MASTER'S THESIS

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"Subadditive Ergodic Theorem and its Applications"

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Abstract

In this master thesis, we introduce the concept of subadditivity for sequences of random variables and study two different versions of the subadditive ergodic theorem. First, we state the older version by Kingman and then proceed to the improved version by Liggett, which we will also prove. Subsequently, in the last chapter we examine five different applications of Liggett's version of the theorem. We start with the products of random matrices, after which we will see how the subadditive ergodic theorem is used for proofs of the shape theorems for three various models. These are the frog model, the branching random walks and the interlacement set. Last, but not least we look at some results for the Parabolic Anderson model.

Zusammenfassung

In dieser Masterarbeit stellen wir das Konzept der Subadditivität für Sequenzen von Zufallsvariablen vor und untersuchen zwei verschiedene Versionen des subadditiven Ergodentheorems. Zuerst geben wir die ältere Version von Kingman an und dann gehen wir zu der verbesserten Version von Liggett über, die wir ebenfalls beweisen werden. Im letzten Abschnitt untersuchen wir fünf verschiedene Anwendungen der Version von Liggett des subadditiven Ergodentheorems. Wir beginnen mit den Produkten von Zufallsmatrizen, nach denen wir sehen werden, wie das subadditive Ergodentheorem für Beweise der Formsätze für drei verschiedene Modelle verwendet wird. Dies sind das so genannte Frog Model, die verzweigenden Random Walks und das Interlacement Set. Anchließend schauen wir uns einige Ergebnisse für das Parabolische Anderson Modell an.

Contents

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1 Introduction

The subadditive ergodic theory was first studied by Hammersley and Welsh (1965). They noted that some problems of percolation theory could be written as a family of random variables satisfying the subadditivity condition. Thus, the theory of the subadditive ergodic processes was born.

This theory was later expanded by J.F.C. Kingman (1968), when he used the condition of Hammersley and Welsh; and proved a convergence result for the subadditive ergodic processes, the so called subadditive ergodic theorem. Subsequently, Thomas M. Liggett (1985) managed to weaken Kingmans' assumptions. Let us quickly look at these different conditions.

Let $\{X_{m,n}\}$ be a family of random variables, where the indices m, n run over the set of non-negative integers and $m < n$. The classical subadditive condition, used in Kingman's version, says

$$X_{l,n} \leq X_{l,m} + X_{m,n}, \quad \text{for } 0 \leq l < m < n. \tag{1}$$

We compare this with the modified subadditivity condition of Liggett.

$$X_{0,n} \leq X_{0,m} + X_{m,n}, \quad \text{for } 0 < m < n. \tag{2}$$

A process which satisfies the stronger condition by Kingman, trivially also satisfies Liggett's modified condition. Interestingly, even though Liggett weakens Kingman's assumptions, we will see in this thesis that the conclusions of the theorem remain valid.

The aim of this master thesis is to understand the subadditive processes for the families of random variables. We examine two different versions of the subadditive ergodic theorem and find an example which shows that the assumptions of these theorems are not equivalent, as one can find a case which satisfies the assumptions of exactly one of the versions of the theorem. Finally, we introduce and study how the subadditive ergodic theorem is used in applications. In this master thesis, we describe five different applications. During my research of the topic, I have encountered many more interesting ones. Unfortunately, it is not possible to put them all in one paper.

This thesis is split between two major chapters, Subadditive Ergodic Theorem and Applications. The first of these major chapters further consists of three parts. In the first section, we introduce the concept of subadditivity for general deterministic sequences and prove an important result, Fekete's lemma. Next, we define subadditivity, as well as stationarity, for families of random variables and prove a variant of Fekete's lemma. Last result of this section will be a so called maximal ergodic theorem. The second section shortly mentions the more special additive case. In the third and most important part, we state two different versions of the subadditive ergodic theorem, the older by Kingman and the newer by Liggett. The latter, we will also prove. The last major chapter of this thesis deals with some applications of the subadditive ergodic theorem.

2 Subadditive Ergodic Theorem

In this chapter, we first define what is a subadditivity in general, state few easy examples and prove a well known result, Fekete's lemma. Next, we will proceed to the processes of random variables, prove some results about them and also quickly mention the more special additive case. Afterwards, we get to the most important part of this chapter, the subadditive ergodic theorem. We state two versions, the one by Kingman and one by Liggett, the latter we will prove in detail. At the end of this chapter we find an example which satisfies the assumptions of Liggett's theorem, but not those of Kingman.

2.1 Subadditivity

Definition 2.1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called subadditive if for all $m, n \in \mathbb{N}$ holds that

$$a_{m+n} \leq a_m + a_n. \tag{3}$$

To acquaint ourselves better with the concept of subadditivity, we look at a few (very simple) examples. It is trivial to see that they all fulfill the condition (3) above.

Example 2.2. These sequences are subadditive:

- $a_n = c$ with c a non-negative constant
- $a_n = n + 1$
- $a_n = \sqrt{n}$

A very important result about the subadditive sequences is called Fekete's lemma.

Lemma 2.3 (Fekete's Lemma)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a subadditive sequence. Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to*

$$L = \inf_{n\geq 1} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n}. \tag{4}$$

Proof. Let $L = \inf_{n\geq 1} \frac{a_n}{n}$. We fix an $m \geq 1$ and define $n = km + l$ such that $0 \leq l < m$ and $k, l, m, n \in \mathbb{N}$. Then by the definition of a subadditive sequence and induction over k

$$\begin{aligned} a_n = a_{km+l} &\leq a_{km} + a_l \leq a_{(k-1)m+m} + a_l \\ &\leq a_{(k-1)m} + a_m + a_l \leq ... \leq k a_m + a_l. \end{aligned} \tag{5}$$

Since m is fixed and $l < m$, l must also be finite. Thus $n \to \infty$ implies $k \to \infty$. Therefore,

$$\frac{a_n}{n} = \frac{a_{km+l}}{km+l} \leq \frac{ka_m + l}{km+l} \xrightarrow{k\to\infty \text{ as } n\to\infty} \frac{a_m}{m}. \tag{6}$$

Since m was chosen as an arbitrary natural number, we can conclude

$$L \leq \liminf_{n\to\infty} \leq \limsup_{n\to\infty} \leq L. \tag{7}$$

This concludes the proof. □

We modify the definition of subadditivity for a family X of random variables $\{X_{m,n},\ m < n\}$, with the indices m and n in the set of non negative integers T. Throughout this entire thesis, we assume by convention that $X_{m,m} = 0$. We call X a subadditive process if for all $l, m, n \in T$ with $0 \leq l < m < n$ holds that

$$X_{l,n} \leq X_{l,m} + X_{m,n}. \tag{8}$$

A process is called superadditive if the reverse inequality holds. Therefore, X is superadditive if and only if $-X$ is subadditive and any result about subadditive processes can also be applied to the superadditive ones. A process X is called additive if it is both subadditive and superadditive, so for all $l, m, n \in T, 0 \leq l < m < n$:

$$X_{l,n} = X_{l,m} + X_{m,n}. \tag{9}$$

Definition 2.4. A subadditive process X is called stationary if the distributions of the process $\{X^*_{m,n}, m < n\} := \{X_{m+1,n+1}, m < n\}$ are the same as those of $\{X_{m,n}, m < n\}$.

Example 2.5. If $X_{m,n}$ are i.i.d random variables then the process X is stationary.

For the expectation of subadditive stationary processes holds a result similar to Fekete's lemma. This proposition will be used in the following section to help prove the most important result of this thesis, the subadditive ergodic theorem.

Proposition 2.6. *Let $\{X_{m,n}, m < n\}$ be a stationary subadditive process and assume that $\mathbb{E}[X] < \infty$. Then*

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[X_{0,n}] = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}[X_{0,n}]. \tag{10}$$

We call γ a time constant of the subadditive process.

Proof. By the subadditivity condition (8),

$$X_{0,n} \leq X_{0,m} + X_{m,n}, \tag{11}$$

for $0 < m < n$. The expectation is linear and therefore

$$\mathbb{E}[X_{0,n}] \leq \mathbb{E}[X_{0,m}] + \mathbb{E}[X_{m,n}]. \tag{12}$$

However, $\mathbb{E}[X_{m,n}] = \mathbb{E}[X_{0,n-m}]$ because the process X is stationary. Therefore,

$$\mathbb{E}[X_{0,n}] \leq \mathbb{E}[X_{0,m}] + \mathbb{E}[X_{0,n-m}], \tag{13}$$

for $0 < m < n$. Therefore, the sequence $\mathbb{E}[X_{0,n}]$ is subadditive and we only need to apply the already proven Fekete's lemma 2.3 to complete the proof. □

Before we proceed to the next section, we state and prove the so called maximal subadditive ergodic theorem. One of the main results of this thesis, Kingman's subadditive ergodic theorem, can be proved with the help of this theorem.

Theorem 2.7 (Maximal Subadditive Ergodic Theorem)**.** *Let $X = \{X_{m,n}, m < n\}$ be a family of random variables on a probability space Ω. Let us assume that X is stationary and subadditive with time constant γ. We define*

$$A = \{X_{0,n} \geq 0 \textit{ for some } n \geq 1\} \tag{14}$$

and suppose that A has a positive probability. Then the conditional expected value

$$\mathbb{E}[X_{0,1}|A] \geq 0. \tag{15}$$

Proof. We will use the notation $X_n = X_{0,n}$ for all $n \geq 1$. Define

$$M_n = \max_{1 \leq m \leq n} X_m. \tag{16}$$

Applying the subadditivity condition (8) with $l = 0, n = m + 1, m = 1$, we get

$$X_{m+1} \leq X_1 + X_{1,m+1}. \tag{17}$$

Then,

$$\begin{aligned} M_n \leq M_{n+1} &= \max_{1 \leq m \leq n+1} X_m \\ &= \max_{0 \leq m \leq n} X_{m+1} \\ &\leq \max_{0 \leq m \leq n} (X_1 + X_{1,m+1}) \text{ by (17)} \\ &= X_1 + \max_{0 \leq m \leq n} X^*_m, \text{ since } X_1 \text{ does not depend on the maximum} \\ &= X_1 + \max(0, M^*_n), \end{aligned} \tag{18}$$

where X^* is the shifted process from Definition 2.4 and M^*_n is defined as the maximum over this shifted process,

$$M^*_n = \max_{1 \leq m \leq n} X^*_m. \tag{19}$$

Let A_n be the set where the maximum M_n is non-negative, $A_n = \{M_n \geq 0\}$. Since integral is linear, we can use the computation (18).

$$\begin{aligned} \int_{A_n} M_n dP &\leq \int_{A_n} X_1 dP + \int_{A_n} \max(0, M^*_n) dP \\ &\leq \int_{A_n} X_1 dP + \int_{\Omega} \max(0, M^*_n) dP \\ &= \int_{A_n} X_1 dP + \int_{\Omega} \max(0, M_n) dP, \text{ by stationarity of X} \\ &= \int_{A_n} X_1 dP + \int_{A_n} M_n dP. \end{aligned} \tag{20}$$

Subtracting $\int_{A_n} M_n dP$ from both sides of the inequality, we obtain

$$\int_{A_n} X_1 dP \geq 0. \tag{21}$$

Finally we let $n \to \infty$ and obtain

$$\int_{A} X_1 dP \geq 0, \tag{22}$$

which is exactly what we wanted to prove. □

2.2 Additive Case

Before we get to the subadditive ergodic theorem itself, we first take a quick look at the simpler additive case. Let in this section $\{X_{m,n}, m < n\}$, be an additive process. We define a new process $\{Y_n, n \geq 1\}$, as

$$Y_n = X_{n-1,n}. \tag{23}$$

Using (9) repeatedly,

$$\begin{aligned} X_{m,n} &= X_{m,m+1} + X_{m+1,n} \\ &= Y_{m+1} + X_{m+1,m+2} + X_{m+2,n} \\ &= ... \\ &= Y_{m+1} + ... + Y_{n-1} + Y_n \\ &= \sum_{k=m+1}^{n} Y_k. \end{aligned} \tag{24}$$

Therefore, the additive processes can simply be interpreted as partial sums of a sequence of random variables. In the following, assume that the process $\{Y_n, n \geq 1\}$ is stationary. We take equation (24) with $m = 0$. Then,

$$X_{0,n} = \sum_{k=1}^{n} Y_k. \tag{25}$$

We can use Birkhoff's ergodic theorem for stationary sequences.

Theorem 2.8 (Birkhoff's Ergodic Theorem)**.** *Let $\{X_{m,n}, m < n\}$ be an additive process, $X_m \in L^1$ for all m. We define $\{Y_n, n \geq 1\}$ to be the process defined by (23). Then the limit*

$$X = \lim_{n\to\infty} \frac{1}{n} X_{0,n} \tag{26}$$

is finite and exists with probability 1 and in mean. Moreover,

$$\mathbb{E}[X] = \gamma, \tag{27}$$

with $\gamma = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}[X_{0,n}]$.

Since this is a standard result and we are more interested in the subadditive sequences anyway, we will not prove this theorem. The proof can be found in a variety of the books on an ergodic theory, for example in [15]. Instead, we will proceed to the next section and generalize this to a result valid for subadditive sequences.

2.3 Subadditive Ergodic Theorem

In this section we state two different versions of the subadditive ergodic theorem. Both of them reach the same conclusions, therefore these two versions differ only in their assumptions. We begin with the older version by J.F.C. Kingman.

Theorem 2.9 (Kingman's Subadditive Ergodic Theorem)**.** *Let $\{X_{m,n}\}$ be a family of random variables, where the indices m, n run over the set of non-negative integers and $m < n$. Assume that the following three conditions hold:*

(a) ***Subadditivity:*** *If $0 \leq l < m < n$ then $X_{l,n} \leq X_{l,m} + X_{m,n}$.*

(b) ***Stationarity:*** *The joint distribution of the process $(X_{m+1,n+1}; 0 \leq m < n)$ are the same as those of $(X_{m,n}; 0 \leq m < n)$.*

(c) For all n holds $\mathbb{E}[|X_{0,n}|] < \infty$ and $\mathbb{E}[X_{0,n}] \geq -Cn$ for some constant C.

Then we can conclude the following:

$$\gamma = \inf_{n \geq 1} \frac{1}{n} \mathbb{E}[X_{0,n}] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[X_{0,n}], \tag{28}$$

$$X = \lim_{n \to \infty} \frac{1}{n} X_{0,n} \text{ exists a.s. and is in } L_1, \tag{29}$$

$$\mathbb{E}[X] = \gamma. \tag{30}$$

Proof. As we only use the version of Liggett for applications and the proof itself is rather lengthy, we will not state Kingman's proof of the theorem. It can be found in the paper of Kingman himself, see [2]. Most importantly, Liggett's version of the subadditive ergodic theorem, which we will prove shortly, has stronger assumptions than Kingman's theorem and therefore it implies it. □

Theorem 2.10 (Liggett's Subadditive Ergodic Theorem)**.** *Let $\{X_{m,n}\}$ be a family of random variables, where the indices m, n run over the set of non-negative integers and $m < n$. Suppose that the following four conditions hold:*

(a) If $0 < l < m$ then $X_{0,m} \leq X_{0,l} + X_{l,m}$.

(b) The joint distribution of the process $(X_{m+1,m+k+1}; k \geq 1)$ are the same as those of $(X_{m,m+k}; k \geq 1)$ for each $m \geq 0$.

(c) For each $k \geq 1$, $(X_{nk,(n+1)k}; n \geq 1)$, is a stationary process.

(d) *For all n holds $\mathbb{E}[|X_{0,n}|] < \infty$ and $\mathbb{E}[X_{0,n}] \geq -Cn$ for some constant C.*

Then the following are true:

$$\gamma = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}[X_{0,n}] = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[X_{0,n}], \tag{31}$$

$$X = \lim_{n\to\infty} \frac{1}{n} X_{0,n} \text{ exists a.s. and is in } L_1, \tag{32}$$

$$\mathbb{E}[X] = \gamma. \tag{33}$$

Additionally, if the stationary processes of condition (c) are ergodic, then $X = \gamma$ a.s.

Proof. Define $X_n = X_{0,n}, \gamma_n = \mathbb{E}[X_n]$,

$$\overline{X} = \limsup_{n\to\infty} \frac{1}{n} X_n; \quad \underline{X} = \liminf_{n\to\infty} \frac{1}{n} X_n. \tag{34}$$

We will cut the proof into four parts:

(P1) $\gamma = \lim_{n\to\infty} \frac{1}{n}\gamma_n = \inf_{n\geq 1} \frac{1}{n}\gamma_n$,

(P2) $\mathbb{E}[\overline{X}] \leq \gamma$ and if for each $k \geq 1$, the stationary process $(X_{nk,(n+1)k}; n \geq 1)$, is ergodic, then $\overline{X} \leq \gamma$ a.s.,

(P3) $\mathbb{E}[\underline{X}] \geq \gamma$,

(P4) $\lim_{n\to\infty} \mathbb{E}[|\frac{1}{n}X_n - X|] = 0$,

where X is the common value of $\underline{X}$ and $\overline{X}$.

Proof of (P1):
We apply Proposition 2.6 to the process γ_n. This process is subadditive by conditions (a) and (b) of the theorem and the definition of γ_n. Therefore the statement (P1) holds.

Proof of (P2):
We fix $k \geq 1$ and recall the notation $X_{kn} = X_{0,kn}$. We now repeatedly use the condition (a), $X_{0,m} \leq X_{0,l} + X_{l,m}$ for all $0 < l < m$, with initial values $m = kn$ and $l = k(n-1)$. In the second step we take the inequality with $m = k(n-1)$, $l = k(n-2)$. Iteratively in each step we will replace n by $n-1$ until we reach 0.

$$\begin{aligned} X_{kn} &\leq X_{k(n-1)} + X_{k(n-1),kn} \\ &\leq X_{k(n-2)} + X_{k(n-2),k(n-1)} + X_{k(n-1),kn} \\ &\leq X_{k(n-3)} + X_{k(n-3),k(n-2)} + X_{k(n-2),k(n-1)} + X_{k(n-1),kn} \leq \ldots \end{aligned} \tag{35}$$

We rewrite this in a sum notation

$$X_{kn} \le \sum_{j=1}^{n} X_{k(j-1),kj}. \tag{36}$$

Now, we apply Birkhoff's ergodic theorem and condition (c) to conclude that

$$\frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj} \tag{37}$$

converges a.s. and in L_1 to a random variable with mean value γ_k. Therefore,

$$\mathbb{E}[\limsup_{n\to\infty} \frac{X_{kn}}{kn}] \le \frac{1}{k}\gamma_k. \tag{38}$$

Using condition (a) again for $m = kn + j$ and $l = kn$ we get

$$X_{kn+j} \le X_{kn} + X_{kn,kn+j}. \tag{39}$$

By condition (b) the distribution of $X_{kn,kn+j}$ must only depend on j. Then using condition (d) we know that this distribution has a finite expected value and therefore we can apply the Borel-Cantelli lemma and conclude that

$$\lim_{n\to\infty} \frac{X_{kn,kn+j}}{n} = 0 \text{ a.s.} \tag{40}$$

By (38), (39) and (40),

$$\mathbb{E}[\overline{X}] \le \frac{1}{k}\gamma_k. \tag{41}$$

Now we only have to let $k \to \infty$ to obtain $\mathbb{E}[\overline{X}] \le \gamma$. Moreover, if the stationary processes $(X_{nk,(n+1)k}, n \ge 1)$ are ergodic for each $k \ge 1$, then

$$\frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj} \to \gamma_k \text{ a.s.} \tag{42}$$

Therefore, $\overline{X} \le \gamma$ a.s. holds by the already proven part (P1).

Proof of (P3):
We define a new random variable U_n to be independent of the variables $X_{l,m}$ and uniformly distributed on the set $\{1, 2, ..., n\}$. Then $\mathbb{E}[U_n] = \frac{n+1}{2}$ and by the independence of U_n and $X_{l,m}$ also

$$\mathbb{E}[X_{k+U_n}] = \frac{1}{n}\sum_{l=1}^{n} \mathbb{E}[X_{k+l}]. \tag{43}$$

Moreover, define random variables

$$Y_k^n = X_{k+U_n} - X_{k+U_n-1}. \tag{44}$$

Then taking expected value of this equation and using the linearity of the expectation, we compute

$$\begin{aligned}
\mathbb{E}[Y_k^n] &= \frac{1}{n}\sum_{l=1}^{n}\mathbb{E}[X_{k+l} - X_{k+l-1}] \\
&= \frac{1}{n}\sum_{l=1}^{n}(\mathbb{E}[X_{k+l}] - \mathbb{E}[X_{k+l-1}]) \\
&= \frac{1}{n}(\mathbb{E}[X_{k+n}] - \mathbb{E}[X_k]) \\
&= \frac{1}{n}(\gamma_{k+n} - \gamma_k).
\end{aligned} \tag{45}$$

Using the notation $X^+ = \max\{0, X\}$ we also get

$$\begin{aligned}
\mathbb{E}[(Y_k^n)^+] &= \frac{1}{n}\sum_{l=1}^{n}\mathbb{E}[(X_{k+l} - X_{k+l-1})^+] \\
&\leq \frac{1}{n}\sum_{l=1}^{n}\mathbb{E}[X^+_{k+l-1,k+l}],
\end{aligned} \tag{46}$$

by using the condition $X_{0,t} \leq X_{0,s} + X_{s,t}$, with $s = k + l - 1$ and $t = k + l$. By the assumption of the theorem, the joint distributions of $\{X_{k+l-1,k+l}, l \geq 1\}$ are the same for each $k \geq 0$. Therefore,

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}[X^+_{k+l-1,k+l}] \leq \frac{1}{n}n\mathbb{E}[X^+_{0,1}] = \mathbb{E}[X^+_1]. \tag{47}$$

Collecting the previous two results and the already proven part (P1) of this theorem we conclude

$$\sup_n \mathbb{E}[|Y_k^n|] < \infty \tag{48}$$

and

$$\lim_{n\to\infty} \mathbb{E}[Y_k^n] = \gamma \text{ for each } k \geq 1. \tag{49}$$

Since $\sup_n \mathbb{E}[|Y_k^n|]$ is bounded, there must exist a subsequence $(n_i) \subseteq (n)$, such that the joint distributions of $\{Y_k^{n_i}, k \geq 1\}$ converge as $i \to \infty$ to the joint distributions of some collection of random variables $\{Y_k, k \geq 1\}$. We take an arbitrary bounded continuous function f on R, such that f depends on only finitely many coordinates.

$$\begin{aligned}
&\mathbb{E}[f(Y_1, Y_2, Y_3, ...)] \\
&= \lim_{i\to\infty}\frac{1}{n_i}\sum_{l=1}^{n_l}\mathbb{E}[f(X_{l+1} - X_l, X_{l+2} - X_{l+1}, X_{l+3} - X_{l+2}, ...)].
\end{aligned} \tag{50}$$

This implies that $\{Y_k, k \geq 1\}$ is a stationary sequence. By the condition (a) of this theorem with $m = U_n + 1$ and $l = U_n$ we get

$$Y_1^n = X_{U_n+1} - X_{U_n} \leq X_{U_n,U_n+1}. \tag{51}$$

Since the joint distributions are the same we can also conclude

$$X_{U_n,U_n+1} =_d X_1, \tag{52}$$

with $=_d$ the notation for the equality in distribution. By the condition (d) $\{(Y_1^n)^+, n \geq 1\}$ is uniformly integrable. By Fatou's lemma and $\lim_{n\to\infty} \mathbb{E}[Y_k^n] = \gamma$ for each $k \geq 1$ we conclude

$$\mathbb{E}[Y_1] \geq \limsup_{n\to\infty} \mathbb{E}[Y_1^n] = \gamma. \tag{53}$$

We apply Birkhoff's ergodic theorem. Therefore,

$$Y = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k \tag{54}$$

exists a.s. and

$$\mathbb{E}[Y] = \mathbb{E}[Y_1] \geq \gamma. \tag{55}$$

We say that a random variable A is stochastically less than a random variable B if $P(A > x) \leq P(B > x)$ for all x.
To be proved: $\underline{X}$ is stochastically larger than Y. Therefore, we only have to show

$$(Y_1, Y_1 + Y_2, Y_1 + Y_2 + Y_3, ...) \leq_d (X_1, X_2, X_3, ...), \tag{56}$$

with $\leq_d$ denoting the joint stochastic monotonicity. We use the equation (50). Then

$$\begin{aligned}
&\mathbb{E}[f(Y_1, Y_1 + Y_2, Y_1 + Y_2 + Y_3, ...)] \\
&\quad = \lim_{i\to\infty} \frac{1}{n_i} \sum_{l=1}^{n_l} \mathbb{E}[f(X_{l+1} - X_l, X_{l+2} - X_{l+1}, X_{l+3} - X_{l+2}, ...)] \\
&\quad \leq \lim_{i\to\infty} \frac{1}{n_i} \sum_{l=1}^{n_l} \mathbb{E}[f(X_{l,l+1}, X_{l,l+2}, X_{l,l+3}, ...)] \\
&\quad = \mathbb{E}[f(X_1, X_2, X_3, ...)].
\end{aligned} \tag{57}$$

In this calculation we used conditions (a) and (b). Therefore, this holds for any bounded continuous function on $\mathbb{R}^n$ depending on a finite number of coordinates.

Proof of (P4):
By previous two parts of the proof, we have

$$\mathbb{E}[\overline{X}] \leq \gamma \leq \mathbb{E}[\underline{X}]. \tag{58}$$

However, by the definition of $\underline{X}$ and $\overline{X}$, see (34), we have $\underline{X} \leq \overline{X}$ and by monotonicity of expectation

$$\mathbb{E}[\underline{X}] \leq \mathbb{E}[\overline{X}]. \tag{59}$$

Taking these together, we get $\underline{X} = \overline{X}$ a.s. Also $\mathbb{E}[X] = \gamma$. Using the inequality (36) with $k = 1$,

$$X_n \leq \sum_{j=1}^{n} X_{j-1,j}, \tag{60}$$

one obtains that,

$$\Big\{ \Big(\frac{1}{n} X_n^+ \Big), n \geq 1 \Big\} \tag{61}$$

is uniformly integrable. Then

$$\lim_{n \to \infty} \mathbb{E}\Big[\frac{1}{n} X_n - X \Big]^+ = 0. \tag{62}$$

By (31) we have,

$$\gamma = \inf_{n \geq 1} \frac{1}{n} \mathbb{E}[X_{0,n}] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[X_{0,n}].$$

Finally, by using the last two equations together we obtain

$$\lim_{n \to \infty} \mathbb{E}\Bigg[\Big| \frac{1}{n} X_n - X \Big| \Bigg] = 0, \tag{63}$$

as desired. □

2.4 Difference between the Assumptions of Kingman and Liggett

Now that we have finished proving Liggett's version of the subadditive ergodic theorem, which we will use for the applications in the next section, we may ask, whether there is a difference between this version and the one of Kingman. The short answer is yes. There are processes which satisfy the modified subadditive condition of Liggett's version, but not that of Kingman. One such process will be described in the simple example below.

Example 2.11. Let

$$I = \Big\{ (m,n) \in \mathbb{Z}^2 : n \geq 0 \text{ with } m+n \text{ an even number} \Big\}. \tag{64}$$

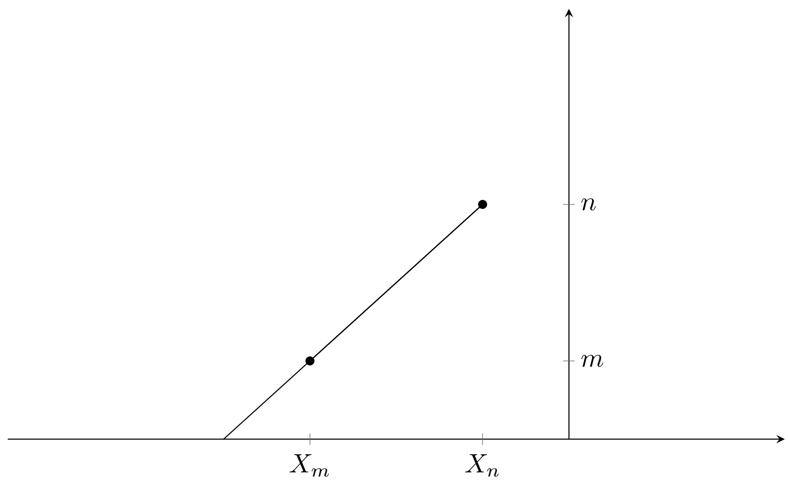
We define a path in I as a sequence $(m_1, n_1), ..., (m_k, n_k)$, such that for each i:

$$\begin{aligned} n_{i+1} &= n_i + 1 \\ m_{i+1} &= m_i \pm 1. \end{aligned} \tag{65}$$

Each edge from (m,n) to either $(m+1, n+1)$ or $(m-1, n+1)$ is labelled as open or closed, independently from each other, with probability p or $1-p$, respectively. We call a path active if all the edges forming a path through neighboring points are open. For $n \geq 0$, we define

$$\begin{aligned} X_n = \max\{k : \ &\text{there exists an active path from } (l,0) \\ &\text{to } (k,n) \text{ for some } l \leq 0\}. \end{aligned} \tag{66}$$

Figure 1: Case 1



For $0 \leq m < n$, define

$$X_{m,n} = \max\{k : \text{there exists an active path from } (l,m) \text{ to } (k,n) \text{ for some } l \leq X_m\} - X_m. \tag{67}$$

Substituting $m = 0$ in the equation above, we get $X_{0,n} = X_n$.

Now we want to verify that while the modified subadditivity condition (a) of Liggett holds, the classic subadditivity condition of Kingman can fail. So for $0 < m < n$, one must check

$$X_n \leq X_m + X_{m,n}. \tag{68}$$

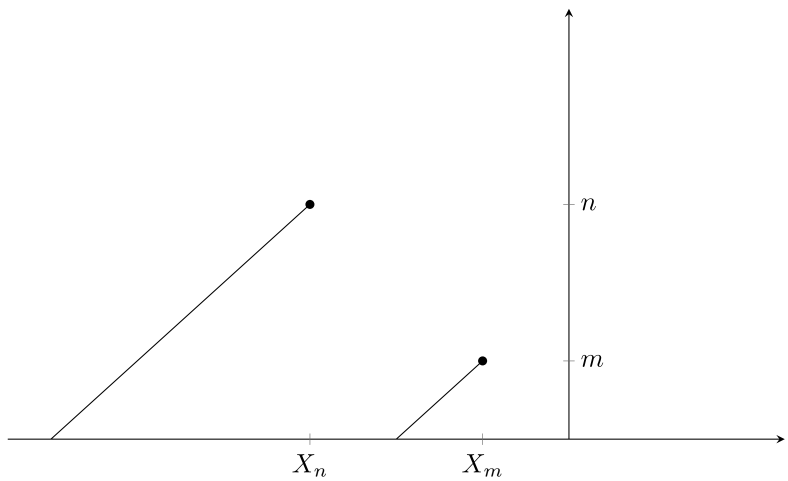
To see that this inequality holds, we make a case distinction. First, fix an arbitrary m, n with $0 < m < n$. Now, we look at the following two cases. In the first case, assume that there is an active path from $(l_1, 0), l_1 \leq 0$, to some point (k_1, n), such that all the active paths, starting at $(l_2, 0), l_2 \leq 0$, which join this point with some (k_2, m), satisfy $l_2 \leq l_1$. The situation is sketched in Figure 1. In this case, we have $X_m < X_n$ and

$$\begin{aligned} X_{m,n} &= \max\{k : \exists \text{ active path from } (l,m) \text{ to } (k,n) \text{ for } l \leq X_m\} - X_m \\ &= X_n - X_m. \end{aligned} \tag{69}$$

Therefore, we obtain

$$\begin{aligned} & X_n \leq X_m + X_{m,n} \\ \Leftrightarrow \quad & X_n \leq X_m + X_n - X_m \\ \Leftrightarrow \quad & 0 \leq 0. \end{aligned} \tag{70}$$

Figure 2: Case 2



and Liggett's subadditivity condition is satisfied. In fact, in this case $X_{m,n}$ is an additive progress.

We get to the second case. Let $(l_1, 0)$, $l_1 \le 0$, be the starting point of an active path to some (k, m) with a maximal possible k. We assume that there is no active path from some point $(l_2, 0)$, $l_2 \le 0$, joined to a point (k_2, n), such that $l_2 < l_1$. Similarly as before, we sketch the situation in Figure 2. We are interested in the position of $X_{m,n}$. An active path from point $(l_2, 0)$ to (k_2, n), must pass through some point (l_3, m) since $m < n$. The active path starting at (l_3, m) will end at the same position as the one starting at $(l_2, 0)$. Therefore, it holds

$$\begin{aligned} X_{m,n} &= \max\{k : \exists \text{ active path from } (l, m) \text{ to } (k, n) \text{ for } l \le X_m\} - X_m \\ &\ge X_n - X_m \end{aligned} \tag{71}$$

Rewriting this, we obtain Liggett's subadditivity condition

$$X_n \le X_m + X_{m,n}. \tag{72}$$

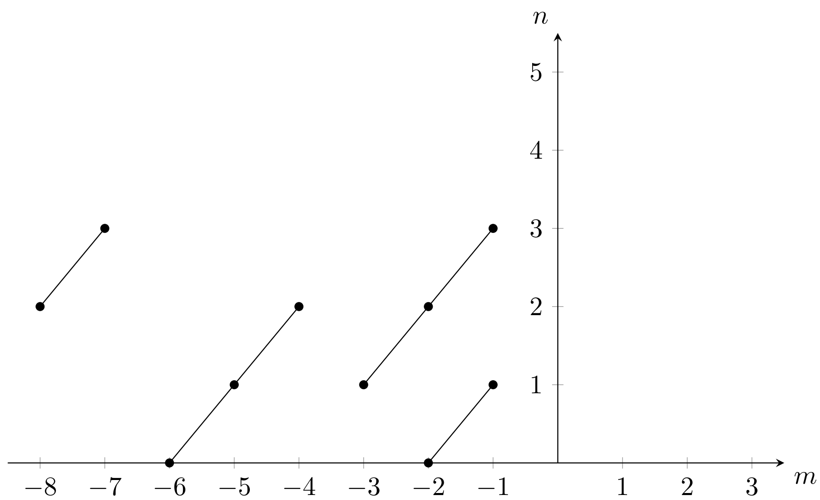
To show that the Kingman's subadditivity condition,

$$X_{l,n} \le X_{l,m} + X_{m,n}, \quad \text{for } 0 \le l < m < n \tag{73}$$

does not have to hold, we will look at the following simple example. We assume that all the open edges are in the following list

$$\begin{aligned} &(-2, 0) \mapsto (-1, 1), \\ &(-3, 1) \mapsto (-2, 2) \mapsto (1, 3), \\ &(-6, 0) \mapsto (-5, 1) \mapsto (-4, 2), \\ &(-8, 2) \mapsto (-7, 3). \end{aligned} \tag{74}$$

Figure 3: Counterexample to Kingman's subadditivity condition



The rest of the edges are assumed to be closed. The situation is sketched in Figure 3, with the open edges illustrated as lines between points. The closed edges are not drawn.

We need to compute all the $X's$ next. For X_1, we need to look at all the open edges starting at the points $(l,0), l \leq 0$ and connecting these to some point $(k,1)$. There are two possibilities here, either we take the open edge $(-2,0) \mapsto (-1,1)$ or $(-6,0) \mapsto (-5,1)$. We want to take the maximum and therefore

$$X_1 = \max\{k : \exists \text{ active path from } (l,0) \text{ to } (k,1) \text{ for some } l \leq 0\} = -1. \quad (75)$$

For X_2, one needs to find an active path starting at $(l,0), l \leq 0$ and ending at some point $(k,2)$. There is only one such path $(-6,0) \mapsto (-5,1) \mapsto (-4,2)$. Hence,

$$X_2 = \max\{k : \exists \text{ active path from } (l,0) \text{ to } (k,2) \text{ for some } l \leq 0\} = -4 \quad (76)$$

There are two possibilities of traveling from a point $(l,1)$, with $l \leq X_1 = -1$, to some point $(k,2)$. We either take the edge $(-3,1) \mapsto (-2,2)$, or $(-5,1) \mapsto (-4,2)$. Since both of these edges satisfy the inequality $l \leq -1$, we take the maximal possible k and obtain,

$$X_{1,2} = \max\{k : \exists \text{ active path from } (l,1) \text{ to } (k,2) \text{ for some } l \leq -1\} + 1 = -1. \quad (77)$$

There is only one possibility of going from a point $(l,1)$ to some point $(k,3)$. That is the active path $(-3,1) \mapsto (-2,2) \mapsto (-1,3)$. Luckily, this path satisfies the condition $l \leq X_m = -1$. So, we get

$$X_{1,3} = \max\{k : \exists \text{ active path from } (l,1) \text{ to } (k,3) \text{ for some } l \leq -1\} + 1 = 0. \quad (78)$$

Finally, we want to find an open edge from some point $(l, 2)$ to $(k, 3)$. There are two possibilities, either $(-8, 2) \mapsto (-7, 3)$ or $(-2, 2) \mapsto (-1, 3)$. The latter path has the maximal m coordinate of the ending point $(k, 3)$. However, we also require the m coordinate of the starting point $(l, 2)$ to satisfy the condition $l \leq X_2 = -4$. Hence, we have to take the first path and obtain,

$$X_{2,3} = \max\{k : \exists \text{ active path from } (l, 2) \text{ to } (k, 3) \text{ for some } l \leq -4\} + 4 = -4. \tag{79}$$

We summarize our results,

$$X_1 = -1, X_2 = -4, X_{1,2} = -1, X_{1,3} = 0 \text{ and } X_{2,3} = -4. \tag{80}$$

Finally, we obtain

$$\begin{aligned} & X_{1,3} > X_{1,2} + X_{2,3} \\ \Leftrightarrow \quad & 0 > -5 \end{aligned} \tag{81}$$

and therefore Kingman's subadditivity condition does not hold. However, Liggett's subadditivity condition is satisfied, for example

$$\begin{aligned} & X_2 \leq X_1 + X_{1,2} \\ \Leftrightarrow \quad & -4 \leq -1. \end{aligned} \tag{82}$$

2.5 Liggett's Theorem in the Terminology of Ergodic Theory

Before we get to the applications of the subadditive ergodic theorem let us first mention few remarks. We can rewrite Liggett's subadditive ergodic theorem in a standard language of ergodic theory. First, we define our setting. Let Ω be a non-empty space, $\mathcal{A}$ a σ-algebra on the space Ω and μ a probability measure. The quadruple $(\Omega, \mathcal{A}, \mu, T)$ then represents a measure space equipped with an $\mathcal{A}/\mathcal{A}$ measurable function $T : \Omega \to \Omega$.

Definition 2.12. Let $\mathcal{B} \subset \mathcal{A}$ be a σ-algebra. We say that the function T is measure preserving with respect to $\mathcal{B}$ if

$$\mu(T^{-n}(A)) = \mu(A), \tag{83}$$

for all $A \in \mathcal{B}$ and all $n \geq 0$.

Next, we state Liggett's subadditive ergodic theorem in the usual language of ergodic theory.

Theorem 2.13. *Let $f^n : \Omega \to \mathbb{R}$ be a sequence of $L^1(\mu)$ functions. We assume the following*

(E1) $f_n(x) \leq f_m(x) + f_{n-m}(T^m(x))$, *for each* $0 < m < n$.

(E2) The function T is measure preserving with respect to the measure $\sigma(f_k : k \geq 1)$.

(E3) For each $k \geq 1$, T^k is measure preserving with respect to the measure $\sigma(f_k(T^{kn}), n \geq 1)$. This condition is equivalent to: for each $k \geq 1$, $\{f_k(T^{kn}), n \geq 1\}$ is a stationary process.

(E4) For all n holds $\mathbb{E}[f_n] \geq -Cn$ for some constant C.

Then it holds that

$$\gamma = \inf_{n \geq 1} \frac{1}{n}\mathbb{E}[f_n] = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[f_n], \tag{84}$$

and

$$\lim_{n\to\infty} \frac{f_n}{n} \text{ exists a.s.} \tag{85}$$

Proof. Full proof of this theorem can be found in [16]. □

To establish a connection with the notation of Liggett, we assume the following:

(C1) $\Omega \subset \mathbb{R}^G$ with $G = \{(m, n), 0 < m < n\}$,

(C2) T is a transformation defined on Ω, such that $T : x \to y$, with $y(m, n) = x(m + 1, n + 1)$,

(C3) $f_n = X_{0,n}$, for all n,

(C4) $f_{n-m}(T^m(x)) = X_{m,n}$ for all $0 < m < n$.

With this notation, we see that the condition

$$f_n(x) \leq f_m(x) + f_{n-m}(T^m(x)), \tag{86}$$

is equivalent to the condition (a) of Liggett,

$$X_n \leq X_m + X_{m,n}. \tag{87}$$

Therefore, we have established the connection between the theory of subadditive processes, examined in this thesis, and the theory of probability measures defined on measurable space Ω, invariant under the transformation T.

3 Applications

In this chapter of this thesis, we will describe five models, all of which use the subadditive ergodic theorem in some way. The common theme for these models is, that we will always use the version by Liggett as this is the one that we previously proved. First, let us quickly name these applications: frog model, branching random walks, interlacement set, products of random matrices and Lyapunov exponents for the Parabolic Anderson model. The first three are all models defined on a d-dimensional lattice $\mathbb{Z}^d$, where the dimension d may vary. In all of them there will be some kind of random walk, either simple or branching, defined on the lattice. The subadditive ergodic theorem will be then used for the proof of a shape theorem, the result about the shape of sites visited by the given random walk. The other two applications use Liggett's theorem in a different way. In both of these, we use the theorem for the proof of existence of the Lyapunov exponent. We begin with arguably the simplest application of the subadditive ergodic theorem, the products of random matrices.

3.1 Products of Random Matrices

The convergence of the product of random matrices had actually been studied even before Kingman's version of the subadditive ergodic theorem was proved. Furstenberg and Kesten proved a result in 1960. The original proof of their theorem of course didn't depend on a subadditive ergodic theorem, but with it already proved their theorem becomes its easy consequence.

Before we mention the Kesten-Furstenberg result, we will simply try to look for a convergence of the following product

$$\Pi_n = A_{i_1} A_{i_2} \cdots A_{i_n} \tag{88}$$

as $n \to \infty$. The matrices A_{i_j}, $1 \le j \le n$ are chosen from the set of invertible matrices $\{A_1, A_2, ..., A_m\}$ and we define $(p_1, ..., p_m)$ to be a probability vector associated to this matrix collection. Then let $i_j = l$ with probability p_l. Now let us demonstrate on a following easy example that the product Π_n from equation (129) does not have a straightforward convergence at all.

Example 3.1. We define the following collection of 2×2 matrices.

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}.$$

The 2×2 matrix A_i will take the values A_1, A_2 with following probabilities

$$\begin{aligned} p_1 &= P(A_i = A_1) = p \\ p_2 &= P(A_i = A_2) = 1 - p. \end{aligned} \tag{89}$$

It is an easy matrix multiplication to calculate that A_2 is the inverse to the matrix A_1, $A_2 = A_1^{-1}$ and thus

$$A_1 A_2 = I = A_2 A_1, \tag{90}$$

where I is the 2×2 identity matrix. Consider the n-matrix product Π_n of equation (88). Let $N \le n$ be the number of instances where $i_j = 1$, $1 \le j \le n$. Conversely, let $M = n - N$ be the number of times $i_j = 2$ for $1 \le j \le n$. We consider three cases: $N < M, N > M$ and $N = M$. In general, a matrix multiplication is not a commutative operation, $AB \ne BA$. In fact, if the matrices are not squares, one of those multiplications is not even defined. However, in the case of square diagonal matrices one can show that matrix multiplication is commutative. As an illustration, we quickly look at the 2-dimensional case. Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

with the entries $a, b, c, d \in \mathbb{R}$. Then

$$AB = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & cd \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = BA.$$

Therefore, we can reorder the terms in the matrix product Π_n as we wish.

If $N > M$, there are more terms A_1 than A_2 in the matrix product Π_n. Since we can reorder terms and we know that $A_1 A_2 = I$, each of the terms A_2 will cancel itself with a term A_1. The matrix product Π_n will then only consist of $N - M$ terms A_1.

$$\Pi_n = A_1^{N-M} = \begin{bmatrix} 2^{N-M} & 0 \\ 0 & 2^{-(N-M)} \end{bmatrix}.$$

If $N < M$, the matrix product Π_n consists of more terms A_2 than A_1. All the terms A_1 will then get canceled with terms A_2, of which $M - N$ will remain in the matrix product Π_n.

$$\Pi_n = A_2^{M-N} = \begin{bmatrix} 2^{-(M-N)} & 0 \\ 0 & 2^{M-N} \end{bmatrix}.$$

Finally if $N = M$, there are equal number of terms A_1, A_2 in Π_n. Then

$$\Pi_n = A_1^N A_2^N = A_1^N A_1^{-N} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that the probability $p > \frac{1}{2}$. Then $p_1 > p_2$, which means that the matrix A_1 is more likely to appear in Π_n than A_2. Therefore,

$$\Pi_n \sim \begin{bmatrix} 2^{n(p_1 - p_2)} & 0 \\ 0 & 2^{-n(p_1 - p_2)} \end{bmatrix},$$

which does not have to converge as $n \to \infty$.

We have seen from the simple example above, that it is not possible to find a general result for the product of the matrices itself. To be able to formulate Furstenberg and Kesten result, we first need to define a norm for the matrices.

Definition 3.2. The matrix norm of a $k \times k$ matrix A is defined as a maximum of the absolute row sums of entries of A, i.e.

$$\|A\| = \max_{i=1,\dots,k} \sum_{j=1}^{k} |A_{ij}|. \tag{91}$$

This norm also has a submultiplicative property.

Lemma 3.3. *If A and B are two $k \times k$ matrices then*

$$\|AB\| \leq \|A\| \|B\|. \tag{92}$$

Proof. The property follows from the following easy computation.

$$\begin{aligned}\|AB\| &= \max_{i=1,\dots,k}\sum_{j=1}^{k}|A_{ij}B_{ij}| \leq \max_{i=1,\dots,k}\sum_{j=1}^{k}|A_{ij}||B_{ij}| \\ &\leq \max_{i=1,\dots,k}\sum_{j=1}^{k}|A_{ij}| \max_{i=1,\dots,k}\sum_{j=1}^{k}|B_{ij}| = \|A\|\|B\|.\end{aligned}$$

Therefore this norm is submultiplicative. □

We already know that the matrix product

$$\Pi_n = A_{i_1}A_{i_2}\cdots A_{i_n}, \tag{93}$$

does not converge in general as $n \to \infty$. Instead, we will take the norm from Definition 3.2 of the product Π_n and consider the following limit

$$\lambda = \lim_{n\to\infty}\frac{1}{n}\log\|\Pi_n\|. \tag{94}$$

We call λ the Lyapunov exponent. That it exists is the result first proved by Furstenberg and Kesten in 1960. However, we will simply prove it by the use of the subadditive ergodic theorem.

Theorem 3.4 (Furstenberg-Kesten)**.** *Let $\{A_i\}_{i\geq 1}$ be a sequence of $k\times k$ independent identically distributed random matrices taking values in GL_k, the group of $k\times k$ invertible matrices. We assume $\mathbb{E}[\log\|A_i\|] < \infty$ for all $i \geq 1$. As before, we define the product*

$$\Pi_n = A_1\cdots A_n. \tag{95}$$

Then

$$\lambda = \lim_{n\to\infty}\frac{1}{n}\log\|\Pi_n\|. \tag{96}$$

exists a.s. and in L_1.

Proof. We would like to apply Liggett's subadditivity condition (a)

$$X_{0,n} \leq X_{0,m} + X_{m,n} \text{ for } 0 < m < n. \tag{97}$$

We want to show that this condition holds for $X_{m,n} = \log\|\Pi_{m,n}\|$, with the notation $\Pi_{m,n} = A_mA_{m+1}\cdots A_n$. By Lemma 3.3, we get the submultiplicative condition

$$\|\Pi_n\| = \|\Pi_{m,n}\Pi_m\| \leq \|\Pi_{m,n}\|\|\Pi_m\|. \tag{98}$$

We know that the logarithm of a product is a sum of logarithms of the factors,

$$\log(xy) = \log(x) + \log(y), \tag{99}$$

provided that x and y are positive. We apply this property to our matrix products and obtain

$$\log \|\Pi_n\| \leq \log \|\Pi_{m,n}\| + \log \|\Pi_m\|. \tag{100}$$

Therefore, we have shown Liggett's subadditivity condition as we wanted. The conditions (b) and (c) of Liggett's subadditive ergodic theorem follow from the assumption that the matrices A_i's are i.i.d. Only the condition (d) remains to be shown. Since $\mathbb{E}[\log \|A_i\|] < \infty$ for all $i \geq 1$ and the matrices A_i are independent by assumptions of the Furstenberg-Kesten theorem, we obtain

$$\mathbb{E}[\log \|A_1\| + ... + \log \|A_n\|] < \infty. \tag{101}$$

By applying the submultiplicative property of the matrix norm as in the equation (98) we get

$$\|\Pi_n\| = \|A_1...A_n\| \leq \|A_1\|...\|A_n\|. \tag{102}$$

Taking the logarithm of this equation

$$\log \|\Pi_n\| \leq \log \|A_i\| + ... + \log \|A_n\|. \tag{103}$$

Since the expectation is monotone, we obtain

$$\mathbb{E}[\log \|\Pi_n\|] \leq \mathbb{E}[\log \|A_i\| + ... + \log \|A_n\|] < \infty. \tag{104}$$

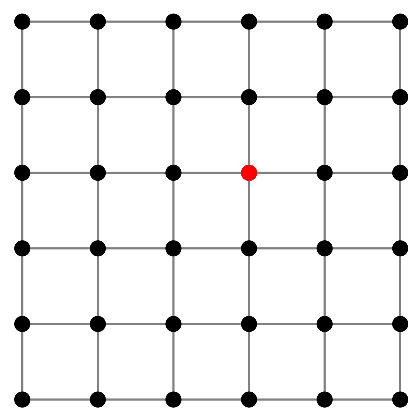
□

3.2 Frog Model

In this section we will define a discrete model on $\mathbb{Z}^d$ in which particles move over the lattice as independent random walks. We can interpret these particles as frogs jumping around, hence the name "frog model". At time 0, we have one frog per each site of which only one is awake and all the others are asleep. Our active frog then starts jumping around, awakening other frogs the moment it jumps on them. The newly awoken frogs then start jumping around as well, performing independent random walks and waking up other sleeping frogs. The length of the jump is exactly 1, meaning each active frog must perform a jump at each discrete time t and its destination must be one of its direct neighbors. Once a frog is awoken it will never fall asleep again. Therefore, the number of jumping particles will never decrease. Only allowed interaction between frogs is an active one waking up a dormant one, two active frogs ignore each other and can share one site freely. We do not care if more than one frog is active at the same site at one time.

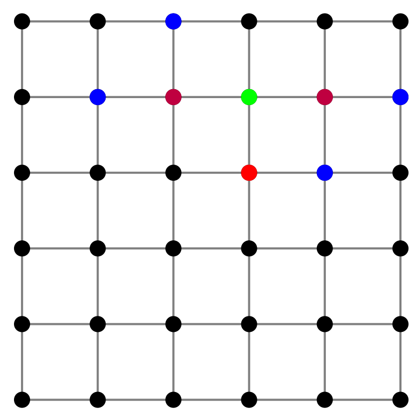
Example 3.5. As an illustration, we sketch the situation in the two-dimensional lattice $\mathbb{Z}^2$. Let $(0, 0)$ be the coordinates of the top-left corner. First we will start with the situation at time $t = 0$. The red point at $(4, 2)$ marks the origin with the active frog, the sleeping frogs are marked with black points, see Figure 4.

Figure 4: Situation at $t = 0$



Next let us sketch few time periods into one picture. At time $t = 1$ the only active frog jumps up from $(4, 2)$ to point $(4, 1)$ and it wakes up the frog sleeping at this position. This point is colored green in our Figure 5. We now have two active frogs ready to jump at time $t = 3$. Lets say the first frog jumps left to $(3, 1)$ and wakes up a third frog, the second frog jumps right to $(5, 1)$ and gives us our fourth active frog (purple points in Figure 5). At time $t = 4$ all four frogs jump again and awaken more frogs (blue points in Figure 5).

Figure 5: First few steps of the frog model



In the next periods, the frogs will continue jumping, coloring more and more points. What we haven't illustrated in the picture above is that the frogs can jump back to the position already visited by an active frog. This will in all probability happen, the only consequence is that this jump will not awaken another frog. However, since the active frogs cannot stop jumping, the number of frogs can never decrease, so the process will never stop.

The main result of this application is the so called shape theorem, a result about the area of sites visited by the jumping frogs until some discrete time point n. Let x be a starting position of our process. We denote by $\xi_{n,x} \subset \mathbb{Z}^d$ the set of sites visited by some active frog between the times $t = 0$ and $t = n$. If the starting point x is equal to the origin 0, we shorten to ξ_n. The area $\xi_{n,x}$ will be described formally later on. However, to be able to state the shape theorem, we only need to define the following

$$\overline{\xi_{n,x}} = \Big\{ y + \Big(-\frac{1}{2}, \frac{1}{2} \Big]^d : y \in \xi_{n,x} \Big\}, \tag{105}$$

where d represents the dimension. Similarly, for $x = 0$ we only write $\overline{\xi_n}$.

Theorem 3.6. *For arbitrary dimension $d \geq 1$ one can find a non-empty convex set $A = A(d) \subset \mathbb{R}^d$ such that, for any $0 < \epsilon < 1$, it holds a.s.*

$$(1-\epsilon)A \subset \frac{\overline{\xi_n}}{n} \subset (1+\epsilon)A \tag{106}$$

for all n large enough.

To be able to prove the shape theorem, we need to introduce more notation and define the frog model in a formal way. As a reminder, we first state the definition of the simple random walk. Let e_i be a d-dimensional standard basis vector with 1 at position i and 0 at all the other positions. Let $\{X_i\}$ be a family of d-dimensional i.i.d. random vectors with

$$X_i = \begin{cases} e_j & \text{with probability } p = \frac{1}{2d} \\ -e_j & \text{with probability } p = \frac{1}{2d} \end{cases} \tag{107}$$

for each $1 \leq j \leq d$. The vector X_i represents the movement at time period i to $i+1$. Since the walk is symmetric, all $2d$ possible directions have the same probability $\frac{1}{2d}$ to be the next destination visited by the random walk.

Then we can define the simple random walk $\{S_n\}$, started at the point x, as

$$S_0 = x, S_n = x + \sum_{i=0}^{n} X_i. \tag{108}$$

We are ready to define this model formally. First, we place the original active frog at point x. Now we define

$$\Big\{\{S_{n,x}\}_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\Big\} \tag{109}$$

to be a family of independent simple random walks with the superscript x denoting the starting point of the random walk. The random walk represents the trajectory of the jumps of the frog placed at time $n = 0$ at the point x. We shorten $S_n = S_{n,0}$ when the random walk starts at the origin 0. Let

$$t(x, z) = \inf\{n : S_{n,x} = z\} \tag{110}$$

be the hitting time for the random walk started at the point x reaches a point z. We will also define a process started at the originally non-active frog y. Let $T(x, z)$ be the shortest possible passage time from the starting point x to the end point z, i.e.

$$T(x, z) = \inf\Big\{\sum_{i=1}^{k} t(x_{i-1}, x_i) : x = x_0, x_1, ..., x_k = z \text{ for some } k\Big\} \tag{111}$$

In the equation above the points $x_0, x_1, ..., x_k \in \mathbb{Z}^d$ represent the sequence of frogs woken up on a way from x to z, k is the length of this sequence. If we take x as the starting point of the process, then x represents the only active frog and $T(x, z)$ is the minimal time it takes to wake up the frog at point z. However, the frog does not have to be woken up by the original frog from x.

Let us take an arbitrary point $y \in \mathbb{Z}^d$ and define $Z_{y,x}(n)$ to be a position of a frog originally placed at point y, at time n. This frog was sleeping at time $n = 0$ as the only active frog at the beginning was placed at site x. $Z_{y,x}(n)$ is a function of discrete time $n \in \mathbb{N}$, the starting positions x, y are fixed. Of course, if $y = x$ then $Z_{y,x}(n)$ represents the trajectory of the original active frog and therefore it holds

$$Z_{x,x}(n) = S_{n,x}. \tag{112}$$

Now let y be one of the sleeping frogs at time $n = 0$, i.e. $y \neq x$. We will look at the position at time n of the frog starting at y, based on the shortest number of steps the original frog at x must take to wake y up. This is represented by $T(x, y)$. If $T(x, y)$ requires more steps than n, the frog at y is still asleep, so its position does not change. If $T(x, y) = n$ the frog at y wakes up precisely at time n. However, it only takes its first jump at time $n + 1$, so it still holds that $Z_{y,x}(n) = y$. This leaves $T(x, y) < n$, the case where the frog has become active before n. In this case, the frog has already woken up at time $T(x, y)$ and jumped precisely $n - T(x, y)$ times. This can be interpreted as a simple random walk, starting at site y, with length $n - T(x, y)$.

We rewrite this neatly as

$$Z_{y,x}(n) = \begin{cases} y & \text{if } T(x, y) \geq n, \\ S_{n-T(x,y),y} & \text{if } T(x, y) < n. \end{cases} \tag{113}$$

The random variables $Z_{y,x}(n)$ are constructed in the same way from random variables $\big\{\{S_{n,x}\}_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\big\}$. Therefore we can define a coupling of processes,

$$\{Z_x, x \in \mathbb{Z}^d\}, \tag{114}$$

with

$$Z_x = \{Z_{y,x} : y \in \mathbb{Z}^d, n \in \mathbb{N}\}. \tag{115}$$

The interpretation behind this is when the frog wakes up from its slumber, it will jump with the same trajectory in all the processes in Z_x.

We are interested in the shape of the area, consisting of points whose frogs, placed here before the beginning of the jumping process, have been awakened by time n. Do not forget that we start the process with only active frog at point x. With all this notation, we can now define the area $\xi_{n,x}$ formally as

$$\xi_{n,x} = \{y \in \mathbb{Z}^d : T(x, y) \leq n\}. \tag{116}$$

Recall that, when the starting point of the process is equal to the origin 0, we shorten the notation to $\xi_n := \xi_{n,0}$. This is the situation which interests us the most and we want to analyze its behavior as time n goes to infinity. Also recall the next definition,

$$\overline{\xi_{n,x}} = \Big\{y + \Big(-\frac{1}{2}, \frac{1}{2}\Big]^d : y \in \xi_{n,x}\Big\}. \tag{117}$$

Again for the starting point at the origin 0, we will use the notation $\overline{\xi_n} := \overline{\xi_{n,0}}$. Now we have everything prepared to prove the main result of this application, the shape theorem for the frog process.

Proof of the Shape Theorem. We will only prove the theorem in $d = 1$. For the proof in higher dimensions, $d \geq 2$, see [9]. We would like to use Liggett's subadditive ergodic theorem. To be able to apply it, we have to show that the conditions (a),(b),(c) and (d) are satisfied for the random variables $X_{m,n}$, $m < n$. In our model we define

$$X_{m,n} = T(mx, nx) \tag{118}$$

for each fixed $x \in \mathbb{Z}$ (since we are only proving the theorem in one dimension). Let us begin with the condition (a). So we need to show

$$T(0, y) \leq T(0, x) + T(x, y) \tag{119}$$

for all $x, y \in \mathbb{Z}$ and all the realizations of the random variables $S_{n,x}$. Recall that $T(x, y)$ is defined as the shortest possible passage time from point x to the point y. If the distance from the origin 0 to point y is shorter than to point x, we reach y before x and therefore

$$T(0, y) \leq T(0, x) \tag{120}$$

holds, which trivially gives us (119) because $T(x, y)$ is non-negative.
Now assume the other case is true, that is the point x is reached sooner than the point y. In this case, we remember that the random variables $T(x, y)$ are constructed using the same family of random variables $S_{n,x}$. Therefore, each of the particles will have the same trajectory as soon as it wakes up from its slumber. We couple the process started from only frog at x awakened, giving us the passage time $T(x, y)$, with the original process started at the origin 0. Of course, the original process can have more frogs awakened at time $T(0, y)$, beside the frog originally placed at point y. $T(0, y) - T(0, x)$ represents the remaining time to reach point y if the process started at the origin 0. We recall the definition

$$T(x, y) = \inf \Big\{ \sum_{i=1}^{k} t(x_{i-1}, x_i) : x = x_0, x_1, ..., x_k = y \text{ for some } k \Big\} \tag{121}$$

and see that this time is longer or equal to $T(0, y) - T(0, x)$ and thus proving (119).
The condition (b) of the Liggett's version of the subadditive ergodic theorem, follows immediately from the definition of the passage time $T(x, y)$. This definition also implicates the stationarity of the sequence

$$\big\{T((n-1)kx, nkx), n \geq 1\big\} \tag{122}$$

for fixed $k \in \mathbb{N}$ and $x \in \mathbb{Z}$. For the condition (c) we also need to prove the ergodicity of the sequence

$$\big\{T((n-1)kn, nkx), n \geq 1\big\}. \tag{123}$$

We will show that this sequence is in fact strongly mixing. From the ergodic theory we know that this is an even stronger property than the ergodicity. As a reminder, we quickly state the definition of strong mixing sequence. Let $Y = \{Y_j\}_{j\in\mathbb{Z}}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, P)$. We consider the Borel σ-algebra generated by some subset these random variables. So, for $-\infty \leq I \leq K \leq \infty$ we define

$$\mathcal{F}_I^K = \sigma(Y_j, I \leq j \leq L, j \in \mathbb{Z}). \tag{124}$$

For arbitrary σ-algebras $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, we define the measure of dependence as

$$\alpha(\mathcal{A}, \mathcal{B}) = |P(A \cap B) - P(A)P(B)|. \tag{125}$$

Now define the strong mixing or α coefficient

$$\alpha(s) = \sup_{t\in\mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{s+t}^\infty) \tag{126}$$

The sequence Y is called strongly mixing if

$$\alpha(s) \to 0 \text{ as } s \to \infty. \tag{127}$$

Taking $\mathcal{A} = \mathcal{F}_{-\infty}^t$ and $\mathcal{B} = \mathcal{F}_{s+t}^\infty$ in equation (125), we can rewrite the strong mixing condition (126) as

$$\alpha(s) = \sup_{t\in\mathbb{Z}} \big\{|P(A \cap B) - P(A)P(B)|, A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{s+t}^\infty\big\} \tag{128}$$

If the events A, B are independent, then $P(A \cap B) = P(A)P(B)$ and consequently $\alpha(s) = 0$ for all s. We want the events

$$\big\{T(n_1kx, (n_1+1)kx) = a\big\}, \big\{T(n_2kx, (n_2+1)kx) = b\big\} \tag{129}$$

to be independent of each other. For this we only need to assume the following inequality

$$a + b < \big|(n_1 - n_2)kx\big|. \tag{130}$$

Time a represents the minimum passage time between n_1kx to $(n_1+1)kx$, b is the passage time between n_2kx and $(n_2+1)kx$. We consider the sum of these passage times $a + b$. If this sum is smaller than $|(n_1 - n_2)kx|$, the distance between the two endpoints, then these two paths do not interact and the events in (129) are independent. The sequence $\{T((n-1)kn, nkx), n \geq 1\}$ is strongly mixing and therefore also ergodic.

For the last condition, we define τ to be the first return time to the origin of the simple random walk. For this time we have

$$P(\tau > t) \leq C\frac{1}{\sqrt{t}} \tag{131}$$

for some constant C. Combining this with the fact that in a random time with exponential time one can find at least three active frog jumping independently, we get $\mathbb{E}[T(0,1)] < \infty$. Therefore, we get

$$\frac{T(0,n)}{n} \to \gamma \text{ a.s.} \tag{132}$$

and with this we have proved the shape theorem for the frog model in one dimension, with $A = \Big[-\frac{1}{\gamma}, \frac{1}{\gamma} \Big]$.

$\square$

Remark 3.7. The frog model is an example of Richardson's model. Let us again consider the model in $\mathbb{Z}^d$. The Richardson model mimics either the spread of a population or an infection. At time $n = 0$ the only occupied/infected site is the origin, for simplicity lets take origin at point 0. In our frog model all sites were occupied by frogs at time 0, however only the one at the origin was awake. One could interpret the frog model as a contact transmission of a disease by jumping. Let p be the probability that the site will get occupied/infected by one of its already occupied/infected neighbors in one time step. We define $k < 2d$ as the number of occupied neighbors. The Richardson's model can be defined by the following two transitions

$$\begin{aligned} &x \in \xi_n \Rightarrow x \in \xi_{n+1} \\ &x \notin \xi_n \Rightarrow P(x \notin \xi_{n+1} | \xi_n) = (1-p)^k \end{aligned} \tag{133}$$

The first line means that once a site has been occupied it will also remain that way. In case of an infection it means that the disease is not lethal and there are no deaths. The second line handles a site x which is not occupied/infected at time n. The probability that it will not get occupied/infected by its already occupied/infected neighbor is $(1 - p)$. The transmission from each site is independent, therefore the probability of not getting occupied/infected $(1-p)^k$. As in the frog model, one could also come up with the same shape theorem for the spread of the process.

3.3 Branching Walks in Random Environment

In the following application, we will look at random walks in random environment on a $d-$dimensional lattice, with $d \geq 1$. The process starts at some fixed site. Then at each discrete time n, each of particles starts branching, i.e. it is substituted by more than one particle and those are then independently from each other placed at some of the neighboring sites of the lattice $\mathbb{Z}^d$. The location of the particle is the only thing influencing its offspring generation. The collection of these rules will be called the environment. The environment is randomly chosen at the beginning of the process and kept fixed during the subsequent discrete time steps. We will distinguish between the recurrent and transient case, and we will culminate this application by using Liggett's version of the subadditive ergodic theorem to prove a shape theorem for the recurrent case.

Let us describe this model properly. We define $\mathcal{U}$ to be a finite set in $\mathbb{Z}^d$ satisfying $\pm e_i \in \mathcal{U}$, for all $i = 1, .., d$, with e_i's denoting the coordinate vectors of the space $\mathbb{Z}^d$. Further, we define

$$\mathcal{V} = \Big\{ v = (v_x, x \in \mathcal{U}) : v_x \in \mathbb{N}, \sum_{x \in \mathcal{U}} v_x \geq 1 \Big\} \tag{134}$$

where the natural numbers $\mathbb{N}$ contain the number 0. For $v \in \mathcal{V}$, we will use the notation

$$|v| = \sum_{x \in \mathcal{U}} v_x. \tag{135}$$

It holds that $|v| \geq 1$ for all $v \in \mathcal{V}$. We define

$$\mathcal{M} = \Big\{\omega = (\omega(v), v \in \mathcal{V}) : \omega(v) \geq 0 \text{ for all } v \in \mathcal{V}, \sum_{x \in \mathcal{U}} \omega(v) = 1\Big\} \tag{136}$$

to be the set of all probability measures ω on $\mathcal{V}$.

We take a probability measure $\mathcal{Q}$ on $\mathcal{M}$. Then for each $x \in \mathbb{Z}^d$, we independently choose a random element $\omega_x \in \mathcal{M}$ according to the probability measure $\mathcal{Q}$. The environment is defined as the collection $\Omega = (\omega_x, x \in \mathbb{Z}^d)$.

For a particular environment Ω, one can describe the evolution of the process. First, one starts at some fixed site of $\mathbb{Z}^d$. At each time $n \geq 1$ the particles branch independently in a following way. For a particle at $x \in \mathbb{Z}^d$, we choose a random element $v = (v_y, y \in \mathcal{U})$ with probability $\omega_x(v)$. The particle at x is then substituted by v_y particles at site $x + y$ for all $y \in \mathcal{U}$. We do not assume that the immediate descendants of particle jump independently. We look at a simple example in the two dimensional case. If a particle at site x generated five offspring, then with probability 1 at least two of the offspring have to jump in the same direction.

The probability and expectation with respect to the environment Ω will be denoted by $\mathcal{P}$ and $\mathbb{E}$, respectively. Since the environment Ω is i.i.d., then

$$\mathcal{P} = \bigotimes_{x \in \mathbb{Z}^d} \mathcal{Q}_x, \tag{137}$$

where $\mathcal{Q}_x$ are copies of $\mathcal{Q}$. In a fixed environment Ω we will denote the so-called "quenched" probability and expected value of the process starting at site x by $\mathcal{P}^x_\omega$ and $\mathbb{E}^x_\omega$, respectively. Finally, we will use the notation $\boldsymbol{P}^x() = \mathbb{E}[\mathcal{P}^x_\omega]$ for the annealed law of the branching random walk in random environment. With $\boldsymbol{E}$ we will denote its expectation.

During this entire application, we will assume that the following two conditions are satisfied. First condition ensures that there is a particle, which will produce at least two offspring and therefore, the branching random walk will never become a simple random walk, where each particle produces exactly one offspring.

Condition 3.8.

$$\mathcal{Q}\Big(\omega : \text{ there exist } v \in \mathcal{V} \text{ s.t. } \omega(v) > 0, |v| \geq 2\Big) > 0 \tag{138}$$

The second condition is often called a natural ellipticity condition and thanks to it we know that our branching random walk is really d-dimensional.

Condition 3.9.

$$\mathcal{Q}\Big(\omega : \sum_{v : v_e \geq 1} \omega(v) > 0 \text{ for any } e \in \{\pm e_1, ..., \pm e_d\}\Big) = 1 \tag{139}$$

Sometimes we will even assume the stronger uniform ellipticity condition.

Condition 3.10. For some $\epsilon_0 > 0$ it holds,

$$\mathcal{Q}\Big(\omega : \sum_{v: v_e \geq 1} \omega(v) \geq \epsilon_0 \text{ for any } e \in \{\pm e_1, ..., \pm e_d\}\Big) = 1. \tag{140}$$

We want to prove a shape theorem for the model, i.e. a result for the set of sites visited by a branching random walk in the first n steps. To be able to do this, we first have to construct the process formally. Let x_0 be the starting point of the process. We define $\eta_n^{x_0}(x)$ to be the number of particles at site $x \in \mathbb{Z}^d$ at time $n \geq 0$. Fix a particular environment Ω. Then for all $x \in \mathbb{Z}^d$, consider a family of i.i.d random variables of $\mathcal{V}$; $v^{x,i}(n)$, for $i = 1, 2, 3, ...$ and $n = 0, 1, 2, ...$, satisfying

$$\mathcal{P}\big(v^{x,i}(n) = v\big) = \omega_x(v). \tag{141}$$

Next we construct a collection of branching random walks, indexed by their starting point, using the same realization of $v^{x,i}(n), x \in \mathbb{Z}^d$, with $i = 1, 2, 3, ...$ and $n = 0, 1, 2, ...$

Let us consider the process starting at the origin 0. At time $n = 0$, we get

$$\eta_0^0(y) = \begin{cases} 1 & \text{for } y = 0 \\ 0 & \text{for } y \neq 0 \end{cases} \tag{142}$$

Inductively, we define for $y \in \mathbb{Z}^d$ and $n \geq 1$

$$\eta_{n+1}^0(y) = \sum_{x: y \in \mathcal{U}+x} \sum_{i=1}^{\eta_n^0(x)} v_{y-x}^{x,i}(n). \tag{143}$$

Assuming that the process started at the origin 0, we define $T(0, y)$ to be the first hitting time, at which a site y is first entered by some particle. If there is a particle which hits site y at some time n, we define

$$T(0, y) = \inf_{n \geq 0} \eta_n^0(y) \geq 1. \tag{144}$$

If no particle ever reaches the site y, we write

$$T(0, y) = \infty. \tag{145}$$

Now, assume that the process starts at site $z \neq 0$. Similarly, as in the case above, it holds for $n = 0$

$$\eta_0^z(y) = \begin{cases} 1 & \text{for } y = z \\ 0 & \text{for } y \neq z \end{cases} \tag{146}$$

We will distinguish two cases. The first when the branching random walk started at site 0 never reaches the site z . In this case, $T(0,z)=\infty$. Therefore, the process started at z is independent from the process started at the origin 0 and

$$\eta^z_{n+1}(y)=\sum_{x:y\in\mathcal{U}+x}\sum_{i=1}^{\eta^z_n(x)}v^{x,i}_{y-x}(n). \tag{147}$$

In the second case, one hits the site y in a finite time m_0 if the process started from 0. The hitting time $T(0,z)=m_0<\infty$. Therefore, we have a shift from time n to time $n+m_0$ and obtain

$$\eta^z_{n+1}(y)=\sum_{x:y\in\mathcal{U}+x}\sum_{i=1}^{\eta^z_n(x)}v^{x,i}_{y-x}(n+m_0) \tag{148}$$

We also define the hitting time, when a particle first reaches the site y, of the process started from the site z as

$$T(z,y)=\inf_{n\geq 0}\{\eta^z_n(y)\geq 1\}. \tag{149}$$

The hitting times fulfill the subadditivity condition (a). of Liggett's subadditive ergodic theorem.

Lemma 3.11. *For any $y,z\in\mathbb{Z}^d$, with $0<z<y$ and for all realizations of $v^{u,i}(n),u\in\mathbb{Z}^d,i=1,2,3,...,n=0,1,2,...$ it holds that*

$$T(0,y)\leq T(0,z)+T(z,y). \tag{150}$$

Proof. We consider two cases, $T(0,z)=\infty$ and $T(0,z)<\infty$.

If $T(0,z)=\infty$, the condition (150) trivially holds. For the other case we assume $m_0=T(0,z)<\infty$. For an arbitrary $x\in\mathbb{Z}^d$, one can show by induction that

$$\eta^z_n(x)\leq\eta^0_{n+m_0}(x) \tag{151}$$

for all $n\geq 0$. For $n=0$ and for $x\neq z$ we get

$$0=\eta^z_0(x)\leq\eta^0_{m_0}(x), \tag{152}$$

which is obviously true as the number of particles at any site cannot be negative. For $x=z$, one has

$$1=\eta^z_0(z)\leq\eta^0_{m_0}(z). \tag{153}$$

This inequality holds, since we assume that $T(0,z)=\inf_{n\geq 0}\{\eta^0_n(z)\geq 1\}=m_0$, the first hitting time a particle enters z if it started at the origin 0. Therefore, at least this one particle will be present at the site z at time m_0, proving $\eta^0_{m_0}(z)\geq 1$.

□

To continue with the preparation of the shape theorem, we now want to define the area of sites visited by the branching random walk by the time n. We will denote the start of the branching random walk by x.

$$B_n^x = \{y \in \mathbb{Z}^d : \text{ there is } m \leq n \text{ such that } \eta_m^x(y) \geq 1\} \tag{154}$$

This definition can also be rewritten with help of the hitting time $T(x, y)$, as

$$B_n^x = \{y : T(x, y) \leq n\}. \tag{155}$$

We define $\overline{B}_n^x$ to be a set of sites, at which at least one particle is present precisely at time n,

$$\overline{B}_n^x = \{y \in \mathbb{Z}^d : \eta_n^x(y) \geq 1\}. \tag{156}$$

Finally, by $\tilde{B}_n^x$ we will denote the set of sites, which starting with the time n will always contain a minimum of one particle,

$$\tilde{B}_n^x = \{y \in \mathbb{Z}^d : \eta_m^x(y) \geq 1 \text{ for all } m \geq n\} \tag{157}$$

Naturally, the following inclusion holds for all x and for all n,

$$\tilde{B}_n^x \subset \overline{B}_n^x \subset B_n^x. \tag{158}$$

If x is a d-dimensional vector, $x = (x_{1,...,x_d}) \in \mathbb{Z}^d$, then the 1-norm is defined in the usual way as

$$\|x\|_1 = |x_1| + ... + |x_d|. \tag{159}$$

With this we can define the aperiodicity condition.

Condition 3.12 (Aperiodicity)**.** There exist $x \in \mathcal{U}$, $v \in \mathcal{V}$ with $\|x\|_1$ even and $v_x \leq 1$ such that

$$Q\big(\omega \in \mathcal{M} : \omega(v) > 0\big) > 0. \tag{160}$$

This condition ensures that the process starting at the origin, does not live on even sites at even times, and conversely on odd sites at odd times. Instead, it is aperiodic and hence the name the aperiodicity condition.
For an arbitrary set $A \subset \mathbb{Z}^d$, we define new set $F(A)$ as the fill in of the spaces between the points in the set A:

$$F(A) = \Big\{y + \Big(-\frac{1}{2}, \frac{1}{2}\Big]^d : y \in A\Big\}. \tag{161}$$

The set $F(A)$ is not contained in $\mathbb{Z}^d$. Obviously, it holds $F(A) \in \mathbb{R}^d$.

We will now introduce the notions of recurrence and transience for the branching random walk.

Definition 3.13. We fix a particular realization of the random environment Ω. The branching walk is recurrent if

$$\mathcal{P}_\omega^0(\text{the origin is visited infinitely often}) = 1 \tag{162}$$

The branching random walk is otherwise called transient.

However, since the conditional probability distribution of the future site of the branching random walk depends only on the present state, the branching random walk satisfies the Markov property. The definition of the recurrence is therefore equivalent to

$$\mathcal{P}_{\omega}^{0}(\text{the origin is visited at least once}) = 1. \tag{163}$$

We define a notation

$$K_n = [-n, n]^d \tag{164}$$

and state the next lemma, which ensures that the set of sited visited by the branching random walk grows at least linearly.

Lemma 3.14. *We assume $d \geq 2$, the recurrence of the branching random walk in random environment and that the condition 3.10 is satisfied. Then the following holds:*

1. *There exist $\delta_0, \theta_0 > 0$ such that*

$$\boldsymbol{P}(K_{\delta_0} n \subset B_n^0) \geq 1 - exp\{-\theta_0 \log^d(n)\} \tag{165}$$

for all large enough n.

2. *Additionally, assume that the aperiodicity condition 3.12 is satisfied. Then there exist $\delta_1, \theta_1 > 0$ such that*

$$\boldsymbol{P}(K_{\delta_1} n \subset \tilde{B}_n^0) \geq 1 - exp\{-\theta_1 \log^d(n)\} \tag{166}$$

for all sufficiently large n.

Proof of Lemma. Since the proof of this lemma does not depend on the subadditive ergodic theorem, we will not state it here. The proof can be found in [13], Lemma 5.1., pages 103-105. □

We are now ready to state the shape theorem for the branching random walk. For simplicity we will only deal with the recurrent case.

Theorem 3.15. *We assume that the branching random walk in random environment is recurrent and the uniform ellipticity condition 3.10 holds. Then there is a deterministic compact set $B \subset \mathbb{R}^d$, whose interior contains the point 0, such that P-a.s., it holds that for any $0 < \epsilon < 1$,*

$$(1-\epsilon)B \subset \frac{F\big(B_n^0\big)}{n} \subset (1+\epsilon)B \tag{167}$$

for sufficiently large n.

Moreover, if Condition 3.12 is satisfied, then the same result also holds for the sets of sites $\overline{B}_n^0$ and $\tilde{B}_n^x$. So for $\overline{B}_n^0$, we have

$$(1-\epsilon)B \subset \frac{F\big(\overline{B}_n^0\big)}{n} \subset (1+\epsilon)B. \tag{168}$$

Similarly for the other set.

The way this shape theorem is proved differs depending on the dimension of the branching random walk. If $d = 1$, one cannot guarantee that

$$\mathbb{E}[T(0,1)] < \infty \tag{169}$$

and the subadditive ergodic theorem is not applicable in this case. Therefore, we will not prove the one-dimensional result in this thesis. The proof can be found in [13], page 109.

For $d \geq 2$, the proof of the theorem above goes via the Liggett's version of the subadditive ergodic theorem. To be able to apply this theorem, we first have to construct the process and the collection of the random variables which will fulfill the four assumptions in Liggett's subadditive ergodic theorem.

Proof of the Theorem in $d \geq 2$. Take an arbitrary $x_0 \in \mathbb{Z}^d \backslash \{0\}$ and define a family of random variables $\{Y^{x_0}(m,n)\}_{0 \leq m < n}$ as

$$Y^{x_0}(m,n) = T(mx_0, nx_0), \text{ for } 0 \leq m < n. \tag{170}$$

We want to show that these random variables fulfill the four assumptions of the Liggett's theorem. For condition (a), we need to show

$$Y^{x_0}(0,n) \leq Y^{x_0}(0,m) + Y^{x_0}(m,n) \quad \text{for all } 0 \leq m < n. \tag{171}$$

By Lemma 3.11, it holds that

$$T(0,z) \leq T(0,y) + T(y,z). \tag{172}$$

Taking $z = nx_0$ and $y = mx_0$, we obtain (170). The assumption (b) follows directly from the construction of the random variables $T(\cdot,\cdot)$.

We would like to prove the condition (c) next. Unfortunately, the sequence of random variables

$$\Big(Y^{x_0}\big((n-1),n\big)\Big), \quad n = 1,2,3,... \tag{173}$$

isn't necessarily stationary, even though it is identically distributed by its construction. For example, assuming that the branching random walk is conditioned on an environment Ω, the random variables $T(0,x_0)$ and $T(0,2x_0)$ are independent. This is because, if we use the notation $T(0,x_0) = r$, then the first hitting time $T(x_0,2x_0)$ depends on $v^{x,i}(n)$ for $n \geq r$. However, the hitting times $T(x_0,2x_0)$ and $T(2x_0,3x_0)$ do not have to be independent. Fortunately, the following lemma shows us that the sequence of random variables (173) satisfies the law of large numbers.

Lemma 3.16. *Define $\beta_{x_0} = \boldsymbol{E}[Y^{x_0}(0,1)]$. Then for any $\epsilon > 0$ there is θ_2, which depend on the chosen ϵ, such that it holds for all n*

$$\boldsymbol{P}\left(|\frac{1}{n}\sum_{i=1}^{n} Y^{x_0}(i-1,i) - \beta_{x_0}| > \epsilon\right) \leq exp\{-\theta_2 \log_d(n)\}. \tag{174}$$

Particularly, it holds that

$$\frac{1}{n}\sum_{i=1}^{n} Y^{x_0}(i-1,i) \to \beta_{x_0}, \quad \boldsymbol{P}\text{-a.s. and in } L^p, p \geq 1. \tag{175}$$

Proof of Lemma. Can be found in [13], pages 106-107. □

We would like to substitute the strong law of the large numbers above, for the condition (c) of Liggett's theorem. We will try to show that this is, in fact, possible in the following proposition, in which we will use the notation from the Liggett's theorem.

Proposition 3.17. *Liggett's subadditive ergodic theorem still holds if we substitute the assumption (c),*

- *for each $k \geq 1$*
$$\left(X_{nk,(n+1)k}, n \geq 1\right) \tag{176}$$
is a stationary process.

by the following

- *for all $k \geq 1$ there exists a constant γ_k such that the sequence of random variables*
$$\left(\frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj}\right)_{n\geq 0} \tag{177}$$
converges a.s. and in L_1 to γ_k.

Proof. When reading the proof of Liggett's subadditive ergodic theorem 2.10, one notices that the condition (c) of the said theorem is actually used in the proof only once. That is in the part (P2) of the proof, in the equation (37), which uses the stationarity and the Birkhoff's theorem to show that

$$\frac{1}{n}\sum_{j=1}^{n} X_{k(j-1),kj} \tag{178}$$

converges a.s and in L_1 to a random variable with mean γ_k. We cannot assume stationarity here, but we do not need to as the condition (177) is exactly (37). Therefore, the proposition is proved.

□

The condition (177) holds for the sequence of random variables $Y(\cdot,\cdot)$ due to Lemma 3.16. Also,

$$Y^{x_0}(mk, nk) = Y^{kx_0}(m, n). \tag{179}$$

To satisfy the final condition (d) of the Liggett's theorem we have to show that

$$\mathbb{E}[Y(0,1)] < \infty. \tag{180}$$

The condition holds due to the following proposition.

Proposition 3.18. *Let $d \geq 1$ and assume that the branching random walk in random environment is recurrent. For an arbitrary $x_0 \in \mathbb{Z}^d$ we can find θ, depending on the x_0 and the probability measure $\mathcal{Q}$ on the set $\mathcal{M}$, such that*

$$\boldsymbol{P}(T(0,x_0) > n) \leq exp\{-\theta \log^d(n)\} \tag{181}$$

for all sufficiently large n.
Define $\mathcal{G} \subset \mathcal{M}$ to be the set of ω's without branching, that means

$$\mathcal{G} = \left\{ \omega \in \mathcal{M} : \sum_{v \in \mathcal{V}:|v|=1} \omega(v) = 1 \right\}. \tag{182}$$

So if at the site x the environment belongs to $\mathcal{G}$, then the particles only jump without creating new particles. For an $\omega \in \mathcal{Q}$, we define the drift as

$$\Delta_\omega = \sum_{x \in \mathcal{U}} x\omega(\delta_x). \tag{183}$$

Moreover, we suppose that $\mathcal{Q}(\mathcal{G}) > 0$ and the origin belongs to the interior of the convex hull of

$$\{\Delta_\omega : \omega \in \mathcal{G} \cap supp(\mathcal{Q})\}. \tag{184}$$

Then for an arbitrary $x_0 \in \mathbb{Z}^d$, there exists θ_2, depending on x_0 and the probability measure $\mathcal{Q}$, such that

$$\boldsymbol{P}(T(0,x_0) < n) \geq exp\{-\theta_2 \log^d(n)\} \tag{185}$$

for all large enough n.

Proof of the Proposition. Can be found in [13], page 96. ☐

The proposition then implies

$$\boldsymbol{P}\Big(T(0,x_0) < \infty \text{ for all x}\Big) = 1 \Longrightarrow \mathbb{E}[T(0,x)] < \infty \text{ for all } x \tag{186}$$

in $d \geq 2$. In dimension $d = 1$, the constant θ_2 can be less than 1 and

$$\mathbb{E}[T(0,1)] \to \infty. \tag{187}$$

With the (modified) assumption of Liggett's theorem satisfied, we can conclude that for an arbitrary $x \in \mathbb{Z}^d$, there exists a number $\mu(x)$ such that

$$\frac{T(0,nx)}{n} \to \mu(x) \quad \boldsymbol{P}\text{-a.s. as } n \to \infty. \tag{188}$$

For the rest of the proof, we will only mention the main steps of the rest of the proof. First, for an arbitrary $x \in \mathbb{Z}^d$ and $a \in \mathbb{N}$, we have

$$\mu(ax) = a\mu(x). \tag{189}$$

Next, we want to extend $\mu(x)$ on $x \in \mathbb{R}^d$. To do this, we first extend to the rational coefficients, we take $x \in \mathbb{Q}^d$ and $ax \in \mathbb{Z}^d$, with $a \in \mathbb{N}$, and we get

$$\mu(x) = \frac{\mu(ax)}{a}. \tag{190}$$

Finally, we use the subadditivity and Lemma 3.14, part 1 to extend to the entire $\mathbb{R}^d$. The limiting shape B can be defined by

$$B = \Big\{x \in \mathbb{R}^d : \mu(x) \leq 1\Big\}. \tag{191}$$

The shape B is convex, since the subadditivity conditions is inherited from the random variables $T(\cdot,\cdot)$ to μ, and therefore

$$\mu(x+y) \leq \mu(x) + \mu(y). \tag{192}$$

The shape B does not have to be symmetric, since in general

$$\mu(x) \neq \mu(-x). \tag{193}$$

We cover B and a sufficiently large annulus(a region bounded by two concentric circles) of B by balls of a sufficiently small radius δ'. Again, we use Lemma 3.14, part 1. This completes the proof for B_n^0. Now we recall the relation

$$\tilde{B}_n^x \subset \overline{B}_n^x \subset B_n^x \tag{194}$$

to see that all we have to do to complete the proof in $d \geq 2$ is to show that for an arbitrary $\epsilon >$, it holds that

$$(1-\epsilon)B \in \mathcal{F}(\tilde{B}_n^0) \tag{195}$$

for all sufficiently large n. However, this follows directly from Lemma 3.14, part 2 and the shape result for B_n^0. □

3.4 Lyapunov Exponents for the Parabolic Anderson Model

We want to define a probability space on $(\Omega, \mathcal{F}, P)$. For this, we take a collection of independent, one-dimensional Brownian motions

$$\big\{B_x(t), x \in \mathbb{Z}^d, t \geq 0\big\}. \tag{196}$$

Now let us choose a fixed constant $k > 0$ and define

$$\{X(t), t \geq 0, \mathcal{F}_t, P_x\} \tag{197}$$

to be a symmetric random walk on $\mathbb{Z}^d$ with jump rate equal to the constant k. In the following, we will denote $X(t) = X_t$. Moreover, we assume the independence between this random walk and the field $\{B_x(t), x \in \mathbb{Z}^d, t \geq 0\}$. Then under the probability measure P_x, the random walk $\{X_t, t \geq 0\}$ is the pure jump Markov process on $\mathbb{Z}^d$ with origin at the point 0 with generator $k\Delta$, where Δ is the discrete Laplace operator defined by

$$\Delta f(x) = \frac{1}{2d} \sum_{|y-x|=1} \big(f(y) - f(x)\big). \tag{198}$$

We will define the following stochastic equations,

$$\begin{aligned} u(t,x) =& u_0(x) + k\int_0^t \Delta u(s,x)ds + \int_0^t u(s,x)\delta B_x(s) \\ u(0,x) &= u_0(x) \end{aligned} \tag{199}$$

where δ is the Stratonovich differential.

Remark 3.19 (Stratonovich Integral)**.** In the financial mathematics course we have learned about the Ito-integral. Its disadvantage is the difficult manipulation with it under the change of variables. Stratonovich integral is another type of stochastic integral, where the change of variables is much easier to handle. The integral is in some way similar to the classical Riemann integral, as it can also be defined as a limit of the Riemann sums.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space equipped with filtration. We take a Brownian motion B_t and a semi-martingale X_t which we take as adapted to the filtration of the Brownian motion B_t. Moreover, we take a partition of the interval $[0,T]$, $0 < t_0 < t_1 < ... < t_n = T$. Let $|\alpha| = \max_i t_i - t_{i-1}$.
We can now define the Stratonovich integral

$$\int_0^T X_s dB_s = \lim_{|\alpha|\to 0} \sum_{i=0}^{n-1} \frac{X_{t_{i+1}} + X_{t_i}}{2}(B_{t_{i+1}} - B_{t_i}). \tag{200}$$

If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then

$$\int_0^T f'(B_t)dB_t = f(B_T) - f(B_0). \tag{201}$$

We need this formula in stochastic equations (199) in two dimension so let us quickly mention the formula here. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function then the following holds

$$\int_0^T \frac{\delta f}{\delta B}(B_t,t)dB_t + \int_0^T \frac{\delta f}{\delta t}(B_t,t)dt = f(B_T,T) - f(B_0,0). \tag{202}$$

Carmona and Molchanov proved in 1994, see [14], that u_0 is bounded and the solution can be computed with the help of Feynman-Kac formula as

$$u(t,x) = \mathbb{E}_x\big[u_0(X_t)e^{\int_0^t dB_{X_s}(t-s)}\big]. \tag{203}$$

When $k = 0$ the solution simplifies to

$$u(t,x) = u_0(x)e^{-B_X(t)}. \tag{204}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \log(u(t,x)) = 0. \tag{205}$$

If $k > 0$ the situation is more complicated. By Carmona-Molchanov result, when u_0 has compact support, then

$$\lim_{t\to\infty} \frac{1}{t} \log\big(u(t,x)\big) = \lambda(k) \text{ P-a.s.} \tag{206}$$

We call the positive constant $\lambda(k)$ the Lyapunov exponent. It was also shown that there are constants c_1 and c_2 such that

$$\frac{c_1}{\log \frac{1}{k}} \leq \lambda(k) \leq \frac{c_2}{\log \frac{1}{k}}. \tag{207}$$

We want to generalize (206) to the case where u_0 isn't necessarily required to have compact support. Instead, we only want u_0 to be bounded non-negative functions that is different from 0. This requires more than our subadditivity theorem, but since we are only interested in its applications we only prove a result which used the Liggett's version of the theorem.

Theorem 3.20. *There exist a $\lambda_1(k)$ such that for all $x \in \mathbb{Z}^d$ holds*

$$\lambda_1(k) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}_x\big[e^{\int_0^t dB_{X_s}(s)}\big], \ \textit{P-a.s.} \tag{208}$$

Proof. We can take x equal to the point 0 by using the translation invariance. Next define a function

$$v(t,y) = \mathbb{E}_0\big[e^{\int_0^t dB_{X_s}(s)} \mathbf{1}_y(X_t)\big], \tag{209}$$

where $\mathbf{1}_y$ is the indicator function, i.e.

$$\mathbf{1}_y(X_t) = \begin{cases} 1 & \text{if } y = X_t \\ 0 & \text{if } y \neq X_t \end{cases} \tag{210}$$

We will denote $\mathbb{E}[\cdot] = \mathbb{E}_0[\cdot]$. Also define

$$Z(t) = \max_{y\in\mathbb{Z}} v(t,y). \tag{211}$$

We get

$$Z(t+u) \geq Z(t)Z(u), \tag{212}$$

so $Z(\cdot)$ is a supermultiplicative process. Now take logarithm of the functional $Z(t)$ and use its property that for all x,y holds $\log(xy) = \log(x) + \log(y)$. Then we obtain the following inequality

$$\log Z(t+u) \geq logZ(t) + logZ(u). \tag{213}$$

Therefore, the process $\log((Z(t)))$ is superadditive. To get a subadditive process, we simply take $-\log(Z(t))$. Now we can use the subadditive ergodic theorem for this process. Therefore,

$$\lim_{t\to\infty}\frac{\log(Z(t))}{t}=\lambda_1(k),\ \text{P-a.s.} \tag{214}$$

for some constant, which we will denote $\lambda_1(k)$. Next we compute

$$\begin{aligned}Z(t)&=\max_{y\in\mathbb{Z}}v(t,y)\\&=\max_{y\in\mathbb{Z}}\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_y(X_t)]\\&\geq\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_0(X_t)],\end{aligned} \tag{215}$$

where the inequality holds since $Z(t)$ is the maximum. Then by Carmona-Molchanov, $\lambda_1(k)\geq\lambda(k)\geq 0$.

$$\log\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}]\geq\log(Z(t)) \tag{216}$$

Taking lim inf and dividing by t, the equation (216) transforms into

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}]\geq\lambda_1(k),\text{P-a.s.} \tag{217}$$

It also holds that

$$\begin{aligned}\frac{1}{t}\log\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}]&=\frac{1}{t}\log(\mathbb{E}[\sum_{|x|<t^2}e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_x(X_t)])\\&+\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(t^2,\infty)}(|X_t|)]\\&\leq\frac{1}{t}(\log(ct^2e^{Z(t)}+\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(t^2,\infty)}(|X_t|)])\end{aligned} \tag{218}$$

But we can also compute

$$\begin{aligned}\mathbb{E}_P[\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(\frac{t^2}{2},\infty)}(|X_t|)]]&=\mathbb{E}[\mathbb{E}_P[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(\frac{t^2}{2},\infty)}(|X_t|)]]\\&=e^{\frac{t^2}{2}}P(|X_t|\geq\frac{t^2}{2})\\&\leq e^{-ct^2},\end{aligned} \tag{219}$$

with c some positive constant. By using Chebyshev's inequality **??** we obtain

$$P(\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(\frac{t^2}{2},\infty)}(|X_t|)]>\frac{1}{K})\leq e^{-ct^2}, \tag{220}$$

with K a large fixed constant. By the first Borel-Cantelli lemma **??**

$$P(\mathbb{E}[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(\frac{t^2}{2},\infty)}(|X_t|)]>\frac{1}{K},\text{ infinitely often})=0. \tag{221}$$

We will define a random variable T to be a stopping time, with respect to the filtration

$$\mathbb{F}_t = \sigma(B_x(s) : x \in \mathbb{Z}, 0 \le s < t), \tag{222}$$

as

$$T = \inf\Big\{t \ge N_0 : \mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(t^2,\infty)}(|X_t|)\big] \ge 1\Big\}. \tag{223}$$

By two-moment argument, if $\{T < \infty\}$ for $T \in (t-1, T]$, we get

$$P\Big(\mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(\frac{t^2}{2},\infty)}(|X_t|)\big] < \frac{1}{K}\big|\mathbb{F}_t\Big) \to 0 \text{ as } N_0 \to \infty. \tag{224}$$

We let $N_0 \to \infty$. Then P-a.s., there is a t_0 such that

$$\mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\mathbf{1}_{(t^2,\infty)}(|X_t|)\big] \le 1 \text{ for } t \ge t_0. \tag{225}$$

Taking t large,

$$\begin{aligned}\frac{1}{t}\log\mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\big] &\le \frac{1}{t}\log(ct^2e^{Z(t)}+1)\\ &\le \frac{Z(t)}{t} + \frac{O(\log t)}{t}.\end{aligned} \tag{226}$$

Then it holds

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\big] \le \lambda_1(k), \text{ P-a.s.} \tag{227}$$

and also

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\big[e^{\int_0^t dB_{X_s}(s)}\big] = \lim_{t\to\infty}\frac{1}{t}\log Z(t). \tag{228}$$

□

3.5 Interlacement Set

In the next application, we define an interlacement set on $\mathbb{Z}^d$, $d \ge 3$, which consists of a countable collection of doubly infinite trajectories of simple random walks. The number of trajectories is given by a non-negative parameter u. The set created by the trajectories of these random walks is called the interlacement set $\mathcal{I}^u$. In the following, we will define this model formally and at the end of this application we will see how the subadditive ergodic theorem is applied to the proof of its shape theorem.

We define $\mathbb{N} = \{0, 1, 2, ...\}$ to be the set of natural numbers. Let $e_1, ..., e_d$ be the standard basis vectors of the set $\mathbb{Z}^d$. We use the notation $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for the Euclidean, L_1 and L_∞ norms, respectively. We define

$$B(x, r) = \big\{y \in \mathbb{R}^d : \|y - x\|_\infty \le r\big\}, \tag{229}$$

the closed L_∞ ball centered at x with non-negative radius r. If the center of the ball x is the origin 0, we use the abbreviation

$$B(r) = B(0, r). \tag{230}$$

Let A be a subset of $\mathbb{Z}^d$ and $x, y \in A$ be arbitrary points in this set. We say that the set A is connected if there is a nearest-neighbor path connecting the points x and y, which is fully contained in A. We use the notation $|A|$ for the cardinality of A, $diam(A) = \max_{x,y\in A} \|x - y\|_\infty$ for its diameter in the L_∞ norm, $\partial A = \{x \in A : \exists y \in A^C, \|x - y\| = 1\}$ for its internal boundary.

Let $(X_n)_{n\in\mathbb{N}}$ be a discrete time simple random walk on the set $\mathbb{Z}^d$ and the point x its origin. We will denote its law by P_x. For a set $A \in \mathbb{Z}^d$, we define

$$\begin{aligned} H_A &= \inf\{n \geq 0 : X_n \in A\}, \\ \widetilde{H}_A &= \inf\{n \geq 1 : X_n \in A\}, \\ T_A &= \inf\{n \geq 0 : X_n \notin A\}, \end{aligned} \tag{231}$$

to be the entrance time in A, the hitting time of A and the exit time from A, respectively. If $A \in \mathbb{Z}^d$ is finite, we define the equilibrium measure of A by

$$e_A(x) = P_x(\widetilde{H}_A = \infty)\mathbf{1}_A(x) \tag{232}$$

and denote its total mass by

$$cap(A) = \sum_{x\in A} e_A(x). \tag{233}$$

We can now start to describe the model of the interlacement set. Let W be the space of doubly infinite, nearest neighbor trajectories on $\mathbb{Z}^d$, these trajectories tend to infinity as the time goes to either positive or negative infinity. We define W^* to be the space of equivalence classes of trajectories ω in W modulo time shift,

$$W^* = W_{/\sim}, \text{ where } \omega_1 \sim \omega_2 \text{ if } \omega_1(.) = \omega_2(. + k) \text{ for } k \in \mathbb{Z}. \tag{234}$$

We now consider a Poisson point process

$$\omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)}, \omega_i^* \in \Omega, u_i \geq 0 \tag{235}$$

taking values in the space of point measures Ω, on $W^* \times [0, \infty)$ with the intensity measure $\nu \otimes du$. Let P be the law of this process.

To understand, what the measure ν of the Poisson point process is, we denote by μ_A^u, with $A \subset \mathbb{Z}^d$ and $u \geq 0$, the mapping from Ω to the space of point measures on W. This mapping chooses from $\omega \in \Omega$ the trajectories whose labels are smaller than u intersecting the set A and then parametrizes them in a way that they enter A at time 0. Formally, we define for ω of the form in (235),

$$\mu_K^u(w) = \sum_{i\geq 0} \delta_{s_A(\omega_i^*)} \mathbf{1}_{\{Ran(\omega_i^*) \cup A \neq \emptyset, u_i \leq u\}}, \tag{236}$$

where

$$Ran(\omega^*) = \bigcup_{n\in\mathbb{Z}} \omega(n) \tag{237}$$

for an arbitrary ω from the equivalence class ω^* and $s_A(\omega^*)$ is the unique $\omega \in W$ such that $\omega_0 \in A$, $\omega_{-n} \in A$, $n > 0$. The measure ν is uniquely determined by the following two properties.

1. For each finite set $A \in \mathbb{Z}^d$, under P, the number $\eta_A^u = \eta_A^u(\omega)(W)$ of trajectories in ω, with labels smaller than u when entering A, is Poisson distributed with parameters $u \cdot \text{cap}(A)$.

2. Let

$$\mu_A^u(\omega) = \sum_{i=1}^{\eta_A^u} \delta_{\omega_i}, \omega_i \in W. \tag{238}$$

Then under the measure P, ω_i are i.i.d and independent of the number η_A^u, with the law defined as

$$P\big((\omega_i(n))_{n\geq 0} \in F\big) = \sum_{x\in A} \frac{e_A(x)}{e_A(A)} P_x(F) \tag{239}$$

for any measurable set F in the space of single-infinite nearest neighbor paths. Therefore, if we restrict ourselves to non-negative times, ω_i are i.i.d. simple random walk trajectories with the start at the normalized equilibrium measure $\frac{e_A(\cdot)}{e_A(A)}$.

The proof of the properties above can be found in [12] and we will not state it here. Instead, we proceed to the definition of the interlacement set at level u,

$$\mathcal{I}^u(\omega) = \bigcup_{i\geq 0} Ran(w_i^*) \mathbf{1}_{\{u_i \leq u\}}, \tag{240}$$

the trace of all trajectories in ω with labels smaller than u. We assume that the origin 0 is located in the interlacement set $\mathcal{I}^u$. Then we denote by

$$P_0^u = P(\cdot | 0 \in \mathcal{I}^u) \tag{241}$$

the conditional distribution. Let $x, y \in \mathcal{I}^u$. By $\rho_u(x, y)$ we define the internal distance of the two points x and y within the interlacement set $\mathcal{I}^u$,

$$\begin{aligned} \rho_u(x, y) = \min\{n : \exists x_0, x_1, ..., x_n \in \mathcal{I}^u \text{ s.t. } x_0 = x, x_n = y, \\ \text{and } \forall k = 1, ..., n : \|x_k - x_{k-1}\|_1\} \end{aligned} \tag{242}$$

with $\|\cdot\|_1$ denoting the L_1 norm in $\mathbb{Z}^d$. If $x \in \mathcal{I}^u$, we define

$$\Lambda^u(x, n) = \{y \in \mathcal{I}^u : \rho_u(x, y) \leq n\}, \tag{243}$$

to be the ball with center x of radius $n \geq 0$ in the internal distance within the interlacement set $\mathcal{I}^u$. If the ball is centered at the point 0, we shorten the notation to

$$\Lambda^u(n) = \Lambda^u(0, n). \tag{244}$$

Without proof, we will state the following result about the distance within the interlacement set. Interestingly, it is usually of the same of the same order as a usual distance. This is an important technical fact needed for the proof of the shape theorem for the interlacement set.

Theorem 3.21. *For every $u > 0$ and $d \geq 3$ one can find constants $C^1, C^2 < \infty$ and $\delta \in (0,1)$ such that*

$$P_0^u\big(\exists x \in \mathcal{I}^u \cap [-n,n]^d \text{ such that } \rho_u(0,x) > C^1 n\big) \leq C^2 e^{-n^\delta}. \tag{245}$$

With this result out of the way, we are finally ready to state the shape theorem.

Theorem 3.22 (Shape theorem)**.** *For all $u > 0$ and $d \geq 3$, there is a compact convex set $D_u \in \mathbb{R}^d$, such that for an arbitrary $\epsilon > 0$, we can find a P_0^u-a.s. finite random variable N such that*

$$((1-\epsilon)nD_u \cap \mathcal{I}^u) \subset \Lambda^u(n) \subset (1+\epsilon)nD_u \tag{246}$$

for all $n \geq N$.

Before we get to the proof of the shape theorem we define a notation used throughout the proof. For $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$, let

$$\begin{aligned} \zeta_0^{(x)}(y) &= \max\{m \leq 0 : mx + y \in \mathcal{I}^u\} \\ \zeta_{k+1}^{(x)}(y) &= \min\{m > \zeta_k^{(x)}(y) : mx + y \in \mathcal{I}^u\}, \quad k \geq 0. \end{aligned} \tag{247}$$

For the site on the interlacement set $\mathcal{I}^u$ corresponding to $\zeta_k^{(x)}(y)$, we will use the notation

$$\psi_k^{(x)}(y) = y + (\zeta_k^{(x)}(y)x. \tag{248}$$

If $y = 0$ or $x = e_1$, we simply shorten to $\zeta_k = \zeta_k^{(e_1)}(0)$.

Proof of the shape theorem. We want to apply the Liggett's version of the subadditive ergodic theorem for a sequence of random variables

$$X_{m,n} = \rho_u(\psi_m^{(x)}, \psi_n^{(x)}) \tag{249}$$

under the measure P_0^u. To be able to do this, we have to check if the four assumptions of Liggett's theorem hold for the sequence $X_{m,n}$. For condition (a) we note that ρ_u is a metric. By the triangle inequality

$$\rho_u(\psi_0^{(x)}, \psi_m^{(x)}) \leq \rho_u(\psi_0^{(x)}, \psi_l^{(x)}) + \rho_u(\psi_l^{(x)}, \psi_m^{(x)}). \tag{250}$$

The conditions (b) and (c) follow from Theorem 2.1. of [12]. The condition (d) follows from the estimate

$$P_0^u(\rho_u(0, \psi_1^{(x)}) > n) \leq \mathbf{s.e.}(n), \tag{251}$$

with **s.e**. the abbreviation for stretched-exponentially. We say that a function $f(n)$ is stretched-exponentially-small if for all $n \geq 1$, it holds that

$$0 \leq f(n) \leq c_1 e^{-c_2 n^{c_3}} \tag{252}$$

with c_1, c_2, c_3 constant. We write $f(n) = \mathbf{s.e.}(n)$.
That this holds can be proven by using Lemma 6.2 and Proposition 4.2. of [12].

With the four conditions satisfied, we can apply Liggett's subadditive ergodic theorem. Therefore, for all $x \in \mathbb{Z}^d$ one can find a positive number $\sigma_u^{'}(x)$ such that

$$P_0^u\left(\lim_{n\to\infty} \frac{\rho_u(0, \psi_n^x)}{n} = \sigma_u^{'}(x)\right) = 1. \tag{253}$$

We define

$$\begin{aligned} \sigma_u(x) &= \frac{\sigma_u^{'}(x)}{\mathbb{E}_0^u[\zeta_1^{(x)}]}, \text{ if } x \neq 0 \\ \sigma_u(x) &= 0, \text{ if } x = 0 \end{aligned} \tag{254}$$

There are two possibilities for the value of $\psi_0^{(x)}(nx)$. In case, of $nx \in \mathcal{I}^u$, one has $\psi_0^{(x)}(nx) = nx$. On the other hand, if nx is not in the interlacement set $\mathcal{I}^u$, $\psi_0^{(x)}(nx)$ is the last site on the discrete ray $\{kx, k \geq 0\}$, visited before nx. With

$$\lim_{n\to\infty} \frac{\rho_u\Big(0, \psi_0^{(x)}(nx)\Big)}{n} = \sigma_u(x), \quad P_0^u\text{-a.s.} \tag{255}$$

For an arbitrary $m \in \mathbb{Z}$ and $x \in \mathbb{Z}^d$, it holds

$$\sigma_u(mx) = m\sigma_u(x). \tag{256}$$

This property allows us to extend σ_u to the field of d-dimensional rational numbers $\mathbb{Q}^d$ by

$$\sigma_u(x) = m^{-1}\sigma_u(mx), \tag{257}$$

with $mx \in \mathbb{Z}^d$. Also, for any $x \in \mathbb{Z}^d$, $\sigma_u(x) \geq \|x\|_1$.

Lemma 3.23. *For an arbitrary $x, y \in \mathbb{Z}^d$, the following subadditivity condition holds*

$$\sigma_u(x+y) \leq \sigma_u(x) + \sigma_u(y). \tag{258}$$

Proof. We will use the notation $b_x = \mathbb{E}_0^u[\zeta_1^{(x)}]$. By the ergodic theorem, it holds

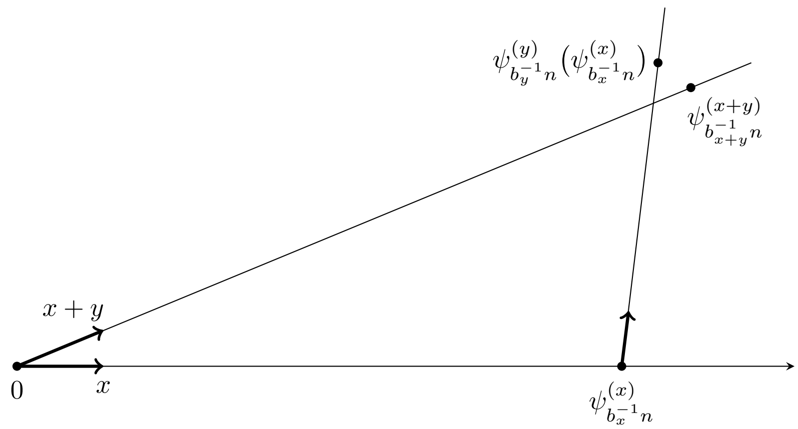
$$\lim_{n\to\infty} \frac{\left\| \psi^{(x)}_{b_x^{-1}n} - nx \right\|}{n} = 0, \quad P_0^u\text{-a.s.} \tag{259}$$

We already know that ρ_u is metric and looking at the picture below, we see that the following inequality

$$\frac{\rho_u\Big(0, \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big)\Big)}{n} \leq \frac{\rho_u\Big(0, \psi^{(x)}_{b_x^{-1}n}\Big)}{n} + \frac{\rho_u\Big(\psi^{(x)}_{b_x^{-1}n}, \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big)\Big)}{n} \tag{260}$$

holds for all n.

Figure 6



We look at the limit of (260) in probability. For the first term on the right-hand side of the inequality (260), we only need to apply (253) and (254) to see that

$$\frac{\rho_u\Big(0, \psi^{(x)}_{b_x^{-1}n}\Big)}{n} \to \sigma_u(x), \quad P_0^u\text{-a.s.}, \tag{261}$$

which implies convergence in probability. For the second term on the same side of the inequality, we notice that under the measure P_0^u, that the terms $\big(\psi^{(x)}_{b_x^{-1}n}, \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big)\big)$ and $\rho_u\big(0, \psi^{(y)}_{b_y^{-1}n}\big)$ are equal in distribution, and therefore

$$\frac{\rho_u\Big(\psi^{(x)}_{b_x^{-1}n}, \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big)\Big)}{n} \to \sigma_u(y) \tag{262}$$

in distribution and probability. Finally, for the term on the left-hand side of the inequality (260), we write

$$\frac{\rho_u\Big(0, \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big)\Big)}{n} \geq \frac{\rho_u\Big(0, \psi^{(x+y)}_{b_{x+y}^{-1}n}\Big)}{n} - \frac{\rho_u\Big(\psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big), \psi^{(x+y)}_{b_{x+y}^{-1}n}\Big)}{n}. \tag{263}$$

Again, we get for the first term on the right-hand side

$$\frac{\rho_u\Big(0, \psi^{(x+y)}_{b_{x+y}^{-1}n}\Big)}{n} \to \sigma_u(x+y), \quad P_0^u\text{-a.s.} \tag{264}$$

We want to show that the second term on the right hand side converges to 0 in probability. To see this, we consider the following inequality

$$\frac{\left\| \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big) - \psi^{(x+y)}_{b_{x+y}^{-1}n} \right\|}{n} \leq \frac{\left\| (x+y)n - \psi^{(x+y)}_{b_{x+y}^{-1}n} \right\|}{n} + \frac{\left\| nx - \psi^{(x)}_{b_x^{-1}n} \right\|}{n} + \frac{\left\| \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big) - \psi^{(x)}_{b_x^{-1}n} - ny \right\|}{n}. \tag{265}$$

The last term on the right of the inequality (265) is equal in distribution to $\frac{\left\| ny - \psi^{(y)}_{b_y^{-1}n} \right\|}{n}$ and therefore by (259),

$$\frac{\left\| \psi^{(y)}_{b_y^{-1}n}\big(\psi^{(x)}_{b_x^{-1}n}\big) - \psi^{(x+y)}_{b_{x+y}^{-1}n} \right\|}{n} \to 0 \quad \text{in probability.} \tag{266}$$

Hence, by Theorem 3.21 the second term on the right-hand side of (263) converges to 0. This completes the proof of the Lemma. □

We continue with the proof of the shape theorem. Define the set

$$D_u = \{x \in \mathbb{R}^d : \sigma_u(x) \leq 1\}. \tag{267}$$

Let

$$\begin{aligned} \epsilon_1 &= (1-\epsilon)^{-1} - 1 \\ \epsilon_2 &= 1 - (1+\epsilon)^{-1} \end{aligned} \tag{268}$$

We only need to prove for all sufficiently large n that

$$nD_u \cap \mathcal{I}^u \subset \Lambda^u((1+\epsilon_1)n) \text{ and } \Lambda^u((1-\epsilon_2)n) \subset nD_u \tag{269}$$

holds P-a.s. The set D_u is compact by definition and therefore we can find a finite set $F = \{x_1, ..., x_k\} \subset D_u \cap \mathbb{Q}^d$ such that $\sigma_u(x_i) < 1$ for $i = 1, ..., k$ and the constant C^1 from Theorem 3.21

$$D_u \cap \mathcal{I}^u \subset \bigcup_{i=1}^{k} B\big(x_i, C^{-1}\epsilon_1\big). \tag{270}$$

For an arbitrary $x_i \in F$, we define m_i to be the minimal positive integer with $m_i x_i \in \mathbb{Z}^d$. Let $n = jm_i + s$ with $0 \leq s \leq m_i - 1$. Then, for all sufficiently large n it holds by (255) that

$$\psi_0^{(m_i x_i}(jm_i x_i) \in \Lambda^u(n), \quad P_0^u\text{-a.s.} \tag{271}$$

We apply Theorem 3.21 and the Borel-Cantelli lemma and obtain that P_0^u-a.s. it holds for all sufficiently large n

$$B\big(nx_i, C^{-1}n\epsilon_1\big) \cap \mathcal{I}^u \subset \Lambda^u(jm_i x_i, n\epsilon_1), \tag{272}$$

for all $i = 1, 2, ..., k$. Therefore, we get $nD_u \cap \mathcal{I}^u \subset \Lambda^u((1+\epsilon_1)n)$ and the first part of the proof is finished. Now, we choose

$$G = \{y_1, ..., y_k\} \subset (2D_u \backslash D_u) \cap \mathbb{Z}^d, \tag{273}$$

such that

$$2D_u \backslash D_u \subset \bigcup_{i=1}^{k} B(y_1, \epsilon_2 \delta). \tag{274}$$

For $i = 1, ..., k$, it holds $\sigma_u(y_i) > 1$. Further, we have $nG \cap \Delta^u(n) = \emptyset$ for all sufficiently large n, P_0^u-a.s. Similarly, it holds by Theorem 3.21 and the Borel-Cantelli lemma, for all sufficiently large n, that $\Lambda^u((1-\epsilon_2)n) \cap n(2D_u \backslash D_u) \neq \emptyset$, implies $\Lambda^u(n) \cap nG \neq \emptyset$. Therefore, $\Lambda_u((1-\epsilon_2)n) \subset nD_u$ for all sufficiently large n, P_0^u-a.s., and completes the proof. $\square$

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