases of X_m and Y_m, respectively. We have

$$\begin{aligned}(f_b \otimes_{\mathrm{k}} e_a) \otimes_{\mathrm{k}[\boldsymbol{y}]} \theta_1^{j_1} \dots \theta_{m_1+m_2}^{j_{m_1+m_2}} &\xmapsto{\rho\circ\tilde{\iota}} \delta_{j_1,0} \dots \delta_{j_{m_1+m_2},0}.(f_b \otimes_{\mathrm{k}} e_a) \\ &\xmapsto{\beta_{(X,Y)}} (-1)^{|e_a||f_b|}\delta_{j_1,0} \dots \delta_{j_{m_1+m_2},0}.(e_a \otimes_{\mathrm{k}} f_b) \\ &\xmapsto{\chi_l\circ\iota(\lambda^{-1})} (-1)^{|e_a||f_b|}\delta_{j_1,0} \dots \delta_{j_{m_1+m_2},0}.\lambda^{-1}_{X\square Y}(e_a \otimes_{\mathrm{k}} f_b).\end{aligned} \tag{3.3.5}$$

Using the formula (2.5.10) for λ^{-1} and setting $\widetilde{\boldsymbol{z}} = (\boldsymbol{z}_1, \boldsymbol{z}_2)$, the last line of (3.3.5) becomes

$$(-1)^{|e_a||f_b|}\delta_{j_1,0} \dots \delta_{j_{m_1+m_2},0}.\sum_{l\geq 0}\sum_{i_1<\dots<i_l}\sum_{(a',b')\in I\times J}$$
$$\{\partial^{\widetilde{\boldsymbol{z}},\widetilde{\boldsymbol{z}}'}_{[i_l]} d_{X\square Y} \dots \partial^{\widetilde{\boldsymbol{z}},\widetilde{\boldsymbol{z}}'}_{[i_1]} d_{X\square Y}\}_{e_{a'}\otimes f_{b'}, e_a\otimes f_b}.\theta_{i_1}\dots\theta_{i_l} \otimes_{\mathrm{k}[\widetilde{\boldsymbol{z}}]} (e_{a'} \otimes_{\mathrm{k}} f_{b'}),$$

Note that independently of the 1-morphisms X or Y the composition (3.3.4) is a 2-morphism to which Lemma 3.1.10 applies.

Proof of Lemma 3.3.5. First, by its definition, for all 1-morphisms X and Y the map $b_{(X,Y)}$ is an isomorphism which is natural in X and Y. Indeed, $\tilde{\iota}$, $\beta_{(X,Y)}$ and χ_l are natural by definition. Also, that ρ is natural in $\iota_{-V_2-V_1+W,-V+W}(Y\square X)$ implies that it is natural in both X and Y since $\iota_{-V_2-V_1+W,-V+W}(-\square-)$ is functorial in both arguments. Similarly, $\lambda^{-1}_{X\square Y}$ is natural in $X\square Y$ and therefore in X and Y since $\square$ is functorial in both arguments. Consequently $\iota(\lambda^{-1})$ is natural in X and Y.

Secondly, $b_{(X,Y)}$ is compatible with compositions, cf. (A.1.5), since by Lemma 3.1.10 for $X_1 \in \mathcal{LG}(U_1, V_1)$, $Y_1 \in \mathcal{LG}(V_1, W_1)$ and $X_2 \in \mathcal{LG}(U_2, V_2)$, $Y_2 \in \mathcal{LG}(V_2, W_2)$ the following two concatenations in which we suppress associators α for the horizontal composition $\otimes_{\mathrm{k}[-]}$ of $\mathcal{LG}$ result in the same morphism:

$$\begin{aligned}((Y_2\square Y_1)\otimes(X_2\square X_1))\otimes b_{(U_1,U_2)} &\xrightarrow{(\square\circ\tau)\otimes 1} ((Y_2\otimes X_2)\square(Y_1\otimes X_1))\otimes b_{(U_1,U_2)} \\ &\xrightarrow{b_{(Y_1\otimes X_1, Y_2\otimes X_2)}} b_{(W_1,W_2)}\otimes((Y_1\otimes X_1)\square(Y_2\otimes X_2)),\end{aligned}$$

$$
\begin{aligned}
((Y_2\Box Y_1)\otimes(X_2\Box X_1))\otimes b_{(U_1,U_2)} &\xrightarrow{1\otimes b_{(X_1,X_2)}} (Y_2\Box Y_1)\otimes(b_{(V_1,V_2)}\otimes(X_1\Box X_2))\\
&\xrightarrow{b_{(Y_1,Y_2)}\otimes 1} (b_{(W_1,W_2)}\otimes(Y_1\Box Y_2))\otimes(X_1\Box X_2)\\
&\xrightarrow{1\otimes\Box} b_{(W_1,W_2)}\otimes((Y_1\otimes X_1)\Box(Y_2\otimes X_2)).
\end{aligned}
$$

Indeed, as $\tilde{\iota}$, β and χ are defined as special cases of Lemma 3.1.6 $b_{(X,Y)}$ is a map to which Lemma 3.1.10 can be applied.

Likewise, it follows from Lemma 3.1.10 that b respects unit 1-morphisms, i.e. that the following two composites of 2-morphisms agree, cf. (A.1.6):

$$
\begin{aligned}
&I_{W+V}\otimes b_{(V,W)} \xrightarrow{\lambda} b_{(V,W)} \xrightarrow{\rho^{-1}} b_{(V,W)}\otimes I_{V+W} \xrightarrow{1\otimes\Box} b_{(V,W)}\otimes(I_V\Box I_W),\\
&I_{W+V}\otimes b_{(V,W)} \xrightarrow{(\Box\circ\tau)\otimes 1} (I_W\Box I_V)\otimes b_{(V,W)} \xrightarrow{b_{(I_V,I_W)}} b_{(V,W)}\otimes(I_V\Box I_W).
\end{aligned}
$$

□

Remark 3.3.7. The 1-morphism component of the braiding b of the potential $W=z^N$, $N\in\mathbb{N}_{\geq 2}$, cf. Examples 2.1, with itself is the matrix factorization

$$b_{(W,W)} = \iota_{-z_1^N-z_2^N+z_3^N+z_4^N,-z_1^N-z_2^N+z_4^N+z_3^N}(I_{W+W}), \tag{3.3.6}$$

where we use indices rather than primes on the variables. This is unambiguous since we consider a polynomial in a single variable. Applying $\iota(\Box_{(W,W)})$ to (3.3.6) yields $\iota(I_W\Box I_W)$. This is the matrix factorization which is assigned to the "virtual crossing" in [KR2, (A.7)], where the isomorphism of categories ι is left implicit. This matrix factorization features prominently in the complexes of matrix factorizations [KR2, (A.53), (A.54)]. According to [KR1, §10] these complexes give a braided monoidal structure on a bicategory whose objects are potentials and 1-morphisms are suitable complexes of matrix factorizations. The 2-morphisms of this bicategory are equivalence classes of homomorphisms of such complexes.

3.3.2. Three modifications for a braided monoidal bicategory

According to the "algebraic" spirit of the definition of a braiding in [SP1] b is an adjoint equivalence. However, as noted in [SP1, Remark 2.4] since we are dealing with a symmetric monoidal bicategory, we need not specify an adjoint to b separately. It is given by the same morphisms as b itself:

$$b^-:\Box\circ\tau\to\Box,\quad b^-_{(V,W)} := b_{(W,V)},\quad b^-_{(X,Y)} := b_{(Y,X)}.$$

Thus, it is enough to provide the following modification. As we verify later on this satisfies the required conditions to simultaneously play the role of a syllepsis. According to [SP1, Remark 2.4] this equivalently verifies that b and b^- form an adjoint equivalence.

Lemma 3.3.8. Set $U := V + W$. There is an invertible modification $\sigma : 1_\Box \to b^- \circ b$ with $\sigma_{(V,W)}$ defined as

$$\begin{aligned} I_{V+W} &= \iota_{-U+W+V,-U+U}(\iota_{-U+U,-U+W+V}(I_{V+W})) \\ &\xrightarrow{\iota(\lambda^{-1})} \iota_{-U+W+V,-U+U}(I_{W+V} \otimes \iota_{-U+U,-U+W+V}(I_{V+W})) \\ &\xrightarrow{\chi_l} \iota_{-W-V+W+V,-W-V+U}(I_{W+V}) \otimes \iota_{-U+U,-U+W+V}(I_{V+W}) \\ &= b_{(W,V)} \otimes b_{(V,W)} \\ &= b^-_{(V,W)} \otimes b_{(V,W)}. \end{aligned}$$

Definition 3.3.9. The modification σ in Lemma 3.3.8 is the *syllepsis*.

Proof of Lemma 3.3.8. The components of σ are defined as isomorphisms which makes σ invertible. It is a modification since the following two sequences of maps are equal, where $X \in \mathcal{LG}(V_1, V_2)$, $Y \in \mathcal{LG}(W_1, W_2)$, cf. (A.1.7):

$$\begin{aligned} (X \Box Y) \otimes I_{V_1+W_1} &\xrightarrow{\lambda^{-1}\circ\rho} I_{V_2+W_2} \otimes (X \Box Y) \\ &\xrightarrow{\sigma\otimes 1} (b_{(W_2,V_2)} \otimes b_{(V_2,W_2)}) \otimes (X \Box Y), \end{aligned}$$

$$\begin{aligned} (X \Box Y) \otimes I_{V_1+W_1} &\xrightarrow{1\otimes\sigma} (X \Box Y) \otimes (b_{(W_1,V_1)} \otimes b_{(V_1,W_1)}) \\ &\xrightarrow{(b\otimes 1)\circ\alpha^{-1}} (b_{(W_2,V_2)} \otimes (Y \Box X)) \otimes b_{(V_1,W_1)} \\ &\xrightarrow{(1\otimes b)\circ\alpha} b_{(W_2,V_2)} \otimes (b_{(V_2,W_2)} \otimes (X \Box Y)) \\ &\xrightarrow{\alpha^{-1}} (b_{(W_2,V_2)} \otimes b_{(V_2,W_2)}) \otimes (X \Box Y). \end{aligned}$$

This follows from Lemma 3.1.10 as also the components of σ are 2-morphisms to which this Lemma applies. □

In introducing the final two modifications completing a symmetric monoidal structure on $\mathcal{LG}$ we make use of the following 2-isomorphisms.

Definition 3.3.10. Set $U_1 := W + V$, $U_2 := V + W$. Then the isomorphism $\mathscr{I}_{(V,W)} : \iota_{-U_1+U_1,-U_2+U_1}(I_{U_1}) \to b_{(V,W)}$ is

$$\begin{aligned} \iota_{-U_1+U_1,-U_2+U_1}(I_{U_1}) &\xrightarrow{\iota(\Box)} \iota_{-U_1+U_1,-U_2+U_1}(I_W \Box I_V) \\ &\xrightarrow{\beta} \iota_{-U_2+U_2,-U_2+U_1}(I_V \Box I_W) \\ &\xrightarrow{\iota(\Box^{-1})} \iota_{-U_2+U_2,-U_2+U_1}(I_{V+W}) \\ &= b_{(V,W)}. \end{aligned}$$

Set $T_1 := U + V + W$, $T_2 := V + U + W$ and $T_3 := U + W + V$. Then the isomorphism $\mathscr{I}_{(U,V,W),(V,U,W)} : \iota_{-T_1+T_1,-T_2+T_2}(I_{T_1}) \to I_{T_2}$ is

$$\begin{aligned}\iota_{-T_1+T_1,-T_2+T_2}(I_{T_1}) \xrightarrow{\iota((\square_{(U,V)}\square 1)\circ\square_{(U+V,W)})} & \iota_{-T_1+T_1,-T_2+T_2}((I_U \square I_V)\square I_W) \\ & \cong (I_V \square I_U)\square I_W \\ \xrightarrow{\square^{-1}_{(V+U,W)}\circ(\square^{-1}_{(V,U)}\square 1)} & I_{T_2},\end{aligned}$$

where the second isomorphism follows from Lemma 3.1.6. The isomorphism $\mathscr{I}_{(U,V,W),(U,W,V)} : \iota_{-T_1+T_1,-T_3+T_3}(I_{T_1}) \to I_{T_3}$ is

$$\begin{aligned}\iota_{-T_1+T_1,-T_3+T_3}(I_{T_1}) \xrightarrow{\iota((1\square\ \square_{(V,W)})\circ\square_{(U,V+W)})} & \iota_{-T_1+T_1,-T_3+T_3}(I_U \square(I_V \square I_W)) \\ & \cong I_U \square(I_W \square I_V) \\ \xrightarrow{\square^{-1}_{(U,W+V)}\circ(1\square\ \square^{-1}_{(W,V)})} & I_{T_3},\end{aligned}$$

where the second isomorphism is an instance of Lemma 3.1.6.

The following completes a symmetric monoidal structure, cf. Definition A.4.1, on $\mathcal{LG}$:

Lemma 3.3.11. There are invertible modifications $R : a\circ b\circ a \to (1\square b)\circ a\circ(b\square 1)$ and $S : a^- \circ b \circ a^- \to (b\square 1) \circ a^- \circ (1\square b)$ with components $R_{((U,V),W)}$:

$$a_{((V,W),U)} \otimes b_{(U,V+W)} \otimes a_{((U,V),W)}$$

$$\|$$

$$I_{V+W+U} \otimes \iota_{-U-V-W+U+V+W,-U-V-W+V+W+U}(I_{U+V+W}) \otimes I_{U+V+W}$$

$$\Big\downarrow \square_{(V,W+U)} \otimes 1 \otimes \square_{(U+V,W)}$$

$$(I_V \square I_{W+U}) \otimes \iota_{-U-V-W+U+V+W,-U-V-W+V+W+U}(I_{U+V+W}) \otimes (I_{U+V}\square I_W)$$

$$\Big\downarrow \cong$$

$$\iota_{-V-U-W+V+W+U,-V-W-U+V+W+U}(I_V \square \iota_{-W-U+W+U,-U-W+W+U}(I_{W+U}))$$

$$\otimes \iota_{-U-V-W+U+V+W,-U-V-W+V+W+U}(I_{U+V+W})$$

$$\otimes \iota_{-U-V-W+V+U+W,-U-V-W+U+V+W}(\iota_{-U-V+U+V,-U-V+V+U}(I_{U+V})\square I_W)$$

$$
\Big\downarrow{\scriptstyle (\tilde{\iota}^{-1}\otimes 1)\circ(1\otimes\tilde{\iota})}
$$

$$
(I_V\Box\iota_{-W-U+W+U,-U-W+W+U}(I_{W+U}))
$$

$$
\otimes\iota_{-U-V-W+U+V+W,-V-U-W+V+U+W}(I_{U+V+W}))\otimes(b_{(U,V)}\Box I_W)
$$

$$
\Big\downarrow{\scriptstyle (1\Box\mathscr{I}_{(U,W)})\otimes\mathscr{I}_{(U,V,W),(V,U,W)}\otimes 1}
$$

$$
(I_V\Box b_{(U,W)})\otimes I_{V+U+W}\otimes(b_{(U,V)}\Box I_W)
$$

$$
\|
$$

$$
(I_V\Box b_{(U,W)})\otimes a_{((V,U),W)}\otimes(b_{(U,V)}\Box I_W)
$$

and $S_{((U,V),W)}$:

$$
a^{-}_{((W,U),V)}\otimes b_{(U+V,W)}\otimes a^{-}_{((U,V),W)}
$$

$$
\|
$$

$$
I_{W+U+V}\otimes\iota_{-U-V-W+U+V+W,-U-V-W+W+U+V}(I_{U+V+W})\otimes I_{U+V+W}
$$

$$
\Big\downarrow{\scriptstyle \Box_{(W+U,V)}\otimes 1\otimes\Box_{(U,V+W)}}
$$

$$
(I_{W+U}\Box I_V)\otimes\iota_{-U-V-W+U+V+W,-U-V-W+W+U+V}(I_{U+V+W})\otimes(I_U\Box I_{V+W})
$$

$$
\Big\downarrow{\scriptstyle \cong}
$$

$$
\iota_{-U-W-V+W+U+V,-W-U-V+W+U+V}(\iota_{-W-U+W+U,-U-W+W+U}(I_{W+U})\Box I_V)
$$

$$
\otimes\iota_{-U-V-W+U+V+W,-U-V-W+W+U+V}(I_{U+V+W})
$$

$$
\otimes\iota_{-U-V-W+U+W+V,-U-V-W+U+V+W}(I_U\Box\iota_{-V-W+V+W,-V-W+W+V}(I_{V+W}))
$$

$$
\Big\downarrow{\scriptstyle (\tilde{\iota}^{-1}\otimes 1)\circ(1\otimes\tilde{\iota})}
$$

$$
(\iota_{-W-U+W+U,-U-W+W+U}(I_{W+U})\Box I_V)
$$

$$
\otimes\iota_{-U-V-W+U+V+W,-U-W-V+U+W+V}(I_{U+V+W})\otimes(I_U\Box b_{(V,W)})
$$

$$
\Big\downarrow{\scriptstyle (\mathscr{I}_{(U,W)}\Box 1)\otimes\mathscr{I}_{(U,V,W),(U,W,V)}\otimes 1}
$$

$$
(b_{(U,W)}\Box I_V)\otimes I_{U+W+V}\otimes(I_U\Box b_{(V,W)})
$$

$$\parallel$$

$$(b_{(U,W)}\square I_V)\otimes a^{-}_{((U,W),V)}\otimes (I_U\square b_{(V,W)})$$

where the unlabeled isomorphisms are instances of Lemma 3.1.6.

Using the basis (2.5.5) for all unit matrix factorizations involved, in particular also for the 1-morphism components of a and b, the action of $R_{((U,V),W)}$ is given on a general basis element by

$$\begin{aligned}
&\theta_1^{\alpha_1}\dots\theta_{m+n+l}^{\alpha_{m+n+l}}\otimes\theta_1^{\beta_1}\dots\theta_{l+m+n}^{\beta_{l+m+n}}\otimes\theta_1^{\gamma_1}\dots\theta_{l+m+n}^{\gamma_{l+m+n}}\\
\mapsto\ &\Big(\theta_1^{\alpha_1}\dots\theta_m^{\alpha_m}\otimes_{\mathrm{k}}(-1)^{\sum_{p=1}^{n}\alpha_{m+p}\sum_{q=1}^{l}\alpha_{m+n+q}}\theta_1^{\alpha_{m+n+1}}\dots\theta_l^{\alpha_{m+n+l}}\theta_{l+1}^{\alpha_{m+1}}\dots\theta_{l+n}^{\alpha_{m+n}}\Big)\\
&\otimes(-1)^{\sum_{r=1}^{l}\beta_r\sum_{s=1}^{m}\beta_{l+s}}\theta_1^{\beta_{l+1}}\dots\theta_m^{\beta_{l+m}}\theta_{m+1}^{\beta_1}\dots\theta_{m+l}^{\beta_l}\theta_{m+l+1}^{\beta_{l+m+1}}\dots\theta_{m+l+n}^{\beta_{l+m+n}}\\
&\otimes(\theta_1^{\gamma_1}\dots\theta_{l+m}^{\gamma_{l+m}}\otimes_{\mathrm{k}}\theta_1^{\gamma_{l+m+1}}\dots\theta_n^{\gamma_{l+m+n}}).
\end{aligned}$$

Similarly, $S_{((U,V),W)}$ acts as

$$\begin{aligned}
&\theta_1^{\alpha_1}\dots\theta_{n+l+m}^{\alpha_{n+l+m}}\otimes\theta_1^{\beta_1}\dots\theta_{l+m+n}^{\beta_{l+m+n}}\otimes\theta_1^{\gamma_1}\dots\theta_{l+m+n}^{\gamma_{l+m+n}}\\
\mapsto\ &\Big((-1)^{\sum_{p=1}^{n}\alpha_p\sum_{q=1}^{l}\alpha_{n+q}}\theta_1^{\alpha_{n+1}}\dots\theta_l^{\alpha_{n+l}}\theta_{l+1}^{\alpha_1}\dots\theta_{l+n}^{\alpha_n}\otimes_{\mathrm{k}}\theta_1^{\alpha_{n+l+1}}\dots\theta_m^{\alpha_{n+l+m}}\Big)\\
&\otimes(-1)^{\sum_{r=1}^{m}\beta_{l+r}\sum_{t=1}^{n}\beta_{l+m+t}}\theta_1^{\beta_1}\dots\theta_l^{\beta_l}\theta_{l+1}^{\beta_{l+m+1}}\dots\theta_{l+n}^{\beta_{l+m+n}}\theta_{l+n+1}^{\beta_{l+1}}\dots\theta_{l+n+m}^{\beta_{l+m}}\\
&\otimes(\theta_1^{\gamma_1}\dots\theta_l^{\gamma_l}\otimes_{\mathrm{k}}\theta_1^{\gamma_{l+1}}\dots\theta_{m+n}^{\gamma_{l+m+n}}).
\end{aligned}$$

Proof of Lemma 3.3.11. Since we have defined both R and S via components which are compositions of isomorphisms, both R and S are invertible. We show that they are modifications. For R, this means that the two sequences of maps displayed momentarily are equal, cf. (A.1.7). In these we do not spell out bracketings and associators for the horizontal composition $\otimes$ of $\mathcal{LG}$. Also, we use $\tilde{\tilde{\imath}} := (\tilde{\imath}^{-1}\otimes 1)\circ(1\otimes\tilde{\imath})$.

$$(Y\square(Z\square X))\otimes a_{((V_1,W_1),U_1)}\otimes b_{(U_1,V_1+W_1)}\otimes a_{((U_1,V_1),W_1)}$$

$$\Big\downarrow\ a_{((Y,Z),X)}\otimes 1\otimes 1$$

$$a_{((V_2,W_2),U_2)}\otimes((Y\square Z)\square X)\otimes b_{(U_1,V_1+W_1)}\otimes a_{((U_1,V_1),W_1)}$$

$$\Big\downarrow\ 1\otimes b_{(X,Y\square Z)}\otimes 1$$

$$a_{((V_2,W_2),U_2)}\otimes b_{(U_2,V_2+W_2)}\otimes(X\square(Y\square Z))\otimes a_{((U_1,V_1),W_1)}$$

$$\Big\downarrow{\scriptstyle 1\otimes 1\otimes a_{((X,Y),Z)}}$$

$$a_{((V_2,W_2),U_2)}\otimes b_{(U_2,V_2+W_2)}\otimes a_{((U_2,V_2),W_2)}\otimes((X\Box Y)\Box Z)$$

$$\Big\downarrow{\scriptstyle (((1\Box\mathscr{I})\otimes\mathscr{I}\otimes 1)\circ\tilde{i}\circ(\Box\otimes 1\otimes\Box))\otimes 1}$$

$$(I_{V_2}\Box b_{(U_2,W_2)})\otimes a_{((V_2,U_2),W_2)}\otimes(b_{(U_2,V_2)}\Box I_{W_2})\otimes((X\Box Y)\Box Z),$$

$$(Y\Box(Z\Box X))\otimes a_{((V_1,W_1),U_1)}\otimes b_{(U_1,V_1+W_1)}\otimes a_{((U_1,V_1),W_1)}$$

$$\Big\downarrow{\scriptstyle 1\otimes(((1\Box\mathscr{I})\otimes\mathscr{I}\otimes 1)\circ\tilde{i}\circ(\Box\otimes 1\otimes\Box))}$$

$$(Y\Box(Z\Box X))\otimes(I_{V_1}\Box b_{(U_1,W_1)})\otimes a_{((V_1,U_1),W_1)}\otimes(b_{(U_1,V_1)}\Box I_{W_1})$$

$$\Big\downarrow{\scriptstyle (\Box\circ((\lambda^{-1}\circ\rho)\Box b_{(X,Z)})\circ\Box)\otimes 1\otimes 1}$$

$$(I_{V_2}\Box b_{(U_2,W_2)})\otimes(Y\Box(X\Box Z))\otimes a_{((V_1,U_1),W_1)}\otimes(b_{(U_1,V_1)}\Box I_{W_1})$$

$$\Big\downarrow{\scriptstyle 1\otimes a_{((Y,X),Z)}\otimes 1}$$

$$(I_{V_2}\Box b_{(U_2,W_2)})\otimes a_{((V_2,U_2),W_2)}\otimes((Y\Box X)\Box Z)\otimes(b_{(U_1,V_1)}\Box I_{W_1})$$

$$\Big\downarrow{\scriptstyle 1\otimes 1\otimes(\Box\circ(b_{(X,Y)}\Box(\lambda^{-1}\circ\rho))\circ\Box)}$$

$$(I_{V_2}\Box b_{(U_2,W_2)})\otimes a_{((V_2,U_2),W_2)}\otimes(b_{(U_2,V_2)}\Box I_{W_2})\otimes((X\Box Y)\Box Z).$$

These maps (including the associators not displayed) are equal by Lemma 3.1.10.

One can show analogously that the diagram exhibiting S as a modification, cf. (A.1.7), commutes. $\Box$

3.3.3. The theorem

Theorem 3.3.12. $\mathcal{LG}$ equipped with the above structure is a symmetric monoidal bicategory.

Proof. It remains to show that the data specified above fulfill the identities of pasting diagrams

$$\text{(A.4.1)} = \text{(A.4.2)},\ \text{(A.4.3)} = \text{(A.4.4)},\ \text{(A.4.5)} = \text{(A.4.6)},\ \text{(A.4.7)} = \text{(A.4.8)}$$

and the equalities in (A.4.9), (A.4.10) and (A.4.11). Note that all morphisms which constitute the symmetric monoidal structure of $\mathcal{LG}$ are maps to which Lemma 3.1.10 applies. Moreover, the equations which these morphisms have to satisfy do not include any other maps. Therefore it follows from Lemma 3.1.10 that they are satisfied. □

4. Fully dualizable objects of $\mathcal{LG}$

In this chapter we show that every object of the symmetric monoidal bicategory $\mathcal{LG}$ specified in Chapter 3 is fully dualizable. Our definition of the structure exhibiting every object of $\mathcal{LG}$ as dualizable adapts the discussion in [CR2, §4] and is in accord with [KR1].[1] The definition of a fully dualizable object in a symmetric monoidal bicategory is included in Appendix A.5.

We keep using the notation introduced at the beginning of Section 3.1.1, cf. Table 3.1 and the surrounding text.

Definition 4.1. For $(\boldsymbol{z}, W) \in \mathcal{LG}$ its *dual object* $(\boldsymbol{z}, W)^* \in \mathcal{LG}$ is $(\boldsymbol{z}, -W)$.

If $W \in \mathrm{k}[\boldsymbol{z}]$ is a potential it follows from Definition 2.1.1 being indifferent to the sign of W that so is $-W \in \mathrm{k}[\boldsymbol{z}]$. Hence Definition 4.1 is well-defined.

To justify Definition 4.1 we introduce (co-)evaluation 1-morphisms shortly. First, however, we clarify objects of which categories these 1-morphisms are:

$$\begin{aligned} \mathcal{LG}(\square(W,-W),0) &= \mathcal{LG}(W(\boldsymbol{z}) - W(\boldsymbol{z}'),0) \\ &= \mathrm{hmf}^{\oplus}_{-(W(\boldsymbol{z})-W(\boldsymbol{z}'))+0} = \mathrm{hmf}^{\oplus}_{-W(\boldsymbol{z})+W(\boldsymbol{z}')}, \\ \mathcal{LG}(0,\square(-W,W)) &= \mathcal{LG}(0,-W(\boldsymbol{z})+W(\boldsymbol{z}')) = \mathrm{hmf}^{\oplus}_{-W(\boldsymbol{z})+W(\boldsymbol{z}')}. \end{aligned} \tag{4.1}$$

Note that (4.1) exemplifies that saying that a matrix factorization X is a 1-morphism in $\mathcal{LG}(V,W)$ includes two pieces of information: $X \in \mathrm{hmf}^{\oplus}_{-V+W}$ and that X has source V and target W. Conversely, a given matrix factorization $X \in \mathrm{hmf}^{\oplus}_{-V+W}$ can be viewed as a 1-morphism in $\mathcal{LG}$ in several ways, e.g. also as $X \in \mathcal{LG}(0,-V+W)$. To accommodate this we use the following notation. For $X \in \mathcal{LG}(V,W)$ we denote by $X_{\mathrm{mf}} \in \mathrm{hmf}^{\oplus}_{-V+W}$ its underlying matrix factorization (not viewed as a 1-morphism and therefore without source and target objects).

Definition 4.2. The 1-morphism $\mathrm{ev}_W \in \mathcal{LG}(W(\boldsymbol{z}) - W(\boldsymbol{z}'),0)$ has $(\mathrm{ev}_W)_{\mathrm{mf}} := (I_W)_{\mathrm{mf}}$ and $\mathrm{coev}_W \in \mathcal{LG}(0,-W(\boldsymbol{z})+W(\boldsymbol{z}'))$ has $(\mathrm{coev}_W)_{\mathrm{mf}} := (I_W)_{\mathrm{mf}}$.

We prepare for the cusp isomorphisms, cf. Definition A.5.1, proving that coev_W and ev_W exhibit W^* as a dual object to W by introducing some isomorphisms of matrix factorizations. The following Definition 4.3 and Definition 4.4 are well-defined by Lemma 3.1.6.

[1] The figure [KR1, Figure 11] shows string diagrams for duals of objects in the homotopy category of $\mathcal{LG}$. These duals are inherited from the duals of objects in $\mathcal{LG}$ which we define.

Definition 4.3. Let $X_1 \in \mathcal{LG}(0, \square(V, W_1))$, $X_2 \in \mathcal{LG}(U_1, U_2)$, $Y_2 \in \mathcal{LG}(W_1, W_2)$, $Y_1 \in \mathcal{LG}(\square(U_2, V), 0)$ and define

$\tilde{X_1} \in \mathcal{LG}(-V, W_1)$ by $(\tilde{X_1})_{\mathrm{mf}} := (X_1)_{\mathrm{mf}}$ and $\tilde{Y_1} \in \mathcal{LG}(U_2, -V)$ by $(\tilde{Y_1})_{\mathrm{mf}} := (Y_1)_{\mathrm{mf}}$.

Then $\mathcal{C}_{Y_2,X_1,Y_1,X_2}$ is the following isomorphism in $\mathrm{hmf}^{\oplus}_{-U_1+W_2}$:

$$\begin{aligned}(Y_1 \square Y_2) \otimes_{\mathrm{k}[\boldsymbol{x}_2,\boldsymbol{y},\boldsymbol{z}_1]} (X_2 \square X_1) &\to (Y_2 \otimes_{\mathrm{k}[\boldsymbol{z}_1]} \tilde{X_1}) \otimes_{\mathrm{k}[\boldsymbol{y}]} (\tilde{Y_1} \otimes_{\mathrm{k}[\boldsymbol{x}_2]} X_2)\\ (g \otimes_k h) \otimes_{\mathrm{k}[\boldsymbol{x}_2,\boldsymbol{y},\boldsymbol{z}_1]} (f \otimes_{\mathrm{k}} e) &\mapsto (-1)^{|g||h|+|e|(|g|+|f|)}(h \otimes_{\mathrm{k}[\boldsymbol{z}_1]} e) \otimes_{\mathrm{k}[\boldsymbol{y}]} (g \otimes_{\mathrm{k}[\boldsymbol{x}_2]} f).\end{aligned}$$

The prescription defining $\mathcal{C}_{Y_2,X_1,Y_1,X_2}$ respects the $\mathrm{k}[\boldsymbol{x}_1, \boldsymbol{z}_2]$-module structures since these are such that in both the source and target of $\mathcal{C}_{Y_2,X_1,Y_1,X_2}$ scalars in $\mathrm{k}[\boldsymbol{x}_1]$ multiply f and those in $\mathrm{k}[\boldsymbol{z}_2]$ multiply h. Moreover the assignment defining $\mathcal{C}_{Y_2,X_1,Y_1,X_2}$ respects the tensor products in the sense introduced preceding Lemma 3.1.7.

Definition 4.4. Let $X_1 \in \mathcal{LG}(U_1, U_2)$, $X_2 \in \mathcal{LG}(0, \square(W_1, V))$, $Y_1 \in \mathcal{LG}(W_1, W_2)$, $Y_2 \in \mathcal{LG}(\square(V, U_2), 0)$ and define

$$\begin{aligned}\tilde{X_2} \in \mathcal{LG}(-V, W_1) &\text{ by } (\tilde{X_2})_{\mathrm{mf}} := \iota_{W_1+V,V+W_1}((X_2)_{\mathrm{mf}}),\\ \tilde{Y_2} \in \mathcal{LG}(U_2, -V) &\text{ by } (\tilde{Y_2})_{\mathrm{mf}} := \iota_{-V-U_2,-U_2-V}((Y_2)_{\mathrm{mf}}),\end{aligned}$$

where we use Notation 3.1.2. Then $\widehat{\mathcal{C}}_{Y_1,X_2,Y_2,X_1}$ is the following isomorphism in $\mathrm{hmf}^{\oplus}_{-U_1+W_2}$:

$$\begin{aligned}(Y_1 \square Y_2) \otimes_{\mathrm{k}[\boldsymbol{z}_1,\boldsymbol{y},\boldsymbol{x}_2]} (X_2 \square X_1) &\to (Y_1 \otimes_{\mathrm{k}[\boldsymbol{z}_1]} \tilde{X_2}) \otimes_{\mathrm{k}[\boldsymbol{y}]} (\tilde{Y_2} \otimes_{\mathrm{k}[\boldsymbol{x}_2]} X_1)\\ (g \otimes_{\mathrm{k}} h) \otimes_{\mathrm{k}[\boldsymbol{z}_1,\boldsymbol{y},\boldsymbol{x}_2]} (f \otimes_{\mathrm{k}} e) &\mapsto (-1)^{|h||f|}(g \otimes_{\mathrm{k}[\boldsymbol{z}_1]} f) \otimes_{\mathrm{k}[\boldsymbol{y}]} (h \otimes_{\mathrm{k}[\boldsymbol{x}_2]} e).\end{aligned}$$

The prescription defining $\widehat{\mathcal{C}}_{Y_1,X_2,Y_2,X_1}$ is a module map since the functors of restriction of scalars are such that scalars in $\mathrm{k}[\boldsymbol{x}_1] \subset \mathrm{k}[\boldsymbol{x}_1, \boldsymbol{z}_2]$ multiply e while those in $\mathrm{k}[\boldsymbol{z}_2] \subset \mathrm{k}[\boldsymbol{x}_1, \boldsymbol{z}_2]$ multiply g on both sides and also the tensor products are respected.

There is one more ingredient of the cusp isomorphism in Lemma 4.6 which we introduce separately:

Lemma 4.5. The matrix factorizations $\iota_{-W(\boldsymbol{z}')+W(\boldsymbol{z}),W(\boldsymbol{z})-W(\boldsymbol{z}')}(I_W)$ and I_{-W} are isomorphic in $\mathrm{hmf}^{\oplus}_{W(\boldsymbol{z})-W(\boldsymbol{z}')}$.

Proof. Rather than providing an explicit isomorphism we prove Lemma 4.5 using some facts from the literature on matrix factorizations. The first of these is that [CR1, §2.4], cf. [KR1, Proposition 23], exhibits unit matrix factorizations $\widetilde{I_W} \in \mathcal{LG}(W, W)$ of the form

$$(\widetilde{I_W})_{\mathrm{mf}} := \bigotimes_{i=1}^{n} X_i,\ X_i := \big(\mathrm{k}[\boldsymbol{z}', \boldsymbol{z}] \oplus \mathrm{k}[\boldsymbol{z}', \boldsymbol{z}]\theta_i,\ (z_i - z_i').\theta_i^* + \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'}(W).\theta_i \wedge (-)\big)$$
$$\in \mathrm{hmf}^{\oplus}_{\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}],W(z_1',\ldots,z_{i-1}',z_i,\ldots,z_n)-W(z_1',\ldots,z_i',z_{i+1},\ldots,z_n)},$$

where $\bigotimes \equiv \bigotimes_{\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]}$ and we use θ as in Lemma 2.5.3. Since units in bicategories are unique up to isomorphism there is an isomorphism $I_W \cong \widetilde{I_W}$. This implies, writing $\varphi : \mathrm{k}[\boldsymbol{z},\boldsymbol{z}'] \to \mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]$ for the canonical ring isomorphism,

$$\iota_{-W(\boldsymbol{z}')+W(\boldsymbol{z}),W(\boldsymbol{z})-W(\boldsymbol{z}')}(I_W) \cong \bigotimes_{i=1}^{n} \Big(\varphi_*(\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}] \oplus \mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]\theta_i), \\ (z_i - z_i').\theta_i^* + \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'}(W).\theta_i \wedge (-) \Big), \tag{4.2}$$

where now $\bigotimes \equiv \bigotimes_{\mathrm{k}[\boldsymbol{z},\boldsymbol{z}']}$. A change of basis $1 \mapsto 1$, $\theta_i \mapsto \vartheta_i := -\theta_i$, $i \in \mathbb{N}_{\leq n}$, shows that the matrix factorization (4.2) is the same as

$$\bigotimes_{i=1}^{n} (\varphi_*(\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}] \oplus \mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]\vartheta_i), -(z_i - z_i').\vartheta_i^* - \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'}(W).\vartheta_i \wedge (-)), \tag{4.3}$$

where the differential is minus that of (4.2). Via the canonical isomorphism of $\mathrm{k}[\boldsymbol{z},\boldsymbol{z}']$-modules $\varphi_*(\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}]) \cong \mathrm{k}[\boldsymbol{z},\boldsymbol{z}']$ this is isomorphic to the matrix factorization

$$\bigotimes_{i=1}^{n} (\mathrm{k}[\boldsymbol{z},\boldsymbol{z}'] \oplus \mathrm{k}[\boldsymbol{z},\boldsymbol{z}']\vartheta_i, -(z_i - z_i').\vartheta_i^* - \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'}(W).\vartheta_i \wedge (-)). \tag{4.4}$$

According to [KR2, Theorem 2.1] the matrix factorization (4.4) is isomorphic to one where the second summand of the differential is altered as long as it stays a matrix factorization of the same potential. In particular, (4.4) is isomorphic to

$$\bigotimes_{i=1}^{n} (\mathrm{k}[\boldsymbol{z},\boldsymbol{z}'] \oplus \mathrm{k}[\boldsymbol{z},\boldsymbol{z}']\vartheta_i, (z_i' - z_i).\vartheta_i^* + \partial_{[i]}^{\boldsymbol{z}',\boldsymbol{z}}(-W).\vartheta_i \wedge (-)). \tag{4.5}$$

As (4.5) is $\widetilde{I_{-W}}$ this concludes our proof. □

In Lemma 4.6 we use the notion of duality for objects of a symmetric monoidal bicategory as it is recalled in Definition A.5.1.

Lemma 4.6. ev_W and coev_W make W^* the dual of W in $\mathcal{LG}$.

Proof. Throughout the proof we omit brackets and associators for the horizontal composition in $\mathcal{LG}$ as justified by Lemma 3.1.10. Recall the unitor pseudonatural transformations for $\mathcal{LG}$ from Lemma 3.2.11 and Lemma 3.2.15. We show that there is an isomorphism as in (A.5.1) for ev_W and coev_W:

$$\begin{aligned}
&l_{(*,W)} \otimes (\mathrm{ev}_W \Box I_W) \otimes a^-_{((W,-W),W)} \otimes (I_W \Box \mathrm{coev}_W) \otimes r^-_{(W,*)} \\
=&I_W \otimes_{\mathrm{k}[\boldsymbol{z}]} (\mathrm{ev}_W \Box I_W) \otimes_{\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}'',\boldsymbol{z}''']} I_{W-W+W} \otimes_{\mathrm{k}[\boldsymbol{z}'''',\boldsymbol{z}''''',\boldsymbol{z}'''''']} (I_W \Box \mathrm{coev}_W) \otimes_{\mathrm{k}[\boldsymbol{z}]} I_W \\
\xrightarrow{\lambda\otimes\rho}&(\mathrm{ev}_W \Box I_W) \otimes_{\mathrm{k}[\boldsymbol{z},\boldsymbol{z}',\boldsymbol{z}'']} I_{W-W+W} \otimes_{\mathrm{k}[\boldsymbol{z}''',\boldsymbol{z}'''',\boldsymbol{z}''''']} (I_W \Box \mathrm{coev}_W) \\
\xrightarrow{1\otimes\lambda}&(\mathrm{ev}_W \Box I_W) \otimes_{\mathrm{k}[\boldsymbol{z},\boldsymbol{z}',\boldsymbol{z}'']} (I_W \Box \mathrm{coev}_W) \\
\xrightarrow{\mathcal{C}}&(I_W \otimes_{\mathrm{k}[\boldsymbol{z}]} I_W) \otimes_{\mathrm{k}[\boldsymbol{z}]} (I_W \otimes_{\mathrm{k}[\boldsymbol{z}]} I_W) \xrightarrow{\cong} I_W,
\end{aligned}$$

where the last isomorphism consists merely of structure morphisms of $\mathcal{LG}$.

We move on to (A.5.2):

$$\begin{aligned}
&r_{(-W,*)} \otimes (I_{-W} \Box \mathrm{ev}_W) \otimes a_{((-W,W),-W)} \otimes (\mathrm{coev}_W \Box I_{-W}) \otimes l^{-}_{(*,-W)} \\
=&I_{-W} \otimes_{\mathrm{k}[\boldsymbol{z}]} (I_{-W} \Box \mathrm{ev}_W) \otimes_{\mathrm{k}[\boldsymbol{z}',\boldsymbol{z}'',\boldsymbol{z}''']} I_{-W+W-W} \\
&\otimes_{\mathrm{k}[\boldsymbol{z}'''',\boldsymbol{z}''''',\boldsymbol{z}'''''']} (\mathrm{coev}_W \Box I_{-W}) \otimes_{\mathrm{k}[\boldsymbol{z}]} I_{-W} \\
\xrightarrow{\lambda\otimes\rho}&(I_{-W} \Box \mathrm{ev}_W) \otimes_{\mathrm{k}[\boldsymbol{z},\boldsymbol{z}',\boldsymbol{z}'']} I_{-W+W-W} \otimes_{\mathrm{k}[\boldsymbol{z}''',\boldsymbol{z}'''',\boldsymbol{z}''''']} (\mathrm{coev}_W \Box I_{-W}) \\
\xrightarrow{1\otimes\lambda}&(I_{-W} \Box \mathrm{ev}_W) \otimes_{\mathrm{k}[\boldsymbol{z},\boldsymbol{z}',\boldsymbol{z}'']} (\mathrm{coev}_W \Box I_{-W}) \\
\xrightarrow{\widehat{c}}&(I_{-W} \otimes_{\mathrm{k}[\boldsymbol{z}]} \iota_{-W(\boldsymbol{z})+W(\boldsymbol{z}'),W(\boldsymbol{z}')-W(\boldsymbol{z})}((\mathrm{coev}_W)_{\mathrm{mf}})) \\
&\otimes_{\mathrm{k}[\boldsymbol{z}']} (\iota_{-W(\boldsymbol{z}')+W(\boldsymbol{z}),W(\boldsymbol{z})-W(\boldsymbol{z}')}((\mathrm{ev}_W)_{\mathrm{mf}}) \otimes_{\mathrm{k}[\boldsymbol{z}]} I_{-W}) \\
\xrightarrow{\lambda\otimes\rho}&\iota_{-W(\boldsymbol{z})+W(\boldsymbol{z}'),W(\boldsymbol{z}')-W(\boldsymbol{z})}((\mathrm{coev}_W)_{\mathrm{mf}}) \otimes_{\mathrm{k}[\boldsymbol{z}']} \iota_{-W(\boldsymbol{z}')+W(\boldsymbol{z}),W(\boldsymbol{z})-W(\boldsymbol{z}')}((\mathrm{ev}_W)_{\mathrm{mf}}) \\
\cong&I_{-W} \otimes_{\mathrm{k}[\boldsymbol{z}]} I_{-W} \xrightarrow{\lambda} I_{-W},
\end{aligned}$$

where the unlabeled isomorphism in the last line comes from applying Lemma 4.5 to both factors of the horizontal composite separately, using that $(\mathrm{coev}_W)_{\mathrm{mf}} = (I_W)_{\mathrm{mf}} = (\mathrm{ev}_W)_{\mathrm{mf}}$. □

Combining Lemma 4.6 with the fact proven in [CM2] that every 1-morphism in $\mathcal{LG}$ – and therefore in particular also coev_W and ev_W – has both left and right adjoints yields

Corollary 4.7. Every object of $\mathcal{LG}$ is fully dualizable.

5. Extended 2d TQFTs in $\mathcal{LG}$

In this chapter we apply the explicit symmetric monoidal structure on $\mathcal{LG}$ established in Chapter 3 and the duals for objects of $\mathcal{LG}$ from Chapter 4 to determine extended two-dimensional TQFTs with values in $\mathcal{LG}$.

In Section 5.1 we recall some background on the two-dimensional cobordism hypothesis and its cousin for oriented bordisms. These motivate our subsequent studies in $\mathcal{LG}$ and allow for the interpretations of our results: Corollary 5.2.1 states that every object of $\mathcal{LG}$ gives rise to a framed extended 2d TQFT while according to Corollary 5.2.4 precisely those objects of $\mathcal{LG}$ with an even number of variables are the value of an oriented extended 2d TQFT $\mathrm{Bord}_2^{\mathrm{or}} \to \mathcal{LG}$ in the positively oriented point. In Subsection 5.3.1 we introduce a bicategory LG which is closely related to $\mathcal{LG}$. We show that every object of $\mathcal{LG}$ gives rise to an oriented extended 2d TQFT valued in LG. Subsection 5.3.2 discusses an example of such a TQFT realizing Khovanov and Rozansky's prescriptions in [KR1, §9].

5.1. An account of extended 2d TQFTs

5.1.1. Framed extended 2d TQFTs

As we recall in the introduction the *cobordism hypothesis* formulated by John Baez and James Dolan in [BD] classifies symmetric monoidal n-functors $\mathrm{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$. Here $\mathcal{C}$ is an arbitrary symmetric monoidal n-category and $\mathrm{Bord}_n^{\mathrm{fr}}$ is a symmetric monoidal n-category whose objects are disjoint unions of framed points, 1-morphisms are framed 1-bordisms, i.e. 1-dimensional manifolds whose boundaries are determined by their source and target objects, 2-morphisms are framed 2-bordisms etc. up to equivalence classes of n-bordisms as n-morphisms. Such functors are called *fully extended n-dimensional TQFTs*.

The cobordism hypothesis is formalized by Jacob Lurie in [Lu] using the language of (∞, n)-categories. This inspires [P] to formulate a 2d version of the cobordism hypothesis in terms of symmetric monoidal bicategories. This cobordism hypothesis is proven through explicit computations in [P]. In this section we give a non-technical account of this approach to the cobordism hypothesis and some related work.

We start by sketching the definition of the unframed bordism bicategory Bord_2. Throughout, unless specified differently, when we write manifold we implicitly mean a smooth manifold. As a first approximation, the objects of Bord_2 are 0-

manifolds, its 1-morphisms are 1-bordisms and 2-morphisms are diffeomorphism classes of 2-bordisms.

There is however an issue with horizontal composition of 1-bordisms. Indeed, gluing 1-bordisms along a common boundary results in a well-defined topological manifold which can be equipped with a smooth structure. But this smooth structure is unique only up to a non-unique diffeomorphism. This is dealt with in 1-categories of bordisms by taking diffeomorphism classes of bordisms as morphisms. Since Bord_2 also involves 2-bordisms whose source and target are 1-bordisms rather than diffeomorphism classes of such, this subtlety needs to be solved differently to construct the bordism bicategory.

Roughly, objects of the bicategory Bord_2 are 0-manifolds equipped with an "infinitesimal neighborhood of a 2-manifold" [P]. Similarly, 1-morphisms are 1-bordisms with such a 2-dimensional neighborhood and 2-morphisms are isomorphism classes relative to the boundary of 2-bordisms with a 2-dimensional neighborhood. Horizontal and vertical composition are given by gluing which can be defined using the 2-dimensional neighborhoods. The symmetric monoidal structure of Bord_2 comes from disjoint union. This has been worked out in [SP1, §3.2.3] and is reviewed in [P, §4.1].

To state the cobordism hypothesis Bord_2 needs to be refined to a bicategory of framed bordisms. Informally, this is achieved in [P] by equipping the 2-dimensional infinitesimal neighborhoods of 0-, 1- and 2-manifolds in Bord_2 with framings, i.e. trivializations of the tangent bundles. Such a trivialization is a linear isomorphism between the tangent space in each point and $\mathbb{R}^2$ varying smoothly with the base point. The framing of a bordism is required to be compatible with the framings on its source and target. The resulting bicategory $\mathrm{Bord}_2^{\mathrm{fr}}$ has the framed versions of the objects and 1-morphisms of Bord_2 as objects and 1-morphisms, respectively, and certain equivalence classes of framed 2-bordisms as 2-morphisms. Horizontal and vertical composition are given by gluing, and the symmetric monoidal structure comes from the disjoint union. We refer to [P, Theorem 5.11] for the details.

On the "purely algebraic" side of the cobordism hypothesis we use the following notation. Let $\mathcal{B}$ be a bicategory. We denote by $K(\mathcal{B})$ the *core* of $\mathcal{B}$.[1] This is the 2-groupoid with the same objects as $\mathcal{B}$, whose 1-morphisms are the 1-equivalences in $\mathcal{B}$ and whose 2-morphisms are 2-isomorphisms in $\mathcal{B}$. Moreover, if $\mathcal{B}$ is a symmetric monoidal bicategory we write $\mathcal{B}^{\mathrm{fd}}$ for the subbicategory of $\mathcal{B}$ built on its fully dualizable objects, cf. Definition A.5.2.

In quoting the 2d cobordism hypothesis, [P, Theorem 8.1], we use the following notation. Let $\mathcal{B}$, $\widetilde{\mathcal{B}}$ be symmetric monoidal bicategories. We denote by $\mathrm{Fun}_{\mathrm{sym\ mon}}(\mathcal{B}, \widetilde{\mathcal{B}})$ the bicategory of symmetric monoidal 2-functors $\mathcal{B} \to \widetilde{\mathcal{B}}$, symmetric monoidal pseudonatural transformations and symmetric monoidal modifi-

[1] $K(\mathcal{B})$ is referred to as the underlying groupoid of $\mathcal{B}$ in [P] and as its maximal subgroupoid in [H].

cations as it is defined in [SP1, §2.3]. Furthermore, we write pt^+ for the positively framed point.

Theorem 5.1.1. Let $\mathcal{B}$ be a symmetric monoidal bicategory. There is an equivalence of 2-groupoids

$$\mathrm{Fun}_{\text{sym mon}}(\mathrm{Bord}_2^{\mathrm{fr}}, \mathcal{B}) \to K(\mathcal{B}^{\mathrm{fd}}) \tag{5.1.1}$$

which acts on objects as $Z \mapsto Z(pt^+)$.

A *framed extended 2d TQFTs valued in* $\mathcal{B}$ is a symmetric monoidal 2-functor $Z : \mathrm{Bord}_2^{\mathrm{fr}} \to \mathcal{B}$. Hence Theorem 5.1.1 states that every fully dualizable object B of $\mathcal{B}$ gives rise to a framed extended 2d TQFT, whose value in the positively framed point is equivalent to B.

5.1.2. Oriented extended 2d TQFTs

Similarly to Theorem 5.1.1 there is a version of the 2d cobordism hypothesis in which the bordisms are equipped with orientations. This is [H, Corollary 5.9]. We review this result in the following. First, we give an overview of the developments that led to the proof of this cobordism hypothesis. This is intended to motivate the structures we study subsequently, beginning with Definition 5.1.2.

As in the case of the 2d cobordism hypothesis for framed bordisms treated in the preceding Section 5.1.1, also its oriented version uses low-dimensional higher category theory to realize some ideas advocated in [Lu] concretely.

The first idea drawn from [Lu] is that there is an $SO(2)$-action on framed 2-manifolds rotating the framing pointwise which extends to an action on the framed 2d bordism category. According to [Lu] precomposing with this action on bordisms yields an $SO(2)$-action on the left hand side of (5.1.1). This is predicted to translate into an action of $SO(2)$ on the right hand side of (5.1.1). The intuition that oriented manifolds correspond to fixed points of the $SO(2)$-action on framed manifolds leads [Lu] to state that oriented TQFTs correspond to homotopy fixed points of the $SO(2)$-action[2] on the algebraic side of the cobordism hypothesis.

The ideas outlined in the previous paragraph are worked out in [H] and [HV] building also on [HSV]. Indeed, extending [HSV], [HV] defines the relevant kind of action of $SO(2)$ on symmetric monoidal bicategories. This allows for a detailed description of the corresponding homotopy fixed point bicategory.

One particular $SO(2)$-action on the core of fully dualizable objects of an arbitrary symmetric monoidal bicategory $\mathcal{B}$ is suggested in [Lu], cf. [SP2] and [P]. This is realized in [HV]. It is then proven in [H] that the bicategory of homotopy fixed points under this action is equivalent to the bicategory $\mathrm{Fun}_{\text{sym mon}}(\mathrm{Bord}_2^{\mathrm{or}}, \mathcal{B})$ of oriented extended 2d TQFTs with values in $\mathcal{B}$. Here $\mathrm{Bord}_2^{\mathrm{or}}$ is a bicategory of

[2]Homotopy fixed points generalize fixed points of group actions on sets to group actions on (higher) categories.

oriented bordisms analogous to $\mathrm{Bord}_2^{\mathrm{fr}}$, cf. Section 5.1.1, which we describe below. The proof in [H] uses that these TQFTs can be described explicitly thanks to a presentation of $\mathrm{Bord}_2^{\mathrm{or}}$ in terms of generators and relations worked out in [SP1].

Finally, [HV] also verifies that the particular action on the bicategory of homotopy fixed points used in the cobordism hypothesis indeed can be obtained from an action on the framed bordism bicategory inspired by changing the framing. This hinges on a presentation of the framed bordism bicategory detailed in [P].

We recall the definition of the relevant group actions on bicategories and touch upon their bicategories of homotopy fixed points next. Thereafter we comment on the bicategory of oriented bordisms in order to then state the cobordism hypothesis for oriented 2d bordisms as it is proven in [H].

We start with [HSV, Definition 3.5]:

Definition 5.1.2. Let G be a topological group.[3] The *fundamental 2-groupoid* $\Pi_2(G)$ of G is the monoidal bicategory whose objects are the points in G, with paths in G as 1-morphisms between their endpoints and homotopy classes of homotopies between paths as 2-morphisms. The monoidal product is given by the group multiplication (pointwise for paths and homotopies) and the monoidal structure is trivial.

An action of a group G on a bicategory $\mathcal{B}$ in particular assigns an auto-equivalence of $\mathcal{B}$ to each group element. In which sense this assignment is to be functorial is formalized using the bicategory $\mathrm{Aut}(\mathcal{B})$ of auto-equivalences of $\mathcal{B}$, pseudonatural isomorphisms and invertible modifications. This bicategory is monoidal via composition, cf. [HSV, Remark 3.7].

Definition 5.1.3. Let G be a topological group and $\mathcal{B}$ a bicategory. A G-action on $\mathcal{B}$ is a monoidal 2-functor $\Pi_2(G) \to \mathrm{Aut}(\mathcal{B})$.

The data and conditions implicit in Definition 5.1.3 are spelled out in [HSV, Remark 3.8].

We are aiming at stating the oriented 2d cobordism hypothesis. Therefore we are interested in $SO(2)$-actions. Thus, it is worth noting that $\Pi_2(SO(2))$ is particularly simple. Since $SO(2)$ is topologically a circle, $\Pi_2(SO(2))$ has just a single equivalence class of objects. Furthermore $\pi_1(SO(2)) \cong \mathbb{Z}$ and all homotopies between two given paths in $SO(2)$ are homotopic. Altogether, $\Pi_2(SO(2))$ is equivalent to a bicategory $\mathrm{B}\underline{\mathbb{Z}}$ with a single object, integer numbers as 1-morphisms and solely identity 2-morphisms. 1-morphisms are composed by adding them. The notation which we use for this bicategory is in line with that introduced in Definition 5.1.5 and immediately below Definition 5.1.6.

Similarly to our discussion in Remark 2.10.5 – but one level higher on the categorical ladder – since $\Pi_2(SO(2))$ and $\mathrm{B}\underline{\mathbb{Z}}$ are equivalent bicategories, the

[3] That $SO(2)$ is regarded as a topological group in the oriented 2d cobordism hypothesis is inspired by [Lu].

monoidal structure on the former induces one on the latter bicategory such that the two are monoidally equivalent. Since $\mathrm{B}\underline{\mathbb{Z}}$ only has identity 2-morphisms it follows from an Eckmann-Hilton argument, cf. [BD, (5)] or [SP2, Example 2.5], that the monoidal product of 1-morphisms in $\mathrm{B}\underline{\mathbb{Z}}$ necessarily agrees with their composition.

Thus, specifying a monoidal 2-functor $\mathrm{B}\underline{\mathbb{Z}} \to \mathrm{Aut}(\mathcal{B})$ determines an $SO(2)$-action on a bicategory $\mathcal{B}$. Such an action is essentially given by the pseudonatural equivalence of $1_{\mathcal{B}}$ assigned to the 1-morphism 1 in $\mathrm{B}\underline{\mathbb{Z}}$, cf. [HV, Definition 4.1].

It can be inferred from [Lu, Remark 4.2.5] that the pseudonatural transformation defining the $SO(2)$-action on the core of fully dualizable objects of a symmetric monoidal bicategory which is relevant for the oriented 2d cobordism hypothesis has the following components:

Definition 5.1.4. Let $\mathcal{B}$ be a symmetric monoidal bicategory and $X \in \mathcal{B}$ a fully dualizable object. The *Serre automorphism* S_X of X is the 1-morphism in $\mathcal{B}$ composed as

$$X \xrightarrow{r^-} X \otimes 1 \xrightarrow{1\otimes \mathrm{ev}^\dagger} X \otimes X \otimes X^* \xrightarrow{b\otimes 1} X \otimes X \otimes X^* \xrightarrow{1\otimes \mathrm{ev}} X \otimes 1 \xrightarrow{r} X,$$

where b is the braiding for the monoidal product on $\mathcal{B}$, cf. Definition A.4.1, r is the right unitor for this product and r^- the latter unitor's adjoint, cf. Definition A.3.1. The 1-morphism $\mathrm{ev} \equiv \mathrm{ev}_X$ is the evaluation morphism associated to X, cf. Definition A.5.1, and $\mathrm{ev}^\dagger$ is its right adjoint, cf. Definition A.2.1. The order in which these 1-morphisms are composed is irrelevant by the coherence theorem for bicategories, cf. e.g. [SP1, §A.2].

Some details concerning this action can be found in [SP2, §§15–18]. That the Serre automorphisms indeed assemble into a monoidal pseudonatural endotransformation of the identity 2-functor on the core of fully dualizable objects of a symmetric monoidal bicategory is proven as [HV, Proposition 2.12].

Eventually, the oriented 2d cobordism hypothesis only features homotopy fixed points of the $SO(2)$-action on the core of fully dualizable objects of a symmetric monoidal bicategory by the Serre automorphism. Therefore, it is adequate to recall the definition of a bicategory of homotopy fixed points next. However, we do not provide the details here, cf. [HSV, Definition 3.9], since this definition uses notions of the theory of tricategories which we do not need elsewhere in this thesis. Rather than introducing these higher categorical objects we prefer to give some impression of the line of thought leading to the definition of the bicategory of homotopy fixed points by presenting its 1-categorical analogue.

We begin working towards the category of homotopy fixed points of a group action on a category with the following analogue of Definition 5.1.2.

Definition 5.1.5. Let G be a discrete group. $\underline{G}$ is the monoidal category whose objects are the elements of G and which has only got identity morphisms. The horizontal composition of objects is the group operation.

Using this we proceed similarly as in Definition 5.1.3.

Definition 5.1.6. Let G be a finite group and $\mathcal{C}$ a category. A *G-action on $\mathcal{C}$* is a monoidal functor from $\underline{G}$ to the category of auto-equivalences of $\mathcal{C}$.

Note that a G-action as in Definition 5.1.6 entails the same information as a 2-functor $\mathrm{B}\underline{G} \to \mathrm{Cat}$, $* \mapsto \mathcal{C}$. Here $\mathrm{B}\underline{G}$ is the *delooping* of $\underline{G}$, i.e. the bicategory with a single object $*$ whose endomorphism category is $\underline{G}$. By Cat we denote the bicategory of categories.

The category of homotopy fixed points of an action $\mathrm{B}\underline{G} \to \mathrm{Cat}$ is defined as a bicategorical limit of this 2-functor, cf. [HSV, Definition 3.3]. Thus, it suggests itself that the bicategory of homotopy fixed points of a group action on a bicategory be defined as the trilimit of the trifunctor which is basically given by the monoidal 2-functor defining the action, cf. [HV, (4.1)]. Unpacking this definition and applying it to the $SO(2)$-action featuring in the oriented 2d cobordism hypothesis yields the following, cf. [HV, Corollary 4.4].

Theorem 5.1.7. Let $\mathcal{B}$ be a symmetric monoidal bicategory. The bicategory $K(\mathcal{B}^{\mathrm{fd}})^{\mathrm{h}SO(2)}$ of homotopy fixed points of the $SO(2)$-action on $K(\mathcal{B}^{\mathrm{fd}})$ by the Serre automorphism is equivalent to the following bicategory. Objects are pairs (X, λ_X), where $X \in \mathcal{B}^{\mathrm{fd}}$ and λ_X is a 2-isomorphism $S_X \to 1_X$. 1-morphisms $(X, \lambda_X) \to (Y, \lambda_Y)$ are 1-morphisms $f \in \mathcal{B}(X, Y)$ compatible with the 2-morphisms of the pseudonatural transformation determined by the Serre automorphism in the sense that the diagram [HV, (4.14)] commutes. 2-morphisms are 2-isomorphisms in $\mathcal{B}$.

If there is an isomorphism $S_X \to 1_X$ we say that the Serre automorphism for X is *trivializable*.

This concludes the description of the algebraic side of the oriented 2d cobordism hypothesis. We turn to the geometric part next.

Stating the 2d cobordism hypothesis for oriented TQFTs requires an oriented bordism bicategory $\mathrm{Bord}_2^{\mathrm{or}}$. This is constructed in [SP1] similarly to the bicategory of framed bordisms which we sketch in the previous Section 5.1.1. In this case the 2d "infinitesimal neighborhoods" are equipped with an orientation. Again, these orientations have to be compatible with sources and targets of bordisms. Horizontal and vertical composition are once more given by gluing, and the symmetric monoidal structure is based on disjoint union. In [SP1, Theorem 3.50] a generators-and-relations presentation of the resulting symmetric monoidal bicategory $\mathrm{Bord}_2^{\mathrm{or}}$ is given.

The presentation of $\mathrm{Bord}_2^{\mathrm{or}}$ in [SP1, Theorem 3.50] enables [H] to explicitly show that the object assigned to the positively oriented point by a TQFT $Z : \mathrm{Bord}_2^{\mathrm{or}} \to \mathcal{B}$ is fully dualizable and that its Serre automorphism is trivializable. Vice versa, [H] shows that starting from a fully dualizable object in a symmetric monoidal bicategory $\mathcal{B}$ whose Serre automorphism is trivializable one can construct an oriented 2d extended TQFT which sends the positively oriented point to this object. Moreover, [H] proves similar relations between morphisms of $\mathrm{Fun}_{\mathrm{sym\ mon}}(\mathrm{Bord}_2^{\mathrm{or}}, \mathcal{B})$ and those of $K(\mathcal{B}^{\mathrm{fd}})$. This is the content of [H, Theorem 5.5 and Theorem 5.8] which taken together result in [H, Corollary 5.9]. We quote the latter as

Theorem 5.1.8. Let $\mathcal{B}$ be a symmetric monoidal bicategory. There is an equivalence of bigroupoids

$$\mathrm{Fun}_{\mathrm{sym\ mon}}(\mathrm{Bord}_2^{\mathrm{or}}, \mathcal{B}) \cong K(\mathcal{B}^{\mathrm{fd}})^{\mathrm{h}SO(2)}$$

which acts on objects as $Z \mapsto Z(pt^+)$, where pt^+ denotes the positively oriented point.

5.2. Framed and oriented extended 2d TQFTs valued in $\mathcal{LG}$

In this section we determine framed and oriented extended 2d TQFTs valued in $\mathcal{LG}$.

The results collected above immediately allow us to state which objects of $\mathcal{LG}$ give rise to framed extended 2d TQFTs. To wit, in light of the cobordism hypothesis Theorem 5.1.1, i.e. [P, Theorem 8.1], Corollary 4.7 entails

Corollary 5.2.1. Every object of $\mathcal{LG}$ determines a framed extended 2d TQFT with values in $\mathcal{LG}$.

Next, we show that every object of $\mathcal{LG}$ with an even number of variables gives rise to an oriented extended 2d TQFT. We do so building on the results of [H] and [HV] reviewed in Section 5.1.2. That is, we prove that the Serre automorphism of an object of $\mathcal{LG}$ – which is well-defined since according to Corollary 4.7 every object of $\mathcal{LG}$ is fully dualizable – is isomorphic to the identity 1-morphism if and only if the object has an even number of variables. By the work of [H] summarized in Theorem 5.1.8 it follows that precisely these objects of $\mathcal{LG}$ give rise to oriented extended TQFTs. They are the latter's value in the positively oriented point.

We first introduce an isomorphism of matrix factorizations in Definition 5.2.2 which we utilize thereafter to arrive at the result we mention above. To set the stage for Definition 5.2.2 let

$$X_1 \in \mathcal{LG}(U_1, U_2),\ X_2 \in \mathcal{LG}(0, W_1 + V),\ Y_1 \in \mathcal{LG}(W_1, W_2),\ Y_2 \in \mathcal{LG}(U_2 + V, 0),$$

$\widetilde{X_2} \in \mathcal{LG}(-V, W_1)$ with $(\widetilde{X_2})_{\mathrm{mf}} := \iota_{W_1+V,V+W_1}((X_2)_{\mathrm{mf}})$ and $\widetilde{Y_2} \in \mathcal{LG}(U_2, -V)$ with $(\widetilde{Y_2})_{\mathrm{mf}} := (Y_2)_{\mathrm{mf}}$. Set furthermore $T_1 := -U_1 + U_2 + W_1 + V$ and $T_2 := -U_1 + W_1 + U_2 + V$. The following is well-defined by Lemma 3.1.6.

Definition 5.2.2. $\mathcal{T}_{Y_1,X_2,Y_2,X_1}$ is the isomorphism in $\mathrm{hmf}^{\oplus}_{-U_1+W_2}$ given by

$$\begin{aligned}(Y_1\square Y_2)\otimes_{\mathrm{k}[\boldsymbol{z}_1,\boldsymbol{x}_2,\boldsymbol{y}]}\iota_{T_1,T_2}(X_1\square X_2)&\to (Y_1\otimes_{\mathrm{k}[\boldsymbol{z}_1]}(\widetilde{X_2\otimes_{\mathrm{k}[\boldsymbol{y}]}Y_2}))\otimes_{\mathrm{k}[\boldsymbol{x}_2]}X_1\\ (g\otimes_{\mathrm{k}}h)\otimes_{\mathrm{k}[\boldsymbol{z}_1,\boldsymbol{x}_2,\boldsymbol{y}]}(e\otimes_{\mathrm{k}}f)&\mapsto(-1)^{|f|(|h|+|e|)}(g\otimes_{\mathrm{k}[\boldsymbol{z}_1]}(f\otimes_{\mathrm{k}[\boldsymbol{y}]}h))\otimes_{\mathrm{k}[\boldsymbol{x}_2]}e.\end{aligned}\tag{5.2.1}$$

The prescription (5.2.1) defines a map of $\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{z}_2]$-modules as elements of $\mathrm{k}[\boldsymbol{x}_1]\subset\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{z}_2]$ multiply e and scalars in $\mathrm{k}[\boldsymbol{z}_2]\subset\mathrm{k}[\boldsymbol{x}_1,\boldsymbol{z}_2]$ multiply g on both sides. Furthermore, the restriction of scalars on the left hand side is such that the tensor products are respected in the sense introduced preceding Lemma 3.1.7.

Recall the Serre automorphism from Definition 5.1.4.

Lemma 5.2.3. The Serre automorphism $S_W\in\mathcal{LG}(W,W)$ is isomorphic to I_W if and only if n is even.

Proof. Recall the braiding b for the monoidal product on $\mathcal{LG}$ from Lemma 3.3.5 as well as the right unitor r and its adjoint r^- from Lemma 3.2.15. By definition, using

$$\begin{aligned}V_1&:=-W(\boldsymbol{z})-W(\boldsymbol{z}')+W(\boldsymbol{z}'')+W(\boldsymbol{z}'''),\\ V_2&:=-W(\boldsymbol{z})-W(\boldsymbol{z}')+W(\boldsymbol{z}''')+W(\boldsymbol{z}'')\end{aligned}$$

and omitting indices on $\otimes$ (and bracketings as well as associators for $\otimes$ which are not specified in the definition),

$$\begin{aligned}S_W&=r_{(W,*)}\otimes(I_W\square\mathrm{ev}_W)\otimes(b_{(W,W)}\square I_{-W})\otimes(I_W\square(\mathrm{ev}_W)^\dagger)\otimes r^-_{(W,*)}\\ &=I_W\otimes(I_W\square\mathrm{ev}_W)\otimes(\iota_{V_1,V_2}(I_{W+W})\square I_{-W})\otimes(I_W\square(\mathrm{ev}_W)^\dagger)\otimes I_W.\end{aligned}$$

Starting with this and setting

$$\begin{aligned}U_1&:=-W(\boldsymbol{z})-W(\boldsymbol{z}')+W(\boldsymbol{z}'')+W(\boldsymbol{z}''')+W(\boldsymbol{z}'''')-W(\boldsymbol{z}'''''),\\ U_2&:=-W(\boldsymbol{z})-W(\boldsymbol{z}')+W(\boldsymbol{z}'')+W(\boldsymbol{z}'''')+W(\boldsymbol{z}''')-W(\boldsymbol{z}'''''),\\ T_1&:=-W(\boldsymbol{z})+W(\boldsymbol{z}')+W(\boldsymbol{z}'')-W(\boldsymbol{z}'''),\\ T_2&:=-W(\boldsymbol{z})+W(\boldsymbol{z}'')+W(\boldsymbol{z}')-W(\boldsymbol{z}'''),\end{aligned}$$

we have the following chain of isomorphisms

$$\begin{aligned}S_W&\xrightarrow{\lambda\otimes\rho}(I_W\square\mathrm{ev}_W)\otimes(\iota_{V_1,V_2}(I_{W+W})\square I_{-W})\otimes(I_W\square(\mathrm{ev}_W)^\dagger)\\ &\cong(I_W\square\mathrm{ev}_W)\otimes\iota_{U_1,U_2}(I_{W+W}\square I_{-W})\otimes(I_W\square(\mathrm{ev}_W)^\dagger)\\ &\xrightarrow{1\otimes\iota(\square^{-1})\otimes1}(I_W\square\mathrm{ev}_W)\otimes\iota_{U_1,U_2}(I_{W+W-W})\otimes(I_W\square(\mathrm{ev}_W)^\dagger)\\ &\xrightarrow{1\otimes\chi_l^{-1}}(I_W\square\mathrm{ev}_W)\otimes\iota_{T_1,T_2}(I_{W+W-W}\otimes(I_W\square(\mathrm{ev}_W)^\dagger))\\ &\xrightarrow{1\otimes\iota(\lambda)}(I_W\square\mathrm{ev}_W)\otimes\iota_{T_1,T_2}(I_W\square(\mathrm{ev}_W)^\dagger)\\ &\xrightarrow{\mathcal{T}}\big(I_W\otimes(\widetilde{(\mathrm{ev}_W)^\dagger\otimes\widetilde{\mathrm{ev}_W}})\big)\otimes I_W\\ &\xrightarrow{\lambda\circ\rho}\widetilde{(\mathrm{ev}_W)^\dagger}\otimes\widetilde{\mathrm{ev}_W}.\end{aligned}\tag{5.2.2}$$

The second isomorphism in (5.2.2) is an instance of Lemma 3.1.6. Indeed, according to the definitions of the functors of restriction of scalars and $\square$ there is a module map

$$(\iota_{V_1,V_2}(I_{W+W})\square I_{-W})_m \to (\iota_{U_1,U_2}(I_{W+W}\square I_{-W}))_m,\ e\otimes_{\mathrm{k}} f \mapsto e\otimes_{\mathrm{k}} f.$$

By Lemma 3.1.6 this is an isomorphism ψ of matrix factorizations. The isomorphism in the second line of (5.2.2) is $1\otimes\psi\otimes 1$.

To proceed we recall from Definition 4.2 that $\mathrm{ev}_W \in \mathcal{LG}(W(\boldsymbol{z})-W(\boldsymbol{z}'),0)$ is defined by $(\mathrm{ev}_W)_{\mathrm{mf}} := (I_W)_{\mathrm{mf}}$. Replacing Y_2 in the setting of Definition 5.2.2 by ev_W shows that $\widetilde{\mathrm{ev}_W} \in \mathcal{LG}(W,W)$ is defined by $(\widetilde{\mathrm{ev}_W})_{\mathrm{mf}} = (\mathrm{ev}_W)_{\mathrm{mf}}$. This implies $\widetilde{\mathrm{ev}_W} = I_W$. Thus, post-composing (5.2.2) with ρ yields $S_W \cong \widetilde{(\mathrm{ev}_W)^\dagger}$.

It remains to show $\widetilde{(\mathrm{ev}_W)^\dagger} \cong I_W$ if and only if n is even. Replacing X_2 in the setting of Definition 5.2.2 by $\mathrm{ev}_W^\dagger$ shows that $\widetilde{(\mathrm{ev}_W)^\dagger} \in \mathcal{LG}(W,W)$ is defined by

$$\left(\widetilde{(\mathrm{ev}_W)^\dagger}\right)_{\mathrm{mf}} = \iota_{W(\boldsymbol{z})-W(\boldsymbol{z}'),-W(\boldsymbol{z}')+W(\boldsymbol{z})}\left(\left((\mathrm{ev}_W)^\dagger\right)_{\mathrm{mf}}\right). \tag{5.2.3}$$

Moreover, $(\mathrm{ev}_W)^\dagger \in \mathcal{LG}(0,W(\boldsymbol{z})-W(\boldsymbol{z}')) = \mathrm{hmf}^{\oplus}_{W(\boldsymbol{z})-W(\boldsymbol{z}')}$ is obtained from $\mathrm{ev}_W \in \mathcal{LG}(W(\boldsymbol{z})-W(\boldsymbol{z}'),0) = \mathrm{hmf}^{\oplus}_{-W(\boldsymbol{z})+W(\boldsymbol{z}')}$ via $((\mathrm{ev}_W)_{\mathrm{mf}})^\vee \in \mathrm{hmf}^{\oplus}_{W(\boldsymbol{z})-W(\boldsymbol{z}')}$ as

$$\begin{aligned}\left((\mathrm{ev}_W)^\dagger\right)_{\mathrm{mf}} &:= \iota_{W(\boldsymbol{z})-W(\boldsymbol{z}')+0,0+W(\boldsymbol{z})-W(\boldsymbol{z}')}\left(((\mathrm{ev}_W)_{\mathrm{mf}})^\vee[2n]\right)\\ &= ((\mathrm{ev}_W)_{\mathrm{mf}})^\vee[2n] = ((\mathrm{ev}_W)_{\mathrm{mf}})^\vee.\end{aligned} \tag{5.2.4}$$

Furthermore, $(\mathrm{ev}_W)_{\mathrm{mf}} = (I_W)_{\mathrm{mf}}$ such that inserting (5.2.4) in (5.2.3) and using $I_W \in \mathcal{LG}(W,W)$ we have

$$(\widetilde{(\mathrm{ev}_W)^\dagger})_{\mathrm{mf}} = \iota_{W(\boldsymbol{z})-W(\boldsymbol{z}'),-W(\boldsymbol{z}')+W(\boldsymbol{z})}(((I_W)_{\mathrm{mf}})^\vee) = ((I_W)^\dagger[n])_{\mathrm{mf}}.$$

This shows $\widetilde{(\mathrm{ev}_W)^\dagger} = (I_W)^\dagger[n]$. Since in every monoidal category with duals for objects such as e.g. $\mathcal{LG}(W,W)$ duals are unique up to an isomorphism and the unitors make the unit object dual to itself we have $(I_W)^\dagger \cong I_W$, cf. [CR2, Remark 2.8 (iii)]. Thus, as shifting is a functor and hence maps isomorphisms to isomorphisms $\widetilde{(\mathrm{ev}_W)^\dagger} \cong I_W[n]$. Therefore we have in total $S_W \cong I_W[n]$ which implies $S_W \cong I_W$ for n even and $S_W \ncong I_W$ for n odd. □

Therefore, according to our discussion in Section 5.1.2 we can state our result as

Corollary 5.2.4. An object $(\boldsymbol{z},W) \in \mathcal{LG}$ is the value of an oriented extended two-dimensional TQFT in the positively oriented point if and only if $\boldsymbol{z}$ has an even number of variables.

5.3. On Khovanov and Rozansky's TQFT

Finally, we return to the original inspiration for this thesis. As alluded to in the introduction, in [KR1, §9] Khovanov and Rozansky claim to define an oriented "2d TQFT with corners" which takes values in a bicategory very similar to $\mathcal{LG}$. In Subsection 5.3.1 we define a bicategory LG which refines $\mathcal{LG}$ to mirror the bicategory of [KR1, §9] even more closely. Subsequently, we show that indeed, every object of LG gives rise to an oriented extended 2d TQFT valued in LG. In Subsection 5.3.2 we discuss one particular example of such a TQFT which corresponds to the construction in [KR1, §9].

5.3.1. Oriented extended 2d TQFTs valued in LG

We begin by defining categories which serve as categories of morphisms for the bicategory LG which we assemble thereafter. In the following we use the differential $\delta_{X,Y}$ on $\mathrm{MF}_{\mathcal{R},W}(X,Y)$ from Lemma 2.6.1. We write $\mathrm{H}^{\bullet}_{\delta_{X,Y}}(\mathrm{MF}_{\mathcal{R},W}(X,Y))$ for the (full) $\delta_{X,Y}$-cohomology of $(\mathrm{MF}_{\mathcal{R},W}(X,Y))$ as opposed to just its degree zero part as in Remark 2.6.3.

Lemma 5.3.1. Let $\mathcal{R}$ be a commutative ring, $W \in \mathcal{R}$. The following specifies a category $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}$.

1. The set $(\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W})_{\mathrm{O}}$ of objects is $(\mathrm{hmf}^{\oplus}_{\mathcal{R},W})_{\mathrm{O}}$.

2. For all pairs X, Y of objects the set $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)$ of morphisms is $\mathrm{H}^{\bullet}_{\delta_{X,Y}}(\mathrm{MF}_{\mathcal{R},W}(X,Y))$.

Composition is given by the composition of representatives of morphisms in $\mathrm{MF}_{\mathcal{R},W}$. Identity morphisms are the equivalence classes of the identities.

Proof. We show that the composition respects the equivalence classes in the first argument. One can proceed analogously for the second argument. Let ψ be a representative of $[\psi] \in \widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(Y,Z)$ and ϕ be a representative of $[\phi] \in \widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)$, both of homogeneous $\mathbb{Z}_2$-degree. Let $h \in \mathrm{MF}_{\mathcal{R},W}(X,Y)^{(|\psi|+1)\mathrm{mod}2}$. Then we have

$$\begin{aligned}(\psi + \delta_{Y,Z}(h))\phi &= (\psi + d_Z h - (-1)^{|h|} h d_Y)\phi = \psi\phi + d_Z h\phi - (-1)^{|h|} h d_Y \phi \\ &= \psi\phi + d_Z h\phi - (-1)^{|h|+|\phi|} h\phi d_X = \psi\phi + \delta_{X,Z}(h\phi).\end{aligned}$$

□

As we intend to build a bicategory whose categories of morphisms are based on categories of the form $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$, we need to define candidates for the functors of horizontal composition next. At this point we meet the situation discussed below

Remark 2.4.3. Morphisms in $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$ can have both even and odd $\mathbb{Z}_2$-degree. Therefore we cannot define $\mathbb{Z}_2$-graded tensor product-functors satisfying (2.4.1) whose source is the cartesian product of two categories of the type $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$. This does not respect the composition of morphisms, cf. (2.4.4).

We follow the solution to the mismatch in (2.4.4) suggested in [KR1]. For this it is useful to observe, that defining $\mathbb{Z}_2$-graded tensor products on cartesian products of categories of matrix factorizations featuring both $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd morphisms is only obstructed by signs in front of morphisms. Led by [KR1] we circumvent this issue by identifying each morphism of matrix factorizations with its negative.

Remark 5.3.2. We comment on another, more conceptual way of dealing with the discrepancy in signs in (2.4.4) that lies outside the focus of this thesis, and outside the author's expertise. We parallel the reasoning underlying some of the explanation in [C2, §2.4.5].

Presume a variant of a bicategory where in the source of the horizontal composition we replace the cartesian product by the tensor product of dg categories. Let us call that a bicategory in this remark. The following results indicate that there is a bicategory whose horizontal composition is given by Definition 2.4.6.

First, [BFK, Definition A.10] substantiates that there is a bicategory $\widetilde{\mathcal{LG}}^{\mathrm{dg}}$ of the following form. Its objects are those of $\mathcal{LG}$. The category of morphisms from V to W consists of a certain kind of dg functors $\mathrm{MF}_V \to \mathrm{MF}_W$ and the corresponding natural transformations.

Next, according to [Dyc, Theorem 6.1], there is a dg equivalence between a category of a particular sort of dg functors $\mathrm{MF}_V \to \mathrm{MF}_W$ with the matching natural transformations as morphisms and MF_{-V+W}. This suggests that there is a bicategory similar to $\widetilde{\mathcal{LG}}^{\mathrm{dg}}$ whose category of morphisms from V to W is MF_{-V+W}. It is reasonable that the horizontal composition is given by Definition 2.4.6. Imitating Chapter 2, we expect to extract a bicategory whose categories of morphisms are $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$.

For every pair of objects X, Y, we have a $\mathbb{Z}_2$-action

$$\mathbb{Z}_2 \to \mathrm{Aut}(\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)),\ 0 \mapsto \mathrm{id},\ 1 \mapsto (-\mathrm{id} : [\phi] \mapsto [-\phi]). \tag{5.3.1}$$

We use this instantaneously.

Definition 5.3.3. The category $\mathrm{hmf}^{\bullet,\oplus}_{\mathcal{R},W}$ is the following.

1. The set $(\mathrm{hmf}^{\bullet,\oplus}_{\mathcal{R},W})_{\mathrm{O}}$ of objects is $(\mathrm{hmf}^{\oplus}_{\mathcal{R},W})_{\mathrm{O}}$.
2. For all pairs X, Y of objects the set $\mathrm{hmf}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)$ of morphisms is $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)/\mathbb{Z}_2$, where we quotient by the $\mathbb{Z}_2$-action of (5.3.1).

Explicitly, if ϕ represents a morphism in $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)$, then ϕ and $-\phi$ represent the same morphism in $\mathrm{hmf}^{\bullet,\oplus}_{\mathcal{R},W}(X,Y)$.

The $\mathbb{Z}_2$-action is compatible with the composition of morphisms in $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$ since $(-\psi)\circ\phi = -(\psi\circ\phi) = \psi\circ(-\phi)$ are all representatives of the equivalence class $[\psi\circ\phi] \in \mathrm{hmf}^{\bullet,\oplus}(X,Z)$. Therefore, $\mathrm{hmf}^{\bullet,\oplus}_{\mathcal{R},W}$ is well-defined.

Next, we show that the functors for horizontal composition of morphisms in $\mathcal{LG}$ induce functors on the categories $\mathrm{hmf}^{\bullet,\oplus}$.

Lemma 5.3.4. The functors $\otimes_{\Bbbk[\boldsymbol{y}]}$ of Corollary 2.9.8 induce functors $\otimes_{\Bbbk[\boldsymbol{y}]}$: $\mathrm{hmf}^{\bullet,\oplus}_{-V+W} \times \mathrm{hmf}^{\bullet,\oplus}_{-U+V} \to \mathrm{hmf}^{\bullet,\oplus}_{-U+W}$.

Proof. Note that the functors of extension and restriction of scalars for $\mathrm{MF}_{\mathcal{R},W}$ descend to $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}$ analogously to Lemma 2.6.5. Indeed, the latter's proof does not use that morphisms in $\mathrm{HMF}_{\mathcal{R},W}$ have $\mathbb{Z}_2$-degree zero. Moreover, neither of the two kinds of functors changes the actions of module maps representing morphisms in $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}_{\mathcal{R},W}$ on elements. Therefore they respect the equivalence classes under the $\mathbb{Z}_2$-action (5.3.1). So it suffices to show that the tensor products are functors on $\mathrm{hmf}^{\bullet,\oplus}$. Let ψ and ϕ be module maps of homogeneous $\mathbb{Z}_2$-degree representing morphisms in $\mathrm{hmf}^{\bullet,\oplus}_{-V+W}(Y,Y')$ and $\mathrm{hmf}^{\bullet,\oplus}_{-U+V}(X,X')$, respectively. Then $\psi\otimes_{\Bbbk[\boldsymbol{y}]}\phi$ represents an element of $\mathrm{hmf}^{\bullet,\oplus}_{-U+W}(Y\otimes X, Y'\otimes X')$ since it is $\delta_{Y\otimes X,Y'\otimes X'}$-closed:

$$\begin{aligned} d_{Y'\otimes X'}(\psi\otimes\phi) &= (d_{Y'}\otimes 1 + 1\otimes d_{X'})(\psi\otimes\phi) = d_{Y'}\psi\otimes\phi + (-1)^{|\psi|}\psi\otimes d_{X'}\phi \\ &= (-1)^{|\psi|}\psi d_Y\otimes\phi + (-1)^{|\psi|+|\phi|}\psi\otimes\phi d_X \\ &= (-1)^{|\psi|+|\phi|}(\psi\otimes\phi)(d_Y\otimes 1) + (-1)^{|\psi|+|\phi|}(\psi\otimes\phi)(1\otimes d_X) \\ &= (-1)^{|\psi|+|\phi|}(\psi\otimes\phi)d_{Y\otimes X}. \end{aligned}$$

For $h \in \mathrm{MF}_{-V+W}(Y,Y')^{(|\psi|+1)\mathrm{mod}2}$, the following shows that the tensor product respects the homological equivalence classes of morphisms in the first argument.

$$\begin{aligned} \psi\otimes\phi \sim& (\psi + d_{Y'}h - (-1)^{|h|}hd_Y)\otimes\phi \\ =&\psi\otimes\phi + d_{Y'}h\otimes\phi - (-1)^{|h|}hd_Y\otimes\phi \\ =&\psi\otimes\phi + (d_{Y'}\otimes 1)(h\otimes\phi) - (-1)^{|h|+|\phi|}(h\otimes\phi)(d_Y\otimes 1) \\ =&\psi\otimes\phi + (d_{Y'}\otimes 1 + 1\otimes d_{X'})(h\otimes\phi) - (1\otimes d_{X'})(h\otimes\phi) \\ &- (-1)^{|h|+|\phi|}(h\otimes\phi)(d_Y\otimes 1) \\ =&\psi\otimes\phi + d_{Y'\otimes X'}(h\otimes\phi) - (-1)^{|h|}(h\otimes d_{X'}\phi) - (-1)^{|h|+|\phi|}(h\otimes\phi)(d_Y\otimes 1) \\ =&\psi\otimes\phi + d_{Y'\otimes X'}(h\otimes\phi) - (-1)^{|h|+|\phi|}(h\otimes\phi d_X) \\ &- (-1)^{|h|+|\phi|}(h\otimes\phi)(d_Y\otimes 1) \\ =&\psi\otimes\phi + d_{Y'\otimes X'}(h\otimes\phi) - (-1)^{|h|+|\phi|}(h\otimes\phi)d_{Y\otimes X} \\ =&\psi\otimes\phi + \delta_{Y\otimes X,Y'\otimes X'}(h\otimes\phi). \end{aligned}$$

An analogous computation verifies that the homological equivalence classes in the second factor are respected. That the tensor products are compatible with the $\mathbb{Z}_2$-equivalence classes is analogous to the compatibility of composition and the $\mathbb{Z}_2$-action. Finally, (2.4.4) shows that $\otimes_{\mathrm{k}[\boldsymbol{y}]} : \mathrm{hmf}^{\bullet,\oplus}_{-V+W} \times \mathrm{hmf}^{\bullet,\oplus}_{-U+V} \to \mathrm{hmf}^{\bullet,\oplus}_{-U+W}$ is compatible with the composition of morphisms and preserves identity morphisms. □

We can now assemble the bicategory LG.

Lemma 5.3.5. There is a bicategory LG consisting of the following.

1. The set $\mathrm{LG_O}$ of objects is $\mathcal{LG}_\mathrm{O}$.
2. For every pair $(\boldsymbol{y}, V)$, $(\boldsymbol{z}, W)$ of objects the category $\mathrm{LG}((\boldsymbol{y}, V), (\boldsymbol{z}, W))$ of morphisms is $\mathrm{hmf}^{\bullet,\oplus}_{-V+W}$.
3. For objects $(\boldsymbol{x}, U)$, $(\boldsymbol{y}, V)$, $(\boldsymbol{z}, W)$ of LG, the horizontal composition is given by the functors
$$\otimes_{\mathrm{k}[\boldsymbol{y}]} : \mathrm{hmf}^{\bullet,\oplus}_{-V+W} \times \mathrm{hmf}^{\bullet,\oplus}_{-U+V} \to \mathrm{hmf}^{\bullet,\oplus}_{-U+W}$$
of Lemma 5.3.4.
4. Associators, unit 1-morphisms and unitors are those of $\mathcal{LG}$.

Proof. Note that every 2-morphism in $\mathcal{LG}$ represents a 2-morphisms in LG and 2-isomorphisms in $\mathcal{LG}$ are representatives of 2-isomorphisms in LG. Thus, the associators and unitors as defined in Lemma 5.3.5 have 2-isomorphisms as components. Furthermore, they are natural. The only possible obstruction to this are signs coming from the rule (2.4.1). For the unitors no such signs can prevent them from being natural since they project the unit matrix factorizations in their sources to $\mathbb{Z}_2$-degree zero. For the associators no such sign issues can occur, since the order of the factors of the tensor products of the source and target coincide.

The pentagon and triangle axioms, cf. (A.1.1), (A.1.2), are satisfied since they hold in $\mathcal{LG}$. □

Since the 1-morphisms of LG coincide with those of $\mathcal{LG}$ one can define their adjoints as in $\mathcal{LG}$. As the surjection from 2-morphisms in $\mathcal{LG}$ to 2-morphisms in LG respects the horizontal and vertical composition of 2-morphisms, also the (co-)evaluation morphisms of $\mathcal{LG}$ induce such morphisms in LG. Hence, LG has adjoints.

The symmetric monoidal structure on $\mathcal{LG}$ gives a symmetric monoidal structure on LG. Moreover, since $\mathrm{LG_O} = \mathcal{LG}_\mathrm{O}$, 1-morphisms in $\mathcal{LG}$ are 1-morphisms in LG, and every 2-morphism in $\mathcal{LG}$ determines a 2-morphism in LG, also the duals of $\mathcal{LG}$ carry over to duals in LG.

We summarize these observations in

Proposition 5.3.6. Every 1-morphism in LG has adjoints and there is a symmetric monoidal structure on LG such that every object has a dual.

Remark 5.3.7. We pick up on Remark 5.3.2 about a bicategory with horizontal composition $\widetilde{\mathrm{hmf}}^{\bullet,\oplus} \otimes \widetilde{\mathrm{hmf}}^{\bullet,\oplus} \to \widetilde{\mathrm{hmf}}^{\bullet,\oplus}$. To establish that this bicategory is monoidal, the functors σ in Lemma 3.1.6, which permute the factors in cartesian products of categories are to be replaced by graded versions which multiply morphisms with a factor of (-1) for each pair of $\mathbb{Z}_2$-odd factors whose order is reversed. For example, in Lemma 3.3.5 we have to replace the 2-functor $\tau : \mathcal{LG} \times \mathcal{LG} \to \mathcal{LG} \times \mathcal{LG}$ which exchanges the order of the tuples of objects, 1- and 2-morphisms by a 2-functor which agrees with τ on objects, but acts on 1- and 2-morphisms as the following functors[4]:

$$\widetilde{\mathrm{hmf}}_W^{\bullet,\oplus} \otimes \widetilde{\mathrm{hmf}}_V^{\bullet,\oplus} \to \widetilde{\mathrm{hmf}}_V^{\bullet,\oplus} \otimes \widetilde{\mathrm{hmf}}_W^{\bullet,\oplus},\ (Y,X) \mapsto (X,Y),\ \psi \otimes_{\mathrm{k}} \phi \mapsto (-1)^{|\phi||\psi|} \phi \otimes_{\mathrm{k}} \psi. \tag{5.3.2}$$

This is necessary, e.g. for the morphisms $\beta_{(X,Y)}$ of Definition 3.3.4 to represent natural isomorphisms in $\widetilde{\mathrm{hmf}}^{\bullet,\oplus}$. To see this, let $e \in X_m$, $f \in Y_m$, ψ represent $[\psi] \in \mathrm{H}^1_{\delta_{Y,Y'}}(\mathrm{MF}(Y,Y'))$ and ϕ represent $[\phi] \in \mathrm{H}^1_{\delta_{X,X'}}(\mathrm{MF}(X,X'))$. Then we have

$$f \otimes_{\mathrm{k}} e \xrightarrow{\psi \otimes_{\mathrm{k}} \phi} (-1)^{|f||\phi|} \psi(f) \otimes_{\mathrm{k}} \phi(e) \xrightarrow{\beta_{X,Y}} (-1)^{(|f|+|\psi|)(|e|+|\phi|)+|\phi||f|} \phi(e) \otimes_{\mathrm{k}} \psi(f)$$
$$f \otimes_{\mathrm{k}} e \xrightarrow{\beta_{X',Y'}} (-1)^{|e||f|} e \otimes_{\mathrm{k}} f \xrightarrow{(-1)^{|\phi||\psi|} \phi \otimes_{\mathrm{k}} \psi} (-1)^{|e||f|+|\psi||e|+|\phi||\psi|} \phi(e) \otimes_{\mathrm{k}} \psi(f).$$

The factor of $(-1)^{|\phi||\psi|}$ on the second arrow in the lower line, which is essential for the two lines to coincide, comes from (5.3.2).

Proceeding along the lines of the proof of Lemma 5.2.3, we get that for every object $W \in \mathrm{LG}$ the associated Serre automorphism satisfies $S_W \cong I_W[n]$. Recall from Remark 2.7.4 that we have an isomorphism $\mathrm{H}^0_{\delta_{I,I}}(\mathrm{MF}(I_W, I_W)) \cong \mathrm{H}^1_{\delta_{I,I[1]}}(\mathrm{MF}(I_W, I_W[1]))$. Under this isomorphism the identity 2-morphism on I_W is sent to an isomorphism $I_W \cong I_W[1]$ in $\mathrm{H}^1_{\delta_{I,I[1]}}(\mathrm{MF}(I_W, I_W[1]))$. This isomorphism is a 2-isomorphism in LG. Therefore, for every object of LG the corresponding Serre automorphism is trivializable. Thus, by the 2d cobordism hypothesis for oriented bordisms, Theorem 5.1.8, we have

Corollary 5.3.8. Every object of LG is the value of the positively oriented point under an oriented extended 2d TQFT valued in LG.

Since $\mathcal{LG}_{\mathrm{O}} = \mathrm{LG}_{\mathrm{O}}$, Proposition 5.3.8 implies that despite Corollary 5.2.4 every object of $\mathcal{LG}$ does determine an oriented extended 2d TQFT valued in LG.

[4]The functors (5.3.2) are the natural choice for components of the well-known symmetric braiding for the tensor product of dg categories, cf. [Ke, T]. This features explicitly e.g. in [Mo].

Remark 5.3.9. It is noteworthy that we are led to deviate from the philosophy underlying Remark 2.6.4 in order to get oriented extended 2d TQFTs (valued in LG) that associate to the positively oriented point an object of $\mathcal{LG}$ with an odd number of variables. Indeed, apparently it is not sufficient for all considerations involving $\mathcal{LG}$ that $\mathrm{H}^1(\mathrm{MF}(X,Y))$ can in principle be recovered from $\mathrm{H}^0(\mathrm{MF}(X,Y))$. Another hint of a similar nature is in [M]. There it is shown to be useful to include the shift functor for matrix factorizations as part of the data associated to $\mathcal{LG}$ by regarding $\mathcal{LG}$ as a superbicategory, a structure defined in [M, Definition 2.6]. Arguably, it is also closer to the physical inspiration for studying $\mathcal{LG}$, to keep track of both $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd morphisms of matrix factorizations. To wit, from the quantum field theoretic point of view, the $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd morphisms are interpreted as bosons respectively fermions.

5.3.2. A TQFT à la Khovanov and Rozansky

In this section we discuss one explicit example of an oriented extended 2d TQFT valued in LG. Namely, from now on we specialize to the potential (cf. Examples 2.1)

$$W := z^{m+1} \in \mathrm{k}[z],\ m \in \mathbb{N}$$

and present some details of its associated oriented extended 2d TQFT. We call this TQFT Z_{KR} and compare it with the construction in [KR1, §9]. To enable this we first review the latter.

Remark 5.3.10. Recall from Remark 2.9.4 that in [KR1] Khovanov and Rozansky work with $\mathbb{Z}$-graded matrix factorizations. Furthermore they consider matrix factorizations over formal power series rings. We are not aware of any obstacles to rephrasing our discussion below in those terms. Still we prefer to translate the parts of [KR1] that are relevant to the following into statements concerning LG.

In summarizing the assignments in [KR1, §9] and thereafter we adopt a piece of notation deviating from our previous conventions. To wit, instead of primes on variables as we use them above, in [KR1] the variables are indexed. This is possible since the focus of this paper is on the potential $z^{m+1} \in \mathrm{k}[z]$, which depends only on a single variable.

We paraphrase the claim at the end of [KR1, §9] as saying that the following assignments are those of an oriented extended 2d TQFT $\widetilde{Z_{\mathrm{KR}}} : \mathrm{Bord}_2^{\mathrm{or}} \to \mathrm{LG}$:

1. The positively oriented point pt^+ is assigned (z, z^{m+1}) and $\widetilde{Z_{\mathrm{KR}}}(pt^-) = (z, -z^{m+1})$.

2. An interval as e.g. those in (5.3.3) is assigned the unit matrix factorization $I_{\pm z^{m+1}}$ of $(\mathrm{k}[z], \pm z^{m+1})$. It is implicit that the signs can be chosen consistently.

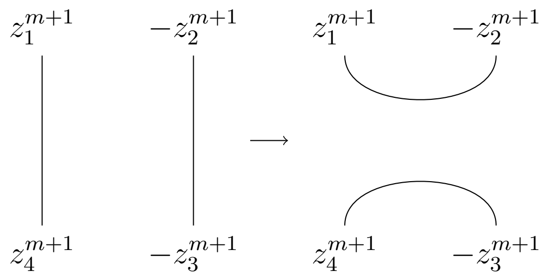
3. To every circle $\widetilde{Z_{\mathrm{KR}}}$ assigns the matrix factorization $A := \big(\{0\} \oplus \mathrm{k}[z]/(z^m), 0\big)$.

4. To several disjoint 1-manifolds $\widetilde{Z_{\mathrm{KR}}}$ assigns the $\square$-product, cf. Definition 3.1.3, of the associated matrix factorizations.

5. To the cup 2-bordism from the empty set to a circle $\widetilde{Z_{\mathrm{KR}}}$ associates the 2-morphism in LG represented by the linear map $\iota : \mathrm{k} \to \mathrm{k}[z]/(z^m)$, $1 \mapsto 1$.

6. To the cap 2-bordism from a circle to the empty set $\widetilde{Z_{\mathrm{KR}}}$ associates the 2-morphism in LG represented by the linear map $\epsilon : \mathrm{k}[z]/(z^m) \to \mathrm{k}$, $z^i \mapsto \frac{1}{m+1}\delta_{i,m-1}$.

7. Consider the saddle 2-bordism (whose source and target we prefer to draw without orientations)



(5.3.3)

This is associated the $\mathbb{Z}_2$-even 2-morphism in LG which is represented by the map η of matrix factorizations that is given by the following matrices[5] with respect to bases that are to be inferred from [KR1]:

$$\left(\begin{pmatrix} \frac{1}{2}(e_{123} + e_{124} + (z_4 - z_3)r) & 1 \\ \frac{1}{2}(-e_{134} - e_{234} + (z_1 - z_2)r) & 1 \end{pmatrix}, \right. \tag{5.3.4}$$
$$\left. \begin{pmatrix} -1 & 1 \\ \frac{1}{2}(-e_{123} - e_{234} + (z_1 - z_4)r) & \frac{1}{2}(-e_{134} - e_{124} + (z_3 - z_2)r) \end{pmatrix} \right).$$

Here $r \in \mathrm{k}[z_1, z_2, z_3, z_4]$ is an arbitrary polynomial of degree $m - 2$ and

$$e_{ijk} := \sum_{(p,q,s) \in \mathbb{N}^3,\ p+q+s=m-1} z_i^p z_j^q z_k^s.$$

8. To a 2-bordism obtained by gluing identity bordisms on intervals and saddle bordisms along boundaries which are neither source nor target 1-bordisms, $\widetilde{Z_{\mathrm{KR}}}$ associates the horizontal composite of the associated maps in LG.

[5] We rescale the non-scalar entries of the matrices given in [KR1] by a factor of $\frac{1}{2}$. This is necessary for η to be closed with respect to the differential specified in [KR1] and therefore for η to be a 2-morphism in LG.

9. To the gluing of 2-bordisms along source and target 1-manifolds $\widetilde{Z_{\mathrm{KR}}}$ associates the composition of maps in LG.

For later reference, we insert $r = \sum_{(i,j,k,l)\in\mathbb{N}^4, i+j+k+l=m-2} z_1^i z_2^j z_3^k z_4^l$ in the matrices (5.3.4) representing the map η:

$$\begin{pmatrix} \frac{1}{2}(e_{123}+e_{124}+(z_4-z_3)r) & 1 \\ \frac{1}{2}(-e_{134}-e_{234}+(z_1-z_2)r) & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e_{123}+e_{124}+e_{124}-e_{123}) & 1 \\ \frac{1}{2}(-e_{134}-e_{234}+e_{134}-e_{234}) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e_{124} & 1 \\ -e_{234} & 1 \end{pmatrix}, \tag{5.3.5}$$

$$\begin{pmatrix} -1 & 1 \\ \frac{1}{2}(-e_{123}-e_{234}+(z_1-z_4)r) & \frac{1}{2}(-e_{134}-e_{124}+(z_3-z_2)r) \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 1 \\ \frac{1}{2}(-e_{123}-e_{234}+e_{123}-e_{234}) & \frac{1}{2}(-e_{134}-e_{124}+e_{134}-e_{124}) \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 1 \\ -e_{234} & -e_{124} \end{pmatrix}. \tag{5.3.6}$$

Note the following consequence of the 2d cobordism hypothesis for oriented bordisms. If there exists an oriented extended 2d TQFT $\widetilde{Z_{\mathrm{KR}}}$ specified by the above list of assignments, then it has to be equivalent to Z_{KR}. With this in mind, we compare the assignments of $\widetilde{Z_{\mathrm{KR}}}$ to those of Z_{KR} in the remainder of this section.

The TQFT Z_{KR} agrees with $\widetilde{Z_{\mathrm{KR}}}$ on 0-manifolds. Both assign (z, z^{m+1}) to the positively oriented point. As the negatively oriented point is the dual to the positively oriented point in $\mathrm{Bord}_2^{\mathrm{or}}$, it is mapped to $(z, -z^{m+1}) = (z, z^{m+1})^*$ by Z_{KR}. This is in line with $\widetilde{Z_{\mathrm{KR}}}$. Since $\widetilde{Z_{\mathrm{KR}}}$ associates the $\square$-product to disjoint unions of 1-manifolds, it must map the disjoint union of points to $\square_{\mathrm{O}}$ of their corresponding objects in LG. Also this matches Z_{KR}.

Next we assert that Z_{KR} agrees with $\widetilde{Z_{\mathrm{KR}}}$ on 1-manifolds. Every connected 1-bordism with non-empty boundary is an interval. To each such 1-bordism $\widetilde{Z_{\mathrm{KR}}}$ associates the unit matrix factorization of $(\mathrm{k}[z], \pm z^{m+1})$. Since the TQFT Z_{KR} respects unit 1-morphisms it also maps each interval viewed as identity 1-bordism to this matrix factorization. Intervals regarded as 1-bordisms with the empty set as either source or target object, cf. the right hand side of (5.3.3), are the (co-)evaluation 1-morphisms in $\mathrm{Bord}_2^{\mathrm{or}}$. They are called "elbows". The TQFT Z_{KR} associates the respective (co-)evaluation 1-morphisms in LG. Their underlying matrix factorizations are unit matrix factorizations. As the specification of $\widetilde{Z_{\mathrm{KR}}}$ in [KR1] is not explicit about the source and target of matrix factorizations viewed as 1-morphisms, we can say that Z_{KR} coincides with $\widetilde{Z_{\mathrm{KR}}}$ on all connected 1-bordisms with nonempty boundary.

To a circle the TQFT Z_{KR} associates $\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger$. This is dictated by viewing the circle as the composition of two elbow 1-bordisms. As for the assignment in part 3 of the specification of $\widetilde{Z_{\mathrm{KR}}}$ the situation is more intricate. In fact, our way of reading [KR1] suggests that we are to identify the matrix factorizations $A = (\{0\} \oplus \mathrm{k}[z]/(z^m), 0)$ and $(\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger)_{\mathrm{mf}}$. Indeed, we have the following

Lemma 5.3.11. There is an isomorphism in $\mathrm{hmf}^{\oplus}_{\mathrm{k},0}$

$$\alpha : (\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger)_{\mathrm{mf}} \to (\mathrm{k}[z]/(z^m) \oplus \{0\}, 0).$$

Proof. Inserting the definitions of ev_W and $\mathrm{ev}_W^\dagger$, we have

$$(\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger)_{\mathrm{mf}} = (\mathrm{ev}_W)_{\mathrm{mf}} \otimes (\mathrm{ev}_W^\dagger)_{\mathrm{mf}} = (I_W)_{\mathrm{mf}} \otimes_{\mathrm{k}[z,z']} (I_W)^\vee_{\mathrm{mf}},$$

where as in the proof of Lemma 5.2.3 we note that $(\mathrm{ev}_W^\dagger)_{\mathrm{mf}} = (\mathrm{ev}_W)^\vee_{\mathrm{mf}}$ since the source object of ev_W has an even number of variables. Next, we use a morphism in $\mathrm{HMF}_{\mathrm{k}[z,z'],0}$ specified in [CM2, (2.39)]:

$$\begin{aligned}(I_W)_{\mathrm{mf}} \otimes_{\mathrm{k}[z,z']} (I_W)^\vee_{\mathrm{mf}} &\to \big(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m), \delta_{I_W,I_W}\big). \\ e \otimes f &\mapsto \big(g \mapsto f(g).e\big)\end{aligned} \tag{5.3.7}$$

The differential $\delta_{I,I}$ is as in Lemma 2.6.1. Note that the conditions for (5.3.7) to be an isomorphism mentioned in [CM2] are satisfied as $(I_W)_m$ is a free $\mathrm{k}[z,z']$-module of finite rank. Now we use that, e.g. by the proof of [CR1, Lemma 2.9], which carries over directly to our setting, the $\delta_{I,I}$-cohomology of $\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m)$ is

$$\begin{aligned}\mathrm{H}^0_{\delta_{I,I}}(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m)) &\cong \mathrm{k}[z]/(z^m), \\ \mathrm{H}^1_{\delta_{I,I}}(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m)) &= \{0\}.\end{aligned} \tag{5.3.8}$$

Before spelling out the isomorphism of k-vector spaces (5.3.8) we note that $(I_W)_m = \mathrm{k}[z]^e \oplus \mathrm{k}[z]^e\theta$. This implies that the components ϕ^i, $i \in \{0,1\}$ of $(\phi^0, \phi^1) \in \mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m)^0$, cf. (2.2.1) for the notation, can be identified with polynomials $p_i \in \mathrm{k}[z,z']$ for $i \in \{0,1\}$. Under this identification the isomorphism (5.3.8) acts on a representative of $[(\phi,\phi)] \in \mathrm{H}^0_{\delta_{I,I}}(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m))$ as

$$(\phi,\phi) \mapsto p|_{z=z'}. \tag{5.3.9}$$

By (5.3.8), $\big(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m), \delta_{I,I}\big)$ and $(\mathrm{k}[z]/(z^m) \oplus \{0\}, 0)$ are quasi-isomorphic chain complexes of k-vector spaces. Hence the two chain complexes are isomorphic in $\mathrm{HMF}_{\mathrm{k},0}$. This holds e.g. by [KR1, Proposition 8] applied to matrix factorizations of $(\mathrm{k}, 0)$. In this case the notion of cohomology of matrix factorizations of [KR1] reduces to ordinary cohomology of complexes of vector spaces. Altogether, since $\mathrm{HMF}_{\mathrm{k}[z,z'],0}$ is a subcategory of $\mathrm{HMF}_{\mathrm{k},0}$, we have

$$(\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger)_{\mathrm{mf}} \cong (\mathrm{k}[z]/(z^m) \oplus \{0\}, 0)$$

in $\mathrm{HMF}_{\mathrm{k},0}$. Now $(\mathrm{k}[z]/(z^m) \oplus \{0\}, 0) \in \mathrm{hmf}_{\mathrm{k},0}$ and by Theorem 2.9.3 $\mathrm{ev}_W \otimes \mathrm{ev}_W^\dagger \in \mathrm{hmf}^{\oplus}_{\mathrm{k},0}$. As $\mathrm{hmf}^{\oplus}_{\mathrm{k},0}$ is a full subcategory of $\mathrm{HMF}_{\mathrm{k},0}$ this concludes the proof. □

Since LG includes $\mathbb{Z}_2$-odd 2-morphisms we have that $A = (\{0\} \oplus \mathrm{k}[z]/(z^m), 0) \cong A[1]$ in LG. Thus it follows from Lemma 5.3.11 that the values of $\widetilde{Z_{\mathrm{KR}}}$ and Z_{KR} on the circle are isomorphic 1-morphisms in LG.

For later reference we assemble the isomorphism α from Lemma 5.3.11:

$$(\mathrm{ev}_W \otimes_{\mathrm{k}[z,z']} \mathrm{ev}_W^\dagger)_{\mathrm{mf}} \to \big(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m), \delta_{I,I}\big) \to (\mathrm{k}[z]/(z^m) \oplus \{0\}, 0).$$

$$z^a(z')^b.e \otimes f^* \mapsto \begin{cases} \dots \mapsto 0, \ e \neq f \\ \big(g \mapsto \quad z^a(z')^b f^*(g).e\big) \mapsto \quad z^{a+b}, \quad e = f \end{cases} \tag{5.3.10}$$

We move on to compare part 4 of the specification of $\widetilde{Z_{\mathrm{KR}}}$ with Z_{KR}. To the disjoint union of 1-bordisms, the TQFT Z_{KR} associates the $\square$-product of the respective matrix factorizations. This is in line with $\widetilde{Z_{\mathrm{KR}}}$.

The gluing of bordisms described in part 8 of our list above is their horizontal composition in $\mathrm{Bord}_2^{\mathrm{or}}$. The TQFT Z_{KR} maps this to the horizontal composition in LG. This agrees with $\widetilde{Z_{\mathrm{KR}}}$.

The gluing of bordisms described in part 9 of the above list is their concatenation as 2-morphisms in $\mathrm{Bord}_2^{\mathrm{or}}$. This gets mapped to the composition of morphisms of matrix factorizations by the TQFT Z_{KR}. This coincides with $\widetilde{Z_{\mathrm{KR}}}$.

It is left to compare the 2-morphisms assigned to the "saddle", "cup" and "cap" 2-bordisms by Z_{KR} with the maps that $\widetilde{Z_{\mathrm{KR}}}$ associates to these bordisms.

We start with the saddle 2-bordism. Its source as depicted on the left hand side of (5.3.3) is to be read as the identity bordism on the disjoint union of a positively and a negatively oriented point. Its target is the horizontal composite (drawn vertically) of first the evaluation elbow and then its adjoint, i.e. a reflected elbow. Applying the TQFT Z_{KR} to this bordism yields, in the notation of Proposition 2.8.7, the 2-morphism in LG represented by

$$\widetilde{\mathrm{coev}}_{\mathrm{ev}_W} : I_{W(z_4)-W(z_3)} \to \mathrm{ev}_W^\dagger \otimes \mathrm{ev}_W$$

Recall the following formula from Proposition 2.8.7:

$$\widetilde{\mathrm{coev}}_X(\gamma) = \sum_{(i,j)\in I^2} (-1)^{(l+1)|e_j|+s} \{\partial_{[b_l]}^{\boldsymbol{x},\boldsymbol{x}'} d_X \dots \partial_{[b_1]}^{\boldsymbol{x},\boldsymbol{x}'} d_X\}_{ji}.e_i^* \widetilde{\otimes}_{\mathrm{k}[\boldsymbol{y}]} e_j, \tag{5.3.11}$$

where $X \in \mathrm{hmf}_{\mathrm{k}[x_1\dots,x_M,y_1,\dots,y_N],V(\boldsymbol{y})-U(\boldsymbol{x})}$, $\{e_i\}_{i\in I}$, I some index set, is a basis of X_m with dual basis $\{e_i^*\}_{i\in I}$, and $b_1 < \dots < b_l$ and s are such that $\gamma \wedge \theta_{b_1} \dots \theta_{b_l} = (-1)^s \theta_1 \dots \theta_M$. Specializing to $X = \mathrm{ev}_W \in \mathrm{hmf}_{\mathrm{k}[z_4,z_3],-z_4^{m+1}+z_3^{m+1}}$, cf. Definition 4.2, yields:

$$\widetilde{\mathrm{coev}}_{\mathrm{ev}_W}(\gamma) = \sum_{(e,f)\in\{1,\theta\}^2} (-1)^{(l+1)|f|+s} \{\partial_{[b_l]}^{\boldsymbol{z},\boldsymbol{z}'} d_{I_W} \dots \partial_{[b_1]}^{\boldsymbol{z},\boldsymbol{z}'} d_{I_W}\}_{fe}.e^* \otimes_{\mathrm{k}} f. \tag{5.3.12}$$

Here we use the basis $\{1, \theta\}$ with $\deg_{\mathbb{Z}_2}(1) = 0$, $\deg_{\mathbb{Z}_2}(\theta) = 1$ for $(\mathrm{ev}_W)_m = (I_W)_m$ as in Lemma 2.5.3. Furthermore, we note that the horizontal composition of $\mathrm{ev}_W^\dagger$ and ev_W is indeed $\mathrm{ev}_W^\dagger \otimes_{\mathrm{k}} \mathrm{ev}_W$. Finally, we write $\boldsymbol{z} \equiv (z_4, z_3)$.

Now l, $b_1 < \cdots < b_l$ and s in (5.3.12) are specified by $\gamma \wedge \theta_{b_1} \ldots \theta_{b_l} = (-1)^s \theta_1\theta_2$. In the present case, the basis element $\gamma \in (I_{-W(z_4)+W(z_3)})_m$ can take four values, $\gamma \in \{1, \theta_1, \theta_2, \theta_1\theta_2\}$. Thus, we have the options

$$\begin{aligned} &\gamma = 1 \Rightarrow b_1 = 1,\ b_2 = 2,\ s = 0, && \gamma = \theta_1 \Rightarrow b_1 = 2, s = 0, \\ &\gamma = \theta_2 \Rightarrow b_1 = 1,\ s = 1, && \gamma = \theta_1\theta_2 \Rightarrow l = 0,\ s = 0. \end{aligned}$$

To compute $\widetilde{\mathrm{coev}}_{\mathrm{ev}_W}(\gamma)$ we also need:

$$\begin{aligned} \partial^{z,z'}_{[1]} d_{I_W} &= \frac{(z_3 - z_4) - (z_3 - z_4')}{z_4 - z_4'} .\theta^* + e_{3,4,4'}.\theta \wedge (-) = -\theta^* + e_{3,4,4'}.\theta \wedge (-), \\ \partial^{z,z'}_{[2]} d_{I_W} &= \frac{(z_3 - z_4') - (z_3' - z_4')}{z_3 - z_3'} .\theta^* + e_{3,3',4'}.\theta \wedge (-) = \theta^* + e_{3,3',4'}.\theta \wedge (-). \end{aligned}$$

Here we write $e_{3,4,4'} = \sum_{(a,b,c)\in\mathbb{N}^3, a+b+c=m-1} z_3^a z_4^b (z_4')^c$ and similarly for $e_{3,3',4}$. In line with (5.3.3) we identify $z_4 \to z_1$, $z_4' \to z_4$, $z_3 \to z_2$, $z_3' \to z_3$. Under this identification we replace $e_{3,4,4'}$ by e_{214} (note that $e_{214} = e_{124}$) and $e_{3,3',4'}$ by e_{234} in the following.

We assemble $\widetilde{\mathrm{coev}}_{\mathrm{ev}_W}$:

$$\begin{aligned} 1 &\mapsto \sum_{(e,f)\in\{1,\theta\}^2} (-1)^{3|f|} \big\{(\theta^* + e_{234}.\theta \wedge (-))(-\theta^* + e_{124}.\theta \wedge (-))\big\}_{fe}.e^* \otimes_{\mathrm{k}} f \\ &= e_{124}.1^* \otimes_{\mathrm{k}} 1 + e_{234}.\theta^* \otimes_{\mathrm{k}} \theta, \\ \theta_1 &\mapsto \sum_{(e,f)\in\{1,\theta\}^2} \{\theta^* + e_{234}.\theta \wedge (-)\}_{fe}.e^* \otimes_{\mathrm{k}} f \\ &= e_{234}.1^* \otimes_{\mathrm{k}} \theta + \theta^* \otimes_{\mathrm{k}} 1, \\ \theta_2 &\mapsto - \sum_{(e,f)\in\{1,\theta\}^2} \{-\theta^* + e_{124}.\theta \wedge (-)\}_{fe}.e^* \otimes_{\mathrm{k}} f \\ &= -e_{124}.1^* \otimes_{\mathrm{k}} \theta + \theta^* \otimes_{\mathrm{k}} 1, \\ \theta_1\theta_2 &\mapsto \sum_{(e,f)\in\{1,\theta\}^2} (-1)^{|f|} \delta_{fe}.e^* \otimes_{\mathrm{k}} f = 1^* \otimes_{\mathrm{k}} 1 - \theta^* \otimes_{\mathrm{k}} \theta. \end{aligned}$$

The matrix representing $\widetilde{\mathrm{coev}}_{\mathrm{ev}_W}$ with respect to the bases $\{1, \theta_1\theta_2, -\theta_1, \theta_2\}$ of $(I_{W(z_4)-W(z_3)})_m$ and $\{1^* \otimes 1, -\theta^* \otimes \theta, \theta^* \otimes 1, 1^* \otimes \theta\}$ of $(\mathrm{ev}_W^\dagger \otimes \mathrm{ev}_W)_m$ is

$$\begin{pmatrix} e_{124} & 1 & 0 & 0 \\ -e_{234} & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -e_{234} & -e_{124} \end{pmatrix}. \tag{5.3.13}$$

This coincides with (5.3.5), (5.3.6).

Remark 5.3.12. We can equivalently view the saddle bordism (5.3.3) as the coevaluation 2-morphism of the coevaluation 1-morphism of the negatively oriented point, $\mathrm{coev}_{\mathrm{coev}_{pt-}}$. Computing $\mathrm{coev}_{\mathrm{coev}_{-W}}$ using the formula in Proposition 2.8.7 and representing it with respect to suitable bases we again get the matrix (5.3.13).

Next, we compare the 2-morphism ϵ in part 6 of the list defining $\widetilde{Z_{\mathrm{KR}}}$ to $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$. Indeed, the cap 2-bordism represents the evaluation of the evaluation elbow 1-bordism of the positively oriented point. Thus, Z_{KR} maps this 2-bordism to the 2-morphism in LG represented by $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$. Recall the formula for

$$\widetilde{\mathrm{ev}}_X : X \otimes X^\dagger \to I_V,$$

$X \in \mathcal{LG}\big(((x_1, \dots, x_M), U), ((y_1, \dots, y_N), V)\big)$ a matrix factorization of finite rank, $p \in \mathrm{k}[\boldsymbol{x}]$, from Proposition 2.8.7:

$$p.e_j \otimes_{\mathrm{k}[\boldsymbol{x}]} e_i^* \mapsto \sum_{l \geq 0} \sum_{i_1 < \dots < i_l} (-1)^{l+(M+1)|e_j|} \theta_{i_1} \dots \theta_{i_l} \operatorname{Res}_{\mathrm{k}[\boldsymbol{x}]/\mathrm{k}} \left[\frac{\{\partial_{[i_l]}^{\boldsymbol{y},\boldsymbol{y}'} d_X \dots \partial_{[i_1]}^{\boldsymbol{y},\boldsymbol{y}'} d_X \Lambda^{(x)}\}_{ij} p \,\mathrm{d}\, x}{\partial_{x_1} U, \dots, \partial_{x_M} U} \right].$$

Adapting this to the case $X = \mathrm{ev}_W \in \mathcal{LG}(W(z_1) - W(z_2), 0)$ yields

$$\begin{aligned} \widetilde{\mathrm{ev}}_{\mathrm{ev}_W} : \mathrm{ev}_W \otimes_{\mathrm{k}[z_1, z_2]} \mathrm{ev}_W^\dagger &\to I_0 \\ p.f \otimes_{\mathrm{k}[z_1, z_2]} e^* &\mapsto (-1)^{|f|} \operatorname{Res}_{\mathrm{k}[\boldsymbol{z}]/\mathrm{k}} \left[\frac{\{\partial_{z_1} d_{I_W} \partial_{z_2} d_{I_W}\}_{ef} \; p \,\mathrm{d}\, z}{\partial_{z_1} W, -\partial_{z_2} W} \right]. \end{aligned}$$

As a first step to compute this, we have

$$\begin{aligned} \{\partial_{z_1} d_{I_W} \partial_{z_2} d_{I_W}\}_{ef} &= \Big\{ \big(-\theta^* + \sum_{a=0}^m a z_1^{a-1} z_2^{m-a}.\theta \wedge (-) \big) \big(\theta^* + \sum_{b=0}^m b z_1^{m-b} z_2^{b-1}.\theta \wedge (-) \big) \Big\}_{ef} \\ &= \begin{cases} -\sum_{b=0}^m b z_1^{m-b} z_2^{b-1}, & e = f = 1 \\ \sum_{a=0}^m a z_1^{a-1} z_2^{m-a}, & e = f = \theta \\ 0, & e \neq f \end{cases} . \end{aligned}$$

In the following calculation we use two of the main properties of residues, cf. [CM2, §2.4]: they are linear as in (5.3.14) and they behave like residues from complex analysis in the sense that we have the equality (5.3.15) below.

$$\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}(z_1^c\ z_2^d\ 1 \otimes 1^*) = \mathrm{Res}_{\mathrm{k}[\boldsymbol{z}]/\mathrm{k}} \left[\frac{-\sum_{b=0}^{m} b z_1^{m-b} z_2^{b-1}\ z_1^c\ z_2^d\ \mathrm{d}\, z_1\ \mathrm{d}\, z_2}{(m+1)z_1^m, -(m+1)z_2^m} \right]$$

$$= \sum_{b=0}^{m} \frac{b}{(m+1)^2} \mathrm{Res}_{\mathrm{k}[\boldsymbol{z}]/\mathrm{k}} \left[\frac{z_1^{m-b+c}\ z_2^{b-1+d}\ \mathrm{d}\, z_1\ \mathrm{d}\, z_2}{z_1^m, z_2^m} \right] \tag{5.3.14}$$

$$= \sum_{b=0}^{m} \frac{b}{(m+1)^2}\ \delta_{m-b+c,m-1}\delta_{b-1+d,m-1} \tag{5.3.15}$$

$$= \frac{c+1}{(m+1)^2}\delta_{c+d,m-1}, \tag{5.3.16}$$

$$\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}(z_1^c\ z_2^d\ \theta \otimes \theta^*) = -\,\mathrm{Res}_{\mathrm{k}[\boldsymbol{z}]/\mathrm{k}} \left[\frac{\sum_{a=0}^{m} a z_1^{a-1} z_2^{m-a}\ z_1^c\ z_2^d\ \mathrm{d}\, z_1\ \mathrm{d}\, z_2}{(m+1)z_1^m, -(m+1)z_2^m} \right]$$

$$= \sum_{a=0}^{m} \frac{a}{(m+1)^2} \mathrm{Res}_{\mathrm{k}[\boldsymbol{z}]/\mathrm{k}} \left[\frac{z_1^{a-1+c}\ z_2^{m-a+d}\ \mathrm{d}\, z_1\ \mathrm{d}\, z_2}{z_1^m, z_2^m} \right]$$

$$= \sum_{a=0}^{m} \frac{a}{(m+1)^2}\ \delta_{a-1+c,m-1}\delta_{m-a+d,m-1}$$

$$= \frac{d+1}{(m+1)^2}\delta_{c+d,m-1}. \tag{5.3.17}$$

In view of Lemma 5.3.11 we pre-compose the map ϵ that $\widetilde{Z_{\mathrm{KR}}}$ associates to the cap, cf. part 6 of our list, with the isomorphism α, cf. (5.3.10), in order to compare it with $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$. This yields

$$\begin{aligned} \mathrm{ev}_W \otimes_{\mathrm{k}[z_1,z_2]} \mathrm{ev}_W^\dagger &\to A[1] &&\to I_0 \\ z_1^c\ z_2^d\ e \otimes e^* &\mapsto z^{c+d} &&\mapsto \frac{1}{m+1}\delta_{c+d,m-1}. \end{aligned}$$

To see that $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$ and $\epsilon \circ \alpha$ are not homotopic and therefore do not coincide as 2-morphisms in LG, we show that they induce different maps on cohomology, cf. [We, Lemma 1.4.5]. In light of (5.3.9) we see that $\epsilon \circ \alpha$ induces the map

$$z^a \mapsto (\epsilon \circ \alpha)(z_1^a(1 \otimes 1^* + \theta \otimes \theta^*)) = \epsilon(2z^a) = \frac{2}{m+1}\delta_{a,m-1}$$

on cohomology. The map $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$ induces

$$z^a \mapsto \widetilde{\mathrm{ev}}_{\mathrm{ev}_W}(z_1^a(1 \otimes 1^* + \theta \otimes \theta^*)) = \frac{a+1}{(m+1)^2}\delta_{a,m-1} + \frac{1}{(m+1)^2}\delta_{a,m-1} = \frac{a+2}{(m+1)^2}\delta_{a,m-1}.$$

Thus, we see that the maps which $\epsilon \circ \alpha$ and $\widetilde{\mathrm{ev}}_{\mathrm{ev}_W}$ induce on cohomology disagree. Therefore, we can say that the value that $\widetilde{Z_{\mathrm{KR}}}$ takes on the cap 2-bordism differs from the one of Z_{KR} by an invertible scalar prefactor.

Finally, we turn to the map ι in part 5 of our list specifying $\widetilde{Z_{\mathrm{KR}}}$. This is associated to the cup 2-bordism, which is the coevaluation of the coevaluation 1-bordism of the positively oriented point. Hence we compute

$$\widetilde{\mathrm{coev}}_{\mathrm{coev}_W} : I_0 \to \mathrm{coev}_W^\dagger \otimes \mathrm{coev}_W.$$

The corresponding formula in Proposition 2.8.7 is

$$\gamma \mapsto \sum_{(i,j)\in I^2} (-1)^{(l+1)|e_j|+s}\{\partial_{[b_l]}^{\boldsymbol{x},\boldsymbol{x}'} d_X \dots \partial_{[b_1]}^{\boldsymbol{x},\boldsymbol{x}'} d_X\}_{ji}.e_i^* \otimes_{\mathrm{k}[\boldsymbol{y}]} e_j,$$

where $X \in \mathrm{hmf}_{\mathrm{k}[x_1 \dots, x_M, y_1, \dots, y_N], V(\boldsymbol{y})-U(\boldsymbol{x})}$ with a basis $\{e_i\}_{i\in I}$, I some index set, of X_m, etc. cf. below (5.3.11). In the case of $X = \mathrm{coev}_W$ this reduces to

$$\widetilde{\mathrm{coev}}_{\mathrm{coev}_W}(1) = \sum_{(e,f)\in\{1,\theta\}^2} (-1)^{|f|}\delta_{e,f}.e^* \otimes f = \sum_{e\in\{1,\theta\}} (-1)^{|e|}.e^* \otimes e = 1^* \otimes 1 - \theta^* \otimes \theta.$$

To compare this with the specification of $\widetilde{Z_{\mathrm{KR}}}$ we need to compose $\widetilde{\mathrm{coev}}_{\mathrm{coev}_W}$ with an isomorphism

$$\widetilde{\alpha} : \mathrm{coev}_W^\dagger \otimes \mathrm{coev}_W \to (\mathrm{k}[z]/(z^m) \oplus \{0\}, 0)$$

in $\mathrm{hmf}_{\mathrm{k},0}$. To get $\widetilde{\alpha}$ we can replace the isomorphism in [CM2, (2.39)] by that in [CM2, (2.37)] in our proof of Lemma 5.3.11. This introduces an additional minus sign in the formula (5.3.10) if $e = f = \theta$. Thus, we get the map $\widetilde{\alpha}$:

$$(\mathrm{coev}_W^\dagger \otimes_{\mathrm{k}[z,z']} \mathrm{coev}_W)_{\mathrm{mf}} \to \big(\mathrm{Mod}_{\mathrm{k}[z,z']}((I_W)_m, (I_W)_m), \delta_{I,I}\big) \to (\mathrm{k}[z]/(z^m) \oplus \{0\}, 0).$$

$$z^a(z')^b.e^* \otimes f \mapsto \begin{cases} \dots \mapsto 0, \ e \neq f \\ \big(g \mapsto \ \ z^a(z')^b e^*(g).f\big) \mapsto \ \ z^{a+b}, \ \ e = f = 1^* \\ \big(g \mapsto -z^a(z')^b e^*(g).f\big) \mapsto -z^{a+b}, \ e = \theta = f \end{cases}.$$

Composing $\widetilde{\mathrm{coev}}_{\mathrm{coev}_W}$ with $\widetilde{\alpha}$ leaves us with the k-linear map $1 \mapsto 2$. This is simultaneously the map it induces on cohomology. This shows that the map ι in part 5 of our list above is not the same 2-morphism in LG as $\widetilde{\alpha} \circ \widetilde{\mathrm{coev}}_{\mathrm{coev}_W}$.

Hence we can say that $\widetilde{Z_{\mathrm{KR}}}$ disagrees from Z_{KR} on the cup 2-bordism by a factor of 2.

We summarize the results of this subsection. Denote by $Z_{\mathrm{KR}} : \mathrm{Bord}_2^{\mathrm{or}} \to \mathrm{LG}$ the oriented extended 2d TQFT determined by $Z_{\mathrm{KR}}(pt^+) = (z, z^{m+1})$. The assignments in [KR1, §9], which we refer to as $\widetilde{Z_{\mathrm{KR}}}$, coincide with Z_{KR} on objects as well as on 1- and 2-bordisms which do not involve the circle. The two rules disagree on the circle by a 2-isomorphism α. Taking this into account $\widetilde{Z_{\mathrm{KR}}}$ still differs from Z_{KR} on the cap and cup 2-bordisms by invertible scalar prefactors.

We end by discussing some consequences of our findings summarized in the preceding paragraph. The value of an oriented extended 2d TQFT on the circle is dictated by what it assigns to the elbow 1-bordisms and the natural isomorphisms expressing how it is compatible with the horizontal composition of bordisms. For $\widetilde{Z_{\mathrm{KR}}}$ the latter isomorphisms are identities, as is the case for Z_{KR}. Since $\widetilde{Z_{\mathrm{KR}}}$ also agrees with Z_{KR} on the elbow 1-bordisms, $\widetilde{Z_{\mathrm{KR}}}$ needs to be modified by the 2-isomorphism α in order to potentially describe an oriented extended 2d TQFT.

According to the classification of oriented extended 2d TFQTs in [SP1] the 2-morphisms associated by such a TQFT to the cap and cup 2-bordisms need to satisfy relations which also involve the maps associated to the saddle 2-bordism, cf. [SP1, Figure 3.13]. Since Z_{KR} is an oriented extended 2d TQFT it satisfies these relations. As Z_{KR} agrees with $\widetilde{Z_{\mathrm{KR}}}$ on all ingredients of these relations except the cap and cup 2-bordisms, it follows that $\widetilde{Z_{\mathrm{KR}}}$ can only satisfy these relations if it is adjusted by the invertible scalar prefactors in which it differs from Z_{KR} on the cap and cup 2-bordisms. Thus, $\widetilde{Z_{\mathrm{KR}}}$ needs to be modified by these prefactors in order to define an oriented extended 2d TQFT "up to composing the 2-morphism associated to the circle with a 2-isomorphism".

A. Some higher categorical notions

We assume the reader is familiar with the definition of a category, of a functor and of a natural transformation. The higher categorical notions needed in the body of the thesis are summarized below. We mostly follow [SP1, §2.3, Appendix A, Appendix C]. All the diagrams below are adapted from there.

Notation A.0.1. Let $\mathfrak{C}$ be a category. We denote the set of objects of $\mathfrak{C}$ as $\mathfrak{C}_{\mathrm{O}}$. For an object $C \in \mathfrak{C}_{\mathrm{O}}$ we also write $C \in \mathfrak{C}$. For A, B objects of $\mathfrak{C}$, we denote the set of morphisms from A to B in $\mathfrak{C}$ by $\mathfrak{C}(A, B)$.

A.1. Bicategories, 2-functors, and more

Given three categories $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ denote by $\mathfrak{a} : (\mathfrak{A} \times \mathfrak{B}) \times \mathfrak{C} \to \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C})$ the functor which acts on objects as $((A, B), C) \mapsto (A, (B, C))$ and analogously on morphisms.

Definition A.1.1. A *bicategory* $\mathcal{B}$ consists of

1. a collection $\mathcal{B}_{\mathrm{O}}$ of *objects* $A \in \mathcal{B}$,
2. for every pair A, B of objects a category $\mathcal{B}(A, B)$ whose objects are called *1-morphisms* (from A to B) and whose morphisms are called *2-morphisms*,
3. for every triple A, B, C of objects a functor

$$\otimes_{A,B,C} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$$

called *horizontal composition*,

4. for every object A a distinguished 1-morphism $1_A \in \mathcal{B}(A, A)$ called the *unit 1-morphism*,
5. for all quadruples A, B, C, D of objects natural isomorphisms

$$\alpha_{A,B,C,D} : \otimes_{A,B,D} \circ (\otimes_{B,C,D} \times 1_{\mathcal{B}(A,B)}) \to \otimes_{A,C,D} \circ (1_{\mathcal{B}(C,D)} \times \otimes_{A,B,C}) \circ \mathfrak{a}$$

referred to as *associators*,

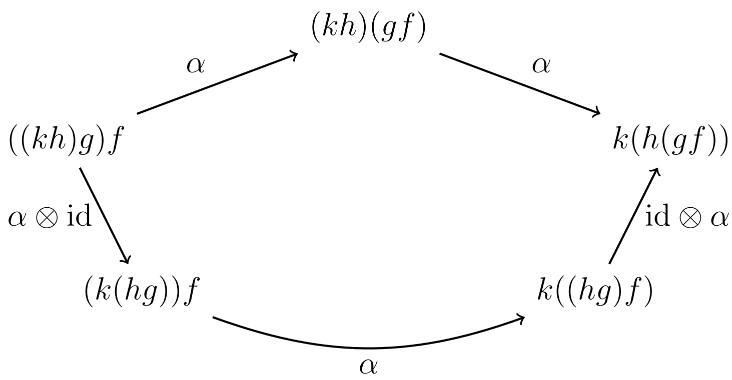
6. for all pairs A, B of objects natural isomorphisms

$$\lambda^{A,B} : \otimes_{A,B,B}(1_B, -) \to 1_{\mathcal{B}(A,B)},\ \rho^{A,B} : \otimes_{A,A,B}(-, 1_A) \to 1_{\mathcal{B}(A,B)}$$

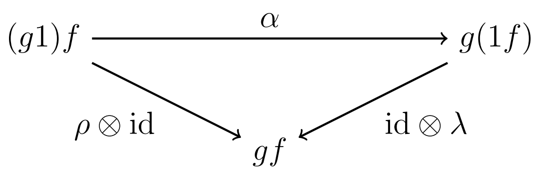
called *unitors*

such that the diagrams depicting the pentagon identity, cf. (A.1.1), and the triangle identity, cf. (A.1.2), commute for all suitably composable 2-morphisms f, g, h, k.

In the diagrams relevant to Definition A.1.1 we suppress $\otimes$ in the composition of 1-morphisms and indices as well as superscripts on α and λ.



(A.1.1)



(A.1.2)

Definition A.1.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bicategories. A *2-functor* $F : \mathcal{B} \to \mathcal{B}'$ consists of

1. a function $F_{\mathrm{O}} : \mathcal{B}_{\mathrm{O}} \to \mathcal{B}'_{\mathrm{O}}$,
2. for every pair A, B of objects in $\mathcal{B}$ a functor

$$F_{A,B} : \mathcal{B}(A, B) \to \mathcal{B}'(F_{\mathrm{O}}(A), F_{\mathrm{O}}(B)),$$

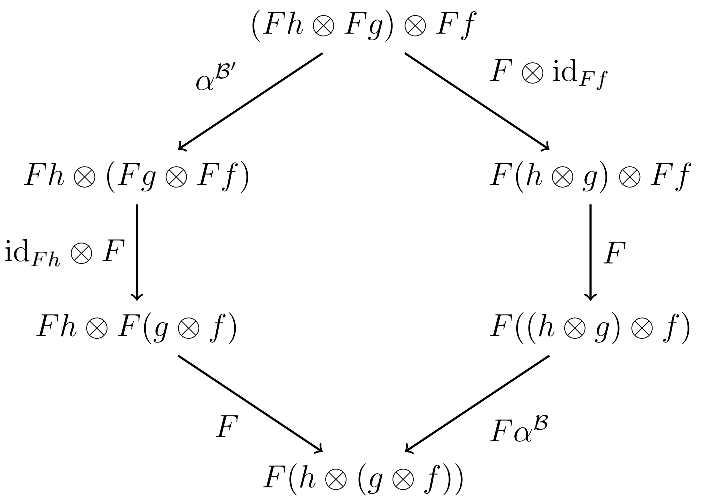
3. for every triple A, B, C of objects in $\mathcal{B}$ a natural isomorphism

$$F_{A,B,C} : \otimes^{\mathcal{B}'}_{F_{\mathrm{O}}(A),F_{\mathrm{O}}(B),F_{\mathrm{O}}(C)} \circ (F_{B,C} \times F_{A,B}) \to F_{A,C} \circ \otimes^{\mathcal{B}}_{A,B,C},$$

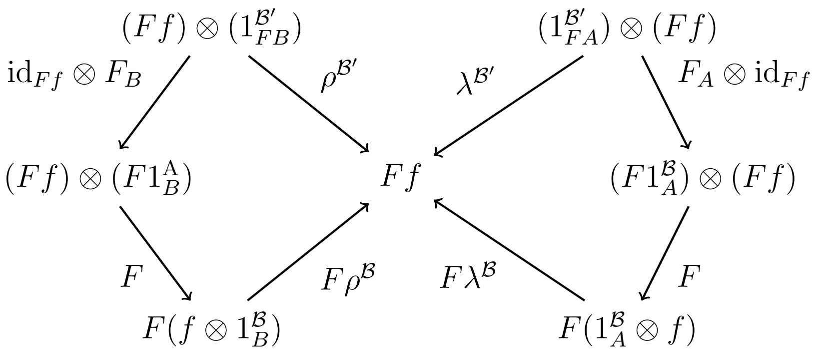
4. for every object A in $\mathcal{B}$ an isomorphism $F_A : 1^{\mathcal{B}'}_{F_{\mathrm{O}}(A)} \to F_{A,A}(1^{\mathcal{B}}_A)$

such that the diagrams (A.1.3) and (A.1.4) commute for all objects A and B in $\mathcal{B}$ and suitably composable 1-morphisms f, g, h in $\mathcal{B}$.

In the diagrams depicting the axioms on a 2-functor we write $FA := F_O(A)$ for $A \in \mathcal{B}$ and similarly for 1- and 2-morphisms. Furthermore, as in most of this thesis, we suppress indices on the components of 2-functors.



(A.1.3)



(A.1.4)

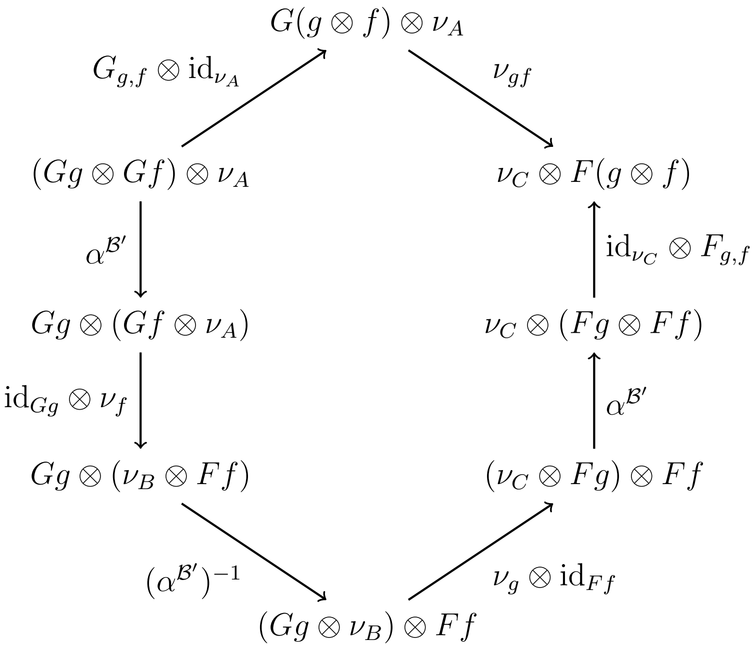
Remark A.1.3. 2-functors according to Definition A.1.2 are referred to as "homomorphisms" in [SP1]. A 2-functor $F : \mathcal{B} \to \mathcal{B}'$ is called *strict* if F_A and $F_{A,B,C}$ are identities for all objects A, B, C of $\mathcal{B}$.

Definition A.1.4. Let $\mathcal{B}$, $\mathcal{B}'$ be bicategories and F, G be 2-functors $\mathcal{B} \to \mathcal{B}'$. A *pseudonatural transformation* $\nu : F \to G$ consists of

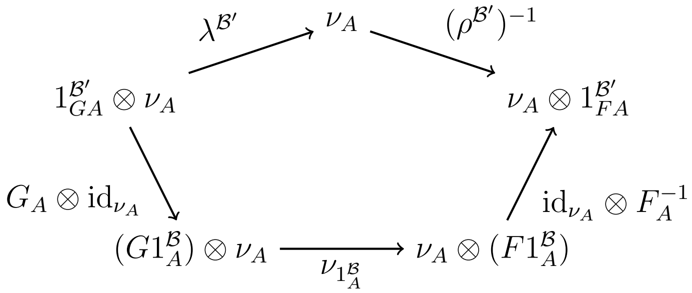
1. for every object A of $\mathcal{B}$ a 1-morphism $\nu_A \in \mathcal{B}'(F(A), G(A))$,
2. for every pair A, B of objects of $\mathcal{B}$ a natural isomorphism

$$\nu_{A,B} : G_{A,B}(-) \otimes^{\mathcal{B}'} \nu_A \to \nu_B \otimes^{\mathcal{B}'} F_{A,B}(-)$$

such that the diagrams (A.1.5) and (A.1.6) commute for all objects A, B, C of $\mathcal{B}$ and 1-morphisms $f \in \mathcal{B}(A, B)$, $g \in \mathcal{B}(B, C)$.

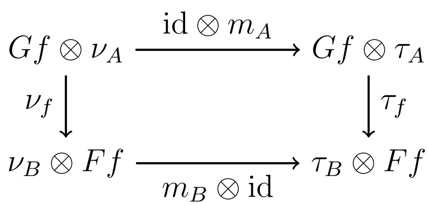


(A.1.5)



(A.1.6)

Definition A.1.5. Let $\mathcal{B}$, $\mathcal{B}'$ be bicategories, F, G be 2-functors $\mathcal{B} \to \mathcal{B}'$ and ν, τ be pseudonatural transformations $F \to G$. A *modification* $m : \nu \to \tau$ consists of a 2-morphism $m_A : \nu_A \to \tau_A$ for every $A \in \mathcal{B}$ such that the diagram (A.1.7) commutes for all objects A, B of $\mathcal{B}$. A modification is *invertible* if its components are 2-isomorphisms.



(A.1.7)

2-functors can be composed as follows.

Definition A.1.6. Let $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ be bicategories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be 2-functors. The composite $G \circ F : \mathcal{A} \to \mathcal{C}$ is defined to be the 2-functor with components

1. the function $(G \circ F)_{\mathrm{O}} := G_{\mathrm{O}} \circ F_{\mathrm{O}} : \mathcal{A}_{\mathrm{O}} \to \mathcal{C}_{\mathrm{O}}$,
2. the functor
$$(G \circ F)_{A,B} := G_{F(A),F(B)} \circ F_{A,B} : \mathcal{A}(A, B) \to \mathcal{C}((G \circ F)(A), (G \circ F)(B)),$$
3. the natural transformations $1_{(G \circ F)(A)} \xrightarrow{G_{F(A)}} G(1_{F(A)}) \xrightarrow{G(F_A)} (G \circ F)(1_A)$ and
$$\begin{array}{c} \otimes^{\mathcal{C}}_{GF(A),GF(B),GF(C)} \circ (G_{F(B),F(C)} \times G_{F(A),F(B)}) \circ (F_{B,C} \times F_{A,B}) \\ \Big\downarrow G_{F(A),F(B),F(C)} \otimes (F_{B,C} \times F_{A,B}) \\ G_{F(A),F(C)} \circ \otimes^{\mathcal{B}}_{F(A),F(B),F(C)} \circ (F_{B,C} \times F_{A,B}) \\ \Big\downarrow G_{F(A),F(C)} \circ F_{A,B,C} \\ G_{F(A),F(C)} \circ F_{A,C} \circ \otimes^{\mathcal{A}}_{A,B,C} \,. \end{array}$$

Pseudonatural transformations can be composed according to

Definition A.1.7. Let $\mathcal{B}$, $\mathcal{B}'$ be bicategories, F, G and H be 2-functors $\mathcal{B} \to \mathcal{B}'$ and $\nu : F \to G$, $\tau : G \to H$ be pseudonatural transformations. The composite $\tau \circ \nu : F \to H$ is the pseudonatural transformation with components

1. the 1-morphism $\tau_A \otimes \nu_A$ for every $A \in \mathcal{B}$
2. the natural isomorphism $H(-) \otimes (\tau \circ \nu)_A \to (\tau \circ \nu)_B \otimes F(-)$ given by

$$H_{A,B} \otimes (\tau_A \otimes \nu_A) \xrightarrow{\alpha^{-1}} (H_{A,B} \otimes \tau_A) \otimes \nu_A \xrightarrow{\tau} (\tau_B \otimes G_{A,B}) \otimes \nu_A$$
$$\xrightarrow{\alpha} \tau_B \otimes (G_{A,B} \otimes \nu_A) \xrightarrow[\nu]{} \tau_B \otimes (\nu_B \otimes F_{A,B}) \xrightarrow[\alpha^{-1}]{} (\tau_B \otimes \nu_B) \otimes F_{A,B}.$$

For modifications there are the following compositions

Definition A.1.8. Let $\mathcal{B}$, $\mathcal{B}'$ be bicategories, F and G 2-functors $\mathcal{B} \to \mathcal{B}'$, ν, σ, τ pseudonatural transformations $F \to G$, m a modification from ν to σ and n a modification from σ to τ. The *vertical composite* $n \circ m$ is the modification from ν to τ given by $(n \circ m)_A = n_A \circ m_A$ for all $A \in \mathcal{B}$. Let there be a third 2-functor $H : \mathcal{B} \to \mathcal{B}'$, pseudonatural transformations $\widetilde{\nu}$, $\widetilde{\sigma}$ from G to H and a modification $\widetilde{m}$ from $\widetilde{\nu}$ to $\widetilde{\sigma}$. The *horizontal composite* of $\widetilde{m}$ and m is the modification $\widetilde{m} \otimes m$ from $\widetilde{\nu} \circ \nu$ to $\widetilde{\sigma} \circ \sigma$ with components $(\widetilde{m} \otimes m)_A = \widetilde{m}_A \otimes m_A$ for all $A \in \mathcal{B}$.

According to [G, §6.3.1] for every pair of bicategories $\mathcal{B}$, $\mathcal{B}'$ one obtains a bicategory $\mathrm{Bicat}(\mathcal{B}, \mathcal{B}')$ whose objects are 2-functors, 1-morphisms are pseudonatural transformations and 2-morphisms are modifications.

We end our exposition of background on bicategories with two notions of equivalence.

Definition A.1.9. Let $\mathcal{B}$ be a bicategory and A, B objects of $\mathcal{B}$. A and B are *equivalent* if there are 1-morphisms $f \in \mathcal{B}(A, B)$, $g \in \mathcal{B}(B, A)$ together with 2-isomorphisms $f \otimes g \cong 1_B$, $g \circ f \cong 1_A$.

Definition A.1.10. Let $\mathcal{B}$, $\mathcal{B}'$ be bicategories. A 2-functor $F : \mathcal{B} \to \mathcal{B}'$ is an *equivalence of bicategories* if the following conditions are satisfied.

1. For every object $B \in \mathcal{B}'$ there exists an object $A \in \mathcal{B}$ such that $F(A)$ is equivalent to B in $\mathcal{B}'$.
2. For all objects A, B of $\mathcal{B}$ the functor $F_{A,B} : \mathcal{B}(A, B) \to \mathcal{B}'(F(A), F(B))$ is an equivalence, i.e. essentially surjective and fully faithful.

If for a 2-functor $F : \mathcal{B} \to \mathcal{B}'$ for all objects A, B of $\mathcal{B}$ the functor $F_{A,B}$ is an equivalence we say that F is *locally an equivalence.*

A.2. Adjoints in bicategories

Let $\mathcal{B}$ be a bicategory, A, B objects of $\mathcal{B}$.

Definition A.2.1. An *adjunction* in $\mathcal{B}$ is quadruple $(f, g, \mathrm{ev}, \mathrm{coev})$ consisting of two 1-morphisms $f \in \mathcal{B}(A, B)$, $g \in \mathcal{B}(B, A)$ and two 2-morphisms $\mathrm{ev} : g \otimes f \to 1_A$, $\mathrm{coev} : 1_B \to f \otimes g$ such that the following two equalities are satisfied:

$$\begin{aligned} &\lambda_g \circ (\mathrm{ev} \otimes 1) \circ \alpha^{-1} \circ (1 \otimes \mathrm{coev}) \circ \rho_g^{-1} = 1_g, \\ &\rho_f \circ (1 \otimes \mathrm{ev}) \circ \alpha \circ (\mathrm{coev} \otimes 1) \circ \lambda_f^{-1} = 1_f. \end{aligned} \tag{A.2.1}$$

In this case we say that f is *right adjoint* to g, respectively g *left adjoint* to f and call ev and coev the *evaluation* respectively *coevaluation* map (for the adjunction of f and g).

Definition A.2.2. A bicategory $\mathcal{B}$ *has adjoints* if for every 1-morphism f in $\mathcal{B}$ there is an adjunction in which f is the right adjoint and an adjunction in which f is the left adjoint. We say that $\mathcal{B}$ is a *bicategory with adjoints* if $\mathcal{B}$ has adjoints and for every 1-morphism f there are chosen adjunctions in which f is the left adjoint respectively the right adjoint.

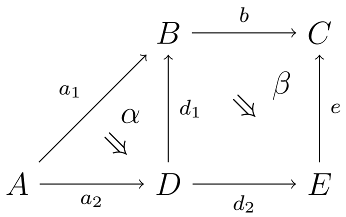
Definition A.2.3. An *adjoint equivalence* is an adjunction $(f, g, \mathrm{ev}, \mathrm{coev})$ in which both ev and coev are 2-isomorphisms.

A.3. Monoidal bicategories

Given three bicategories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote by $\mathfrak{A} : (\mathcal{A}\times\mathcal{B})\times\mathcal{C} \to \mathcal{A}\times(\mathcal{B}\times\mathcal{C})$ the 2-functor which acts as $((A,B),C) \mapsto (A,(B,C))$ on objects and analogously on morphisms. Also, we write $\mathbf{1}$ for the bicategory with a single object $*$ and only identity morphisms.

Recall from Appendix A.1 that for every pair of bicategories $\mathcal{B}$, $\mathcal{B}'$ there is a bicategory $\mathrm{Bicat}(\mathcal{B},\mathcal{B}')$ of 2-functors, pseudonatural transformations and modifications. Therefore we can consider adjoints for pseudonatural transformations in the sense of Definition A.2.1.

The axioms on the structure of a monoidal bicategory are represented by pasting diagrams below. We illustrate how to read them by an example and refer to [SP1, §A.4] for more details. Let $\mathcal{B}$ be a bicategory. We can consider the following diagram in $\mathcal{B}$:



Here the vertices are labeled by objects of $\mathcal{B}$, the edges by 1-morphisms and the faces by 2-morphisms. This diagram represents a 2-morphism in $\mathcal{B}(A,C)$ from $b \circ a_1$ to $e \circ d_2 \circ a_2$, namely $(\beta \otimes 1_{a_2}) \circ (1_b \otimes \alpha)$.

Definition A.3.1. A *monoidal bicategory* $\mathcal{B}$ consists of

1. a bicategory $\mathcal{B}$,

2. a 2-functor $\Box : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ called *monoidal product*,

3. an adjoint equivalence $(a, a^-, \mathrm{ev}_a, \mathrm{coev}_a)$ in which the pseudonatural transformation
$$a : \Box \circ (\Box \times 1) \to \Box \circ (1 \times \Box) \circ \mathfrak{A}$$
is called *associator*,

4. an invertible modification π called *pentagonator* as displayed in the following diagram, where we suppress the 2-functor $\Box$ writing e.g. AB for $\Box_O(A,B)$ for objects A, B of $\mathcal{B}$ and similarly for morphisms:

$$
\begin{array}{ccccc}
 & & (AB)(CD) & & \\
 & \overset{a}{\nearrow} & & \overset{a}{\searrow} & \\
((AB)C)D & & & & \\
a1_D \big\downarrow & & \Uparrow \pi & & A(B(CD)) \;, \\
(A(BC))D & & & & \\
 & \underset{a}{\searrow} & & \underset{1_A a}{\nearrow} & \\
 & & A((BC)D) & &
\end{array}
$$

5. an object I called *monoidal unit*, which determines the 2-functor $\mathbf{I} : \mathbf{1} \to \mathcal{B}$ by $\mathbf{I}(*) := I$,

6. two pseudonatural transformations $l : \Box \circ (\mathbf{I} \times \mathrm{id}_{\mathcal{B}}) \to \mathrm{id}_{\mathcal{B}}$ called *left unitor* and $r : \Box \circ (\mathrm{id}_{\mathcal{B}} \times \mathbf{I}) \to \mathrm{id}_{\mathcal{B}}$ called *right unitor* which come with the data of adjoint equivalences $(l, l^-, \mathrm{ev}_l, \mathrm{coev}_l)$ and $(r, r^-, \mathrm{ev}_r, \mathrm{coev}_r)$,

7. three invertible modifications λ, ρ, μ called *2-unitors* as in the following diagrams:

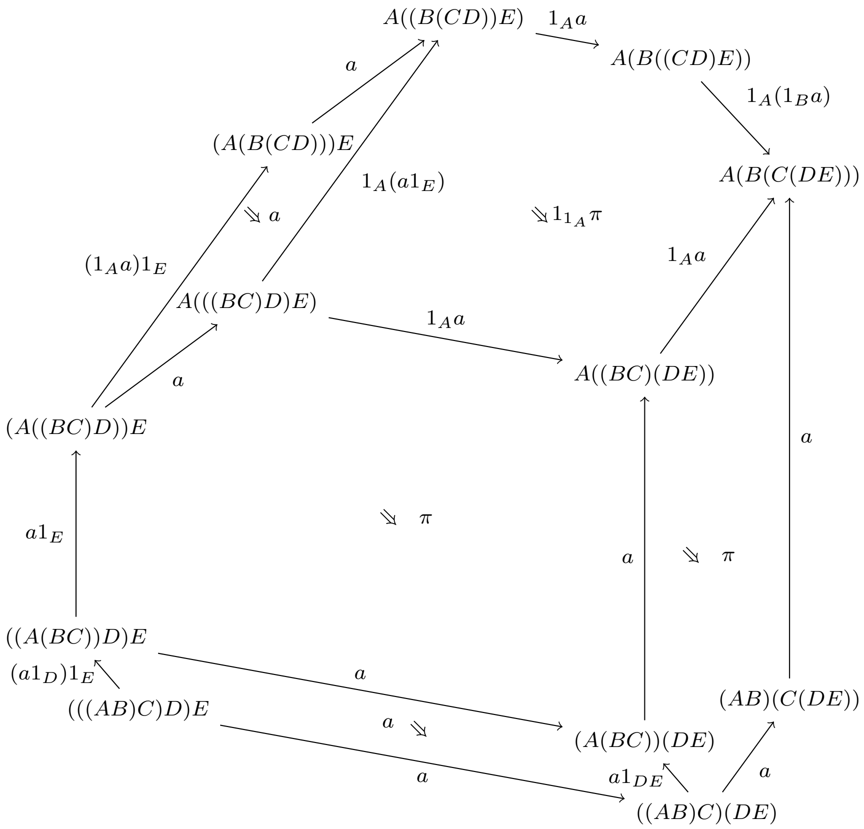
$$
\begin{array}{ccc}
(IA)B & \overset{a}{\searrow} & \\
l1_B \big\downarrow & \Rightarrow \lambda & I(AB) \\
AB & \underset{l}{\swarrow} &
\end{array}
$$

$$
\begin{array}{ccc}
(AI)B & \overset{a}{\searrow} & \\
r1_B \big\downarrow & \Rightarrow \mu & A(IB) \\
AB & \underset{1_A l}{\swarrow} &
\end{array}
$$

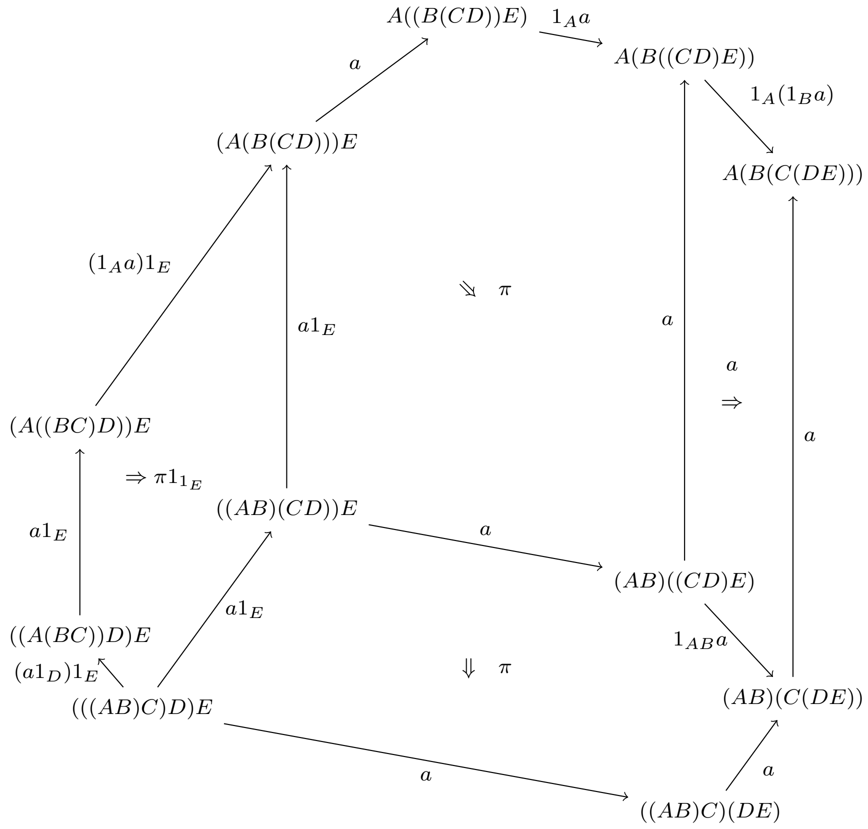
$$
\begin{array}{ccc}
(AB)I & \overset{a}{\searrow} & \\
r \big\downarrow & \Rightarrow \rho & A(BI) \\
AB & \underset{1_A r}{\swarrow} &
\end{array}
$$

such that the identity of pasting diagrams (A.3.1)=(A.3.2) holds and the equalities in (A.3.3) and (A.3.4) below are satisfied.

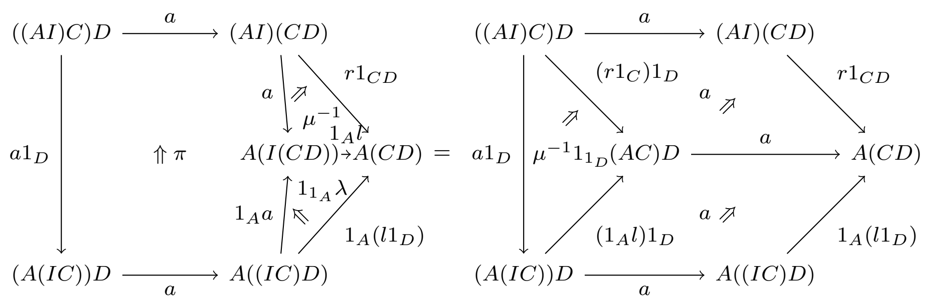
In the diagrams showing the axioms for a monoidal bicategory we do not depict 2-morphism components of □ as the coherence theorem for 2-functors assures that there is a unique way to insert these, cf. [G, Remark 3.1.6].



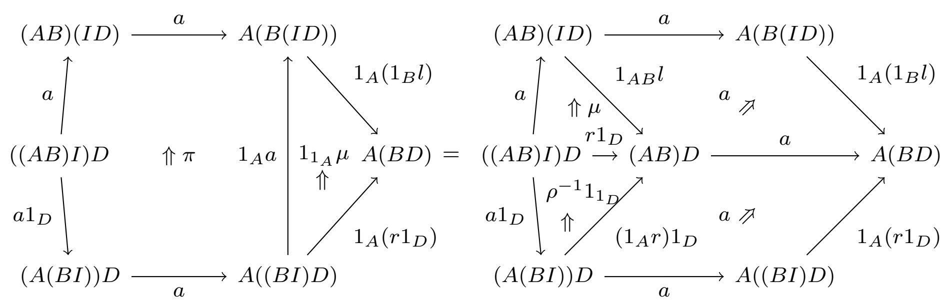
(A.3.1)


A((B(CD))E)
1_A a
A(B((CD)E))
a
1_A(1_B a)
(A(B(CD)))E
A(B(C(DE)))
(1_A a)1_E
⇘ π
a1_E
a
a
⇒
a
(A((BC)D))E
⇒ π1_{1_E}
((AB)(CD))E
a1_E
a
(AB)((CD)E)
((A(BC))D)E
a1_E
1_{AB}a
(a1_D)1_E
⇓ π
(((AB)C)D)E
(AB)(C(DE))
a
a
((AB)C)(DE)

(A.3.2)


((AI)C)D
a
(AI)(CD)
r1_{CD}
a
μ^{-1}
1_A l
a1_D
⇑ π
A(I(CD)) → A(CD) =
1_{1_A}λ
1_A a
1_A(l1_D)
(A(IC))D
a
A((IC)D)
((AI)C)D
a
(AI)(CD)
(r1_C)1_D
r1_{CD}
a
a
a1_D
μ^{-1}1_{1_D}(AC)D
A(CD)
(1_A l)1_D
a
1_A(l1_D)
(A(IC))D
a
A((IC)D)

(A.3.3)


(AB)(ID)
a
A(B(ID))
1_A(1_B l)
a
((AB)I)D
⇑ π
1_A a
1_{1_A}μ
A(BD) =
a1_D
1_A(r1_D)
(A(BI))D
a
A((BI)D)
(AB)(ID)
a
A(B(ID))
1_{AB}l
1_A(1_B l)
a
⇑ μ
a
r1_D
((AB)I)D → (AB)D
a
A(BD)
ρ^{-1}1_{1_D}
a1_D
a
(1_A r)1_D
1_A(r1_D)
(A(BI))D
a
A((BI)D)

(A.3.4)

A.4. Symmetric monoidal bicategories

Let $(\mathcal{B}, \Box, a, \pi, I, l, r, \lambda, \rho, \mu)$ be a monoidal bicategory. We denote by $\tau : \mathcal{B}\times\mathcal{B} \to \mathcal{B} \times \mathcal{B}$ the 2-functor which acts on objects as $(A, B) \mapsto (B, A)$ and similarly on morphisms. Furthermore, we stick to the notational conventions used in the diagrams in Definition A.3.1. In particular, we write AB for the monoidal product of two objects A and B of $\mathcal{B}$ and we do not display components of the monoidal product 2-functor $\Box$.

In the diagrams depicting the axioms on the structure morphisms of a symmetric monoidal bicategory below, we use indices on modifications as follows. Let F, G, H and K be 2-functors which all share the same source and target bicategories. Let the following be pseudonatural transformations:

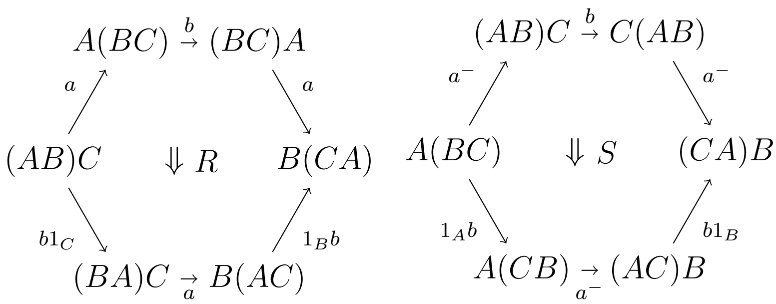
$$c : F \to G,\ d : G \to K,\ e : F \to H,\ f : H \to K.$$

Assume that c^- and f^- are pseudonatural transformations left adjoint to c and f, respectively. Then [SP1, Proposition A.30] states that there is a bijection between the modifications $d \otimes c \to f \otimes e$ and $f^- \otimes d \to e \otimes c^-$. Given a modification $m : d \otimes c \to f \otimes e$ we denote the modification obtained from m under the preceding bijection by m_j for some $j \in \mathbb{N}$. The number in the index carries no information. Since the correspondence between modifications $d \otimes c \to f \otimes e$ and $f^- \otimes d \to e \otimes c^-$ is bijective, m_j is determined unambigously by m and knowing that m_j is obtained from m using the given adjunctions.

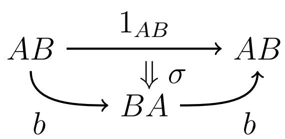
Definition A.4.1. A *symmetric monoidal bicategory* is a monoidal bicategory $\mathcal{B}$ equipped with

1. an adjoint equivalence $(b, b^-, \mathrm{ev}_b, \mathrm{coev}_b)$ in which the pseudonatural transformation $b : \Box \to \Box \circ \tau$ is called *braiding*,

2. two invertible modifications



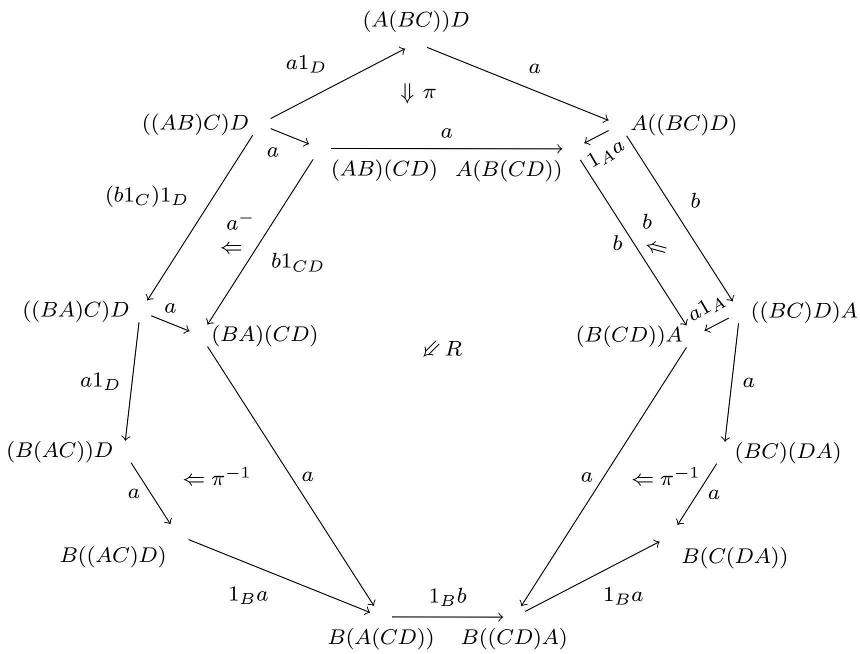
3. an invertible modification called *syllepsis* as follows



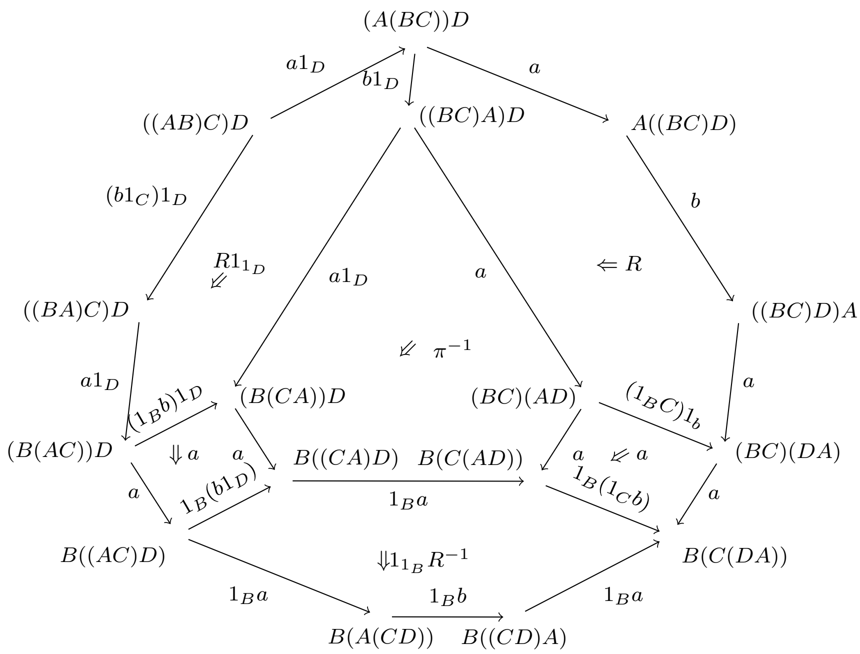
such that the identities of pasting diagrams

$$\text{(A.4.1)} = \text{(A.4.2)},\ \text{(A.4.3)} = \text{(A.4.4)},\ \text{(A.4.5)} = \text{(A.4.6)},\ \text{(A.4.7)} = \text{(A.4.8)}$$

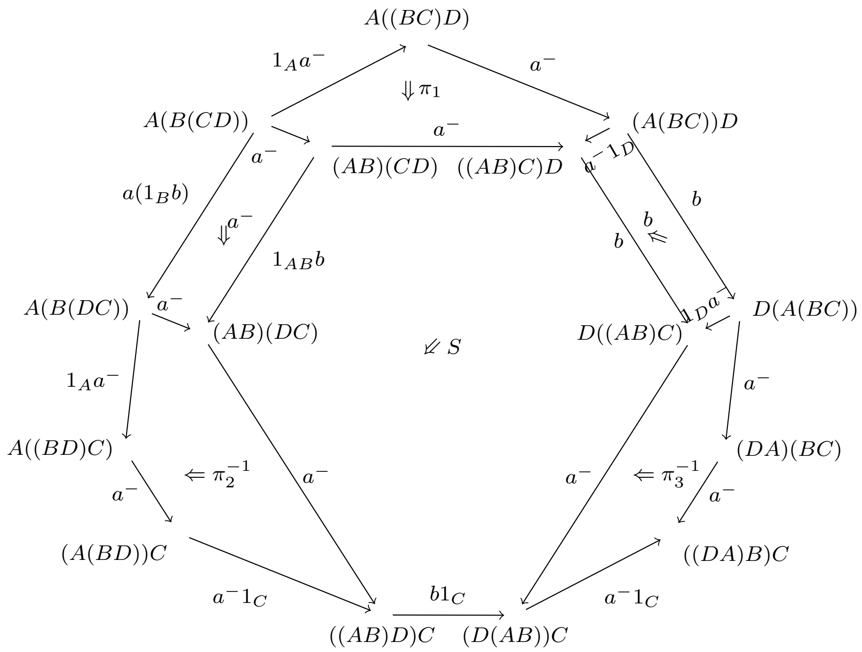
are satisfied and the equalities (A.4.9), (A.4.10) and (A.4.11) below hold.



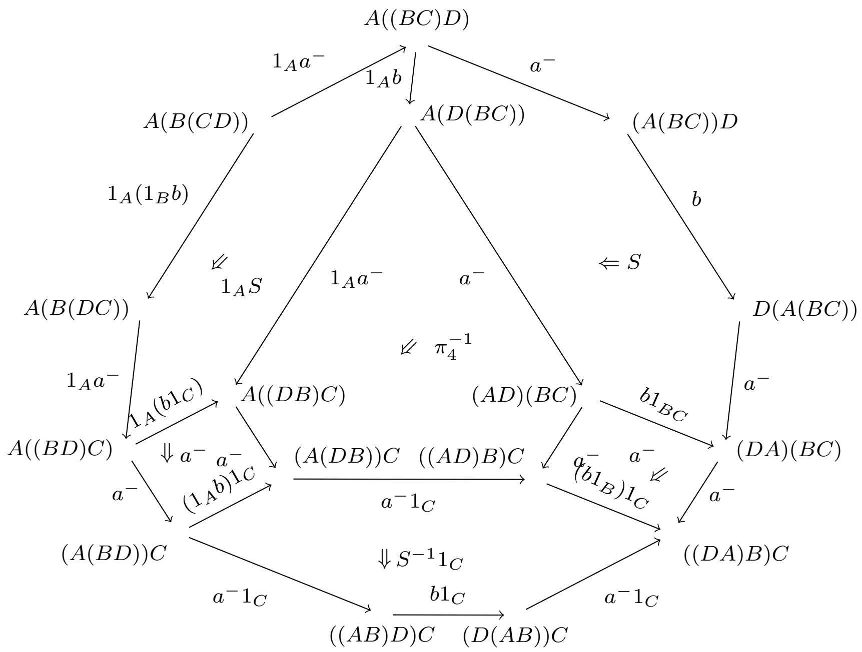
(A.4.1)



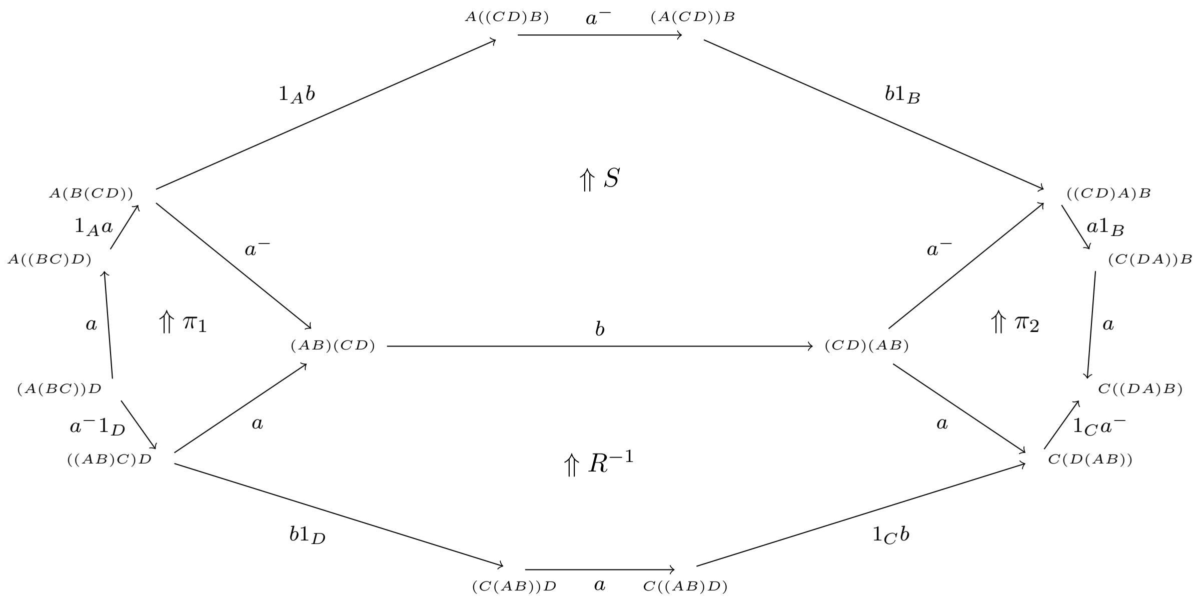
(A.4.2)


A((BC)D)
1_A a^-
a^-
⇓π_1
A(B(CD))
(A(BC))D
a^-
a^-
(AB)(CD)
((AB)C)D
a^-1 1_D
a(1_B b)
⇓a^-
1_AB b
b
b
b ⇐
A(B(DC))
a^-
(AB)(DC)
⇙ S
D((AB)C)
1_D a^-
D(A(BC))
1_A a^-
a^-
A((BD)C)
⇐ π_2^-1
a^-
a^-
⇐ π_3^-1
(DA)(BC)
a^-
a^-
(A(BD))C
((DA)B)C
a^-1 1_C
b1_C
a^-1 1_C
((AB)D)C
(D(AB))C

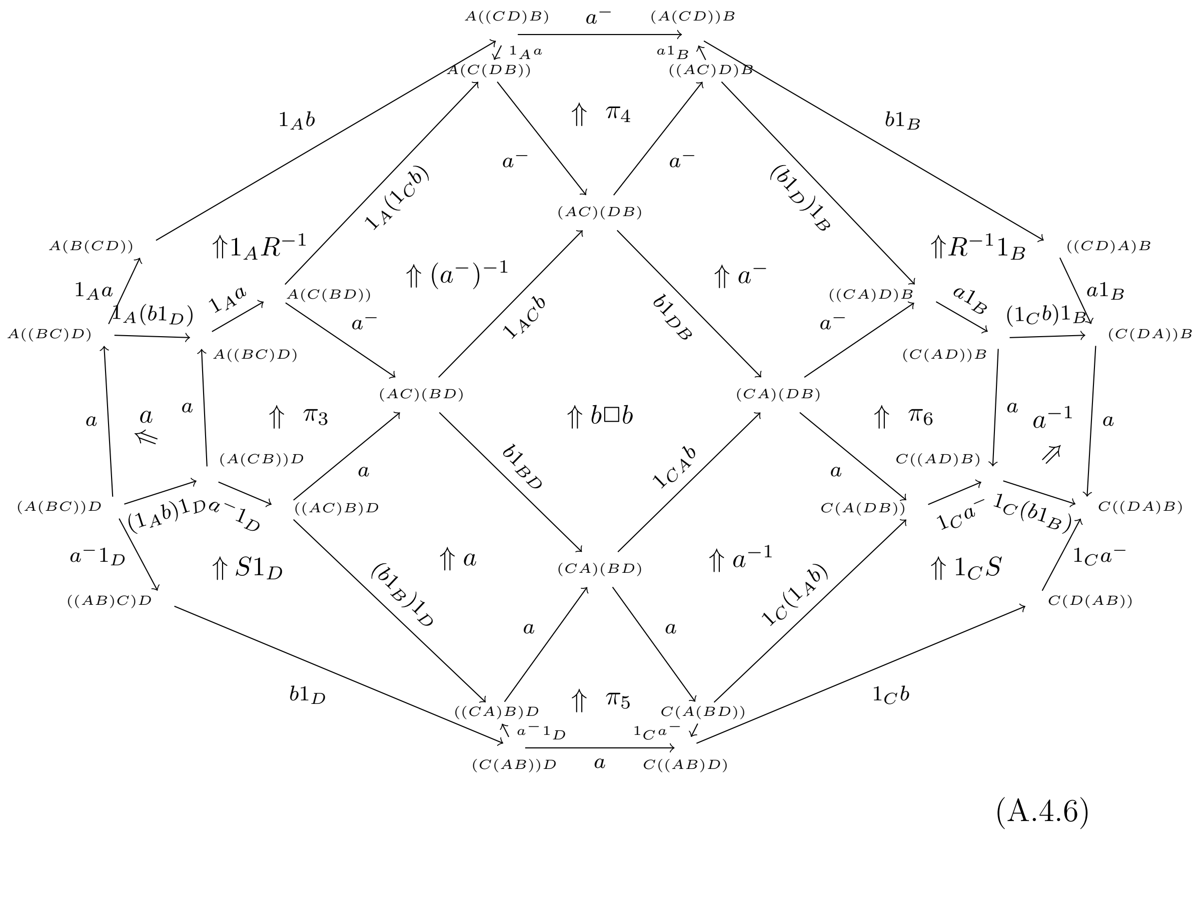
(A.4.3)


A((BC)D)
1_A a^-
1_A b
a^-
A(B(CD))
A(D(BC))
(A(BC))D
1_A(1_B b)
b
⇙ 1_A S
1_A a^-
a^-
⇐ S
A(B(DC))
D(A(BC))
⇙ π_4^-1
1_A a^-
1_A(b1_C)
A((DB)C)
(AD)(BC)
b1_BC
a^-
A((BD)C)
⇓ a^-
a^-
(A(DB))C
((AD)B)C
a^-
a^-
⇙
(DA)(BC)
a^-
(1_A b)1_C
a^-1 1_C
(b1_B)1_C
a^-
(A(BD))C
⇓ S^-1 1_C
((DA)B)C
a^-1 1_C
b1_C
a^-1 1_C
((AB)D)C
(D(AB))C

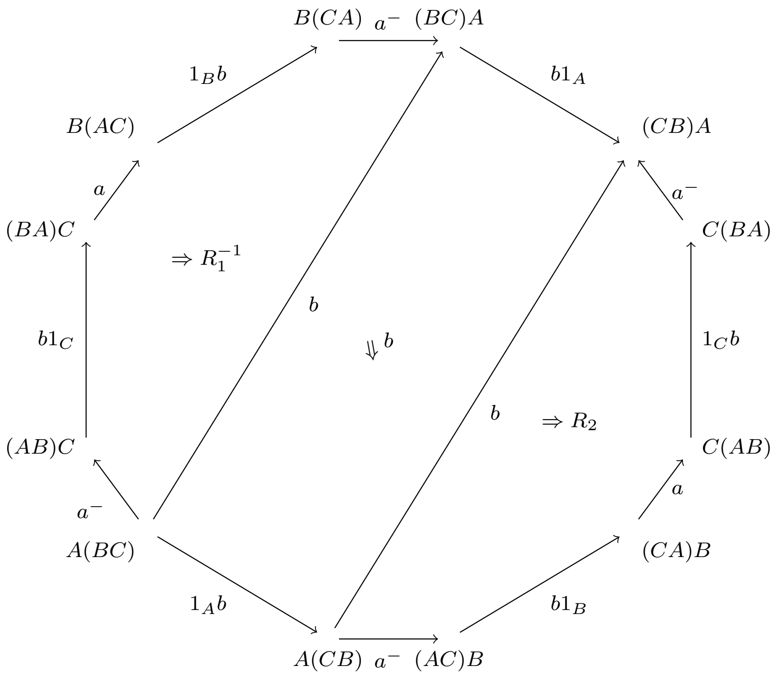
(A.4.4)



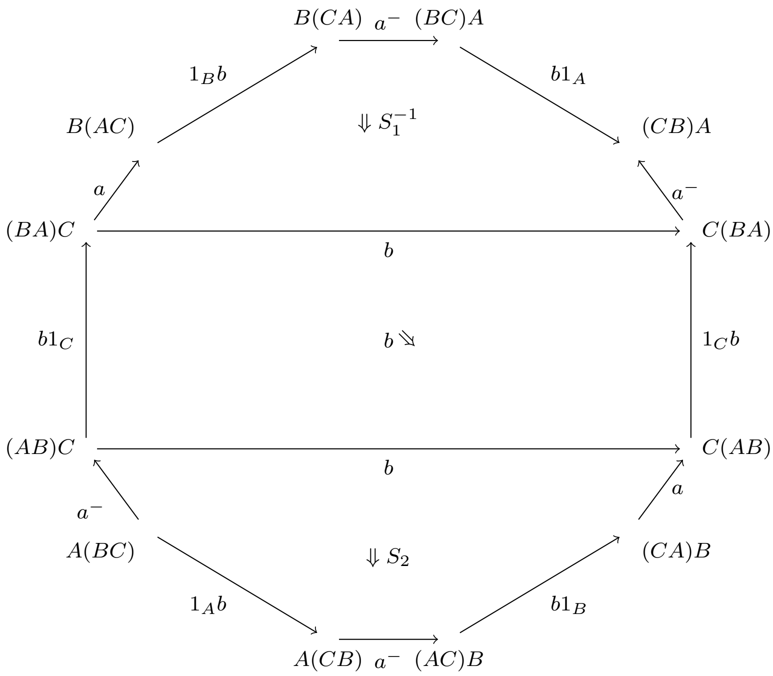
(A.4.5)



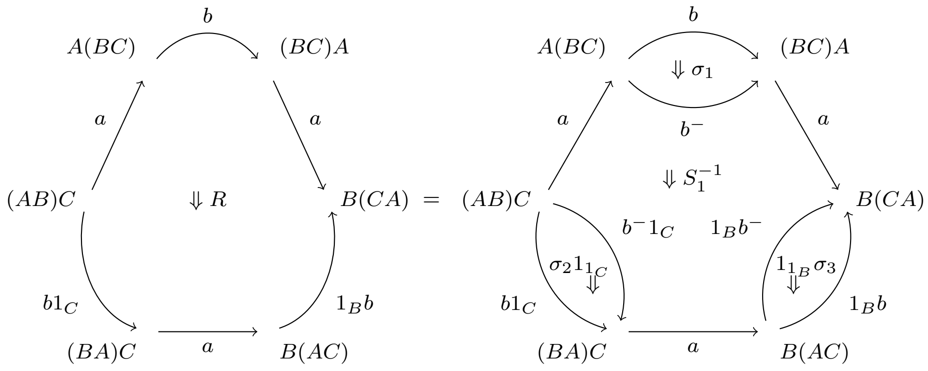
(A.4.6)


B(CA)
a⁻
(BC)A
1_B b
b1_A
B(AC)
(CB)A
a
a⁻
(BA)C
C(BA)
⇒ R_1^{-1}
b
⇓ b
b1_C
1_C b
b
⇒ R_2
(AB)C
C(AB)
a⁻
a
A(BC)
(CA)B
1_A b
b1_B
A(CB)
a⁻
(AC)B

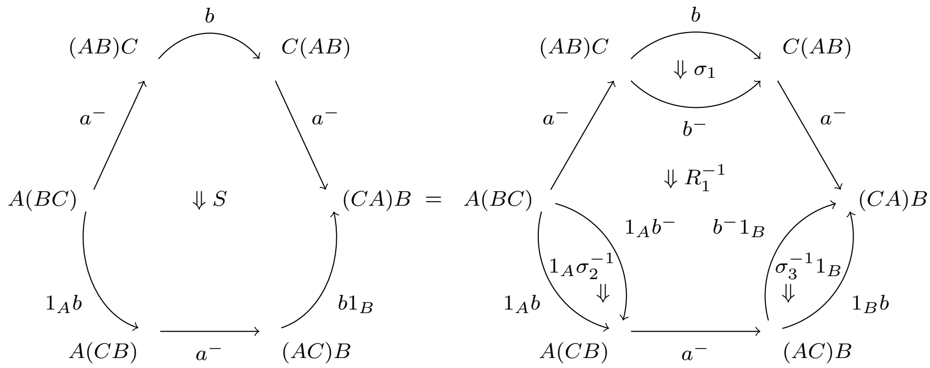
(A.4.7)


B(CA)
a⁻
(BC)A
1_B b
b1_A
B(AC)
⇓ S_1^{-1}
(CB)A
a
a⁻
(BA)C
C(BA)
b
b1_C
b ⇘
1_C b
(AB)C
C(AB)
b
a⁻
a
A(BC)
⇓ S_2
(CA)B
1_A b
b1_B
A(CB)
a⁻
(AC)B

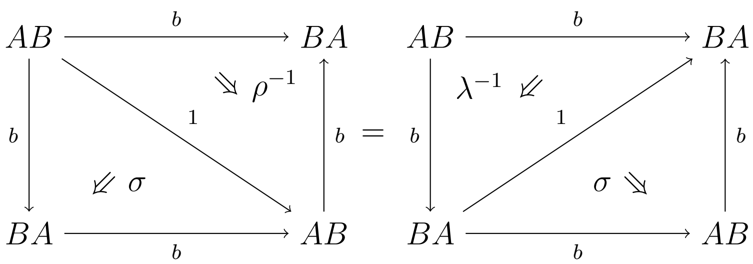
(A.4.8)



(A.4.9)



(A.4.10)



(A.4.11)

A.5. Duals in monoidal bicategories

Some background on duals in monoidal bicategories can be found e.g. in [P, §2,§3].

Definition A.5.1. Let $\mathcal{B}$ be a monoidal bicategory and $A \in \mathcal{B}$. An object $A^* \in \mathcal{B}$ is called *right dual* to A if there are *(co)evaluation* 1-morphisms $\mathrm{ev}_A : A \Box A^* \to I$ and $\mathrm{coev}_A : I \to A^* \Box A$ together with *cusp isomorphisms*

$$l \otimes (\mathrm{ev}_A \Box 1) \otimes a^- \otimes (1 \Box \mathrm{coev}_A) \otimes r^- \to 1_A, \tag{A.5.1}$$

$$r \otimes (1 \Box \mathrm{ev}_A) \otimes a \otimes (\mathrm{coev}_A \Box 1) \otimes l^- \to 1_{A^*}. \tag{A.5.2}$$

Analogously to Definition A.5.1 one can define left duals for objects of a monoidal bicategory. In a symmetric monoidal bicategory every right dual A^* of an object A is simultaneously a left dual to A. Indeed, $\mathrm{ev}_A \otimes b$ respectively

$b \otimes \mathrm{coev}_A$ serve as the necessary 1-morphisms. Since we are only concerned with symmetric monoidal bicategories, we adapt the convention that we refer to the right dual A^* of an object A as the *dual object* to A.

Definition A.5.2. Let $\mathcal{B}$ be a symmetric monoidal bicategory. An object $A \in \mathcal{B}$ is *fully dualizable* if it has a dual object such that both the evaluation and coevaluation 1-morphism have both left and right adjoints.

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Zusammenfassung

Diese Arbeit bestimmt gerahmte und orientierte erweiterte zweidimensionale topologische Quantenfeldtheorien (TQFTs) mit Werten in der Bikategorie der Landau-Ginzburg Modelle $\mathcal{LG}$. Diese Bikategorie ist zum Beipiel in der Knotentheorie, String-Theorie und homologischer mirror symmetry von Bedeutung. Insbesondere ist diese Arbeit durch eine Veröffentlichung zu homologischen (oder kategorifizierten) Knoteninvarianten inspiriert.

Zunächst wiederholen wir die Definition der Bikategorie mit Adjungierten $\mathcal{LG}$. Grob sind deren Objekte Polynome mit einer isolierten Singularität im Ursprung. Die Kategorie von Morphismen zwischen zwei Polynomen dieser Art ist eine Homotopiekategorie von Matrixfaktorisierungen ihrer Differenz. Wir stellen $\mathcal{LG}$ einschließlich mancher Details dar, die in der Literatur nicht erwähnt werden.

Daraufhin arbeiten wir eine explizite symmetrisch monoidale Struktur auf $\mathcal{LG}$ heraus. Das monoidale Produkt zweier Objekte ist im Wesentlichen die Summe zweier Polynome. Für die Morphismen, die Teil der symmetrisch monoidalen Struktur sind, spielen die Identitäts-1-Morphismen in $\mathcal{LG}$ und deren Unitoren eine entscheidende Rolle. Außerdem sind Funktoren zur Einschränkung von Skalaren entlang von Ring-Isomorphismen von großer Bedeutung.

Im dritten Abschnitt des Hauptteils dieser Arbeit definieren wir das Duale eines Objektes in $\mathcal{LG}$, im Wesentlichen das negative des Polynoms. Auf den Identitäts-1-Morphismen in $\mathcal{LG}$ aufbauend, bestimmen wir Coevaluations- und Evaluations-Morphismen für diese Duale. Als Folgerung daraus ergibt sich, dass jedes Objekt von $\mathcal{LG}$ vollständig dualisierbar ist.

Bevor wir uns den letzten Untersuchungen dieser Arbeit zuwenden, erinnern wir an die bikategorielle Cobordismus-Hypothese und ihr Analogon für orientierte Bordismen. Aus der erstgenannten Cobordismus-Hypothese folgt zusammen mit dem Fazit des vorangegangenen Kapitels, dass jedes Objekt eine gerahmte erweiterte TQFT mit Werten in $\mathcal{LG}$ festgelegt. Danach zeigen wir, dass genau für jene Objekte von $\mathcal{LG}$, die durch Polynome in einer geraden Anzahl von Variablen geben sind, der zugehörige Serre-Automorphismus trivialisierbar ist. Folglich bestimmen diese Objekte orientierte erweiterte TQFTs mit Werten in $\mathcal{LG}$. Letztlich definieren wir eine Bikategorie LG, die $\mathcal{LG}$ sehr ähnlich ist. Insbesondere hat sie die gleichen Objekte und 1-Morphismen. Im Gegensatz zu $\mathcal{LG}$ hat LG 2-Morphismen sowohl geraden als auch ungeraden Grades (wobei jeder dieser Morphismen mittels einer $\mathbb{Z}_2$-Wirkung mit minus sich selbst identifiziert wird). Das ermöglicht, dass jedes der Objekte eine orientierte erweiterte 2d TQFT mit Werten in LG bestimmt. Wir geben ein Beispiel einer solchen TQFT an, das eng an unsere knotentheoretische Inspiration angelehnt ist.

Zur starken Vereinfachung unserer Beweise, führen wir diese auf wenige Kohärenz-Resultate zurück. In diesen wiederum verallgemeinern wir bekannte Kohärenz-Resultate derart, dass sie zusätzlich die Auswirkungen von Funktoren zur Einschränkung von Skalaren entlang von Ring-Isomorphismen einschließen.