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„The fundamental energy costs of letting a clock tick“

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BSc Emanuel Schwarzhans

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Abstract

Quantum thermodynamics is the field that tries to build the bridge between the theory of quantum mechanics, describing the world on a microscopic scale, and the theory of thermodynamics, which neglects microscopic degrees of freedom to describe the world from a coarse grained point of view. While statistical physics mainly considers systems at equilibrium, quantum thermodynamics also deals with system out of equilibrium.
The genesis of thermodynamics not only introduced us to an operational way of thinking about problems, but also brought us a fundamentally different concept of time. Quantum mechanics on the other hand struggles with the concept of time. Combining these two fields in order to analyze the passing of time from another perspective is the goal of this thesis.
The main objective of this work is to calculate the fundamental energy costs that are necessary for letting a clock tick from a quantum mechanical perspective, using the concepts of thermodynamics. To that end we introduce the notion of quantum thermal machines in order to construct a quantum clock which is driven by a thermodynamic heat flow. The thermal machine drives a ladder qudit to its most exited energy level. This level couples to a photon field. When it decays it sends out a photon which can be detected as a tick. As the clock performance depends on the energy costs imposed by the thermal machine, the trade off between performance and energy cost is of main interest. The performance of the clock is defined by its accuracy and resolution. In order to control these figures of merit we generalize the clock with respect to the number of participating machines and find a general expression for the occupation probability of the most exited level. Finally numerical integration leads us to the intimate relationship of energy costs, accuracy and resolution.

Zusammenfassung

Quantenthermodynamik ist das Forschungsgebiet, dass versucht Quantenmechanik, welche die Welt auf einer mikroskopischen Größenskala beschreibt, und Thermodynamik, die im Gegenzug mikroskopische Freiheitsgrade vernachlässigt um die Welt von einer makroskopischen Perspektive aus zu beschreiben, zu kombinieren. Im Gegensatz zur statistischen Physik, welche sich hauptsächlich mit Systemen im Equilibrium befasst, beschäftigt sich Quantenthermodynamik auch mit Systemen die nicht im Equilibrium sind.
Die Entdeckung der Thermodynamik brachte uns nicht nur einen operationelleren Zugang zu Problemlösungen nahe, sondern veränderte auch unser fundamentales Verständnis von Zeit. In der Quantenmechanik gibt es, andererseits große Probleme mit dem Konzept Zeit. Das Ziel dieser These is es die genannten Forschungsgebiete zu kombinieren um die Natur der Zeit von einem neuen Blickwinkel aus zu beschreiben.
Das Hauptaugenmerk liegt dabei auf den fundamentalen Energiekosten die notwendigerweise aufgebracht werden müssen um eine Uhr ticken zu lassen. Diese werden mittels quantenmechanischen Methoden und unter der Anwendung thermodynamischer Konzepte berechnet. Zu diesem Zweck führen wir den Begriff der quantenthermischen Maschinen ein, welche uns bei der Konstruktion einer Uhr, die durch einen thermodynamischen Wärmefluss angetrieben wird, behilflich sind. Die Idee ist, dass die quantenthermische Maschine eine Quantenleiter in ihren höchsten angeregten Zustand treibt. Dieser Zustand ist instabil und lässt die Leiter, unter Emission eines Photons, in seinen Grundzustand zurückfallen. Das emittierte Photon kann folglich als Tick detektiert werden. Da die Qualität der Uhr von den Energiekosten, die durch die thermalen Maschinen generiert werden, abhängt, ist der Tradeoff zwischen Qualität und Energiekosten von zentraler Bedeutung. Die Qualität der Uhr wird dabei von den Größen Genauigkeit und Auflösung definiert. Um diese Qualitätsmerkmale steuern zu können, generalisieren wir die Uhr im Bezug auf die Anzahl der teilnehmenden Maschinen und finden einen generellen Ausdruck für die Besetzungszahl des höchsten angeregten Levels der Leiter. Schlussendlich

wenden wir die Methoden der numerischen Integration an, welche uns zur innigen Beziehung zwischen Energiekosten, Genauigkeit und Auflösung führen.

1 Introduction

1.1 Time and its oddities

> *I stop and do nothing. Nothing happens. I am thinking about nothing. I listen to the passing of time.*
>
> *The Order of Time, Carlo Rovelli*

The concept of time comes quite naturally to us on an intuitive level. As time passes, the sun changes its position in the sky, we collect new experiences and grow older, and we forget old experiences. It seems like our intuitive concept of time gives us a notion of what it means for the evolution of single objects, like the sun in the sky, or a ball being thrown towards us, when time passes. We can predict when the ball will reach us and catch it. But at the same time we seem to have an intuitive understanding of future and past, of the arrow of time. We age, and when we remember past days, we not only know that they lie behind us, but also that we can not turn around and go back.
But what is behind these intuitions? What is the nature of time? These are the questions that we should ask before diving into further technicalities. We will not be able to answer them in the course of this thesis, but we will try to shed some more light on them, reduce the oddity of time a little more and bring us a little closer to understanding the nature, and our conception of time.
The motion of the sun in the sky, or a ball falling can be described up to high accuracy using classical mechanics. Time enters this theory as a parameter t inducing the dynamics of the system. The same parameter is used in quantum mechanics to describe the passage of time by using the Schrödinger equation. It is this classical parameter that we mean, when we talk about an "external", or "background" time.
If time only is a parameter controlling the dynamics of the world, how can we have a notion of it? How can my wristwatch tell me how late it is? We can observe the passing of time after all, therefore shouldn't it be viewed

as a physical observable?
In 1926 Wolfgang Pauli showed that it is not possible to introduce a self adjoint operator conjugate to the Hamilton operator[1], i.e. a time observable, within the framework of quantum mechanics. Since then many attempts have been made to quantize time [1], without much success.

This notion of time covers our intuition quite well when thinking about the dynamics of systems. But what about the direction of time? The dynamics of classical mechanics, general relativity and quantum mechanics are time reversible. There is no distinction between future and past. Still we clearly can distinguish between what already happened and what has not happened yet. We can only remember the past, not the future. Time, if you want, leaves its traces on us [2] [3].
One possible explanation of this aspect of time can be given by the theory of thermodynamics. Within this framework we encounter the notion of irreversibility, which is directly connected to the second thermodynamic law, stating that the total entropy of a closed system only can increase or stay the same, but never decrease. Irreversibility and the increase of entropy are therefore different sides of the same coin. The passing of time, in this picture, can be seen as an emergent property of the world, not as something fundamental, where the notions of future and past are defined by the entropy of a system. Further more, since entropy is a quantity that strongly depends on the observers knowledge, time looses its universality and becomes relational.
We grow older and remember the past because time left its irreversible traces on us, or to formulate it differently, the irreversible traces left on us by the world define time for us.

1.2 Clocks

Now that we introduced the different facets of the concept of time in quantum mechanics, let us introduce an operational way of dealing with it, namely clocks. To that end we will define what we mean by clocks, dis-

[1] See Appendix B

cuss the role that irreversibility plays when measuring time, and introduce ways of quantifying the performance of clocks.

Our every day experience naturally provides us with some intuition of what constitutes a clock. First of all, a clock is a measurement device which enables us to make assertions about time. To be more precise, we should distinguish two kinds of such measurement devices. A stopwatch denotes a device which measures the time that passed in between the occurrence of two temporally separated events. Therefore with a stopwatch we know the time that passed only after the second event, or after we pressed the stop button. In contrast to the stopwatch stands the clock, which provides us with a constant time reference, i.e. it periodically outputs equally timed ticks and stores them. A clock therefore consists of two parts, a pointer and a register, depicted in figure 1 [4]. The pointer provides ticks, i.e. information about the passage of time, and the register stores this information classically, enabling an external observer to read it out [5].

Intuitively one might think that a clock can easily be realized by a cascade of stopwatches. This is not possible because a stopwatch provides us with time only after the second event, i.e. after pressing stop. But without having a reference time we wouldn't know when to press stop, we'd only know the time that passed aposteriori. Therefore, we could not provide equally timed ticks.

Including a register which stores information classically presupposes irreversibility. Excluding the register would render the clock useless, because it would not be possible to read out the any information about how much time has passed since the last tick. We conclude that clocks always must induce a certain irreversibility in order to be useful. More precisely, there must exist an irreversible flow of information from the pointer to the register. Due to Landauer's principle [6] [7] we can conclude, that the supply of a steady time reference is directly connected to an increase of entropy. Consequently the problem of the second law of thermodynamics as well as the arrow of time are naturally connected with clocks. Further more we are led to the conclusion that a clock requires some kind of out of equilib-

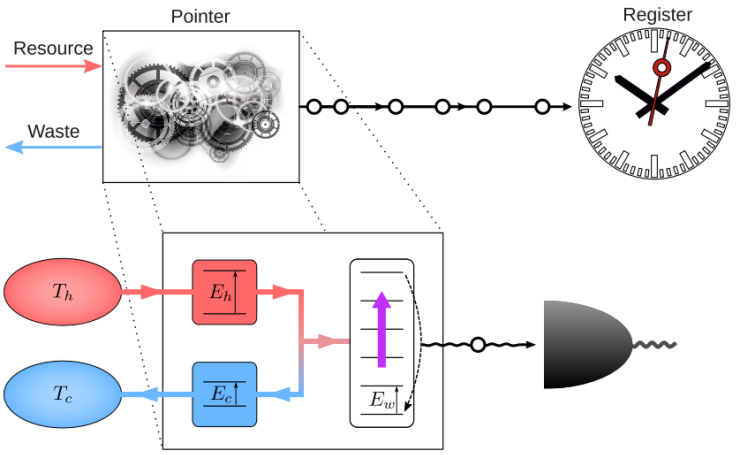


Figure 1: Illustration of the components of a clock. On the left the pointer, also illustrated as a quantum system which will be discussed, taking in a resource to produce a signal. On the right the register measuring the time information send by the Pointer and storing it classically.
This illustration was taken from [5]

rium resource, because if the whole system was in equilibrium a flow of information from the pointer to the register would not exist.
In the remaining work we will take the whole dynamics of the register, i.e. the interaction with the pointer and the observer, for granted and only consider the pointer.

We will now define the figures of merit necessary to discuss clock quantitatively. The performance of a clock, or more precisely the performance of the pointer is defined by its *resolution*, i.e. the frequency of its ticks,

$$R = \frac{1}{\bar{t}} \tag{1}$$

and its *accuracy*, i.e. how many ticks the clock provides before its uncertainty becomes bigger than the average time between ticks [5].

$$A = \left(\frac{\bar{t}}{\Delta\bar{t}}\right)^2 \tag{2}$$

In chapter 3.4 these two quantities are discussed in more detail. In order to build an ideal clock we would have to let both accuracy, and resolution

grow infinitely. The resulting signal emitted by the pointer approaches the form of periodic delta functions whose frequency approaches infinity. As you will see later, increasing one of these performance criteria influences the other, leading to a trade off between them, making it impossible to actually build an ideal clock.

In the following chapters we will define a clock model enabling us to control the accuracy and the resolution within the framework of quantum thermodynamics, and connect them to energy costs, which are strongly related to entropy production.

1.3 The field of quantum thermodynamics

In this chapter we will introduce the very young filed of quantum thermodynamics, but before we do so, let us consider the question: What are the fundamental elements of the world?
Depending on the theory you consult and who you ask, different answer can and will be given. For example in classical mechanics you might get the answer that the most fundamental entity is a particle with a certain mass, a position in space and a velocity, who's dynamics is determined by Newton's laws. Adding electrodynamics you additionally get the charge of a particle as fundamental element and Maxwell's equations to describe the motion. When going towards relativistic regimes Einstein's theory of general relativity comes into play. The former concepts of space and time have to be overthrown. Finally quantum mechanics enters the field. The question about what is fundamental suddenly takes new dimension. Since the beginning of quantum mechanics its ontology was debated heavily, and still is. Many different interpretations have been proposed, almost none of them ever disappeared again [8] [9].
Considering the most pertinent of these theories, quantum mechanics and general relativity, compatibility becomes an issue, as these two theories are contradicting each other.
All of these theories developed from a reductionist viewpoint, in other words a believe that the world consists of certain fundamental entities

and every other theory should, in principle, be reducible to these fundamental elements. [10]
Opposing this frenzy of ontological entities and fundamental concepts one other theory exists. The theory of thermodynamics envelopes the laws of nature, like a cloud surrounding the peak of a mountain.
Since its founding 1824 by Sadi Carnot and refinement by Rudolf Clausius, thermodynamics remained valid. It not only survived the advent of new theories like quantum mechanics and the theory of relativity, but also enriches them by adding different perspectives and new approaches to problems in each of the domains of these theories [11] [12]. Thermodynamics never tried to give an explanation for the most fundamental aspects of the world, it only gives us the means to navigate through the mountain peaks that constitute the world. It helps us to exploit the given resources in the most efficient way. Thermodynamics is a theory ignorant of the detailed structure of the world, ignorant of the microscopic details. This is where it universal applicability originates from [13] .

The field of quantum thermodynamics, a rather young field, investigates the connection of the microscopic description that quantum mechanics provides us with and the the macroscopic law that thermodynamics predicts. Unlike statistical mechanics, which also deals with the emergence of thermodynamic laws from a microscopic description, quantum thermodynamics strongly emphasizes the importance of non-equilibrium processes. In the following part we will try to give you an overview of the research landscape of quantum thermodynamics:

- **The problem of thermalization and equilibration**
  According to quantum mechanics, the universe evolves unitarily, which implies that its entropy does not increase. Thermodynamics on the other hand states, that the entropy of the universe should increase if it is a closed system. The task of this subfield is to understand how the reversible unitary evolution of quantum mechanics can make systems equilibrate and evolve towards steady, and thermal states. [14]

- **Resource theories**
  Using thermodynamics as a resource theory can help to better understand the role of thermodynamic quantities at the quantum scale. [15]

- **Entanglement theory in thermodynamic settings**
  One very prominent resource in quantum mechanics is entanglement. It is of vast importance to the field of quantum information processing. As this field is becoming more applicable, it becomes necessary to think about the fundamental restrictions to it emerging from thermodynamic considerations.

- **Quantum Fluctuation relations and quantum information**
  In the domain of macroscopic objects thermodynamics has been used to describe the equilibrium properties of systems. As we approach the quantum regime, we diverge from equilibrium. In the domain where quantum fluctuations begin to dominate, how can thermodynamics be described?

- **Quantum thermal machines**
  The genesis of thermodynamics started with asking how we can utilize heat to perform work. Quantum thermal machines are the quantum versions of heat engines or refrigerators. It is this subfield of quantum thermodynamics which will interest us through out the remainder of this thesis.

2 Theoretical Background

2.1 Mathematical tools

In this section we will introduce basic mathematical tools which are of big importance to understand all the derivations[2].

[2]Some of the definitions are based on the master thesis of Jessica Bavaresco [16]

Throughout the whole thesis we will be dealing with quantum states, which are postulated by quantum mechanics. They will be defined here

Definition 1. *(Quantum States)*
Let $\mathcal{H}^d$ be a d-dimensional Hilbert space. A pure quantum state $|v\rangle \epsilon \mathcal{H}^d$ is a d-dimensional, normalized vector on the Hilbert space $\mathcal{H}^d$.

$$\langle v|v\rangle = 1 \tag{3}$$

A quantum state contains all the accessible information in a quantum system. It can be written as a linear combination of basis vectors of the Hilbert space as

$$|v\rangle = \sum_{i=0}^{d-1} c_i |i\rangle. \tag{4}$$

where $c_i \epsilon \mathbb{C}$.
Considering statistical mixtures of quantum states the density matrix formalism needs to be introduced.

Definition 2. *(Density Matrix)*
A quantum state can be written as a density matrix $\rho : \mathcal{H}^d \rightarrow \mathcal{H}^d$, a positive semidefinite linear operator of unit trace acting on $\mathcal{H}^d$.

$$\begin{aligned} Tr[\rho] &= 1 && (\textit{Normalization}) \\ \rho &\geq 0 && (\textit{Positiv semi definite}) \end{aligned}$$

The density matrix of a pure state can be written as the outer product of the state with it self,

$$\rho = \left|\psi\right\rangle\left\langle\psi\right|.$$

Statistical mixture of quantum states can be written as convex mixtures of density matrices.

$$\rho_{mix} = \sum_{i=1}^{n} p_i \rho_i$$

with $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$.

Definition 3. *(Multipartite states)*
Given N single systems of parties $1, 2, ..., N$ *of dimension* $d_1, d_2, ..., d_N$*, acting on the Hilbert spaces* $\mathcal{H}_1^{d_1}, \mathcal{H}_2^{d_2}, ..., \mathcal{H}_N^{d_N}$*, the global n-partite state of this system* $\rho_{1,2,..,N} : \mathcal{H}_{1,2,...,N} \rightarrow \mathcal{H}_{1,2,...,N}$ *is a quantum state acting on the Hilbert space* $\mathcal{H}_{1,2,...,N} = \mathcal{H}_1^{d_1} \otimes \mathcal{H}_2^{d_2} \otimes ... \otimes \mathcal{H}_N^{d_N}$

Definition 4. *(Reduced states)*
Given a multipartite state $\rho_{1,2...,N}$ *the reduced state of one of these parties is given by the partial trace over all the other parties:*

$$\rho_1 = Tr_{2,3,...,N}[\rho_{1,2,...,N}],$$

analogously for all the other parties.

The partial trace is defined in the following way:

Definition 5. *(Partial trace)*
Given a state ρ_{AB} *on the composite Hilbert space* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$*, where* $\mathcal{H}_B$ *can be a composition of further Hilbert spaces, i.e.* $\mathcal{H}_B = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$*. Let* $I = \{|i\rangle\}_B$ *be a orthonormal basis on* $\mathcal{H}_B$*. Then the partial trace is defined as*

$$Tr_B[\rho_{AB}] = \sum_{i \epsilon I} \langle i| \rho_{AB} |i\rangle. \tag{5}$$

Definition 6. *(Projective measurement*[3]*)*
The projective measurement is a set of O positive semidefinite, linear operators $\Pi_a : \mathcal{H}^d \rightarrow \mathcal{H}^d$ *with a* $\epsilon \{1, 2, ..., O\}$ *who's elements are projectors on the Hilbert space* $\mathcal{H}^d$ *and sum to identity*

$$\Pi_a \geq 0,$$
$$\sum_a \Pi_a = \mathbb{1}.$$

The born rule tells us what the probability for a certain outcome of a quantum measurement is.

[3]The projective measurement is a special case of POVMs, the generalization will not be discussed here.

Definition 7. *(Born rule)*
Let $\{\Pi_a\}$ *be a set quantum measurements acting on the Hilbert space* $\mathcal{H}^d$ *with a* $\epsilon\{1, 2, ..., O\}$ *and let* ρ *be a quantum state on the same Hilbert space. Then the probability of obtaining the outcome a when performing the measurement* Π_a *is given by*

$$p(a) = Tr[\Pi_a \rho].$$

In quantum mechanics the time evolution of a system is given by the Schrödinger equation.

$$i\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle, \tag{6}$$

where $\hbar$ is set to 1.
Solving it for a time independent Hamilton operator H yields the unitary time evolution operator.

Definition 8. *(Unitary time evolution)*
Given a Hamilton operator H acting on the Hilbert space H^d, the time evolution of a quantum system is given by a unitary operator $U(t)$ *which satisfies*

$$U(t) = e^{-iHt}.$$

To calculate the time evolution of a particular quantum stat ρ we apply the $U(t)$ in the following way:

$$\rho(t) = U(t)\rho U^{\dagger}(t),$$

where $U^{\dagger}(t)$ is the hermitian adjoint of $U(t)$.

2.2 Heat baths

The basic concept of thermal machines builds on the existence of out of equilibrium resources like for example thermal baths, or thermal reservoirs. The idea is, that heat flows from a hot bath, i.e. a bath at higher temperature, into a machine, which utilizes the heat to perform work and dumps the rest of the heat into the cold bath. Therefore a thermal machine utilizes the flow of heat from a hot to a cold bath. The description of such

as system becomes extremely complicate, if the temperature change of the heat baths due to the heat flow is taken into account. An easy way around this problem is to assume, that the heat bath is very big. As the size of the system increases, also its heat capacity grows. Consequently the heat flow out of, or into the system induces a neglect able amount of temperature change in the bath.

Consider now one single, very big, isolated system with a given total energy. Experimentally we know that the system will come to a steady state if we wait long enough. This means that the macroscopic quantities of the system don't change any more in time, apart from fluctuations. Such a system has come to an equilibrium, it evolved into a Gibbs state.
If you, on the other hand consider a system which has some sort of machine attached to it. This machine extracts all the energy that can possibly be extracted from the system. The final state of the system can then be described by a so called passive state.
The set of all Gibbs states forms a subset of the set of all passive states.

2.2.1 Passive states

The notion of passive states was first defined by Pusz and Woronowicz [17]. Here we rewrite its definition in a more operational way.

Definition 9. *(Passive state)*
Let the Hamilton operator of a system acting on the Hilbert space $\mathcal{H}^d$ be given by $H \in \mathcal{H}^d$. ρ_p is a passive state if it is not possible to find a Unitary transformation such that

$$Tr[\rho_p H] > Tr[U\rho_p U^\dagger H]. \tag{7}$$

Now one can ask the question, what happens if we have more than one passive state? Assume two systems which are each in a passive state.

$$\rho_{comp} = \rho_p \otimes \rho_p \tag{8}$$

We can ask now: is the composite system also in a passive state?
In general the property of passivity is not invariant under the tensor prod-

uct, i.e. there exist passive states, such that they loose their passivity under the tensor product. States for which passivity is invariant are called completely passive states and will be discussed in the next chapter.

Passive states under the tensor product

A way to see that there are passive states who's passivity is not invariant under the tensor product is to explicitly calculate the the density matrix. In dimension two, all passive states are completely passive, therefore the calculation will be done in three dimensions. Therefore, assume a system described by the state ρ in the Hilbert space $\mathcal{H}^3$ with the Hamilton operator H given as:

$$\rho_p = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix},$$

where representation of the state is in the energy eigenbasis with energy levels orderd in a ascending way $E_1 \leq E_2 \leq E_3$.
Taking the tensor product of this state results in

$$\rho_p \otimes \rho_p = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \overbrace{\begin{bmatrix} \to 0 \\ \to 1 \\ \to 1.5 \\ \to 1 \\ \to 2 \\ \to 2.5 \\ \to 1.5 \\ \to 2.5 \\ \to 3 \end{bmatrix}}^{Energy}.$$

Now you can see in the subspace $\{|02\rangle, |11\rangle\}$, the probability for the energy $E_{02} = 1.5$ is $p_{02} = 0$ and for the energy $E_{11} = 2$ is $p_{11} = 1/4$ thus

$$p_{02} \leq p_{11}, \qquad E_{02} \leq E_{11}.$$

Thus we can find a unitary transformation (a simple rotation between those two eigenstates) that violate the passivity condition.

2.2.2 Jaynes principle and complete passivity

In his paper *Information Theory and Statistical Mechanics* [18] Edwin Thompson Jaynes shows that the Gibbs distribution of probabilities can be derived from purely statistical considerations, i.e. independent of physical arguments. He concludes, that it is not necessary to have additional physical assumptions, like egoridcity, metric transivity, equal apriori probabilities, e.t.c., in order to make the best estimates possible. For this purpose he uses the maximum-entropy principle.

Definition 10. *(Maximum entropy principle)*
Given a system can be described by a state ρ on the Hilbert space $\mathcal{H}^d$ unknown to us. The maximum entropy principle says that the best representation of our current state of knowledge about this system is the one which maximizes its entropy, constrained only by the properties O_i we do know, i.e.

$$\rho_{best} = Max_{\rho}[S(\rho)] \qquad s.t. \;\; Tr[\rho O_i] = \overline{O_i} \quad \forall i. \tag{9}$$

When regarding probabilities as an subjective notion, one could reformulate this principle as the principle of maximum ignorance. Our knowledge of the system thus determines the best possible description of it.
In the following we will not give the exact derivation as it was done in Jaynes paper, but use the notion of density matrices.
Suppose we have a system which is described by a quantum state given by the density matrix ρ of which we know the average energy, ie.

$$Tr[H\rho] = \langle H \rangle = Const. \tag{10}$$

Where H is the Hamiltonian of the system.
We also know, that density matrices need to be of trace one, ie.

$$Tr[\rho] = \sum_{i=0}^{d-1} \langle i|\rho|i\rangle = 1. \tag{11}$$

The entropy of our system can be written as the Von Neumann entropy, which in therm can be reduced to the Shannon entropy of the eigenvalues.

$$S(\rho) = -Tr[\rho ln(\rho)] = -\sum_i p_i ln(p_i), \tag{12}$$

where the p_i are the eigenvalues of ρ.
In order to maximize the entropy conditioned on (10) and (11) we make use of the method of Lagrange multipliers. Our Lagrange function is then given by

$$\Lambda = -Tr[\rho ln(\rho)] + \beta(Tr[\rho H] - \langle H \rangle). \tag{13}$$

Or, since we can rewrite ρ in a diagonal eigenbasis

$$\Lambda = -\sum_i p_i ln(p_i) + \beta(Tr[\rho H] - \langle H \rangle). \tag{14}$$

By deriving Λ partially for every eigenvalue p_i of ρ and setting it zero, we get $n = dim(\mathcal{H})$ equations which can again be rewritten in matrix notation as

$$-\mathbb{1} - ln(\rho) + \beta H = 0. \tag{15}$$

Thus resulting in

$$\rho = Ce^{-\beta H}. \tag{16}$$

With $C = const.$. Now using (11) we get

$$\boxed{\rho = \frac{e^{-\beta H}}{Z}} \tag{17}$$

where $Z = Tr[e^{-\beta H}]$ is the partition function. We can now identify β with the inverse temperature $\beta = \frac{1}{T}$, where we set the Boltzmann constant k_B to 1. Thus we derived the thermal state from purely statistical considerations.

The thermal state fulfills the criterion to be a completely passive state. When considering thermal baths, complete passivity is required by the first thermodynamic law, because it would be possible to extract an infinite amount of work out of two systems. Work extraction would be still possible even when they reached a steady state if these states where not passive. Therefore we will use thermal, or gibbs states to describe thermal baths.

2.3 Thermal machines

Thermal machines are devices that utilize a heat flow between two systems to perform work (heat engine), to increase the temperature of a system (heat pump) or to cool a system down (refrigerator). One of the first quantum thermal machines was introduced as the MASER [19]. Since then many other models have been proposed [20] [21] [22]. It is even possible to construct a quantum thermal machine that is able to reach Carnot efficiency[4] with as little as 3 qubits [23]. [5]

The main objective of this thesis is to analyze the fundamental energy costs of autonomous quantum clocks. In order to construct such a clock it is instructive to understand the notion of quantum thermal machines and in which regimes they operate. The simplest version of such a machine consisting only of qubits is the three qubit thermal machine. Assume there exist two non-interacting thermal baths at different temperatures. As we concluded before, after a sufficient thermalization time these baths can be assumed to have a Boltzmann distribution of energy eigenstates. Let us call one of them the heat bath at temperature T_H and the other environment or cold bath at temperature T_C. One single such bath exhibits by it self only rather trivial behavior. The thermal machine accesses both baths, therefore allowing a heat flow between them. More specifically one of the qubits, the "hot" qubit is coupled to the hot bath, whilst the other two qubits, called "cold" qubit and "target", or "ladder" qubit are coupled to the environment. By letting the qubits interact with each non-trivial processes emerge. The coupling between the qubits and the environment is assumed to be very weak compared to the coupling between the qubits, i.e. the thermalization time of the qubits is much bigger than the timescale at which interactions happen between them. As the names of the qubits already suggest, each of them has a distinct role. The cold qubit simply connects the machine in a certain way to the cold bath. The target qubit constitutes the system upon which the machine acts. For simplicity we assumed here that the target system is of dimension two, this will be gen-

[4] The maximal possible efficiency any machine can reach.

[5] There are also quantum machine models consisting only of a single qutrit or 2 qubits [24]. We won't cover them because they are not relevant for this thesis.

eralized to higher dimensions later on.
The energy structure of such a system is given by the free Hamiltonians of each qubit.

$$\begin{aligned} H_H &= E_H |1\rangle\langle 1| \\ H_C &= E_C |1\rangle\langle 1| \\ H_T &= E_T |1\rangle\langle 1| \end{aligned} \tag{18}$$

The ground state energy of each system is set to zero. The total free Hamiltonian is thus given by

$$H = H_C \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes H_H \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes H_T. \tag{19}$$

As the coupling between the heat baths and the qubits is weak, the interactions between the qubits have to obey energy conservation. Therefore in order to allow for non trivial transitions, degeneracy in the energy spectrum has to be introduced. The energy gaps are thus chosen to be

$$E_T = E_H - E_C. \tag{20}$$

The interaction between these qubits can now described by the interaction Hamiltonian

$$H_{int} = g(|101\rangle\langle 010| + |010\rangle\langle 101|). \tag{21}$$

Where g is the coupling constant, a parameter controlling the coupling strength between the qubits.
Such a system shows a much richer structure than could be achieved with a single heat bath.

2.3.1 The Virtual qubit

In order to characterize the different working regimes, we will introduce the notion of virtual qubit for the simple system above. Now let's say that the hot qubit is coupled to a bath with temperature T_H and the cold qubit is coupled to a bath at temperature T_C. We now can look at the composite system of these two heat baths as one system, introducing the

virtual qubit. It is then possible to assign a virtual temperature to this new qubit. Using this temperature we will characterize the different working regimes of the thermal machine. To be more specific, let us introduce the virtual qubit explicitly now.
The composite system of both heat baths can be written as tensor product of the two qubits with the temperatures of the respective heat baths they are coupled to

$$\tau_M(\beta_V) = \tau_C(\beta_C) \otimes \tau_H(\beta_H).$$

There are four possible configurations of this composite system, namely

$$|00\rangle\langle 00|, \qquad |01\rangle\langle 01|, \qquad |10\rangle\langle 10|, \qquad |11\rangle\langle 11|.$$

Since we require that only energy conserving interactions can take place, the two states which are of most interest to us are $|01\rangle\langle 01|$ and $|10\rangle\langle 10|$. Using these two states we introduce a virtual qubit as:

$$|V_0\rangle = |10\rangle, \qquad |V_1\rangle = |01\rangle, \tag{22}$$

where $\left|V_{0/1}\right\rangle$ denotes the virtual ground/excited state.
In order to find the "virtual temperature" we look at the ratio of populations of p_V^1 and p_V^0, of the excited and ground state.

$$p_V^0 = \frac{1}{Z_C Z_H} e^{-\beta_C E_C} \qquad p_v^1 = \frac{1}{Z_C Z_H} e^{-\beta_H E_H}, \tag{23}$$

resulting in

$$e^{\beta_V E_V} = \frac{p_V^1}{p_V^0} = e^{\beta_C E_C - \beta_H E_H}. \tag{24}$$

Thus the virtual temperature is

$$T_V = \frac{E_H - E_C}{\frac{E_H}{T_H} - \frac{E_C}{T_C}}. \tag{25}$$

The virtual temperature now can take all the values outside of the range of T_C and T_H, even negative ones, depending on the difference of the energy spacings of the two qubits and their temperatures. It is simple to analyze

the behavior of the machine in this picture. The interaction of the target with the virtual qubit can now simply be described by the thermalization of the target with the virtual qubit. Therefore we can now extract different working regimes of such a system using the notion of virtual temperature.

Cooling
If the virtual temperature is smaller than T_C and T_H, i.e. $0 \leq T_V \leq T_C, T_H$, the machine acts in the cooling regime, reducing the temperature of the target qubit.

Heat pump
If the virtual temperature is higher than T_C and T_H, i.e. $T_V \geq T_C, T_H$ the machine acts as a heat pump, increasing the temperature of the target qubit.

Heat machine
Having a negative virtual temperature $T_V < 0$ leads to a "state inversion" in the target qubit, meaning theta the probability to be in the excited state becomes higher then to be in the ground state. In contrast to just heating up a system, in such a process work is being done on the target qubit.

In the following we will only consider the regime of heat machines, as we want to utilize state inversion to produce a reliable source of ticks.

3 Autonomous quantum clocks

Before we introduce the actual model which we will be working with, let us motivate why we are considering thermal clocks in the first place. As we saw in chapter 1.2, a clock can only be realized if there exists some out of equilibrium resource. Letting a clock run, therefore always comes at some thermodynamical costs. Thus, each tick of a clock should lead to an inevitable increase of entropy. Such an out of equilibrium resource can be realized by introducing a thermal bath at a temperature which is higher than the environmental temperature. The convenience of using thermal baths as resources is, that they can be prepared easily (chapter 2.2.2) and that the costs for preparing them are well known. Since we want to make assertions about the fundamental energy costs connected to the generation of single ticks of a clock, thermal baths are the most sensible choice of resources.
In the following we will introduce a clock model which utilizes the heat flow from a hot to a cold bath in order to produce equally timed, periodic ticks. This model will be based on the thermal machine introduced in chapter 2.3. Later we will generalize this model with respect to certain quantities we will define soon.

3.1 The minimal thermal clock

3.1.1 The Model

In this section we will discuss the simplest model of a thermal clock i.e. a clock that is driven by the heat flow from a hot bath to a cold environment as it was introduced in [5] by Erker *et. al.* in 2017. Looking at the limit of the temperatures we will derive a simplified version of this clock which can be later used to define energy costs of time measurements.
First, to get a more intuitive perspective of the scenario in question, we consider the classical situation. Imagine a thermal machine, a heat engine for example, which performs work on a weight by pulling it up against a gravitational field, thus increasing the potential energy of the weight. For simplicity we assume, that the weight is lifted very slowly, so that we

can neglect the kinetic energy of the weight. This way the process can be thought of as an adiabatic process, which means that energy is transferred to the target system only in the form of work. Let us assume now, that the weight can only be situated at certain, equidistant heights, which means it can only take discrete, equidistant potential energies. We can then label the energy eigenstates with $|n\rangle$, with corresponding eigenvalues $E_n = nE_L$. [23]

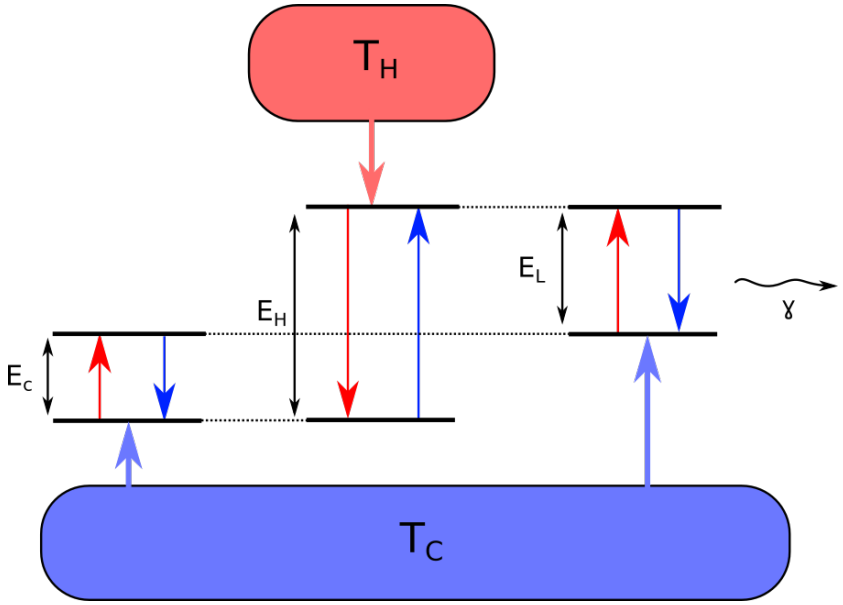


Figure 2: Due to weak coupling between the qubits and the baths, only energy conserving transitions are allowed. These transitions are depicted here. The red arrows indicate the transition where the ladder becomes excited. The blue arrows show the reverse transition. Since the qubit with the biggest energy gap is coupled to the hot bath, there exists a bias towards the transition indicated by the red arrows.

With this rather intuitive picture in mind, we will now introduce the concept of autonomous quantum clocks more rigorously. The simplest autonomous quantum clock consists of three coupled systems. Two of these systems constitute the thermal machine which performs work on the third system and therefore drives the clock. Each of the machine systems can

be described by states on a two dimensional Hilbert space, i.e. qubits, which are coupled to independent thermal baths. The first qubit is coupled to the environment, or cold bath at temperature T_C and is called "cold qubit". The second qubit is coupled to a hot bath at temperature $T_H > T_C$ and called "hot qubit". The cold qubit has an energy gap of E_C. The hot one has an energy gap of $E_H > E_C$.
Let us assume for now, that the qubits are only in contact with their respective heat baths, i.e. not interacting with each other. After letting them interact with the heat bath long enough, the qubits will thermalize to the temperature of their respective baths. After thermalization the qubits can be described by thermal states

$$\tau_{E_{H/E_c}}(\beta_{H/C}) = \frac{1}{Z} e^{-\beta_{H/C} H_{H/C/L}}. \tag{26}$$

Another way to think about the thermalization is to imagine the heat bath to be a reservoir of qubits at corresponding temperature. Since we assume that our baths are very big compared to our system, picking a qubit from the bath at random will most probably yield a thermal state at the temperature of the bath we picked it from.
The machine is coupled to the third system, called "ladder" (the analogy in the upper example is the weight), and consequently exert work on it. In general, this ladder can be a d-dimensional system. For the sake of clarity, we will consider only a 2-dimensional ladder for the introduction of this clock. The ladder also couples to the environment and thus can initially be described by a thermal state at temperature T_C. In figure 2 the energy-level-structure of our simplified system is depicted. Assuming weak coupling between the qubits and the thermal baths, which means that the thermalization time of the qubits is much bigger then the timescale at which interactions between our qubits take place, we allow only for energy conserving unitaries to describe the dynamics.
The whole system's initial state can now be described as a tensor product of the machine qubits and the ladder

$$\rho(t=0) = \tau_{E_C}(\beta_C) \otimes \tau_{E_C}(\beta_C) \otimes \tau_{E_L}(\beta_C). \tag{27}$$

Up to this point the clock is not able not "tick". By making the highest level of the ladder, in our simple case the first exited state, instable, i.e.

couple it to a photon field of infinitely many modes, we recover the law of exponential decay for a time dependent population

$$\frac{dN}{dt} = cNP_{top}(t). \tag{28}$$

Where $P_{top}(t)$ is the occupation probability of the most exited energy eigenstate, in our scenario of the first excitation. Thus with a some probability the ladder decays to the ground state, sending out a photon at energy $E_\gamma = (d-1)E_L$. The emitted photon can then be detected as a tick from a register. Since all of these interactions can be described by unitary time evolutions, the reverse process also is possible. If we now assume that the $k_B T_C \ll E_\gamma$, due to Planck's Law, such a reverse process is highly unlikely and can thus be neglected.

3.1.2 Energy cost

After a tick has occurred the clock system rethermalizes with the heat baths, returning to its initial state. Due to energy conservation, we know that the cold qubit must be in the exited state after a tick has occurred (see figure 2). Therefore, as it rethermalizes with the cold bath, the energy of the cold qubit, E_C gets dumped into the cold bath. This energy is lost, i.e. can not be utilized any more an can thus be counted as energy costs. Since the average energy of the total system is conserved and the free energy change is equal the the energy dumped into the cold bath, we can say that every tick is connected to an entropy production of at least

$$\Delta S_{tick} \geq \Delta E_C \beta_C. \tag{29}$$

The energy of the emitted photon is here not taken into account, because it is the target of the operation and not "waste" heat, like the energy that is dissipated into the cold bath. The process of measuring and storing the information of this photon, as well as analyzing the respective energy costs would exceed the scope of this thesis. Therefore we will not consider it in the remaining parts. If you however are further interested in this topic we can recommend the work of Yelena Guryanova, Nicolai Friis and Marcus Huber [25].

3.1.3 Limiting case

As we are interested in the fundamental assertions, we want to consider the the best possible machines that can be achieved. As we take the limit of $T_C \to 0$ and $T_H \to \infty$ we know that these machines approach Carnot efficiency, which is the best efficiency a machine can reach. Conveniently this limes also eliminates further parameters, making our calculations much easier.

In these limits, the thermal states become

$$\tau(\beta_H) \to \frac{1}{2}\mathbb{1},$$
$$\tau(\beta_C) \to |0\rangle\langle 0|.$$

Consequently the state of the total system becomes

$$\begin{aligned}\rho(t=0) &= |0\rangle\langle 0| \otimes \frac{1}{2}\mathbb{1} \otimes |0\rangle\langle 0| \\ &= \frac{1}{2}(|000\rangle\langle 000| + |010\rangle\langle 010|).\end{aligned}$$

As you will see later on, the advantage of taking this limes is that the final results don't explicitly depend on the energy gaps of the hot and cold qubits anymore.

3.1.4 The dynamics

In figure 2 all possible, non trivial transitions of our clock model are depicted. We will now introduce the dynamics of this system explicitly. The Hamiltonian of the clock system is given by

$$H = H_0 + H_{int}, \tag{30}$$

where H_0 is the free and H_{int} the interaction Hamiltonian.

$$H_0 = E_C\,|1\rangle\langle 1| \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes E_H\,|1\rangle\langle 1| \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes |1\rangle\langle 1|\,E_L \tag{31}$$
$$H_{int} = g\,(|010\rangle\langle 101| + |101\rangle\langle 010|) \tag{32}$$
$$= g(\sigma_{d_1}^{+} \otimes \sigma_1^{-} \otimes \sigma_L^{+} + h.c), \tag{33}$$

where g is the coupling constant. The subscripts denote which Hilbert space is acted upon, with d_1 being the Hilbert space of the cold qubit, 1 denotes the hot qubit and L the ladder. [6]
As we have seen in chapter 2.3 in order to have any non trivial interactions that are energy conserving, there needs to be some degeneracy in H_0. Therefore we require that $E_H - E_C = E_L$.
Using the Schrödinger equation we calculate the unitary, energy conserving time evolution of our clock

$$U(t) = e^{-iH_{int}t}. \tag{34}$$

Here we used the fact that the free Hamiltonian only induces trivial interactions, which means it can be represented by the unit operator. Resulting in the following explicit time evolution

$$U(t)_{S_1} = -i\sin(gt)(|010\rangle\langle 101| + |101\rangle\langle 010|) + \cos(gt)(|010\rangle\langle 010| + |101\rangle\langle 101|). \tag{35}$$

Here the subscript S_1 indicates that we only write the non trivially acting part of the unitary, i.e. the part that acts on the subspace $S_1 = Span(|010\rangle, |101\rangle)$.
Applying this unitary to $\rho(t=0)$ gives us the time evolution of our system

$$\rho(t) = U(t)\rho(t=0)U^{\dagger}(t) = \frac{1}{2}(|000\rangle\langle 000| + cos^2(gt)|010\rangle\langle 010| + \sin^2(gt)|101\rangle\langle 101|).$$

By taking the partial trace we find the time dependent populations of the reduced ladder state.

$$Tr_{E_c,E_H}(\rho(t)) = \frac{1 + cos^2(gt)}{2}|0\rangle\langle 0| + \frac{\sin^2(gt)}{2}|1\rangle\langle 1|$$

In particular we find the probability for the ladder to be in the most exited state by using the Born rule

$$\boxed{P_{top}(t) = \frac{1}{2}\sin^2(gt)} \tag{36}$$

Analyzing this equation, we can make significant simplifications to our model. It is clear that P_{top} does not depend on E_C. In fact, if we let $E_C \to 0$,

[6] This notation may seem not very useful now, but it will come in handy later on.

which implies by eq. 20 that $E_H \to E_L$, $P_{top}(t)$ retains its form.

As E_C goes towards zero, also the energy costs get arbitrarily close to zero, which might seem counterintuitive. Coming back to our performance criteria from chapter 1.2 we can ask now how accurate this clock is. $P_{top}(t)$ never exceeds $\frac{1}{2}$, which means that this machine produces no state inversion. Thus the probability distribution for a tick to occur at time t has a very big variance.

Calculating the unitary time evolution

Here we will show how the explicit form of the unitary time evolution is calculated.
Calculating the eigenvectors and eigenvalues of H_{int} yields

Eigenvectors $\left\|E_j\right\rangle$	Eigenvalues
$\frac{1}{\sqrt{2}}(\|010\rangle + \|101\rangle)$	g
$\frac{1}{\sqrt{2}}(\|010\rangle - \|101\rangle)$	$-g$
All other eigenvectors	0

Using the spectral decomposition and the tailor expansion of $e^{-iH_{int}t}$ leads to:

$$U(t)_{S_1} = \sum_{\forall \left|E_j\right\rangle} e^{E_j t} \left|E_j\right\rangle\left\langle E_j\right| =$$

$$\frac{1}{2}e^{-gt}(|010\rangle\langle 010| + |101\rangle\langle 101| + |010\rangle\rangle\langle 101| + |101\rangle\langle 010|)$$
$$+ \frac{1}{2}e^{gt}(|010\rangle\rangle\langle 010| + |101\rangle\langle 101| - |010\rangle\langle 101| - |101\rangle\langle 010|)$$
$$= -i\sin(gt)(|010\rangle\langle 101| + |101\rangle\langle 010|) + \cos(gt)(|010\rangle\langle 010| + |101\rangle\langle 101|)$$

3.1.5 Decay and probability of a tick

We will now calculate the tick probability distribution that is achievable with such a clock. In order to do so, we will, as mentioned above, use the law of exponential decay to describe the probability for emitting a photon.

$$\frac{dN}{dt} = -cNP_{top}(t) \tag{37}$$

where N usually is refereed to as the population that occupies a state. Here, since we have a time dependent occupation number, we write our decay rate as product of the time independent and the time dependent population of the top level. The decay rate can be controlled by the coupling constant c. It is a free parameter which influences the resolution and accuracy of our clock. Changes in c affect all clock models equally strong, we will assume it to be fixed. Solving equation 37 yields

$$N(t) = e^{-c\int_0^t dt' P_{top}(t')} \tag{38}$$

Which is the probability that no decay happened after time t. To calculate the probability that a tick occurred after time t, we take $1-N(t)$. Since this is a cumulative probability, we can calculate the probability density for a single tick to occur at time t by taking the derivative of this with respect to t

$$P_{tick}(t) = e^{-c\int_0^t dt' P_{top}(t')} \frac{d}{dt}\left(-c\int_0^t dt' P_{top}(t')\right) \tag{39}$$

Using eq. 36 the probability for a single tick becomes.

$$\boxed{p_{tick}(t) = c\frac{\sin^2(gt)}{2} e^{-c\left(\frac{t}{4} - \frac{\sin(2gt)}{8g}\right)}} \tag{40}$$

To calculate the expectation value and the variance of $P_{tick}(t)$, we have to calculate the following integrals

$$\overline{t} = \int_0^\infty t P_{tick}(t) dt \tag{41}$$

$$(\Delta\overline{t})^2 = \overline{t^2} - \overline{t}^2 = \int_0^\infty t^2 P_{tick}(t) dt - \left(\int_0^\infty t P_{tick}(t) dt\right)^2 \tag{42}$$

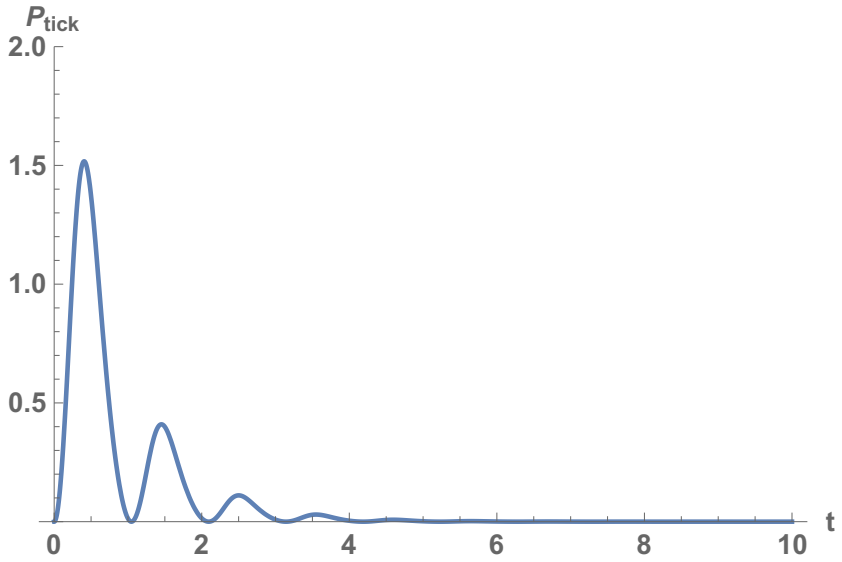


Figure 3: Tick probability distribution versus time.

Non of these integrals are analytically solvable. Thus, a simple analytic expression connecting the energy costs, the resolution and the accuracy is not achievable this way. We will nevertheless make use of these integrals later on, as we discuss more general clocks.

3.2 Multiple Machines

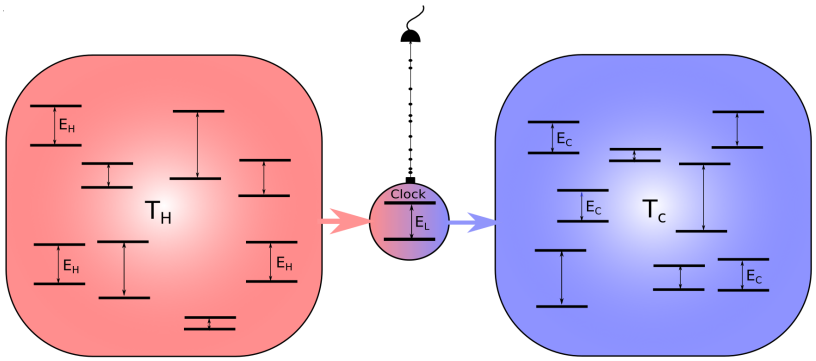


Figure 4: Illustration of a clock accessing multiple systems of the hot and cold heat bath.

In this section we will introduce an extension to the minimal clock

which allows us to achieve state inversion in the limits $T_H \to \infty, T_C \to 0$, which consequently increases the performance of our clock.
Since we assume, that the hot bath is very big compared to the clock system, we can assume, that it consists of many differently spaced subsystems. Many of them are qubits with an energy spacing of E_H. The same holds for the cold bath, of which many qubits have energy gaps E_C (see figure 4). The system can then be described by the initial state

$$\rho_M = \underbrace{(|0\rangle\langle 0| \otimes ... \otimes |0\rangle\langle 0|)}_{M_c \text{ times}} \otimes \underbrace{\left(\frac{1}{2}|0\rangle\langle 0| \otimes \mathbb{1} \otimes ... \otimes \frac{1}{2}|0\rangle\langle 0| \otimes \mathbb{1}\right)}_{M_H \text{ times}} \otimes |0\rangle\langle 0| \tag{43}$$

Let us now say that part of these qubits couple to the ladder and each other. A general interaction Hamiltonian can then be written as

$$H_{int} = g \sum_{j=1}^{M} \left[\bigotimes_{i=1}^{M} |0\rangle \left|\delta_{ij}\right\rangle \left\langle\delta_{ij}\right| \langle 0| \right] \otimes |0\rangle\langle 1| + h.c. \tag{44}$$

Where δ_{ij} stands for the Kronecker-Delta and M denotes the number of hot qubits involved. Note, that the energy cost of such a clock still can be taken arbitrarily close to zero. Accordingly it turns out, that P_{top} does not exceed $\frac{1}{2}$. In Fact the maximal top level probability of this clock does not even exceed $\frac{M}{2^M}$, which means that this clock performs worse than the minimal clock in terms of resolution and accuracy.

3.2.1 Introducing energy costs for M = 2

The same argument as above holds for cold qubits of other energy gaps, i.e. we can assume that there are many qubits with energy gap E_D in thermal states at temperature T_C. With E_D being the energy gap of a family of qubits we will call "heat dumps", given by

$$E_D = nE_H - E_L \tag{45}$$

with $n \epsilon \mathbb{N}$.
Let us now consider what happens if we increase the energy costs. One

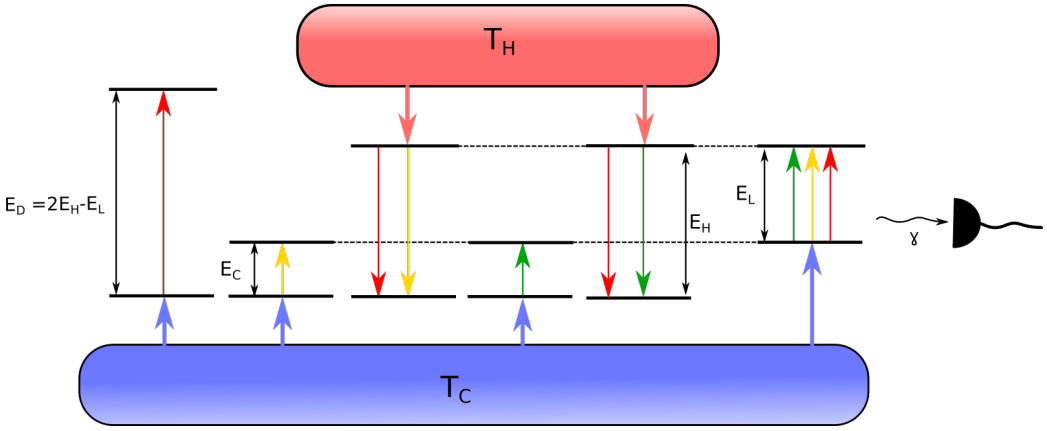


Figure 5: Illustration of the energy gap structure of a clock consisting of two machines, a heat dump (qubit on the left) and the ladder (qubit on the right). The arrows indicate the different possible transitions. Each color stands for one single transition. The advantage of this clock comes from the transition indicated by the red arrows. Both hot qubits decay while sending the ladder bit to its exited state and dumping the remaining energy into the heat dump thus exiting it as well.

way of doing so is to introduce an interaction Hamiltonian that couples M machines, M_D heat dumps and the ladder qubit, allowing all unitary, energy conserving time evolutions that induce rotations on a 2 dimensional subspace. For simplicity we will consider now the case of $M = 2$ and $M_D = 1$, which is depicted in figure 5. Let H_{int} be the interaction Hamiltonian acting on the Hilbert space

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{C,1} \otimes \mathcal{H}_{H,1} \otimes \mathcal{H}_{C,2} \otimes \mathcal{H}_{H,2} \otimes \mathcal{H}_L \tag{46}$$

where the $\mathcal{H}_D$ is the Hilbert space of the heat dump, $\mathcal{H}_{C,i}$ and $\mathcal{H}_{H,i}$ with $i = \{1, 2\}$ are the Hilbert spaces of the hot and cold qubits of the machines and $\mathcal{H}_L$ is the Hilbert space of the ladder.

Then, using equation 44 and adding the heat dump in the tensor product and interaction terms, the interaction Hamiltonian can be written as

$$H_{int} = g(|000010\rangle\langle 000101| + |001000\rangle\langle 010001| + |001010\rangle\langle 100001| + h.c.) \tag{47}$$

or simpler

$$H_{int} = g(\sigma^+_{C,2}\otimes\sigma^-_{H,2}\otimes\sigma^+_L + \sigma^+_{C,1}\otimes\sigma^-_{H,1}\otimes\sigma^+_L + \sigma^+_D\otimes\sigma^-_{H,1}\otimes\sigma^-_{H,2}\otimes\sigma^+_L + h.c.) \quad (48)$$

where the trivial action of the Hamiltonian has been neglected.
Calculating the unitary evolution which induces rotations on the three distinct 2 dimensional subspaces

$$S_1 = Span(|000010\rangle, |000101\rangle)$$
$$S_2 = Span(|001000\rangle, |010001\rangle)$$
$$S_3 = Span(|001010\rangle, |100001\rangle)$$

yields

$$\begin{aligned}
&U_{S_1,S_2,S_3}(t) = \\
&-i\sin(gt)(|000010\rangle\langle 000101| + |001000\rangle\langle 010001| + |001010\rangle\langle 100001| + h.c.) \\
&+\cos(gt)(|000010\rangle\langle 000010| + |001000\rangle\langle 001000| + |001010\rangle\langle 001010| \\
&+|000101\rangle\langle 000101| + |010001\rangle\langle 010001| + |100001\rangle\langle 100001|)
\end{aligned}$$

or using the sigma notation

$$\begin{aligned}
&U(t) = \\
&-i\sin(gt)\left(\sigma^+_{C,2}\otimes\sigma^-_{H,2}\otimes\sigma^+_L + \sigma^+_{C,1}\otimes\sigma^-_{H,1}\otimes\sigma^+_L + \sigma^+_D\otimes\sigma^-_{H,1}\otimes\sigma^-_{H,2} + h.c.\right) \\
&+\cos(gt)(\mathbb{1}_{S_1} + \mathbb{1}_{S_2} + \mathbb{1}_{S_3})
\end{aligned}$$

The state of our system at time $t = 0$, i.e. directly after thermalization is given by

$$\begin{aligned}
&\rho(t=0) = \\
&\frac{1}{2}(|000000\rangle\langle 000000| + |001000\rangle\langle 001000| + |000010\rangle\langle 000010| + |001010\rangle\langle 001010|)
\end{aligned}$$

In analogy to the previous calculation in section 3.1.4 we calculate the probability of the ladder to be exited

$$\boxed{P_{top}(t) = \frac{3}{4}\sin^2(gt)} \quad (49)$$

The first thing we can immediately recognize is, that this clock achieves state inversion. As we will later see, this has a strong effect on the resolution of the clock and a minor effect on the accuracy. Having state inversion increases the resolution compared to having no state inversion. Therefore we can say, that increasing the energy cost of a tick increases the performance of our clock.
This gain of performance of course comes at a certain cost, which is no surprise since we introduced them in the hope of some gain. As all transitions are equally likely to produce a tick, in order to calculate the energy costs, we have to calculate the average energy that is dumped into the cold bath.

$$E_{cost} = \frac{2E_C + E_D}{3} = \frac{E_H}{3} + 3E_C \tag{50}$$

or as $E_C \to 0$

$$E_{cost} = \frac{E_H}{3} \tag{51}$$

3.2.2 Adding more machines, M=3

If we now add a third machine we can ask how many possible combinations of hot qubits are there. In figure 6 the number of possible combinations is illustrated. Again each machine by it self can interact with the ladder "for free" as $E_C \to 0$, i.e. without costs. There are seven different combinations of two hot qubits to interact simultaneously with the ladder. If we label them by 1, 2 and 3, the following combinations are possible: Since we are, looking at all rotations in 2 dimensional subspaces, we need to introduce three energy dumps with energy gap E_L in order to maintain energy conservation. Each of these dumps participates in one of the three interactions mentioned above. Since now there also is the possibility that all three hot qubits decay at the same time we either have to couple with this transition to two more heat dumps, or equally, to one more heat dump with an energy gap of $2E_L$. The number of two dimensional subspace rotations that are possible with this model can be calculated using the binomial coefficient.

$$\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 7$$

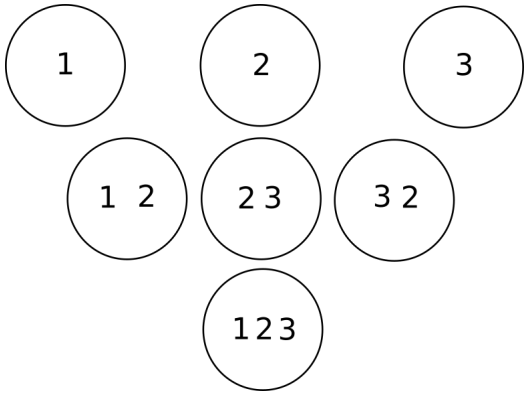


Figure 6: Illustration of all possible combinations of transitions, where the different transitions are labeled by the numbers 1,2 and 3.

Which is, as you can easily see, equal to the number of circles in the upper diagram. The resulting excitation probability $P_{top}(t)$, calculated in the same way as before, is

$$P_{top}(t) = \frac{7}{8}\sin^2(gt) \tag{52}$$

Again we assumed, that the coupling of each of the processes mentioned above is the same, therefore the probability for each process to occur is the same, namely $\frac{1}{4}$. Since only one of them actually results in a tick, to calculate the energy cost we take the average of the energy gaps of all heat dumps involved.

$$E_{cost} = \frac{5}{4}E_H \tag{53}$$

Concluding this section, we know now, that we not only have to add more heat dumps for every further machine we add to the interaction, but also that the energy gap of the heat dumps has to increase. This already gives a hint at the relation of clock "performance" and energy costs, namely that the behavior is non-linear.

3.2.3 General expression for M hot qubits

In the following part we will generalize this clock with respect to the number of machines M that are involved in the process of generating a tick. First of all let us specify our system. Given M hot qubits, what is the

number of heat dumps M_D that maximizes the number of possible two dimensional subspace rotations. In general, given a set of M elements, the number of all subsets that can be formed within this set can be calculated using the binomial coefficient the following way

$$M_D = \sum_{k=1}^{M} \binom{M}{k} = 2^M - 1 \tag{54}$$

We assume from now on that $E_C = 0$, such that only energy dumps produce energy costs.
As we concluded before, not all heat dumps have an equally sized energy gap. Therefore, in order to conserve energy, for the different transitions, we need heat dumps with corresponding energy gaps. In detail this means for a transition where k hot qubits go to their ground state, a heat dump with an energy gap of

$$E_D = (k-1)E_L \tag{55}$$

is needed to maintain energy conservation. Our system can thus be described by a Hamiltonian of the form.

$$H = H_0 + H_{int} \tag{56}$$

where H_0 denotes the free Hamiltonian, which is given by

$$\begin{aligned} H_0 =& \binom{M_H}{2} E_L \left(|1\rangle_1 \langle 1| + |1\rangle_2 \langle 1| + ... + |1\rangle_{\binom{M_H}{2}} \langle 1| \right) \\ &+ \binom{M_H}{3} 2E_L \left(|1\rangle_{\binom{M_H}{2}} \langle 1| + ... + |1\rangle_{\binom{M_H}{2}+\binom{M_H}{3}} \langle 1| \right) + ... \\ &+ (M_H - 1) E_L \left(|1\rangle_{\binom{M_H}{M_H-1}} \langle 1| + ... + |1\rangle_{\binom{M_H}{M_H-1}+1} \langle 1| \right) \end{aligned}$$

and H_{int} denotes the interaction Hamiltonian. We can construct the interaction Hamiltonian by splitting it up into all interaction Hamiltonians that are non trivial on a two dimensional subspace using the sigma notation we introduced earlier, i.e.

$$H_\alpha = g \mathbb{1}^{\otimes(2^M - 1 + M - |\alpha| - 1)} \left(\otimes \sigma^+_{d_\alpha} \bigotimes_{i \epsilon \alpha} \sigma_i^- \otimes \sigma_L^+ + h.c. \right) \tag{57}$$

where $\alpha \subseteq \{1, 2, ..., M\}$ is the set numbers counting all machines, where each i stands for a different machine and $|\alpha|$ denotes the cardinality of this set, i.e. how many elements are in the set. The total interaction Hamiltonian can be calculated by summing over all possible two dimensional subspace Hamiltonians.

$$H_{int} = \sum_{\alpha \subseteq \{1,2,...,M\}} H_\alpha \tag{58}$$

The unitaries corresponding to each sub space rotation, when we only consider the non trivial part, turn out to be

$$U_\alpha = -i \sin(gt) \left(\sigma_{d_\alpha}^+ \bigotimes_{i \epsilon \alpha} \sigma_i^- \otimes \sigma_L^+ + h.c. \right) \tag{59}$$

$$+ \cos(gt) \left(|0\rangle_{d_\alpha} \langle 0| \bigotimes_{i \epsilon \alpha} |1\rangle_i \langle 1| \otimes |0\rangle_L \langle 0| + |1\rangle_{d_\alpha} \langle 1| \bigotimes_{i \epsilon \alpha} |0\rangle_i \langle 0| \otimes |1\rangle_L \langle 1| \right) \tag{60}$$

The state of the system at time $t = 0$ can accordingly be described by taking the tensor product of all heat dumps, which are in the ground state due to the limit $T_C \to 0$, all hot qubits, which are totally mixed states due to $T_H \to \infty$ and the ladder, which is also in the ground state.

$$\rho(t=0) = \frac{1}{2^M} \underbrace{|0\rangle\langle 0|^{\otimes M_C}}_{\text{Heat dumps}} \otimes \underbrace{(|0\rangle\langle 0| \otimes \mathbb{1})^{\otimes M}}_{\text{Machines}} \otimes |0\rangle\langle 0| \tag{61}$$

In the same way as in the previous examples we calculate the unitary time evolution of this interaction Hamiltonian and apply it to the state. After tracing out all heat dumps and machines, the top level excitation probability of the ladder calculates to

$$\boxed{P_{top}(t) = \left(1 - \frac{1}{2^M}\right) \sin^2(gt)} \tag{62}$$

Taking the limit of $M \to \infty$ we can see, that this becomes $\sin^2(gt)$, which means that the amount of state inversion increases with the number of hot qubits. In the limit of infinitely many hot qubits maximal state inversion is achieved. In order to bring the state inversion in relation with the energy costs, we need to find a general expression for the latter.

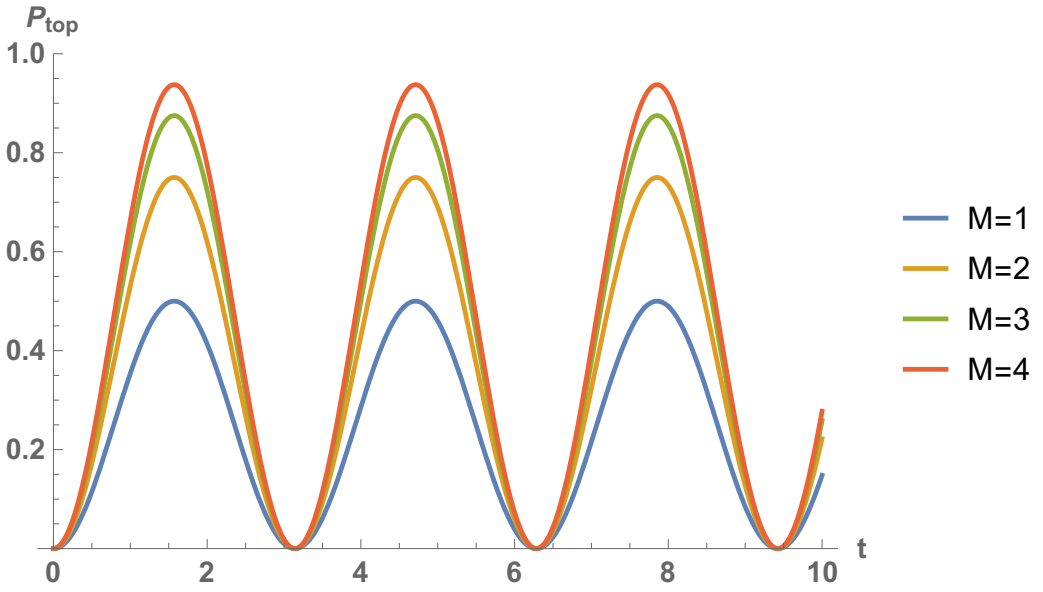


Figure 7

The number of transitions where k hot qubits interact with the ladder and the heat dumps can be calculated as before by using the binomial coefficient. The mathematical problem corresponds to finding all possible sub sets containing k elements in a set of M elements. From equation 55 we know the energy gap of a heat dump involved in a transition with k hot qubits. Therefore summing the energy gaps of all heat dumps and the energy gaps of the cold machine qubits yields

$$\begin{aligned}
&\sum_{k=1}^{M} \binom{M}{k} [(k-1)E_L] \\
&= ... \\
&= E_L \left(1 - 2^M + M2^{M-1}\right)
\end{aligned}$$

As we argued before, since all transitions are equally likely, but only one of them is realized, in order to calculate the average energy cost we have to take the average over the number of heat dumps, which is given by equation 54. Therefore the energy costs of a clock with M hot qubits is

$$E_{cost} = \frac{E_L}{2^M - 1} \left(1 - 2^M + M2^{M-1}\right) \tag{63}$$

The energy cost increases linearly with the number of machines for $M >> 1$. In the regime of few machines, it grows non-linear.
In figure 7 the effect of changing M on the top level probability is depicted. We can see, that the only effect of increasing M is, that the maximal top level occupation increases. The spread around each maximum stays unchanged, or to put it differently, only the amplitude changes, but the function stays proportional to $\sin^2(gt)$.

3.3 Higher ladder dimension

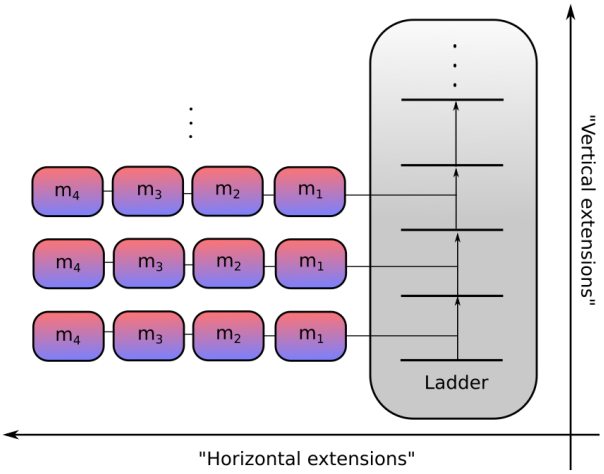


Figure 8: On the horizontal axis you can see the addition of machines coupling to one single energy gap of the ladder, which was introduced in chapter 3.2, which will be called "horizontal extensions". The vertical axis points towards increasing ladder dimension. We assume that each ladder level couples to equally many machines, increasing the number of machines in this direction will be called "vertical extension".

As we argued in section 1.2 an optimal clock produces signals which correspond to delta functions, i.e. very accurate peaks, whilst maintaining a good resolution. In order to produce such a signal, the top level probability P_{top} of our clock needs not only to achieve state inversion, but also has to have a small "variance", meaning that the peaks should be very sharp.

By increasing the dimension of the ladder, it is possible to achieve this. In the following chapters we will therefore derive the top level probability for D dimensional ladders.

3.3.1 For dimension 3

For the sake of simplicity let us first assume the ladder is of dimension 3 with equidistant energy gaps. Its free Hamiltonian is then given by

$$H_{0,L} = E_L |1\rangle\langle 1| + 2E_L |2\rangle\langle 2| \tag{64}$$

We assume now, that the machines only couple to one "gap", or to be more precise only to two adjacent energy levels of the ladder. As you can see in the appendix C it turns out that the minimal clock with ladder dimension two must consist of at least 2 machines, where one of them couples to the ground and first exited level, and the other to first exited and top level of the ladder, because with only one machine the probability $P_{top}(t)$ is equal to zero for all times. We will assume that the coupling strength g of all machines with their respective energy levels is equal.
To distinguish this addition of machines from the one before more clearly, let us say, that the machines we add here are called "vertical extensions", and the ones added in chapter 3.2 are called "horizontal extensions" (see figure 8).
Assuming a very simple form[7] of this clock we can describe its initial state by

$$\rho(t=0) = \frac{1}{4}(\underbrace{|0\rangle\langle 0| \otimes \mathbb{1}}_{\text{machine 1}} \otimes \underbrace{|0\rangle\langle 0| \otimes \mathbb{1}}_{\text{machine 2}} \otimes |0\rangle\langle 0|)$$

$$= \frac{1}{4}(|00000\rangle\langle 00000| + |01000\rangle\langle 01000| + |00010\rangle\langle 00010| + |01010\rangle\langle 01010|)$$

[7] In fact it is true, that there is a simpler form of this clock, excluding the cold machine qubits. We don't use this form here, because we wanted to maintain continuity in our description of the clock in order to make our arguments easier to follow.

Since we have two machines acting on different energy levels, let us first write the interaction Hamiltonian for each machine separately.

$$H_{int} = H_{int,1} + H_{int,2} \tag{65}$$
$$H_{int,1} = g(\mathbb{1} \otimes \mathbb{1} \otimes |010\rangle\langle 101| + h.c.) \tag{66}$$
$$H_{int,2} = g(|01\rangle\langle 10| \otimes \mathbb{1} \otimes \mathbb{1} \otimes |1\rangle\langle 2| + h.c) \tag{67}$$

These Hamiltonians also generate two dimensional subspace rotations. In an analogous way as before, we calculate the unitary time evolutions, acting only non-trivially on the subspaces

$$\begin{aligned} V_1 = \quad & \{Span(|00010\rangle, |00101\rangle), Span(|01010\rangle, |01101\rangle), \\ & Span(|10010\rangle, |10101\rangle), Span(|11010\rangle, |11101\rangle)\} \\ V_2 = \quad & \{Span(|01001\rangle, |10002\rangle), Span(|01011\rangle, |10012\rangle), \\ & Span(|01101\rangle, |10102\rangle), Span(|01111\rangle, |10112\rangle)\} \end{aligned}$$

to be

$$\begin{aligned} & U_{1,V_1}(t) = \\ & -i\sin(gt)(|00010\rangle\langle 00101| + |01010\rangle\langle 01101| + |10010\rangle\langle 10101| + |11010\rangle\langle 11101| + h.c) \\ & + \cos(gt)(|00010\rangle\langle 00010| + |01010\rangle\langle 01010| + |10010\rangle\langle 10010| + |11010\rangle\langle 11010| \\ & + |00101\rangle\langle 00101| + |01101\rangle\langle 01101| + |10101\rangle\langle 10101| + |11101\rangle\langle 11101|) \\ & = \boxed{-i\sin(gt)(\mathbb{1}_4 \otimes |010\rangle\langle 101| + h.c.) + \cos(gt)(\mathbb{1}_4 \otimes (|101\rangle\langle 101| + |010\rangle\langle 010|))} \end{aligned}$$

$$\begin{aligned} & U_{2,V_2}(t) = \\ & -i\sin(gt)(|01001\rangle\langle 10002| + |01011\rangle\langle 10012| + |01101\rangle\langle 10102| + |01111\rangle\langle 10112| + h.c.) \\ & + \cos(gt)(|01001\rangle\langle 01001| + |01011\rangle\langle 01011| + |01101\rangle\langle 01101| + |01111\rangle\langle 01111| + \\ & |10002\rangle\langle 10002| + |10012\rangle\langle 10012| + |10102\rangle\langle 10102| + |10112\rangle\langle 10112|) \\ & \boxed{= -i\sin(gt)(|01\rangle\langle 10| \otimes \mathbb{1}_4 \otimes |1\rangle\langle 2| + h.c.) + \cos(gt)(|01\rangle\langle 01| \otimes \mathbb{1}_4 \otimes |1\rangle\langle 1| + |10\rangle\langle 10| \otimes \mathbb{1}_4 \otimes |2\rangle\langle 2|)} \end{aligned}$$

Here it should be mentioned that the total interaction Hamiltonian H_{int} can, in general not be split up as we do it above, due to the Baker-Campbell-Hausdorff formula and the non-commutativity of $H_{int,1}$ and $H_{int,2}$, i.e. $[H_{int,1}, H_{int,2}] \neq 0$. However, in appendix A we show for $N = 3$ and $N = 4$ with $M = 1$ that it is possible to fine-tune the coupling constants g_1 and

g_2 of $H_{int,1}$ and $H_{int,2}$ such that the resulting top level probability resembles the top level probability that is achieved by the method we use here exactly. The, rather plausible assumption we make here, is that this fine-tuning can be done for arbitrary N. The reason for splitting up the Hamiltonian is, that it simplifies the mathematical complexity vastly.

Applying this time evolution to the initial state $\rho(t = 0)$, i.e. applying $U_2(t)U_1(t)\rho U_1^\dagger(t)U_2^\dagger(t) = \rho(t)$, tracing all machines out and considering only the top level of the ladder results a occupation probability $P_{top}(t)$ of

$$P_{top}(t) = \frac{1}{4}\sin^4(gt). \tag{68}$$

As you can see, the shape the top level probability changed to a function with "sharper" peaks, which increases the performance of our clock. On the other hand we can observe, that the amplitude of P_{top}, the state inversion decreased which leads to a reduction of clock performance. In order to understand this interplay in more detail we will derive a general expression for the top level probability with respect to the ladder dimension D.

3.3.2 Generalization to a N dimensional ladder

In the following we will generalize the clock with respect to the ladder dimension. We assume now that the ladder consists of N equally spaced energy gaps, giving us the free Hamiltonian for the ladder as

$$H_{0,L} = \sum_{n=1}^{N} nE_L |n\rangle\langle n|. \tag{69}$$

To every energy gap of the ladder we couple one machine. The initial state of this system can then be written as

$$\rho(t = 0) = \underbrace{\frac{1}{2^N}(|0\rangle\langle 0| \otimes \mathbb{1})^{\otimes N}}_{\text{Machines}} \otimes \underbrace{|0\rangle\langle 0|}_{\text{ladder}}. \tag{70}$$

The total Hilbert space dimension of this system is given by $D = N2^{2(N-1)}$. The coupling strength g between machine and ladder is assumed equal for

all levels, otherwise the effect would be weaker, i.e. the maximal top level probability would be smaller, as the rotations would be unsynchronized. The time evolution the clock is generated by an interaction Hamiltonian which can be written as a sum of the interaction Hamiltonians of machine with its respective energy level

$$H_{int} = \sum_{n=1}^{N} H_n, \tag{71}$$

where each H_n generates a rotation on a two dimensional subspace and can be written like

$$H_n = \mathbb{1}_{D-4N} \otimes (\underbrace{|01\rangle\langle 10|}_{\mathcal{H}_n} \otimes \underbrace{|n-1\rangle\langle n|}_{\mathcal{H}_L} + h.c), \tag{72}$$

where $\mathcal{H}_n$ denotes the Hilbert space of the n_{th} machine and $\mathcal{H}_L$ the Hilbert space of the ladder. As you can see, the subspace of interest, i.e. on which the non trivial interactions are generated is $V_n = Span(|01n\rangle, |10(n+1)\rangle)$.

Side note on the dimensions of the Unitary operators

$$U_n(t) = e^{-iH_n t} = e^{\mathbb{1}_{(D-4N)} \otimes H_{n,V_n} t} = \mathbb{1}_{(D-4N)} e^{-iH_{n,V_n} t},$$

where V_n in the subscript of H_{n,V_n} denotes the subspace to which H_n is reduced, i.e. what is left of H_n after tracing out all systems except the ones in $\mathcal{H}_n \otimes \mathcal{H}_L$. Because

$$e^{\mathbb{1} \otimes A} = \sum_{i=0}^{\infty} \frac{(\mathbb{1} \otimes A)^n}{n!} = \sum_{i=0}^{\infty} \frac{\mathbb{1}^n \otimes A^n}{n!} = \mathbb{1} \otimes e^A \tag{73}$$

Therefore this unitary only acts on the subspace V_n non trivially, which means we can trace out the rest and focus only on this subspace.

Now we can calculate N different unitary evolutions $U_n(t)$, each of which will describe a rotation in V_n. For the sake of simpler notation we will write only the non trivial part of the unitaries, i.e. the part acting on

V_n

$$U_n(t) = -i\sin(gt)(|01(n-1)\rangle\langle 10n| + |10n\rangle\langle 01(n-1)|) \quad (74)$$

$$+\cos(gt)(|10n\rangle\langle 10n| + |01(n-1)\rangle\langle 01(n-1)|) \quad (75)$$

In order to calculate $P_{top}(t)$ we will now successively apply the unitaries to our initial state ρ. To simplify the calculations we will only consider the subspace that is acted non trivially upon, i.e. trace out the rest of the system. This way for each level we only consider

$$\rho_n = \underbrace{\frac{1}{2}(|0\rangle\langle 0| \otimes \mathbb{1})}_{n_{th} \text{ machine } m_n} \otimes \rho_L \quad (76)$$

$$\rho_1'(t) = U_1 \rho_1 U_1^\dagger = \frac{1}{2}(|000\rangle\langle 000| + \cos^2(gt)|010\rangle\langle 010| + \sin^2(gt)|101\rangle\langle 101|) \quad (77)$$

Now we trace the machine m_1 out

$$\rho_{L,1}' = Tr_{m1}(\rho_1'(t)) = \frac{1+\cos^2(gt)}{2}|0\rangle\langle 0| + \frac{\sin^2(gt)}{2}|1\rangle\langle 1| \quad (78)$$

If the dimension of the ladder N were 2, we would already have arrived at the end and could conclude that the top level probability $P_{top,2}$ would be

$$P_{top,1} = \frac{1}{2}\sin^2(gt) \quad (79)$$

for higher dimensions we can now extend $\rho_{L,1}'$ with the machine of the next level.

$$\rho_2 = \frac{1}{2}(|0\rangle\langle 0| \otimes \mathbb{1}) \otimes \left(\frac{1+\cos^2(gt)}{2}|0\rangle\langle 0| + \frac{\sin^2(gt)}{2}|1\rangle\langle 1|\right)$$
$$= \frac{1+\cos^2(gt)}{4}|000\rangle\langle 000| + \frac{1+\cos^2(gt)}{4}|010\rangle\langle 010|$$
$$+ \frac{\sin^2(gt)}{4}|001\rangle\langle 001| + \frac{\sin^2(gt)}{4}|011\rangle\langle 011|$$

Applying U_2 and tracing the machine out leads to

$$\rho_{L,2}' = \underbrace{\frac{1+\cos^2(gt)}{2}}_{p_0}|0\rangle\langle 0| + \underbrace{\frac{\sin^2(gt)+\sin^2(gt)\cos^2(gt)}{4}}_{p_1}|1\rangle\langle 1| + \underbrace{\frac{\sin^4(gt)}{4}}_{p_2}|2\rangle\langle 2| \quad (80)$$

We labeled the probabilities for the different energy levels by p_n. As you can see, applying the unitary U_{n+1} only affects the probability p_n and p_{n+1}. As indicated above we rewrite the the ladder state now in a more general way

$$\rho_{L,n} = \sum_{j=1}^{n} p_j |j\rangle\langle j| \tag{81}$$

to find $\rho'_{L,n+1}$ we do the same as before, i.e. attach the corresponding machine and apply the unitary evolution at this level.

$$\begin{aligned}
&\rho'_{n+1} = U_{n+1}\rho'_n U^\dagger_{n+1} \\
&= U_{n+1}\left(\sum_{j=0}^{n-1} p_j \frac{|01j\rangle\langle 01j| + |00j\rangle\langle 00j|}{2} + p_n \frac{|01n\rangle\langle 01n| + |00n\rangle\langle 00n|}{2}\right) U^\dagger_{n+1} \\
&= \sum_{j=0}^{n-1} p_j \frac{|01j\rangle\langle 01j| + |00j\rangle\langle 00j|}{2} \\
&+ p_n\left(\frac{\sin^2(gt)}{2}|10(n+1)\rangle\langle 10(n+1)| + \frac{\cos^2(gt)}{2}|01n\rangle\langle 01n| + \frac{1}{2}\right)
\end{aligned}$$

And tracing out the machine leaves us with

$$\boxed{\sum_{j=0}^{n-1} p_j |j\rangle\langle j| + p_n\left(\frac{\cos^2(gt)+1}{2}|n\rangle\langle n| + \frac{\sin^2(gt)}{2}|n+1\rangle\langle n+1|\right)} \tag{82}$$

From this we can now construct a recursive formula connecting the top level probabilities of all intermediate steps $P_{top,n}$

$$P_{top,n+1} = P_{top,n}\frac{\sin^2(gt)}{2} \tag{83}$$

Using equation 79 and 83 we can calculate a general expression for the top level probability letting n go to N.

$$\boxed{P_{top}(t) = \frac{1}{2^N}\sin^N(gt)} \tag{84}$$

This general expression now helps us to understand the interplay of "sharpness" and amount of state inversion in our top level probability, which are,

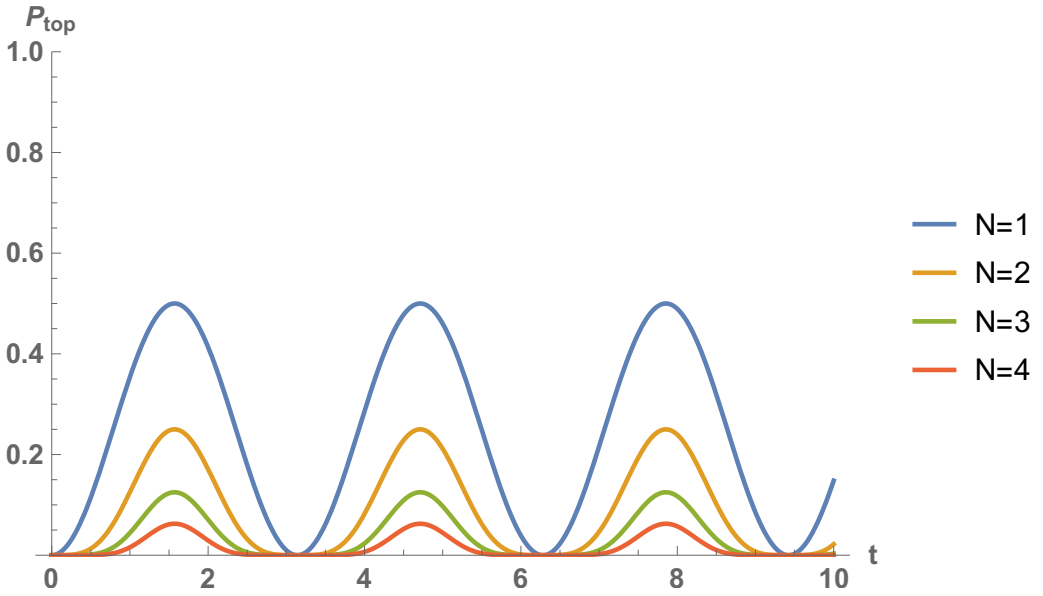


Figure 9

as we will show later, strongly connected to the notions of accuracy and resolution of our clock. In figure 9 you can see how the top level probability changes its shape with increasing N. The trade off between "sharpness" and amplitude is clearly observable. Let us consider the limit $N \to \infty$. The top maximal level probability gets exponentially smaller and goes towards 0 in the limes. At the same time as N gets bigger, the peaks of $P_{top}(t)$ get "sharper". As we have seen in chapter 3.2.3 adding machines at one level increases the amount of state inversion, or the amplitude of $P_{top}(t)$. So before we start quantifying the trade off between the "sharpness" and the amplitude in terms of accuracy and resolution, it is advisable to combine the two strategies of adding machines "horizontally", i.e. increasing the number of machines per level, and adding machines vertically, i.e. increasing the dimension of the ladder.

3.3.3 Combining "horizontal" and "vertical" extension

We assume now, that there are M machines for each of the N energy levels of the ladder, i.e. M "horizontal" extensions, and that each energy level couples exactly to M machines with equal coupling strength g. This means

that there are $D \times M$ machines in total. The dimension of the total Hilbert space of this system thus becomes $dim(\mathcal{H}) = N2^{MN}$. The initial state of our system can now be written as

$$\rho(t=0) = \underbrace{\left(\underbrace{(|0\rangle\langle 0|)^{\otimes(2^M-1)}}_{\text{heat dumps}} \otimes \underbrace{\frac{1}{2^N}(|0\rangle\langle 0| \otimes \mathbb{1})^{\otimes M}}_{\text{"horizontal" extensions}} \right)^{\otimes N}}_{\text{"vertical"extensions}} \otimes \underbrace{|0\rangle\langle 0|}_{\text{ladder}} \tag{85}$$

We can now write the total interaction Hamiltonian as combination of all interaction Hamiltonians generating two dimensional sub space rotations. Again, the fine-tuning assumption of the previous chapter is applied (see appendix A).

$$H_{tot} = \sum_{n=1}^{N} \sum_{\alpha \epsilon \{1,2,\ldots,M\}} H_{\alpha,n} \tag{86}$$

Where we adapted the Hamiltonian of equation 57 by extending "vertically", i.e.

$$H_{\alpha,n} = g\mathbb{1}^{\otimes(N(2^M-1+M)-|\alpha|+1)} \otimes \left(\sigma^+_{d_\alpha,n} \bigotimes_{i\epsilon\alpha} \sigma^-_{i,n} \otimes |n\rangle\langle n-1| + h.c. \right) \tag{87}$$

All of these subspace Hamiltonians commute with each other. Therefore the corresponding unitaries commute, which means we can calculate a general expression for the top level probability with respect to the number of "horizontal" and "vertical" extension following the same procedure as in chapter 3.3.2. In the previous chapters we already calculated the non trivial part of the unitaries of all of these Hamiltonians separately. Now we can just use what we already have and adapt in an analogous way we adapted the interaction Hamiltonian. Thus from equation 60 we get the non trivial part of the unitary evolution.

$$\begin{aligned} U_{\alpha,n} = &-i\sin(gt)\left(\sigma^+_{d_\alpha} \bigotimes_{i\epsilon\alpha} \sigma^-_i \otimes \sigma^+_L + h.c. \right) \\ &+ \cos(gt)\left(|0\rangle_{d_\alpha}\langle 0| \bigotimes_{i\epsilon\alpha} |1\rangle_i\langle 1| \otimes |0\rangle_L\langle 0| + |1\rangle_{d_\alpha}\langle 1| \bigotimes_{i\epsilon\alpha} |0\rangle_i\langle 0| \otimes |1\rangle_L\langle 1| \right) \end{aligned} \tag{88}$$

The time dependent clock state can now be written as

$$\rho(t) = U_{1,1}U_{1,2}...U_{1,N}U_{2,1}...U_{\{1,2,..,M\},N}\rho(t=0)U^{\dagger}_{1,1}U^{\dagger}_{1,2}...U^{\dagger}_{1,N}U^{\dagger}_{2,1}...U^{\dagger}_{\{1,2,..,M\},N} \tag{89}$$

To get an analogy to equation 79 of chapter 3.3.2 we set $n = 1$ and apply all remaining unitaries to ρ. This yields

$$P_{top,1} = (1 - \frac{1}{2^M})\sin^2(gt) \tag{90}$$

and is nothing different than the top level probability for a system consisting of M "horizontal" extensions and a ladder of dimension 1. Continuing to follow the afore mentioned procedure we arrive at a recursive formula for the top level probability by using equation 88 and applying all unitaries with fixed at $n+1$, i.e.

$$U_{\{1\},n+1}...U_{\{1,2...,M\},n+1}\rho U^{\dagger}_{\{1\},n+1}...U^{\dagger}_{\{1,2...,M\},n+1} \tag{91}$$

arriving at

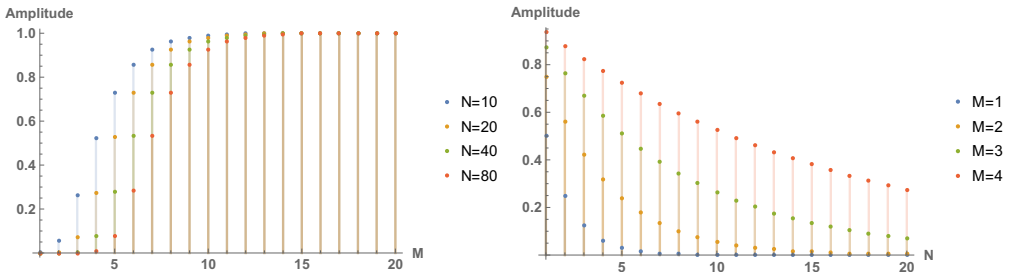
$$P_{top,n+1} = P_{top,n}(1 - \frac{1}{2^M})\sin^2(gt) \tag{92}$$

Which, using equation 90 yields for a N dimensional ladder with M machines coupled to the respective energy gap

$$\boxed{P_{top} = (1 - \frac{1}{2^M})^N \sin^{2N}(gt)} \tag{93}$$

Now this is the quantity from which our final results will arise. Before we go into more details, let's take a moment to think about it. The trade off between amplitude and "sharpness" becomes very tangible with this equation. In fact, before we consider quantities like resolution and accuracy, we should take a look at this trade off in more detail. Let us first look at the amplitude and analyze its behavior with changing number of "horizontal extensions" M and "vertical extensions" N. In figure 10a we can see the how the Amplitude changes with respect to to M for different ladder dimensions N and in figure 10b with respect to N for different M.

What we can see in fig. 10a is that the amplitude increases for with M non linearly. In fact after adding a certain amount of "horizontal extensions" the amplitude saturates. At this point increasing the amplitude



(a) (b)

Figure 10:
(a): Amplitude of $P_{top}(t)$ vs number of "horizontal extensions" M for different ladder dimensions N.
(b): Amplitude of $P_{top}(t)$ vs number of "vertical extensions" N for different numbers of "horizontal extensions" N.

further becomes much more expensive in terms of additional machines and therefore in terms of energy costs. Considering different ladder dimensions N you can see, that the saturation happens at a bigger number of "horizontal extensions" M , which means that the energy cost for having the same amplitude increases with the ladder dimension. This can also be seen in fig. 10b. Keeping the number of "horizontal extensions" fixed, we increase the ladder dimension, which in therm decreases the amplitude exponentially. Thus we can conclude, that the amplitude depends on M and N with a certain trade-off.
In order to talk about the behavior of the "sharpness" more quantitatively, we look at the full width at half maximum (FWHM) of one peak of $P_{top}(t)$, which means we only consider the region $t = [0, \pi/g]$. The "sharpness" behaves inverse proportional to the FWHM. In figure 11 you can see that the FWHM plotted over the ladder dimension. With increasing ladder dimension the FWHM of $P_{top}(t)$ decreases non linearly. It further more becomes linear for high N, which tells us, that the trade off between "sharpness" and amplitude behaves non trivially as these two quantities change in very distinct ways. On the other hand changing the number of "horizontal extensions" does not affect the FWHM at all. As we will see later, the FWHM of one peak is strongly connected to the accuracy.

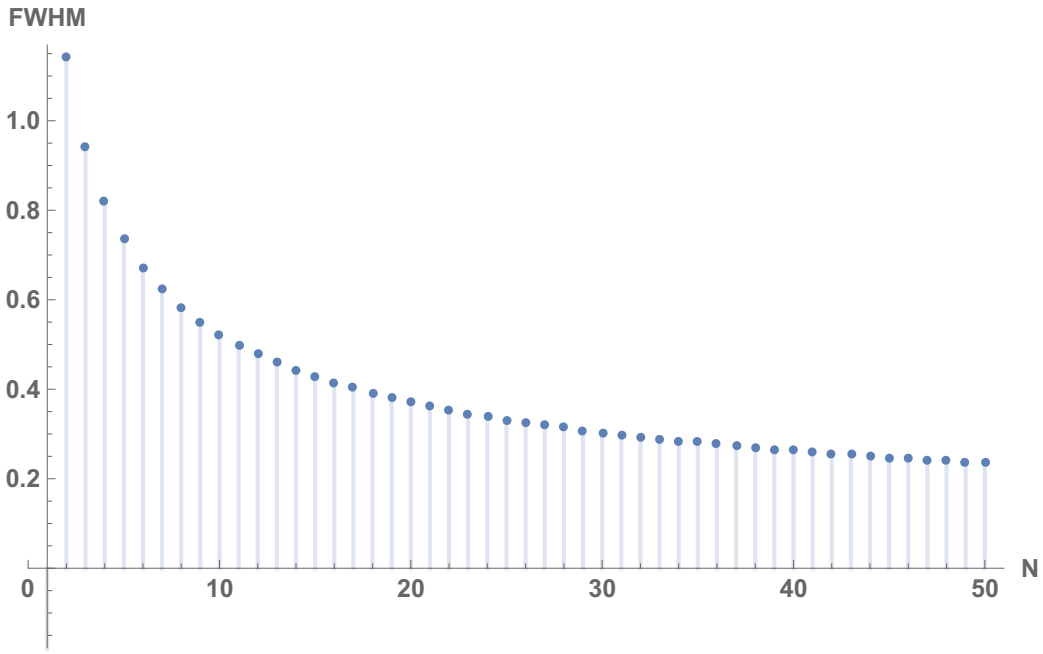


Figure 11: Full width half maximum of one peak of $P_{top}(t)$ versus ladder dimension N.

3.3.4 General energy costs

In order to make more sense of the costs, we will now derive a general expression for the energy cost with respect to M and N. Deriving these costs is rather straight forward. We assumed that every energy gap has an equal number of "horizontal extensions". The energy cost of a clock with a N dimensional ladder can simply be calculated by summing up the energy cost of every energy level of the ladder (equation 63) , which leads us to

$$E_{cost} = D\frac{E_L}{2^M - 1}\left(1 - 2^M + M2^{M-1}\right) \tag{94}$$

Which means that the energy cost grows, for large M linear in M and in general linear in D.

3.4 Calculating the resolution and accuracy

In this chapter we will calculate the quantities resolution and accuracy explicitly. We will investigate the relation of these quantities with respect to the energy cost and analyze the trade off between them.
In chapter 3.1.5 we introduced the probability density for having a "tick" at time t for the minimal clock model. We will generalize this now with respect to "horizontal" and "vertical extensions".
Using the general top level probability from equation 93 and insert it into equation 39 we arrive at

$$P_{tick}(t) = \frac{d}{dt}\left(c\int_0^t (1-\frac{1}{2^M})^N \sin^{2N}(gt')dt\right) e^{-c\int_0^t (1-\frac{1}{2^M})^N \sin^{2N}(gt')dt}$$

Solving this integral is highly non trivial as it can only be solved using the hyper geometric function, which can be written as the series expansion

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)} \frac{z^k}{k!} \tag{95}$$

with Γ being the gamma function defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt \tag{96}$$

where $z\epsilon\mathcal{C}$ and $Re(Z) \geq 0$.
Using the hyper geometric function the anti derivative of powers of the sine function can be obtained

$$\int \sin^n(x)dx = -\cos(x)\frac{\sin^{n+1}(x)}{\sin^2(x)^{\frac{n+1}{2}}}\ {}_2F_1\left(\frac{1}{2},\frac{1-n}{2},\frac{3}{2},\cos^2(x)\right) \tag{97}$$

which gives us the solution

$$\int_0^t P_{top}(t)dt = c\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2}-d;\frac{3}{2};\cos^2(gt)\right)sin(2gt)sin^2(gt)^{-d-\frac{1}{2}}\left((1-2^{-m})sin^2(gt)\right)^d}{2g} \tag{98}$$

this solution we use to calculate the generalized top level probability.

$$\boxed{\begin{aligned} &P_{top}(t) = \\ &c\left((1-2^{-m})\sin^2(gt)\right)^d e^{\left(\frac{c\sin(2gt)\sin^2(gt)^{-d-\frac{1}{2}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}-d;\frac{3}{2};\cos^2(gt)\right)\left((1-2^{-m})\sin^2(gt)\right)^d}{2g}\right)} \end{aligned}} \tag{99}$$

Having the generalized form of $P_{top}(t)$ we can now calculate the accuracy and the resolution using their definitions from chapter 1.2. First we recall that the resolution was defined as the number of "ticks" per unit time, in other words the one over the average (Schrödinger) time $\bar{t}$ it takes for the clock to tick once. To calculate this average time of a single tick we take the integral:

$$\bar{t} = \int_0^\infty t P_{tick}(t) dt \tag{100}$$

The resolution, defined above can thus be expressed as

$$R = \frac{1}{\bar{t}} \tag{101}$$

Secondly we recall the definition of the accuracy as being the number of ticks A such that the standard deviation of the A_{th} tick is equal to average tick time $\bar{t}$.

$$A = \left(\frac{\bar{t}}{\Delta \bar{t}} \right)^2 \tag{102}$$

To calculate the variance we use

$$(\Delta \bar{t})^2 = \overline{t^2} - \bar{t}^2 \tag{103}$$

with

$$\overline{t^2} = \int_0^\infty t^2 P_{tick}(t) dt \tag{104}$$

leading to

$$A = \frac{1}{R^2 \overline{t^2} - 1} \tag{105}$$

We will now analyze the behavior of resolution and accuracy with respect to the energy cost in a similar way as we analyzed the amplitude and the "sharpness" at the end of chapter 3.3.3. In figure 12 and 13 the resolution and the accuracy are plotted over the total energy costs. Comparing these two plots there are several observations we can make. Both the accuracy and the resolution converge to a certain value as the energy cost increases. This value, as you can see in the plots, depends on the dimension of the ladder. In figure 13 you can see, that a higher ladder dimension

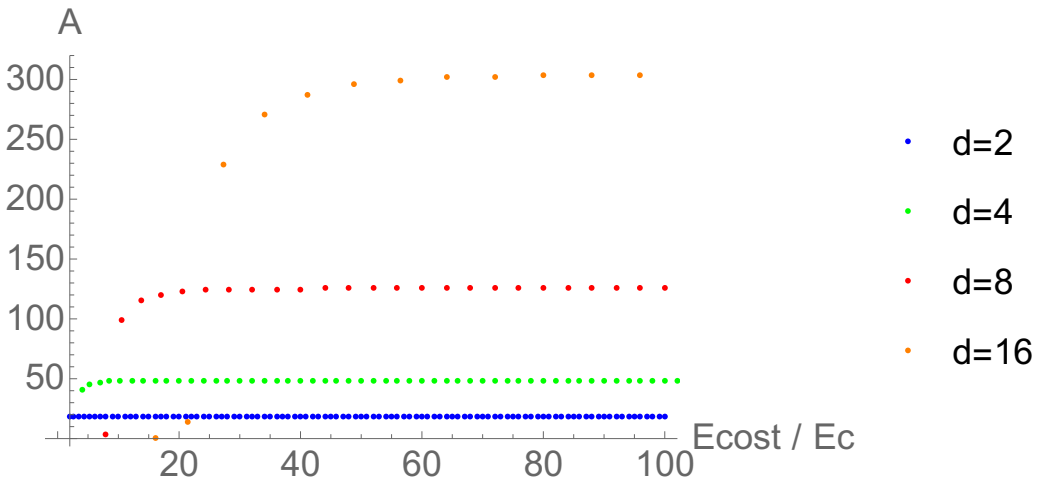


Figure 12: asdf

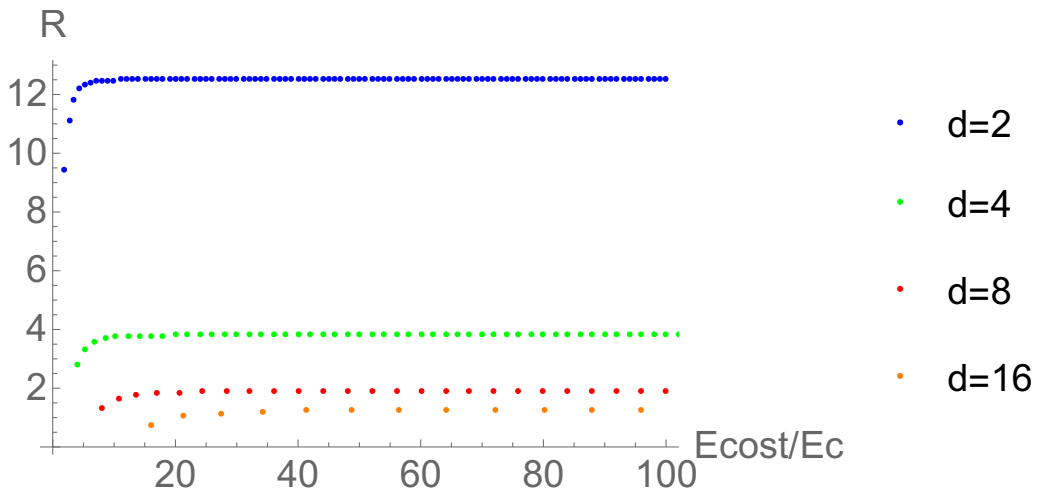


Figure 13: 2 Figures side by side

leads to a lower resolution. Conversely looking at the accuracy in figure 12 this behavior gets inverted, i.e. increasing ladder dimension leads to higher accuracy in the limit of high energy costs. This trade off exemplifies the intimate relationship between resolution, accuracy and energy cost. If we investigate the accuracy for low energy costs instead of considering the limit of many machines, we find regimes where the accuracy behaves differently. For certain energy costs we find, that the most accurate machine is one of lower ladder dimension. This way it is possible to optimize the clock with at given energy costs with respect to the accuracy by changing the ladder dimension.

As the entropy production is directly connected to the energy costs we can conclude, that the fundamental entropy production of a single tick of a clock depends on the accuracy and the resolution of this clock. Furthermore there exists a trade off between them, which leads to a direct connection between the overall clock performance and the entropy production of this clock.

4 Conclusion

The concept of time has been elusive ever since humans first started thinking about it. In this thesis we tried to shine some light on the connection between our intuitive picture of time provided by clocks, and the physics that lies on the foundations of this picture. For this purpose we introduced the idea of thermal time and motivated how quantum thermodynamics might help to understand it. The basic idea was, that entropy production is on a fundamental level, is responsible for the emergence of time as we perceive it, or from a different viewpoint, that the measurement of time inevitably leads to an increase of entropy.

Building on the work of Paul Erker, Mark T. Mitchison, Ralph Silva, Mischa P. Woods, Nicolas Brunner and Marcus Huber [5] we introduced quantum thermal machines and autonomous quantum clocks. In contrast to their work, we analyzed the behavior of such a clock in a single shot manner, which means that we looked at the energy conserving unitary time evolution of it that leads to the generation of ticks. In order to do that we had to construct a clock model which enables us to manipulate the resolution and the accuracy of the clock with respect to its energy consumption. This energy consumption then can be linked directly to the entropy production of a single tick. We found, that the energy consumption is connected to the accuracy and resolution of the clock in a non-trivial way. Analyzing the interplay between these figures of merit showed that, given a fixed amount of energy to spend, either accuracy can be increased at the cost of resolution, and the other way around.

Next steps to be taken from here are the analyze the trade off between resolution and accuracy in more detail and trying to find a general analytic expression for this trade off.

The resources that were used to drive this clock where thermal baths at different temperatures. The heat flow from hot to cold bath is utilized by the clock to produce a periodic signal. The advantage of using such resources is that they are easy to prepare and that they are ubiquitous in the world. Nevertheless it might be interesting to look into other kinds of resources and quantum systems to drive a clock.

In order to completely quantify the energy costs of time measurement one also would have to take the costs of measurement and storing, i.e. the cost of converting quantum information into classical information, into account.
It moreover might be interesting to see what happens, when correlations and entanglement are allowed as resources. Could these purely quantum effects increase the performance of clocks?
The concept of entropic time gives us an idea of the direction of time and what future means, therefore we can say with clear conscience that we are looking forward to future research in that direction.

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Appendix

A Fine tuning of coupling constants in N=3 and N=4 with M=1

Here we will show that it is possible to fine tune the coupling constants such that top level probability after sequentially applying the unitary time evolution of each partial Hamiltonian $H_{int,i}$, is equal to applying the total unitary evolution of H_{int}. We will do this for clocks of the dimensions $N = 3$ and $N = 4$ with $M = 1$.

First let us calculate the top level probability for the case of $N = 3$. The top level probability achieved by applying the unitary evolution of each Hamiltonian separately is given in equation 68. The unitary time evolution of the total Hamiltonian is calculated using: $U_{tot} = e^{-i(H_1+H_2)t}$, where we assume this time, that the two Hamiltonians can have different coupling strengths. Similar to before we trace out the two machine systems and use the Born rule to achieve the top level probability of the ladder,

$$P_{top,tot}(t) = \frac{g_1^2 g_2^2 \sin^4\left(\frac{1}{2}t\sqrt{g_1^2+g_2^2}\right)}{\left(g_1^2+g_2^2\right)^2}. \tag{106}$$

In this simple case we can assume that $g_1 = g_2$, which yields

$$\frac{1}{4}\sin^4\left(\frac{g_1 t}{\sqrt{2}}\right). \tag{107}$$

Thus fine-tuning g simply means setting $g = \frac{g_1}{\sqrt{2}}$.

It for $N = 4$ we calculate $P_{top}(t)$ in the same way, resulting in

$$P_{top}(t) = g_2^2 \left(\frac{\sqrt{g_1^2 + \sqrt{g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)^2} + g_2^2 + g_3^2} \sin\left(\frac{t\sqrt{g_1^2 - \sqrt{g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)^2} + g_2^2 + g_3^2}}{\sqrt{2}} \right)}{16\left(g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)^2\right)} \right.$$
$$\left. - \frac{\sqrt{g_1^2 - \sqrt{g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)^2} + g_2^2 + g_3^2} \sin\left(\frac{t\sqrt{g_1^2 + \sqrt{g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)^2} + g_2^2 + g_3^2}}{\sqrt{2}} \right)}{16\left(g_1^4 + 2g_1^2(g_2 - g_3)(g_2 + g_3) + \left(g_2^2 + g_3^2\right)\right)^2} \right)^2, \tag{108}$$

which can be simplified by setting $g_1 = g_3$ to

$$\frac{\left(\sqrt{g_2\left(\sqrt{g_2^2 + 4g_3^2} + g_2\right) + 2g_3^2} \sin\left(\frac{t\sqrt{-g_2\sqrt{g_2^2 + 4g_3^2} + g_2^2 + 2g_3^2}}{\sqrt{2}} \right) - \sqrt{-g_2\sqrt{g_2^2 + 4g_3^2} + g_2^2 + 2g_3^2} \sin\left(\frac{t\sqrt{g_2\left(\sqrt{g_2^2 + 4g_3^2} + g_2\right) + 2g_3^2}}{\sqrt{2}} \right) \right)^2}{16\left(g_2^2 + 4g_3^2\right)} \tag{109}$$

Fine-tuning g_3 to be $g_3 = \sqrt{g}\sqrt{g + g_2}$, and further assuming $g_2 = 2g$ yields

$$\frac{\left(\sqrt{g_2\left(4\sqrt{g_2^2} + 5g_2\right)} \sin\left(\frac{1}{2}\sqrt{g_2\left(5g_2 - 4\sqrt{g_2^2}\right)}t \right) - \sqrt{g_2\left(5g_2 - 4\sqrt{g_2^2}\right)} \sin\left(\frac{1}{2}\sqrt{g_2\left(4\sqrt{g_2^2} + 5g_2\right)}t \right) \right)^2}{128g_2^2} \tag{110}$$

As such, this formula doesn't look much like the form we would like it to look, but it turns out, that by setting $g_2 = 2$ it is possible to obtain

$$\frac{\sin^6(t)}{8}. \tag{111}$$

Therefore this assumption also holds for $N = 4$. These findings therefore give evidence for the general possibility to achieve equality of the sequential approach and the application of the total interaction Hamiltonian with respect to the top level probability, by fine tuning the coupling constants.

B Pauli's Theorem

In his work of 1926 Wolfgang Pauli shows, that it is not possible to construct a Hermitian time operator. Inspired by the master Thesis of Sofia Qvarfort [26] we will redo Pauli's proof here as it was done by Galapon in [27].

Theorem (Pauli): *The use of a Hermitian time operator $T = T^\dagger$ such that $[T, H] = i$ is forbidden when the Hamiltonian H is bounded from below.*

Proof:
Let H be a self-adjoint Hamiltonian and assume, that there exists a self-adjoint time operator T conjugate to H, with the commutation relation

$$[T, H] = i \tag{112}$$

As T is a self-adjoint operator, it generates a unitary $U_\beta = e^{-i\beta T}$, where $\beta \epsilon \mathbb{R}$.
By making use of the commutator relation $[A, B^n] = n[A, B^n]B^{n-1}$, which holds when $[A, B] = const.$ (for proof we refer to [26]), we recover the commutation relation

$$[U_\beta, H] = \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!}[T^k, H] = -\beta U_\beta \tag{113}$$

Assuming now that $\left|\phi_E\right\rangle$ is an energy eigenstate, we examine $HU_\beta \left|\phi_E\right\rangle$.

$$HU_\beta \left|\phi_E\right\rangle = (U_\beta H + \beta U_\beta)\left|\phi_E\right\rangle = (E + \beta)U_\beta \left|\phi_E\right\rangle, \tag{114}$$

which means for one, that $U_\beta \left|phi_E\right\rangle$ is a eigenstate of H with the eigenvalue $E + \beta$. This means for the spectrum of H to be continuous. Furthermore since $\beta \epsilon \mathbb{R}$ this eigenvalue it spans the entire reals space, making it unbounded from below, which is in contradiction with the requirement that H is bounded from below. Therefore it is not possible to construct a time operator like this.

C Why one machine is not enough

In this section we will show why it is necessary that there needs to be at least one distinct machine coupled to each energy gap of the ladder in order to recover any clock like behavior. We will show this for the limiting case $T_H \to \infty$ and $T_C \to 0$. In the case of finite temperatures, thermal ticks would occur. We will show this only for the case of $N = 3$, but with this proof it can easily shown to hold for arbitrary N.

Assume one thermal machine coupling to both energy gaps of a 3 dimensional ladder. The initial state is then given by

$$\rho = \frac{1}{2}(|000\rangle\langle 000| + |010\rangle\langle 010|). \tag{115}$$

The possible transitions are given by the energy conserving interaction Hamiltonian

$$\begin{aligned} H_{int} = g\,(|010\rangle\langle 101| + |011\rangle\langle 102| + h.c) &= H_{int_1} + H_{int_2} \\ H_{int,1} &= g(|010\rangle\langle 101| + h.c.) \\ H_{int,1} &= g(|011\rangle\langle 102| + h.c.), \end{aligned}$$

where the commutator $[H_{int_1}, H_{int_2}]$ vanishes.
We can write the unitary time evolution as

$$U(t) = e^{-iH_{int}t} = e^{-iH_{int_1}t} e^{-iH_{int_2}} = U_1 U_2, \tag{116}$$

which results again in rotations in two dimensional subspaces, i.e.

$$U_1 = -isin(g_1 t)(|010\rangle\langle 101| + |101\rangle\langle 010|) + cos(g_1 t)(|010\rangle\langle 010| + |101\rangle\langle 101| \tag{117}$$

$$U_2 = -isin(g_2 t)(|011\rangle\langle 102| + |102\rangle\langle 011|) + cos(g_2 t)(|011\rangle\langle 011| + |102\rangle\langle 102| \tag{118}$$

applying the unitary results in

$$\begin{aligned} &U_2 U_1 \rho U_1^\dagger U_2^\dagger = \\ &\rho(t) = \frac{1}{2}(|000\rangle\langle 000| + sin^2(g_1 t)|101\rangle\langle 101| + cos^2(g_1 t)|010\rangle\langle 010|). \end{aligned} \tag{119}$$

By calculating the reduced state of the ladder and using Born's rule to calculate the top level probability we recover

$$\langle 2|\rho_L(t)|2\rangle = 0, \tag{120}$$

for all time. This means that this machine can never rotate our system in the top level. Thus no tick will ever occur.