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CONTENTS

ZUSAMMENFASSUNG

Man kennt und versteht Supergravitation seit ungefähr 40 Jahren, Newtonsche Gravitation sogar seit über 300 Jahren. Trotzdem ist es bis heute niemandem gelungen, eine supersymmetrische Erweiterung der Newtonschen Gravitationstheorie zu konstruieren. Das ist, gelinde gesagt, beachtenswert.

In dieser Arbeit legen wir einige Forschungsergebnisse über nichtrelativistische Geometrie – sogenannte Newton-Cartan Geometrie – dar, die den Weg zu Newtonscher Supergravitation in vier Dimensionen bereiten. Newton-Cartan Geometrie ist charakterisiert durch eine Blätterung der Raumzeit, der eine absolute Zeitrichtung entspricht. Insbesondere betrachten und verallgemeinern wir Resultate der letzten Jahre, in denen die Theorie durch ein sogenanntes Eichprinzip aus einer nichtrelativistischen Raumzeitsymmetrie Superalgebra abgeleitet wurde. Wir haben eine alternative Methode angewandt, die auf einer speziellen Art der Dimensionsreduktion – sogenannter Null Reduktion – beruht. Dadurch war es möglich, die Theorie besser zu verstehen, sodass der Übergang zu allgemeinen Dimensionen in Reichweite scheint.

Stichwörter: Supergravitation, Newton-Cartan Gravitation, Differentialgeometrie

ABSTRACT

Supergravity has been known for about 40 years, Newtonian gravity for over 300 years. Nevertheless, to this day nobody has constructed a supersymmetric extension of Newtonian gravity. This is remarkable, to say the least.

In this work we present some developments in nonrelativistic geometry – so-called Newton-Cartan geometry – leading the way to Newtonian supergravity in four dimensions. Newton-Cartan geometry is characterized by a foliation of spacetime, corresponding to an absolute time direction. In particular, we review and generalize recent results, where the theory has been obtained via a gauging of a nonrelativistic spacetime superalgebra. We applied an alternative technique building on a special type of dimensional reduction called Null Reduction. In the process we gained a better understanding of the theory making the transition to general dimensions seem within range.

Keywords: Supergravity, Newton-Cartan gravity, Differential Geometry

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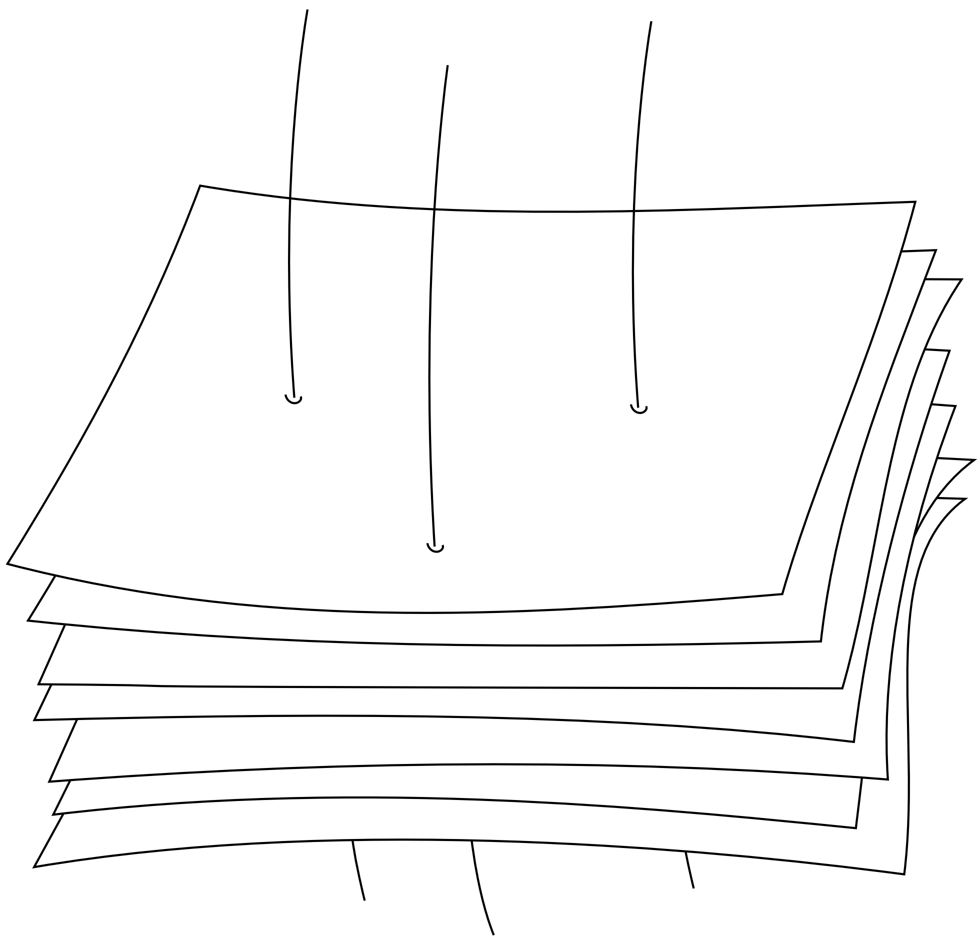
CHAPTER 1

INTRODUCTION



a) Relativitiätsprinzip: Die Naturgesetze sind nur Aussagen über zeiträumiche Koinzidenzen; sie finden deshalb ihren einzig natürlichen Ausdruck in allgemein kovarianten Gleichungen. [...]
Wenn es nämlich auch richtig ist, daß man jedes empirische Gesetz in allgemein kovariante Form muß bringen können, so besitzt das Prinzip a) doch eine bedeutende heuristische Kraft, die sich am Gravitationsproblem ja schon glänzend bewährt hat [...] Man bringe einmal die Newtonsche Gravitationsmechanik in die Form von absolut kovarianten Gleichungen (vierdimensional) und man wird sicherlich überzeugt sein, daß das Prinzip a) diese Theorie zwar nicht theoretisch, aber praktisch ausschließt! (A. Einstein 1918, [1])

En gardant toutes les notions de la Méchanique classique, on a ainsi une réduction à la Géometrie de la gravitation newtonienne, réduction qui est exactement de la même nature que celle qui est fournie par la théorie d'Einstein et qui montre, comme elle, l'interdependence de la connexion affine de l'Univers et de la matrière dans l'espace. (E. Cartan 1924, [2]) Einstein's theory of gravitation is a theory describing the dynamics of Riemannian geometry. Geometrodynamics – as J.A. Wheeler [3] called it – makes statements about the curvature of spacetime rather than forces. In this respect it is conceptually incomparable to Newtonian gravity. As a result, Einstein succeeded in combining gravity with special relativity (SR) and showed that:

$$(SR) + (GRAV) \Rightarrow (GEOMETRY).$$

In other words, relativistic gravity cannot be separated from geometry. But is the opposite also true? What happens to geometry when removing relativity, that is taking $c \to \infty$? Or:

How does the "geometry" in Geometrodynamics change when $c \to \infty$?

Answering this question is a central aspect of this master's thesis. In part it will be about describing nonrelativistic geometry (as opposed to Riemannian geometry) and its dynamics, described by the so-called Newton-Cartan theory (NC). The physical predictions are exactly those of Newtonian gravity [4] but presented in a coordinate invariant way as pioneered by E. Cartan in 1924 [2]. We will argue that NC is the correct nonrelativistic limit of GR. Recently, Newton-Cartan geometries have been considered in contexts other than gravity, too. Two examples are: Fractional Quantum Hall physics as done by D.T. Son [5] and collaborators and nonrelativistic holography – so-called Lifshitz holography – as done by J. Hartong and N. Obers [6, 7] and collaborators.

Building on the above, it is possible to consider supersymmetric extensions of NC. We will refer to those theories as supersymmetric Newton-Cartan (SNC) or nonrelativistic supergravity (NSUGRA). Nonrelativistic supersymmetry is not yet fully understood. The field of SNC is only in its early days, with some recent progress in three dimensions [8] that we will review here. One motivation to consider NSUGRA is certainly of an academic nature, since there is no known supersymmetric extension of Newtonian gravity in four dimensions. Moreover, it is conceivable that nonrelativistic supergravity gives rise to powerful localization techniques for supersymmetric field theories on arbitrary NC backgrounds. In the relativistic setting this has been done with great success [9, 10]. An essential ingredient to taking these steps is the construction of nonrelativistic off-shell supergravity. Indeed, we have been able to formulate a torsionful $D = 3$, $\mathcal{N} = 2$ off-shell nonrelativistic supergravity. Some of the results are presented in this master's thesis, but there is also an upcoming publication [11].

This work is structured as follows. Chapter 2 introduces many of the central notions of later chapters in the context of free particles, both relativistic and nonrelativistic. These include the nonrelativistic symmetry algebra, first order actions and dimensional reduction along lightlike directions. Chapter 3 is mainly about introducing both technical and physical concepts that are used in later chapters. On the technical side, these include some notions of differential geometry and group theory. On the physical side, we introduce General Relativity as the dynamics of geometry (GR or geometrodynamics) and similarly Newtonian gravity as the dynamics of nonrelativistic geometry (Newton Cartan theory – NC). In particular we will see how to obtain the kinematics of the latter from the nonrelativistic isometry algebra

barg. Chapter 4 extends the results of the preceding chapter to supergravity (SUGRA) and supersymmetric Newton-Cartan theory (SNC). Again, the focus will be on the symmetry structure of supersymmetric relativistic **spoin** and supersymmetric nonrelativistic **sbarg** spacetime algebras. In chapter 5 we present a method to obtain nonrelativistic geometrodynamics directly from GR by means of a special type of dimensional reduction. This approach extends to the supersymmetric case as done in chapter 6. In fact, the results are stronger than those obtained in chapter 4 and previous work.

The main objective of this thesis is to understand supersymmetric Newton-Cartan gravity (SNC). The following diagram summarizes the different strategies pursued:

$$
\begin{tikzcd}
 & \mathbf{poin} \arrow[rr, "3.2.2" description] \arrow[dl, "3.1.4" description] \arrow[dd, dash] & & \mathbf{barg} \arrow[dl, "3.2.3" description] \arrow[dd, dash] \\
GR \arrow[rr, "5.2.1" description] \arrow[dd, dash] & & NC \arrow[dd, dash] & \\
 & \mathbf{spoin} \arrow[rr, "4.2.2" description] \arrow[dl, "4.1.3" description] & & \mathbf{sbarg} \arrow[dl, "4.2.3" description] \\
SUGRA \arrow[rr, "6.1" description] & & SNC &
\end{tikzcd}
$$

CHAPTER 2

THE PARTICLE

In this chapter we introduce some central concepts of this master's thesis on the basis of the free particle. These include gauge invariance, nonrelativistic limits not involving $c \to \infty$, first order actions and compactification along null-like directions. All of these will feature in more generality in the later chapters. What is more, the nonrelativistic particle as formulated here can serve as a basis for considering nonrelativistic limits of extended objects, as done in [11–14]. The bulk of this chapter follows [15].

2.1 RELATIVISTIC

The particle is an abstract concept that has been formative for theoretical physics from the early days of Newton through to contemporary research. Despite the long history, mathematicians and physicists do not agree on the definition of an elementary particle:

1. (maths) A unitray irreducible representation of the Poincaré group,
2. (physics) A particle without structure whose dynamics depend only on the underlying geometry.

We first follow the latter viewpoint and will see glimpses of the first in chapter 3. Let us consider an (elementary) particle of mass m. The action describing motion along paths of minimal distance in Minkowski spacetime reads

$$S[X] = -mc \int \sqrt{-ds^2} = -mc \int d\tau \sqrt{-\dot{X} \cdot \dot{X}}, \qquad \dot{X}^M = \frac{dX^M}{d\tau}, \tag{2.1}$$

where τ is an arbitrary parameter along the worldline $X^M(\tau)$ in D-dimensional Minkowski spacetime and $\dot{X} \cdot \dot{X} = \eta_{MN} \dot{X}^M \dot{X}^N$. Arbitrary means, that the action (2.1) is invariant under reparametrizations of the worldline parameter τ. Explicitly: $\tau \mapsto \tau'$ and $X^M(\tau) = X'^M(\tau')$. If we restrict to infinitesimal changes in reparametrization $\tau - \tau' = \xi(\tau)$, we see that

$$\begin{aligned} \delta_\xi X^M(\tau) &= X'^M(\tau) - X^M(\tau) \\ &= \left(X^M(\tau) + \xi(\tau)\dot{X}^M(\tau)\right) - X^M(\tau) = \xi(\tau)\dot{X}^M(\tau). \end{aligned} \tag{2.2}$$

It is straightforward to check that $\delta_\xi \sqrt{-\dot{X} \cdot \dot{X}} = d/dt(\xi \sqrt{-\dot{X} \cdot \dot{X}})$ and thus $\delta_\xi S = 0$. It is important to note that this kind of transformation is not a symmetry in the sense of mapping solutions to solutions. Two worldlines with different parametrizations related by eq. (2.2) represent the same solution and are hence to be seen as the same physical worldline. We encounter this type of redundancies all over theoretical physics where they are known as gauge invariances. Whenever we use the term gauge invariance we mean some type of redundancy in the theory. Reparametrization invariance is a specific example that we will generalize to diffeomorphism invariance in the next chapter.

Since reparametrization invariance is an abundance of variables we can fix it by choosing a gauge for $X = (X^0, \mathbf{X})$, e.g.

$$\begin{aligned} X^0(\tau) = c\tau \quad \longrightarrow \qquad S[X] &= -mc^2 \int d\tau \sqrt{1 - c^{-2}\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}} \\ &= \int d\tau \Big\{ -mc^2 + \frac{1}{2} m \dot{\mathbf{X}} \cdot \dot{\mathbf{X}} (1 + \mathcal{O}(c^{-2}\dot{\mathbf{X}} \cdot \dot{\mathbf{X}})) \Big\}, \end{aligned} \tag{2.3}$$

where in the last line we made an expansion in orders of $c^{-2}\dot{\mathbf{X}}\cdot\dot{\mathbf{X}}$. We will come back to this when discussing the nonrelativistic limit $c\to\infty$. Contrary to the reparametrization invariant form (2.1), Lorentz invariance is no longer manifest in (2.3). By fixing the gauge we have not changed the physics but just the form of the action.

The point particle action (2.1) is invariant under other transformations other than reparametrizations. Again, we display just the transformation under small transformations $\delta X^M = X'^M - X^M$

$$\delta(\lambda,\zeta)X^M = \zeta^M - \lambda^M{}_N X^N, \tag{2.4}$$

where ζ is a constant vector and $\lambda_{MN} = \eta_{MM'}\lambda^{M'}_N = -\lambda_{NM}$. This kind of transformations are called spacetime symmetries as they act on the coordinates X^M themselves. We will meet transformations like this later on in this work in much more detail in the guise of name Poincaré transformations. Note that both forms of the action (2.1) and (2.3) are invariant under Poincaré transformations. However, since it is much easier to see this in eq. (2.2), we say the symmetry is manifest.

Let us now try to make a Hamiltonian analysis of the particle $\mathcal{L} = -mc\sqrt{-\dot{X}\cdot\dot{X}}$. This will allow us to make the transition to the nonrelativistic particle more transparent and also to treat the massless particle correctly as it is not clear what the limit $m\to 0$ means for (2.1). We start by deriving the canonical momentum

$$P^M = \frac{\partial\mathcal{L}}{\partial\dot{X}_M} = (-\dot{X}\cdot\dot{X})^{-1/2}\Big(mc\dot{X}^M\Big) \quad\longrightarrow\quad P\cdot P + m^2c^4 = 0. \tag{2.5}$$

So we see that not all components of the momentum are independent but rather subject to a condition that is called the mass-shell condition or relativistic dispersion relation as it can also be read as $P_0 = \pm\sqrt{\mathbf{P}\cdot\mathbf{P}c^2 + m^2c^4}$. As a consequence, we cannot solve for $\dot{X} = \dot{X}(P)$, since not all components of P are independent. We resolve this issue by replacing the dependent $P = P(\dot{X})$ by independent variables P^M. To account for the mass-shell condition, we impose it via a Lagrange multiplier E. So we really introduced two extra fields and call this approach a first order formulation (in later chapters we will encounter a very similar type of approach to General Relativity and supergravity). This approach to relativistic dynamics was pioneered by Dirac [16] in the 1950s. From now on we will set $c = 1$ for simplicity.

$$S[X,P,E] = \int d\tau\Big\{\dot{X}\cdot P - \frac{1}{2}E\Big(P\cdot P + m^2\Big)\Big\}, \tag{2.6}$$

from which we infer three equations of motion. Note, however, that not all of them will be interpreted as equations of motion, rather some of them as constraints:

$$E: \qquad P\cdot P + m^2 = 0, \tag{2.7a}$$

$$P^M: \qquad P^M = E^{-1}\dot{X}^M, \tag{2.7b}$$

$$X^M: \qquad \dot{P}^M = 0. \tag{2.7c}$$

We use the algebraic 'P' and 'E' equations (2.7b)/(2.7a) to eliminate the respective variables from the action:

$$S\Big[X, P^M = E^{-1}\dot{X}^M, E = m^{-1}\sqrt{-\dot{X}\cdot\dot{X}}\Big] = -m\int d\tau\sqrt{-\dot{X}\cdot\dot{X}}, \tag{2.8}$$

as in our starting point (2.1). So, we see that the first order action (2.6) is equivalent to the geometric action (2.1). As an intermediate step we could use just one of the equations and leave E as an independent field leading to the so-called Polyakov type particle action (2.9):

$$S[X, P = E^{-1}\dot{X}, E] = \frac{1}{2}\int d\tau\Big\{E^{-1}\dot{X}\cdot\dot{X} - Em^2\Big\} =: S[X, E], \tag{2.9}$$

which is very useful in many applications, as the massless case is now well-defined:

$$S_{ml}[X, E] = \frac{1}{2}\int d^D x E^{-1}\dot{X}\cdot\dot{X}. \tag{2.10}$$

Let us again see what the invariances of the first-order equation are. It is again invariant under reparametrizations and Poincaré transformations, where the fields transform in the following way

$$\delta X^M = \xi\dot{X}^M + \zeta^M - \lambda^M_{\ N}X^N, \tag{2.11a}$$
$$\delta P^M = \xi\dot{P}^M - \lambda^M_{\ N}P^N, \tag{2.11b}$$
$$\delta E = \dot{\xi}E + \xi\dot{E}. \tag{2.11c}$$

The way we have formulated relativistic particle dynamics here, it is very straightforward to couple the particle to a nontrivial background geometry, that is gravity. We simply replace

$$A\cdot B = \eta_{MN}A^M B^N \longmapsto g_{MN}A^M B^N \equiv A\cdot B, \tag{2.12}$$

so, we can leave all the expressions as they are with the dot properly understood. For general backgrounds, we will not have the Poincaré symmetry. The action (2.9), coupled to an arbitrary background is an example of what is known as a non-linear sigma model having many applications in string theory and condensed matter physics. In this work we will introduce a form of nonrelativistic non-linear sigma model by coupling the nonrelativistic particle to arbitrary background geometries.

2.2 NONRELATIVISTIC

In this section we construct an action for a nonrelativistic particle prepared to be coupled to arbitrary background geometries, following the procedures of the preceding chapter. Ultimately, we want to learn something about the background geometry itself. This will be the topic of chapters-to-come. To distinguish relativistic from nonrelativistic we use lower-case letters, see appendix A.

It is straightforward to find an action giving the Newtonian equations of motion for a free particle $\ddot{\mathbf{x}} = 0$. We will, however not do that but follow a more top-down approach to put the heavy machinery of the last section at work. Going back to what we called first order action, we see that the relativistic dispersion relation $P\cdot P + m^2 = 0$ can easily be replaced by the nonrelativistic analogue $\delta_{ij}\mathbf{p}^i\mathbf{p}^j - 2mp^0 = 0$ to yield the equivalent non-relativistic particle action

$$S[x, p, e] = \int d\tau\Big\{ - \dot{x}^0 p^0 + \delta_{ij}\dot{\mathbf{x}}^i\mathbf{p}^j - \frac{1}{2}e\Big(\delta_{ij}\mathbf{p}^i\mathbf{p}^j - 2mp^0\Big)\Big\}, \tag{2.13}$$

from which we derive the following equations of motion/constraints

$$x^0: \quad \dot{p}_0 = 0, \qquad \mathbf{x}^i: \quad \dot{\mathbf{p}}^i = 0, \tag{2.14a}$$
$$e: \quad p^0 = (2m)^{-1}\delta_{ij}\mathbf{p}^i\mathbf{p}^j, \tag{2.14b}$$
$$p^0: \quad \dot{x}^0 = em, \qquad \mathbf{p}^i: \quad \dot{\mathbf{x}}^i = e\mathbf{p}^i. \tag{2.14c}$$

Again we can partially use these to find the following action

$$S[x, p^0 = (2m)^{-1}\mathbf{p}^i\mathbf{p}_i, \mathbf{p}^i = m(\dot{x}^0)^{-1}\dot{\mathbf{x}}^i, e = m^{-1}\dot{x}^0] = m\int d\tau(\dot{x}^0)^{-1}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} =: S[x]. \tag{2.15}$$

The action is invariant under reparametrizations $\delta_\xi \tau = \xi(\tau)$ and

$$\delta_\xi x^\mu = \xi \dot{x}^\mu, \tag{2.16}$$

where again we can fix some of this gauge freedom by e.g. $(\dot{x}^0)^{-1} = 2^{-1}$ and thereby obtaining the second term in the $\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}$-expansion in eq. (2.3). The interpretation is that of kinetic energy. Let us now try to determine the spacetime symmetries of the nonrelativistic action (2.15). The expression $\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = \delta_{ij}\dot{\mathbf{x}}^i\dot{\mathbf{x}}^j$ is manifestly symmetric under rotations and spatial translations $\delta(\zeta,\lambda)x^i = \zeta^i - \lambda^i{}_j x^j$. Also, we have the trivial time translation symmetry $\delta(\zeta)x^0 = \zeta^0$. More interestingly the action is also invariant under nonrelativistic boosts $\delta(\lambda^i)x^i = -\lambda^i x^0$ under which

$$\delta(\lambda^i)S[x] = \int d\tau \frac{d}{d\tau}\Big\{-2m\lambda^i x_i\Big\} = 0. \tag{2.17}$$

Contrary to relativistic physics, the boost symmetry is not manifest. The action is, but the Lagrangian transforms with a total derivative. We will comment on this fact shortly, but first here is the full set of symmetry transformations

$$\delta x^0 = \xi\dot{x}^0 + \zeta^0, \tag{2.18a}$$
$$\delta x^i = \xi\dot{x}^i + \zeta^i - \lambda^i{}_j x^j - \lambda^i x^0. \tag{2.18b}$$

The fact that the nonrelativistic particle Lagrangian transforms with a total derivative $\delta\mathcal{L} = \dot{\Theta}$ under boosts alters the form of the associated Noether charges $Q_T = -\pi_\mu \delta_T x^\mu + \Theta$ slightly:

$$\text{time translations } H: \qquad Q_H = -\pi_0\zeta^0, \tag{2.19a}$$
$$\text{spatial translations } P: \qquad Q_P = -\pi_i\zeta^i, \tag{2.19b}$$
$$\text{rotations } J: \qquad Q_J = \pi_i\lambda^i{}_j x^j, \tag{2.19c}$$
$$\text{Galilean boosts } G: \qquad Q_G = \pi_i\lambda^i x^0 - 2m\lambda_i x^i. \tag{2.19d}$$

These generate transformations via $\delta_T x^\mu = -\{Q_T, x^\mu\}_{PB}$, but more importantly it allows us to consider consecutive actions of the symmetries via

$$\{Q_R, Q_S\}_{PB} = f^T_{RS}Q_T, \tag{2.20}$$

which has the structure of a Lie algebra with structure constants f^T_{RS}. Indeed, it is the symmetry algebra of nonrelativistic – that is Galilean – relativity: the Galilei algebra. In fact it is a little bit more, namely the centrally extended version thereof – the Bargmann algebra, for which translations and Galilean boosts commute into a central charge transformation. This crucial difference is a consequence of the nontrivial term proportional to m in Q_G, coming from the total derivative in the variation of the Lagrangian:

$$\{Q_G, Q_P\}_{PB} = 2m\lambda^i\zeta^j\{x_i, p_j\}_{PB} - \lambda^i\zeta^j\{p_i x^0, p_j\}_{PB} = 2m\lambda_i\zeta^i. \tag{2.21}$$

So translations and nonrelativistic boost commute only up to a central charge that we will eventually call M, when going to abstract Lie algebras, see table 3.2. Already at this stage it is apparent that it relates to the concept of mass (or particle number) conservation in nonrelativistic physics.

2.2.1 Lightcone Embedding – Null Reduction

We may ask ourselves whether we can change the action (2.15) so that the Lagrangian is itself invariant under nonrelativistic boosts. Note that we can always add a total derivative to the Lagrangian $\mathcal{L} \mapsto \mathcal{L} + \dot{z}(x,\dot{x})$ without changing the dynamics of the theory. To cancel the unwanted term it must transform as

$$\delta z = \zeta^z + 2m\lambda^i x_i, \tag{2.22}$$

and invariance under the other spacetime symmetries. Since the transformation rule follows from those of x, this imposes a set of differential conditions on $z(x,\dot{x})$ which cannot be solved (see [17]).

There is, however, an alternative to that approach namely to interpret z not as a functional term added to Lagrangian but as an extra coordinate added to the spacetime, with spacetime symmetry (2.22), see [12, 17]. Hence the action reads

$$S_{ext}[x,z] = m\int d\tau(\dot{x}^0)^{-1}\Big\{\dot{x}^i\dot{x}^j\delta_{ij} + \dot{z}\dot{x}^0\Big\}, \tag{2.23}$$

where one sees that the c-number m is the conjugate momentum to the extra coordinate z. This is remarkably similar to the action of a massless particle in $D+1$ dimensions, represented in lightcone coordinates.

Let us take this similarity serious for a moment and consider the action of the massless particle (2.10). Equivalently we could consider massive particles at infinite momentum [18, 19]. Restricting movement to one of the lightlike directions induces an emergent Galilei structure. Remarkably, it is the high energy limit where some of the nonrelativistic intuition holds. A similar analysis can be done on the symmetry algebra level, by embedding the Bargmann algebra in D dimensions as a subalgebra in the Poincaré algebra in $(D+1)$ dimensions, see appendix C and [20, 21]. What is remarkable about this kind of embedding is that they do not involve taking limits $c\to\infty$, unlike so-called Inönü-Wigner contraction that we will meet later on.

$$\begin{aligned} S_{ml}[X] &= \int d\tau E^{-1}\dot{X}^M\dot{X}^N\eta_{MN}, \\ &= \int d\tau E^{-1}\Big\{\dot{X}^i\dot{X}^j\delta_{ij} + \dot{X}^+\dot{X}^- + \dot{X}^-\dot{X}^+\Big\}, \end{aligned} \tag{2.24}$$

meaning that the metric in light cone coordinates reads

$$(\eta_{MN}) = \begin{pmatrix} \delta_{ij} & 0 & \\ & 0 & 1 \\ 0 & & \\ & 1 & 0 \end{pmatrix} = \left(\mathbb{1}_{D-1} \oplus \sigma_1\right).$$

Relating this to the action (2.23) we have to make the following field identifications

$$X^i = x^i, \quad E = m^{-1}\dot{X}^- = m^{-1}\dot{x}^0, \quad X^+ = 2^{-1}z.$$

Note how the second constraint $mE = \dot{X}^- = \dot{x}^0$ breaks some of the relativistic symmetry, by singling out one direction of spacetime. It is consistent with the fact that that the mass m is the canonical momentum of the extra coordinate X^+. This is how the genuinely nonrelativistic effect of mass conservation is obtained. This formalism is reflected by what is called Null-Reduction, see chapter 5. A careful analysis of the symmetries, taking into account $mE = \dot{X}^-$ yields the Bargmann algebra again. See appendix C.

An action that correctly reproduces the coupling of a particle to a gravitational potential $\phi(x)$ is the following

$$S[x,\phi] = \frac{m}{2}\int d\tau \Big\{ (\dot{x}^0)^{-1}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - 2\phi(x)\dot{x}^0 \Big\}, \tag{2.25}$$

yielding $\ddot{\mathbf{x}}^i = -\delta^{ij}\partial_j\phi$, when gauge fixing $x^0 = \tau$. We know that the field ϕ itself obeys a Poisson equation $\delta^{ij}\partial_i\partial_j\phi = 4\pi G_N\rho$. This equation will look different in different coordinate systems. A covariant reformulation of the Poisson equation will naturally lead to a geometric understanding of nonrelativistic gravity. This is the content of Newton Cartan gravity. It is then consistent to interpret it as a geometric theory of gravity, just as you would do for General Relativity. Also the coupling to nonrelativistic gravity should be understood as a coupling to nonrelativistic geometry. In chapter 5, we will derive a coordinate invariant version of eq. (2.25) seen as minimal coupling to an arbitrary NC background geometry $(\tau_\mu, h_{\mu\nu}, m_\mu)$

$$S[x,h,\tau,m] = m\int d\tau \Big[\Big(\tau_\mu \dot{x}^\mu\Big)^{-1} \dot{x}^\mu\dot{x}^\nu h_{\mu\nu} - 2m_\mu\dot{x}^\mu \Big].$$

We will clarify the role of the fields later on, but we would like to stress the role of the m_μ. It is associated to the previously introduced central charge M. Its term has the form of an electromagnetic field coupled to a charged particle. The field m_μ is thus analogous to the photon field coupling to mass instead of electric charge. More generally, coupling general field theories to nonrelativisitic background geometry is indeed widely used in applications of NC [5].

CHAPTER 3

GEOMETRY FROM SYMMETRY

3.1 GR FROM **POIN**

The purpose of this section is twofold. On the one hand we want to introduce some of the basics of General Relativity from a specific point of view, apt for the following sections. This will involve some elements of Poincaré representation theory and Poincaré gauge theory and the Vielbein and Palatini formalism. It is this formulation of GR that carries over to nonrelativistic geometry best. The literature is immense, we adhere most closely to [22] and [23].

On the other hand, we will want to consider a supersymmetric theory of gravity called supergravity (SUGRA). Even more than for nonrelativistic geometry, it is imperative to formulate GR in a Palatini form. Only then can we describe spinors in curved spacetimes. The treatment in chapter 4 will follow the strategies outlined in the present chapter.

In section 3.1.1 we start by introducing General Relativity from a geometric perspective. This will involve introducing local bases (Vielbeine) allowing to describe fermionic objects in curved spacetime. That notion of local bases will carry over to the nonrelativistic case. In section 3.1.2 we consider flat Minkowski space and its isometry group **Poin**. The structure is exemplary not only for nonrelativistic isometries but also for supersymmetric generalizations thereof. There is, however, a third reason to consider **Poin** and its Lie algebra **poin**. In section 3.1.4 we construct the kinematics of GR by a so-called gauging procedure. Again, this is exemplary for both the nonrelativistic and supersymmetric deformations thereof, as done in section 3.2 and chapter 4.

3.1.1 About GR

3.1.1.1 The Vielbein Formalism

It is common to introduce General Relativity in terms of a metric tensor g. As such it has an immediate interpretation as a theory of geometry – geometrodynamics [3]. All the objects appearing such as the Levi-Civita connection and the Riemann curvature tensors are functions of it and can explicitly be expressed as $\Gamma(g)$ and $R(\Gamma(g))$. While this is close to an intuitive and geometric understanding of GR, there is a different formulation where the metric is no longer the fundamental variable. Roughly speaking, we take the square root E as the fundamental object:

$$E^2 \sim g.$$

Actually this reformulation makes explicit the all-encompassing foundation of General Relativity - the equivalence principle. Spacetime is a Lorentzian manifold, as such it is locally indistinguishable from Minkowski spacetime. So, the metric is locally always that of Minkowski spacetime meaning that there exists an orthonormal basis $\{E_A(p)\} = \{E_0(p), E_1(p), E_2(p), \cdots, E_{D-1}(p)\}$ obeying

$$g(E_A, E_B)(p) = \eta_{AB} = \mathrm{diag}(-1, +1, \cdots, +1). \tag{3.1}$$

Note that on the left-hand side, both the metric and the local basis depend on p, whereas the right hand side is just a constant. It is worth mentioning that it is not always possible to introduce these vector fields smoothly, one prominent counterexample with euclidean signature is the 2-sphere. Nevertheless we will for the most part not be concerned with global issues. As $\{E_A\}$ are vector fields, we can also expand them in a coordinate basis $\{\partial_M\}$:

$$E_A = E_A^{\ M}\partial_M, \tag{3.2}$$

and we rewrite eq. (3.1) as $g_{MN}E_A^{\ M}E_B^{\ N} = \eta_{AB}$. Hence, the component matrices $(E_A^{\ M})$ can be seen as local similarity transformations taking $g \to \eta$. To this point we have just introduced D vector fields, establishing a local, orthonormal basis at every point of the manifold. It is possible to introduce a dual basis $\{E^A\}$ at every point, obeying $E^A(E_B)(p) = \delta^A_{\ B}$. Similar to the above, we expand the 1-forms in a coordinate basis $E^A = E_M^{\ A}dx^M$ yielding $E_A^{\ M}E_M^{\ B} = \delta_A^{\ B}$ and $E_M^{\ A}E_A^{\ N} = \delta_M^{\ N}$. The objects $\{E^A\}$ are called frame fields or Vielbeine*. Correspondingly the objects $\{E_A\}$ are called inverse frame fields and inverse Vielbeine. While the name frame field is closest to a proper, geometric interpretation, for some reason Vielbein is the name most used in the physics literature, where it is even common to also call the square matrices $(E_M^{\ A})$ Vielbeine. We will mostly stick to that same abuse of notation. With that, we can also derive that

$$g_{MN}(p) = \eta_{AB}E_M^{\ A}(p)E_N^{\ B}(p). \tag{3.3}$$

Schematically it has the form $g \sim E^2$. This is the reason why people like to make the somewhat delphic statement: Vielbeine are the square root of the metric. We can ask ourselves whether it is possible to understand the metric as a derived object and reformulate all of GR in terms of the Vielbein. Indeed, it is possible and is exactly what we are going to do in section 3.1.1.5.

We have introduced two kinds of indices. The ‘early’ Latin indices $(A, B, \cdots)$ we will call flat indices, while the ‘middle’ capital Latin and Greek ones $(M, N, \cdots)$ we will call curved (see appendix A). Note that any vector can be expressed as

$$V = V^A E_A = V^A E_A^{\ M}\partial_M = V^M\partial_M \quad \Rightarrow \quad V^M = V^A e_A^{\ M} \ / \ V^A = V^M E_M^{\ A}. \tag{3.4}$$

Similarly for 1-forms we find

$$\omega = \omega_A E^A = \omega_A E_M^{\ A}dx^M = \omega_M dx^M \quad \Rightarrow \quad \omega_M = \omega_A E_M^{\ A} \ / \ \omega_A = \omega_M E_A^{\ M}. \tag{3.5}$$

In some sense the Vielbeine turn flat into curved and curved into flat indices. More mathematically they induce an isomorphism between the tangent spaces and flat Minkowski space.

We defined the Vielbein as a local similarity transformation, it is however not clear how unique this choice really is. Indeed, there is some redundancy in the definition. We can redefine the Vielbein by a local rotation $\left(\Lambda^A_{\ B}\right) \in SO(1, D-1)$, such that

$$E'^{\ A}_M = \Lambda^A_{\ B}E_M^{\ B}, \tag{3.6}$$

meaning that physically $E' \cong E$ since

$$\text{if} \quad E^2 = g \quad \text{then also} \quad E'^2 = g,$$

as can be seen from eq. (3.1) and the definition of $SO(1, D-1)$. In fact, there must be some redundancy, as the metric is a symmetric tensor, whereas the Vielbein matrix is unconstrained. In a general dimension D, there is a mismatch of $D^2 - 2^{-1}D(D+1) = 2^{-1}D(D-1)$ components. This corresponds exactly to the $2^{-1}D(D-1)$ components in $\Lambda^A_{\ B}$. Apparently by going to the Vielbein formulation we introduced more variables than we really needed. Hence, the two Vielbeine in eq. (3.6) should really be considered physically equivalent. Any construction of GR in terms of a Vielbein will have to take into account this redundancy and be formulated in a way that is covariant under $SO(1, D-1)$.

*This is a german word, in dimensions 1-4 people also use Einbein, Zweibein, Dreibein and Vierbein. The correct german plural is Vielbeine, rather than Vielbeins.

3.1.1.2 Symmetries of the Vielbein

In this section we gather results about the symmetries of the theory. Here and in what follows we restrict ourselves to infinitesimal transformations. While not distorting the relevant symmetry structure this simplification is of great practical use. The final result eq. (3.11) will serve as a starting point for many of the deformations and generalizations in the later chapters.

General Relativity is a theory that is invariant under general coordinate transformations (GCT). All the physical quantities are represented by tensors, such that the form of the equations of motion does not depend on the coordinate system. If all the physical quantities are represented by geometric objects, the theory is guaranteed to be manifestly covariant. Hence we will use the words invariance under general coordinate transformations, general covariance and diffeomorphism (diffeo) invariance interchangeably.

However, as we have introduced the Vielbein as the fundamental object of the gravitational theory and with it another redundancy in the description, there is really two transformations the physical equations of motion have to respect. Any object must be a tensor with respect to diffeomorphisms and local Lorentz transformations. In particular, the indices will help to avoid confusion, see appendix A. Compare [22], [23] and [24] for clear expositions.

Let us consider infinitesimal coordinate transformations $x^M \mapsto x^M + \delta x^M = x^M + \xi^M$, neglecting $\mathcal{O}\Big[(\xi^M)^2\Big]$. A (p,q)-tensor T transforms as

$$\delta_\xi T \equiv T'(x) - T(x) = \mathcal{L}_\xi T(x), \tag{3.7}$$

where $\mathcal{L}_\xi$ is the Lie derivative, defined as:

$$\begin{aligned}
\mathcal{L}_\xi T_{M_1\cdots M_p}{}^{N_1\cdots N_q} &= \xi^P \partial_P T_{M_1\cdots M_p}{}^{N_1\cdots N_q} \\
&\quad + \Big(\partial_{M_1}\xi^P\Big) T_{PM_2\cdots M_p}{}^{N_1\cdots N_q} + \Big(\partial_{M_2}\xi^P\Big) T_{M_1P\cdots M_p}{}^{N_1\cdots N_q} + \cdots \\
&\quad - \Big(\partial_P \xi^{N_1}\Big) T_{M_1\cdots M_p}{}^{PN_2\cdots N_q} - \Big(\partial_P \xi^{N_2}\Big) T_{M_1\cdots M_p}{}^{N_1 P\cdots N_q} - \cdots
\end{aligned} \tag{3.8}$$

It is worth mentioning that the Lie derivative acts only on the spacetime (curved) indices. The Vielbein E_M^A is a $(1,0)$ and not a $(1,1)$ tensor under diffeomorphisms:

$$\delta_\xi E_M^A = \mathcal{L}_\xi E_M^A = \xi^P \partial_P E_M^A + \Big(\partial_M \xi^P\Big) E_P^A. \tag{3.9}$$

Along similar lines we can introduce small local Lorentz transformations $\Lambda^A{}_B = \delta^A{}_B - \lambda^A{}_B$. The Vielbein has one upper index and thus looks like a vector to local Lorentz transformations, hence

$$\delta(\lambda) E_M^A = (E')_M^A - E_M^A = -\lambda^A{}_B E_M^B. \tag{3.10}$$

To summarize, instead of working with a metric, we can formulate GR in terms of Vielbeine as defined in (3.1). This has to be done at the expense of introducing local Lorentz transformations such that the full transformation under GCT and local Lorentz transformations is given by

$$\begin{aligned}
\delta E_M^A &= \mathcal{L}_\xi E_M^A + \delta(\lambda) E_M^A \\
&= \xi^P \partial_P E_M^A + \Big(\partial_M \xi^P\Big) E_P^A - \lambda^A{}_B E_M^B.
\end{aligned} \tag{3.11}$$

3.1.1.3 Covariant Quantities

Considering the objects E^A we observe that they are not just a basis of some vector space but also 1-forms. Hence, we have a natural differentiable structure: the exterior derivative $dE^A = (\partial_M E_N{}^A)dx^M \wedge dx^N$. This object is a 2-form and transforms properly under general coordinate transformations, it is not a tensor under local Lorentz transformations though. It transforms with an inhomogeneous term $\delta dE^A = -(d\lambda^A{}_B)E^B$. We introduce another 1-form $\Omega^A{}_B$ to cancel the unwanted terms:

$$dE^A + \Omega^A{}_B \wedge E^B \equiv DE^A \equiv T^A. \tag{3.12}$$

We have introduced two new objects, a 1-form Ω and a 2-form T. The latter is called torsion, we will see how it relates to the well-known notion of torsion appearing in the metric formulation. It is clear that the 1-form Ω transforms with a derivative of the parameter, such that DE^A transforms in the same way under full local Lorentz transformations as does E^A

$$\delta E^A = -\lambda^A{}_B E^B, \tag{3.13a}$$

$$\delta\Omega_{AB} = d\lambda^A{}_B + 2\lambda^{[A}{}_C\Omega^{B]C}, \tag{3.13b}$$

$$\delta DE^A = \delta T^A = -\lambda^A{}_B T^B. \tag{3.13c}$$

Now, the object $\Omega^A{}_B$ is the gauge field of local Lorentz transformations, as such we require it to be antisymmetric in the flat indices $\Omega^{AB} = -\Omega^{BA} = \eta^{BB'}\Omega^A{}_{B'}$. In the physics literature this is called spin connection. It is not obvious, but we can construct another tensor from this object by imitating the structure of eq. (3.12):

$$d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B \equiv D\Omega^A{}_B \equiv R^A{}_B. \tag{3.14}$$

It is straightforward to show that under local Lorentz transformations

$$\delta R^{AB} = 2\lambda^{[A}{}_C R^{B]C}, \tag{3.15}$$

Equations (3.12) and (3.14) are known as Cartan's first and second structure equation, respectively.

Hence we have four objects: $(E^A, \Omega^A{}_B, T^A, R^A{}_B)$, two of which are 1-forms, two are 2-forms and similarly (E, T, R) are tensors under local Lorentz transformations, while Ω is not. Apparently, $R^A{}_B = R^A{}_B(\Omega)$. The spin connection can be expressed in terms of Vielbein and torsion $\Omega = \Omega(E, T)$. It can then be shown that two of the flat indices can be turned into curved indices in the curvature tensor $R_{MN}{}^A{}_B E_{AP} E_Q{}^B = R_{MNPQ}$. From this point of view the whole Vielbein formalism is useful for explicit calculations of the curvature tensor. The procedure would be something like: read off the Vielbein from eq. (3.3), calculate its exterior derivative and determine the spin connection from this, then calculate the Riemann tensor via eq. (3.14)[23, pp. 49].

There are however more reasons to consider the Vielbein formulation, indeed for us it is indispensable. First of all it unveils the relation to gauge theory, as we will see in the upcoming chapters. Secondly, whenever we want to define spinors on curved spacetimes, we have to introduce representation theory of the rotation (or Lorentz) group. The diffeomorphism group of the metric formulation has no spinorial representations. By introducing the redundant Vielbein formalism we effectively make space for a definition of spinors. Thus, whenever we want to consider fermions in curved spacetime we have to introduce the Vielbein formalism. In particular, supergravity requires the Vielbein formalism.

So far we have constructed one notion of covariant derivative in (3.12). It is defined only on p-forms, hence it is sometimes called exterior covariant derivative

$$D = d + \Omega(E,T)\wedge, \tag{3.16}$$

where we have suppressed all the indices. Any $(p{+}1)$-form $D\alpha^{AB\cdots C}$ transforms as $\delta D\alpha^{AB\cdots C} = \mathcal{L}_\xi D\alpha^{AB\cdots C} - \lambda^A{}_{A'}\alpha^{A'B\cdots C} - \cdots - \lambda^C{}_{C'}\alpha^{AB\cdots C'}$. With indices eq.(3.12) becomes

$$D_{[M}E^A_{N]} = \partial_{[M}E^A_{N]} + \Omega^{AB}_{[M}(E,T)E_{N]B}, \tag{3.17}$$

such that $(DE)^A_{MN} = 2D_{[M}E^A_{N]}$.

It is possible to construct a more general covariant derivative D_M that maps tensors into tensors. It must be said that objects can be tensors with respect to spacetime and with respect to the tangent space. This is reflected in the curved and flat index structure. Let us first assume a spacetime scalar – that is, an object with just flat indices. Then $D_M T^{AB\cdots C}$ transforms under local Lorentz transformations in the same way as $T^{AB\cdots C}$ itself:

$$D_M T^{AB\cdots C} = \partial_M T^{AB\cdots C} + \Omega_M^{AA'}T_{A'}{}^{B\cdots C} + \Omega_M^{BB'}T^A{}_{B'}{}^{\cdots C} + \cdots\,. \tag{3.18}$$

It is not yet clear how D_M acts on spacetime tensors*, that is objects with curved indices. It must be such that $D_M T$ transforms as a $(p+1,q)$ spacetime tensor under diffeomorphisms (3.8) if T is a (p,q) tensor. We can, however, map all spacetime tensors to tangent space tensors by means of the Vielbein. In other words, by turning (p,q) curved into $(p+q)$ flat indices. It is thus sufficient to know how the covariant derivative D_M acts on the Vielbein, with the usual Levi-Civita connection Γ

$$D_M E_N{}^A = \partial_M E_N{}^A + \Omega_M^{AB}E_{NB} - \Gamma^P_{MN}E_P{}^A, \tag{3.19}$$

by definition we take $\Gamma^P_{MN} = \Gamma^P_{NM}$, such that indeed $2D_{[M}E^A_{N]} = (DE)^A_{MN}$. Similar to metric compatibility we require the so-called Vielbein postulate

$$D_M E_N{}^A = 0, \tag{3.20}$$

from which also follows that $D_M E_A{}^N = 0$. Note that this allows one to solve for the Levi-Civita connection $\Gamma = \Gamma(E,\Omega(E,T))$.

$$\Gamma^P_{MN}(E) = E_A{}^P\Big(\partial_M E_N{}^A + \Omega_M^{AB}E_{NB}\Big). \tag{3.21}$$

It is then straightforward to determine the action on e.g. vectors V^M

$$D_M V^N = E_A{}^N D_M V^A - \Gamma^N_{MP}(E)V^P \tag{3.22}$$

It should be clear how to generalize this to general tensors with possibly mixed index structure. Also, one can introduce curvatures by

$$[D_M, D_N]\phi = 0, \tag{3.23a}$$
$$[D_M, D_N]V^A = R_{MN}{}^A{}_B V^B, \tag{3.23b}$$
$$[D_M, D_N]\alpha_A = -R_{MN}{}^B{}_A\alpha_B, \tag{3.23c}$$

*It should be mentioned that some authors (e.g. [22]) introduce a separate symbol ∇_M for the full covariant derivative. We refrain from doing so, because in this work all the covariant derivatives either act on spacetime scalars or appear as exterior covariant derivative.

with $R_{MN}{}^{A}{}_{B}$ being that of eq. (3.14), for more see [22, pp. 162].

3.1.1.4 Spin Connection

Eq. (3.12) can be solved for $\Omega = \Omega(E,T)$, the calculation essentially consists of writing out all the indices explicitly and permuting them. It is useful to introduce the following symbol, called anholonomy coefficients

$$C_{MN}^{\ \ A} \equiv 2\partial_{[M}E_{N]}^{\ A}, \tag{3.24}$$

such that eq. (3.12) reads

$$\Big(\underbrace{C_{MN}^{\ \ A} - T_{MN}^{\ \ A}}_{\tilde{C}_{MN}^{\ \ A}(E,T)}\Big) + 2\Omega_{[M}^{\ AB}(E,T)E_{N]B} = 0 \tag{3.25}$$

and by permuting $\sum_{(MNP)}\Big\{\tilde{C}_{MNP} + \Omega_{MPN} - \Omega_{NPM}\Big\} = 0$ we find

$$\begin{aligned}
\Omega_M^{\ AB}(E,T) =& \Omega_M^{\ AB}(E) + \Omega_M^{\ AB}(T) = \\
=& \Big(E^{AN}\partial_{[M}E_{N]}^{\ B} - E^{BN}\partial_{[M}E_{N]}^{\ A} - E_{MC}E^{AN}E^{BP}\partial_{[N}E_{P]}^{\ C}\Big) \\
& - \frac{1}{2}\Big(E^{AN}T_{MN}^{\ \ B} - E^{NB}T_{MN}^{\ \ A} - E_{MC}E^{AN}E^{BP}T_{NP}^{\ \ C}\Big) \\
=& \tilde{C}_M^{\ [AB]}(E,T) - \frac{1}{2}\tilde{C}^{[AB]}{}_M(E,T)
\end{aligned} \tag{3.26}$$

When coming to supergravity we will see an explicit expression for T.

3.1.1.5 Palatini or First Order Formalism

After having introduced the kinematics and symmetries of General Relativity, it is now time to introduce dynamics. Note that at this point we have found essentially four objects, a Vielbein, a connection, a torsion and a curvature, of which just the Vielbein and the torsion are independent. For the moment we will assume zero torsion, as is in General Relativity, though we will have to relax this condition when going to supergravity. Hence General Relativity depends on the Vielbein only,

$$\Big(E_M^{\ A}, \Omega_M^{\ AB}(E), R_{MN}{}^{AB}(\Omega(E))\Big). \tag{3.27}$$

Giving dynamics to this set of variables means imposing either an action or equations of motion*, in this case the Einstein-Hilbert action:

$$S_{EH}[E,\Omega(E)] = \int d^Dx|E|E_A^{\ M}E_B^{\ N}R_{MN}{}^{AB}(\Omega(E)), \tag{3.28}$$

where $|E| = |\det(E_M^{\ A})|$. It is important to note that by itself, the Riemann curvature depends just on the connection.

Varying the action yields two contributions, one from the Vielbeine, one from the determinant $\delta|E| = -|E|E_M^{\ A}\delta E_A^{\ M}$

*Note, that later in this thesis it will not even be possible to introduce an action.

$$\delta S_{EH}[E,\Omega(E)] = 2\int d^Dx|E|\Big(\underbrace{R_{MN}{}^{AB}E_B{}^N - \frac{1}{2}E_M{}^A R}_{\equiv G_M^A}\Big)\delta E_A{}^M, \tag{3.29}$$

here $R = R_{MN}{}^{AB}E_A{}^M E_B{}^N$ is the Ricci scalar. Hence the vacuum Einstein equations $G_M^A = 0$ follow. A detailed discussion on how the variation works in detail can be found in any book on GR, we follow especially [23] and [22].

We have imposed a condition (eq. (3.12)) on the connection, rendering it a dependent field in the theory. Note, that alternatively we could also consider the action (3.28) and take the spin connection Ω as an independent field

$$S_P[E,\Omega] = \int d^Dx|E|E_A{}^M E_B{}^N R_{MN}{}^{AB}(\Omega). \tag{3.30}$$

This is called Palatini action. Conversely there are two sets of equations of motion. Variation gives

$$\delta S_P[E,\Omega] = 2\int d^Dx|E|\Big[G_M^A\delta E_A{}^M + 3\Big((D_N E_P{}^A)E_{[A}^N E_B{}^P E_{C]}^M\Big)\delta\Omega_M{}^{BC}\Big], \tag{3.31}$$

yielding

$$\Omega: \quad \partial_{[M}E_{N]}^A + \Omega_{[M}^{AB}E_{N]B} = 0, \tag{3.32a}$$

$$E: \quad G_M^A = R_{MN}{}^{AB}E_B{}^N - \frac{1}{2}E_M^A R = 0. \tag{3.32b}$$

The second is the vacuum Einstein field equations We get one more equation, though, which turns out to be nothing but the constraint eq. (3.12). To conclude: generalizing the Hilbert-Einstein action to the Palatini or first-order action for gravity, we get two equations of motion, one of which we can solve explicitly to obtain $\Omega(E)$. In this uncoupled, torsionless gravity theory we find that both formulations are equivalent:

$$S_P[E,\Omega]|_{\Omega=\Omega(E)} = S_{EH}[E,\Omega(E)], \tag{3.33}$$

where $\Omega = \Omega(E)$ is an explicit solution to eq. (3.32a), given by expression (3.26).

3.1.2 About **poin**

In this section we go back to flat space. We develop some technical tools to describe the isometry structure of Minkowski spacetime. Since there is no curvature, there is no need to distinguish between flat and curved indices. For convenience we will use the flat $(A, B, C, \cdots)$ ones. In principle we could allow for arbitrary signature, but we will only consider $(1, D-1)$-dimensional Minkowski space $\mathbb{M}^{1,D-1}$, see appendix A.

The symmetries of this space are represented by what is known as the Poincaré group **Poin**. We will discuss some basic notions of group and representation theory. This will help us to deform General Relativity to Newton-Cartan, supergravity and ultimately supersymmetric Newton-Cartan. The basic idea is to do the deformation at the level of flat space symmetries and then go back to a gravitational theory in curved space. Roughly speaking, we make the isometries local by translating the flat space symmetry to a tangent space symmetry. More on this so-called gauging of spacetime symmetries in section 3.1.4.

Most of this section is standard and mainly to be seen as a statement of results and starting point for what is to follow. The presentation adheres to [22], in particular in terms of conventions. Other useful references are [25–28]

3.1.2.1 The Group, its Action and its Algebra

The Poincaré group **Poin** is the isometry group of Minkowski spacetime. As such it acts on points $x \in \mathbb{M}^{1,D-1}$, such that $\eta_{AB}dx^A dx^B = \eta_{AB}dx'^A dx'^B$. This is left invariant by Lorentz transformations $(\Lambda^A{}_B) \in SO(1, D-1)$ and constant translations $(a^A) \in \mathbb{R}^D$. The action on points in $\mathbb{M}^{1,D-1}$ reads

$$x^A \longmapsto (\Lambda^{-1})^A{}_B \Big(x^B - a^B \Big) = x'^A, \tag{3.34}$$

where $\eta_{A'B'}\Lambda^{A'}{}_A\Lambda^{B'}{}_B = \eta_{AB}$. **Poin** is a Lie group consisting of elements (Λ, a) with the following law of composition

$$(\Lambda', a')(\Lambda, a) = (\Lambda'\Lambda, \Lambda' a + a'). \tag{3.35}$$

Hence Lorentz transformations and translations do not act independently of each other, rather they interfere. This kind of behaviour is known as a semidirect product of groups: **Poin** $= \mathrm{Iso}(\mathbb{M}^{1,D-1}) = SO(1, D-1) \ltimes \mathbb{R}^D$.

Typically we will restrict to infinitesimal transformations*. This means that we expand Lorentz transformations and translation in orders of infinitesimal parameters (λ, ξ). By defining $\delta x^A = x'^A - x^A$:

$$\delta(\lambda, \xi)x^A = -\lambda^A{}_B x^B + \xi^A \tag{3.36}$$

we see that $\lambda_{AB} = \eta_{AA'}\lambda^{A'}{}_B = -\lambda_{BA}$. An immediate consequence of which is, that the Poincaré algebra – and thus the group – is described by $D + 2^{-1}D(D-1) = 2^{-1}D(D+1)$ parameters. In four dimensions this amounts to ten parameters, which we can physically interpret as three translations, rotations and boosts plus one time translation (see appendix C).

Abstractly the infinitesimal action of **Poin** is represented by a Lie algebra **poin**†. Geometrically this is the tangent space to the group manifold at the identity element. This corresponds to the intuitive notion of small transformations close to the identity. The group composition law eq. (3.35) is reflected at the algebra level schematically by

$$[(\lambda, \xi), (\lambda', \xi')] \in (\mathbf{poin}). \tag{3.37}$$

So the commutator of two infinitesimal transformations is again an infinitesimal transformation. This is a nontrivial and fundamental statement in the theory of Lie groups and Lie algebras.

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*In fact, this is an example where we lose some of the structure. Discrete symmetries as time $(t \mapsto -t)$ and space $(\mathbf{x} \mapsto -\mathbf{x})$ reversal belong to **Poin**, can however not be reached by consecutive action of small transformations. We will not consider these and thus be content with the algebra.

†We use lowercase, bold typeface for the corresponding algebra: **poin** is the Lie algebra of **Poin**. Just as **g** is the Lie algebra of **G**

A little detour on Lie algebra generalities and realization of symmetries is due*.
First of all, a Lie algebra is a vector space $\mathbf{g}$. As such it has a basis and thus a notion of dimension. But moreover it has a bilinear, antisymmetric map:

$$[-,-]: \quad \mathbf{g}\times\mathbf{g}\longrightarrow\mathbf{g}. \tag{3.38}$$

This is called the Lie bracket and satisfies the Jacobi identity:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \quad \text{w/ } X,Y,Z\in\mathbf{g}. \tag{3.39}$$

Whenever we encounter Lie algebras they represent some kind of symmetry. They represent symmetry insofar as they represent symmetry groups on an infinitesimal level. That is, understanding Lie algebras allows us to understand how objects change under small deformations. To do so, we need to elaborate how the group and the algebra acts on physical objects. This is the subject of the next section. For example, we will see how Lorentz transformations act differently on scalar fields, vector fields and spinors. This is crucial for all branches of theoretical physics. It allows us to understand supersymmetry and SUGRA in a most efficient way.

Usually we introduce symmetry algebras via a fundamental action – for spacetime symmetries it is the action on spacetime itself, see eq. (3.36). However, often it is important to understand also how the symmetry affects more general objects. In particular we will have to see how **poin** acts on different fields to be able to understand GR and SUGRA from a group theoretic point of view. Let us denote by $\Gamma_r(t_A)$ the representative of some transformation $t_A\in\mathbf{g}$ on the space of fields†. Also, we introduce a parameter ϵ^A. An infinitesimal transformation of some field ϕ by 'ϵ' reads

$$\delta(\epsilon)\phi=\epsilon^A\Gamma_r(t_A)\phi. \tag{3.40}$$

The index 'A' here is completely generic, in fact all that is spelled out here can be applied to both internal and spacetime symmetries. We will see in due time what it actually can be.

As indicated above, the set $\{t_A\}$ forms a basis for the abstract Lie algebra. A lot of the structure not only of the Lie algebra, but also the respective Lie group is captured by the structure constants‡§ ${f_{AB}}^C$

$$\Big[t_A,t_B\Big]={f_{AB}}^C t_C. \tag{3.41}$$

We can imagine having a specific representation, that is a set of matrices ${(t_A)^i}_j$, such that the above is nothing but a matrix equation. Suppose we have a field ϕ^i with one free index, then the Lie algebra will act

$$\Gamma_v(t_A)\phi^i=-{(t_A)^i}_j\phi^j, \tag{3.42}$$

*This detour follows chapter 11 of [22] and [26].

†Here we are digressing a bit from [22]'s notation, they use a capital T_A for the same thing. Even though this makes sense, we want to minimize the change in notation in this work. Hence we stick to [29]'s notation right away. The label on $\Gamma_\square$ signifies the specific form of the representation involved.

‡Remember that we are following the conventions of [22]. In this context they adhere to the mathematics literature, where the elements t_A are anti-hermitian rather than hermitian. The upshot is the fact, that considerably fewer factors of 'i' appear.

§In this work we will try to ignore the global structure as much as possible. For most applications this is legitimate as we can construct all the group elements that are path-connected to the identity element by the exponential mapping: $\mathcal{G}\ni U(\epsilon^A t_A)=\exp(-\epsilon^A t_A)$. Again, here the minus sign is related to our conventions trying to circumvent the appearance of factors of 'i'.

hence we call Γ_v the vector representation. The minus sign has been introduced for later convenience. It is important to distinguish these three objects $\{t_A,\ (t_A)^i{}_j,\ \Gamma_r(t_A)\}$, the notation is kept similar intentionally, though. As mentioned above, most of the particular symmetry structure unveils when we consider the above commutator, the Lie bracket. This allows us to determine the structure constants.

$$\begin{aligned}\delta(\epsilon_1)\delta(\epsilon_2)\phi^i &= \epsilon_1^A\Gamma_v(t_A)\Big(\epsilon_2^B\Gamma_v(t_B)\phi^i\Big)\\ &= \epsilon_1^A\Gamma_v(t_A)\Big(-\epsilon_2^B(t_B)^i{}_j\phi^j\Big)\\ &= -\epsilon_1^A\epsilon_2^B(t_B)^i{}_j\Gamma_v(t_A)\phi^j\\ &= \epsilon_1^A\epsilon_2^B(t_B)^i{}_j(t_A)^j{}_k\phi^k\\ &= \epsilon_1^A\epsilon_2^B(t_Bt_A)^i{}_j\phi^j.\end{aligned}$$

Actually the above is completely general. Applying the transformations in the opposite order yields the commutator:

$$\begin{aligned}\Big[\delta(\epsilon_1),\delta(\epsilon_2)\Big]\phi^i &= \epsilon_1^A\epsilon_2^B\Gamma_v(t_A)\Gamma_v(t_B)\phi^i - \epsilon_2^B\epsilon_1^A\Gamma_v(t_B)\Gamma_v(t_A)\phi^i\\ &= \epsilon_1^A\epsilon_2^B\Big[\Gamma_v(t_A),\Gamma_v(t_B)]\phi^i\\ &= \epsilon_1^A\epsilon_2^B([t_A,\Gamma_v(t_B)]\phi)^i\\ &= -\epsilon_1^A\epsilon_2^B f_{AB}{}^C(t_C\phi)^i\\ &= \epsilon_1^A\epsilon_2^B f_{AB}{}^C\Gamma_v(t_C)\phi^i. \end{aligned} \tag{3.43}$$

Actually, from the first to the second line we made a crucial assumption, that we are going to drop when coming to supersymmetry, namely that the parameters commute $\epsilon_1\epsilon_2 = \epsilon_2\epsilon_1$. Hence, in the remainder of this detour we restrict to bosonic symmetries. The structure of eq. (3.43) can be further abstracted, as the field ϕ^i has been nothing but a passenger:

$$\begin{aligned}[\delta(\epsilon_1),\delta(\epsilon_2)] &= \delta(\epsilon_3 = \epsilon_1^A\epsilon_2^B f_{AB}{}^C)\\ \leftrightarrow\quad [\Gamma_r(t_A),\Gamma_r(t_B)] &= f_{AB}{}^C\Gamma_r(t_C).\end{aligned} \tag{3.44}$$

So the generators on the fields $\Gamma_r(t_A)$ satisfy the same algebra as do the abstract Lie algebra elements t_A. This means that whatever representation on the fields we consider, it must satisfy the symmetry algebra as is given on an abstract level by eq. (3.41). This must be true for any realization of the symmetry. In fact it is possible to guess a particular set of transformation rules on fields, without knowing about the abstract algebra. This is not the way we are going to proceed. Moreover, it is important that all the fields in a multiplet satisfy the same algebra. We will see this at work when coming to supersymmetry.

Following the logic of the general statements in the above we now go back to $\mathbf{g} = \mathbf{poin}$ and consider the commutators $[\delta(\lambda_1,\xi_1),\delta(\lambda_2,\xi_2)]$ in the defining representation given by (3.36):

$$\begin{aligned}[\delta(\lambda_1,\xi_1),\delta(\lambda_2,\xi_2)]x^A &= \delta(\lambda_1,\xi_1)\Big(\delta(\lambda_2,\xi_2)x^A\Big) - (1\leftrightarrow 2)\\ &= -[\lambda_1\lambda_2-\lambda_2\lambda_1]^A{}_Bx^B + [\lambda_1\xi_2-\lambda_2\xi_1]^A\\ &= \delta(\lambda_3 = \lambda_1\lambda_2-\lambda_2\lambda_1, \xi_3 = \lambda_1\xi_2-\lambda_2\xi_1)x^A\end{aligned} \tag{3.45}$$

We see that the action on spacetime closes in the above sense. The underlying Lie algebra **poin** is spanned by the basis $\{P_A, J_{[AB]}\}$. For more general applications, in particular field theory and supergravity, we must understand how it acts on fields. We will denote representations, that is operators implementing the abstract structure, by $\{\Gamma_r(J_{[AB]}), \Gamma_r(P_A)\}$ *. A general infinitesimal transformation reads

$$\delta(\xi, \lambda) = \xi^A \Gamma_r(P_A) - \frac{1}{2}\lambda^{AB}\Gamma_r(J_{[AB]}), \tag{3.46}$$

hinting at eq. (3.40). As a first example, we choose the so-called vector representation v:

$$\Gamma_v(J_{[AB]})^{A'}{}_{B'} = 2\delta^{A'}{}_{[A}\eta_{B]B'}, \tag{3.47}$$

$$\Gamma_v(P_A)^{A'}{}_{B'} = \delta^{A'}{}_{B'}\partial_A, \tag{3.48}$$

note that only ${}^{A'}{}_{B'}$ are matrix indices, constructed such that $\frac{1}{2}\lambda^{AB}\Gamma_v(J_{[AB]})^{A'}{}_{B'}x^{B'} = \lambda^{A'}{}_{B'}x^{B'}$ and $\xi^A\Gamma_v(P_A)^{A'}{}_{B'}x^{B'} = \xi^{A'}$. Since these are matrices, we can compute commutators of the form $[\Gamma_v, \Gamma_v]$ reflecting the abstract **poin** algebra

$$\Big[P_A, P_B\Big] = 0, \tag{3.49a}$$

$$\Big[P_A, J_{[BC]}\Big] = 2\eta_{A[B}P_{C]} = -\Gamma_v(J_{[BC]})^{A'}{}_{A}P_{A'}, \tag{3.49b}$$

$$\Big[J_{[AB]}, J_{[CD]}\Big] = 4\eta_{[A[C}J_{D]B]} = -\Gamma_v(J_{[CD]})^{A'}{}_{A}J_{[A'B]} - \Gamma_v(J_{[CD]})^{B'}{}_{B}J_{[AB']}. \tag{3.49c}$$

The explicit representation Γ_v indeed defines vectors under Lorentz transformations, hence we conclude from (3.49b) that P_A is a (co)vector.

It is useful to read off the structure constants, as given by eq. (3.41), where the indices were generic. To avoid confusion, in spacetime algebras there are always rotation generators with two instead of one index. Hence we should see $[AB]$ as one lump index taking $2^{-1}D(D-1)$ values.

$$f^C_{AB} = 0, \qquad f^D_{A[BC]} = 2\eta_{A[B}\delta^D{}_{C]}, \qquad f^{[EF]}_{[AB][CD]} = 8\eta_{[A[C}\delta^{[E}{}_{D]}\delta^{F]}{}_{B]}. \tag{3.50}$$

These expressions will be of some use later on†. Note that $f^D_{A[BC]} = \Gamma_v(J_{[BC]})^D{}_A$, meaning that the structure constants themselves furnish a representation, called the adjoint representation equivalent to the matric representation. This adjoint representation will be implicitly used in section 3.1.3.

3.1.2.2 Rudiments of Representation Theory

In the preceding section we have introduced the Poincaré algebra as the the isometry algebra of Minkowski spacetime, by giving its defining representation (3.36). Moreover, it has been possible to discern the abstract symmetry structure (3.49). Also, we have constructed an explicit representation, capturing the defining spacetime transformations, the so-called vector representation (3.47). In this section we will consider the notion of **poin**-representations

*The notation comes from [29].

†Observe that we are using antisymmetrization with 'weight one', so for any antisymmetrization of n indices we get a factor of $n!$. In the last expression we have three antisymmetrizations of two indices each, hence $(2!)^3 = 8$.

more generically. In particular we will discuss the spinor-representation as introduced in appendix B and put it into context. Finally, we will ask ourselves how to classify all irreducible representations. Since it is true, that all physical representation decompose into irreducible ones, we thereby gain an organizing principle for relativistic (field) theory. This will facilitate the transition to supersymmetric field theories.

We have seen how the Poincaré algebra has a natural action on spacetime $\mathbb{M}^{1,D-1}$, as given by eq. (3.36). Fields are functions of spacetime taking values in some representation space. Transforming spacetime will have an effect on the fields, irrespective of the particular representation. On top of that, they can themselves transform non-trivially under the action of **poin**. There is, however, a persistent 'minimal' action induced by a transformation of the underlying spacetime. These so-called transport terms are given by

$$\delta(\xi^A, 0)\phi = \xi^A \Gamma_f(P_A)\phi = \xi^A \partial_A \phi, \tag{3.51a}$$

$$\delta(0, \lambda^{AB})\phi = -\frac{1}{2}\lambda^{AB}\Gamma_f(J_{[AB]})\phi = -\lambda^{AB} x_A \partial_B \phi, \tag{3.51b}$$

where ϕ is a scalar field not transforming under **poin** itself. For other fields we have to add additional terms, for vectors V^A

$$\begin{aligned}\delta(\xi, \lambda)V^A &= \xi^B \Gamma_f(P_B)V^A - \frac{1}{2}\lambda^{BC}\Big(\Gamma_f(J_{[BC]}) + \Gamma_v(J_{[BC]})\Big)V^A \\ &= \Big(\xi^B - \lambda^{BC} x_C\Big)\partial_B V^A - \lambda^A{}_B V^B,\end{aligned} \tag{3.52}$$

observe the extra term Γ_v, rightfully called vector representation. Note that it also appeared in the fundamental representation and would appear in representations of higher order tensors. There is, however, another, in some sense 'smaller' representation. This is called spinor representation Γ_s, defined in appendix B, especially eq. (B.17)

$$\begin{aligned}\delta(\xi, \lambda)\Psi &= \xi^A \Gamma_f(P_A)\Psi - \frac{1}{2}\lambda^{AB}\Big(\Gamma_f(J_{[AB]} + \Gamma_s(J_{[AB]})\Big)\Psi \\ &= \Big(\xi^A - \lambda^{AB} x_B\Big)\partial_A \Psi - \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi.\end{aligned} \tag{3.53}$$

Different kinds of fields will transform differently under **poin**. It is a mathematically well-posed question how to classify fields by their transformation under the algebra. This procedure is known by the name of Wigner classification. As we will see it is done by means of the so-called method of induced representations [30, 31]. Representations of the full group are induced by representations of a compact subgroup. We will not go into detail, following essentially [27], for more detail, see [25, 32].

The strategy is roughly as follows. We try to find Casimir operators of the **poin**, that is elements that commute with the full algebra $[C_i, \mathbf{g}] = 0$. Indeed there are two such elements $C_1 = P^A P_A$ and $C_2 = W^A W_A = (2^{-1}\varepsilon^{ABCD} P_B J_{[CD]})^2$, where the second is the square of what is known as Pauli-Lubansky vector. Schur's lemma teaches us that operators that commute with all irreducible representations (irreps) are proportional to the identity. Hence we can label irreps by two numbers. Moreover, naturally irreps will be labelled by the P_A eigenvalue p_A, having the physical interpretation of a momentum eigenstate. Hence we have three labels (c_1, c_2, p_A). It is natural to identify c_1 as the mass of the respective irrep. The interpretation of c_2 is less immediate, though. The first step to an understanding is fixing whether $c_1 = 0$ or not. Depending on that choice we fix P_A and look at the subgroup respecting that choice - the little group. Finite dimensional unitary representations of the little group induce infinite dimensional unitary representations of the full group indexed by the on-shell momentum p. Explicitly these are carried by a function of p with values in the little group representation $\phi(p)$. By Fourier transforming one gets fields in Minkowski spacetime subject to classical equations of motion.

For massive representations the little group is $Spin(3) = SU(2)$, whose representations are labelled by spin j, hence a massive representation of **poin** characterized by (p_A, m, j). In words: momentum, mass and spin. The building blocks of relativistic quantum field theories are thus labelled just by their mass and spin.

Similar statements can be made for massless representations where the little group is the two-dimensional euclidean group $Spin(2) \ltimes \mathbb{R}^2$. Its physical representations are labelled by a half-integer called helicity $h \in \frac{1}{2}\mathbb{Z}$. CPT reverses helicity meaning that any massless multiplet will have two states usually interpreted as polarizations $(-h, h)$.

All of the above is very abstract, it is however very powerful from a conceptual point of view. In relativistic field theories, all physical objects can be decomposed into irreducible representations. Hence they should be seen as elementary building blocks of field theories. Moreover, representation theory will help us in later chapters to get an intuitive understanding of supersymmetry and supergravity. Whereas in one **poin** multiplet there can only be one scalar or one vector or one fermion, in SUSY field theories, multiplets are allowed to contain e.g. a scalar, two fermions and one vector. It is remarkable, how symmetry shapes the space of possible theories. Even though we can do the same for nonrelativistic algebras, it is unfortunately not true that every nonrelativistic field decomposes into irreducible blocks. This is one of the reasons that make nonrelativistic field theory harder to understand.

3.1.3 Generic Gauge Theory

In this section we present the home base of high energy physics. It is the backbone not only of the standard model but also of beyond the standard model physics. It is one of the major reasons why high energy physicists are so excited about what they are doing. And the mathematicians like it, too.*

The word gauge has a rich history in physics. We are, however, going to be rather generic and use it only in the sense of redundancy. It is thus not to be understood as a symmetry but an abundance of variables that has no effect on the underlying physics. Hence it is also fair to call general coordinate invariance a gauge symmetry. On a technical level, gauging means making the parameters of the theory local $\epsilon \to \epsilon(x)$.

In trying to understand GR as a gauge theory, we are mimicking Yang-Mills theory as far as possible. Hence it is crucial to understand the basic logic of it. Just as in the preceding chapters we denote by $\Gamma_r(X)^i{}_j$ representations of the Lie algebra involved with matrix indices explicitly shown, such that

$$\delta(\epsilon)\phi^i = \epsilon^A \Gamma_r(t_A)^i{}_j \phi^j. \tag{3.54}$$

The parameter in the above is a constant, so whenever we make a transformation it is the same in all of spacetime. This kind of infinitesimal transformation we will call rigid.

*For an excellent, modern account of problems in gauge fixing, see [33].

Conversely, we can allow for a dependence $\epsilon^A \to \epsilon^A(x)$. Clearly, this does not change the form of eq. (3.54). Problems occur when we want to transform derivatives of the fields:

$$\delta(\epsilon)\phi^i = \epsilon^A \Gamma_r(t_A)^i{}_j \phi^j, \tag{3.55a}$$
$$\delta(\epsilon)\partial_M\phi^i = \epsilon^A \Gamma_r(t_A)^i{}_j \partial_M\phi^j + (\partial_M\epsilon)^A \Gamma_r(t_A)^i{}_j \phi^j. \tag{3.55b}$$

The second term is problematic; $\partial_M\phi^i$ does not transform in the same way as does ϕ^i. The situation is very much reminiscent of partial derivatives in differential geometry. There we had to introduce another field, the spin connection, to cancel the unwanted terms. Similarly:

$$\delta(\epsilon)D_M\phi^i = \delta(\epsilon)\partial_M\phi^i + \delta(\epsilon)\Big(h_M\phi\Big)^i = \epsilon^A\Gamma_r(t_A)^i{}_j(D_M\phi)^j, \tag{3.56}$$

a consequence of which is the transformation property of the newly introduced gauge field h_M. It takes values in the representation of the Lie algebra* $h_M = h_M^A\Gamma_r(t_A)$ and transforms with a derivative of the parameter

$$\delta(\epsilon)h_M^A = \partial_M\epsilon^A + h_M^B\epsilon^C f_{BC}^A. \tag{3.57}$$

The same thing can be expressed with all indices suppressed $\delta h = d\epsilon + [h, \epsilon]$. Note that we have omitted a coupling constant as we try to make the expressions as simple as possible, the full expression can e.g. found in [34, p. 88].

Gauge theory takes a simple form when all structure constants vanish, $f_{AB}^C = 0$. This means all the objects in the algebra commute and we speak of an abelian gauge theory. Indeed Maxwell's theory of electromagnetism is a an abelian gauge theory with $\mathbf{G} = U(1)$. It has been natural to consider non-abelian analogons of electromagnetism by choosing $\mathbf{G} = SU(N)$. These theories go under the name of Yang-Mills theories and represent the basic structure of the standard model of particle physics, where $\mathbf{G} = SU(3) \times SU(2) \times U(1)$.

So far we know only the field content (ϕ^i, h_M^A) of the theory† together with the transformation properties. This we call the kinematics. We also want to consider dynamics in the form of an action and thus equations of motion. The gauge field h_M^A is a bad choice though, as it transforms non-covariantly under gauge transformations.

Covariant will henceforth mean that the infinitesimal transformation does not contain a derivative of the parameter. Gauge fields transform non-covariantly.

The physics should be invariant under gauge transformations. The best way to proceed is to organize everything into covariant objects, transforming without a term $\partial_M\epsilon^A$. The easiest way to do so is by considering

$$[D_M, D_N]\phi^i = F_{MN}(t^A)\Gamma_r(t_A)^i{}_j\phi^j, \tag{3.58}$$

since the left hand side transforms properly, it is guaranteed that also the right hand side and in particular the new object $F_{MN}(t^A)$ transforms covariantly. By doing the above construction explicitly, we find

$$F_{MN}(t^A) = \partial_M h_N^A - \partial_N h_M^A + h_M^B h_N^C f_{BC}^A \tag{3.59}$$

*It is common practice to suppress indices, usually we will explicitly show the form-index and suppress matrix indices.

†The matter fields ϕ^i can be in any representation of **poin**, typically they will be in some spinor representation, though.

Due to the similarity to the construction of the Riemann tensor (see in particular eq. (3.23b) and (3.14)), we call this new object a curvature. Indeed, while not being a curvature in spacetime, it is legitimate to do so, since it defines a notion of parallel transport. For a neat exposition of the similarities, see [35] (also referencing to the beautiful mathematics of gauge theory).

The covariant derivative and curvature can also formally be expressed as

$$D_M = \partial_M - \delta(h_M), \tag{3.60a}$$
$$[D_M, D_N] = -\delta(F_{MN}) \tag{3.60b}$$
$$w/ \quad F_{MN}(t^A) = 2\partial_{[M} h_{N]}^A + h_M^B h_N^C f_{BC}^A. \tag{3.60c}$$

3.1.4 Gauging the **poin**

In this section we see how to obtain the kinematics of General Relativity from a gauging procedure. The logic developed here will later be applied both in the supersymmetric and nonrelativistic case. Historically this was first done by [36]. We follow [29] and [22].

The Poincaré algebra has two kinds of basis elements P_A and $J_{[AB]}$, representing translations and Lorentz transformations, respectively. According to the logic of the last section we have to associate two kinds of local parameters $(\xi^A(x), \lambda^{AB}(x))$ and gauge fields accordingly (E_M^A, Ω_M^{AB}). It is worth mentioning that the index 'A' has a meaning different from the generic 'A' in the last section. To summarize

$$\begin{array}{lclcl}
(GEN) & & (PAR) & & (GAU) \\
P_A & \leftrightarrow & \xi^A & \leftrightarrow & E_M^A \\
J_{[AB]} & \leftrightarrow & \lambda^{AB} & \leftrightarrow & \Omega_M^{AB}
\end{array}$$

What is more, from the **poin** structure constants (3.50) and eq. (3.57) we find that

$$\delta(\xi,\lambda)E_M^A = \partial_M \xi^A + \Omega_M^{AB}\xi_B - \lambda^A{}_B E_M^B = D_M\xi^A - \lambda^A{}_B E_M^B, \tag{3.61a}$$
$$\delta(\xi,\lambda)\Omega_M{}^{AB} = \partial_M \lambda^{AB} + 2\Omega_M{}^{C[A}\lambda^{B]}{}_C = D_M \lambda^{AB}. \tag{3.61b}$$

Accordingly we can construct covariant quantities, either directly from the transformation rules or via eq. (3.60c):

$$R_{MN}(P^A) = 2\partial_{[M} E_{N]}^A + 2\Omega_{[M}^{AB} E_{N]B} = 2D_{[M}E_{N]}^A, \tag{3.62a}$$
$$R_{MN}(J^{AB}) = 2\partial_{[M}\Omega_{N]}{}^{AB} + 2\Omega_{[M}{}^{AC}\Omega_{N]C}{}^B = 2D_{[M}\Omega_{N]}^{AB}. \tag{3.62b}$$

These are Cartan's first and second structure equation, (3.12)/(3.14). Hence we are led to interpret the gauge field of translations as a Vielbein and that of local Lorentz transformations as the spin connection. Moreover, we can also see torsion as a translational curvature. The Riemann tensor is then essentially the Lorentz curvature.

3.1.4.1 Where to Get the Diffeomorphisms from?

Note that even though the gauging procedure correctly reproduced the covariant quantities of GR, it has not reproduced diffeomorphisms. Also the gauge field transform under **poin** translations which is not a sensible concept in curved spacetime. The strategy is roughly as follows: we redefine some of the parameters for the purpose of exchanging translations for general coordinate transformations. It will turn out that this can only be done at the expense of setting the translational curvature – that is torsion – to zero:

$$\delta(\xi) \;\text{------}\; R_{MN}(P^A) = 0 \longrightarrow \delta_\xi$$

From this point on it is crucial to make the distinction between different kinds of indices, that is algebra indices $A, B, C, \cdots$ and spacetime indices $M, N, P, \cdots$. These will correspond to flat and curved indices, respectively. Diffeomorphisms act with a parameter ξ^M, which a priori has nothing to do with ξ^A (remember, that the gauge field of translation E_M^A does not have the interpretation of a Vielbein yet):

$$\delta_\xi x^M = \xi^M, \tag{3.63a}$$

$$\begin{aligned}
\delta_\xi E_M^A &= \mathcal{L}_\xi E_M^A = \xi^P \partial_P E_M^A + \Big(\partial_M \xi^P\Big) E_P{}^A \\
&= \partial_M\Big(\xi^P E_P^A\Big) + \Omega_M^{AB}\Big(\xi^P E_{PB}\Big) - \xi^P\Big(\partial_M E_P{}^A - \partial_P E_M^A + \Omega_M{}^A{}_B E_P^B\Big) \\
&= D_M\Big(\xi^P E_P{}^A\Big) - \Big(\xi^P \Omega_P{}^{AB}\Big) E_{MB} - \xi^P R_{MP}(P^A) \\
&= \delta(\xi^A := \xi^P E_P{}^A, \lambda^{AB} := \xi^P \Omega_P{}^{AB}) E_M^A - \xi^P R_{MP}(P^A),
\end{aligned} \tag{3.63b}$$

$$\delta_\xi \Omega_M^{AB} = \mathcal{L}_\xi \Omega_M^{AB}. \tag{3.63c}$$

So we see that diffeomorphisms can be interpreted as the action of a translation and Lorentz transformation with parameters $\xi \cdot E^A$ and $\xi \cdot \Omega^{AB}$, respectively. All provided that $R_{MN}(P^A) = 0$. That seems like too strong a condition. However, when going back to the differential geometry results we see that it is nothing but the no-torsion condition. So, imposing $R_{MN}(P^A) = 0$ is not at all unnatural from a GR point of view. It is however, from the original Yang-Mills point of view, where we never encounter such a restriction in kinematics.

To conclude, the important result is, that we can get diffeomorphisms in our **poin**-gauge theory, by setting the translation curvature $R_{MN}(P^A)$ to zero. What is more, this condition will allow us to solve for $\Omega^{AB} = \Omega^{AB}(E)$, making the translational gauge field the only independent variable. Also we used the E-gauge field to relate algebra to spacetime indices allowing us to interpret it as the Vielbein, so the relation reads

$$\begin{aligned}
R_{MN}(P^A) = 0 &\leftrightarrow T^A = 0 \\
R_{MN}(J^{[AB]}) &\leftrightarrow R_{MN}{}^{AB}.
\end{aligned}$$

3.1.4.2 Problems w/ Gauging

It is remarkable how we got all the kinematics of GR from group theoretical bricolage, there is still a few caveats, for a more complete discussion see [29]. The most obvious is the fact that we have no means to determine the dynamics from group theory. All we can do is guess a gauge-invariant action from the curvature. That is, a scalar both under diffeomorphisms and local Lorentz transformations. In particular it is easy to run into actions that lead to trivial dynamics. What is more, while we can now interpret Vielbeine as gauge fields, it is not clear from this why they should be invertible.

3.2 NC FROM **BARG**

In this section we introduce nonrelativistic geometrodynamics – or Newton-Cartan (NC) theory. The theory will be formulated in terms of Vielbein components akin to the Palatini formulation of GR. In fact it is a gauging of the Bargmann algebra. It is worth mentioning that there is a metric formulation as can be found e.g. in [3, chpt. 12]. The gauging of **barg** was historically first done in [37]. In this Vielbein formulation the structure of NC theory becomes most transparent. It is in particular the symmetry structure that is spelled out much clearer than in the older approaches. And indeed, for us it is indispensable, as we want to go to supergravity eventually.

3.2.1 About NC

Let's consider a particle $\mathbf{x}(\tau)$ moving in a gravitational potential $\phi(\mathbf{x})$, the equation of motion is

$$\ddot{\mathbf{x}}^a = -\delta^{ab}\partial_b\phi(\mathbf{x}), \tag{3.64}$$

with its solutions being curved paths in space. We attribute this deviation from straight paths to the gravitational force $F_G = -m\partial\phi$. Interestingly enough, the paths taken do not depend on the mass. That is, every particle starting with the same initial conditions will take the same path, irrespective of mass. Einstein has taken this curious fact seriously and changed the perspective radically. He used this to develop a geometric formulation of gravitation.

The path taken is not a consequence of some properties of the particle but of spacetime itself. What is more, the particle actually moves in perfectly straight lines. It is the geometry of spacetime causing the apparent bending of trajectories. Straight lines in a curved spacetime are described by the geodesic equations

$$\ddot{x}^0 + \Gamma^0_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{3.65a}$$
$$\ddot{\mathbf{x}}^a + \Gamma^a_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{3.65b}$$

From which it is easy to read off the only nonzero Christoffel symbols

$$\Gamma^a_{00} = \delta^{ab}\partial_b\phi(\mathbf{x}), \tag{3.66}$$

where we have fixed $x^0 = \tau$. Hence the Riemann tensor has the following non-vanishing coefficients

$$R^a{}_{0c0} = \delta^{ab}\partial_b\partial_c\phi(\mathbf{x}), \tag{3.67}$$

The equation of motion $R^a{}_{0a0} = 4\pi G_N\rho$ imposes the Poisson equation. Note that this rewriting of Newtonian mechanics cannot yet be interpreted as a geometric theory as no statement about the underlying metric structure have been made yet.

The Newtonian Christoffel symbols (3.66) cannot come from a nondegenerate metric. One way to see this is by looking at the index symmetries of the corresponding Riemann tensor (3.67). These are not the ones we would expect from a generic Riemann tensor derived from a nondegenerate metric. Another, more immediate way to see the degeneracy of nonrelativistic metrics is by going to flat space

$$(\tau_{\mu\nu}) = (c^{-2}\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & c^{-2}\mathbb{1}_{D-1} \end{pmatrix}, \qquad (h^{\mu\nu}) = (\eta^{\mu\nu}) = \begin{pmatrix} -c^{-2} & 0 \\ 0 & \mathbb{1}_{D-1} \end{pmatrix}. \tag{3.68}$$

It is clear that with $c \to \infty$ both become degenerate with rank 1 and $(D-1)$ respectively. The Galilei group preserves both metrics separately, so it is natural to associate this degeneracy with the splitting of space and time in nonrelativistic physics.

Note that the degeneracy of the metric in flat space eq. (3.68) carries over to curved spacetime as

$$h^{\mu\nu}\tau_{\mu\nu} = 0, \tag{3.69}$$

since $\tau_{\mu\nu}$ is of rank one, i.e. a 1×1 matrix effectively, we can express $\tau_{\mu\nu} = \tau_\mu\tau_\nu$. On these the degeneracy implies

$$h^{\mu\nu}\tau_\nu = 0. \tag{3.70}$$

It is possible to discuss NC in the metric formalism, see [3] for a textbook treatment. Since we want to derive the theory from a gauging principle alluding to **poin**-gauging, it is useful to formulate the theory in a Vielbein language. Moreover, this will facilitate the transition to supersymmetric NC. For a treatment of both formalisms see [17].

We will see in due time that one of the defining properties of NC is the foliation of spacetime, represented by the existence of a time function $t(x)$ such that

$$\tau_\mu = \partial_\mu t(x), \tag{3.71}$$

very similar to the situation in globally hyperbolic spacetimes in GR. The difference lies in the fact that the foliation in NC geometry is observer-independent – that is, the above condition is preserved under all nonrelativistic symmetries. It is then useful to introduce locally inertial coordinates $\{(y^a)|a = 1, \cdots, D-1\}$ on the spatial slices Σ_t. At every point $(x_0) = (x_0^0, x_0^a) \in \Sigma_t \subset \mathcal{M}$ we define so-called spatial Vielbeine as cotangent vectors to the spatial slices

$$e_\mu{}^a := \left.\frac{\partial y^a(x^0, \mathbf{x})}{\partial x^\mu}\right|_{x=x_0}. \tag{3.72}$$

To avoid confusion, these are not Vielbeine in the conventional sense of section 3.1.1.1, but rather rectangular $D \times (D-1)$ matrices representing the locally inertial spatial geometry similar to the ADM decomposition of GR [3]. The pushforward of h along the embedding unveils the locally flat structure*:

$$h^{\mu\nu}e_\mu{}^a e_\nu{}^b = \delta^{ab}. \tag{3.73}$$

*We will see later that the spatial slices are globally flat in an appropriate sense in NC gravity.

We have translated the metric structure $(\tau_{\mu\nu}, h^{\mu\nu})$ into a Vielbein like structure $(\tau_\mu, e_\mu{}^a)$. The degeneracy of the metrics $h^{\mu\nu}\tau_{\mu\nu} = 0$ translates naturally into orthogonality relations $e_a{}^\mu \tau_\mu = 0$ by defining projective inverses $(e_a{}^\mu, \tau^\mu)$

$$e_a{}^\mu e_\mu{}^b = \delta_a{}^b, \quad e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu - \tau_\mu \tau^\nu, \quad \tau^\mu \tau_\mu = 1, \quad \tau^\mu e_\mu{}^a = 0, \quad e_a{}^\mu \tau_\mu = 0. \tag{3.74}$$

The second should really be interpreted as a projector on the spatial hypersurface. Hence we see that indeed

$$h^{\mu\nu} = e_a{}^\mu e_b{}^\nu \delta^{ab}. \tag{3.75}$$

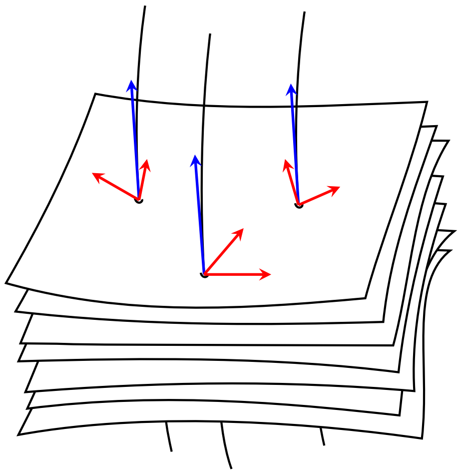

Figure 3.1: Nonrelativistic geometry. There is a distinct time (indicated by blue arrows) and spatial direction (indicated by red arrows). We can describe these by two degenerate metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$. Equivalently we can introduce $(D-1)$ basis vectors $(e_a{}^\mu)$ on the spatial slices and one (τ^μ) in the time direction.

Note that we can choose coordinates adapted to flat space

$$e_a{}^\mu = \delta_a{}^\mu, \qquad \tau_\mu = \delta_\mu{}^0. \tag{3.76}$$

It is conceivable that space and time do not mix under local rotations and time translations. Similarly not under diffeomorphisms. Just as in flat space it is the local boost symmetry that will somehow fuse space and time. To make this explicit we will now introduce the kinematics of NC via a gauging principle.

3.2.2 About **barg**

The isometry algebra of nonrelativstic physics is the Bargmann algebra **barg**. We have introduced it by means of looking at the Poisson structure of the nonrelativistic particle in section 2. We will go back and derive it by another method, essentially mimicking the nonrelativistic limit on the Poincaré algebra. The main motivation to do so is the fact that this kind of Inönü-Wigner contraction carries over to the supersymmetric case. A third method to obtain **barg**(D) as subalgebra of **poin**$(D+1)$ can be found in appendix C.

We will obtain the abstract symmetry algebra of nonrelativistic physics in a way alluding to the $c \to \infty$ limit $P_0 = \sqrt{P_a P^a c^2 + m^2 c^4} \cong mc^2 + (2m)^{-1}\mathbf{P}_a \mathbf{P}^a$. The contraction of the Poincaré algebra to the Bargmann algebra goes roughly as follows:

1. Trivially extend the Poincaré algebra by adding a generator Z,
2. Introduce a real parameter c and perform a basis change akin to eq. (C.1),
3. Consider the algebra in the new basis and take the limit $c \to \infty$. The resulting algebra is the Bargmann algebra **barg**.

Formally step one is nothing but

$$\mathbf{poin} \longrightarrow \mathbf{poin} \oplus \mathbf{g}_Z, \tag{3.77}$$

where $\mathbf{g}_Z$ is a trivial extension, that is $[\mathbf{poin}, Z] = 0$. Apparently the extended algebra $\mathbf{poin} \oplus \mathbf{g}_Z$, spanned by the generators $\{P_A, J_{AB}, Z\}$, has exactly the same nonzero commutators. Now we make a basis change $\{P_A, J_{AB}, Z\} \mapsto \{H, \mathbf{P}_a, \mathbf{G}_a, \mathbf{J}_{ab}, M\}$:

$$\begin{aligned} &P_0 \to \tfrac{1}{2c}H + cM, \quad Z \to \tfrac{1}{2c}H - cM, \quad J_{a0} \to c\mathbf{G}_a, \\ &P_a \to \mathbf{P}_a, \qquad\qquad\qquad\qquad\qquad\quad J_{ab} \to \mathbf{J}_{ab}, \end{aligned} \tag{3.78}$$

where $a, b = 1, \cdots, D-1$ are the purely spatial indices. Their commutation relations will have terms of order $\mathcal{O}(c^n)$ on the right hand side. The choice of reparametrization is motivated by the nonrelativistic dispersion relation. Now it is just a mechanical task to see how the algebra transforms under $c \to \infty$. This is an example of an Inönü Wigner contraction [38].

$$\begin{aligned} [\mathbf{J}_{ab}, \mathbf{J}_{cd}] &= 4\delta_{[a[c}\mathbf{J}_{d]b]}, \\ [\mathbf{P}_a, \mathbf{J}_{bc}] &= 2\delta_{a[b}\mathbf{P}_{c]}, \\ [\mathbf{G}_a, \mathbf{J}_{bc}] &= 2\delta_{a[b}\mathbf{G}_{c]}, \\ [H, \mathbf{G}_a] &= \mathbf{P}_a, \\ [\mathbf{P}_a, \mathbf{G}_b] &= \delta_{ab}M. \end{aligned} \tag{3.79}$$

$biso(D-1,1)$	$iso(D-1)$	$so(D-1)$ $[\mathbf{J}_{ab}, \mathbf{J}_{cd}] = 4\delta_{[a[c}\mathbf{J}_{d]b]}$
	$[\mathbf{P}_a, \mathbf{J}_{bc}] = 2\delta_{a[b}\mathbf{P}_{c]}$	
$[\mathbf{G}_a, \mathbf{J}_{bc}] = 2\delta_{a[b}\mathbf{P}_{c]}$	$[\mathbf{G}_a, \mathbf{P}_b] = \delta_{ab}M$	$[H, \mathbf{G}_a] = P_a$

Figure 3.2: The Bargmann algebra **barg** and its subalgebras.

For the subalgebra structure, see table 3.2. It is instructive to compare this to the Poincaré algebra in the split up form - eq.s (C.3f) through (C.3h). **barg** builds on the kinematical algebra $iso(D-1)$ just as does **poin**, it is the commutators (C.3f) and (C.3g) that make the difference (we use $\mathbf{G}$ for Galilean boosts and $\mathbf{K}$ for relativistic boosts):

$$[\mathbf{K}_a, \mathbf{K}_b] = -\mathbf{J}_{ab} \qquad \leftrightarrow \qquad [\mathbf{G}_a, \mathbf{G}_b] = 0, \tag{3.80a}$$

$$[\mathbf{P}_a, \mathbf{K}_b] = \delta_{ab}H \qquad \leftrightarrow \qquad [\mathbf{P}_a, \mathbf{G}_b] = \delta_{ab}M. \tag{3.80b}$$

Note that the physical effect of Thomas precession is not present with nonrelativistic boosts $\mathbf{G}_a$. Moreover we broke the Lorentz invariance explicitly by not having H, the generator of time translations as the result of consecutively boosting and translating. In other words, time translations boost to spatial translations but spatial translations do not boost to time translations. Instead we have another generator M, commuting with all other elements, appearing as the result of boosting spatial translations. In case the central extension is absent the resulting algebra is called Galilei algebra.

3.2.3 Gauging the **barg**

In section 3 we showed how to obtain the kinematics of General Relativity by gauging the Poincaré algebra. In this section we emulate that very same logic for the Bargmann algebra. That is we associate gauge fields to the generators $(J_{ab}, \mathbf{P}_a, \mathbf{G}_a, H, M)$ and derive transformation properties and associated curvatures.

The strategy can be summarized by diagram 3.3.

$$
\begin{tikzcd}
 & \mathbf{poin} \arrow[rr, "IW"] \arrow[dl, "G"'] & & \mathbf{barg} \arrow[dl, "G"] \\
GR \arrow[rr, "0R"] & & NC &
\end{tikzcd}
$$

Figure 3.3: The strategies applied to obtain NC. In this section we follow a gauging procedure indicated by the blue arrows, consisting of an Inönü Wigner contraction (IW) and a gauging (G) of **barg**. The alternative red path will be described in chapter 5.

3.2.3.1 NC from **barg**

Gauging an algebra as outlined in section 3.1.4 means associating local parameters, gauge fields and curvatures to every generator. In the case of **barg**, there are five:

$$
\begin{array}{lllllll}
(gen) & & (par) & & (gau) & & (cur) \\
H & \leftrightarrow & \xi^0 & \leftrightarrow & \tau_\mu & \leftrightarrow & r_{\mu\nu}(H) \\
\mathbf{P}_a & \leftrightarrow & \xi^a & \leftrightarrow & e_\mu{}^a & \leftrightarrow & r_{\mu\nu}(P^a) \\
\mathbf{G}_a & \leftrightarrow & \lambda^a & \leftrightarrow & \omega_\mu{}^a & \leftrightarrow & r_{\mu\nu}(G^a) \\
\mathbf{J}_{ab} & \leftrightarrow & \lambda^{ab} & \leftrightarrow & \omega_\mu{}^{ab} & \leftrightarrow & r_{\mu\nu}(J^{ab}) \\
M & \leftrightarrow & \sigma & \leftrightarrow & m_\mu & \leftrightarrow & r_{\mu\nu}(M)
\end{array}
$$

The notation is adapted to that of the last section and we will eventually call the gauge fields of time and space translations time- and spacelike Vielbeine. Gauge fields of boosts and rotations will play the role of spin connections, similar to the relativistic case. It is only the central charge gauge field, whose interpretation is yet to become clear. And indeed it is the achievement of [37] to make clear the significance of m_μ.

We have seen that the symmetry algebra determines the transformation properties of the gauge theory, see eq.s (3.57) and (3.60c). The only input is the structure constants, which can be read off from table (3.79)

$$
\begin{aligned}
&f_{[ab][cd]}^{[ef]} = 8\delta_{[a[c}\delta_{d]}^{[e}\delta_{b]}^{f]}, \quad f_{(P_a)[bc]}^{(P_d)} = 2\delta_{a[b}\delta_{c]}^{(P_d)}, \\
&f_{(G_a)[bc]}^{(G_d)} = 2\delta_{a[b}\delta_{c]}^{(G_d)}, \quad f_{(G_a)(P_b)}^{(M)} = \delta_{ab}^{(M)}, \\
&f_{(H)(G_a)}^{(P_b)} = \delta_{(G_a)}^{(P_b)}.
\end{aligned}
$$

From this and eq. (3.57) we can identify how the gauge fields transform under the Bargmann algebra

$$
\begin{aligned}
\delta\tau_\mu &= \partial_\mu \xi^0, && (3.81a)\\
\delta e_\mu{}^a &= D_\mu \xi^a - \lambda^a{}_c e_\mu{}^c - \lambda^a \tau_\mu - \tau\omega_\mu{}^a, && (3.81b)\\
\delta\omega_\mu{}^a &= D_\mu \lambda^a - \lambda^a{}_c \omega_\mu{}^c, && (3.81c)\\
\delta\omega_\mu{}^{ab} &= D_\mu \lambda^{ab}, && (3.81d)\\
\delta m_\mu &= \partial_\mu \sigma - \xi_c \omega_\mu{}^c - \lambda_c e_\mu{}^c. && (3.81e)
\end{aligned}
$$

Here the covariant derivative $D_\mu \phi^A = \partial_\mu \phi^A - \phi^C \omega_\mu{}^{ab} f_{[ab]C}^A$ is covariant just with respect to spatial rotations, that is, it only contains $\omega_\mu{}^{ab}$.

It is straightforward to construct the associated curvatures

$$
\begin{aligned}
r_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]}, && (3.82a)\\
r_{\mu\nu}(P^a) &= 2D_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^a \tau_{\nu]}, && (3.82b)\\
r_{\mu\nu}(G^a) &= 2D_{[\mu}\omega_{\nu]}^a, && (3.82c)\\
r_{\mu\nu}(J^{ab}) &= 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{ac}\omega_{\nu]}^{db}\delta_{cd}, && (3.82d)\\
r_{\mu\nu}(M) &= 2\partial_{[\mu}m_{\nu]} + 2e_{[\mu}^a \omega_{\nu]a}. && (3.82e)
\end{aligned}
$$

Note that the transformations (3.81) do not contain diffeomorphism but translations. This is the same situation we had in the relativistic case 3.1.4.1. The strategy is again to exchange (τ, ζ^a) for ξ^μ. In the process we will have to set $r_{\mu\nu}(H) = 0 = r_{\mu\nu}(P^a) = r_{\mu\nu}(M)$.

$$
\begin{aligned}
\delta_\xi \tau_\mu &= \xi^\rho \partial_\rho \tau_\mu + (\partial_\mu \xi^\rho)\tau_\rho = \partial_\mu(\xi^\rho \tau_\rho) - \xi^\rho r_{\mu\rho}(H) = \delta(\xi^0 = \xi^\rho\tau_\rho)\tau_\mu - \xi^\rho r_{\mu\rho}(H), && (3.83a)\\
\delta_\xi e_\mu{}^a &= \delta(\xi^a = \xi^\rho e_\rho{}^a, \lambda'^{ab} = \lambda^{ab} + \xi^\rho \omega_\rho{}^{ab}, \lambda'^a = \lambda^a - \xi^\rho \omega_\rho{}^a) e_\mu{}^a - \xi^\rho r_{\mu\rho}(P^a), && (3.83b)\\
\delta_\xi m_\mu &= \delta(\sigma = \xi^\rho m_\rho, \lambda'^a = \lambda^a - \xi^\rho \omega_\rho{}^a, \xi^a = \xi^\rho e_\rho{}^a) m_\mu - \xi^\rho r_{\mu\rho}(M), && (3.83c)
\end{aligned}
$$

so we see how diffeomorphism are induced by fixing translations and altering the other gauge-symmetries appropriately. On top of that we have to set curvatures to zero. In the relativistic case, however, it turned out that $R_{MN}(P^A) = 0$ was nothing but the no-torsion constraint allowing to solve for $\Omega_M^{AB}(E)$.

The logic is similar in the nonrelativistic case, because both the central charge and spatial translation curvature (3.82e) and (3.82b) are algebraic in the spin connections. Thus they are called conventional constraints. It is possible to solve those in terms of $\omega_\mu{}^a(e, \tau, m)$ and $\omega_\mu{}^{ab}(e, \tau, m)$, meaning that the conventional constraints are zero identically. The time translation curvature (3.82a), though, plays a different role: it represents foliation of spacetime. This can explicitly be seen by solving $r_{\mu\nu}(H) = 0$ by $\tau_\mu = \partial_\mu t$. In other words, there is a globally defined time function $t(x)$. We will come back to the role of the foliation constraint.

3.2.3.2 Torsionless NC

Newton-Cartan theory has the following independent fields $(e_\mu{}^a, \tau_\mu, m_\mu)$ together with projective inverses defined by eq.s (3.74) and the following transformation rules

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu, && (3.84a)\\ \delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu, && (3.84b)\\ \delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu{}^a. && (3.84c) \end{aligned}$$

Let's first of all use the conventional constraints $r_{\mu\nu}(M) = 0 = r_{\mu\nu}(P^a)$ and solve for the spin connections. The strategy is similar to what we did in section 3.1.1.4 leading up to eq. (3.26), however, we will get additional terms:

$$r_{\mu\nu}(P^c)e_{c\rho} - r_{\nu\rho}(P^c)e_{c\mu} + r_{\rho\mu}(P^c)e_{c\nu} = 0, \qquad (3.85)$$

which allows to solve for the Lorentz spin connection just along the same lines as above (see eq. (3.24), where we introduce the following anholonomy coefficients:

$$c_{\mu\nu}^{\ a} \equiv 2\partial_{[\mu} e_{\nu]}^{a}, \qquad c_{\mu\nu} \equiv 2\partial_{[\mu} m_{\nu]}, \qquad (3.86)$$

so that $r_{\mu\nu}(P^a) = c_{\mu\nu}^{\ a} + 2\omega_{[\mu}^{\ ab} e_{\nu]b} - 2\omega_{[\mu}^{\ a}\tau_{\nu]} = 0$ and $r_{\mu\nu}(M) = c_{\mu\nu} + 2e_{[\mu}^{\ a}\omega_{\nu]a} = 0$ and thus

$$\omega_\mu{}^{ab}(e,\tau,\omega^a) = e^{\nu[a} c_{\mu\nu}^{\ b]} - \frac{1}{2} c^{ab}{}_\mu + \tau_\mu e^{[a|\nu|}\omega_\nu{}^{b]} \qquad (3.87)$$

Now, apparently this result is somewhat unsatisfactory as it contains $\omega_\mu{}^a$ explicitly. However we still have the conventional constraint $r_{\mu\nu}(M) = 0$, also we insert result (3.87) into $f_{\mu\nu}(P^a)$ and contract with $e^{\mu b}$ and τ^ν, this gives us the following constraints on $\omega_\mu{}^a$:

$$\begin{aligned} 2r_{\mu\nu}(P^{(a})e^{b)\mu}\tau^\nu = 0: &\qquad \omega_\mu{}^{(a}e^{b)\mu} = c_{\mu\nu}^{(a} e^{b)\mu}\tau^\nu = c^{(a\ b)}{}_{0}, && (3.88a)\\ 2r_{\mu\nu}(M)e^{a\mu}e^{b\nu} = 0: &\qquad \omega_\mu{}^{[a}e^{b]\mu} = \frac{1}{2}e^{a\mu}e^{b\nu}c_{\mu\nu} = \frac{1}{2}c^{ab}, && (3.88b)\\ 2r_{\mu\nu}(M)\tau^\mu e^{a\nu} = 0: &\qquad \tau^\mu\omega_\mu{}^a = \tau^\mu e^{a\nu}c_{\mu\nu} = c_0{}^a. && (3.88c) \end{aligned}$$

Together with eq. (3.74) this yields $\omega_\mu{}^a(e,\tau,m)$ and hence also $\omega_\mu{}^{ab}(e,\tau,m)$ (via. eq. (3.87)

$$\begin{aligned} \omega_\mu{}^{ab}(e,\tau,m) &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{\ b]} - e_{\mu c}e^{\nu a}e^{\rho b}\partial_{[\nu}e_{\rho]}^{\ c} - 2\tau_\mu e^{\nu a}e^{\rho b}\partial_{[\nu}m_{\rho]}, && (3.89a)\\ \omega_\mu{}^a(e,\tau,m) &= e^{\nu a}\partial_{[\mu}m_{\nu]} - \tau^\nu\partial_{[\mu}e_{\nu]}^{\ a} + e_{\mu b}\tau^\nu e^{\rho a}\partial_{[\nu}e_{\rho]}^{\ b} + \tau_\mu\tau^\nu e^{\rho a}\partial_{[\mu}m_{\nu]}, && (3.89b) \end{aligned}$$

or $\omega_\mu{}^{ab} = c_\mu{}^{[ab]} - 2^{-1}c^{ab}{}_\mu - \tau_\mu c^{ab}$ and $2\omega_{\mu a} = c_{a\mu} - c_{\mu 0 a} + c_{0 a\mu} + \tau_\mu c_{0a}$.

In the rotation spin connection we recognize the naive term eq. (3.26) with an additional term $\tau_\mu c^{ab}$ coming from the contraction of the full Poincaré group.

The only thing we are missing are the correct transformation rules for the spin connections. Note that it is in general not true that the transformation rules can be deduced from the **barg** algebra, since we have already put constraints on the fields and thereby made the spin connections dependent fields. In other words, $\delta\omega_\mu{}^{ab}$ and $\delta\omega_\mu{}^a$ are such that $\delta r_{\mu\nu}(P^a) = 0 = \delta r_{\mu\nu}(M)$ or

$$\delta\omega_\mu{}^{ab} = \delta c_\mu{}^{[ab]} - 2^{-1}\delta c^{ab}{}_\mu - \delta\tau_\mu c^{ab} - \tau_\mu \delta c^{ab},$$
$$2\delta\omega_{\mu a} = \delta c_{a\mu} - \delta c_{\mu 0a} + \delta c_{0a\mu} + \delta\tau_\mu c_{0a} + \tau_\mu \delta c_{0a},$$

where all the variations on the right hand side can be done from (3.84). We will not give this result here, but show the same for a more general theory.

3.2.3.3 Torsional NC

Aside from the conventional constraints (3.82e) and (3.82b) we also imposed the foliation constraint

$$r_{\mu\nu}(H) \equiv \tau_{\mu\nu} = 0, \tag{3.90}$$

it is an interesting question how much one can weaken this constraint on the geometry without loosing the foliation of spacetime. And indeed, the Frobenius theorem (see e.g. [23], [24]), states that not $\tau_{\mu\nu} = 0$ but $\tau_{[\mu}\tau_{\nu\rho]} = 0$ is the weakest condition for a manifold to foliate. In our setting this is equivalent to requiring $\tau^{ab} = 0$. In the literature this is known as twistless torsion [39]. It has recently been proposed that this is the correct way to do post-Newtonian expansion, where strong gravitational fields are represented by non-vanishing torsion, see [40]. In fact, when considering theories with scale invariance it is even necessary to weaken the no-torsion to a twistless torsion condition, see [6].

Note that for zero torsion we can always solve for $\tau_\mu = \partial_\mu t$, such that the time difference

$$T = \int_{\mathcal{C}} dx^\mu \tau_\mu \tag{3.91}$$

is independent of the path $\mathcal{C}$. Thus it is fair to speak of absolute time in nonrelativistic geometry. The situation changes when generalizing to twistless torsion. Even though spacetime is still foliated and τ_μ is hypersurface orthogonal, it is no longer true that the time difference is path-independent. Torsional nonrelativistic geometry (TNC) does not have a well-defined notion of absolute time.

Variation with respect to boosts will give torsion terms $\delta c^{\,a}_{\mu\nu} = \cdots + \lambda^a \tau_{\mu\nu} + \cdots$, explicitly

$$\delta\omega_\mu{}^{ab}(\tau, e, m) = \mathcal{L}_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} - 2\omega_\mu^{[a|c|}\lambda_c{}^{b]} + \lambda^c e_{\mu c}\tau^{ab} + 2\lambda^{[a} e^{b]\rho}\tau_{\mu\rho}. \tag{3.92a}$$
$$\delta\omega_\mu{}^a(\tau, e, m) = \mathcal{L}_\xi \omega_\mu{}^a + \partial_\mu \lambda^a - \omega_\mu{}^{ac}\lambda_c - \omega_\mu{}^c \lambda_c{}^a - \lambda^c e_{\mu c}\tau_0{}^a - \lambda^a e_{\mu c}\tau_0{}^c, \tag{3.92b}$$

From this knowledge it is easy to construct the corresponding curvatures or field strengths, see eq. (3.57) and (3.60c)

$$\tilde{r}_{\mu\nu}(G^a) = 2\partial_{[\mu}\omega_{\nu]}^{\ a} - 2\omega_{[\mu}^{\ ab}\omega_{\nu]b} + 2\omega_{[\mu}^{\ b} e_{\nu]b}\tau_0{}^a + 2\omega_{[\mu}^{\ a} e_{\nu]b}\tau_0{}^b, \tag{3.93a}$$
$$\tilde{r}_{\mu\nu}(J^{ab}) = 2\partial_{[\mu}\omega_{\nu]}^{\ ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 2\omega_{[\mu}{}^c e_{\nu]c}\tau^{ab} - 4\omega_{[\mu}^{\ [a} e^{b]\rho}\tau_{\nu]\rho}. \tag{3.93b}$$

Note how this generalizes the naive curvatures (3.82c) and (3.82d) to arbitrary torsion $\tau_{\mu\nu}$. This theory is more general that NC gravity and we will see how it arises by the method of Null Reduction in a later chapter. Upon setting $\tau_{\mu\nu} = 0$ we get back the older results.

3.2.3.4 Galilean Gravity via Gauge Fixing

In this work we are mainly concerned with kinematical aspects of nonrelativistic geometry. Notwithstanding, it is vital that the theory is able to describe Newtonian gravity upon gauge fixing. All the equations of motion are covariant, that is equations involving only curvatures. In this section we show what these equations are and how to choose coordinates such that we arrive at the Poisson equation. See [8] for more details.

We had imposed two equations akin to the first Cartan structure equation (3.12)

$$r_{\mu\nu}(P^a) = 0, \qquad\qquad r_{\mu\nu}(M) = 0, \tag{3.94}$$

these are called conventional constraints as they allow to solve for the spin connections. On top of that we impose the zero-torsion constraint (for a treatment of twistless torsion see [39]) and flatness of the spatial slices:

$$r_{\mu\nu}(H) = \tau_{\mu\nu} = 0, \qquad\qquad r_{\mu\nu}(J^{ab}) = 0. \tag{3.95}$$

These two constraints are not conventional but geometric. It turns out that the second is needed to obtain Newtonian gravity in flat space. We will see that it is a consequence of the zero-torsion condition in the supersymmetric theory. Hence all the curvatures apart from (3.82) are set to zero and indeed it is the boost curvature that carries the Poisson equation

$$r_{0a}(G^a) = \tau^{\mu} e_a{}^{\nu} r_{\mu\nu}(G^a) = 0, \tag{3.96}$$

this is the dynamical equation and all other components are zero identically.

Let us now try to solve some of the equations by gauge fixing to specific coordinates adapted to flat space and zero torsion:

$$\tau_\mu = \delta_\mu{}^0, \qquad e_\mu{}^a = \delta_\mu{}^a, \qquad \omega_\mu{}^{ab} = 0. \tag{3.97}$$

For a more general Ansatz, see [8]. The above breaks some of the symmetries to rigid ones

$$\xi^0(x) = \xi^0, \qquad \lambda^{ab}(x) = \lambda^{ab}, \qquad \xi^a(x) = \xi^a(t) - \lambda^a{}_b \delta_\mu{}^b x^\mu, \qquad \lambda^a(x) = -\dot{\xi}^a(t). \tag{3.98}$$

The only independent field that is not constrained is m_μ and hence it must carry the Newtonian potential:

$$m_a = \delta_a{}^\mu m_\mu = 0, \quad m_0(x) = \delta_0{}^\mu m_\mu = \phi(x) \quad \Longrightarrow \quad \omega_0{}^a = -\delta^{ab}\partial_b\phi(x). \tag{3.99}$$

It is then straightforward to see that eq. (3.96) indeed reduces to the vacuum Poisson (Laplace) equation:

$$\Delta\phi(x) = \delta^{ab}\partial_a\partial_b\phi(x) = 0. \tag{3.100}$$

It is worth pointing out the significance of the field m_μ, carrying the Newtonian potential along. This shows the importance of taking into account the central charge in the full Bargmann algebra rather than just the Galilei algebra.

CHAPTER 4

SUPERGEOMETRY FROM SUPERSYMMETRY

4.1 SUGRA FROM **SPOIN**

Supersymmetry (SUSY) is a very special type of symmetry, relating bosonic and fermionic degrees of freedom. In supergravity (SUGRA) this means it maps the graviton (Vielbein) to a fermionic field called the gravitino. Contrary to bosons, the mathematical description of fermions is very sensitive to the number of spacetime dimensions (see appendix B). SUGRA has to be constructed in every spacetime dimension separately and is thus not a unique theory but a host of many gravitational theories with supersymmetry. SUSY imposes strong restrictions on physical theories delaying divergences in quantum field theories to higher loop orders in perturbation theory. This fuelled the hope that $D = 4,\ \mathcal{N} = 8$ SUGRA turns out to be a finite theory of quantum gravity. In fact it has not been ruled out that this maximally supersymmetric theory of gravity is indeed finite.

Another motivation for studying SUGRA is that it is the low energy limit of superstring theories. Many see them as the most promising contender for a theory of quantum gravity. A lot of work in this area of research is devoted to discerning properties of the full superstring theories from their low energy effective SUGRA theories.
Apart from the above, supergravity has found many applications disconnected from the field of quantum gravity. See e.g. [9, 41].

This short review introduces minimal $D = 4,\ \mathcal{N} = 1$ SUGRA. We follow [22] most closely. Other useful references include [42–45]

4.1.1 About SUGRA

Supersymmetric theories are such that an action describing the dynamics is invariant under a transformation with an anticommuting, spinorial parameter. Moreover we want the consecutive action to describe a spacetime transformation. Hence supersymmetry should be understood as a spacetime symmetry. There are other transformations with anticommuting parameters, like BRST, that are not supersymmetries in this above sense.

Kinetic terms for bosons contain two derivatives whereas those of fermions contain just one. Examples are the Maxwell, Klein-Gordon and Einstein-Hilbert action on the one side and the Dirac and Rarita-Schwinger action on the other. The last of which will be introduced in this section and describes fermions with spin-$\frac{3}{2}$ called gravitini. It follows from the above that the mass dimensions differ by 1/2:

$$[ferm] - [bos] = \frac{1}{2}. \tag{4.1}$$

This will lead us to supersymmetry (SUSY). Schematically, the bosonic degrees of freedom transform as $\delta(bos) \sim \epsilon(ferm)$, meaning that the parameter ϵ itself has to have mass dimension $[\epsilon] = -1/2$. This forbids a transformation of the form $\delta(ferm) \sim \epsilon(bos)$, but rather

$$\delta(bos) \sim \epsilon(ferm), \qquad \delta(ferm) \sim \partial\Big(\epsilon(bos)\Big). \tag{4.2}$$

It turns out that the parameter has to have spin$-\frac{1}{2}$ and be anticommuting. Hence the fields in the SUSY multiplet differ by units of spin$-\frac{1}{2}$. The possible minimal multiplets are $(2 - \frac{3}{2}, \frac{3}{2} - 1, 1 - \frac{1}{2}, \frac{1}{2} - 0)^*$, these are indeed the multiplets of massless $\mathcal{N} = 1$ SUSY, as we

*Note that for massless fields we always restrict to spin≤ 2, since gauge invariance becomes too restrictive will see.

Supersymmetric field theories have to match bosonic and fermionic degrees of freedom. In more complicated situations there may be a host of different bosonic and fermionic fields, thus the actual form of transformation rules may become very complicated. We will consider minimal SUGRA only, where we are by the above forced to consider a graviton (spin-2) and a gravitino (spin$-\frac{3}{2}$). What is more, there may also be auxiliary fields representing no physical degrees of freedom. We will meet a so-called off-shell multiplet in later chapters.

In supersymmetric field theories, SUGRA in particular, we are dealing with spinors. These are very sensible to the number of spacetime dimensions, see appendix B. Hence there is a large number of inequivalent SUGRAs. We will only present the easiest (and oldest), $\mathcal{N} = 1$ SUGRA in four spacetime dimensions.

Following the lines of the above, we know that it has to contain at least a Vielbein E_M^A and a gravitino Ψ_M. We also know that the Vielbein transforms to the gravitino and the gravitino transforms to a total derivative. But since we are in a geometric theory, the total derivative should better be a covariant one:

$$\delta(grav) \sim \epsilon(ferm) + \cdots , \qquad \delta(ferm) \sim D(\cdots) + \cdots . \tag{4.3}$$

As we know the minimal content (E_M^A, Ψ_M), we could now guess an action, invariant under some SUSY transformation rules. Apparently this is not guaranteed to work and typically very tedious. Another route to SUGRA is to start from a SUGRA in higher dimensions (when available) and perform a dimensional reduction. We will see how to do this in the next chapter. As a third option, we may gauge a SUSY algebra, just as we did for **poin** to obtain GR.

This last option manifestly conceives SUGRA as a theory of local supersymmetry or a gauge theory of superalgebras. The logic is very simple to that of Poincaré gauge theory in chpt. 3. We will mostly follow that strategy as it allows us to find the symmetry transformations constructively. It should however be noted that not every SUGRA can be obtained from such a gauging procedure.

In a first step, let us explore some of the structure by writing the minimal content of SUGRA:

$$S[E, \Psi, \cdots] = \frac{1}{2} \int d^D x E\Big(R - \bar{\Psi}_M \Gamma^{MNP} D_N\big(\Omega(E)\big)\Psi_P + \cdots\Big), \tag{4.4}$$

where $\Gamma^{MNP} \equiv E^{MA}E^{NB}E^{PC}\Gamma_{ABC}$ and Γ_{ABC} is a numeric Gamma matrix as introduced in appendix B. We see that $S[E, 0]$ is the Einstein Hilbert action. It turns out that in dimension three and four the fields (E, Ψ) already form a sufficiently large multiplet. So the $\cdots$-terms can at most be built from these, e.g. four-point gravitino interactions $\sim (\bar{\Psi}\Psi)^2$. In general we would have to introduce additional fields with appropriate actions.

The form of the transformation rules to lowest order is always of the form (modulo

for higher spin fields. These are then necessarily free. In fact the same reasoning allows to show that gravity is universal, see [46].

conventions)

$$\delta E_M^A = \frac{1}{2}\bar{\epsilon}\Gamma^A\Psi_M, \tag{4.5a}$$

$$\delta\Psi_M = D_M\epsilon = \partial_M\epsilon + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}(E)\epsilon + \cdots . \tag{4.5b}$$

It is straightforward to show that indeed (4.4) is invariant under (4.5a) and (4.5b), when ignoring higher orders in the gravitini. We will not do this here but it can be found in [22].

It should be clear that the equations of motion for the Vielbein are Einstein's field equations with the energy-momentum tensor depending on the fermions Ψ_M. We will explore some of the properties of this field itself in the next section.

4.1.1.1 The Free Rarita Schwinger Field

We will go one step back and consider the fermionic part of (4.4) in a flat background. It is a relativistic theory of a spin$-\frac{3}{2}$ field called Rarita-Schwinger field [47].

$$S[\delta, \Psi] = -\frac{1}{2}\int d^D x \bar{\Psi}_M \Gamma^{MNP}\partial_N\Psi_P. \tag{4.6}$$

It should be noted that as of yet no elementary particle with spin $\frac{3}{2}$ has been found. The action eq. (4.6) is invariant under a gauge transformation

$$\delta\Psi_M = \partial_M\epsilon, \tag{4.7}$$

which is of course the flat space remnant of eq. (4.5b) with ϵ a Majorana spinor. The form of the action is almost fixed by requiring it to be first order in spacetime derivatives and symmetries. In fact, the Lagrangian itself is invariant only up to a total derivative as is typical for supersymmetric field theories.

The equations of motion following from the action (4.6) read

$$\Gamma^{MNP}\partial_N\Psi_P = 0. \tag{4.8}$$

By virtue of Gamma matrix manipulations (B.33) we can rewrite the equations of motion as $\partial\!\!\!/\Psi_M = \partial_M \Psi\!\!\!/$ or by writing $\Psi_{MN} = 2\partial_{[M}\Psi_{N]}$ as

$$\Gamma^N\Psi_{MN} = 0. \tag{4.9}$$

4.1.1.2 Matching Degrees of Freedom

Here we follow [43, lect. 4]. See also [22, pp. 130].
There is a distinction between counting independent components of a field and counting physical, propagating degrees of freedom. The first is called off-shell, the second on-shell counting.

Since gauge symmetries are to be understood as redundancies in the description, we have to subtract the number of gauge transformations. It is, however not always clear whether a specific gauge choice fully fixes the redundancy. Hence even off-shell counting can be rather involved. We refer to [22] for details.

On-shell degrees of freedom on the other hand is the number of independent components after imposing equations of motion. It will turn out that not all equations of motion reduce the number of degrees of freedom. On-shell degrees of freedom are typically understood as the propagating, physical degrees of freedom.

In nonrelativistic physics it is often hard to distinguish between the two as it is not always clear what is an equation of motion and what not. Usually, nonrelativistic degrees of freedom are not propagating in the relativistic sense. For the relativistic counting, see table 4.1.1.2.

(dim)		$\phi(x)$	$S(x)$	$A_M(x)$	$E_M^A(x)$	$\Psi(x)$	$\Psi_M(x)$	$A^{(p)}(x)$
D	OFF	1	1	$D-1$	$D(D-1)/2$	$2^{[\frac{D}{2}]}$	$2^{[\frac{D}{2}]}(D-1)$	$C(D-1,p)$
D	ON	1	0	$D-2$	$D(D-3)/2$	$\frac{1}{2}2^{[\frac{D}{2}]}$	$\frac{1}{2}2^{[\frac{D}{2}]}(D-3)$	$C(D-2,p)$

Figure 4.1: The (relativistic) fields from which supergravity multiplets are built with their respective off/on-shell degrees of freedom.

4.1.1.3 Minimal SUGRA in 4D

We see from table 4.1.1.2 that Ψ_M and E_M^A have the same number of on-shell degrees of freedom, namely 2. Thus there is the same amount of bosonic and fermionic physics, they can form a susy multiplet, described at leading order by:

$$S_0[E,\Psi] = \frac{1}{2}\int d^4x E\Big(E_A^{\ M} E_B^{\ N} R_{MN}^{\ \ \ AB}(\Omega(E)) - \bar{\Psi}_M \Gamma^{MNP} D_N(\Omega(E))\Psi_P\Big), \tag{4.10}$$

with $\Omega(E)$ as in eq. (3.26).

It turns out that the above minimal action is not invariant under supersymmetry (4.5b) / (4.5a). In particular there will be corrections from $\delta\Omega_N^{\ AB}(E)\bar{\Psi}\Gamma^{MNP}\Gamma_{AB}\Psi \sim \mathcal{O}(\Psi^3)$, which we can in principle cancel with additional terms of $\mathcal{O}(\Psi^4)$ in the action. Even though this is very non-trivial, it is how Freedman, van Nieuwenhuizen and Ferrara constructed the first supergravity in 1976 [48]. Nowadays, this approach is referred to as the second order formalism. There is more modern approach called 1.5 order formalism that we are sketching now. It is again a second order approach, in the sense that the spin connection is a dependent field and indeed the action will take the same form as above. However, it is also a bit first order formalism as we solve for the spin connection as an independent field and thus allowing for torsion contributions.

The basic idea is the following, assume the action to take the above form but change $\Omega(E) \mapsto \hat{\Omega}(E,\Psi)$, such that all the higher order terms are stored in the spin connection and thus the (super)covariant derivative, that is

$$S[E,\Psi] = \frac{1}{2}\int d^Dx E\Big(E_A^{\ M} E_B^{\ N} R_{MN}^{\ \ \ AB}(\hat{\Omega}(E,\Psi)) - \bar{\Psi}_M \Gamma^{MNP} \hat{D}_N(\hat{\Omega}(E,\Psi))\Psi_P\Big), \tag{4.11}$$

which now satisfies, similar to the procedure in the Palatini/first order formulation of GR in section (3.1.1.5)

$$\delta S = \int d^D x \Bigg\{ \frac{\partial \mathcal{L}}{\partial E}\Big|_{\Omega=\hat{\Omega}(E,\Psi)} \delta E + \frac{\partial \mathcal{L}}{\partial \Psi}\Big|_{\Omega=\hat{\Omega}(E,\Psi)} \delta \Psi + \underline{\frac{\partial \mathcal{L}}{\partial \Omega}}\Big|_{\Omega=\hat{\Omega}(E,\Psi)} \Big(\frac{\partial \hat{\Omega}(E,\Psi)}{\partial E} \delta E + \frac{\partial \hat{\Omega}(E,\Psi)}{\partial \Psi} \delta \Psi \Big) \Bigg\}. \tag{4.12}$$

Note how the first two variations will give rise to properly coupled Einstein and Rarita Schwinger field equations. The third term we would like to vanish. But we can achieve that by solving $\partial \mathcal{L}/\partial \Omega = 0$ for $\hat{\Omega}$. In this sense we are following a first order approach, regarding the spin connection as an independent field, but reinserting the solution to have a simpler second order action, where the spin connection is dependent. This is known as 1.5 order formalism as it stands somewhat in between the two.

What has to be done in practice is the following, find an analogous term to the no-torsion constraint eq. (3.12) and solve it for $\hat{\Omega}$. In fact, what we can do is to super-covariantize $2D_{[M}E_{N]}^{A} = 0$ to

$$2\hat{D}_{[M}E_{N]}^{A} \equiv 2D_{[M}E_{N]}^{A} - \frac{1}{2}\bar{\Psi}_M \Gamma^A \Psi_N = 0. \tag{4.13}$$

It is straightforward to prove that this transforms covariantly (that is without $\partial\epsilon$ terms). Note that we can reinterpret the quadratic gravitino term as a torsion contribution, so we see how supergravity necessitates torsion. The above can be solved for the super-spin connection (following the procedure around eq. (3.26)):

$$\begin{aligned}\hat{\Omega}_M^{AB}(E,\Psi) &= \Omega_M^{AB}(E) + \Omega_M^{AB}(\Psi) \\ &= \Big(\partial_{[M}E_{N]}^{A} E^{BN} - \partial_{[M}E_{N]}^{B} E^{AN} + E_{MC}E^{AN}E^{BP}\partial_{[N}E_{P]}^{C} \Big) \\ &+ \frac{1}{4}\Big(\bar{\Psi}_M\Gamma^A\Psi_N E^{BN} - \bar{\Psi}_M\Gamma^B\Psi_N E^{AN} + E_{MC}E^{AN}E^{BP}\bar{\Psi}_N\Gamma^C\Psi_P \Big).\end{aligned} \tag{4.14}$$

Introducing supersymmetric anholonomy coefficients similar to what we did in the bosonic case eq. (3.24):

$$\hat{C}_{MN}^{A} = 2\partial_{[M}E_{N]}^{A} - \frac{1}{2}\bar{\Psi}_M\Gamma^A\Psi_N, \tag{4.15}$$

we can write the torsionful connection as $\hat{\Omega} = \hat{C}_M^{[AB]} - 2^{-1}\hat{C}^{[AB]}{}_M$.

Inserting the explicit form into the action (4.11) it is straightforward to find the four-Fermi terms

$$\begin{aligned}S[E,\Psi] &= S_0[E,\Psi] + S_4[E,\Psi] \\ &= \int d^4xE \Bigg[\frac{1}{2}\{ E_A{}^M E_B{}^N R_{MN}{}^{AB}(\Omega(E)) - \bar{\Psi}_M\Gamma^{MNP}D_N(\Omega(E))\Psi_P \} \\ &- \frac{1}{32}\Big\{ (\bar{\Psi}^M\Gamma^N\Psi^P)(\bar{\Psi}_M\Gamma_N\Psi_P + 2\bar{\Psi}_M\Gamma_P\Psi_N) - 4(\bar{\Psi}^M\Gamma^P\Psi_P)(\bar{\Psi}_M\Gamma^P\Psi_P) \Big\} \Bigg].\end{aligned} \tag{4.16}$$

Note that this is equivalent to the compact form (4.11). Moreover we have to change the transformation rules of the gravitini slightly

$$\delta\Psi_M = \mathcal{L}_\xi\Psi_M - \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi_M + \hat{D}_M(\hat{\Omega})\epsilon. \tag{4.17a}$$

Since the supercovariant spin connection $\hat{\Omega}$ is not an independent field, we can in principle derive its transformation rules from eq. (4.14). In fact it is often more efficient to derive it from

$$\delta(\epsilon)2\hat{D}_{[M}(\hat{\Omega})E_{N]}^{A}=0, \tag{4.18}$$

yielding (after a lengthy calculation akin to (3.26))

$$\delta(\epsilon)\hat{\Omega}_M^{AB}(E,\Psi)=-\frac{1}{4}\bar{\epsilon}\Gamma_M\hat{\Psi}^{AB}+\frac{1}{2}\bar{\epsilon}\Gamma^{[A}\hat{\Psi}^{B]}{}_M, \tag{4.19}$$

where $\hat{\Psi}_{MN}=2\hat{D}_{[M}\Psi_{N]}$ and $\hat{\Psi}_{AB}\equiv E_A{}^M E_B{}^N\hat{\Psi}_{MN}$. These are called gravitino curvatures as they are built from the gravitini in a similar manner as are torsion and Riemann tensor from the Vielbein and the spin connection, respectively.

4.1.2 About **spoin**

In the 1960s S. Coleman and J. Mandula [49] investigated the relation between spacetime and internal symmetries. They formulated a no-go theorem, stating that one cannot extend the spacetime symmetry group without making the dynamics trivial. Indeed if we restrict to massive particles the Poincaré group is maximal. For massless particles there is a bigger group, though, the conformal group. As is often the case with no-go theorems, most interesting things can be extracted from violations*.

In the proof of the theorem they made a crucial assumption about the possible symmetries of the S-matrix. The trick is not to relax some of the assumptions but to redefine the very understanding of symmetry to encompass symmetries relating bosons to fermions. A more general structure called graded Lie algebra admits to introduce supersymmetry from a top-down perspective. The resulting graded symmetry algebra is called super Poincaré – or **spoin**. Haag, Łopuzański and Sohnius reexamined the results of Coleman and Mandula in a seminal paper [50]. They characterized the most general Lie superalgebras of S-matrix symmetries. Following the lines of Poincaré gauge theory, we will therefore derive SUGRA as the gauge theory of the super Poincaré algebra **spoin**.

4.1.2.1 Graded Lie Algebras

Describing the abstract invariance structure of supersymmetry requires the introduction of a new mathematical object. It builds on the previously defined Lie algebras, where we replace the commutator by a new bracket $[-,-\}$ being either symmetric or antisymmetric in the arguments. Roughly speaking we allow every element in the algebra to have a grade in $\mathbb{Z}_2$, such that the parity of the grades decides whether $[-,-\}$ is a commutator or an anticommutator. The main example will be that of a graded version of the Poincaré algebra - the super Poincaré algebra **spoin** - leading to a definition of SUGRA via a gauging procedure. In fact, the Clifford algebra introduced in appendix B is an example of a graded algebra.

In appendix chapter B we have argued/postulated that spinorial objects have to be Grassmannian, that is two Majorana parameters (ϵ_1,ϵ_2) obey

$$\epsilon_1\epsilon_2=-\epsilon_2\epsilon_1. \tag{4.20}$$

*Jokesters speak of *go-go* theorems.

When considering consecutive actions of infinitesimal transformations in section 3, specifically in (3.43), we exchanged two parameters. Hence, we see that we no longer consider the commutator $[\Gamma_m(t_A), \Gamma_m(t_B)] = f_{AB}^C \Gamma_m(t_C)$ but rather the anticommutator

$$\{\Gamma_m(t_A), \Gamma_m(t_B)\} \equiv \Gamma_m(t_A)\Gamma_m(t_B) + \Gamma_m(t_B)\Gamma_M(t_A) = f_{AB}^C \Gamma_m(t_C). \tag{4.21}$$

Since supersymmetry deals with fermionic parameters we are thus led to a generalization of Lie algebras called graded Lie algebras (sometimes also super Lie algebras).

To every element of the graded Lie algebra $\mathbf{g}$ with basis $\{t_A\}$ we associate a weight $|A| \in \mathbb{Z}_2$ and call the elements even and odd, respectively. Thus as a vector space $\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1$. Moreover, we introduce the graded bracket

$$[-, -\} : \quad \mathbf{g}_I \times \mathbf{g}_J \longrightarrow \mathbf{g}_{I+J(mod2)}, \qquad I, J \in \{0, 1\}, \tag{4.22}$$

that is it takes $[odd, odd\} \to even \leftarrow [even, even\}$ and $[odd, even\} \to odd \leftarrow [even, odd\}$. Moreover, it has the following symmetry property:

$$[t_A, t_B\} = -(-)^{|A||B|}[t_B, t_A\}. \tag{4.23}$$

For parity $|A||B|$ even, this is just the ordinary commutator and the anticommutator for odd parity. What is more, these have to obey a super Jacobi

$$[[t_A, t_B\}, t_C\} = [t_A, [t_B, t_C\}\} - (-)^{|A||B|}[t_B, [t_A, t_C\}\}, \tag{4.24}$$

where the minus sign comes from exchanging the 'A' and the 'B' basis element.

4.1.2.2 Super Poincaré Algebra: **spoin**

The minimal supersymmetric extension of the Poincaré algebra contains a so-called SUSY charge Q. This is a Majorana spinor and carries a grading of $|Q| = 1$. Let us interpret grade-0 charges as bosonic and grade-1 charges as fermionic, then according to eq. (4.23) $[B, B]$, $[B, F]$ and $\{F, F\}$, schematically. Contrary to most of the other parts in this work, it is convenient to display spinor indices explicitly: Q_α. The **spoin** is

$$[P_A, J_{[BC]}] = 2\eta_{A[B}P_{C]}, \qquad [J_{[AB]}, J_{[CD]}] = 4\eta_{[A[C}J_{D]B]}, \tag{4.25a}$$

$$[Q_\alpha, J_{[AB]}] = \frac{1}{2}\Big(\Gamma_{AB}\Big)_\alpha^{\ \beta} Q_\beta, \qquad [P_A, Q_\alpha] = 0, \tag{4.25b}$$

$$\Big\{Q_\alpha, Q_\beta\Big\} = -\frac{1}{2}\Big(\Gamma^A C^{-1}\Big)_{\alpha\beta} P_A. \tag{4.25c}$$

The second line tells us that the SUSY charges are spinors under four dimensional Lorentz transformations. It is the last line that contains the most nontrivial information. As said in words earlier, supersymmetry transformations are such that they square to spacetime transformations.

We can now state the following remarkable theorem: In any theory realizing the above SUSY algebra with invertible translations, the bosonic and fermionic degrees of freedom match.
To prove this, observe the action of the last commutator on the space of bosons:

$$(bos) \xrightarrow{Q} (ferm) \xrightarrow{Q} P_A(bos) \simeq (bos), \tag{4.26}$$

that is, bosons go to fermions go to translated bosons. But since we assumed translations to be invertible, all three spaces are of the same size. Hence there is an equal number of bosonic and ferionic degrees of freedom. See [26].

There is an important result by Haag, Łopuszánski and Sohnius [50] generalizing the result of Coleman and Mandula to SUSY algebras. Remarkably, when restricting to a single SUSY charge Q and massive representations the **spoin** algebra (4.25) is the only choice not leading to trivial dynamics. In this sense the above is not only minimal but also maximal. There are two ways how to generalize this. On the one hand one may go to massless realizations where we can extend the bosonic part to the conformal algebra. On the other hand we may allow for more than one SUSY charge $\{Q^I|I = 1, \cdots, N\}$, these are called $\mathcal{N} = N$ extended SUSY algebras or **espoin**s.

To discuss extended (in particular $\mathcal{N} = 2$) super Poincaré algebras, it is convenient to switch to Weyl spinors instead of Majorana spinors. See appendix B, especially (B.29). We start with N Majorana charges Q_I and define the left and right hand components by the index positions

$$q_I \equiv P_L Q_I, \qquad q^I \equiv P_R Q_I. \tag{4.27}$$

This means that the Majorana spinors are $q^I + q_I$ and $(q_I)^C = q^I$, see [22] for more details. It is then straightforward to show that

$$\Big\{q_{I\alpha}, q^J_\beta\Big\} = -\delta_I^J \frac{1}{2}\Big(P_L \Gamma^A C^{-1}\Big)_{\alpha\beta} P_A, \qquad \Big\{q^I_\alpha, q_{\beta J}\Big\} = -\delta^I_J \frac{1}{2}\Big(P_R \Gamma^A C^{-1}\Big)_{\alpha\beta} P_A. \tag{4.28}$$

All the other commutators generalize naturally. For $\mathcal{N} = 2$ there is, however, an interesting generalization of the $\{Q, Q\}$ anticommutators, by introducing a complex central charge Z:

$$\Big\{q_{I\alpha}, q_{J\beta}\Big\} = -\frac{1}{2}\epsilon_{IJ} P_{L\alpha\beta} Z, \qquad \Big\{q^I_\alpha, q^J_\beta\Big\} = -\frac{1}{2}\epsilon^{IJ} P_{R\alpha\beta} \bar{Z}. \tag{4.29}$$

Bogomol'nyi, Prasad and Sommerfield (BPS) have shown [51] that the mass is bound from below by the central charge: $M \geq |Z|$. We have seen in chapter 3 how a central charge relates to the mass of nonrelativistic fields. We will see how its role changes when going to nonrelativistic superalgebras.

It is possible to translate this back to Majorana fermions, suppressing spinor indices

$$\Big\{Q_I, Q_J\Big\} = -\frac{1}{2}\delta_{IJ}\Gamma^A C^{-1} P_A - \frac{1}{2}\epsilon_{IJ} C^{-1} Z. \tag{4.30}$$

There are higher N extended superalgebras with more general central charges, but we will not need them, see [22] for more.

4.1.2.3 **spoin** Representations

It is possible to apply the method of induced representations for **spoin**. The logic is very similar to that of section 3.1.2.2. Again, one would look for Casimir elements serving as labels for different representations. And indeed one finds that $P^A P_A$ is a Casimir again with the interpretation of mass. The Pauli-Lubansky $W^A W_A$ does not commute with the supercharges, though. This means that not all the fields in a multiplet have the same spin or helicity, respectively. Details of the construction can be found in [27, 32, 52].

It turns out that the different fields in a multiplet differ by units of spin-$\frac{1}{2}$. Similar to the bosonic treatment, there is a big difference between massive and massless representations. Since this work is mainly about supergravity, we are interested in massless multiplets only. For $\mathcal{N} = 1$, there are just two helicity states $(h, h + 1/2)$. Since the resulting theory should be CPT invariant, we always consider the four states $(\pm h, \pm h \pm 1/2)$. For us, the most important example is that of $(\pm 3/2, \pm 2)$, called the supergravity multiplet. The helicity states are appropriately interpreted as degrees of freedom of a graviton and a gravitino. In general, the number of fields in a super-multiplet will be $2^{\mathcal{N}}$ plus the CPT conjugates.

4.1.3 Gauging the **spoin**

As mentioned at the beginning of this chapter: Supergravity can be seen as a theory of local supersymmetry. Mathematically, this means that the SUSY parameter is a function of spacetime. This procedure is known as gauging. In section 3.1.4 we have shown how to do this for a spacetime symmetry algebra. Similarly, we will show how to derive the kinematics of SUGRA from a similar approach towards **spoin**. We will see that the gravitini can be seen as gauge fields of the supercharges. References include [22, 29].

4.1.3.1 Transformation Rules

In section 3.1.4 we extracted the kinematics of GR from the flat space isometry algebra **poin**. Similarly, we will show in this section how to get transformation rules and super covariant quantities of $\mathcal{N} = 1$, $D = 4$ SUGRA from **spoin**. Indeed, the general theory of gauging as laid out in section 3.1.3 is powerful enough to work for graded Lie algebras too.

The four dimensional $\mathcal{N} = 1$ super Poincaré algebra is spanned by three generators $\{P_A, J_{[AB]}, Q\}$ to which we associate gauge fields and parameters

$$
\begin{array}{lllll}
(GEN) & & (PAR) & & (GAU) \\
P_A & \leftrightarrow & \xi^A & \leftrightarrow & E_M^{\ A} \\
J_{[AB]} & \leftrightarrow & \lambda^{AB} & \leftrightarrow & \Omega_M^{\ AB} \\
Q & \leftrightarrow & \bar{\epsilon} & \leftrightarrow & \Psi_M
\end{array}
$$

The minimal **spoin** (4.25) defines four non-zero structure constants, where the form of the indices indicates the respective algebra element:

$$
\begin{aligned}
&f^{D}_{A[BC]} = 2\eta_{A[B}\delta^{D}_{C]}, \qquad f^{[EF]}_{[AB][CD]} = 8\eta_{[A[C}\delta^{[E}_{D]}\delta^{F]}_{B]}, \\
&f^{\beta}_{\alpha[AB]} = \frac{1}{2}\Big(\Gamma_{AB}\Big)_{\alpha}^{\ \beta}, \\
&f^{A}_{\alpha\beta} = -\frac{1}{2}\Big(\Gamma^{A}C^{-1}\Big)_{\alpha\beta}
\end{aligned}
$$

From eq. (3.57) we find for the translation and SUSY gauge field (eventually, Ω will become dependent and thus the naive transformation rule will be obsolete):

$$\delta E_M^{\ A} = D_M \xi^A - \lambda^A{}_B E_M^{\ B} + \frac{1}{2}\bar{\epsilon}\Gamma^A \Psi_M, \tag{4.31a}$$

$$\delta \Psi_M = D_M \epsilon - \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi_M. \tag{4.31b}$$

Equivalently, either using eq. (3.60c) or constructing covariant quantities from the above rules, we find the curvatures

$$\hat{R}_{MN}(P^A) = 2\hat{D}_{[M}E_{N]}^{\ A} = 2D_{[M}E_{N]}^{\ A} - \frac{1}{2}\bar{\Psi}_M\Gamma^A\Psi_N, \tag{4.32a}$$

$$\hat{R}_{MN}(Q) = 2\hat{D}_{[M}\Psi_{N]}. \tag{4.32b}$$

Again we can trade translational symmetry for diffeomorphism invariance (see eq. (3.63)) at the cost of

$$\hat{R}_{MN}(P^A) = \hat{C}_{MN}^{\ \ A} + 2\hat{\Omega}_{[M}^{\ AB}(E,\Psi)E_{N]}^{\ B} = 0, \tag{4.33}$$

where we have introduced the supercovariant anholonomy coefficients $\hat{C}_{MN}^{\ \ A}$ as in eq. (4.15). Also we have indicated that the curvature constraint can be solved for the torsionful spin connection $\hat{\Omega} = \hat{\Omega}(E,\Psi)$ to yield (4.14). To conclude, by emulating the strategy taken in **poin** gauging we obtained a theory with the field content and transformation rules (see eq.s (4.17)/(4.19)) of $\mathcal{N} = 1$ SUGRA in four dimensions.

4.1.3.2 SUGRA Algebra Closure

Even though the transformation rules (4.17) meet the formal requirements of SUSY transformation rules there is one thing left to check. In every (super)symmetry structure there is not just transformation rules but also a commutator representing the consecutive action of (super)symmetries. If the fields $(E_M^{\ A}, \Psi_M)$ together with eq.s (4.17) are to realize supersymmetry, they must obey

$$\Big[(susy), (susy)\Big] \in (\mathbf{spoin}).$$

In other words, the consecutive action of supersymmetry transformations on fields must again be a transformation in **spoin**. In the minimal setting, the fields transform under general coordinate transformations δ_ξ, local Lorentz transformations δ_λ and supersymmetry δ_ϵ, hence we require that

$$[\delta(\epsilon_1), \delta(\epsilon_2)]E_M^{\ A} = \Big(\delta_\xi - \delta_\lambda - \delta_\epsilon\Big)E_M^{\ A}, \tag{4.34a}$$

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\Psi_M = \Big(\delta_\xi - \delta_\lambda - \delta_\epsilon\Big)\Psi_M + \Big(\cdots\Big), \tag{4.34b}$$

where the additional terms $(\cdots)$ vanish if Ψ_M obeys its equations of motion. This is called on-shell closure of the algebra.

For the Vielbein we see that

$$
\begin{aligned}
\left[\delta_1, \delta_2\right] E_M^{\ A} &= \frac{1}{2}\bar{\epsilon}_2 \Gamma^A \hat{D}_M \epsilon_1 - (1 \leftrightarrow 2) \\
&= \frac{1}{2}\bar{\epsilon}_2 \Gamma^A \partial_M \epsilon_1 + \hat{\Omega}_M^{\ BC} \bar{\epsilon}_2 \delta^A_{[B} \Gamma_{C]} \epsilon_1 + \frac{1}{2}\hat{\Omega}_M^{\ BC} \bar{\epsilon}_2 \Gamma^A{}_{BC} \epsilon_1 - (1 \leftrightarrow 2) \\
&= \partial_M \xi^A + 2\hat{\Omega}_M^{\ AB} \xi_B \\
&= \mathcal{L}_\xi E_M^{\ A} - \lambda'^A_{\ B} + \frac{1}{2}\bar{\epsilon}' \Gamma^A \Psi_M - \xi^N \hat{R}_{MN}(P^A),
\end{aligned} \tag{4.35}
$$

where $\xi^M = \frac{1}{2} E_A^{\ M} \bar{\epsilon}_2 \Gamma^A \epsilon_1$, $\lambda'^{AB} = -\xi^M \hat{\Omega}_M^{\ AB}$ and $\varepsilon' = -\xi^M \Psi_M$. So we see that that the conventional constraint $\hat{R}_{MN}(P^A) = 0$ is again a condition of the symmetry algebra. See eq.s (4.34) for the structure. In the second line we have used gamma matrix rearrangements (B.35) and eq.s (B.25d).

The above result is remarkable as it generalizes our usual understanding of symmetry algebras. There we see that the commutator of two symmetries goes to another transformation described by structure constants. In the present case, however, these structure constants are no longer constant but dependent on the fields themselves. Some people use the name structure functions or soft algebras [22]. This is a characteristic of susy field theories.

We have shown that the (soft) algebra closes on the Vielbein. When checking the closure on the gravitini we would find that it closes only modulo equations of motion. That, too, is characteristic for SUSY field theories. A detailed calculation can be found in [42]. A SUSY field theory where the equations of motion have to be used is called on-shell. By adding auxiliary fields to the multiplet cancelling the respective terms one gets the so-called off-shell multiplet. The name comes from the fact that no equations of motion have to be imposed on the fields.

4.2 SNC FROM **SBARG**

In this section, we finally come to nonrelativistic supergravity (NSUGRA) – or: supersymmetric Newton-Cartan theory (SNC). Supergravity has been known for more than forty years, Newtonian gravity has been developed more the three hundred years ago. Still there is no supersymmetric extension of Newtonian gravity in four dimensions. In the recent years[8], some considerable headway has been made by constructing three dimensional SNC as a gauge theory of the supersymmetric extension of the Bargmann algebra **sbarg**. In principle this technique is applicable to arbitrary dimensions.

Nonrelativistic supersymmetry is not as well developed as relativistic SUSY. The possible physical realizations of the latter can be found by means of representation theory and counting arguments. Unfortunately, the situation for nonrelativistic field theories is worse as matching degrees of freedom is not a well defined concept.

In this section we review the **sbarg**-gauging construction of [8, 53]. As is indicated in the diagram, this is just one way to SNC. In chapter 6, we will see an alternative method obtaining SNC directly from a relativistic SUGRA.

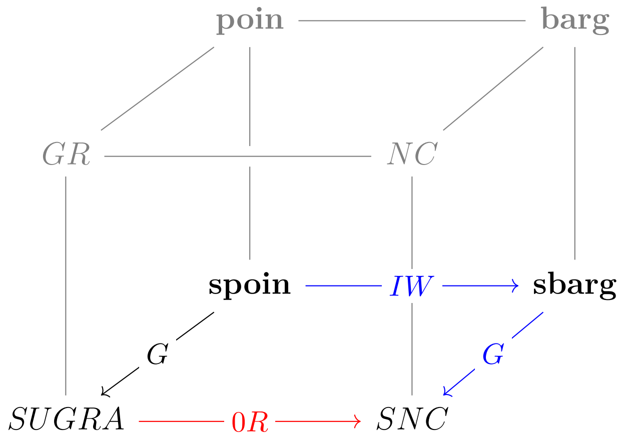

Figure 4.2: The strategies applied to obtain SNC. In this section we follow the blue path. The logic is to obtain **sbarg** via an Inönü Wigner contraction (IW) from **spoin** and gauging (G) it to get the kinematics of SNC. The other red path is the subject of chapters 5 and 6. Compare diagram 3.3.

4.2.1 About SNC

This section first introduces a nonrelativistic superalgebra in three spacetime dimension denoted as **sbarg**. The construction of SNC is alluding to the gauging procedures applied to **poin**, **barg** and **spoin** obtaining the kinematics of GR, NC and SUGRA, respectively. The main focus lies on the symmetry structure and geometrical aspects and not so much on the dynamics.

In introducing the supersymmetric Bargmann algebra, we will see that $\mathcal{N} = 2$ rather than $\mathcal{N} = 1$ is minimal. The reason is that the algebra with just one SUSY generator leads to unsatisfying commutation relations, in particular $\{Q, Q\} \sim M$, where M does not have the interpretation of a spacetime translation. Hence $\mathcal{N} = 1$ **sbarg** cannot be called a supersymmetric algebra. Conversely there are two gravitini in $\mathcal{N} = 2$ SNC, transforming asymmetrically under boosts. This can be seen as a characteristic of nonrelativistic SUGRA.

4.2.2 About **sbarg**

Here the supersymmetric extension of the Bargmann algebra is constructed. It is done by virtue of an Inönü-Wigner contraction of an $\mathcal{N} = 2$ super Poincaré algebra in three dimensions. It is conceivable that the logic of light cone embedding (laid out in appendix C) can be applied to an $\mathcal{N} = 1$ **poin** in four dimensions.

This construction follows the logic of 3.2.2. Remember that we had to add a central charge to **poin** to make the contraction to **barg**. In the setting of extended supersymmetric algebras, the occurence of central extensions related to mass is natural [51]. Remember that the $\mathcal{N} = 2$ **spoin** in three dimensions has the following QQ-commutator (see (4.30)

$$\left\{Q_I, Q_J\right\} = -\frac{1}{2}\delta_{IJ}\Gamma^A C^{-1} P_A - \frac{1}{2}\epsilon_{IJ} C^{-1} Z, \qquad (I, J = 1, 2).$$

The central extension Z is real. We define new SUSY charges as follows

$$Q_{\pm} \equiv 2^{-1/2}(Q_1 \pm \gamma_0 Q_2), \tag{4.36}$$

and make the following basis change:

$$\begin{aligned} &P_0 \to \tfrac{1}{2c}H + cM, \quad Z \to \tfrac{1}{2c}H - cM, && J_{a0} \to c\mathbf{G}_a, \\ &P_a \to \mathbf{P}_a, && J_{ab} \to \mathbf{J}_{ab}, \\ &Q_- \to \sqrt{c}Q_-, && Q_+ \to \tfrac{1}{\sqrt{c}}Q_+. \end{aligned} \tag{4.37}$$

It is then a straightforward task to rewrite the **spoin** and take the limit $c \to \infty$. The result is the $3D$ $\mathcal{N}=2$ super Bargmann algebra **sbarg**:

$$\begin{aligned} &[\mathbf{P}_a, \mathbf{J}_{bc}] = 2\delta_{a[b}\mathbf{P}_{c]}, && [\mathbf{G}_a, \mathbf{J}_{bc}] = 2\delta_{a[b}\mathbf{G}_{c]}, \\ &[H, \mathbf{G}_a] = \mathbf{P}_a, && [\mathbf{G}_a, \mathbf{P}_b] = \delta_{ab}M, \\ &[Q_\pm, \mathbf{J}_{ab}] = \tfrac{1}{2}\gamma_{ab}Q_\pm, && [Q_+, \mathbf{G}_a] = \tfrac{1}{2}\gamma_{a0}Q_-, \end{aligned}$$

$$\begin{aligned} &\left\{Q_+, Q_+\right\} = -\gamma^0 C^{-1}H, \quad \left\{Q_-, Q_-\right\} = -2\gamma^0 C^{-1}M, \\ &\left\{Q_+, Q_-\right\} = -\gamma^a C^{-1}\mathbf{P}_a \end{aligned} \tag{4.38}$$

Observe how $\{\mathbf{J}_{ab}, \mathbf{G}_a, \mathbf{P}_a, H, M, Q_-\}$ form a $\mathcal{N}=1$ subalgebra with the unsatisfying property that the SUSY charges do not anticommute to a spacetime transformation. This is the reason why $\mathcal{N}=2$ is taken as the minimal super algebra.

4.2.3 3D $\mathcal{N}=2$ SNC from Gauging the **sbarg**

The **sbarg** has seven basis elements to which we associate gauge fields and parameters as follows

$$\begin{array}{lclcl} (gen) & & (par) & & (gau) \\ H & \leftrightarrow & \xi^0 & \leftrightarrow & \tau_\mu \\ \mathbf{P}_a & \leftrightarrow & \xi^a & \leftrightarrow & e_\mu{}^a \\ \mathbf{G}_a & \leftrightarrow & \lambda^a & \leftrightarrow & \omega_\mu{}^a \\ \mathbf{J}_{[ab]} & \leftrightarrow & \lambda^{ab} & \leftrightarrow & \omega_\mu{}^{ab} \\ M & \leftrightarrow & \sigma & \leftrightarrow & m_\mu \\ Q_+ & \leftrightarrow & \bar{\epsilon}_+ & \leftrightarrow & \psi_{\mu+} \\ Q_- & \leftrightarrow & \bar{\epsilon}_- & \leftrightarrow & \psi_{\mu-} \end{array} \tag{4.39}$$

From the structure constants in (4.38) and (3.57) we infer the transformations rules of the independent bosonic fields. Similarly to the above we turn translations into diffeomorphisms by setting (as found from (3.60c))

$$\hat{r}_{\mu\nu}(\mathbf{P}^a) = 0 = 2\partial_{[\mu}e^a_{\nu]} + 2\hat{\omega}_{[\mu}{}^{ab}e_{\nu]b} + 2\hat{\omega}_{[\mu}{}^{a}\tau_{\nu]} - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}, \tag{4.40a}$$

$$\hat{r}_{\mu\nu}(M) = 0 = 2\partial_{[\mu}m_{\nu]} - 2\hat{\omega}_{[\mu}{}^{a}e_{\nu]a} - 2^{-1/2}\bar{\psi}_{\mu-}\gamma_0\psi_{\nu-}, \tag{4.40b}$$

$$\hat{r}_{\mu\nu}(H) = 0 = \hat{\tau}_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]} + 2^{-1/2}\bar{\psi}_{\mu+}\gamma_0\psi_{\nu+}, \tag{4.40c}$$

note that the last equation has a special role as it is the super covariant version of the no torsion condition and relates to the foliation of spacetime. The first and second are called conventional constraints and can be solved for ${\omega_\mu}^{ab}(e,\tau,m,\psi_\pm)$ and ${\omega_\mu}^{a}(e,\tau,m,\psi_\pm)$ as we will see later.
The transformation rules read

$$\delta {e_\mu}^a = \mathcal{L}_\xi {e_\mu}^a - \lambda^a{}_b {e_\mu}^b - \lambda^a\tau_\mu + 2^{-1}\Big(\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \bar{\epsilon}_-\gamma^a\psi_{\mu+}\Big), \tag{4.41a}$$

$$\delta\tau_\mu = \mathcal{L}_\xi\tau_\mu - 2^{-1/2}\bar{\epsilon}_+\gamma_0\psi_{\mu+}, \tag{4.41b}$$

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + e_{\mu a}\lambda^a - 2^{-1/2}\bar{\epsilon}_-\gamma_0\psi_{\mu-}. \tag{4.41c}$$

Similarly for the gravitino transformation rules

$$\delta\psi_{\mu+} = \mathcal{L}_\xi\psi_{\mu+} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \hat{D}_\mu(\hat{\omega})\epsilon_+, \tag{4.42a}$$

$$\delta\psi_{\mu-} = \mathcal{L}_\xi\psi_{\mu-} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu-} + 2^{-1/2}\lambda^a\gamma_{a0}\psi_{\mu+} + \hat{D}_\mu(\hat{\omega})\epsilon_- - 2^{-1/2}{\hat{\omega}_\mu}{}^a\gamma_{a0}\epsilon_+, \tag{4.42b}$$

with $\hat{D}_\mu(\hat{\omega}) = \partial_\mu + 4^{-1}{\hat{\omega}_\mu}{}^{ab}(e,\tau,m,\psi_\pm)\gamma_{ab}$ implicitly containing higher order fermion contributions. Observe how the $-$gravitino is boosted to the $+$gravitino but not the other way around. This is a characteristic feature of this supersymmetric Newton-Cartan theory (SNC). It breaks the exchange symmetry between ψ_+ and ψ_-.

${\hat{\omega}_\mu}{}^{ab}$ and ${\hat{\omega}_\mu}{}^{a}$ and thus $\hat{D}(\hat{\omega})$ have yet to be determined by virtue of conventional constraints. When introducing the notation $2\partial_{[\mu}e^a_{\nu]} \mapsto \hat{c}^{\,a}_{\mu\nu} \equiv 2\partial_{[\mu}e^a_{\nu]} - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}$ and $2\partial_{[\mu}m_{\nu]} \mapsto \hat{c}_{\mu\nu} \equiv 2\partial_{[\mu}m_{\nu]} - 2^{-1/2}\bar{\psi}_{\mu-}\gamma_0\psi_{\nu-}$, which are essentially super covariantized anholonomy coefficients, we find that

$${\hat{\omega}_\mu}{}^{ab}(e,\tau,m,\psi_\pm) = e^{\nu[a}\hat{c}^{\,b]}_{\mu\nu} - \frac{1}{2}e_{\mu c}e^{\nu a}e^{\rho b}\hat{c}^{\,c}_{\nu\rho} - \tau_\mu e^{\nu a}e^{\rho b}\hat{c}_{\nu\rho}, \tag{4.43a}$$

$${\hat{\omega}_\mu}{}^{a}(e,\tau,m,\psi_\pm) = \frac{1}{2}e^{\nu a}\hat{c}_{\mu\nu} - \frac{1}{2}\tau^\nu\hat{c}^{\,a}_{\mu\nu} + \frac{1}{2}e_{\mu b}\tau^\nu e^{\rho a}\hat{c}^{\,b}_{\nu\rho} + \frac{1}{2}\tau_\mu\tau^\nu e^{\rho a}\hat{c}_{\nu\rho}, \tag{4.43b}$$

or ${\hat{\omega}_\mu}{}^{ab} = {\hat{c}_\mu}{}^{[ab]} - 2^{-1}\hat{c}^{ab}{}_\mu - \tau_\mu\hat{c}^{ab}$ and $2\hat{\omega}_{\mu a} = \hat{c}_{a\mu} - \hat{c}_{\mu 0a} + \hat{c}_{0a\mu} + \tau_\mu\hat{c}_{0a}$.

It is possible to construct super covariant curvatures from the gravitini transformations (4.42)

$$\hat{r}_{\mu\nu}(Q_+) = \hat{\psi}_{\mu\nu+} = 2\hat{D}_{[\mu}\psi_{\nu]+}, \tag{4.44a}$$

$$\hat{r}_{\mu\nu}(Q_-) = \hat{\psi}_{\mu\nu-} = 2\hat{D}_{[\mu}\psi_{\nu]-} - 2^{1/2}\hat{\omega}_{[\mu}{}^{a}\gamma_{a0}\psi_{\nu]+} \tag{4.44b}$$

We have imposed curvature constraints, two conventional ones to solve for the spin connections and one additional no torsion constraint. We will see in later chapters how to relax that last condition. In SUSY it is crucial to check whether constraints are consistent:

$$\delta\hat{r}_{\mu\nu}(\mathbf{P}^a) = 0, \qquad \delta\hat{r}_{\mu\nu}(M) = 0, \qquad \delta\hat{r}_{\mu\nu}(H) = \delta\hat{\tau}_{\mu\nu} = 0. \tag{4.45}$$

Similar to the relativistic theory, the first two can be seen as a definition of $\delta\hat{\omega}_\mu{}^{ab}$ and $\delta\hat{\omega}_\mu{}^a$. The last however, is indeed quite nontrivial and indeed varies to

$$\begin{aligned}
& & & & &\xmapsto{\delta(\epsilon_-)} \hat{r}_{ab}(Q_-) = 0 \\
\hat{r}_{\mu\nu}(H) = 0 &\xmapsto{\delta(\epsilon_+)} \hat{r}_{\mu\nu}(Q_+) = 0 & & & &\xmapsto{\delta(\epsilon_+)} \hat{r}_{\mu\nu}(\mathbf{J}^{ab}) = 0 \\
& & & & &\xmapsto{\delta(\epsilon_+)} \gamma^a\hat{r}_{a0}(Q_-) = 0 \xmapsto{\delta(\epsilon_+)} \hat{r}_{0a}(\mathbf{G}^a) = 0\ .
\end{aligned} \tag{4.46}$$

See [53] for more details. Note how rich the constraint structure is. The first variation of the no torsion constraint forces the +gravitino curvature to zero. Variation of this yields that the spatial curvature, that is the curvature of the spatial slices has to vanish. Remember that we previously took that as an additional assumption. In SUGRA it follows from the foliation constraint. The other results can be seen as equations of motion for the −gravitino and metric, since $\hat{r}_{a0}(Q_-)$ is the only nonzero gravitino curvature left and containing a time derivative. On the other hand, as we saw, $\hat{r}_{0a}(\mathbf{G}^a)$ is the super covariant version of the Poisson equation (3.96).

There is a remarkably large amount of structure contained in the no torsion constraint. In the last years, a lot of effort has been made to weaken the no torsion constraint and thus generalize SNC to a nonrelativistic SUGRA, where the Poisson equation is not an immediate consequence of the constraint structure. This is the field of off-shell nonrelativistic SUGRA, see chapter 6.

4.2.3.1 Galilean SUGRA and the Dual Newtonian Potential

Similar to what we did in the bosonic theory, see section 3.2.3.4, it is now possible to go to coordinates adapted to the constraints in (4.46). The procedure for the bosonic fields goes through accordingly but on top of that we impose

$$\psi_{\mu+} = 0, \qquad \delta_a{}^\mu\psi_{\mu-} = 0, \tag{4.47}$$

braking the ϵ_+-symmetry to a rigid SUSY. One then finds the following nonzero components

$$m_0(x) = \delta_0{}^\mu m_\mu = \phi(x), \qquad \hat{\omega}_0{}^a = \delta_0{}^\mu\hat{\omega}_\mu{}^a = -\delta^{ab}\partial_b\phi(x), \qquad \psi_{0-} = \delta_0{}^\mu\psi_{\mu-} =: \Psi(x). \tag{4.48}$$

These obey equations $\gamma^a\hat{r}_{a0}(Q_-) = 0$ and $\hat{r}_{0a}(G^a) = 0$ (see eq. (4.46):

$$\gamma^a\partial_a\Psi = 0, \qquad \delta^{ab}\partial_a\partial_b\phi = 0, \tag{4.49}$$

related by supersymmetry. For a full discussion see [8].

CHAPTER 5

$\{0, \pm 1\}$ - REDUCTION

5.1 DIMENSIONAL REDUCTION

There is a two-fold motivation to consider dimensional reduction of higher dimensional theories. On the one hand it has turned out that superstring theories are consistent only in dimension $D = 10$. SUGRA theories occur as low-energy limits of them. When wanting to describe four dimensional physics, it is necessary to perform a dimensional reduction. In this *top-down* approach, the dimensional reduction is an imperative ingredient.

On the other hand, the dimensional reduction is itself a powerful technique for constructing theories in four dimensions. In this *bottom-up* approach one typically embeds a complicated theory into a simpler theory in higher dimensions.

In fact we apply dimensional reduction in the latter sense, allowing to embed nonrelativistic gravity in D dimensions into relativistic gravity in $(D+1)$ dimensions. This *bottom-up* approach can be read either way, as an embedding or a reduction. It was historically first done by [54, 55] and is called Null Reduction (ORED).

The idea of dimensional reduction in the first place entered physics almost a hundred years ago with work of Kałuza and Klein [56, 57]. They considered General Relativity in five spacetime dimensions and showed that the effective theory in four dimensions unifies gravity and electromagnetism.

5.1.1 Compact Dimensions and Kaluza-Klein Reduction

The crucial idea of Kaluza Klein (KK) theories is the following. Spacetime has more than four dimensions, which are not observable for some reason. Spacetime symmetries in these extra dimensions leave a trace on four dimensional physics, that can be interpreted as gauge invariances.

In this way, KK theories unify geometry with gauge theory. It is important to note the all-encompassing role of symmetry. Here we assume the extra dimension to be small. There are, however, other reasons why it is not observable, e.g. we could live on a four dimensional membrane embedded in a higher dimensional space to which we simply have no access, see [58] and references therein.

We will take the classical approach of assuming the extra dimension to be small, to be concrete we assume an extra circular dimension of small radius $\mathcal{M} = \mathbb{M}^{1,3} \times S^1$. By small we mean sufficiently small but since there is only one natural scale when talking about gravity – the Planck scale – we assume the length to be of the order of 10^{-33} cm. So there is a small fifth coordinate z with

$$z \sim z + 2\pi R. \tag{5.1}$$

Let us a massless scalar field in five dimensions $\phi(x,z)$. Since the compact manifold is a circle, we expand in Fourier modes $\phi(x,z) = \sum \exp(inR^{-1}z)\phi^{(n)}(x)$ and thus the massless KG equation in five dimensions induces field equations on the Fourier modes

$$\Box_5 \phi(x,z) = 0 \Longrightarrow \left(\Box_4 - n^2 R^{-2}\right)\phi^{(n)}(x) = 0. \tag{5.2}$$

The Fourier modes $\phi^{(n)}$ can be interpreted as massive Klein-Gordon fields with $m_n = |n/R|$. In this way, the five dimensional physics entails the dynamics of infinitely many fields in four dimensions. It is then a reasonable hope to organize dirty four-dimensional theories into a simple theory in higher dimensions.

The masses are suppressed by the size R of the extra dimension, though. Hence, when lowering the radius the excitations go to higher and higher energies. It is thus reasonable to truncate the spectrum to only the very lowest – that is $n = 0$ – excitation.

5.1.2 Symmetries

In chapter 3 we have shown that the dynamics of GR can be described in terms of a Vielbein E_M^A, with the following symmetries

$$\delta E_M^{\ A} = \xi^P \partial_P E_M^{\ A} + \left(\partial_M \xi^P\right) E_P^{\ A} - \lambda^A{}_B E_M^{\ B}. \tag{5.3}$$

A dimensional reduction as described above will break some of these symmetries.

Note that we cannot apply the above analysis naively to the gravitational field E_M^A. Whereas it is straightforward to interpret the Fourier modes of $(D+1)$–dimensional scalar field as scalar fields in D dimensions, it is not possible to interpret the Fourier modes of a $(D+1)$–dimensional Vielbein as D–dimensional Vielbeine. The reason is simply the fact that E_M^A and $E_\mu{}^a$ do not have the same number of components*. The same applies to all fields that are not scalars – e.g. vectors and spinors. It will however be possible to organize $D+1$ dimensional (irreducible) representations of **poin** into D dimensional, reducible representations of **poin**. Symmetries and index structures are vital in organizing the fields appropriately.

It is important to note that in KK type reduction we have to adapt coordinates to the compact manifold, that is separate the coordinates $(M) = (\mu, z)$, where the extra coordinate z is periodic as above. Since we are only interested in the lowest, massless mode we impose

$$\partial_z E_M^{\ A} = 0, \tag{5.4}$$

it is, however, useful to rephrase this condition more geometrically by imposing the existence of an isometry along the circle. This can be done by virtue of a Killing vector Ξ along the circle

$$\mathcal{L}_\Xi E_M^{\ A} = 0, \tag{5.5}$$

in adapted coordinates $\Xi = \partial_z$ (or $\Xi^M = \delta_z{}^M$) this reduces to the previously given independence of z. Note that this is a local constraint, from which it is not clear that the integral curves indeed close. In fact, we mostly do not use this condition at all, even though it guarantees consistency. Since this is the natural starting point for Null Reduction, let's rephrase it once more

Kaluza Klein reduction can be mathematically implemented by imposing the existence of a spacelike Killing vector, whose integral curves are S^1s.

D–dimensional hypersurfaces orthogonal to Ξ are then the spacetime of the reduced theory. When going to the metric formulation this induces a metric on the hypersurfaces by

*Note that in this section we suspend the usual convention of "lower-case indices for nonrelativistic quantities". Under dimensional reduction $(M) = (\mu, z)$, similarly the flat components $(A) = (a, \star)$.

$\Pi_{MN} = g_{MN} - g(\Xi,\Xi)^{-1}\Xi_M\Xi_N$. Since we are going to take the null-like limit $g(\Xi,\Xi) = 0$ eventually, this geometric projection does not work. It gives an indication of the degenerate metric structure of NC, though.

The existence of a Killing vector in adapted coordinates $\Xi = \partial_z$ breaks some of the diffeomorphism invariance

$$\mathcal{L}_\xi \Xi = 0 \quad \Longrightarrow \quad \partial_z \xi^M = 0. \tag{5.6}$$

So we see that not only the metric, but also the parameter of diffeomorphisms is independent of z. What is more, since the left hand side in $\delta E_M^A = \cdots$ (see eq. (5.3)) does not depend on the extra coordinate, we learn that $\partial_z \lambda^A{}_B = 0$. So we see that everything is independent of z.

Note that the diffeomorphisms are described by $(\xi^M)(x) = (\xi^\mu(x), \xi^z(x))$. We will see that ξ^μ naturally describes diffeomorphisms on the D–dimensional hypersurfaces, it is however less clear what the role of ξ^z is. Naively applying the rules

$$\delta_\xi E_\mu{}^A = \xi^\rho \partial_\rho E_\mu{}^A + \Big(\partial_\mu \xi^\rho\Big) E_\rho{}^A + \Big(\partial_\mu \xi^z\Big) E_z{}^A, \tag{5.7a}$$
$$\delta_\xi E_z{}^A = \xi^\rho \partial_\rho E_z{}^A, \tag{5.7b}$$

so we see how ξ^z appears as a kind of $U(1)$-gauge parameter in the lower-dimensional theory. In addition to that we require that the space-likeness of Ξ is preserved under all the symmetries, that is $\delta\Big(g(\Xi,\Xi) = 1\Big) = 0$, leading to the following constraints

$$E_z{}^a = 0, \qquad \lambda^a{}_\star = 0, \tag{5.8}$$

or, read differently: we have used part of the local Lorentz invariance ($\lambda^a{}_\star = 0$) to set some of the Vielbein components to zero ($E_z{}^a = 0$). It is worth mentioning that this choice is invariant under all the other symmetries.

Let us gather the results. We have imposed the existence of spacelike Killing vector Ξ on a $(D+1)$–dimensional geometry described by a Vielbein E_M^A. As a consequence of which we have found that all the Vielbein components and parameters are independent of the $(+1)$ dimension z and moreover that $E_z{}^a = 0$ and thus $\lambda^a{}_\star = 0$. Hence we have a theory in D dimensions described by the irreducible fields $(E_\mu{}^a, \phi \equiv E_z{}^\star, \phi A_\mu \equiv E_\mu{}^\star)$ transforming under symmetries $(\xi^\mu, \xi^z, \lambda^a{}_b)$ as

$$\delta E_\mu{}^a = \xi^\rho \partial_\rho E_\mu{}^a + \Big(\partial_\mu \xi^\rho\Big) E_\rho{}^a - \lambda^a{}_b E_\mu{}^b, \tag{5.9a}$$
$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + \Big(\partial_\mu \xi^\rho\Big) A_\rho + \partial_\mu \xi^z, \tag{5.9b}$$
$$\delta \phi = \xi^\rho \partial_\rho \phi. \tag{5.9c}$$

These symmetries show that the fields are tensors under diffeomorphisms of the D–dimensional hypersurfaces and tensors under local Lorentz transformations. Moreover, we see that A_μ is a $U(1)$-gauge field. The appearance can intuitively be understood as the stabilizer of the extra S^1-dimension. This kind of geometrization of $U(1)$–gauge theory has fuelled interest in KK–type theories from the very beginning. We will see how these three fields couple dynamically and thereby realize a unified theory of geometry, electromagnetism and some scalar field. In fact the above structure can be summarized by the following Ansätze

$$\left(E_M^{\ A}\right) = \begin{array}{c} \\ \mu \\ z \end{array}\begin{array}{c} \begin{array}{cc} a & \star \end{array} \\ \begin{pmatrix} E_\mu^{\ a} & \phi A_\mu \\ 0 & \phi \end{pmatrix} \end{array}, \qquad \left(E_A^{\ M}\right) = \begin{array}{c} \\ a \\ \star \end{array}\begin{array}{c} \begin{array}{cc} \mu & z \end{array} \\ \begin{pmatrix} E_a^{\ \mu} & -A_a \\ 0 & \phi^{-1} \end{pmatrix} \end{array}. \tag{5.10}$$

5.1.3 Kaluza Klein Reduction

In chapter 3 we showed that both the spin connection Ω_M^{AB} and the curvature R_{MN}^{AB} depend only on the Vielbein. Consequently, we could compute the objects $\Omega_\mu^{\ ab}(E, A, \phi)$, $\Omega_z^{\ ab}(E, A, \phi)$, $\Omega_\mu^{\ a\star}(E, A, \phi)$ and $\Omega_z^{\ a\star}(E, A, \phi)$ explicitly from eq. (3.26). Similarly for $R_{\mu\nu}^{\ \ ab}(\Omega_\mu^{\ ab}, \Omega_z^{\ ab}, \cdots)$ etc. Since we will do the somewhat more complicated case of Null Reduction in some detail, we will not do this here explicitly, but give the final result:

$$\begin{aligned} &\int d^{(D+1)}x |\det(E_M^{\ A})| E_A^{\ M} E_B^{\ N} R_{MN}^{\ \ AB} \\ &\qquad\qquad \propto \int d^D x |\det(E_\mu^{\ a})| \phi \Big[E_a^{\ \mu} E_b^{\ \nu} R_{\mu\nu}^{\ \ ab} - \frac{1}{4}\phi^2 F_{\mu\nu} F^{\mu\nu} \Big], \end{aligned} \tag{5.11}$$

note that this can be brought to other forms where kinetic terms for ϕ are explicit, for details see [29]. There are further subtleties and useful gauge choices which can be found in the same reference.

5.2 0 REDUCTION

In this section we introduce the mechanics of Null Reduction, building on the ideas of the last section to reproduce NC theory by virtue of a special type of dimensional reduction. The starting point is again the presence of a Killing vector representing the compact dimension. Contrary to Kaluza Klein reduction the Killing vector is no spacelike, though. It is assumed to be lightlike $\Xi^2 = 0$. The procedure is called Null Reduction and should be understood as a *bottom-up* approach to nonrelativistic geometry. It is worth pointing out that dimensional reduction with a timelike Killing vector leads to euclidean theories, see e.g. [59].

The idea of Null Reduction was introduced by [55] and [54]. We are following the presentation of [39]. Indeed the technique is very powerful and we will see in the next chapter how to apply it to SUGRA. Even for the purely bosonic case it is of merit to consider 0RED as it gives torsionful $\tau_{\mu\nu} \neq 0$ NC theory, see section 3.2.3.3.

To see how dimensional reduction can lead to nonreltativistic physics let us consider a complex massless scalar field in five spacetime dimensions as in section 5.1.1, see [12, 60]. We rewrite the equations of motion in lightcone coordinates, such that there are two explicit null directions (x^+, x^-). Without loss of generality we take x^+ to be the compact dimension. The framework of so-called generalized dimensional reduction [61] allows us to take a more general Ansatz, parametrized by a continuous parameter m. The $U(1)$-symmetry of the five-dimensional theory permits a phase factor dependent on the compact dimension $\phi(x^a, x^-, x^+) = \exp(-imx^+)\ \psi(x^a, x^-)$, yielding

$$\begin{aligned} \Box_5 \phi(x^a, x^-, x^+) &= \delta^{ab}\partial_a\partial_b\phi + 2\partial_-\partial_+\phi = 0 \\ &\Longrightarrow -i\dot{\psi}(x,t) + (2m)^{-1}\Delta\psi(x,t) = 0, \end{aligned} \tag{5.12}$$

where we relabelled $x^- \equiv t$. We see how the Schrödinger equation is embedded in the massless Klein-Gordon equation. Observe how a lightlike direction becomes a time direction in the four-dimensional theory. In general, the lightlike compactification of field theories will be more involved, though, see [62].

5.2.1 The Ansatz and NC from 0 Reduction

In this section we construct Null Reduction as a method to embed Galilean physics into relativistic physics in one dimension higher. Phrased differently, it is a compactification along a lightlike direction. In this section we consider the bosonic part, i.e. reduction of the Vielbein. The most natural starting point for Null Reduction is the existence of a Killing vector. But contrary to the previous section we take the Killing vector to be light – or null – rather than spacelike:

$$\mathcal{L}_\Xi E_M^{\ A} = 0, \qquad \text{and} \qquad \Xi^M \Xi^N E_M^{\ A} E_N^{\ B} \eta_{AB} = 0. \tag{5.13}$$

This geometric constraint on $(D+1)$–dimensional spacetime will implement nonrelativistic geometry. Intuitively, this can be understood as follows: the Bargmann group **Barg** is the stabilizer of a lightlike direction, see appendix C for a more concrete treatment.

Note that one cannot introduce just one null direction. When compactifying along one of these, there is still one distinguished direction of spacetime. In this way we implement the splitting of space and time as one of the lightcone coordinates will become the time coordinate in the nonrelativistic theory. We will introduce an adapted coordinate z just as in the spacelike case but make the distinction clearer in the flat, tangent space indices $(A) \mapsto (a, -, +)$, where $(+)$ corresponds to the compact spacetime coordinate z and $(-)$ to the new time coordinate. It should be clear from this that the Ansatz takes the following form

$$\left(E_M^{\ A}\right) = \begin{array}{c} \\ \mu \\ z \end{array} \!\! \begin{array}{c} \quad a \qquad - \qquad + \\ \begin{pmatrix} \square & \square & \square \\ 0 & \Xi^2 & \square \end{pmatrix} \end{array}, \tag{5.14}$$

where $E_z^{\ -} = \Xi^2 = 0$ indicates that the Killing vector is null-like, so there is really two zeros rather than just one in the KK ansatz (5.10). This will turn out to be of great importance in the supersymmetric theory. And indeed it is what makes the distinction to the ordinary KK reduction. The metric conventions are as follows $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$. In principle we can label the nonzero elements as we want but since we want to reproduce results from **barg**-gauging

$$\left(E_M^{\ A}\right) = \begin{array}{c} \\ \mu \\ z \end{array} \!\! \begin{array}{c} \quad a \qquad\quad - \qquad\quad + \\ \begin{pmatrix} e_\mu^{\ a} & S^{-1}\tau_\mu & -Sm_\mu \\ 0 & 0 & S \end{pmatrix} \end{array}, \qquad \left(E_A^{\ M}\right) = \begin{array}{c} \\ a \\ - \\ + \end{array} \!\! \begin{array}{c} \quad \mu \qquad z \\ \begin{pmatrix} e_a^{\ \mu} & m_a \\ S\tau^\mu & Sm_0 \\ 0 & S^{-1} \end{pmatrix} \end{array}, \tag{5.15}$$

where $m_a \equiv e_a^{\ \mu} m_\mu$ and $m_0 \equiv \tau^\mu m_\mu$. Note how this Ansatz implies the projective invertibility relations (3.74)

$$e_a^{\ \mu} e_\mu^{\ b} = \delta_a^{\ b}, \quad e_\mu^{\ a} e_a^{\ \nu} = \delta_\mu^{\ \nu} - \tau_\mu \tau^\nu, \quad \tau^\mu \tau_\mu = 1, \quad \tau^\mu e_\mu^{\ a} = 0, \quad e_a^{\ \mu} \tau_\mu = 0. \tag{5.16}$$

Note that the form of the Killing vector remains unchanged, so the discussion in the previous chapter leading to the diffeomorphism-reduction has not changed, hence we see that also the lower dimensional theory has GCT invariance mathematically represented by $\delta_\xi = \mathcal{L}_\xi$. Moreover, the field m_μ varies with a central charge $\partial_\mu \xi^z$.

On the side of local Lorentz transformations, we see that preservation of the "zeros" $(E_z{}^a, E_z{}^-)$ implies that

$$\delta E_z{}^a = 0 \qquad \Rightarrow \qquad \lambda^{a-} = 0. \tag{5.17}$$

Note that we had a similar thing in the spacelike reduction where $\lambda^{a\star}$ had to vanish. In this case, however, it has an effect on non-compact dimensions too and thus on the physics involved. By virtue of relabelling $\lambda^a{}_- \equiv \lambda^a$, $\lambda^{-+} \equiv \lambda$ and $\sigma \equiv -\xi^z$ we find

$$\delta \tau_\mu = \mathcal{L}_\xi \tau_\mu, \tag{5.18a}$$
$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - S^{-1} \lambda^a \tau_\mu, \tag{5.18b}$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + S^{-1} \lambda_a e_\mu{}^a, \tag{5.18c}$$
$$\delta S = -\lambda S. \tag{5.18d}$$

At this point we make a gauge fix by scaling $S \mapsto 1$ and consequently $\lambda \mapsto 0$. Note that this is a consistent gauge fix not interfering with the other symmetries*. With this we see that the transformation rules reduce to what we got from **barg**-gauging (3.84) and

$$\left(E_M{}^A\right) = \begin{array}{c} \\ \mu \\ z \end{array} \!\!\begin{array}{c} \begin{array}{ccc} a & - & + \end{array} \\ \begin{pmatrix} e_\mu{}^a & \tau_\mu & -m_\mu \\ 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{5.19}$$

And the inverse accordingly. This form of the Vielbein plays a role very analogous to eq. (5.10) containing the lower dimensional theory implicitly. In the same sense the above contains NC gravity in the sense of a compact object making manifest all the dirty nonrelativistic physics. One may be tempted to tackle the Einstein-Hilbert action (3.28) directly. In fact this will not work. The reason is that by $g(\Xi, \Xi) = 0$ one of the equations of motion, namely that coming from $(Ric)_{zz}$, goes missing. And unfortunately this is exactly the Poisson equation defining nonrelativistic gravity. This may be a coincidence, but it is unlikely. See [55] for more details.

An immediate application of eq. (5.19) is the following. In chapter 2 we showed how to embed the nonrelativistic particles into a massless particle in one dimension higher, see (2.24). It is then straightforward to coupled that to to arbitrary backgrounds by replacing $\eta_{MN} \to \eta_{AB} E_M^A E_N^B$ yielding

$$\begin{aligned} S[X(x^0, \mathbf{x}, z); e, \tau, m] &= \int d\tau E^{-1} \dot{X}^M \dot{X}^N E_M^A E_N^B \eta_{AB} \\ &= m \int d\tau \Big[(\dot{x}^\mu \tau_\mu)^{-1} \dot{x}^\mu \dot{x}^\nu e_\mu{}^a e_\nu{}^b \delta_{ab} - 2 m_\mu \dot{x}^\mu \Big], \end{aligned}$$

which reduces to eq. (2.25) upon gauge fixing to Galilean gravity as in section 3.2.3.4. Observe how m_μ appears as a Wess-Zumino term usually describing the coupling of an electromagnetic field to a electrically charged particle. Hence we see again, that m_μ can be interpreted as the gravitational analogon of the photon field coupling to particles with they charge being mass.

*This is no longer true when considering conformal gravity, see [39] for details.

5.2.2 Fermions

Fermions are very sensitive to the number of spacetime dimensions (and signature). We have seen in appendix B that the number of spinor-components doubles at every jump from an odd to the succeeding even spacetime dimension. It thus follows that spinors have two components in three dimensions and four in four dimensions. Under a dimensional reduction the number of fermions doubles:

$$(D=4,\mathcal{N}=1)\Longrightarrow(D=3,\mathcal{N}=2). \tag{5.20}$$

Hence, the nonrelativistic supergravity is by default an extended supergravity. We have come to the same conclusion via group theoretic arguments in section 4.2.3.

Four dimensional spinors Ψ will decompose into two-component $\Psi_\pm$ as follows

$$\Psi=\Psi_+\otimes\chi_-+\Psi_-\otimes\chi_+, \tag{5.21}$$

where $\chi_\pm$ are constant, two-component spinors on the compact space. In fact, we have to be cautious with the three dimensional objects, as it is not guaranteed that they will transform properly under 3-dimensional diffeomorphisms.

Another thing to consider is the decomposition of corresponding Γ-matrices. It is of the form $\Gamma_A\sim\gamma_a\otimes\sigma_i$. Indeed, we are working with three spinor spaces and hence three representations of the Clifford algebra. For the four-dimensional theory we use Γ_A, for the three-dimensional theory γ_a and σ_i for the internal, reducible space. It is natural to go to a lightcone-basis:

$$\Big\{\Gamma_a=\gamma_a\otimes\mathbb{1}_2,\qquad\Gamma_0=\gamma_0\otimes\sigma_1,\qquad\Gamma_3=\gamma_0\otimes\Big(i\sigma_2\Big)\Big\} \tag{5.22a}$$

$$\downdownarrows\quad\downdownarrows\quad\downdownarrows\quad\downdownarrows\quad\downdownarrows\quad\downdownarrows\quad\downdownarrows$$

$$\Big\{\Gamma_a=\gamma_a\otimes\mathbb{1}_2,\qquad\Gamma_\pm=2^{-1/2}\Big(\Gamma_0\pm\Gamma_3\Big)=\gamma_0\otimes\sigma_\pm\Big\}, \tag{5.22b}$$

where $\sigma_\pm=2^{-1/2}(\sigma_1\pm i\sigma_2)$. In the end, this decomposition is engineered so that the $so(1,3)$-generators decompose nicely:

$$\Gamma_{ab}=\gamma_{ab}\otimes\mathbb{1}_2,\qquad\Gamma_{a\pm}=\gamma_{a0}\otimes\sigma_\pm,\qquad\Gamma_{+-}=-\mathbb{1}_2\otimes\sigma_3. \tag{5.23}$$

It then follows that the charge conjugation matrix decomposes as follows

$$C^{(4)}=c^{(3)}\otimes\sigma_1. \tag{5.24}$$

Hence Majorana spinors in four dimensions will turn into Majorana spinors in three dimensions. Another useful object is

$$\Gamma_*=i\gamma_0\otimes\sigma_3\qquad\text{obeying}\qquad\{\Gamma_\pm,\Gamma_*\}=0. \tag{5.25}$$

We now pick a basis in the internal space by

$$\chi_+=\begin{pmatrix}1\\0\end{pmatrix},\qquad\chi_-=\begin{pmatrix}0\\1\end{pmatrix} \tag{5.26}$$

obeying

$$\sigma_\pm\chi_\pm=0,\qquad\sigma_\pm\sigma_\mp=\sqrt{2}\chi_\pm \tag{5.27}$$

where σ_i are the Pauli matrices. From the form of the charge conjugation and the internal spinors $\chi_\pm$ it follows that

$$\varepsilon = \epsilon_- \oplus \epsilon_+, \qquad \text{but} \qquad \bar{\varepsilon} = \left(\bar{\epsilon}_+ \oplus \bar{\epsilon}_-\right)^t. \tag{5.28}$$

We will see in the next section that the reduction of spin$-\frac{3}{2}$ needs some more attention.

Some of these reductions come out of the blue, apparently. In fact they were found from the perspective of reproducing results in section 4.2. References [63, 64] have been useful in obtaining the correct Clifford matrix reduction.

CHAPTER 6

NONRELATIVISTIC OFF-SHELL SUPERGRAVITY

In this section we construct nonrelativistic supergravity from Null Reduction. We not only reproduce results from section 4.2.3, but make way for nonrelativistic off-shell supergravity. Off-shell multiplets include auxiliary fields that cannot be derived from a gauging principle. The resulting theory is more general and reduces to the earlier results upon setting all the auxiliary fields to zero. It is worth mentioning that in an off-shell theory we do not impose equations of motion, in particular not those of NC. Hence the name SNC is no longer justified, rather it should be seen as the construction of a consistent nonrelativistic multiplet, to which we will refer as nonrelativistic supergravity (NSUGRA). What is more, the off-shell theory turns out to be consistent even with non-zero torsion. This surprising fact puts the theory on a more general level than the previously considered multiplets of [8, 53, 65].

6.1 0 REDUCTION OF $4D$, $\mathcal{N}=1$ SUGRA

Just as in the purely bosonic case, we will start with a Null Reduction of the transformation rules. Everything else will follow from there. So the natural starting point are eq.s (4.17). Just considering the diffeomorphism invariance we learn that the two-component objects $\Psi_{\mu\pm}$ in $\Psi_\mu = \Psi_{\mu+}\otimes\chi_- + \Psi_{\mu-}\otimes\chi_+$ do not transform as vectors under diffeomorphisms in the three-dimensional theory but $\delta_\xi\Psi_{\mu\pm} = \mathcal{L}_\xi\Psi_{\mu\pm} + (\partial_\mu\xi^z)\Psi_{z\pm}$. From eq.s (5.18) it should be clear that $\psi_{\mu\pm}$ in the Ansatz $\Psi_{\mu\pm}\equiv\psi_{\mu\pm} - m_\mu\psi_{z\pm}$ transforms as a vector. Alternatively we could transform to flat indices $\Psi_M E_A^{\ M}$ where the reduciton can be done naively. The decomposition of the gravitini reads

$$
\begin{aligned}
&\Psi_z = \psi_{z-}\otimes\chi_+ + \psi_{z+}\otimes\chi_- \\
\leftrightarrow\quad &\bar\Psi_z = \bar\psi_{z+}\otimes\chi_+^t + \bar\psi_{z-}\otimes\chi_-^t
\end{aligned}
\tag{6.1a}
$$

$$
\begin{aligned}
&\Psi_\mu = \Big(\psi_{\mu-} - m_\mu\psi_{z-}\Big)\otimes\chi_+ + \Big(\psi_{\mu+} - m_\mu\psi_{z+}\Big)\otimes\chi_- \\
\leftrightarrow\quad &\bar\Psi_\mu = \Big(\bar\psi_{\mu+} - m_\mu\bar\psi_{z+}\Big)\otimes\chi_+^t + \Big(\bar\psi_{\mu-} - m_\mu\bar\psi_{z-}\Big)\otimes\chi_-^t,
\end{aligned}
\tag{6.1b}
$$

or short $\Psi_z = \psi_{z-}\oplus\psi_{z+}$ and $\Psi_\mu = (\psi_{\mu-} - m_\mu\psi_{z-})\oplus(\psi_{\mu+} - m_\mu\psi_{z+})$. Note that the order is different on the conjugate ones.

We are now in the position to reduce the transformation rules as given in eq.s (4.5a) and (4.17). In the purely bosonic reduction it was the variation of $E_z^{\ a} = 0$ giving rise to the breaking of relativistic symmetries to nonrelativistic ones, so let's start with

$$\delta E_z^{\ a} = -\lambda^a_{\ +} + 2^{-1}\Big(\bar\epsilon_+\gamma^a\psi_{z-} + \bar\epsilon_-\gamma^a\psi_{z-}\Big) = 0, \tag{6.2a}$$

$$\delta E_z^{\ -} = -2^{-1/2}\bar\epsilon_+\gamma_0\psi_{z+} = 0, \tag{6.2b}$$

$$\delta E_z^{\ +} = -\lambda^+_{\ +} - 2^{-1/2}\bar\epsilon_-\gamma_0\psi_{z-} = 0. \tag{6.2c}$$

Note that the second condition can be true for arbitrary variations only if $\psi_{z+}=0$, which is a direct consequence of the nullity of the Killing vector. Hence it should not be interpreted as a gauge-fixing condition, but a proper feature of nonrelativistic (super)geometry. Note, however, that there is no other condition setting ψ_{z-} to zero and in fact it will play a crucial role in the off-shell construction. For convenience,

$$\psi_{z+} = 0, \qquad\qquad \psi\equiv\psi_{z-}. \tag{6.3}$$

The other conditions in eq.s (6.2) can be solved similarly. Contrary to the bosonic case, we cannot assume that $\lambda^a{}_+ = 0$ and $\lambda^{+-} = 0$ but rather impose the weaker

$$\lambda^a{}_+ = 2^{-1}\bar{\epsilon}_+\gamma^a\psi, \tag{6.4a}$$
$$\lambda^{+-} = 2^{1/2}\bar{\epsilon}_-\gamma_0\psi. \tag{6.4b}$$

The fact that constraints on the fields lead to nontrivial constraints on the parameters is common in SUSY field theories. The transformations induced by parameters $(\lambda^a{}_+(\psi), \lambda^{+-}(\psi))$ are known as compensating transformations.
Taking these into account we find the altered transformation rules of the independent bosonic fields

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a - \lambda^a{}_b e_\mu{}^b - \lambda^a\tau_\mu + 2^{-1}\bar{\epsilon}_+\gamma^a\psi_{\mu-} + 2^{-1}\bar{\epsilon}_-\gamma^a\psi_{\mu+}, \tag{6.5a}$$
$$\delta\tau_\mu = \mathcal{L}_\xi\tau_\mu - 2^{-1/2}\bar{\epsilon}_+\gamma_0\psi_{\mu+} - 2^{-1}e_\mu{}^a\bar{\epsilon}_+\gamma_a\psi - 2^{-1/2}\tau_\mu\bar{\epsilon}_-\gamma_0\psi, \tag{6.5b}$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + \lambda_a e_\mu{}^a + 2^{-1/2}\bar{\epsilon}_+\gamma_0\psi. \tag{6.5c}$$

Where we set $\lambda^a{}_- = \lambda^a$ and $\sigma = -\xi^z$. It is instructive to compare these to the **sbarg** gauging results eq.s (4.41). Clearly, the extra terms come from the compensating transformations. Upon setting $\psi = 0$ they reduce to the earlier results. In going further, we will see that ψ should really be interpreted as a fermionic auxiliary field.

From this extended form of transformation rules we can construct extended curvatures

$$\hat{\tau}_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]} + 2^{-1/2}\bar{\psi}_{\mu+}\gamma_0\psi_{\nu+} - e_{[\mu}^{\ a}\bar{\psi}_{\nu]+}\gamma_a\psi - 2^{1/2}\tau_{[\mu}\bar{\psi}_{\nu]-}\gamma_0\psi, \tag{6.6a}$$
$$\hat{r}_{\mu\nu}(\mathbf{P}^a) = 2\partial_{[\mu}e_{\nu]}^{\ a} + 2\hat{\omega}_{[\mu}^{\ ab}e_{\nu]b} + 2\hat{\omega}_{[\mu}^{\ a}\tau_{\nu]} - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}, \tag{6.6b}$$
$$\hat{r}_{\mu\nu}(M) = 2\partial_{[\mu}m_{\nu]} - 2\hat{\omega}_{[\mu}^{\ a}e_{\nu]a} - 2^{-1/2}\bar{\psi}_{\mu-}\gamma_0\psi_{\nu-}, \tag{6.6c}$$

where only the timelike curvature (torsion) differs from the gauging results (4.40). Since the conventional constraints match it is clear that the spin connections will be the same as those obtained from gauging (4.43).

As in the purely bosonic theory see eq.s (D.1) we can equally well reduce the four dimensional spin connections $\hat{\Omega}_M^{AB}(E,\Psi)$ as given in (4.14)

$$\hat{\Omega}_\mu{}^{ab} = \hat{\omega}_\mu{}^{ab} - 2^{-1}m_\mu\hat{\tau}^{ab}, \tag{6.7a}$$
$$\hat{\Omega}_\mu{}^{a+} = -\hat{\omega}_\mu{}^a - 2^{-1}m_\mu\hat{\tau}^{a0}, \tag{6.7b}$$
$$\hat{\Omega}_\mu{}^{a-} = 2^{-1}\hat{\tau}_\mu{}^a - 2^{-1}\bar{\psi}_{\mu+}\gamma^a\psi, \tag{6.7c}$$
$$\hat{\Omega}_\mu{}^{+-} = 2^{-1}\hat{\tau}_\mu{}^0 + 2^{-1/2}\bar{\psi}_{\mu-}\gamma_0\psi, \tag{6.7d}$$
$$\hat{\Omega}_z{}^{ab} = 2^{-1}\hat{\tau}^{ab}, \qquad \hat{\Omega}_z{}^{a+} = 2^{-1}\hat{\tau}^{a0}, \tag{6.7e}$$
$$\hat{\Omega}_z{}^{a-} = 0 = \hat{\Omega}_z{}^{+-}, \tag{6.7f}$$

where $\hat{\Omega} = \hat{\Omega}(e,\tau,m,\psi_\pm,\psi)$, but $\hat{\omega} = \hat{\omega}(e,\tau,m,\psi_\pm)$. Note that we have not imposed any condition on $\hat{\tau}_{\mu\nu}$ whatsoever. When considering the transformation rules of the gravitini fields, especially $\delta\psi_{z+} = 0$ we will see that torsion is tied to the persistence of the extra

ψ. The gravitino transformation rules can be found from the relativistic $\delta_\varepsilon \Psi_M = \partial_M \varepsilon + 4^{-1}\hat{\Omega}_M^{AB}(E,\Psi)\varepsilon$:

$$\delta\psi_{z+} = \frac{1}{8}\Big(\hat{\tau}^{ab}\epsilon_{ab} - 2\sqrt{2}\bar{\psi}\psi\Big)\gamma_0\epsilon_+ = 0, \tag{6.8a}$$

$$\delta\psi = \mathcal{L}_\xi\psi - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi + \frac{\sqrt{2}}{4}\hat{\tau}^{a0}\gamma_{a0}\epsilon_+ + \frac{1}{8}\Big(\hat{\tau}^{ab}\epsilon_{ab} + \sqrt{2}\bar{\psi}\psi\Big)\gamma_0\epsilon_-. \tag{6.8b}$$

The first constraint shows that with $\psi \to 0$ we immediately get twistless torsion $\hat{\tau}^{ab} = 0$:

$$\hat{\tau}^{ab}\epsilon_{ab} = 2\sqrt{2}\bar{\psi}\psi. \tag{6.9}$$

Note that this equation will give further constraints on geometric objects under SUSY variation. The hope is that this sequence of variations closes at some order, that is

$$\delta_\epsilon \circ \cdots \circ \delta_\epsilon(E_z^{\ -} = 0) = 0 \tag{6.10}$$

identically. We will see that in the off-shell construction it actually closes at order three. In the on-shell case we have checked it up to fifth order but stopped eventually and went over to the off-shell case.

So we see that setting $\psi = 0$ forces $\hat{\tau}^{ab} = 0$. Requiring consistency by varying $\delta(\psi = 0) = 0$ as in eq. (6.8b) we infer that in fact twistless torsion implies zero torsion $\hat{\tau}_{\mu\nu} = 0$. Hence it is the extra fermion ψ that carries the non-zero torsion. The other transformation rules read

$$\begin{aligned}\delta\psi_{\mu+} &= \mathcal{L}_\xi\psi_{\mu+} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \hat{D}_\mu\epsilon_+ + \frac{\sqrt{2}}{4}\hat{\tau}_\mu^{\ a}\gamma_{a0}\epsilon_- + \frac{1}{4}\hat{\tau}_\mu^{\ 0} \\ &\quad + \frac{\sqrt{2}}{4}(\bar{\epsilon}_-\psi)\gamma_0\psi_{\mu+} - \frac{\sqrt{2}}{4}(\bar{\psi}_{\mu-}\psi)\gamma_0\epsilon_+,\end{aligned} \tag{6.11a}$$

$$\begin{aligned}\delta\psi_{\mu-} &= \mathcal{L}_\xi\psi_{\mu-} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu-} + \frac{1}{\sqrt{2}}\lambda^a\gamma_{a0}\psi_{\mu+} + \hat{D}_\mu\epsilon_- - \frac{1}{\sqrt{2}}\hat{\omega}_\mu^{\ a}\gamma_{a0}\epsilon_+ \\ &\quad + e_\mu^{\ a}\lambda_a\psi - \frac{1}{4}\hat{\tau}_\mu^{\ 0}\epsilon_- - \frac{\sqrt{2}}{4}(\bar{\epsilon}_-\psi)\gamma_0\psi_{\mu-} + \frac{\sqrt{2}}{4}(\bar{\psi}_{\mu-}\psi)\gamma_0\epsilon_-.\end{aligned} \tag{6.11b}$$

Again, $\hat{D}_\mu = \partial_\mu + 4^{-1}\hat{\omega}_\mu^{\ ab}(e,\tau,m,\psi_\pm)\gamma_{ab}$.

6.2 OLD MINIMAL SUGRA IN $D = 4$

We have seen that the soft algebra of SUGRA closes only if the gravitino equations of motion are used. This kind of behaviour is characteristic of supersymmetric field theories and is called on-shell closure. The corresponding multiplet is given the name on-shell alike. This is in agreement with the name on-shell used in counting degrees of freedom, see table 4.1.1.2. In an on-shell multiplet, the on-shell degrees of freedom match.

It is, however, possible to add auxiliary fields to the on-shell multiplet. These will transform nontrivially under SUSY such that the algebra closes without using equations of motion. Such a closure is called off-shell, the corresponding multiplet accordingly. In four dimensions, the minimal on-shell SUGRA multiplet is (E_M^A, Ψ_M) with two real on-shell degrees of freedom (see table 4.1.1.2). The off-shell degrees do not match, though

$$\#_{off}\Psi_M - \#_{off}E_M^{\ A} = 12 - 6 = 6, \tag{6.12}$$

hence we have to add auxiliary fields with $\#_{off}BOS - \#_{off}FERM = 6$. The simplest such choice is a vector A_M and a complex scalar F. This multiplet is called *old-minimal*

$$\Big\{E_M^{\ A}, \Psi_M, A_M, F, \bar{F}\Big\}. \tag{6.13}$$

The transformation rules are taken and adapted from [22, p. 335]* and [42]:

$$\delta E_M^{\ A} = \mathcal{L}_\xi E_M^{\ A} - \lambda^a{}_B E_M^{\ B} + \frac{1}{2}\bar{\varepsilon}\Gamma^A\Psi_M, \tag{6.14a}$$

$$\begin{aligned}\delta P_L\Psi_M &= \mathcal{L}_\xi\Psi_M - \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi_M \\ &\quad + P_L\Big(\partial_M + \frac{1}{4}\hat{\Omega}_M^{\ AB}\Gamma_{AB} - \frac{3i}{2}A_M + \frac{i}{2}\Gamma_M \not{A} + \frac{1}{2\sqrt{3}}\Gamma_M\bar{F}\Big)\varepsilon,\end{aligned} \tag{6.14b}$$

$$\delta A_M = \mathcal{L}_\xi A_M - \frac{i}{4}\bar{\varepsilon}\Gamma_*\Big(- \Gamma^N\hat{R}_{MN}(Q) + \frac{1}{6}\Gamma_M\Gamma^{AB}\hat{R}_{AB}(Q)\Big), \tag{6.14c}$$

$$\delta F = \mathcal{L}_\xi F + \frac{\sqrt{3}}{6}\bar{\varepsilon}P_R\Gamma^{AB}\hat{R}_{AB}(Q), \tag{6.14d}$$

where $P_L\hat{R}_{MN}(Q) = 2P_L\Big(\partial_{[M} + \frac{1}{4}\hat{\Omega}_{[M}^{\ AB}\Gamma_{AB} - \frac{3i}{2}A_{[M} + \frac{i}{2}\Gamma_{[M}\not{A} + \frac{1}{2\sqrt{3}}\Gamma_{[M}\bar{F}\Big)\Psi_{N]}$ and $\not{A} \equiv \Gamma^A E_A^{\ M} A_M$.

By virtue of $P_{L/R} = 2^{-1}(\mathbb{1} \pm \Gamma_*)$ (see eq. (B.29) we can rewrite

$$\begin{aligned}\delta\Psi_M &= \mathcal{L}_\xi\Psi_M - \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi_M + \Big(\partial_M + \frac{1}{4}\Omega_M^{\ AB}\Gamma_{AB} - \frac{3i}{2}A_M\Gamma_*\Big)\varepsilon \\ &\quad + \frac{1}{2}\Gamma_M\Big(\frac{1}{2\sqrt{3}}(F + \bar{F}) + \frac{1}{2\sqrt{3}}(F - \bar{F})\Gamma_* - i\Gamma_*\not{A}\Big)\varepsilon\end{aligned} \tag{6.15}$$

and correspondingly

$$\begin{aligned}\hat{R}_{MN}(Q) = 2\Big(&\partial_{[M} + \frac{1}{4}\hat{\Omega}_{[M}^{\ AB}\Gamma_{AB} - \frac{3i}{2}A_{[M}\Gamma_* + \frac{i}{2}\Gamma_{[M}\not{A}\Gamma_* \\ &+ \frac{1}{4\sqrt{3}}\Gamma_{[M}\Big(F + \bar{F}\Big) + \frac{1}{4\sqrt{3}}\Gamma_{[M}\Big(F - \bar{F}\Big)\Gamma_*\Big)\Psi_{N]}.\end{aligned} \tag{6.16}$$

In all the above $\hat{\Omega}$ is a solution to the conventional constraint $\hat{R}_{MN}(P^A) = 0$ just as in (4.14).

For completeness, the corresponding action reads:

$$\begin{aligned}S_{o-m}[E, \Psi, A, M] = \frac{1}{2}\int d^4x E\ \Big[&\Big(R(\Omega(E, \Psi)) \\ &- \bar{\Psi}_M\Gamma^{MNP}(\partial_N + \frac{1}{4}\hat{\Omega}_N^{\ AB}(E, \Psi)\Gamma_{AB})\Psi_P + 6A^C A_C\Big) - F\bar{F}\Big].\end{aligned} \tag{6.17}$$

*Also discussing other off-shell realizations in $D = 4$ like *new-minimal* and $16 + 16$. All of them are such that the auxiliary fields are purely bosonic. In principle one could add fermionic auxiliary fields, too. We will see that in the nonrelativistic setting. It should be noted that it is hard to find valid off-shell realizations in general. For instance, there is no known off-shell multiplet for $D = 11$, $\mathcal{N} = 1$ SUGRA.

6.3 NONRELATIVISTIC OFF-SHELL SUGRA IN $D = 3$

From the above it is in principle straightforward to discern the nonrelativistic kinematics via Null Reduction. Since there is a vector appearing in the multiplet, we have to be careful with the reduction:

$$A_z \longmapsto a, \tag{6.18}$$

$$A_\mu \longmapsto a_\mu - m_\mu a. \tag{6.19}$$

Hence the field content must be the following: the on-shell multiplet together with an auxiliary fermion ψ, a vector a_μ and three real scalars $(f, \bar{f}, a)$:

$$\Big\{E_M^{\ A}, \Psi_M, A_M, F, \bar{F}\Big\} \overset{ORED}{\longrightarrow} \Big\{e_\mu^{\ a}, \tau_\mu, m_\mu, \psi_{\mu+}, \psi_{\mu-}, \psi, a_\mu, f, \bar{f}, a\Big\}, \tag{6.20}$$

apparently the kinematics will be much more complicated than what we have seen for the relativistic multiplet. It should however be noted that only the fermionic transformation rules will change. Consequently also the conventional constraints and thus the spin connections will be unaltered. This means that results (6.5), (6.6) and (6.7) are completely unaltered.

We have seen that the crucial constraint on geometry is represented by $\Xi^2 = 0$, that is the Killing vector being null-like. This had as immediate consequence that $E_z^{\ -} = 0$. It is not clear a priori, whether this condition is a consistent one. So we want to check whether the series of variations

$$\delta_\epsilon \circ \cdots \circ \delta_\epsilon\Big(\Xi^2 = 0\Big),$$

vanishes at sufficiently low order. The worst case scenario would be that variation yields to an overconstrained system. The consequence of which would be a trivial theory of supergravity. Luckily this does not happen. In fact, $\delta_\epsilon \circ \delta_\epsilon(\Xi^2 = 0)$ yields

$$\hat{\tau}^{ab}\epsilon_{ab} = 2\sqrt{2}\bar{\psi}\psi + 12a, \tag{6.21}$$

being an immediate generalization of eq. (6.9). It turns out that variation of this is zero identically. So the chain of variations closes at order three:

$$\delta_\epsilon \circ \delta_\epsilon \circ \delta_\epsilon\Big(\Xi^2 = 0\Big) = 0. \tag{6.22}$$

The corresponding variations can be found in appendix D.

The resulting theory with the multiplet as given in eq. (6.20) is an $D = 3$, $\mathcal{N} = 2$ nonrelativistic off-shell supergravity. The theory has arbitrary torsion, distinguishing it from earlier results [8, 53, 65]. In the relativistic context it has been possible to classify 3-manifolds allowing the introduction of supersymmetric field theories [9]. Building on earlier results of [53], J. Liu and collaborators [66] have applied the same analysis to nonrelativistic curved spaces. It is conceivable that this generalized, torsional supergravity gives rise to a richer spectrum of curved nonrelativistic spaces, generalizing the results of [66]. A publication is in preparation [11].

CHAPTER 7

OUTLOOK

In this master's thesis we have discussed how three-dimensional supersymmetric Newton-Cartan theory can be constructed both from a gauging principle and from a Null Reduction. The latter method has not been used in the context of nonrelativistic supergravity before. It reproduces all the results obtained in earlier work [53]. What is more, it has been possible to construct an off-shell theory of nonrelativistic supergravity. This theory has arbitrary torsion. Remarkably the multiplet contains a fermionic auxiliary field allowing for a consistently torsional theory.

With that, we not only gain a deeper understanding of the theory, but open it up for interesting applications. An upcoming publication [11] will generalize earlier results on rigid supersymmetry on curved NC backgrounds by [66]. Ultimately, the objective is to extend the powerful machinery of SUSY localization techniques to nonrelativistic field theories. This is a direction of research that has not been pursued so far, but it is likely to give rise to many exciting results in the upcoming years.

The construction of off-shell NSUGRA is also very encouraging from an academic point of view. We believe that four-dimensional SNC is within reach. In particular, the method of Null Reduction is powerful enough to generalize results obtained in this work to arbitrary dimensions. This would include finding the supersymmetric extension of Newtonian gravity for the first time.

In the first chapter we presented some aspects of the nonrelativistic particle, it is possible to extend this approach to extended objects, e.g. consider nonrelativistic string theory [67–69]. It would be interesting to use methods developed here and consider the interplay of SNC with nonrelativistic p-branes and learn more about the role of supersymmetry[70].

Another extension we would like to mention is nonrelativistic holography [6, 7]. These realizations do not consider supersymmetric theories, it would be interesting to learn about possible applications of SNC in this context, bearing in mind that SUGRA features prominently in the currently best understood example of holography AdS/CFT.

In this context it would also be of interest to clarify the relation to the extended Bargmann supergravity [71] and Hořava-Lifshitz gravity[72]. The off-shell NSUGRA is, as mentioned, not tied to any equations of motion. Hence it should be seen as a consistent nonrelativistic SUSY multiplet that can be used to construct three-dimensional theories other than SNC, such as higher derivative theories. In particular it may even be possible to construct supersymmetric extensions of Hořava-Lifshitz gravity, which has not been possible yet.

APPENDIX A

CONVENTIONS

We use the so-called mostly plus metric convention:

$$(\eta_{AB}) = \mathrm{diag}(-1, +1, \cdots, +1). \tag{A.1}$$

A consistent change to the mostly minus metric involves $g \mapsto -g$ and $\Gamma_A \mapsto i\Gamma_A$. When using light cone coordinates we choose them such that $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$.

There are five kinds of re-occurring index conventions, in general we use capital for relativistic and lower-case for nonrelativistic quantities:

1. $(A, B, C, \cdots)$ referred to as *flat indices*; mostly for tangent space tensors; transforming under local Lorentz transformations. Also: Lie algebra indices.

2. $(M, N, P, \cdots)$ referred to as *curved indices*; for spacetime tensors; transforming under diffeomorphisms/GCT.

3. $(a, b, c, \cdots)$ referred to as *flat indices*; tangents to the spatial slices in NC geometries, that is on $e_\mu{}^a$; transforming under local rotations and Galilean boosts; range $1, \cdots, D-1$.

4. $(\mu, \nu, \rho, \cdots)$ referred to as *curved indices*; nonrelativistic spacetime tensors; transforming under diffeomorphisms/GCT; range $0, 1, \cdots, D-1$.

5. $(\alpha, \beta, \cdots)$ referred to as *spinor indices*; spinors; transforming under rotations in the spinor representation; range $\lfloor D/2 \rfloor$.

We antisymmetrize and symmetrize with *weight one*:

$$M_{[AB]} = \frac{1}{2}M_{AB} - \frac{1}{2}M_{BA}, \tag{A.2a}$$

$$M_{(AB)} = \frac{1}{2}M_{AB} + \frac{1}{2}M_{BA}, \tag{A.2b}$$

and similarly for higher (anti)symmetrizations: $M_{[A_1A_2\cdots A_n]} = (n!)^{-1}(M_{A_1A_2\cdots A_n} - +)$.

Exterior derivatives are defined as follows

$$d\alpha^{(p)} = \frac{1}{p!}\partial_M \alpha^{(p)}_{N_1\cdots N_p} dx^{N_1} \wedge \cdots \wedge dx^{N_p}, \tag{A.3}$$

or $(d\alpha)_{MN_1\cdots N_p} = \partial_{[M}\alpha_{N_1\cdots N_p]}$.

The Levi-Civita tensor density is defined as

$$\varepsilon_{01\cdots(D-1)} = -1, \qquad \varepsilon^{01\cdots(D-1)} = +1, \tag{A.4}$$

note that here we differ from [26]. The contraction identity reads

$$\varepsilon_{A_1\cdots A_r B_1\cdots B_s}\varepsilon^{A_1\cdots A_r C_1\cdots C_s} = -r!s!\delta^{C_1}_{[B_1}\cdots\delta^{C_s}_{B_s]}, \qquad (r+s=D). \tag{A.5}$$

For tensor products of matrices/vector spaces we take the convention that the last factor determines the large scale structure, that is e.g.

$$\begin{bmatrix} u \\ v \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xu \\ xv \\ yu \\ yv \end{bmatrix}, \qquad [\mathbb{1} \otimes \sigma_1] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{A.6}$$

We will, however, almost never write out matrices explictly.

APPENDIX B

SPINORS IN ARBITRARY DIMENSIONS

Here we present some results from spinor algebra and Clifford algebras. This is crucial for supersymmetry and supergravity. Since in this work we always consider spacetimes with exactly one time direction, some of the results presented here are not in the most general form. We will try to mention as little mathematical details as possible while not doing the field theoretic derivation, though. So we will not introduce the spinor as a necessity in the Dirac equation, but from a more top-down perspective as a representation of $Spin(3) = SU(2)$, the double cover of $SO(3)$. It turns out that the Clifford algebra generates it infinitesimally. We follow [26] both in spirit and notation, where generic signatures are considered. Other useful references are [22, 27].

B.1 GAMMA MATRICES AND THEIR SYMMETRIES

The Clifford algebra is generated by D objects $\{\Gamma_A | A = 0, \cdots, D-1\}$, subject to the condition

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}\mathbb{1}, \tag{B.1}$$

or $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}\mathbb{1}$. As mentioned above, we consider spacetime with the following signature $(\eta) = \mathrm{diag}(-, +, +, \cdots, +)$.

It is possible to represent the abstract algebra by a set of D matrices $(\Gamma_A)^\alpha{}_\beta$. It is nontrivial to see how big these matrices have to be at mimimum such that the representation is faithful. We will find out shortly. First let us try to guess a representation in $D = 2$, that is $(\eta) = \mathrm{diag}(-1, +1)$. It is clear that it is not possible with 1×1-matrices, that is c-numbers. So we find that it must at least be 2×2-matrices:

$$\Gamma_0 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \Gamma_1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{B.2}$$

where we really only need the following $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}, i, j, k = 1, 2, 3$ with $\sigma_3 = \mathrm{diag}(1, -1)$. With that we see that adjoining the third Pauli matrix we can also represent the 3-dimensional Clifford algebra. Going to higher spacetime dimensions the representation matrices will have to be bigger. It is easy to check that the following gives an explicit representation for arbitrary dimensions

$$\begin{aligned}
\Gamma_0 &= i\sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \\
\Gamma_1 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \\
\Gamma_2 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \cdots \\
\Gamma_3 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \cdots \\
\Gamma_4 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \cdots \\
\cdots &= \cdots,
\end{aligned} \tag{B.3}$$

where $\mathbb{1}$ is the two dimensional unit matrix. It is straightforward to check that this indeed satisfies eq. (B.1). What we learn from this specific representation consist of $2^{\lfloor D/2 \rfloor} \times 2^{\lfloor D/2 \rfloor}$-matrices, where $\lfloor - \rfloor$ is the floor function rounding down. Hence we see that for any even and the following odd spacetime dimension the gamma matrices are of the same size.

The above is one representation of the Clifford algebra. We may ask ourselves whether every representation can be brought to this form by changing basis. In other words: how many inequivalent classes of representations are there, where we take two representations to be unitary equivalent if

$$\Gamma'_A = U^\dagger \Gamma_A U, \tag{B.4}$$

where U is unitary. It is clear that this preserves eq. (B.1). It turns out that the answer is different for even and odd dimensions respectively.

In the explicit representation the timelike matrix is antihermitian $\Gamma_0^\dagger = -\Gamma_0$, whereas the spacelike ones are hermitian $\Gamma_a^\dagger = \Gamma_a$. Since we only consider unitary equivalence, this property is one of the whole equivalence class. In general this can be expressed as

$$\Gamma_A^\dagger = \Gamma_0 \Gamma_A \Gamma_0. \tag{B.5}$$

Let us now restrict to **even dimensions** for the moment and consider arbitrary products of gamma matrices, it is clear from the defining property eq. (B.1) that everything can be reduced to antisymmetric products. This, however, limits the number of possible elements so we have a finite number of basis elements $\{\Gamma^{(n)} | n = 0, \cdots, D\}$ with

$$\Gamma^{(n)} \equiv \Gamma_{A_1 \cdots A_n} \equiv \Gamma_{[A_1} \cdots \Gamma_{A_n]}. \tag{B.6}$$

In particular $\Gamma_{AB} = 2^{-1}\Gamma_A\Gamma_B - 2^{-1}\Gamma_B\Gamma_A$ and $\Gamma_A\Gamma_B = \eta_{AB} + \Gamma_{AB}$. This last equation is an example how we decompose general products of gamma matrices into elements $\Gamma^{(n)}$, we will later see a method for the generic case.
Let us first of all focus on the **even dimensional case**. Then $\{\Gamma^{(n)}\}$ forms a basis for the vector space of $2^{\lfloor D/2 \rfloor} \times 2^{\lfloor D/2 \rfloor}$-matrices.
One can define a top element $\Gamma_* \propto \Gamma^{(D)}$, obeying

$$\{\Gamma_*, \Gamma_A\} = 0, \qquad \Gamma_*^2 = \mathbb{1}. \tag{B.7}$$

by the following

$$\Gamma_* = (-i)^{D/2+1} \Gamma_0 \Gamma_1 \cdots \Gamma_{D-1}. \tag{B.8}$$

The second property in (B.7) will prove to be of great importance for physical applications. In the representation which we started with

$$\Gamma_* = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots . \tag{B.9}$$

In even dimensions it is possible to make an identification of the form $\Gamma_{(r)} = \varepsilon \Gamma_* \Gamma^{(D-r)}$, explicitly

$$\Gamma_{A_1 \cdots A_r} = \frac{i^{D/2+1}}{(D-r)!} \varepsilon_{A_1 \cdots A_r B_{(r+1)} \cdots B_D} \Gamma_* \Gamma^{B_D \cdots B_{(r+1)}}, \tag{B.10}$$

this will prove to be useful in rearrange large arrays of Γ-matrices.

To this point we have focussed on the even dimensional case. The top element Γ_* commutes with all the other elements and squares to one. Hence, if $\{\Gamma_0, \cdots, \Gamma_{D-1}\}$ is a basis in D even dimensions, then

$$\{\Gamma_0, \cdots, \Gamma_{D-1}, \Gamma_*\} \tag{B.11}$$

forms a natural basis in $(D+1)$ **odd dimensions**.

Given a representation $\{\Gamma_A\}$ satisfying eq. (B.1) it is possible to construct equivalent ones by virtue of a basis change U, see eq. (B.4). It is clear, that the hermitian conjugate, the transpose and the complex conjugate $\{\Gamma_A^\dagger, \Gamma_a^t, \Gamma_A^*\}$ too satisfy the Clifford algebra (B.1). It is, however, less clear whether they are unitary equivalent to $\{\Gamma_A\}$. The most general transformation preserving eq. (B.1) is induced by virtue of matrices (B, C) and signs (ϵ, η):

$$\Gamma_A^t = -\eta C \Gamma_A C^{-1}, \qquad C^t = -\epsilon C, \tag{B.12a}$$
$$\Gamma_A^* = +\eta B \Gamma_A B^{-1}, \tag{B.12b}$$

where the two conventional signs $\epsilon = \pm 1$ and $\eta = \pm 1$ depend on the spacetime dimension D. These will play an important role shortly. From the fact that $\Gamma^* = (\Gamma^\dagger)^t$ and (B.5) it is clear that

$$B^t = -C\Gamma_0, \qquad \text{and} \qquad B^* B = \epsilon\eta. \tag{B.13}$$

It is a fair question to ask whether all representations of the Clifford algebra are unitary equivalent (B.4). We will not prove this here, but the statement is: in even dimensions there is just one equivalence class of representations, while in odd dimensions there are two. The standard reference is [73].

It is relevant for supersymmetry applications to know for which values of n the matrix $C\Gamma^{(n)}$ is (anti)symmetric, that is to determine the values $\theta(\epsilon, \eta, n)$:

$$\left(C\Gamma^{(n)}\right)^t = -\theta(\epsilon, \eta, n) C\Gamma^{(n)}, \qquad \theta = (-)^{n(n-1)/2}(-\eta)^n \epsilon. \tag{B.14}$$

It turns out that for odd dimensions there is always one unique solution whereas there are two for even dimensions. We do not prove it here but reproduce the results in table B.1 as given in [22] for various n and dimensions D

The full derivation can be found in [26]. It should be noted that the numbers under $\theta = \pm 1$ are the values of n with (as it turns out) a periodicity of four. We have boxed the choices that are especially useful for supersymmetry. In practice we would choose the values of ϵ and η to obtain symmetries of $C\Gamma^{(n)}$.

B.2 CLIFFORD ALGEBRAS AND ROTATIONS

Now, the gamma matrices furnish a representation of the Clifford algebra but at the same time they naturally act on a complex vector space $\mathcal{S}$ of dimension $2^{\lfloor D/2 \rfloor}$. The gamma matrices generate endomorphisms on $\mathcal{S}$. The vector space $\mathcal{S}$ - the representation space - is called spinor space and its elements $\lambda \in \mathcal{S}$ are called spinors. In fact the name spinor is meant exclusively for dimension three, however we will not be dogmatic and use it more generally.

Among the elements $\{\Gamma^{(n)}\}$ there are two of particular interest, one we already mentioned: Γ_*. The other is $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]} = 2^{-1}[\Gamma_A, \Gamma_B]$. It is a tedious, but straightforward task to check that the $2^{-1}D(D-1)$ matrices

$$\left(\frac{1}{4}\Gamma_{AB}\right)^\alpha_{\ \beta} \text{ obey } \left[-\frac{1}{4}\Gamma_{AB}, -\frac{1}{4}\Gamma_{CD}\right] = 4\eta_{[A[C}\Big(-\frac{1}{4}\Gamma_{D]B]}\Big), \tag{B.15}$$

so it furnishes a representation of $so(1, D-1)$ eq. (3.49c). Hence it is fair to label

D(mod 8)	$\theta = +1$	$\theta = -1$	ϵ	η
0	0, 3	1, 2	–1	+1
	0, 1	2, 3	–1	–1
1	0, 1	2, 3	–1	–1
2	0, 1	2, 3	–1	–1
	1, 2	0, 3	+1	+1
3	1, 2	0, 3	+1	+1
4	1, 2	0, 3	+1	+1
	2, 3	0, 1	+1	–1
5	2, 3	0, 1	+1	–1
6	2, 3	0, 1	+1	–1
	0, 3	1, 2	–1	+1
7	0, 3	1, 2	–1	+1

Figure B.1: A useful collection of possibles choices of ϵ and η and their consequences on symmetries of $C\Gamma^{(n)}$. The values of n in the second and third column are to be understood modulo four. The boxed rows are especially apt for supersymmetry applications.

$$\Gamma_s(J_{[AB]}) = -\frac{1}{4}\Gamma_{AB}, \tag{B.16}$$

and call it the spinor representation of $so(1, D-1)$ (indeed this is true for arbitrary signature). Hence, spinorial objects – fermions – transform under infinitesimal Lorentz transformations as

$$\delta(\lambda)\Psi = -\frac{1}{4}\lambda^{AB}\Gamma_{AB}\Psi. \tag{B.17}$$

An example of particular interest is three dimensional eulidean spacetime with the rotation algebra $so(3)$ and $[J_i, J_j] = \epsilon_{ijk}J_k$. Right when we began the gamma matrix chapter we saw that the Pauli matrices obey exactly the same algebra. So they represent the rotation algebra. But indeed they are already the generators of another algebra $su(2)$. While $so(3) \equiv su(2)$, the respective groups are not fully equivalent, though. In fact, $SU(2)$ is known as the double cover of $SO(3)$, which is a topological rather than a local distinction [74].

More generally, the double cover of $SO(p, q)$ is called $Spin(p, q)$. This can be classified by virtue of Clifford algebras (B.17). Representation spaces of $Spin(p, q)$ are called spinor spaces $\mathcal{S}$, see [73].

B.3 IRREDUCIBLE SPINORS

Actually we have not yet shown whether the objects $\lambda \in \mathcal{S}$ in the representation space are irreducible representations of $Spin(1, D-1)$. We will try to find conditions and projections reducing the objects further.

In even dimensions we have the highest element Γ_*, squaring to the identity by definition. In odd dimensions the highest element is itself trivial by construction. So it only makes sense to define projections in even dimensions

$$P_L = \frac{1}{2}(\mathbb{1} + \Gamma_*), \qquad P_R = \frac{1}{2}(\mathbb{1} - \Gamma_*). \tag{B.18}$$

It is important to note that both projections commute with Γ_{AB}, hence $\lambda \in \mathcal{S}$ cannot be irreducible as $P_L\lambda$ and $P_R\lambda$ are *smaller* representations. We call the resulting $(P_L\lambda, P_R\lambda)$ spinors Weyl (or left/right chiral spinors). For our applications these are of less importance, it should be noted, however, that the use of chiral spinors is crucial in the standard model of particle physics.

Another possibility is a kind of reality condition:

$$\lambda^* = R\lambda, \tag{B.19}$$

for some matrix R. Consistency with Lorentz transformations (B.17) $(\delta\lambda)^* = R\delta\lambda$ yields:

$$B\Gamma_{AB}B^{-1}R = R\Gamma_{AB} \qquad \longrightarrow R = \alpha B, \tag{B.20}$$

by virtue of Schur's lemma. Also, we require that $\lambda^{**} = \lambda$ and thus $R^*R = \mathbb{1}$. Hence α is just a phase but more importantly $B^*B = \mathbb{1}$ in eq. (B.13). This means (for our choice of signature with just one time-direction) that $\epsilon\eta = 1$. From the table we presented earlier we discern that we can impose such a condition in dimensions $D = 0, 1, 2, 3, 4 \mod 8$. We do not care about most of them but are happy that in both three and four dimensions we can impose the so-called Majorana condition (B.19) consistently.

It should be noted that even though condition (B.19) is called a reality condition it does not necessarily mean that the object λ is real. What it means is that there exists a basis in which Γ_{AB} is real and thus there is a basis in which the spinors are real. We now present two explicit representations in four dimensions apt for Majorana and chiral spinors respectively.

B.4 FOUR DIMENSIONS

Going back to table B.1, we see that we have to choose $\epsilon = +1 = \eta$ as $\epsilon\eta = +1$ by the Majorana condition (B.19). There exists a basis (which is not that of eq. (B.3)) yielding really real basis:

$$\begin{aligned}
&\Gamma_0 = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix}, \\
&\Gamma_3 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \qquad\qquad \Gamma_* = \begin{bmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix}.
\end{aligned} \tag{B.21}$$

Which we could also write as (applying the conventions (A.6)): $\Gamma_0 = \sigma_3 \otimes i\sigma_2$, $\Gamma_1 = \sigma_1 \otimes (-\mathbb{1})$, $\Gamma_2 = \sigma_2 \otimes \sigma_2$, $\Gamma_3 = \sigma_3 \otimes \mathbb{1}$ and $\Gamma_* = \sigma_2 \otimes \sigma_1$.

We can now determine the matrix C (almost uniquely) from eq.s (B.12a). It has to be $C = \beta\Gamma_0$ with $|\beta| = 1$. From this and eq. (B.13) it follows that $B \propto \mathbb{1}$ with the factor of proportionality again just a phase, which we now fix to be 1. Hence the Majorana condition is indeed just a reality condition $\lambda^* = \lambda$.

This basis was constructed to obtain a really real representation, that is all the matrices Γ_a are real and thus particularly apt for Majorana spinors. It may also be favourable to have a diagonal Γ_*. Such a basis makes chiral projections particularly natural:

$$\begin{aligned} \Gamma_0 &= i\mathbb{1} \otimes \sigma_1, \quad \Gamma_a = \sigma_a \otimes \sigma_2, \\ \Gamma_* &= \mathbb{1} \otimes \sigma_3, \quad C = i\sigma_2 \otimes \sigma_3. \end{aligned} \tag{B.22}$$

We present another basis that is adapted to dimensional reduction as is done in chapter 5, in particular eq. (5.22).

B.5 MORE ON THE MAJORANA CONDITION

We introduced the Majorana condition as a reality condition eq. (B.19). Another way to introduce it is by stating that Majorana equals Dirac conjugate. To make this statement a sensible one we must first introduce the notion of a spinor conjugate. Just as we use a dual vector space to build scalars out of vectors, we can introduce a sort of dual spinor space by an action of conjugation. In effect, an object $\bar{\chi}\varphi$ should transform as a scalar under local Lorentz transformations, that is $\delta(\bar{\chi}\varphi) = 0$. Essentially there are two options

$$\bar{\chi}^M \equiv \chi^t C, \tag{B.23a}$$
$$\bar{\chi}^D \equiv \alpha^{-1}\chi^\dagger\Gamma_0 = i\chi^\dagger\Gamma_0, \tag{B.23b}$$

called Dirac and Majorana conjugation respectively. At this point we fixed $\alpha = -i$. It is easy to see that the Majorana condition (B.19) is equivalent to the requirement $\bar{\chi}^M = \bar{\chi}^D \equiv \bar{\chi}$. We will assume Majorana spinors for the remainder of the section.
Let us now see that $\bar{\chi}\lambda$ indeed transforms as a scalar under Lorentz transformations. First we will derive the transformation law $\delta\bar{\chi}$ from $\delta\chi = -4^{-1}\lambda \cdot \Gamma\chi$, by putting table B.1 to work:

$$\begin{aligned} \delta(\lambda)\bar{\chi} &= (\delta\chi)^t C = -\frac{1}{4}\lambda^{AB}(\Gamma_{AB}\chi)^t C \\ &= -\frac{1}{4}\lambda^{AB}\chi^t\Big(\Gamma^t_{AB}C\Big) = +\frac{1}{4}\lambda^{AB}\chi^t\Big(C\Gamma_{AB}\Big)^t \\ &= +\frac{1}{4}\lambda^{AB}\chi^t C\Gamma_{AB} = \frac{1}{4}\lambda^{AB}\bar{\chi}\Gamma_{AB}. \end{aligned} \tag{B.24}$$

It is then straightforward to prove that $\delta(\bar{\chi}\varphi) = (\delta\bar{\chi})\varphi + \chi\bar{\delta}\varphi = 0$.

Majorana spinors are widely used in supergravity for various reasons, one of the most important is the simple symmetry properties of spinor bilinears $\bar{\chi}\Gamma^{(n)}\varphi$, e.g.

$$\bar{\chi}\varphi = \bar{\varphi}\chi, \tag{B.25a}$$
$$\bar{\chi}\Gamma_A\varphi = -\bar{\varphi}\Gamma_A\chi, \tag{B.25b}$$
$$\bar{\chi}\Gamma_{AB}\varphi = -\bar{\varphi}\Gamma_{AB}\chi \tag{B.25c}$$
$$\bar{\chi}\Gamma^{(n)}\varphi = -\theta(n)\bar{\varphi}\Gamma^{(n)}\chi, \tag{B.25d}$$

with θ as in table B.1. The examples are for the case $\epsilon = +1 = \eta$ in four dimensions. The general expression would just be $\bar{\chi}\Gamma^{(n)}\varphi = -\theta(\epsilon, \eta, n)\bar{\varphi}\Gamma^{(n)}\chi$. In deriving these rules we used the fact that spinors are Grassmann valued that is, moving them through each other induces an extra minus sign. Here we just impose that, but it is in fact a consequence of the spin statistics theorem in relativistic quantum field theory.

To this point we have introduced the Majorana condition in two forms, eq. (B.19) and $\bar{\chi}^D = \bar{\chi}^M$. Let us introduce another one, that is apt for specific problems, we introduce complex conjugation as

$$\varphi^C \equiv iB^{-1}\varphi^*. \tag{B.26}$$

The Majorana condition is then nothing but $\varphi^C = \varphi$. Note that $(\overline{\varphi^C})^M = \bar{\varphi}^D$, as in eq.s (B.23b) and (B.23b).
For matrices we define an adjoint action $M^C = B^{-1}M^*B$, yielding

$$\Gamma_A^C = \eta\Gamma_A, \qquad \Gamma_*^C = (-)^{D/2+1}\Gamma_*. \tag{B.27}$$

Note that this operation does not change the order $(M_1M_2)^C = M_1^C M_2^C$. See [26] for details.

So we learn that in four dimensions $\Gamma_*^C = -\Gamma_*$ and thus $P_R^C = P_L$ and vice versa. For an arbitrary Dirac spinor Ψ, which can of course be decomposed into chiral components: $\Psi = P_R\Psi + P_L\Psi$, we find that

$$(P_L\Psi)^C = P_R\Psi^C, \qquad (P_R\Psi)^C = P_L\Psi^C. \tag{B.28}$$

We can now take a Weyl fermion $P_L\Psi$ and build from it a Majorana fermion

$$\chi = P_L\Psi + (P_L\Psi)^C = \chi^C. \tag{B.29}$$

More details on this kind of decomposition can be found in [22].

B.6 MORE PRACTICAL GAMMA MATRIX MANIPULATIONS

In supergravity it is crucial to be able to simplify complicated Gamma matrix structures. We said that $\{\Gamma^{(n)}|n = 0, \cdots, D-1\}$ is a basis for all $2^{\lfloor D/2\rfloor} \times 2^{\lfloor D/2\rfloor}$-matrices. Hence any matrix M in spinor space is expressible as a sum of the form $M = \sum a^{(n)}\Gamma_{(n)}$:

$$M = 2^{-\lfloor D/2\rfloor} \sum_{n=0}^{[D]} \frac{(-)^{n(n-1)/2}}{n!}\Gamma_{(n)}\mathrm{Tr}\Big(\Gamma^{(n)}M\Big), \tag{B.30}$$

where $\Gamma_{(n)} = \Gamma_{A_1\cdots A_n}$ is to be distinguished from $\Gamma^{(n)} = \Gamma^{A_1\cdots A_n}$.

If we consider the tensor product of a spinor with an adjoined spinor $\varphi \otimes \bar{\chi} \sim \varphi_\alpha\bar{\chi}_\beta \sim \varphi\bar{\chi}$ we also get a matrix, which can be expressed

$$\varphi\bar{\chi} = -2^{-\lfloor D/2 \rfloor} \sum_{n=0}^{[D]} \frac{(-)^{n(n-1)/2}}{n!} \Gamma_{A_1 \cdots A_n} \bar{\chi} \Gamma^{A_1 \cdots A_n} \varphi. \tag{B.31}$$

This type of decomposition is known as Fierz rearrangement (sloppily also called Fierzing). In two dimensions it reads

$$\varphi\bar{\chi} = -\frac{1}{2}\bar{\chi}\varphi\mathbb{1} - \frac{1}{2}\Gamma_A \bar{\chi}\Gamma^A \varphi - \frac{1}{2}\Gamma_* \bar{\chi}\Gamma_* \varphi. \tag{B.32}$$

Fierz rearrangements are of great practical use in supersymmetry and supergravity. Another rearrangement, that is used regularly in this context is the following: a product of two general gamma matrix generators $\Gamma^{(n)}$ should again be expressable in that same basis: $\Gamma^{(n)} \cdot \Gamma^{(m)} = \sum c^{(p)}(n,m)\Gamma^{(p)}$. The simplest example is $\Gamma_A \Gamma_B = \delta_{AB}\mathbb{1} + \Gamma_{AB}$. It is rather hard technically to generalize this, but here is the result (taken from [26]):

$$\Gamma_{A_1 \cdots A_i} \Gamma^{B_1 \cdots B_j} = \sum_{k=|i-j| \in 2\mathbb{N}}^{i+j} \frac{i!j!}{s!t!u!} \delta_{[A_i}^{[B_1} \cdots \delta_{A_{t+1}}^{B_s} \Gamma_{A_1 \cdots A_t]}{}^{B_{s+1} \cdots B_j]}, \tag{B.33}$$

$$s = \frac{1}{2}(i+j-k), \qquad t = \frac{1}{2}(i-j+k), \qquad u = \frac{1}{2}(-i+j+k).$$

In practice we write down the gamma matrix with all antisymmetrized and subtract lower rank gamma matrices where two indices go into a metric (or Kronecker delta) until there is nothing left, schematically

$$\Gamma_{(m)}\Gamma_{(n)} = \Gamma_{(m+n)} + c_1 \eta \Gamma_{(m+n-2)} + \cdots + c_{\min(m,n)} \eta \cdots \eta \Gamma_{|m-n|}. \tag{B.34}$$

We determine the factors by explicit choices of indices.
Let's see some examples to get the intuition right:

$$\Gamma_A \Gamma_B = \frac{1!1!}{1!0!0!}\eta_{AB} + \frac{1!1!}{0!1!1!}\Gamma_{AB} = \eta_{AB} + \Gamma_{AB}, \tag{B.35a}$$

$$\Gamma_A \Gamma^{BC} = \delta_A^{[B}\Gamma^{C]} + \Gamma_A{}^{BC} = \delta_A{}^B \Gamma^C - \delta_A{}^C \Gamma^B + \Gamma_A{}^{BC}, \tag{B.35b}$$

$$\begin{aligned} \Gamma_{AB}\Gamma^{CD} &= \delta_{[B}^{[C}\delta_{A]}^{D]} + \delta_{[B}^{[C}\Gamma_{A]}^{D]} + \Gamma_{AB}^{CD} \\ &= 2\delta_{[B}^{[C}\delta_{A]}^{D]} + 4\delta_{[B}^{[C}\Gamma_{A]}^{D]} + \Gamma_{AB}^{CD}, \end{aligned} \tag{B.35c}$$

$$\Gamma_A \Gamma^{BCD} = 6\delta_A^{[B}\Gamma^{CD]} + \Gamma_A{}^{BCD}. \tag{B.35d}$$

APPENDIX C

POIN IN DISGUISE

In this appendix we rewrite the Poincaré algebra **poin** in different bases. First we split space and time and thereby see how explicitly what distinguishes the relativistic **poin** from the nonrelativistic **barg**, see eq. (3.79).

Then we extend the spacetime dimension by one and expand the algebra in a lightcone basis. The surprising fact will be that the Bargmann algebra appears as a subalgebra. Similar work has been done by [20, 21].

C.1 SPACE + TIME

We now make a step backwards and decompose the nicely packed objects $(P_A, J_{[AB]})$. For the moment we consider four dimensional spacetime only and separate the $(D-1)$-vector parts:

$$(P, J) \longrightarrow (H, \mathbf{P}, \mathbf{J}, \mathbf{K}), \tag{C.1}$$

where H is a scalar, $\mathbf{P}$ and $\mathbf{K}$ are vectors and $\mathbf{J}$ is a $(D-1)$-matrix. We split the index range $(A) = (0, a)$ and define

$$\begin{aligned} H &:= P_0, & \mathbf{P}_a &:= P_a, \\ \mathbf{K}_a &:= J_{a0}, & \mathbf{J}_{[ab]} &:= J_{[ab]}. \end{aligned}$$

An infinitesimal transformation then reads (defining $\lambda^a \equiv \lambda^{a0}$):

$$\delta(\xi^0, \xi^a, \lambda^a, \lambda^{ab}) = \xi^0 \Gamma_r(H) + \xi^a \Gamma_r(\mathbf{P}_a) - \lambda^a \Gamma_r(\mathbf{K}_a) - \frac{1}{2}\lambda^{ab}\Gamma_r(\mathbf{J}_{[ab]}). \tag{C.2}$$

Note that we have not done anything but relabel the generators, which in turn satisfy the Poincaré algebra in their specific form:

$$\left[\mathbf{J}_{[ab]}, \mathbf{J}_{[cd]}\right] = 4\delta_{[a[c}\mathbf{J}_{d]b]}, \tag{C.3a}$$
$$\left[\mathbf{P}_a, \mathbf{J}_{[bc]}\right] = 2\delta_{a[b}\mathbf{P}_{c]}, \tag{C.3b}$$
$$\left[H, \mathbf{J}_{[ab]}\right] = 0, \tag{C.3c}$$
$$\left[\mathbf{P}_a, \mathbf{P}_b\right] = 0 = \left[H, P_a\right] = \left[H, H\right] \tag{C.3d}$$

$$\left[\mathbf{K}_a, \mathbf{J}_{[bc]}\right] = 2\delta_{a[b}\mathbf{K}_{c]}, \tag{C.3e}$$
$$\left[\mathbf{K}_a, \mathbf{K}_b\right] = -\mathbf{J}_{[ab]}, \tag{C.3f}$$
$$\left[\mathbf{P}_a, \mathbf{K}_b\right] = \delta_{ab}H, \tag{C.3g}$$
$$\left[H, \mathbf{K}_a\right] = \mathbf{P}_a. \tag{C.3h}$$

It is worth noting that eq.s (C.3a) and (C.3b) form a subalgebra that looks remarkably similar to (3.49). Indeed it is the isomety algebra of the $(D-1)$ dimensional euclidean plane $iso(D-1)$. It forms the backbone of all spacetime symmetry algebras, see [28] (and references therein) for a classification of all possible *kinematical Lie algebras*. This part of the algebra will be unaltered when going to the nonrelativistic algebra via a contraction.

From a more physical point of view, eq. (C.3b) tells us that what we called a 3-vector is indeed a vector. Similarly, eq. (C.3c) justifies the name scalar for H. Apparently, more nontrivial structure is captured by eq.s (C.3e) through (C.3h). And indeed these will change when we go to nonrelativistic symmetry.
While the first tells us that $\mathbf{K}$ is a 3-vector too, the second is much less trivial. It suggests that a successive transformation by $\mathbf{K}$ induces a rotation. This is nothing but Thomas precession from a symmetry point of view. Hence we can interpret $\mathbf{K}$ as the generator of Lorentz boosts. We will see in the following chapters that this is a genuinely relativistic effect. Equations (C.3g) and (C.3h) represent the symmetry between space and time as spatial translations are boosted to time translations just as time translations are boosted to spatial translations. When going to nonrelativistic algebras we expect one of these (eq. (C.3g), to be specific) to vanish.

C.2 LIGHTCONE POINCARÉ

Expanding **poin** in a basis adapted to lightcone coordinates uncovers Galilean structures when restricting to motion along one null-like direction [12, 18–20]. We now consider a Poincaré algebra in $D+1$ dimensions and label

$$\mathbf{K}_a = J_{a0}, \qquad \mathbf{L}_a = J_{aD}, \qquad \Delta = J_{0D}. \tag{C.4}$$

The algebra is very similar to what we had in eq.s (C.3), with $\mathbf{L}_a$ dual to $\mathbf{K}_a$ and P_0 to P_D. The only undualled generator Δ translates between the two sides. Ultimately we are not interested in this form of the algebra but in one where we go to a light-cone basis $X^\pm = 2^{-1/2}(X^0 \pm X^D)$

$$\begin{aligned}
(P_0, \mathbf{P}_a, P_D) \quad &\longrightarrow \quad (\mathbf{P}_a, M = -2^{-1/2}(P_0 + P_D), H = 2^{-1/2}(P_0 - P_D)), && \text{(C.5a)}\\
(\mathbf{K}_a, \mathbf{L}_a) \quad &\longrightarrow \quad (\mathbf{G}_a = 2^{-1/2}(\mathbf{K}_a + \mathbf{L}_a), \mathbf{S}_a = 2^{-1/2}(\mathbf{K}_a - \mathbf{L}_a)), && \text{(C.5b)}\\
\Delta \quad &\longrightarrow \quad \Delta. && \text{(C.5c)}
\end{aligned}$$

The subalgebra spanned by $(J_{ab}, \mathbf{P}_a, \mathbf{G}_a, M, H)$ is the Bargmann algebra. So we see how a nonrelativistic structure arises in the Poincaré algebra without even mentioning the speed of light. The nonzero commutation relations read

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}, \quad [\mathbf{P}_a, J_{bc}] = 2\delta_{a[b}P_{c]}, \quad [H, \mathbf{G}_a] = \mathbf{P}_a, \quad [\mathbf{G}_a, \mathbf{P}_b] = \delta_{ab}M, \tag{C.6}$$

which is as promised the Bargmann algebra. The remaining commutators containing $(\mathbf{S}_a, \Delta)$ read

$$[\mathbf{S}_a, J_{bc}] = 2\delta_{a[b}S_{c]}, \quad [M, \mathbf{S}_a] = -\mathbf{P}_a, \quad [\mathbf{S}_a, \mathbf{P}_b] = -\delta_{ab}H, \quad [\mathbf{S}_a, \mathbf{G}_b] = -(J_{ab} + \Delta\delta_{ab}), \tag{C.7a}$$

$$[\Delta, M] = M, \quad [\Delta, H] = -H, \quad [\Delta, \mathbf{G}_a] = \mathbf{G}_a, \quad [\Delta, \mathbf{S}_a] = -\mathbf{S}_a. \tag{C.7b}$$

Note that this is the isometry algebra of $D+1$-dimensional Minkowski spacetime in lightcone coordinates

$$\eta = 2dX^- dX^+ + \delta_{ab}dX^a dX^b. \tag{C.8}$$

If we make the additional assumption that not just $\delta\eta = 0$ but separately also $\delta dX^- = 0$ we are forced to set $\mathbf{S}_a = 0 = \Delta$, thus yielding the Bargmann algebra. It is then natural to interpret X^- as nonrelativistic time. So it seems we can embed nonrelativistic symmetry into relativistic relativistic symmetry of a lightcone-direction. Note the similarity to ambient formulations of conformal field theories [75].

APPENDIX D

RESULTS

D.1 BOSONIC RESULTS FROM 0 REDUCTION

Here we present some results that are of great technical but of less conceptual interest. Thus they are not in the main text but in this separate section. Essentially these are the results found in [39] constructing torsionful Newton Cartan gravity by means of a Null Reduction.

The idea of 0 Reduction is implemented by the Ansätze (5.15). Since we are not dealing with conformal structures here, it is consistent to gauge-fix $S = 1$ and thus use (5.19). It is then straightforward to derive the transformation rules of the independent fields, as done in eq.s (5.18).

The technically challenging part is to deduce the forms of the eight spin connection blocks $\Omega_M^{AB}(e^a, \tau, m)$ with $A = a, +, -$ and $M = \mu, z$ from the relativistic (3.26):

D.1.1 Spin Connection Components

We will use the nonrelativistic anholonomy symbols as defined before eq. (3.87): $c_{\mu\nu}^{\ a} = 2\partial_{[\mu}e_{\nu]}^{\ a}$ and $c_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}$:

$$\Omega_\mu{}^{ab} = e^{\mu[a}c_{\mu\nu}{}^{b]} - \frac{1}{2}e_{\mu c}c^{abc} - \frac{1}{2}\tau_\mu c^{ab} - \frac{1}{2}m_\mu\tau^{ab}, \tag{D.1a}$$

$$\Omega_\mu{}^{a+} = \frac{1}{2}c_{\mu 0}{}^a - \frac{1}{2}c_\mu{}^a - \frac{1}{2}e_{\mu b}c_0{}^{ab} - \frac{1}{2}\tau_\mu c_0{}^a + \frac{1}{2}m_\mu\tau^{a0}, \tag{D.1b}$$

$$\Omega_\mu^{a-} = \frac{1}{2}\tau_\mu{}^a, \qquad \Omega_\mu{}^{+-} = \frac{1}{2}\tau_\mu{}^0, \tag{D.1c}$$

$$\Omega_z{}^{ab} = \frac{1}{2}\tau^{ab}, \qquad \Omega_z{}^{a+} = \frac{1}{2}\tau^{a0}, \tag{D.1d}$$

$$\Omega_z{}^{a-} = 0 = \Omega_z{}^{+-}. \tag{D.1e}$$

When comparing these results with those obtained from a gauging procedure (3.89a) and (3.89b) we see that

$$\Omega_\mu{}^{ab}(e, \tau, m) = \omega_\mu{}^{ab}(e, \tau, m) - m_\mu\Omega_z{}^{ab}(e, \tau, m), \tag{D.2a}$$

$$\Omega_\mu{}^{a+}(e, \tau, m) = -\Big(\omega_\mu{}^a(e, \tau, m) - m_\mu\Omega_z{}^{a+}(e, \tau, m)\Big). \tag{D.2b}$$

Note the particular structure $V_\mu \sim v_\mu - m_\mu V_z$, guaranteeing that v_μ transforms as a vector under diffeomorphisms. All the results reduce to those obtained by gauging (3.89a) and (3.89b).

In principle one could reduce the Riemann tensor as given in (3.14) or (3.62b). The results can be found in [39] and consist of curvatures as derived in (3.82a) through (3.82e) and the torsionful (3.93a) and (3.93b).

D.2 OFF-SHELL NR SUGRA

D.2.1 Gravitini Transformations

The transformation rules for the bosonic NC fields $(e_\mu{}^a, \tau_\mu, m_\mu)$ remain unaltered. The fermion transformation rules read:

$$\begin{aligned}\delta\psi = {}& \mathcal{L}_\xi\psi - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi + \frac{3}{16}\hat{\tau}^{ab}\gamma_{ab}\epsilon_- + \frac{\sqrt{2}}{4}\hat{\tau}^{a0}\gamma_{a0}\epsilon_+ + \frac{1}{\sqrt{2}}a_a\gamma^a\epsilon_+ - \frac{1}{4}a\gamma_0\epsilon_- \\ & + \frac{1}{2\sqrt{6}}(f+\bar{f})\gamma_0\epsilon_+ + \frac{\mathrm{i}}{2\sqrt{6}}(f-\bar{f})\epsilon_+ ,\end{aligned} \tag{D.3}$$

$$\begin{aligned}\delta\psi_{\mu+} = {}& \mathcal{L}_\xi\psi_{\mu+} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu+} + D_\mu\epsilon_+ + \frac{\sqrt{2}}{4}\hat{\tau}_\mu{}^a\gamma_{a0}\epsilon_- + \frac{1}{4}\hat{\tau}_\mu{}^0\epsilon_+ \\ & - \frac{3}{2}a_\mu\gamma_0\epsilon_+ + \frac{1}{2}e_\mu{}^a a_b\gamma_a\gamma^b\gamma_0\epsilon_+ + \tau_\mu a_0\gamma_0\epsilon_+ - \frac{1}{\sqrt{2}}\tau_\mu a_a\gamma^a\epsilon_- - \frac{1}{\sqrt{2}}a\, e_\mu{}^a\gamma_a\epsilon_- \\ & + \frac{1}{4\sqrt{3}}e_\mu{}^a(f+\bar{f})\gamma_a\epsilon_+ + \frac{\sqrt{2}}{4\sqrt{3}}\tau_\mu(f+\bar{f})\gamma_0\epsilon_- - \frac{\mathrm{i}}{4\sqrt{3}}e_\mu{}^a(f-\bar{f})\gamma_{a0}\epsilon_+ \\ & - \frac{\mathrm{i}\sqrt{2}}{4\sqrt{3}}\tau_\mu(f-\bar{f})\epsilon_- + \frac{\sqrt{2}}{4}\left(\bar{\epsilon}_-\psi\right)\gamma_0\psi_{\mu+} - \frac{\sqrt{2}}{4}\left(\bar{\psi}_{\mu-}\psi\right)\gamma_0\epsilon_+\end{aligned} \tag{D.4a}$$

$$\begin{aligned}\delta\psi_{\mu-} = {}& \mathcal{L}_\xi\psi_{\mu-} - \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu-} + \frac{1}{\sqrt{2}}\lambda^a\gamma_{a0}\psi_{\mu+} + \lambda^a e_{\mu a}\psi \\ & + D_\mu\epsilon_- - \frac{1}{\sqrt{2}}\omega_\mu{}^a\gamma_{a0}\epsilon_+ - \frac{1}{4}\hat{\tau}_\mu{}^0\epsilon_- + \frac{3}{2}a_\mu\gamma_0\epsilon_- - \frac{1}{2}e_\mu{}^a a_b\gamma_a\gamma^b\gamma_0\epsilon_- \\ & + \frac{1}{\sqrt{2}}e_\mu{}^a a_0\gamma_a\epsilon_+ + \frac{1}{4\sqrt{3}}e_\mu{}^a(f+\bar{f})\gamma_a\epsilon_- + \frac{\mathrm{i}}{4\sqrt{3}}e_\mu{}^a(f-\bar{f})\gamma_{a0}\epsilon_- \\ & + \frac{\sqrt{2}}{4}\left(\bar{\psi}_{\mu-}\psi\right)\gamma_0\epsilon_- - \frac{\sqrt{2}}{4}\left(\bar{\epsilon}_-\psi\right)\gamma_0\psi_{\mu-} ,\end{aligned} \tag{D.4b}$$

where we have introduced

$$a_0 = \tau^\mu a_\mu , \qquad a_a = e^\mu{}_a a_\mu . \tag{D.5}$$

D.2.2 Gravitini Curvatures

It is possible to derive gravitini curvatures from the above transformation rules

$$\begin{aligned}
\hat{r}_{\mu\nu}(Q_+) &= 2\partial_{[\mu}\psi_{\nu]+} + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} + \frac{1}{\sqrt{2}}\hat{\tau}_{[\mu}{}^{a}\gamma_{a0}\psi_{\nu]-} + \frac{1}{2}\hat{\tau}_{[\mu}{}^{0}\psi_{\nu]+} - 3a_{[\mu}\gamma_0\psi_{\nu]+} \\
&\quad + e_{[\mu}{}^{a}a_b\gamma_a\gamma^b\gamma_0\psi_{\nu]+} + 2\tau_{[\mu}a_0\gamma_0\psi_{\nu]+} - \sqrt{2}\tau_{[\mu}a_a\gamma^a\psi_{\nu]-} - \sqrt{2}ae_{[\mu}{}^{a}\gamma_a\psi_{\nu]-} \\
&\quad + \frac{1}{2\sqrt{3}}e_{[\mu}{}^{a}(f+\bar{f})\gamma_a\psi_{\nu]+} + \frac{\sqrt{2}}{2\sqrt{3}}\tau_{[\mu}(f+\bar{f})\gamma_0\psi_{\nu]-} - \frac{\mathrm{i}}{2\sqrt{3}}e_{[\mu}{}^{a}(f-\bar{f})\gamma_{a0}\psi_{\nu]+} \\
&\quad - \frac{\mathrm{i}\sqrt{2}}{2\sqrt{3}}\tau_{[\mu}(f-\bar{f})\psi_{\nu]-} - \frac{1}{\sqrt{2}}(\bar{\psi}_{[\mu-}\psi)\gamma_0\psi_{\nu]+} \qquad \text{(D.6a)} \\
\hat{r}_{\mu\nu}(Q_-) &= 2\partial_{[\mu}\psi_{\nu]-} + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - \sqrt{2}\omega_{[\mu}{}^{a}\gamma_{a0}\psi_{\nu]+} - 2\omega_{[\mu}{}^{a}e_{\nu]a}\psi - \frac{1}{2}\hat{\tau}_{[\mu}{}^{0}\psi_{\nu]-} \\
&\quad + 3a_{[\mu}\gamma_0\psi_{\nu]-} - e_{[\mu}{}^{a}a_b\gamma_a\gamma^b\gamma_0\psi_{\nu]-} + \sqrt{2}e_{[\mu}{}^{a}a_0\gamma_a\psi_{\nu]+} + \frac{1}{2\sqrt{3}}e_{[\mu}{}^{a}(f+\bar{f})\gamma_a\psi_{\nu]-} \\
&\quad + \frac{\mathrm{i}}{2\sqrt{3}}e_{[\mu}{}^{a}(f-\bar{f})\gamma_{a0}\psi_{\nu]-} + \frac{1}{\sqrt{2}}(\bar{\psi}_{[\mu-}\psi)\gamma_0\psi_{\nu]-}, \qquad \text{(D.6b)} \\
\hat{r}_{\mu}(Q_-) &= \partial_\mu\psi + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi - \frac{3}{16}\hat{\tau}^{ab}\gamma_{ab}\psi_{\mu-} - \frac{\sqrt{2}}{4}\hat{\tau}^{a0}\gamma_{a0}\psi_{\mu+} - \frac{1}{\sqrt{2}}a_a\gamma^a\psi_{\mu+} + \frac{1}{4}a\gamma_0\psi_{\mu-} \\
&\quad - \frac{1}{2\sqrt{6}}(f+\bar{f})\gamma_0\psi_{\mu+} - \frac{\mathrm{i}}{2\sqrt{6}}(f-\bar{f})\psi_{\mu+}, \qquad \text{(D.6c)}
\end{aligned}$$

these are related to four-dimensional gravitino curvatures

$$\hat{R}_{MN}(Q) = \hat{R}^{(+)}_{MN}(Q)\otimes\chi_- + \hat{R}^{(-)}_{MN}(Q)\otimes\chi_+ \qquad \text{(D.7)}$$

by

$$\begin{aligned}
\hat{R}^{+}_{ab}(Q) &= \hat{r}_{ab}(Q), \qquad \hat{R}^{+}_{a-}(Q) = \hat{r}_{a0}(Q_+)\,, \\
\hat{R}^{+}_{a+}(Q) &= \frac{\sqrt{2}}{4}\hat{\tau}_a{}^{b}\gamma_{b0}\psi - \frac{1}{\sqrt{2}}a\gamma_a\psi\,, \\
\hat{R}^{+}_{+-}(Q) &= -\frac{\sqrt{2}}{4}\hat{\tau}_0{}^{a}\gamma_{a0}\psi + \frac{1}{\sqrt{2}}a_a\gamma^a\psi \\
&\quad - \frac{1}{2\sqrt{6}}(f+\bar{f})\gamma_0\psi + \frac{\mathrm{i}}{2\sqrt{6}}(f-\bar{f})\psi, \\
\hat{R}^{-}_{ab}(Q) &= \hat{r}_{ab}(Q_-)\,, \qquad \hat{R}^{-}_{a-}(Q) = \hat{r}_{a0}(Q_-)\,, \\
\hat{R}^{-}_{a+}(Q) &= \hat{r}_a(Q_) - \frac{1}{4}\hat{\tau}_a{}^{0}\psi + \frac{3}{2}a_a\gamma_0\psi - \frac{1}{2}a_b\gamma_a\gamma^b\gamma_0\psi \\
&\quad + \frac{1}{4\sqrt{3}}(f+\bar{f})\gamma_a\psi + \frac{\mathrm{i}}{4\sqrt{3}}(f-\bar{f})\gamma_{a0}\psi\,, \\
\hat{R}^{-}_{+-}(Q) &= -\hat{r}_0(Q_-) - \frac{3}{2}a_0\gamma_0\psi\,. \qquad \text{(D.8)}
\end{aligned}$$

D.2.3 Auxiliary Field Transformation Rules

With the above we can reduce the transformation rules of the auxiliary fields (6.14c) and (6.14d):

$$
\begin{aligned}
\delta f = {} & \mathcal{L}_\xi f + \frac{\mathrm{i}}{4\sqrt{3}}\bar{\epsilon}_+ \left(1 - \mathrm{i}\gamma_0\right) \epsilon^{ab}\hat{\psi}_{ab-} - \frac{\mathrm{i}}{4\sqrt{3}}\bar{\epsilon}_- \left(1 + \mathrm{i}\gamma_0\right) \epsilon^{ab}\hat{\psi}_{ab+} \\
& + \frac{1}{\sqrt{6}}\bar{\epsilon}_+ \left(1 - \mathrm{i}\gamma_0\right) \gamma^{a0}\hat{\psi}_{a0+} - \frac{1}{\sqrt{6}}\bar{\epsilon}_- \left(1 + \mathrm{i}\gamma_0\right) \gamma^{a}{}_{0}\hat{D}_a\psi - \frac{1}{2\sqrt{3}}\bar{\epsilon}_+ \left(1 - \mathrm{i}\gamma_0\right) \hat{D}_0\psi \\
& - \frac{\sqrt{3}}{4}\bar{\epsilon}_+ \left(1 - \mathrm{i}\gamma_0\right) a_0\gamma_0\psi + \frac{1}{2\sqrt{6}}\bar{\epsilon}_- \left(1 + \mathrm{i}\gamma_0\right) \gamma^{a0}\hat{\tau}_{a0}\psi + \frac{1}{4\sqrt{2}}(f + \bar{f})\bar{\epsilon}_- \left(1 + \mathrm{i}\gamma_0\right) \gamma_0\psi \\
& - \frac{\mathrm{i}}{4\sqrt{2}}(f - \bar{f})\bar{\epsilon}_- \left(1 + \mathrm{i}\gamma_0\right) \psi. \qquad \text{(D.9)}
\end{aligned}
$$

$$
\begin{aligned}
\delta a_\mu = {} & \mathcal{L}_\xi a_\mu + \frac{1}{4}\bar{\epsilon}_-\gamma^{a0}\hat{\psi}_{\mu a+} + \frac{1}{6}\bar{\epsilon}_-\gamma^{a0}\tau_\mu\hat{\psi}_{a0+} - \frac{1}{24}\bar{\epsilon}_- e_\mu{}^a\gamma_a\epsilon^{bc}\hat{\psi}_{bc+} \\
& + \frac{1}{12\sqrt{2}}\bar{\epsilon}_-\tau_\mu\gamma_0\epsilon^{ab}\hat{\psi}_{ab-} - \frac{1}{4}\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{\mu a-} - \frac{1}{2\sqrt{2}}\bar{\epsilon}_+\hat{\psi}_{\mu 0+} + \frac{1}{24}\bar{\epsilon}_+ e_\mu{}^a\gamma_a\epsilon^{bc}\hat{\psi}_{bc-} \\
& + \frac{1}{6\sqrt{2}}\bar{\epsilon}_+ e_\mu{}^a\gamma_a\gamma^b\hat{\psi}_{b0+} - \frac{1}{6\sqrt{2}}\bar{\epsilon}_-\tau_\mu\hat{D}_0\psi + \frac{1}{2\sqrt{2}}\bar{\epsilon}_-\hat{D}_\mu\psi + \frac{1}{6\sqrt{2}}\bar{\epsilon}_- e_\mu{}^a\gamma_a\gamma^b\hat{D}_b\psi \\
& - \frac{1}{12}\bar{\epsilon}_+ e_\mu{}^a\gamma_{0a}\hat{D}_0\psi + \frac{3\sqrt{2}}{8}\bar{\epsilon}_-\gamma_0 a_\mu\psi + \frac{1}{8\sqrt{2}}\bar{\epsilon}_-\hat{\tau}_{\mu 0}\psi - \frac{1}{12\sqrt{2}}\bar{\epsilon}_- e_\mu{}^a\gamma_{0a}\gamma^b a_b\psi \\
& + \frac{1}{6\sqrt{6}}\bar{\epsilon}_- e_\mu{}^a\gamma_a(f + \bar{f})\psi - \frac{\mathrm{i}}{6\sqrt{6}}\bar{\epsilon}_- e_\mu{}^a\gamma_{0a}(f - \bar{f})\psi - \frac{1}{4\sqrt{2}}\bar{\epsilon}_-\tau_\mu a_0\gamma_0\psi \\
& + \frac{1}{\sqrt{2}}\bar{\epsilon}_- a\gamma_0\psi_{\mu-} - \frac{1}{8}\bar{\epsilon}_+ e_\mu{}^a\gamma_a a_0\psi\ , \qquad \text{(D.10)} \\
\delta a = {} & \mathcal{L}_\xi a - \frac{1}{12}\bar{\epsilon}_+\gamma^{0a}\hat{D}_a\psi - \frac{\sqrt{2}}{24}\bar{\epsilon}_+\gamma^{ab}\hat{\psi}_{ab+} - \frac{1}{8}\bar{\epsilon}_+\gamma^a a_a\psi + \frac{5}{48}\bar{\epsilon}_+\hat{\tau}^{0a}\gamma_{0a}\psi \\
& + \frac{1}{8\sqrt{3}}\bar{\epsilon}_+(f + \bar{f})\gamma_0\psi - \frac{1}{8\sqrt{3}}\,\mathrm{i}\,\bar{\epsilon}_+(f - \bar{f})\psi + \frac{1}{8\sqrt{2}}\bar{\epsilon}_-(\hat{\tau}^{ab}\epsilon_{ab} + 4a)\gamma_0\psi\ . \qquad \text{(D.11)}
\end{aligned}
$$

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