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## „A conjectured relation between fusion ideals "


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**Abstract**

The study of Lie algebras is of utmost importance to modern day mathematical physics. Within the setting of affine Lie algebras which are an infinite dimensional extension of finite dimensional Lie algebras a conjecture about fusion ideals of affine Lie algebras with application in string theory has been established. In this thesis we present a proof of a special case of the conjecture, namely the case of the regular embedding of the special unitary Lie algebra $\mathfrak{su}(n)$ in $\mathfrak{su}(m)$ where $n \leq m$ are arbitrary positive integers. Furthermore we give an outlook on the case of symplectic Lie algebras.

**Keywords:** String theory, D-branes, fusion rings, fusion ideals, Lie algebras, representation theory

**Zusammenfassung**

Das Studium von Liealgebren ist von großer Bedeutung für die moderne mathematische Physik. Für affine Liealgebren, im Wesentlichen unendlichdimensionale Erweiterungen endlicher Liealgebren, existiert eine Vermutung über Fusionsideale. Diese Vermutung findet innerhalb der Stringtheorie Anwendung. In dieser Arbeit stellen wir einen Beweis für einen Spezialfall dieser Vermutung vor: Der Fall der regulären Einbettung der speziellen unitären Algebra $\mathfrak{su}(n)$ in $\mathfrak{su}(m)$, wobei $n \leq m$ beliebige positive ganze Zahlen sind. Des Weiteren betrachten wir den Fall von symplektischen Liealgebren.

**Schlagwörter:** Stringtheorie, D-Branen, Fusionsringe, Fusionsideale, Liealgebren, Darstellungstheorie

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# Contents

# Chapter 1

# Introduction

Within the theory of Lie algebras one is confronted with a certain type of Lie algebras, called *affine Lie algebras.* They are infinite dimensional and can be described as the central extension of *loop algebras.* Loop algebras are essentially the tensor product of a finite dimensional Lie algebra $\mathfrak{g}$ with the associative algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$. Most of the theory of finite dimensional Lie algebras may be carried over to the affine case. We may introduce affine weights and roots, the affine Weyl group and so on. Now when looking at representations of Lie algebras there is some additional structure which may be defined in the finite dimensional case: the *representation ring* $\mathrm{Rep}(\mathfrak{g})$. The ring structure of the representation ring is given by the direct sum and tensor product of representations. In the affine case such a ring is called *fusion ring.* However the tensor product is replaced by a different multiplicative structure, called the *fusion rules.* It is possible to formalize the notion of the fusion ring independently of Lie algebra representations. Now the representation ring of a Lie algebra $\mathfrak{g}$ and the fusion ring of its corresponding affine Lie algebra $\hat{\mathfrak{g}}$ are related by dividing out a certain ideal out of the representation ring known as the *fusion ideal.* One could ask some questions about the fusion ideal, e.g. what happens when embedding a Lie algebra $\mathfrak{h}$ into another Lie algebra $\mathfrak{g}$. The question arises how the fusion ideals of $\mathfrak{h}$ and $\mathfrak{g}$ are related. Considering the fusion ideal $I_{\mathfrak{h},k'}$ at level $k'$ (we will specify the term 'level' below) and the fusion ideal $I_{\mathfrak{g},k}$ at level $k$, the two levels $k'$ and $k$ are related by some formula. By the theory of representations we may under certain assumptions decompose the representations in the fusion ideal of $\mathfrak{g}$ into representations of $\mathfrak{h}$. Doing this for all representations in $I_{\mathfrak{g},k}$ will thus yield a set of representations of $\mathfrak{h}$ which however not necessarily form an ideal in $\mathrm{Rep}(\mathfrak{h})$ but we may look at the ideal this set generates. The conjectured relation is now that the fusion ideal $I_{\mathfrak{h},k'}$ is a subset of the ideal generated by the decomposed representations of $I_{\mathfrak{g},k}$.

The physical background where this conjecture arises is string theory. When considering objects called $D$-branes which can be understood as $(p+1)$-dimensional objects to which strings can couple we can define a group of formal $D$-brane configurations. On the set of branes we can define *dynamical processes* as transitions between branes. Here we want to describe the *charge group* $\mathcal{C}$ which is crucial for understanding the dynamical processes between branes. The structure of the charge group is in certain models constructed from Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ (called coset models) the tensor product of fusion rings divided by some ideal. If the conjecture holds true then the expression for $\mathcal{C}$ takes a simpler form: We would have $\mathcal{C}_0 \cong \otimes(\mathrm{Rep}(\mathfrak{h})/\langle I_{\mathfrak{g}|_{\mathfrak{h}}}\rangle)$.

Goal of this master thesis is to investigate the question whether this relation is valid or not. As a general proof (if it exists) would require deeper understanding of the theory behind fusion rings we attempt to show the conjecture for special cases and take a more pedestrian approach. Namely we will first show that the conjecture holds true for $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$. Having a set of generators of $\mathfrak{su}(n)$ ideals at hand we will decompose them into representations of $\mathfrak{su}(n-1)$ and check if the fusion ideal of $\mathfrak{su}(n-1)$ lies within the ideal generated by those representations. The generators have been provided by Bouwknegt and Ridout, see [BDR02]. Luckily this easily generalizes to the case of the regular embedding of $\mathfrak{su}(n) \subset \mathfrak{su}(n+m)$ where $m$ is a nonnegative integer. Furthermore we will look at the principal embedding of $\mathfrak{su}(2)$ in $\mathfrak{su}(n)$. Concerning other Lie algebras than $\mathfrak{su}$ we will then look at the case of $\mathfrak{sp}$, i.e. symplectic Lie algebras.

The structure of the thesis is as follows: First we will give a very basic and short introduction to the general theory of Lie algebras with emphasis on embeddings, branching rules and the decomposition of tensor products of representations. Seminal work on Lie algebra embeddings and the decomposition has been done by Dynkin, see [Dyn52b] and [Dyn52a]. A good introduction to the matter can be found in Cahn, see [Cah84]. Proofs are mostly omitted as they are well-known and can be found in introductory books. On the other hand we will give many examples in order to build up intuition on the tools needed later for the treatment of the conjecture in the cases mentioned above. The subsequent section is devoted to the theory of affine Lie algebras which while nowadays being rather well-known needs a deeper introduction. We will prove results if necessary and refer to them if they are not important to the understanding of the concepts. Fusion rings will be treated as well on the basis of the works by Jürgen Fuchs. We will introduce them in an abstract way (i.e. without referring to Lie algebras) and then see how they are applicable to affine algebras. After this theoretical introduction we will state the conjecture and turn to the examples already mentioned. At the end a brief outlook will discuss what could be done in the future.

# Chapter 2

# Preliminaries on Lie algebras

Lie algebras arise in practically every branch of physics. In Quantum Mechanics they encode the relations of observables, for example the commutation relation between the position operator $\hat{x}$ and the momentum operator $\hat{p}$ via $[\hat{x}, \hat{p}] = i\hbar\mathrm{id}$. The *Pauli matrices* form a basis of the special unitary Lie algebra $\mathfrak{su}(2)$ and describe spin whereas the *Gell-Mann matrices* form a basis of the Lie algebra $\mathfrak{su}(3)$ and describe color charge within Quantum Chromodynamics. Branching rules of Lie algebra representations are used when calculating *Clebsch-Gordan coefficients*. But also more recent developments rely profoundly on Lie theory. For example the *conformal algebra* plays a central role in *Conformal Field Theories*.

In the following we will give a brief introduction on Lie algebras and representation theory. Starting off with the most basic definitions which include solvability, (semi–)simplicity, representations and the Killing form we also define roots, Cartan matrices, Dynkin diagrams and Coxeter numbers. Highest weights and highest weight representations which become more important in the subsequent chapters will be discussed as well as tensor products of representations and branching rules. The presented facts are mostly well-known, the main sources of this section are [FH91], [EW06] and [DFMS97].

## 2.1 Basic definitions

**Definition 2.1.1.** *A* **Lie algebra** *is a vector space* $\mathfrak{g}$ *over a field* $k$ *together with a map* $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(called the* **Lie bracket***) which has the following properties:*

*(i)* $[\ ,\ ]$ *is* **bilinear***, i.e. for* $X, Y, Z \in \mathfrak{g}$ *and* $\alpha \in k$ *we have* $[X, Y + \alpha Z] = [X, Y] + \alpha[Y, Z]$

*(ii)* $[\ ,\ ]$ *is* **skew-symmetric***, i.e.* $[X, Y] = -[Y, X]$ *for all* $X, Y \in \mathfrak{g}$

*(iii) The* **Jacobi identity** *holds, i.e. for all* $X, Y, Z \in \mathfrak{g}$ *we have*

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \tag{2.1.1}$$

*A* **(Lie) subalgebra** *is a subspace* $\mathfrak{h}$ *such that* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. *It is denoted by* $\mathfrak{h} \leq \mathfrak{g}$. *An* **ideal** *is a subalgebra* $\mathfrak{k}$ *such that* $[\mathfrak{g}, \mathfrak{k}] \subset \mathfrak{k}$. *A Lie algebra is called* **Abelian** *if* $[X, Y] = 0$ *for all* $X, Y \in \mathfrak{g}$. *A* **homomorphism** *of Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is a linear map* $\rho\colon \mathfrak{g} \to \mathfrak{h}$ *that is compatible with the Lie bracket, i.e.* $\rho([X, Y]) = [\rho(X), \rho(Y)]$ *for* $X, Y \in \mathfrak{g}$. *A bijective homomorphism is called an* **isomorphism**.

Somewhat analogously to groups we can define a *presentation* of a Lie algebra:

**Definition 2.1.2.** *Let* $\mathfrak{g}$ *be a Lie algebra. A set of* **generators** *for* $\mathfrak{g}$ *is a subset of* $\mathfrak{g}$ *such that any element* $Y \in \mathfrak{g}$ *can be written as a linear combination of Lie brackets of the generators. The Lie brackets of the generators* $X_i$ *are then called the* **Lie bracket relations** *and they are defined by*

$$[X_i, X_j] = \sum_{k=1}^{n=\dim \mathfrak{g}} f_{ij}^k X_k \tag{2.1.2}$$

*where the coefficients* ${f_{ij}}^k$ *are called* **structure constants** *of* $\mathfrak{g}$. *The generators together with their relations are called a* **presentation** *of the Lie algebra* $\mathfrak{g}$.

In the appendix we provide the definition for the *standard generators* of Lie algebras. Next we define the concepts of *semisimplicity* and *simplicity* In order to do that we have to define solvability first. This will be done via the *derived series* of $\mathfrak{g}$. In order to do that, note that the subspace $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$. This ideal is called the ***commutator ideal***.

**Definition 2.1.3.** *The* **derived series** *of a Lie algebra* $\mathfrak{g}$ *is a sequence of subspaces* $\mathfrak{g}^{(k)}$ *defined inductively as:* $\mathfrak{g}^{(1)} := \mathfrak{g}$ *and* $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. *It holds that every* $\mathfrak{g}^{(k+1)}$ *is an ideal in* $\mathfrak{g}^{(k)}$ *and therefore we have a decreasing sequence*

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset ... \supset \mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)} \supset ... \tag{2.1.3}$$

*and* $\mathfrak{g}$ *is called* **solvable** *if the derived series terminates after some* $k \in \mathbb{N}$, *i.e.* $\mathfrak{g}^{(k)} = 0$ *for some* $k \in \mathbb{N}$.

The concept of solvability gives rise to semisimple, simple and reductive Lie algebras which are amongst the most important types of Lie algebras.

**Definition 2.1.4.** *Let* $\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g}\colon [X, Y] = 0 \ \forall Y \in \mathfrak{g}\}$ *be the* **center** *of a Lie algebra* $\mathfrak{g}$. $\mathfrak{g}$ *is called*

*(i)* **semisimple** *if there is no solvable ideal except* $\{0\}$.

*(ii)* **simple** *if* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ *and there are no ideals except* $\{0\}$ *and* $\mathfrak{g}$ *itself.*

*(iii)* **reductive** *if any solvable ideal is contained in* $\mathfrak{z}(\mathfrak{g})$.

Note that simple Lie algebras form a subset of semisimple Lie algebras. In fact we will see later that semisimple Lie algebras are always a direct sum of simple subalgebras. Furthermore, since the center $\mathfrak{z}(\mathfrak{g})$ forms an Abelian ideal of $\mathfrak{g}$ (meaning that the Lie bracket vanishes on $\mathfrak{z}(\mathfrak{g})$ implying solvability) any semisimple Lie algebra has a trivial center. Let us give some most basic examples for Lie algebras.

**Examples 2.1.5.** (i) Let $V$ be a $k$-vector space. If we define $[\ ,\ ]$ to be the commutator of two elements $X, Y \in V$, i.e. $[X, Y] = XY - YX$ we obtain the ***general linear Lie algebra*** of $V$, denoted by $\mathfrak{gl}(V)$. Note that $V$ can also be $\mathfrak{g}$ since $\mathfrak{g}$ is a vector space.

(ii) We define the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ (the ***special linear Lie algebra***) to be the vector space of all tracefree complex $n \times n$-matrices, i.e.

$$\mathfrak{sl}(n, \mathbb{C}) := \left\{ A \in \mathbb{C}^{n \times n} \colon \operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = 0 \right\}. \tag{2.1.4}$$

Defining the Lie bracket to be the commutator, we see that $\operatorname{tr}([X, Y]) = \operatorname{tr}(XY - YX) = \operatorname{tr}(XY) - \operatorname{tr}(YX) = 0$ since the trace is invariant under the exchange of $X$ and $Y$, so $\mathfrak{sl}(n, \mathbb{C})$ is well-defined. Also note that $\mathfrak{sl}(n, \mathbb{C})$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. Furthermore, $\mathfrak{sl}(n, \mathbb{C})$ is a fundamental example of a simple Lie algebra. In physics one often writes $\mathfrak{su}$ which, albeit being a different *real* Lie algebra, has the same complexification (more precisely: its complexification is isomorphic to the one of $\mathfrak{sl}$). Later we will switch to the $\mathfrak{su}$-notation.

*Representations* of Lie algebras are homomorphisms from the Lie algebra to the general linear algebra of a given vector space $V$. That means we can in some sense confer the commutation relations of a given Lie algebra (which are preserved under a homomorphism) to the general linear algebra of $V$ where we are equipped with the commutator as the Lie bracket. In practice it is desirable to represent a Lie algebra on some space of matrices, i.e. the commutation relations should be the same between the elements of the algebra and the matrices of the matrix representation respectively.

**Definition 2.1.6.** *A* **representation** *of a Lie algebra $\mathfrak{g}$ is a homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. We often call the representation space $V$ the representation of $\mathfrak{g}$, too. A* **subrepresentation** *or* **invariant subspace**[1] *is a subspace $W \subset V$ such that $W$ is invariant under $\rho$, i.e. $\rho(X)(w) \in W$ for all $X \in \mathfrak{g}$ and $w \in W$. A representation is called* **irreducible** *if there are no subrepresentations except $\{0\}$ and $V$ itself. It is called* **indecomposable** *if there are no two invariant subspaces $W_1, W_2$ such that $V = W_1 \oplus W_2$.*[2]

[1] Already here the confusion of representation as a map and as a space is manifest.
[2] Note that irreducibility implies indecomposability.

A very important representation which we will use frequently is the ***adjoint representation***. The map $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $\mathrm{ad}(X)(Y) := [X, Y]$. This means the adjoint representation maps an element $Y$ to its commutator with a given element $X$. The Jacobi identity (2.1.1) guarantees that this indeed defines a representation. Let us look at an example of the adjoint representation:

**Examples 2.1.7.** The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is 3-dimensional. A basis of $\mathfrak{sl}(2,\mathbb{C})$ is given by:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.1.5}$$

The Lie brackets or commutators are then given by

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H. \tag{2.1.6}$$

The maps $\mathrm{ad}(X_i)$ for $X_i = E, F, H$ are then represented by matrices in the basis $\{E, F, H\}$:

$$\mathrm{ad}(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \mathrm{ad}(E) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}(F) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.1.7}$$

Another construction made from representations is their tensor product which can be defined quite naturally.

**Definition 2.1.8.** *Let $\rho$ and $\rho'$ be representations of a Lie algebra $\mathfrak{g}$. Then the* **tensor product of $\rho$ and $\rho'$** *is defined by*

$$\rho \otimes \rho' \colon \mathfrak{g} \to \mathfrak{gl}(V \otimes W) \tag{2.1.8}$$

$$(\rho \otimes \rho')(X)(v \otimes w) := (\rho(X)(v)) \otimes w + v \otimes (\rho'(X)(w)). \tag{2.1.9}$$

*Higher order tensor products are then iteratively defined:*

$$(\rho_1 \otimes \cdots \otimes \rho_k)(X)(v_1 \otimes \cdots \otimes v_k) := \sum_{i=1}^{k} v_1 \otimes \cdots \otimes \rho_i(X)(v_i) \otimes \cdots \otimes v_k \tag{2.1.10}$$

If it is clear from the context, we will simply write $X(v \otimes w) = Xv \otimes w + v \otimes Xw$. The following concept of the *Killing form* provides us with a tool that has many applications.

**Definition 2.1.9.** *The* **Killing form** $B \equiv B_{\mathrm{ad}} \equiv \kappa$ *is a bilinear form on $\mathfrak{g}$ (i.e. it maps $\mathfrak{g} \times \mathfrak{g}$ to $k$ and is linear in both slots) that is defined as*

$$\kappa(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)). \tag{2.1.11}$$

The first application of the Killing form is the following theorem that characterizes solvability. It goes back to Cartan.

**Theorem 2.1.10.** (Cartan)

*(i)* *Let $V$ be a vector space and $\mathfrak{g}$ a subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if $\kappa$ is zero on $\mathfrak{g}$.*

*(ii)* *Any $\mathfrak{g}$ is solvable iff $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

*(iii)* *Any $\mathfrak{g}$ is semisimple iff $\kappa$ is non-degenerate.*

*Proof.* For a proof see for example [EW06]. □

As mentioned above we have the result that any semisimple Lie algebra $\mathfrak{g}$ can be written as the direct sum of simple ideals in $\mathfrak{g}$.

**Theorem 2.1.11.** *Let $\mathfrak{g}$ be semisimple. Then there exist unique simple ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Furthermore, the image of a semisimple Lie algebra under a homomorphism is semisimple.*

*Proof.* See for example [EW06]. □

## 2.2 Weights and roots

Weights and roots are of utmost importance to the theory of Lie algebras. They can be seen as eigenvalues or more precisely as functionals on a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$. We will see that most of the theory of semisimple Lie algebras boils down to the study of its weights and roots. Roots are fundamental to the classification of simple Lie algebras. That is, every simple Lie algebra has a characteristic set of roots whose properties directly tell us a lot about the Lie algebra itself. We will also introduce a very common notation of representations in terms of weights which often simplifies calculations.

After having introduced basic definitions we will define abstract root systems and state how they serve the aim of characterization of Lie algebras. Furthermore we introduce highest weights, highest weight representations and the Weyl group where we also elaborate on examples.

**Definition 2.2.1.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We call an element $X \in \mathfrak{g}$* **semisimple** *if $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. A* **Cartan subalgebra** *$\mathfrak{h}$ of $\mathfrak{g}$ is a maximal commutative subalgebra that contains only semisimple elements. The* **centralizer** *$\mathfrak{c}(H)$ of a semisimple element $H$ is defined as $\mathfrak{c}(H) := \{X \in \mathfrak{g}\colon [X, H] = 0\}$. The* **rank** *of $\mathfrak{g}$ is defined as $\mathrm{rank}(\mathfrak{g}) := \min\{\dim \mathfrak{c}(H),\ H \text{ semisimple}\}$. A semisimple element $H$ is said to be* **regular** *if it minimizes $\dim \mathfrak{c}(H)$.*

These definitions are motivated by the following considerations: Take some semisimple elements $X_i \in \mathfrak{g}$. From linear algebra we know that the maps $\mathrm{ad}(X_i)$ are simultaneously diagonalizable if they commute:

$$0 = [\mathrm{ad}(X_i), \mathrm{ad}(X_j)] = \mathrm{ad}([X_i, X_j]) \tag{2.2.1}$$

since ad is a representation (and hence a homomorphism). But since ad is injective ($\ker(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g}) = \{0\}$ for a semisimple Lie algebra) it is convenient to look at subalgebras that consist of semisimple elements and in order to maximize the number of $\mathrm{ad}(X_i)$ that are simultaneously diagonalizable, we choose the subalgebra to be maximal. Having the definition of Cartan subalgebras at hand we can now define weights and roots.

**Definition 2.2.2.** *(i)* (Weights) *Let $\rho$ be a finite dimensional complex representation and $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. On a simultaneous eigenspace the eigenvalues of $\rho(H)$ depend linearly on $H$ and are thus described by a linear functional $\lambda\colon \mathfrak{h} \to \mathbb{C}$. If the eigenspace is nontrivial, we call $\lambda$ a* **weight** *of $\rho$ or rather $V$. In other words, a weight is a linear functional $\lambda\colon \mathfrak{h} \to \mathbb{C}$ with a nonzero vector $v \in V$ such that $\rho(H)(v) = \lambda(H)v$ for all $H \in \mathfrak{h}$. The* **weight space** *$V_\lambda$ is defined as $\{v \in V : \rho(H)(v) = \lambda(H)v \ \forall H \in \mathfrak{h}\}$. The dimension of $V_\lambda$ is called the* **multiplicity** *of the weight $\lambda$. The set of weights of $V$, denoted by $\mathrm{wt}(V)$, is a finite subset of the dual space of $\mathfrak{h}$, i.e. $\mathrm{wt}(V) \subset \mathfrak{h}^*$.*

*(ii)* (Roots) *The nonzero weights of the adjoint representation are called* **roots** *of the Lie algebra $\mathfrak{g}$. The weight space $\mathfrak{g}_\alpha$ for a root $\alpha$ is called* **root space**. *The set of all roots is denoted by $\Delta$. Since $\mathfrak{g}_0 = \{X \in \mathfrak{g}\colon \mathrm{ad}(H)(X) = [H, X] = 0\} = \mathfrak{h}$ is the Cartan subalgebra, one obtains the* **root decomposition** *$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$.*

In other words, weights are functionals describing the eigenvalues of the image of an element $H$ of the Cartan subalgebra under a given representation. Thus we will have to specify the representation when we talk about weights. The weights of the adjoint representation however are sufficiently important to deserve a proper name. The following definition introduces two important objects, namely a bilinear form on $\mathfrak{h}^*$ and the *root reflection.*

**Definition 2.2.3.** *(i) There is a unique element $H_\alpha$ in $\mathfrak{h}$ such that*

$$\kappa(H, H_\alpha) = \alpha(H) \tag{2.2.2}$$

*for all $H \in \mathfrak{h}$.*

*(ii) On $\mathfrak{h}^*$,*

$$\langle\varphi, \psi\rangle := \kappa(H_\varphi, H_\psi) = \varphi(H_\psi) = \psi(H_\varphi) \tag{2.2.3}$$

*defines a non-degenerate complex bilinear form.*

*(iii)* *Let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the real subspace spanned by $H_\alpha$ for $\alpha \in \Delta$. Then the* **root reflection** *$s_\alpha \colon \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ is defined as*

$$s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{2.2.4}$$

The bilinear form $\langle\ ,\ \rangle$ for weights from above has no particular name but shows up very often, e.g. in the definition of the root reflection. The root reflection $s_\alpha$ has the desirable property that it maps $\alpha$ to $-\alpha$, whence the name reflection:

$$s_\alpha(\alpha) = \alpha - \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = \alpha - 2\alpha = -\alpha. \tag{2.2.5}$$

The following proposition collects crucial information about roots. First of all a root $\alpha$ has no nonzero multiples except $-\alpha$ that are themselves roots. Furthermore the root spaces $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \colon [H, X] = \alpha(H)X\}$ are one-dimensional. Finally the root system of a semisimple Lie algebra is finite.

**Proposition 2.2.4.** *(i) Let $\alpha \in \Delta$. Then the only nonzero complex multiple of $\alpha$ that is a root is $-\alpha$.*

*(ii) The root space $\mathfrak{g}_\alpha$ is one-dimensional, the subalgebra $\mathfrak{s}_\alpha$ spanned by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*

*(iii) Let $\alpha, \beta \in \Delta$, $\beta \neq -\alpha$ and $\alpha + \beta \in \Delta$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Otherwise $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$.*

*(iv) Let $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$, $z \in \mathbb{C}$. Then $\beta + z\alpha$ is a root if and only if $z \in \mathbb{Z}$ and the roots form an* **unbroken string** *$\beta - p\alpha, \beta - (p-1)\alpha, ..., \beta + q\alpha$ for $p, q \geq 0$ and $p - q = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$.*

*Proof.* A proof can be found in introductory books, e.g. [EW06]. □

The properties obtained from the proposition above suggest to define root systems independently of the definition of roots as subsets of the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ with certain properties. Those systems are called *abstract root systems.*

**Definition 2.2.5.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \leq \mathfrak{g}$ a Cartan subalgebra. An* **abstract root system** *$\Delta$ is a finite subset of the Euclidean vector space $(\mathfrak{h}_0^*, \langle\ ,\ \rangle)$ such that:*

*(i)* $\mathrm{span}(\Delta) = \mathfrak{h}_0^*$

*(ii)* $s_\alpha(\Delta) = \Delta$

*(iii)* $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$

*It is called* **reduced** *if for $\alpha \in \Delta$ it follows that $2\alpha \notin \Delta$. It is called* **irreducible** *if there are no subsystems $\Delta_1, \Delta_2$ such that $\Delta = \Delta_1 \sqcup \Delta_2$.*

In fact a complex semisimple Lie algebra is simple if and only if its root system is irreducible. We can go further in the abstraction of roots and root systems. Based on the bilinear form $\langle \ , \ \rangle$ we can define a matrix that encodes the relations of the roots amongst themselves. Such matrices are called *Cartan matrices*. They also give rise to *Dynkin diagrams* which visualize the root system of a Lie algebra. But before the definition of those objects we need the notion of simple roots which form a very basic subset of the root system.

**Definition 2.2.6.** *Let $V$ be a vector space and $\Delta \subset V$ an abstract root system. We define $V^+$ as some subspace that is stable under addition and multiplication of positive scalars such that $V^+ \sqcup \{0\} \sqcup (-V^+)$. The the* **positive roots** *$\Delta^+$ are defined as $\Delta^+ := \Delta \cap V^+$. The* **simple roots** *are the positive roots which cannot be written as the sum of two positive roots. One defines also an* **ordering** *on $\Delta$ where $\alpha \leq \beta$ iff $\beta - \alpha \in \Delta^+ \sqcup \{0\}$.*

**Definition 2.2.7.** *Let $\Delta$ be an abstract root system and $\Delta^0 = \{\alpha_1, ..., \alpha_n\}$ the corresponding set of simple roots. The* **Cartan matrix** *is defined as:*

$$A = (a_{ij}), \quad a_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}. \tag{2.2.6}$$

Note that the entries on the main diagonal are always 2 and we have $a_{ij} \leq 0$ if $i \neq j$. We set

$$\alpha_i^\vee = \frac{2\alpha_i}{|\alpha_i|^2}, \quad \alpha_i \in \Delta. \tag{2.2.7}$$

$\alpha_i^\vee$ is called the ***coroot*** associated to the root $\alpha_i$. This simplifies notation of the entries of the Cartan matrix to $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$.

**Definition 2.2.8.** *The* **Dynkin diagram** *of a Lie algebra is graph that is constructed in the following way:*

1. *To each simple root $\alpha_i$ associate a vertex (represented by a white dot).*
2. *The number of edges between $\alpha_i$ and $\alpha_j$ is given by $n_{ij} := a_{ij}a_{ji}$*
3. *If $n_{ij} > 1$ then $|\alpha_j| = \sqrt{n_{ij}}|\alpha_i|$ and the fact that a root is shorter is notated by a black dot for the short root.*

The Dynkin diagram encodes the root system in a simple way. Every vertex represents one root and their relations are expressed by the number of edges joining them. Some properties of the Dynkin diagram as a graph directly correspond to properties of the root system and consequently for the corresponding Lie algebra. For example one can prove that an abstract root system is irreducible iff its associated Dynkin diagram is connected. Therefore we obtain the following result:

**Proposition 2.2.9.** *A complex semisimple Lie algebra $\mathfrak{g}$ is simple iff its Dynkin diagram is connected.*

The striking result of Lie algebra theory is now that we can classify the connected Dynkin diagrams restricted by the premises for root systems. As it can also be shown that there is a one-to-one correspondence between simple Lie algebras and connected Dynkin diagrams the classification of connected Dynkin diagrams leads also to a full classification of simple Lie algebras. We quote the following theorem:

**Theorem 2.2.10.** *Let $\mathfrak{g}$ be a complex simple Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the* **classical Lie algebras** $\mathfrak{sl}(n, \mathbb{C})$*,* $\mathfrak{so}(2n, \mathbb{C})$*,* $\mathfrak{so}(2n+1, \mathbb{C})$ *and* $\mathfrak{sp}(2n, \mathbb{C})$ *or to the one of the five* **exceptional Lie algebras** $\mathfrak{e}_6$*,* $\mathfrak{e}_7$*,* $\mathfrak{e}_8$*,* $\mathfrak{f}_4$ *and* $\mathfrak{g}_2$*.*

*Proof.* The rather involved proof can be found in [FH91]. □

In the following figure we present the Dynkin diagrams for all simple Lie algebras. It is taken from [FK12].

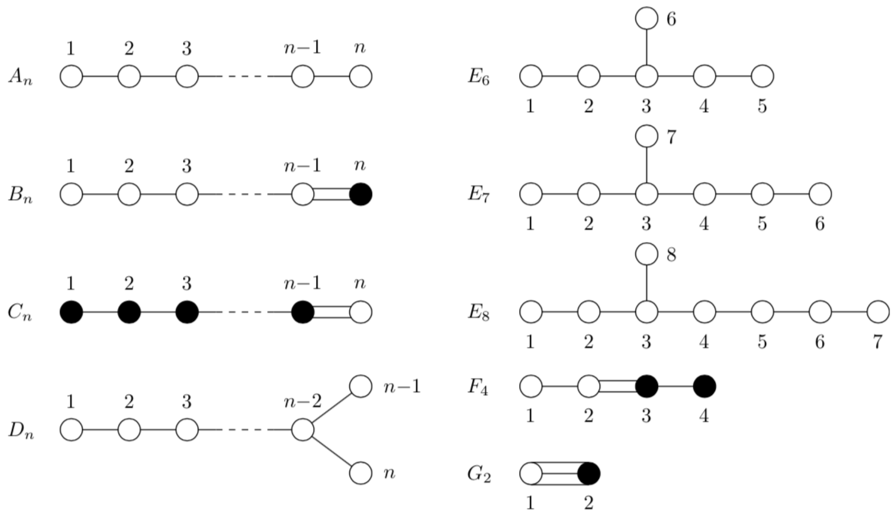


Figure 2.1: The Dynkin diagrams for the simple Lie algebras. The names translate as follows: $A_n$: $\mathfrak{su}(n+1)$, $B_n$: $\mathfrak{so}(2n+1)$, $C_n$: $\mathfrak{sp}(2n)$, $D_n$: $\mathfrak{so}(2n)$, $E_i$: $\mathfrak{e}_i$ for $i = 6, 7, 8$, $F_4$: $\mathfrak{f}_4$, $G_2$: $\mathfrak{g}_2$.

Another important concept is the ***Cartan-Weyl basis***. Having chosen a Cartan subalgebra $\mathfrak{h}$ generated by $r \equiv \text{rank}(\mathfrak{h})$ elements $H^i$ we define the ***ladder operators*** $E^\alpha$ by the eigenvalue equation

$$[H^i, E^\alpha] = \alpha^i E^\alpha \tag{2.2.8}$$

where the $\alpha^i$ are the corresponding roots. The full set of the defining commutators for the Cartan-Weyl basis is then given by

$$[H^i, H^j] = 0 \tag{2.2.9}$$

$$[H^i, E^\alpha] = \alpha^i E^\alpha \tag{2.2.10}$$

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha\beta}E^{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta \\ \frac{2}{\langle\alpha,\alpha\rangle}\sum_{i=1}^r \alpha^i H^i & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases}. \tag{2.2.11}$$

The following example discusses in detail the root system of $\mathfrak{sl}(n)$.

**Examples 2.2.11.** (i) The root system of $\mathfrak{sl}(n, \mathbb{C})$. For general $n \in \mathbb{N}$ we choose as the Cartan subalgebra the subspace of tracefree diagonal matrices:

$$\mathfrak{h} := \{A \in \mathfrak{sl}(n, \mathbb{C})\colon A = \mathrm{diag}(a_{11}, ..., a_{nn}) \wedge \mathrm{tr}(A) = \sum_i a_{ii} = 0\}. \tag{2.2.12}$$

If we define the ***elementary matrix*** $E_{ij}$ to be the matrix with 1 at the $ij$-position and 0 elsewhere we obtain

$$[H, E_{ij}] = (x_{ii} - x_{jj})E_{ij} \tag{2.2.13}$$

where the $x_{ii}$ are the diagonal elements of $H$. This equation means that the eigenvalues of $\mathrm{ad}(H)$ are precisely the functionals $e_i - e_j \in \mathfrak{h}^*$ where $e_i$ extracts the $i$-th diagonal entry from $H$. From this observation we obtain information on the general root system structure of $\mathfrak{sl}(n, \mathbb{C})$ which is in accord with Proposition 2.2.4:

**Proposition 2.2.12.** *(i) The root system of $\mathfrak{sl}(n, \mathbb{C})$ is given by $\Delta = \{e_i - e_j\colon i \neq j\}$.*

*(ii) Every root space $\mathfrak{g}_{e_i - e_j}$ is one-dimensional and $\mathfrak{g}_{e_i-e_j} = \mathrm{span}(E_{ij})$. The only nonzero multiple of $\alpha \in \Delta$ is $-\alpha$.*

*(iii) The roots $\Delta$ span the dual $\mathfrak{h}^*$.*

*(iv) The rank of $\mathfrak{sl}(n, \mathbb{C})$ is $n - 1$.*

The positive roots of $\mathfrak{sl}(n, \mathbb{C})$ are then given by $\Delta^+ = \{e_i - e_j\colon i < j\}$. The simple roots are given by $\Delta^0 = \{\alpha_1, ..., \alpha_{n-1} | \alpha_i := e_i - e_{i+1}\}$[3]. Since $\langle\alpha_i, \alpha_i\rangle = 2$, $\langle\alpha_i, \alpha_{i+1}\rangle = -1$ and $\langle\alpha_i, \alpha_j\rangle = 0$ otherwise, the Cartan matrix is given by $A_{ii} = 2$, $A_{i,i+1} = A_{i+1,i} = -1$ and $A_{ij} = 0$ otherwise. The Dynkin diagram has $n - 1$ vertices where the $i$-th vertex is incident[4] with the $(i-1)$-th and $(i+1)$-th vertex.

[3]Since the simple roots form a basis of $\mathfrak{h}^*$ their number must equal the rank of $\mathfrak{sl}(n, \mathbb{C})$.

[4]i.e has one edge with

For the special case $\mathfrak{sl}(2,\mathbb{C})$ we choose the matrix $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the basis of the one-dimensional Cartan subalgebra $\mathfrak{h}$. The two root spaces are then spanned by $E_{12} = E$ and $E_{21} = F$. Since $\mathrm{ad}(H)(E) = [H,E] = 2E$ and $\mathrm{ad}(H)(F) = [H,F] = -2F$ we obtain the root system $\Delta = \{2,-2\}$ and the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$.

**Definition 2.2.13.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $V$ a representation, $\mathfrak{h}$ a Cartan subalgebra and $\alpha \in \Delta^+$ a positive root. We call $v \in V$ a* **highest weight vector** *if $X(v) = 0$ for all $X \in \mathfrak{g}_\alpha$. For a finite dimensional $V$ there are finitely many weights. Together with the ordering on $\mathfrak{h}_0^*$ from Definition 2.2.6 one concludes that there is a maximal weight $\theta \in \mathrm{wt}(V)$ such that $\lambda \leq \theta$ for all $\lambda \in \mathrm{wt}(V)$. We call $\theta$ the* **highest weight** *of $V$.*

Note that if $\theta$ is a highest weight, any $v \in V_\theta$, $v \neq 0$ is a highest weight vector since $E_i(v) \neq 0$ would imply that $v$ is a weight vector of weight strictly bigger than $\theta$ which is not possible. The expansion of a highest weight $\theta$ in terms of simple roots and simple coroots is given by

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee, \quad a_i, a_i^\vee \in \mathbb{N}. \tag{2.2.14}$$

The numbers $a_i$ are called ***marks*** or ***Kac labels***, the numbers $a_i^\vee$ are called ***comarks*** or ***dual Kac labels***. They are related by $a_i = a_i^\vee \frac{2}{|\alpha_i|^2}$. Let now $a_i^{vee}$ be the comarks of the ***highest root*** of $\mathfrak{g}$. Then th ***dual Coxeter number*** $h^\vee$ is defined as

$$h^\vee = \sum_{i=1}^{r} a_i^\vee + 1. \tag{2.2.15}$$

**Definition 2.2.14.** *The* **Weyl group** *$W = W(\Delta)$ of an abstract root system $\Delta$ is the subgroup $W \leq O(V)$ of the orthogonal group generated by the root reflections $s_\alpha$:*

$$W(\Delta) := \langle s_\alpha | \alpha \in \Delta \rangle \leq O(V). \tag{2.2.16}$$

**Definition 2.2.15.** *An element $\lambda \in \mathfrak{h}_0^*$ is called* **algebraically integral** *if $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta$. A weight $\lambda$ is called dominant if $\langle\lambda,\alpha_i\rangle \geq 0$ for all $\alpha_i \in \Delta^0$. The set of all dominant elements of $\lambda \in \mathfrak{h}_0^*$ is called the closed* **dominant Weyl chamber**. *Generally, an open* **Weyl chamber** *is the connected component of the complement of the hyperplanes perpendicular to the roots. In other words the Weyl chambers $C_w$ are given by:*

$$C_w = \{\lambda | \langle w(\lambda), \alpha_i \rangle \geq 0, \alpha_i \text{ simple}\},\ w \in W \tag{2.2.17}$$

In particular any highest weight is dominant and algebraically integral. A rather involved result is the existence of highest weights for finite dimensional complex semisimple Lie algebras:

**Theorem 2.2.16.** *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\lambda \in \mathfrak{h}_0^*$ be a dominant algebraically integral weight. Then there exists a unique*[5] *finite dimensional irreducible representation with highest weight $\lambda$. It will be denoted by $\mathcal{L}_\lambda$.*

The set of these representations will be denoted by $P_+^{\mathfrak{g}}$. The expansion of weights in terms of simple roots is not useful for irreducible finite-dimensional representations since the coefficients are in general not in $\mathbb{Z}$. We therefore introduce *fundamental weights* as the dual basis to the simple coroot basis:

**Definition 2.2.17.** *Let $\{\alpha_i^\vee\}$ be a basis of simple coroots. Its dual basis $\{\omega_i\}$ is defined by:*

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{2.2.18}$$

*The $\{\omega_i\}$ are called* **fundamental weights**. *The expansion coefficients $\lambda_i$ of a weight $\lambda$ with respect to the fundamental weights $\omega_i$ are called* **Dynkin labels** *and have the following form*

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \quad \Leftrightarrow \quad \lambda_i = \langle \lambda, \alpha_i^\vee \rangle. \tag{2.2.19}$$

We know from above that the Dynkin labels of integral weights are always in $\mathbb{Z}$. We can thus write a weight $\lambda$ in the following form

$$\lambda = (\lambda_1, ..., \lambda_r) \tag{2.2.20}$$

where the $\lambda_i$ are the respective Dynkin labels. The Dynkin labels of the simple roots $\alpha_i$ are precisely the entries of the Cartan matrix:

$$\alpha_i = \sum_j A_{ij} \omega_j. \tag{2.2.21}$$

An important consequence of this observation is that the rows of the Cartan matrix give a basis of simple roots. For example in the case of $\mathfrak{su}(2)$ the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.2.22}$$

The set of simple roots is thus given by $\{(2,-1),(-1,2)\}$ in Dynkin label notation. Another object we will use later is the ***Weyl vector*** $\rho$. It is defined as the weight where all Dynkin labels are one:

$$\rho = \sum_i \omega_i = (1, 1, ..., 1). \tag{2.2.23}$$

Equivalently it is the half of the sum of all positive roots. Let us now introduce a frequently used notation:

[5] Here "unique" means *unique up to isomorphism.*

**Notation:** When given a highest weight representation with highest weight $\lambda$ we will refer to that representation by simply writing down the Dynkin labels of its highest weight. For example the highest weight of the adjoint representation of $\mathfrak{su}(2)$ is given by $\lambda = \alpha_1 + \alpha_2$ where $\alpha_1$, $\alpha_2$ are the simple roots of $\mathfrak{su}(3)$. The expansion of $\lambda$ in terms of fundamental weights then reads

$$\lambda = \omega_1 + \omega_2. \tag{2.2.24}$$

Thus $\lambda$ can be denoted as

$$\lambda = (1, 1). \tag{2.2.25}$$

We will then refer to the adjoint representation (which is the highest weight representation with highest weight $\theta$) as the representation $(1, 1)$. Now let us give examples for the introduced concepts:

**Examples 2.2.18.** (i) $\mathfrak{su}(2)$. The root system of the Lie algebra $\mathfrak{su}(2)$ (being of rank 1) has the simple root $\alpha_1$. The Cartan matrix consequently boils down to $A = (2)$. The fundamental weight $\omega_1$ which has to fulfill $\frac{2\langle\omega_1,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle} = 1$ is then given by $\omega_1 = \frac{1}{2}\alpha_1$. The Weyl group is generated by the simple reflections $s_i$. In this case there is only one simple root, therefore $s_1 = s_{\alpha_1}$. The action on a weight $\lambda = \lambda_1\omega_1$ (all weights are only multiples of the fundamental weight) is then given by $s_1(\lambda_1\omega_1) = \lambda_1\omega_1 - \lambda_1\langle\omega_1, \alpha_1\rangle\alpha_1 = \lambda_1(\omega_1 - \alpha_1) = -\lambda_1\omega_1$. But then of course $s_1^2 = 1$ so $W$ is only of order 2 and $W = \{1, s_1\}$. The fundamental Weyl chamber is thus given by the set

$$C_1 = \{\lambda_1\omega_1, \lambda_1 \in \mathbb{Z}_{\geq 0}\}. \tag{2.2.26}$$

(ii) $\mathfrak{su}(3)$. A more sophisticated example is the special unitary algebra in three dimensions. This algebra has rank 2 (since the elements $H_1 = \mathrm{diag}(1, -1, 0)$ and $H_2 = \mathrm{diag}(0, 1, -1)$ form a Cartan subalgebra) and has two simple roots $\alpha_1$ and $\alpha_2$. They both have the same length and thus the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.2.27}$$

Since the length of $\alpha_1$ and $\alpha_2$ is two, they coincide with their respective coroots. The Dynkin labels of the roots are the entries of the Cartan matrix so $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -\omega_1 + 2\omega_2$ where $\omega_1$ and $\omega_2$ are the fundamental weights.

## 2.3 Branching rules and embeddings

We will need the concept of embeddings and the decomposition of $\mathfrak{g}$-representations into representations of a given subalgebra $\mathfrak{h}$ in the treatment of the conjecture. Therefore we will briefly introduce embeddings of Lie algebras, branching rules and how to compute some of them. References for this section are mainly [DFMS97], [Cah84] and [Fuc95].

**Definition 2.3.1.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \leq \mathfrak{g}$ a subalgebra. $\mathfrak{h}$ is called* **maximal** *if there is no strict subalgebra $\mathfrak{k}$ such that $\mathfrak{h} \leq \mathfrak{k} \leq \mathfrak{g}$.*

**Definition 2.3.2.** *Let $\mathfrak{g}$ be simple. $\mathfrak{g}' \leq \mathfrak{g}$ is called* **regular** *if the roots of $\mathfrak{g}'$ are contained in the root system of $\mathfrak{g}$, i.e. if $\Delta' \subset \Delta$. A subalgebra $\mathfrak{s}$ not having this property is called* **special**. *A subalgebra contained in a regular subalgebra is called a* **R-subalgebra** *and the others are called* **S-subalgebras**.

Dynkin (see [Dyn52b] and [Dyn52a]) has provided an algorithm to find the regular and special subalgebras, in particular those which are maximal, i.e. not contained in any other regular or special proper subalgebra respectively. We will discuss the algorithm how to find regular subalgebras only. Let $\mathfrak{g}'$ be a regular subalgebra of $\mathfrak{g}$ where $\mathfrak{g}$ is taken to be simple. Let $\Delta' \subset \Delta$ be the set of their respective roots and choose a basis $\Delta'_0$ of simple roots of $\mathfrak{g}'$. For two simple roots $\alpha', \beta' \in \Delta'_0$ their difference $\alpha' - \beta'$ does not lie in $\Delta'$. By an easy argument it follows that $\alpha' - \beta' \notin \Delta$. Finding regular subalgebras thus reduces to seeking subsets $\Delta'_0 \subset \Delta$ such that for $\alpha', \beta' \in \Delta'_0$ it follows that $\alpha' - \beta' \notin \Delta$. Having found this set $\mathfrak{g}'$ ***is defined as the subalgebra generated by*** $E_{\alpha'}, E_{-\alpha'}, H_{\alpha'}$ ***for*** $\alpha' \in \Delta'_0$. The algorithm for this goes as follows:

(i) Add to $\Delta_0$ of $\mathfrak{g}$ the negative of the highest root $\theta$, i.e. $-\theta$. Denote this set by $\overline{\Delta}_0$.

(ii) This set has the property that for $\alpha', \beta' \in \overline{\Delta}_0$ it holds that $\alpha' - \beta' \notin \Delta$.

(iii) Since $\overline{\Delta}_0$ is a linearly dependent set (since $\Delta_0$ is a basis) eliminate one (or several) vectors from it. Notate the so obtained set by $\Delta'_0$.

(iv) $\Delta'_0$ generates a (semi-)simple regular subalgebra of $\mathfrak{g}$.

**Remark 2.3.3.** This procedure is easily visualized by Dynkin diagrams. Define the ***extended Dynkin diagram*** associated with $\overline{\Delta}_0$ by adding the dot representing $-\theta$. Since the Dynkin lables of $-\theta$ are known, the respective entries of the Cartan matrix are easily calculated and therefore also the number of lines connecting $-\theta$ and the other roots. This gives some immediate embeddings. Removing $m+1$ where $m \leq n$ dots from $\bar{A}_n$, i.e. the extended Dynkin diagram corresponding to $\mathfrak{su}(n+1)$ will yield $A_{n-m}$. Therefore $\mathfrak{su}(n-m) \leq \mathfrak{su}(n)$ is a regular subalgebra. Similarly for $C_n$ which corresponds to $\mathfrak{sp}(2n)$ removing two (or more) dots will yield the diagram $C_{n-1}$ such that $\mathfrak{sp}(2n-2)$ is a regular subalgebra of $\mathfrak{sp}(2n)$.

**Definition 2.3.4.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}' \leq \mathfrak{g}$ a subalgebra. We say that $\mathfrak{g}'$ is* **embedded** *into $\mathfrak{g}$ if there is a Lie algebra homomorphism $f\colon \mathfrak{g}' \to \mathfrak{g}$ such that $f$ is a homeomorphism onto its image $f(\mathfrak{g}') \subset \mathfrak{g}$.*

It is possible to map the Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}'$ into the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by $f$. Also if $\rho$ is a representation of $\mathfrak{g}$ then $\rho \circ f$ is a representation of $\mathfrak{g}'$ since the composition of two Lie algebra homomorphisms is again a Lie algebra homomorphism. It is customary to omit mentioning the specific mapping $f$ and to simply speak of an embedding $\mathfrak{g}' \leq \mathfrak{g}$. The proof of the following theorem can be found in [Cah84] and for completeness it is stated in the appendix.

**Proposition 2.3.5.** *Let $f$ be an embedding from $\mathfrak{g}'$ to $\mathfrak{g}$ with respective Cartan subalgebras $\mathfrak{h}'$ and $\mathfrak{h}$ and root systems $\Delta'$ and $\Delta$. Define a mapping $f^*\colon \Delta \to \Delta'$ as follows: For any root $\alpha \in \Delta$ and any $H' \in \mathfrak{h}'$ it holds that*

$$(f^*(\alpha))(H') = \alpha(f(H')). \tag{2.3.1}$$

*Then $f^*$ is a projection from $\Delta$ to $\Delta'$. Furthermore, if $\lambda$ is a weight of the representation $\rho$ of $\mathfrak{g}$, then $f^*(\lambda)$ is a weight of the representation $\rho \circ f$ of $\mathfrak{g}'$.*

In other words, the weights of the representation of the subalgebra are simply the projections of the weights of the bigger algebra. As an immediate corollary we state:

**Corollary 2.3.6.** *Let $\mathfrak{g}' \leq \mathfrak{g}$ be a subalgebra. If the rank of $\mathfrak{g}'$ coincides with the rank of $\mathfrak{g}$, then $\mathfrak{g}'$ is a regular subalgebra.*

*Proof.* If the ranks are equal then $\mathfrak{h}' = \mathfrak{h}$ and thus the projection $f^*$ is simply the identity. $\square$

The following discussion is about the questions how a subalgebra acts on the representations of the ambient algebra. The decomposition of representations with respect to the smaller algebra is called a *branching rule*. We will give definitions and an algorithm to calculate the branching rules going back to Dynkin before discussing examples in detail.

**Definition 2.3.7.** *Let $\mathfrak{g}' \leq \mathfrak{g}$ be an embedding. Let $\mathcal{L}_\lambda$ be a irreducible representation of $\mathfrak{g}$ and $\mathcal{L}_\mu$, $\mu \in P_+^{\mathfrak{g}'}$, be irreducible representations of $\mathfrak{g}'$. The decomposition of $\mathcal{L}_\lambda$ into the representations $\mathcal{L}_\mu$ is called a* **branching rule***. We write*

$$\mathcal{L}_\lambda \mapsto \bigoplus_{\mu \in P_+^{\mathfrak{g}'}} b_{\lambda\mu} \mathcal{L}_\mu. \tag{2.3.2}$$

*If it is clear that the notation $\lambda$ represents the corresponding highest weight representation $\mathcal{L}_\lambda$, we simply write*

$$\lambda \mapsto \bigoplus_{\mu \in P_+^{\mathfrak{g}'}} b_{\lambda\mu} \mu. \tag{2.3.3}$$

*The numbers $b_{\lambda\mu}$ are called* **branching coefficients***.*

The projection we have discussed above plays a central role in the treatment of branching rules. Given an embedding $\mathfrak{p} \subset \mathfrak{g}$ the ***projection matrix*** $\mathcal{P}$ projects the weights of $\mathfrak{g}$ onto weights of $\mathfrak{p}$. The branching rules are then calculated by projecting the weights of a given irreducible representation of $\mathfrak{g}$ into $\mathfrak{p}$-weights and then regrouping those into irreducible $\mathfrak{p}$-representations.

**Definition 2.3.8.** *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of Lie algebras with corresponding projection matrix $\mathcal{P}$. Let $\theta$ be the highest root of $\mathfrak{g}$ and $\vartheta$ the highest root of $\mathfrak{g}'$. The* **embedding index** *$x_e$ of the embedding $\mathfrak{p} \subset \mathfrak{g}$ is defined as:*

$$x_e := \frac{|\mathcal{P}\theta|^2}{|\vartheta|^2}. \tag{2.3.4}$$

Following [Cah84] we may describe a formalism to compute the projections of the representations of the bigger algebra for regular subalgebras. Similarly to finding the regular subalgebras by adding the most negative root $-\theta$ we find the decomposition of a given representation of $\mathfrak{g}$ into representations of $\mathfrak{g}'$ by adding $-\theta$ and striking one root. Then the remaining set forms a basis of simple roots for the subalgebra. Calculating Dynkin labels of weights with respect to the new basis is easy as the Dynkin labels of $-\theta$ with respect to the old basis are known.

Since it soon becomes cumbersome to calculate branching rules by hand programs to calculate them by computer have been developed. The most common program to do so is LieART by Feger and Kephart (see [FK12]) which is running on Mathematica. Branching rules in LieART have been implemented for all simple Lie algebras except $C_n$. However extensive tables of branching rules encompassing $C_n$ can be found in [MP81] and [LP11]. In the following examples we will calculate some branching rules by hand in order to become familiar with the concept but later on we will take them for granted and refer to the sources above.

**Examples 2.3.9.** (i) $\mathfrak{su}(2)$ embedded in $\mathfrak{su}(3)$. We consider the regular embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$. This is explicitly given by taking a matrix $A \in \mathfrak{su}(2)$ and mapping it to $\mathfrak{su}(3)$ in the following way:

$$A \mapsto \begin{pmatrix} A & \\ & 0 \end{pmatrix}. \tag{2.3.5}$$

This map is well-defined since $\mathrm{tr}(A)+0=0$. Hence if we take the basis element $H$ for the one-dimensional standard Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{su}(2)$ given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3.6}$$

it will be mapped to one of the two basis elements $H_1, H_2$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(3)$, namely

$$H \mapsto \begin{pmatrix} H & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} =: H_1 \tag{2.3.7}$$

Take now a highest wight representation $V$ of $\mathfrak{su}(3)$ labelled by Dynkin labels. This means for a given highest weight $\lambda$ with the expansion

$$\lambda = \sum_{i=1}^{r=2} \lambda_i \omega_i \tag{2.3.8}$$

in terms of the fundamental weights $\omega_i$ we denote the corresponding highest weight representation by

$$(\lambda_1, \lambda_2). \tag{2.3.9}$$

Consider now a highest weight vector $v_{(\lambda_1,\lambda_2)} \in V$. Then $H_1$ acts on $v_{(\lambda_1,\lambda_2)}$ as follows:

$$H_1 v_{(\lambda_1,\lambda_2)} = \lambda_1 v_{(\lambda_1,\lambda_2)}. \tag{2.3.10}$$

This is because $\langle \alpha_1, \lambda \rangle = \langle \alpha_1, \lambda_1\omega_1 + \lambda_2\omega_2 \rangle$. The last expression is equal to $\lambda_1$ since $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$ by definition. Eventually the weight $(\lambda_1, \lambda_2)$ decomposes to $\lambda_1$ with respect to the embedding from above. In other words, given a weight $\lambda = (\lambda_1, \lambda_2)$ we just extract the first entry in its Dynkin label notation. Thus, if we now are given an explicit representation of $\mathfrak{su}(3)$, we will know how it decomposes into representations of $\mathfrak{su}(2)$ with respect to the embedding. As an easy example take the first fundamental representation with highest weight $\omega_1$. The weights of this representation are given by $\omega_1$, $\omega_1 - \alpha_1$ and $\omega_1 - \alpha_1 - \alpha_2$. In Dynkin label notation this translates to $(1, 0)$, $(-1, 1)$ and $(0, -1)$ respectively. By our observation we have the following correspondences:

$$\begin{aligned} (1,0) &\mapsto 1 \\ (-1,1) &\mapsto -1 \\ (0,-1) &\mapsto 0 \end{aligned} \tag{2.3.11}$$

We can now regroup those weights of $\mathfrak{su}(2)$ into representations. The weights corresponding to $\lambda_1 = 1$ and $\lambda_1 = -1$ give the standard two-dimensional representation of $\mathfrak{su}(2)$ labelled by $(1)$. The remaining representation is the one-dimensional trivial representation corresponding to $\lambda_1 = 0$, denoted by $(0)$. Taking the direct sum $(1) \oplus (0)$ we see that the dimensions match and we thus have found the decomposition of the three-dimensional representation $(1, 0)$ of $\mathfrak{su}(3)$ into a two- and a one-dimensional representation of $\mathfrak{su}(2)$:

$$(1, 0) \mapsto (1) \oplus (0). \tag{2.3.12}$$

As a general rule we state the following:

**Corollary 2.3.10.** *Let $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ be the regular embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$. Let $(n, 0)$ denote the highest weight representation of $\mathfrak{su}(3)$ with highest weight $(n, 0)$, $n \in \mathbb{N}$, labelled by Dynkin labels. Then $(n, 0)$ decomposes into $\mathfrak{su}(2)$ representations as follows:*

$$(n, 0) \mapsto \bigoplus_{k=0}^{n} (k). \tag{2.3.13}$$

(ii) We can conduct the calculation from above for other representations of $\mathfrak{su}(3)$ and see how they decompose into $\mathfrak{su}(2)$ representations. Take the eight-dimensional representation of $\mathfrak{su}(3)$ denoted by its highest weight $(1,1)$ (in terms of simple roots it is given by $\alpha_1 + \alpha_2 = \omega_1 + \omega_2$). The other nontrivial weights of this representation are given by $(2,-1)$, $(1,-2)$, $(-1,-1)$, $(-2,1)$ and $(-1,2)$. Extracting the first entries gives $-2,-1,-1,1,1,2$ and $0$ for the trivial weight. The highest weight representation corresponding to 2 is then denoted by $(2)$. Furthermore there are two representations $(1)$ and the trivial representation $(0)$. Therefore we obtain

$$(1,1) \to (2) \oplus (1) \oplus (1) \oplus (0) \tag{2.3.14}$$

The $(0)$-representation was one-dimensional, the two $(1)$-representations together yield four dimensions and the $(2)$-representation gives the remaining three dimensions to add up to eight dimensions as it should.

(iii) The representation $(1,1)$ has another decomposition which corresponds to the special embedding. This is given by

$$(\lambda_1, \lambda_2) \mapsto 2(\lambda_1 + \lambda_2). \tag{2.3.15}$$

This can be seen as follows. Take the generators of $\mathfrak{su}(3)$ corresponding to the roots as well as the two basis elements of the Cartan subalgebra $\mathfrak{h}$. Those are

$$E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+\alpha_2}, F_{\alpha_1}, F_{\alpha_2}, F_{\alpha_1+\alpha_2}, H_1, H_2. \tag{2.3.16}$$

Now define $E = E_{\alpha_1} + E_{\alpha_2}$ and $F = F_{\alpha_1} + F_{\alpha_2}$. Those will be the ladder operators for $\mathfrak{su}(2)$ embedded in $\mathfrak{su}(3)$. The generating element of the Cartan subalgebra $\mathfrak{h}'$ is then given by the commutator $[E,F]$:

$$\begin{aligned} H := [E,F] &= [E_{\alpha_1} + E_{\alpha_2}, F_{\alpha_1}, F_{\alpha_2}] \\ &= [E_{\alpha_1}, F_{\alpha_1}] + [E_{\alpha_1}, F_{\alpha_2}] + [E_{\alpha_2}, F_{\alpha_1}] + [E_{\alpha_2}, F_{\alpha_2}] \\ &= [E_{\alpha_1}, F_{\alpha_1}] + [E_{\alpha_2}, F_{\alpha_2}] \\ &= 2(H_1 + H_2) \end{aligned} \tag{2.3.17}$$

by the Serre relations Theorem B.0.1. Using this relation we can now easily find out how the roots of $\mathfrak{su}(3)$ are projected onto roots of $\mathfrak{su}(2)$. After quick calculation we are left with $-4,-2,-2,0,2,2$ and $4$. The first highest weight representation corresponds to the weight 4 and encompasses $-4,-2,0,2,4$. It is of course five-dimensional and denoted by $(4)$. The remaining representation is $-2,0,2$ and denoted by $(2)$. As seen above it is three-dimensional. The decomposition of $\mathfrak{su}(3)$ with respect to the special embedding is then found to be

$$(2,-1) \mapsto (4) \oplus (2). \tag{2.3.18}$$

(iv) Similarly the first fundamental representation may be decomposed: The weights $(1,0)$, $(-1,1)$ and $(-1,0)$ are mapped to $2(1+0)=2$, $2(1-1)=0$ and $(2(-1+0)=-2$ thus we are left with only one representation, namely $(2)$. That is

$$(1,0) \mapsto (2). \tag{2.3.19}$$

(v) Higher dimensional embeddings are also handled easily. Consider the regular embedding $\mathfrak{su}(3) \subset \mathfrak{su}(4)$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(3)$ is generated by two elements:

$$H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{2.3.20}$$

The regular embedding now very similarly to the case $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ puts the elements $H_1$ and $H_2$ in the top left corner of a four-dimensional matrix with zeros otherwise, i.e.

$$\bar{H}_1 := f(H_1) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \bar{H}_2 := f(H_2) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}. \tag{2.3.21}$$

The embedding map is denoted by $f\colon \mathfrak{su}(3) \to \mathfrak{su}(4)$. For completeness the third generating element of the Cartan subalgebra $\bar{\mathfrak{h}}$ is given by

$$\bar{H}_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{2.3.22}$$

Now by again looking at a highest weight representation $\mathcal{L}_\lambda$ with highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ we observe that $\bar{H}_1$ sends a weight vector $v \in \mathcal{L}_\lambda$ to $\lambda_1 v$ and $\bar{H}_2$ sends it to $\lambda_2 v$. Hence the highest weight $(\lambda_1, \lambda_2, \lambda_3)$ is projected to the $\mathfrak{su}(3)$-weight $(\lambda_1, \lambda_2)$. Let us illustrate this by an example: The first fundamental representation of $\mathfrak{su}(4)$ is given by the weights $(1,0,0)$, $(-1,1,0)$, $(0,-1,1)$ and $(0,0,-1)$. By the rule above we send those weights to the $\mathfrak{su}(3)$-weights $(1,0)$, $(-1,1)$, $(0,-1)$ and $(0,0)$. Regrouping into irreducible representations we obtain the highest weight representation $(1,0)$ and the trivial representation such that

$$(1,0,0) \mapsto (1,0) \oplus (0,0). \tag{2.3.23}$$

This may be generalized to the regular embedding of arbitrary dimension $n$, i.e. $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$. We have

**Corollary 2.3.11.** *Let $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$ be the regular embedding. The representation $(m, 0, ..., 0)_{\mathfrak{su}(n+1)}$ decomposes into $\mathfrak{su}(n)$-representations as follows:*

$$(m, 0, ..., 0)_{\mathfrak{su}(n+1)} \to \sum_{\ell=0}^{m} (\ell, 0, ..., 0)_{\mathfrak{su}(n)} \tag{2.3.24}$$

*where the representations on the right hand side have $n-1$ entries.*

(vi) Consider now the Lie algebra $\mathfrak{so}(2n+1)$, i.e. the special orthogonal algebra of odd dimensions. If we set $n = 3$ we obtain $\mathfrak{so}(7)$. We will find the decomposition by using the method of adding the most negative root $-\theta$. The seven-dimensional highest-weight representation with highest weight $(1, 0, 0)$ of $B_3$ is given by $(1, 0, 0)$, $(-1, 1, 0)$, $(0, -1, 2)$, $(0, 0, 0)$, $(0, 1-2)$, $(1, -1, 0)$ and $(-1, 0, 0)$. The highest root of $B_3$ is given by $(0, 1, 0)$. Looking at the Cartan matrix will give us the expansion of $-\theta$ in terms of the simple roots $\alpha_i$. Explicitly, $-\theta = \alpha_1 + 2\alpha_2 + \ 2\alpha_3$. Let $\lambda = \sum a_i \alpha_i$ be the expansion of an arbitrary weight in terms of the $\alpha_i$. Then the coefficient with respect to $-\theta$ is given by $-a_2 - 2a_2 - a_3$. The *extended weight scheme* of $B_3$ is thus given by

$$\begin{aligned}&(1, 0, 0, -1), (-1, 1, 0, -1), (0, -1, 2, 0), (0, 0, 0, 0),\\&(0, 1, -2, 0), (1, -1, 0, 1), (-1, 0, 0, 1).\end{aligned}$$

Deleting the third entries gives us the regular subalgebra $A_3$ (i.e. $\mathfrak{su}(4)$). We have

$$\begin{aligned}&(1, 0, -1), (-1, 1, -1), (0, -1, 0), (0, 0, 0),\\&(0, 1, 0), (1, -1, 1), (-1, 0, 1).\end{aligned}$$

One candidate for a highest weight representation is $(0, 1, 0)$. The other is $(0, 0, 0)$. Therefore we obtain two highest weight representations of $\mathfrak{su}(4)$, namely $(0, 1, 0)$ and $(0, 0, 0)$. Thus

$$\left((1, 0, 0)_{\mathfrak{so}(7)}\right)|_{\mathfrak{su}(4)} = \left((0, 1, 0) \oplus (0, 0, 0)\right)_{\mathfrak{su}(4)} . \tag{2.3.25}$$

## 2.4 Tensor products of representations

Given two highest weight representations $\mathcal{L}_\lambda$ and $\mathcal{L}_\mu$ of a semisimple Lie algebra $\mathfrak{g}$ we wish to find out how to calculate the tensor product $\mathcal{L}_\lambda \otimes \mathcal{L}_\mu$ which we will write as $\lambda \otimes \mu$ for convenience. The idea is to write it as a direct sum of irreducible highest weight representations of $\mathfrak{g}$, denoted by $P_+$. This is possible since $\mathfrak{g}$ is semisimple. That means we write

$$\lambda \otimes \mu = \bigoplus_{\nu \in P_+} \mathcal{N}_{\lambda\mu}^{\nu} \nu. \tag{2.4.1}$$

The numbers $\mathcal{N}_{\lambda\mu}^{\nu}$ are called ***tensor product coefficients***. They have some obvious properties which are stated below. In the proposition $\lambda^*$ stands for the ***conjugate representation*** of $\lambda$ which is defined as $\lambda^* := -w_0\lambda$ where $w_0$ is the longest element of the Weyl group $W$:

**Proposition 2.4.1.** *The tensor product coefficients from 2.4.1 have the following properties:*

*(i)* $\mathcal{N}_{\lambda 0}^{\nu} = \delta_{\lambda}^{\nu}$,

*(ii)* $\mathcal{N}_{\lambda\lambda^*}^{0} = 1$,

*(iii)* $\mathcal{N}_{\lambda\mu}^{\nu} = \mathcal{N}_{\lambda\nu^*}^{\mu^*}$,

*(iv)* $\mathcal{N}_{\lambda\mu}^{\nu} = \mathcal{N}_{\lambda\mu\nu^*}$.

*Proof.* (i) $\lambda \otimes 0 = \bigoplus \mathcal{N}_{\lambda 0}^{\nu} = \bigoplus \delta_{\lambda}^{\nu}\nu$

(ii) The tensor product $\lambda \otimes \lambda^*$ contains the scalar representation (i.e. 0) which is obtained by pairing the elements of $\lambda$ with their negatives in $\lambda^*$. Therefore $\mathcal{N}_{\lambda\lambda^*}^{0} = 1$.

(iii) Follows from (i) and (ii).

(iv) $\mathcal{N}_{\lambda\mu\nu}$ corresponds to the multiplicity of 0 in the product $\lambda \otimes \mu \otimes \nu$. Conclude by (iii).

□

We will now give an algorithm for the calculation of the tensor product by the *character method*. The ***character*** of a highest weight representation $\lambda$ is defined as

$$\chi_\lambda := \sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') e^{\lambda'} \tag{2.4.2}$$

where $\Omega_\lambda$ denotes the set of all weights in the representation $\mathcal{L}_\lambda$ and $e^{\lambda'}$ a formal exponential. Let now $\epsilon(w)$ denote the ***signature*** of $w \in W$, defined as $\epsilon(w) := (-1)^{\ell(w)}$ where $\ell(w)$ is the ***length*** of $w$, i.e. the minimum number of all reflections $s_i$ in a decomposition $w = \prod_i s_i$. Then the *character method* for the calculation of the tensor product coefficients $\mathcal{N}_{\lambda\mu}^{\nu}$ is given by

$$\mathcal{N}_{\lambda\mu}^{\nu} = \sum_{w \in W} \epsilon(w)\mathrm{mult}_\mu(w \cdot \nu - \lambda). \tag{2.4.3}$$

Here, $w \cdot \lambda$ denotes the *shifted Weyl reflection* defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$ ($\rho$ denotes the Weyl vector). For a detailed derivation of 2.4.3 we refer to [DFMS97]. This gives way to an algorithm for the calculation of $\mathcal{N}_{\lambda\mu}^{\nu}$.

1. Let $\lambda \otimes \mu$ be the tensor product of the representations $\lambda$ and $\mu$.

2. Write down all weights $\mu'$ in the representation $\mu$. Add those weights to $\lambda+\rho$.

3. Distinguish between two types of the sum $\mu' + \lambda + \rho$.

   (i) $\mu' + \lambda + \rho$ can be reflected into dominant weight by some $w \in W$.

   (ii) $\mu' + \lambda + \rho$ lies in the orbit of a weight with some vanishing Dynkin labels.

4. Type-(i) weights contribute $\epsilon(w)$-times to the tensor product coefficients $\mathcal{N}_{\lambda\mu}^{\nu}$. The sum of all contributions is then $\mathcal{N}_{\lambda\mu}^{\nu}$.

5. Type-(ii) weights do not contribute.

We illustrate this by examples.

**Examples 2.4.2.** (i) $\mathfrak{su}(2)$: Consider the eight-dimensional representation (7) and the three-dimensional representation (2) of $\mathfrak{su}(2)$. The weight lattice of (7) is

$$(-7\omega_1, -5\omega_1, ..., 7\omega_1). \tag{2.4.4}$$

The Weyl vector is in this case equal to the fundamental weight, so $\rho = \omega_1$. The highest weight $\lambda$ is then $2\omega_1$. The only weight that is of type (ii) is $-\omega_1$ since it lies in the orbit of 0 under the shifted Weyl reflection: $w \cdot 0 = -\omega_1$. By reflection, the non-dominant weights $-5\omega_1$ and $-3\omega_1$ are mapped to $3\omega_1$ and $\omega_1$ which cancel with the contributions from (1) and (3). The only remaining contributions are thus (5), (7) and (9). That means

$$(2) \otimes (7) = (5) \oplus (7) \oplus (9). \tag{2.4.5}$$

The dimension of $(2) \otimes (7)$ is $3 \cdot 8 = 24$. On the other hand

$$\dim((5) \oplus (7) \oplus (9)) = 6 + 8 + 10 = 24, \tag{2.4.6}$$

so the dimensions match. Generally we obtain the following result:

**Corollary 2.4.3.** *Let $(n)$ and $(m)$ for $n, m \in \mathbb{N}$ be fundamental representations of $\mathfrak{su}(2)$ labelled by Dynkin labels. Then their tensor product $(n) \otimes (m)$ decomposes into a direct sum as follows:*

$$(n) \otimes (m) = \bigoplus_{\substack{\ell=|n-m| \\ \ell+n+m \text{ even}}}^{n+m} (\ell). \tag{2.4.7}$$

For example $(3) \otimes (5)$ decomposes into a sum from $|3-5| = 2$ to $3+5 = 8$ with double integer steps, i.e.

$$(3) \otimes (5) = (2) \oplus (4) \oplus (6) \oplus (8). \tag{2.4.8}$$

(ii) $\mathfrak{su}(3)$: We take the representations $(1,0)$ and $(2,0)$. Their tensor product is $(1,0)\otimes(2,0)$. By step (2) we write down the weights of the representation $(2,0)$ and add $(1,0)$ to each of them.

$$\begin{gathered}(2,0),\ (0,1),\ (1,-1),\ (-2,2),\ (-1,0),\ (0,-2)\\ \downarrow +(1,0)\\ (3,0),\ (1,1),\ (2-1),\ (-1,2),\ (0,0),\ (1,-2).\end{gathered}$$

The weights $(2,-1)$ and $(-1,2)$ are in the $W$-orbit of some weight with vanishing Dynkin labels, thus of type (ii) and to be ignored. The shifted action $\cdot$ of $s_2$ on $(1,-2)$ is given by

$$s_2\cdot(1,-2)=(0,0) \tag{2.4.9}$$

therefore canceling with $(0,0)$. The remaining contributions are $(3,0)$ and $(1,1)$ thus yielding

$$(1,0)\otimes(2,0)=(3,0)\oplus(1,1). \tag{2.4.10}$$

We could of course have gone the other way and consider the tensor product $(2,0)\otimes(1,0)$. For later use we state the general rule of how to decompose the tensor product $(k+1,0)\otimes(1,0)$ for some integer $k\geq 0$:

**Corollary 2.4.4.** *Let $(k+1,0)$ and $(1,0)$ be representations of $\mathfrak{su}(3)$ labelled by Dynkin labels. Then their tensor product $(k+1,0)\otimes(1,0)$ decomposes into a direct sum of $\mathfrak{su}(3)$ representations as follows:*

$$(k+1,0)\otimes(1,0)=(k+2,0)\oplus(k,1). \tag{2.4.11}$$

Together with the tensor product of representations we will now introduce the concept of the *representation ring* which we will use later.

**Definition 2.4.5.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $P_+^{\mathfrak{g}}$ be the set of dominant highest weight representations of $\mathfrak{g}$. The* **representation ring** *of $\mathfrak{g}$ is defined as the set of formal sums of representations $\rho\in P_+^{\mathfrak{g}}$ with integer coefficients.*

$$\mathrm{Rep}(\mathfrak{g}):=\left\{\sum_{\rho\in P_+^{\mathfrak{g}}} n_\rho(\rho)\,|\,n_\rho\in\mathbb{Z}, n_\rho=0\,\text{for all but finitely many }\rho\right\}. \tag{2.4.12}$$

*The multiplication on this ring is well-defined by the tensor product of representations.*

In fact there is quite a subtlety here concerning the additive structure. Representations as such cannot be 'negative'. It therefore remains to specify what we mean by subtraction of representations (or by the additive inverse in the language of algebra). It is possible to *formally* define the negative of a representation by just defining $\rho+(-\rho)=0$ where 0 denotes the trivial representation. If we can establish an equation such as $\rho-\lambda=\sigma$ then $\sigma$ is interpreted as the representation such that $\rho=\lambda\oplus\sigma$.

# Chapter 3

# Affine Lie algebras and fusion rings

## 3.1 Affine Lie algebras

Affine Lie algebras extend a given finite-dimensional (often just referred to as finite) Lie algebra $\mathfrak{g}$ to an infinite-dimensional Lie algebra $\hat{\mathfrak{g}}$. Generally they are a special case of so-called *Kac-Moody algebras*. Their physical applications can be found in string theory, most notably in *Wess-Zumino-Witten models* or shortly *WZW models*. The following discussion of affine Lie algebras can be retraced in [KMPS90] and in [DFMS97]. The construction of affine Lie algebras goes as follows: First we define the *loop algebra* $\tilde{\mathfrak{g}}$ for a given Lie algebra $\mathfrak{g}$.

**Definition 3.1.1.** *Let $\mathbb{C}[t, t^{-1}]$ be the associative algebra of Laurent polynomials, i.e. the ring of polynomials over one variable $t$ and its negative power $t^{-1}$ over $\mathbb{C}$. Then the* **loop algebra** $\tilde{\mathfrak{g}}$ *is defined as the tensor product of $\mathfrak{g}$ with $\mathbb{C}[t, t^{-1}]$, i.e.*

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]. \tag{3.1.1}$$

.

If $J^a$ are generators for $\mathfrak{g}$ and $n, m \in \mathbb{Z}$, we can naturally define a Lie bracket on $\tilde{\mathfrak{g}}$ turning it into a Lie algebra:

$$[J^a \otimes t^n, J^b \otimes t^m] := i \sum_c f_c^{ab} J^c \otimes t^{n+m}. \tag{3.1.2}$$

We will make use of the short notation $J_n^a := J^a \otimes t^n$ for the generators of the affine algebra.

**Examples 3.1.2.** (i) Let $\mathfrak{gl}_n(\mathbb{C})$ be the Lie algebra of complex $n \times n$ matrices. Then the loop algebra $\widehat{\mathfrak{gl}_n(\mathbb{C})}$ is given by

$$\widehat{\mathfrak{gl}_n(\mathbb{C})} = \left\{ \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ & \vdots & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \middle| p_{ij}(t) \in \mathbb{C}[t, t^{-1}] \quad \forall i, j \right\}. \tag{3.1.3}$$

Each of the matrices is called a ***loop*** in $\mathfrak{gl}_n(\mathbb{C})$. The Lie bracket is the usual commutator from $\mathfrak{gl}_n(\mathbb{C})$ but the coefficients are in $\mathbb{C}[t,t^{-1}]$.

(ii) The loop algebra $\widehat{\mathfrak{sl}_n(\mathbb{C})}$ of $\mathfrak{sl}_n(\mathbb{C})$ is given by the loops of $\mathfrak{gl}_n(\mathbb{C})$ with vanishing trace, i.e.

$$\widehat{\mathfrak{sl}_n(\mathbb{C})} = \left\{ X \in \widehat{\mathfrak{gl}_n(\mathbb{C})} | p_{11}(t) + ... + p_{nn}(t) = 0 \right\} \tag{3.1.4}$$

The next step is to construct a ***central extension*** of $\tilde{\mathfrak{g}}$ by adjoining a ***central element*** $k$, i.e. an element that commutes with any element of $\tilde{\mathfrak{g}}$, up to some factor:

$$[J^a \otimes t^n, J^b \otimes t^m] := i \sum_c f_c^{ab} J^c \otimes t^{n+m} + kn \cdot \kappa(J^a, J^b)\delta_{n+m,0}. \tag{3.1.5}$$

Now we have all the ingredients for the definition of *affine Lie algebras*.

**Definition 3.1.3.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ be its loop algebra where $\mathbb{C}[t,t^{-1}]$ denotes the associative algebra*[1] *of Laurent polynomials in one variable $t$. Let $k$ be such that $[k, X] = 0$ for all $X \in \tilde{\mathfrak{g}}$. The* **affine Lie algebra** $\hat{\mathfrak{g}}$ *of $\mathfrak{g}$ is then defined as:*

$$\hat{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \mathbb{C}k. \tag{3.1.6}$$

Note that this algebra is infinite-dimensional. Since the central extension adds a commuting element, the roots of the affine Lie algebra $\hat{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ are essentially the same. We have $\hat{\mathfrak{g}}_\alpha = \tilde{\mathfrak{g}}_\alpha$ for $\alpha \neq 0$ and $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{h}} = \tilde{\mathfrak{g}}_0 + \mathbb{C}k = \tilde{\mathfrak{h}} + \mathbb{C}k$ for $\alpha = 0$. We can thus write

$$\hat{\mathfrak{g}} = \oplus_{\alpha \in \hat{\Delta}} \hat{\mathfrak{g}}_\alpha. \tag{3.1.7}$$

We will discuss weights and roots in affine Lie algebras further below. The defining commutation relations for the affine Lie algebra in terms of the Cartan-Weyl basis can be given explicitly. For the elements of the Lie algebra $\mathfrak{g}$ the Killing forms read

$$\kappa(H^i, H^j) = \delta^{ij}, \qquad \kappa(E^\alpha, E^{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle} \tag{3.1.8}$$

which leads then to the commutation relations

$$[H^i_n, H^j_m] = kn\delta^{ij}\delta_{n+m,0} \tag{3.1.9}$$

$$[H^i, E^\alpha] = \alpha^i E^\alpha_{n+m} \tag{3.1.10}$$

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha\beta} E^{\alpha+\beta}_{n+m} & \text{if } \alpha + \beta \in \Delta \\ \frac{2}{\langle \alpha, \alpha \rangle} \left( \sum_{i=1}^r \alpha^i H^i_{n+m} + kn\delta_{n+m,0} \right) & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \tag{3.1.11}$$

[1] i.e. a vector space $A$ together with an associative bilinear map $\mu \colon A \times A \to A$

where $\Delta$ refers to the set of roots of $\mathfrak{g}$. Remember that $H_n^i = H^i \otimes t^n$ and $E_n^\alpha = E^\alpha \otimes t^n$. This gives us some insight on the set of generators $\{H_0^1, ..., H_0^r, k\}$ of $\hat{\mathfrak{g}}$. The eigenvalues of the adjoint representation of $H_0^i$ and $k$ can be read from the commutation relations above, they are $\alpha^i$ and 0 respectively. Thus the eigenvector $(\alpha^1, ..., \alpha^r, 0)$ is independent of $n$ and therefore the same for all $E_m^\alpha$ no matter what $m$ is. This means that $(\alpha^1, ..., \alpha^r, 0)$ is infinitely degenerate and hence the set $\{H_0^1, ..., H_0^r, k\}$ cannot be a maximal abelian subalgebra. This inconvience is resolved by adding an extra (differential) operator depending on $n$, denoted by $L_0$, which is defined as

$$L_0 := -t\frac{\mathrm{d}}{\mathrm{d}t}. \tag{3.1.12}$$

Its adjoint action on the generators $J_n^a = J^a \otimes t^n$ is then given by

$$\mathrm{ad}(L_0)(J^a \otimes t^n) = [L_0, J^a \otimes t^n] = -t\frac{\mathrm{d}}{\mathrm{d}t}J^a \otimes t^n = -nJ^a \otimes t^n \tag{3.1.13}$$

or, in short notation

$$[L_0, J_n^a] = -nJ_n^a. \tag{3.1.14}$$

This expression also shows that $L_0$ is a derivation[2] of the Lie algebra $\hat{\mathfrak{g}}$. The Cartan subalgebra is then generated by the set $\{H_0^1, ..., H_0^r, k, L_0\}$. Eventually the affine Lie algebra has the form

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}k \oplus \mathbb{C}L_0. \tag{3.1.15}$$

Naturally we wish to extend the concepts of the finite dimensional case to affine Lie algebras. The question is how to describe root systems and weights in the affine case since we have seen in the first section that roots contain much knowledge about the Lie algebra. First of all we have to introduce a scalar product for the affine case. The most important feature we use in order to equip $\hat{\mathfrak{g}}$ with such a scalar product is the extension of the Killing form $\kappa$ of $\mathfrak{g}$. We have seen that a characteristic property of the Killing form is its invariance. This means that for all $X, Y, Z \in \mathfrak{g}$, $\kappa([Z, X], Y) + \kappa(X, [Z, Y]) = 0$. Let us apply the invariance of $\kappa$ to the generators $J_n^a$ and $L_0$:

$$\kappa([L_0, J_n^a], J_m^b) + \kappa(J_n^a, [L_0, J_m^b]) = -(n+m)\kappa(J_n^a, J_m^b) = 0 \tag{3.1.16}$$

so we conclude that unless $n + m = 0$ the Killing form of $J_n^a$ and $J_m^b$ is the same as of $J^a$ and $J^b$:

$$\kappa(J_n^a, J_m^b) = \delta^{ab}\delta_{n+m,0}. \tag{3.1.17}$$

Similarly one finds the other Killing forms:

$$\kappa(J_n^a, k) = 0 \qquad \kappa(k, k) = 0 \tag{3.1.18}$$

$$\kappa(J_n^a, L_0) = 0 \qquad \kappa(L_0, k) = -1. \tag{3.1.19}$$

[2]A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ such that $D([X, Y]) = [D(X), Y] + [X, D(Y)]$. The space $\mathfrak{der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

One chooses the unspecified Killing form of $L_0$ with itself to be zero, i.e.

$$\kappa(L_0, L_0) = 0. \tag{3.1.20}$$

In order to make the scalar product based on the Killing form precise we have to generalize weights to the affine case which is done in the following definiton.

**Definition 3.1.4.** *Let $\lambda = (\lambda(H_0^1), ..., \lambda(H_0^r))$ be the vector of the eigenvalues of a simultaneous eigenvector of the generating elements of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which extends to a simultaneous eigenvector of the $H_0^i$, $k$ and $L_0$ with eigenvalues $k_\lambda$ and $n_\lambda$ respectively. Then*

$$\hat{\lambda} := (\lambda; k_\lambda; n_\lambda) \tag{3.1.21}$$

*is called an* **affine weight**.

Now we are ready to define the scalar product $\langle \hat{\lambda}, \hat{\mu} \rangle$ between to affine weights $\hat{\lambda}$ and $\hat{\mu}$. The scalar product of the finite part of the weights, namely $\langle \lambda, \mu \rangle$ will show up as an individual summand whereas the eigenvalues for the additional generators will play a different role.

**Definition 3.1.5.** *Let $\hat{\lambda}$, $\hat{\mu}$ be affine weights. The scalar product induced by the affine Killing form is defined by*

$$\langle \hat{\lambda}, \hat{\mu} \rangle := \langle \lambda, \mu \rangle + k_\lambda n_\mu + k_\mu n_\lambda. \tag{3.1.22}$$

Now ***affine roots*** are just the affine weights of the adjoint representation so this definition works for them as well. However it is heavily simplified by the fact that $k$ commutes with all generators and thus has zero eigenvalues in the adjoint representation. Therefore their affine scalar product is equal to the finite case. We can thus always write an affine root $\hat{\beta}$ in the form

$$\hat{\beta} = (\beta; 0; n) \tag{3.1.23}$$

and so the scalar product induced by the affine Killing form is given by

$$\langle \hat{\beta}, \hat{\alpha} \rangle = \langle \beta, \alpha \rangle. \tag{3.1.24}$$

The roots of the generators $E_n^\alpha$ are easily determined: The first roots are of course the same as in the finite case, i.e. $\alpha$. The adjoint action on $k$ is, as said, zero, whereas the adjoint action on $L_0$ gives the eigenvalues $n$ as we have found out in the foregoing section. All in all we have

$$\hat{\alpha} = (\alpha; 0; n). \tag{3.1.25}$$

The roots of the generators $H_n^i$ are found by a similar reasoning. Since they are commuting elements their adjoint action on each other is zero. On $L_0$, they act again by multiplication by $n$. If we define $\delta := (0; 0; 1)$ we may denote the root associated with $H^i_n$ by $n\delta$. If we use the notation $\hat{\alpha} = \alpha + n\delta$ we may write the ***full set of roots*** $\hat{\Delta}$ of the affine Lie algebra $\hat{\mathfrak{g}}$ in the following way:

$$\hat{\Delta} = \{\alpha + n\delta | n \in \mathbb{Z}, \alpha \in \Delta\} \cup \{n\delta | n \in \mathbb{Z}^*\} = \{\alpha + n\delta | n \in \mathbb{Z}, \alpha \neq 0\} \cup \mathbb{Z}\delta. \quad (3.1.26)$$

One often calls the first part the ***real roots*** and the second part the ***imaginary roots***. We may consequently define *positive* and *simple roots* within the setting of affine Lie algebras.

**Definition/Lemma 3.1.6.** *Let $\mathfrak{g}$ be a Lie algebra with highest root $\theta$. Define the root $\alpha_0$ to be $\alpha_0 := (-\theta; 0; 1) = -\theta + \delta$. Then the set $\{\alpha_i\}, i = 0, ..., r$ is a basis of* **simple roots** *(denoted by $\hat{\Delta}^0$) for the affine Lie algebra $\hat{\mathfrak{g}}$. The corresponding set of* **positive roots** *is given by*

$$\hat{\Delta}^+ = \{\alpha + n\delta | n \in \mathbb{Z}_{>0}, \alpha \in \Delta\} \cup \{\alpha | \alpha \in \Delta^+\}. \quad (3.1.27)$$

*Proof.* The basis of simple roots must contain $r + 1$ elements, $r$ of them must necessarily be the finite simple roots $\alpha_i, i = 1, ..., r$. The additional simple root is then given by the one defined above. To check that the set of positive roots has the desired property of nonnegative coefficients in a basis expansion, consider for $n > 0$ and $\alpha \in \Delta$:

$$\alpha + n\delta = \alpha + n\alpha_0 + n\theta = n\alpha_0 + (n-1)\theta + (\theta + \alpha) \quad (3.1.28)$$

where the coefficients are nonnegative. □

**Corollary 3.1.7.** *The adjoint representation of any affine Lie algebra is never a highest weight representation.*

*Proof.* Consider equation (3.1.28) which assures that there cannot be a highest root. □

**Definition 3.1.8.** *Let $\hat{\mathfrak{g}}$ be an affine Lie algebra, $\hat{\Delta}^0$ a basis of affine simple roots and $\langle\ ,\ \rangle$ a scalar product on $\hat{\mathfrak{h}}^*$. The* **extended Cartan matrix** *is defined as:*

$$\hat{A}_{ij} := \langle \alpha_i, \alpha_j^\vee \rangle \quad (3.1.29)$$

*where $\hat{\alpha}^\vee = \frac{2}{\langle \hat{\alpha}, \hat{\alpha} \rangle}(\alpha; 0; n)$ is the* **affine coroot** *corresponding to $\hat{\alpha}$. The* **extended Dynkin diagram** *is then obtained by adding an extra node representing $\alpha_0$ which is connected to the other $\alpha_i$-nodes by $\hat{A}_{0i}\hat{A}_{0i}$ lines.*

The following figure gives some of the affine Dynkin diagrams for the simple Lie algebras. It is taken from [DFMS97]:

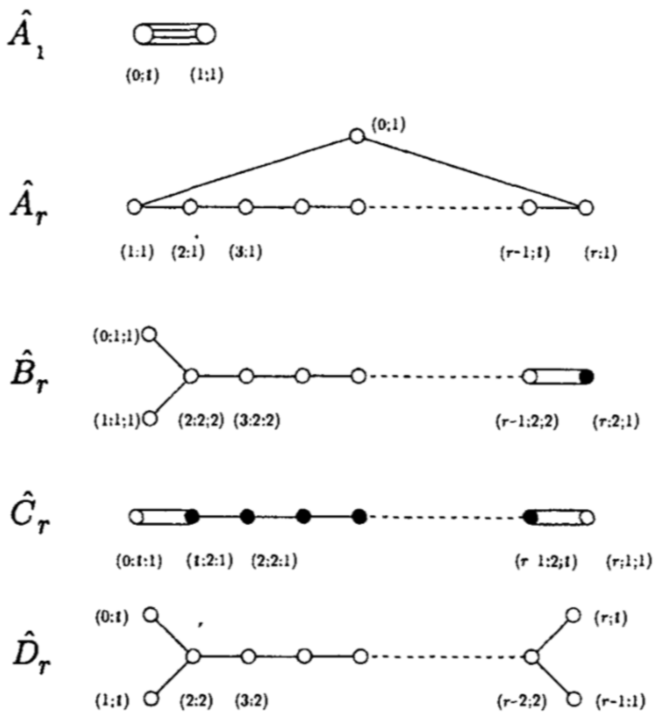


Figure 3.1: The numbers give the ordering of the roots followed by their mark and comark (if they don't coincide).

We proceed by introducing *fundamental weights* and *levels.*

**Definition 3.1.9.** *Let $\hat{\mathfrak{g}}$ be an affine Lie algebra and $\hat{\Delta}^0$ a basis of affine simple roots. The* **affine fundamental weights** *$\{\hat{\omega}_i\}, i = 0, ..., r$ are defined to be the basis dual to the simple coroots with eigenstates of $L_0$ with zero eigenvalue. That is:*

$$\hat{\omega}_i = (\omega_i; a_i^\vee; 0), \quad i \neq 0 \qquad\qquad \hat{\omega}_0 = (0; 1; 0), \quad i = 0. \tag{3.1.30}$$

This needs some explanation. For the finite part the choice is clear (meaning the $\omega_i$ for $i = 1, ..., r$). The eigenvalue corresponding to $k$ is determined by the condition $\langle \hat{\omega}_i, \alpha_0^\vee \rangle = 0$ for $i \neq 0$ whereas the zeroth affine fundamental weight $\hat{\omega}_0$ satisfies the condition $\langle \hat{\omega}_0, \alpha_0^\vee \rangle = 1$ and should have zero scalar product with all finite $\alpha_i$ which only leaves the choice $\hat{\omega}_0 = (0; 1; 0)$.[3]. Regarding notation, we may write $\omega_i \equiv (\omega_i; 0; 0)$, such that a general affine fundamental weight $\hat{\omega}_i$ is given as

$$\hat{\omega}_i = a_i^\vee \hat{\omega}_0 + \omega_i. \tag{3.1.31}$$

[3]Since $\alpha_0^\vee = (-\theta; 0; 1)$, we have $\langle \hat{\omega}_0, \alpha_0^\vee \rangle = -\langle \omega_0, \theta \rangle + k_{\omega_0} \cdot 1$ which should equal one. Together with the other constraint this yields $\omega_0 = 0$ and $k_{\omega_0} = 1$

Subsequently we may expand any affine weight $\hat{\lambda}$ as a linear combination of affine fundamental weights $\hat{\omega}_i$ and $\delta$ as

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + m\delta, \quad m \in \mathbb{R}. \tag{3.1.32}$$

By the definition of fundamental weights each $\hat{\omega}_i$ contributes to the $k$ eigenvalue a factor of $a_i^\vee$. This allows us to define *levels*:

**Definition/Lemma 3.1.10.** *Let $\hat{\lambda}$ be an affine weight and $\{\hat{\omega}_i\}, i = 0, ..., r$ a complete set of affine fundamental weights. The eigenvalue corresponding to $k$, i.e. $\hat{\lambda}(k)$, is then given by*

$$\tilde{k} \equiv k := \hat{\lambda}(k) = \sum_{i=0}^{r} a_i^\vee \lambda_i \tag{3.1.33}$$

*and will subsequently be called* **level** *of the weight $\hat{\lambda}$. Note that we will use the same letter for the central element $k$ and for its eigenvalue, the level $k$.*

*Proof.* Let $\hat{\lambda}$ be as above. Then $\langle \hat{\lambda}, \delta \rangle = \langle \lambda, 0 \rangle + \hat{\lambda}(k) \cdot 1 = \hat{\lambda}(k)$. We can expand $\delta$ as $\delta = \sum_{i=0}^{r} a_i^\vee \alpha_i^\vee$ from which we conclude $\hat{\lambda}(k) = \sum_{i=0}^{r} a_i^\vee \langle \hat{\lambda}, \alpha_i^\vee \rangle$. But since $\lambda_i = \langle \hat{\lambda}, \alpha_i^\vee \rangle$ we obtain formula (3.1.33). $\square$

We will say that an affine Lie algebra is *at level $k$* and denote it by $\hat{\mathfrak{g}}_k$ depending on the choice of the central element $k$. In analogy to the finite counterpart we will now introduce the *affine Weyl group*:

**Definition 3.1.11.** *Let $\hat{\alpha}$ be a real affine root. Its corresponding* **Weyl reflection** *is defined as $s_{\hat{\alpha}}(\hat{\lambda}) := \hat{\lambda} - \langle \hat{\lambda}, \hat{\alpha}^\vee \rangle \hat{\alpha}$. Then the* **affine Weyl group** *$\widehat{W}$ is defined as the group generated by all reflections together with the composition of mappings as the group multiplication.*

Note that the definition of the root reflection only makes sense for real roots since $\hat{\alpha}^\vee$ does not exist for imaginary roots. Similarly we define *affine Weyl chambers*:

**Definition 3.1.12.** *Let $\widehat{W}$ be the affine Weyl group of an affine Lie algebra $\hat{g}$ and $\{\alpha_i\}$ be a basis of simple roots. Then the* **affine Weyl chambers** *$\widehat{C}_w$ for each $w \in \widehat{W}$ are the subsets in the vector space of affine weights*

$$\widehat{C}_w := \{\lambda | \langle w(\hat{\lambda}), \alpha_i \rangle \geq 0, i = 0, 1, ..., r\}. \tag{3.1.34}$$

*The* **fundamental chamber** *is the chamber $\widehat{C}_1$ corresponding to the identity element $w = 1$.*

Let us illustrate the abstract concepts introduced in this section by some examples:

**Examples 3.1.13.** [(i)]

1. $\widehat{\mathfrak{su}}(2)$. We wish to find the root system of the affine Lie algebra of $\mathfrak{su}(2)$. We have found out above that it has only one simple root $\alpha_1$ which is then also the highest root $\theta$. Now $\alpha_0$ was defined as $\alpha_0 = -\theta + \delta$ so $\alpha_0 = -\alpha_1 + \delta$. The extended Cartan matrix is then calculated by the scalar products $\langle \alpha_i, \alpha_j^\vee \rangle$ for $i, j = 0, 1$. This gives four scalar products: $\langle \alpha_0, \alpha_0^\vee \rangle = 2$, $\langle \alpha_0, \alpha_1^\vee \rangle = -2$, $\langle \alpha_1, \alpha_0^\vee \rangle = 2$ and $\langle \alpha_1, \alpha_1^\vee \rangle = -2$ since $\langle \alpha_0, \alpha_1^\vee \rangle = \langle \alpha_1, \alpha_0^\vee \rangle = \langle \alpha_1, \alpha_0 \rangle = -\alpha_1^2 = -2$. The Cartan matrix is then given by

$$\widehat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{3.1.35}$$

We have found out above that the Dynkin lables of the simple roots are simply the entries of the extended Cartan matrix, thus yielding

$$\alpha_0 = [2, -2] \qquad\qquad \alpha_1 = [-2, 2]. \tag{3.1.36}$$

Since the highest root is at the same time the only simple root, the marks and comarks (the expansion coefficients of $\theta$ in terms of the simple roots) are one. The level was defined by $\tilde{k} = \sum_{i=0}^{r} a_i^\vee \lambda_i$ where $\lambda_i$ denote the Dynkin labels. They add up to zero for the simple roots of $\widehat{\mathfrak{su}}(2)$. The gathered information allows us to write down the complete set of roots of $\widehat{\mathfrak{su}}(2)$:

$$\hat{\Delta} = \{-\alpha_1, \alpha_1\} \cup \{-\alpha_1 + n\delta, \alpha_1 + n\delta | n \in \mathbb{Z}^*\}. \tag{3.1.37}$$

From $\alpha_0 = -\alpha_1 + \delta$ we know that $\delta = \alpha_0 + \alpha_1$ so we can rewrite the set above as

$$\hat{\Delta} = \{n\alpha_0 + m\alpha_1 | \; |n - m| \leq 1, n, m \in \mathbb{Z}\}. \tag{3.1.38}$$

2. $\widehat{\mathfrak{su}}(3)$. A more interesting example is the affine algebra of the three-dimensional special unitary Lie algebra. The highest root of $\mathfrak{su}(3)$ is the sum of the two simple roots $\alpha_1$ and $\alpha_2$ (the rank of $\mathfrak{su}(3)$ is equal to two). The highest weight of the adjoint representation, i.e. the highest root is given by

$$\theta = \alpha_1 + \alpha_2. \tag{3.1.39}$$

Then simple calculation gives the Cartan matrix

$$\hat{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{3.1.40}$$

As above we can read off the Dynkin labels of the simple roots of $\widehat{\mathfrak{su}}(3)$:

$$\begin{aligned} \alpha_0 &= [2, -1 - 1] \\ \alpha_1 &= [-1, 2, -1]. \\ \alpha_2 &= [-1, -1, 2] \end{aligned} \tag{3.1.41}$$

One can imagine this set of roots as an infinite pile of hexagons, each hexagon being the hexagon from the finite algebra $\mathfrak{su}(3)$ and $\delta$ counting the layers. Furthermore we can describe the root reflections generated by the simple roots by using the formula $s_{\alpha_i}(\hat{\lambda}) = \hat{\lambda} - \langle \hat{\lambda}, \hat{\alpha}_i \rangle \alpha_i$.

Another interesting application of affine Lie algebras is that they can be used to find branching rules. In [Que02] a closed formula for the branching coefficients is given which makes use of the affine extension of $\mathfrak{g}$.

## 3.2 Fusion rings

In this section we want to introduce *fusion rings*. Having discussed representation rings earlier it is now natural to ask about a ring structure for the representations of affine Lie algebras. Here the fusion ring is the preferred choice. On the one hand there exists a completely abstract algebraic definition of these rings, on the other hand we want to apply the concept to affine Lie algebras. In the latter case the fusion ring is a ring based on highest weight representations of affine Lie algebras where the multiplication is given by the *fusion rules*. In the former case we define the fusion as a commutative, associative, positive ring with a finite basis with some additional technical properties. However we will not go too deep into the connection between the two approaches. The following discussion is mainly inspired by [Fuc94] and by [BDR02].

**Definition 3.2.1.** *Let $\mathcal{F}$ be a set. Denote by $\star$ a well-defined multiplication on $\mathcal{F}$. $\mathcal{F}$ is called a* **fusion ring** *if it is a ring over the integers $\mathbb{Z}$ together with the following properties:*

*(i)* **Commutativity:** *For all $\phi_i, \phi_j \in \mathcal{F}$ it holds that*

$$\phi_i \star \phi_j = \phi_j \star \phi_i \tag{3.2.1}$$

*(ii)* **Associativity:** *For all $\phi_i, \phi_j, \phi_k \in \mathcal{F}$ it holds that*

$$(\phi_i \star \phi_j) \star \phi_k = \phi_i \star (\phi_j \star \phi_k) \tag{3.2.2}$$

*(iii)* **Positivity:** *There is a finite basis $\{\phi_i\}_{i\in I}$ with nonnegative structure constants.*

*(iv)* **Conjugation:** *There is an element of the basis $\{\phi_i\}_{i\in I}$ from (iii) such that the evaluation of the product with respect to this basis element furnishes an involutive automorphism.*

Since especially the last two axioms need explanation we will go a little into detail. If we write the product $\star$ between two elements of $\mathcal{F}$ in the following way:

$$\phi_i \star \phi_j = \sum_{k\in I} N_{ij}{}^{k} \phi_k \tag{3.2.3}$$

where the numbers ${N_{ij}}^k$ are called ***structure constants*** or ***fusion coefficients***. We may reformulate the axioms from 3.2.1. Commutativity would simply read

$$ {N_{ij}}^k = {N_{ji}}^k \tag{3.2.4} $$

whereas associativity would translate into

$$ \sum_{k\in I} {N_{ij}}^k {N_{kl}}^m = \sum_{k\in I} {N_{jl}}^k {N_{ik}}^m. \tag{3.2.5} $$

Also, nonnegativity can be notated in a concise way: There is a basis $\{\phi_i\}$ such that

$$ {N_{ij}}^k \in \mathbb{Z}_{\geq 0} \quad \forall i,j,k \in I. \tag{3.2.6} $$

Finally the fourth axiom can be translated in the following way: For the basis for which the structure constants are nonnegative there is an index $i_0$ such that the ***conjugation map*** $i_C\colon \phi_i \mapsto \sum_{j\in I} C_{ij}\phi_j$ where $C_{jk} = N_{jk}^{i_0}$ is an involution[4] which implies that the conjugation map $i_C$ is a permutation of order two and such that $i_C$ is an automorphism, i.e. $i_C \in \mathrm{Aut}(\mathcal{F})$ or in other words

$$ i_C(\phi_i) \star i_C(\phi_j) = \sum {N_{ij}}^k i_C(\phi_k). \tag{3.2.7} $$

Furthermore it can be proven that $\mathcal{F}$ is a unital ring, i.e. there is an element $\mathbf{1} \in \mathcal{F}$ such that $\mathbf{1} \star \phi = \phi \star \mathbf{1} = \phi$ for all $\phi \in \mathcal{F}$. A proof of this statement can be found in the appendix. Let us now look at the fusion coefficients ${N_{ij}}^k$ and their properties. They naturally give rise to a set of matrices $N_i$ defined by:

$$ {(N_i)_j}^k := {N_{ij}}^k. \tag{3.2.8} $$

By commutativity of $\mathcal{F}$ those matrices mutually commute and are thus simultaneously diagonalizable. This is done by some matrix $S$. The fusion coefficients can be expressed by the entries of $S$ by the famous ***Verlinde formula***:

$$ {N_{ij}}^k = \sum_{m\in I} \frac{S_{im} S_{jm} S^*_{km}}{S_{1m}}. \tag{3.2.9} $$

It is further known (see [Gep91]) that a fusion ring $\mathcal{F}$ is isomorphic to the quotient of a free polynomial ring of $N$ variables over the complex numbers with respect to a certain ideal $I$, i.e.

$$ \mathcal{F} \cong \mathbb{C}[x_1, ..., x_N]/I. \tag{3.2.10} $$

The ideal $I$ is called ***fusion ideal***. So far this has been an abstract discussion. Now we will give the definition of the fusion ring for affine Lie algebras and then explain how to construct the fusion ideal in this case. The notion of the fusion ring is similar to the representation ring. The question arises whether for a given Lie algebra $\mathfrak{g}$ and its representation ring we may define a similar structure for the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level $k$ of $\mathfrak{g}$. The addition is given by the fusion coefficients.

[4] An *involution* is a matrix $C$ that fulfills $C^2 = \mathbf{1}$

**Definition 3.2.2.** *Let $\hat{\mathfrak{g}}_k$ be an affine Lie algebra at level $k \in \mathbb{N}$. Let $P_+^{\mathfrak{g},k}$ be the set of dominant highest weight representations of $\hat{\mathfrak{g}}_k$. The* **fusion ring** *of $\hat{\mathfrak{g}}_k$ is defined as the set of formal sums of representations $\rho \in P_+^{\mathfrak{g},k}$ with integer coefficients.*

$$\mathrm{Fus}(\hat{\mathfrak{g}}_k) \equiv \mathcal{F}_{\mathfrak{g},k} := \left\{ \sum_{\rho \in P_+^{\hat{\mathfrak{g}},k}} n_\rho(\rho) | \, n_\rho \in \mathbb{Z}, n_\rho = 0 \, \text{for all but finitely many } \rho \right\}. \tag{3.2.11}$$

*The multiplication on this ring is defined by the* **fusion rules**

$$(\rho) \times (\sigma) := \sum_{\tau \in P_+^{\hat{\mathfrak{g}},k}} N_{\rho\sigma}^{\tau}(\tau) \tag{3.2.12}$$

*where the $N_{\rho\sigma}^{\tau} \in \mathbb{N}_0$ are called* **fusion coefficients***.*

Now we can identify the representation ring with $\mathbb{C}[x_1, ..., x_N]$. This is because we can identify the $x_i$ with the fundamental weights $\omega_i$, at least in the simple case. This means we have the following statement:

**Corollary 3.2.3.** *Let $\mathfrak{g}$ be a Lie algebra and $\hat{\mathfrak{g}}_k$ be the associated level $k$ affine algebra. Let $\mathrm{Rep}(\mathfrak{g})$ be the representation ring of $\mathfrak{g}$ and $\mathrm{Fus}(\hat{\mathfrak{g}}_k)$ be the fusion ring of $\hat{\mathfrak{g}}_k$. Then there exists a ideal $I_{\mathfrak{g},k}$, called the* **fusion ideal** *or* **Verlinde ideal** *such that:*

$$\mathrm{Fus}(\hat{\mathfrak{g}}_k) = \mathrm{Rep}(\mathfrak{g})/I_{\mathfrak{g},k}. \tag{3.2.13}$$

Let us now try to characterize the fusion ideal in this case. To this end recall the definition of the ***character*** $\chi_\lambda$ of the finite dimensional irreducible representation $\mathcal{L}_\lambda$ with highest weight $\lambda$:

$$\chi_\lambda(\sigma) = \frac{\sum_{w \in W} \epsilon(w) \exp\{\langle w(\lambda + \rho), \sigma \rangle\}}{\sum_{w \in W} \epsilon(w) \exp\{\langle w(\rho), \sigma \rangle\}}, \qquad \sigma \in \mathfrak{h}^*. \tag{3.2.14}$$

If we set

$$\zeta_\mu := 2\pi i \frac{\mu + \rho}{k + h_{\mathfrak{g}}^{\vee}} \tag{3.2.15}$$

and use the Verlinde formula (3.2.9) together with the fact that

$$\frac{S_{\mu\lambda}^*}{S_{\mu 1}} = \chi_\lambda(\zeta_\mu) \tag{3.2.16}$$

we obtain the connection between characters $\chi$ and the fusion coefficients ${N_{\lambda\mu}}^{\nu}$:

$$\chi_\lambda(\zeta_\alpha)\chi_\mu(\zeta_\alpha) = \sum_{\nu \in P_+^{\mathfrak{g},k}} {N_{\lambda\mu}}^{\nu} \chi_\nu(\zeta_\alpha) \tag{3.2.17}$$

where $\alpha$ is taken in $P_+^{\mathfrak{g},k}$. From this we conclude that we can characterize the fusion ring $\mathcal{F}_k$ of an affine Lie algebra $\hat{\mathfrak{g}}$ at level $k$ as follows:

$$\mathcal{F}_k = \mathbb{Z}[\chi_1, ..., \chi_n]/I_k. \tag{3.2.18}$$

Here, $\chi_i$ denotes the character of the fundamental representation of highest weight $\omega_i$. The fusion ideal is then the set of elements in the Grothendieck ring of characters vanishing at the points $\zeta_\mu$ for $\mu \in P_+^{\mathfrak{g},k}$ (see [BDR02] for more details). The following lemma gives a useful identity for the computation of the fusion ideal.

**Lemma 3.2.4.** *Let $w \in \widehat{W}$ be some element of the affine Weyl group of $\hat{\mathfrak{g}}$. Let $\zeta_\mu$ for some weight $\mu$ be defined as in equation (3.2.15). Then the following identity holds:*

$$\chi_{w\cdot\lambda}(\zeta_\mu) = \epsilon(w)\chi_\lambda(\zeta_\mu) \tag{3.2.19}$$

*where $\cdot$ denotes the shifted Weyl reflection $w \cdot \lambda := w(\lambda + \rho) - \rho$.*

*Proof.* See [DFMS97], Chapter 14.4.1. □

By the use of this identity we see that $\chi_{w\cdot\lambda} - \epsilon(w)\chi_\lambda \in I_k$ for all $P_+$ and $w \in \widehat{W}$ (since the fusion ideal is characterized by those elements vanishing on $\zeta_\mu$). On the other hand $\chi_\lambda \in I_k$ if $\chi(\zeta_\mu)$ for all $\mu \in P_+^{\mathfrak{g},k}$. This happens exactly when $s_\alpha \cdot \lambda = \lambda$ which in turn is implied if there is a positive root $\alpha \in \Delta^+$ such that

$$\langle \lambda + \rho, \alpha \rangle \equiv 0 \pmod{k + h^\vee}. \tag{3.2.20}$$

**Definition 3.2.5.** *Let $\lambda$ be a weight and $\theta$ be the highest root of $\mathfrak{g}$ and $k \in \mathbb{Z}_{\geq 0}$ be the level of $\hat{\mathfrak{g}}$. Then $\lambda$ is called a* **boundary weight** *if*

$$\langle \lambda, \theta \rangle = k + 1. \tag{3.2.21}$$

By the observations above all characters which correspond to a boundary weight $\lambda$ at level $k$ are contained in $I_k$. It is believed (see [BDR02]) that those characters actually generate the fusion ideal $I_k$ for all simple Lie algebras. This statement has been proven for $A_n$ and $C_n$, see [BDR02] for further reference.

# Chapter 4

# The conjecture and its verification

## 4.1 Introductory remarks

In this chapter we want to state the conjecture and verify it for some examples of Lie algebras. In the previous chapter we developed the necessary theory of representations and fusion rings and the central object in the conjecture, the fusion ideal. The conjecture can be described as follows: In the last part of the foregoing chapter we observed that the fusion ring is obtained by the quotient $\mathrm{Rep}(\mathfrak{g})/I_{\mathfrak{g},k}$. Passing to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ the corresponding fusion and representation ring can be defined. That means that one obtains another fusion ideal $I_{\mathfrak{h},k'}$ with some other level $k'$ that has yet to be specified. In Section 2.3 we found out how to decompose representations with respect to a given subalgebra (i.e. branching rules). Applying these branching rules to the representations in the fusion ideal $I_{\mathfrak{g},k}$ with respect to the subalgebra $\mathfrak{h}$ we obtain a set in $\mathrm{Rep}(\mathfrak{h})$ which we denote by $I_{\mathfrak{g},k}|_{\mathfrak{h}}$. Here $|_{\mathfrak{h}}$ denotes the decomposition map which decomposes $\mathfrak{g}$-representations into $\mathfrak{h}$-representations. Note that this set is however not necessarily an ideal in $\mathrm{Rep}(\mathfrak{h})$. The question that arises now is how the fusion ideal of $\mathfrak{h}$, i.e. $I_{\mathfrak{h},k'}$ is related to $I_{\mathfrak{g},k}|_{\mathfrak{h}}$. The conjecture now is that the fusion ideal $I_{\mathfrak{h},k'}$ is contained in the ideal that $I_{\mathfrak{g},k}|_{\mathfrak{h}}$ *generates.* This ideal will be denoted by $\langle I_{\mathfrak{g},k}|_{\mathfrak{h}}\rangle$ in the following. On abstract grounds which we will not describe here the right choice for the level $k'$ of the subalgebra $\mathfrak{h}$ for a given level $k$ of the ambient algebra $\mathfrak{g}$ is such that it fulfils the equation

$$k' + h^\vee_{\mathfrak{h}} = x_{\mathfrak{g},\mathfrak{h}}(k + h^\vee_{\mathfrak{g}}) \tag{4.1.1}$$

where $x_{\mathfrak{g},\mathfrak{h}}$ is the embedding index as defined in (2.3.4) and $h^\vee_{\mathfrak{g}}$, $h^\vee_{\mathfrak{h}}$ are the dual Coxeter numbers as defined in (2.2.15) respectively.

**Conjecture 4.1.1.** *Let $\mathfrak{g}$ be a Lie algebra corresponding to a compact, simply-connected Lie group $G$ and $\mathfrak{h} \leq \mathfrak{g}$ be a simple subalgebra. Let*

$$|_{\mathfrak{h}} \colon \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{h}) \tag{4.1.2}$$

*be the map which decomposes* $\mathfrak{g}$*-representations into* $\mathfrak{h}$*-representations. Let* $I_{\mathfrak{g},k}$*,* $I_{\mathfrak{h},k'}$ *be the fusion ideals of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *respectively where* $k'+h^\vee_{\mathfrak{h}} = x_{\mathfrak{g},\mathfrak{h}}(k+h^\vee_{\mathfrak{g}})$ *and let* $\langle I_{\mathfrak{g},k}|_{\mathfrak{h}}\rangle$ *be the ideal generated by* $I_{\mathfrak{g},k}|_{\mathfrak{h}}$*. Then it is conjectured that*

$$I_{\mathfrak{h},k'} \subset \langle I_{\mathfrak{g},k}|_{\mathfrak{h}}\rangle. \tag{4.1.3}$$

Let us give some physical motivation for the conjecture. In string theory, ***D*-branes** arise which can be seen as $(p+1)$-dimensional objects to which open strings can couple. In **coset models** one can label branes by a tuple $[L,\sigma,l]$ corresponding to the representation $\mathcal{H}^{(L,\sigma,l)}$ of the **chiral algebra** $\mathcal{A}$ defined by the quotient

$$\mathcal{A} = (\hat{\mathfrak{g}}_k \otimes \widehat{\mathfrak{so}}(d)_1)/\hat{\mathfrak{h}}_{k'} \tag{4.1.4}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras corresponding to the Lie groups $H \subset G$ where $G$ is semisimple and compact and $H$ is reductive. The level $k'$ is given by the relation defined in the conjecture. The spectrum of strings between the two branes is given by the fusion rules:

$$\mathcal{H}^{[L_2,\sigma_2,l_2]}_{[L_1,\sigma_1,l_1]} = \bigoplus_{[L',\sigma',l']} N^{[L_2,\sigma_2,l_2]}_{[L',\sigma',l'],[L_1,\sigma_1,l_1]} \mathcal{H}^{[L',\sigma',l']}. \tag{4.1.5}$$

Now we want to consider dynamics between two brane configurations $\alpha$ and $\beta$ which are formal sums of elementary branes (labelled by $[L,\sigma,l]$) with integer coefficients $\alpha_i$ and $\beta_i$:

$$\alpha = \sum_i \alpha_i [L_i,\sigma,l_i] \qquad\qquad \beta = \sum_i \beta[L_i,\sigma,l_i]. \tag{4.1.6}$$

The label $\sigma$ of $\widehat{\mathfrak{so}}(d)_1$ should be the same for all branes. Now we say that $\alpha$ and $\beta$ are related by a **dynamical process** if the corresponding representations $V^{\alpha}$[1] and $V^{\beta}$ of $\mathfrak{g} \oplus \mathfrak{h}$ are isomorphic as representations of the diagonal subalgebra $\mathfrak{h}_{\text{diag}} \subset \mathfrak{g} \oplus \mathfrak{h}$. Let us now look at **conserved charges**. To this end define the abelian group $\mathcal{B}$ of formal $D$-brane configurations: Let $I$ be a finite index set and let $B_i$ be the elementary branes. Then $\mathcal{B}$ consists of combinations of the $B_i$ with integer coefficients $\alpha_i$:

$$\alpha = \sum_i \alpha_i B_i. \tag{4.1.7}$$

If there are no dynamical processes, then $\mathcal{B}$ would be the maximal group of conserved charges such that the **charge** $\alpha_i$ would simply count the number of branes of type $B_i$. However when there are transitions $\alpha \to \beta$ of two branes corresponding to a dynamical process we need to identify the charge of the configurations $\alpha$ and $\beta$. In the case that all dynamics are known (i.e. all transitions) the maximal charge group $\mathcal{C}$ is given as the qoutient

$$\mathcal{C} = \mathcal{B}/\mathcal{I} \tag{4.1.8}$$

[1] $V^{\alpha}$ is given as $\bigoplus_i \alpha_i V^{L_i} \otimes V^{l_i}$ for the representations $V^{L_i}$ and $V^{l_i}$ of the finite dimensional Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

where $\mathcal{I}$ is the subgroup generated by configurations of zero charge (which are differences $\alpha-\beta$ for some transition $\alpha \to \beta$). Let us look more into the charge group $\mathcal{C}$. For some particular coset models (called $N=1$ *supersymmetric, rational coset models with charge conjugated modular invariant*) the group of maximally symmetric $D$-brane configurations $\mathcal{B}$ is given by the fusion ring $\mathcal{F}_{\mathfrak{g}/\mathfrak{h}}$. In this case the subgroup $\mathcal{I}$ is actually an ideal. In some cases (where the *identification group* $G_{\mathrm{id}}$ is trivial) $\mathcal{F}_{\mathfrak{g}/\mathfrak{h}}$ can be written as a tensor product:

$$\mathcal{F}_{\mathfrak{g}/\mathfrak{h}} = \mathcal{F}_{\mathfrak{g}} \otimes \mathcal{F}_{\mathfrak{so}} \otimes \mathcal{F}_{\mathfrak{h}}. \tag{4.1.9}$$

Hence $\mathcal{C}$ can be written as

$$\mathcal{C} = \left(\mathcal{F}_{\mathfrak{g}} \otimes \mathcal{F}_{\mathfrak{so}} \otimes \mathcal{F}_{\mathfrak{h}}\right)/\mathcal{I}. \tag{4.1.10}$$

Under some further assumptions and some technical manipulation we can rewrite this as

$$\mathcal{C} \cong \mathcal{F}_{\mathfrak{so}} \otimes \left(\mathrm{Rep}(\mathfrak{h})/\langle I_g|_{\mathfrak{h}} \oplus I_{\mathfrak{h}}\rangle\right) \tag{4.1.11}$$

where $I_{\mathfrak{g}}$ and $I_{\mathfrak{h}}$ are the fusion ideals of $\mathfrak{g}$ and $\mathfrak{h}$. Now if the conjecture holds true, we would have

$$\mathcal{C} \cong \mathcal{F}_{\mathfrak{so}} \otimes \left(\mathrm{Rep}(\mathfrak{h})/\langle I_g|_{\mathfrak{h}}\rangle\right). \tag{4.1.12}$$

An application of this formula would be WZW models based on the Lie group $G$. Say that $\mathfrak{h}$ is trivial such that $\mathrm{Rep}(\mathfrak{h}) \cong \mathbb{Z}$. The restriction of $\mathrm{Rep}(\mathfrak{g})$ to $\mathbb{Z}$ maps a given representation $\rho$ to its dimension $\dim \rho$. Hence

$$\mathrm{Rep}(\mathfrak{h})/\langle I_g|_{\mathfrak{h}}\rangle \cong \mathbb{Z}/\langle \dim I_{\mathfrak{g}}\rangle \tag{4.1.13}$$

and so

$$\mathcal{C} \cong \mathcal{F}_{\mathfrak{so}} \otimes \left(\mathbb{Z}/\langle \dim I_{\mathfrak{g}}\rangle\right). \tag{4.1.14}$$

The strategy in the following case studies is to test whether the conjecture holds true for certain classes of Lie algebras. A general treatment of the conjecture, i.e. an attempt to prove it in its totality (assuming it is true) would go beyond the scope of this thesis. Therefore we limit ourselves to testing the conjecture for some classes (e.g. $\mathfrak{su}(n)$) by decomposing the generators of the fusion ideal (as far as they are known) into representations of certain subalgebras. Then we check if the generators of the fusion ideal of the subalgebra lie in the set generated by the decomposition of the generators of the ambient algebra by exploiting the additive group property of ideals.

## 4.2 The algebras $\mathfrak{su}(n)$

### 4.2.1 The case $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$

The simplest example to start with are the algebras $\mathfrak{su}(n)$ since the structure of their representations and tensor products is very simple. In order to warm up a little bit consider as a first example the regular embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$. Let $(1,0)$ be the first fundamental representation of $\mathfrak{su}(3)$. This representation decomposes under the regular embedding into $\mathfrak{su}(2)$ representations as

$$(1,0) \to (0) \oplus (1) \tag{4.2.1}$$

or, in the notation from above,

$$(1,0)|_{\mathfrak{su}(2)} = (0) \oplus (1). \tag{4.2.2}$$

The embedding index $x_{\mathfrak{su}(3),\mathfrak{su}(2)} \equiv x$ is given by $x = 1$ so we immediately conclude that the level $k'$ must be

$$k' = k + 1 \tag{4.2.3}$$

since $h^\vee_{\mathfrak{su}(n+1)} - h^\vee_{\mathfrak{su}(n)} = n + 1 - n = 1$. The next step is to understand the fusion ideal $I_{\mathfrak{su}(2),k+1}$ in terms of $\mathfrak{su}(3)$-representations decomposed under $|_{\mathfrak{su}(2)}$. From Section A.1 we know that the fusion ideal $I_{\mathfrak{su}(3),k}$ of $\mathfrak{su}(3)$ at level $k$ is generated by the representations $(k+1,0)$ and $(k,1)$. However we may replace the second generator with $(k+2,0)$ for the following reason: If $(k+1,0)$ lies in $I_{\mathfrak{su}(3),k}$ then so does $(k+1,0) \otimes (1,0) = (k+2,0) \oplus (k,1)$. Therefore $(k,1) \in I_{\mathfrak{su}(3),k}$ implies $(k+2,0) \in I_{\mathfrak{su}(3),k}$ and vice versa since $I_{\mathfrak{su}(3),k}$ is a group with respect to addition. This justifies the replacement. $(k+1,0)$ and $(k+2,0)$ are decomposed under $|_{\mathfrak{su}(2)}$ as follows:

$$(k+1,0)|_{\mathfrak{su}(2)} = (0) \oplus (1) \oplus \cdots \oplus (k+1) \tag{4.2.4}$$

$$(k+2,0)|_{\mathfrak{su}(2)} = (0) \oplus (1) \oplus \cdots \oplus (k+1) \oplus (k+2). \tag{4.2.5}$$

In the sense of the additive structure within $\mathrm{Rep}(\mathfrak{su}(3))$ we may take the difference of those direct sums to obtain

$$(k+2) = \bigoplus_{n=0}^{k+2}(n) - \bigoplus_{n=0}^{k+1}(n) = ((k+2,0) - (k+1,0))\,|_{\mathfrak{su}(2)}. \tag{4.2.6}$$

Now since the fusion ideal $I_{\mathfrak{su}(2),k+1}$ is generated by $(k'+1) = (k+2)$ we immediately conclude that

$$I_{\mathfrak{su}(2),k+1} = \langle (k+2) \rangle = \langle ((k+2,0) - (k+1,0))\,|_{\mathfrak{su}(2)} \rangle. \tag{4.2.7}$$

Since an ideal is by definition a subgroup of the additive group of the underlying ring, the linear combination $(k+2,0) - (k+1,0)$ and all linear combinations it generates necessarily lie in $I_{\mathfrak{su}(3),k}$ and therefore $((k+2,0) - (k+1,0))\,|_{\mathfrak{su}(2)}$ and every element it generates lie in $I_{\mathfrak{su}(3),k}|_{\mathfrak{su}(2)}$. This ultimately yields that

$$I_{\mathfrak{su}(2),k+1} \subseteq \langle I_{\mathfrak{su}(3),k}|_{\mathfrak{su}(2)} \rangle. \tag{4.2.8}$$

It is possible to generalize this to the case of the regular embedding of $\mathfrak{su}(n)$ in $\mathfrak{su}(n+1)$ for $n \geq 2$. The idea is the following: The fusion ideal $I_k$ at level $k$ of $\mathfrak{su}(n+1)$ is generated by the representations $(k+1, 0, ..., 0)$, $(k, 1, 0, ..., 0)$, ... $(k, 0, ..., 0, 1)$. We will show that we may replace this generating set by the representations $(k+1, 0, ..., 0)$, ... $(k+n, 0, ..., 0)$. As stated before the $\mathfrak{su}(n+1)$ representation $(m, 0, ..., 0)$ decomposes into $\mathfrak{su}(n)$ representations as follows:

$$\underbrace{(m, 0, ..., 0)}_{n-1 \text{ entries}}|_{\mathfrak{su}(n)} = \bigoplus_{\ell=0}^{m} \underbrace{(\ell, 0, ..., 0)}_{n-1 \text{ entries}}. \tag{4.2.9}$$

Given that the embedding index of the regular embedding $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$ is always $x = 1$ we know that the relationship between $k$ and $k'$ is $k' = k+1$ as in the example above. Thus it is possible to subtract the representations in an analogous way to $\mathfrak{su}(2) \subset \mathfrak{su}(3)$. We will make this formal in the following.

**Proposition 4.2.1.** *Let $n$ be a positive integer. Let $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$ be the regular embedding. Then*

$$I_{\mathfrak{su}(n),k'} \subset \langle I_{\mathfrak{su}(n+1),k}|_{\mathfrak{su}(n)} \rangle. \tag{4.2.10}$$

*Proof.* The level $k'$ of the algebra $\mathfrak{su}(n)$ is given by $k' = k+1$ for the regular embedding of $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$ since the embedding index for this embedding is $x = 1$. Now by Section A.1 we have that the fusion ideal $I_{\mathfrak{su}(n+1),k}$ at level $k$ is generated by the elements

$$\{(k+1, 0, ..., 0), (k, 1, 0, ..., 0), ..., (k, 0, ..., 0, 1)\} \tag{4.2.11}$$

where we have $n$ entries for $\mathfrak{su}(n+1)$-representations. We may replace this generating set (similarly to the considerations above) by

$$\{(k+1, 0, ..., 0), (k+2, 0, ..., 0), ..., (k+n, 0, ..., 0)\}. \tag{4.2.12}$$

The decomposition of those representations under the embedding $\mathfrak{su}(n) \subset \mathfrak{su}(n+1)$ is given by

$$\begin{aligned} (k+1, 0, ..., 0)_{\mathfrak{su}(n+1)} &\mapsto (0, ..., 0)_{\mathfrak{su}(n)} \oplus \cdots \oplus (k+1, 0, .., 0)_{\mathfrak{su}(n)} \\ &\vdots \\ (k+n, 0, ..., 0)_{\mathfrak{su}(n+1)} &\mapsto (0, ..., 0)_{\mathfrak{su}(n)} \oplus \cdots \oplus (k+n, 0..., 0)_{\mathfrak{su}(n)} \end{aligned} \tag{4.2.13}$$

where the representations in the right hand side have $n-1$ entries as representations of $\mathfrak{su}(n)$. However the fusion ideal $I_{\mathfrak{su}(n),k+1}$ at level $k' = k+1$ is generated by the elements

$$\{(k'+1, ..., 0), ..., (k'+n-1, 0, .., 0)\} \tag{4.2.14}$$

which by $k' = k+1$ is the same as

$$\{(k+2, 0, ..., 0), ..., (k+n, 0, ..., 0)\}. \tag{4.2.15}$$

Then we may substract the suitable projections of the $\mathfrak{su}(n+1)$-representations to obtain

$$\begin{aligned}(k+n,0,...,0)_{\mathfrak{su}(n)} &= ((k+n,0,...,0)-(k+n-1,0,...,0))\,|_{\mathfrak{su}(n)}\\ &\vdots \\ (k+2,0,...,0)_{\mathfrak{su}(n)} &= ((k+2,0,...,0)-(k+1,0,...,0))\,|_{\mathfrak{su}(n)}.\end{aligned} \tag{4.2.16}$$

Hence any $\mathfrak{su}(n)$-representation that generates the fusion ideal $I_{\mathfrak{su}(n),k+1}$ can be written as the projection of a difference of two $\mathfrak{su}(n+1)$-representations which generate $I_{\mathfrak{su}(3),k}$ and thus, by the group property of ideals, lie within $\langle I_{\mathfrak{su}(n+1)}|_{\mathfrak{su}(n)}\rangle$. □

### 4.2.2 The case $\mathfrak{su}(n) \subset \mathfrak{su}(n+m)$

So far we have only considered the embedding where we raised the dimension of $\mathfrak{su}(n)$ by one. The question arises if it is possible to generalize this result to the regular embedding of $\mathfrak{su}(n) \subset \mathfrak{su}(n+m)$ where $m$ is some other positive integer. It turns out that this is a very easy generalization of the previous result, at least for the regular embedding. Namely when we are decomposing $\mathfrak{su}(n+1)$-representations into $\mathfrak{su}(n)$-representations we may easily iterate this procedure in the sense that we decompose the $\mathfrak{su}(n)$-representations into $\mathfrak{su}(n-1)$-representations and thus obtain the decomposition of $\mathfrak{su}(n+1)$ into $\mathfrak{su}(n-1)$ representations. It is clear that we may go on like this for arbitrary dimensions. So for example take the regular embedding $\mathfrak{su}(2) \subset \mathfrak{su}(4)$. One decomposes the representation $(k+1,0,0)$ of $\mathfrak{su}(4)$ into representations $(k+1,0) \oplus \cdots \oplus (0,0)$ of $\mathfrak{su}(3)$ and then those representations into representations of $\mathfrak{su}(2)$. The important observation now is that many terms cancel conveniently. In order to get acquainted with the idea we calculate this case step by step. The situation is as follows: The level $k'$ is now given by $k' = k+2$ since $n+m=4$ where $n=2$ and $m=2$. Now if we set $k=1$ we obtain $\mathfrak{su}(2)$ at level $k'=3$ and $\mathfrak{su}(4)$ at level $k=1$. The fusion ideal $I_{\mathfrak{su}(2),3}$ is generated by the element $(k'+1)=(4)$. On the other hand the fusion ideal $I_{\mathfrak{su}(4),1}$ is generated by $(k+1,0,0)$, $(k+2,0,0)$ up to $(k+n+m-1,0,0)$ which explicitly are $(2,0,0)$, $(3,0,0)$ and $(4,0,0)$. Thus we have to construct $(4)$ as a $\mathfrak{su}(2)$ representation by projections of the three generating elements of $I_{\mathfrak{su}(4),1}$. Looking at $(4,0,0)$ we see that

$$(4,0,0) \to (4,0) \oplus (3,0) \oplus (2,0) \oplus (1,0) \oplus (0,0). \tag{4.2.17}$$

Consequently we have the two identities

$$((4,0,0)-(3,0,0))\,|_{\mathfrak{su}(3)} = (4,0) \tag{4.2.18}$$

$$((3,0,0)-(2,0,0))\,|_{\mathfrak{su}(3)} = (3,0). \tag{4.2.19}$$

But we also know that

$$((4,0)-(3,0))\,|_{\mathfrak{su}(2)} = (4) \tag{4.2.20}$$

such that eventually

$$((4,0,0)-2(3,0,0)+(2,0,0))\,|_{\mathfrak{su}(2)}=(4). \tag{4.2.21}$$

This proves the case for the regular embedding $\mathfrak{su}(2)\subset\mathfrak{su}(4)$ with $\mathfrak{su}(4)$ at level $k=1$. Obviously it has become more difficult to construct the right combination of representations. As is easily seen this does not really depend on the level but only on the difference $m$ between $n+m$ and $n$. Indeed take again the regular embedding $\mathfrak{su}(2)\subset\mathfrak{su}(4)$ but now with arbitrary level $k$ and $k'+2$. Then as stated above $I_{\mathfrak{su}(2),k+2}$ is generated by $(k'+1)=(k+3)$ and $I_{\mathfrak{su}(4),k}$ is generated by $(k+1,0,0)$, $(k+2,0,0)$ and $(k+3,0,0)$. Thus

$$((k+3,0,0)-(k+2,0,0)))_{\mathfrak{su}(3)}=(k+3,0) \tag{4.2.22}$$

and

$$((k+2,0,0)-(k+1,0,0)))\,|_{\mathfrak{su}(3)}=(k+2,0). \tag{4.2.23}$$

Furthermore

$$((k+3,0)-(k+2,0))\,|_{\mathfrak{su}(2)}=(k+3) \tag{4.2.24}$$

and hence

$$((k+3,0,0)-2(k+2,0,0)+(k+1,0,0))\,|_{\mathfrak{su}(2)}=(k+3). \tag{4.2.25}$$

Let us now turn to the general case $\mathfrak{su}(n)\subset\mathfrak{su}(n+m)$. The question we ask is whether the fusion ideal $I_{\mathfrak{su}(n)}$ at level $k'=k+m$ lies in the ideal generated by $I_{\mathfrak{su}(n+m),k}$. The fusion ideal $I_{\mathfrak{su}(n),k+m}$ is generated by the set

$$\left\{(k+m+1,0,...,0)_{\mathfrak{su}(n)},...,(k+m+n-1,0,...,0)_{\mathfrak{su}(n)}\right\} \tag{4.2.26}$$

whereas $I_{\mathfrak{su}(n+m),k}$ is generated by

$$\left\{(k+1,0,...,0)_{\mathfrak{su}(n+m)},...,(k+n+m-1,0,...,0)_{\mathfrak{su}(n+m)}\right\} \tag{4.2.27}$$

So $I_{\mathfrak{su}(n+m),k}$ is generated by $n+m-1$ elements and $I_{\mathfrak{su}(n),k+m}$ is generated by $n-1$ elements. In the spirit of the example before we eliminate representations of $I_{\mathfrak{su}(n+m),k}$ by projection on the next smaller dimension until we reach $n$. Hence this is done in $n+m-1-(n-1)=m$ steps. So in the first step we go from $\mathfrak{su}(n+m)$ to $\mathfrak{su}(n+m-1)$ and project the differences

$$\begin{aligned}
(\underbrace{(k+m+n-1,0,...)}_{\mathfrak{su}(n+m)}-\underbrace{(k+m+n-2,0,...)}_{\mathfrak{su}(n+m)})|_{\mathfrak{su}(n+m-1)} &= \underbrace{(k+m+n-1,0,...)}_{\mathfrak{su}(n+m-1)}\\
&\vdots\\
(\underbrace{(k+2,0,...)}_{\mathfrak{su}(n+m)}-\underbrace{(k+1,0,...)}_{\mathfrak{su}(n+m)})|_{\mathfrak{su}(n+m-1)} &= \underbrace{(k+2,0,...)}_{\mathfrak{su}(n+m-1)}.
\end{aligned} \tag{4.2.28}$$

Thus we have eliminated one element. Iteratively we eliminate more and more elements until we arrive at the desired representations. That is that after $m$ steps we will have arrived at $\mathfrak{su}(n)$ and have eliminated $m$ representations such that we are left with $n$ ones (of $\mathfrak{su}(n)$). The formula for the element $(k+m+n-1,0,..,0)$ of $\mathfrak{su}(n)$ in terms of projections of $\mathfrak{su}(n+m)$ reads then:

$$\begin{aligned}&(k+m+n-1,0,...,0)_{(n)} = \\ &= \left.\left((k+m+n-1,0,...)_{(n+m)} + \sum_{j=1}^{m}(-1)^j\binom{m}{j}(k+m+n-1-j,0,...)_{(n+m)}\right)\right|_{\mathfrak{su}(n)}\end{aligned} \tag{4.2.29}$$

where we have omitted the explicit notion of $\mathfrak{su}$ in the subscript and only mentioned the dimension in order to improve readability. Of course this can be done for all elements, namely for the smallest one:

$$\begin{aligned}&(k+m+1,0,...,0)_{(n)} = \\ &= \left.\left((k+m+1,0,...)_{(n+m)} + \sum_{j=1}^{m}(-1)^j\binom{m}{j}(k+m+1-j,0,...)_{(n+m)}\right)\right|_{\mathfrak{su}(n)} .\end{aligned} \tag{4.2.30}$$

As before we have shown that every generating element of $I_{\mathfrak{su}(n),k+m}$ can be written as a linear combination with integer coefficients of projections of generating elements of $I_{\mathfrak{su}(n+m),k}$ and thus we have proved:

**Proposition 4.2.2.** *Let $n, m$ be positive integers. Let $\mathfrak{su}(n) \subset \mathfrak{su}(n+m)$ be the regular embedding. Then*

$$I_{\mathfrak{su}(n),k+m} \subset \langle I_{\mathfrak{su}(n+m),k}|_{\mathfrak{su}(n)}\rangle. \tag{4.2.31}$$

### 4.2.3 The principal embedding $\mathfrak{su}(2) \subset \mathfrak{su}(n)$

Another important embedding is the **principal embedding** where the three-dimensional Lie algebra $\mathfrak{sl}(2)$ is embedded in $\mathfrak{sl}(n)$. There are multiple ways to define the principal embedding for $\mathfrak{sl}(2)$ for an arbitrary Lie algebra $\mathfrak{g}$. One definition states that for the principal embedding the number of irreducible $A_1$-submodules of $\mathfrak{g}$ is equal to the rank $r$ of $\mathfrak{g}$ since for all other embeddings the number is strictly greater than the rank (see for example [LS12]). Another way is to define the **principal subalgebra** if it contains a **regular nilpotent**[2] element (see for example [GOMV94]). However

[2] *Nilpotent element* in this context means the nilpotent part in the Jordan decomposition of an element $X \in \mathfrak{g}$ viewed as a linear map.

we will give an explicit realization of the principal embedding. Define the generators $J_+$ and $J_-$ as follows:

$$J_+ := \begin{pmatrix} 0 & & & \\ -1 & 0 & & \\ & \ddots & \ddots & \\ & & -1 & 0 \end{pmatrix} \tag{4.2.32}$$

$$J_- := \begin{pmatrix} 0 & (n-1) & & & \\ & 0 & 2(n-2) & & \\ & & \ddots & \ddots & \\ & & & 0 & (n-1) \end{pmatrix}. \tag{4.2.33}$$

The Cartan element $H$ is then defined as

$$H := \frac{1}{2}[J_+, J_-]. \tag{4.2.34}$$

We may calculate this matrix explicitly:

$$H = \frac{1}{2} \begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & -n+1 \end{pmatrix} \tag{4.2.35}$$

Hence we can read off how the defining representation $(1, 0, ..., 0)$ of $\mathfrak{su}(n)$ decomposes under this embedding from the entries of the Cartan element $H$. That is,

$$(1, 0, ..., 0)|_{\mathfrak{su}(2)} = (n-1). \tag{4.2.36}$$

Since we also know how the weights in the representation $(1, 0, ..., 0)$ are projected onto the weights in $(n-1)$ we may read off how a weight $(\lambda) = (\lambda_1, ..., \lambda_{n-1})$ of $\mathfrak{su}(n)$ is projected. The formula is given by

$$(\lambda_1, ..., \lambda_{n-1}) \mapsto ((n-1)\lambda_1 + 2(n-2)\lambda_2 + \cdots + (n-1)\lambda_{n-1}) \tag{4.2.37}$$

The embedding index of $\mathfrak{su}(2) \subset \mathfrak{su}(n)$ can be found by looking at the highest roots. The highest root of $\mathfrak{su}(n)$ is given by $\theta = (1, 0, ..., 0, 1)$ with the exception of $\mathfrak{su}(2)$ where it is given by $\vartheta = (2)$. Now the projection of the highest root of $\mathfrak{su}(n)$ onto a weight of $\mathfrak{su}(2)$ is given by

$$\theta = (1, 0, ..., 0, 1) \mapsto (2(n-1)) \tag{4.2.38}$$

by equation (4.2.37). The embedding index was defined as the ratio

$$x_e := \frac{|\mathcal{P}\theta|^2}{|\vartheta|^2}. \tag{4.2.39}$$

Hence we have

$$x_{\mathfrak{su}(2),\mathfrak{su}(n)} = \frac{4(n-1)^2}{4} = (n-1)^2. \tag{4.2.40}$$

The level $k'$ in relation to $k$ is then also easily calculated. The Coxeter number of $\mathfrak{su}(n)$ is given by $h^\vee_{\mathfrak{su}(n)} = n$. Therefore, $h^\vee_{\mathfrak{su}(2)} = 2$ and hence

$$k' = (n-1)^2(k+n) - 2. \tag{4.2.41}$$

The fusion ideal of $\mathfrak{su}(2)$ at level $k'$ is generated by $(k'+1)$ which is given in terms of $k$ and $n$ as

$$(k'+1) = ((n-1)^2(k+n) - 1). \tag{4.2.42}$$

So this is the representation we wish to construct from the decomposition of $\mathfrak{su}(n)$-representations. For later use we manipulate this expression a little bit and write it as follows:

$$(k'+1) = \big(((n-1)^2 - 1)(k+n) + (k+n-1)\big) \tag{4.2.43}$$

This means that if we can show that $(k+n-1) \in \langle I_{A_{n-1}}|_{A_1}\rangle$ then we know that $\big(((n-1)^2-1)(k+n) + (k+n-1)\big) \in \langle I_{A_3}|_{A_1}\rangle$ for the following reason: By multiplying $(k+n-1)$ with $(1)$ precisely $(k+n)$ times we see that $(k+n+k+n-1) = (2(k+n)-1)$ lies in $\langle I_{A_3}|_{A_1}\rangle$. But then $(m(k+n) + (k+n-1)) \in \langle I_{A_3}|_{A_1}\rangle$ for every $m \in \mathbb{N}$, in particular for $m = (n-1)^2 - 1$ as in the formula above.

Let us first look at the case $\mathfrak{su}(2) \subset \mathfrak{su}(4)$. We have the generating set $(k+1,0,0)$, $(k+2,0,0)$ and $(k+3,0,0)$. When $n=4$ the formula for $k'$ reduces to

$$k' + 1 = 9(k+4) - 1. \tag{4.2.44}$$

Hence we wish to show by the considerations above that $(k+n-1) = (k+3)$ lies in $\langle I_{\mathfrak{su}(4),k}|_{\mathfrak{su}(2),k'}\rangle$. We suppose that the way to do this is the following:

$$(k+3,0,0)|_{\mathfrak{su}(2)} \oplus (k+2,0,0)|_{\mathfrak{su}(2)} \otimes (-(3)\oplus(1)) \oplus (k+1,0,0)|_{\mathfrak{su}(2)} \stackrel{?}{=} (k+3). \tag{4.2.45}$$

How can we show that this holds true for any level without having to compute the general branching rules for representations of the form $(\lambda,0,0)$? It turns out that the characters are very powerful here. For representation of this particular form of $\mathfrak{su}(4)$ the character formula takes a rather simple form:

$$\chi_{(\lambda,0,0)} = \frac{(e^{(\lambda+1)\tau} - e^{-(\lambda+1)\tau})(e^{(\lambda+2)\tau} - e^{-(\lambda+2)\tau})(e^{(\lambda+3)\tau} - e^{-(\lambda+3)\tau})}{(e^{\tau} - e^{-\tau})(e^{2\tau} - e^{-2\tau})(e^{3\tau} - e^{-3\tau})} \tag{4.2.46}$$

Here the character is understood to be evaluated at $2\tau\rho$. For the derivation of this formula we refer to the appendix. Now by taking $(\lambda,0,0)$ to be the generators of $I_{\mathfrak{su}(4)}$ and inserting in (4.2.45) we obtain the following expression:

$$\begin{aligned}
&\chi_{(k+3,0,0)} + \chi_{(k+2,0,0)}(-\chi_3 + \chi_1) + \chi_{(k+1,0,0)} = \\
&= \frac{1}{(e^{\tau} - e^{-\tau})(e^{2\tau} - e^{-2\tau})(e^{3\tau} - e^{-3\tau})} \times \\
&\times \Big\{ (e^{(k+4)\tau} - e^{-(k+4)\tau})(e^{(k+5)\tau} - e^{-(k+5)\tau})(e^{(k+6)\tau} - e^{-(k+6)\tau}) \\
&+ (e^{(k+3)\tau} - e^{-(k+3)\tau})(e^{(k+4)\tau} - e^{-(k+4)\tau})(e^{(k+5)\tau} - e^{-(k+5)\tau})(-e^{3\tau} - e^{-3\tau}) \\
&+ (e^{(k+2)\tau} - e^{-(k+2)\tau})(e^{(k+3)\tau} - e^{-(k+3)\tau})(e^{(k+4)\tau} - e^{-(k+4)\tau}) \Big\} .
\end{aligned} \tag{4.2.47}$$

We can immediately see that we can pull out the terms involving $k+4$. This is advantageous since then the fraction

$$\frac{e^{(k+4)\tau} - e^{-(k+4)\tau}}{e^{\tau} - e^{-\tau}} \tag{4.2.48}$$

in front of the brackets would cancel conveniently to $e^{(k+3)\tau} + e^{(k+1)\tau} + \cdots e^{-(k+3)\tau}$ which is the character corresponding to the representation $(k+3)$ which is exactly what we want. Hence we must get rid of the other two terms in the denominator in front. As it turns out the remaining terms within the brackets collapse precisely to those two terms such that (4.2.48) is left. The calculation is straightforward and will not be repeated here. But now we have shown that equation (4.2.45) indeed holds true and this proves the original claim that $I_{\mathfrak{su}(2),k'} \subset \langle I_{\mathfrak{su}(4)}|_{\mathfrak{su}(2)} \rangle$ by the considerations we made earlier in this section.

However the procedure for higher dimensional embeddings does not go as smoothly as for those two cases. Already for $\mathfrak{su}(2) \subset \mathfrak{su}(5)$ a simple equation like (4.2.45) is not immediately obvious. However the case $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ could point in the direction since it works a bit differently than proposed in the general discussion above. The fundamental representation $(1,0)$ decomposes into $(1,0)|_{\mathfrak{su}(2)} = (2)$. The embedding index is $x = 4$ so we have

$$k' + 2 = 4(k+3) = 4k + 12 \tag{4.2.49}$$

and therefore

$$k' = 4k + 10. \tag{4.2.50}$$

Let's look at the decomposition of the representations under this embedding. The fusion ideal of $\mathfrak{su}(3)$ is still generated by $(k+1,0)$ and $(k+2,0)$. These are decomposed as follows:

$$(k+1,0) \to (2k+2) \oplus (2k-2) \oplus \cdots \oplus (1 + (-1)^k) \tag{4.2.51}$$

$$(k+2,0) \to (2k+4) \oplus (2k) \oplus \cdots \oplus (1 - (-1)^k) \tag{4.2.52}$$

Bear in mind that the fusion ideal at level $4k + 10$ of $\mathfrak{su}(2)$ is generated by $(4k+10+1) = (4k+11)$. We have to find a way to build it from sums and products of projections of $\mathfrak{su}(3)$-representations. If we multiply (4.2.52) by the fundamental representation (1) of $\mathfrak{su}(2)$ we obtain

$$(2k+3) \oplus (2k+1) \oplus (2k-1) \oplus (2k-3) \cdots \tag{4.2.53}$$

$$(2k+5) \oplus (2k+3) \oplus (2k+1) \oplus (2k-1) \cdots \tag{4.2.54}$$

Therefore $(2k+5) = ((k+2,0) - (k+1,0))_{\mathfrak{su}(2)} \otimes (1)$. Now $(4k+11)$ can be obtained as the highest term in the tensor product $(2k+6) \otimes (2k+5)$ since $2k+6+2k+5 = 4k+11$. However we must subtract the other terms popping up which are precisely the ones in $(2k+5) \otimes (2k+4)$. Hence

$$(4k+11) = ((2k+6) - (2k+4)) \otimes (2k+5). \tag{4.2.55}$$

## 4.3 Symplectic Lie algebras

In this section we turn to the question how to test the conjecture for other Lie algebras than $\mathfrak{su}$. As the generators for the fusion ideal are only known for $\mathfrak{su}$ and $\mathfrak{sp}$, we will consider here the case of symplectic Lie algebras. As we shall soon see our methods for $\mathfrak{su}$ will not work analogously for $\mathfrak{sp}$. However we are in the good situation that the generators of $I_{\mathfrak{sp}}$ are quite similar to the generators of $I_{\mathfrak{su}}$. In this section we will rely on numerical computation of branching rules and tensor products as can either be found in tables [MP81] or can be calculated with programs such as LieART. However branching rules for $\mathfrak{sp}$ are unfortunately not implemented in LieART.

The strategy for this section is as follows: We will look at some lower-dimensional examples of regular embeddings, e.g. $\mathfrak{sp}(2) \times \mathfrak{sp}(2) \times \mathfrak{sp}(2) \subset \mathfrak{sp}(6)$ and try to test the conjecture here. In the beginning we will also specify some level $k = 1$. Then we will try to deduce some kind of pattern if possible. Let us for now look at the regular embedding $A_1 \times A_1 \subset C_2$. The embedding $A_1 \subset C_2$ is actually *not* regular despite its simple appearance. We will deal with this case later. The embedding index of $A_1 \times A_1 \subset C_2$ is equal to one and hence $k' = k + 1$. The fusion ideal of $C_2$ at level $k$ is generated by $(k+1, 0)$ and $(k, 1)$ as can be seen from the appendix. So far we have not dealt with product algebras and how the fusion ideal of such an algebra is generated but this is rather simple. If we take a generator of one of the two factors, we have to tensor it with the neutral element of the other which is the trivial representation. In the case of $A_1$ the fusion ideal is generated by $(k'+1) = (k+2)$ and hence the two generating elements are $(k+2) \otimes (0)$ and $(0) \otimes (k+2)$. At level $k = 1$ we have the following situation: The fusion ideal of $C_2$ is generated by $(2, 0)$ and $(1, 1)$ which are decomposed into:

$$(2,0) \mapsto (2)(0) \oplus (1)(1) \oplus (0)(2) \tag{4.3.1}$$

and

$$(1)(1) \mapsto (2)(1) \oplus (1)(2) \oplus (1)(0) \oplus (0)(1). \tag{4.3.2}$$

Tensoring equation (4.3.1) with the fundamental representation $(1, 0)$ gives

$$(3)(0) \otimes (1)(0) \oplus (2)(1) \oplus (0)(1) \oplus (1)(2). \tag{4.3.3}$$

Subtracting (4.3.2) from this equation yields $(3)(0)$. However, $(3)(0)$ is a generator of $I_{\mathfrak{sp}(2)\times\mathfrak{sp}(2),3}$ and since $k = 1$, we have $k + 2 = 3$. In a similar way (i.e. by tensoring with $(0, 1)$ instead of $(1, 0)$) we obtain the other generator $(0)(3)$. As a general rule for $A_1 \times A_1 \subset C_2$ we may consider the following: The representation $(k + 1, 0)$ decomposes under the embedding by the branching rule

$$(k+1, 0) \mapsto \sum_{j=0}^{k+1} (k + 1 - j)(j) \tag{4.3.4}$$

whereas for $(k, 1)$ we have

$$(k, 1) \mapsto \sum_{j=0}^{k} \left((k + 1 - j)(j + 1) \oplus (k - j)(j)\right). \tag{4.3.5}$$

Therefore by tensoring (4.3.4) with $(1)(0)$ we obtain

$$\sum_{j=0}^{k+1} (k + 1 - j)(j) \otimes (1)(0) = \sum_{j=0}^{k+1} ((k + 1 - j) \otimes (1))(j). \tag{4.3.6}$$

Evaluating those tensor products gives

$$\begin{aligned} &(k + 2)(0) \oplus (k)(0)\oplus \\ &(k + 1)(1) \oplus (k - 1)(1)\oplus \\ &\quad (k)(2) \oplus (k - 2)(2)\oplus \\ &\qquad \vdots \\ &(3)(k - 1) \oplus (1)(k - 1)\oplus \\ &\quad (2)(k) \oplus (0)(k)\oplus \\ &\qquad \oplus (1)(k + 1). \end{aligned} \tag{4.3.7}$$

When subtracting (4.3.5) from this term the only term remaining is $(k + 2)(0)$ which is the first generator. By a similar argument (tensoring by $(0)(1)$) we obtain the other generator $(0)(k + 2)$. We have thus shown that

$$I_{A_1\times A_1,k+2} \subset \langle I_{C_2,k}|_{A_1\times A_1}\rangle. \tag{4.3.8}$$

The other embedding we have mentioned before is $A_1 \subset C_2$. Although seeming regular at first glance it is actually a special embedding. Therefore we have to work a little more in order to find the right combination which yields the correct result. Let us first find out what the related level $k'$ is. For a weight $(\lambda_1, \lambda_2)$ its decomposition under the embedding $A_1 \subset C_2$ is given by

$$(\lambda_1, \lambda_2) \mapsto 3\lambda_1 + 4\lambda_2. \tag{4.3.9}$$

This can be seen for example by looking at how the fundamental representations $(1, 0)$ and $(0, 1)$ decompose. The highest root of $\mathfrak{sp}(4)$ is $\theta = (2, 0)$ which by the rule above is projected by the projection matrix $\mathcal{P}$ to $(6)$. The embedding index was defined by the ratio of the square length of the projection of the highest root, i.e. $|\mathcal{P}\theta|^2$, by the square length of the highest root of $\mathfrak{g}' = \mathfrak{sp}(2)$, i.e. $|\vartheta|^2$. This gives

$$x_{A_1,C_2} = \frac{|\mathcal{P}\theta|^2}{|\vartheta|^2} = 9. \tag{4.3.10}$$

Hence by the relation between $k'$ and $k$ from above we have

$$k' + 2 = 9(k + 3) \tag{4.3.11}$$

since 2 and 3 are the Coxeter numbers of $A_1$ and $C_2$ respectively. For instance by setting $k = 1$ we obtain

$$k' = 9 \cdot 4 - 2 = 34. \tag{4.3.12}$$

That means the fusion ideal $I_{A_1,34}$ is generated by $(k' + 1) = (35)$. Hence we have to find a way how to write the representation $(35)$ as a linear combination such that we see that it lies in the right ideal which is given by $\langle I_{\mathfrak{sp}(4),1}|_{\mathfrak{sp}(2)}\rangle$. Let us at first show that $(3)$ lies in the ideal. The generators $(2, 0)$ and $(1, 1)$ decompose under this special embedding by the rules

$$(2, 0) \mapsto (6) \oplus (2) \tag{4.3.13}$$

and

$$(1, 1) \mapsto (7) \oplus (5) \oplus (1). \tag{4.3.14}$$

By tensoring the first equation with $(1)$ we obtain

$$((6) \oplus (2)) = (1) \oplus (3) \oplus (5) \oplus (7) \tag{4.3.15}$$

and hence by subtracting the second equation from it we get $(3)$. Now from $(3)$ we are able to reach $(35)$ by the following steps. First, multiply $(3)$ by $(1)$ and obtain $(2) \oplus (4)$ which then lies in the ideal. Again multiplying by $(1)$ gives $(1) \oplus 2(3) \oplus (5)$. But we already know that $(3)$ is in the ideal so we conclude that $(1) \oplus (5)$ lies in it. Carrying on like this yields that $(7)$ is in the ideal. Then we repeat the same procedure and see that $(15)$ is in the ideal and consequently by always going 4 steps we reach the next representation. Hence $(15 + 5 \cdot 4) = (35)$. So we have that $(35) \in \langle I_{\mathfrak{sp}(4),1}|_{\mathfrak{sp}(2)}\rangle$. The situation for the other levels is similar. One can show that the representation $(k + 2)$ is in $\langle I_{\mathfrak{sp}(4),k}|_{\mathfrak{sp}(2)}\rangle$. By multiplying repeatedly by $(1)$ we reach the next representation with only one entry in $k+3$ steps. This representation is then $(k+2+k+3) = (2k+5)$. We can always add $k+3$ until we reach a suitable representation. That means $(k+2+m(k+3))$ where $m$ is some nonnegative integer is always in the ideal. The formula for the level $k'$ was $k' = 9(k+3)-2$ and so the generator is always $(k'+1) = ((9(k+3)-1) = (9k+26)$. When setting $m = 8$ in the term above we see that $(k+2+8\cdot(k+3)) = (9k+26)$. That means it is always possible to obtain the generator $(9k+26)$. Thus we have

$$I_{\mathfrak{sp}(2),9k+25} \subset \langle I_{\mathfrak{sp}(4),k}|_{\mathfrak{sp}(2)}\rangle. \tag{4.3.16}$$

In fact we believe that for the embedding $A_1 \subset C_n$ we can always achieve that $(k+n) \in \langle I_{C_n,k}|_{A_n}\rangle$. If this is true, then we could argue that the shifted level $k'+1$ would assume the value

$$k'+1 = (2n-1)^2(k+n+1) - 1 \tag{4.3.17}$$

and thus by some manipulation of this formula

$$k'+1 = \left((2n-1)^2 - 1\right)(k+n+1) + (k+n). \tag{4.3.18}$$

In fact the exact value of the embedding index, i.e $x_e = (2n-1)^2$ does not really play a role here. Since $(k+n) \in \langle I_{C_n,k}|_{A_n}\rangle$ we can multiply by (1) (in the spirit of the above) and reach a representation with only one summand in $(k+n+1)$ steps. Hence the representation

$$(k'+1) = \left(((2n-1)^2-1)(k+n+1) + (k+n)\right) \tag{4.3.19}$$

lies in $\langle I_{C_n,k}|_{A_n}\rangle$ provided $(k+n) \in \langle I_{C_n,k}|_{A_n}\rangle$. This would then show the claim for the embedding $A_1 \subset C_n$.

# Appendices

# Appendix A

# Lie algebras: Notation and data

## A.1 Generators of fusion ideals

The following gives a list of generators of fusion ideals of selected Lie algebras, extracted from [BDR02]. Denote by $\omega_i$ the fundamental weights of the respective Lie algebra. Let $k$ be the level of the respective affine Lie algebra.

- $\mathfrak{su}(n+1)$ or $A_n$:

  $$\{k\omega_1 + \omega_i : i = 1, ..., n\} \tag{A.1.1}$$

  In Dynkin label notation:

  $$\{(k+1, 0, ..., 0), (k, 1, 0, ..., 0), ..., (k, 0, ..., 0, 1)\} \tag{A.1.2}$$

  where the vectors have $n$ entries.

- $\mathfrak{so}(2n+1)$ or $B_n$:

  $$\begin{aligned} &\{(k+1)\omega_1, (k-1)\omega_1 + 2\omega_n\} \\ \cup&\{(k-1)\omega_1 + \omega_i, (k-2)\omega_1 + \omega_i + \omega_n : i = 2, ..., n-1\} \end{aligned} \tag{A.1.3}$$

- $\mathfrak{sp}(2n)$ or $C_n$:

  $$\{k\omega_1 + \omega_i : i = 1, ..., n\} \tag{A.1.4}$$

  In Dynkin label notation:

  $$\{(k+1, 0, ..., 0), (k, 1, 0, ..., 0), ..., (k, 0, ..., 0, 1)\} \tag{A.1.5}$$

  where the vectors have $n$ entries.

# Appendix B

# Definitions, lemmata, proofs

**Definition B.0.1.** *(i) Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\Delta^0 = \{\alpha_1, ..., \alpha_r\}$ be a basis of simple roots. For any $j = 1, ..., r$ choose elements $E_j$ and $F_j$ in the root space $\mathfrak{g}_{\alpha_j}$ and $\mathfrak{g}_{-\alpha_j}$ such that they fulfil $\kappa(E_j, F_j) = \frac{2}{\langle \alpha_j, \alpha_j \rangle}$. Define $H_j := [E_j, F_j]$. Then the set*

$$\{E_j, F_j, H_j, j = 1, ..., r\} \tag{B.0.1}$$

*is called a set of* **standard generators** *of $\mathfrak{g}$.*

*(ii) Let $A = (a_{ij})$ be the Cartan matrix of $\mathfrak{g}$. The relations*

$$[E_i, F_j] = \delta_{ij} H_i \qquad [H_i, H_j] = 0 \tag{B.0.2}$$

$$[H_i, E_j] = a_{ij} E_j \qquad [H_i, F_j] = -a_{ij} F_j \tag{B.0.3}$$

$$ad(E_i)^{-a_{ij}+1}(E_j) = 0 \qquad ad(F_i)^{-a_{ij}+1}(F_j) = 0 \tag{B.0.4}$$

*where in the last line $i \neq j$ are called* **Serre relations**. *They form a complete set of relations.*

Now we give the derivation of the character formula used in the section about the principal embedding. Let $\rho$ be the Weyl vector and $t$ be some real parameter. Then the character $\chi_\lambda(t\rho)$ can be expressed as follows:

$$\chi_\lambda(t\rho) = \prod_{\alpha > 0} \frac{\sinh\left(\langle \alpha, (\lambda + \rho)t/2 \rangle\right)}{\sinh\left(\langle \alpha, \rho t/2 \rangle\right)} \tag{B.0.5}$$

where $\alpha$ goes over all positive roots. In the case of $\mathfrak{su}(4)$ the positive roots are $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\alpha_1 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$. The Weyl vector for $\mathfrak{su}(4)$ is given by $\rho = (1, 1, 1)$. When taking a representation that has only the first entry nonzero the only terms remaining when evaluating the scalar products are then

$$\chi(t\rho) = \frac{\sinh\left((\lambda_1 + 1)t/2\right) \sinh\left((\lambda_1 + 2)t/2\right) \sinh\left((\lambda_1 + 3)t/2\right)}{\sinh\left(t/2\right) \sinh\left(t\right) \sinh\left(3t/2\right)}, \tag{B.0.6}$$

Finally after replacing $t/2$ by $\tau$ and expressing the hyperbolic sine by exponentials we arrive at

$$\frac{(e^{(\lambda+1)\tau}-e^{-(\lambda+1)\tau})(e^{(\lambda+2)\tau}-e^{-(\lambda+2)\tau})(e^{(\lambda+3)\tau}-e^{-(\lambda+3)\tau})}{(e^{\tau}-e^{-\tau})(e^{2\tau}-e^{-2\tau})(e^{3\tau}-e^{-3\tau})}. \tag{B.0.7}$$

In the following we give proofs of theorems of foregoing sections which were left out due to readability.

*Proof.* (Theorem 2.3.5). Since from the definition of the embedding we have that the Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{l}$ is contained in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and since there is a one-to-one correspondence between elements of $\mathfrak{h}$, respectively $\mathfrak{h}'$ and the roots of $\mathfrak{g}$, respectively $\mathfrak{l}$ we can decompose $\Delta$ as the sum of $\Delta'$ and its orthogonal complement $\Delta'^{\perp}$ with respect to the Killing form. Indeed, take $\alpha' \in \Delta'$ and $\tau \in \Delta'^{\perp}$, then $\langle \tau, \alpha' \rangle = 0$. Now take some $H_\tau \in \mathfrak{h}$ corresponding to $\tau \in \Delta'^{\perp}$ and some $H_{\alpha'} \in \mathfrak{h}'$ corresponding to $\alpha \in \Delta'$. Then it follows that $\tau(H_{\alpha'}) = \langle \tau, \alpha' \rangle = 0$. Thus elements of $\Delta'^{\perp}$ are functionals sending elements of $\mathfrak{h}'$ to zero.

Let now $\alpha$ be an element of $\Delta$ with decomposition $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in \Delta'$ and $\alpha_2 \in \Delta'^{\perp}$. Then for an arbitrary $H' \in \mathfrak{h}'$ we have that

$$(f^*(\alpha))(H') = \alpha(f(H')) = (\alpha_1 + \alpha_2)(H') = \alpha_1(H') \tag{B.0.8}$$

Therefore $f^*(\alpha) = \alpha_1$. Furthermore $f^*(f^*(\alpha)) = f^*(\alpha)$ since the application of $f$ to $f(H')$ is the identity. This shows that $f^*$ is a projection with the desired properties.

It remains to show the second statement. Therefore take a weight $\lambda$ of $\rho$ together with an eigenvector $X \in \mathfrak{g}$. This means that $\rho(H)(X) = \lambda(H)(X)$. For $H \in \mathfrak{h}' \subset \mathfrak{h}$ we then have

$$\rho(f(H))(X) = \lambda(f(H))(X) = f^*(\lambda(H))(X) \tag{B.0.9}$$

by extending $f^*$ from $\Delta$ to the vector space of all weights, i.e. the real dual of $\mathfrak{h}$, i.e. $\mathfrak{h}_0^*$. □

**Lemma B.0.2.** *$\mathcal{F}$ is a unital ring.*

*Proof.* The unit element is given by the generator associated to the index $i_0$. Indeed, $N_{i_0 j}^{k} = N_{i_0 j k_C} = N_{j k_C i_0} = N_{j k_C}^{i_0,C} = N_{j_C k}^{i_0} = C_{j_C k} = \delta_{jk}$. Therefore $\phi_{i_0}$ is a right unit. Analogously one shows that it is also a left unit. One writes $\phi_{i_0} = \mathbf{1}$. □

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