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(In)stability and asymptotic analysis of the Zakharov-
Rubenchik system and related wave type equations

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Zusammenfassung

In dieser Thesis behandeln wir einige mathematische Probleme betreffend des Chauchy-Problems für einige zweidimensionale nichtlineare dispersiver Gleichungen, und Störungen derselben durch ein Linien-Soliton, mit Bezug auf die transversale Stabilität dieser Modelle. Wir betrachten diese Modelle als asymptotische Modelle des sehr allgemeinen “Wasserwellen-Systems”, wie etwa das Zakharov-Rubenchik-System. Dies führt auf viele interessante Fragen zur Langzeitexistenz einer Lösung (bezüglich eines kleinen Parameters).

In Kapitel 1 stellen wir die Probleme vor, geben einen Ueberblick über bekannte Resultate sowie die Literatur für transverse Stabilität/instabilität, und die von uns behandelten Modelle.

In Kapitel 2 studieren wir das zwei-dimensionale Zakharov-Rubenchik-system. Eine Störung des Zakharovsystems durch sein Linien-Soliton enthält neue lokale und nichtlokale lineare Terme, dies führt zu Schwierigkeiten bei Verwenden der “dispersiven Methode”, die auf Strichartz-Estimates und Dispersionseigenschaften des Schrödinger-Operators basiert. Zuerst verwenden wir die Methode von Bourgain, um lokale Wohlgestelltheit in einer großen Klasse von Sobolev-Räumen zu erhalten. Dann verwenden wir die Methode von Schochet und Weinstein, um eine weitere Version der lokalen Wohlgestelltheit in einer kleineren Klasse von Sobolev-Räumen zu erhalten. Diese ergibt den Grenzwert zur korrespondierenden Schrödinger-Gleichung, wenn ein Parameter der Gleichung gegen Unendlich geht.

In Kapitel 3 untersuchen wir das Zakharov-Rubenchik/Benney-Roskes-System (Z-R/B-R). Das Z-R/B-R-System ist fundamental, es ist ein “generisches” asymptotisches System im sogenannten modulation regime (langsam variierende Einhüllende eines schnell oszillierenden Wellenzugs), das in mehreren physikalischen Zusammenhängen hergeleitet wurde. Außerdem enthält es als Grenzwerte das klassische (skalare) Zakharov-System und die Davey-Stewartson (DS) -Systeme. Somit ist Z-R/B-R reicher als die einfacheren Modelle und sollte mehr der ursprünglichen Dynamik beschreiben.

In Kapitel 4 behandeln wir ein asymptotisches Modell der (Oberflächen – oder Internen) Wasserwellen-Gleichungen, das Full Dispersion Davey-Stewartson (FDDS) – System. Das FDDS wurde vor kurzem von David Lannes hergeleitet, und im Gegensatz zum herkömmlichen DS-System erhält es die lineare Dispersionsrelation der vollen Wasserwellengleichung. Es sollte daher eine bessere Beschreibung der originalen Wellenbewegung ergeben als die klassische Näherung. Leider ist über das Cauchy-Problem für FDDS viel weniger bekannt, aufgrund der Nicht-invertierbarkeit der zweiten Gleichung. Wir verwenden eine Frequenzschranke, um eine Approximation an FDDS herzuleiten, und beweisen dann, dass deren Lösung gegen die Lösung des entsprechenden klassischen DS – Systems tendiert, wenn der kleine Parameter gegen null geht.

In Kapitel 5 betrachten wir das zweidimensionale “(a,b,c,d)” Boussinesq-System, das ein asymptotisches Modell für Oberflächenwasserwellen im Bereich langer Wellen und kleiner Amplituden ist. Es ist bekannt, dass für spezielle Fälle (der Werte von “(a,b,c,d)”) ein-dimensionale Solitärwellen des Boussinesq-Systems existieren. Wir untersuchen das Cauchy-Problem für die Störung des zweidimensionalen Boussinesq-Systems durch seine ein-dimensionale Solitärwelle in diesen Fällen.

In Kapitel 6 behandeln wir numerische Simulationen. Wir verwenden die Time Splitting Fourier Spectral-Methode, um das Z-R/B-R System aus Kapitel 3 zu untersuchen. Die Ergebnisse sind nicht völlig eindeutig, aber sie sind konsistent mit transversaler Instabilität des Linien-Solitons. Außerdem geben sie Hinweise auf den Zusammenhang zwischen (In-)Stabilität und Wellengeschwindigkeit des Solitons und ein generelles Bild über das Verhalten der Lösung des gestörten Systems.

Abstract

In this thesis, we address various mathematical problems concerning the Cauchy problem for various two-dimensional nonlinear dispersive equations and their perturbations by a line soliton, related to the problem of transverse stability for these models.

We consider these models as the asymptotic models of the very general "water wave equation system" like the Zakharov-Rubenchik system. This leads to many interesting questions on the long time existence of solution (with respect to some small parameters).

In the Chapter 1, we present the problems, review existing results and the literature for the transverse stability/instability problems and the models which we consider.

In the Chapter 2, we study the two-dimensional Zakharov system. The perturbation of the Zakharov system by its line soliton contains some new local and nonlocal linear terms, that gives trouble if we use the "dispersive method" based on the Strichartz's estimates and smoothing effect of the Schrödinger operator. We first use the method of Bourgain to provide a local well-posedness result in a large class of Sobolev space. Then, we use the method inspired by Schochet and Weinstein, to get another version of local well-posedness result in a smaller class of Sobolev spaces, but it gives us the limit to the corresponding Schrödinger equation when a certain parameter of the equation tends to infinity.

In the Chapter 3, we study the Zakharov-Rubenchik/Benney-Roskes (Z-R/B-R) system. The Z-R/B-R system is fundamental, being a "generic" asymptotic system in the so-called modulation regime (slowly varying envelope of a fast oscillating train) and it was actually derived in various physical contexts. Moreover it contains in various limits the classical (scalar) Zakharov system and the Davey-Stewartson systems (DS). Thus, the Z-R/B-R is richer than those simpler models and should capture more of the original dynamics.

In the Chapter 4, we consider an asymptotic model of the (surface or internal) water wave equations - the full dispersion Davey-Stewartson system (FDDS). The FDDS was recently derived by David Lannes, and in contrast to the standard DS system, the full dispersion system preserves the linear dispersion relation of the full water wave equations, and should therefore give a richer description of the original wave dynamics than the classical approximation. Unfortunately, the Cauchy problem for the FDDS is very less known due to the non-invertibility of the second equation. We use a cutoff in the frequency to derive an approximation of the FDDS, then prove that its solution tends to the solution of the corresponded classical DS system when the small parameter tends to zero.

In the Chapter 5, we consider the two dimensional "(a,b,c,d)" Boussinesq system which is an asymptotic model of surface water waves in the long wave, small amplitude regime. It is known for some particular cases (due to the value of "(a,b,c,d)"), there exists the one-dimensional solitary wave of the Boussinesq system. We then study the Cauchy problem for the perturbation of the two-dimensional Boussinesq system by its one-dimensional solitary wave in these cases.

In the chapter 6, we finally deal with numerical simulations of the above PDE systems. We use time splitting Fourier spectral methods to study the Z-R/B-R system introduced in the Chapter 3. The results are not totally conclusive but they are consistent with the transverse instability of the line soliton. They also give a hint about the relation of the (in-)stability and the wave speed of the line soliton and a general picture about the behavior of the solution of the perturbed system.

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Chapter 1

Introduction

In this thesis we deal with the analysis of nonlinear time dependent Partial Differential Equations (PDEs) that arise in models of (water) wave phenomena. Typically, such PDEs are “hyperbolic” (like the standard “wave equation”) or “dispersive” (like the Schrödinger equation”) - in our context we also deal with “elliptic” PDEs (like the Poisson equation). In fact, nonlinear PDE systems like the Zakharov-Rubenchik system with several parameters contain all these types of PDE. The “existence and uniqueness analysis”, i.e. the question if the nonlinear PDE has unique solutions in which sense, is absolutely non-trivial. For such time dependent PDEs the question of “global in time” vs “local in time” solutions is a deep issue - sometimes “finite time blow up” can occur, i.e. the solution “explodes” at a certain time (e.g. the “energy space norm” tends to infinity). And this can depend on tiny details, one change of sign can create completely different behaviour, as it is well known for the easiest case of PDEs that we consider, the one dimensional cubic NLS (Nonlinear Schrödinger equation):

$$\begin{aligned} iu_t + \partial_{xx}u + \lambda|u|^2u &= 0, \\ u(t=0) &= u_I \end{aligned} \tag{1.0.1}$$

We call (1.0.1) de-focusing or focusing NLS if $\lambda = -1$ or 1 respectively. By simple calculation, one can prove (formally) the conservation (in time) of the following quantities which we call the mass and the energy of (1.0.1) respectively

$$\begin{aligned} M(u(t)) &= \int_{\mathbb{R}} |u|^2 dx = M(u_I), \\ E(u(t)) &= \int_{\mathbb{R}} \left(\frac{1}{2} |u_x|^2 - \frac{\lambda}{4} |u|^4 \right) dx = E(u_I). \end{aligned}$$

In the defocusing case, when $\lambda = -1$ the Sobolev norm $H^1(\mathbb{R})$ of the solution u is bounded by the energy then the solution can not “explode” in finite time and thus we have the global existence (in time). However in the focusing case, we do not have that bound. In fact, if the initial data is large enough, the “blow-up” phenomena will occur.

In this thesis we will often use the terminology “stability” which is somewhat ambiguous in mathematics - we will precise the meaning of (in)stability in each context.

In general, “stability” means that the perturbed solutions “stay close” to the original solution if the perturbation is small enough. Then the definition of the “stability” depends on how we measure the distance between to solutions, for example, in the context of “orbital stability” we measure the distance between the “orbits” of solutions.

1.1 Transverse stability/instability problem

There are many two-dimensional dispersive PDEs that possess one-dimensional solitary waves which are stable when submitted to one-dimensional perturbations but which are unstable when submitted to general two-dimensional perturbations. This fact is so-called the transverse instability of a line solitary waves. Transverse stability and instability of a solitary waves correspond to a class of perturbations that are travelling in a direction transverse to the direction of the basic solitary wave.

The transverse instability of the line solitary wave for some two dimensional models such as nonlinear Schrödinger equation (NLS), Kadomtsev-Petviashvili equation and for some general "abstract" problems have been studied extensively in [60, 58, 59].

In order to precise the definition of the transverse stability (instability), we will recall the result of the transverse instability in the case of two-dimensional cubic NLS in [58].

The two-dimensional NLS takes the form

$$iu_t + \Delta_{x,y}u + |u|^2u = 0, \quad (1.1.1)$$

where u is a complex valued function.

First, we start with the following definitions from [21]. We denote that $U(t)u_0$ is the unique solution at time t of (1.1.1) with the initial data u_0 and $T^*(u_0)$ is the existence time of this solution. For any $X \subset H^1(\mathbb{R}^n)$, $n = 1, 2$ and $\delta > 0$, we define $\mathcal{V}(X, \delta)$ the δ -neighborhood of X in H^1 by

$$\mathcal{V}(X, \delta) = \bigcup_{v \in X} B_\delta(v),$$

where $B_\delta(v)$ is the ball in H^1 , $B_\delta(v) = \{u \in H^1 \mid \|u - v\|_{H^1} < \delta\}$.

Note that: H^s denotes the usual Sobolev space.

Definition 1.1.1. We say that $X \subset H^1$ is stable by the flow of U if and only if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $u_0 \in \mathcal{V}(X, \delta)$, $T^*(u_0) = +\infty$ and $U(t)u_0 \in \mathcal{V}(X, \varepsilon)$, $\forall t \geq 0$.

If u is a periodic in time solution of (1.1.1), then orbital stability of u means stability of the closed orbit $\{u(t) \mid t \in \mathbb{R}\}$ by the flow of U .

In the case, $u = e^{i\omega t}\varphi$ is a standing waves solution of (1.1.1), due to the invariances under translation and multiplication by $e^{i\theta}$, we need to introduce the following extension of the closed orbit $\{e^{i\omega t}\varphi \mid t \in \mathbb{R}\}$:

$$\Omega_\varphi = \{e^{i\gamma}\tau_a\varphi \mid \gamma \in \mathbb{R}, a \in \mathbb{R}^n\},$$

where τ_a is the translation operator defined by $\tau_a v = v(\cdot + a)$. Then, the orbital stability of u means stability of Ω_φ by the flow of U .

Now, we go back to (1.1.1), This equation has a famous one-dimensional solitary wave $Q(x)e^{it}$ with Q given by

$$Q(x) = \frac{\sqrt{2}}{\cosh(x)}.$$

This solitary wave is orbitally stable when submitted to one-dimensional perturbations (see [16]), that means

$$\forall \varepsilon > 0, \exists \delta > 0 : \|u(0, \cdot) - Q\|_{H^1(\mathbb{R})} < \delta \Rightarrow \inf_{a \in \mathbb{R}, \gamma \in \mathbb{R}} \|u(t, \cdot) - e^{i\gamma}e^{it}Q(\cdot - a)\|_{H^1(\mathbb{R})} < \varepsilon.$$

Next, we consider the two-dimensional version of (1.1.1), set $u = e^{it}v$ and it leads to the equation

$$iv_t + \Delta v - v + |v|^2v = 0, \quad (1.1.2)$$

for $(x, y) \in \mathbb{R} \times \mathbb{T}_L$ where $\mathbb{T}_L$ is the flat torus $\mathbb{R}/2\pi L\mathbb{Z}$. A stationary solution of this equation is now given by the ground state $Q(x)$. The following result states that $Q(x)$ is unstable when it is submitted to (1.1.2).

Theorem 1.1.2. *There exists L_0 such that for $L \geq L_0$, for every $s \geq 2$, there exists $\eta > 0$ such that for every $\delta > 0$, there exists $v_0^\delta \in H^s$ and a time $T^\delta \simeq |\log \delta|$ such that*

$$\|v_0^\delta - Q\|_{H^s(\mathbb{R} \times \mathbb{T}_L)} < \delta,$$

and the solution v^δ of (1.1.2) with initial value v_0^δ belongs to $C([0, T^\delta], H^s)$ and satisfies

$$\inf_{a \in \mathbb{R}, \gamma \in \mathbb{R}} \|v^\delta(T^\delta, \cdot) - Q(\cdot - a)e^{i\gamma}\|_{L^2(\mathbb{R} \times \mathbb{T}_L)} \geq \eta.$$

Remark 1. *i) As we can see from the above example, solving transverse stability/instability of a line solitary wave under the time evolution of a two dimensional model involves two independent steps:*

1. *Establish the well-posedness (local but if possible global) of the Cauchy problem in the suitable functional framework, that is in $H^s(\mathbb{R} \times \mathbb{T})$ if one considers y -periodic perturbations of the line solitary wave ϕ_c , or in $\phi_c + H^s(\mathbb{R}^2)$ if one consider a localized perturbation of ϕ_c .*
2. *Prove the linear (spectral) transverse stability/instability of the line solitary wave and then the corresponding nonlinear properties.*

ii) In this thesis, I will mainly focus on the first step where there are many interesting problems are to be solved.

1.2 Zakharov system

The Zakharov system is introduced in [72] to describe the propagation of Langmuir waves in plasma. For more details about the derivation and the physical background of Zakharov system we refer to [66, 72].

We are interested in the following two-dimensional Zakharov system

$$\begin{cases} i\partial_t u + \Delta u = nu, \\ \frac{1}{\lambda^2} \partial_t^2 n - \Delta n = \Delta(|u|^2), \end{cases} \quad (1.2.1)$$

where $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$, λ is a fixed number, u is a complex valued function, n is a real function. We denote $e^{it}Q(x)$, the line soliton of the two-dimensional cubic focusing NLS as in the previous section. Therefore, $(e^{it}Q(x), -Q(x)^2)$ is a line soliton of (1.2.1), it can be considered as a two dimensional (constant in y) object, then a natural question is that of its transverse (with respect to y) stability. When looking for *localized* perturbations, one is lead, as a first step, to study the Cauchy problem for the perturbed system below.

$$\begin{cases} i\partial_t(u + Q) - (u + Q) + \Delta(u + Q) = (n - Q^2)(u + Q), \\ \frac{1}{\lambda^2} \partial_t^2(n - Q^2) - \Delta(n - Q^2) = \Delta(|u + Q|^2), \end{cases} \quad (1.2.2)$$

with initial data

$$u(x, y, 0) = u_0(x, y), \quad n(x, y, 0) = n_0(x, y), \quad n_t(x, y, 0) = n_1(x, y). \quad (1.2.3)$$

Note that, in the above system, we used the change of variable $\tilde{u}(x, y, t) := e^{it}u(x, y, t)$.

Another possibility, looking for y -periodic perturbations, would be to study (1.2.1) as a system posed on $\mathbb{R} \times \mathbb{T}$, the 1-d torus case was studied in [69, 14]. On the other hand, the Zakharov system in spatial space $\mathbb{R}^2$ and $\mathbb{R}^3$ have been studied by many authors [34, 4, 5, 8, 13, 36, 35, 41, 63, 65].

It is also worth to notice that if $n_t = \lambda^2 \nabla \cdot \mathbf{v}$ then the solution of (1.2.2) satisfies the following energy conservation

$$E(t) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2 + \frac{n^2}{2} + \frac{\lambda^2 |\mathbf{v}|^2}{2} - Q^2 |u|^2 + n(|u|^2 + 2Q \operatorname{Re}(u)) \right) dx dy,$$

but we do not have the L^2 -conservation.

The local well-posedness of (1.2.2)-(1.2.3) which is proved in Chapter 2 can be viewed as the first step to study the transverse stability (or instability) of the line soliton $(Q(x), -Q(x)^2)$. Concerning *global* issues we can quote [35, 36] where the authors prove the existence of self-similar blow-up solutions and the instability by the blow-up of periodic (in time) solutions (solitary wave solutions) of the Zakharov system with data in $H^1(\mathbb{R}^2)$. Their method uses the radial symmetry of the system which is broken when one writes the system satisfied by a localized perturbation of the line solitary wave.

If we reduce (1.2.2) to a nonlinear Schrödinger equation with loss of derivative in the nonlinearity, then we can think of using the smoothing effect of the Schrödinger operator as in [41]. In our case, the linear terms Qn and Q^2u give trouble since the function Q does not decay in y at infinity. In order to treat the new *linear* terms, we use the method of Bourgain (actually the techniques developed in [34, 13]), It, then, leads to the following result.

Theorem 1. *Let k and l satisfy*

$$k > l \geq 0, \quad l + 1 \geq k \geq \frac{l+1}{2}.$$

Then the system (1.2.2) with initial data $(u_0, n_0, n_1) \in H^k \times H^l \times H^{l-1}$ is locally well posed in $X_1^{k, b_1} \times X_2^{l, b} \times X_2^{l-1, b}$ for suitable b, b_1 close to $1/2$.

Furthermore the solutions satisfy

$$(u, n, \partial_t n) \in \mathcal{C}([0, T], H^k \times H^l \times H^{l-1}),$$

where T is the existence time.

The Bourgain spaces X_1^{k, b_1} and $X_2^{l, b}$ are associated to two operators with symbols $\phi_1(\xi) = |\xi|^2$ and $\phi_2(\xi) = \pm|\xi|$, which are given by the following definition

$$\|u\|_{X_j^{s, b}} = \left\| \langle \xi \rangle^s \langle \tau + \phi_j(\xi) \rangle^b \hat{u}(\xi, \tau) \right\|_{L^2},$$

where $\hat{u}$ denotes the space-time Fourier transform of u , $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ with $\xi \in \mathbb{R}^2$.

Otherwise, we use the method inspired by Schochet and Weinstein ([63]) to obtain another version of the local-wellposedness result. In comparing to Bourgain method, Schochet-Weinstein's method which is based on rewriting (1.2.2) in the form of a perturbation of a symmetric hyperbolic system, provides well-posedness in smaller Sobolev space but allow us to obtain the Schrödinger limit when λ tends to infinity. The results are given in the following theorems.

Theorem 2. *Let $s > 2$ and consider the initial value problem (1.2.2) with initial data of the form*

$$u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = \nabla \cdot f_0.$$

Suppose that

$$\|u_0(\lambda)\|_{H^{s+1}} + \|n_0(\lambda)\|_{H^s} + \frac{1}{\lambda} \|f_0(\lambda)\|_{H^s} \leq C_1.$$

Then (1.2.2) has a unique solution

$$(u, n) \in L^\infty([0, T]; H^{s+1}(\mathbb{R}^2)) \times L^\infty([0, T]; H^s(\mathbb{R}^2)),$$

where $[0, T]$ is the time interval of existence, T depends only on C_1 . In addition, the solution (u, n) satisfies

$$\begin{aligned} & \|u(t, \lambda)\|_{H^{s+1}} + \|u_t(t, \lambda)\|_{H^{s-1}} + \|n(t, \lambda)\|_{H^s} + \frac{1}{\lambda} \|n_t(t, \lambda)\|_{H^{s-1}} \\ & + \frac{1}{\lambda^2} \|n_{tt}(t, \lambda)\|_{H^{s-2}} \leq C_2, \end{aligned}$$

for all $t \in [0, T]$.

Before stating the next theorem, we consider the following family of system labeled by the parameter λ

$$\begin{cases} i\partial_t u_\lambda - u_\lambda + \Delta u_\lambda = n_\lambda u_\lambda - Q^2 u_\lambda + Q n_\lambda \\ \frac{1}{\lambda^2} \partial_t^2 n_\lambda - \Delta n_\lambda = \Delta(|u_\lambda|^2) + 2\Delta(Q \operatorname{Re}(u_\lambda)) \\ \partial_t n_\lambda + \nabla \cdot f_\lambda = 0, \end{cases} \quad (1.2.4)$$

with initial data given by

$$u_\lambda(0) = u_0, \quad n_\lambda(0) = n_0, \quad f_\lambda(0) = f_0. \quad (1.2.5)$$

We study the behaviour of the solution (u_λ, n_λ) when $\lambda \rightarrow \infty$, that is the theorem

Theorem 3. *Let $s > 2$ and $(u_\lambda, n_\lambda, f_\lambda) \in L^\infty(0, T, H^{s+1}) \times L^\infty(0, T, H^s) \times L^\infty(0, T, H^s)$ be the local solution of the system (1.2.4) given by the Theorem 2 with $t \in [0, T]$ and initial data (1.2.5). Then as $\lambda \rightarrow \infty$, (u_λ, n_λ) converges to $(u, -|u|^2)$ in $L^\infty(0, T, H^1) \times L^\infty(0, T, L^2)$ weak star, where u is the unique solution of the corresponded perturbed nonlinear Schrödinger equation with initial data $u_0 \in H^{s+1}$.*

1.3 Zakharov-Rubenchik/Benney-Roskes system

The Zakharov-Rubenchik system is fundamental, being a "generic" asymptotic system in the so-called modulation regime (slowly varying envelope of a fast oscillating train) and it was actually derived in various physical contexts. Moreover it contains in various limits the classical (scalar) Zakharov system (coupling a nonlinear Schrödinger equation and a wave equation, see (1.3.3) below) and the Davey-Stewartson systems (coupling a nonlinear Schrödinger equation and an elliptic equation). We refer to [73] for more details on the formal derivation of those systems and on the physical background.

The Davey-Stewartson system was first derived formally in the context of water waves in [29, 3, 30] (see also [23, 22] for a derivation of Davey-Stewartson systems in a different context). However, as noticed in [45] it is less general than the Benney-Roskes system (1.3.6) below in the sense that the initial conditions for the acoustic type components have to be prepared to obtain an approximation of the full water waves system.

We refer to [24] for a rigorous justification of the Zakharov limit of the Zakharov-Rubenchik system and to [53] for the Schrödinger limit of the Zakharov-Rubenchik system in the one-dimensional case and for well-prepared initial data.

The Zakharov-Rubenchik/Benney-Roskes system is thus richer than those simpler models and should capture more of the original dynamics. It was introduced in [74] (see also the survey article [73]) to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. The analysis is general and carried out in the Hamiltonian formalism and yields the following *universal* system

$$\begin{cases} \psi_t + v_g \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_g}{2k} \Delta_{\perp} \psi - i(q|\psi|^2 + \beta\rho + \alpha\phi_x)\psi = 0, \\ \rho_t + \rho_0 \Delta \phi + \alpha(|\psi|^2)_x = 0, \\ \phi_t + \frac{c^2}{\rho_0} \rho + \beta|\psi|^2 = 0, \end{cases} \quad (1.3.1)$$

where $v_g, \omega'', k, q, \beta, \alpha, \rho_0, c$ are parameters. The two last equations describe the acoustic type waves and $\Delta_{\perp} = \partial_y^2 + \partial_z^2$ or ∂_y^2 , $\Delta = \Delta_{\perp} + \partial_x^2$.

In two space dimensions a more specific (formal) derivation in the context of surface water waves is displayed in [40] and rigorously justified in [45], see below for a more precise description.

In the notations of [57] (see also [68] where it is used in the context of Alfvén waves in dispersive MHD), the Zakharov-Rubenchik system has the form

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} - i\sigma_1 \Delta_{\perp} \psi + i\{\sigma_2 |\psi|^2 + W(\rho + D\phi_x)\} \psi = 0, \\ \rho_t + \Delta \phi + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0, \end{cases} \quad (1.3.2)$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, $\rho, \phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d = 2, 3$ describe the fast oscillating and, resp., acoustic type waves.

Here $\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the strength of the coupling with acoustic type waves, $M > 0$ is a Mach number, $D \in \mathbb{R}$ is associated to the Doppler shift due to the medium velocity and $\delta \in \mathbb{R}$ is a nondimensional dispersion coefficient.

When $\alpha = 0$ (resp. $D = 0$) in (1.3.1) (resp. (1.3.2)) the Zakharov-Rubenchik system reduces to the classical (scalar) Zakharov system (see *eg* Chapter V in [66]). More precisely, in the framework of (1.3.1), one gets

$$\begin{cases} \psi_t + v_g \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_g}{2k} \Delta_{\perp} \psi = i(q|\psi|^2 + \beta\rho)\psi, \\ \rho_{tt} - c^2 \Delta \rho = \beta \rho_0 \Delta |\psi|^2, \end{cases} \quad (1.3.3)$$

which is a form of the 2 or 3D Zakharov system. Note however that the second order operator in the first equation is not necessarily elliptic.

The local well-posedness in $H^s(\mathbb{R}^d) \times H^{s-1/2}(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$ with $s > \frac{d}{2}$, $d = 2, 3$ for (1.3.2), (1.3.1) was obtained in [57] by using the local smoothing property of the free Schrödinger operator after reducing the system to a quasilinear (non local) Schrödinger equation. Since it uses dispersive properties of the free Schrödinger group that are valid only in the whole space the proof does not extend to the Cauchy problem posed on the torus $\mathbb{T}^d$ or "semi-periodically" in $\mathbb{R}^{d-1} \times \mathbb{T}$, the latter situation being relevant for transverse stability issues. On the other hand, when applied to the Benney-Roskes system (1.3.6) below, it provides ¹ an existence time of order $O(1)$, while an

¹Roughly speaking, the idea in [57] is to reduce the system to a (nonlocal) quasilinear Schrödinger equation. When ϵ is taken into account, the crucial dispersive smoothing estimate on the Schrödinger group has a $1/\epsilon$ factor while the nonlinear term has a ϵ factor.

existence time of order $O(1/\epsilon)$ is needed to fully justify the Benney-Roskes as a water wave model on the correct time scales (see [44]).

Local well-posedness of the Zakharov-Rubenchik/Benney-Roskes system was also obtained in [49], for $s > 2$ with the additional condition $\delta\sigma_1 > 0$ (that is the second order operator in the first equation of (1.3.2), (1.3.1) is elliptic) by using an energy method inspired by the work of Schochet-Weinstein in [63] on the nonlinear Schrödinger limit of the Zakharov system. The method used in [49] and [63] consists in rewriting the Zakharov system (or the Zakharov-Rubenchik system) as a dispersive (skew-adjoint) perturbation of a symmetric nonlinear hyperbolic system and it uses only the algebraic structure of the system. A shortcoming of the method is that one has to prepare the initial data.

We will see that, when the small parameter ϵ is included, this method provides also the existence on the time scale $O(1)$ in the context of water waves (see the Benney-Roskes system (1.3.6) below) and moreover that it can be applied to the system obtained from (1.3.2) which is satisfied by a (localized) perturbation of a line soliton. Also, since it does not use any dispersive property of the Schrödinger group, it applies to the Cauchy problem in $\mathbb{T}^d$ or $\mathbb{R}^{d-1} \times \mathbb{T}$, a situation that has not been addressed before (see on the other hand [14, 13] for the periodic Zakharov system).

Thus, none of the two aforementioned methods seems to give the expected existence time scale for the Benney-Roskes system. Nevertheless they provide different results for Zakharov-Rubenchik type systems. The "dispersive method" used in [57] works only in $\mathbb{R}^d$ but does not need the Schrödinger part of the system to be "elliptic" (that is it does not need the condition $\delta\sigma_1 > 0$). Also it lowers the regularity on the initial data (an effect of the dispersive smoothing effect) and could be applied as well to (possibly non physical) nonlinear perturbations of the system.

On the other hand, the Schochet-Weinstein type "hyperbolic like" methods allow to deal with the periodic or semi-periodic cases, but are relatively rigid (they rely on the algebraic structure of the system) and require initial data in the "hyperbolic space" $H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + 1$.

The situation is better understood in spatial dimension one. Oliveira [52] proved the local (thus global using the conservation laws below) well-posedness in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. This result was improved in [46] where in particular global well-posedness was established in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

It is worth noticing that (1.3.2) possesses two conserved quantities, the L^2 norm

$$\int |\psi(x, y_\perp, t)|^2 = \int |\psi(x, y_\perp, 0)|^2,$$

where $y_\perp = y$ or (y, z) , and, after the change of variable $(x, t) \rightarrow (x + \sigma_3 t, t)$, the Hamiltonian

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^d} \left(\frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\nabla_\perp \psi|^2 + \frac{\sigma_2}{4} |\psi|^4 + \frac{W}{4M^2} \rho^2 + \frac{W}{4} |\nabla \phi|^2 + \frac{W}{2} (\rho + D\phi_x) |\psi|^2 \right) \\ &= E(0). \end{aligned} \tag{1.3.4}$$

The conservation laws are used in [57] to obtain global weak solutions under suitable assumptions on the coefficients. Note also that the conservation laws can be used to get the global well-posedness of the Zakharov-Rubenchik, Benney-Roskes system in space dimension one (see [52]).

As aforementioned, in the context of water waves, the Zakharov-Rubenchik system is known as the Benney-Roskes system and it was formally derived in [40]. We follow here the notations in [45], where a rigorous derivation is performed.

$$\begin{aligned} \mathbf{k} &= |\mathbf{k}| \mathbf{e}_x, \quad \omega(\mathbf{k}) = \underline{\omega}(|\mathbf{k}|), \\ \omega &= \underline{\omega}(|\mathbf{k}|), \quad \omega' = \underline{\omega}'(|\mathbf{k}|), \quad \omega'' = \underline{\omega}''(|\mathbf{k}|), \\ \text{where } \underline{\omega}(|\xi|) &= \left(\left(g + \frac{\sigma}{\rho} |\xi|^2 \right) |\xi| \tanh(\sqrt{\mu} |\xi|) \right)^{1/2} \end{aligned} \tag{1.3.5}$$

is the dispersion relation of water waves and where $|\mathbf{k}|$ is a fixed wave number, g is the gravity, $\sigma \geq 0$ is a surface tension coefficient, ρ is the density of the water and μ is the shallowness parameter (square of the typical fluid depth over a typical horizontal scale) which is large or infinite in the deep water models and $\alpha = -\frac{9}{8\sigma^2}(1 - \sigma^2)^2$.

The small parameter ϵ is the *wave steepness* that is the ratio of a typical amplitude of the wave over a typical horizontal scale. Recall ([45]) that the typical time scale for the solutions of (1.3.6) below is $1/\epsilon$ and so it is crucial to establish the well-posedness on those time scales.

The Benney-Roskes equations can then be written in 2 dimensions as follows

$$\begin{cases} \partial_t \psi_{01} + \omega' \partial_x \psi_{01} - i\epsilon \frac{1}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_{01} \\ \quad + i\epsilon \left(|\mathbf{k}| \partial_x \psi_{00} + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta_{10} + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_{01}|^2 \right) \psi_{01} = 0, \\ \partial_t \zeta_{10} + \sqrt{\mu} \Delta \psi_{00} = -2\omega |\mathbf{k}| \partial_x (|\psi_{01}|^2), \\ \partial_t \psi_{00} + \zeta_{10} = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi_{01}|^2. \end{cases} \quad (1.3.6)$$

It is known (see eg [45] Chapter 8) that $\omega' > 0$, while for purely gravity waves ($\sigma = 0$) ω is a concave function, thus $\omega'' < 0$ and the Schrödinger equation in the Benney-Roskes system is "non elliptic".

On the other hand, in the presence of a surface tension, the condition $\omega'' > 0$ is possible. In order to apply the Schochet-Weinstein method we will need the condition $\delta\sigma_1 > 0$ and we will only consider the Zakharov-Rubenchik (or Benney-Roskes) system of the form of (1.3.2) satisfying this condition.

The one-dimensional Zakharov system possesses solitary wave solutions and in [52], Oliveira proved their orbital stability. One motivation of the present thesis was the study of their *transverse* stability. It is thus of interest to study the transverse stability of the line soliton for the two dimensional model (1.3.2) and the first step is to study the Cauchy problem of a localized perturbation of (1.3.2) by a line soliton. Another possibility is to consider y or (y, z) -periodic perturbations of the line solitary wave, a first step being to establish the well-posedness of the Cauchy problem for the Zakharov-Rubenchik, Benney-Roskes system in $\mathbb{R}^d \times \mathbb{T}$, $d = 1, 2$, which could not result from the methods used in [57], but in the way we achieve here and also in the pure periodic case $\mathbb{T}^{d+1}$.

The results on the Zakharov-Rubenchik/Benney-Roskes system on this thesis can be summarized as follows. First, we reformulate the existence of one-dimensional solitary waves (bright and dark) in our framework. Next, we use the Schochet-Weinstein method to prove a local existence for the Benney-Roskes/ Zakharov-Rubenchik system, keeping the small parameter ϵ which is relevant for deep water waves. Then, we consider the case of a localized perturbation of a line solitary wave. Finally we prove the global existence of weak solutions perturbing a dark solitary wave.

1.4 Full dispersion Davey-Stewartson system

The Davey-Stewartson system was introduced in [29] (see also [3, 30]) to model the evolution of wave packets of surface water waves. It is actually derived for a more general system, the Benney-Roskes system.

Our interest in this thesis is a full dispersion version of the classical Davey-Stewartson system introduced by David Lannes in [45] and that we describe now.

The framework is that of the motion of a irrotational inviscid fluid in an infinite layer of finite depth of an Euler fluid bounded by a free surface and a flat bottom.

The specific regime here is the so-called modulation ("Schrödinger") one. In order to define the key small parameter we introduce h a typical depth of the fluid layer, a a typical amplitude of the

wave and λ a typical wavelength in the horizontal directions (assumed to be isotropic). Then we set $\epsilon = a/h$ and $\mu = h^2/\lambda^2$.

In the modulation regime, the relevant small parameter is the *wave steepness*, $\varepsilon = a/\lambda = \epsilon\sqrt{\mu}$. The starting point of the analysis is the water waves system coupling the wave elevation ζ and the trace ψ of the velocity potential at the free surface, written in dimensionless variables (see [45])

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} \mathcal{G} \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \varepsilon \frac{\left(\frac{1}{\sqrt{\mu}} \mathcal{G} \psi + \varepsilon \nabla \zeta \cdot \nabla \psi \right)^2}{2(1 + \varepsilon^2 |\nabla \zeta|^2)} = 0, \end{cases} \quad (1.4.1)$$

where $\mathcal{G}$ is the Dirichlet to Neumann operator which links the trace of the velocity potential at the free surface to its normal derivative at the same surface. We denote $X = (x, y)$ or x the spatial variable.

One introduces a fixed wave vector $\mathbf{k} \in \mathbb{R}^d, d = 1, 2$. Setting $\omega = \omega(\mathbf{k}) = (|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|))^{1/2}$ the dispersion relation of surface gravity waves, one looks for an approximate solution of the water wave system of the form

$$U_{\text{app}}(X, t) = U_0(X, t) + \varepsilon U_1(X, t) + \varepsilon U^2(X, t), \quad (1.4.2)$$

where $X = (x, y)$ or x and U_0 is a sum of a modulated wave packet and of an induced mean mode ϕ

$$U_0(X, t) = \begin{pmatrix} i\omega\psi(\varepsilon X, \varepsilon t) \\ \psi(\varepsilon X, \varepsilon t) \end{pmatrix} e^{i(\mathbf{k} \cdot X - \omega t)} + \text{c.c.} + \begin{pmatrix} 0 \\ \phi(\varepsilon X, \varepsilon t) \end{pmatrix}. \quad (1.4.3)$$

In [45], D. Lannes derived the following Full Dispersion Benney-Roskes (FDBR) system coupling ψ, ϕ and the leading term ζ of the surface elevation (see [50] for the case of capillary-gravity waves)

$$\begin{cases} \partial'_t \psi + i \frac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon} \psi \\ + \varepsilon i [\mathbf{k} \cdot \nabla' \phi + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2] \psi = 0, \\ \partial'_t \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2\omega \mathbf{k} \cdot \nabla' (|\psi|^2), \\ \partial'_t \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2, \end{cases} \quad (1.4.4)$$

where $\mathbf{k}$ is the given wave vector, $\sigma = \tanh(\sqrt{\mu}|\mathbf{k}|)$ and $\alpha = -\frac{9}{8\sigma^2} (1 - \sigma^2)^2$.

Here the time and space derivatives (indicated by a $'$) are taken with respect to the slow time and space variables $t' = \varepsilon t$ and $X' = \varepsilon X$ and D' denotes the operator $\frac{1}{i} \nabla'$.

Without loss of generality one may assume that $\mathbf{k}$ is oriented in the x direction, that is $\mathbf{k} = |\mathbf{k}| e_x$. We will thus write (1.4.4) as

$$\begin{cases} \partial'_t \psi + i \frac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon} \psi \\ + \varepsilon i [|\mathbf{k}| \partial'_x \phi + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2] \psi = 0, \\ \partial'_t \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2\omega |\mathbf{k}| \partial'_x (|\psi|^2), \\ \partial'_t \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2, \end{cases} \quad (1.4.5)$$

Remark 2. It is proven in [45] that (4.1.4) (or (1.4.5)) is consistent with the full water wave system. We recall that the notion of consistency is a "static" (not dynamic) one. It amounts to saying that an assumed solution of the water waves system on some time interval $[0, T]$ solves the BRFD or FDBR system on the same time interval up to some error term. A full justification of the BRFD system requires in particular that solutions of (1.4.4) exist on a time scale of order $O(1/\varepsilon)$ which is an open problem.

It also necessitates to prove the well-posedness of the full water -wave system in the modulation regime on the correct time scales, which has been proven in one and two horizontal spatial dimensions in [71, 70] in the infinite depth case. The finite depth case is up to our knowledge open.

The obvious advantage of (4.1.4) (or (1.4.5)) on the classical ² Benney-Roskes system (see [40, 45]) is its a priori validity over an extended range of frequencies.

The full dispersion Benney-Roskes system with surface tension (capillary-gravity waves) is given by (see [50])

$$\begin{cases} \partial_t' \psi + i \frac{\omega_S(\mathbf{k} + \varepsilon D') - \omega_S(\mathbf{k})}{\varepsilon} \psi \\ + i \varepsilon [|\mathbf{k}| \partial_x' \phi + \frac{|\mathbf{k}|^2 \kappa}{2\omega_S} (1 - \sigma^2) \zeta - \frac{2|\mathbf{k}|^4}{\omega_S} \gamma |\psi|^2] \psi = 0, \\ \partial_t' \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2 \frac{\omega_S}{\kappa} |\mathbf{k}| \partial_x' (|\psi|^2), \\ \partial_t' \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2. \end{cases} \quad (1.4.6)$$

Here, the dispersion relation for the water waves with surface tension is given by

$$\omega_S(\xi) = [|\xi| \tanh(\sqrt{\mu} |\xi|) (1 + B |\xi|^2)]^{1/2},$$

where $B > 0$ measures the surface tension effects (Bond number).

As for purely gravity waves, $\mathbf{k} = |\mathbf{k}| e_x$ is a given wave vector. We will use the notations $\omega_S = \omega_S(\mathbf{k})$, $\kappa = \kappa(\mathbf{k}) = 1 + B |\mathbf{k}|^2$ and $\sigma = \tanh(\sqrt{\mu} |\mathbf{k}|)$. γ is a constant depending on $\mathbf{k}$ and B .

The classical Benney-Roskes system in [40] is obtained from (1.4.4) or (1.4.6) by approximating the nonlocal operators $eg i\omega(\mathbf{k} + \varepsilon D')$ and $\frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon}$ as (denoting $\mathcal{H}_\omega(\mathbf{k})$ the Hessian of ω at $\mathbf{k}$)

$$i\omega(\mathbf{k} + \varepsilon D') \sim \varepsilon \nabla \omega(\mathbf{k}) \cdot \nabla' - \frac{i\varepsilon^2}{2} \nabla' \mathcal{H}_\omega(\mathbf{k}) \nabla',$$

and

$$|D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \sim \sqrt{\mu} |D'|^2.$$

The Cauchy problem for the Benney-Roskes system³ is proven in [57] to be locally well-posed, but on time scales which do not reach the order $O(1/\varepsilon)$ that would be required (together with uniform bounds in ε) for the full justification of the system, see [45]. We refer to [49] for some results on the Cauchy problem for the full dispersion Benney-Roskes system.

As explained in [45] in the "classical" case, the Davey-Stewartson model is obtained from the Benney-Roskes system by observing that at leading order, ψ travels at the group speed velocity $c_g = \nabla \omega(\mathbf{k})$. One thus looks for solutions of (1.4.4) under the form

$$\psi(X', t') = \tilde{\psi}(X' - c_g t, \varepsilon t'),$$

²We will use the term "classical" to refer to those systems where the nonlocal dispersion has been Taylor-expanded yielding *local* differential operators instead of *nonlocal* ones.

³a similar system is derived in [74] in the general context of interactions between low and high-frequency waves. See also [66], Chap. 15, for a similar system describing the interaction of a laser beam with plasma low-frequency fluctuations driven by ponderomotive forces.

$$\begin{aligned}\phi(X', t') &= \tilde{\phi}(X' - c_g t', \varepsilon t') + \phi^*(X', t', \varepsilon t'), \\ \zeta(X', t') &= \tilde{\zeta}(X' - c_g t', \varepsilon t') + \zeta^*(X', t', \varepsilon t').\end{aligned}$$

Neglecting $O(\varepsilon)$ correctors, one can replace ∂_t' by $-c_g \cdot \nabla'$, and one is finally led (see details in [45]) to the *Full Dispersion Davey-Stewartson System* (FDDS) for gravity waves, where $\tau = \varepsilon t' = \varepsilon^2 t$ (from now on, we will drop the tildas in the FDDS systems)

$$\begin{cases} \partial_\tau \psi + \frac{i}{\varepsilon^2} [\omega(\mathbf{k} + \varepsilon D') - \omega - \varepsilon \omega' D'_x] \psi \\ + i(\beta \partial_x' \phi + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \tilde{\alpha}) |\psi|^2) \psi = 0, \\ \left(\frac{|D'| \tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} + \omega'^2 \partial_x'^2 \right) \phi = 2\omega \beta \partial_x' (|\psi|^2). \end{cases} \quad (1.4.7)$$

We have used here the following notations (see [45])

$$\begin{cases} \mathbf{k} = |\mathbf{k}| \mathbf{e}_x, \quad \omega(\mathbf{k}) = \tilde{\omega}(|\mathbf{k}|), \quad \text{with} \quad \tilde{\omega}(r) = (r \tanh(\sqrt{\mu} r))^{1/2}, \\ \omega = \tilde{\omega}(|\mathbf{k}|), \quad \omega' = \tilde{\omega}'(|\mathbf{k}|), \quad \omega'' = \tilde{\omega}''(|\mathbf{k}|), \\ \sigma = \tanh(\sqrt{\mu} |\mathbf{k}|), \quad \alpha = -\frac{9}{8\sigma^2} (1 - \sigma^2)^2, \quad \tilde{\alpha} = \alpha + \frac{1}{4} (1 - \sigma^2)^2, \\ \beta = |\mathbf{k}| \left(1 + (1 - \sigma^2) \frac{\omega' |\mathbf{k}|}{2\omega} \right). \end{cases} \quad (1.4.8)$$

We also recall that the Fourier multiplier operator $\omega(\mathbf{k} + \varepsilon D')$ is given by

$$\omega(\mathbf{k} + \varepsilon D') = (|\mathbf{k} + \varepsilon D'| \tanh(\sqrt{\mu} |\mathbf{k} + \varepsilon D'|))^{1/2}.$$

Note (see [45]) that the complete rigorous justification of the full dispersion Davey-Stewartson system requires just the well-posedness of the Cauchy problem for (1.4.16) on time scales $\tau = O(1)$.

For capillary-gravity waves, the full dispersion Davey-Stewartson system writes

$$\begin{cases} \psi_\tau + \frac{i}{\varepsilon^2} [\omega_S(\mathbf{k} + \varepsilon D') - \omega_S - \varepsilon \omega'_S D'_x] \psi \\ + i(\beta_S \partial_x' \phi - 2 \frac{|\mathbf{k}|^4}{\omega_S} \tilde{\gamma} |\psi|^2) \psi = 0, \\ \left(|D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} + (\omega'_S)^2 \partial_x'^2 \right) \phi = 2 \frac{\omega_S}{\kappa} \beta_S \partial_x' (|\psi|^2), \end{cases} \quad (1.4.9)$$

with

$$\beta_S = |\mathbf{k}| (1 + (1 - \sigma^2) \frac{\omega'_S |\mathbf{k}|}{2\omega_S}), \quad \tilde{\gamma} = \gamma + \frac{\kappa}{4} (1 - \sigma^2)^2.$$

The classical Davey-Stewartson system which was derived formally in [29] for gravity waves and in [3, 30] for capillary-gravity waves has been justified rigorously in the sense of consistency in [26, 27, 25].⁴ It is obtained by approximating the nonlocal operators in (1.4.16) by the relevant local ones as indicated above for the Full dispersion Benney-Roskes system.

We recall (see [33]) that the Davey-Stewartson systems have the general form

$$\begin{cases} i \partial_t \psi + a \partial_x^2 \psi + b \partial_y^2 \psi = (\nu_1 |\psi|^2 + \nu_2 \partial_x \phi) \psi, \\ c \partial_x^2 \phi + \partial_y^2 \phi = -\delta \partial_x (|\psi|^2). \end{cases} \quad (1.4.10)$$

⁴Its "universal" character as an asymptotic system of quadratic hyperbolic systems in the modulation regime has been proved in [21].

We refer to [3] for an explicitation of $a, b, c, \nu_1, \nu_2, \delta$ in terms of the physical parameters. It is also convenient to use the normalized form (see [33])

$$\begin{aligned} i\partial_t\psi + \partial_x^2\psi + \delta\partial_y^2\psi &= (\chi|\psi|^2 + b\partial_x\phi)\psi, \\ \partial_x^2\phi + m\partial_y^2\phi &= \partial_x|\psi|^2, \end{aligned} \quad (1.4.11)$$

where the parameters δ, χ, b and m are real and can assume both signs, and δ, χ have been normalized in such a way that $|\delta| = |\chi| = 1$.

The Davey-Stewartson systems can be classified as “elliptic-elliptic”, “hyperbolic-elliptic”, “elliptic-hyperbolic” and “hyperbolic-hyperbolic” depending on the sign of δ, m .

The “hyperbolic-hyperbolic” case does not occur in the context of water waves. The “elliptic-elliptic” case may lead to a blow-up phenomenon reminiscent of that of the cubic focusing NLS (see [33] for a proof of blow-up and [55] for numerical simulations analyzing the dynamics of the singularity formation).

In the case of purely gravity waves, one has $b > 0$ and $\delta > 0$. Moreover, $a = \frac{1}{2}\omega'' < 0$ since $\omega'' = \tilde{\omega}''(|\mathbf{k}|) < 0$ by the concavity of $\tilde{\omega}$.

On the other hand, $c = \sqrt{\mu} - \omega'^2 > 0$ (see Section 2). This is a *subsonic* condition. Actually this amounts to saying that $M < 1$ where the Mach number M is defined as $M = C_g/\sqrt{gh}$, where $c_g = \frac{d}{dr}\sqrt{gr \tanh(hr)}|_{r=|\mathbf{k}|}$.

One can thus invert in this case the equation for ϕ , writing (one can assume $c = 1$ without loss of generality) $\phi_x = \delta R_1^2(|\psi|^2)$, where R_1 is the Riesz transform $R_1 = \partial_x(-\Delta)^{-1/2}$, and reduce (1.4.19) to a single semilinear (cubic) Schrödinger type equation

$$i\partial_t\psi + a\partial_x^2\psi + b\partial_y^2\psi = (\nu_1|\psi|^2 + \nu_2\delta R_1^2(|\psi|^2))\psi, \quad (1.4.12)$$

which is studied in [33]. In particular, whatever the signs of a and b , the Cauchy problem is locally well-posed for initial data in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$ and globally for small data, as the corresponding cubic NLS. The global Cauchy problem is much less understood. For instance, a global well-posedness result for the *hyperbolic-elliptic* case ($b > 0, a < 0$) in (1.4.12) is still missing for arbitrary large initial data.

However, for a very specific choice of coefficients⁵, (1.4.12) is integrable by the inverse scattering transform method leading to qualitative informations on the flow. After scaling, this integrable version of the DS system writes

$$\begin{cases} i\partial_t\psi + \psi_{xx} - \psi_{yy} = \alpha|\psi|^2\psi + \beta\psi\phi_x, \\ \Delta\phi = \partial_x(|\psi|^2), \end{cases} \quad (1.4.13)$$

with

$$\alpha + \frac{\beta}{2} = 0.$$

It is then known as the DS II system, with two subcases, the focussing ($\beta > 0$) and the defocussing one ($\beta < 0$). For the defocussing DS II, one has global solutions that behaves asymptotically in time as the solution of the linear case (see [67, 56, 1]). In particular global well-posedness (and scattering) is proven in [1] with arbitrary L^2 initial data.

On the other hand, in the focussing integrable DS II system, the presence of an explicit localized *lump* solitary wave leads to blow-up in finite time by a pseudo-conformal law [54]. This later behavior does not seem to be generic, that is does not seem to persist in the non integrable case (see the numerical simulations in [43]). In particular, one can prove rigorously that no localized solitary waves exist “not too close” to the integrable case (see [32]). However, the “scattering”

⁵with a limited physical relevance

behavior of arbitrary large solutions to the defocussing integrable DS II system proven in [67, 56] might be generic. In both situations, the numerical simulations in [42, 43] are illuminating.

When surface tension is strong enough however, c is no more positive (see [3]) and ϕ is inverted (with suitable conditions at infinity) via a wave equation and $\partial_x \phi$ is expressed in terms of $|\psi|^2$ via an order one nonlocal operator. The integrable version is known as the DS I system. We refer to [2] and the references therein for a description of the results obtained on DS I system by inverse scattering techniques, in particular the existence of special localized solutions called *dromions*. The resulting Schrödinger type equation is no more semilinear (it involves an order one nonlinear term) which makes the study by PDE techniques delicate. The local associated Cauchy problem is studied in particular in [47, 37, 39]. Scattering of small solutions is proven in [38].

For the FDDS system, in contrast to the classical system (1.4.19) for gravity waves ("hyperbolic-elliptic"), one cannot invert in a straightforward fashion the equation for ϕ which makes the associated Cauchy problem delicate.

As suggested in [45] one could consider as well the full dispersion version of only the first or the second of the two equations of the FDDS system (1.4.16), namely

$$\begin{cases} \partial_\tau \psi + \frac{i}{\varepsilon^2} [\omega(\mathbf{k} + \varepsilon D') - \omega - \varepsilon \omega' D'_x] \psi \\ + i(\beta \partial'_x \phi + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \tilde{\alpha}) |\psi|^2) \psi = 0, \\ [(\sqrt{\mu} - \omega'^2) \partial_x^2 + \sqrt{\mu} \partial_y^2] \phi = -2\omega \beta \partial'_x (|\psi|^2), \end{cases} \quad (1.4.14)$$

or

$$\begin{cases} \partial_\tau \psi - \frac{i}{2} (\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial'_x \phi + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \tilde{\alpha}) |\psi|^2) \psi = 0, \\ \left(\frac{|D'| \tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} + \omega'^2 \partial_x^2 \right) \phi = 2\omega \beta \partial'_x (|\psi|^2). \end{cases} \quad (1.4.15)$$

This has the advantage of "splitting the difficulties": in the first case, one can invert the equation for ϕ as for the hyperbolic-elliptic classical DS case, but one keeps the "bad" dispersive properties of the Schrödinger part. In the later case one has the "good" properties of the hyperbolic NLS and the "bad" invertibility properties of the equation for ϕ .

We consider here the full dispersion Davey-Stewartson system written as

$$\begin{cases} \partial_t \psi - \frac{i}{2} (\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial_x \phi + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2) \psi = 0, \\ \left(\frac{|D| \tanh(\varepsilon \sqrt{\mu} |D|)}{\varepsilon} + \omega'^2 \partial_x^2 \right) \phi = 2\omega \beta \partial_x (|\psi|^2). \end{cases} \quad (1.4.16)$$

Which is a 2-d model in space for the unknown $\psi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ and $\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

As noticed in [51] the "subsonic" condition $\sqrt{\mu} - \omega'^2 > 0$ does not ensures the solvability of the equation for ϕ since, denoting ξ the dual variable of X' , the frequencies ξ such that

$$\frac{\tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon |\xi|} = \omega'^2 \frac{\xi_1^2}{|\xi|^2}$$

will cancel the symbol.

The parameters $\omega', \omega'', |\mathbf{k}|, \alpha, \beta, \mu$ are defined as previously.

We will only consider the case $\sqrt{\mu} > \omega'^2$, so that we can find r_0 such that:

$$\tanh(\sqrt{\mu} r_0) = \omega'^2 r_0.$$

In order to invert easily the equation for ϕ we introduce an approximate system by truncating the high frequencies.

More precisely, let δ be fixed such that $0 < \delta < r_0$, for each $\varepsilon > 0$, we define $R_\varepsilon = \frac{r_0 - \delta}{\varepsilon}$. We then replace the mean mode ϕ by $\phi_\varepsilon := \mathcal{F}^{-1}(\chi(|\xi| \leq R_\varepsilon)\widehat{\phi})$ so that ⁶

$$\text{supp } \widehat{\phi_\varepsilon} \subset \{\xi \in \mathbb{R}^2; |\xi| \leq R_\varepsilon\} =: \Omega_\varepsilon, \quad (1.4.17)$$

to get a family of systems depending on the parameter ε as follows

$$\begin{cases} \partial_t \psi_\varepsilon - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_\varepsilon + i(\beta \partial_x \phi_\varepsilon + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_\varepsilon|^2) \psi_\varepsilon = 0, \\ (\frac{|D| \tanh(\varepsilon \sqrt{\mu} |D|)}{\varepsilon} + \omega'^2 \partial_x^2) \phi_\varepsilon = 2\omega \beta \partial_x (|\psi_\varepsilon|^2). \end{cases} \quad (1.4.18)$$

Since the supports of $\widehat{\phi_\varepsilon}$ and $\widehat{|\psi_\varepsilon|^2}$ are in $\{\xi \in \mathbb{R}^2; |\xi| < R_\varepsilon\}$, we need to construct a suitable initial data for the Cauchy problem for the $\varepsilon - FDDS$ which converge to the initial data of the Cauchy problem for the corresponding classical DS system. We proceed as follows.

First, let $\psi_0 \in H^1$ be the initial data of the classical DS system, then we define

$$\psi_{0\varepsilon} = f_\varepsilon \star \psi_0 \text{ where } f_\varepsilon = \mathcal{F}^{-1}(\chi(|\xi| < \frac{R_\varepsilon}{2})).$$

Then we have that $\text{supp } \widehat{\psi_{0\varepsilon}} \subset \{|\xi| < \frac{R_\varepsilon}{2}\}$. It follows

$$\text{supp } \widehat{|\psi_{0\varepsilon}|^2} \subset \text{supp } \widehat{\psi_{0\varepsilon}} + \text{supp } \overline{\widehat{\psi_{0\varepsilon}}(-)} \subset \{|\xi| < R_\varepsilon\}.$$

Our purpose is to prove that when ε tends to 0, the solution of (1.4.18) with initial data $\psi_{0\varepsilon}$ tends to the solution of the following classical Davey-Stewartson system with initial data ψ_0 :

$$\begin{cases} \partial_t \psi - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial_x \phi + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2) \psi = 0, \\ (\sqrt{\mu} |D|^2 + \omega'^2 \partial_x^2) \phi = 2\omega \beta \partial_x (|\psi|^2). \end{cases} \quad (1.4.19)$$

1.5 Boussinesq system

We are concerned with the (a,b,c,d) Boussinesq systems introduced in [9] to model surface water waves. They couple the elevation η and the horizontal velocity $\mathbf{u}$ of the wave:

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \varepsilon \nabla \cdot (\eta \mathbf{u}) + \mu [a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0, \\ \mathbf{u}_t + \nabla \eta + \varepsilon \frac{1}{2} \nabla |\mathbf{u}|^2 + \mu [c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \quad \text{in } \mathbb{R}^d \times \mathbb{R}, d = 1, 2. \end{cases} \quad (1.5.1)$$

Here μ and ε are the small parameters (shallowness and nonlinearity parameters respectively) defined as

$$\mu = \frac{h^2}{\lambda^2}, \quad \varepsilon = \frac{\alpha}{h}$$

where α is a typical amplitude of the wave, h a typical depth and λ a typical horizontal wavelength.

⁶ $\chi(A)$ denote the characteristic function of the set A
 $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in space variable.

In the Boussinesq regime, ϵ and μ are supposed to be of same order, $\epsilon \sim \mu \ll 1$, and we will take for simplicity $\epsilon = \mu$, writing (1.5.1) as

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon[\nabla \cdot (\eta \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon[\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \quad (1.5.2)$$

The class of systems (1.5.1), (1.5.2) models water waves on a flat bottom propagating in both directions in the aforementioned regime (see [9, 10, 12]).

One could also derive similar systems with a non trivial bathymetry (non flat bottom), see [17], and one has then to distinguish between the case when the bottom varies slowly and the case where it is strongly varying. In the former case, (1.5.2) has to be slightly modified and becomes

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon[\nabla \cdot ((\eta - \beta) \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0, \\ \mathbf{u}_t + \nabla \eta + \epsilon[\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \quad (1.5.3)$$

where β is a smooth function on $\mathbb{R}^d$, $d = 1, 2$, bounded together with its derivatives measuring the bathymetry. In the second case one gets much more complicated systems [17].

Also, systems similar to (1.5.2) can be derived for internal waves, see for instance [11, 7, 31].

Recall (see [9, 28]) that the modeling parameters are constrained by the relation

$$a + b + c + d = \frac{1}{3} - \tau,$$

where $\tau \geq 0$ is the surface tension parameter (Bond number).

Recall also [9] that (1.5.2) is linearly well-posed when

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0, \quad (1.5.4)$$

and when

$$a = c, b \geq 0, d \geq 0. \quad (1.5.5)$$

It has been established in [12] that the (a,b,c,d) systems are good approximations of the full water waves system, in the relevant regime, with an error of order $O(\epsilon^2 t)$. The complete justification of the Boussinesq systems needs thus to establish the *long time existence* for the Cauchy problem, that is on time scales of order $O(1/\epsilon)$. This has been achieved for all linearly well-posed Boussinesq systems in [48, 62, 15, 61]. Note that no global well-posedness result seems to be known for the Cauchy problem of (1.5.2) in the spatial two-dimensional case (however the global existence of solutions is established in a few one-dimensional cases, see [10, 6, 64]).

It has been proved by Zakharov [75] that the full water waves system is Hamiltonian. This is not the case in general for the (a,b,c,d) systems due to the transformations made to derive them starting from the following (ill-posed) "original" Boussinesq system obtained by expanding the Dirichlet-Neumann operator with respect to ϵ (see [9, 45]):

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \frac{\epsilon}{3} \Delta \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0 \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 = 0. \end{cases} \quad (1.5.6)$$

Nevertheless, an Hamiltonian structure is found when $b = d$. More precisely, denoting by J_ϵ the skew adjoint matrix operator

$$J_\epsilon = \begin{pmatrix} 0 & \partial_x(I - \epsilon b \Delta)^{-1} & \partial_y(I - \epsilon b \Delta)^{-1} \\ \partial_x(I - \epsilon b \Delta)^{-1} & 0 & 0 \\ \partial_y(I - \epsilon b \Delta)^{-1} & 0 & 0 \end{pmatrix},$$

and

$$U = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix},$$

the Boussinesq systems write in this case

$$\partial_t U = -J_\epsilon(\text{grad } \mathcal{H}_\epsilon)(U),$$

where $\mathcal{H}_\epsilon(U)$ is the Hamiltonian given by

$$\mathcal{H}_\epsilon(U) = \frac{1}{2} \int_{\mathbb{R}^2} (-c\epsilon|\nabla\eta|^2 - a\epsilon|\nabla\mathbf{u}|^2 + \eta^2 + |\mathbf{u}|^2 + \epsilon\eta|\mathbf{u}|^2) dx dy,$$

so that $\mathcal{H}_\epsilon(U)$ is (formally) conserved by the flow.

Note that when $d = 1$ the following quantity (momentum) is also formally conserved

$$\mathcal{I}_\epsilon(U) = \int_{\mathbb{R}} (\eta u + \epsilon b \eta_x u_x) dx.$$

We are interested here in *localized* solitary wave solutions to (1.5.2). No such solutions are known to exist in the two-dimensional case (see however [20] for the existence of asymmetric periodic two-dimensional wave patterns).

We will thus focus on one-dimensional solitary waves and more specifically to their transverse stability. To start with we review the known results on the existence of one dimensional traveling waves to (1.5.2). We thus look for solutions of (5.1.2) of the form $\eta = \eta(x - \omega t), u = u(x - \omega t)$ where $x \in \mathbb{R}$ yielding the system

$$\begin{cases} -\omega\eta + u + \epsilon u\eta + \epsilon a u_{xx} - \epsilon b \eta_{xx} = 0 \\ -\omega u + \eta + \frac{\epsilon}{2} u^2 + \epsilon c u_{xx} - \epsilon d \eta_{xx} = 0. \end{cases} \quad (1.5.7)$$

Min Chen [18, 19] has found in a few cases exact solutions when η and u are proportional but it turns out that the only case compatible with the well posedness conditions (1.5.4) and (1.5.5) are

$$a - b + 2d = 0, \quad a = c, \quad d > 0 \quad (1.5.8)$$

and

$$\begin{cases} a - b + 2d \neq 0, \\ p = \frac{-b + c + 2d}{a - b + 2d} > 0, \\ (p - \frac{1}{2})((b - a)p - b) > 0. \end{cases} \quad (1.5.9)$$

The exact solutions have a sech profile, namely

$$\eta(x, t) = \eta_0 \text{sech}^2(\lambda(x + x_0 - c_s t))$$

and

$$u(x, t) = \pm \sqrt{\frac{3}{\eta_0 + 3}} \eta_0 \text{sech}^2(\lambda(x + x_0 - c_s t)),$$

where λ and c_s are appropriate constants.

Note that conditions (1.5.8), (1.5.9) can hold in the non Hamiltonian case $b \neq d$.

On the other hand no general *non existence result* seems to be available. One has nevertheless the following remark which excludes the existence of solitary waves with small velocity when the well-posedness condition (1.5.5) holds true with moreover $b = d$.

Proposition. *Let $b = d \geq 0$. Then any H^1 solitary wave solution (η, u) satisfies the identity*

$$\int_{\mathbb{R}} (u^2 + \eta^2 + 5\epsilon(au_x^2 + c\eta_x^2))dx = 2\omega \int_{\mathbb{R}} (3u\eta - 5\epsilon bu_x\eta_x)dx.$$

As a consequence, one obtains that no non trivial solitary waves with ω small exist when

$$a \geq 0, c \geq 0, b = 0,$$

or

$$a > 0, c > 0, b > 0.$$

We recall that one need the condition $a = c$ to get a well-posed system when a, c are not negative.

In this thesis, we survey the known existing results on existence (and stability) of one-dimensional traveling waves to (1.5.2). Then, we prove the suitable long time existence result for the systems derived from (1.5.2) by a localized perturbation of a line solitary wave.

1.6 Numerics for the two dimensional Zakharov-Rubenchik/Benney-Roskes system

Finally we want to illustrate and complement the analytical studies by numerical simulations in some special situations. Such numerical simulations can guide the analytical work and give insight in the qualitative behaviour of solutions (e.g. [78], [79] and many more).

As we already introduced in Section 1.3, the two dimensional Z-R/B-R system is a very general equation with many open and difficult problems such as the transverse stability of the line soliton or the global existence of solution.

We first recall some notations and definitions from Section 1.3. The following is the two dimensional Z-R/B-R system equation:

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\delta\psi_{xx} - i\sigma_1\psi_{yy} + i\left\{\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right\}\psi = 0, \\ \rho_t + \Delta\phi + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2}\rho + |\psi|^2 = 0, \end{cases} \quad (1.6.1)$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, $\rho, \phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\sigma_3, \delta, \sigma_1, \sigma_2, W, \rho, D, M$ are real parameters which depend in the context of problem. In the context of water wave problem, it is known as Benney-Roskes equation and in the MHD, it is called Zakharov-Rubenchik equation. The local Cauchy problem for the Z-R/B-R equation and related problems are studied in [57] and recently in [76] when the equation is perturbed by a line solitary wave.

The one dimensional Z-R/B-R system has the form

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \phi_{xx} + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0. \end{cases} \quad (1.6.2)$$

Let

$$a = \frac{-(1 + cD)}{1/M^2 - c^2} \quad \text{and} \quad b = \frac{-(c + D/M^2)}{1/M^2 - c^2}.$$

If

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) < 0, \\ \frac{1}{\delta} (\sigma_2 + W(a + bD)) < 0, \end{cases}$$

From the Chapter 3, there is a solitary wave of (1.6.2) of the form

$$Q = (e^{i\lambda t} e^{i(c+\sigma_3)(x-ct)/2\delta} R(x-ct), a R^2(x-ct), b P(x-ct)). \quad (1.6.3)$$

where

$$R(x) = \alpha \operatorname{sech}(\beta x), \quad P(x) = \frac{\alpha^2}{\beta} \tanh(\beta x),$$

$$\alpha = \sqrt{\frac{2}{\sigma_2 + W(a + bD)} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{-\frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}.$$

In one dimensional case, it is proved rigorously in [52] that Q is orbitally stable and this result is also verified numerically in [77].

We use a finite domain in x with periodic boundary conditions, which is the good setting to use a time-splitting Fourier Pseudo-Spectral method (see [77], [78]).

However, since the initial data of ϕ in (1.6.3) is not compatible with the periodic boundary condition setting, we will differentiate the last equation of (1.6.1) and consider two new variables those are ϕ_x and ϕ_y . Hence we consider the following system

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} - i\sigma_1 \psi_{yy} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \partial_x(\phi_x) + \partial_y(\phi_y) + D(|\psi|^2)_x = 0, \\ \partial_t(\phi_x) + \frac{1}{M^2} \rho_x + (|\psi|^2)_x = 0, \\ \partial_t(\phi_y) + \frac{1}{M^2} \rho_y + (|\psi|^2)_y = 0. \end{cases} \quad (1.6.4)$$

In Chapter 6, we will present the numerical method in detail which is used to study the equation (1.6.4) and present our numerical results.

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Chapter 2

Local well-posedness for the Zakharov system on the background of a line soliton

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Abstract: We prove that the Cauchy problem for the two-dimensional Zakharov system is locally well-posed for initial data which are localized perturbations of a line solitary wave. Furthermore, for this Zakharov system, we prove a weak convergence to a nonlinear Schrödinger equation.

2.1 Introduction

In this paper, we study the initial value problem for the scalar version of the two dimensional Zakharov system

$$i\partial_t u + \Delta u = nu, \quad (2.1.1)$$

$$\frac{1}{\lambda^2} \partial_t^2 n - \Delta n = \Delta(|u|^2), \quad (2.1.2)$$

where $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$, λ is a fixed number, u is a complex valued function, n is a real function, with initial data

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) + Q(x), \quad n(x, y, 0) = n_0(x, y) - Q(x)^2, \\ n_t(x, y, 0) &= n_1(x, y). \end{aligned} \quad (2.1.3)$$

The above function $Q(x) = 2\sqrt{2}/(e^x + e^{-x})$ is the unique positive solution of the equation

$$Q_{xx} - Q + Q^3 = 0,$$

and

$$(e^{it}Q(x), -Q(x)^2) \quad (2.1.4)$$

is a line soliton of (2.1.1)-(2.1.2).

And also note that $e^{it}Q(x)$ is a line soliton of the cubic focusing nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0. \quad (2.1.5)$$

The Zakharov system is introduced in [21] to describe the propagation of Langmuir waves in plasma. For more details about the derivation and the physical background of Zakharov system we refer to [19] and [21].

In (6.1.3), the soliton is considered as a two dimensional (constant in y) object. A natural question is that of its transverse (with respect to y) stability. When looking for *localized* perturbations, one is lead, as a first step, to study the Cauchy problem for the perturbed system below.

Another possibility, looking for *y-periodic* perturbations, would be to study (2.1.1)-(2.1.2) as a system posed on $\mathbb{R} \times \mathbb{T}$, the 1-d torus case was studied in [5, 20]. On the other hand, the Zakharov system in spatial space $\mathbb{R}^2$ and $\mathbb{R}^3$ have been studied by many authors [1, 2, 3, 4, 6, 7, 8, 9, 16, 18]

The local well-posedness of (2.1.1)-(2.1.2) with initial data given by (2.1.3) which will be proved in this paper can be viewed as the first step to study the transverse stability (or instability) of the line soliton (6.1.3). As far as we know, this problem is still an open problem. Concerning *global* issues we can quote [7, 8] where the authors prove the existence of self-similar blow-up solutions and the instability by blow-up of periodic (in time) solutions (solitary wave solutions) of the Zakharov system with data in $H^1(\mathbb{R}^2)$. Their method uses the radial symmetry of the system which is broken when one writes the system satisfied by a localized perturbation of the line solitary wave.

The transverse instability of the line solitary wave for some two dimensional models such as nonlinear Schrödinger equation, Kadomtsev-Petviashvili equation and for some general “abstract” problems have been studied extensively in [14, 13, 15] but the framework of those papers does not seem to include the case of the Zakharov system.

In other way, instead of the system (2.1.1)-(2.1.2) with the initial data of the form (2.1.3) we can consider the following perturbed system

$$i\partial_t(u + Q) - (u + Q) + \Delta(u + Q) = (n - Q^2)(u + Q), \quad (2.1.6)$$

$$\frac{1}{\lambda^2}\partial_t^2(n - Q^2) - \Delta(n - Q^2) = \Delta(|u + Q|^2), \quad (2.1.7)$$

with initial data

$$u(x, y, 0) = u_0, \quad n(x, y, 0) = n_0, \quad n_t(x, y, 0) = n_1. \quad (2.1.8)$$

Note that in the above system, we used the change of variable $\bar{u}(x, y, t) := e^{it}u(x, y, t)$.

The main purpose of this paper is to prove that the Cauchy problem (2.1.6)-(2.1.7), (2.1.8) is locally well-posed in a suitable functional framework. The main differences between the Zakharov system and its perturbation lie in the new terms containing “ Q ”. More precisely,

i) If we reduce system (2.1.6)-(2.1.7) to a nonlinear Schrödinger equation with loss of derivative in the nonlinearity, then we can think of using the smoothing effect of Schrödinger operator as in [9]. In our case, the linear terms Qn and Q^2u will give trouble because the function Q does not decay in y at infinity.

ii) We use the method of Bourgain (actually the techniques developed in [4, 6]) and the method of Schochet-Weinstein (see [16]) to obtain two versions of the local well-posedness result for system (2.1.6)-(2.1.7). The differences with the case of the unperturbed Zakharov system are the estimates for the new linear terms and the effects of Q in each method.

Compared to Bourgain method, Schochet-Weinstein method provides well-posedness in smaller Sobolev spaces but allows, when a suitable small parameter is included, to obtain the Schrödinger limit. For the unperturbed system see [2, 12, 16].

iii) There are some difficulties concerning blow-up and global existence issues. In the setting of (2.1.6)-(2.1.7), the approach in [8, 7] does not make sense because we do not have radial symmetric solutions. We also do not have L^2 conservation or at least a bound on L^2 norm that makes the usual method to extend the local solution impossible to apply.

The rest of this paper is organized as follows.

In Section 2.2, we present the energy conservation of system (2.1.6)-(2.1.7).

In Section 2.3, we assume that $\lambda = 1$ and use the Bourgain’s method to prove the local well-posedness of system (2.1.6)-(2.1.7), more precisely we have the following theorem.

Theorem 2.1.1. *Let k and l satisfy*

$$k > l \geq 0, \quad l + 1 \geq k \geq \frac{l + 1}{2}. \quad (2.1.9)$$

Then the system (2.1.6)-(2.1.7) with initial data $(u_0, n_0, n_1) \in H^k \times H^l \times H^{l-1}$ is locally well posed in $X_1^{k, b_1} \times X_2^{l, b} \times X_2^{l-1, b}$ for suitable b, b_1 close to $1/2$.

Furthermore the solutions satisfy

$$(u, n, \partial_t n) \in \mathcal{C}([0, T], H^k \times H^l \times H^{l-1}), \quad (2.1.10)$$

where T is the existence time.

(The Bourgain space $X_j^{s, b}$ will be defined as (2.3.8) in section 2.3.)

In Section 2.4, we prove the local existence by using the method of Schochet and Weinstein. In particular, we have the following theorem.

Theorem 2.1.2. *Let $s > 2$ and consider the initial value problem (2.1.6)-(2.1.7) with initial data of the form*

$$u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = \nabla \cdot f_0.$$

Suppose that

$$\|u_0(\lambda)\|_{H^{s+1}} + \|n_0(\lambda)\|_{H^s} + \frac{1}{\lambda} \|f_0(\lambda)\|_{H^s} \leq C_1. \quad (2.1.11)$$

Then (2.1.6)-(2.1.7) has a unique solution

$$(u, n) \in L^\infty([0, T]; H^{s+1}(\mathbb{R}^2)) \times L^\infty([0, T]; H^s(\mathbb{R}^2)),$$

where $[0, T]$ is the time interval of existence, T depends only on C_1 . In addition, the solution (u, n) satisfies

$$\begin{aligned} & \|u(t, \lambda)\|_{H^{s+1}} + \|u_t(t, \lambda)\|_{H^{s-1}} + \|n(t, \lambda)\|_{H^s} + \frac{1}{\lambda} \|n_t(t, \lambda)\|_{H^{s-1}} \\ & + \frac{1}{\lambda^2} \|n_{tt}(t, \lambda)\|_{H^{s-2}} \leq C_2, \end{aligned} \quad (2.1.12)$$

for all $t \in [0, T]$.

In Section 2.5, using the uniform bound in the Theorem 2.1.2, we establish a weak convergence result stating that the local solution $(u(\lambda), n(\lambda))$ of (2.1.6)-(2.1.7) with initial data (2.1.8) tends to $(u, -|u|^2)$ weakly when λ tends to ∞ . Where u is the unique solution of the following perturbation of the nonlinear Schrödinger equation

$$i(u + Q)_t - (u + Q) + \Delta(u + Q) + |u + Q|^2(u + Q) = 0 \quad (2.1.13)$$

with $u(x, y, 0) = u_0$.

Remark 3. The local well-posedness of (2.1.13) in a Sobolev space H^s with $s \geq 1$ can be obtained by using Strichartz estimate. Unfortunately, the global existence in the focusing case is still unknown to us.

Notations. $\mathcal{F}$, $\mathcal{F}_t$, $\mathcal{F}_x$, $\mathcal{F}_y$ and $\mathcal{F}^{-1}$ denote the Fourier transform of a function in spacetime, time, space variable and the inverse Fourier transform respectively. We also use “ $\wedge$ ” as the short notation of the space-time Fourier transform.

H^s is the usual Sobolev space.

$|(\xi_1, \xi_2)| = (\xi_1^2 + \xi_2^2)^{1/2}$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$.

$D_x = -i d/dx$.

$\|\cdot\|_2$: The L^2 norm in space and time.

$\|u; X\|$ or $\|u\|_X$: The norm of function u in functional space X .

C will be a general constant unless otherwise explicitly indicated. $f \lesssim g$ means that there exists a constant C such that $f \leq Cg$.

- ∂_x or $(\cdot)_x$ will be used to denote the derivative with respect to variable x .
- $H^s(D)$, $s \in \mathbb{R}$ denotes the classical Sobolev space in the domain D .
- $\|\cdot\|_X$: The norm in a functional space X .
- $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transform respectively.
- $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$ and $\sigma(D)$ denotes the Fourier multiplier with the symbol $\sigma(\xi)$.
- $\Re$ and $\Im$ denote the real part and imaginary part of a complex number respectively.

2.2 Conservation law

The system (2.1.6)-(2.1.7) can be rewritten as follows

$$i\partial_t u + \Delta u - u = nu + Qn - Q^2 u \quad (2.2.1)$$

$$n_t = \lambda^2 \nabla \cdot \mathbf{v} \quad (2.2.2)$$

$$\partial_t \mathbf{v} - \nabla n = \nabla(|u|^2) + 2\nabla(QRe(u)). \quad (2.2.3)$$

Proposition 1. *Let $(u, n, \mathbf{v})$ be a solution of system (2.2.1)-(2.2.3) obtained in Theorem 2.1.1, defined in the time interval $[0, T]$. Then*

$$\frac{d}{dt} E(t) = 0, \quad 0 \leq t \leq T, \quad (2.2.4)$$

where

$$E(t) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2 + \frac{n^2}{2} + \frac{\lambda^2 |\mathbf{v}|^2}{2} - Q^2 |u|^2 + n(|u|^2 + 2QRe(u)) \right) dx dy \quad (2.2.5)$$

Proof. To obtain (2.2.4) we proceed formally. A rigorous proof can be obtained by smoothing the initial data and passing to the appropriate limit.

We multiply (2.2.1) by $\partial_t \bar{u}$, integrate and take its real part to get

$$\int \partial_t (|\nabla u|^2 + |u|^2 - Q^2 |u|^2) + n \partial_t (|u|^2 + 2QRe(u)) = 0. \quad (2.2.6)$$

We also multiply (2.2.3) by $\nabla^{-1} n_t$ or $\lambda^2 \mathbf{v}$ and integrate to get

$$\int \partial_t \left(\frac{\lambda^2 |\mathbf{v}|^2}{2} + \frac{n^2}{2} \right) + (|u|^2 + 2QRe(u)) n_t = 0. \quad (2.2.7)$$

Combining (2.2.6) and (2.2.7) we obtain (2.2.4). $\square$

Remark 4. *The energy space of system (2.1.6)-(2.1.7) is $H^1 \times L^2 \times H^{-1}$ and the Theorem 2.1.1 gives the local well-posedness of (2.1.6)-(2.1.7) in the energy space. It is expected to get a global solution with small initial data in the energy space, but in our case, the difficulty is the lack of L^2 conservation.*

2.3 Bourgain method

2.3.1 Linear estimate

Throughout this Section we will assume that $\lambda = 1$ and split the function n in (2.1.6)-(2.1.7) into two parts

$$n = n_+ + n_-,$$

where $n_{\pm} = n \pm i\omega^{-1} \partial_t n$, $\omega = (-\Delta)^{1/2}$, then the system (2.1.6)-(2.1.7) can be rewritten as

$$i\partial_t u = -\Delta u + u + \frac{n_+ + n_-}{2} u + \frac{n_+ + n_-}{2} Q - Q^2 u, \quad (2.3.1)$$

$$(i\partial_t \mp \omega) n_{\pm} = \pm \omega(|u|^2) \pm 2\omega(QRe(u)). \quad (2.3.2)$$

The equations (2.3.1) and (2.3.2) have the form

$$i\partial_t u = \phi(-i\nabla)u + f(u), \quad (2.3.3)$$

where ϕ is a real function defined in $\mathbb{R}^2$ and f some nonlinear function. The Cauchy problem for (2.3.3) with initial data u_0 is rewritten as the integral equation

$$\begin{aligned} u(t) &= U(t)u_0 - i \int_0^t dt' U(t-t') f(u(t')) \\ &= U(t)u_0 - iU *_R f(u), \end{aligned} \quad (2.3.4)$$

where $U(t) = e^{-it\phi(-i\nabla)}$ and $*_R$ denotes the retarded convolution in time.

Let $\psi_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ be even with $0 \leq \psi_1 \leq 1$, $\psi_1(t) = 1$ for $|t| < 1$, $\psi_1(t) = 0$ for $|t| \geq 2$ and let $\psi_T = \psi_1(t/T)$ for $0 < T \leq 1$.

One then replaces equation (2.3.4) by the cut off equation

$$u(t) = \psi_1(t)U(t)u_0 - i\psi_T(t) \int_0^t dt' U(t-t') f(u(t')). \quad (2.3.5)$$

Then, we have the cut off integral equations associated with (2.3.1) and (2.3.2) namely

$$\begin{aligned} u(t) &= \psi_1(t)U(t)u_0 - (i/2)\psi_T(t) \int_0^t dt' U(t-t') (u + (u+Q)(n_+ + n_-) \\ &\quad - 2|Q|^2 u)(t'), \end{aligned} \quad (2.3.6)$$

$$n_\pm(t) = \psi_1(t)V_\pm(t)n_{\pm 0} \mp i\psi_T(t) \int_0^t dt' V_\pm(t-t') \omega(|u|^2 + 2Q\text{Re}(u))(t'), \quad (2.3.7)$$

where $U(t) = e^{it\Delta}$ and $V_\pm(t) = e^{\mp i\omega t}$.

We use a standard contraction method on the two operators on the right hand side of (2.3.6)-(2.3.7) with $u \in X_1^{k,b_1}$ and $n_\pm \in X_2^{l,b}$. X_1^{k,b_1} and $X_2^{l,b}$ are the Bourgain spaces associated to two operators with symbols $\phi_1(\xi) = |\xi|^2$ and $\phi_2(\xi) = \pm|\xi|$, which are given by the following definition

$$\|u; X_j^{s,b}\| = \left\| \langle \xi \rangle^s \langle \tau + \phi_j(\xi) \rangle^b \widehat{u}(\xi, \tau) \right\|_2, \quad (2.3.8)$$

where $\phi_j(\xi)$ is the symbol of the associated differential operator.

We will also use the following definition of space time Sobolev space,

$$\|u; H^{s,b}\| = \left\| \langle \xi \rangle^s \langle \tau \rangle^b \widehat{u}(\xi, \tau) \right\|_2.$$

The linear estimate is given by the following lemma

Lemma 2.3.1. (cf. [6, Lemma 2.1]) (i) Let $b' \leq 0 \leq b \leq b' + 1$ and $T \leq 1$. Then

$$\begin{aligned} &\left\| \psi_T U *_R f; X^{s,b} \right\| \\ &\leq C\sqrt{2}\{T^{1-b+b'} \|f; X^{s,b'}\| + T^{1/2-b} \|\mathcal{F}^{-1}\chi(|\tau|T \geq 1)\mathcal{F}f; Y^s\|\}, \end{aligned} \quad (2.3.9)$$

where

$$\|f; Y^s\| = \left\| \langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^{-1} \widehat{f}(\xi, \tau); L_\xi^2 L_\tau^1 \right\|.$$

(ii) Suppose in addition that $b' > -1/2$. Then

$$\left\| \psi_T U *_R f; X^{s,b} \right\| \leq CT^{1-b+b'} \|f; X^{s,b'}\|. \quad (2.3.10)$$

For $b > 1/2$, it is clear that $X^{s,b} \subset \mathcal{C}(\mathbb{R}, H^s)$. This is no longer true if $b \leq 1/2$ and we shall need the following Lemma for that result.

Lemma 2.3.2. (cf. [6, Lemma 2.2]). Let $f \in Y^s$, then $\int_0^t dt' U(t-t')f(t') \in \mathcal{C}(\mathbb{R}, H^s)$.

Remark 5. 1. In the case: $k < l + 1$ we shall be able to take (2.3.10) with $b, b_1 < 1/2$ and suitable $0 < c, c_1 < 1/2$ to estimate the following terms

- i) $n_{\pm}u, n_{\pm}Q, Q^2u$ and u in $X_1^{k, -c_1}$,
- ii) $\omega|u|^2$ and $\omega(Q\text{Re}(u))$ in $X_2^{l, -c}$.

Furthermore, in order to get solution in $\mathcal{C}([0, T], H^k \times H^l \times H^{l-1})$, we also need to estimate the terms on Y_1^k and Y_2^l respectively. (Y_j corresponds to the symbol ϕ_j , $j = 1, 2$)

- 2. In the limit case: $k = l + 1$ we shall be forced to take $b_1 = 1/2$. The estimates for Q^2u , $Qn_{\pm}$, $n_{\pm}u$ and u in Y_1^k is also needed for the proof of the local well-posedness.
- 3. We are allowed to assume that u and $n_{\pm}$ have compact support in t by using additional cutoffs inside f in (2.3.5) and consider the equation.

$$u(t) = \psi_1(t)U(t)u_0 - \psi_T(t) \int_0^t dt' U(t-t') f(\psi_{2T}(t')u(t')). \quad (2.3.11)$$

For the effect of those factors in the spaces $X^{s,b}$, we refer [6] Lemma 2.5.

The rest of this Section will be organized as follows: In section 2.3.2, we will prove the estimates for linear terms which are new in this context. The estimates for nonlinear terms were proved completely in [6] and we will recall them in section 2.3.3. Finally, we give the final step of the proof of Theorem 2.1.1.

2.3.2 Estimates for linear terms

We may assume that $n_{\pm} \in X_2^{l,b}$ and $u \in X_1^{k,b_1}$ then they can be rewritten in the form

$$\begin{aligned} \widehat{n_{\pm}} &= \langle \xi \rangle^{-l} \langle \tau \pm |\xi| \rangle^{-b} \widehat{v}, \\ \widehat{u} &= \langle \xi \rangle^{-k} \langle \tau + |\xi|^2 \rangle^{-b_1} \widehat{w}, \\ \widehat{\bar{u}} &= \langle \xi \rangle^{-k} \langle \tau - |\xi|^2 \rangle^{-b_1} \widehat{\bar{w}}, \end{aligned}$$

where $v, w \in L^2(\mathbb{R}^2)$. Then

$$\begin{aligned} \widehat{n_{\pm}Q}(\xi, \tau) &= \mathcal{F}_x(Q(x)\mathcal{F}_{yt}(n_{\pm}))(\xi_1) \\ &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{n_{\pm}}(\xi_1 - \xi'_1, \xi_2, \tau) d\xi'_1 \\ &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-l} \\ &\quad \times \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{-b} d\xi'_1, \\ \widehat{Qu}(\xi, \tau) &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1, \end{aligned}$$

and

$$\begin{aligned} \widehat{|Q|^2u}(\xi, \tau) &= \int \mathcal{F}_x\{|Q|^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1. \end{aligned}$$

Furthermore, in order to estimate $\omega(QRe(u))$ we will rewrite it as $\omega(Q \frac{u+\bar{u}}{2})$, then we also need the following form

$$\begin{aligned} \widehat{Q\bar{u}}(\xi, \tau) &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1. \end{aligned}$$

It is important to note that in our arguments that $\mathcal{F}_x\{Q\}$ and $\mathcal{F}_x\{Q^2\}$ are positive functions.

In order to estimate $n_{\pm}Q$ in $X_1^{k, -c_1}$, we take its scalar product with a generic function in X_1^{-k, c_1} with Fourier transform $\langle \xi \rangle^k \langle \tau + |\xi|^2 \rangle^{-c_1} \widehat{v_1}$ and $v_1 \in L^2$. The required estimate of $n_{\pm}Q$ in $X_1^{k, -c_1}$ then takes the form

$$|I_1| \leq CT^{\theta_1} \|v\|_2 \|v_1\|_2, \quad (2.3.12)$$

where

$$I_1 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_1}(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^b \langle \tau + |\xi|^2 \rangle^{c_1}} d\xi d\tau d\xi'_1,$$

with the notation $\xi = (\xi_1, \xi_2)$.

In order to estimate Q^2u in $X_1^{k, -c_1}$, the required estimate takes the form

$$|I_2| \leq CT^{\theta_2} \|w\|_2 \|v_2\|_2, \quad (2.3.13)$$

where

$$I_2 = \int \frac{\mathcal{F}_x\{Q^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_2}(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau + |\xi|^2 \rangle^{c_1}} d\xi d\tau d\xi'_1,$$

with $v_2 \in L^2$. The required estimate for u in $X_1^{k, -c_1}$ is simpler than (2.3.13) since we do not have the term Q and the variable ξ'_1 .

In order to estimate $\omega(QRe(u))$ in $X_2^{l, -c}$, we will estimate $\omega(Qu)$ and $\omega(Q\bar{u})$ in $X_2^{l, -c}$. We take its scalar product with a generic function in $X_2^{-l, c}$ with Fourier transform $\langle \xi \rangle^l \langle \tau \pm |\xi|^2 \rangle^{-c} \widehat{v_2}$ and $v_2 \in L^2$. The required estimates for $\omega Re(Qu)$ then takes the form

$$|I_3| \leq CT^{\theta_3} \|w\|_2 \|v_2\|_2, \quad (2.3.14)$$

$$|I_4| \leq CT^{\theta_4} \|w\|_2 \|v_2\|_2, \quad (2.3.15)$$

where

$$I_3 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_2}(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi|^2 \rangle^c} d\xi d\tau d\xi'_1,$$

$$I_4 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_2}(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi|^2 \rangle^c} d\xi d\tau d\xi'_1.$$

We now consider the estimates for the linear terms in Y_1^k and Y_2^l . The estimates for Q^2u and u are similar, then we will only estimate Q^2u in Y_1^k . Similarly to the previous part, the required estimates for Q^2u in Y_1^k will take the form :

$$|I_5| \leq CT^{\theta_5} \|w\|_2 \|v_3\|_{L_x^2}, \quad (2.3.16)$$

where

$$I_5 = \int \frac{\mathcal{F}_x\{Q^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_3}(\xi) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau + |\xi|^2 \rangle^c} d\xi d\tau d\xi'_1.$$

Estimate for $Qn_{\pm}$ in Y_1^k :

$$|I_6| \leq CT^{\theta_6} \|v\|_2 \|v_3\|_{L_x^2}, \quad (2.3.17)$$

where

$$I_6 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_3}(\xi) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + |\xi|^2 \rangle} d\xi d\tau d\xi'_1.$$

Estimate for $\omega(Qu)$ in Y_2^l :

$$|I_7| \leq CT^{\theta_7} \|w\|_2 \|v_3\|_{L_x^2}, \quad (2.3.18)$$

where

$$I_7 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_3}(\xi) \langle \xi \rangle^l |\xi|}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle} d\xi d\tau d\xi'_1.$$

Estimate for $\omega(Q\bar{u})$ in Y_2^l :

$$|I_8| \leq CT^{\theta_8} \|w\|_2 \|v_3\|_{L_x^2}, \quad (2.3.19)$$

where

$$I_8 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_3}(\xi) \langle \xi \rangle^l |\xi|}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle} d\xi d\tau d\xi'_1.$$

Because of the effect of cutoff function, in (2.3.12)-(2.3.19) we are allowed to assume that the following functions

$$\mathcal{F}^{-1}(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{-b} \widehat{v}), \mathcal{F}^{-1}(\langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} \widehat{w})$$

will be supported in a region $|t| < CT$.

Preparing for the proofs of (2.3.12)-(2.3.17), we first recall the Strichartz estimate and some elementary inequalities which we will need.

Lemma 2.3.3. (cf. [6, Lemma 3.1]) Let $b_0 > 1/2$, let $a \geq 0$, $a' \geq 0$, let $0 \leq \gamma \leq 1$. Assume in addition that $(1 - \gamma)a \leq b_0$ and $\gamma a \leq a'$. Let $0 < \eta \leq 1$ and define q and r by

$$2/q = 1 - \eta(1 - \gamma)a/b_0 \quad (2.3.20)$$

$$1 - 2/r = (1 - \eta)(1 - \gamma)a/b_0. \quad (2.3.21)$$

Let $v \in L^2$ be such that $\mathcal{F}^{-1}(\langle \tau + |\xi|^2 \rangle^{-a'} \widehat{v})$ has support in $|t| \leq CT$. Then

$$\left\| \mathcal{F}^{-1}(\langle \tau + |\xi|^2 \rangle^{-a} |\widehat{v}|); L^q L^r \right\| \leq CT^{\theta} \|v\|_2, \quad (2.3.22)$$

with $\theta \geq 0$. Note that $\theta = 0$ if and only if $a = 0$ or $\gamma = 0$.

Remark 6. In particular, in this paper, in order to estimate the linear terms we shall use only the special case $q = r = 2$, $\gamma = 1$ for both Schrödinger equation and wave equation.

Lemma 2.3.4. Let $\xi_1, \xi'_1, \xi_2, \tau \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2)$, then there exists $C > 0$ such that these following estimates hold,

$$\langle \xi \rangle^2 \leq C \left(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + |\xi|^2 \rangle + \langle \xi'_1 \rangle \right), \quad (2.3.23)$$

$$\langle \xi \rangle^2 \leq C \left(\langle \tau \pm |\xi| \rangle + \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle + \langle \xi'_1 \rangle^2 \right), \quad (2.3.24)$$

$$\langle \xi \rangle \leq C \langle \xi'_1 \rangle \langle (\xi_1 - \xi'_1, \xi_2) \rangle \quad (2.3.25)$$

Proof. Proof of (2.3.23). By Cauchy-Schwarz inequality

$$\begin{aligned}
& \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 \\
&= 3 + (\tau \pm |(\xi_1 - \xi'_1, \xi_2)|)^2 + (\tau + |\xi|^2)^2 + \xi_1'^2 \\
&\geq 3 + (|\xi|^2 + |\xi'_1| \mp |(\xi_1 - \xi'_1, \xi_2)|)^2 / 3,
\end{aligned}$$

and

$$\begin{aligned}
|(\xi_1 - \xi'_1, \xi_2)| &= \sqrt{(\xi_1 - \xi'_1)^2 + \xi_2^2} \\
&\leq (|\xi_1 - \xi'_1| + |\xi_2|) \\
&\leq (|\xi_1| + |\xi'_1| + |\xi_2|).
\end{aligned}$$

If $|\xi_1| + |\xi_2| \geq 4$ then

$$\begin{aligned}
|\xi|^2 + |\xi'_1| &\geq \frac{(|\xi_1| + |\xi_2|)^2}{2} + |\xi'_1| \\
&\geq \frac{(|\xi_1| + |\xi_2|)^2}{4} + (|\xi_1| + |\xi_2| + |\xi'_1|).
\end{aligned}$$

Therefore,

$$|\xi|^2 + |\xi'_1| \mp |(\xi_1 - \xi'_1, \xi_2)| \geq \frac{(|\xi_1| + |\xi_2|)^2}{4},$$

and

$$\begin{aligned}
\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 &\geq 3 + (|\xi_1| + |\xi_2|)^4 / 48 \\
&\geq 3 + (\xi_1^2 + \xi_2^2)^2 / 48 \\
&> \langle \xi \rangle^4 / 96.
\end{aligned}$$

If $|\xi_1| + |\xi_2| < 4$ or $|\xi| < 4$ then there exists a constant C large enough such that

$$\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 \geq 3 > \langle \xi \rangle^4 / C.$$

Therefore, there exists $C > 0$ such that (2.3.23) holds. The proof of (2.3.24) is similar.

Proof of (2.3.25),

$$\begin{aligned}
\langle \xi'_1 \rangle^2 \langle \xi_1 - \xi'_1, \xi_2 \rangle^2 &= (1 + \xi_1'^2) (1 + (\xi_1 - \xi'_1)^2 + \xi_2^2) \\
&\geq 1 + \xi_1'^2 + (\xi_1 - \xi'_1)^2 + \xi_2^2 \\
&\geq 1 + \xi_1^2 / 2 + \xi_2^2 \\
&> \langle \xi \rangle^2 / 2,
\end{aligned}$$

then (2.3.25) holds. □

Lemma 2.3.5. (i) (cf. [6, Lemma 3.3 (i)]) Let $y_1, y_2 \in \mathbb{R}$ and $z = y_1 - y_2$. Then for any $\nu > 1$

$$|z| \leq \nu |y_2| + \frac{\nu}{\nu - 1} |y_1| \chi(|z| \geq \nu |y_2|) \chi \left(\frac{\nu}{\nu + 1} \leq \frac{|z|}{|y_1|} \leq \frac{\nu}{\nu - 1} \right), \quad (2.3.26)$$

where $\chi(\Omega)$ is the characteristic function of the set Ω .

(ii) Let $\xi_1, \xi_2, \xi'_1 \in \mathbb{R}$, $\xi = (\xi_1, \xi_2)$ and let $|\xi| \geq 2|\xi'_1|$. Then

$$\langle \xi \rangle^2 \leq C \left(\langle \xi'_1 \rangle + 2 \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + 2 \langle \tau + |\xi|^2 \rangle \chi(\mathcal{B}) \right), \quad (2.3.27)$$

where

$$\mathcal{B} = \left((\tau, \xi) : \frac{1}{2}(|\xi|^2 - \frac{3}{2}|\xi|) \leq |\tau + |\xi|^2| \leq \frac{3}{2}(|\xi|^2 + \frac{3}{2}|\xi|) \right).$$

Proof. Proof of (i): See [6] lemma 3.3.

Proof of (ii): Using (i) with $z = |\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)|$, $y_1 = \tau + |\xi|^2$, $y_2 = \tau \pm |(\xi_1 - \xi'_1, \xi_2)|$ and $\nu = 2$, we have

$$\begin{aligned} ||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)|| &\leq 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| \\ &+ 2|\tau + |\xi|^2| \chi \left(\frac{2}{3} \leq \frac{||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)||}{\tau + |\xi|^2} \leq 2 \right). \end{aligned} \quad (2.3.28)$$

Now we will take a close look on the set

$$\mathcal{A} = \left((\tau, \xi) : \frac{2}{3} \leq \frac{||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)||}{\tau + |\xi|^2} \leq 2 \right).$$

By using triangle inequality and the fact that $|\xi| \geq 2|\xi'_1|$, we have

$$\begin{aligned} ||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)|| &\leq |\xi|^2 + |(\xi_1 - \xi'_1, \xi_2)| \\ &\leq |\xi|^2 + |\xi| + |\xi'_1| \\ &\leq |\xi|^2 + \frac{3}{2}|\xi|, \end{aligned}$$

and

$$\begin{aligned} ||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)|| &\geq |\xi|^2 - |(\xi_1 - \xi'_1, \xi_2)| \\ &\geq |\xi|^2 - \frac{3}{2}|\xi|. \end{aligned}$$

Therefore,

$$\mathcal{A} \subset \mathcal{B}.$$

Then (2.3.28) follows

$$\begin{aligned} |\xi|^2 &\leq |(\xi_1 - \xi'_1, \xi_2)| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2| \chi(\mathcal{B}) \\ &\leq |\xi| + |\xi'_1| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2| \chi(\mathcal{B}). \end{aligned}$$

Set $A = |\xi'_1| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2| \chi(\mathcal{B}) + \frac{1}{4}$, then

$$|\xi| \leq \sqrt{A} + \frac{1}{2},$$

and then

$$|\xi|^2 \leq A + \sqrt{A} + \frac{1}{4}.$$

Therefore, (2.3.27) holds with suitable constant C . □

Remark 7. (2.3.27) will be used to cancel the logarithmic singularities which will appear in (2.3.17).

In the next Lemma, we will prove (2.3.12), (2.3.13), (2.3.14) and (2.3.15).

Lemma 2.3.6. *Let $k, l \geq 0$ and $k = l + \epsilon$ with $0 < \epsilon \leq 1$. Let b, b_1, c, c_1 satisfy:*

(i) *If $0 < \epsilon < 1$.*

$$\frac{\epsilon}{2} < b < \frac{1}{2}, \quad \frac{1-\epsilon}{2} < b_1 < \frac{1}{2} \quad (2.3.29)$$

$$\frac{\epsilon}{2} \leq c_1 < \frac{1}{2}, \quad \frac{1-\epsilon}{2} \leq c < \frac{1}{2}. \quad (2.3.30)$$

(ii) *If $\epsilon = 1$.*

$$\frac{1}{2} < b < 1, \quad b_1 = \frac{1}{2} \quad (2.3.31)$$

$$0 < c < 1 - b, \quad c_1 = \frac{1}{2} \quad (2.3.32)$$

then the estimates (2.3.12), (2.3.13), (2.3.14) and (2.3.15) hold and $\theta_1, \theta_2, \theta_3, \theta_4 > 0$.

Proof. i) *Proof of (2.3.12).*

By using symbol inequalities (2.3.23), (2.3.25) and the fact that $k = l + \epsilon$, we have

$$\begin{aligned} |I_1| &= \left| \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v_1}(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} d\xi d\tau d\xi'_1 \right| \\ &\leq C \int \frac{|\langle \xi'_1 \rangle^{l+1/2} \mathcal{F}_x\{Q\}(\xi'_1) |\widehat{v}| |\widehat{v_1}|}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} \\ &\quad \times \left(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + \xi^2 \rangle + \langle \xi'_1 \rangle \right)^{\epsilon/2} d\xi d\tau d\xi'_1. \end{aligned}$$

Let us note that

$$\begin{aligned} &\frac{(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + \xi^2 \rangle + \langle \xi'_1 \rangle)^{\epsilon/2}}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} \\ &\approx \frac{\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{\epsilon/2} + \langle \tau + \xi^2 \rangle^{\epsilon/2} + \langle \xi'_1 \rangle^{\epsilon/2}}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}}. \end{aligned}$$

Then by subtracting $\epsilon/2$ from one of $(b, c_1, 1/2)$ and using the hypothesis $b > \epsilon/2$ ($> 1/2$ if $\epsilon = 1$), we only need to consider the terms of the following form, with $\alpha > 0, \beta \geq 0$ and $s \geq 0$.

$$\begin{aligned} \bar{I}_1 &= \int \frac{|\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1) |\widehat{v}| |\widehat{v_1}|}{\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^\alpha \langle \tau + |\xi|^2 \rangle^\beta} d\xi d\tau d\xi'_1, \\ &\leq \int \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1) \left(\langle \tau \pm |\xi| \rangle^{-\alpha} \widehat{v} \right) (\xi_1 - \xi'_1) d\xi'_1 \right) \\ &\quad \times \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v_1}| \right) d\xi d\tau. \end{aligned}$$

By using Plancherel identity, Hölder inequality, we have

$$\begin{aligned} \bar{I}_1 &\leq \left\| \mathcal{F}^{-1} \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1) \left(\langle \tau \pm |\xi| \rangle^{-\alpha} \widehat{v} \right) (\xi_1 - \xi'_1) d\xi'_1 \right) \right\|_2 \\ &\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v_1}| \right) \right\|_2. \end{aligned}$$

The first term on the right hand side is rewritten as follow

$$\begin{aligned}
& \mathcal{F}^{-1} \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right) (\xi_1 - \xi'_1, \xi_2, \tau) | d\xi'_1 \right) \\
&= \mathcal{F}^{-1} \left(\int |\mathcal{F}_x (\langle D_x \rangle^s Q) (\xi'_1)| \left| \left(\langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right) (\xi_1 - \xi'_1, \xi_2, \tau) \right| d\xi'_1 \right) \\
&= \mathcal{F}^{-1} \left(|\mathcal{F}_x (\langle D_x \rangle^s Q)| * \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| (\xi_1) \right) \\
&= \mathcal{F}_{yt}^{-1} \left(\mathcal{F}_x^{-1} |\mathcal{F}_x (\langle D_x \rangle^s Q)| \cdot \mathcal{F}_x^{-1} \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| \right) \\
&= \mathcal{F}_{yt}^{-1} \left((\langle D_x \rangle^s Q) \cdot \mathcal{F}_x^{-1} \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| \right) \\
&= (\langle D_x \rangle^s Q(x)) \mathcal{F}^{-1} \left(\langle \tau + \phi(\xi) \rangle^{-b} |\widehat{v}| \right).
\end{aligned}$$

We used the fact that $\mathcal{F}_x (\langle D_x \rangle^s Q) (\xi_1) = \langle \xi_1 \rangle^s Q(\xi_1) > 0$. With $\phi(\xi) = \pm|\xi|$, it implies

$$\begin{aligned}
\bar{I}_1 &\leq C \left\| (\langle D_x \rangle^s Q(x)) \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\alpha} |\widehat{v}| \right) \right\|_2 \times \left\| \mathcal{F}^{-1} \left(\left\langle \tau + |\xi|^2 \right\rangle^{-\beta} |\widehat{v}_1| \right) \right\|_2 \\
&\leq C \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\alpha} |\widehat{v}| \right) \right\|_2 \times \left\| \mathcal{F}^{-1} \left(\left\langle \tau + |\xi|^2 \right\rangle^{-\beta} |\widehat{v}_1| \right) \right\|_2 \\
&\leq CT^{\theta_1} \|v\|_2 \|v_1\|_2.
\end{aligned}$$

We used the fact that $\langle D_x \rangle^s Q(x) \in L_x^\infty$ for any $s \geq 0$, and $\theta_1 > 0$ comes from the Strichartz estimate (2.3.22) with $\alpha > 0$. Thus, (2.3.12) holds even in the limit case $c_1 = 1/2$.

ii) The proof of (2.3.13) is easier than (2.3.12), because we only need to use the estimate (2.3.25) to remove $\langle \xi \rangle^k$ in the numerator and using: $\mathcal{F}_x(Q^2) > 0$, $\langle D_x \rangle^s Q^2(x) \in L_x^\infty$ for any $s \geq 0$.

iii) *Proof of (2.3.14) and (2.3.15).* We only prove (2.3.14), (2.3.15) is treated similarly, because, with $a > 0$

$$\begin{aligned}
\left\| \left\langle \tau - \xi^2 \right\rangle^{-a} \widehat{w}(\tau, \xi); L_\tau^2 L_\xi^2 \right\| &= \left\| \left\langle \tau - \xi^2 \right\rangle^{-a} \overline{\widehat{w}(-\tau, -\xi)}; L_\tau^2 L_\xi^2 \right\| \\
&= \left\| \left\langle \tau + \xi^2 \right\rangle^{-a} \widehat{w}(\tau, \xi); L_\tau^2 L_\xi^2 \right\|.
\end{aligned}$$

By using symbol inequalities (2.3.24), (2.3.25) and the fact that $k = l + \epsilon$, we have

$$\begin{aligned}
|I_3| &= \left| \int \frac{\mathcal{F}_x \{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_2(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1 \right| \\
&\leq C \int \frac{|\langle \xi'_1 \rangle^k \mathcal{F}_x \{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2| \langle \xi \rangle^{1-\epsilon}}{\langle \tau + |(\xi_1 - x'_1, x_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1 \\
&\leq C \int \frac{|\langle \xi'_1 \rangle^k \mathcal{F}_x \{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2|}{\langle \tau + |(\xi_1 - x'_1, x_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} \\
&\quad \times \left(\langle \tau \pm |\xi| \rangle + \left\langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \right\rangle + \langle \xi'_1 \rangle^2 \right)^{\frac{1-\epsilon}{2}}.
\end{aligned}$$

Subtracting $\frac{1-\epsilon}{2}$ from one of $(b_1, c, 1/2)$ and using the fact that $b_1 > \frac{1-\epsilon}{2}$, we only need to consider the terms of the following form with $\alpha > 0$, $\beta \geq 0$ and $s > 0$.

$$\bar{I}_3 = \int \frac{|\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2|}{\langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^\alpha \langle \tau \pm |\xi| \rangle^\beta} d\xi d\tau d\xi'_1.$$

Similarly as in the previous part, we have

$$\begin{aligned}
\bar{I}_3 &\leq \int \left(\int \left| \langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} \widehat{w} \right) (\xi_1 - \xi'_1) \right| d\xi'_1 \right) \\
&\quad \times \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) d\xi d\tau \\
&\leq \left\| \mathcal{F}^{-1} \left(\int \left| \langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} \widehat{w} \right) (\xi_1 - \xi'_1) \right| d\xi'_1 \right) \right\|_2 \\
&\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq C \left\| (\langle D_x \rangle^s Q(x)) \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} |\widehat{w}| \right) \right\|_2 \\
&\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq C \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} |\widehat{w}| \right) \right\|_2 \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq CT^{\theta_3} \|w\|_2 \|v_2\|_2.
\end{aligned}$$

Therefore, (2.3.14) holds with $\theta_3 > 0$. $\square$

Remark 8. In the case $\epsilon = 1$, we take $b > 1/2$ since we need a positive power of T on (2.3.12) and also on (2.3.17).

We now prove (2.3.16) and (2.3.17).

Lemma 2.3.7. Let $k, l \geq 0$ and $k = l + \epsilon$ with $0 < \epsilon \leq 1$. Let b, b_1 satisfy:

(i) If $0 < \epsilon < 1$.

$$\frac{\epsilon}{2} < b < \frac{1}{2}, \quad \frac{1-\epsilon}{2} < b_1 < \frac{1}{2}. \quad (2.3.33)$$

(ii) If $\epsilon = 1$.

$$\frac{1}{2} < b < 1, \quad b_1 = \frac{1}{2}. \quad (2.3.34)$$

Then (2.3.16), (2.3.17), (2.3.18) and (2.3.19) hold and $\theta_5, \theta_6, \theta_7, \theta_8 > 0$.

Proof. i) If $0 < \epsilon < 1$ then the proofs of (2.3.16), (2.3.17), (2.3.18) and (2.3.19) are very similar as the proofs of (2.3.12), (2.3.13), (2.3.14) and (2.3.15). It remains to prove two inequalities of the form

$$\left\| \langle \tau + |\xi|^2 \rangle^{-1/2-\delta} \widehat{v}_3(\xi); L_\xi^2 L_\tau^2 \right\| \leq C \|v_3\|_{L_x^2},$$

and

$$\left\| \langle \tau + \pm |\xi| \rangle^{-1/2-\delta} \widehat{v}_3(\xi); L_\xi^2 L_\tau^2 \right\| \leq C \|v_3\|_{L_x^2},$$

where $-1/2 - \delta = 1 - \epsilon/2$ or $1 - (1 - \epsilon)/2$. Therefore $\delta > 0$ and the inequalities hold. The positive power of T comes from the Strichartz estimate for remaining term.

ii) If $\epsilon = 1$ we only consider (2.3.17), because in (2.3.16) I_5 does not depend on l . In (2.3.18) and (2.3.19) we can use (2.3.25). We separate the integration region of I_6 into two subregions:

Region $|\xi| \leq 2|\xi'_1|$: The proof is the same as in part i) for (2.3.16) since we can eliminate the term $\langle \xi \rangle^k$.

Region $|\xi| \geq 2|\xi'_1|$: By using (2.3.27), it remains to estimate the following term

$$\begin{aligned}
&\left\| \langle \tau + |\xi|^2 \rangle^{-\frac{1}{2}} \widehat{v}_3(\xi) \chi \left(\frac{1}{2}(|\xi|^2 - \frac{3}{2}|\xi|) \leq |\tau + |\xi|^2| \leq \frac{3}{2}(|\xi|^2 + \frac{3}{2}|\xi|) \right); L_\xi^2 L_\tau^2 \right\| \\
&\leq C(\ln(3))^{1/2} \|v_3\|_{L_x^2}.
\end{aligned}$$

Therefore (2.3.17) holds, and $\theta_6 > 0$ comes from Strichartz estimate for the remaining term. $\square$

2.3.3 Estimates for nonlinear terms

Remark 9. In the following Lemmas, we are allowed to assume that $n_{\pm}$ and u have compact support in time.

Lemma 2.3.8. (*Estimate for $n_{\pm}u$ in $X_1^{k,-c_1}$*) (cf. [6, Lemma 3.4]). Let $b_0 > 1/2$ and $0 < b, c_1, b_1 \leq b_0 < b + c_1 + b_1 - c_0$, with $0 < c_0 \leq \text{Min}(b, c_1, b_1)$. And

$$\begin{aligned} 2b_0 &< b + c_1 + b_1, \\ l &\geq 0, |k| \leq l + 2c_0. \end{aligned}$$

Then

$$\|n_{\pm}u; X_1^{k,-c_1}\| \leq CT^{\theta} \|n_{\pm}; X_2^{l,b}\| \|u; X_1^{k,b_1}\| \quad (2.3.35)$$

holds, with $\theta > 0$.

Lemma 2.3.9. (*Estimate for $\omega|u|^2$ in $X_2^{l,-c}$*) (cf. [6, Lemma 3.5]). Let $b_0 > 1/2$ and let $0 < c, b_1 \leq b_0 < c + 2b_1 - \bar{c}_0$ with $0 < \bar{c}_0 \leq \text{Min}(c, b_1)$. And

$$\begin{aligned} 2b_0 &< c + 2b_1, \\ 2k &\geq l + 1, \quad k \geq l + 1 - 2\bar{c}_0, \quad k \geq 0. \end{aligned}$$

Then

$$\|\omega|u|^2; X_2^{l,-c}\| \leq CT^{\theta} \|u; X_1^{k,b_1}\|^2 \quad (2.3.36)$$

holds, with $\theta > 0$.

Lemma 2.3.10. (*Estimate for $n_{\pm}u$ in Y_1^k*) (cf. [6, Lemma 3.6]). Let $b_0 > 1/2$, let $0 < c_1 < 1/2$ and let $0 < b, b_1 \leq b_0 < b + c_1 + b_1 - c_0$, with $0 < c_0 \leq \text{min}(b, 1/2, b_1)$. And

$$\begin{aligned} l &\geq 0, \quad |k| \leq l + 2c_0, \\ 2b_0 &< b + c_1 + b_1. \end{aligned}$$

Then

$$\|n_{\pm}u; Y_1^k\| \leq CT^{\theta} \|n_{\pm}; X_2^{l,b}\| \|u; X_1^{k,b_1}\| \quad (2.3.37)$$

holds, with $\theta > 0$.

Lemma 2.3.11. (*Estimate for $\omega|u|^2$ in Y_2^l*) (cf. [6, Lemma 3.7]). Let $b_0 > 1/2$, let $0 < c < 1/2$ and let $0 < b_1 \leq b_0 < c + 2b_1 - \bar{c}_0$ where $0 < \bar{c}_0 \leq \text{Min}(1/2, b_1)$. And

$$\begin{aligned} 2b_0 &< c + 2b_1, \\ 2k &\geq l + 1, \quad k \geq l + 1 - 2\bar{c}_0. \end{aligned}$$

Then

$$\|\omega|u|^2; Y_2^l\| \leq T^{\theta} \|u; X_1^{k,b_1}\|^2 \quad (2.3.38)$$

holds, with $\theta > 0$.

Proof of Theorem. We finish the proof of Theorem 2.1.1 by combining the estimates for the linear and nonlinear terms in Sections 2.3.2, 2.3.3 and the contraction argument described in Section 2.3.1.

If $k < l+1$, by the Lemmas 2.3.1, 2.3.2, 2.3.6 (i) and 2.3.7 (i), it is sufficient to find $1/2 > c, c_1 \geq 0$ such that the estimates (2.3.35)-(2.3.38) hold. In the Lemmas 2.3.8, 2.3.9, 2.3.10 and 2.3.11 we choose

$$\begin{aligned} c_1 &= \text{Min}(1 - b_1, 1/2), & c &= \text{Min}(1 - b, 1/2), \\ c_0 &= \text{Min}(b, 1 - b_1, b_1), & \bar{c}_0 &= \text{Min}(1 - b, b_1, 1/2). \end{aligned} \quad (2.3.39)$$

Then with b, b_1 close enough to $1/2$, the conditions in the Lemmas 2.3.8, 2.3.9, 2.3.10 and 2.3.11 will be fulfilled.

Remember that in this case, we only need the estimates (2.3.16)-(2.3.19), (2.3.10) and (2.3.11) to get a solution $(u, n, \partial_t n) \in \mathcal{C}(H^k \times H^l \times H^{l-1})$.

If $k = l + 1$, by the Lemmas 2.3.1 and 2.3.7 (ii), we are forced to take $b_1 = 1/2$ and $b > 1/2$. For the estimates (2.3.12)-(2.3.15) we use the Lemma 2.3.6 (ii), for the estimates (2.3.16)-(2.3.17) we use the Lemma 2.3.7 (ii), for the estimates (2.3.35)-(2.3.36) we use the same choice of $(c, c_1, c_0, \bar{c}_0)$ as in (2.3.39), for (2.3.37) we choose $c_0 = c_1 = 1/2$. If we choose b close enough to $1/2$ then all the required estimates for linear and nonlinear terms hold.

Furthermore, in order to extend the solution to $\mathcal{C}(H^k \times H^l \times H^{l-1})$ we also need (2.3.18), (2.3.19) and (2.3.38) given by the Lemmas 2.3.7 (ii) and 2.3.11.

Up to now we proved the well-posedness of the cut off integral equation (2.3.6)-(2.3.7), for the proof of the independence of the cut off function we refer to [6]. $\square$

2.4 Schochet-Weinstein method

2.4.1 Zakharov system as a dispersive perturbation of a symmetric hyperbolic system.

In this section, we will follow the method of Schochet and Weinstein ([16]) to rewrite Zakharov system (2.1.1)-(2.1.2) as a dispersive perturbation of a symmetric hyperbolic system.

First, we define some auxiliary functions

$$V = -\frac{1}{\lambda} \Delta^{-1} \nabla(n_t), \quad (2.4.1)$$

$$P = n + |u|^2, \quad (2.4.2)$$

then (2.1.1)-(2.1.2) will become

$$iu_t + \Delta u + |u|^2 u - Pu = 0, \quad (2.4.3)$$

$$P_t + \lambda \nabla \cdot V - (|u|^2)_t = 0, \quad (2.4.4)$$

$$V_t + \lambda \nabla P = 0. \quad (2.4.5)$$

Multiplying (2.4.3) by $\bar{u}$ and taking the imaginary part of the resulting equation to get

$$(|u|^2)_t = i(\bar{u} \Delta u - u \Delta \bar{u}). \quad (2.4.6)$$

Next, we take the gradient of (2.4.3) and get

$$i \nabla u_t + \Delta \nabla u + |u|^2 \nabla u + (u \nabla \bar{u} + \bar{u} \nabla u) u - P \nabla u - u \nabla P = 0. \quad (2.4.7)$$

Now let $\sqrt{2}u = F + iG$ and $\sqrt{2}\nabla u = H + iL$. Then, the use of (2.4.6) in (2.4.4) leads to the

following system equivalent to (2.4.3)-(2.4.5) , (2.4.7)

$$P_t + \lambda \nabla \cdot V + F \nabla \cdot L - G \nabla \cdot H = 0, \quad (2.4.8)$$

$$V_t + \lambda \nabla P = 0, \quad (2.4.9)$$

$$F_t + \frac{1}{2}(F^2 + G^2)G - PG = -\Delta G, \quad (2.4.10)$$

$$G_t - \frac{1}{2}(F^2 + G^2)F + PF = \Delta F, \quad (2.4.11)$$

$$H_t - G \nabla P + \frac{1}{2}(F^2 + G^2)L + (FH + GL)G - PL = -\Delta L, \quad (2.4.12)$$

$$L_t + F \nabla P - \frac{1}{2}(F^2 + G^2)H - (FH + GL)F + PH = \Delta H. \quad (2.4.13)$$

Introducing the 9 - component vector function $U = (P, V, F, G, H, L)^T$, equations (2.4.8)-(2.4.13) can be rewritten as follows

$$U_t + \sum_{j=1}^2 (A^j(U) + \lambda C^j) U_{x_j} + B(U)U = K \Delta U, \quad (2.4.14)$$

where A^j and C^j are symmetric 9×9 matrices , K is a antisymmetric matrix. C^j and K are constant matrices.

In the next section, we will use the above argument to rewrite (2.1.6)-(2.1.7) in a similar form.

2.4.2 Perturbed Zakharov system as a dispersive perturbation of a symmetric hyperbolic system

In Section 2.3, we already considered the perturbation of (2.1.1)-(2.1.2) by $Q(x) = 2\sqrt{2}/(e^x + e^{-x})$ that is (2.1.6)-(2.1.7). In this section, we will consider the perturbation of (2.1.1)-(2.1.2) by $e^{it}Q(x)$. This trick will make the calculation simpler by using the previous calculation.

Furthermore, in order to make computation transparent, we will denote $(e^{it}Q(x), -|Q(x)|^2)$ by (ϕ, ϕ_1) ¹ and change the notation of spatial variables from (x, y) to (x_1, x_2) . Then, the perturbed system will have the form

$$i\partial_t(u + \phi) + \Delta(u + \phi) = (n + \phi_1)(u + \phi), \quad (2.4.15)$$

$$\frac{1}{\lambda^2} \partial_t^2(n + \phi_1) - \Delta(n + \phi_1) = \Delta(|u + \phi|^2). \quad (2.4.16)$$

(ϕ, ϕ_1) is a 1-D solution of (2.1.1) - (2.1.2), then if we denote

$$V_r = -\frac{1}{\lambda} \Delta^{-1} \nabla(\partial_t \phi_1) = -\frac{1}{\lambda} \Delta^{-1}(\partial_{x_1} \partial_t \phi_1, 0),$$

$$P_r = \phi_1 + |\phi|^2,$$

then (ϕ, Q_r, V_r) is also a solution of (2.4.3)-(2.4.5).

We denote that

$$\tilde{V} = V + V_r = -\frac{1}{\lambda} \Delta^{-1} \nabla(\partial_t(n + \phi_1)),$$

$$\tilde{P} = P + P_r = (n + \phi_1) + |u + \phi|^2,$$

then because of (2.4.15)-(2.4.16), $(u + \phi, \tilde{P}, \tilde{V})$ is a solution of (2.4.3)-(2.4.5).

¹ These notations are not the same as in Section 2.3 where ϕ and ϕ_1 denote something different.

Now applying the argument in the previous section, if we denote

$$\begin{aligned}\sqrt{2}\phi &= F_r + iG_r, & \sqrt{2}\nabla\phi &= H_r + iL_r, \\ \sqrt{2}(u + \phi) &= \tilde{F} + i\tilde{G} = F + F_r + i(G + G_r), \\ \sqrt{2}\nabla(u + \phi) &= \tilde{H} + i\tilde{L} = H + H_r + i(L + L_r),\end{aligned}$$

then $(P_r, V_r, F_r, G_r, H_r, L_r)^T$ and $(\tilde{P}, \tilde{V}, \tilde{F}, \tilde{G}, \tilde{H}, \tilde{L})^T$ are the solutions of (2.4.8)-(2.4.13). Therefore, we can get a system solved by $(P, V, F, G, H, L)^T$ by eliminating the terms only depend on $(P_r, V_r, F_r, G_r, H_r, L_r)$, that is the following system

$$P_t + \lambda \nabla \cdot V + (F + F_r) \nabla \cdot L - (G + G_r) \nabla \cdot H + \mathcal{R}_1 = 0, \quad (2.4.17)$$

$$V_t + \lambda \nabla P = 0, \quad (2.4.18)$$

$$F_t + \mathcal{R}_2 = -\Delta G, \quad (2.4.19)$$

$$G_t + \mathcal{R}_3 = \Delta F, \quad (2.4.20)$$

$$H_t - (G + G_r) \nabla P + \mathcal{R}_4 = -\Delta L, \quad (2.4.21)$$

$$L_t + (F + F_r) \nabla P + \mathcal{R}_5 = \Delta H. \quad (2.4.22)$$

Where the residuals are

$$\begin{aligned}\mathcal{R}_1 &= F \nabla \cdot L_r - G \nabla \cdot H_r, \\ \mathcal{R}_2 &= \frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)G + \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)G_r - \tilde{P}G - PG_r, \\ \mathcal{R}_3 &= -\frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)F - \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)F_r + \tilde{P}F + PF_r, \\ \mathcal{R}_4 &= \frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)L + \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)L_r \\ &\quad + (\tilde{F}\tilde{H} + \tilde{G}\tilde{L})G + (FH + GL + FH_r + F_rH + GL_r + G_rL)G_r \\ &\quad - PL_r - P_rL - PL, \\ \mathcal{R}_5 &= -\frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)H - \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)H_r \\ &\quad - (\tilde{F}\tilde{H} + \tilde{G}\tilde{L})F - (FH + GL + F_rH + FH_r + G_rL + GL_r)F_r \\ &\quad + PH + P_rH + PH_r.\end{aligned}$$

Therefore, we now can rewrite (2.4.15)-(2.4.16) as a dispersive perturbation of a symmetric hyperbolic system

$$U_t + \sum_{j=1}^2 (A^j(U) + B^j(\phi) + \lambda C^j) U_{x_j} + (D^1(U) + D^2(\phi))U = K \Delta U, \quad (2.4.23)$$

where $U = (P, V, F, G, H, L)^T$, A^j, B^j, C^j are symmetric 9×9 matrices, K is an anti-symmetric 9×9 matrix.

$$A^1(U) = \begin{pmatrix} 0 & 0_{1 \times 4} & -G & 0 & F & 0 \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ -G & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix},$$

$$\begin{aligned}
B^1(\phi) &= \begin{pmatrix} 0 & 0_{1 \times 4} & -G_r & 0 & F_r & 0 \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ -G_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}, \\
A^2(U) &= \begin{pmatrix} 0 & 0_{1 \times 4} & 0 & -G & 0 & F \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ -G & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}, \\
B^2(\phi) &= \begin{pmatrix} 0 & 0_{1 \times 4} & 0 & -G_r & 0 & F_r \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ -G_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}.
\end{aligned}$$

Now, instead of studying the Cauchy problem for the perturbed Zakharov system (2.4.15)-(2.4.16) we will study the Cauchy problem for (2.4.23).

Theorem 2.4.1. *Let $s > 2$ and $U_0 \in (H^s(\mathbb{R}^2))^9$, then there exists $T = T(\|U_0\|_{(H^s(\mathbb{R}^2))^9})$ such that the equation (2.4.23) has a unique solution*

$$U \in L^\infty([0, T], (H^s(\mathbb{R}^2))^9),$$

with $U(0) = U_0$.

Proof. The proof of the existence proceeds via a classical iteration scheme. We first regularize the initial data by taking a family of self-adjoint regularization operators $\mathcal{J}_\epsilon$ as following.

We choose $j \in C_0^\infty(\mathbb{R}^2)$, $\text{supp } j \subset \{X = (x_1, x_2) \in \mathbb{R}^2; |X| < 1\}$, $\int j = 1$ and set $j_\epsilon(X) = \epsilon^{-2}j(X/\epsilon)$. Then, we define $\mathcal{J}_\epsilon u \in C^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ by

$$\mathcal{J}_\epsilon u = j_\epsilon * u.$$

Set $\epsilon_k = 2^{-k}$, $U_0^k = \mathcal{J}_{\epsilon_k} U_0$. We construct a local solution of (2.4.23) by considering the iteration scheme :

$$U^0(x_1, x_2, t) = U_0^0(x_1, x_2), \quad (2.4.24)$$

$$U_t^{k+1} + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi) + \lambda C^j) U_{x_j}^{k+1} + (D^1(U^k) + D^2(\phi)) U^{k+1} = K \Delta U^{k+1}, \quad (2.4.25)$$

$$U^{k+1}(x_1, x_2, 0) = U_0^{k+1}(x_1, x_2). \quad (2.4.26)$$

Since $\langle \xi \rangle^s \sim \langle \xi_1 \rangle^s + \langle \xi_2 \rangle^s$, $\forall s \geq 0$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we have $\|\cdot\|_{H^s} \sim \|J_{x_1}^s \cdot\|_{L^2} + \|J_{x_2}^s \cdot\|_{L^2}$, where $J_{x_1} = \mathcal{F}^{-1} \langle \xi_1 \rangle \mathcal{F}$, $J_{x_2} = \mathcal{F}^{-1} \langle \xi_2 \rangle \mathcal{F}$.

Therefore, in order to estimate $\|U^{k+1}\|_{H^s}$ we will estimate $\|J_{x_1}^s U^{k+1}\|_{L^2}$ and $\|J_{x_2}^s U^{k+1}\|_{L^2}$ separately.

Estimate for $\|J_{x_1}^s U^{k+1}\|_{L^2}$. Applying $J_{x_1}^s$ to (4.25), we get

$$\begin{aligned} & \partial_t(J_{x_1}^s U^{k+1}) + \sum_{j=1}^2 J_{x_1}^s \left((A^j(U^k) + B^j(\phi)) U_{x_j}^{k+1} \right) + \sum_{j=1}^2 \lambda C^j J_{x_1}^s U_{x_j}^{k+1} \\ & + J_{x_1}^s \left((D^1(U^k) + D^2(\phi)) U^{k+1} \right) \\ & = K \Delta J_{x_1}^s U^{k+1}. \end{aligned}$$

Or

$$\begin{aligned} & \partial_t(J_{x_1}^s U^{k+1}) + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi)) J_{x_1}^s U_{x_j}^{k+1} + \sum_{j=1}^2 \lambda C^j J_{x_1}^s U_{x_j}^{k+1} \\ & + (D^1(U^k) + D^2(\phi)) J_{x_1}^s U^{k+1} + \sum_{j=1}^2 [J_{x_1}^s, A^j(U^k) + B^j(\phi)] U_{x_j}^{k+1} \\ & + [J_{x_1}^s, D^1(U^k) + D^2(\phi)] U^{k+1} \\ & = K \Delta J_{x_1}^s U^{k+1}, \end{aligned}$$

with notation for commutators $[F, G] = FG - GF$.

We multiply (4.28) by $J_{x_1}^s U^{k+1}$ and integrating in $\mathbb{R}^2$, with integration by parts and using the symmetry of A^j and B^j we obtain

$$\begin{aligned} & \frac{1}{2} \partial_t \|J_{x_1}^s U^{k+1}\|_{L^2}^2 + \sum_{j=1}^2 \left(-\frac{1}{2} \left\langle (A^j(U_{x_j}^k) + B^j(\phi_{x_j})) J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \right. \\ & + \left\langle [J_{x_1}^s, A^j(U^k) + B^j(\phi)] U_{x_j}^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \Big) \\ & + \left\langle (D^1(U^k) + D^2(\phi)) J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & + \left\langle [J_{x_1}^s, D^1(U^k) + D^2(\phi)] U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & = 0. \end{aligned} \tag{2.4.27}$$

Note that $\langle \cdot, \cdot \rangle$ denotes $L^2(\mathbb{R}^2)$ scalar product and $\langle \cdot, \cdot \rangle_1$ denotes $L^2(\mathbb{R})$ scalar product with respect to x_1 .

Before going to estimate (2.4.27), we need the following lemma (see [10]. B2).

Definition 2.4.2. We say that a Fourier multiplier $\sigma(D)$ is of order s ($s \in \mathbb{R}$) and write $\sigma \in \mathcal{S}^s$ if $\xi \in \mathbb{R}^d \mapsto \sigma(\xi) \in \mathcal{C}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta| - s} |\partial^\beta \sigma(\xi)| < \infty.$$

Lemma 2.4.3. Let $t_0 > d/2$, $s \geq 0$ and $\sigma \in \mathcal{S}^s$. If $f \in H^s(\mathbb{R}^d) \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$ then

$$\|[\sigma(D), f]g\|_{L^2} \leq C(\sigma) (\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}} \|g\|_{L^\infty}), \tag{2.4.28}$$

where $C(\sigma)$ depends only on σ .

First, using (5.3.24) with $d = 2$, $\sigma(D) = J_{x_1}^s$, Sobolev embedding inequality and note that $D^1(U)$ contains linear and quadratic terms on U and $A^j(U)$ depends linearly on U , we get

$$\begin{aligned} & \sum_{j=1}^2 \left\langle [J_{x_1}^s, A^j(U^k)] U_{x_j}^{k+1}, J_{x_1}^s U^{k+1} \right\rangle + \left\langle [J_{x_1}^s, D^1(U^k)] U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & \lesssim \left(\|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|J_{x_1}^s U^{k+1}\|_{L^2}^2 \\ & \lesssim \left(\|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|U^{k+1}\|_{H^s}^2 \end{aligned} \quad (2.4.29)$$

Now using (5.3.24) with $d = 1$, $\sigma(D) = J_{x_1}^s$ and Sobolev embedding inequality, we have

$$\begin{aligned} & \sum_{j=1}^2 \left\langle [J_{x_1}^s, B^j(\phi)] U_{x_j}^{k+1}, V^{k+1} \right\rangle_1 + \left\langle [J_{x_1}^s, D^2(\phi)] U^{k+1}, V^{k+1} \right\rangle_1 \\ & \leq C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) \left(\|U_{x_2}^{k+1}\|_{H_{x_1}^{s-1}} + \|U_{x_2}^{k+1}\|_{L_{x_1}^\infty} + \|U^{k+1}\|_{H_{x_1}^s} \right) \\ & \quad \left\| J_{x_1}^s U^{k+1} \right\|_{L_{x_1}^2} \\ & \leq C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) \left(\|U_{x_2}^{k+1}\|_{H_{x_1}^{s-1}} + \|J_{x_1}^s U_{x_2}^{k+1}\|_{L_{x_1}^2} + \|U^{k+1}\|_{H_{x_1}^s} \right) \\ & \quad \left\| J_{x_1}^s U^{k+1} \right\|_{L_{x_1}^2}. \end{aligned} \quad (2.4.30)$$

Then, integrating both side of (2.4.30) in $\mathbb{R}$ with respect to x_2 we obtain

$$\begin{aligned} & \sum_{j=1}^2 \left\langle [J_{x_1}^s, B^j(\phi)] U_{x_j}^{k+1}, V^{k+1} \right\rangle + \left\langle [J_{x_1}^s, D^2(\phi)] U^{k+1}, V^{k+1} \right\rangle \\ & \lesssim C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) \|U^{k+1}\|_{H^s}^2. \end{aligned} \quad (2.4.31)$$

It is not hard to see that

$$\begin{aligned} & \sum_{j=1}^2 -\frac{1}{2} \left\langle (A^j(U^k) + B^j(\phi_{x_j})) J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & \lesssim \left(\|U^k\|_{H^s} + \|\nabla \phi\|_{W^{s,\infty}} \right) \|U^{k+1}\|_{H^s}^2. \end{aligned} \quad (2.4.32)$$

Combining (2.4.27), (2.4.29), (2.4.31) and (2.4.32) we get

$$\frac{1}{2} \partial_t \|J_{x_1}^s U^{k+1}\|_{L^2}^2 \leq C \left(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|U^{k+1}\|_{H^s}^2. \quad (2.4.33)$$

Estimate for $\|J_{x_2}^s U^{k+1}\|_{L^2}$. We do similarly as previous step but note that ϕ does not depend on x_2 then

$$\begin{aligned} & \partial_t (J_{x_2}^s U^{k+1}) + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi)) J_{x_2}^s U_{x_j}^{k+1} + \sum_{j=1}^2 \lambda C^j J_{x_2}^s U_{x_j}^{k+1} \\ & + \sum_{j=1}^2 [J_{x_2}^s, A^j(U^k)] U_{x_j}^{k+1} + (D^1(U^k) + D^2(\phi)) J_{x_2}^s U^{k+1} \\ & + [J_{x_2}^s, D^1(U^k)] U^{k+1} \\ & = K \Delta J_{x_2}^s U^{k+1}, \end{aligned}$$

Multiplying the above expression with $J_{x_2}^s U^{k+1}$, integrating in $\mathbb{R}^2$, using integral by parts and the symmetry of A^j, B^j ($j = 1, 2$). We get

$$\begin{aligned}
& \frac{1}{2} \partial_t \|J_{x_2}^s U^{k+1}\|_{L^2}^2 + \sum_{j=1}^2 \left(-\frac{1}{2} \left\langle (A^j(U_{x_j}^k) + B^j(\phi_{x_j})) J_{x_2}^s U^{k+1}, J_{x_2}^s U^{k+1} \right\rangle \right. \\
& \quad \left. + \left\langle [J_{x_2}^s, A^j(U^k)] U_{x_j}^{k+1}, J_{x_2}^s U^{k+1} \right\rangle \right) \\
& + \left\langle (D^1(U^k) + D^2(\phi)) J_{x_2}^s U^{k+1}, J_{x_2}^s U^{k+1} \right\rangle \\
& + \left\langle [J_{x_2}^s, D^1(U^k)] U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\
& = 0.
\end{aligned} \tag{2.4.34}$$

We can see that there is no “ ϕ ” term in the commutator part of (2.4.34), that make it is easier to estimate than (2.4.27). Indeed, we also have

$$\frac{1}{2} \partial_t \|J_{x_2}^s U^{k+1}\|_{L^2} \leq C(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2) \|U^{k+1}\|_{H^s}^2. \tag{2.4.35}$$

Thanks to (2.4.33) and (2.4.35) we have

$$\frac{1}{2} \|U^{k+1}\|_{H^s}^2 \leq C \left(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|U^{k+1}\|_{H^s}^2. \tag{2.4.36}$$

We now consider a closed ball in $(H^s(\mathbb{R}^2))^9$

$$\mathcal{B} = \{U \in (H^s(\mathbb{R}^2))^9 : \|U\|_{H^s}^2 \leq \|U_0\|^2 + 1\}.$$

Therefore by combining (2.4.36), Gronwall inequality and induction method we can prove that

$$\|U^k(t)\|_{H^s}^2 \leq \|U_0\|_{H^s}^2 + 1, \tag{2.4.37}$$

for $t \in [0, T]$ and $T = T(\|U_0\|_{H^s})$.

Equation (2.4.25) then give us the uniform bound on U_t^k

$$\|U_t^k\|_{L^\infty([0, T]; (H^{s-2})^9)} \leq C(\|U_0\|_{H^s}).$$

By the Aubin-Lions theorem (Lemma 4.4.1), there is a subsequence of $\{U^{k_n}\}$ (with the same notation) converging strongly in $L^\infty([0, T]; (H_{loc}^{s-\delta})^9)$ with $0 < \delta < 2$. This allows to take the limit in the nonlinear terms of (2.4.23).

The uniqueness of solution of (2.4.23) is done by using energy method for the difference of two solutions since the dispersive part contributes nothing to the energy. $\square$

Proof of Theorem 2. Theorem 2.1.2 is a consequence of Theorem 2.4.1 as follows:

First, set $u = (1/\sqrt{2})(F + iG)$ then (2.4.18)-(2.4.20) implies

$$i\partial_t(u + \phi) + \Delta(u + \phi) + |u + \phi|^2(u + \phi) - \tilde{P}(u + \phi) = 0, \tag{2.4.38}$$

and

$$\tilde{V}_t + \lambda \nabla \tilde{P} = 0. \tag{2.4.39}$$

Where $\tilde{P} = P + P_r$ and $\tilde{V} = V + V_r$.

Next, From (2.4.19)-(2.4.21) we can derive an L^2 energy estimate of $W = (\nabla F - H, \nabla G - L)$, that implies

$$\|W(t)\|_{L^2}^2 \leq e^{CT} \|W(0)\|_{L^2}^2 = 0.$$

Therefore, $\nabla u = \frac{1}{\sqrt{2}}(H + iL)$. it leads to

$$(|u + \phi|^2)_t = i((\overline{u + \phi})\Delta(u + \phi) - (u + \phi)\Delta(\overline{u + \phi})),$$

and

$$\tilde{P}_t + \lambda \nabla \cdot \tilde{V} - (|u + \phi|^2)_t = 0.$$

Set $\tilde{n} = \tilde{P} - |u + \phi|^2$ we have that

$$\tilde{n}_t + \lambda \nabla \cdot \tilde{V} = 0. \quad (2.4.40)$$

Combining (2.4.39) and (2.4.40), we obtain (2.4.16), it is also clear that (2.4.38) implies (2.4.15). Estimate (2.1.12) follows similarly. $\square$

2.5 A weak convergence result

We consider the following family of system labeled by the parameter λ

$$i\partial_t u_\lambda - u_\lambda + \Delta u_\lambda = n_\lambda u_\lambda - Q^2 u_\lambda + Q n_\lambda \quad (2.5.1)$$

$$\frac{1}{\lambda^2} \partial_t^2 n_\lambda - \Delta n_\lambda = \Delta(|u_\lambda|^2) + 2\Delta(Q \operatorname{Re}(u_\lambda)) \quad (2.5.2)$$

$$\partial_t n_\lambda + \nabla \cdot f_\lambda = 0, \quad (2.5.3)$$

with initial data given by

$$u_\lambda(0) = u_0, \quad n_\lambda(0) = n_0, \quad f_\lambda(0) = f_0. \quad (2.5.4)$$

We study the behaviour of the solution (u_λ, n_λ) when λ tends to ∞ , that is the theorem

Theorem 2.5.1. *Let $s > 2$ and $(u_\lambda, n_\lambda, f_\lambda) \in L^\infty(0, T, H^{s+1}) \times L^\infty(0, T, H^s) \times L^\infty(0, T, H^s)$ be the local solution of the system (2.5.1)-(2.5.3) given by the Theorem 2.1.2 with $t \in [0, T]$ and initial data (2.5.4). Then as $\lambda \rightarrow \infty$, (u_λ, n_λ) converges to $(u, -|u|^2)$ in $L^\infty(0, T, H^1) \times L^\infty(0, T, L^2)$ weak star, where u is the unique solution of the perturbed nonlinear Schrödinger equation (2.1.13) with initial data $u_0 \in H^s$.*

We use a classical compactness method and we will follow the ideas of H. Added and S. Added in [2]. Before giving the proof of theorem 2.5.1 we want to recall the two following lemma in [11].

Lemma 2.5.2 (Aubin-Lions's theorem). *Let B_0, B, B_1 be three reflexive Banach spaces. Assume that B_0 is compactly embedded in B and B is continuously embedded in B_1 . Let*

$$W = \{V \in L^{p_0}(0, T, B_0), \frac{\partial V}{\partial t} \in L^{p_1}(0, T, B_1)\}, \quad T < \infty, \quad 1 < p_0, p_1 < \infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}(0, T, B_0)} + \|V_t\|_{L^{p_1}(0, T, B_1)}.$$

Then the embedding $W \hookrightarrow L^{p_0}(0, T, B)$ is compact.

When $p_0 = \infty, p_1 > 1$, the above statement is also true, see [17].

Lemma 2.5.3. *Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that*

$$g_\varepsilon \rightarrow g \text{ a.e in } \Omega \text{ and } \|g_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then $g_\varepsilon \rightarrow g$ weakly in $L^p(\Omega)$.

Proof of Theorem 2.5.1. (i) By using Theorem 2.1.2, we have that the quantities

$\|u_\lambda\|_{L^\infty(0,T,H^1)}$, $\|n_\lambda\|_{L^\infty(0,T,L^2)}$ and $\|(1/\lambda)f_\lambda\|_{L^\infty(0,T,L^2)}$ are bounded uniformly in λ . So, some subsequence of $(u_\lambda, n_\lambda, (1/\lambda)f_\lambda)$ (still denoted by λ) has a weak limit (u, n, w) . More precisely

$$u_\lambda \rightarrow u \text{ in } L^\infty(0, T, H^1) \text{ weak star (w.s.)}, \quad (2.5.5)$$

$$n_\lambda \rightarrow n \text{ in } L^\infty(0, T, L^2) \text{ w.s.} \quad (2.5.6)$$

and

$$(1/\lambda)f_\lambda \rightarrow w \text{ in } L^\infty(0, T, L^2) \text{ w.s.} \quad (2.5.7)$$

Let us note that the following maps are continuous

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow L^4(\mathbb{R}^2) \\ u &\mapsto u, \end{aligned} \quad (2.5.8)$$

$$\begin{aligned} H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}^2) \\ (u, v) &\mapsto uv, \end{aligned} \quad (2.5.9)$$

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}) \\ u &\mapsto Q^2 u, \end{aligned} \quad (2.5.10)$$

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow L^2(\mathbb{R}^2) \\ u &\mapsto Qu \end{aligned} \quad (2.5.11)$$

and

$$\begin{aligned} L^2(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}^2) \\ v &\mapsto Qv. \end{aligned} \quad (2.5.12)$$

Since Q and ∇Q are bounded.

It then follows from (2.5.5)-(2.5.12) that the quantities $\|Q^2 u_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\|Q n_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\|\Delta u_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\| |u_\lambda|^2 \|_{L^\infty(0,T,L^2)}$, $\|Q Re(u_\lambda)\|_{L^\infty(0,T,L^2)}$ and $\|n_\lambda u_\lambda\|_{L^\infty(0,T,H^{-1})}$ are bounded uniformly in λ . So it can be assumed that

$$n_\lambda u_\lambda \rightarrow g \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (2.5.13)$$

$$|u_\lambda|^2 \rightarrow h \text{ in } L^\infty(0, T, L^2) \text{ w.s.}, \quad (2.5.14)$$

$$Q Re(u_\lambda) \rightarrow Q Re(u) \text{ in } L^\infty(0, T, L^2) \text{ w.s.}, \quad (2.5.15)$$

$$Q^2 u_\lambda \rightarrow Q^2 u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (2.5.16)$$

$$Q n_\lambda \rightarrow Q n \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (2.5.17)$$

and

$$\Delta u_\lambda \rightarrow \Delta u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.} \quad (2.5.18)$$

Finally, taking into account (2.5.5), (2.5.13), (2.5.16), (2.5.17) and (2.5.18), equation (2.5.1) implies that

$$\partial_t u_\lambda \rightarrow \partial_t u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s..} \quad (2.5.19)$$

Using the above results, this proof will be complete if we establish that

$$n + |u|^2 + 2QRe(u) = 0, \quad (2.5.20)$$

and

$$g + u|u|^2 + 2QuRe(u) = 0. \quad (2.5.21)$$

(ii) *Proof of (2.5.20)*. Let Ω be any bounded subdomain of $\mathbb{R}^2$. We notice that the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, and for any Banach space X ,

$$\text{the embedding } L^\infty(0, T, X) \hookrightarrow L^2(0, T, X) \text{ is continuous.} \quad (2.5.22)$$

Hence, according to (2.5.22), (2.5.5), (2.5.19) and applying Lemma 4.4.1 to $B_0 = H^1(\Omega)$, $B = L^4(\Omega)$, $B_1 = H^{-1}(\Omega)$, $p_0 = p_1 = 2$, it implies that some subsequence of u_λ (on Ω) (also denoted by λ) converges strongly to u on the domain Ω in $L^2(0, T, L^4(\Omega))$. So we can assume that

$$u_\lambda \rightarrow u \text{ strongly in } L^2(0, T, L^4_{loc}(\mathbb{R}^2)), \quad (2.5.23)$$

and thus,

$$u_\lambda \rightarrow u \text{ almost everywhere in } (t, x, y) \in (0, T, \mathbb{R}^2). \quad (2.5.24)$$

Then, using Lemma 2.5.3, (2.5.14) and (2.5.24) implies that $h = |u|^2$.

Taking into account that $\|(1/\lambda)f_\lambda\|_{L^\infty(0, T, L^2)}$ is bounded uniformly in λ , equations (2.5.2) and (2.5.3) follow that

$$\nabla(n_\lambda + |u_\lambda|^2 + 2QRe(u_\lambda)) \rightarrow 0 \text{ in } \mathcal{D}'(0, T, L^2).$$

So $\nabla(n + |u|^2 + 2QRe(u)) = 0$. Since $n + |u|^2 + 2QRe(u) \in L^\infty(0, T, L^2)$, it follows that

$$n + |u|^2 + 2QRe(u) = 0.$$

(iii) *Proof of (2.5.21)*. We shall prove that $n_\lambda u_\lambda \rightarrow -u|u|^2 - 2QuRe(u)$ in $L^2(0, T, H^{-1})$ w.s., which combined with (2.5.13) and (2.5.22) will ensure that $g = -u|u|^2 - 2QuRe(u)$. First, let ψ be some test function in $L^2(0, T, H^1)$ vanishing out of a compact set $\Omega \in \mathbb{R}^2$. Then

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} (n_\lambda u_\lambda + u|u|^2 + 2QuRe(u)) \psi \, dx \, dt \\ &= \int_0^T \int_\Omega n_\lambda (u_\lambda - u) \psi \, dx \, dt + \int_0^T \int_\Omega (n_\lambda + |u|^2 + 2QRe(u)) u \psi \, dx \, dt. \end{aligned}$$

Let $I_\lambda^1(\psi)$ and $I_\lambda^2(\psi)$ denote the first and the second integral on the right side respectively. Then

$$|I_\lambda^1(\psi)| \leq \|n_\lambda\|_{L^\infty(0, T, L^2)} \|\psi\|_{L^2(0, T, L^4(\Omega))} \|u_\lambda - u\|_{L^2(0, T, L^4(\Omega))}.$$

Since Ω is bounded we deduce from (2.5.6) and (2.5.23) that $I_\lambda^1(\psi)$ goes to 0 when λ tends to ∞ .

Using Sobolev embedding theorem and Hölder inequality, we see that $\psi u \in L^1(0, T, L^2)$. Therefore, using that $n_\lambda \rightarrow -|u|^2 - 2QRe(u)$ in $L^\infty(0, T, L^2)$ w.s., it follows that $I_\lambda^2(\psi)$ goes to 0 when λ tends to ∞ .

Hence,

$$\int_0^T \int_{\mathbb{R}^2} (n_\lambda u_\lambda + u|u|^2 + 2QuRe(u)) \psi \, dx \, dt \rightarrow 0$$

for every test function ψ then the proof is complete. $\square$

Conclusions. 1) We proved the local well-posedness in the usual Sobolev spaces for the 2-d Zakharov system perturbed by its 1-d soliton solution. This is the preliminary step to studying transverse stability (or instability) of that 1-d soliton under the 2-d Zakharov flow. Moreover, Theorem 2.5.1 shows that one obtains an NLS type equation in an appropriate limit.

It is also interesting to consider the same problem for the general vectorial Zakharov system. 2) Schochet-Weinstein method is also interesting if we consider a perturbation of a 3-d Zakharov system by its 2-d soliton. We think the same method should work there, too, since we only used the algebraic structure of the system and the fact that the soliton is smooth and bounded.

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Chapter 3

On the Cauchy problem for the Zakharov-Rubenchik/Benney-Roskes system

Joint work with Norbert J. Mauser and Jean-Claude Saut. (*Communications on Pure and applied analysis. Volume 17, Number 4, July 2018*).

Abstract: We address various issues concerning the Cauchy problem for the Zakharov-Rubenchik system (known as the Benney-Roskes system in water waves theory), which models the interaction of short and long waves in many physical situations. Motivated by the transverse stability/instability of the one-dimensional solitary wave (line solitary), we study the Cauchy problem in the background of a line solitary wave.

3.1 Introduction

This paper is concerned with various issues concerning the Cauchy problem for the two or three-dimensional Zakharov-Rubenchik (or Benney-Roskes) system and its perturbation by a line soliton. The Zakharov-Rubenchik system is fundamental, being a "generic" asymptotic system in the so-called modulation regime (slowly varying envelope of a fast oscillating train) and it was actually derived in various physical contexts. Moreover it contains in various limits the classical (scalar) Zakharov system (coupling a nonlinear Schrödinger equation and a wave equation, see (3.1.3) below) and the Davey-Stewartson systems (coupling a nonlinear Schrödinger equation and an elliptic equation). We refer to [40] for more details on the formal derivation of those systems and on the physical background.

The Davey-Stewartson system was first derived formally in the context of water waves in [9, 1, 10] (see also [7, 6] for a derivation of Davey-Stewartson systems in a different context). However, as noticed in [18] it is less general than the Benney-Roskes system (3.1.6) below in the sense that the initial conditions for the acoustic type components have to be prepared to obtain an approximation of the full water waves system.

We refer to [8] for a rigorous justification of the Zakharov limit of the Zakharov-Rubenchik system and to [29] for the Schrödinger limit of the Zakharov-Rubenchik system in the one-dimensional case and for well-prepared initial data.

The Zakharov-Rubenchik/Benney-Roskes system is thus richer than those simpler models and should capture more of the original dynamics. It was introduced in [41] (see also the survey article [40]) to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. The analysis is general and carried out in the Hamiltonian formalism and yields the following *universal* system

$$\begin{cases} \psi_t + v_g \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_g}{2k} \Delta_{\perp} \psi - i(q|\psi|^2 + \beta \rho + \alpha \phi_x) \psi = 0, \\ \rho_t + \rho_0 \Delta \phi + \alpha(|\psi|^2)_x = 0, \\ \phi_t + \frac{c^2}{\rho_0} \rho + \beta |\psi|^2 = 0, \end{cases} \quad (3.1.1)$$

where $v_g, \omega'', k, q, \beta, \alpha, \rho_0, c$ are parameters. The two last equations describe the acoustic type waves and $\Delta_{\perp} = \partial_y^2 + \partial_z^2$ or ∂_y^2 , $\Delta = \Delta_{\perp} + \partial_x^2$.

In two space dimensions a more specific (formal) derivation in the context of surface water waves is displayed in [15] and rigorously justified in [18], see below for a more precise description.

In the notations of [32] (see also [39] where it is used in the context of Alfvén waves in dispersive MHD), the Zakharov-Rubenchik system has the form

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i \delta \psi_{xx} - i \sigma_1 \Delta_{\perp} \psi + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D \phi_x) \right\} \psi = 0, \\ \rho_t + \Delta \phi + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0, \end{cases} \quad (3.1.2)$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, $\rho, \phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d = 2, 3$ describe the fast oscillating and, resp., acoustic type waves.

Here $\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the strength of the coupling with acoustic type waves, $M > 0$ is a Mach number, $D \in \mathbb{R}$ is associated to the Doppler shift due to the medium velocity and $\delta \in \mathbb{R}$ is a nondimensional dispersion coefficient.

When $\alpha = 0$ (resp. $D = 0$) in (3.1.1) (resp. (3.1.2)) the Zakharov-Rubenchik system reduces to the classical (scalar) Zakharov system (see *eg* Chapter V in [38]). More precisely, in the framework of (3.1.1), one gets

$$\begin{cases} \psi_t + v_g \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_g}{2k} \Delta_{\perp} \psi = i(q|\psi|^2 + \beta \rho) \psi, \\ \rho_{tt} - c^2 \Delta \rho = \beta \rho_0 \Delta |\psi|^2, \end{cases} \quad (3.1.3)$$

which is a form of the 2 or 3D Zakharov system. Note however that the second order operator in the first equation is not necessarily elliptic.

The local well-posedness in $H^s(\mathbb{R}^d) \times H^{s-1/2}(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$ with $s > \frac{d}{2}$, $d = 2, 3$ for (3.1.2), (3.1.1) was obtained in [32] by using the local smoothing property of the free Schrödinger operator after reducing the system to a quasilinear (non local) Schrödinger equation. Since it uses dispersive properties of the free Schrödinger group that are valid only in the whole space the proof does not extend to the Cauchy problem posed on the torus $\mathbb{T}^d$ or "semi-periodically" in $\mathbb{R}^{d-1} \times \mathbb{T}$, the latter situation being relevant for transverse stability issues. On the other hand, when applied to the Benney-Roskes system (3.1.5) below, it provides ¹ an existence time of order $O(1)$, while an existence time of order $O(1/\epsilon)$ is needed to fully justify the Benney-Roskes as a water wave model on the correct time scales (see [17]).

Local well-posedness of the Zakharov-Rubenchik/Benney-Roskes system was also obtained in [24], for $s > 2$ with the additional condition $\delta \sigma_1 > 0$ (that is the second order operator in the first equation of (3.1.2), (3.1.1) is elliptic) by using an energy method inspired by the work of Schochet-Weinstein in [36] on the nonlinear Schrödinger limit of the Zakharov system. The method used in [24] and [36] consists in rewriting the Zakharov system (or the Zakharov-Rubenchik system) as a dispersive (skew-adjoint) perturbation of a symmetric nonlinear hyperbolic system and it uses only the algebraic structure of the system. A shortcoming of the method is that one has to prepare the initial data.

We will see that, when the small parameter ϵ is included, this method provides also the existence on the time scale $O(1)$ in the context of water waves (see the Benney-Roskes system (3.1.6) below) and moreover that it can be applied to the system obtained from (3.1.2) which is satisfied by a (localized) perturbation of a line soliton. Also, since it does not use any dispersive property of the Schrödinger group, it applies to the Cauchy problem in $\mathbb{T}^d$ or $\mathbb{R}^{d-1} \times \mathbb{T}$, a situation that has not been addressed before (see on the other hand [3, 2] for the periodic Zakharov system).

Thus, none of the two aforementioned methods seems to give the expected existence time scale for the Benney-Roskes system. Nevertheless they provide different results for Zakharov-Rubenchik type systems. The "dispersive method" used in [32] works only in $\mathbb{R}^d$ but does not need the Schrödinger part of the system to be "elliptic" (that is it does not need the condition $\delta \sigma_1 > 0$). Also it lowers the regularity on the initial data (an effect of the dispersive smoothing effect) and could be applied as well to (possibly non physical) nonlinear perturbations of the system.

On the other hand, the Schochet-Weinstein type "hyperbolic like" methods allow to deal with the periodic or semi-periodic cases, but are relatively rigid (they rely on the algebraic structure of the system) and require initial data in the "hyperbolic space" $H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + 1$.

¹Roughly speaking, the idea in [32] is to reduce the system to a (nonlocal) quasilinear Schrödinger equation. When ϵ is taken into account, the crucial dispersive smoothing estimate on the Schrödinger group has a $1/\epsilon$ factor while the nonlinear term has a ϵ factor.

The situation is better understood in spatial dimension one. Oliveira [28] proved the local (thus global using the conservation laws below) well-posedness in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. This result was improved in [19] where in particular global well-posedness was established in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

It is worth noticing that (3.1.2) possesses two conserved quantities, the L^2 norm

$$\int |\psi(x, y_\perp, t)|^2 = \int |\psi(x, y_\perp, 0)|^2,$$

where $y_\perp = y$ or (y, z) , and, after the change of variable $(x, t) \rightarrow (x + \sigma_3 t, t)$, the Hamiltonian

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^d} \left(\frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\nabla_\perp \psi|^2 + \frac{\sigma_2}{4} |\psi|^4 + \frac{W}{4M^2} \rho^2 + \frac{W}{4} |\nabla \phi|^2 + \frac{W}{2} (\rho + D\phi_x) |\psi|^2 \right) \\ &= E(0). \end{aligned} \quad (3.1.4)$$

The conservation laws are used in [32] to obtain global weak solutions under suitable assumptions on the coefficients. We will use them in Section 5 to prove the global existence of weak solutions of the systems obtained by perturbing a line (dark) soliton. Note also that the conservation laws can be used to get the global well-posedness of the Zakharov-Rubenchik, Benney-Roskes system in space dimension one (see [28]).

As aforementioned, in the context of water waves, the Zakharov-Rubenchik system is known as the Benney-Roskes sytem and it was formally derived in [15]. We follow here the notations in [18], where a rigorous derivation is performed.

$$\begin{aligned} \mathbf{k} &= |\mathbf{k}| \mathbf{e}_x, \quad \omega(\mathbf{k}) = \underline{\omega}(|\mathbf{k}|), \\ \omega &= \underline{\omega}(|\mathbf{k}|), \quad \omega' = \underline{\omega}'(|\mathbf{k}|), \quad \omega'' = \underline{\omega}''(|\mathbf{k}|), \\ \text{where } \underline{\omega}(|\xi|) &= \left(\left(g + \frac{\sigma}{\rho} |\xi|^2 \right) |\xi| \tanh(\sqrt{\mu} |\xi|) \right)^{1/2} \end{aligned} \quad (3.1.5)$$

is the dispersion relation of water waves and where $|\mathbf{k}|$ is a fixed wave number, g is the gravity, $\sigma \geq 0$ is a surface tension coefficient, ρ is the density of the water and μ is the shallowness parameter (square of the typical fluid depth over a typical horizontal scale) which is large or infinite in the deep water models and $\alpha = -\frac{9}{8\sigma^2} (1 - \sigma^2)^2$.

The small parameter ϵ is the *wave steepness* that is the ratio of a typical amplitude of the wave over a typical horizontal scale. Recall ([18]) that the typical time scale for the solutions of (3.1.6) below is $1/\epsilon$ and so it is crucial to establish the well-posedness on those time scales.

The Benney-Roskes equations can then be written in 2 dimensions as follows

$$\begin{cases} \partial_t \psi_{01} + \omega' \partial_x \psi_{01} - i\epsilon \frac{1}{2} (\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_{01} \\ \quad + i\epsilon \left(|\mathbf{k}| \partial_x \psi_{00} + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta_{10} + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_{01}|^2 \right) \psi_{01} = 0, \\ \partial_t \zeta_{10} + \sqrt{\mu} \Delta \psi_{00} = -2\omega |\mathbf{k}| \partial_x (|\psi_{01}|^2), \\ \partial_t \psi_{00} + \zeta_{10} = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi_{01}|^2. \end{cases} \quad (3.1.6)$$

It is known (see *eg* [18] Chapter 8) that $\omega' > 0$, while for purely gravity waves ($\sigma = 0$) ω is a concave function, thus $\omega'' < 0$ and the Schrödinger equation in the Benney-Roskes system is "non elliptic".

On the other hand, in the presence of a surface tension, the condition $\omega'' > 0$ is possible as shown in the following computation.

For simplicity of notations, we will consider ω of the following form instead of (3.1.5)

$$\omega(r) = \left((1 + \gamma r^2) r \tanh(\sqrt{\mu} r) \right)^{1/2}$$

where $\gamma > 0$ which depends on (g, ρ) and is proportional to σ and $r = |\mathbf{k}|$. We have

$$\begin{aligned} \omega'(r) &= \left((1 + 3\gamma r^2) \tanh(\sqrt{\mu} r) + \sqrt{\mu}(r + \gamma r^3) \operatorname{sech}^2(\sqrt{\mu} r) \right) \\ &\quad \times \frac{1}{2} \left((r + \gamma r^3) \tanh(\sqrt{\mu} r) \right)^{-1/2} \end{aligned}$$

and

$$\begin{aligned} \omega''(r) &= -\frac{1}{4} \left((1 + 3\gamma r^2) \tanh(\sqrt{\mu} r) + \sqrt{\mu}(r + \gamma r^3) \operatorname{sech}^2(\sqrt{\mu} r) \right)^2 \left((r + \gamma r^3) \tanh(\sqrt{\mu} r) \right)^{-3/2} \\ &\quad + \frac{1}{2} \left((r + \gamma r^3) \tanh(\sqrt{\mu} r) \right)^{-1/2} \left(6\gamma r \tanh(\sqrt{\mu} r) + 2\sqrt{\mu}(1 + 3\gamma r^2) \operatorname{sech}^2(\sqrt{\mu} r) \right. \\ &\quad \left. - 2\mu(r + \gamma r^3) \tanh(\sqrt{\mu} r) \operatorname{sech}^2(\sqrt{\mu} r) \right). \end{aligned}$$

We see that $\omega'(r) > 0$ with $\gamma, r > 0$, we thus will look for r such that with fixed γ , $\omega''(r) > 0$. We assume that $\sqrt{\mu} r \gg 1$ implying that $\operatorname{sech}(\sqrt{\mu} r) \approx 0$ and $\tanh(\sqrt{\mu} r) \approx 1$. Therefore we only need to choose r large enough so that

$$12\gamma r > \frac{(1 + 3\gamma r^2)^2}{r + \gamma r^3}$$

or

$$3\gamma^2 r^4 + 6\gamma r^2 > 1.$$

In order to apply the Schochet-Weinstein method we will need the condition $\delta\sigma_1 > 0$ and we will only consider the Zakharov-Rubenchik (or Benney-Roskes) system of the form of (3.1.2) satisfying this condition.

The one-dimensional Zakharov system possesses solitary wave solutions and in [28], Oliveira proved their orbital stability. One motivation of the present paper was the study of their *transverse* stability. The transverse instability of the line solitary wave for some two dimensional models such as the nonlinear Schrödinger equation (NLS), the Kadomtsev-Petviashvili equation (KP) and some general "abstract" Hamiltonian systems have been carried out extensively in [35, 33, 34, 22, 23].

It is thus of interest to study the transverse stability of the line soliton for the two dimensional model (3.1.2) and the first step is to study the Cauchy problem of a localized perturbation of (3.1.2) by a line soliton. Another possibility is to consider y or (y, z) -periodic perturbations of the line solitary wave, a first step being to establish the well-posedness of the Cauchy problem for the Zakharov-Rubenchik, Benney-Roskes system in $\mathbb{R}^d \times \mathbb{T}$, $d = 1, 2$, which could not result from the methods used in [32], but in the way we achieve here and also in the pure periodic case $\mathbb{T}^{d+1}$.

In order to unify the notation, we will rewrite the Benney-Roskes system (3.1.6) in the form of (3.1.2). We replace $(\psi_{01}, \frac{\zeta_{10}}{|\mathbf{k}|^2(1-\sigma^2)\sqrt{\mu}}, \frac{\psi_{00}}{|\mathbf{k}|^2(1-\sigma^2)})$ by (ψ, ρ, ϕ) and after calculating the corresponding coefficients, we have:

$$\begin{cases} \sigma_3 = -\omega', \delta = \frac{\epsilon\omega''}{2}, \sigma_1 = \frac{\epsilon\omega'}{2|\mathbf{k}|}, \sigma_2 = \frac{2\epsilon|\mathbf{k}|^4(1-\alpha)}{\omega}, \\ W = \frac{\epsilon|\mathbf{k}|^4(1-\sigma^2)^2\sqrt{\mu}}{2\omega}, D = \frac{2\omega}{|\mathbf{k}|(1-\sigma^2)\sqrt{\mu}}, M = \mu^{-1/4}. \end{cases} \quad (3.1.7)$$

The paper is organized as follows. In the next Section we reformulate the existence of one-dimensional solitary waves (bright and dark) in our framework. In Section 3 we use the Schochet-Weinstein method to prove a local existence for the Benney-Roskes/ Zakharov-Rubenchik system, keeping the small parameter ϵ which is relevant for deep water waves. In Section 4 we consider the case of a localized perturbation of a line solitary wave. Finally we prove in Section 5 the global existence of weak solutions perturbing a dark solitary wave.

We conclude the paper by a list of open questions.

Notation. • ∂_x or $(\cdot)_x$ will be used to denote the derivative with respect to variable x .

- $H^s(D)$, $s \in \mathbb{R}$ denotes the classical Sobolev space in the domain D .
- $\|\cdot\|_X$: The norm in a functional space X .
- $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transform respectively.
- $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$ and $\sigma(D)$ denotes the Fourier multiplier with the symbol $\sigma(\xi)$.
- $\Re$ and $\Im$ denote the real part and imaginary part of a complex number respectively.

3.2 Existence of one dimensional solitary waves

In this section, we reframe the proof of the existence of 1-d solitary waves in [28] in our setting. The 1-d Zakharov-Rubenchik system has the form

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \phi_{xx} + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0. \end{cases} \quad (3.2.1)$$

Setting $\tilde{\phi} = \phi_x$, (6.1.2) becomes

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\tilde{\phi}) \right\} \psi = 0, \\ \rho_t + \tilde{\phi}_x + D(|\psi|^2)_x = 0, \\ \tilde{\phi}_t + \frac{1}{M^2} \rho_x + (|\psi|^2)_x = 0. \end{cases} \quad (3.2.2)$$

Let $c \geq 0$, we look for solutions of the system (3.2.2) of the form

$$(e^{i\lambda t} K(x - ct), a|K(x - ct)|^2, b|K(x - ct)|^2).$$

From the last two equations of (3.2.2) we deduce that

$$a = \frac{-(1 + cD)}{1/M^2 - c^2} \quad \text{and} \quad b = \frac{-(c + D/M^2)}{1/M^2 - c^2}. \quad (3.2.3)$$

Then the first equation of (3.2.2) is equivalent to

$$\delta \ddot{K} - i(c + \sigma_3) \dot{K} - \lambda K = (\sigma_2 + W(a + bD)) |K|^2 K.$$

Set

$$R(x) = e^{-i(c + \sigma_3)x/2\delta} K(x),$$

then

$$\delta \ddot{R} + \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) R = (\sigma_2 + W(a + bD)) |R|^2 R. \quad (3.2.4)$$

The equation (3.2.4) has a positive solution, which is unique up to a translation, if :

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) < 0, \\ \frac{1}{\delta} (\sigma_2 + W(a + bD)) < 0 \end{cases}$$

or equivalently

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) < 0, \\ \frac{1}{\delta} \left(\sigma_2 - \frac{W(1 + D^2/M^2 + 2cD)}{1/M^2 - c^2} \right) < 0. \end{cases} \quad (3.2.5)$$

We see that if $c \rightarrow (1/M)^-$ and λ is large enough then (3.2.5) holds assuming that $W > 0$ and $\delta > 0$ which holds true in both models (3.1.2) and (3.1.6).

In this case,

$$R(x) = \sqrt{\frac{2}{\sigma_2 + W(a + bD)} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)} \operatorname{sech} \left(\sqrt{-\frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)} x \right). \quad (3.2.6)$$

Otherwise, if

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) > 0, \\ \frac{1}{\delta} (\sigma_2 + W(a + bD)) > 0, \end{cases}$$

or equivalently

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) > 0, \\ \frac{1}{\delta} \left(\sigma_2 - \frac{W(1 + D^2/M^2 + 2cD)}{1/M^2 - c^2} \right) > 0. \end{cases} \quad (3.2.7)$$

In the context of water waves (Benney-Roskes system) there is a regime where the condition (3.2.7) holds. In particular, if we choose $c > \frac{1}{M}$ and $\lambda < \frac{(c + \sigma_3)^2}{4\delta}$ then (3.2.7) holds, since from (3.1.7) we know that δ , σ_2 , W and D are positive. If $c = 0$, then (3.2.7) is equivalent to

$$\begin{cases} \frac{\sigma_3^2}{4\delta} > \lambda, \\ \frac{2|\mathbf{k}|^4(1 - \alpha)}{\omega} - \frac{|\mathbf{k}|^4(1 - \sigma^2)^2}{2\omega} - \frac{2\omega|\mathbf{k}|^2}{\sqrt{\mu}} > 0. \end{cases}$$

Since $\alpha < 0$, and in the context of water wave, the surface tension is small, then one has $\frac{2|\mathbf{k}|^4(1 - \alpha)}{\omega} > \frac{|\mathbf{k}|^4(1 - \sigma^2)^2}{2\omega}$. Therefore, if μ is large enough (which occurs in the context of deep water waves) then the above conditions hold.

In this case,

$$R(x) = \sqrt{\frac{1}{\sigma_2 + W(a + bD)} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)} \tanh \left(\sqrt{\frac{1}{2\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)} x \right) \quad (3.2.8)$$

Then the system (3.2.2) has two kind of solitary waves corresponding to the two conditions (3.2.5) and (3.2.7):

$$(e^{i\lambda t} e^{i(c+\sigma_3)x/2\delta} R(x-ct), a R^2(x-ct), b R^2(x-ct)).$$

Recalling that $\tilde{\phi} = \phi_x$, the solutions of system (6.1.2) should have thus the form

$$Q = (e^{i\lambda t} e^{i(c+\sigma_3)x/2\delta} R(x-ct), a R^2(x-ct), b P(x-ct)), \quad (3.2.9)$$

where

$$P(x) = \frac{\alpha^2}{\beta} \tanh(\beta x)$$

with

$$\alpha = \sqrt{\frac{2}{\sigma_2 + W(a+bD)} \left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{-\frac{1}{\delta} \left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda \right)}$$

in the case $R(x)$ is given by (3.2.6),

$$P(x) = \frac{\alpha^2}{\beta} (\beta x - \tanh(\beta x))$$

with

$$\alpha = \sqrt{\frac{1}{\sigma_2 + W(a+bD)} \left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{\frac{1}{2\delta} \left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda \right)}$$

in the case $R(x)$ is given by (3.2.8).

Remark 10. Similarly to the case of the cubic nonlinear Schrödinger equation, we will call the 1-d solitary wave corresponding to the condition (3.2.5) and (3.2.7) the “bright”, “dark” soliton respectively.

3.3 The Z-R/B-R system

As aforementioned the asymptotic model (3.1.6) is a good approximation of the full water wave system on a time scale $O(1/\epsilon)$ (see [18] page 233). It is thus crucial to prove the well-posedness of the Cauchy problem on time scales of order $1/\epsilon$. However, the existence time obtained by using the method in [32] does not reach the $O(1/\epsilon)$ time scale (as we already mentioned it is of order $O(1)$)

In this section, we give the proof of the local well-posedness for (3.1.2) in two dimensional case by using Schochet-Weinstein method in [24] but keeping the parameter ϵ in (3.1.6) to estimate the existence time obtained by this method. It turns out, however, that one does not improve upon the previously known $O(1)$ result (see however the comments in Introduction).

We consider the following system

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \Delta \phi + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0, \end{cases} \quad (3.3.1)$$

with initial data $(\psi, \rho, \phi)(t=0) = (\psi_0, \rho_0, \phi_0)$ for which we obtain a local existence result:

Theorem 3.3.1. *Let $\delta \sigma_1 > 0$, $s > 2$. For any initial data $(\psi_0, \rho_0, \phi_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exist $T > 0$ independent of $\epsilon \in (0, 1]$ such that (6.1.1) admits a unique solution $(\psi, \rho, \phi) \in C(0, T; H^{s+1}(\mathbb{R}^2)) \times C(0, T; H^s(\mathbb{R}^2)) \times C(0, T; H^{s+1}(\mathbb{R}^2))$.*

Remark 11. *With some minor changes, one obtains the same result in the three-dimensional case, that is ψ_{yy} replaced by $\Delta_{\perp}\psi$.*

Remark 12. *The above theorem and its proof are valid mutatis mutandi in a periodic $(\mathbb{T}^d)$, $d = 2, 3$, or semi-periodic $(\mathbb{R}^{d-1} \times \mathbb{T})$ setting.*

Proof. We follow closely the proof in [24], Section 3.3, but we keep track of the parameter ϵ .

We first rewrite (6.1.1) as a dispersive perturbation of a symmetric hyperbolic system. We take the time derivative of the second and the third equation of the system (3.1.2). This allows to decouple the linear parts of those equations

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left(\sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right) \psi = 0, \\ \rho_{tt} - \Delta \left(\frac{1}{M^2} \rho + |\psi|^2 \right) + D(|\psi|^2)_{xt} = 0, \\ \phi_{tt} - \frac{1}{M^2} \left(\Delta \phi + D(|\psi|^2)_x \right) + (|\psi|^2)_t = 0. \end{cases} \quad (3.3.2)$$

We set

$$\mathcal{U} = W\rho + WD\phi_x. \quad (3.3.3)$$

We then get a coupled system for ψ and $\mathcal{U}$

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left(\sigma_2 |\psi|^2 + \mathcal{U} \right) \psi = 0 \\ \mathcal{U}_{tt} - \frac{1}{M^2} \Delta \mathcal{U} - W \Delta(|\psi|^2) + 2DW(|\psi|^2)_{xt} - \frac{WD^2}{M^2} (|\psi|^2)_{xx} = 0. \end{cases} \quad (3.3.4)$$

Following the idea in [36] for the Zakharov system, we define the following auxiliary (vector valued) function V as the unique solution of the following equation

$$V_t = \frac{1}{M} \nabla \mathcal{U} + WM \nabla(|\psi|^2) + \left(\frac{WD^2}{M} (|\psi|^2)_x, 0 \right)^T, \quad (3.3.5)$$

with the initial data given by

$$V(t=0) = V_0 = -MW \nabla \phi_0 - \frac{WD}{M} (\rho_0, 0)^T. \quad (3.3.6)$$

Therefore, we obtain the equivalent first order system

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left(\sigma_2 |\psi|^2 + \mathcal{U} \right) \psi = 0, \\ \mathcal{U}_t - \frac{1}{M} \nabla \cdot V + 2WD(|\psi|^2)_x = 0, \\ V_t - \frac{1}{M} \nabla \mathcal{U} - WM \nabla(|\psi|^2) - \left(\frac{WD^2}{M} (|\psi|^2)_x, 0 \right)^T = 0. \end{cases} \quad (3.3.7)$$

We now set

$$\mathcal{D} = \mathcal{U} + \frac{\sigma_1}{2} |\psi|^2,$$

then the system (3.3.7) becomes

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left(\left(\sigma_2 - \frac{\sigma_1}{2} \right) |\psi|^2 + \mathcal{D} \right) \psi = 0, \\ \mathcal{D}_t - \frac{1}{M} \nabla \cdot V - \frac{\sigma_1}{2} \left((|\psi|^2)_t - \sigma_3 (|\psi|^2)_x \right) + \left(2WD - \frac{\sigma_1 \sigma_3}{2} \right) (|\psi|^2)_x = 0, \\ V_t - \frac{1}{M} \nabla \mathcal{D} - \left(\frac{2W(D^2 + M^2) - \sigma_1}{2M} (|\psi|^2)_x, \frac{2WM^2 - \sigma_1}{2M} (|\psi|^2)_y \right)^T = 0. \end{cases} \quad (3.3.8)$$

For simplicity of notation we set

$$c_1 = 2WD - \frac{\sigma_1\sigma_3}{2}, \quad c_2 = \frac{2W(D^2 + M^2) - \sigma_1}{2M}, \quad c_3 = \frac{2WM^2 - \sigma_1}{2M}.$$

Furthermore, we split the ψ in real and imaginary part

$$\psi = F + iG$$

and

$$\nabla\psi = H + iL = (H_1, H_2)^T + i(L_1, L_2)^T.$$

Multiplying the first equation of (3.3.8) by $\bar{\psi}$ and taking the real part, we deduce that

$$\begin{aligned} (|\psi|^2)_t - \sigma_3(|\psi|_x^2) &= i\epsilon\delta\psi_{xx}\bar{\psi} - i\epsilon\delta\bar{\psi}_{xx}\psi + i\epsilon\sigma_1\psi_{yy}\bar{\psi} - i\epsilon\sigma_1\bar{\psi}_{yy}\psi \\ &= 2\epsilon\delta(G\partial_x H_1 - F\partial_x L_1) + 2\epsilon\sigma_1(G\partial_y H_2 - F\partial_y L_2). \end{aligned} \quad (3.3.9)$$

We insert (3.3.9) into (3.3.8) and then separate the real and imaginary parts of the first equation of the system. We furthermore apply the spatial gradient to the first equation of (3.3.8) to get the equations satisfied by H and L . This leads to the following system

$$H_t - \sigma_3 H_x + \epsilon\delta L_{xx} + \epsilon\sigma_1 L_{yy} - \epsilon G\nabla\mathcal{D} \quad (3.3.10)$$

$$- \epsilon \left(\mathcal{D}L + \left(\sigma_2 - \frac{\sigma_1}{2} \right) \left((F^2 + G^2)L + 2G(FH + GL) \right) \right) = 0,$$

$$L_t - \sigma_3 L_x - \epsilon\delta H_{xx} - \epsilon\sigma_1 H_{yy} + \epsilon F\nabla\mathcal{D} \quad (3.3.11)$$

$$+ \epsilon \left(\mathcal{D}H + \left(\sigma_2 - \frac{\sigma_1}{2} \right) \left((F^2 + G^2)H + 2F(FH + GL) \right) \right) = 0,$$

$$F_t - \sigma_3 F_x + \epsilon\delta G_{xx} + \epsilon\sigma_1 G_{yy} - \epsilon \left(\left(\sigma_2 - \frac{\sigma_1}{2} \right) (F^2 + G^2) + \mathcal{D} \right) G = 0, \quad (3.3.12)$$

$$G_t - \sigma_3 G_x - \epsilon\delta F_{xx} - \epsilon\sigma_1 F_{yy} + \epsilon \left(\left(\sigma_2 - \frac{\sigma_1}{2} \right) (F^2 + G^2) + \mathcal{D} \right) F = 0, \quad (3.3.13)$$

$$\begin{aligned} \mathcal{D}_t - \frac{1}{M} \nabla \cdot V - \epsilon\sigma_1 \delta (G\partial_x H_1 - F\partial_x L_1) - \epsilon\sigma_1^2 (G\partial_y H_2 - F\partial_y L_2) \\ + 2c_1 (H_1 F + L_1 G) = 0, \end{aligned} \quad (3.3.14)$$

$$V_t - \frac{1}{M} \nabla\mathcal{D} - 2(c_2(H_1 F + L_1 G), c_3(H_2 F + L_2 G))^T = 0. \quad (3.3.15)$$

Since $\sigma_1\delta > 0$, we can perform the following change of variables

$$H^* = (\sqrt{\delta\sigma_1}H_1, \sigma_1 H_2)^T \text{ and } L^* = (\sqrt{\delta\sigma_1}L_1, \sigma_1 L_2)^T,$$

and we then set $U = (H^*, L^*, F, G, \mathcal{D}, V)^T$.

Therefore, (6.1.1) is rewritten as a dispersive (skew adjoint) perturbation of a symmetric hyperbolic system given by

$$\begin{aligned} U_t + (\epsilon A_1(U) + B_1)U_x + (\epsilon A_2(U) + B_2)U_y + C(U)U \\ = -K_1 U_{xx} - K_2 U_{yy}, \end{aligned} \quad (3.3.16)$$

where A_1, A_2, B_1 and B_2 are symmetric matrices, K_1 and K_2 are skew symmetric matrices.

$$A_1(U) = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & M_1(U)^T & 0_{3 \times 2} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 2} \\ M_1(U) & 0_{1 \times 3} & 0 & 0_{1 \times 2} \\ 0_{2 \times 3} & 0_{2 \times 3} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix},$$

with

$$M_1(U) = (-\sqrt{\delta\sigma_1}G, 0, \sqrt{\delta\sigma_1}F).$$

$$A_2(U) = \begin{pmatrix} 0 & 0_{1 \times 3} & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\ 0_{3 \times 1} & 0_{3 \times 3} & 0_{3 \times 2} & M_2(U)^T & 0_{3 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \\ 0 & M_2(U) & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix}$$

with

$$M_2(U) = (-\sigma_1 G, 0, \sigma_1 F).$$

Now note that $C(U)$ contains the term that is independent of ϵ which is

$$C_1(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ 0 & 0 & 0 & 0 & 0_{1 \times 5} \\ \frac{-2c_1}{\sqrt{\delta\sigma_1}}F & 0 & \frac{-2c_1}{\sqrt{\delta\sigma_1}}G & 0 & 0_{1 \times 5} \\ \frac{-2c_2}{\sqrt{\delta\sigma_1}}F & 0 & \frac{-2c_2}{\sqrt{\delta\sigma_1}}G & 0 & 0_{1 \times 5} \\ 0 & \frac{-2c_3}{\sigma_1}F & 0 & \frac{-2c_3}{\sigma_1}G & 0_{1 \times 5} \end{pmatrix}.$$

Next, we prove that if the initial data $U(0, x, y) = U_0 \in (H^s(\mathbb{R}^2))^9, s > 2$, then there exists $T = T(\|U_0\|_{(H^s(\mathbb{R}^2))^9})$ such that equation (3.3.16) has a unique solution in $L^\infty([0, T], (H^s(\mathbb{R}^2))^9)$.

The proof of the existence of a solution is standard and proceeds via a classical iteration scheme for symmetric hyperbolic system (see [21, 16]). The presence of $C_1(U)$, unfortunately, leads to the existence time of order $O(1)$.

The uniqueness of solution of (3.3.16) is classically obtained by estimating the difference of two solutions since the dispersive part does not contribute to the L^2 energy estimate.

The last step will be to recover the solution of (6.1.1) from the solution of (3.3.16). To this end, the initial data $U_0 = (H_0, L_0, F_0, G_0, \mathcal{D}_0, V_0)$ should be chosen as $F_0 + iG_0 = \psi_0$, $H_0 + iL_0 = \nabla\psi_0$, $\mathcal{D}_0 = W(\rho_0 + D\phi_{0x})$ and $V_0 = -MW\nabla\phi_0 - \frac{WD}{M}(\rho_0, 0)^T$.

First, set $\psi = F + iG$, then (3.3.12) and (3.3.13) imply that

$$\psi_t - \sigma_3\psi - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left((\sigma_2 - \frac{\sigma_1}{2})|\psi|^2 + \mathcal{D}\right)\psi = 0.$$

From (3.3.10)-(3.3.13) we can derive an L^2 estimate of $\mathcal{W} = (\nabla F - H, \nabla G - L)$, that implies

$$\|\mathcal{W}(t)\|_{L^2}^2 \leq e^{Ct} \|\mathcal{W}(0)\|_{L^2}^2,$$

thus, $\nabla\psi = H + iL$.

Then (3.3.14)-(3.3.15) implies the last two equations of (3.3.8).

Next, we are going to recover (ρ, ϕ) . Let ϕ be the unique solution of the following linear wave equation

$$\phi_{tt} - \frac{1}{M^2} \left(\Delta\phi + D(|\psi|^2)_x \right) + (|\psi|^2)_t = 0,$$

with initial data given by $\phi(t=0) = \phi_0$ and $\phi_t(t=0) = -\frac{1}{M^2}\rho_0 - |\psi_0|^2$.

Define

$$\rho = \frac{1}{W}(\mathcal{U} - WD\phi_x),$$

with

$$\mathcal{U} = \mathcal{D} - \frac{\sigma_1}{2}|\psi|^2.$$

Then we get that (ψ, ρ, ϕ) solves (3.3.2) uniquely with respect to the given initial data.

Next, let $\tilde{\phi}(t, x, y)$ be the unique solution of the differential equation

$$\tilde{\phi}_t + \frac{1}{M^2}\rho + |\psi|^2 = 0,$$

with initial data $\tilde{\phi}(t=0) = \phi_0$.

From the second equation in (3.3.2) we get

$$\rho_{tt} + \Delta\tilde{\phi}_t + D|\psi|_{xt}^2 = 0,$$

and integrating this in time and using the initial data, we get the following system

$$\begin{cases} \rho_t + \Delta\tilde{\phi} + D(|\psi|^2)_x = 0, \\ \tilde{\phi}_t + \frac{1}{M^2}\rho + |\psi|^2 = 0. \end{cases} \quad (3.3.17)$$

Taking the time derivative of the second equation in (3.3.17) we get

$$\tilde{\phi}_{tt} - \frac{1}{M^2}(\Delta\tilde{\phi} + D(|\psi|^2)_x) + (|\psi|^2)_t = 0.$$

Note that the initial data is also given by $\tilde{\phi}(t=0) = \phi_0$ and $\tilde{\phi}_t(t=0) = -\frac{1}{M^2}\rho_0 - |\psi_0|^2$. Therefore we have $\tilde{\phi} = \phi$, achieving to prove that (ψ, ρ, ϕ) solves the original Zakharov-Rubenchik system. $\square$

3.4 The perturbed Z-R/B-R system

In this section, we consider the Cauchy problem for (3.1.2) when it is perturbed by the line solitary wave Q given by (6.1.3). That means, we will find solutions of (3.1.2) of the form $(\psi + \phi_1, \rho + \phi_2, \phi + \phi_3)$, where we denote $Q = (\phi_1, \phi_2, \phi_3)$. The new system reads

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left\{\sigma_2|\psi|^2 + 2\sigma_2\Re(\phi_1\bar{\psi}) + W(\rho + D\phi_x)\right\}\phi_1 \\ \quad + i\epsilon\{\sigma_2|\psi + \phi_1|^2 + W(\rho + \phi_2 + D(\phi + \phi_3)_x)\}\psi = 0, \\ \rho_t + \Delta\phi + D(|\psi|^2 + 2\Re(\phi_1\bar{\psi}))_x = 0, \\ \phi_t + \frac{1}{M^2}\rho + |\psi|^2 + 2\Re(\phi_1\bar{\psi}) = 0. \end{cases} \quad (3.4.1)$$

Remark 13. A natural way to solve the Cauchy problem for (3.4.1) would be to use the "dispersive method" in [32]. However the fact that the line soliton does not decay to 0 in the transverse direction leads to a difficulty when dealing with a new nonlocal linear term and seems to preclude to extend this method in a straightforward way. Therefore we will apply the method of the previous section to this case.

In the first step, we need to rewrite (3.4.1) in the form of a skew-adjoint perturbation of a symmetric hyperbolic system.

Using the fact that $Q = (\phi_1, \phi_2, \phi_3)$ is also a solution of (6.1.1) with $\epsilon = 1$ then by the same calculations as in the previous section, we obtain that $(H_r, L_r, F_r, G_r, D_r, V_r)^T$ is a solution of (3.3.10)-(3.3.15) with $\epsilon = 1$, where

$$\begin{cases} H_r = \nabla(\Re\phi_1), & L_r = \nabla(\Im\phi_1), \\ F_r = \Re\phi_1, & G_r = \Im\phi_1, \\ \mathcal{D}_r = \mathcal{U}_r + \frac{\sigma_1}{2}|\phi_1|^2 \text{ with } \mathcal{U}_r = W\phi_2 + WD(\phi_3)_x, \\ V_r = -WM((\phi_3)_x, 0)^T - \frac{WD}{M}((\phi_2)_x, 0)^T. \end{cases} \quad (3.4.2)$$

Similarly, if (ψ, ρ, ϕ) is a solution of (3.4.1) or if $(\psi + \phi_1, \rho + \phi_2, \phi + \phi_3)$ is a solution of (6.1.1) (with $\epsilon = 1$), then $(\tilde{H}, \tilde{L}, \tilde{F}, \tilde{G}, \tilde{\mathcal{D}}, \tilde{V})^T$ is a solution of (3.3.10)-(3.3.15) with $\epsilon = 1$, where

$$\begin{cases} \tilde{H} = \nabla(\Re(\psi + \phi_1)), & \tilde{L} = \nabla(\Im(\psi + \phi_1)), \\ \tilde{F} = \Re(\psi + \phi_1), & \tilde{G} = \Im(\psi + \phi_1), \\ \tilde{\mathcal{D}} = \tilde{\mathcal{U}} + \frac{\sigma_1}{2}|\psi + \phi_1|^2 \text{ with } \tilde{\mathcal{U}} = W(\rho + \phi_2) + WD(\phi + \phi_3)_x, \\ \tilde{V}_t - \frac{1}{M}\nabla\tilde{\mathcal{U}} - WM\nabla(|\psi + \phi_1|^2) - \left(\frac{WD^2}{M}(|\psi + \phi_1|^2)_x, 0\right)^T = 0 \\ \text{with } \tilde{V}(t=0) = \left(-WM\nabla(\phi + \phi_3) - \frac{WD}{M}(\rho + \phi_2, 0)^T\right)(t=0). \end{cases} \quad (3.4.3)$$

We now set $(H, L, F, G, \mathcal{D}, V)^T = (\tilde{H}, \tilde{L}, \tilde{F}, \tilde{G}, \tilde{\mathcal{D}}, \tilde{V})^T - (H_r, L_r, F_r, G_r, D_r, V_r)^T$, more precisely, we have

$$\begin{cases} H = \nabla(\Re\psi), & L = \nabla(\Im\psi), \\ F = \Re\psi, & G = \Im\psi, \\ \mathcal{D} = \mathcal{U} + \frac{\sigma_1}{2}(|\psi|^2 + 2\Re(\phi_1\bar{\psi})) \text{ with } \mathcal{U} = W\rho + WD\phi_x, \\ V_t - \frac{1}{M}\nabla\mathcal{U} - WM\nabla(|\psi|^2 + 2\Re(\phi_1\bar{\psi})) - \left(\frac{WD^2}{M}(|\psi|^2 + 2\Re(\phi_1\bar{\psi}))_x, 0\right)^T = 0 \\ \text{with } V(t=0) = V_0 = \left(-WM\nabla\phi - \frac{WD}{M}(\rho, 0)^T\right)(t=0). \end{cases} \quad (3.4.4)$$

Combining (3.3.10)-(3.3.15), (3.4.2) and (3.4.3), it transpires that $(H, L, F, G, \mathcal{D}, V)^T$ is a solution of

$$\begin{cases} H_t - \sigma_3 H_x + \delta L_{xx} + \sigma_1 L_{yy} - (G + G_r)\nabla\mathcal{D} + \mathcal{R}_1 = 0, \\ L_t - \sigma_3 L_x - \delta H_{xx} - \sigma_1 H_{yy} + (F + F_r)\nabla\mathcal{D} + \mathcal{R}_2 = 0, \\ F_t - \sigma_3 F_x + \delta G_{xx} + \sigma_1 G_{yy} + \mathcal{R}_3 = 0, \\ G_t - \sigma_3 G_x - \delta F_{xx} - \sigma_1 F_{yy} + \mathcal{R}_4 = 0, \\ \mathcal{D}_t - \frac{1}{M}\nabla \cdot V - \sigma_1 \delta ((G + G_r)\partial_x H_1 - (F + F_r)\partial_x L_1) \\ \quad - \sigma_1^2 ((G + G_r)\partial_y H_2 - (F + F_r)\partial_y L_2) + \mathcal{R}_5 = 0, \\ V_t - \frac{1}{M}\nabla\mathcal{D} + \mathcal{R}_6 = 0, \end{cases} \quad (3.4.5)$$

where

$$\begin{aligned}
\mathcal{R}_1 &= -G\nabla\mathcal{D}_r - \tilde{\mathcal{D}}L - \mathcal{D}L_r \\
&\quad - \left(\sigma_2 - \frac{\sigma_1}{2}\right) \left((\tilde{F}^2 + \tilde{G}^2)L + (F^2 + G^2 + 2FF_r + 2GG_r)L_r \right. \\
&\quad \left. + 2G(\tilde{H}\tilde{F} + \tilde{G}\tilde{L}) + 2G_r(\tilde{H}F + HF_r + \tilde{L}G + LG_r) \right), \\
\mathcal{R}_2 &= F\nabla\mathcal{D}_r + \tilde{\mathcal{D}}H + \mathcal{D}H_r \\
&\quad + \left(\sigma_2 - \frac{\sigma_1}{2}\right) \left((\tilde{F}^2 + \tilde{G}^2)H + (F^2 + G^2 + 2FF_r + 2GG_r)H_r \right. \\
&\quad \left. + 2F(\tilde{H}\tilde{F} + \tilde{G}\tilde{L}) + 2F_r(\tilde{H}F + HF_r + \tilde{G}L + GL_r) \right), \\
\mathcal{R}_3 &= - \left(\left(\sigma_2 - \frac{\sigma_1}{2}\right) (\tilde{F}^2 + \tilde{G}^2) + \tilde{\mathcal{D}} \right) G \\
&\quad - \left(\left(\sigma_2 - \frac{\sigma_1}{2}\right) (F^2 + G^2 + 2FF_r + 2GG_r) + \mathcal{D} \right) G_r, \\
\mathcal{R}_4 &= \left(\left(\sigma_2 - \frac{\sigma_1}{2}\right) (\tilde{F}^2 + \tilde{G}^2) + \tilde{\mathcal{D}} \right) F \\
&\quad + \left(\left(\sigma_2 - \frac{\sigma_1}{2}\right) (F^2 + G^2 + 2FF_r + 2GG_r) + \mathcal{D} \right) F_r, \\
\mathcal{R}_5 &= \left(2c_1\tilde{H}_1 + \sigma_1\delta\Im(\phi_1)_{xx} \right) F + \left(2c_1\tilde{L} - \sigma_1\delta\Re(\phi_1)_{xx} \right) G \\
&\quad + 2c_1(H_1F_r + L_1G_r), \\
\mathcal{R}_6 &= -2 \left(c_2(\tilde{H}_1F + F_rH_1 + \tilde{L}_1G + G_rL_1), c_3(\tilde{H}_2F + \tilde{L}_2G + F_rH_2 + G_rL_2) \right)^T.
\end{aligned}$$

Similarly to the last section, if $\rho_1\delta > 0$, we can change variables as follows

$$H^* = (\sqrt{\delta\sigma_1}H_1, \sigma_1H_2)^T \text{ and } L^* = (\sqrt{\delta\sigma_1}L_1, \sigma_1L_2)^T,$$

and then we set $U = (H^*, L^*, F, G, \mathcal{D}, V)^T$. Therefore, the perturbation of (3.1.2) by the line solitary wave Q is rewritten as a dispersive perturbation of a symmetric hyperbolic system given by

$$\begin{aligned}
U_t + (A_1(U) + B_1(\phi_1) + C_1)U_x + (A_2(U) + B_2(\phi_1) + C_2)U_y \\
+ C(U, Q)U = -K_1U_{xx} - K_2U_{yy},
\end{aligned} \tag{3.4.6}$$

where, with $j \in \{1, 2\}$, A_j, B_j, C_j are symmetric matrices, K_j are skew symmetric and C_j are constant matrices. A_j have the same form as in the proof of Theorem 3.3.1 and B_j have the form

$$B_1(\phi_1) = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & N_1(\phi_1)^T & 0_{3 \times 2} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 2} \\ N_1(\phi_1) & 0_{1 \times 3} & 0 & 0_{1 \times 2} \\ 0_{2 \times 3} & 0_{2 \times 3} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix},$$

with

$$N_1(\phi_1) = (-\sqrt{\delta\sigma_1}G_r, 0, \sqrt{\delta\sigma_1}F_r),$$

$$B_2(\phi_1) = \begin{pmatrix} 0 & 0_{1 \times 3} & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\ 0_{3 \times 1} & 0_{3 \times 3} & 0_{3 \times 2} & N_2(\phi_1)^T & 0_{3 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \\ 0 & N_2(\phi_1) & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix}$$

with

$$N_2(\phi_1) = (-\sigma_1 G_r, 0, \sigma_1 F_r).$$

Furthermore, note that the matrix $C(U, Q)$ contains the term which depends only on $(\phi_1, \phi_2, \partial_x \phi_3)$ making the following analysis hold for both cases when Q is the bright or the dark soliton.

We have written the perturbation of (3.1.2) by the line solitary wave Q in the form of a symmetric hyperbolic system. Applying the same method as in the proof of Theorem 3.3.1 we obtain the following result.

Theorem 3.4.1. *Let $\delta\sigma_1 > 0$ and $s > 2$. For any initial data $(\psi_0, \rho_0, \phi_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exists $T > 0$ such that (3.1.2), when it is perturbed by the line soliton $Q = (\phi_1, \phi_2, \phi_3)$, admits a unique solution $(\psi, \rho, \phi) \in C([0, T]; H^{s+1}(\mathbb{R}^2)) \times C([0, T]; H^s(\mathbb{R}^2)) \times C([0, T]; H^{s+1}(\mathbb{R}^2))$.*

Proof. The proof of Theorem 3.4.1 and Theorem 3.3.1 are essentially the same except the estimates for the terms $B_j(\phi_1)$ and $\tilde{C}(Q)$.

In the proof of Theorem 3.3.1, in order to estimate the derivative of order s , we use the commutator estimate and the Bessel potential $J^s = (1 - \Delta)^{s/2}$. Although, in this case, the 1-D soliton solution Q does not decay in the “ y ” direction which makes that argument not true. Therefore, we use the fact that $J^s \sim J_x^s + J_y^s$, where $J_x = \mathcal{F}^{-1} \langle \xi_1 \rangle \mathcal{F}$, $J_y = \mathcal{F}^{-1} \langle \xi_2 \rangle \mathcal{F}$. Hence, we only need to estimate $J_x^s U$ and $J_y^s U$ instead of $J^s U$. Since Q is independent of y , J_y^s is harmless and since $\|J_x^s U\|_{L^2} = \left\| \|J_x^s U\|_{L_x^2} \right\|_{L_y^2}$, we can apply the commutator estimate in one dimensional case.

The rest of the proof proceeds exactly as in the proof of Theorem 3.3.1. Again we emphasize that the same result holds true *mutatis mutandi* in a $\mathbb{R} \times \mathbb{T}$ setting, a framework that would be needed to study the stability of the line soliton with respect to periodic transverse perturbations. $\square$

3.5 Global solution

In this section we will establish the conservation of energy for the perturbation of (3.1.2) by the line soliton Q given in Section 3.2 and as a consequence, the existence of a global weak solution when Q is the dark soliton.

In order to make the calculation easier, we will consider the solution of the form

$$(e^{i\lambda t} e^{i\frac{\sigma_3}{2\delta} x} \psi(x, y, t), \rho(x, y, t), \phi(x, y, t)),$$

then the 1-d solitary wave has the following form

$$Q = (\phi_1, \phi_2, \phi_3) = (R(x), aR^2(x), bP(x)),$$

with $R(x)$, $P(x)$ are given in Section 3.2 (note that this trick will not affect the analysis in section 3.4). Then the system (3.4.1) becomes

$$\begin{cases} \psi_t + i(\lambda - \frac{\sigma_3^2}{4\delta})\psi - i\delta\psi_{xx} - i\sigma_1\psi_{yy} \\ \quad + i\left\{\sigma_2|\psi|^2 + W(\rho + D\rho_x) + 2\sigma_2\phi_1\Re(\psi) + \sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right\}\psi \\ \quad + i\{\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1\Re(\psi)\}\phi_1 = 0, \\ \rho_t + \Delta\phi + D(|\psi|^2 + 2\phi_1\Re(\psi))_x = 0, \\ \phi_t + \frac{1}{M^2}\rho + |\psi|^2 + 2\phi_1\Re(\psi) = 0. \end{cases} \quad (3.5.1)$$

In this section we will establish the energy conservation for (3.5.1) when it is perturbed by a line soliton Q and the existence of a global weak solution when Q is the dark soliton.

Theorem 3.5.1. *Let (ψ, ρ, ϕ) be a solution of the system (3.5.1) obtained in Theorem 3.4.1, defined in the time interval $[0, T]$. Then the quantity*

$$\begin{aligned} E = & (\lambda - \frac{\sigma_3^2}{4\delta}) \|\psi\|_{L^2}^2 + \delta \|\psi_x\|_{L^2}^2 + \sigma_1 \|\psi_y\|_{L^2}^2 \\ & + \frac{\sigma_2}{2} \left\| |\psi|^2 + 2\phi_1\Re(\psi) \right\|_{L^2}^2 + \sigma_2 \|\phi_1\psi\|_{L^2}^2 + \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \frac{W}{2} \|\nabla\phi\|_{L^2}^2 \\ & - \int W(M^2 + D^2)|\phi_1|^2|\psi|^2 + \int W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1\Re(\psi)), \end{aligned} \quad (3.5.2)$$

is conserved for $t \in [0, T]$.

Proof. We multiply the first equation in (3.5.1) by $\partial_t\bar{\psi}$, integrate the result and take its imaginary part to get successively

$$(\lambda - \frac{\sigma_3^2}{4\delta})\Re \int \psi\bar{\psi}_t = \frac{1}{2}(\lambda - \frac{\sigma_3^2}{4\delta}) \int (|\psi|^2)_t, \quad (3.5.3)$$

$$- \delta\Re \int \psi_{xx}\bar{\psi}_t = \delta\Re \int \psi_x\bar{\psi}_{xt} = \frac{1}{2}\delta \int (|\psi_x|^2)_t, \quad (3.5.4)$$

$$- \sigma_1\Re \int \psi_{yy}\bar{\psi}_t = \frac{1}{2}\sigma_1 \int (|\psi_y|^2)_t, \quad (3.5.5)$$

$$\begin{aligned} & \Re \int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1\Re(\psi) + \sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3) \right) \psi\bar{\psi}_t \\ = & \frac{1}{2} \int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1\Re(\psi) + \sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3) \right) (|\psi|^2)_t \\ = & \frac{1}{2} \int \frac{1}{2} \sigma_2 (|\psi|^4)_t + W(\rho + D\phi_x)(|\psi|^2)_t + 2\sigma_2\phi_1\Re(\psi)(|\psi|^2)_t \\ & + \left((\sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)) |\psi|^2 \right)_t, \end{aligned} \quad (3.5.6)$$

$$\begin{aligned} & \Re \int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1\Re(\psi) \right) \phi_1\bar{\psi}_t \\ = & \int \sigma_2|\psi|^2(\phi_1\Re(\psi))_t + W(\rho + D\phi_x)(\phi_1\Re(\psi))_t + \sigma_2(|\phi_1\Re(\psi)|^2)_t. \end{aligned} \quad (3.5.7)$$

Combining (3.5.3), (3.5.4), (3.5.5), (3.5.6) and (3.5.7) we obtain

$$\begin{aligned}
0 = & \frac{d}{dt} \int \left(\frac{1}{2} \left(\lambda - \frac{\sigma_3^2}{4\delta} \right) |\psi|^2 + \frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\psi_y|^2 + \frac{\sigma_2}{4} |\psi|^4 \right. \\
& + \sigma_2 |\psi|^2 \phi_1 \Re(\psi) + \frac{1}{2} \left(\sigma_2 |\phi_1|^2 + W(\phi_2 + D\partial_x \phi_3) \right) |\psi|^2 + \sigma_2 |\phi_1 \Re(\psi)|^2 \\
& + \frac{1}{2} W(\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)) \Big) \\
& - \frac{1}{2} \int W(\rho_t + D\phi_{xt}) (|\psi|^2 + 2\phi_1 \Re(\psi)).
\end{aligned}$$

From the second and the third equation in (3.5.1), we get

$$\int \rho_t (|\psi|^2 + 2\phi_1 \Re(\psi)) = - \int \rho_t (\phi_t + \frac{1}{M^2} \rho) = - \int \rho_t \phi_t - \frac{1}{2M^2} \int \rho_t^2,$$

and

$$\begin{aligned}
\int D\phi_{xt} (|\psi|^2 + 2\phi_1 \Re(\psi)) &= - \int \phi_t D(|\psi|^2 + 2\phi_1 \Re(\psi))_x \\
&= \int \phi_t (\rho_t + \Delta \phi) \\
&= \int \phi_t \rho_t - \frac{1}{2} \int (|\nabla \phi|^2)_t.
\end{aligned}$$

That implies

$$\begin{aligned}
0 = & \frac{d}{dt} \int \left(\frac{1}{2} \left(\lambda - \frac{\sigma_3^2}{4\delta} \right) |\psi|^2 + \frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\psi_y|^2 + \frac{\rho_2}{4} |\psi|^4 \right. \\
& + \sigma_2 |\psi|^2 \phi_1 \Re(\psi) + \frac{1}{2} \left(\sigma_2 |\phi_1|^2 + W(\phi_2 + D\partial_x \phi_3) \right) |\psi|^2 + \sigma_2 |\phi_1 \Re(\psi)|^2 \\
& + \frac{1}{2} W(\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)) \\
& \left. + \frac{W}{4M^2} \rho^2 + \frac{W}{4} |\nabla \phi|^2 \right).
\end{aligned}$$

Finally, we get the conserved energy

$$\begin{aligned}
E = & \left(\lambda - \frac{\sigma_3^2}{4\delta} \right) \|\psi\|_{L^2}^2 + \delta \|\psi_x\|_{L^2}^2 + \sigma_1 \|\psi_y\|_{L^2}^2 + \frac{\sigma_2}{2} \|\psi\|_{L^4}^4 \\
& + 2\sigma_2 \int |\psi|^2 \phi_1 \Re(\psi) + \sigma_2 \|\phi_1 \psi\|_{L^2}^2 + 2\sigma_2 \|\phi_1 \Re(\psi)\|_{L^2}^2 \\
& + \int W(\phi_2 + D\partial_x \phi_3) |\psi|^2 + \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \frac{W}{2} \|\nabla \phi\|_{L^2}^2 \\
& + \int W(\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)) \\
= & \left(\lambda - \frac{\sigma_3^2}{4\delta} \right) \|\psi\|_{L^2}^2 + \delta \|\psi_x\|_{L^2}^2 + \sigma_1 \|\psi_y\|_{L^2}^2 \\
& + \frac{\sigma_2}{2} \left\| |\psi|^2 + 2\phi_1 \Re(\psi) \right\|_{L^2}^2 + \sigma_2 \|\phi_1 \psi\|_{L^2}^2 + \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \frac{W}{2} \|\nabla \phi\|_{L^2}^2 \\
& - \int W(M^2 + D^2) |\phi_1|^2 |\psi|^2 + \int W(\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)).
\end{aligned}$$

Note that we got rid of the terms involving ϕ_2 and ϕ_3 since by (3.2.3) one has $a + Db = -(M^2 + D^2)$ implying that $\phi_2 + D\partial_x \phi_3 = -(M^2 + D^2)\phi_1^2$. $\square$

Theorem 3.5.2. Assume that Q is the dark soliton given by (3.2.8) with wave speed $c = 0$.

i) Let (ψ, ρ, ϕ) be the solution of (3.5.1) obtained by Theorem 3.4.1 with existence time interval $[0, T]$. Then for all $t \in [0, T]$ we have

$$\|\psi(t)\|_{H^1} + \|\rho(t)\|_{L^2} + \|\phi(t)\|_{H^1} \leq C(t). \quad (3.5.8)$$

ii) For any $(\psi_0, \rho_0, \phi_0) \in H^1 \times L^2 \times H^1$, there exists a global weak solution (ψ, ρ, ϕ) of (3.5.1) such that for any $T > 0$

$$\begin{aligned} \psi, \phi &\in L^\infty([0, T]; H^1), \quad \rho \in L^\infty([0, T]; L^2) \\ \psi_t, \rho_t &\in L^\infty([0, T]; H^{-1}), \quad \phi_t \in L^\infty([0, T]; L^2). \end{aligned} \quad (3.5.9)$$

Proof. i) For any $\varepsilon \in (0, 1)$, using Cauchy's inequality we have

$$\left| \int W \rho (|\psi|^2 + 2\phi_1 \Re(\psi)) \right| \leq \frac{W}{2M^2} (1 - \varepsilon) \int \rho^2 + \frac{M^2 W}{2(1 - \varepsilon)} \int (|\psi|^2 + 2\phi_1 \Re(\psi))^2$$

and

$$\left| \int W D \phi_x (|\psi|^2 + 2\phi_1 \Re(\psi)) \right| \leq \frac{W}{2} (1 - \varepsilon) \int |\phi_x|^2 + \frac{W D^2}{2(1 - \varepsilon)} \int (|\psi|^2 + 2\phi_1 \Re(\psi))^2,$$

then

$$\begin{aligned} &\left| \int W (\rho + D \phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)) \right| \\ &\leq \frac{W(M^2 + D^2)}{2(1 - \varepsilon)} \left\| |\psi|^2 + 2\phi_1 \Re(\psi) \right\|_{L^2}^2 + \frac{W}{2M^2} (1 - \varepsilon) \|\rho\|_{L^2}^2 \\ &\quad + \frac{W}{2} (1 - \varepsilon) \|\nabla \phi\|_{L^2}^2. \end{aligned}$$

Note that we are considering the stationary dark soliton, so that the condition (3.2.7) becomes

$$\begin{cases} \frac{1}{\delta} \left(\frac{\sigma_3^2}{4\delta} - \lambda \right) > 0, \\ \frac{1}{\delta} \left(\sigma_2 - W(M^2 + D^2) \right) > 0, \end{cases}$$

so there exists $\varepsilon > 0$ small enough such that

$$\sigma_2 > \frac{W(M^2 + D^2)}{1 - \varepsilon}.$$

Therefore, the conservation law (3.5.2) implies

$$\begin{aligned} &\frac{1}{2} \left(\sigma_2 - \frac{W(M^2 + D^2)}{1 - \varepsilon} \right) \left\| |\psi|^2 + 2\phi_1 \Re(\psi) \right\|_{L^2}^2 + \varepsilon \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \varepsilon \frac{W}{2} \|\nabla \phi\|_{L^2}^2 \\ &\leq E + \left(\frac{\sigma_3^2}{4\delta} - \lambda \right) \|\psi\|_{L^2}^2. \end{aligned} \quad (3.5.10)$$

From now we will fix such an ε and define

$$c_1 = \frac{1}{2} \left(\sigma_2 - \frac{W(M^2 + D^2)}{1 - \varepsilon} \right), \quad c_2 = \varepsilon \frac{W}{2M}, \quad c_3 = \varepsilon \frac{W}{2}.$$

The first equation of (3.5.1) implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 &= \Im \int (\sigma_2(|\psi|^2 + 2\phi_1 \Re(\psi)) + W(\rho + D\phi_x)) \phi_1 \bar{\psi} \\ &\leq c_1 \left\| |\psi|^2 + 2\phi_1 \Re(\psi) \right\|_{L^2}^2 + c_2 \|\rho\|_{L^2}^2 + c_3 \|\phi_x\|_{L^2}^2 \\ &\quad + \left(\frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2 D^2}{4c_3} \right) \|\phi_1 \Im(\psi)\|_{L^2}^2. \end{aligned}$$

Combining with (3.5.10) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 &\leq E + \left(\frac{\sigma_3^2}{4\delta} - \lambda \right) \|\psi\|_{L^2}^2 \\ &\quad + \left(\frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2 D^2}{4c_3} \right) \|\phi_1 \Im(\psi)\|_{L^2}^2. \end{aligned}$$

Since $\phi_1 \in L^\infty$, $\|\phi_1 \Im(\psi)\|_{L^2}$ is under control and we recall that $\frac{\sigma_3^2}{4\delta} - \lambda > 0$. Hence, by Gronwall's inequality we can bound $\|\psi(t)\|_{L^2}$ by a constant depending on t . The bounds on $\|\nabla \psi\|_{L^2}$, $\|\rho\|_{L^2}$, $\|\nabla \phi\|_{L^2}$ then follow from the energy conservation.

ii) We shall use a classical compactness method (see for instance [20]). The estimates in (i) prove that regular solutions of (3.5.1) are uniformly bounded in $H^1 \times L^2 \times H^1$. To obtain global weak solutions, as in [37], we first implement a Galerkin approximation process (possibly after smoothing the initial data), yielding a sequence of approximate solutions (ψ_m, ρ_m, ϕ_m) of (3.5.1). Using part i), we have that for any $T > 0$, $(\psi_\eta, \rho_\eta, \phi_\eta)$ is bounded, independently of η , in the space

$$L^\infty((0, T); H^1) \times L^\infty((0, T); L^2) \times L^\infty((0, T); H^1). \quad (3.5.11)$$

Therefore, using (3.5.1) and Sobolev's theorem we infer that $(\partial_t \psi_m, \partial_t \rho_m, \partial_t \phi_m)$ is bounded in

$$L^\infty((0, T); H^{-1}) \times L^\infty((0, T); H^{-1}) \times L^\infty((0, T); L^2).$$

Hence, up to a subsequence, one can assume that

$$\begin{aligned} \psi_m &\rightarrow \psi \text{ in } L^\infty((0, T); H^1) - \text{weak}^*, \\ \rho_m &\rightarrow \rho \text{ in } L^\infty((0, T); L^2) - \text{weak}^*, \\ \phi_m &\rightarrow \phi \text{ in } L^\infty((0, T); H^1) - \text{weak}^*. \end{aligned} \quad (3.5.12)$$

By Aubin-Lions lemma one can furthermore assume that up to a subsequence

$$\psi_m \rightarrow \psi \text{ in } L_{loc}^p([0, T]; L_{loc}^q(\mathbb{R}^2)), \quad (3.5.13)$$

for any $2 \leq p, q < \infty$. Similar convergence results hold true for ϕ_m, ρ_m .

These convergences allow to pass to the limit in the distribution sense in (3.5.1) for (ψ_m, ρ_m, ϕ_m) , proving that (ψ, ρ, ϕ) satisfies (3.5.1) in $L^\infty((0, T); H^{-1}) \times L^\infty((0, T); H^{-1}) \times L^\infty((0, T); L^2)$. The initial condition makes sense since

$$(\psi, \rho, \phi) \in C_w([0, T]; H^1) \times C_w([0, T]; L^2) \times C_w([0, T]; H^1).$$

□

3.6 Solitary wave solutions in higher dimension

Let consider now solitary waves solutions of (3.1.2). i.e. solutions of the form $(e^{i\omega t}\psi(x+\sigma_3 t, y), \phi(x+\sigma_3 t, y), \eta(x+\sigma_3 t, y))$, $\omega \in \mathbb{R}$, $\psi \in H^1(\mathbb{R}^2)$, yielding the system

$$\begin{cases} -\omega\psi + \delta\psi_{xx} + \sigma_1\psi_{yy} - (\sigma_2 - WM^2)|\psi|^2\psi - c\phi_x\psi = 0, \\ (1 - \sigma_3^2 M^2)\phi_{xx} + \phi_{yy} + (\sigma_2 - WM^2)(|\psi|^2)_x = 0, \end{cases} \quad (3.6.1)$$

which is similar to the equation for the solitary wave solutions of the elliptic/hyperbolic-elliptic Davey-Stewartson systems in the terminology of [12]. By Pohojaev type arguments one obtains (see [11] for similar arguments) that non trivial solutions to (3.6.1) cannot exist when $\delta\sigma_1 < 0$.

On the other hand the existence of non trivial solutions to (3.6.1) has been established ([4]) in the *focusing case*

$$\delta\sigma_1 > 0, \quad c < 0, \quad cW(\sigma_2 - WM^2) < 0.$$

Various stability and instability results of solutions to (3.6.1) have been obtained in [5, 27, 26, 25] in the context of the Davey-Stewartson systems, but no similar results seemed to be known when they are viewed as solutions to the Zakharov-Rubenchik systems. In particular one does not know if the solutions of (3.6.1) are constrained minimizers of the Zakharov-Rubenchik system.

According to the Davey-Stewartson case, one could conjecture that those localized solitary waves are unstable.

3.7 Conclusion and open questions

We have addressed in this paper some issues on the Zakharov-Rubenchik, Benney-Roskes systems. Many questions remain unsolved for those important systems and we indicate a few below.

1. Justify rigorously the limit of ZR (BR) systems to the Davey-Stewartson systems. This is non trivial (because of a boundary layer at $t = 0$) in analogy (but more delicate) to the Schrödinger limit of the Zakharov system (see [30, 31]).

2. The present work can be viewed as a preliminary step towards the study of the transverse stability/instability of the ZR or BR one dimensional dark or bright solitary wave. Perturbations could be localized in (x, y) or periodic in y . The Cauchy problem was addressed in the present paper in both cases.

In both the functional settings, we plan to come back to those transverse stability issues a subsequent work, in the spirit of [35, 33, 34, 22, 23].

3. It is known ([14, 13]) that (radially symmetric) solutions of the Zakharov system may blow up in finite time. Such a result is unknown for the Zakharov-Rubenchik, Benney-Roskes system and it would be interesting to see if the results in [14, 13] extend to (3.1.1).

4. The existence result Theorem 3.3.1 is established when $\delta\sigma_1 > 0$, that is when the Schrödinger equation in (3.1.1) is not an "non elliptic" one in the terminology of [12]. This condition is never satisfied in the context of purely gravity waves water waves (see [18] and the discussion above) and it would interesting to relax it.

5. We recall that an existence result on time scales of order $1/\epsilon$ is needed to fully justify the Benney-Roskes system. Obtaining such a result is still a challenging open problem.

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Chapter 4

On a regularized full dispersion Davey-Stewartson system

Joint work with Jean-Claude Saut. (*Work in progress*).

Abstract: This paper which continues [24] is concerned with the Cauchy problem for the Full-Dispersion Davey-Stewartson system introduced in [19]. More precisely, in order to invert the equation for the mean mode ϕ we introduce an approximate system by a suitable truncation of the high frequencies of ϕ and we prove that the solutions to the truncated systems converge to the solutions of the classical Davey-Stewartson system.

4.1 Introduction

The Davey-Stewartson system was introduced in [9] (see also [3, 10]) to model the evolution of wave packets of surface water waves. It is actually derived for a more general system, the Benney-Roskes system, introduced in [16] in the context of water waves and also known as the Zakharov-Rubenchik system, introduced in [34] as a universal Hamiltonian system to describe the interaction of short and (acoustic type) long waves.

Our interest in the present paper is a full dispersion version of the classical Davey-Stewartson system introduced by David Lannes in [19] and that we describe now.

The framework is that of the motion of a irrotational inviscid fluid in an infinite layer of finite depth of an Euler fluid bounded by a free surface and a flat bottom.

The specific regime here is the so-called modulation ("Schrödinger") one. In order to define the key small parameter we introduce h a typical depth of the fluid layer, a a typical amplitude of the wave and λ a typical wavelength in the horizontal directions (assumed to be isotropic). Then we set $\epsilon = a/h$ and $\mu = h^2/\lambda^2$.

In the modulation regime, the relevant small parameter is the *wave steepness*, $\varepsilon = a/\lambda = \epsilon\sqrt{\mu}$. The starting point of the analysis is the water waves system coupling the wave elevation ζ and the trace ψ of the velocity potential at the free surface, written in dimensionless variables (see [19])

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} \mathcal{G} \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \varepsilon \frac{\left(\frac{1}{\sqrt{\mu}} \mathcal{G} \psi + \varepsilon \nabla \zeta \cdot \nabla \psi \right)^2}{2(1 + \varepsilon^2 |\nabla \zeta|^2)} = 0, \end{cases} \quad (4.1.1)$$

where $\mathcal{G}$ is the Dirichlet to Neumann operator which links the trace of the velocity potential at the free surface to its normal derivative at the same surface. We denote $X = (x, y)$ or x the spatial variable.

One introduces a fixed wave vector $\mathbf{k} \in \mathbb{R}^d, d = 1, 2$. Setting $\omega = \omega(\mathbf{k}) = (|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|))^{1/2}$ the dispersion relation of surface gravity waves, one looks for an approximate solution of the water wave system of the form

$$U_{\text{app}}(X, t) = U_0(X, t) + \varepsilon U_1(X, t) + \varepsilon U^2(X, t), \quad (4.1.2)$$

where $X = (x, y)$ or x and U_0 is a sum of a modulated wave packet and of an induced mean mode ϕ

$$U_0(X, t) = \begin{pmatrix} i\omega\psi(\varepsilon X, \varepsilon t) \\ \psi(\varepsilon X, \varepsilon t) \end{pmatrix} e^{i(\mathbf{k} \cdot X - \omega t)} + \text{c.c.} + \begin{pmatrix} 0 \\ \phi(\varepsilon X, \varepsilon t) \end{pmatrix}. \quad (4.1.3)$$

In [19], D. Lannes derived the following Full Dispersion Benney-Roskes (FDBR) system coupling

ψ, ϕ and the leading term ζ of the surface elevation (see [23] for the case of capillary-gravity waves)

$$\begin{cases} \partial_t' \psi + i \frac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon} \psi \\ + \varepsilon i [\mathbf{k} \cdot \nabla' \phi + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2] \psi = 0, \\ \partial_t' \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2\omega \mathbf{k} \cdot \nabla' (|\psi|^2), \\ \partial_t' \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2, \end{cases} \quad (4.1.4)$$

where $\mathbf{k}$ is the given wave vector, $\sigma = \tanh(\sqrt{\mu} |\mathbf{k}|)$ and $\alpha = -\frac{9}{8\sigma^2} (1 - \sigma^2)^2$.

Here the time and space derivatives (indicated by a ') are taken with respect to the slow time and space variables $t' = \varepsilon t$ and $X' = \varepsilon X$ and D' denotes the operator $\frac{1}{i} \nabla'$.

Without loss of generality one may assume that $\mathbf{k}$ is oriented in the x direction, that is $\mathbf{k} = |\mathbf{k}| e_x$. We will thus write (4.1.4) as

$$\begin{cases} \partial_t' \psi + i \frac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon} \psi \\ + \varepsilon i [|\mathbf{k}| \partial_x' \phi + \frac{|\mathbf{k}|^2}{2\omega} (1 - \sigma^2) \zeta + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2] \psi = 0, \\ \partial_t' \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2\omega |\mathbf{k}| \partial_x' (|\psi|^2), \\ \partial_t' \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2, \end{cases} \quad (4.1.5)$$

Remark 14. It is proven in [19] that (4.1.4) (or (4.1.5)) is consistent with the full water wave system. We recall that the notion of consistency is a "static" (not dynamic) one. It amounts to saying that an assumed solution of the water waves system on some time interval $[0, T]$ solves the BRFD or FDBR system on the same time interval up to some error term. A full justification of the BRFD system requires in particular that solutions of (4.1.4) exist on a time scale of order $O(1/\varepsilon)$ which is an open problem.

It also necessitates to prove the well-posedness of the full water -wave system in the modulation regime on the correct time scales, which has been proven in one and two horizontal spatial dimensions in [33, 32] in the infinite depth case. The finite depth case is up to our knowledge open.

The obvious advantage of (4.1.4) (or (4.1.5)) on the classical¹ Benney-Roskes system (see [16, 19]) is its a priori validity over an extended range of frequencies.

The full dispersion Benney-Roskes system with surface tension (capillary-gravity waves) is given by (see [23])

$$\begin{cases} \partial_t' \psi + i \frac{\omega_S(\mathbf{k} + \varepsilon D') - \omega_S(\mathbf{k})}{\varepsilon} \psi \\ + i \varepsilon [|\mathbf{k}| \partial_x' \phi + \frac{|\mathbf{k}|^2 \kappa}{2\omega_S} (1 - \sigma^2) \zeta - \frac{2|\mathbf{k}|^4}{\omega_S} \gamma |\psi|^2] \psi = 0, \\ \partial_t' \zeta - |D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} \phi = -2 \frac{\omega_S}{\kappa} |\mathbf{k}| \partial_x' (|\psi|^2), \\ \partial_t' \phi + \zeta = -|\mathbf{k}|^2 (1 - \sigma^2) |\psi|^2. \end{cases} \quad (4.1.6)$$

Here, the dispersion relation for the water waves with surface tension is given by

$$\omega_S(\xi) = [|\xi| \tanh(\sqrt{\mu} |\xi|) (1 + B |\xi|^2)]^{1/2},$$

¹We will use the term "classical" to refer to those systems where the nonlocal dispersion has been Taylor-expanded yielding *local* differential operators instead of *nonlocal* ones.

where $B > 0$ measures the surface tension effects (Bond number).

As for purely gravity waves, $\mathbf{k} = |\mathbf{k}|e_x$ is a given wave vector. We will use the notations $\omega_S = \omega_S(\mathbf{k})$, $\kappa = \kappa(\mathbf{k}) = 1 + B|\mathbf{k}|^2$ and $\sigma = \tanh(\sqrt{\mu}|\mathbf{k}|)$. γ is a constant depending on $\mathbf{k}$ and B .

The classical Benney-Roskes system [16] is obtained from (4.1.4) or (4.1.6) by approximating the nonlocal operators $eg i\omega(\mathbf{k} + \varepsilon D')$ and $\frac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon}$ as (denoting $\mathcal{H}_\omega(\mathbf{k})$ the Hessian of ω at $\mathbf{k}$)

$$i\omega(\mathbf{k} + \varepsilon D') \sim \varepsilon \nabla \omega(\mathbf{k}) \cdot \nabla' - \frac{i\varepsilon^2}{2} \nabla' \mathcal{H}_\omega(\mathbf{k}) \nabla',$$

and

$$|D'| \frac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} \sim \sqrt{\mu}|D'|^2.$$

The Cauchy problem for the Benney-Roskes system² is proven in [28] to be locally well-posed, but on time scales which do not reach the order $O(1/\varepsilon)$ that would be required (together with uniform bounds in ε) for the full justification of the system, see [19]. We refer to [22] for some results on the Cauchy problem for the full dispersion Benney-Roskes system.

As explained in [19] in the "classical" case, the Davey-Stewartson model is obtained from the Benney-Roskes system by observing that at leading order, ψ travels at the group speed velocity $c_g = \nabla \omega(\mathbf{k})$. One thus looks for solutions of (4.1.4) under the form

$$\begin{aligned} \psi(X', t') &= \tilde{\psi}(X' - c_g t, \varepsilon t'), \\ \phi(X', t') &= \tilde{\phi}(X' - c_g t, \varepsilon t') + \phi^*(X', t', \varepsilon t'), \\ \zeta(X', t') &= \tilde{\zeta}(X' - c_g t', \varepsilon t') + \zeta^*(X', t', \varepsilon t'). \end{aligned}$$

Neglecting $O(\varepsilon)$ correctors, one can replace ∂_t by $-c_g \cdot \nabla'$, and one is finally led (see details in [19]) to the *Full Dispersion Davey-Stewartson System* (FDDS) for gravity waves, where $\tau = \varepsilon t' = \varepsilon^2 t$ (from now on, we will drop the tildas in the FDDS systems)

$$\begin{cases} \partial_\tau \psi + \frac{i}{\varepsilon^2} [\omega(\mathbf{k} + \varepsilon D') - \omega - \varepsilon \omega' D'_x] \psi \\ + i(\beta \partial'_x \phi + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \tilde{\alpha}) |\psi|^2) \psi = 0, \\ \left(\frac{|D'| \tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} + \omega'^2 \partial_x'^2 \right) \phi = 2\omega \beta \partial'_x (|\psi|^2). \end{cases} \quad (4.1.7)$$

We have used here the following notations (see [19])

$$\begin{cases} \mathbf{k} = |\mathbf{k}|e_x, \quad \omega(\mathbf{k}) = \tilde{\omega}(|\mathbf{k}|), \quad \text{with} \quad \tilde{\omega}(r) = (r \tanh(\sqrt{\mu}r))^{1/2}, \\ \omega = \tilde{\omega}(|\mathbf{k}|), \quad \omega' = \tilde{\omega}'(|\mathbf{k}|), \quad \omega'' = \tilde{\omega}''(|\mathbf{k}|), \\ \sigma = \tanh(\sqrt{\mu}|\mathbf{k}|), \quad \alpha = -\frac{9}{8\sigma^2} (1 - \sigma^2)^2, \quad \tilde{\alpha} = \alpha + \frac{1}{4} (1 - \sigma^2)^2, \\ \beta = |\mathbf{k}| \left(1 + (1 - \sigma^2) \frac{\omega' |\mathbf{k}|}{2\omega} \right). \end{cases} \quad (4.1.8)$$

We also recall that the Fourier multiplier operator $\omega(\mathbf{k} + \varepsilon D')$ is given by

$$\omega(\mathbf{k} + \varepsilon D') = (|\mathbf{k} + \varepsilon D'| \tanh(\sqrt{\mu}|\mathbf{k} + \varepsilon D'|))^{1/2}.$$

²a similar system is derived in [34] in the general context of interactions between low and high-frequency waves. See also [30], Chap. 15, for a similar system describing the interaction of a laser beam with plasma low-frequency fluctuations driven by ponderomotive forces.

Note (see [19]) that the complete rigorous justification of the full dispersion Davey-Stewartson system requires just the well-posedness of the Cauchy problem for (4.1.16) on time scales $\tau = O(1)$. For capillary-gravity waves, the full dispersion Davey-Stewartson system writes

$$\begin{cases} \psi_\tau + \frac{i}{\varepsilon^2}[\omega_S(\mathbf{k} + \varepsilon D') - \omega_S - \varepsilon \omega'_S D'_x] \psi \\ + i(\beta_S \partial'_x \phi - 2 \frac{|\mathbf{k}|^4}{\omega_S} \tilde{\gamma} |\psi|^2) \psi = 0, \\ \left(|D'| \frac{\tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} + (\omega'_S)^2 \partial_x'^2 \right) \phi = 2 \frac{\omega_S}{\kappa} \beta_S \partial'_x (|\psi|^2), \end{cases} \quad (4.1.9)$$

with

$$\beta_S = |\mathbf{k}|(1 + (1 - \sigma^2) \frac{\omega'_S |\mathbf{k}|}{2\omega_S}), \quad \tilde{\gamma} = \gamma + \frac{\kappa}{4}(1 - \sigma^2)^2.$$

The classical Davey-Stewartson system which was derived formally in [9] for gravity waves and in [3, 10] for capillary-gravity waves has been justified rigorously in the sense of consistency in [7, 8, 6].³ It is obtained by approximating the nonlocal operators in (4.1.16) by the relevant local ones as indicated above for the Full dispersion Benney-Roskes system.

We recall (see [12]) that the Davey-Stewartson systems have the general form

$$\begin{cases} i\partial_t \psi + a\partial_x^2 \psi + b\partial_y^2 \psi = (\nu_1 |\psi|^2 + \nu_2 \partial_x \phi) \psi, \\ c\partial_x^2 \phi + \partial_y^2 \phi = -\delta \partial_x (|\psi|^2). \end{cases} \quad (4.1.10)$$

We refer to [3] for an explication of $a, b, c, \nu_1, \nu_2, \delta$ in terms of the physical parameters. It is also convenient to use the normalized form (see [12])

$$\begin{cases} i\partial_t \psi + \partial_x^2 \psi + \delta \partial_y^2 \psi = (\chi |\psi|^2 + b \partial_x \phi) \psi, \\ \partial_x^2 \phi + m \partial_y^2 \phi = \partial_x (|\psi|^2), \end{cases} \quad (4.1.11)$$

where the parameters δ, χ, b and m are real and can assume both signs, and δ, χ have been normalized in such a way that $|\delta| = |\chi| = 1$.

The Davey-Stewartson systems can be classified as "elliptic-elliptic", "hyperbolic-elliptic", "elliptic-hyperbolic" and "hyperbolic-hyperbolic" depending on the sign of δ, m .

The "hyperbolic-hyperbolic" case does not occur in the context of water waves. The "elliptic-elliptic" case may lead to a blow-up phenomenon reminiscent of that of the cubic focusing NLS (see [12] for a proof of blow-up and [26] for numerical simulations analyzing the dynamics of the singularity formation).

In the case of purely gravity waves, one has $b > 0$ and $\delta > 0$. Moreover, $a = \frac{1}{2}\omega'' < 0$ since $\omega'' = \tilde{\omega}''(|\mathbf{k}|) < 0$ by the concavity of $\tilde{\omega}$.

On the other hand, $c = \sqrt{\mu} - \omega'^2 > 0$ (see Section 2). This is a *subsonic* condition. Actually this amounts to saying that $M < 1$ where the Mach number M is defined as $M = C_g / \sqrt{g h}$, where $c_g = \frac{d}{dr} \sqrt{gr \tanh(hr)}|_{r=|\mathbf{k}|}$.

One can thus invert in this case the equation for ϕ , writing (one can assume $c = 1$ without loss of generality) $\phi_x = \delta R_1^2(|\psi|^2)$, where R_1 is the Riesz transform $R_1 = \partial_x (-\Delta)^{-1/2}$, and reduce (4.1.20) to a single semilinear (cubic) Schrödinger type equation

$$i\partial_t \psi + a\partial_x^2 \psi + b\partial_y^2 \psi = (\nu_1 |\psi|^2 + \nu_2 \delta R_1^2(|\psi|^2)) \psi, \quad (4.1.12)$$

³Its "universal" character as an asymptotic system of quadratic hyperbolic systems in the modulation regime has been proved in [5].

which is studied in [12]. In particular, whatever the signs of a and b , the Cauchy problem is locally well-posed for initial data in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$ and globally for small data, as the corresponding cubic NLS. The global Cauchy problem is much less understood. For instance, a global well-posedness result for the *hyperbolic-elliptic* case ($b > 0$, $a < 0$) in (4.1.12) is still missing for arbitrary large initial data.

However, for a very specific choice of coefficients⁴, (4.1.12) is integrable by the inverse scattering transform method leading to qualitative informations on the flow. After scaling, this integrable version of the DS system writes

$$\begin{cases} i\partial_t\psi + \psi_{xx} - \psi_{yy} = \alpha|\psi|^2\psi + \beta\psi\phi_x, \\ \Delta\phi = \partial_x(|\psi|^2), \end{cases} \quad (4.1.13)$$

with

$$\alpha + \frac{\beta}{2} = 0.$$

It is then known as the DS II system, with two subcases, the focussing ($\beta > 0$) and the defocussing one ($\beta < 0$). For the defocussing DS II, one has global solutions that behaves asymptotically in time as the solution of the linear case (see [31, 27, 1]). In particular global well-posedness (and scattering) is proven in [1] with arbitrary L^2 initial data.

On the other hand, in the focussing integrable DS II system, the presence of an explicit localized *lump* solitary wave leads to blow-up in finite time by a pseudo-conformal law [25]. This later behavior does not seem to be generic, that is does not seem to persist in the non integrable case (see the numerical simulations in [18]). In particular, one can prove rigorously that no localized solitary waves exist "not too close" to the integrable case (see [11]). However, the "scattering" behavior of arbitrary large solutions to the defocussing integrable DS II system proven in [31, 27] might be generic. In both situations, the numerical simulations in [17, 18] are illuminating.

When surface tension is strong enough however, c is no more positive (see [3]) and ϕ is inverted (with suitable conditions at infinity) via a wave equation and $\partial_x\phi$ is expressed in terms of $|\psi|^2$ via an order one nonlocal operator. The integrable version is known as the DS I system. We refer to [2] and the references therein for a description of the results obtained on DS I system by inverse scattering techniques, in particular the existence of special localized solutions called *dromions*. The resulting Schrödinger type equation is no more semilinear (it involves an order one nonlinear term) which makes the study by PDE techniques delicate. The local associated Cauchy problem is studied in particular in [20, 13, 15]. Scattering of small solutions is proven in [14].

For the FDDS system, in contrast to the classical system (4.1.20) for gravity waves ("hyperbolic-elliptic"), one cannot invert in a straightforward fashion the equation for ϕ which makes the associated Cauchy problem delicate.

As suggested in [19] one could consider as well the full dispersion version of only the first or the second of the two equations of the FDDS system (4.1.16), namely

$$\begin{cases} \partial_\tau\psi + \frac{i}{\varepsilon^2}[\omega(\mathbf{k} + \varepsilon D') - \omega - \varepsilon\omega' D'_x]\psi \\ + i(\beta\partial'_x\phi + 2\frac{|\mathbf{k}|^4}{\omega}(1 - \tilde{\alpha})|\psi|^2)\psi = 0, \\ [(\sqrt{\mu} - \omega'^2)\partial_x'^2 + \sqrt{\mu}\partial_y'^2]\phi = -2\omega\beta\partial'_x(|\psi|^2), \end{cases} \quad (4.1.14)$$

⁴with a limited physical relevance

or

$$\begin{cases} \partial_t \psi - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial_x \phi + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \tilde{\alpha}) |\psi|^2) \psi = 0, \\ \left(\frac{|D'| \tanh(\varepsilon \sqrt{\mu} |D'|)}{\varepsilon} + \omega'^2 \partial_x^2 \right) \phi = 2\omega \beta \partial_x (|\psi|^2). \end{cases} \quad (4.1.15)$$

This has the advantage of "splitting the difficulties": in the first case, one can invert the equation for ϕ as for the hyperbolic-elliptic classical DS case, but one keeps the "bad" dispersive properties of the Schrödinger part. In the later case one has the "good" properties of the hyperbolic NLS and the "bad" invertibility properties of the equation for ϕ .

We consider here the full dispersion Davey-Stewartson system written as

$$\begin{cases} \partial_t \psi - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial_x \phi + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2) \psi = 0, \\ \left(\frac{|D| \tanh(\varepsilon \sqrt{\mu} |D|)}{\varepsilon} + \omega'^2 \partial_x^2 \right) \phi = 2\omega \beta \partial_x (|\psi|^2). \end{cases} \quad (4.1.16)$$

Which is a 2-d model in space for the unknown $\psi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ and $\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

As noticed in [24] the "subsonic" condition $\sqrt{\mu} - \omega'^2 > 0$ does not ensures the solvability of the equation for ϕ since, denoting ξ the dual variable of X' , the frequencies ξ such that

$$\frac{\tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon |\xi|} = \omega'^2 \frac{\xi_1^2}{|\xi|^2}$$

will cancel the symbol.

The parameters $\omega', \omega'', |\mathbf{k}|, \alpha, \beta, \mu$ are defined as previously.

We will only consider the case $\sqrt{\mu} > \omega'^2$, so that we can find r_0 such that:

$$\tanh(\sqrt{\mu} r_0) = \omega'^2 r_0.$$

In order to invert easily the equation for ϕ we introduce an approximate system by truncating the high frequencies.

More precisely, let δ be fixed such that $0 < \delta < r_0$, for each $\varepsilon > 0$, we define $R_\varepsilon = \frac{r_0 - \delta}{\varepsilon}$. We then replace the mean mode ϕ by $\phi_\varepsilon := \mathcal{F}^{-1}(\chi(|\xi| \leq R_\varepsilon) \hat{\phi})$ so that ⁵

$$\text{supp } \widehat{\phi_\varepsilon} \subset \{\xi \in \mathbb{R}^2; |\xi| \leq R_\varepsilon\} =: \Omega_\varepsilon, \quad (4.1.17)$$

to get a family of systems depending on the parameter ε as follows

$$\begin{cases} \partial_t \psi_\varepsilon - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_\varepsilon + i(\beta \partial_x \phi_\varepsilon + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_\varepsilon|^2) \psi_\varepsilon = 0, \\ \left(\frac{|D| \tanh(\varepsilon \sqrt{\mu} |D|)}{\varepsilon} + \omega'^2 \partial_x^2 \right) \phi_\varepsilon = 2\omega \beta \partial_x (|\psi_\varepsilon|^2). \end{cases} \quad (4.1.18)$$

Since the supports of $\widehat{\phi_\varepsilon}$ and $|\widehat{\psi_\varepsilon}|^2$ are in $\{\xi \in \mathbb{R}^2; |\xi| < R_\varepsilon\}$, we need to construct a suitable initial data for the Cauchy problem for the $\varepsilon - FDDS$ which converge to the initial data of the Cauchy problem for the corresponding classical DS system. We proceed as follows.

First, let $\psi_0 \in H^1$ be the initial data of the classical DS system, then we define

$$\psi_{0\varepsilon} = f_\varepsilon \star \psi_0 \text{ where } f_\varepsilon = \mathcal{F}^{-1}(\chi(|\xi| < \frac{R_\varepsilon}{2})). \quad (4.1.19)$$

⁵ $\chi(A)$ denote the characteristic function of the set A .

Then we have that $\widehat{\psi_{0\varepsilon}} \subset \{|\xi| < \frac{R_\varepsilon}{2}\}$. It follows

$$\widehat{\supp |\psi_{0\varepsilon}|^2} \subset \widehat{\supp \psi_{0\varepsilon}} + \widehat{\supp \overline{\psi_{0\varepsilon}}(-\cdot)} \subset \{|\xi| < R_\varepsilon\}.$$

Our purpose is to prove that when ε tends to 0, the solution of (4.1.18) with initial data $\psi_{0\varepsilon}$ tends to the solution of the following classical Davey-Stewartson system with initial data ψ_0 :

$$\begin{cases} \partial_t \psi - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi + i(\beta \partial_x \phi + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^2) \psi = 0, \\ (\sqrt{\mu} |D|^2 + \omega'^2 \partial_x^2) \phi = 2\omega \beta \partial_x (|\psi|^2). \end{cases} \quad (4.1.20)$$

The paper is organized as follows. As a preliminary step we solve the equation for $\partial_x \phi_\varepsilon$ reducing the approximate FDDS system to a NLS type equation. The corresponding Cauchy problem is classically solved in the next section, on a time interval independent of ε . We then pass to the limit as $\varepsilon \rightarrow 0$ by compactness arguments.

1.1. Notations. The norm in L^2 based Sobolev spaces H^s will be denoted $\|u\|_s$. The norm in the Lebesgue spaces L^p will be denoted $\|u\|_p$. The Fourier transform in spatial space will be denoted by $\widehat{(\cdot)}$ or $\mathcal{F}$ and by $\mathcal{F}^{-1}$ for the inverse Fourier transform. $|D|$ will denote the Fourier multiplier defined by $\widehat{|D|f}(\xi) = |\xi| \widehat{f}(\xi)$. Throughout this paper, C denotes a general constant independent of ε if there is no further specification.

4.2 Solving the equation for $\partial_x \phi_\varepsilon$

We first rewrite the second equation of (4.1.18) in Fourier space

$$\left(\frac{|\xi| \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon} - \omega'^2 \xi_1^2 \right) \widehat{\phi_\varepsilon}(\xi) = 2i\omega \beta \xi_1 \widehat{|\psi_\varepsilon|^2}(\xi),$$

then, since the support of $\widehat{\phi_\varepsilon}$ does not contain any zero of the equation $\frac{|\xi| \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon} - \omega'^2 \xi_1^2 = 0$, one can write

$$\widehat{\partial_x \phi_\varepsilon}(\xi) = \frac{2\omega \beta \xi_1^2}{\omega'^2 \xi_1^2 - \frac{|\xi| \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon}} \widehat{|\psi_\varepsilon|^2}(\xi). \quad (4.2.1)$$

We now consider the function

$$\begin{aligned} f(\xi) &= \left| \omega'^2 - \frac{|\xi| \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon \xi_1^2} \right| \\ &= \left| \omega'^2 - \frac{|\xi|^2 \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\xi_1^2 \varepsilon |\xi|} \right|, \end{aligned}$$

when $|\xi| \leq R_\varepsilon$.

Since the function: $x \mapsto \frac{\tanh x}{x}$ is strictly decreasing in $\mathbb{R}_+$ and $R_\varepsilon < \frac{r_0}{\varepsilon}$ we have

$$\sqrt{\mu} \geq \frac{\tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon |\xi|} \geq \frac{\tanh(\varepsilon \sqrt{\mu} R_\varepsilon)}{\varepsilon R_\varepsilon} > \frac{\tanh(\sqrt{\mu} r_0)}{r_0} = \omega'^2.$$

Then

$$f(\xi) \geq \frac{\tanh(\varepsilon \sqrt{\mu} R_\varepsilon)}{\varepsilon R_\varepsilon} - \omega'^2 = \frac{\tanh(\sqrt{\mu} (r_0 - \delta))}{r_0 - \delta} - \omega'^2 > 0.$$

Therefore,

$$|\widehat{\partial_x \phi_\varepsilon}| \leq C_0 |\widehat{|\psi_\varepsilon|^2}|, \quad (4.2.2)$$

where C_0 is independent of ε , namely

$$C_0 = \frac{|2\omega\beta|}{\frac{\tanh(\sqrt{\mu}(r_0-\delta))}{r_0-\delta} - \omega'^2}.$$

The above observation leads to the following lemma.

Lemma 4.2.1. *The second equation in (4.1.18) is uniquely solvable and*

$$\|\partial_x \phi_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C_0 \|\psi_\varepsilon\|_{L^4(\mathbb{R}^2)}^2. \quad (4.2.3)$$

Note that besides the L^2 norm, by a similiar calculation as in the case of the classical Davey-Stewartson system (see [12]), the energy $E(\psi)$ is formally conserved by the flow of (4.1.16) where

$$\begin{aligned} E(\psi) = \int_{\mathbb{R}^2} & \left(\omega'' |\psi_x|^2 + \frac{\omega'}{|\mathbf{k}|} |\psi_y|^2 + 2 \frac{|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi|^4 + \frac{\omega'^2}{4\omega} (\phi_x)^2 \right. \\ & \left. - \frac{1}{4\omega} \left(\frac{\tanh(\epsilon\sqrt{\mu}|D|)}{\epsilon} \phi \right) (|D|\phi) \right) dx dy. \end{aligned}$$

Setting $\partial_x \phi_\varepsilon = F_\varepsilon(|\psi_\varepsilon|^2)$, one can rewrite (4.1.18) as a single NLS type equation,

$$\partial_t \psi_\varepsilon - \frac{i}{2} (\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_\varepsilon + i(\beta F_\varepsilon(|\psi_\varepsilon|^2) + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_\varepsilon|^2) \psi_\varepsilon = 0. \quad (4.2.4)$$

4.3 Cauchy problem for the ε -FDDS

In this section, we will consider the Cauchy problem for (4.1.18) (or (4.2.4)) with initial data $\psi_\varepsilon(t=0) = \psi_{0\varepsilon} \in H^1(\mathbb{R}^2)$ defined as in (4.1.19). In particular we obtain the following theorem.

Theorem 4.3.1. *Let $\psi_\varepsilon(t=0) = \psi_{0\varepsilon} \in H^1(\mathbb{R}^2)$, then there exists a unique maximal solution $(\psi_\varepsilon, \phi_\varepsilon)$ solution of (4.1.18) on $[0, T_\varepsilon^*]$ where $T_\varepsilon^* = O(1/\|\psi_{0\varepsilon}\|_{H^1}^2)$, such that*

$$\begin{aligned} \psi_\varepsilon & \in L_t^\infty(0, T_\varepsilon^*; H^1(\mathbb{R}^2)), \quad \nabla \psi_\varepsilon \in L_t^4(0, T_\varepsilon^*; L^4(\mathbb{R}^2)), \\ \partial_x \phi_\varepsilon & \in L_t^\infty(0, T_\varepsilon^*; L^2(\mathbb{R}^2)) \cap L_t^2(0, T_\varepsilon^*; H^1(\mathbb{R}^2)) \end{aligned}$$

One has the conservation of the L^2 norm and of the energy :

$$|\psi_\varepsilon(\cdot, t)|_2 = |\psi_{0\varepsilon}|_2, \quad E(\psi_\varepsilon(\cdot, t)) = E(\psi_{0\varepsilon}), \quad 0 \leq t < T_\varepsilon^*.$$

Moreover one has the persistency property : if $\psi_0 \in H^s(\mathbb{R}^2)$, $s > 1$, then

$$\psi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; H^s(\mathbb{R}^2)), \quad \partial_x \phi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; H^s(\mathbb{R}^2)).$$

Proof. Let us denote

$$U(t) := e^{it(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2)}, \quad (\text{unitary in all Sobolev spaces})$$

then we can rewrite (4.2.4) using the Duhamel formulation as an integral equation

$$\psi_\varepsilon = U(t) \psi_{0\varepsilon} - i \int_0^t U(t-s) (\beta F_\varepsilon(|\psi_\varepsilon|^2) + \frac{2|\mathbf{k}|^4}{\omega} (1 - \alpha) |\psi_\varepsilon|^2) \psi_\varepsilon(s) ds. \quad (4.3.1)$$

We denote the operator on the right hand side by $\mathcal{T}(\psi_\varepsilon)$ and recall the two-dimensional Strichartz estimate for $U(t)$

$$\left\| \int_0^t U(t-s)f(s)ds \right\|_{L_t^\infty(0,T;L^2) \cap L_t^4(0,T;L^4)} \leq C \|f\|_{L_t^{4/3}(0,T;L^{4/3})}. \quad (4.3.2)$$

Set

$$X = \{u \in L_t^\infty(0,T;H^1), \nabla u \in L_t^4(0,T;L^4)\},$$

and

$$B = \{\|u\|_X \leq 3 \|\psi_{0\varepsilon}\|_{H^1}\},$$

T can be chosen later.

The proof is standard and relies classically on the Banach fixed point theorem. We only derive here suitable bounds in order to emphasize that the existence time and the norm of solution are independent of ε .

We now prove that B is stable under $\mathcal{T}$. Using estimate (4.2.3), Strichartz's estimate (4.3.2), Sobolev's embedding and Hölder's inequality, we get the following estimates:

$$\begin{aligned} & \|\mathcal{T}(\psi_\varepsilon)\|_X \\ &= \|\mathcal{T}(\psi_\varepsilon)\|_{L_t^\infty(0,T;L^2)} + \|\nabla \mathcal{T}(\psi_\varepsilon)\|_{L_t^\infty(0,T;L^2)} + \|\nabla \mathcal{T}(\psi_\varepsilon)\|_{L_t^4(0,T;L^4)} \\ &\leq 2 \|\psi_{0\varepsilon}\|_{H^1} + C \left(\|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon\|_{L_t^{4/3}(0,T;L^{4/3})} + \|\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon)\|_{L_t^{4/3}(0,T;L^{4/3})} \right. \\ &\quad \left. + \| |\psi_\varepsilon|^2 \psi_\varepsilon \|_{L_t^{4/3}(0,T;L^{4/3})} + \|\nabla(|\psi_\varepsilon|^2 \psi_\varepsilon)\|_{L_t^{4/3}(0,T;L^{4/3})} \right). \end{aligned} \quad (4.3.3)$$

And,

$$\begin{aligned} \|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon\|_{L_t^{4/3}(0,T;L^{4/3})} &\leq \|F_\varepsilon(|\psi_\varepsilon|^2)\|_{L_t^2(0,T;L^2)} \|\psi_\varepsilon\|_{L_t^4(0,T;L^4)} \\ &\leq C_0 T^{3/4} \|\psi_\varepsilon\|_{L_t^\infty(0,T;L^4)}^3 \\ &\leq C T^{3/4} \|\psi_\varepsilon\|_{L_t^\infty(0,T;H^1)}^3, \end{aligned} \quad (4.3.4)$$

$$\| |\psi_\varepsilon|^2 \psi_\varepsilon \|_{L_t^{4/3}(0,T;L^{4/3})} \leq C T^{3/4} \|\psi_\varepsilon\|_{L_t^\infty(0,T;H^1)}, \quad (4.3.5)$$

$$\begin{aligned} \|\nabla(|\psi_\varepsilon|^2 \psi_\varepsilon)\|_{L_t^{4/3}(0,T;L^{4/3})} &= \|\psi_\varepsilon^2 \nabla \bar{\psi}_\varepsilon + 2|\psi_\varepsilon|^2 \nabla \psi_\varepsilon\|_{L_t^{4/3}(0,T;L^{4/3})} \\ &\leq C T^{1/2} \|\psi_\varepsilon\|_{L_t^\infty(0,T;H^1)}^2 \|\nabla \psi_\varepsilon\|_{L_t^4(0,T;L^4)}. \end{aligned} \quad (4.3.6)$$

And, for $h \in \mathbb{R}^2$

$$\begin{aligned} & \|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon(X+h) - F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon(X)\|_{L_t^{4/3}(0,T;L^{4/3})} \\ &= \left\| (\psi_\varepsilon(X+h) - \psi_\varepsilon(X)) F_\varepsilon(|\psi_\varepsilon(X+h)|^2) \right. \\ &\quad \left. + (F_\varepsilon(|\psi_\varepsilon(X+h)|^2) - F_\varepsilon(|\psi_\varepsilon(X)|^2)) \psi_\varepsilon(X) \right\|_{L_t^{4/3}(0,T;L^{4/3})} \\ &\leq 3C_0 T^{1/2} \|\psi_\varepsilon\|_{L_t^\infty(0,T;L^4)}^2 \|\psi_\varepsilon(X+h) - \psi_\varepsilon(X)\|_{L_t^4(0,T;L^4)}, \end{aligned}$$

Dividing by $|h|$ and letting $|h|$ tends to 0, we deduce that $\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon) \in L_t^{4/3}(0,T;L^{4/3})$ and

$$\|\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon)\|_{L_t^{4/3}(0,T;L^{4/3})} \leq 3C_0 T^{1/2} \|\psi_\varepsilon\|_{L_t^\infty(0,T;L^4)}^2 \|\nabla \psi_\varepsilon\|_{L_t^4(0,T;L^4)} \quad (4.3.7)$$

Combining (4.3.3)-(4.3.7), we see that there exists $T = O(1/\|\psi_{0\varepsilon}\|_{H^1}^2)$ such that B is invariant under $\mathcal{T}$.

Proving that $\mathcal{T}$ is a contraction on B proceeds as in the previous step by using the following expression.

$$\psi_1 F_\varepsilon(|\psi_1|^2) - F_\varepsilon(|\psi_2|^2) \psi_2 = (\psi_1 - \psi_2) F_\varepsilon(|\psi_1|^2) + \psi_2 F_\varepsilon((|\psi_1| - |\psi_2|)(|\psi_1| + |\psi_2|))$$

and

$$\psi_1 |\psi_1|^2 - \psi_2 |\psi_2|^2 = (\psi_1 - \psi_2) |\psi_1|^2 + \psi_2 (|\psi_1| - |\psi_2|)(|\psi_1| + |\psi_2|)$$

where ψ_1 and ψ_2 are in X .

Therefore from these two steps, we can choose the existence time $T_\varepsilon^* = O(1/\|\psi_{0\varepsilon}\|_{H^1}^2)$. $\square$

Remark 15. From (4.1.19) we have

$$\|\psi_{0\varepsilon}\|_{H^1} = \left\| \sqrt{1 + |\xi|^2} \chi(|\xi| < \frac{R_\varepsilon}{2}) \widehat{\psi_0} \right\|_{L^2} \leq \|\psi_0\|_{H^1}.$$

Then, we can choose an existence time T^* of the solution of (4.1.18) which is independent of ε and the solutions are also bounded by $3\|\psi_0\|_{H^1}$ in that time interval. Therefore, in the next section, we will prove that the solutions tend to a solution of the classical Davey-Stewartson system when ε tends to zero.

4.4 From ε -FDDS to classical DS

4.4.1 Weak convergence

We prove here that the solutions of the ε -FDDS system converge to that of the classical Davey-Stewartson system as ε tends to 0. We first recall a very classical result in [21], and some useful estimates.

Lemma 4.4.1. (Aubin-Lions's Lemma) Let B_0, B, B_1 be three reflexive Banach spaces. Assume that B_0 is compactly embedded in B and B is continuously embedded in B_1 . Let

$$W = \{V \in L^{p_0}(0, T, B_0), \frac{\partial V}{\partial t} \in L^{p_1}(0, T, B_1)\}, \quad T < \infty, 1 < p_0, p_1 < \infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}(0, T, B_0)} + \|V_t\|_{L^{p_1}(0, T, B_1)}.$$

Then the embedding $W \hookrightarrow L^{p_0}(0, T, B)$ is compact.

When $p_0 = \infty, p_1 > 1$, the above statement is also true, see [29].

Lemma 4.4.2. Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that

$$g_\varepsilon \rightarrow g \text{ almost everywhere in } \Omega \text{ and } \|g_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then $g_\varepsilon \rightarrow g$ weakly in $L^p(\Omega)$.

Lemma 4.4.3. If $t_0 > 1$ and $s \geq 0$, one has

$$\|fg\|_{H^s} \leq C \left(\|f\|_{H^{t_0}} \|g\|_{H^s} + \langle \|f\|_{H^s} \|g\|_{H^{t_0}} \rangle_{s > t_0} \right), \quad \forall f, g \in H^s \cap H^{t_0}(\mathbb{R}^2). \quad (4.4.1)$$

Note that, the condensed notation $A_s = B_s + \langle C_s \rangle_{s > s^*}$ means that $A_s = B_s$ if $s \leq s^*$ and $A_s = B_s + C_s$ if $s > s^*$.

Theorem 4.4.4. Let $\psi_0 \in H^1(\mathbb{R}^2)$, $\psi_{0\varepsilon}$ is defined as in (4.1.19) and $(\psi_\varepsilon, \phi_\varepsilon)$ be the solution of (4.1.18) given by theorem 4.3.1. Then $(\psi_\varepsilon, \partial_x \phi_\varepsilon)$ tends to $(\psi, \partial_x \phi)$ in $L_t^\infty(0, T^*; H^1(\mathbb{R}^2)) \times L_t^\infty(0, T^*; L^2(\mathbb{R}^2))$ weak-star, where (ψ, ϕ) is the solution of (4.1.20) with initial data ψ_0 .

Furthermore, If $\psi_0 \in H^s(\mathbb{R}^2)$, $s > 3$ then $(\psi_\varepsilon, \partial_x \phi_\varepsilon)$ tends to $(\psi, \partial_x \phi)$ strongly in $L_t^\infty(0, T^*; H_{loc}^{s-\delta}(\mathbb{R}^2))$ for $0 < \delta < 2$.

(T^* is the existence time interval for all ε and is defined as in Remark 15.)

Proof. We first denote

$$F(u) = \mathcal{F}_X^{-1} \left(\frac{2\omega\beta\xi_1^2}{\omega'^2\xi_1^2 - \sqrt{\mu}|\xi|^2} \hat{u}(\xi) \right).$$

It is not difficult to see that $\psi_{0\varepsilon} \rightarrow \psi_0$ in H^1 when ε tends to zero.

From theorem 4.3.1 and Remark 15, we know that $\|\psi_\varepsilon\|_{L_t^\infty(0, T^*; H^1)}$, $\|\partial_x \phi_\varepsilon\|_{L_t^\infty(0, T^*; L^2)}$ are uniformly bounded in ε , then by using Hölder inequality and Sobolev embedding theorem, we also have that $\| |\psi_\varepsilon|^2 \psi_\varepsilon \|_{L_t^\infty(0, T^*; L^2)}$, $\| (\partial_x \phi_\varepsilon) \psi_\varepsilon \|_{L_t^\infty(0, T^*; H^{-1})}$ are uniformly bounded in ε . Using the first equation of (4.1.18), it is also true for $\|\partial_t \psi_\varepsilon\|_{L_t^\infty(0, T^*; H^{-1})}$. Therefore, we have (up to extraction of subsequences)

$$\psi_\varepsilon \rightarrow \psi \text{ in } L_t^\infty(0, T^*; H^1) \quad \text{weak-star.} \quad (4.4.2)$$

$$\partial_x \phi_\varepsilon \text{ or } F_\varepsilon(|\psi_\varepsilon|^2) \rightarrow l \text{ in } L_t^\infty(0, T^*; L^2) \quad \text{weak-star.} \quad (4.4.3)$$

$$\partial_t \psi_\varepsilon \rightarrow \psi \text{ in } L_t^\infty(0, T^*; H^{-1}) \quad \text{weak-star.} \quad (4.4.4)$$

$$(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi_\varepsilon \rightarrow (\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|} \partial_y^2) \psi \text{ in } L_t^\infty(0, T^*; H^{-1}) \quad \text{weak-star.} \quad (4.4.5)$$

$$|\psi_\varepsilon|^2 \psi_\varepsilon \rightarrow h \text{ in } L_t^\infty(0, T^*; L^2) \quad \text{weak-star.} \quad (4.4.6)$$

$$(\partial_x \phi_\varepsilon) \psi_\varepsilon \text{ or } F_\varepsilon(|\psi_\varepsilon|^2) \psi_\varepsilon \rightarrow g \text{ in } L_t^\infty(0, T^*; H^{-1}) \quad \text{weak-star.} \quad (4.4.7)$$

Thus, we need to prove

$$h = |\psi|^2 \psi, \quad (4.4.8)$$

$$g = F(|\psi|^2) \psi, \quad (4.4.9)$$

and

$$l = F(|\psi|^2) \text{ or } \partial_x \phi. \quad (4.4.10)$$

We will use, for any bounded open subset Ω of $\mathbb{R}^2$, the compact embedding

$$H^1(\Omega) \hookrightarrow L^4(\Omega),$$

and the continuous injection

$$L_t^\infty(0, T^*; X) \hookrightarrow L_t^2(0, T^*; X), \text{ for any Banach space } X.$$

Let denote

$$W = \{v \in L_t^2(0, T^*; H^1(\Omega)); \frac{\partial v}{\partial t} \in L_t^2(0, T^*; H^{-1}(\Omega))\}.$$

By Aubin-Lions theorem, we have

$$W \hookrightarrow L_t^2(0, T^*; L^4(\Omega)) \text{ compactly .}$$

Hence, $\{\psi_\varepsilon\}$ is relatively compact in $L_t^2(0, T^*; L^4(\Omega))$ for any bounded subdomain Ω in $\mathbb{R}^2$. Therefore, $\psi_\varepsilon \rightarrow \psi$ strongly in $L_t^2(0, T^*; L_{loc}^4(\mathbb{R}^2))$ up to a subsequence. Thus,

$$\psi_\varepsilon \rightarrow \psi \text{ almost everywhere in } (0, T^*) \times \mathbb{R}^2,$$

or

$$|\psi_\varepsilon|^2 \psi_\varepsilon \rightarrow |\psi|^2 \psi \text{ almost everywhere in } (0, T^*) \times \mathbb{R}^2.$$

Combining with lemma 4.4.2 we get (4.4.8).

We now prove (4.4.9) by proving that $F_\varepsilon(|\psi_\varepsilon|^2 \psi_\varepsilon) \rightarrow F(|\psi|^2) \psi$ in $L_t^2(0, T^*; H^{-1})$ weak star. First, let v be some test function in $L_t^2(0, T^*; H^1)$ vanishing outside of a compact set $\Omega \in \mathbb{R}^2$. Then

$$\begin{aligned} & \int_0^{T^*} \int_{\mathbb{R}^2} (F_\varepsilon(|\psi_\varepsilon|^2) \psi_\varepsilon - F(|\psi|^2) \psi) v \, dX \, dt \\ &= \int_0^{T^*} \int_\Omega F_\varepsilon(|\psi_\varepsilon|^2) (\psi_\varepsilon - \psi) v \, dX \, dt + \int_0^{T^*} \int_{\mathbb{R}^2} (F_\varepsilon(|\psi_\varepsilon|^2) - F(|\psi|^2)) \psi v \, dX \, dt \\ &= I_\varepsilon^1 + I_\varepsilon^2 \end{aligned}$$

Estimate for I_ε^1 :

$$|I_\varepsilon^1| \leq \left\| F_\varepsilon(|\psi_\varepsilon|^2) \right\|_{L_t^\infty(0, T^*; L^2(\mathbb{R}^2))} \|\psi_\varepsilon - \psi\|_{L_t^2(0, T^*; L^4(\Omega))} \|v\|_{L_t^2(0, T^*; L^4)}.$$

Since $\psi_\varepsilon \rightarrow \psi$ strongly in $L_t^2(0, T^*; L^4)$, $I_\varepsilon^1 \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Estimate for I_ε^2 : Note that we can rewrite

$$\begin{aligned} & F_\varepsilon(|\psi_\varepsilon|^2) - F(|\psi|^2) \\ &= C \mathcal{F}^{-1} \left(\frac{\widehat{|\psi_\varepsilon|^2}(\xi)}{\omega'^2 - \frac{|\xi| \tanh(\varepsilon \sqrt{\mu} |\xi|)}{\varepsilon \xi_1^2}} - \frac{\widehat{|\psi|^2}(\xi)}{\omega'^2 - \sqrt{\mu} \frac{|\xi|^2}{\xi_1^2}} \right) \\ &:= C \mathcal{F}^{-1} \left(q_\varepsilon(\xi) \widehat{|\psi_\varepsilon|^2}(\xi) - q_0(\xi) \widehat{|\psi|^2}(\xi) \right) \\ &= C \mathcal{F}^{-1} \left((q_\varepsilon - q_0)(\xi) \widehat{|\psi_\varepsilon|^2}(\xi) \right) + C \mathcal{F}^{-1} \left(q_0(\xi) (\widehat{|\psi_\varepsilon|^2} - \widehat{|\psi|^2})(\xi) \right) \end{aligned}$$

Then by Plancherel theorem, we have

$$\begin{aligned} & |I_\varepsilon^2| \\ & \leq C \left| \int_0^{T^*} \int_{\mathbb{R}^2} (q_\varepsilon - q_0)(\xi) \widehat{|\psi_\varepsilon|^2}(\xi) \overline{\widehat{\psi v}}(\xi) \, d\xi \, dt \right| + C \left| \int_0^{T^*} \int_{\mathbb{R}^2} q_0(\xi) (\widehat{|\psi_\varepsilon|^2} - \widehat{|\psi|^2})(\xi) \overline{\widehat{\psi v}}(\xi) \, d\xi \, dt \right| \\ & := I_\varepsilon^3 + I_\varepsilon^4 \end{aligned}$$

Estimate for I_ε^3 : By Hölder inequality and

$$I_\varepsilon^3 \leq \left\| \widehat{|\psi_\varepsilon|^2} \right\|_{L_t^2(0, T^*; L^2)} \left\| \chi(\Omega_\varepsilon) (q_\varepsilon - q_0) \widehat{\psi v} \right\|_{L_t^2(0, T^*; L^2)}.$$

Since $\chi(\Omega_\varepsilon) (q_\varepsilon - q_0) \widehat{\psi v}(t, \xi) \rightarrow 0$ pointwise in $(0, T^*) \times \mathbb{R}^2$ and $|\chi(\Omega_\varepsilon) (q_\varepsilon - q_0) \widehat{\psi v}| < C |\widehat{\psi v}|$ and $\widehat{\psi v} \in L_t^2(0, T^*; L^2)$, dominated convergence theorem follows

$$\left\| \chi(\Omega_\varepsilon) (q_\varepsilon - q_0) \widehat{\psi v} \right\|_{L_t^2(0, T^*; L^2)} \rightarrow 0$$

when ε tends to 0. Hence, $I_\varepsilon^3 \rightarrow 0$ when ε tends to 0.

Estimate for I_ε^4 : By Plancherel identity

$$I_\varepsilon^4 = \left| \int_0^{T^*} \int_{\mathbb{R}^2} (|\psi_\varepsilon|^2 - |\psi|^2) \overline{\mathcal{F}^{-1}(q_0(\xi) \widehat{\psi v}(\xi))} \, dX \, dt \right|.$$

Since $\|\psi_\varepsilon\|^2_{L_t^\infty(0,T^*;L^2)}$ is uniformly bounded then by similar argument of getting (4.4.8), we get that $|\psi_\varepsilon|^2 \rightarrow |\psi|^2$ in $L_t^\infty(0,T^*;L^2)$ weak-star. Therefore we need to show that $\mathcal{F}^{-1}(q_0(\xi)\widehat{\psi v}(\xi)) \in L_t^1(0,T^*;L^2)$. Indeed, by Plancherel's identity

$$\begin{aligned} \left\| \mathcal{F}^{-1}(q_0(\xi)\widehat{\psi v}(\xi)) \right\|_{L_t^1(0,T^*;L^2)} &= \left\| q_0(\xi)\widehat{\psi v}(\xi) \right\|_{L_t^1(0,T^*;L^2)} \\ &\leq C \left\| \widehat{\psi v}(\xi) \right\|_{L_t^1(0,T^*;L^2)} \\ &= C \|\psi v\|_{L_t^1(0,T^*;L^2)} \\ &\leq C \|\psi\|_{L_t^2(0,T^*;L^4)} \|v\|_{L_t^2(0,T^*;L^4)} < \infty. \end{aligned}$$

The proof of (4.4.10) is similar to that of (4.4.9), and this concludes the proof of weak convergence.

In order to prove the strong convergence, we note that by a similar argument as above, $(\psi_\varepsilon, \partial_x \phi_\varepsilon)$ tends to $(\psi, \partial_x \phi)$ in $L_t^\infty(0,T^*;H^s(\mathbb{R}^2))$ weak-star. Furthermore, by (4.4.1), Plancherel identity and using the fact that $s > 3$, we have

$$\begin{aligned} \|\partial_t \partial_x \phi_\varepsilon\|_{L_t^\infty(0,T^*;H^{s-2})} &= \|F_\varepsilon(2\operatorname{Re}(\partial_t \psi_\varepsilon \overline{\psi_\varepsilon}))\|_{L_t^\infty(0,T^*;H^{s-2})} \\ &\leq C \left\| \langle \xi \rangle^{s-2} \mathcal{F}(\operatorname{Re}(\partial_t \psi_\varepsilon \overline{\psi_\varepsilon})) \right\|_{L_t^\infty(0,T^*;L^2)} \\ &= C \left\| J^{s-2} \operatorname{Re}(\partial_t \psi_\varepsilon \overline{\psi_\varepsilon}) \right\|_{L_t^\infty(0,T^*;L^2)} \\ &\leq C \left(\left\| J^{s-2}(\partial_t \psi_\varepsilon \overline{\psi_\varepsilon}) \right\|_{L_t^\infty(0,T^*;L^2)} + \left\| J^{s-2}(\overline{\partial_t \psi_\varepsilon} \psi_\varepsilon) \right\|_{L_t^\infty(0,T^*;L^2)} \right) \\ &\leq C \|\psi_\varepsilon\|_{L_t^\infty(0,T^*;H^s)} \|\partial_t \psi_\varepsilon\|_{L_t^\infty(0,T^*;H^{s-2})}. \end{aligned}$$

Therefore, Simon's lemma 4.4.2 and the identity of the weak and strong limit follow that $(\psi_\varepsilon, \partial_x \phi_\varepsilon)$ tends to $(\psi, \partial_x \phi)$ in $L_t^\infty(0,T^*;H_{loc}^{s-\delta}(\mathbb{R}^2))$ strongly, where $0 < \delta < 2$. $\square$

4.5 Solitary waves

We consider in this section solitary wave solutions to the ε -FDDS system, that is solutions of the form $(u_\varepsilon(x, y)e^{ict}, \phi_\varepsilon(x, y))$ where $u_\varepsilon \in H^1(\mathbb{R}^2)$, $\nabla \phi_\varepsilon \in L^2(\mathbb{R}^2)$ are real valued functions.

The profile u_ε should thus satisfies the system

$$\begin{cases} cu_\varepsilon - \frac{1}{2}(\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)u_\varepsilon + (\beta\partial_x \phi_\varepsilon + \delta|u_\varepsilon|^2)u_\varepsilon = 0 \\ L_\varepsilon \phi_\varepsilon = 2\omega\beta\partial_x(|u|^2) \end{cases} \quad (4.5.1)$$

where $\delta = \frac{2|\mathbf{k}|^4}{\omega}(1 - \alpha)$ and $L_\varepsilon = (\frac{|D|\tanh(\varepsilon\sqrt{\mu}|D|)}{\varepsilon} + \omega'^2\partial_x^2)$. or equivalently

$$cu_\varepsilon - \frac{1}{2}(\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)u_\varepsilon = \frac{2\omega\beta^2}{\sqrt{\mu} - \omega'^2}G_\varepsilon(|u_\varepsilon|^2)u_\varepsilon - \delta|u_\varepsilon|^2u_\varepsilon. \quad (4.5.2)$$

where

$$G_\varepsilon(|u_\varepsilon|^2) = \mathcal{F}^{-1} \left(\frac{(\sqrt{\mu} - \omega'^2)\xi_1^2}{\frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon} - \omega'^2\xi_1^2} |\widehat{u_\varepsilon|^2}(\xi) \right)$$

For the classical Davey-Stewartson systems, the non-existence of solitary waves is classified in [11] according to the nature of the system and the values of the coefficients.

More precisely, a solitary wave $(e^{i\omega t}u(x), q(x))$ of the elliptic-elliptic Davey-Stewartson system (4.1.11) with $\delta = m = 1$ should satisfy the system :

$$\begin{aligned} -\omega u + \Delta u &= (\chi|u|^2 + bq_x)u, \\ \Delta q &= \partial_x|u|^2, \end{aligned} \quad (4.5.3)$$

By classical Pohojaev type arguments, one obtains

Proposition 2. *The elliptic-elliptic Davey-Stewartson systems do not possess non trivial solitary waves whenever*

$$\omega \leq 0$$

$$\chi \geq 0 \quad \text{and} \quad b \geq 0$$

$$b \leq 0 \quad \text{and} \quad \chi + b \geq 0.$$

On the other hand the following result has been obtained in [4]

Theorem 4.5.1. *Let $\chi + b < 0$. Then there exists a smooth (C^∞) , exponentially decaying ground state solution.*

According to this result one might expect that (5.1.7) possesses non trivial solutions provided $\omega', \omega'' > 0$ and $\frac{2\omega\beta^2}{\sqrt{\mu}-\omega'^2} - \delta > 0$, at least for sufficiently small ε . However, we also have the following remark on these conditions.

Remark 16. *We have:*

$$\begin{aligned} \mu &= h^2/\lambda^2, \\ \omega &= (|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|))^{1/2}; \quad \sigma = \tanh(\sqrt{\mu}|\mathbf{k}|), \\ \omega' &= \frac{1}{2} \left(|\mathbf{k}| \sqrt{\mu} (1 - \tanh^2(\sqrt{\mu}|\mathbf{k}|)) + \tanh(\sqrt{\mu}|\mathbf{k}|) \right) (|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|))^{-1/2}, \\ \alpha &= -\frac{9}{8\sigma^2} (1 - \sigma^2)^2, \quad \beta = |\mathbf{k}| \left(1 + (1 - \sigma^2) \frac{\omega'|\mathbf{k}|}{2\omega} \right). \end{aligned}$$

First, we see that if $|\mathbf{k}| \gg 1$ then $\tanh(\sqrt{\mu}|\mathbf{k}|) \simeq 1$. It follows

$$\begin{aligned} \omega'^2 &\simeq 1/4 |\mathbf{k}|^{-1}, \\ \frac{2\omega\beta^2}{\sqrt{\mu}-\omega'^2} &\simeq \frac{|\mathbf{k}|^{7/2}(5-\sigma^2)^2}{2(4|\mathbf{k}|\sqrt{\mu}-1)}, \end{aligned}$$

and

$$\delta \simeq 2|\mathbf{k}|^{7/2}(1-\alpha).$$

That means if $|\mathbf{k}|$ is large enough then the assumption $\sqrt{\mu} > \omega'^2$ holds but the assumption $\frac{2\omega\beta^2}{\sqrt{\mu}-\omega'^2} - \delta > 0$ fails.

4.6 Remarks on the complete full dispersion system

One can of course apply the same truncation process on the ϕ equation for the complete full dispersion Davey-Stewartson systems (4.1.16), (4.1.9) yielding approximations of them. Using Lemma 2.1, the fact that the linear dispersive part is skew-adjoint and standard arguments, one obtains straightforwardly that those two systems are locally well posed for initial data in $H^s(\mathbb{R}^2)$, $s > 1$ on a time interval $[0, T^*)$ where T^* does not depend on ε . Note that this result does not depend on the dispersion.

In presence of surface tension, the resulting equation in ψ_ε is reminiscent of a fractional nonlinear Schrödinger equation.

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Chapter 5

The Boussinesq systems on the background of a line solitary wave

Joint work with Jean-Claude Saut. (*Work in progress*).

Abstract: This paper is motivated by the transverse stability properties of the line solitary wave solutions of the (a,b,c,d) class of Boussinesq systems introduced in [12]. We first review the known results on existence of one-dimensional solitary waves. Then we address the question of long time existence of the systems satisfied by a localized perturbation of a line solitary wave.

5.1 Introduction

The motivation of this paper is the study of transverse stability issues of line solitary waves to the (a,b,c,d) Boussinesq systems introduced in [12] to model surface water waves. They couple the elevation η and the horizontal velocity $\mathbf{u}$ of the wave:

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) + \mu [a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0, \\ \mathbf{u}_t + \nabla \eta + \epsilon \frac{1}{2} \nabla |\mathbf{u}|^2 + \mu [c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \text{ in } \mathbb{R}^d \times \mathbb{R}, d = 1, 2. \quad (5.1.1)$$

Here μ and ϵ are the small parameters (shallowness and nonlinearity parameters respectively) defined as

$$\mu = \frac{h^2}{\lambda^2}, \quad \epsilon = \frac{\alpha}{h}$$

where α is a typical amplitude of the wave, h a typical depth and λ a typical horizontal wavelength.

In the Boussinesq regime, ϵ and μ are supposed to be of same order, $\epsilon \sim \mu \ll 1$, and we will take for simplicity $\epsilon = \mu$, writing (5.1.1) as

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon [\nabla \cdot (\eta \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon [\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \quad (5.1.2)$$

The class of systems (5.1.1), (5.1.2) models water waves on a flat bottom propagating in both directions in the aforementioned regime (see [12, 13, 11]).

One could also derive similar systems with a non trivial bathymetry (non flat bottom), see [18], and one has then to distinguish between the case when the bottom varies slowly and the case where it is strongly varying. In the former case, (5.1.2) has to be slightly modified and becomes

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon [\nabla \cdot ((\eta - \beta) \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0, \\ \mathbf{u}_t + \nabla \eta + \epsilon [\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \quad (5.1.3)$$

where β is a smooth function on $\mathbb{R}^d, d = 1, 2$, bounded together with its derivatives measuring the bathymetry. In the second case one gets much more complicated systems [18, 47].

Also, systems similar to (5.1.2) can be derived for internal waves, see for instance [14, 30, 35].

Recall (see [12, 31]) that the modeling parameters are constrained by the relation

$$a + b + c + d = \frac{1}{3} - \tau,$$

where $\tau \geq 0$ is the surface tension parameter (Bond number).

Recall also [12] that (5.1.2) is linearly well-posed when

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0, \quad (5.1.4)$$

and when

$$a = c, b \geq 0, d \geq 0. \quad (5.1.5)$$

It has been established in [11] that the (a,b,c,d) systems are good approximations of the full water waves system, in the relevant regime, with an error of order $O(\epsilon^2 t)$. The complete justification of the Boussinesq systems needs thus to establish the *long time existence* for the Cauchy problem, that is on time scales of order $O(1/\epsilon)$. This has been achieved for all linearly well-posed Boussinesq systems in [48, 55, 16, 56]. Note that no global well-posedness result seems to be known for the Cauchy problem of (5.1.2) in the spatial two-dimensional case (however the global existence of solutions is established in a few one-dimensional cases, see [13, 2, 57]).

It has been proved by Zakharov [62] that the full water waves system is Hamiltonian. This is not the case in general for the (a,b,c,d) systems due to the transformations made to derive them starting from the following (ill-posed) "original" Boussinesq system obtained by expanding the Dirichlet-Neumann operator with respect to ϵ (see [12, 44]):

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \frac{\epsilon}{3} \Delta \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0 \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 = 0. \end{cases} \quad (5.1.6)$$

Nevertheless, an Hamiltonian structure is found when $b = d$. More precisely, denoting by J_ϵ the skew adjoint matrix operator

$$J_\epsilon = \begin{pmatrix} 0 & \partial_x(I - \epsilon b \Delta)^{-1} & \partial_y(I - \epsilon b \Delta)^{-1} \\ \partial_x(I - \epsilon b \Delta)^{-1} & 0 & 0 \\ \partial_y(I - \epsilon b \Delta)^{-1} & 0 & 0 \end{pmatrix},$$

and

$$U = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix},$$

the Boussinesq systems write in this case

$$\partial_t U = -J_\epsilon(\text{grad } \mathcal{H}_\epsilon)(U),$$

where $\mathcal{H}_\epsilon(U)$ is the Hamiltonian given by

$$\mathcal{H}_\epsilon(U) = \frac{1}{2} \int_{\mathbb{R}^2} (-c\epsilon |\nabla \eta|^2 - a\epsilon |\nabla \mathbf{u}|^2 + \eta^2 + |\mathbf{u}|^2 + \epsilon \eta |\mathbf{u}|^2) dx dy,$$

so that $\mathcal{H}_\epsilon(U)$ is (formally) conserved by the flow.

Note that when $d = 1$ the following quantity (momentum) is also formally conserved

$$\mathcal{I}_\epsilon(U) = \int_{\mathbb{R}} (\eta u + \epsilon b \eta_x u_x) dx.$$

We are interested here in *localized* solitary wave solutions to (5.1.2). No such solutions are known to exist in the two-dimensional case (see however [26] for the existence of asymmetric periodic two-dimensional wave patterns).

We will thus focus on one-dimensional solitary waves and more specifically to their transverse stability. To start with we review the known results on the existence of one dimensional traveling waves to (5.1.2).

We thus look for solutions of (5.1.2) of the form $\eta = \eta(x - \omega t)$, $u = u(x - \omega t)$ where $x \in \mathbb{R}$ and $(\eta(x), u(x)) \rightarrow (0, 0)$ as $|x| \rightarrow \infty$ yielding the system

$$\begin{cases} -\omega \eta + u + \epsilon u \eta + \epsilon a u_{xx} - \epsilon b \eta_{xx} = 0 \\ -\omega u + \eta + \frac{\epsilon}{2} u^2 + \epsilon c u_{xx} - \epsilon d \eta_{xx} = 0. \end{cases} \quad (5.1.7)$$

Min Chen [19, 20] has found in a few cases exact solutions when η and u are proportional but it turns out that the only case compatible with the well posedness conditions (5.1.4) and (5.1.5) are

$$a - b + 2d = 0, \quad a = c, \quad d > 0 \quad (5.1.8)$$

and

$$a - b + 2d \neq 0 \text{ and } p > 0, \quad (p - \frac{1}{2})((b - a)p - b) > 0 \text{ where } p = (-b + c + 2d)/(a - b + 2d) > 0. \quad (5.1.9)$$

The exact solutions have a sech profile, namely

$$\eta(x, t) = \eta_0 \text{sech}^2(\lambda(x + x_0 - c_s t))$$

and

$$u(x, t) = \pm \sqrt{\frac{3}{\eta_0 + 3}} \eta_0 \text{sech}^2(\lambda(x + x_0 - c_s t)),$$

where λ and c_s are appropriate constants.

Note that conditions (5.1.8), (5.1.9) can hold in the non Hamiltonian case $b \neq d$.

On the other hand no general *non existence result* seems to be available. One has nevertheless the following remark which excludes the existence of solitary waves with small velocity when the well-posedness condition (5.1.5) holds true with moreover $b = d$.

Proposition 3. *Let $b = d \geq 0$. Then any H^1 solitary wave solution (η, u) satisfies the identity*

$$\int (u^2 + \eta^2 + 5\epsilon(au_x^2 + c\eta_x^2))dx = 2\omega \int (3u\eta - 5\epsilon bu_x \eta_x)dx$$

Proof. This is a standard Pohojaev type argument. The following computations can be justified by a regularizing procedure.

We multiply the first equation in (5.1.7) by xu_x , the second one by $x\eta_x$. Integrating the resulting equations and adding one gets

$$\frac{1}{2} \int (u^2 + \eta^2 + a\epsilon u_x^2 + c\epsilon \eta_x^2) = \int (\omega u\eta - \epsilon b \omega u_x \eta_x - \frac{\epsilon}{2} u^2 \eta). \quad (5.1.10)$$

We then multiply the first equation in (5.1.7) by u the second one by η . Integrating the resulting equations and adding one gets

$$\int (u^2 + \eta^2 - a\epsilon u_x^2 - c\epsilon \eta_x^2) = \epsilon \int (2\omega b u_x \eta_x - \frac{3}{2} \eta u^2). \quad (5.1.11)$$

The proposition results from subtracting three times (5.1.10) from (5.1.11). □

As a consequence, one obtains that no non trivial solitary waves with ω small exist when

$$a \geq 0, \quad c \geq 0, \quad b = 0,$$

or

$$a > 0, \quad c > 0, \quad b > 0.$$

We recall that one needs the condition $a = c$ to get a well-posed system when a, c are not negative.

The paper is organized as follows. In the next section we survey the known existing results on existence (and stability) of one-dimensional traveling waves to (5.1.2). In section 3 we prove the suitable long time existence result for the systems derived from (5.1.2) by a localized perturbation of a line solitary wave. Then we focus on the transverse stability of line solitary waves. Finally, in a last section we consider various variants of the (a,b,c,d) systems arising mainly in the modeling of internal waves.

Notation. Vectors will be denoted in bold letters, for example $\mathbf{v}$. For any $s \in \mathbb{R}$, the L^2 based Sobolev spaces are denoted by $H^s(\mathbb{R}^n)$. If X is a Banach space, the X norm is $\|\cdot\|_X$. The $L^2(\mathbb{R}^n)$ scalar product is denoted by $(u, v)_2$. We use the notation $\Lambda = (I - \Delta)^{1/2}$ and the notation $f \lesssim g$ means that there exists a constant C such that $f \leq Cg$. The condensed notation $A_s = B_s + \langle C_s \rangle_{s \geq \tilde{s}}$ says that $A_s = B_s$ if $s \leq \tilde{s}$ and $A_s = B_s + C_s$ if $s > \tilde{s}$.

if A, B are two operators, $[A, B] = AB - BA$ denotes their commutator. Throughout the paper, C will always denote a universal constant which may be different from line to line but is independent of the parameters involved.

5.2 Known results on one-dimensional solitary waves

We now turn to the existence results for solitary waves. Except the aforementioned explicit cases aforementioned, most of the results concern the Hamiltonian case $b = d$.

When

$$b = d > 0, \quad a < 0, \quad c < 0,$$

the existence of solitary waves with small propagation speeds is obtained in [28] by using the concentration-compactness method to minimize in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ the functional (where $\mu > 0$ is fixed)

$$\mathcal{E}_\mu(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} (-\epsilon c u_x^2 - \epsilon a \eta_x^2 + \eta^2 + u^2) dx - \mu \int_{\mathbb{R}} (\eta u + \epsilon b \eta_x u_x) dx$$

under the constraint

$$P(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} \eta u^2 dx = p$$

Assuming moreover that

$$ac > b^2, \tag{5.2.1}$$

the solitary waves are shown in [27] to be orbitally stable. Note that (5.2.1) implies that $a+b+c+d < 0$, that is $\tau > \frac{1}{3}$, corresponding to strong surface tension.

We now focus on the case

$$a < 0, c < 0, b = d = 0.$$

Oliveira [49] has proven in this case the existence of a family of non-negative, radially decreasing, exponentially decaying, solitary waves with small velocity and having a uniform (in velocity) L^2 bound.

5.2.1 Boussinesq and Euler-Korteweg

Of special interest is the case $a = b = d = 0, c < 0$ since it appears to be a particular case of Euler-Korteweg systems that write

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla(g(\rho)) = \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right), \end{cases} \quad (5.2.2)$$

where $K(\rho) > 0$ is the capillarity coefficient. See Benzoni-Danchin-Descombes [9] for the LWP of the Cauchy problem for "general" $g(\rho)$ and $K(\rho)$ and also [10] where a long time existence result is obtained for Euler-Korteweg systems in a Boussinesq scaling.

Setting $K(\rho) = K > 0, g(\rho) = \rho, \rho = 1 + \eta, \mathbf{u} = \nabla \psi$, one gets

$$\begin{cases} \partial_t \eta + \nabla \cdot \mathbf{u} + \nabla \cdot (\eta \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2 + \nabla \eta = K \nabla \Delta \eta, \end{cases} \quad (5.2.3)$$

which is the $(0, 0, -K, 0)$ Boussinesq system.

To recover exactly an (abcd) Boussinesq system involving the small parameter ϵ , one has to consider for (5.2.2) the *Boussinesq regime* (see [10], that is

$$\eta(x, y, t) = \epsilon \tilde{\eta}(\sqrt{\epsilon}x, \sqrt{\epsilon}y, \sqrt{\epsilon}t), \quad u(x, y, t) = \epsilon \tilde{u}(\sqrt{\epsilon}x, \sqrt{\epsilon}y, \sqrt{\epsilon}t).$$

When applied to (5.2.3) this yields the $(0, 0, 0, -K)$ Boussinesq for $(\tilde{\eta}, \tilde{u})$ in the variables $(X, Y, T) = (\sqrt{\epsilon}x, \sqrt{\epsilon}y, \sqrt{\epsilon}t)$.

One-dimensional solitary waves for (5.2.2) have been studied in [5, 6, 8, 3, 50]. When applied to the Boussinesq system (5.2.3), they yield the existence of solitary waves of velocity $|\omega| < 1$. The existence of solitary waves follows from a phase portrait analysis of the governing ODEs. They can also be viewed as critical points of the Hamiltonian

$$H(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} (K \eta_x^2 + \eta^2 + u^2 + u^2 \eta) dx$$

under the constraint

$$\mathcal{Q}(\eta, u) = \int_{\mathbb{R}} \eta u dx = \text{const},$$

see [6].

The stability of solitary waves of speed ω is governed by the convexity of the Boussinesq momentum of instability

$$m(\omega) = \mathcal{H}_\epsilon(\eta, u) - \omega \mathcal{Q}(\eta, u)$$

More precisely, the solitary wave (η_ω, u_ω) is orbitally stable when

$$\frac{\partial^2 m}{\partial \omega^2}(\omega) > 0$$

and linearly unstable when

$$\frac{\partial^2 m}{\partial \omega^2}(\omega) < 0.$$

The transverse stability of one-dimensional solitary waves of (5.2.2) is studied in [52, 50]. Again we restrict those results to the particular case (5.2.3). The linear instability of solitary waves with

velocity $|\omega| < 1$ is proved in [6, 52]. The nonlinear instability (for localized perturbations in (x, y)) is established in [50].

5.2.2 Further studies

Other studies concern the weaker notion of spectral stability. Numerical investigations of the spectral stability (both for one-dimensional and transverse perturbations) are displayed in [29] for the cases

$$a = -\frac{1}{9}, \quad b = \frac{1}{3}, \quad c = -\frac{1}{9}, \quad d = \frac{2}{9}$$

and

$$a = -\frac{8}{9}, \quad c = -\frac{8}{9}, \quad b = \frac{10}{9}, \quad d = 1,$$

for which the solitary waves are explicit ([19, 20]).

Rigorous spectral stability results are proven in [37] for the explicit solitary waves known to exist in the cases

$$a = c = -b, \quad b = d > 0$$

and

$$a = c < 0, \quad b = d > 0.$$

On the other hand, localized solitary waves have been proved to exist for the Bona-Smith class of Boussinesq systems. The original Bona-Smith system derived in [15] corresponds to $a = 0, b = d = \frac{1}{3}, c = -\frac{1}{3}$. The existence of solitary waves in this case is due to Toland [58].

A general class of Bona-Smith systems corresponds to $a = 0, b = d = \frac{3\theta^2-1}{6}, c = \frac{2-3\theta^2}{3}, \frac{2}{3} \leq \theta^2 \leq 1$ where θ parametrizes the height of the fluid layer where the horizontal velocity is taken, see [15]. Then, the techniques of Toland ([59]) can be used to prove the existence of solitary waves of arbitrary velocity $c > 1$ (see [33, 32]). The uniqueness of such solitary waves is considered in [60].

Remark 17. *The Boussinesq systems of the Bona-Smith class are among the few for which one can get the global existence of solutions to the Cauchy problem. Actually (see [15, 13]), such a result is obtained for initial data satisfying the non cavitation condition $1 + \epsilon\eta_0 > 0$ and having a sufficiently small Hamiltonian $\mathcal{H}(\eta_0, u_0)$.*

Remark 18. *The Boussinesq system corresponding to $a = c = b = 0, d > 0$ has a particular importance since it is the BBM version of the natural (ill-posed) system obtained by expanding the Dirichlet-Neumann operator with respect to ϵ in the full water wave system (see [44]).*

By considering it as a dispersive perturbation of the hyperbolic Saint-Venant system, Schonbek and Amick ([57, 2]) have proved that the global existence of the Cauchy problem for the one-dimensional $a = c = d = 0, b > 0$ system has global solutions for arbitrary large initial data. On the other hand this system is not Hamiltonian and to our knowledge no (positive or negative) existence result for solitary waves is known.

The case $a = c = d = 0, b > 0$ is also particular. It is linearly ill-posed but in the one-dimensional case this ill-posed system is also known as the Kaup-Kupperschmidt system and it is completely integrable ([39, 41]).

5.3 Long time existence

We show here how to extend the long time existence results in [55, 56] to the systems obtained by perturbing a line soliton by a localized two-dimensional perturbation.

Remark 19. *Since it does not rely on dispersive techniques but rather on "hyperbolic" symmetrization arguments, the method used in [55, 56] works as well for periodic perturbations in y , that is for the problem posed on $\mathbb{R} \times \mathbb{T}$. It works also for the purely periodic Cauchy problem posed on $\mathbb{T}^2$ which is the right framework to investigate the transverse stability of cnoïdal waves which we do not address here.*

We will consider of course only the values of the coefficients (a, b, c, d) for which the existence of line solitary waves is known. Since existence of solitary waves is known only when

$$a < 0, c < 0, b > 0, d > 0, \quad (5.3.1)$$

(that is the *generic case* in the terminology of [55]),

or

$$a < 0, c < 0, b = 0, d > 0 \quad (5.3.2)$$

or

$$a = 0, c < 0, b = 0, d > 0 \quad (5.3.3)$$

or

$$a = b = d = 0, c < 0 \quad (5.3.4)$$

we will from now on assume that (5.3.1) or (5.3.2) or (5.3.3) holds.

Let $\tilde{U}_\omega = (\tilde{\eta}_\omega, \tilde{\mathbf{u}}_\omega)$ a one dimensional solitary wave of velocity ω . As aforementioned, studying the transverse stability of $\tilde{U}_c$ with respect to periodic transverse perturbations in y would necessitate to solve the Cauchy problem for (5.1.1) in a $H^s(\mathbb{R} \times \mathbb{T})$ setting and this can be done as in [55, 56] for the $H^s(\mathbb{R}^2)$ case. We consider now localized perturbations of $\tilde{U}_\omega$. They satisfy the system

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon[\nabla \cdot ((\eta \mathbf{u}) + \tilde{\eta}_\omega \mathbf{u} + \tilde{\mathbf{u}}_\omega \eta) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon[\nabla(\mathbf{u} \cdot \tilde{\mathbf{u}}_\omega) + \frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases} \quad (5.3.5)$$

with initial condition $(\eta_0, \mathbf{u}_0)$.

Before stating the main result of this section we need to introduce the space (see [55]) :

Definition 5.3.1. *For any $s \in \mathbb{R}$, $k \in \mathbb{N}$, $\epsilon \in (0, 1)$, the Banach space $X_{\epsilon^k}^s(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm:*

$$\|u\|_{X_{\epsilon^k}^s}^2 = \|u\|_{H^s}^2 + \epsilon^k \|u\|_{H^{s+k}}^2. \quad (5.3.6)$$

The solutions of system (5.3.5) will belong to some space $X_{\epsilon^k}^s(\mathbb{R}^n) \times X_{\epsilon^{k'}}^s(\mathbb{R}^n)$ with k and k' determined by a, b, c, d as follows :

Definition 5.3.2. *For any a, b, c, d satisfying (5.3.1), (5.3.2), (5.3.3) we define a pair of numbers $(k, k') = (k(a, b, c, d), k'(a, b, c, d))$ according to the admissible sets of (a, b, c, d) 's as follows:*

- (3) $(k, k') = (3, 4)$ for $b = 0, d > 0, a, c < 0$;
- (4) $(k, k') = (3, 3)$ for $b \neq d, b, d > 0, a, c < 0$;
- (7) $(k, k') = (2, 1)$ for $b = d > 0, a = 0, c < 0$;

Theorem 5.3.3. *Let $t_0 > \frac{n}{2}$, $n = 1, 2$, $s \geq t_0 + 2$ if $b \neq d$ or $b = d > 0$, $s \geq t_0 + 4$ if $b = d = 0$. Assume that $\eta_0 \in X_{\epsilon^k}^s(\mathbb{R}^n)$, $\mathbf{u}_0 \in X_{\epsilon^{k'}}^s(\mathbb{R}^n)$ satisfy the (non-cavitation) condition*

$$1 - \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1), \quad (5.3.7)$$

where (k, k') is defined in Definition 5.3.2. Then there exists a constant $\tilde{c}_0$ such that for any $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\eta_0|_{X_{\epsilon^k}^s} + |\mathbf{u}_0|_{X_{\epsilon^{k'}}^s})}$, there exists $T > 0$, such that (5.3.5) has a unique solution $(\epsilon, \mathbf{v})^T$ with $\eta \in C([0, T]; X_{\epsilon^k}^s(\mathbb{R}^n))$ and $\mathbf{u} \in C([0, T]; X_{\epsilon^{k'}}^s(\mathbb{R}^n))$. Moreover,

$$\max_{t \in [0, T]} (\|\eta\|_{X_{\epsilon^k}^s} + \|\mathbf{u}\|_{X_{\epsilon^{k'}}^s}) \leq \tilde{c}(\|\eta_0\|_{X_{\epsilon^k}^s} + \|\mathbf{u}_0\|_{X_{\epsilon^{k'}}^s}), \quad (5.3.8)$$

here $\tilde{c} = C(H^{-1})$ and $\tilde{c}_0 = C(H^{-1})$ are nondecreasing functions of their argument.

More precisely, the existence time scale, $T = O(1/\sqrt{\epsilon})$ if $b \neq d$, $b, d > 0$, $a, c < 0$. And $T = O(1/\epsilon)$ if $b = d > 0$, $a = 0$, $c < 0$.

5.3.1 Linearization of (5.3.5)

We will use similar idea as in [55]. Setting $\mathbf{U} = (\eta, \mathbf{u})^T$, $\mathbf{V} = (\zeta, \mathbf{v})^T = \epsilon \mathbf{U}$, we rewrite (5.1.2) as

$$\begin{cases} (1 - b\epsilon\Delta)\partial_t \zeta + \nabla \cdot \mathbf{v} + \nabla \cdot (\zeta \mathbf{v}) + a\epsilon \nabla \cdot \Delta \mathbf{v} = 0, \\ (1 - d\epsilon)\partial_t \mathbf{v} + \nabla \zeta + \frac{1}{2} \nabla(|\mathbf{v}|^2) + c\epsilon \nabla \Delta \zeta = \mathbf{0}. \end{cases} \quad (5.3.9)$$

If $b > 0$, $d \geq 0$ or $b = d = 0$, let $g(D) = (1 - b\epsilon\Delta)(1 - d\epsilon\Delta)^{-1}$, then (5.3.9) is equivalent after applying $g(D)$ to the second equation to the condensed system

$$(1 - b\epsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V}, D)\mathbf{V} = \mathbf{0}, \quad (5.3.10)$$

where

$$M(\mathbf{V}, D) = \begin{pmatrix} \mathbf{v} \cdot \nabla & (1 + \zeta + a\epsilon\Delta)\partial_{x_1} & (1 + \zeta + a\epsilon\Delta)\partial_{x_2} \\ g(D)(1 + c\epsilon\Delta)\partial_{x_1} & g(D)(v_1\partial_{x_1}) & g(D)(v_2\partial_{x_1}) \\ g(D)(1 + c\epsilon\Delta)\partial_{x_2} & g(D)(v_1\partial_{x_2}) & g(D)(v_2\partial_{x_2}) \end{pmatrix} \quad (5.3.11)$$

Now we denote $\tilde{\mathbf{V}}_\omega = (\tilde{\zeta}_\omega, \tilde{\mathbf{v}}_\omega) = \epsilon \tilde{\mathbf{U}}_\omega$, then the equation (5.3.5) is equivalent to the following perturbation of (5.3.10) when we replace $\mathbf{V}$ by $\mathbf{V} + \tilde{\mathbf{V}}_\omega$

$$(1 - b\epsilon\Delta)\partial_t(\mathbf{V} + \tilde{\mathbf{V}}_\omega) + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)(\mathbf{V} + \tilde{\mathbf{V}}_\omega) = \mathbf{0},$$

or

$$(1 - b\epsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = -(M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D))\tilde{\mathbf{V}}_\omega. \quad (5.3.12)$$

In order to solve (5.3.5), we consider the following linearized equation

$$(1 - b\epsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = -(M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D))\tilde{\mathbf{V}}_\omega, \quad (5.3.13)$$

or

$$(1 - b\epsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = F \quad (5.3.14)$$

where $M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)$ and $M(\tilde{\mathbf{V}}_\omega, D)$ are defined in (5.3.11), and note that we identify the notations of $\tilde{\mathbf{V}}_\omega$ and $\underline{\tilde{\mathbf{V}}}_\omega$ since the line soliton is considered as a known function.

The idea is to treat the equation (5.3.14) as a symmetrizable hyperbolic system under some smallness assumption on $\underline{\mathbf{V}}$ and $\tilde{\mathbf{V}}$. It is known in [55] that there exists a symmetrizer $S_{\underline{\mathbf{V}}}(D)$ of $M(\underline{\mathbf{V}}, D)$ such that the principal part of $iS_{\underline{\mathbf{V}}}(\xi)M(\underline{\mathbf{V}}, \xi)$ is self-adjoint, and that of $S_{\underline{\mathbf{V}}}(\xi)$ is positive and self-adjoint under a smallness assumption on $\underline{\mathbf{V}}$. Fortunately, it is also true for the symmetrizer $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$ of $M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)$, so that we can apply that method for solving (5.3.14) with some slight modifications.

From [55], we have

(i) If $b = d$, $g(D) = 1$, $S_{\underline{\mathbf{V}}}(D)$ is defined by

$$\begin{pmatrix} 1 + c\epsilon\Delta & v_1 & v_2 \\ v_1 & 1 + \underline{\zeta} + a\epsilon\Delta & 0 \\ v_2 & 0 & 1 + \underline{\zeta} + a\epsilon\Delta \end{pmatrix}; \quad (5.3.15)$$

then

$$S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}}(D) + S_1, \quad (5.3.16)$$

where

$$S_1 = \begin{pmatrix} 0 & \tilde{v}_{1\omega} & \tilde{v}_{2\omega} \\ \tilde{v}_{1\omega} & \tilde{\zeta}_\omega & 0 \\ \tilde{v}_{2\omega} & 0 & \tilde{\zeta}_\omega \end{pmatrix}.$$

(ii) If $b \neq d$, $S_{\underline{\mathbf{V}}}(D)$ is defined by

$$\begin{pmatrix} (1 + c\epsilon\Delta)^2 g(D) & g(D)(v_1(1 + c\epsilon\Delta)) & g(D)(v_2(1 + c\epsilon\Delta)) \\ g(D)(v_1(1 + c\epsilon\Delta)) & (1 + \underline{\zeta} + a\epsilon\Delta)(1 + c\epsilon\Delta) & 0 \\ g(D)(v_2(1 + c\epsilon\Delta)) & 0 & (1 + \underline{\zeta} + a\epsilon\Delta)(1 + c\epsilon\Delta) \end{pmatrix} \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_1 v_1 & v_1 v_2 \\ 0 & v_1 v_2 & v_2 v_2 \end{pmatrix} (g(D) - 1). \quad (5.3.17)$$

Then

$$S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}} + S_1 + S_2, \quad (5.3.18)$$

where

$$S_1 = \begin{pmatrix} 0 & g(D)(\tilde{v}_{1\omega}(1 + c\epsilon\Delta)) & g(D)(\tilde{v}_{2\omega}(1 + c\epsilon\Delta)) \\ g(D)(\tilde{v}_{1\omega}(1 + c\epsilon\Delta)) & \tilde{\zeta}_\omega(1 + c\epsilon\Delta) & 0 \\ g(D)(\tilde{v}_{2\omega}(1 + c\epsilon\Delta)) & 0 & \tilde{\zeta}_\omega(1 + c\epsilon\Delta) \end{pmatrix} \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{v}_{1\omega}^2 & \tilde{v}_{1\omega}\tilde{v}_{2\omega} \\ 0 & \tilde{v}_{1\omega}\tilde{v}_{2\omega} & \tilde{v}_{2\omega}^2 \end{pmatrix} (g(D) - 1),$$

and

$$S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2v_1\tilde{v}_{1\omega} & v_1\tilde{v}_{2\omega} + v_2\tilde{v}_{1\omega} \\ 0 & v_1\tilde{v}_{2\omega} + v_2\tilde{v}_{1\omega} & 2v_2\tilde{v}_{2\omega} \end{pmatrix} (g(D) - 1).$$

We also need the following expression

$$M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) = M(\tilde{\mathbf{V}}_\omega, D) + \tilde{M}_1, \quad (5.3.19)$$

where

$$\tilde{M}_1 = \begin{pmatrix} \underline{\mathbf{v}} \cdot \nabla & \underline{\zeta} \partial_{x_1} & \underline{\zeta} \partial_{x_2} \\ 0 & g(D)(\underline{v}_1 \partial_{x_1}) & g(D)(\underline{v}_2 \partial_{x_1}) \\ 0 & g(D)(\underline{v}_1 \partial_{x_2}) & g(D)(\underline{v}_2 \partial_{x_2}) \end{pmatrix}.$$

And

$$M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) = M(\underline{\mathbf{V}}, D) + M_1, \quad (5.3.20)$$

where

$$M_1 = \begin{pmatrix} \tilde{\underline{\mathbf{v}}}_\omega \cdot \nabla & \tilde{\underline{\zeta}}_\omega \partial_{x_1} & \tilde{\underline{\zeta}}_\omega \partial_{x_2} \\ 0 & g(D)(\tilde{\underline{v}}_{1\omega} \partial_{x_1}) & g(D)(\tilde{\underline{v}}_{2\omega} \partial_{x_1}) \\ 0 & g(D)(\tilde{\underline{v}}_{1\omega} \partial_{x_2}) & g(D)(\tilde{\underline{v}}_{2\omega} \partial_{x_2}) \end{pmatrix}.$$

Remark 20. Because the solitary wave has the “sech” profile and using the expression (5.3.19), it is not hard to see that the term F on the right hand side of (5.3.14) is in $X_\epsilon^s(\mathbb{R}^2)$ if $\underline{\mathbf{V}} \in X_{\epsilon^k}^s(\mathbb{R}^2)$ for $s, k > 0, s \in \mathbb{N}$.

Next we define the energy functional associated to (5.3.14) as

$$E_s(\underline{\mathbf{V}}) = \left((1 - b\epsilon\Delta)\Lambda^s \underline{\mathbf{V}}, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)\Lambda^s \underline{\mathbf{V}} \right)_2. \quad (5.3.21)$$

We shall show that $E_s(\underline{\mathbf{V}})$ defined in (5.3.21) is equivalent to some $X_{\epsilon^k}^s(\mathbb{R}^2)$ norm.

Remark 21. when $b = 0, d > 0$, (5.3.9) is equivalent after applying $(1 - d\epsilon\Delta)$ to the first equation to the condensed system

$$(1 - d\epsilon\Delta)\partial_t \underline{\mathbf{V}} + M(\underline{\mathbf{V}}, D)\underline{\mathbf{V}} = 0,$$

with $M(\underline{\mathbf{V}}, D)$ defined by

$$\begin{pmatrix} (1 - d\epsilon\Delta)(\underline{\mathbf{v}} \cdot \nabla) & (1 - d\epsilon\Delta)((1 + \underline{\zeta} + a\epsilon\Delta)\partial_{x_1}) & (1 - d\epsilon\Delta)((1 + \underline{\zeta} + a\epsilon\Delta)\partial_{x_2}) \\ (1 + c\epsilon\Delta)\partial_{x_1} & v_1 \partial_{x_1} & v_2 \partial_{x_1} \\ (1 + c\epsilon\Delta)\partial_{x_2} & v_1 \partial_{x_2} & v_2 \partial_{x_2} \end{pmatrix}.$$

Then the equation (5.3.5) is equivalent to

$$(1 - d\epsilon\Delta)\partial_t \underline{\mathbf{V}} + M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)\underline{\mathbf{V}} = - \left(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D) \right) \tilde{\mathbf{V}}_\omega.$$

The symmetrizer $S_{\underline{\mathbf{V}}}(D)$ of $M(\underline{\mathbf{V}}, D)$ is defined by

$$\begin{pmatrix} (1 + c\epsilon\Delta)^2 & \underline{v}_1(1 + c\epsilon\Delta) & \underline{v}_2(1 + c\epsilon\Delta) \\ \underline{v}_1(1 + c\epsilon\Delta) & (1 + c\epsilon\Delta)[(1 + \underline{\zeta} + a\epsilon\Delta)(1 - d\epsilon\Delta)] & 0 \\ \underline{v}_2(1 + c\epsilon\Delta) & 0 & (1 + c\epsilon\Delta)[(1 + \underline{\zeta} + a\epsilon\Delta)(1 - d\epsilon\Delta)] \end{pmatrix} \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & d\epsilon \underline{v}_1 \underline{v}_1 \Delta & d\epsilon \underline{v}_1 \underline{v}_2 \Delta \\ 0 & d\epsilon \underline{v}_1 \underline{v}_2 \Delta & d\epsilon \underline{v}_2 \underline{v}_2 \Delta \end{pmatrix}.$$

Therefore, the symmetrizer $S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)$ of $M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)$ is defined by

$$S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}}(D) + S_1 + S_2,$$

where

$$S_1 = \begin{pmatrix} 0 & \tilde{v}_{1\omega}(1 + c\epsilon\Delta) & \tilde{v}_{2\omega}(1 + c\epsilon\Delta) \\ \tilde{v}_{1\omega}(1 + c\epsilon\Delta) & (1 + c\epsilon\Delta)[\tilde{\zeta}_{\omega}(1 - d\epsilon\Delta)] & 0 \\ \tilde{v}_{2\omega}(1 + c\epsilon\Delta) & 0 & (1 + c\epsilon\Delta)[\tilde{\zeta}_{\omega}(1 - d\epsilon\Delta)] \end{pmatrix} \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & d\epsilon\tilde{v}_{1\omega}\tilde{v}_{1\omega}\Delta & d\epsilon\tilde{v}_{1\omega}\tilde{v}_{2\omega}\Delta \\ 0 & d\epsilon\tilde{v}_{1\omega}\tilde{v}_{2\omega}\Delta & d\epsilon\tilde{v}_{2\omega}\tilde{v}_{2\omega}\Delta \end{pmatrix},$$

and

$$S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2d\epsilon v_1\tilde{v}_{1\omega}\Delta & d\epsilon(v_1\tilde{v}_{2\omega} + v_2\tilde{v}_{1\omega})\Delta \\ 0 & d\epsilon(v_1\tilde{v}_{2\omega} + v_2\tilde{v}_{1\omega})\Delta & 2d\epsilon v_2\tilde{v}_{2\omega}\Delta \end{pmatrix}.$$

Remark 22. We decomposed $S_{\underline{\mathbf{V}}+\underline{\mathbf{V}}_{\omega}}(D)$ into three components: $S_{\underline{\mathbf{V}}}(D)$, S_1 which only depends on $\underline{\mathbf{V}}_{\omega}$, and S_2 which depends on both $\underline{\mathbf{V}}$ and $\underline{\mathbf{V}}_{\omega}$.

Before going to the main results of this section, we recall the following definitions and Lemma which can be found in [55].

Lemma 5.3.4. If $t_0 > \frac{n}{2}$ ($n = 1, 2$) and $s \geq 0$, one has

$$\|fg\|_{H^s} \leq C \left(\|f\|_{H^{t_0}} \|g\|_{H^s} + \langle \|f\|_{H^s} \|g\|_{H^{t_0}} \rangle_{s>t_0} \right), \quad \forall f, g \in H^s \cap H^{t_0}(\mathbb{R}^n). \quad (5.3.22)$$

Lemma 5.3.5. Let $\phi \in H^s(\mathbb{R}_{x_1})$ for every $s \geq 1$, $t_0 > 1$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^2)$. Then

$$\|\phi f\|_{H^s} \leq C \left(\|\phi\|_{H_{x_1}^{t_0}} \|f\|_{H^s} + \left\langle \|\phi\|_{H_{x_1}^s} \cdot \|f\|_{H^{t_0}} \right\rangle_{s>t_0} \right) \quad (5.3.23)$$

Definition 5.3.6. We say that a Fourier multiplier $\sigma(D)$ is of order s ($s \in \mathbb{R}$) and write $\sigma \in \mathcal{S}^s$ if $\xi \in \mathbb{R}^d \mapsto \sigma(\xi) \in \mathbb{C}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta|-s} |\partial^\beta \sigma(\xi)| < \infty.$$

Lemma 5.3.7. Let $t_0 > d/2$, $s \geq 0$ and $\sigma \in \mathcal{S}^s$. If $f \in H^s(\mathbb{R}^d) \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$ then

$$\|[\sigma(D), f]g\|_{L^2} \leq C(\sigma) (\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}} \|g\|_{L^\infty}), \quad (5.3.24)$$

where $C(\sigma)$ depends only on σ .

Lemma 5.3.8. Let $b, d > 0$ and $b \neq d$, $s \in \mathbb{R}$, $\theta \geq 0$, then

(i) for all $f \in H^s(\mathbb{R}^n)$, there hold

$$\min \left\{ 1, \left(\frac{b}{d} \right)^\theta \right\} \|f\|_{H^s} \leq \|g(D)^\theta f\|_{H^s} \leq \max \left\{ 1, \left(\frac{b}{d} \right)^\theta \right\} \|f\|_{H^s}, \quad (5.3.25)$$

$$\|(g(D) - 1)f\|_{H^s} \leq \frac{|b - d|}{d} \|f\|_{H^s}. \quad (5.3.26)$$

(ii) Let $t_0 > \frac{n}{2}$, $-t_0 < r \leq t_0 + 1$, for all $f \in H^{t_0+1}(\mathbb{R}^n)$ and $u \in H^{r-1}(\mathbb{R}^n)$, there holds

$$\| [g(D)^\theta, f] u \|_{H^r} \leq C \|f\|_{H^{t_0+1}} \|u\|_{H^{r-1}}, \quad (5.3.27)$$

where C is a constant independent of ϵ .

5.3.2 Sovability of linearized equation

The case $b \neq d$, $b, d > 0$, $a, c < 0$

Proposition 4. Let $t_0 > 1$, $s \geq t_0 + 1$, $s \in \mathbb{N}$, $T' > 0$. Assume that $F \in C([0, T']; X_\epsilon^s(\mathbb{R}^2))$ and $\underline{\mathbf{V}} \in C^1([0, T']; X_{\epsilon^3}^{s-1}(\mathbb{R}^2)) \cap C([0, T']; X_{\epsilon^3}^s(\mathbb{R}^2))$ satisfies

$$1 + \underline{\zeta} + \underline{\zeta}_\omega \geq H > 0, \quad \|\underline{\mathbf{V}}\|_\infty + \|\tilde{\underline{\mathbf{V}}}_\omega\|_\infty \leq \kappa_H, \quad \|\underline{\mathbf{V}}\|_{H^s} + \|\tilde{\underline{\mathbf{V}}}_\omega\|_{H^s} \leq 1, \quad \forall t \in [0, T'], \quad (5.3.28)$$

with κ_H ensures the equivalence of $E_s(\mathbf{V})$ and $\|\mathbf{V}\|_{X_{\epsilon^3}^s}^2$. (see the Appendix 5.6.1)

$$\frac{1}{c} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2 \leq E_s(\mathbf{V}) \leq C \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.29)$$

Then for any $V_0 \in X_{\epsilon^3}^s(\mathbb{R}^2)$, (5.3.14) has a unique solution $\mathbf{V} \in C^1([0, T']; X_{\epsilon^3}^{s-1}(\mathbb{R}^2)) \cap C([0, T']; X_{\epsilon^3}^s(\mathbb{R}^2))$ and one has for any $t \in (0, T')$.

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) \leq & \tilde{c} \left(\left(1 + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^3}^{s-1}} + \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \right) \|F\|_{X_{\epsilon^3}^s} E_s(\mathbf{V})^{1/2} \right. \\ & \left. + \epsilon^{1/2} \left(1 + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^3}^{s-1}} + \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \right) E_s(\mathbf{V}) \right), \end{aligned} \quad (5.3.30)$$

$$\begin{aligned} \|\mathbf{V}(t)\|_{X_{\epsilon^3}^s}^2 \leq & \tilde{c} \|\mathbf{V}_0\|_{X_{\epsilon^3}^s}^2 + \tilde{c} \int_0^t \left(\left(1 + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^3}^{s-1}} + \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \right) \|F\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s} \right. \\ & \left. + \epsilon^{1/2} \left(1 + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^3}^{s-1}} + \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \right) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2 \right) dt, \end{aligned} \quad (5.3.31)$$

$$\|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}} \leq C \left(\|F\|_{H^s} + \|\mathbf{V}\|_{X_{\epsilon^3}^s} \right). \quad (5.3.32)$$

Remark 23. The assumption (5.3.28) is a smallness condition which holds if ϵ is small enough.

Proof. We derive a priori estimate on (5.3.14). First, we have that

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) = & \left(\Lambda^s F, \left(S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ & - \left(\Lambda^s \left(M(\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega, D) \mathbf{V} \right), \left(S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ & - b\epsilon \left(\left[S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D)^*, \Delta \right] \Lambda^s \mathbf{V}, \Lambda^s \partial_t \mathbf{V} \right)_2 + \left((1 - b\epsilon \Delta) \Lambda^s \mathbf{V}, \left(\partial_t S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D) \right) \Lambda^s \mathbf{V} \right)_2 \\ = & I + II + III + IV, \end{aligned} \quad (5.3.33)$$

where $S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D)^*$ is the adjoint operator of $S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D)$.

Estimate on I. By using integral by parts, we have

$$\begin{aligned} \left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 = & \left(\Lambda^s F_1, (1 + c\epsilon \Delta)^2 g(D) \Lambda^s \zeta \right)_2 + \left(\Lambda^s F_1, g(D) ((\underline{\mathbf{v}} + \tilde{\underline{\mathbf{v}}}_\omega)(1 + c\epsilon \Delta) \Lambda^s \mathbf{v}) \right)_2 \\ & + \sum_{i=1}^2 \left\{ \left(\Lambda^s F_{i+1}, g(D) ((\underline{v}_i + \tilde{v}_{i\omega})(1 + c\epsilon \Delta) \Lambda^s \zeta) \right)_2 \right. \\ & + \left(\Lambda^s F_{i+1}, (1 + \underline{\zeta} + \tilde{\zeta})(1 + c\epsilon \Delta) \Lambda^s v_i \right)_2 - a\epsilon \left(\nabla \Lambda^s F_{i+1}, \nabla (1 + c\epsilon \Delta) \Lambda^s v_i \right)_2 \\ & \left. + \left(\Lambda^s F_{i+1}, (\underline{v}_i + \tilde{v}_{i\omega})(\underline{\mathbf{v}} + \tilde{\underline{\mathbf{v}}}_\omega) \cdot (g(D) - 1) \Lambda^s \mathbf{v} \right)_2 \right\}, \end{aligned}$$

which together with (5.3.28), (5.3.25) and (5.3.26) implies that

$$\left| \left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \right| \lesssim \left(\|F\|_{H^s} + \epsilon^{1/2} \|F\|_{H^{s+1}} \right) \left(\|\mathbf{V}\|_{H^s} + \epsilon \|\mathbf{V}\|_{H^{s+2} + \epsilon^{3/2} \|\mathbf{V}\|_{H^{s+3}}} \right).$$

Similarly we have

$$\left| \left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \Lambda^s \mathbf{V} \right)_2 \right| \lesssim \left(\|\mathbf{V}\|_{H^s} + \epsilon^{1/2} \|\mathbf{V}\|_{H^{s+1}} \right) \left(\|F\|_{H^s} + \epsilon \|F\|_{H^{s+2}} + \epsilon^{3/2} \|F\|_{H^{s+3}} \right),$$

Therefore,

$$|I| \lesssim \|F\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}. \quad (5.3.34)$$

Estimate on II . We have

$$\begin{aligned} II &= - \left(\left[\Lambda^s, M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) \right] \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &\quad - \left(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) \Lambda^s \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &= II_1 + II_2. \end{aligned}$$

Using (5.3.20) and (5.3.18), we can rewrite

$$\begin{aligned} II_1 &= - \left([\Lambda^s, M(\underline{\mathbf{V}}, D)] \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &\quad - \left([\Lambda^s, M_1] \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &= II_{11} + II_{12}. \end{aligned}$$

Estimate on II_{11} . Since $\tilde{\mathbf{V}}_\omega$ is sufficiently smooth and bounded, we can use the same argument as in [55] to get

$$|II_{11}| \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.35)$$

Estimate on II_{12} . M_1 depends only in $\tilde{\mathbf{V}}_\omega$ then there will be a problem if we apply the commutator estimate (5.3.24) directly to Λ^s . We use the expression $\Lambda^s = \Lambda_1^s + \Lambda_2^s$ and estimate II_{12} separately as follows

First, we have

$$\begin{aligned} &\left([\Lambda_1^s, M_1] \mathbf{V}, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \\ &\lesssim \|g(D)([\Lambda_1^s, M_1] \mathbf{V})\|_2 (\|\underline{\mathbf{V}}\|_\infty + \|\tilde{\mathbf{V}}_\omega\|_\infty) \|(1 + c\epsilon\Delta) \Lambda^s \mathbf{V}\|_2 \\ &\quad + \|[\Lambda_1^s, M_1] \mathbf{V}\|_2 (\|(1 + c\epsilon\Delta)g(D) \Lambda^s \mathbf{V}\|_2 + \|(g(D) - 1) \Lambda^s \mathbf{V}\|_2) \\ &\quad - c\epsilon (\nabla([\Lambda_1^s, M_1] \mathbf{V})_1, \nabla(1 + c\epsilon\Delta)g(D) \Lambda^s \mathbf{V})_2 \\ &\quad - a\epsilon (\nabla([\Lambda_1^s, M_1] \mathbf{V})_2, \nabla(1 + c\epsilon\Delta) \Lambda^s v_1)_2 \\ &\quad - a\epsilon (\nabla([\Lambda_1^s, M_1] \mathbf{V})_3, \nabla(1 + c\epsilon\Delta) \Lambda^s v_2)_2. \end{aligned}$$

(where $(\cdot)_j$ is the notation of j^{th} component of the vector)

Using the expression of M_1 we have that

$$\|[\Lambda_1^s, M_1] \mathbf{V}\|_2 \lesssim \|[\Lambda_1^s, \tilde{\mathbf{V}}_\omega] \mathbf{V}\|_2 = \left\| \left\| [\Lambda_1^s, \tilde{\mathbf{V}}_\omega] \mathbf{V} \right\|_{L_{x_1}^2} \right\|_{L_{x_2}^2}$$

and

$$\|\nabla([\Lambda_1^s, M_1] \mathbf{V})_j\|_2 \lesssim \|([\Lambda_1^s, \nabla \tilde{\mathbf{V}}_\omega] \mathbf{V})_j\|_2 \lesssim \left\| \left\| [\Lambda_1^s, \nabla \tilde{\mathbf{V}}_\omega] \mathbf{V} \right\|_{L_{x_1}^2} \right\|_{L_{x_2}^2}.$$

Then, using (5.3.24) with $d = 1$, $\sigma(D) = \Lambda_1^s$ we obtain

$$\left\| [\Lambda_1^s, \tilde{\mathbf{V}}_\omega] \mathbf{V} \right\|_{L_{x_1}^2} \lesssim \left\| \nabla \tilde{\mathbf{V}}_\omega \right\|_{L_{x_1}^\infty} \left\| \mathbf{V} \right\|_{H_{x_1}^{s-1}} + \left\| \nabla \tilde{\mathbf{V}}_\omega \right\|_{H_{x_1}^{s-1}} \left\| \mathbf{V} \right\|_{L_{x_1}^\infty} \lesssim \epsilon \left\| \Lambda_1^s \mathbf{V} \right\|_{L_{x_1}^2},$$

and, with Leibnitz rule

$$\left\| [\Lambda_1^s, \nabla \tilde{\mathbf{V}}_\omega] \mathbf{V} \right\|_{L_{x_1}^2} \lesssim \epsilon \left\| \Lambda_1^{s+1} \mathbf{V} \right\|_{L_{x_1}^2}.$$

Remark: The coefficients on these above estimates depend on $\left\| \tilde{\mathbf{V}}_\omega \right\|_{W^{2,\infty}}$ and do not depend on ϵ since M_1 is independent of ϵ .

Therefore, by combining the above estimates and (5.3.25)-(5.3.26), we have

$$\begin{aligned} & \left([\Lambda_1^s, M_1] \mathbf{V}, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \\ & \lesssim \epsilon \left\| \Lambda_1^s \mathbf{V} \right\|_2 \left(\left\| \mathbf{V} \right\|_{H^s} + \epsilon \left\| \mathbf{V} \right\|_{H^{s+2}} \right) + \left\| \Lambda_1^s \mathbf{V} \right\|_2 \left(\left\| \mathbf{V} \right\|_{H^s} + \epsilon \left\| \mathbf{V} \right\|_{H^{s+2}} \right) \\ & \quad + \epsilon^{3/2} \left\| \Lambda_1^{s+1} \mathbf{V} \right\|_2 \left(\epsilon^{1/2} \left\| \mathbf{V} \right\|_{H^{s+1}} + \epsilon^{3/2} \left\| \mathbf{V} \right\|_{H^{s+3}} \right) \\ & \lesssim \epsilon \left(\left\| \mathbf{V} \right\|_{H^s} + \epsilon^{1/2} \left\| \mathbf{V} \right\|_{H^{s+1}} + \epsilon \left\| \mathbf{V} \right\|_{H^{s+2}} + \epsilon^{3/2} \left\| \mathbf{V} \right\|_{H^{s+3}} \right)^2 \\ & \lesssim \epsilon \left\| \mathbf{V} \right\|_{X_{\epsilon^3}^s}^2. \end{aligned} \tag{5.3.36}$$

The estimate for $\left([\Lambda_1^s, M_1] \mathbf{V}, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^* \Lambda^s \mathbf{V} \right)_2$ follows similarly as (5.3.36).

The estimate for $\left([\Lambda_2^s, M_1] \mathbf{V}, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2$ is easier since $\tilde{\mathbf{V}}_\omega$ does not depend on x_2 . Therefore we have

$$\left([\Lambda^s, M_1] \mathbf{V}, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \lesssim \epsilon \left\| \mathbf{V} \right\|_{X_{\epsilon^3}^s}^2. \tag{5.3.37}$$

Estimate on II_1 : Thanks to (5.3.35) and (5.3.37) we get

$$II_1 \lesssim \epsilon \left\| \underline{\mathbf{V}} \right\|_{X_{\epsilon^3}^s} \left\| \mathbf{V} \right\|_{X_{\epsilon^3}^s}^2. \tag{5.3.38}$$

Estimate on II_2 : In order to estimate II_2 , similarly to the argument in [55], we first calculate

$S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D) := A(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D) = (a_{ij})$ as follows:

$$\begin{aligned}
a_{11} &= (1+c\epsilon\Delta)^2g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla)+g(D)\left((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\epsilon\Delta)^2g(D)\right), \\
a_{12} &= (1+c\epsilon\Delta)^2g(D)\left((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_1}\right)+g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(1+c\epsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\nabla)), \\
a_{13} &= (1+c\epsilon\Delta)^2g(D)\left((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_2}\right)+g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(1+c\epsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\nabla)), \\
a_{21} &= g(D)((\underline{v}_1+\tilde{v}_{1\omega})(1+c\epsilon\Delta)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla)) \\
&\quad +(\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\epsilon\Delta)g(D)(g(D)-1) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)^2g(D)\partial_{x_1}, \\
a_{22} &= g(D)\left((\underline{v}_1+\tilde{v}_{1\omega})(1+c\epsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_1}]\right) \\
&\quad +(\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_1}), \\
a_{23} &= g(D)\left((\underline{v}_1+\tilde{v}_{1\omega})(1+c\epsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_2}]\right) \\
&\quad +(\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1}), \\
a_{31} &= g(D)((\underline{v}_2+\tilde{v}_{2\omega})(1+c\epsilon\Delta)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla)) \\
&\quad +(\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\epsilon\Delta)g(D)(g(D)-1) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)^2g(D)\partial_{x_2}, \\
a_{32} &= g(D)\left((\underline{v}_2+\tilde{v}_{2\omega})(1+c\epsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_1}]\right) \\
&\quad +(\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2}), \\
a_{33} &= g(D)\left((\underline{v}_2+\tilde{v}_{2\omega})(1+c\epsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)\partial_{x_2}]\right) \\
&\quad +(\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla) \\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\epsilon\Delta)(1+c\epsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_2}),
\end{aligned}$$

For a_{11} ,

$$\begin{aligned}
(a_{11}\Lambda^s\zeta, \Lambda^s\zeta)_2 &= \left((1+c\epsilon\Delta)^2g(D)(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta)+g(D)(\underline{\mathbf{v}}\cdot\nabla(1+c\epsilon\Delta)^2g(D)\Lambda^s\zeta), \Lambda^s\zeta\right)_2 \\
&\quad + \left((1+c\epsilon\Delta)^2g(D)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta)+g(D)(\tilde{\mathbf{v}}_\omega\cdot\nabla(1+c\epsilon\Delta)^2g(D)\Lambda^s\zeta), \Lambda^s\zeta\right)_2 \\
&:= B_{11}+B_{12}.
\end{aligned} \tag{5.3.39}$$

From [55] we have

$$B_{11} \lesssim \|\underline{\mathbf{v}}\|_{X_{\epsilon^3}^s} \|\zeta\|_{X_{\epsilon^3}^s}^2, \tag{5.3.40}$$

with B_{12} , we can not use the commutator estimate for $g(D)$ and $\tilde{\mathbf{v}}_\omega$. In order to get the relevant norm we will use that $\tilde{\mathbf{v}}_\omega = \epsilon\tilde{\mathbf{u}}_\omega$. First, using integrating by part, we have

$$\begin{aligned}
B_{12} &= ((1+c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta), (1+c\epsilon\Delta)g(D)\Lambda^s\zeta)_2 \\
&\quad - \left((1+c\epsilon\Delta)^2g(D)\Lambda^s\zeta, \nabla\cdot(\tilde{\mathbf{v}}_\omega g(D)\Lambda^s\zeta)\right)_2 \\
&= \epsilon((1+c\epsilon\Delta)(\tilde{\mathbf{u}}_\omega\cdot\nabla\Lambda^s\zeta), (1+c\epsilon\Delta)g(D)\Lambda^s\zeta)_2 \\
&\quad - \epsilon((1+c\epsilon\Delta)g(D)\Lambda^s\zeta, (1+c\epsilon\Delta)\nabla\cdot(\tilde{\mathbf{u}}_\omega g(D)\Lambda^s\zeta))_2.
\end{aligned} \tag{5.3.41}$$

By using Leibniz rule, it is not hard to see that

$$B_{12} \lesssim (\epsilon^{1/2} + \epsilon^2) \|\tilde{\mathbf{u}}_\omega\|_{W^{2,\infty}} \|\zeta\|_{X_{\epsilon^3}^s}^2 \quad (5.3.42)$$

Combining (5.3.42) and (5.3.40), we obtain

$$(a_{11}\Lambda^s\zeta, \Lambda^s\zeta)_2 \lesssim ((\epsilon^{1/2} + \epsilon^2) \|\tilde{\mathbf{u}}_\omega\|_{W^{2,\infty}} + \|\mathbf{v}\|_{X_{\epsilon^3}^s}) \|\zeta\|_{X_{\epsilon^3}^s}^2 \lesssim \epsilon^{1/2}(1 + \epsilon^{1/2} \|\mathbf{U}\|_{X_{\epsilon^3}^s}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.43)$$

For a_{12} and a_{21} , using integration by part, we get:

$$\begin{aligned} & (a_{12}\Lambda^s v_1, \Lambda^s\zeta)_2 + (a_{21}\Lambda^s\zeta, \Lambda^s v_1)_2 \\ &= - \left((1 + c\epsilon\Delta)(\partial_{x_1}(\zeta + \tilde{\zeta}_\omega)\Lambda^s v_1), g(D)(1 + c\epsilon\Delta)\Lambda^s\zeta \right)_2 \\ &+ \left\{ (g(D)((\mathbf{v} + \tilde{\mathbf{v}}_\omega) \cdot (1 + c\epsilon\Delta)g(D)((v_1 + \tilde{v}_{1\omega})\nabla\Lambda^s v_1), \Lambda^s\zeta)_2 \right. \\ &+ (g(D)((v_1 + \tilde{v}_{1\omega})(1 + c\epsilon\Delta)((\mathbf{v} + \tilde{\mathbf{v}}_\omega) \cdot \nabla\Lambda^s\zeta), \Lambda^s v_1)_2 \\ &+ ((v_1 + \tilde{v}_{1\omega})(\mathbf{v} + \tilde{\mathbf{v}}_\omega) \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s\zeta, \Lambda^s v_1)_2 \left. \right\} \\ &= B_{21} + B_{22}, \end{aligned} \quad (5.3.44)$$

where B_{21} contains only the terms which depends only on $\mathbf{V}$ and

$$\begin{aligned} B_{22} = & - \left((1 + c\epsilon\Delta)(\partial_{x_1}\tilde{\zeta}_\omega\Lambda^s v_1), g(D)(1 + c\epsilon\Delta)\Lambda^s\zeta \right)_2 \\ &+ ((1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1), \mathbf{v}(g(D)\Lambda^s\zeta))_2 \\ &+ ((1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1), \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2 \\ &+ ((1 + c\epsilon\Delta)g(D)(v_1\nabla\Lambda^s v_1), \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2 \\ &+ ((1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega \cdot \nabla\Lambda^s\zeta), v_1g(D)\Lambda^s v_1)_2 \\ &+ ((1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega \cdot \nabla\Lambda^s\zeta), \tilde{v}_{1\omega}g(D)\Lambda^s v_1)_2 \\ &+ ((1 + c\epsilon\Delta)(\mathbf{v} \cdot \nabla\Lambda^s\zeta), \tilde{v}_{1\omega}g(D)\Lambda^s v_1)_2 \\ &+ (v_1\tilde{\mathbf{v}}_\omega \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s\zeta, \Lambda^s v_1)_2 \\ &+ (\tilde{v}_{1\omega}\mathbf{v} \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s\zeta, \Lambda^s v_1)_2 \\ &+ (\tilde{v}_{1\omega}\tilde{\mathbf{v}}_\omega \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s\zeta, \Lambda^s v_1)_2 \end{aligned} \quad (5.3.45)$$

By using Leibnitz rule, Hölder inequality and (5.3.25) we have

$$\begin{aligned}
& \left((1 + c\epsilon\Delta)((\partial_{x_1}\tilde{\zeta}_\omega)\Lambda^s v_1), g(D)(1 + c\epsilon\Delta)\Lambda^s \zeta \right)_2 \lesssim \|\tilde{\zeta}_\omega\|_{W^{3,\infty}} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1), \underline{\mathbf{v}}(g(D)\Lambda^s \zeta) \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{u}_{1\omega}\|_{W^{2,\infty}} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1), \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s \zeta) \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{u}_{1\omega}\|_{W^{2,\infty}} \|\tilde{\mathbf{v}}_\omega\|_\infty \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)g(D)(\underline{v}_1\nabla\Lambda^s v_1), \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s \zeta) \right)_2 = (g(D)(\underline{v}_1\nabla\Lambda^s v_1), (1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega(g(D)\Lambda^s \zeta)))_2 \\
& \quad \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{\mathbf{u}}_\omega\|_{W^{2,\infty}} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega \cdot \nabla\Lambda^s \zeta), \underline{v}_1 g(D)\Lambda^s v_1 \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{\mathbf{u}}_\omega\|_{W^{2,\infty}} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega \cdot \nabla\Lambda^s \zeta), \tilde{v}_{1\omega} g(D)\Lambda^s v_1 \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{\mathbf{u}}_\omega\|_{W^{2,\infty}} \|\tilde{v}_{1\omega}\|_\infty \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left((1 + c\epsilon\Delta)(\underline{\mathbf{v}} \cdot \nabla\Lambda^s \zeta), \tilde{v}_{1\omega} g(D)\Lambda^s v_1 \right)_2 = (\underline{\mathbf{v}} \cdot \nabla\Lambda^s \zeta, (1 + c\epsilon\Delta)(\tilde{v}_{1\omega} g(D)\Lambda^s v_1))_2 \\
& \quad \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{u}_{1\omega}\|_{W^{2,\infty}} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left(\underline{v}_1 \tilde{\mathbf{v}}_\omega \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s \zeta, \Lambda^s v_1 \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{\mathbf{u}}_\omega\|_\infty \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left(\tilde{v}_{1\omega} \underline{\mathbf{v}} \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s \zeta, \Lambda^s v_1 \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{u}_{1\omega}\|_\infty \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \\
& \left(\tilde{v}_{1\omega} \tilde{\mathbf{v}}_\omega \cdot \nabla(1 + c\epsilon\Delta)g(D)(g(D) - 1)\Lambda^s \zeta, \Lambda^s v_1 \right)_2 \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{u}_{1\omega}\|_\infty \|\tilde{\mathbf{V}}_\omega\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2,
\end{aligned}$$

Therefore,

$$B_{22} \lesssim (\epsilon^{1/2} + \epsilon^{3/2}) \|\tilde{\mathbf{u}}_\omega\|_{W^{3,\infty}} (\epsilon^{1/2} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.46)$$

From [55] we also have

$$B_{21} \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.47)$$

Combining (5.3.47)-(5.3.46), it follows

$$(a_{12}\Lambda^s v_1, \Lambda^s \zeta)_2 + (a_{21}\Lambda^s \zeta, \Lambda^s v_1)_2 \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.48)$$

□

The same estimate holds for $(a_{13}\Lambda^s v_1, \Lambda^s \zeta)_2 + (a_{31}\Lambda^s \zeta, \Lambda^s v_2)_2$.

For a_{22} ,

$$\begin{aligned}
(a_{22}\Lambda^s v_1, \Lambda^s v_1)_2 &= \left((\underline{v}_1 + \tilde{v}_{1\omega})(1 + c\epsilon\Delta)(1 + \zeta + \tilde{\zeta}_\omega + a\epsilon\Delta)\partial_{x_1}\Lambda^s v_1, g(D)\Lambda^s v_1 \right)_2 \\
&\quad + \left((1 + \zeta + \tilde{\zeta}_\omega + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)((\underline{v}_1 + \tilde{v}_{1\omega})\partial_{x_1}\Lambda^s v_1), \Lambda^s v_1 \right)_2 \\
&\quad + \left((\underline{v}_1 + \tilde{v}_{1\omega})(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot g(D)(g(D) - 1)((\underline{v}_1 + \tilde{v}_{1\omega})\nabla\Lambda^s v_1), \Lambda^s v_1 \right)_2 \\
&:= B_{31} + B_{32}.
\end{aligned} \quad (5.3.49)$$

Where B_{31} contains only the terms which depends only on $\mathbf{V}$ and

$$\begin{aligned}
B_{32} = & \left(\underline{v}_1(1 + c\epsilon\Delta)(\tilde{\zeta}_\omega \partial_{x_1} \Lambda^s v_1), g(D)\Lambda^s v_1 \right)_2 \\
& + \left(\tilde{v}_{1\omega}(1 + c\epsilon\Delta)(1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)(\partial_{x_1} \Lambda^s v_1), g(D)\Lambda^s v_1 \right)_2 \\
& + \left(\tilde{\zeta}_\omega(1 + c\epsilon\Delta)g(D)((v_1 + \tilde{v}_{1\omega})\partial_{x_1} \Lambda^s v_1), \Lambda^s v_1 \right)_2 \\
& + \left((1 + \underline{\zeta} + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\partial_{x_1} \Lambda^s v_1), \Lambda^s v_1 \right)_2 \\
& + (\tilde{v}_{1\omega}(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot g(D)(g(D) - 1)((v_1 + \tilde{v}_{1\omega})\nabla \Lambda^s v_1), \Lambda^s v_1)_2 \\
& + (v_1 \tilde{\mathbf{v}}_\omega \cdot g(D)(g(D) - 1)((v_1 + \tilde{v}_{1\omega})\nabla \Lambda^s v_1), \Lambda^s v_1)_2 \\
& + (v_1 \underline{\mathbf{v}} \cdot g(D)(g(D) - 1)(\tilde{v}_{1\omega}\nabla \Lambda^s v_1), \Lambda^s v_1)_2.
\end{aligned} \tag{5.3.50}$$

Similarly as (5.3.46) we get

$$B_{32} \lesssim \epsilon^{1/2} \left\| \tilde{\underline{\mathbf{U}}}_\omega \right\|_{W^{2,\infty}} (1 + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}^2 + \|\tilde{\underline{\mathbf{V}}}_\omega\|_\infty^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.51}$$

From [55] we have

$$B_{31} \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \tag{5.3.52}$$

it follows

$$(a_{22}\Lambda^s v_1, \Lambda^s v_1)_2 \lesssim \epsilon^{1/2} (1 + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{V}\|_{X_{\epsilon^3}^s}^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.53}$$

The same estimate holds for $(a_{33}\Lambda^s v_2, \Lambda^s v_2)_2$.

For a_{23} and a_{32} , we have

$$\begin{aligned}
& (a_{23}\Lambda^s v_2, \Lambda^s v_1)_2 + (a_{32}\Lambda^s v_1, \Lambda^s v_2)_2 \\
& = \left(g(D)((v_1 + \tilde{v}_{1\omega})(1 + c\epsilon\Delta)(1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)\partial_{x_2} \Lambda^s v_2), \Lambda^s v_1 \right)_2 \\
& \quad + ((v_1 + \tilde{v}_{1\omega})(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot g(D)(g(D) - 1)((v_2 + \tilde{v}_{2\omega})\nabla \Lambda^s v_2), \Lambda^s v_1)_2 \\
& \quad + \left((1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)((v_2 + \tilde{v}_{2\omega})\partial_{x_1} \Lambda^s v_2), \Lambda^s v_1 \right)_2 \\
& \quad + \left(g(D)((v_2 + \tilde{v}_{2\omega})(1 + c\epsilon\Delta)(1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)\partial_{x_1} \Lambda^s v_1), \Lambda^s v_2 \right)_2 \\
& \quad + ((v_2 + \tilde{v}_{2\omega})(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot g(D)(g(D) - 1)((v_1 + \tilde{v}_{1\omega})\nabla \Lambda^s v_1), \Lambda^s v_2)_2 \\
& \quad + \left((1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)((v_1 + \tilde{v}_{1\omega})\partial_{x_2} \Lambda^s v_1), \Lambda^s v_2 \right)_2 \\
& = B_{41} + B_{42},
\end{aligned} \tag{5.3.54}$$

where B_{41} contains only the terms which depends only on $\mathbf{V}$ and

$$\begin{aligned}
B_{42} = & \left(g(D)(\tilde{v}_{1\omega}(1 + c\epsilon\Delta)(1 + \underline{\zeta} + \tilde{\zeta}_{\omega} + a\epsilon\Delta)\partial_{x_2}\Lambda^s v_2), \Lambda^s v_1 \right)_2 \\
& + \left(g(D)(v_1(1 + c\epsilon\Delta)(\tilde{\zeta}_{\omega}\partial_{x_2}\Lambda^s v_2)), \Lambda^s v_1 \right)_2 \\
& + (\tilde{v}_{1\omega}(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_{\omega}) \cdot g(D)(g(D) - 1)((v_2 + \tilde{v}_{2\omega})\nabla\Lambda^s v_2), \Lambda^s v_1)_2 \\
& + (v_1\tilde{\mathbf{v}}_{\omega} \cdot g(D)(g(D) - 1)((v_2 + \tilde{v}_{2\omega})\nabla\Lambda^s v_2), \Lambda^s v_1)_2 \\
& + (v_1\underline{\mathbf{v}} \cdot g(D)(g(D) - 1)(\tilde{v}_{2\omega}\nabla\Lambda^s v_2), \Lambda^s v_1)_2 \\
& + \left(\tilde{\zeta}_{\omega}(1 + c\epsilon\Delta)g(D)((v_2 + \tilde{v}_{2\omega})\partial_{x_1}\Lambda^s v_2), \Lambda^s v_1 \right)_2 \\
& + \left((1 + \underline{\zeta} + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)(\tilde{v}_{2\omega}\partial_{x_1}\Lambda^s v_2), \Lambda^s v_1 \right)_2 \\
& + \left(g(D)(\tilde{v}_{2\omega}(1 + c\epsilon\Delta)(1 + \underline{\zeta} + \tilde{\zeta}_{\omega} + a\epsilon\Delta)\partial_{x_1}\Lambda^s v_1), \Lambda^s v_2 \right)_2 \\
& + \left(g(D)(v_2(1 + c\epsilon\Delta)(\tilde{\zeta}_{\omega}\partial_{x_1}\Lambda^s v_1)), \Lambda^s v_2 \right)_2 \\
& + (\tilde{v}_{2\omega}(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_{\omega}) \cdot g(D)(g(D) - 1)((v_1 + \tilde{v}_{1\omega})\nabla\Lambda^s v_1), \Lambda^s v_2)_2 \\
& + (v_2\tilde{\mathbf{v}}_{\omega} \cdot g(D)(g(D) - 1)((v_1 + \tilde{v}_{1\omega})\nabla\Lambda^s v_1), \Lambda^s v_2)_2 \\
& + (v_2\underline{\mathbf{v}} \cdot g(D)(g(D) - 1)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1), \Lambda^s v_2)_2 \\
& + \left(\tilde{\zeta}_{\omega}(1 + c\epsilon\Delta)g(D)((v_1 + \tilde{v}_{1\omega})\partial_{x_2}\Lambda^s v_1), \Lambda^s v_2 \right)_2 \\
& + \left((1 + \underline{\zeta} + a\epsilon\Delta)(1 + c\epsilon\Delta)g(D)(\tilde{v}_{1\omega}\partial_{x_2}\Lambda^s v_1), \Lambda^s v_2 \right)_2.
\end{aligned} \tag{5.3.55}$$

Similarly as (5.3.46) and (5.3.51) we have

$$B_{42} \lesssim \epsilon^{1/2} \left\| \tilde{\underline{\mathbf{V}}}_{\omega} \right\|_{W^{2,\infty}} (1 + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} + \left\| \tilde{\underline{\mathbf{V}}}_{\omega} \right\|_{\infty}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2, \tag{5.3.56}$$

from [55] we also have

$$B_{41} \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.57}$$

Therefore,

$$(a_{23}\Lambda^s v_2, \Lambda^s v_1)_2 + (a_{32}\Lambda^s v_1, \Lambda^s v_2)_2 \lesssim \epsilon^{1/2}(1 + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.58}$$

Thanks to (5.3.43), (5.3.48), (5.3.53) and (5.3.58), we obtain

$$\left| \left(S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_{\omega}}(D) M(\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_{\omega}, D) \Lambda^s \mathbf{V}, \Lambda^s \mathbf{V} \right)_2 \right| \lesssim \epsilon^{1/2} (1 + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2.$$

The same estimate holds for the term $\left(M(\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_{\omega}, D) \Lambda^s \mathbf{V}, S_{\underline{\mathbf{V}} + \tilde{\underline{\mathbf{V}}}_{\omega}}(D) \Lambda^s \mathbf{V} \right)_2$, then

$$\|II_2\| \lesssim \epsilon^{1/2} (1 + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.59}$$

Due to (5.3.38) and (5.3.59), we have

$$\|II\| \lesssim \epsilon^{1/2} (1 + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \tag{5.3.60}$$

Estimate on III. Using (5.3.18) and the expression of $(S_1 + S_2)^*$ we only need to estimate

$$\begin{aligned}
& -b\epsilon \left([(S_1 + S_2)^*, \Delta] \Lambda^s \mathbf{V}, \Lambda^s \partial_t \mathbf{V} \right)_2 \\
& \lesssim \epsilon \left\| [\tilde{\underline{\mathbf{V}}}_{\omega}, \Delta] g(D) \Lambda^s \mathbf{V} \right\|_2 \left\| (1 + c\epsilon\Delta) \Lambda^s \partial_t \mathbf{V} \right\|_2 \\
& + \epsilon \sum_{i,j=1}^2 \left\| [\tilde{v}_i \tilde{v}_{j\omega} + \tilde{v}_{i\omega} \tilde{v}_j + \tilde{v}_{i\omega} \tilde{v}_{j\omega}, \Delta] \Lambda^s \mathbf{V} \right\|_2 \left\| (g(D) - 1) \Lambda^s \partial_t \mathbf{V} \right\|_2.
\end{aligned}$$

In order to use the commutator estimate (5.3.24), we use the same argument in the estimate for II_{11} . It follows

$$\begin{aligned} & -b\epsilon((S_1 + S_2)^*, \Delta]\Lambda^s \mathbf{V}, \Lambda^s \partial_t \mathbf{V})_2 \\ & \lesssim \epsilon^2 \|\mathbf{V}\|_{H^{s+1}} (\|\partial_t \mathbf{V}\|_{H^s} + \epsilon \|\partial_t \mathbf{V}\|_{H^{s+2}}) + \epsilon^2 (\|\underline{\mathbf{V}}\| + 1) \|\mathbf{V}\|_{H^{s+1}} \|\partial_t \mathbf{V}\|_{H^s} \\ & \lesssim \epsilon^{3/2} (\|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} + 1) \|\mathbf{V}\|_{X_{\epsilon^3}^s} \|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}} \end{aligned} \quad (5.3.61)$$

From [55] we know that

$$-b\epsilon([\underline{S}_{\mathbf{V}}(D)^*, \Delta]\Lambda^s \mathbf{V}, \Lambda^s \partial_t \mathbf{V})_2 \lesssim \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s} \|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}}, \quad (5.3.62)$$

Therefore,

$$III \lesssim (\epsilon^{3/2} + \epsilon^{3/2} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}) \|\mathbf{V}\|_{X_{\epsilon^3}^s} \|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}}. \quad (5.3.63)$$

Estimate on IV. Plugging the expression of $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$ into IV we get that

$$IV = IV_1 + IV_2, \quad (5.3.64)$$

where $IV_1 = ((1 - b\epsilon\Delta)\Lambda^s \mathbf{V}, (\partial_t S_{\underline{\mathbf{V}}}(D))\Lambda^s \mathbf{V})_2$, and from [55] we know that

$$IV_1 \lesssim \|\partial_t \underline{\mathbf{V}}\|_{H^{s-1}} \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.65)$$

And

$$\begin{aligned} IV_2 = & ((1 - b\epsilon\Delta)\Lambda^s \zeta, g(D)((\partial_t \tilde{v}_{1\omega})(1 + c\epsilon\Delta)\Lambda^s v_1))_2 \\ & + ((1 - b\epsilon\Delta)\Lambda^s \zeta, g(D)(\partial_t(\tilde{v}_{2\omega})(1 + c\epsilon\Delta)\Lambda^s v_2))_2 \\ & + ((1 - b\epsilon\Delta)\Lambda^s v_1, g(D)(\partial_t(\tilde{v}_{1\omega})(1 + c\epsilon\Delta)\Lambda^s \zeta))_2 \\ & + \left((1 - b\epsilon\Delta)\Lambda^s v_1, (\partial_t \tilde{\zeta}_{\omega})(1 + c\epsilon\Delta)\Lambda^s v_1 + (\partial_t(2v_1 \tilde{v}_{1\omega} + \tilde{v}_{1\omega}^2))(g(D) - 1)\Lambda^s v_1 \right)_2 \\ & + ((1 - b\epsilon\Delta)\Lambda^s v_1, (\partial_t(\tilde{v}_{1\omega} \tilde{v}_{2\omega} + v_1 \tilde{v}_{2\omega} + v_2 \tilde{v}_{1\omega}))(g(D) - 1)\Lambda^s v_2)_2 \\ & + ((1 - b\epsilon\Delta)\Lambda^s v_2, g(D)((\partial_t \tilde{v}_{2\omega})(1 + c\epsilon\Delta)\Lambda^s \zeta))_2 \\ & + ((1 - b\epsilon\Delta)\Lambda^s v_2, (\partial_t(\tilde{v}_{1\omega} \tilde{v}_{2\omega} + v_1 \tilde{v}_{2\omega} + v_2 \tilde{v}_{1\omega}))(g(D) - 1)\Lambda^s v_1)_2 \\ & + \left((1 - b\epsilon\Delta)\Lambda^s v_2, (\partial_t \tilde{\zeta}_{\omega})(1 + c\epsilon\Delta)\Lambda^s v_2 + (\partial_t(\tilde{v}_{2\omega}^2 + 2v_2 \tilde{v}_{2\omega}))(g(D) - 1)\Lambda^s v_2 \right)_2. \end{aligned} \quad (5.3.66)$$

Therefore it is not hard to see that

$$IV_2 \lesssim \epsilon(1 + \|\partial_t \underline{\mathbf{V}}\|_{H^{s-1}}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.67)$$

By combining (5.3.65) and (5.3.67) we have

$$IV \lesssim \epsilon(1 + \|\partial_t \underline{\mathbf{V}}\|_{H^{s-1}}) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2. \quad (5.3.68)$$

Combining (5.3.34), (5.3.60), (5.3.63) and (5.3.68) we obtain

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) & \lesssim \|F\|_{X_{\epsilon^3}^s} \|\mathbf{V}\|_{X_{\epsilon^3}^s} + \epsilon^{1/2} (1 + \epsilon^{1/2} \|\partial_t \underline{\mathbf{V}}\|_{H^{s-1}} + \epsilon^{1/2} \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s}^2) \|\mathbf{V}\|_{X_{\epsilon^3}^s}^2 \\ & + \epsilon(1 + \epsilon^{1/2} \|\underline{\mathbf{V}}\|_{X_{\epsilon^3}^s} + \|\underline{\mathbf{U}}\|_{X_{\epsilon^3}^s}) \|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}} \|\mathbf{V}\|_{X_{\epsilon^3}^s}. \end{aligned} \quad (5.3.69)$$

We are going to estimate $\|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}}$, (5.3.14) follows that $\partial_t \mathbf{V} = (1 - b\epsilon\Delta)^{-1}(F - M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V}) := (1 - b\epsilon\Delta)^{-1}G$. Since the solitary wave has the “sech” profile and using in addition the estimate (5.3.23) we have the the same estimate as in [55] as follows,

$$\|\partial_t \mathbf{V}\|_{X_{\epsilon^3}^{s-1}} \lesssim \|F\|_{H^s} + \|\mathbf{V}\|_{X_{\epsilon^3}^s}. \quad (5.3.70)$$

This is the estimate (5.3.32). Plugging (5.3.32) into (5.3.69), using (5.3.29), we obtain (5.3.30) and (5.3.31).

The *existence* and *uniqueness* of the solution of (5.3.14) are obtained similarly as in [55].

The case $b = d > 0$, $a = 0$, $c < 0$

Proposition 5. *Let $t_0 > 1$, $s \geq t_0 + 2$, $T' > 0$. Assume that $F \in C([0, T']; X_\epsilon^s(\mathbb{R}^2))$ and $\mathbf{V} = (\zeta, \mathbf{v}) \in C^1([0, T']; X_{\epsilon^3}^{s-1}(\mathbb{R}^2)) \cap C([0, T']; X_{\epsilon^2}^s(\mathbb{R}^2) \times (X_\epsilon^s(\mathbb{R}^2))^2)$ satisfies*

$$1 + \underline{\zeta} + \tilde{\zeta}_\omega \geq H > 0, \quad \|\underline{\mathbf{V}}\|_\infty + \|\tilde{\mathbf{V}}_\omega\|_\infty \leq \kappa_H, \quad \|\underline{\mathbf{V}}\|_{H^s} + \|\tilde{\mathbf{V}}_\omega\|_{H^s} \leq 1, \quad \forall t \in [0, T'], \quad (5.3.71)$$

with κ_H ensures the equivalence of $E_s(\mathbf{V})$ and $\|\zeta\|_{X_{\epsilon^2}^s}^2 + \|\mathbf{v}\|_{X_\epsilon^s}^2$. (see appendix 5.6.2)

Then for any $V_0 \in X_{\epsilon^3}^s(\mathbb{R}^2)$, (5.3.14) has a unique solution $\mathbf{V} \in C^1([0, T']; X_{\epsilon^3}^{s-1}(\mathbb{R}^2)) \cap C([0, T']; X_{\epsilon^2}^s(\mathbb{R}^2) \times (X_\epsilon^s(\mathbb{R}^2))^2)$ and one has for any $t \in (0, T')$

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) &\lesssim \left(\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^2}^{s-1}} \right) E_s(\mathbf{V})^2 \\ &\quad + \left(\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} \right) \|\partial_t \mathbf{V}\|_{X_{\epsilon^2}^{s-1}} E_s(\mathbf{V}) \\ &\quad + \|F\|_{H^s} E_s(\mathbf{V}). \end{aligned} \quad (5.3.72)$$

$$\begin{aligned} \|\mathbf{v}(t)\|_{X_\epsilon^s}^2 + \|\zeta(t)\|_{X_{\epsilon^2}^s}^2 &\leq \tilde{c} \left(\|\mathbf{v}(0)\|_{X_\epsilon^s}^2 + \|\zeta(0)\|_{X_{\epsilon^2}^s}^2 \right) \\ &\quad + \tilde{c} \int_0^t \left(\|F\|_{H^s} (\|\mathbf{v}(t')\|_{X_\epsilon^s} + \|\zeta(t')\|_{X_{\epsilon^2}^s}) \right. \\ &\quad \left. + (\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s}) (\|\mathbf{v}(t')\|_{X_\epsilon^s} + \|\zeta(t')\|_{X_{\epsilon^2}^s}) \|\partial_t \mathbf{V}\|_{X_{\epsilon^2}^{s-1}} \right. \\ &\quad \left. + (\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^2}^{s-1}}) (\|\mathbf{v}(t')\|_{X_\epsilon^s} + \|\zeta(t')\|_{X_{\epsilon^2}^s})^2 dt' \right). \end{aligned} \quad (5.3.73)$$

$$\|\partial_t \mathbf{V}\|_{X_{\epsilon^2}^{s-1}} \lesssim \|F\|_{H^s} + \|\zeta\|_{X_{\epsilon^2}^s} + \|\mathbf{v}\|_{H^s}. \quad (5.3.74)$$

Proof. We start with the expression (5.3.33),

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) &= \left(\Lambda^s F, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &\quad - \left(\Lambda^s \left(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) \mathbf{V} \right), \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &\quad - b\epsilon \left(\left[S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^*, \Delta \right] \Lambda^s \mathbf{V}, \Lambda^s \partial_t \mathbf{V} \right)_2 + \left((1 - b\epsilon \Delta) \Lambda^s \mathbf{V}, \left(\partial_t S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \right) \Lambda^s \mathbf{V} \right)_2 \\ &= I + II + III + IV, \end{aligned}$$

□

Estimate on I. By using integral by part and the expression (5.3.15) of $S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)$, we obtain

$$\begin{aligned} &\left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \\ &= (\Lambda^s F_1, (1 + c\epsilon \Delta) \Lambda^s \zeta + (v_1 + \tilde{v}_{1\omega}) \Lambda^s v_1 + (v_2 + \tilde{v}_{2\omega}) \Lambda^s v_2)_2 \\ &\quad + \left(\Lambda^s F_2, (v_1 + \tilde{v}_{1\omega}) \Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s v_1 \right)_2 \\ &\quad + \left(\Lambda^s F_3, (v_2 + \tilde{v}_{2\omega}) \Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s v_2 \right)_2, \end{aligned} \quad (5.3.75)$$

with assumption (5.3.71), it is not hard to get that

$$\left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \lesssim \|F\|_{H^s} (\|\zeta\|_{X_{\epsilon^2}^s} + \|\mathbf{v}\|_{H^s}). \quad (5.3.76)$$

The same estimate holds for $\left(\Lambda^s F, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \Lambda^s \mathbf{V} \right)_2$, then we have

$$|I| \lesssim \|F\|_{H^s} (\|\zeta\|_{X_{\epsilon^2}^s} + \|\mathbf{v}\|_{H^s}). \quad (5.3.77)$$

Estimate on II. First we rewrite,

$$\begin{aligned} II &= - \left(\Lambda^s, M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) \right] \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &\quad - \left(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) \Lambda^s \mathbf{V}, \left(S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \right) \Lambda^s \mathbf{V} \right)_2 \\ &:= II_1 + II_2. \end{aligned} \quad (5.3.78)$$

Estimate on II_1 . Using the expression of $M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)$ and $S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)$, we can rewrite

$$\begin{aligned} &\left([\Lambda^s, M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)] \mathbf{V}, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \\ &= ([\Lambda^s, \underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega] \cdot \nabla \zeta + [\Lambda^s, \underline{\zeta} + \tilde{\zeta}_\omega] \partial_{x_1} v_1 + [\Lambda^s, \underline{\zeta} + \tilde{\zeta}_\omega] \partial_{x_2} v_2, (1 + c\epsilon\Delta) \Lambda^s \zeta \\ &\quad + (\underline{v}_1 + \tilde{v}_{1\omega}) \Lambda^s v_1 + (\underline{v}_2 + \tilde{v}_{2\omega}) \Lambda^s v_2)_2 \\ &\quad + \left([\Lambda^s, \underline{v}_1 + \tilde{v}_{1\omega}] \partial_{x_1} v_1 + [\Lambda^s, \underline{v}_2 + \tilde{v}_{2\omega}] \partial_{x_1} v_2, (\underline{v}_1 + \tilde{v}_{1\omega}) \Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s v_1 \right)_2 \\ &\quad + \left([\Lambda^s, \underline{v}_1 + \tilde{v}_{1\omega}] \partial_{x_2} v_1 + [\Lambda^s, \underline{v}_2 + \tilde{v}_{2\omega}] \partial_{x_2} v_2, (\underline{v}_2 + \tilde{v}_{2\omega}) \Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s v_2 \right)_2. \end{aligned}$$

By using the commutator estimate (5.3.24) with the trick of represent $\Lambda^s = \Lambda_1^s + \Lambda_2^s$, where $\Lambda_j^s = \mathcal{F}^{-1}(1 + |\xi_j|^s)$, similarly as in the previous case, we can prove that

$$\left([\Lambda^s, M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)] \mathbf{V}, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) \Lambda^s \mathbf{V} \right)_2 \lesssim \left(\epsilon + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} \right) \left(\|\mathbf{v}\|_{X_\epsilon^s} + \|\zeta\|_{X_{\epsilon^2}^s} \right).$$

The same estimate holds for $\left([\Lambda^s, M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)] \mathbf{V}, S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)^* \Lambda^s \mathbf{V} \right)_2$, then we obtain

$$|II_1| \lesssim \left(\epsilon + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} \right) \left(\|\mathbf{v}\|_{X_\epsilon^s} + \|\zeta\|_{X_{\epsilon^2}^s} \right). \quad (5.3.79)$$

Estimate on II_2 . We need to calculate $S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) := (a_{ij})$ as follows

$$\begin{aligned} a_{11} &= (1 + c\epsilon\Delta)((\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot \nabla) + (\underline{v}_1 + \tilde{v}_{1\omega})(1 + c\epsilon\Delta) \partial_{x_1} + (\underline{v}_2 + \tilde{v}_{2\omega})(1 + c\epsilon\Delta) \partial_{x_2}, \\ a_{12} &= (1 + c\epsilon\Delta)((1 + \underline{\zeta} + \tilde{\zeta}_\omega) \partial_{x_1}) + (\underline{v}_1 + \tilde{v}_{1\omega})^2 \partial_{x_1} + (\underline{v}_2 + \tilde{v}_{2\omega})(\underline{v}_1 + \tilde{v}_{1\omega}) \partial_{x_2}, \\ a_{13} &= (1 + c\epsilon\Delta)((1 + \underline{\zeta} + \tilde{\zeta}_\omega) \partial_{x_2}) + (\underline{v}_1 + \tilde{v}_{1\omega})(\underline{v}_2 + \tilde{v}_{2\omega}) \partial_{x_1} + (\underline{v}_2 + \tilde{v}_{2\omega})^2 \partial_{x_2}, \\ a_{21} &= (\underline{v}_1 + \tilde{v}_{1\omega})((\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot \nabla) + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)(1 + c\epsilon\Delta) \partial_{x_1}, \\ a_{22} &= 2(1 + \underline{\zeta} + \tilde{\zeta}_\omega)(\underline{v}_1 + \tilde{v}_{1\omega}) \partial_{x_1}, \\ a_{23} &= (\underline{v}_1 + \tilde{v}_{1\omega})(1 + \underline{\zeta} + \tilde{\zeta}_\omega) \partial_{x_2} + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)(\underline{v}_2 + \tilde{v}_{2\omega}) \partial_{x_1}, \\ a_{31} &= (\underline{v}_2 + \tilde{v}_{2\omega})(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot \nabla + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)(1 + c\epsilon\Delta) \partial_{x_2}, \\ a_{32} &= (\underline{v}_2 + \tilde{v}_{2\omega})(1 + \underline{\zeta} + \tilde{\zeta}_\omega) \partial_{x_1} + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)(\underline{v}_1 + \tilde{v}_{1\omega}) \partial_{x_2}, \\ a_{33} &= 2(\underline{v}_2 + \tilde{v}_{2\omega})(1 + \underline{\zeta} + \tilde{\zeta}_\omega) \partial_{x_2} + (1 + \underline{\zeta}). \end{aligned}$$

Remark: This case is much simpler than the previous case, since there is no appearance of $g(D)$. We will only treat here the terms that gave the order of $\epsilon^{1/2}$ in the previous case.

For a_{11} , we have

$$\begin{aligned} (a_{11}\Lambda^s\zeta, \Lambda^s\zeta)_2 &= ((1 + c\epsilon\Delta)((\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \cdot \nabla\Lambda^s\zeta), \Lambda^s\zeta)_2 \\ &\quad + ((\underline{v}_1 + \tilde{v}_{1\omega})(1 + c\epsilon\Delta)\partial_{x_1}\Lambda^s\zeta, \Lambda^s\zeta)_2 \\ &\quad + ((\underline{v}_2 + \tilde{v}_{2\omega})(1 + c\epsilon\Delta)\partial_{x_2}\Lambda^s\zeta, \Lambda^s\zeta)_2 \\ &:= C_{11} + C_{12} + C_{13}. \end{aligned} \tag{5.3.80}$$

Using the commutator estimate (5.3.24), integration by part, Sobolev embedding ($s > 3$) and Hölder inequality we get

$$\begin{aligned} &((1 + c\epsilon\Delta)(\underline{\mathbf{v}} \cdot \nabla\Lambda^s\zeta), \Lambda^s\zeta)_2 \\ &= (\underline{\mathbf{v}} \cdot \nabla\Lambda^s\zeta, \Lambda^s\zeta)_2 + c\epsilon([\Delta, \underline{\mathbf{v}}] \cdot \nabla\Lambda^s\zeta, \Lambda^s\zeta)_2 + c\epsilon(\underline{\mathbf{v}} \cdot \Delta\nabla\Lambda^s\zeta, \Lambda^s\zeta)_2 \\ &= -\frac{1}{2}((\nabla \cdot \underline{\mathbf{v}})\Lambda^s\zeta, \Lambda^s\zeta)_2 + c\epsilon([\Delta, \underline{\mathbf{v}}] \cdot \nabla\Lambda^s\zeta, \Lambda^s\zeta)_2 \\ &\quad + c\epsilon\left(\Lambda^s\zeta, \sum_{j=1}^2(\nabla\underline{v}_j \cdot \nabla\partial_{x_j}\Lambda^s\zeta)\right)_2 - \frac{1}{2}c\epsilon(\Delta\Lambda^s\zeta, (\nabla \cdot \underline{\mathbf{v}})\Lambda^s\zeta)_2 \\ &\quad - \frac{3}{2}c\epsilon(\Lambda^s\zeta, \Delta\underline{\mathbf{v}} \cdot \nabla\Lambda^s\zeta)_2 \\ &\lesssim \|\underline{\mathbf{v}}\|_{H^s} \|\zeta\|_{X_{\epsilon^2}^s}^2. \end{aligned} \tag{5.3.81}$$

By using (5.3.24) with $d = 1$ and the $\sigma(D) = \partial_{x_j}^2$, $j = 1, 2$ and similar argument as above, we get

$$((1 + c\epsilon\Delta)(\tilde{\mathbf{v}}_\omega \cdot \nabla\Lambda^s\zeta), \Lambda^s\zeta)_2 \lesssim \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}} \|\zeta\|_{X_{\epsilon^2}^s}.$$

Therefore,

$$|C_{11}| \lesssim (\|\underline{\mathbf{v}}\|_{H^s} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\zeta\|_{X_{\epsilon^2}^s}^2. \tag{5.3.82}$$

For C_{12} ,

$$\begin{aligned} C_{12} &= -\frac{1}{2}(\Lambda^s\zeta, (\partial_{x_1}(\underline{v}_1 + \tilde{v}_{1\omega}))\Lambda^s\zeta)_2 - \frac{1}{2}c\epsilon(\Delta\Lambda^s\zeta, (\partial_{x_1}(\underline{v}_1 + \tilde{v}_{1\omega}))\Lambda^s\zeta)_2 \\ &\quad - \frac{1}{2}c\epsilon(\Lambda^s\zeta, (\Delta(\underline{v}_1 + \tilde{v}_{1\omega}))\partial_{x_1}\Lambda^s\zeta)_2 - \frac{1}{2}c\epsilon(\Lambda^s\zeta, \nabla(\underline{v}_1 + \tilde{v}_{1\omega}) \cdot \nabla\partial_{x_1}\Lambda^s\zeta)_2 \\ &\lesssim (\|\underline{v}_1\|_{H^s} + \|\tilde{v}_{1\omega}\|_{W^{2,\infty}}) \|\zeta\|_{X_{\epsilon^2}^s}^2. \end{aligned} \tag{5.3.83}$$

The same estimate holds for C_{13} , as long side with (5.3.82) and (5.3.83), it follows

$$(a_{11}\Lambda^s\zeta, \Lambda^s\zeta)_2 \lesssim (\|\underline{\mathbf{v}}\|_{H^s} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\zeta\|_{X_{\epsilon^2}^s}^2. \tag{5.3.84}$$

The same estimate holds for $(a_{22}\Lambda^sv_1, \Lambda^sv_1)_2$ and $(a_{33}\Lambda^sv_2, \Lambda^sv_2)_2$.

For a_{23} and a_{32} ,

$$\begin{aligned} &(a_{23}\Lambda^sv_2, \Lambda^sv_1)_2 + (a_{32}\Lambda^sv_1, \Lambda^sv_2)_2 \\ &= -\left(\Lambda^sv_2, \partial_{x_2}((\underline{v}_1 + \tilde{v}_{1\omega})(1 + \underline{\zeta} + \tilde{\zeta}_\omega)\Lambda^sv_1)\right)_2 \\ &\quad - \left(\Lambda^sv_2, \partial_{x_1}((\underline{v}_2 + \tilde{v}_{2\omega})(1 + \underline{\zeta} + \tilde{\zeta}_\omega)\Lambda^sv_2)\right)_2 \\ &\lesssim (\|\underline{\mathbf{v}}\|_{H^s} + \|\tilde{\mathbf{v}}_\omega\|_{W^{1,\infty}}) \|\mathbf{v}\|_{X_\epsilon^s}^2. \end{aligned} \tag{5.3.85}$$

By doing similarly as previous case for remaining terms, it is not hard to get that

$$|II| \lesssim (\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s})(\|\underline{\mathbf{v}}\|_{X_\epsilon^s}^2 + \|\underline{\zeta}\|_{X_{\epsilon^2}^s}^2), \quad (5.3.86)$$

Estimate on III. By using the expression of $S_{\underline{\mathbf{V}}+\underline{\tilde{\mathbf{V}}}_\omega}(D)^*$ and similar argument as before, we get

$$\begin{aligned} III &= -b\epsilon ([v_1 + \tilde{v}_{1\omega}, \Delta] \Lambda^s v_1 + [v_2 + \tilde{v}_{2\omega}, \Delta] \Lambda^s v_2, \Lambda^s \partial_t \zeta)_2 \\ &\quad - b\epsilon ([\underline{\zeta} + \tilde{\zeta}_\omega, \Delta] \Lambda^s v_1 + [v_1 + \tilde{v}_{1\omega}, \Delta] \Lambda^s \zeta, \Lambda^s \partial_t v_1)_2 \\ &\quad - b\epsilon ([v_2 + \tilde{v}_{2\omega}, \Delta] \Lambda^s \zeta + [\underline{\zeta} + \tilde{\zeta}_\omega, \Delta] \Lambda^s v_2, \Lambda^s \partial_t v_2)_2, \end{aligned}$$

then,

$$|III| \lesssim (\epsilon + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s})(\|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s})(\|\partial_t \zeta\|_{H^s} + \|\partial_t \underline{\mathbf{v}}\|_{H^s}). \quad (5.3.87)$$

Estimate on IV. Similarly as (5.3.66), (5.3.67) and (5.3.68), we have

$$\begin{aligned} IV &= ((1 - b\epsilon\Delta) \Lambda^s \zeta, (\partial_t v_1 + \partial_t \tilde{v}_{1\omega}) \Lambda^s v_1 + (\partial_t v_2 + \partial_t \tilde{v}_{2\omega}) \Lambda^s v_2)_2 \\ &\quad + \left((1 - b\epsilon\Delta) \Lambda^s v_1, (\partial_t v_1 + \partial_t \tilde{v}_{1\omega}) \Lambda^s \zeta + (\partial_t \underline{\zeta} + \partial_t \tilde{\zeta}_\omega) \Lambda^s v_1 \right)_2 \\ &\quad + \left((1 - b\epsilon\Delta) \Lambda^s v_2, (\partial_t v_2 + \partial_t \tilde{v}_{2\omega}) \Lambda^s \zeta + (\partial_t \underline{\zeta} + \partial_t \tilde{\zeta}_\omega) \Lambda^s v_2 \right)_2. \end{aligned}$$

Then,

$$|IV| \lesssim (\epsilon + \|\partial_t \underline{\zeta}\|_{H^s} + \|\partial_t \underline{\mathbf{v}}\|_{H^s})(\|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s}^2). \quad (5.3.88)$$

Thanks to (5.3.77), (5.3.86), (5.3.87) and (5.3.88), we obtain

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) &\lesssim (\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\partial_t \underline{\mathbf{V}}\|_{X_{\epsilon^2}^{s-1}})(\|\underline{\zeta}\|_{X_{\epsilon^2}^s}^2 + \|\underline{\mathbf{v}}\|_{X_\epsilon^s}^2) \\ &\quad + \|F\|_{H^s} (\|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s}) \\ &\quad + (\epsilon + \|\underline{\mathbf{v}}\|_{X_\epsilon^s} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s})(\|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{X_\epsilon^s}) \|\partial_t \mathbf{V}\|_{X_{\epsilon^2}^{s-1}}. \end{aligned} \quad (5.3.89)$$

That is (5.3.72) and then (5.3.73).

Returning to (5.3.14) we set $G = F - M(\underline{\mathbf{V}} + \underline{\tilde{\mathbf{V}}}_\omega, D)\mathbf{V}$ then $\partial_t \mathbf{V} = (1 - b\epsilon\Delta)^{-1}G$. By using the expression of $M(\underline{\mathbf{V}} + \underline{\tilde{\mathbf{V}}}_\omega, D)$ in this case we have that

$$\|G\|_{H^{s-1}} \lesssim \|F\|_{H^{s-1}} + \|\underline{\zeta}\|_{X_{\epsilon^2}^s} + \|\underline{\mathbf{v}}\|_{H^s}. \quad (5.3.90)$$

Beside that we also have

$$\begin{aligned} \|\partial_t \mathbf{V}\|_{X_{\epsilon^2}^{s-1}} &\lesssim \|(1 - b\epsilon\Delta) \partial_t \mathbf{V}\|_{H^{s-1}} \\ &= \left\| (1 - b\epsilon\Delta)(1 - b\epsilon\Delta)^{-1} G \right\|_{H^{s-1}} \\ &= \|G\|_{H^{s-1}}. \end{aligned}$$

Therefore, we obtain (5.3.74).

The *existence* and *uniqueness* of the solution of (5.3.14) are obtained similarly as in [55].

5.3.3 The proof of Theorem (5.3.3)

5.3.4 The case $a = b = d = 0, c < 0$

We recall that this case corresponds to a particular version of Euler-Korteweg systems (5.2.2), that is re-introducing the small parameter ϵ and taking without loss of generality $c = -1$

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon[\nabla \cdot (\eta \mathbf{u})] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon[\frac{1}{2} \nabla |\mathbf{u}|^2 - \nabla \Delta \eta] = 0. \end{cases} \quad (5.3.91)$$

Well-posedness on time scales of order $O(1/\epsilon)$ has been established in [56, 10] (actually [10] considers long time existence issues for a general class of Euler-Korteweg systems).

We now consider the system system satisfied by a localized perturbation on a line solitary wave $\tilde{U}_\omega = (\tilde{\eta}, \tilde{\mathbf{u}})$ of velocity ω

that is

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon[\nabla \cdot ((\eta \mathbf{u}) + \tilde{\eta}_\omega \mathbf{u} + \tilde{\mathbf{u}}_\omega \eta)] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon[\nabla (\mathbf{u} \cdot \tilde{\mathbf{u}}_\omega) + \frac{1}{2} \nabla |\mathbf{u}|^2 - \nabla \Delta \eta] = 0. \end{cases} \quad (5.3.92)$$

The proof of longtime existence will follow that of [56] and it differs somewhat of the corresponding proof for the other cases.

5.4 transverse stability issues

We restrict here to the Hamiltonian case $b = d$ and we rely on the general framework of [53]. We recall that the case of the "Euler-Korteweg system" $(0, 0, 0, -1)$ has been considered in [6, 50].

5.5 Final remarks

5.6 Appendix

5.6.1 Appendix 1.

(The case $b \neq d, b, d > 0, a, c < 0$.)

We are going to prove there exists a constant κ_H such that (5.3.29) holds under the assumption (5.3.28), plugging $S_{\mathbf{V}+\tilde{\mathbf{V}}_\omega}(D)$ into (5.3.21) we have

$$\begin{aligned} E_s(\mathbf{V}) &= \left((1 - b\epsilon\Delta)\Lambda^s \zeta, (1 + c\epsilon\Delta)^2 g(D)\Lambda^s \zeta \right)_2 \\ &\quad + \left((1 - b\epsilon\Delta)\Lambda^s \zeta, g(D)[(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega)(1 + c\epsilon\Delta)\Lambda^s \mathbf{v}] \right)_2 \\ &\quad + \left((1 - b\epsilon\Delta)\Lambda^s \mathbf{v}, g(D)[(\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega)(1 + c\epsilon\Delta)\Lambda^s \zeta] \right)_2 \\ &\quad + \left((1 - b\epsilon\Delta)\Lambda^s \mathbf{v}, (1 + \underline{\zeta} + \tilde{\zeta}_\omega + a\epsilon\Delta)(1 + c\epsilon\Delta)\Lambda^s \mathbf{v} \right)_2 \\ &\quad + \sum_{i,j=1}^2 \left((1 - b\epsilon\Delta)\Lambda^s v_i, (\underline{v}_i + \tilde{v}_{i\omega})(\underline{v}_j + \tilde{v}_{j\omega})(g(D) - 1)\Lambda^s v_j \right)_2 \\ &= A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned} \quad (A.1)$$

For A_1 we have the following result from [55]

$$A_1 \geq \min \left\{ 1, \frac{b}{d} \right\} \left(\|\Lambda^s \zeta\|_2^2 + c^2 \epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2 + (b - 2c)\epsilon \|\nabla \Lambda^s \zeta\|_2^2 + bc^2 \epsilon^3 \|\Delta \nabla \Lambda^s \zeta\|_2^2 + (-2bc)\epsilon^2 \|\nabla \nabla \Lambda^s \zeta\|_2^2 \right), \quad (A.2)$$

$$A_1 \leq \max \left\{ 1, \frac{b}{d} \right\} \left(\|\Lambda^s \zeta\|_2^2 + c^2 \epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2 + (b - 2c) \epsilon \|\nabla \Lambda^s \zeta\|_2^2 + bc^2 \epsilon^3 \|\Delta \nabla \Lambda^s \zeta\|_2^2 + (-2bc) \epsilon^2 \|\nabla \nabla \Lambda^s \zeta\|_2^2 \right). \quad (\text{A.3})$$

For A_2 and A_3 , using (5.3.25), we have

$$\begin{aligned} |A_2| + |A_3| &\leq (\|\mathbf{v}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) \|(1 - b\epsilon\Delta)g(D)\Lambda^s \zeta\|_2 \|(1 + c\epsilon\Delta)\Lambda^s \mathbf{v}\|_2 \\ &\quad + (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\underline{\mathbf{v}}}_\omega\|_\infty) \|(1 - b\epsilon\Delta)g(D)\Lambda^s \mathbf{v}\|_2 \|(1 + c\epsilon\Delta)\Lambda^s \zeta\|_2 \\ &\leq \max \left\{ 1, \frac{b}{d} \right\} (\|\mathbf{v}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) \left(2(\|\Lambda^s \zeta\|_2^2 + \|\Lambda^s \mathbf{v}\|_2^2) + (b^2 + c^2) \epsilon^2 (\|\Delta \Lambda^s \zeta\|_2^2 + \|\Delta \Lambda^s \mathbf{v}\|_2^2) \right). \end{aligned} \quad (\text{A.4})$$

For A_4 , we have

$$\begin{aligned} A_4 &= \left(\Lambda^s \mathbf{v}, (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s \mathbf{v} \right)_2 + abc\epsilon^3 \|\nabla \Delta \Lambda^s \mathbf{v}\|_2^2 + ac\epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2 \\ &\quad + b\epsilon^2 \left(\Delta \Lambda^s \mathbf{v}, -[a + c(1 + \underline{\zeta} + \tilde{\zeta}_\omega)] \Delta \Lambda^s \mathbf{v} \right)_2 \\ &\quad + \epsilon \left(\Lambda^s \mathbf{v}, [a + (c - b)(1 + \underline{\zeta} + \tilde{\zeta}_\omega)] \Delta \Lambda^s \mathbf{v} \right)_2 \end{aligned}$$

which along with the assumption $1 + \underline{\zeta} + \tilde{\zeta}_\omega \geq H > 0$ implies that

$$\begin{aligned} A_4 &\geq H \|\Lambda^s \mathbf{v}\|_2^2 + abc\epsilon^3 \|\nabla \Delta \Lambda^s \mathbf{v}\|_2^2 + (ac - b(a + cH)) \epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2 \\ &\quad + (-a - c + b) \epsilon \|\nabla \Lambda^s \mathbf{v}\|_2^2 + (c - b) \epsilon \left(\|\zeta\|_\infty + \|\tilde{\zeta}_\omega\|_\infty \right) \|\Lambda^s \mathbf{v}\|_2 \|\Delta \Lambda^s \mathbf{v}\|_2 \\ &\geq \frac{H}{2} \|\Lambda^s \mathbf{v}\|_2^2 + abc\epsilon^3 \|\nabla \Delta \Lambda^s \mathbf{v}\|_2^2 + (-a - c + b) \epsilon \|\nabla \Lambda^s \mathbf{v}\|_2^2 \\ &\quad + \left(ac - b(a + cH) - \frac{(c - b)^2 (\|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty)^2}{2H} \right) \epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2. \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} A_4 &\leq \left(1 + \|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty \right) \|\Lambda^s \mathbf{v}\|_2^2 + abc\epsilon^3 \|\nabla \Delta \Lambda^s \mathbf{v}\|_2^2 \\ &\quad + \left(ac - ab - bc \left(1 + \|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty \right) \right) \epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2 \\ &\quad + \left(-a + (-c + b) \left(1 + \|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty \right) \right) \epsilon \|\Lambda^s \mathbf{v}\|_2 \|\Delta \Lambda^s \mathbf{v}\|_2. \end{aligned} \quad (\text{A.6})$$

For A_5 , using (5.3.26) we have

$$\begin{aligned} |A_5| &\leq 2 (\|\mathbf{v}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty)^2 \|(1 - b\epsilon\Delta)\Lambda^s \mathbf{v}\|_2 \|(g(D) - 1)\Lambda^s \mathbf{v}\|_2 \\ &\leq \frac{2|b - d|}{d} (\|\mathbf{v}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty)^2 \left(2 \|\Lambda^s \mathbf{v}\|_2^2 + \frac{b^2}{4} \epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2 \right). \end{aligned} \quad (\text{A.7})$$

Combining (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) and (A.7) we obtain that

$$\begin{aligned}
E_s(\mathbf{V}) \geq & \left(\min \left\{ 1, \frac{b}{d} \right\} - 2 \max \left\{ 1, \frac{b}{d} \right\} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) \right) \|\Lambda^s \zeta\|_2^2 + \min \left\{ 1, \frac{b}{d} \right\} b c^2 \epsilon^3 \|\Delta \nabla \Lambda^s \zeta\|_2^2 \\
& + \left(\min \left\{ 1, \frac{b}{d} \right\} (c^2 - 2bc) - \max \left\{ 1, \frac{b}{d} \right\} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) (b^2 + c^2) \right) \epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2 \\
& + \min \left\{ 1, \frac{b}{d} \right\} (b - 2c) \epsilon \|\nabla \Lambda^s \zeta\|_2^2 \\
& + \left(-2 \max \left\{ 1, \frac{b}{d} \right\} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) + \frac{H}{2} - \frac{4|b-d|}{d} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty)^2 \right) \|\Lambda^s \mathbf{v}\|_2^2 \\
& + \left(\left(-\max \left\{ 1, \frac{b}{d} \right\} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) (b^2 + c^2) + ac - b(a + cH) - \frac{(c-b)^2 (\|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty)^2}{2H} \right. \right. \\
& \quad \left. \left. - \frac{b^2|b-d|}{2d} (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty)^2 \right) \epsilon^2 \|\Delta \Lambda^s \mathbf{v}\|_2^2,
\end{aligned}$$

which along with a suitable lower-bound of $\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty$ implies the left-hand side inequality of (5.3.29). Similarly, we get the right hand side inequality of (5.3.29) and therefore we can choose a suitable constant κ_H .

5.6.2 Appendix 2.

(The case $b = d > 0$, $a = 0$, $c < 0$).

We are going to prove the existence of κ_H which ensures the equivalence of $E_s(\mathbf{V})$ and $\|\zeta\|_{X_{\epsilon^2}^s}^2 + \|\mathbf{v}\|_{X_\epsilon^s}^2$ in the assumption of Proposition (5). By doing similarly as previous case, we get

$$\begin{aligned}
E_s(\mathbf{V}) &= ((1 - b\epsilon\Delta)\Lambda^s \zeta, (1 + c\epsilon\Delta)\Lambda^s \zeta + (\underline{v}_1 + \tilde{v}_{1\omega})\Lambda^s v_1 + (\underline{v}_2 + \tilde{v}_{2\omega})\Lambda^s v_2)_2 \\
&\quad + \left((1 - b\epsilon\Delta)\Lambda^s v_1, (\underline{v}_1 + \tilde{v}_{1\omega})\Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)\Lambda^s v_1 \right)_2 \\
&\quad + \left((1 - b\epsilon\Delta)\Lambda^s v_2, (\underline{v}_2 + \tilde{v}_{2\omega})\Lambda^s \zeta + (1 + \underline{\zeta} + \tilde{\zeta}_\omega)\Lambda^s v_2 \right)_2 \\
&= ((1 - b\epsilon\Delta)\Lambda^s \zeta, (1 + c\epsilon\Delta)\Lambda^s \zeta)_2 \\
&\quad + ((1 - b\epsilon\Delta)\Lambda^s \zeta, (\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega)\Lambda^s \mathbf{v})_2 \\
&\quad + \left((1 - b\epsilon\Delta)\Lambda^s \mathbf{v}, (1 + \underline{\zeta} + \tilde{\zeta}_\omega)\Lambda^s \mathbf{v} \right)_2 \\
&\quad + ((1 - b\epsilon\Delta)\Lambda^s \mathbf{v}, (\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega)\Lambda^s \zeta)_2 \\
&= B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

For B_1 , we have

$$B_1 = \|\Lambda^s \zeta\|_2^2 + (b - c)\epsilon \|\nabla \Lambda^s \zeta\|_2^2 - bc\epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2 \quad (\text{B.1})$$

For B_2 , we have

$$|B_2| \leq (\|\underline{\mathbf{v}}\|_\infty + \|\tilde{\mathbf{v}}_\omega\|_\infty) \|\Lambda^s \mathbf{v}\|_2 (\|\Lambda^s \zeta\|_2 + b\epsilon \|\Delta \Lambda^s \zeta\|_2). \quad (\text{B.2})$$

Then,

$$\begin{aligned}
B_2 \geq & -4(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \mathbf{v}\|_2^2 - \frac{1}{4}(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \zeta\|_2^2 \\
& - 4b(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda \mathbf{v}\|_2^2 - \frac{b}{4}\epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2.
\end{aligned} \quad (\text{B.2.1})$$

And

$$B_2 \geq 4(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \mathbf{v}\|_2^2 + \frac{1}{4}(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \zeta\|_2^2 \\ + 4b(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda \mathbf{v}\|_2^2 + \frac{b}{4}\epsilon^2 \|\Delta \Lambda^s \zeta\|_2^2. \quad (\text{B.2.2})$$

For B_3 , by integrating by parts, we have

$$B_3 = \left(\Lambda^s \mathbf{v}, (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s \mathbf{v} \right)_2 - b\epsilon \sum_{j=1}^2 \left(\Delta \Lambda^s v_j, (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s v_j \right)_2 \\ = \left(\Lambda^s \mathbf{v}, (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \Lambda^s \mathbf{v} \right)_2 + b\epsilon \sum_{j=1}^2 \left[\left(\nabla \Lambda^s v_j, (1 + \underline{\zeta} + \tilde{\zeta}_\omega) \nabla \Lambda^s v_j \right)_2 \right. \\ \left. - \frac{1}{2} \sum_{k=1}^2 \left(\Lambda^s v_j, (\partial_{x_k}^2 \underline{\zeta} + \partial_{x_k}^2 \tilde{\zeta}_\omega) \Lambda^s v_j \right)_2 \right]. \quad (\text{B.3})$$

Which alongside with (5) give that

$$B_3 \geq \left(H - \frac{b\epsilon}{2}(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \right) \|\Lambda^s \mathbf{v}\|_2^2 \\ + b\epsilon \sum_{j=1}^2 H \|\nabla \Lambda^s v_j\|_2^2, \quad (\text{B.3.1})$$

and

$$B_3 \leq \left(1 + (1 + \frac{b\epsilon}{2})(\|\underline{\zeta}\|_{W^{2,\infty}} + \|\tilde{\zeta}_\omega\|_{W^{2,\infty}}) \right) \|\Lambda^s \mathbf{v}\|_2^2 \\ + b\epsilon(1 + \|\underline{\zeta}\|_\infty + \|\tilde{\zeta}_\omega\|_\infty) \sum_{j=1}^2 \|\nabla \Lambda^s v_j\|_2^2. \quad (\text{B.3.2})$$

For B_4 , we have

$$B_4 = (\Lambda^s \mathbf{v}, (\underline{\mathbf{v}} + \tilde{\mathbf{v}}_\omega) \Lambda^s \zeta)_2 - b\epsilon \sum_{j=1}^2 \left[\left(\Lambda^s v_j, (\underline{v}_j + \tilde{v}_{j\omega}) \Delta \Lambda^s \zeta \right)_2 \right. \\ \left. + \left(\Lambda^s v_j, (\Delta \underline{v}_j + \Delta \tilde{v}_{j\omega}) \Lambda^s \zeta \right)_2 + 2 \left(\Lambda^s v_j, \nabla(\underline{v}_j + \tilde{v}_{j\omega}) \cdot \nabla \Lambda^s \zeta \right)_2 \right].$$

Then

$$|B_4| \leq (1 + b\epsilon)(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \mathbf{v}\|_2 \|\Lambda^s \zeta\|_2 \\ + 2b\epsilon(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \|\Lambda^s \mathbf{v}\|_2 \|\nabla \Lambda^s \zeta\|_2. \quad (\text{B.4})$$

That follows

$$B_4 \geq -(1 + b\epsilon)(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \left(4 \|\Lambda^s \mathbf{v}\|_2^2 + \frac{1}{4} \|\Lambda^s \zeta\|_2^2 \right) \\ - 2b(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \left(4 \|\Lambda^s \mathbf{v}\|_2^2 + \frac{\epsilon^2}{4} \|\nabla \Lambda^s \zeta\|_2^2 \right), \quad (\text{B.4.1})$$

and

$$B_4 \leq (1 + b\epsilon)(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \left(4 \|\Lambda^s \mathbf{v}\|_2^2 + \frac{1}{4} \|\Lambda^s \zeta\|_2^2 \right) \\ + 2b(\|\underline{\mathbf{v}}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}) \left(4 \|\Lambda^s \mathbf{v}\|_2^2 + \frac{\epsilon^2}{4} \|\nabla \Lambda^s \zeta\|_2^2 \right). \quad (\text{B.4.2})$$

Thanks to (B.1), (B.2.1), (B.3.1) and (B.4.1), we get

$$E_s(\mathbf{V}) \geq \frac{1}{2} \|\Lambda^s \zeta\|_2^2 + \left(\frac{b}{2} - c\right) \epsilon \|\nabla \Lambda^s \zeta\|_2^2 - b\epsilon \left(c + \frac{\|\mathbf{v}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}}{4}\right) \|\Delta \Lambda^s \zeta\|_2^2 \\ + (H - (\|\mathbf{v}\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}_\omega\|_{W^{2,\infty}}))(8 + 12b + \frac{9}{2}b\epsilon) \|\Lambda^s \mathbf{v}\|_2^2.$$

Note that in this case we have $b > 0$ and $c < 0$, so similarly as the case in the appendix 5.6.1 we can choose a suitable constant κ_H .

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Chapter 6

Some numerical results on the two dimensional Zakharov-Rubenchik/Benney-Roskes system.

Abstract: Our aim are simulations of the transverse stability (or instability) of the 1-D solitary waves and the long time behavior of the solution of the 2-D Zakharov-Rubenchik/Benney-Roskes system (Z-R/B-R) which were studied analytically in the Chapter 3.

6.1 Introduction

We first recall some notations and definitions in the Chapter 3. The following is the two dimensional Z-R/B-R system equation:

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} - i\sigma_1 \psi_{yy} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \Delta \phi + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0, \end{cases} \quad (6.1.1)$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, $\rho, \phi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\sigma_3, \delta, \sigma_1, \sigma_2, W, \rho, D, M$ are real parameters which depend in the context of problem. In the context of water wave problem, it is known as Benney-Roskes equation and in the MHD, it is called Zakharov-Rubenchik equation. The local Cauchy problem for the Z-R/B-R equation and related problems are studied in [1] and recently in [2] when the equation is perturbed by a line solitary wave.

From the Chapter 3, we recall that the one dimensional Z-R/B-R system has the form

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \phi_{xx} + D(|\psi|^2)_x = 0, \\ \phi_t + \frac{1}{M^2} \rho + |\psi|^2 = 0. \end{cases} \quad (6.1.2)$$

Let

$$a = \frac{-(1 + cD)}{1/M^2 - c^2} \quad \text{and} \quad b = \frac{-(c + D/M^2)}{1/M^2 - c^2}.$$

If

$$\begin{cases} \frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) < 0, \\ \frac{1}{\delta} (\sigma_2 + W(a + bD)) < 0, \end{cases}$$

Then there is a solitary wave of (6.1.2) of the form

$$Q = (e^{i\lambda t} e^{i(c+\sigma_3)(x-ct)/2\delta} R(x-ct), a R^2(x-ct), b P(x-ct)). \quad (6.1.3)$$

where

$$R(x) = \alpha \operatorname{sech}(\beta x), \quad P(x) = \frac{\alpha^2}{\beta} \tanh(\beta x),$$

$$\alpha = \sqrt{\frac{2}{\sigma_2 + W(a + bD)} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{-\frac{1}{\delta} \left(\frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}.$$

In one dimensional case, it is proved rigorously in [3] that Q is orbitally stable and this result is also verified numerically in [4].

We use a finite domain in x with periodic boundary conditions, which is the good setting to use a time-splitting Fourier Pseudo-Spectral method (see [4], [5]).

However, since the initial data of ϕ in (1.6.3) is not compatible with the periodic boundary condition setting, we will differentiate the last equation of (1.6.1) and consider two new variables those are ϕ_x and ϕ_y . Hence we consider the following system

$$\begin{cases} \psi_t - \sigma_3 \psi_x - i\delta \psi_{xx} - i\sigma_1 \psi_{yy} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \partial_x(\phi_x) + \partial_y(\phi_y) + D(|\psi|^2)_x = 0, \\ \partial_t(\phi_x) + \frac{1}{M^2} \rho_x + (|\psi|^2)_x = 0, \\ \partial_t(\phi_y) + \frac{1}{M^2} \rho_y + (|\psi|^2)_y = 0, \end{cases} \quad (6.1.4)$$

In order to remove the " ψ_x " term in (6.1.4) we use the transformation:

$$\psi(x, y, t) = e^{i\frac{\sigma_3^2}{4\delta}t} e^{i\frac{\sigma_3}{2\delta}x} \tilde{\psi}(x, y, t).$$

After plugging it in (6.1.4) and removing the tilde, the equation (6.1.4) becomes

$$\begin{cases} \psi_t - i\delta \psi_{xx} - i\sigma_1 \psi_{yy} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi = 0, \\ \rho_t + \partial_x(\phi_x) + \partial_y(\phi_y) + D(|\psi|^2)_x = 0, \\ \partial_t(\phi_x) + \frac{1}{M^2} \rho_x + (|\psi|^2)_x = 0, \\ \partial_t(\phi_y) + \frac{1}{M^2} \rho_y + (|\psi|^2)_y = 0, \end{cases} \quad (6.1.5)$$

The corresponding soliton for (6.1.5) now has the form (with the same notation Q)

$$Q = (e^{i(\lambda - \frac{\sigma_3^2}{4\delta})t} e^{i\frac{(c+\sigma_3)(x-ct) - \sigma_3 x}{2\delta}} R(x-ct), a R^2(x-ct), b R^2(x-ct), 0) := (Q_1, Q_2, Q_3, Q_4).$$

For the simulation, we will use time splitting Fourier spectral method which is introduced in [4] to study the one dimensional Zakharov-Rubenchik equation.

We choose the computational domain $\Omega = [-R, R]^2$ which is large enough to eliminate the truncation error (in comparison to the wave speed c). We take the same numbers of grid points $N_x = N_y = N$ for x and y and define the mesh size $dx = dy = \frac{2R}{N}$, then the grid points are denoted by

$$x_j = -R + (j-1)dx, y_k = -R + (k-1)dy, \quad j = 1, \dots, N_x, \quad k = 1, \dots, N_y. \quad (6.1.6)$$

Similarly, we denote the time step size by $\tau > 0$ and the time step $t_n = n\tau$, $n = 0, 1, \dots$ and by T the final time.

From time $t = t_n$ to time $t = t_{n+1}$, the equation (6.1.5) is solved in two steps. One solves

$$\begin{cases} \psi_t + i\Delta\psi = 0, \\ \rho_t + \partial_x(\phi_x) + \partial_y(\phi_y) = 0, \\ \partial_t(\phi_x) + \frac{1}{M^2}\rho_x = 0, \\ \partial_t(\phi_y) + \frac{1}{M^2}\rho_y = 0, \end{cases} \quad (6.1.7)$$

for one time step, followed by solving

$$\begin{cases} \psi_t + i\left\{\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right\}\psi = 0, \\ \rho_t + D(|\psi|^2)_x = 0, \\ \partial_t(\phi_x) + (|\psi|^2)_x = 0, \\ \partial_t(\phi_y) + (|\psi|^2)_y = 0, \end{cases} \quad (6.1.8)$$

again for the same time step. The system (6.1.7) is discretized in space by the Fourier spectral method and integrated in time exactly. For $l = 1, \dots, N_x$, $m = 1, \dots, N_y$, the (l, m) -Fourier mode of $\hat{\psi}$, $\hat{\rho}$ and $\hat{\phi}$ are computed as follows

$$\begin{cases} \hat{\psi}(t) = \exp^{-i(\mu_x^2 + \mu_y^2)(t-t_n)} \hat{\psi}(t_n), \\ \hat{\rho}_{lm}(t) = \cos(\alpha_{lm}(t-t_n)) \hat{\rho}_{lm}(t_n) - \frac{i\mu_x(l, m) \sin(\alpha_{lm}(t-t_n))}{\alpha_{lm}} \hat{\phi}_{xlm}(t_n) \\ \quad - \frac{i\mu_y(l, m) \sin(\alpha_{lm}(t-t_n))}{\alpha_{lm}} \hat{\phi}_{ylm}(t_n), \\ \hat{\phi}_{xlm}(t) = \frac{-i\mu_x(l, m) \sin(\alpha_{lm}(t-t_n))}{\alpha_{lm} M^2} \hat{\rho}_{lm}(t_n) \\ \quad + \frac{\mu_y(l, m)^2 + \mu_x(l, m)^2 \cos(\alpha_{lm}(t-t_n))}{\alpha_{lm}^2 M^2} \hat{\phi}_{xlm}(t_n) \\ \quad - \frac{-\mu_x(l, m)\mu_y(l, m) + \mu_x(l, m)\mu_y(l, m) \cos(\alpha_{lm}(t-t_n))}{\alpha_{lm}^2 M^2} \hat{\phi}_{ylm}(t_n), \\ \hat{\phi}_{ylm}(t) = \frac{-i\mu_y(l, m) \sin(\alpha_{lm}(t-t_n))}{\alpha_{lm} M^2} \hat{\rho}_{lm}(t_n) \\ \quad + \frac{\mu_x(l, m)^2 + \mu_y(l, m)^2 \cos(\alpha_{lm}(t-t_n))}{\alpha_{lm}^2 M^2} \hat{\phi}_{ylm}(t_n) \\ \quad - \frac{-\mu_x(l, m)\mu_y(l, m) + \mu_x(l, m)\mu_y(l, m) \cos(\alpha_{lm}(t-t_n))}{\alpha_{lm}^2 M^2} \hat{\phi}_{xlm}(t_n), \\ \text{for } l^2 + m^2 \neq 2, \\ \hat{\rho}_{11}(t) = \hat{\rho}_{11}(t_n), \hat{\phi}_{x11}(t) = \hat{\phi}_{x11}(t_n), \hat{\phi}_{y11}(t) = \hat{\phi}_{y11}(t_n), \end{cases} \quad (6.1.9)$$

for $t \in [t_n, t_{n+1}]$ and μ_x, μ_y are respectively the Fourier coefficients corresponding to the x and y derivatives.

$$\mu_x(l, m) = \frac{\pi l}{R}, \quad \mu_y(l, m) = \frac{\pi m}{R}.$$

For the system (6.1.8), multiplying the first equation by $\bar{\psi}$ we get

$$|\psi(x, y, t)| = |\psi(x, y, t_n)|, \quad t_n \leq t \leq t_{n+1}.$$

Thus the last three equations of (6.1.8) are exactly integrable as follows

$$\begin{cases} \rho(x, y, t) = \rho(x, y, t_n) - D(t - t_n)(|\psi(x, t_n)|^2)_x, \\ \phi_x(x, y, t) = \phi_x(x, y, t_n) - (t - t_n)(|\psi(x, y, t_n)|^2)_x, \\ \phi_y(x, y, t) = \phi_y(x, y, t_n) - (t - t_n)(|\psi(x, y, t_n)|^2)_y, \end{cases} \quad (6.1.10)$$

for $t_n \leq t \leq t_{n+1}$.

Plugging (6.1.10) into the first equation of (6.1.8) and integrating for t from t_n to t , we get

$$\begin{aligned} \psi(x, y, t) = & \exp\{-i(t - t_n)(\sigma_2 |\psi(x, y, t_n)|^2 + W\rho(x, y, t_n) + WD\phi_x(x, y, t_n))\} \\ & \times \exp\{2i(t - t_n)^2 WD(|\psi(x, y, t_n)|^2)_x\} \psi(x, y, t_n), \end{aligned} \quad (6.1.11)$$

for $t_n \leq t \leq t_{n+1}$.

To compute the spatial derivative in (6.1.10) and (6.1.11), we use the standard Fourier pseudo spectral method.

The structure of the next part is:

- We first make the accuracy test to compare the l^2 errors between the different choices of the wave speed c and the frequency λ of the solitary wave Q . We also compare the accuracy with different space and time steps.
- We then perturb the solitary wave by a localized Gaussian and simulate the perturbed solution corresponding to the choices of (c, λ) and the perturbation.

6.2 Numerical results

In this part, we choose: $\sigma_1 = -1, \sigma_2 = 1, \sigma_3 = 1, \delta = -6, D = 1, M = 1.1, W = 1$, this configuration is compatible with the derivation of the Zakharov-Rubenchik system in the context of MHD. Then, in order to insure the existence of the soliton, we need to choose carefully (c, λ) , indeed we want to make corresponding (α, β) positive.

It is also worth to notice that with this configuration, the first equation of (6.1.5) is similar to the focusing cubic nonlinear Schrödinger equation. Therefore, it suggests the blow-up phenomena if we choose large enough initial data. However, our numerical test shows that it is not easy to observe this phenomenal in short time.

6.2.1 Accuracy test

We first provide some informations of the parameter α with respect to the choice of (c, λ) .

Table 6.1: Value of (α, β) when $c = 1$

c=1	$\lambda = -5$	$\lambda = -10$	$\lambda = -15$
α	0.6476	0.9237	1.1345
β	0.8975	1.2802	1.5723

Table 6.2: Value of (α, β) when $c = 1.2$

$c=1.2$	$\lambda = -5$	$\lambda = -10$	$\lambda = -15$
α	1.1030	1.5761	1.9370
β	0.8943	1.2779	1.5705

Now we compare the accuracy of the methods with different number of grid points. Before that we need the following definitions of the discrete l^2 norm.

For a discrete function $U = \{U(x_j, y_k)\}$, $j = 1, \dots, N_x$, $k = 1, \dots, N_y$ where the grid points are defined as in (6.1.6), we define the discrete l^2 -norm of U as

$$\|U\|_{l^2}^2 = dx dy \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} |U(x_j, y_k)|^2. \quad (6.2.1)$$

Then, we denote

$$\begin{aligned} e_\psi(t_n) &= \|\psi(t_n) - Q_1(t_n)\|_{l^2}, \\ e_\rho(t_n) &= \|\rho(t_n) - Q_2(t_n)\|_{l^2}, \\ e_{\phi_x}(t_n) &= \|\phi_x(t_n) - Q_3(t_n)\|_{l^2}, \\ e_{\phi_y}(t_n) &= \|\phi_y(t_n)\|_{l^2}. \end{aligned} \quad (6.2.2)$$

In this section, $e_\psi(t_n)$, $e_\rho(t_n)$, $e_{\phi_x}(t_n)$ and $e_{\phi_y}(t_n)$ denote the l^2 -errors at time t_n between the exact solution (line soliton) and the numerical solution which starts with the initial given by the line soliton.

Table 6.3: l^2 error, $c = 1$, $T = 6$, $R = 25$, $N_x = N_y = 100$

$c = 1$	$\tau = 1.8750e - 04$, $\lambda = -5$	$\tau = 1.8750e - 04$, $\lambda = -10$	$\tau = 1.8750e - 04$, $\lambda = -15$
e_ρ	0.0184	0.5871	2.6454
e_{ϕ_x}	0.0172	0.5476	2.4640
e_{ϕ_y}	0	0	0
e_ψ	9.7264e-05	0.0974	1.5712

Table 6.4: l^2 error, $c = 1$, $T = 6$, $R = 25$, $N_x = N_y = 200$

$c = 1$	$\tau = 1.8750e - 04$, $\lambda = -5$	$\tau = 1.8750e - 04$, $\lambda = -10$	$\tau = 1.8750e - 04$, $\lambda = -15$
e_ρ	1.1910e-04	4.6040e-04	0.0025
e_{ϕ_x}	1.0870e-04	4.1959e-04	0.0023
e_{ϕ_y}	0	0	0
e_ψ	2.1347e-04	0.0011	0.0031

Table 6.5: l^2 error, $c = 1.2$, $T = 6$, $R = 35$, $N_x = N_y = 100$

$c = 1.2$	$\tau = 1.8750e - 04$, $\lambda = -5$	$\tau = 1.8750e - 04$, $\lambda = -10$	$\tau = 1.8750e - 04$, $\lambda = -15$
e_ρ	0.2020	3.0187	9.1911
e_{ϕ_x}	0.1902	2.8427	8.6494
e_{ϕ_y}	0	0	0
e_ψ	0.0590	6.1672	34.6855

Table 6.6: l^2 error, $c = 1.2$, $T = 6$, $R = 35$, $N_x = N_y = 200$

$c = 1.2$	$\tau = 1.8750e - 04$, $\lambda = -5$	$\tau = 1.8750e - 04$, $\lambda = -10$	$\tau = 1.8750e - 04$, $\lambda = -15$
e_ρ	0.0013	0.0241	0.2194
e_{ϕ_x}	0.0012	0.0225	0.2051
e_{ϕ_y}	0	0	0
e_ψ	0.0038	0.0185	0.0103

From the tables above we can see that if we decrease the number of grid points by half, then the accuracy may decrease by 100 times. It is due to the spectral accuracy given by the time splitting spectral method.

In another point of view, the accuracy is better with smaller wave speed c and smaller frequency λ , but we can see later that with small wave speed the solution changes very slowly and it is not good to observe it in short time. If we choose a smaller time step τ , the loss of accuracy can be gained but the running time will be very long.

Another remark, if we increase the wave speed, we will need a bigger spatial domain to reduce the truncation error, because the wave tends to the boundary faster. That is also the reasons we increase R in the last case.

Therefore we will do the simulations with the time step $\tau = 1.8750e - 04$, $(c, \lambda) = (1, -10)$ or $(1.2, -5)$ and $N_x = N_y = 200$.

The following figure shows the soliton at $T = 0$ and $T = 6$, using the configuration: $c = 1$, $\lambda = -10$, $T = 6$, $R = 25$, $N_x = N_y = 200$ and $\tau = 1.8750e - 04$.

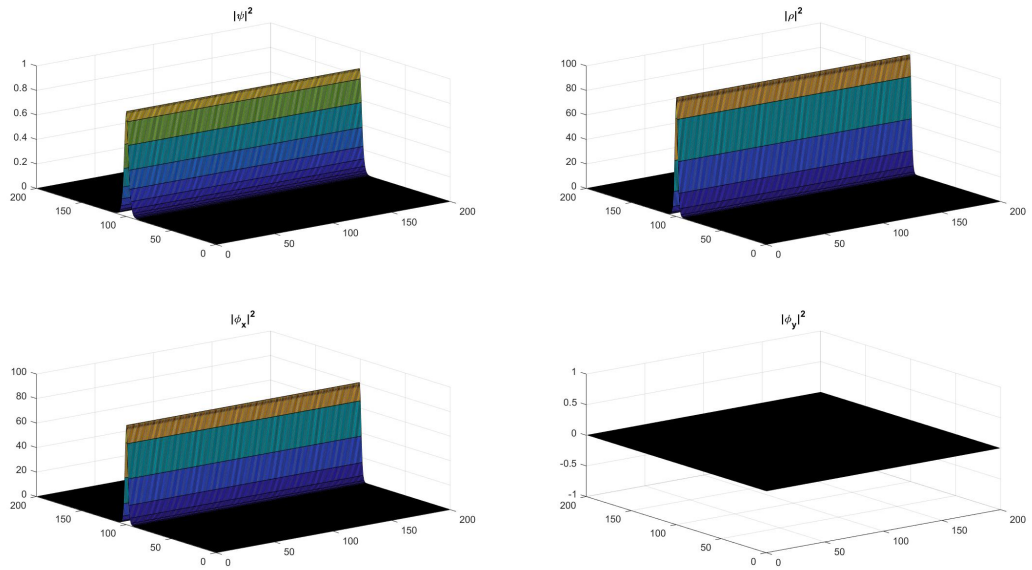


Figure 6.1: Soliton at T=0

and the soliton at $T = 6$

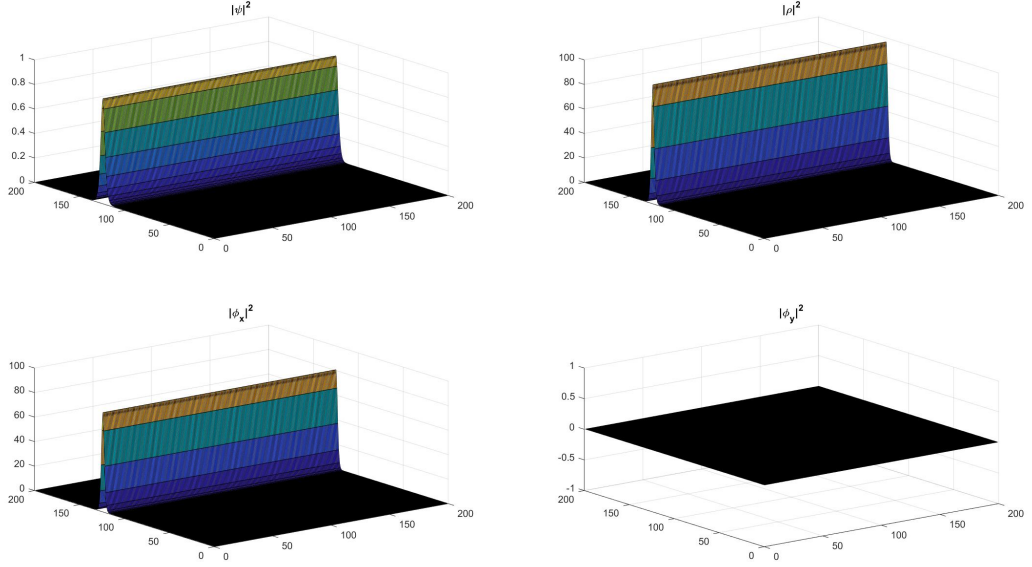


Figure 6.2: Soliton at T=6

6.2.2 Localized perturbation (by a Gaussian function)

In this part we consider the perturbations of the line soliton by a Gaussian function and we will show the behavior of the solution at different time and also the l^2 -norm difference of the perturbation and the exact solution.

We thus propagate initial data of the form

$$\begin{cases} \psi(x, y, 0) = \exp(\frac{icx}{2\delta})R(x) + \epsilon \exp(-\frac{(x+1)^2}{2} - 2(y-1)^2), \\ \rho(x, y, 0) = aR^2(x) + \epsilon \exp(-(x+\frac{1}{2})^2 - (y-1)^2), \\ \phi_x(x, y, 0) = bR^2(x) - \epsilon \frac{x+1}{2} \exp(-\frac{(x+1)^2}{4} - y^2), \\ \phi_y(x, y, 0) = -2\epsilon y \exp(-\frac{(x+1)^2}{4} - y^2). \end{cases} \quad (6.2.3)$$

Note that in this section, $e_\psi(t_n)$, $e_\rho(t_n)$, $e_{\phi_x}(t_n)$ and $e_{\phi_y}(t_n)$ are used to determine the l^2 difference at time t_n of the solution corresponding to initial data given by (6.2.3) and the line soliton Q .

The below figures will show how the perturbed solution changes with different choices of ϵ and (c, λ) .

In the first case, we choose $\epsilon = 1e - 01$, $(c, \lambda) = (1, -10)$, $R = 25$, $N_x = N_y = 200$ and $\tau = 1.8750e - 04$.

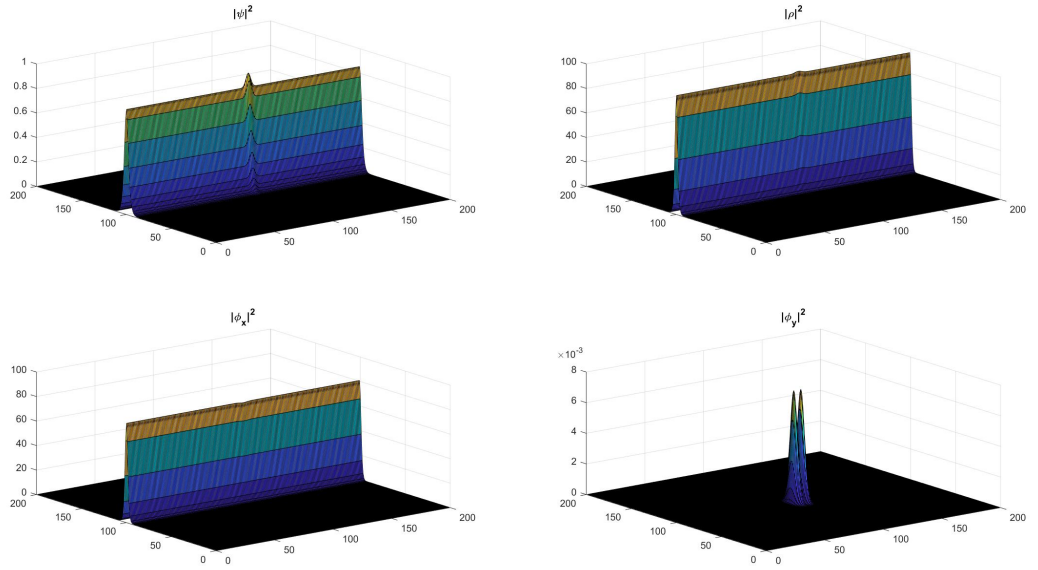


Figure 6.3: Solution at $T=0$

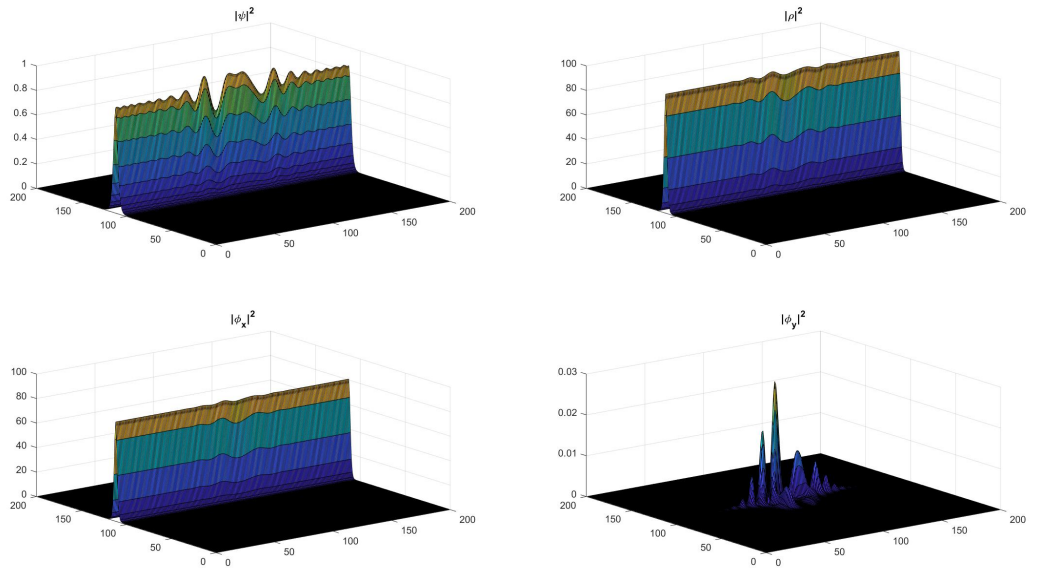


Figure 6.4: Solution at $T=3$

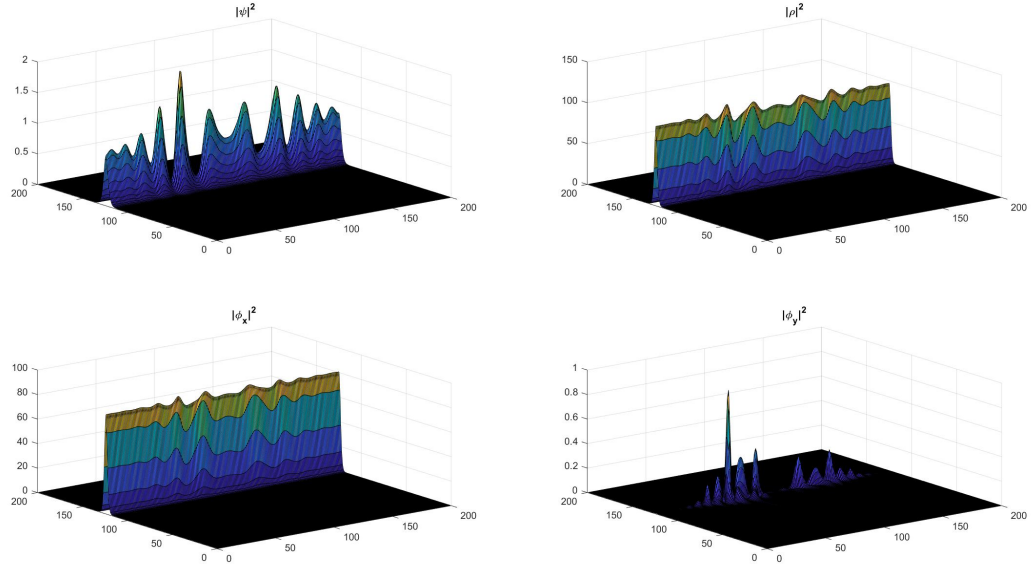


Figure 6.5: Solution at T=6

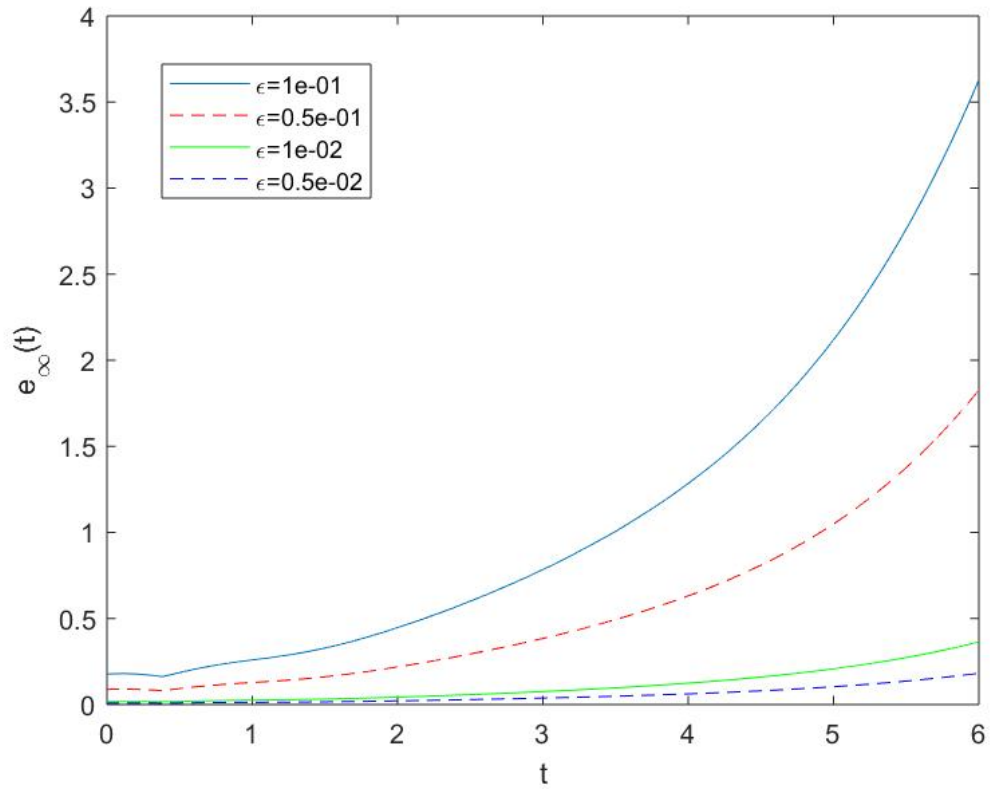


Figure 6.6: Time evolution of $e_{\infty}(t)$: difference between the perturbed solutions and the solitons for $t \in [0, 6]$ at different ϵ

Here we define

$$e_{\infty}(t) = \max(e_{\psi}(t), e_{\rho}(t), e_{\phi_x}(t), e_{\phi_y}(t)).$$

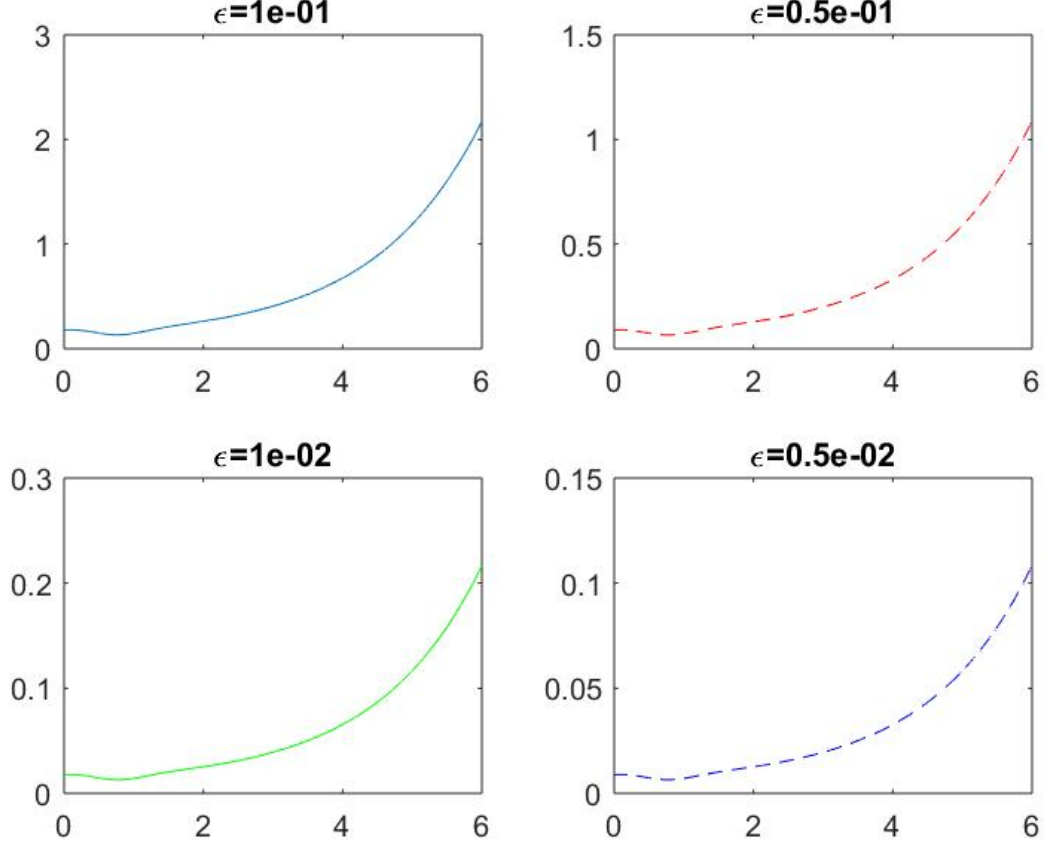


Figure 6.7: Time evolution of $e_{\phi_y}(t)$

In conclusions, we observe the following:

- 1) From the figures 6.4 and 6.5 we see that $|\rho|^2$ and $|\phi_x|^2$ do not change too much at time $T = 6$.
- 2) Figure 6.6 suggests that the one dimensional soliton is L^2 -stable in the time range $[0, 6]$.
- 3) Orbital stability means that the perturbed solution stay close to the closed orbit of the line soliton and since $Q_4 = 0$ for all time, then it make sense if we look at the time evolution of $e_{\phi_y}(t)$. The figure 6.7 shows that $e_{\phi_y}(t)$ behaves like a strictly increasing function in time, so it suggests that ϕ_y is orbitally unstable in the long time.

In the second case, we will choose $\epsilon = 1e - 01$, $(c, \lambda) = (1.2, -5)$, $R = 35$, $N_x = N_y = 200$ and $\tau = 1.8750e - 04$.

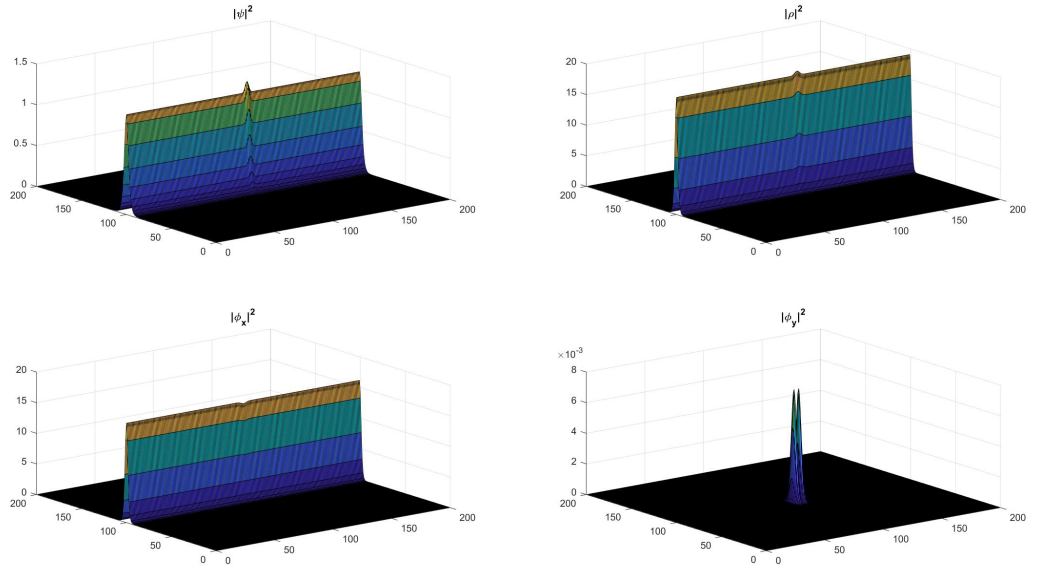


Figure 6.8: Solution at $T=0$

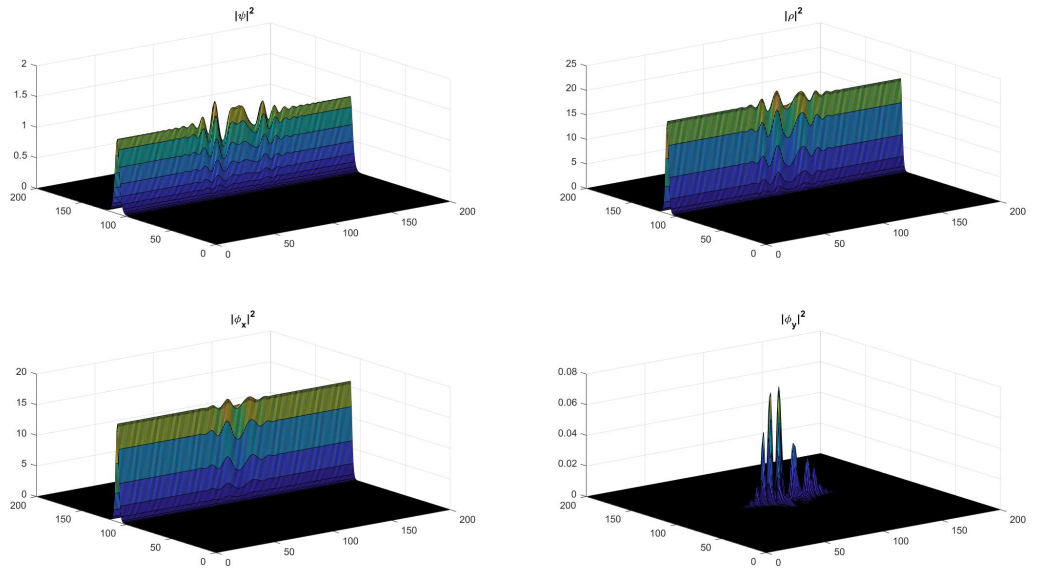


Figure 6.9: Solution at $T=3$

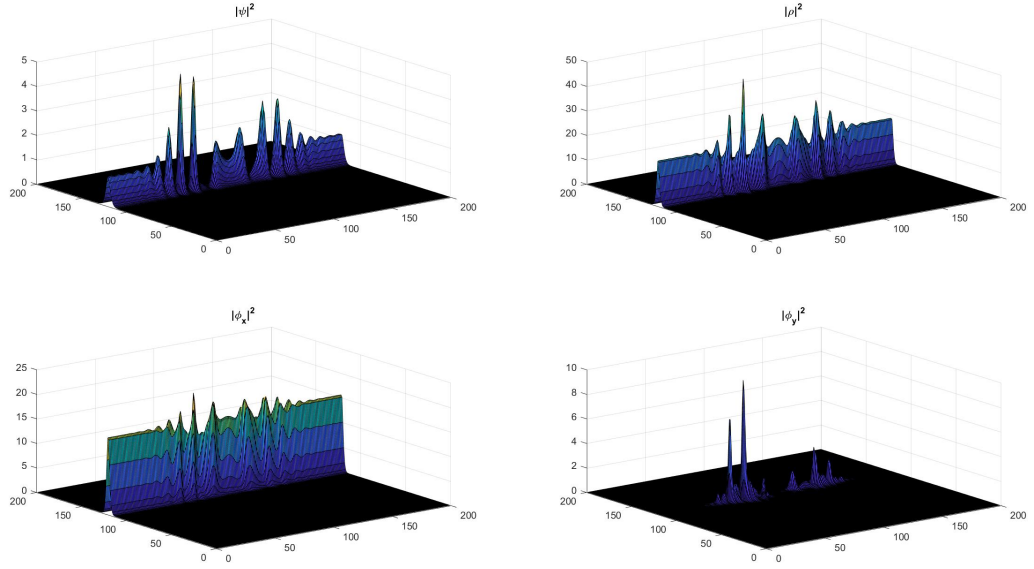


Figure 6.10: Solution at T=6

Comparing the figures 6.5 and 6.10 we see that the perturbed solution changes faster when we increase the wave speed or in general the modulation of the line soliton.

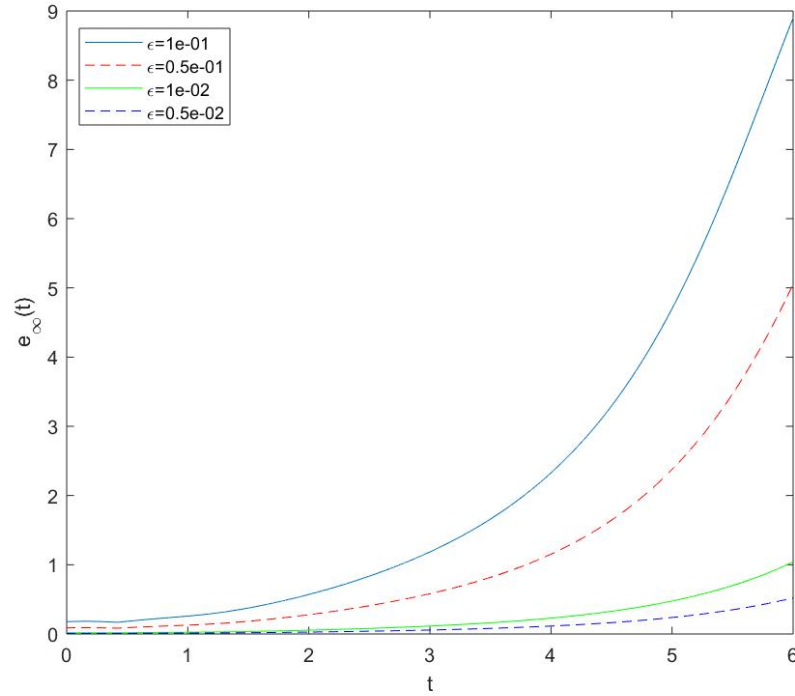


Figure 6.11: Time evolution of $e_{\infty}(t)$: difference between the perturbed solutions and the solitons for $t \in [0, 6]$ at different ϵ

As previous case, the figure 6.6 show that the line soliton is stable in the time interval $[0, 6]$ and we can see the time evolution of $e_{\phi_y}(t)$ behaves similarly.

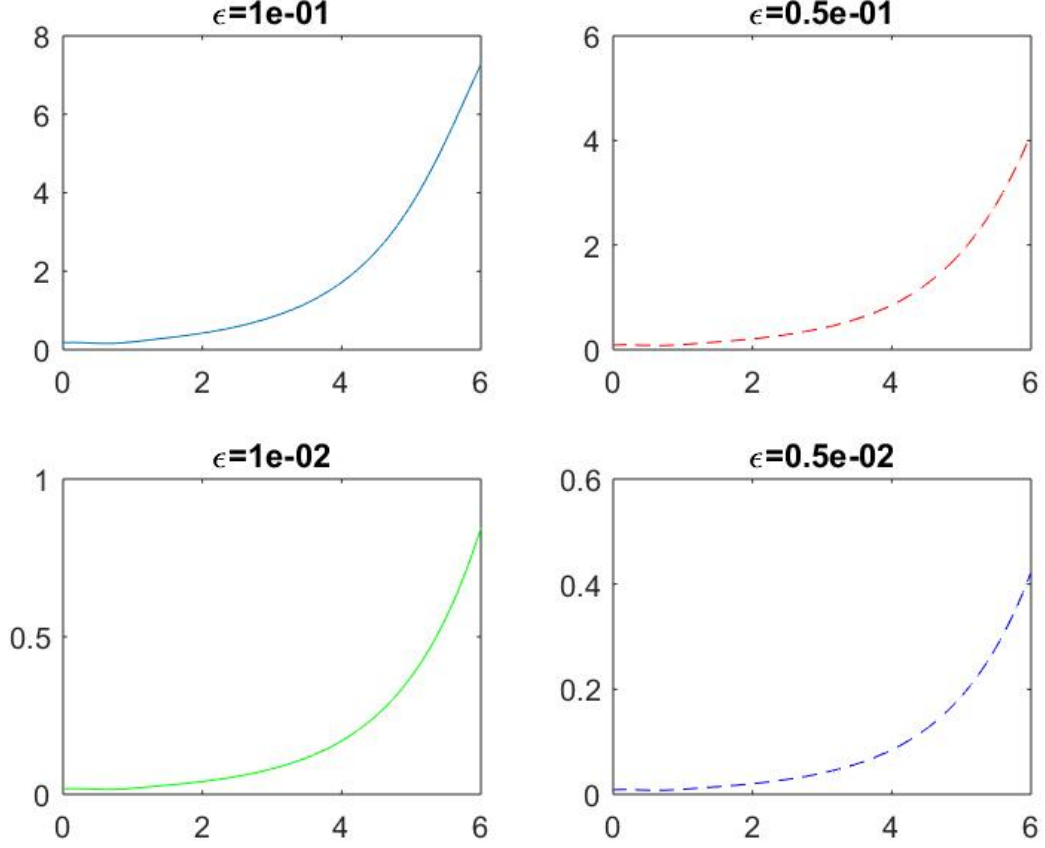


Figure 6.12: Time evolution of $e_{\phi_y}(t)$

Finally, from the above numerical tests, we can summarize our conclusion as follows

- 1) The accuracy of our numerical method depends strongly on the wave speed c and the frequency λ of the line soliton and also the choice of time step and number of grid point. If we choose the bigger wave speed c , the solution changes more quickly and it is good to observe the behavior of the solution (with general initial data). However, in order to maintain a reasonable accuracy, it is necessary to increase the number of grid point or decrease the time step and the time consumption will increase significantly also.
- 2) The figures 6.6 and 6.11 show that the line soliton is stable in the time interval $[0, 6]$. However, since the line soliton of the form of Q_1 is transversely unstable in the case of cubic focusing nonlinear Schrödinger equation, we expect the transverse instability in this case. The figures 6.7 and 6.12 show that the time evolution of $e_{\phi_y}(t)$ behaves as a strictly increasing function in time, then we maintain our opinion that Q is transversely unstable, but it might happen at a larger time and this time might depend on the wave speed of the line soliton.
- 3) With our configuration, the first equation of (6.1.5) is formally a focusing nonlinear Schrödinger equation and the blow-up phenomena may occur. In our preliminary numerical test we have not seen it.

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Curriculum Vitae

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Current position

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- 2014 - 2018 : Predoc research assistant at Univ. of Vienna, funded by FWF projects DK “Dissipation and Dispersion in nonlinear PDEs”, SFB “Vienna Computational Material Science Laboratory” and SFB “Taming Complexity in PDE systems”

Academic Education

- 2014 - 2018: **PhD** student at University of Vienna.
 - PhD Thesis : “(In)stability and asymptotic analysis of the Zakharov-Rubenchik system and related wave type equations”
 - Advisor: Prof. Norbert. J. Mauser (University of Vienna), co-advisor: Prof. Jean-Claude Saut (University Paris 11).
- 2012: **Master degree:** Fundamental Mathematics at University Paris 13, mention: cum laude; advisor : Thomas Dyckaerts.
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Publication

1. H. Luong, *Local well-posedness for the Zakharov system on the background of a line soliton*, Communications on Pure and Applied Analysis. Volume **17**, Number **6**, November **2018**.
2. H. Luong, N.J. Mauser and J. C. Saut, *On the Cauchy problem for the Zakharov-Rubenchik/ Benney-Roskes system*, Communications on Pure and Applied Analysis. Volume **17**, Number **4**, July **2018**.

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4. H. Luong and J. C. Saut, *The Boussinesq systems on the background of a line solitary wave*, manuscript.

Scientific talks

- *On the focusing 3D cubic nonlinear Schrödinger equation with potential*, University of Graz, Austria, 2012.
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Distinctions

- 2008: 3rd prize, Vietnam Mathematical Olympiad for Student.
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