DOCTORAL THESIS

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"Realizable methods in time-frequency analysis based on spline-type constructions"

submitted by

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Zusammenfassung

Diese Arbeit befasst sich einerseits mit realisierbaren biorthogonalen Konstruktionen in *spline-type* Räumen und andererseits mit neuen Charakterisierungen der wichtigsten *dictionaries* der Zeit-Frequenz Analyse, d.h. *shift-invariant, Gabor* und *wavelets* Systemen, bezüglich Stabilität, Konsistenz und Konvergenz. Zweiteres geschieht mittels einer unitären Reformulierung in spline-type Räumen.

Das erste Kapitel behandelt lokalkompakte Gruppen und deren Anwendung im Hinblick auf Verallgemeinerungen von Konzepten für die reelle Gerade. Das zweite Kapitel widmet sich einem der zentralen Belangen von Modellierung und Simulation: Stabilität. Für spline-type Räume ist diese äquivalent zur Beschränktheit der Synthese -und Projektions-Operatoren. Zuerst wird eine Charakterisierung von Stabilität für spline-type Räume, im Sinne von Stabilität von shift-invariant Räumen, gegeben, d.h. lineare Unabhängigkeit charakterisiert Beschränktheit nach unten des Synthese-Operators in Banachräumen von Distributionen. Die konstruktive Natur des Beweises von Theorem 4 ermöglicht uns, das biorthogonale System konstruktiv anzugeben. Inspiriert durch *multiresolution analysis* und den Äquivalenzsatz von Lax für allgemeine Diskretisierungsschemata wird die Stabilität von Folgen von spline-type Räumen mittels gleichmäßiger Beschränktheit von Projektions-Operatoren untersucht. Dies charakterisiert sogar stabile Folgen von spline-type Räumen.

Der zweite Teil der Arbeit erkundet die Realisierbarkeit von spline-type Räumen. Dies ermöglicht einen einheitlichen Ansatz zur Approximation verschiedener Versionen eines gegebenen dictionaries. Zuerst werden Gabor-like Darstellungen eines Signals analysiert, wobei verschiedene Modulationen eines gegebenen Atoms als Erzeuger eines spline-type Raumes verwendet werden. Dieser Rahmen erlaubt es auch die numerische Stabilität von spline-type Systemen zu untersuchen. Weiters wird eine auf Gabor Zeit-Frequenz Filterung basierende Anwendung präsentiert, nämlich eine Verbesserung der anisotropischen Hough Transformation zur Erkennung des mandibulären Kanals in digitalen Dentalröntgenaufnahmen. Darüber hinaus werden *multiresolution Wavelet-like* Systeme eingeführt. Das so konstruierte realisierbare System kann gleichzeitig drei verschieden Eigenschaften eines Signals analysieren, nämlich Zeit, Skalierung und Frequenz.

Die gleiche Konstruktion wird schließlich auch zur Approximation von Pseudodifferentialoperatoren mittels *generalized Gabor multipliers* verwendet. Die Stabilitätstheorie erlaubt uns Deformationen von generalized Gabor multipliern, die stabile mehrstufige Gabor Approximationsschemata induzieren, zu beschreiben. Auch neue Aspekte in Bezug auf computergestützte Berechnung werden untersucht. Zum einen Modulationsselektion, im Sinne einer Zeit-Frequenz Verzögerung des Operators, welche die Approximation des *overspread* Operators ermöglicht, zum anderen die Hough Transformation, welche zur Approximation nicht stationärer Filter angewendet wird. Dies schlägt eine Brücke zwischen Feature Erkennung und harmonischer Analysis, wobei in Spezialfällen die Approximationsgenauigkeit verdoppelt wird.

Abstract

This thesis deals with realizable biorthogonal constructions in spline-type spaces and, via a unitary reformulation in spline-type spaces, with new characterizations of the main fixed dictionaries used in time-frequency signal analysis i.e. shift-invariant, Gabor and wavelets systems, from the point of view of their stability and consistency and their relation with convergence.

In the first chapter, we introduce the basic notation of locally compact groups for an extension of the concepts from the real line and for taking advantage of the bright simplicity of the theory. The second chapter, is devoted to one of the main concerns of the area of scientific modeling and simulations using frames: Stability. For a spline-type space, this quality is equivalent to boundedness of synthesis and projection operators. At first, a characterization of the stability of spline-type spaces is given, in the standard sense of the stability for shift-invariant spaces i.e. linear independence characterize lower boundedness of the synthesis operator in Banach spaces of distributions. The constructive nature of the proof of the Theorem 4 of section2.3, enabled us to constructively realize the biorthogonal system of a given one. Then, inspired by the multiresolution analysis and the Lax equivalence for general discretization schemes, the stability of a sequence of spline-type spaces as uniform boundedness of projection operators is approached; this gives a characterization of stable sequences of spline-type spaces.

The second part of the thesis explores the realizable nature of spline type space giving an unified approach to approximate versions of the fixed dictionaries. First, by using as generators of a spline-type space different modulations of a given atom, we analyze Gabor-like expansions of a signal; this framework also gives the possibility to study the numerical stability of spline-type systems. Following this line, it is presented an application of Gabor based time-frequency filtering: the enhancement of anisotropic generalized Hough transform for the detection of the mandibular canal in digital dental panoramic radiographs. Subsequently, multiresolution wavelet-like system will be developed. The realizable system constructed in this way, can analyze jointly three different features of a signal namely time, scale and frequency.

Finally, the same construction is applied to the approximation of pseudodifferential operators through generalized Gabor multipliers. The stability framework enables us to characterize deformations of a Gabor multiplier that builds up a stable multilevel Gaborian approximation scheme. Also, new computational aspects are explored: first modulations' selection, interpreted as time-frequency lag of the operator, gives us the possibility to approximate overspread operators, then the Hough transform will be applied to the approximation of non-stationary filters; this last procedure creates a bridge between features detections techniques and harmonic analysis methods and in specific cases it doubles the accuracy of approximation.

Acknowledgements

Time has come to thank the people that helped me during the years of the preparation of this thesis.

First and foremost, my supervisor Prof. Dr. Habil. Darian M. Onchis for his constant help and for pushing me over my limits. The duration of my studies coincided in time with, maybe, one of the most trying periods of his life. My thoughts go out to his parents who left too early. Nevertheless, Darian was the pragmatic one, setting goals, schedules and deadline that I needed to stay on track.

This thesis was financed by the Fonds zur Förderung der wissenschaftlichen Forschung. I would like to thank Austria for welcoming me, despite the insane political situation that Europe is going through.

Then, of course, the colleagues of the third floor. Those who left and those who still have to suffer my bouts of boredom. Thanks for the stimulating discussions, concerning both mathematics and something more important: life.

Finally, the people that each abroad student have often in mind: parents and friends (sometimes something else).
You are in my thoughts and you helped me in many ways.

Contents

Acronyms

ΨDO Pseudo Differential Operator. 86

AGHT Anisotropic Generalized Hough Transform. 57

FT Fourier Transform. 11

GGM Generalized Gabor Multiplier. 83, 88

H-S Hilbert-Schmidt. 87

HT Hough Transform. 57

IFT Inverse Fourier Transform. 12

K-N Kohn-Nirenberg. 86

LC Locally Compact. 5

LCA Locally Compact Abelian. 5

SI Shift Invariant. 17

ST Spline-type. 18

STFT Short-Time Fourier Transform. 37

TVF Time-varying filter. 85, 93

List of Figures

List of Algorithms

Preamble and scientific activities

0.1 Motivation and Objectives

In the theory of numerical solution of PDEs, the term stability refers to a property of finite difference equations with increasing finer mesh. Initially coined to express the growth of rounding error, in the Lax theory [53] it has been reformulated as an intrinsic property of the discretization scheme, independent of the particular initial value of the problem.
Rather that PDEs, this thesis is more concerned with studying the properties of more general linear operators, in a framework for which standard discretization will appear as a particular case.
Let Q a linear operator acting between a Banach space of solution X and a Banach space of data Y. The problem

$$Qf = g \qquad g \in Y$$

is said to be well posed if the range of Q is dense in Y and there exists a bounded linear operator $R \in \mathcal{B}(Y, X)$ such that RQ is the identity on the domain of Q, e.g. it is an extended inverse over the whole Y. Note that only to the inverse is required to be bounded.
A discretization scheme is usually defined as a family of problems depending on a real parameter h, the discretization step. The standard contraction operator of real valued function $f : \mathbb{R}^d \to \mathbb{C}$ can bee seen as composition with an automorphism

$$D_h f(x) := f(x/h) = f \circ \left(\frac{1}{h} I\right)(x) \tag{1}$$

A discretization method is a family of approximants depending on the scalar h, hence the study of the asymptotic behavior of the parameter h for $h \to 0$, will define if the family of approximants is *good/bad*.
In the theory of LC groups, it is not possible to define discretization as in (1) since no vector space structure has been defined. To define a multilevel-like approach we will use subsequent application of a contractive automorphism (or a sequence of contractive automorphism if not interested in multiresolution). We hence define discretization schemes as countable families of approximations.

Definition 1. *A discretization scheme is a sequence*

$$\{(X^{(n)}, Y^{(n)}, r^{(n)}, s^{(n)}, Q^{(n)})\}_{n\in\mathbb{N}},$$

being $X^{(n)}$ and $Y^{(n)}$ Banach spaces, $r^{(n)}$ and $s^{(n)}$ uniformly bounded linear operators

$$r^{(n)} : X \mapsto X^{(n)} \qquad s^{(n)} : Y \mapsto Y^{(n)}$$

and $Q^{(n)} : X^{(n)} \mapsto Y^{(n)}$ *such that the problems*

$$Q^{(n)} f^{(n)} = g^{(n)} \qquad g^{(n)} \in Y$$

are well posed with extended inverses $R^{(n)}$.

We say that the discretization scheme converges at $g \in Y$ if

$$\lim_{n\to\infty} \| r^{(n)} Rg - R^{(n)} s^{(n)} g \|_{Y^{(n)}} = 0$$

Proving convergence of a discretization scheme is not an easy task, for this reason two other important qualities of a scheme are introduced. Consistency of a discretization scheme means that

$$\lim_{h\to\infty} \| Q^{(n)} r^{(n)} f - s^{(n)} Qf \| = 0$$

while stability requires that the inverse approximated operators are uniformly bounded

$$\| R^{(n)} \| \leq K$$

for a scalar K.
An equivalence theorem, or Lax-Richtmyer theorem, is the establishment of the equivalence between convergence with stability and consistency.

Theorem 1. *If a method is consistent and stable, then it is convergent. If a method is convergent, then it is stable provided that:*
There exists a constant L *such that, for each* h *and each* $g \in Y_h$ *with* $\| g \| \leq 1$, *there exists an element* $f \in Y$ *such that* $\| f \| < L$ *and* $s_h f = g$.

The term discretization is not well posed in our context since we consider projection onto particular translation invariant spaces called spline-type spaces. In literature, shift invariant (**SI**)[14] and translation invariant(**TI**)[12] define spaces invariant up to shift over a discrete or continuous subgroup, respectively.
The first official mathematical paper working in the setting called spline-type spaces instead of shift-invariant spaces, which is difficult to discern from translation invariant spaces, for the full group action, not a lattice, or wavelet spaces, which come into the picture when also dilations are added, was [1]. We prefer the latter approach since the first can be studied as a particular case.
The Fast Approximation-Projection Iterative Reconstruction Algorithm is considered to be the standard recipe, and the estimates are standard in the sense that the atoms are in the Wiener Algebra $W(C,1)(\mathbb{R}^d)$ of locally continuous-globally integrable functions.
Here we consider the spline-type spaces [35, 60, 31] generated by a finite set of functions or distributions and their shifts over cocompact subgroup which builds up a Riesz basis. This gives an alternative constructive realization to iterative projections algorithms: the construction of the unique biorthogonal system. The main tool will be the inversion of the convolution operator defined over a locally compact group in Fourier domain.
Speaking of convolution, one of the motivations of this thesis relies on the interest on one of the most dynamic areas of research: convolutional neural network. A convolutional neural nerwork[47] consists of several layers which correspond to different operations on a given input signal. Regardless of the specific operation carryed out, each element of the network is called a neuron. The convolutional layer is devoted to the representations

of the inputs using several convolution kernels as neurons, each of them representing a feature map. Each neuron is then connected to a region of neighbouring neurons in the previous layer, building up a hierarchical structure.
For these reasons, with this work, I would like to explore the stability analysis of spline-type spaces for a large class of dictionaries and operators, considering the possibility of developing a multiresolution-like approach of feature extraction, and in such a way that the developed framework can also be used for implementing a consistency theory.
In this thesis, we will explore the stability of a spline-type spaces in the sense of shift invariant spaces literature: boundedness of the synthesis [23, 75].
This will give us the possibility to understand to a greater degree, shift-dependent dictionaries, pseudodifferential operator, and convolution operator.

0.2 Contributions

As the title of the thesis suggests, my work has focused on realizable methods in spline-type spaces theory, i.e. methods which posses a fairly general nature, but that can be implemented in real world applications.
Firstly, I consider in Chapter 2 the problem of establishing stability for spline-type systems as boundedness from above and below of the related synthesis operators. This leads to the standard linear independence also stated for shift invariant spaces, but in the more general framework of locally compact groups.
The constructive nature of the proof for Theorem 4 enabled us to constructively realize the biorthogonal system of a given one. Composing the analysis operator of this system with the synthesis of the original one gives us a reproduction formula for each element of the spline-type space.
Then, inspired by the multiresolution analysis and by the Lax equivalence for general discretization schemes, I consider the stability of a sequence of spline-type spaces

$$S^{\tau} := \{f \circ \tau \; : \; f \in S\}$$

where S is a spline-type space and τ is an arbitrary automorphism of the locally compact group.
The stability of the sequence $S_n := S^{\tau^n}$, i.e. uniform boundedness of the synthesis operators, will lead in Theorem 5 to a simple characterization. This part has been published in [59].

In the Part II of the thesis I will address the problem of reformulating and analyzing well known parameter-dependent dictionaries in signal analysis.
I will start from Gabor-like systems using as generators of the spline-type space modulated versions of a fixed atom. In order to lower the computational load, approximate duals of Gabor-like frames will be computed via a selection procedure of unnecessary modulations. We demonstrate the advantages of this approach in both flexibility and speed while maintaining the accuracy of the approximation. The method is constructive and it allows in a natural way to handle non-standard Gabor constructions like non-uniformity in frequency generated by the reduction of the number of used modulations. Computational complexity, comparison with existing methods, and numerical experiments are presented in support of the algorithm.
This example will also give the possibility to explore the numerical stability of a spline-type system, interpreted as deformation of the frequency content of an atom. We will

provide a theorem for the deformation, in Frobenius norm, of the Gramian of the spline type space. Numerical experiments will reproduce the thesis of the theorem. I personally presented the reformulation of Gabor system through spline-type system during the 18th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC 2016)[61], while the study of numerical stability has been submitted to the 26th European Signal Processing Conference (EUSIPCO 2018) [66].

The Gabor system can be seen as a dictionary depending on two parameters, time and frequency localization. Following this, line I will consider another parametrized dictionary whose connection with harmonic analysis has not been fully formalized yet: the Hough transform. Its parameters are location, scale and orientation of a fixed template and it was shown that such a transform is connected with the Radon transform, being one of its possible discretizations. This usually leads to poor outcome even for input having a low signal-to-noise ratio. Nevertheless, we will consider the case of the detection of the mandibular canal in dental panoramic radiographs, images characterized by a high level of noise. This task will be accomplished through an anisotropic generalized Hough transform, enhanced with a Gabor based time-frequency filtering. The proposed method is based on template matching using the fact that the shape of the mandibular canal is usually the same; the output of the Hough transform, i.e. the accumulator space, is then filtered in the Gabor domain for a precise detection of the position. The proposed procedure consists of an detailed description of the shape of the canal in its canonical form and on preserving Gabor filtering information for sorting the hierarchy of location candidates after applying anisotropically the extraction algorithm. The experimental results show that the proposed procedure is robust to recognition under occlusion and under the presence of additional structures e.g. teeth and projection errors.
My work on the detection of the mandibular canal through Hough transform can be found in [65] and [64].

When it comes to take into account a scale parameter, as for the Hough transform, the most established method in image compression comes in mind: Wavelet system. In Chapter 5 the algorithm is applied to the case of multiresolution wavelet-like systems.
The algorithm is based upon the construction of a sequence of spline-type approximation spaces whose stability is ensured by Theorems 4 and 5.
As for the Gabor case, only the generators that prominently intersect the spectrum of the signal will be considered to build the stable (both theoretically and numerically) approximation schemes. The realizable system constructed in this way jointly analyzes three different features of a signal: time, scale and frequency. We present numerical tests and a detailed complexity analysis to compare the performances of our method against standard constructions. This work has been presented during SYNASC 2017 [63].

Finally, we present the the approximation of the pseudodifferential operator via a generalized Gabor multipliers' dictionary.
Stability with respect to Theorem 5 and the Lax-equivalence is explored. Then the approximation of Hilbert-Schmidt operators will be employed together with two tools

- modulations' selection from Chapter 3: to handle both underspread and overspread operators

- Hough transform of Chapter 4: to handle discontinuities

The last chapter has been has been published in [62].

Original contributions will be marked as

[DO&SZ]

to emphasize that the research was done in an FWF-funded project called "Constructive frames-based realizations in time-frequency analysis", leaded by my supervisor Darian Onchis. The project will be finished at the end of May of 2018.

0.3 Codes

A Matlab implementation of chapters 3 and 7 and related script-examples which reproduce the numerical tests presented in this thesis are provided in the **CREAM (Constructive Realization, Expansion Algorithms and Modeling) toolbox**. The toolbox started from code regarding constructive realizable spline-type spaces written by Darian Onchis and was enhanced and developed with the implementations of the algorithms and constructive results from this thesis. The toolbox was written in the framework of the abovementioned FWF project and it is available for download at the UNIVIE personal page of Darian Onchis. In the last section of the thesis, we present the documentation of the main functions of the toolbox.

0.4 List of Publications

Here is the list of publication that I presented in conferences or published in journals.

0.4.1 Published

Darian M. Onchis, **Simone Zappalà**, Smaranda Laura Goția, Pedro Real, and Marius Pricop.
Detection of the mandibular canal in orthopantomography using a Gabor-filtered anisotropic generalized Hough transform.
Journal Pattern Recognition Letters 83 (2016): 85-90.

Darian M. Onchis, **Simone Zappalà**, Smaranda Laura Goția, and Pedro Real.
Double Hough Transform for Estimating the Position of the Mandibular Canal in Dental Radiographs.
In Special Sessions in Applications of Computer Algebra, pp. 317-327. Springer, Cham, 2015.

Darian M. Onchis and **Simone Zappalà**.
Approximate Duals of Gabor-like Frames Based on Realizable Multi-Window Spline-Type Constructions.
In Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), 2016 18th International Symposium on, pp. 99-104. IEEE, 2016.

Darian M. Onchis and **Simone Zappalà**.
Stability of Spline-Type Systems in the Abelian Case.

Journal Symmetry 10, no. 1 (2017): 7.

Darian M. Onchis and **Simone Zappalà**.
Realizable algorithm for approximating Hilbert Schmidt operators via Gabor Multipliers.
Journal of Computational and Applied Mathematics (2018).

0.4.2 In press

Darian M. Onchis and **Simone Zappalà**.
Constructive realizable wavelet-like systems based on multi-windows spline-type spaces.
In Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), 2017 19th International Symposium on, 2017.

0.4.3 Submitted

Darian M. Onchis and **Simone Zappalà**.
Numerical stability of spline-based Gabor-like systems.
In Signal Processing Conference (EUSIPCO), 2018 26th European. IEEE, 2018.

0.4.4 In progress

Darian M. Onchis and **Simone Zappalà**.
Stability of discretizations for pseudodifferential operators by Generalized Gabor Multipliers.

0.5 Participation to summer schools

To strengthen my preparation on harmonic analysis and signal processing I attended the following summer schools:

- Berlin Mathematical School

"Mathematical and Numerical Methods in Image Processing",

25 July - 5 August 2016, Berlin, Germany.

- University of Genoa - Department of Mathematics & Department of Informatics, Bioengineering, Robotics, and Systems Engineering

"Summer School on Applied Harmonic Analysis"

24 July - 28 July 2017, Genova, Italy.

0.6 Participation to conferences

My work has been presented during the following conferences:

- ACA working group

"ACA 2015: Applications of Computer Algebra 2015"

20-24 July 2015, Kalamata, Greece.

- Department of Computer Science, West University of Timisoara, Romania
- Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria
- Research Institute e-Austria, Timisoara, Romania

"SYNASC 2016: 18th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing "

24-27 September 2016, Timisoara, Romania.

- Department of Computer Science, West University of Timisoara, Romania
- Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria
- Research Institute e-Austria, Timisoara, Romania

"SYNASC 2017: 19th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing "

21-24 September 2017, Timisoara, Romania.

- Department of Mathematics, University of Salzburg

"ANADAY 2017: 13th Austrian Numerical Analysis Day"

4-5 May 2017, Salzburg, Austria.

- AMS - SIAM

"FoCM 2017 Foundations of Computational Mathematics"

July 10 -19 2017, Barcelona, Spain.

0.7 Statement of Originality

I hereby confirm that I have written the accompanying thesis by myself, without contributions from any sources other than those cited in the text, the references, and the acknowledgements. This applies explicitly to all graphics, drawings, maps and images included in the thesis.
With my signature I confirm that I have not manipulated any data.

Vienna Simone Zappalà

Part I

Spline-type spaces theory

Introduction to Part I

Locally compact groups are a natural framework for harmonic analysis, and locally compact Abelian groups are the natural framework for realizable harmonic analysis.
This first sentence, whose meaning will be clear in this part of the present thesis, describes the nature of my work: I am interested only in a theory that can be complemented with algorithms proving its foundation.

In the Chapter 1, I will present the classical notation of locally compact group, in accordance with the most influencial related literature e.g. [74] and [77]. The exposition will be discursive and we will just notice that all the properties of harmonic analysis on the real line can be extended to groups; also this exposition gives the possibility to truly understand the nature and capabilities of computational harmonic analysis.

At the core of the theory of locally compact groups is the Haar measure. In the following Chapter 2, I will introduce the theory of frames and Riesz basis starting again from a more general framework: measurable mappings. This will give us the possibility to understand the meaning of the summation participating in the synthesis operator of a spline-type space.

The main property of an algorithm is its capability of ensuring unique and stable result. For this reason, I focus on the stability of spline-type spaces in both a standard sense, linear independence in Fourier domain, and in a iterative formulation, as multiresolution-like schemes induced by a group automorphism.
This work has been published in [59].

Chapter 1

Locally compact Abelian groups

1.1 Locally compact group

We will work with topological spaces compatible with a group structure.

Definition 2. *A group onto which it is possible to define a topology for winch the operation of the group and the inversion are continuous is called a topological group.*

Using the group operation and inversion we can immediately state two fundamental automorphism of a topological group: the left and right translation.
Given a topological group G, we define the left and shift automorphism as

$$y \mapsto x^{-1}y \qquad y \mapsto yx^{-1} \qquad x, y \in G.$$

It is straightforward that any local topological property defined for any fixed element of the group (e.g. the identity) will hold for each element of the group.

Definition 3. *A topological group is called Locally Compact (LC) if the identity element has a compact neighborhood.*

A group will be called Locally Compact Abelian (LCA) if the group operation is commutative. We will use the acronyms LC and LCA to indicate the class of locally compact and locally compact abelian groups, respectively
Locally compact groups can be a major tool in numerical analysis since usual spaces of standard real analysis can be defined. In this work we are interested in decomposition and approximation of complex valued function defined over a LCA group G.
We denote with $C_c(G)$ the set of compactly supported continuous function defined over G. Since the extreme value theorem holds on general topological space we can define the norm

$$\| f \|_\infty := \max(f), \qquad f \in C_c(G).$$

Another important space of functions is the space of continuous functions vanishing at infinity endowed with the max norm: a function f belongs to the set $C_0(G)$ if for all $\epsilon > 0$ the level sets

$$\{x \in G \, : \, |f(x)| \geq \epsilon\}$$

are compact.
A measure of a LC group is a linear functional on $C_c(G)$ such that for all compact $K \subset G$ and $f \in C_c(G)$ with $supp(f) \subset K$ there exists $M_k > 0$ such that

$$|\mu(f)| = M_k \| f \|_\infty .$$

The number $\mu(f)$ is called the integral of f with respect to μ, and will be also denoted in analogy with the real case as $\int_G f(x)d\mu(x)$.
Any positive measure μ can be extended to general function $F : G \mapsto \overline{\mathbb{R}}_+$ through filtrations of lower semicontinuous functions. The measure of a set with respect to a measure is defined as a the integral of it's characteristic function, a set of measure zero is said negligible and properties are said to hold almost everywere with respect to a measure if the property does not hold on negligible set.
A function is said measurable with respect to a measure μ if for any compact set $K \subset G$ there exist a partition $K = N \cup \{K_n\}_{n\in\mathbb{N}}$ such that N is negligible and the restriction of the function on K_n is continuous.
A measure μ is said to be bounded if

$$\parallel \mu \parallel := inf \{M > 0 \ : \ \mu(f) \leq M \parallel f \parallel, \forall f \in C_c(G)\} < \infty.$$

The number $\parallel \mu \parallel$ is called the norm of μ. Any bounded measure of a non-compact LC group G can be extended to $C_0(G)$. The set of bounded measures form a Banach space with respect to the measure norm. It is called $M^1(G)$ and it is the dual of $C_0(G)$.
Integrable functions are defined through the seminorm

$$\int_G |f(x)|d\mu(x) \tag{1.1}$$

which is zero iff $f = 0$ a.e.
We simply define the space $\mathcal{L}^1(G, \mu)$ as the closure of $C_c(G)$ with respect to this seminorm, and

$$L^1(G, \mu) = \mathcal{L}^1(G, \mu)/\mathcal{R}(G, \mu),$$

being

$$\mathcal{R}(\mathcal{G}, \mu) := \{f \in \mathcal{L}^1(G, \mu) \ : \ \mu(f) = 0\}.$$

For $1 < p < \infty$ analogues $L^p(G, \mu)$ spaces are defined through the seminorm

$$\left(\int_G |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}.$$

The $L^\infty(G, \mu)$ is defined as the same quotient of μ-measurable function having bounded

$$\text{ess sup} f := \inf \left\{M \in \overline{\mathbb{R}}_+ \ : \ |f(x)| \leq M \text{ a.e. with respect to } \mu\right\} \tag{1.2}$$

seminorm.
The space $L^\infty(G, \mu)$ is isomorphic to the topological dual of $L^1(G, \mu)$, and any continuous functional on $L^1(G, \mu)$ can be written as

$$f \mapsto \langle f, \phi\rangle := \int_G f(x)\overline{\phi(x)}d\mu(x) \qquad f \in L^1(G, \mu)$$

for $\phi \in L^\infty(G, \mu)$.
More generally we will use the notation

$$\langle x, x'\rangle_{X\times X^*}$$

to express the application of the linear functional $x' \in X^*$ to the element $x \in X$, or we will omit the subscript if the spaces are fixed. For Hilbert spaces as $L^2(G)$ we will simply write $\langle x, y\rangle_{L^2(G)}$, for $x, y \in L^2(G)$
Starting from the translation automorphisms it is possible to define translation operators acting on function defined over G: for any $x, y \in G$ and f defined on G

$$L_x f(y) := f(x^{-1}y) \tag{1.3}$$

$$R_x f(y) := f(yx^{-1}). \tag{1.4}$$

Many interesting structures are invariants up to translation operators. A space of function X is called left (right) translation invariant if

$$f \in X \to L_x f \in X \quad (f \in X \to R_x f \in X) \qquad x \in G$$

while a measure is called left (right) translation invariant if $\mu(L_x f) = \mu(f)$ $(\mu(R_x f) = \mu(f))$ for all $x \in G$ and $f \in C_c(G)$.
The first important property of a LC group is the existence and the uniqueness, up to a multiplication constant, of a left invariant positive measure. It is called the (left) Haar measure of G and since its integrals have central role it will be denoted as $\int_G f(x) d_G x$ or $\int_G f(x) dx$. The support of the Haar measure is G itself and the measure of G is finite iff G is compact. For this reason, considering (1.1) for the Haar measure, we obtain a norm for $C_c(G)$ hence the whole space of integrable function

$$L^1(G) = \left\{ f : \| f \|_1 := \int_G |f(x)|\, d_G x < \infty \right\} \Big/ \{ f : \| f \|_1 = 0\}$$

can be seen as the completion of $C_c(G)$ for the norm $\| \cdot \|_1$. Likewise for the norm $\| f \|_p := \left(\int_G |f(x)|^p d_G x\right)^{1/p}$ for $1 < p < \infty$.
$L^1(G)$ is a subspace (also a subalgebra with respect with the convolution that will be defined below) of $M^1(G)$. Any integrable function f defines a measure

$$\mu_f(k) = \int_G k(x) f(x) dx. \tag{1.5}$$

Let $d_R x$ and $d_L x$ be right and left Haar measures we have the relationship

$$d_R x = \Delta(x^{-1}) d_L x,$$

where $\Delta : G \mapsto \mathbb{R}_+$ is called the Haar modulus. If the two Haar measure coincide, i.e. if $\Delta(x) = 1$ for all $x \in G$, the group is said to be unimodular. Abelian groups are a particular case of unimodular groups

Every closed subgroup H of a LC group G is LC; hence, any subgroup owns its own Haar measure $d_H x$. Due to left invariance of $d_H x$ over H, for any function $f \in C_c(G)$ and $y \in G$

$$\int_H f(yx) d_H x \qquad = \int_H f(yzx) d_H x$$

for any $z \in H$, i.e. the map $y \mapsto \int_H f(yx) d_H x$ is constant on the any left coset yH. We call this map $T_H : C_c(H) \mapsto C_c(G/H)$

$$T_H f(\dot{x}) = \int_H f(xy) d_H y \qquad x \in \dot{x} \in G/H. \tag{1.6}$$

Remark 1. *Consider the trivial case of the group $\mathbb{R}$, and it's closed subgroup $\mathbb{Z}$. We have that*

$$T_{\mathbb{Z}}f(x) = \sum_{k \in \mathbb{Z}} f(x+k) \qquad x \in \mathbb{T}.$$

From this case we immediately understand that the operator T_H is the LC formulation of the standard periodization operator.

If the subgroup H is normal, the cosets space G/H is a LC group with its own Haar measure $d_{G/H}\dot{x}$.
For any $f \in C_c(G)$

$$\int_{G/H} T_H f(\dot{x}) d_{G/H}\dot{x} \tag{1.7}$$

is a positive linear functional, which is also left invariant. By definition (1.7) is the Haar measure of G. From (1.6) and (1.7) we can normalize the Haar measures in such a way that

$$\int_{G/H} \left(\int_H f(xy) d_H y \right) d_{G/H}\dot{x} = \int_G f(x) d_G x \qquad f \in C_c(G), \tag{1.8}$$

which is called *Weil's formula.* If such a normalization is used, we will say that the Haar measure of G, H and G/H are canonically related. The Weil's formula can be easily extended to $L^1(G)$, so the periodization operator satisfy the inequality

$$\| T_H f \|_1 \leq \| f \|_1 \qquad f \in L^1(G),$$

which tells us that, if H is normal, T_H maps $L^1(G)$ into $L^1(G/H)$.
For the ess sup seminorm we have that

$$\operatorname*{ess\,sup}_{\dot{x} \in G/H} \left(\operatorname*{ess\,sup}_{y \in H} f(xy) \right) \geq \operatorname*{ess\,sup}_{x \in G} f(x)$$

holds in general, while equality holds if G is σ-compact.
The Weil formula can be seen as a particular case of the Lebesgue-Fubini theorem, which extends the product measures

$$\begin{aligned} \int_{X \times Y} k(x,y) d\left(\mu_1 \otimes \mu_2\right)(x,y) &:= \int_X \left(\int_Y k(x,y) d\mu_2(y) \right) \mu_1(x) \\ &= \int_Y \left(\int_X k(x,y) d\mu_1(x) \right) \mu_2(y) \end{aligned}$$

defined over $C_c(X \times Y)$, to L^1 functions.

1.2 Convolution and translation invariant spaces of a locally compact group

In the next chapter we will work and develop the theory of spline-type spaces. Their construction is based on the the convolution operator. It is defined for $C_c(G)$ functions as

$$f *^G g(x) := \int_G f(y) L_y g(x) d_G y, \tag{1.9}$$

and we will omit the superscript if no confusion arises.
Thanks to the Fubini's theorem we get that $*: L^1(G) \times L^1(G) \mapsto L^1(G)$ since

$$\| f * g \|_1 \leq \| f \|_1 \| g \|_1 .$$

More generally, for an unimodular group and for $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ we can extend the convolution as function $*: L^p(G) \times L^q(G) \mapsto L^r(G)$ since

$$\| f * g \|_r \leq \| f \|_p \| g \|_q .$$

The space $(L^1(G), *)$ is an algebra with isometric involution

$$f^* := \overline{f(x^{-1})}\Delta(x^{-1}).$$

We have to analyze the relationship between convolution and closed normal subgroup.
For a normal subgroup H, the periodization T_H is an algebra morphism between the convolution algebras $(L^1(G), *)$ and $(L^1(G/), *)$, meaning that

$$T_H(f *^G g) = T_H f *^{G/H} T_H g$$

and

$$T_H (f^*) = (T_H f)^* .$$

From this, the kernel of T_H is a two side ideal of $(L^1(G), *)$.
Important for spline-type spaces is the following particular case of the convolution: for a closed subgroup H of G, any function in $h \in L^1(H)$ defines a bounded measure on G through

$$\mu_h(k) := \int_H h(y)k(y)dy \qquad k \in C_c(G). \tag{1.10}$$

We can define a convolution between measures defined over H and function in $C_c(G)$ as

$$\mu_h * f(x) := \int_H h(y)L_y f(x)dy, \tag{1.11}$$

which can be extended to the whole $L^p(G)$ spaces.
For general complex bounded measures on G we can define

$$\mu * f(x) := \int_G L_x f d\mu(x) = \mu(L_x f).$$

Spline-type space will be a particular case of spaces, invariant up to a class translations.

Definition 4. *Let $\mathcal{B}$ be a Banach space of function defined over a LC group G and H be a subgroup of G. $\mathcal{S} \subset \mathcal{B}$ is called a left H-translation invariant if for all $y \in H$ and $f \in \mathcal{S}$, $L_y f \in \mathcal{S}$ The space will be called left translation invariant, or just invariant, if $H = G$.*

Since our exposition focuses on left translation operator and LCA groups, we will just say that a space is H-invariant.
The following characterization of closed invariant spaces holds[74, 3.5.10]

Proposition 1. *A subset of $L^1(G)$ is a closed translation invariant subspace of $L^1(G)$ iff it is an ideal of $(L^1(G), *)$.*

For general H-shift invariant spaces $\mathcal{S}$, with H a subgroup of G, we cannot have the same equality. On one side we have that for each $h \in L^1(H)$, the convolution defined in (1.11) is such that

$$\mu_h * \mathcal{S} \subset \mathcal{S};$$

this is due to the fact [74, Proposition 3.5.12] that for any $\epsilon > 0$ there are finitelly many $y_n \in H$ such that

$$\int_G \left| \int_H L_y f(x) h(y) d_H y - \sum_n c_n L_{y_n} f(x) \right| dx < \epsilon$$

being $c_n = \int_{A_n} h(y) d_H y$ and (A_n) a partition of H into measurable subset, hence $\int_H h() y d_H y = \sum_n c_n$.

1.3 Harmonic analysis on LCA groups

The morphisms of a LC group G into the torus $\mathbb{T}$ are called characters of the group. We name the set of continuous characters of a LC group G, $\widehat{G}$. They form a group together with the multiplication over $\mathbb{T}$. We endow this group with the topology defined by the following basis of neighborhood of the unit character: for any compact $K \subset G$ and $\epsilon > 0$ let us we consider

$$U(K, \epsilon) := \left\{ \chi \in \widehat{G} \; : \; \forall x \in K, \; |\chi(x) - 1| < \epsilon \right\}.$$

We call $\widehat{G}$ with such a topology the dual of G.
A LC group have LC dual; a LCA group have LCA dual. Product of LCA group is a LCA group; as a consequence, if $G \in LCA$ then

$$G \times G, \quad \widehat{G} \times \widehat{G}, \quad G \times \widehat{G}, \quad \widehat{G} \times G \in LCA. \tag{1.12}$$

LCA groups are a particularly favorable class of LC group since the Pontryagin-van Kampen's duality theorem holds

Theorem 2. *Let G a LCA. $\widehat{G}$ is LCA and its dual is isomorphic to G.*

The theorem was proved in 1934 by Pontryagin [68] for compact Abelian groups having a countable basis. In the following year, van Kampen extended the duality for all locally compact Abelian groups [88] making use of his structure theorem which states that a LCA group is the product of some $\mathbb{R}^d$ and a group G_0 containing a compact subgroup K such that G_0/K is compact.
Using the aforementioned notation $\langle v, w \rangle$ to express the application of a linear functional $w \in V^*$ to an $v \in V$ of a vector space, a character $\widehat{x} \in \widehat{G}$ can hence be represented as $x \mapsto \langle x, \hat{x} \rangle$ for $x \in G$ and an element of $x \in G$ as $\hat{x} \mapsto \langle x, \hat{x} \rangle$, $\hat{x} \in \widehat{G}$.
For any subgroup H of a LCA group G we can consider

$$H^\perp := \left\{ \hat{x} \in \hat{G} \; : \; \forall x \in H, \; \langle x, \hat{x} \rangle = 1 \right\},$$

the perpendicular (since 1 is the neutral element of $\mathbb{T}$) of H. The only perpendicular element to G is the trivial morphism which maps any element into $1 \in \mathbb{T}$, while for any nontrivial subgroup we can find 'enough' elements in the perpendicular, in the sense that for any closed subgroup H and $x_0 \in G \setminus H$, there exist $\widehat{x}_0 \in H^\perp$ such that $\langle x_0, \hat{x}_0 \rangle \neq 1$. By this we can also obtain that $(H^\perp)^\perp = \overline{H}$
Thanks to these considerations and using the duality theorem the following characterization of the orthogonal of a closed subgroup is available

Proposition 2. *Let G be a LCA group and H a closed subgroup.*

1. $\widehat{G/H} = H^\perp$.
2. $\widehat{H} = \widehat{G}/H^\perp$.

1.3.1 Fourier transform

The Fourier Transform (FT) of a function in $L^1(G)$ is defined as

$$\hat{f}^G(\hat{x}) := \int_G f(x)\overline{\langle x, \hat{x}\rangle} dx \qquad \hat{x} \in \widehat{G}.$$

We will simply denote by $\hat{f}$ the Fourier transform if no confusion with other group can occur.
Using again van Kampen structure theory the Haussdorf-Yong inequality can be extended to LCA group: for $1 \leq p \leq 2$ and p' its Hölder conjugate the inequality

$$\| \widehat{f} \|_{p'} \leq C \| f \|_p$$

holds, being C a sharp constant depending on G and the normalization of the Haar measure [9]. The FT is hence a mapping $\mathcal{F} : L^p(G) \cap L^1(G) \mapsto L^{p'}$.
Given a close subgroup H, we have considered the periodization operator T_H that maps functions in $L^1(G)$ into $L^1(G/H)$, hence for any $f \in L^1(G)$ it is possible to consider the FT of the function $T_H f$ which is a function defined over $\widehat{G/H} = H^\perp$. By applying the Weil formula we easilly get that for any $\widehat{x} \in H^\perp$

$$\begin{aligned} \widehat{T_H f}^{G/H}(\widehat{x}) &= \int_{G/H} T_H f(\dot{x})\overline{\langle \dot{x}, \widehat{x}\rangle} d_{G/H}\dot{x} \\ &= \int_G f(x)\overline{\langle x, \widehat{x}\rangle} d_H x = \widehat{f}^G(\widehat{x}) \end{aligned}$$

which simply means that

$$\widehat{T_H f}^{G/H} = \widehat{f}^G\Big|_{H^\perp}.$$

The FT can be generalized to $M^1(G)$ through

$$\hat{\mu}(\hat{x}) := \int_G \overline{\langle x, \hat{x}\rangle} d\mu(x) \qquad \hat{x} \in \widehat{G}$$

and the property $\| \widehat{\mu} \|_\infty \leq \| f \|$ holds. For a measure defined as in (1.10), we get that its FT is $H^\perp$ periodic. More precisely, given H, a closed subgroup of G, and a measure μ_H obtained as in (1.10), since $\widehat{H} = \widehat{G}/H^\perp$,

$$\widehat{\mu_h}^G = \widehat{h}^H \circ \pi_{H^\perp}$$

where $\pi_{H^\perp}$ is the canonical projection into $\widehat{G}/H^\perp$.
An analogue definition of the FT of a measure is given by defining

$$\widehat{\mu}(k) := \mu(\widehat{k}) \qquad \forall k \in C_c(G).$$

From this definition we easily see that if $\mathcal{F}(\mu) = 0$ then $\mu = 0$. This holds in particular on the subalgebra $L^1(G)$: if $f \in L^1(G)$ is such that $\widehat{f}(\widehat{x}) = 0$, then f=0 almost everywhere. An immediate corollary of this proposition is the so called *inversion formula*: for a function $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$, the relationship

$$f(x) = \int_{\widehat{G}} \widehat{f}(\widehat{x}) \langle x, \widehat{x} \rangle \, d\widehat{x} \tag{1.13}$$

holds a.e.
Formula (1.13) is called the Inverse Fourier Transform (IFT) of the function $\widehat{f}$, denoted with the symbol $\mathcal{F}^{-1}$ or the reversed hat $^\vee$.
The theory of inversion is used to state the Bochner's theorem in a LCA group G: any continuous positive-definite, i.e. complex valued function ϕ such that

$$\sum_{i=1}^{N} \sum_{i=1}^{N} \phi(x_i^{-1} x_j) c_i \overline{c_j} \geq 0$$

for all finite sequences $(x_n)_{n=1}^N \subset G$ and $cx_n)_{n=1}^N \subset \mathbb{C}$, is the IFT of a positive bounded measure on $\widehat{G}$.
Most of the realizable algorithms on LCA group are based upon fact that the FT exchanges convolution with multiplication:

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \qquad f, g \in L^1(G). \tag{1.14}$$

This will give the possibility of inverting the convolution, upon which the spline-type space will be built, as pointwise inversion in Fourier domain.
Property (1.14) is often called *the convolution theorem*; it gives us the possibility of studying $L^1(G)$ through the Fourier algebra

$$\mathcal{F}^1(\widehat{G}) := \mathcal{F}L^1(G)$$

since closed ideals of $L^1(G)$ and $\mathcal{F}^1(\widehat{G})$ are on-to-one through FT. Another operator affected by a particular composition rule with the FT is the multiplication with a character:

$$M_{\widehat{x}} f(x) := \langle x, \widehat{x} \rangle f(x) \qquad \widehat{x} \in \widehat{G}. \tag{1.15}$$

The space $L^1(G)$ is invariant to both translation and character multiplication.
The operator (1.15) is also called modulation, or frequency shift due to the composition formulas

$$\mathcal{F}(L_x f) = M_{x^{-1}} \mathcal{F}(f) \tag{1.16}$$
$$\mathcal{F}(M_{\widehat{x}} f) = L_{\widehat{x}} \mathcal{F}(f). \tag{1.17}$$

for any function $f \in L^1(G)$, and from the particular case $G = \mathbb{R}$, interpreted in application as the domain of a signal depending on a time parameter, and the dual $\widehat{G} = \mathbb{R}$, interpreted

as frequency domain.
Analogous composition rules holds for the IFT:

$$\mathcal{F}^{-1}\left(L_{\widehat{x}}\widehat{f}\right) = M_{\widehat{x}}\mathcal{F}^{-1}\left(\widehat{f}\right)$$
$$\mathcal{F}^{-1}\left(M_{x}\widehat{f}\right) = L_{x^{-1}}\mathcal{F}^{-1}\left(\widehat{f}\right).$$

A particular commutation rule exists between the shift operators

$$M_{\widehat{x}}L_x = \langle x, \widehat{x}\rangle\, L_x M_{\widehat{x}}. \tag{1.18}$$

The combined shift could be interpreted as a time-frequency shift operator: for $\lambda \in G\times\widehat{G}$ we set

$$\pi(\lambda)f(t) := M_{\widehat{x}}L_x f(t) = \langle x, \widehat{x}\rangle\, f(x^{-1}t). \tag{1.19}$$

Since the FT interchange modulation with shift and $L^1(G)$ is invariant up to shift and modulation, $\mathcal{F}^1(\widehat{G})$ is again invariant up to modulation and shift.
Plancherel's theorem holds in general LCA group:

Proposition 3. *There exist a normalization of the Haar measure and a bijective linear map $\mathfrak{F} : L^2(G) \mapsto L^2(\widehat{G})$ such that*

$$\parallel \mathfrak{F}f \parallel_{L^2(\widehat{G})} = \parallel f \parallel_{L^2(G)}$$

and

$$\mathfrak{F}\big|_{L^1(\widehat{G})\cap L^2(\widehat{G})} = \mathcal{F}\big|_{L^1(G)\cap L^2(G)}.$$

From the Plancherel's theorem we get that for $f_1, f_2 \in L^2(G)$

$$\begin{aligned}\langle f_1, f_2\rangle_{L^2(G)} &= \int_G f_1(x)\overline{f_2(x)}dx\\ &= \int_G \widehat{f_1}(\widehat{x})\overline{\widehat{f_2}(\widehat{x})}dx\\ &= \left\langle \widehat{f_1}, \widehat{f_2}\right\rangle_{L^2(\widehat{G})}.\end{aligned}$$

The Plancherel's theorem gives us a method to build well behaved function, e.g. if we want a function which is compactly supported and whose Fourier transform is integrable we can simply take the convolution of two compactly supported function:

$$g, h \in C_c(G) \Rightarrow g, h \in L^2(G) \Rightarrow \widehat{g}, \widehat{h} \in L^2(\widehat{G}) \Rightarrow \widehat{g}\cdot\widehat{h} \in L^1(\widehat{G}).$$

Another strength of LCA group theory is the possibility to develop distribution theory. Schwartz class space was generalized by Bruhat for the case of LCA group by considering the direct limits of elementary LCA group[13]. This space was initially formalized [92] starting from elementary LCA group, i.e. group of the form $G = \mathbb{R}^{a_1}\times\mathbb{Z}^{a_2}\times\mathbb{T}^{a_3}\times F$, a_1, a_2, a_3 integers and being F a finite Abelian group. The schwartz class $\mathcal{S}(G)$ is defined as the functions $f \in C^\infty(G)$ such that $P(\partial)f$ for any polynomial differential operator on the variables $\mathbb{R}^{a_1}\times\mathbb{Z}^{a_2}$. This definition is extended to arbitrary LCA group G as direct limit $\mathcal{S}(G) = \lim_{\rightarrow}\mathcal{S}(H/K)$, being H a compactly generated open subgroup and K compact subgroup, hence H/K elementary.

It is possible to characterize such a space without the use of elementary groups and direct limits, but only by mean of decay property of a function and its FT [67]: we consider the set $\mathcal{A}(G)$ of functions $f \in L^\infty(G)$ for which there exist a compact neighborhood of the identity $C_f \subset G$ such that for all $n \in \mathbb{N}$ there exist $M_n > 0$ such that

$$\| f|_{G \setminus C_f^k} \|_\infty \leq M_n k^{-n}$$

for all $k \geq 1$.
The space $\mathcal{A}(G)$ is a dense translation invariant convolution subalgebra of $L^1(G)$, closed under character and L^∞-function multiplication.
The Schwartz-Bruhat space's definition in [13] is equivalent to the definition

$$\mathcal{S}(G) = \left\{ f \in \mathcal{A}(G) \ : \ \widehat{f} \in \mathcal{A}(G) \right\}$$

and the space of distributions is defined as it's dual. The Schwartz-Bruhat space inherit all the property of $\mathcal{A}(G)$.
The FT is an isomorphism $\mathcal{F} : \mathcal{S}(G) \mapsto \mathcal{S}(\widehat{G})$[91], so $\mathcal{F} : \mathcal{S}(G)^* \mapsto \mathcal{S}(G)^*$, and as a consequence of the Paley-Wiener theorem[67] the FT of a compactly supported distribution is a function defined over $\widehat{G}$ with

$$\langle \langle \cdot, \widehat{x} \rangle , \phi \rangle = \widehat{\phi}^*(\widehat{x}).$$

Translation and convolution are weakly extended to distribution in the following way:

$$L_x : \mathcal{S}(G)^* \mapsto \mathcal{S}(G)^*, \ \langle f, L_x \phi \rangle := \langle L_{x^{-1}} f, \phi \rangle , \ \forall g \in \mathcal{S}(G)$$

and

$$* : \mathcal{S}(G) \times \mathcal{S}(G)^* \mapsto \mathcal{S}(G)^*, \ \langle g, f * \phi \rangle = \left\langle f^\dagger * g, \phi \right\rangle , \ \forall g \in \mathcal{S}(G),$$

being $f^\dagger(x) = f(x^{-1})$. These definitions are well posed also for bounded functions and compactly supported distributions.
The Schwartz-Bruhat space, and its dual, are reflexive spaces.

1.3.2 Wiener's Theorems

We finish this minimal exposition of LCA group by giving the enunciations of the Wiener's theorems. Since we want to invert convolutions, thanks to the convolution theorem (1.14), we can accomplish this goal by pointwise inversion on the Fourier algebra. More generally the Wiener-Lévy theorem holds:

Proposition 4. *Let G be a LCA group. For any $\widehat{f} \in \mathcal{F}^1(\widehat{G})$, $\widehat{K} \subset \widehat{G}$ compact, and A analytic function defined on an open neighborhood of $\widehat{f}(\widehat{K})$, there exist $\widehat{g} \in \mathcal{F}^1(\widehat{G})$ such that $\widehat{g}(\widehat{x}) = A(\widehat{f}(\widehat{x}))$ for all $\widehat{x} \in \widehat{K}$.*

We can also think about an approximation of the inversion in Fourier domain thanks to the relationship between function on $\mathcal{F}^1(\widehat{G})$ and closed ideals in $\mathcal{F}^1(\widehat{G})$.
The cospectrum of a function in $f \in L^1(G)$, $\mathrm{cosp}(f)$, is defined as the set of zeros of $\widehat{f}$, analogously the cospectrum of $\widehat{f} \in \mathcal{F}^1(\widehat{G})$, $\mathrm{cosp}(\widehat{f})$ is defined as the set of zeros. For a subset K of $L^1(G)$ we define its cospectrum as

$$\mathrm{cosp}(K) := \cap_{f \in K} \mathrm{cosp}(f);$$

and analogously for subset in $\mathcal{F}^1(\widehat{G})$.
The local inversion of the convolution is stressed in the relationship between the following generalized Wiener's theorem

Proposition 5. *A closed idea I of $\mathcal{F}^1(\widehat{G})$ contains every function $\widehat{f}$ such that $cospI \subset cosp(f)$ and $\partial(cosp(f)) \cap \partial(cosp(I))$ is a scattered set.*

In the following Chapters 3, 5 and 6 we will apply this intuition: each element of a spline-type space will be expressed starting from a convolution (1.11); its Fourier transform is the product of a periodic function times the FT of the generators of the spline-type space; since ideals are characterized in Fourier domain, we will select only the function which *strongly intersect* the function to be approximated to generate the proper spline-type space to lower the complexity.

Chapter 2

Stability in spline-type spaces

2.1 Problem setting and state of the art

The previous chapter was meant to highlight the particular structure that arbitrary LCA groups have. It was possible to recover most of the property of (not only) harmonic analysis of the additive group $\mathbb{R}^d$. The key property is the Pontryagin duality which gives a fair representation of the group, hence nice constructive formulas on the group, e.g. Poisson's and Plancherel's formulas.
In particular, the convolution operator has a long history of practical application in signal processing and approximation theory. Common techniques in these fields rely on the decomposition of a given *signal* in term of a dictionary whose words have optimal localization properties [71, 94, 70, 86, 95]. The standard decomposition tools in approximation and sampling theory [60, 20, 19] are usually given in underlying Shift Invariant (SI) spaces. Their construction [23, 11, 75] relies on the two main ingredients: a set of window-functions defined over $\mathbb{R}^d$ and the only meaningful (up to isomorphism) proper closed subgroup: $\mathbb{Z}^d$. The SI dictionary is composed by a set of functions and their integer shifts; the SI spaces are the closed linear span over $L^2(\mathbb{R}^d)$. In $L^2(\mathbb{R}^d)$-signal analysis theory [17], for a countable set of functions $\Phi := \{\phi_i\}_{i\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$, the words of a decomposition vocabulary, the analysis operator V_Φ and synthesis operator U_Φ are the main tools. The first measures through the L^2 product the recurrence of each word ϕ_i in a given signal:

$$V_\Phi f := (\langle \phi_i, f\rangle_2)_{i\in I} \tag{2.1}$$

for $f \in L^2(\mathbb{R}^d)$. the synthesis produces a *sentence* starting from a recurrency-sequence $\boldsymbol{c}$ in $L^2(\mathbb{N})$:

$$U_\Phi c := \sum_{i\in N} c_i \phi_i$$

Frames and Riesz basis are countable sets of words which ensure L^2-boundedness of the analysis and synthesis, respectively.
These constructions have been generalized for $L^p(\mathbb{R}^d)$ spaces in [2], $L^2(G)$ for a LCA group G in [14], while in [12] the first two authors have generalized their range function approach introduced in [11] to Translation Invariant (**TI**) space on LCA groups to consider the shifts of a countable set of functions over a cocompact subgroup, i.e. a subgroup that builds a compact quotient space; this leads again to a theory restricted to standard $L^2(\mathbb{R}^d)$-projection techniques.

Here I consider a finite set of generators and their shift over cocompact subgroup; the related closure over a selected translation invariant Banach space will be called Spline-Type space, as a direct continuation of the notation stated in [35, 60] and in [31]. Then duality principle[15], extended to Banach spaces [96], and formulated for continuous systems[29] is used to study the property of ST space through continuous Riesz basis.
The standard computational approach of SI spaces is generalized for a general ST space generated by a vector valued function Φ building the biorthogonal $\tilde{\Phi}$ system which ensures the reproduction formula

$$f = \overline{\sum_{i \in I}} \left\langle \tilde{\phi}_i, f \right\rangle \phi_i$$

being $\overline{\sum}$ the summation over the subgroup that generates the ST space and $\langle \cdot, \cdot \rangle$ the action of a distribution onto a function.
Boundedness of the synthesis operator will be named stability, referring to the literature of SI spaces[23, 75].
Another aspect of ST spaces that could be accounted under the same name stability is related to the analysis of the projection into sequences of ST spaces. This ambiguity comes from our interest of laying the foundations for the stability and the consistency of general discretization schemes [78] in the sense of the Lax-Richtmyer equivalence[53] as stated in the preamble of the thesis, for the space of multipliers over LCA groups. In the equivalence theorem, stability of discretization method is defined as boundedness of the family of operators, independently from the discretization parameter, i.e. uniform boundedness of the family. In analogy with multi-resolution discretization, e.g. wavelets systems, we will build a sequence of ST spaces generated by an original ST space and the operator induced by an automorphism over the LCA group, extending the standard construction of the dyadic contraction operator $f(x) \mapsto f(2x)$ induced by the expansive automorphism $x \mapsto 2x$ over the LCA group $\mathbb{R}^d$ over $L^2(\mathbb{R}^d)$.
The structure of this chapter is the following: firstly ST spaces, continuous measurable mappings, p-frames and q-Riesz basis in Banach spaces [29], which correspond to measurable synthesis operators[87], are defined. Then in Section 2.3 we generalize in this framework the theory of SI space [75] to obtain a constructive realization of the biorthogoonal system which leads to the non-orthogonal expansion of distributions. Theorem 4 is turned into pseudocode in Algorithm 1 to highlight its computational nature. Finally in Section 2.3.1 we introduce the concept of sequence of ST spaces generated by automorphisms of the LCA group, and find the characterization of the induced operators which give stability in the Lax sense.

2.2 Spline-type spaces, Measurable mappings, Frames and Riesz basis

We introduce spline-type spaces as subspaces of translation invariant Banach spaces.

Definition 5. *Given G a LC group, H a subgroup of G, and $\Phi = \{\phi_i : G \to \mathbb{C}\}_{i=1}^{R}$ a finite set of functions or distributions in a translation invariant Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, the collection of left shift $(\Phi, H) := \{L_a \phi_i : a \in H,\ i = 1, \ldots, R\}$ is called* ***Spline-type (ST) system*** *of* ***generating set*** *Φ and subgroup H, while its closed span in $\mathcal{B}$ is called and* ***spline-type space*** *generated by Φ and H, which will be indicated as $\mathcal{S}(\Phi, H)$.*

Following the signal analysis literature, we will call the generating set of a shift invariant space *windows* or *atoms*. If the ST space is generated by only one window it will be called principal. Our formulation of ST space is hence a generalization of multiwindow shift invariant space over arbitrary LC group and considering the shift over arbitrary subgroup. Of course restrictions will come from the requirement of stability of the system. Due to the multiwindow nature of ST space we will consider L^p spaces of vector valued function over the subgroup H. It is actually possible to define Lebesgue space for functions defined on a measure space (Ω, μ) with value in a Banach space X:

$$L^p_X(\Omega,\mu) := \{\boldsymbol{f} : \Omega \mapsto X \ : \\ \| \boldsymbol{f} \|_{L^p_X(\Omega,\mu)} := \left(\int_\Omega \| f \|^p_X \, d\mu(x) \right)^{\frac{1}{p}} < \infty \right\}.$$

For the particular case (G, dx), a Banach space is said to have Fourier type with respect to G if the Haussdorf-Young inequality holds. i.e.

$$\mathcal{F} \ : \ L^p_X(G, dx) \mapsto L^{p'}_X(\widehat{G}, d\widehat{x}) \qquad 1 \leq p \leq 2.$$

In the previous chapter we have explored the basic theory of the Haar measure of a LC group. We have seen three main aspects:

- The relationship of the Haar measures d_G, d_H and $d_{G/H}$.
- Relationship between dual couples of spaces.
- Constructive nature of LCA group given by the Pontryagin duality.

This favorable structure, together with the general nature that we want to give to ST spaces, drove my research to the notion of frame and Riesz basis introduced by measurable mapping as in [29].

Definition 6. *Given a measure space (Ω, μ), a Banach space X and a measurable mapping $F : \Omega \to X^*$ the synthesis of F is weakly defined as $U_F : L^q(\Omega) \to X^*$*

$$U_F \boldsymbol{c}(f) := \int_H c(\omega) \langle f, F(\omega) \rangle \, d_\mu \omega.$$

For the case of a ST space generated by a subgroup H and a finite set of distributions $\Phi = (\phi_i)_{i=1,\ldots,R}$, we consider the measure space $(\Omega, \mu) := (H, d_H)$ and $F(\omega) := (L_\omega \phi_i)_{i=1,\ldots,R}$ so the synthesis operator has form

$$U_{\Phi,H} \boldsymbol{c}(f) := \sum_{i=1}^{R} \int_H c_i(a) \langle f, L_a \phi_i \rangle \, d_H a. \tag{2.2}$$

The operator will be called the synthesis operator of the ST system (Φ, H). In the following part of the thesis we will study where the space $L^q(H)$ is mapped. Predominant role will have $K_{\Phi,H}$, the kernel of the synthesis of the (Φ, H) ST system.
Dual concept of the synthesis operator is the analysis operator.

Definition 7. *Given a measure space (Ω, μ), a Banach space X and a measurable mapping $F : \Omega \to X^*$ the analysis operator is defined as the operator defined on X as*

$$V_F f(\omega) := \langle f, F(\omega) \rangle \qquad f \in X, \, \omega \in \Omega.$$

Since we will consider dual pair of ST spaces, we are interested in mappings $G : \Omega \to X^{**}$ for a reflexive space. Unfortunatelly the Schwartz-Bruhat space is not a Banach space but only a Fréchet space. On the other side, a well known subalgebra of $L^1(G)$, the Feichtinger algebra[30]. This space is composed by the functions $f \in L^1(G)$ such that the norm

$$\int_{\widehat{G}} \| M_{\widehat{x}} f * f \| \widehat{x}$$

is finite. Unfortunatelly this Banach space is not reflexive.
For this reason we will consider function in $L^p(G)$ space for $1 < p < \infty$, and related Hölder dual. We will consider analysis in $L^p(G)$ of a ST space generated by $\Phi^* = (\phi_i^*)_{i=1,\dots,R} \subset L^p(G)^* = L^{p'}(G)$ on a subgroup H as

$$V_{\Phi,H} f := \langle L_a \phi_i^*, f \rangle_{L^{p'}(G) \times L^p(G)} \qquad i = 1, \dots, R,\ a \in H.$$

The scope of system theory is to introduce good sets of functions (or distributions) that are capable to decompose in a meaningful and stable way, a given function defined on the LCA group. Our definition of ST spaces does not ensure boundedness of analysis and synthesis operators between the space of coefficients and the ST space, nor uniqueness of the representation. To introduce the theory of frames and basis, and how everything can be formulated at the operator level, I want to mention the following theorem from [37, Lemma 3.4.1].

Theorem 3. *Consider a bounded linear map $T : B_1 \to B_2$ between Banach spaces. If there exists a bounded linear map $R : B_2 \to B_1$ such that*

$$R \circ T f = f \qquad \text{for all } f \text{ in a dense subspace of } B_1, \tag{2.3}$$

then

(I) R can be extended to $T(B_1)$, i.e. it is a left inverse for T.

(II) T is a Banach space isomorphism from B_1 onto $T(B_1)$; in particular there exists $C_1, C_2 > 0$ such that

$$C_1 \| h \|_{B_1} \leq \| Th \|_{B_2} \leq C_2 \| h \|_{B_1} \qquad \forall h \in B_1. \tag{2.4}$$

(III) R is surjective and such that

$$\text{if } Rh = f \quad \to \quad \| h \|_{B_2} \leq C_2 \| f \|_{B_1}.$$

(IV) $P = T \circ R$ is a bounded projection in B_2 onto $T(B_1)$. In particular $T(B_1)$ is complementable in B_2, i.e. $\exists W \subset B_2$ a closed subspace such that $B_2 = T(B_1) \oplus W$.

The strength of the stability theory of linear operators on Banach spaces relies on the possibility to establish particularly stable algorithms (e.g. projections) without the need of working with Hilbert spaces.
Inequalities as (2.4) are used to characterize the useful dense sets that we are looking for. Historically the research has focused on Hilbert spaces theory [17] studying analysis and synthesis operator as acting between the space of square integrable functions and the space of square integrable coefficients: a (not necessarily [12, Definition 5.1]) countable family of functions in L^2 is called a (continuous) frame if (2.4) holds for the analysis

operator; it is called a (continuous) Riesz basis if the inequalities hold for its synthesis. We could extend the Frame-Riesz terminology to a spline-type space, once convergence of the integral over the subgroup is attained, but we prefer to require the more restricting property (2.3) for the synthesis operator, since standard inequality requirement follows and from property (IV) we see that it is the perfect setting for a possible multirelolution approach. We explicitly define frames and bases in L^p Banach spaces

Definition 8. *A weakly measurable mapping $F : \Omega \to X^*$ is called a* ***continuous p-frame*** *for X if there exists $A, B > 0$ such that*

$$A \parallel x \parallel \leq \left(\int_\Omega |\langle x, f(\omega)\rangle|^p \, d\omega \right)^{1/p} \leq B \parallel x \parallel .$$

A ***Bessel mapping*** *is a weakly measurable mapping which ensures the upper bound. The mapping is called a* ***continuous q Riesz basis*** *for X^* if $\langle x, F(\omega)\rangle \ \forall \omega \in \Omega \to x = 0$ and there exists $A, B > 0$ such that*

$$A \parallel c \parallel_q \leq \parallel U_F c \parallel \leq B \parallel c \parallel_q \qquad c \in L^q(\Omega, \mu). \tag{2.5}$$

The synthesis operator of F on L^q is, for p such that $\frac{1}{p}+\frac{1}{q} = 1$, the dual (up to isometric isomorphism from L^p to L^q) of the analysis with value in L^p. Analogous reasoning can be done for the analysis being the dual of the synthesis for a reflexive Banach space X.
If X is reflexive, then F is a continuous p frame iff U_F is well defined and bounded, and it has bounds $\parallel (U_F^*)^{-1} \parallel$ and $\parallel U_F \parallel$[29, Theorem 2.6].
In the representation of signals through a discrete set of functions, central role have biorthogonal systems [5, 7, 6].

Definition 9. *Given a Banach space X and it's dual X^*, a biorthogonal system in $X \times X^*$ is a family $(\phi_i, \phi_i^*)_{i \in I}$ such that*

$$\left\langle \phi_{i_1}, \phi_{i_2}^* \right\rangle = \delta_{i_1, i_2} \tag{2.6}$$

A biorthogonal system is a projection basis in $X_0 \subset X$ if it is a basis for X_0 and

$$P(f) := \sum_{i \in I} \langle f, \phi_i^* \rangle \, \phi_i \qquad \forall f \in X. \tag{2.7}$$

We extend this definition by saying that a mapping $F : \Omega \mapsto X^*$ and a mapping $G : \Omega \mapsto X^{**}$ are biorthogonal if, in analogy with (2.6), their convolution over Ω is the identity of such convolution.
In the theory of measurable mappings a continuous p-Bessel mapping $F : \Omega \to X^*$ for X and a continuous q-Bessel mapping $G : \Omega \to X^{**}$ for X^* compose a dual pair (F, G) if an analogous of (2.7) holds: For reflexive spaces the Bessel mappings are dual if the composition of the analysis V_G with the synthesis U_F gives the identity on X^* [29, Lemma 2.4 (ii), Therem 5.4 (ii), and Definition 5.5]. Biorthogonal systems ensure the pair is dual. For ST system generated by $\Psi := \{\psi_i\}_{i=1}^R \subset L^p(G)$ and a subgroup H we will shortly say that Ψ is biothogonal to $\Phi := \{\phi_i\}_{i=1}^R \subset L^{p'}(G)$ if $\langle L_a \psi_i, L_b \phi j\rangle = \delta_{i,j}\delta_H$ for all $i, j = 1, \ldots, R$ and $a, b \in H$. Thanks to the Haar measure on H, the summation in (2.7) makes sense.
In the proof of Theorem 4 we will bound from below the synthesis through a constructive procedure which leads to the biorthogonal ST system in $L^{p'}(G)$ for the case of a spline-type system in $S^*(G)$ generated by compactly supported distribution and a proper

subgroup.
Starting from a Bessel system this will leads to the characterization of ST $(L^p(H))^R$-Riesz projection basis for $U_{\Phi,H}\,(L^p(H))^R$ in the translation invariant space in which the generating set is selected, and the related left inverse, which also is used in formula (2.7), is the analysis with respect to the ST biorthogonal system.

2.3 L^p-Stability of spline-type spaces

In the literature of SI spaces the lower bound expressed in (2.5) characterizes the injectivity of the synthesis operator and it is called stability of the SI system [50, 24, 75]. This study has been generalized to $L^p(\mathbb{R}^d)$ spaces in [16] but up to our knowledge no generalization in continuous SI space over LCA groups, i.e. ST spaces, was given. In order to introduce the proposed characterization, we show that the theory developed in [50] can be extended to any ST space over LCA group which consider the shifts of a finite set of function or distribution over an arbitrary cocompact subgroup.
We need to prove first two lemmas; the first characterizes the FT of a particular synthesized distribution, while the second is a straightforward consequence of the first.

Lemma 1. ***[DO&SZ]*** *G be a LCA group, H a closed subgroup, ϕ a compactly supported distribution on G, and $\hat{x} \in \hat{H}$.*
Consider the distribution

$$\phi_{\hat{x}} := \langle \cdot, \hat{x} \rangle *_H \phi = U_{\phi,H} \langle \cdot, \hat{x} \rangle . \tag{2.8}$$

Then $\phi_{\hat{x}}$ define a linear functional for integrable functions and, for every $f \in L^1(G)$, its value can be computed in the dual domain by

$$\langle f, \phi_{\hat{x}} \rangle = \int_{H^\perp} \hat{\phi}(\beta^{-1}\hat{x}) \hat{f}(\hat{x}^{-1}\beta) d_{H^\perp}\beta. \tag{2.9}$$

Proof. Because $\mathcal{A}(G) \cdot L^\infty(G) \subset \mathcal{A}(G)$, definition (2.8) makes sense. Because ϕ is continuous on $\mathcal{S}(G)$, and $\mathcal{S}(G)$ dense in $L^1(G)$, then for all $f \in L^1(G)$

$$\langle f, \phi_{\hat{x}} \rangle = \phi \left(\int_H \langle \alpha, \hat{x} \rangle f(\alpha x) d_H \alpha \right).$$

Fix $x \in G$ and consider the function

$$g(y) = \langle y, \hat{x} \rangle f(yx) \in L^1(G)$$

as function over H. Applying the Poisson's formula we obtain

$$\begin{aligned} \int_H g(\alpha) d_H \alpha &= \int_{H^\perp} \hat{g}(\beta) d_{H^\perp}\beta \\ &= \int_{H^\perp} \left\langle x, \hat{x}^{-1}\beta \right\rangle \hat{f}(\hat{x}^{-1}\beta) d_{H^\perp}\beta. \end{aligned}$$

In this way

$$\begin{aligned} \langle f, \phi_{\hat{x}} \rangle &= \phi \left(\int_{H^\perp} \left\langle x, \hat{x}^{-1}\beta \right\rangle \hat{f}(\hat{x}^{-1}\beta) d_{H^\perp}\beta \right) \\ &= \int_{H^\perp} \phi\left(\left\langle x, \hat{x}^{-1}\beta \right\rangle\right) \hat{f}(\hat{x}^{-1}\beta) d_{H^\perp}\beta \\ &= \int_{H^\perp} \hat{\phi}\left(\beta^{-1}\hat{x}\right) \hat{f}(\hat{x}^{-1}\beta) d_{H^\perp}\beta, \end{aligned}$$

where the last equality is obtained through the definition of the FT of a distribution. □

With the previous lemma we can easily prove whether a character, subsampled to a subgroup H, belongs to the kernel $K_{\phi,H}$ of the synthesis operator of a principal ST space $S(\phi, H)$:

Lemma 2. ***[DO&SZ]*** *G be a LCA group, H a closed subgroup, ϕ a compactly supported distribution on G, then for any $\widehat{x} \in \widehat{H}$*

$$\langle \cdot, \hat{x} \rangle \in K_{\phi,H} \quad \textit{iff} \quad \forall \beta \in H^{\perp},\ \hat{\phi}(\beta^{-1}\hat{x}) = 0$$

Proof. By definition $\langle \cdot, \hat{x} \rangle \in K_{\phi,H}$ if and only if $U_{\phi,H} \langle \cdot, \hat{x} \rangle = 0$. Since the hypothesis of Lemma 1 are fulfilled, this is equivalent to saying that the distribution $\phi_{\widehat{x}}$ built in (2.8) is an annihilator distribution.
Looking at the right hand side of (2.9), this can happen iff $\hat{\phi}(\beta^{-1}\hat{x}) = 0$ for any $\beta \in H^{\perp}$. □

Theorem 4. ***[DO&SZ]*** *G be a LCA group, H a cocompact subgroup and a finite generating set $\Phi = \{\phi_i\}_{i=1}^R$ of compactly supported distributions.*
There is equivalence between

(i) $\nexists \boldsymbol{f}(x) = (f_1(\alpha), \ldots, f_R(\alpha)) \in (L^\infty(H))^R$ *such that* $U_{\Phi,H}\boldsymbol{f} = 0$

(ii) $\nexists \xi \in H^{\perp}$ *such that the functions on* $H^{\perp}$, $\left\{\hat{\phi}_j(\cdot\xi)\right\}_{j=1,\ldots,R}$ *are linearly dependent*

(iii) $\exists \sigma > 0$ *such that* $\forall 1 \leq p \leq \infty, \forall \boldsymbol{f} \in (L^p(H))^R$

$$\sigma \parallel \boldsymbol{f} \parallel_{(L^p(H))^R} \leq \parallel U_{\Phi,H}\boldsymbol{f} \parallel_{L^p(G)} . \tag{2.10}$$

Proof. (i)→(ii): If we suppose that $\exists \xi \in H^{\perp}$ and $\exists (a_1, \ldots, a_R) \in \mathbb{C}^R$ such that

$$\phi = \sum_{j=1}^{R} a_j \hat{\phi}_j(\beta\xi) = 0 \qquad \beta \in H^{\perp}$$

then by Lemma 2 $\langle \cdot, \xi \rangle \in K_{\phi,H}$ so by linearity

$$\sum_{j=1}^{R} a_j \langle \cdot, \xi \rangle \in K_{\Phi,H}$$

which contradicts (i).
(iii)→(i): It is trivial because if we can find $\boldsymbol{f} \in K_{\Phi,H} \setminus \{0\}$, then $\parallel \boldsymbol{f} \parallel_{(L^p(H))^R} > \parallel U_{\Phi,H}\boldsymbol{f} \parallel_{L^p(G)} = 0$
(ii)→(iii): If we consider the correlation matrix

$$b_{j,k}(\alpha) := \int_G \phi_j(\alpha x)\overline{\phi_k(x)} d_G x \tag{2.11}$$

as function on H, because the atoms are compactly supported we can consider the FT on H

$$A_{j,k}(\xi) = \int_H b_{j,k}(\alpha)\overline{\langle \alpha, \xi \rangle} d_H \alpha \qquad \xi \in \hat{H}. \tag{2.12}$$

We prove now that the matrix $A(\xi)$ is positive definite for every $\xi \in \hat{H}$.
Because $A(\xi)$ is trivially Hermitian we only have to prove that

$$I(\xi) = \sum_{j,k=1}^{R} a_j A_{j,k}(\xi)\overline{a_k} > 0 \qquad \forall\,(a_1, \ldots, a_R) \in \mathbb{C}^R \setminus \{0\}.$$

Once again, because we are dealing with finitely generated spline-type scheme, we have to manipulate general combination of such atoms. Consider then

$$\begin{aligned} I(\xi) &= \int_H \sum_{j,k=1}^{R} \left(\int_G a_j\phi_j(\alpha x)\overline{a_k\phi_k(x)} d_G x \right) \overline{\langle \alpha, \xi \rangle} d_H \alpha \\ &= \int_H \left(\int_G \phi(\alpha x)\overline{\phi(x)} d_G x \right) \overline{\langle \alpha, \xi \rangle} d_H \alpha \end{aligned}$$

where $\phi = \sum_{j=1}^{R} a_j \phi_j$.
Now, using the Weil formula and Fubini theorem we obtain

$$\begin{aligned} I(\xi) &= \int_H \left(\int_G \phi(\alpha x)\overline{\phi(x)} d_G x \right) \overline{\langle \alpha, \xi \rangle} d_H \alpha \\ &= \int_H \int_{G/H} \int_H \phi(\alpha x\gamma)\overline{\phi(x\gamma)\langle \alpha, \xi \rangle} d_H \gamma d_{G/H}|x| d_H \alpha \\ &= \int_{G/H} \int_H \int_H \phi(\alpha x\gamma)\overline{\phi(x\gamma)\langle \alpha, \xi \rangle} d_H \gamma d_H \alpha d_{G/H}|x| \\ &= \int_{G/H} |h(x)|^2 \, d_{G/H}|x| \end{aligned}$$

where $h(x) := \int_H \phi(\alpha x)\overline{\langle \alpha, \xi \rangle} d_H \alpha = U_{\phi,H}\left(\langle \cdot, \xi \rangle\right) \neq 0$ by Lemma 2.
Because it is non null in the quotient for LCA group too, we have that $I(\xi)$ is positive definite.
From this we can compute the inverse of A. Since $\Phi = \{\phi_i\}_{i=1}^{R}$ is a finite generating set of compactly supported distributions, A is a matrix in $\mathbb{C}^{R\times R}$ whose entries are trigonometric polynomials, hence each $\left(A^{-1}\right)_{j,k}$ is a quotient of trigonometric polynomial, whose denominator never vanish.
Since H is cocompact, $H^{\perp}$ is compact, thus by Wiener's Lemma, $\left(A^{-1}\right)_{j,k}$ can be expressed as an absolutely convergent integral in $\mathcal{F}^1(H)$

$$\left(A^{-1}\right)_{j,k}(\xi) = \int_H g_{j,k}(\alpha)\overline{\langle \alpha, \xi \rangle} d_H \alpha \tag{2.13}$$

with $g_{l,k} \in L^1(H)$.
Consider the vector valued functions on H, $\boldsymbol{g}_l = (g_{l,1}, \ldots, g_{l,R})$, we build the set of functions

$$\psi_l = U_{\Phi,H}\boldsymbol{g}_l \qquad l = 1, \ldots, R \tag{2.14}$$

Fixing $k \in \{1, \ldots, R\}$ we consider for all $l = 1, \ldots, R$ and for every $\gamma \in H$ the products

$$\begin{aligned} c_{i,j}(\gamma) &:= \left\langle \psi_l, \phi_k(\gamma^{-1}\cdot) \right\rangle \\ &= \sum_{m=1}^{R} \int_G \left(\int_H g_{l,m}(\alpha)\phi_m(\alpha^{-1}x)\overline{\phi_k(\gamma^{-1}x)} d_H\alpha \right) d_G x \\ &= \sum_{m=1}^{R} \int_H g_{l,m}(\alpha) \left(\int_G \phi_m(\alpha^{-1}x)\overline{\phi_k(\gamma^{-1}x)} d_G x \right) d_H\alpha \\ &= \sum_{m=1}^{R} \int_H g_{l,m}(\alpha) b_{m,k}(\alpha^{-1}\gamma) d_H\alpha \end{aligned}$$

hence

$$\begin{aligned} &\int_H c_{i,j}(\gamma)\overline{\langle \gamma, \xi \rangle} d_H\gamma \\ &= \sum_{m=1}^{R} \int_H \int_H g_{l,m}(\alpha) b_{m,k}(\alpha^{-1}\gamma) d_H\alpha \, \overline{\langle \gamma, \xi \rangle} d_H\gamma \\ &= \sum_{m=1}^{R} A_{m,k}(\xi) \left(A^{-1}\right)_{l,m}(\xi) = \delta_{k,l}. \end{aligned}$$

This shows that

$$\langle L_\gamma \psi_l, L_\alpha \phi_k \rangle = \langle L_{\alpha^{-1}\gamma} \psi_l, \phi_k \rangle = \delta_{\alpha,\gamma}\delta_H.$$

Which mean that we have build through the inversion of the matrix A in the Fourier domain, a set of atom $\Psi = \{\psi_1, \ldots, \psi_R\}$ which is the biorthogonal system of Φ.
The reproduction formula holds, then $\forall \zeta \in \mathcal{S}(\phi, H)$

$$\zeta = U_{\Phi,H}\boldsymbol{f} = \sum_{j=1}^{R} \int_H f_j L_\alpha \phi_j d_H\alpha$$

with

$$f_j = \left(\langle \zeta, L_\alpha \psi_j \rangle \right)_{\alpha \in H}.$$

If we fix a p-norm, because $\phi_j, \psi_j \in L^\infty(G)$ for all $j = 1, \ldots, R$ and $g_{jk} \in L^1$ for all $j, k = 1, \ldots, R$ and apply the Young inequality for convolution twice we obtain

$$\begin{aligned} \| f_j \|_{L^p(H)} &\leq \| \zeta \|_{L^p(G)} \| \psi_j \|_{L^\infty(G)} \\ &\leq \| \zeta \|_{L^p(G)} \sum_{m=1}^{R} \| \phi_m \|_{L^\infty(G)} \| g_{j,m} \|_{L^1(G)}. \end{aligned}$$

Then, we obtain (2.10) with

$$\sigma^{-1} = \max_{j=1,\ldots,R} \sum_{m=1}^{R} \| \phi_m \|_{L^\infty(G)} \| g_{j,m} \|_{L^1(G)}.$$

□

The Theorem is particularly appealing since it displays the constructive nature of the theory. We can turn it into Algorithm 1. This algorithm builds the biorthogonal $\tilde{\Phi}$ of the given ST system. If the original preserves also an upper bound for the synthesis operator in $L^q(H)$ then we obtain the reproduction formula for $f \in X^*$

$$f = P_{\Phi,H} f = U_{\Phi,H} V_{\tilde{\Phi},H} f. \tag{2.15}$$

Algorithm 1 [DO&SZ] Computation of the coefficients of the dual system as in (2.14)

Precondition: Windows $\Phi = \{\phi_r\}_{r=1}^{R}$, subgroup H

function COEFFICIENT_COMPUTATION(Φ,H)
  for $r \leftarrow 1$ to R **do**
    $\hat{\phi}_r \leftarrow \mathcal{F}(\phi)$
  end for
  for $r, l \leftarrow 1$ to R **do**
    $G_{r,l} \leftarrow \mathcal{F}^{-1}(\hat{\phi}_r \hat{\bar{\phi}}_l)$ `% Convolutions in` (2.11)
  end for
  $G_{r,l}^{(s)} \leftarrow G_{r,l}(H)$ `% Subsampling to` H
  for $r, l \leftarrow 1$ to R **do**
    $\hat{G}_{r,l}^{(s)} \leftarrow \mathcal{F}_H(G_{r,l}^{(s)})$ `% PD matrix in` (2.12)
  end for

  for $x \in H$ **do**
    $\hat{\boldsymbol{g}}(x) \leftarrow \left(\hat{G}^{(s)}(x)\right)^{-1}$ `% Wiener's inversion in` (2.13)
  end for

  for $r, l \leftarrow 1$ to R **do**
    $\boldsymbol{g}_{r,l} \leftarrow \mathcal{F}_H^{-1}(G_{r,l}^{(s)})$ `% Coefficients` (2.14)
  end for
  return $\boldsymbol{g}$
end function

2.3.1 Stability for sequence of projections

In this section we consider uniform boundedness of the projection operators into a sequence of ST spaces obtained by modifying the generating set and the subgroup through a sequence of automorphisms.

Particularly important feature of an automorphism α of a LC group is its modulus which is the (unique) positive number $\Delta(\alpha)$ such that the composition with the Haar measure dx on G is $d\alpha(x) = \Delta(\alpha)dx$. For each automorphism α on G the adjoint α^* is defined as the automorphism on $\widehat{G}$ such that $\langle \alpha x, \widehat{x} \rangle = \langle x, \alpha^* \widehat{x} \rangle$. Modulus, adjoint and inverse of an automorphism satisfy the following properties:

$$\Delta(\alpha^*) = \Delta(\alpha)$$
$$\Delta(\alpha^{-1}) = \Delta(\alpha)^{-1}$$

and composition with an automorphism and FT follows the property

$$\widehat{f \circ \alpha} = \widehat{f} \circ (\alpha^*)^{-1}.$$

To build orthonormal wavelets over local field in [10] expansive automorphism [57] with respect to a subgroup of the additive group structure were introduced. A slightly more restrictive definition is given in [81] through contractive automorphism: the inverse of a contractive automorphism is expansive, but the inverse of an expansive automorphism is not contractive in general. However both definition display *scaling property*: an expansive automorphism σ has modulus $\Delta(\sigma) < 1$, while a contractive automorphism τ has modulus $\Delta(\tau) > 1$.
On the stage of stability the expansive/contractive nature of an automorphism needs not to be made explicit. Only on a subsequent stage of consistency, i.e. approximation power, this aspect needs to be taken into acconut.
Here we are just interested on the recursive use of an automorphism, hence no assumption on the contractive nature of the automorphism is posed. We plan in the next future to study how a contractive automorphism induces a multiresolution analysis which provide approximation order as described for SI space in [24]. Our approach resemble the one from [10] since it makes use of automorphisms over the LCA group, but does not require a multiresolution framework. This type of analysis seems to be original in the litterature.
We define the operator associated to an automorphism τ on a banach space X

$$D_\tau f = f \circ \tau^{-1} \qquad f \in X.$$

Immediate properties of the operator D_τ are that $(D_\tau)^{-1} = D_{\tau^{-1}}$ and that $\| D_\tau f \| = \Delta(\tau)^{\frac{1}{p}} \| D_\tau f \|$ in $L^p(G)$ for every $1 \leq p \leq \infty$, i.e. the operator is scalar multiple of an isometric isomorphism.
Similarly as for the shift, we weakly define the $D_{*,\tau}$ operator for distribution in X^* as the adjoint of $D_{\tau^{-1}}$:

$$\langle f, D_{*,\tau}\phi \rangle := \left\langle D_\tau^{-1} f, \phi \right\rangle \quad \forall f \in X.$$

The definition is well posed and compatible with the definition of D_τ since for $f \in X \subset X^{**}$,

$$\langle \phi, D_{**,\tau} f \rangle = \left\langle D_{*,\tau}^{-1}\phi, f \right\rangle = \left\langle f, D_{*,\tau}^{-1}\phi \right\rangle = \langle D_\tau f, \phi \rangle .$$

We now build through D_τ a sequences of projections. Given a vector space of function or distribution S, consider the contracted space

$$S^\tau := \{ D_\tau s \ : \ s \in S \} .$$

It is natural to define the sequence

$$S^n := S^{\tau^n} \qquad n \in \mathbb{N}.$$

D_τ do not commute with shift since

$$L_x D_\tau = D_\tau L_{\tau^{-1}x} \quad D_\tau L_x = L_{\tau x} D_\tau,$$

and equivalent properties for $D_{*,\tau}$ and the dual formulation of the shift on on X^*. For this reason we also have that

Proposition 6. ***[DO&SZ]*** *Let (Φ, H) a stable ST system of distribution having a biorthogonal system $(\tilde{\Phi}, H)$,and τ an automorphism.*
Then the contracted space $S(\Phi, H)^\tau$ is the ST space $S(D_{*,\tau}\Phi, \tau H)$ and $(D_\tau \tilde{\Phi}, \tau H)$ is the biothogonal dual.

Proof. Applying the commutation law and the definition of $D_{*,\tau}$ we have for all $x \in \tau H$

$$\left\langle L_x D_\tau \tilde{\phi}_i, D_{*\tau}\phi_j \right\rangle = \left\langle D_{\tau^{-1}} L_x D_\tau \tilde{\phi}_i, \phi_j \right\rangle$$
$$\left\langle L_{\tau^{-1}x} D_{\tau^{-1}} D_\tau \tilde{\phi}_i, \phi_j \right\rangle = \left\langle L_{\tau^{-1}x} \tilde{\phi}_i, \phi_j \right\rangle = \delta_{i,j}\delta_H.$$

□

Corollary 1. ***[DO&SZ]*** *Under the same condition of Proposition 6, the projection operator into the space $S(D_\tau\Phi, \tau H)$ is*

$$P_{D_\tau\Phi,\tau H} = U_{D_\tau\Phi,\tau H} V_{D_\tau\tilde{\Phi},\tau H}$$

The standard dyadic contraction $D_2 f(x) = \frac{1}{\sqrt{2}} f(2x)$ is an unitary operator over $L^2(\mathbb{R}^d)$, e.g. $\langle D_2 f, D_2 g\rangle = \langle f, g\rangle$. The couple $(D_\tau, D_{*,\tau})$ behaves in a similar manner since $\langle D_\tau f, D_{*,\tau}\phi\rangle = \langle f, \phi\rangle$, even if the contraction are not isometries. Since these operators are not isometries we are interested in how a frame or a Riesz basis are influenced by such operators.

Lemma 3. ***[DO&SZ]*** *Let $(\Phi, H) \subset X$ a p-frame for X^* with frame bounds $0 < A < B$. Then $(D_\tau\Phi, \tau H)$ is a p-frame having frames bounds*

$$0 <\| D_{\tau^{-1}} \|^{-1} A <\| D_\tau \| B.$$

Let $(\Phi, H) \subset X^*$ a q-Riesz basis for X^* with bounds $0 < A < B$. Then $(D_\tau\Phi, \tau H)$ is a q-Riesz basis having bounds

$$0 <\| D_\tau \|^{-1} A <\| D_{\tau^{-1}} \| B.$$

Proof. Let $\psi \in X^*$, for the analysis operator of $(D_\tau\Phi, \tau H)$ sums as

$$\begin{aligned}
&\sum_i \int_{\tau H} |\langle L_x D_\tau \phi_i, \psi\rangle|^p \, dx \\
=&\sum_i \int_{\tau H} |\langle D_\tau L_{\tau^{-1}x} \phi_i, \psi\rangle|^p \, dx \\
=&\sum_i \int_{\tau H} |\langle L_{\tau^{-1}x} \phi_i, D_{*,\tau^{-1}}\psi\rangle|^p \, dx \\
=&\sum_i \int_{H} |\langle L_x \phi_i, D_{*,\tau^{-1}}\psi\rangle|^p \, dx
\end{aligned}$$

hence

$$\begin{aligned}
\left(\sum_i \int_{\tau H} |\langle L_x D_\tau \phi_i, \psi\rangle|^p \, dx\right)^{1/p} &= \left(\sum_i \int_{H} |\langle L_x \phi_i, D_{*,\tau^{-1}}\psi\rangle|^p \, dx\right)^{1/p} \\
&\leq B \| D_{*,\tau^{-1}}\psi \|_{X^*} \leq B \| D_{*,\tau^{-1}} \| \| \psi \|_{X^*} \\
&\leq B \| D_\tau \| \| \psi \|_{X^*},
\end{aligned}$$

while for the lower bound, consider $\psi \in X^*$

$$\begin{aligned}
\| \psi \|_{X^*} &= \| (D_\tau D_{\tau^{-1}})^* \psi \|_{X^*} = \| D^*_{\tau^{-1}} D^*_\tau \psi \|_{X^*} \\
&\leq \| D^*_{\tau^{-1}} \| \| D^*_\tau \psi \|_{X^*} \\
&\leq \frac{\| D_{\tau^{-1}} \|}{A} \left(\sum_i \int_H |\langle L_x \phi_i, D^*_\tau \psi \rangle|^p \, dx \right)^{1/p} \\
&= \frac{\| D_{\tau^{-1}} \|}{A} \left(\sum_i \int_{\tau H} |\langle L_x D_\tau \phi_i, \psi \rangle|^p \, dx \right)^{1/p} .
\end{aligned}$$

For the Riesz basis condition we need to consider that for all $f \in X$

$$\begin{aligned}
\langle f, U_{D_{*,\tau}\Phi, \tau H} c \rangle &= \sum_i \int_{\tau H} c(x) \langle f, L_x D_{*\tau} \phi_i \rangle \, dx \\
&= \sum_i \int_H c(\tau x) \langle f, L_{\tau x} D_{*\tau} \phi_i \rangle \, dx \\
&= \sum_i \int_H c(\tau x) \langle f, D_{*\tau} L_x \phi_i \rangle \, dx \\
&= \sum_i \int_H c(\tau x) \langle D_{\tau^{-1}} f, L_x \phi_i \rangle \, dx \\
&= \langle D_{\tau^{-1}} f, U_{\Phi,H} D_{\tau^{-1}} c \rangle \\
&= \langle f, D_{*\tau} U_{\Phi,H} D_{\tau^{-1}} c \rangle
\end{aligned}$$

hence for all $c \in (L^q(\tau H))^r$

$$\begin{aligned}
\| U_{D_{*,\tau}\Phi,\tau H} c \|_{X^*} &= \| D_{*\tau} U_{\Phi,H} D_{\tau^{-1}} c \| \\
&\leq \| D_{*\tau} \| \| U_{\Phi,H} D_{\tau^{-1}} c \|_{X^*} \\
&\leq B \| D_{\tau^{-1}} \| \| D_{\tau^{-1}} c \|_{(L^q(H))^r} \\
&\leq B \| D_{\tau^{-1}} \| \| c \|_{(L^q(\tau H))^r},
\end{aligned}$$

while for the lower bound, considering $c \in (L^q(\tau H))^r$

$$\begin{aligned}
\| c \|_{(L^q(\tau H))^r} &= \| D_{\tau^{-1}} c \|_{(L^q(H))^r} \\
&\leq \frac{1}{A} \| U_{\Phi,H} D_{\tau^{-1}} c \|_{X^*} \\
&= \frac{1}{A} \| D_{*,\tau^{-1}} D_{*,\tau} U_{\Phi,H} D_{\tau^{-1}} c \|_{X^*} \\
&\leq \frac{1}{A} \| D_{*,\tau^{-1}} \| \| U_{D_{*,\tau}\Phi,\tau H} c \|_{X^*} \\
&= \frac{\| D_\tau \|}{A} \| U_{D_{*,\tau}\Phi,\tau H} c \|_{X^*} .
\end{aligned}$$

□

Corollary 2. ***[DO&SZ]*** *Let $(\Psi, H) \subset X$ a p-frame for X^* with bounds $0 < A_1 < B_1$ and $(\Phi, H) \subset X^*$ a q-Riesz basis for X^* with bounds $0 < A_2 < B_2$. Then for all $f \in X^*$*

$$C_\tau^{-1} A_1 A_2 \| f \|_{X^*} \leq \| U_{D_{\tau^n}\Psi, \tau^n H} V_{D_{\tau^n}\Phi, \tau^n H} f \| \leq C_\tau B_1 B_2 \| f \|_{X^*}$$

being $C_\tau = \| D_{\tau^{-1}} \| \| D_\tau \|$.

The previous corollary gives us a strong constraint for the operator D_τ: for a general bounded and invertible operator T on a Banach space we have that $\| T \|\| T^{-1} \| \geq 1$ and equality holds iff T is a scalar multiple of isometric isomorphism. If strong inequality holds, recursive application of D_τ would let the constant C_τ^{-1} tent to zero and C_τ explode. Since we are interested in uniformly boundedness of projection operators we summaries this result in the following theorem

Theorem 5. ***[DO&SZ]*** *For a stable ST space (Φ, H) with the Riesz bounds A_2 and B_2, having a biorthogonal system with frame bounds A_1 and B_1 the frame bounds of the sequence of projection operators into the spaces $S(D_{\tau^n}\Phi, \tau^n H)$ are stable in the following sense:*

$$A_1 A_2 \| f \|_{X^*} \leq \| P_{D_{\tau^n}\Phi, \tau^n H} f \| \leq B_1 B_2 \| f \|_{X^*} \qquad f \in X^*$$

iff D_τ is a multiple of an isometric isomorphism.

Example 1. *Be $G = \mathbb{Z}$ and consider the subgroups $H_k = 2^k\mathbb{Z}$ for $k \in \{4, 5, 6, 7, 8\}$ which can be obtained through the automorphism $x \mapsto 2X$. The dual of G is $\widehat{G} = \mathbb{S}$ and the orthogonals of H_k are the subgroups $H_k^\perp = \{e^{i\frac{l}{2^k}}, l = 0, \ldots, 2^k - 1\}$, respectively.*
As generating set, consider as basic tool the standard Hermite cubic basis on $[0, 1]$

$$\begin{aligned} h_1(t) &= 2t^3 - 3t^2 + 1 \\ h_2(t) &= t^3 - 2t^2 + t \\ h_3(t) &= -2t^3 + 3 * t^2 \\ h_4(t) &= t^3 - t^2. \end{aligned}$$

Ar each level k we have considered the functions $D_{1/2^k} h_i$ and sampled over the integers. On each level k the ST system $(\{D_{1/2^k} h_i,\ i = 1, \ldots, 4\}, 2^k\mathbb{Z})$ satisfied the linear independence condition (ii) of Theorem 4.
In the following table we show the minimum singular value on each level:

k=4	*k=5*	*k=6*	*k=7*	*k=8*
0.4299	*0.7830*	*1.4886*	*2.9000*	*5.7229*

Remark 2. *Theorem 5 can be generalized for an arbitrary sequence of automorphism: given a stable ST space $S(\Phi, H)$ a sequence of automorphism $\{\tau_n\}_{n\geq 1}$, we have stability for the sequence of ST spaces*

$$\begin{cases} S_0 := S(\Phi, H) & \\ S_n := S_{n-1}^{\tau_n} & n \geq 1 \end{cases}$$

iff only a finite number of τ_n are not multiple of isometric isomorphisms.

The approach is original and does not resemble standard construction of wavelets and shearlets system[52] since no multiresolution constraint is required. We plan to apply these results for the analysis of the stability of Petrov-Galerkin discretization schemes.

2.3.2 Remarks about principal spline-type spaces

In SI theory, important role have principal shift invariant spaces, i.e. spaces generated by the integer shift only one function. In particular, they have central role in the computation of the approximation power of a shift invarian space[24].
Since we aim to build a stability-consistency theory for ST space, the same relationship should be studied in ST theory.
As usual our only tool is the explicit formula of the projection of $g \in X^*$ into the ST space $S(\Phi, H) \subset X^*$ and the related projection of $h \in X$ a into the space $S(\tilde{\Phi}, H) \subset X$

$$P_{\Phi,H}g = U_{\Phi,H}V_{\tilde{\Phi},H}g = \sum_i \int_H \left\langle g, L_x\tilde{\phi}_i \right\rangle L_x\phi_i dx$$
$$P_{\tilde{\Phi},H}h = U_{\tilde{\Phi},H}V_{\Phi,H}h = \sum_i \int_H \left\langle L_x\phi_i, h \right\rangle L_x\tilde{\phi}_i dx$$

The strength of a multiwindow system lies on the possibility to have more localized information of a signal in terms of a dictionary built through shifts of fixed atoms. We want to study the mutual relationship between such spaces and a principal ST space $S(g, H)$ built through the same subgroup and the condition of Theorem 4.

Proposition 7. ***[DO&SZ]*** *Let (Φ, H) a stable system and (g, H) a principal stable system having related biorthogonals $\tilde{\Phi}$ and $\tilde{g}$, respectively. Then*

$$P_{\tilde{\Phi},H}\tilde{g} \quad \textit{is biothogonal to} \quad P_{\Phi,H}g.$$

Proof. Let $x \in H$, since the projection onto a H-invariant space is invariant under H-shift invariant, e.g. $L_xP_{\Psi,H} = P_{\Psi,H}L_x$ for all $x \in H$ and arbitrary finite set of distribution (or function) Ψ, we have that

$$\begin{aligned}
\left\langle P_{\Phi,H}g, L_xP_{\tilde{\Phi},H}\tilde{g} \right\rangle &= \left\langle P_{\Phi,H}g, P_{\tilde{\Phi},H}L_x\tilde{g} \right\rangle \\
&= \left\langle \sum_i \int_H \left\langle g, L_y\tilde{\phi}_i \right\rangle L_y\phi_i dy, \sum_j \int_H \left\langle L_z\phi_j, L_x\tilde{g} \right\rangle L_z\tilde{\phi}_j dz \right\rangle \\
&= \sum_i \sum_j \int_H \int_H \left\langle g, L_y\tilde{\phi}_i \right\rangle \left\langle L_y\phi_i, L_z\tilde{\phi}_j \right\rangle \left\langle L_z\phi_j, L_x\tilde{g} \right\rangle dy\, dz \\
&= \sum_i \int_H \left\langle g, L_y\tilde{\phi}_i \right\rangle \left\langle L_z\phi_i, L_x\tilde{g} \right\rangle dy \\
&= \left\langle g, L_x\tilde{g} \right\rangle = \delta_H.
\end{aligned}$$

□

Using the previous proposition we can also prove

Proposition 8. ***[DO&SZ]*** *Let (Φ, H) a stable system and (g, H) a principal stable system having related biorthogonals $\tilde{\Phi}$ and $\tilde{g}$ respectively, then*

$$P_{P_{\Phi,H}g}g = P_{\Phi,H}g.$$

Proof. From the previous lemma

$$P_{P_{\Phi,H}g,H} := U_{P_{\Phi,H}g,H}V_{\widetilde{P_{\tilde{\Phi},H}g},H} = U_{P_{\Phi,H}g,H}V_{P_{\tilde{\Phi},H}\tilde{g},H}$$

hence

$$\begin{aligned}
P_{P_{\Phi,H}g,H}g &= \int_H \left\langle g, L_x P_{\tilde{\Phi},H}\tilde{g}\right\rangle L_x P_{\Phi,H} g dy \\
&= \int_H \left\langle g, P_{\tilde{\Phi},H} L_x \tilde{g}\right\rangle L_x P_{\Phi,H} g dx \\
&= \int_H \left\langle g, \left(\sum_j \int_H \langle L_y \phi_j, L_x \tilde{g}\rangle L_y \tilde{\phi}_j dy\right)\right\rangle L_x P_{\Phi,H} g dy \\
&= \int_H \left(\sum_j \int_H \langle L_y \phi_j, L_x \tilde{g}\rangle \left\langle g, L_x \tilde{\phi}_j\right\rangle dz\right) L_y P_{\Phi,H} g dy \\
&= \int_H \langle g, L_y \tilde{g}\rangle L_y P_{\Phi,H} g dy = \delta_H *_H P_{\Phi,H} g \\
&= P_{\Phi,H} g.
\end{aligned}$$

□

From this it is easy to show that

Proposition 9. ***[DO&SZ]***

$$P_{P_{\Phi,H}g} P_{g,H} = P_{\Phi,H} P_{g,H} \tag{2.16}$$

Part II

Realization and analysis of fixed dictionaries through ST spaces

Introduction to Part II

In the previous part of the thesis, we have explored the theoretical strength of ST spaces. In the present one, we explore some applications of the ST construction to build well known and useful dictionaries, Gabor and Wavelet systems, which can be reformulated[35, 61] as particular cases of SI systems.
Frames are a major tool in signal processing because they are used to decompose and to analyze signals in a stable and redundant manner. The concept of frame starts from a more general idea of giving the numerical analyst a dictionary able to understand a signal by a collection of given *words*. These words are usually characterized by their localization properties into a given parameter space.
SI spaces, which uses a collection of functions and their integer shifts, consider the localization on the function in the domain where the function is defined. Historically, 1D signals have been interpreted as time series, so we say that SI spaces analyze the *time localization* of the signal. Through our generalization of SI space into ST spaces, we will continue saying that localization onto a subgroup of the given LCA group G means time localization.
On the other side Gabor analysis uses a set of time-frequency shifted copies of a given atom obtained through the operator (1.19). The frequency domain is nothing but the Pontryagin dual of the time domain, and for this reason we will consider generalized Gabor-like systems as generated by shifts in time domain, and multiplication with a character, accordingly to property (1.17) which states that character multiplication corresponds to a shift of the FT.
The efficient computation of the dual atoms of a Gabor system analogous to (2.14) which gives the possibility to establish a reproduction formula similar to (2.15) was a long-standing interesting topic in the literature and it was for many years a drawback in the development of the applied field due to the high computational costs involved. Several efficient algorithms exists nowadays for both the computation of the dual atom [89] and for the approximation of it [33]. Nevertheless, even today, some of the most efficient Gabor algorithms are effected by a restrictive constraint: the samples are taken on a rectangular lattice in the time-frequency plane. Here the multi-window nature of spline-type spaces will be used to analyse Gabor systems from a different perspective: shift in frequency will be treated as different generators of our shift invariant space in time. For bandlimited signals, rather than having a redundant system we will select only the useful modulation. In this way, we will be able to weaken the restriction of a lattice structure to only time shifts. I personally presented this work during the 18th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)[61].
This part of the thesis continue analyzing another dictionary depending on discrete paramethers: the Hough transform. The connection with harmonic analysis was found in [25] where the author interprets the Hough transform as particular case of the Radon transform. The formalism was pour as highlighted in [69]. Recently it was shown that the Hough transform of a digital image tends to become the (generalized) Radon transform of the image itself as the discretization of the parameter space becomes finer and finer[3]. Nevertheless we will focus on the basic definition of the generalized Hough transform due to its velocity and versatility; a Gabor-based filter will be used as post-processing of the, notoriously poor, outcome of the Hough transform. Related publications are [65] and [64].
Finally, I present a work about multiresolution-like approximation of signal, in which I

use considerations coming from the Gabor systems section about modulation's selection to speed up the approximation power of the multilevel approximation, with a little time-constant drawback in complexity. This gave me the possibility to implement a dictionary which considers time, frequency and scale localization of the generatig set of the ST space. The work was presented in [63].

Chapter 3

Gabor-like frames based on realizable spline-type constructions

In this chapter, we use the multi-window nature of spline-type spaces to reformulate Gabor systems from a different perspective: different shifts in frequency will be treated as different generators of our shift invariant space in time. In this way, we will be able to weaken the restriction of a lattice structure to only time shifts.
The focus of this section relies on time-frequency analysis [43]. The goal of this research area is to describe the time-frequency behaviour of *signals*, seen as elements of some function space.
We analyse from this perspective Gabor systems, which are build on a different structure than basic shift-invariant systems, but similar to spline-type spaces. In signal analysis, where the function are assumed to be square integrable as condition of finite energy, the most common tool to analyze the local behavior of a function is the continuous Short-Time Fourier Transform (STFT)

$$V_\phi f(x,\omega) := \int_{\mathbb{R}^d)} f(t)e^{2\pi i t\cdot\omega}\overline{\phi(t-x)}dt$$

for some $\phi \in L^2(\mathbb{R}^d)$.
We can generalize this approach by introducing Gabor systems over LCA groups, which are built upon the two different shifts that we introduced in chapter 1: the left shift (1.3) and the frequency shift (1.15).

Definition 10. *Given a function ϕ defined on a LCA group G and a lattice $\Delta \in G \times \widehat{G}$, we define the Gabor system (or Weyl-Heisenberg system) as*

$$\mathcal{G}(\phi,\Delta) = \{\pi(\lambda)\phi \mid \lambda \in \Delta\}.$$

The analysis operator of a Gabor system is the STFT (3.1) with respect to the window function, subsampled on the lattice δ

$$V_\phi f(x,\widehat{x}) := \langle \langle x,\widehat{x}\rangle L_x\phi, f\rangle_{X\times X^*} \tag{3.1}$$

Despite the fact that the commutation property (1.18) is not symmetric, the linear span of a Gabor system generated by a window g and a separable lattice $\Delta = \Delta_{time} \times \Delta_{freq}$, coincides with the spline-type space $\mathcal{S}\left(\{M_i g \,:\, i \in \Delta_{freq}\}, \Delta_{time}\right)$ [43].

The connection between spline-type spaces and constructive Gabor-like systems becomes natural. Given a set of modulations $\{M_i g \,:\, i \in I\}$, we can describe a frequency-generalised Gabor frame i.e. a system of functions generated by the time-frequency shifts on

$$\Delta \times I$$

where Δ is a subgroup of only time.
In this way, we are establishing a theory that can handle non-uniform sampling in frequency. It will become clearer in the next sections, why this will be really helpful once some quality of the spectrum of the signal is known.

Remark 3. *Because we chosen to use only some modulated atoms, we are not bound anymore to the dense structure of a Gabor lattice.*
Several fast ways to implement the Gabor transform are indeed based on the fact that the matrix of the Gabor operator has a particularly nice structure e.g. the matrix operator is block circulant. Through the construction of a non-uniform Gabor frame as proposed in this chapter, we can reach an algorithm having the same complexity but with a major flexibility given by the cancelation of all the unnecessary modulations. In this way, the frequency domain can be still completely discretized as in a classical Gabor transform but the employed modulations are arranged in a non-lattice structure.

The analysis and synthesis operators of a Gabor system build the so called frame operator

$$Sf := TT^* f = \sum_{\lambda \in \Delta} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g.$$

If the frame operator is invertible, thanks to the commutation between frame operator and shifts, we obtain the core of the Gabor theory

Theorem 6. *[43]Reproduction theorem for Gabor frames*
Given a Gabor frame $\mathcal{G}(\phi, \Delta)$ every function in it's shift invariant generated space can be represented as

$$f = \sum_{\lambda \in \Delta} \left\langle f, \pi(\lambda) \left(S^{-1} g\right) \right\rangle \pi(\lambda) g \tag{3.2}$$

$$= \sum_{\lambda \in \Delta} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) \left(S^{-1} g\right). \tag{3.3}$$

We call the function $S^{-1} g$ canonical dual of the window g. It belongs to the space generated by the Gabor system so it posses an expression (3.2).
Any other function h such that

$$\langle h, \pi(\lambda') g \rangle = \delta_{0,\lambda'} \tag{3.4}$$

is called dual of the atom g and can take place of the dual window in formula (3.2).
If we want to build a dual, we can impose biorthogonality between the shifts of the original system and the shift of the biorthogonal window, which possess a representation as in (3.2).

Since the inversion of the frame operator of a Gabor system is quite restricting for applications, especially for dictionaries theory, we can apply the biorthogonality constrain to a Gabor system and obtain

$$\begin{aligned}\delta_{0,\lambda'} &= \langle \tilde{g}, \pi(\lambda')g\rangle \\ &= \left\langle \sum_{\lambda\in\Delta} b_\lambda \pi(\lambda)g, \pi(\lambda')g \right\rangle \\ &= \sum_{\lambda\in\Delta} b_\lambda \langle \pi(\lambda)g, \pi(\lambda')g\rangle \\ &= \sum_{\lambda\in\Delta} b_\lambda \langle x, \widehat{x}' - \widehat{x}\rangle V_g g(\lambda' - \lambda) \qquad (3.5)\end{aligned}$$

for $\lambda = (x, \widehat{x})$ and $\lambda' = (x', \widehat{x}')$.
The previous expression is called the **twisted convolution** of the unknown coefficients with the STFT of the window itself.
The feasibility of the method is ensured by the application of the Wiener's lemma on the Heisenberg group. The theorem is unfortunately not constructive[45], and the authors are not aware of any fast implementation of the inversion for a general framework. The reason relies in the symplectic nature of the Heisenberg-Weyl group, e.g. the commutation rule (1.18) of time and frequency shift operators.
We will use the biorthogonality relations as expressed in (3.4) to build the dual of a Gabor frame via a multi-window spline-type scheme.
This will give us also the possibility to apply the theory of Wiener's lemma on multi-window spline-type spaces for a generating set composed by sparse modulations.

3.1 Constructive reformulation of Gabor-like systems as multi-window spline-type systems

In this section we invert the twisted convolution through multi-window spline-type(MST) reformulation of Gabor systems.
The central topic of spline-type spaces is given by the following theorem in [32], with a trivial generalization in the multi-window setting an in [24].

Theorem 7. *Given a MST Riesz projection basis such that finite sequences in the solid Banach space are dense, then the dual $\mathcal{S}'$ of the generated MST space is also a ST space and it is generated by its dual system.*

The fact that many properties of the system of generators of a spline-type space translate to the dual system give rise to the following reproduction rule in spline-type space.

Theorem 8. *[36]Reproduction theorem in MST space*
Let $\mathcal{S}(\Phi, \Delta)$ the MST space generated by Δ and $\Phi = \{\phi_i\}_{ß\in I}$ that give rise to a Riesz projection basis thrugh its time-shift.
Given any dual system $\Psi = \{\psi_i\}_{ß\in I}$, then $\forall f \in \mathcal{S}(\Phi, \Delta)$

$$f = \sum_{i\in I}\sum_{x\in\Delta} \langle f, L_x\psi_i\rangle L_x\phi_i \qquad (3.6)$$

$$= \sum_{i\in I}\sum_{x\in\Delta} \langle f, L_x\phi_i\rangle L_x\psi_i. \qquad (3.7)$$

If we consider a set of modulations of an atom

$$\mathcal{M} = \left\{ g_i := M_{\widehat{x}_i} g \;:\; i = 1, \ldots, R,\; \widehat{x}_i \in \widehat{G} \right\}$$

and its dual $\tilde{\mathcal{M}} = \{\tilde{g}_i\}_{i=1}^{R}$, imposing biorthogonality

$$\begin{aligned} \delta_{i-j}\delta_{0,x'} &= \langle L_x \tilde{g}_i, L_{x'} g_j \rangle \\ &= \left\langle \sum_{l=1}^{R} \sum_{x \in \Delta} b_x^{(i,l)} L_x g_l, L_{x'} g_j \right\rangle \\ &= \sum_{l=1}^{R} \sum_{x \in \Delta} b_x^{(i,l)} \langle g_l, L_{x'-x} g_j \rangle \end{aligned} \tag{3.8}$$

we obtain the the MST reformulation of the twisted convolution (3.5).
The standard convolution obtained in (3.8) can be easily inverted once the convolution theorem is used. If we consider the system of equations (3.8) in the Fourier domain we obtain the constraints:

$$\forall \widehat{x} \in \hat{\Delta} \qquad Id = \beta(\widehat{x}) G(\widehat{x})$$

where Id is the identity matrix of $\mathbb{R}^{I \times I}$ and the matrix G is the Gramian obtained through the FT $\mathcal{F}\left(\left\langle g_l, L_{(\cdot)} g_j \right\rangle\right)$.
With the particular assumption that the generating set is composed of modulations, $\Phi = \{M_{\widehat{x}_i} g \;: i = 1, \ldots, r\}$, we conclude that the inversion of the numerical scheme is related to the inversion of the matrix

$$G_\Phi = \left([\hat{g}, L_{(\widehat{x}_j - \widehat{x}_i)} \hat{g}] \right)_{i,j=1,\ldots,r} . \tag{3.9}$$

But this procedure is nothing but Theorem 4 as expressed in Algorithm 1.
One should always chose the modulation is such a way that the determinant in (3.9) is nonzero, which is excatly the linear independence condition in theorem 4 . After the inversion of the "small Gramians", we can obtain the coefficients through an inverse FT applied on $\beta(\xi)$.
For uniform sampling in the Fourier domain, we get a circulant structure of the matrix (3.9), exactly the same structure that is used for the global inversion of the Gabor system operator[93].

3.2 Numerical Stability

In this chapter, we analyze the problem of numerical stability of the biorthogonal construction under the assumption of invertible Gramian.
This section was conceived after I started running the numerical example that will be shown in Section 3.4. When dealing with with Gabor system, I asked myself: what would happen to the biorthogonal system is the starting system is deformed in some way? What if we use as analysis window the biorthogonal of the deformed system and as synthesis the original one? What happen in the specular situation?
In my mind there were as model a pass-band filter that fails to take the proper frequency,

so the natural deformation to employ was a character multiplication.
The entries of the Gramian are:

$$(\mathcal{G})_{i,j} = \left\langle \hat{\phi}_i, \hat{\phi}_j \right\rangle$$

To analyze such a matrix, we need to define our concept of numerical stability. As usual, multiplication by a character is a shifts in frequency:

$$\widehat{\langle \cdot, \hat{y} \rangle f} = L_{\hat{y}} \hat{f} \qquad \hat{y} \in \hat{G},$$

so, for every $\hat{\boldsymbol{y}} = (\hat{y}_1, \ldots, \hat{y}_r) \in \hat{G}^r$, we define the deformed Gramian as

$$(L_{\hat{\boldsymbol{y}}}\mathcal{G})_{i,j} := L_{\hat{y}_i}\hat{\phi}_i \, L_{\hat{y}_j}\hat{\phi}_j$$

We aim to control the Frobenius norm of the difference matrix

$$\mathcal{D} := L_{\hat{\boldsymbol{y}}}\mathcal{G} - \mathcal{G} \tag{3.10}$$

hence we need to find the proper neighbourhood $\mathcal{U}$ of $\hat{0} \in \hat{G}^r$ where to choose $\hat{\boldsymbol{y}}$.
We can formulate the following theorem.

Theorem 9. ***[DO&SZ]*** *Given a LCA group G and a generating set $\Phi = \{\phi_1, \ldots, \phi_r\}$ of compactly supported distributions defined over G, then for every $\epsilon > 0$ there exists a vector of shifts in the frequency domain such that the Frobenius norm of the matrix $\mathcal{D}$ defined in* (3.10) *can be bounded as*

$$||\mathcal{D}||_F < 2\epsilon r \max_{j=1,\ldots,r} \parallel \phi_j \parallel_1 \tag{3.11}$$

Proof. Because our atoms are compactly supported distribution, their Fourier transform are uniformly continuous:

$$\forall \epsilon > 0 \exists \mathcal{U}_i \text{ s.t. } \forall y_i \in \mathcal{U}_i \quad \left| L_{y_i}\hat{\phi}_i - \hat{\phi}_i \right| < \epsilon$$

Choosing $\hat{\boldsymbol{y}} \in \bigotimes_{i=1,\ldots,r} \mathcal{U}_i$, we can control the entries of the matrix $\mathcal{D}$

$$\begin{aligned}
\left| (\mathcal{D})_{i,j} \right| &= \left| L_{\hat{y}_i}\hat{\phi}_i \cdot L_{\hat{y}_j}\hat{\phi}_j - \hat{\phi}_i \cdot \hat{\phi}_j \right| \\
&= \left| L_{\hat{y}_i}\hat{\phi}_i \cdot L_{\hat{y}_j}\hat{\phi}_j - L_{\hat{y}_i}\hat{\phi}_i \cdot \hat{\phi}_j + L_{\hat{y}_i}\hat{\phi}_i \cdot \hat{\phi}_j - \hat{\phi}_i \cdot \hat{\phi}_j \right| \\
&\leq \left| L_{\hat{y}_i}\hat{\phi}_i \cdot L_{\hat{y}_j}\hat{\phi}_j - L_{\hat{y}_i}\hat{\phi}_i \cdot \hat{\phi}_j \right| + \left| L_{\hat{y}_i}\hat{\phi}_i \cdot \hat{\phi}_j - \hat{\phi}_i \cdot \hat{\phi}_j \right| \\
&= \left| L_{\hat{y}_i}\hat{\phi}_i \right| \cdot \left| L_{\hat{y}_j}\hat{\phi}_j - \hat{\phi}_j \right| + \left| L_{\hat{y}_i}\hat{\phi}_i - \hat{\phi}_i \right| \cdot \left| \hat{\phi}_j \right| \\
&< \epsilon \left(\parallel \hat{\phi}_i \parallel_\infty + \parallel \hat{\phi}_j \parallel_\infty \right) \\
&\leq \epsilon \left(\parallel \phi_i \parallel_1 + \parallel \phi_j \parallel_1 \right).
\end{aligned}$$

Since the matrix $\mathcal{D}$ is symmetric its 1 and ∞ norms coincide:

$$\begin{aligned}
||\mathcal{D}||_\infty = ||\mathcal{D}||_1 = max_{j=1,\ldots,r} \sum_{i=1}^{r} & \left| L_{\hat{y}_i}\hat{\phi}_i \cdot L_{\hat{y}_j}\hat{\phi}_j - \hat{\phi}_i \cdot \hat{\phi}_j \right| \\
&< max_{j=1,\ldots,r} \sum_{i=1}^{r} \epsilon \left(\parallel \phi_i \parallel_1 + \parallel \phi_j \parallel_1 \right) \\
&= \epsilon \left(r \; max_{j=1,\ldots,r} \parallel \phi_j \parallel_1 + \sum_{i=1}^{r} \parallel \phi_i \parallel_1 \right) \\
&\leq 2\epsilon r \; max_{j=1,\ldots,r} \parallel \phi_j \parallel_1
\end{aligned}$$

hence

$$\begin{aligned} ||\mathcal{D}||_F &\leq \sqrt{||\mathcal{D}||_1 \, ||\mathcal{D}||_\infty} = ||\mathcal{D}||_\infty \\ &< 2\epsilon r \; max_{j=1,\dots,r} \parallel \phi_j \parallel_1 . \end{aligned}$$

□

In Section 3.4.4 we will check the thesis on Theorem 9. We will find that for the deformation parameter ϵ the growth in the deformation of the Gramian will be linear, indicating that such a linear growth is also a lower bound for the deformation. More intricate interpretation will be needed for the dependence on the number of windows of the deformation.

3.3 Description of the proposed procedure

We will run extensive simulations for 1D function using the implementation of the FT in its fast formulation: the fast Fourier transform. This algorithm builds an approximation of the Fourier transform for sampled signal on a periodic domain; it is hence an approximation for the FT related to the LCA group

$$\mathbb{Z}/(L\mathbb{Z})$$

being L the number of sample. Under this assumption, the only subgroups are in the form $\mathbb{Z}/(a\mathbb{Z})$, being a a divisor of L; we will name this scalar also gap_t. The other parameter of our Gabor-like system will be the location in frequency of the characters, which are elements of $\widehat{\mathbb{Z}/(L\mathbb{Z})} = \mathbb{Z}/(L\mathbb{Z})$; in case of standard Gabor system, the character will be taken on a subgroup $\mathbb{Z}/(b\mathbb{Z})$, being b another divisor of L; during the simulations we will refer to this parameter as gap_t or, to stress the relationship with signal processing, to the number of channels $\#c := L/b$.
The dual Basis is computed exactly as in algorithm 1 which was obtained through a transcription in pseudo-code of Theorem 4. The appealing nature of this algorithm relies in 3 features:

(a) The nested **for** runs over the set of windows, not on the lattice as in the common frame-based algorithm

(b) We invert matrices of dimension $R \times R$, where R is the number of used modulations, instead of a large matrix that represents the frame operator

(c) The primarily used tool is the FT, which can be implemented in its fast formulation FFT.

A work that optimally implement the inversion of Gabor system can be found in [93]. While this work aims to reduce the complexity of frame operator inversion through a partition of it that comes from the Walnut representation of Gabor system, our goal is to build a decomposition of the same operator which acts point-wise in Fourier domain.
In this way, our method can handle non-uniform sampling in frequency domain, overcoming the limitation that an integer oversampling introduces.
We finish the analysis of the algorithm proving the stability.

Proposition 10. ***[DO&SZ]****The inversion of the convolution* (3.8) *implemented in the algorithm 1 is stable for any shift in the frequency domain.*

Proof. If we perform a shift of the modulation $\Phi = \{g, M_{\widehat{x}_1}g, \ldots, M_{\widehat{x}_{r-1}}g\}$ along any direction through $M_{\widehat{x}}$ we obtain

$$\begin{aligned} G_{M_{\widehat{x}}\Phi} &= \left([L_{\widehat{x}}L_{\widehat{x}_i}\hat{g}, L_{\widehat{x}}L_{\widehat{x}_j}\hat{g}]\right)_{i,j=0,\ldots,r-1} \\ &= G_\Phi \end{aligned}$$

Assuming that the original matrix is non degenerated, the invertibility is preserved. □

3.3.1 Comparison with existing methods

In [28] and [93] a framework for the inversion of the twisted convolution in the case of rational uniform sampling is presented.
This is done through 2 atoms and building the corresponding frame operator $S_{g,\gamma} = T_\gamma T_g^*$ to obtain a dual basis different from the canonical one. This theory is usually called relaxation of frames.
In the case of integer oversampling this matrix is a block circulant matrix

$$S_{g,\gamma} = \begin{pmatrix} A_0 & A_1 & \cdots & A_{\frac{L}{a}-1} \\ A_{\frac{L}{a}-1} & A_0 & \cdots & A_{\frac{L}{a}-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{pmatrix}$$

with blocks either diagonal or null. The inverse of a block matrix with blocks

$$B_s = \frac{L}{a}\sum_{r=0}^{\frac{L}{a}-1} e^{2\pi i \frac{ars}{L}} \hat{A}_r^{-1}$$

where

$$\hat{A}_r = \sum_{s=0}^{\frac{L}{a}-1} e^{-2\pi i \frac{ars}{L}} A_s.$$

Even if we start from completely different assumptions, similarities are quite evident, with the inversion occurring in the Fourier domain.
Nevertheless, the path that we have chosen is completely different. While the work in [93] aims to reduce the complexity of standard Gabor system through a partition of the whole frame operator, our goal is to build a decomposition of the same operator which acts point-wise.
In this way, our method can handle non-uniform sampling in frequency domain, overcoming the limitation that an integer oversampling introduces.

3.3.2 Complexity discussion

One of the weakness of traditional Gabor algorithm is its complexity. The heaviest computation is the computation of the canonical dual basis, obtained traditionally through the pseudoinverse of the frame operator.
A common way to reduce the complexity is to use the Walnut representation that can speed up the computation (like in the Sondegaard's LTFAT [89]), but this means to be highly dependent on the lattice structure on the time-frequency plane of the sample. With

the reformulation expressed in here, besides the new freedom in the frequency domain, we achieve even a faster code.
For the traditional frame inversion, we consider a matrix having on its rows all the different shifts and modulations of a fixed atom. If we assume that redundancy is $\mathcal{O}(1)$, the matrix is $L \times (\#shift_t \cdot \#shift_f) = \mathcal{O}(L) \times \mathcal{O}(L)$.
Computing the Moore-Penrose pseudo inverse of a $m \times n$ matrix relies in its SVD (Singular-value decomposition), which has complexity $\mathcal{O}(mn^2 + n^3)$, where n is the smaller dimension [40]. So, in our case, we have a $\mathcal{O}(L^3)$ complexity.
There are other ways to invert a matrix, that are based on a fast implementation of multiplication between matrices, but in general we have an $O(L^\alpha)$, with $\alpha > 2$.
Considering the proposed algorithm, naming $|\Delta|$ and $|\hat{\Delta}|$ the cardinality of the shift in time and frequency, we invert $|\Delta|$ matrices of size $|\hat{\Delta}| \times |\hat{\Delta}|$.
Because we are under the hypothesis that the redundancy is constant, $|\Delta| \times |\hat{\Delta}| = \mathcal{O}(L)$. So, if we fix $|\hat{\Delta}| = \mathcal{O}(L^x)$ with $x \in [0, 1]$ (so $|\Delta| = \mathcal{O}(L^{1-x})$), our code has a complexity $\mathcal{O}(L^{1+(\alpha+1)x})$.
We obtain a minimum for $x = 0$, eg. $|\hat{\Delta}| = \mathcal{O}(1)$ and a global complexity of $\mathcal{O}(L)$.
For this reason, in our test runs, we always have fixed $|\hat{\Delta}| \in \{10, \ldots, 50\}$. The following tests are performed using our standard computer in the Faculty of Mathematics of the University of Vienna, an Intel Core i5 4670S @3.10GHz.

3.4 Numerical experiments

In here, we present numerical experiments to explore the performances of the code we described in Algorithm 1.
We have performed extensive testing on the code through different case:

(A) Oversampled Gabor frame on separable lattice

(B) Undersampled Gabor system

(C) Small deformation of Gabor frame on separable lattice

(D) Automatic selection of modulations

To test stability we have used two different types of atoms: bump function and Gaussian function. Numerical tests show that a spline-type reformulation of Gabor system works efficiently for both cases. This was easily foreseeable, since convergence theorems for integrals hold over LC groups. This will give the numerical scheme a really important feature: the capability to select atoms better localized in the Fourier domain than compactly supported distributions.
The generating window of the Gabor system will be normalized in L^2 norm.
To measure the error of the approximation of a signal X through the Gabor-like expansion X_a we used the formula

$$ERR = \frac{\| X - X_a \|}{\| X \|}$$

while to measure the magnitude of a Gramian, which is for us a matrix-valued function over a lattice Δ, we have used the $L^\infty(\Delta)$ norm of the Frobenius norms.
Numbers will be given in the format (Significand)e(base10exponent) as in Matlab notation.

Table 3.1: Error of approximation of a signal for different cases (mean of 10000 runs) **OS**=Oversampled System, **LF**=Low frequency, **FM**=Few modulation, **SF**=Sparse frequency, **FS**=Frequency selection

		MST	Standard Gabor
OS:	L=675 gap_t=27, #c=45	1.5733e-13	3.6865e-16
LF + **FM**:	L=4410 gap_t=30, #c=10	8.9118e-7	2.8554e-2
SF + **FS**:	L=4410 gap_t=30, #c=15	3.3580e-5	1.8194e+3

When explicit measurements of the error are displayed it must be taken into account that, to ensure significance of the outcome, they are a mean of 1000 runs of the approximation or dual construction process.
Comparison between our outcomes and outcomes coming from traditional Gabor analysis are done through the use of the LTFAT toolbox [82].

3.4.1 Oversampled Gabor frame on separable lattice

Separable oversampled Gabor system are defined by a time-frequency lattice

$$\Delta = (\mathbb{Z}/(a\mathbb{Z})) \times (\mathbb{Z}/(b\mathbb{Z}))$$

such that $ab < L$. Hence, through the related discrete short-time transform we get more information than the number of samples.
If we consider the particular case obtained through the parameters L=675, gap_t=27, gap_f=15, and a Gaussian window function, we can see on the first row of Table 3.1 (**OS**) that the loss of precision, with respect with the standard Gabor construction, is negligible.
The plot of the product between the 45 modulations and their duals can be seen in Figure 3.1. The biorthogonality is almost perfect, up to the redundancy as constant.
Indeed, if we consider the ST biorthogonal $\tilde{\phi}$ and the canonical Gabor dual $S^{-1}(\phi)$ we get

$$\frac{\parallel \tilde{\phi} - S^{-1}(\phi) \parallel}{\parallel S^{-1}(\phi) \parallel} = 1.3513e - 13$$

and in Figure 3.2 we cannot distinguish between the two constructions.
Our intuition that the loss of precision is maily due to to the intrinsic error of the FFT is confirmed in Figure 3.3, where the same linear dependence with the length of the signal appear.
An unstable oversampled systems can be built for signal of length L=9072, using a time-frequency separable lattice having time constants gap_t=24 and 48 equispaced modulations, and a bump function having radius' support 7gap_t. Considering the the matrix `GG` whose column are the different modulations, and running the short Matlab code:

```
fourier = fft(GG);
fourier_s = fourier(1:L/gap_t:end,:);
det(fourier_s'*fourier_s)
```

we obtain `ans = 0`. As a consequence, the run of the spline-type reformulation gives NaN results.

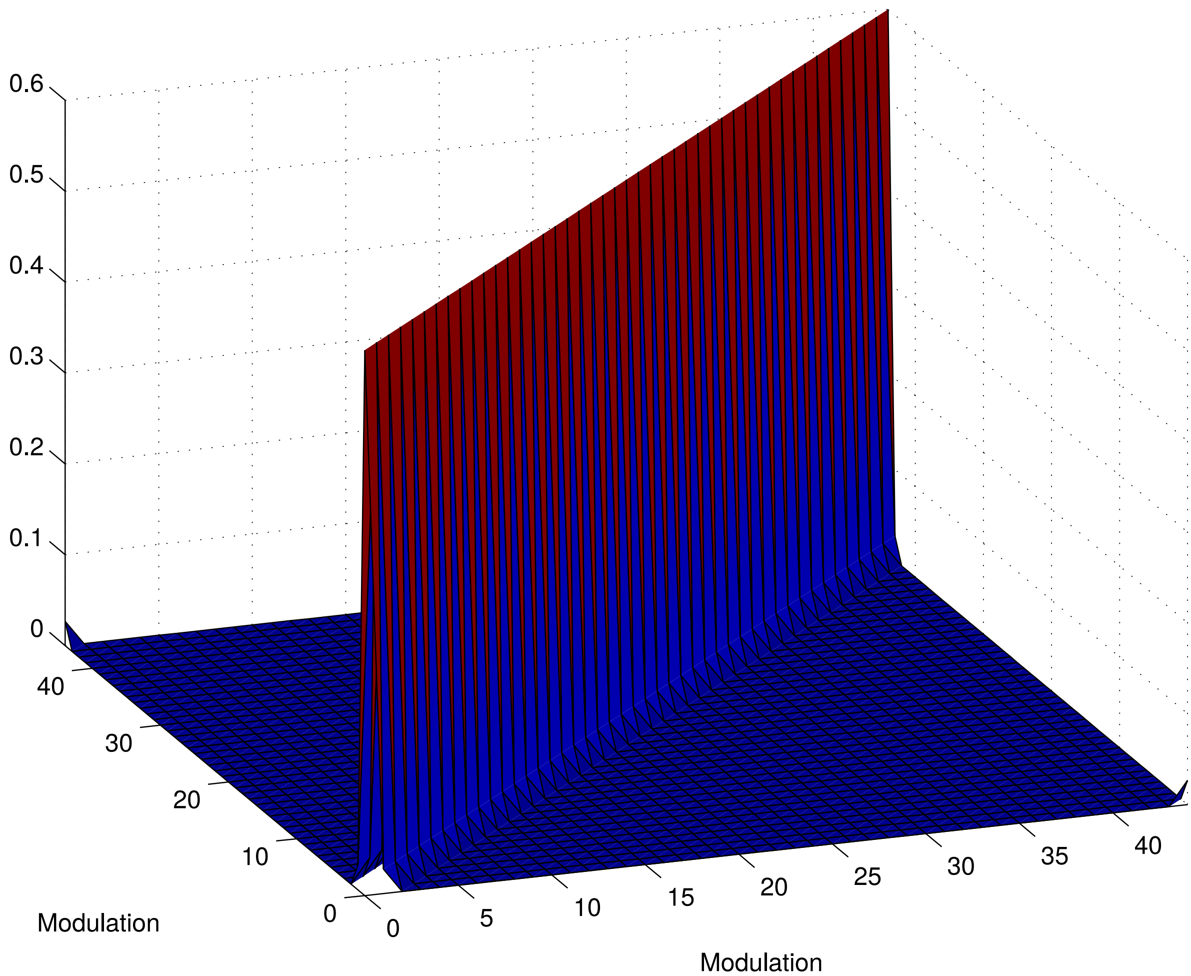


Figure 3.1: Approximation of biorthogonality for the system L=675, gap_t=27, gap_f=15

3.4.2 Undersampled Gabor system on separable lattice

We have performed tests on incomplete Gabor systems defined on a rectangular lattice for both traditional Gabor algorithm and our method as it can be seen in the second line of Table 3.1. In the undersampled case standard Gabor algorithm is highly numerically unstable, giving often a NaN result. We present here 2 of the few example for which Gabor algorithm does not return a NaN. It is possible to appreciate from Figure 3.4a the fact that the biorthogonality reformulation is more stable than standard Gabor dual construction. For an undersampled system of parameters L=4410, gap_t=30, gap_f=441, the approximation for the traditional fast implementation of Gabor duals, gives us a constant approximation error of 2% low frequency bandlimited signal. On the other side, our algorithm give a much more accurate approximation. The two error curves in Figure 3.4a seems to touch for bandlimited signals having maximal frequency $r \approx 70$, which is exactly the Nyquist ratio

$$L = 4410 \quad gap_t = 30 \qquad \Longrightarrow \qquad f_{\max} = 73.5$$

Over this ratio standard algorithm shows instability through an exponential growth of the error, while our algorithm gives an error that is never higher than the energy of the signal (norm 2).

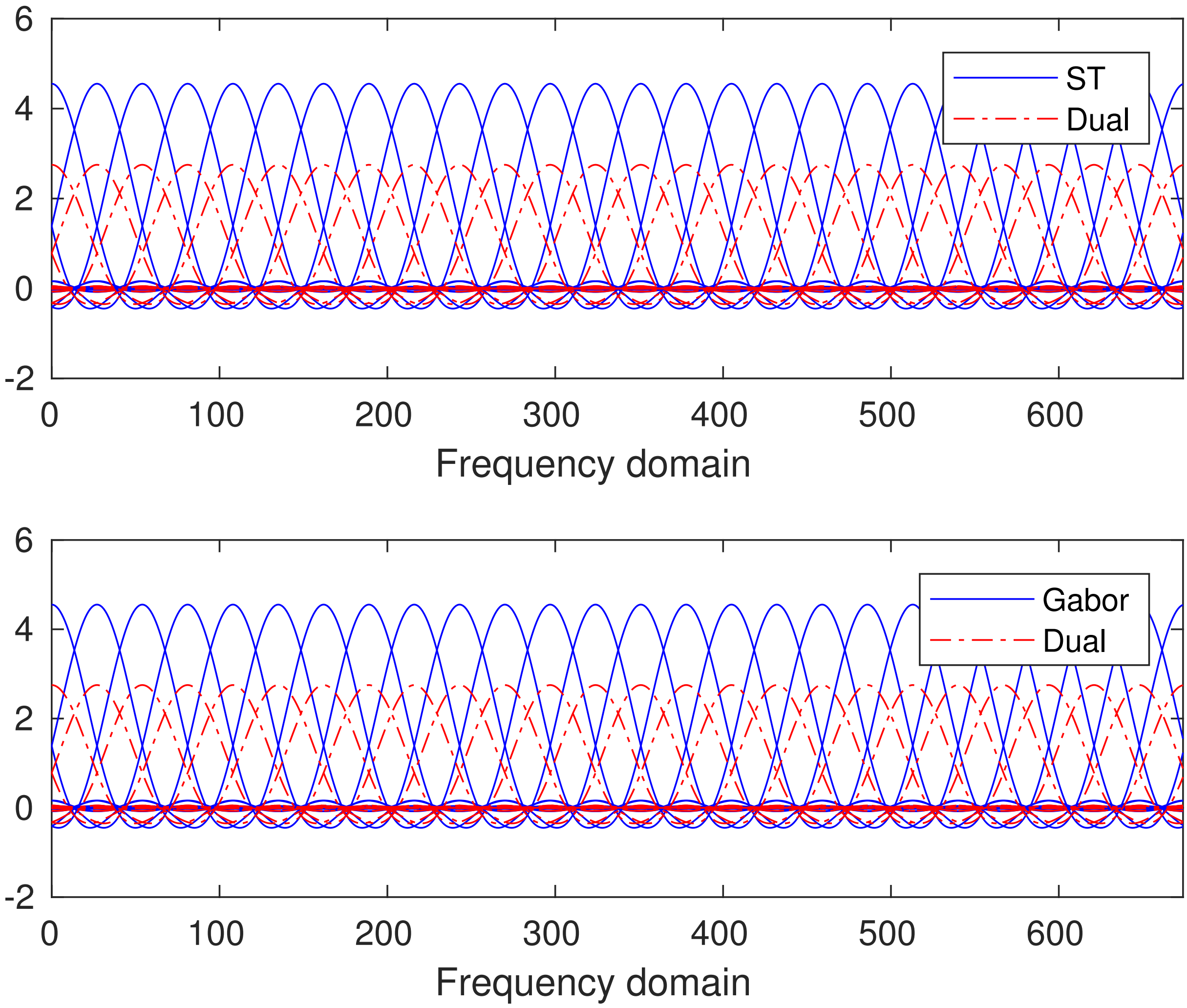


Figure 3.2: Comparison in Fourier domain of ST and Gabor Duals L=675, gap_t=15, gap_f=27

We also show the result coming from a Gabor system generated by a normalised Gaussian over the separable lattice having parameters gap_t=27, gap_f=315 and applied to signal of length L=3780. This is one of the best cases of standard Gabor system, since for low frequency signal it is able to reproduce the same error that we have obtained through MST reformulation, as shown in Figure 3.5a. For signal having bandwidth of radius 70, exactly the sampling rate, Gabor's error display the unstable nature of undersampled Gabor system, while MTS's error is always bounded (Figure 3.4a). From our perspective, problems arise from holes in the frequency domain[61]. Due to the Wiener's inversion theorem, the dual windows belong to the same ideal of the original atoms (characterized by their spectrum), hence they display the same localization in frequency, as shown in the upper part of Figure 3.5b; the same does not occur for different modulations of Gabor's dual, which display the unstable nature on the global inversion of the frame operator as in (3.2).

Wiener's lemma and the characterization of the ideals in convolution subalgebras have finally a fairly direct numerical interpretation through this example: the capability of recovering a signal relies on the possibility of covering its spectrum through the selected modulated atoms from a spline-type space. The sampling theorem tells us that without further adjustment we cannot fully cover this spectrum and for these reason we propose

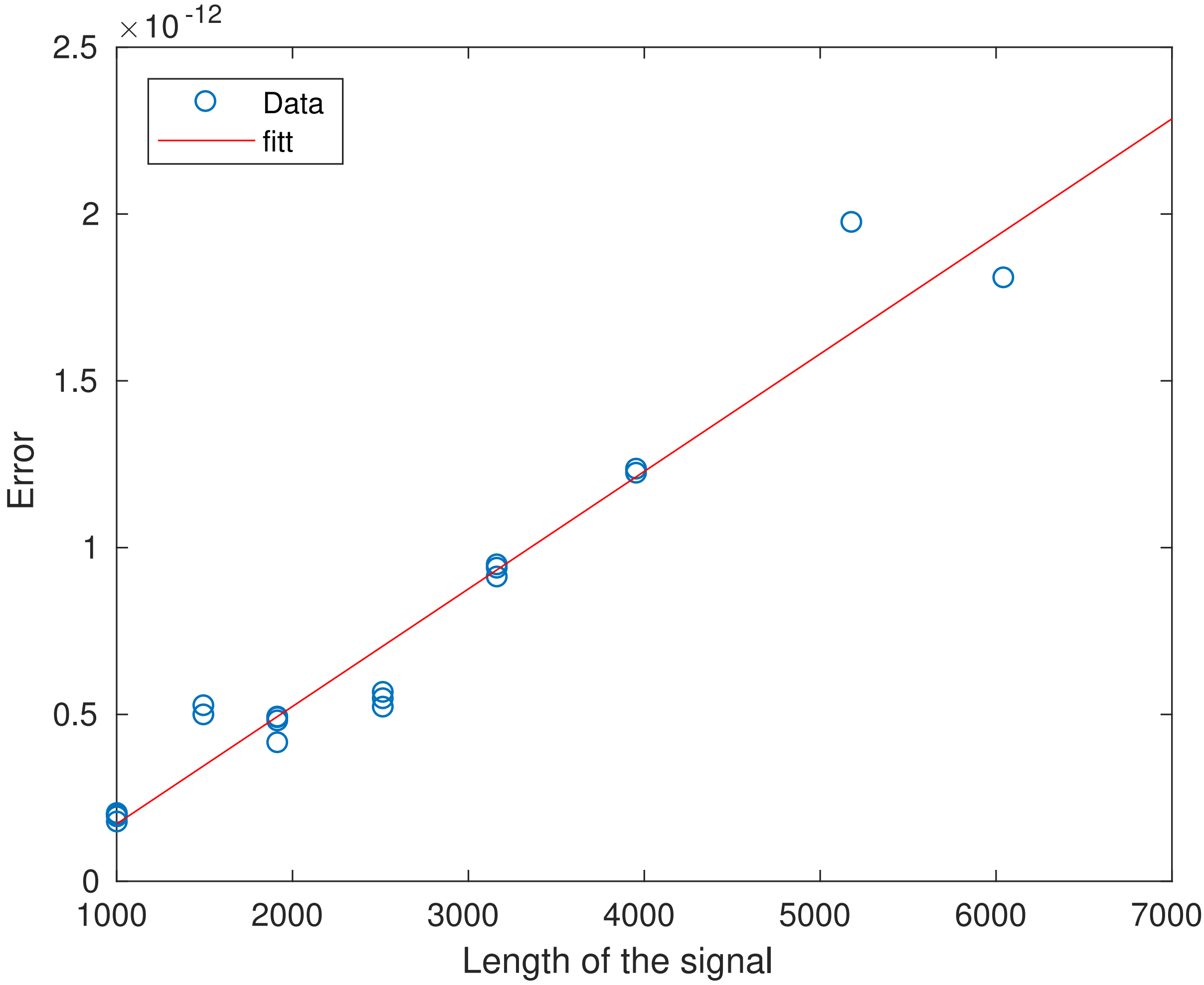


Figure 3.3: Dependence between error and length of the signal for reformulated Gabor ST frame having equal redundancy

in the next Section 3.4.3 a modulations' selection procedure.

3.4.3 The case of the bandlimited signal

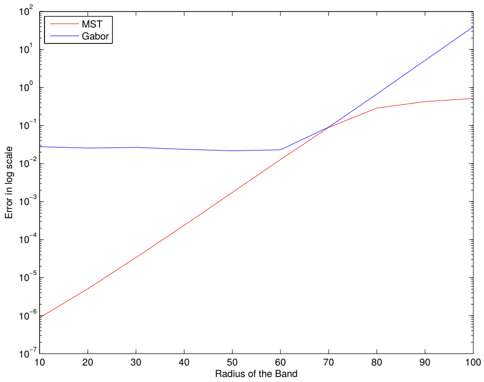
As mentioned already in the introduction, a special relevance have the bandlimited signals, or signals with known cospectrum.
If we consider a function in the Gabor space

$$f = \sum_{(x,\widehat{x})\in\Delta} c_{x,\widehat{x}}\pi(x,\widehat{x})g$$

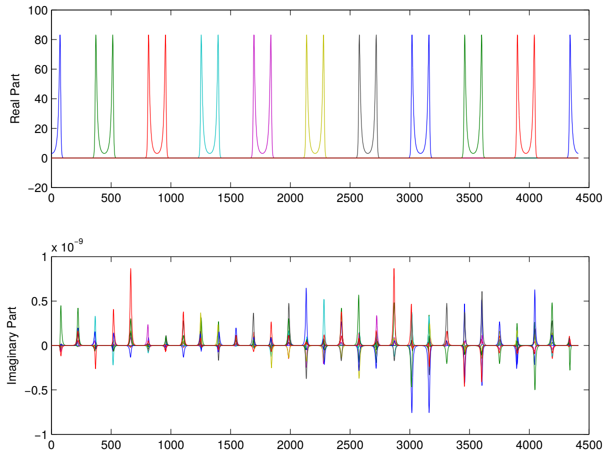
and we apply a FT, through its linearity and properties (1.16) and (1.17), we obtain

$$\begin{aligned}\mathcal{F}f &= \sum_{(x,\widehat{x})\in\Delta} c_{x,\widehat{x}}\mathcal{F}\left(\pi(x,\widehat{x})g\right) \\ &= \sum_{(x,\widehat{x})\in\Delta} c_{x,\widehat{x}}\pi(\widehat{x},-x)\mathcal{F}g.\end{aligned}$$

Consider now a band-limited signal X, or in general a signal having $supp(\mathcal{F}X) \subsetneq [-B,B] \subsetneq \overline{L}$ and a window having the same property, ie. $supp(\mathcal{F}\phi) \subsetneq [-C,C]$.

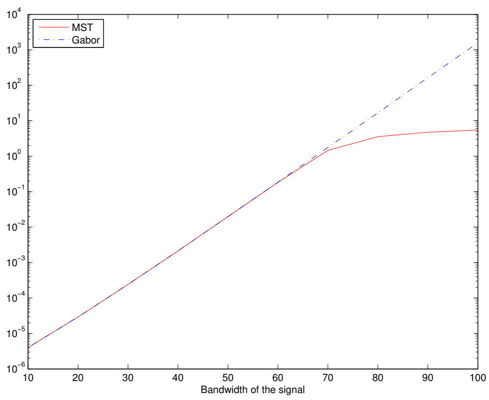


(a) Approximation error for bandlimited signals

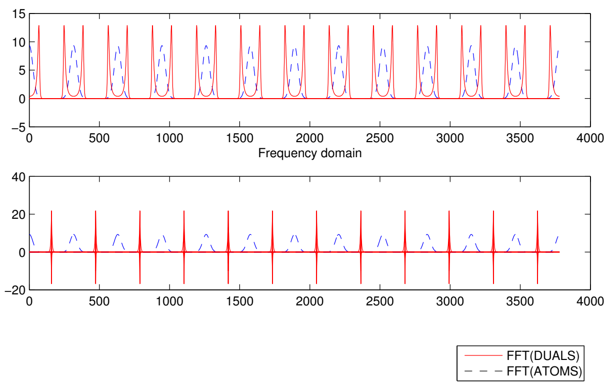


(b) Duals in Fourier Domain.
Top ST, Bottom Gabor

Figure 3.4: Undersampled System L=4410, gap_t=30, gap_f=441



(a) Approximation error for bandlimited signals



(b) Atoms and Duals in Fourier Domain.
Top ST, Bottom Gabor
(Gabor dual rescaled by a factor $\frac{1}{10^9}$)

Figure 3.5: Undersampled System L=3780, gap_t=27, gap_f=315

We can always build an atom such that $C < B$. If $supp(\mathcal{F}\phi') \subsetneq [-C', C']$ we consider a delay $\phi = D_{\frac{1}{\delta}}\phi'$ with $\delta > \frac{C'}{B}$ to have

$$\mathcal{F}D_{\frac{1}{\delta}}\phi')(\xi) = |\delta| D_\delta \mathcal{F}(\phi')(\xi) \Longrightarrow supp(\mathcal{F}\phi) \subsetneq [-C, C] \text{ with } C < B.$$

If we want to reproduce such signal, we need to select only the modulation $M_{\widehat{x}}$ such that

$$\begin{aligned} supp(\mathcal{F}X) \cap supp(\mathcal{F}L_x M_{\widehat{x}} g) &\supseteq supp(\mathcal{F}X) \cap supp(\mathcal{F}M_{\widehat{x}} g) \\ &= supp(\mathcal{F}X) \cap supp(L_{\widehat{x}}\mathcal{F}g) \\ &= supp(\mathcal{F}X) \cap L_{\widehat{x}} supp(\mathcal{F}g) \neq \emptyset. \end{aligned}$$

This is possible because, as said in Section 3.1, we are able to perform an approximated inversion of the convolution in the proper ideal having its spectrum containing the spectrum of the signal.
In the practice, we have often used Gaussian functions as window function due to their good localization properties in both time and frequency. This function has full support and spectrum, so we chose to fix a tolerance under which every function is considered equal to zero. We are selecting a subset of $\overline{L} = \{0, 1, \ldots, L-1\}$, being L the length of the signal.
This comparison between signal and modulations of atom i.e. translation in the Fourier domain, can be seen in Figure 3.6.
Good reproduction of the Fourier domain relies once again in the mildly overlap of the shifts $\mathcal{F}M_{\widehat{x}} g = L_{\widehat{x}}\mathcal{F}g$.
Figure 3.6 is only the first experiment that was performed; I then wrote the short Algorithm 2 to pre-process the signal and to select the proper modulations.

Algorithm 2 [DO&SZ] Selection of the proper modulation

Precondition: Signal X, Window ϕ, frequency steps b

function FIND_MODULATION(X, ϕ, b)
    $L \leftarrow length(\phi)$
    $\mathcal{F}X \leftarrow FFT(X)$
    $\mathcal{F}\phi \leftarrow FFT(\phi)$
    $S \leftarrow supp\,(\mathcal{F}X)$
    $S\_atom \leftarrow supp\,(\mathcal{F}\phi)$
    $\Psi \leftarrow \emptyset$
    for $r \leftarrow 0$ to $\frac{L}{b} - 1$ **do**
        if $S \cap L_{rb}\,(S\_atom) \neq \emptyset$ **then**
            $\Psi \leftarrow \Psi \cup \{r\}$
        end if
    end for
    return Ψ
end function

We applied this selection in 2 different ways:

- Setting to zero the modulations that where not selected.
- Using only the selected modulated atoms.

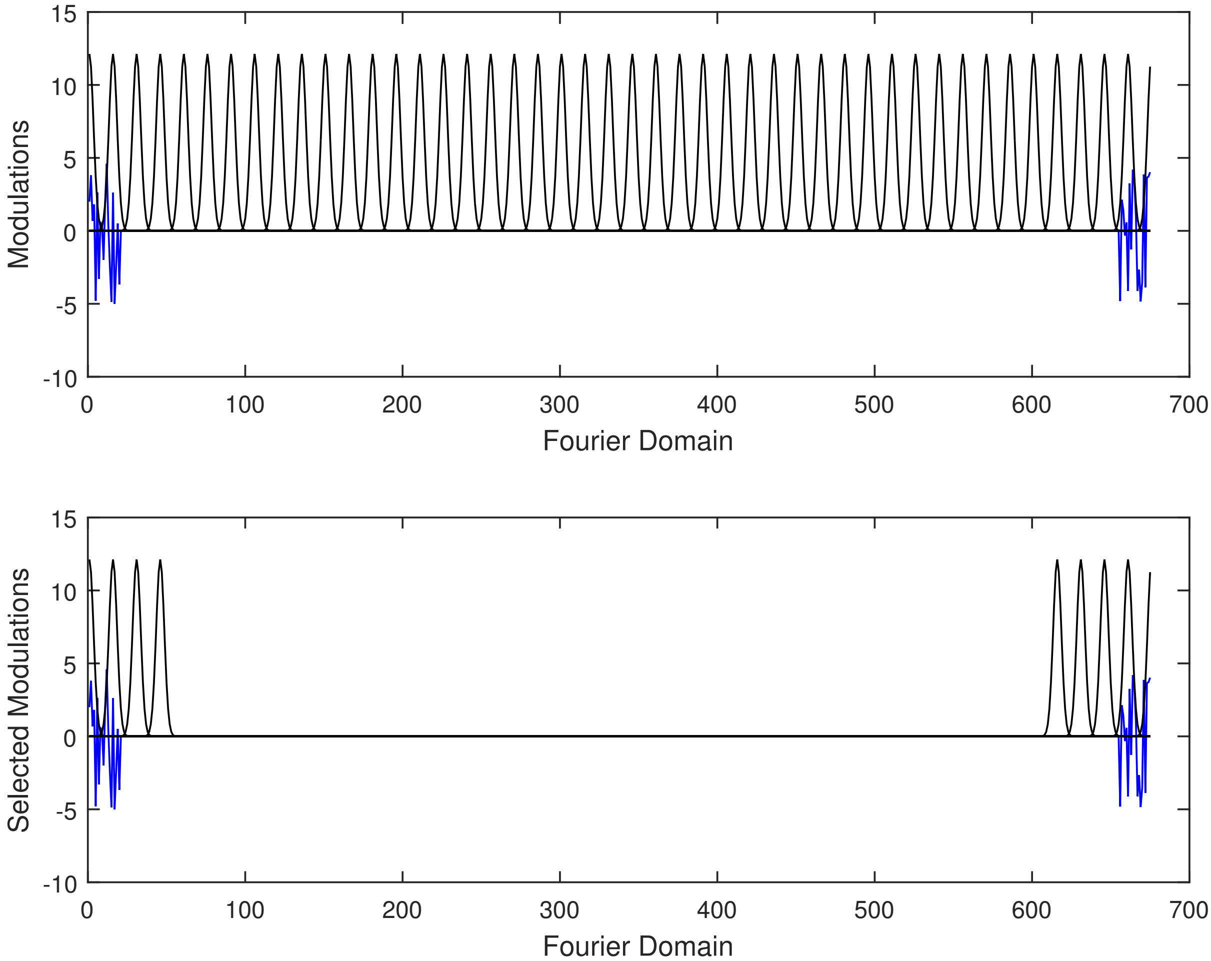


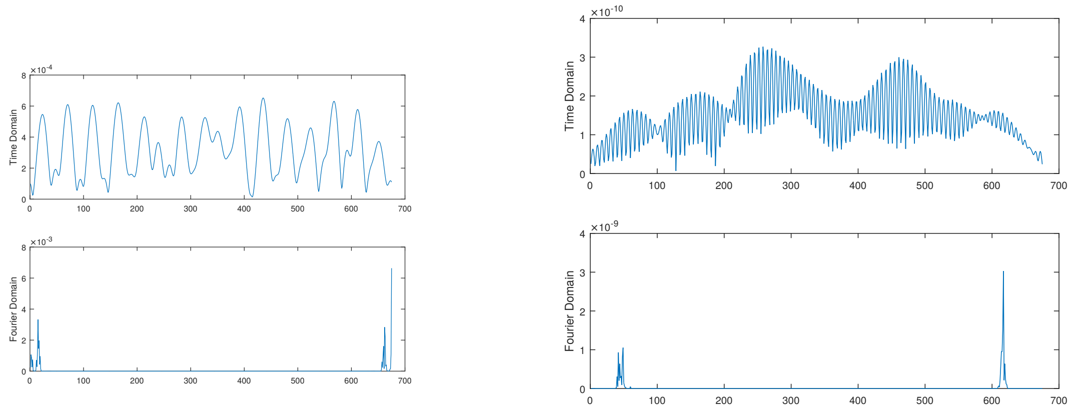
Figure 3.6: Comparison between a band-limited signal and modulation in the Fourier domain. Selection is straightforward (tolerance $= 10^{-5}$)

The second method forces the dual basis to have the same number of elements as the selected modulations, while in the first case we found out that our numerical scheme builds a dual basis having more non zero elements than selected modulations, but still fewer than the whole discretization in frequency.

From these results, the first method seems useful for compression, while the second to speed up the computations. Using the same stable Gabor system we used above, L=675 gap_t=27 and gap_f=15, we generated a 1000 random low-frequency signals having only the first and last 20 non zero components in the Fourier domain. Setting a tolerance of 10^{-5}, the algorithm selects the following modulations of the normalised Gaussian

$$\{M_0 g, M_{15} g, M_{30} g, M_{45} g, M_{630} g, M_{645} g, M_{660} g\}$$

i.e. the 0th modulation, the first 3 and the last 3.
With the second method, we obtained 7 poorly approximated atoms that gives for every randomly generated signal an approximation error of 1%, while in the second case, we obtained 10 non null duals: the one corresponding to the 0th modulation, the duals related to the first 5 modulations (even if the 4th and the 5th were set to zero), and the duals corresponding to the last 4 modulations (even if the 4th to last were set to zero).
In Figure 3.7a and 3.7b, we can understand the difference between the methods: while the second produces a small error near the boundary of the the FT of the original band-



(a) Spread of the error in modulations' selection

(b) Spread of the error with unneeded modulations set to zero

Figure 3.7: Error's spread for modulations' selection starting from Gabor L=675, gap_t=15, gap_f=27
Top:time domain
Bottom: frequency Domain

limited signal, the first produces a much more sparse error.
Theorem 4 tell us that we cannot overcome this loss in a naive way by displacing the modulations denser on boundary of this spectrum (in a Chebyshev-like way) since this would lead us to, at least numerical, linear dependence, hence to inconsistency. This assertion is supported by numerical experiment in which the atoms didn't pass the linear independence test.
This example shows once again that through incomplete Gabor systems, we are reaching a level of compression of a signal that was not accomplished before for such classes of fast algorithms.
Also we understand that we are designing a method that is much more close to traditional sampling theory. Gabor algorithms in undersample situation display their instability through numerical instability of the inversion procedure of the frame operator, while if we formulate the frame operator in spline-type theory we get an error that is never bigger than the magnitude of the input signal. Once again we bump into the usual question in the spline-type research: "Are we covering the proper zones of the Fourier domain?".

3.4.4 Deformation of Gabor system

Finally, we want to numerically explore the stability of our method by testing the usefulness of the Theorem 9. The traditional inversion of Gabor frame is not continuous under the shift parameters [44]. That is the reason why we want to explore the possibility to deform a separable lattice, once a non-uniform reformulation is available.
Before the formulation of Theorem 9 I explored the numerical behaviour of stability under deformation of MST algorithm (C). The standard inversion of Gabor frame is not continuous under the modification of the shift parameters. For our numerical tests, we used the aforementioned frame with parameters L=675, gap_t=27, gap_f=15. We have computed the dual of its deformed version, varying the shifts in frequency, and the probability of a window to be shifted.

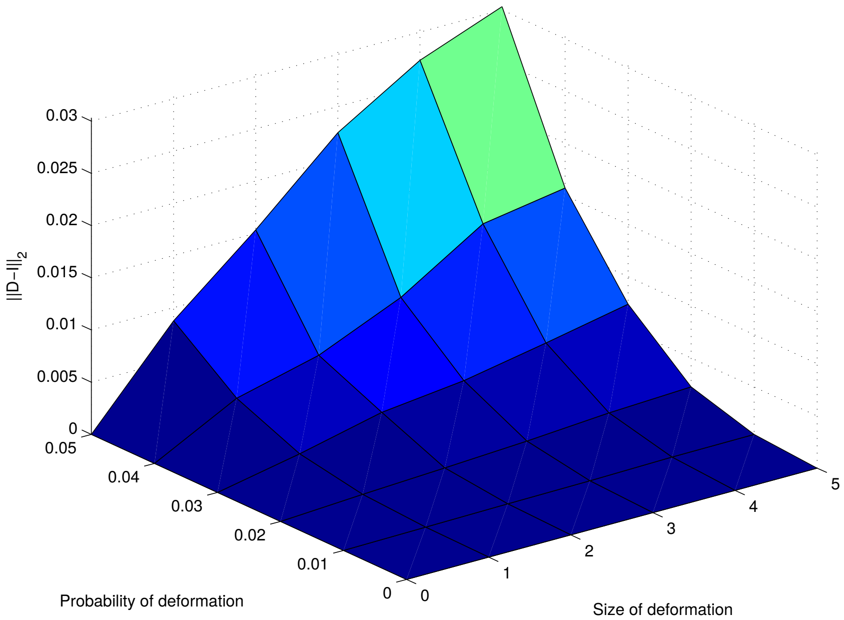


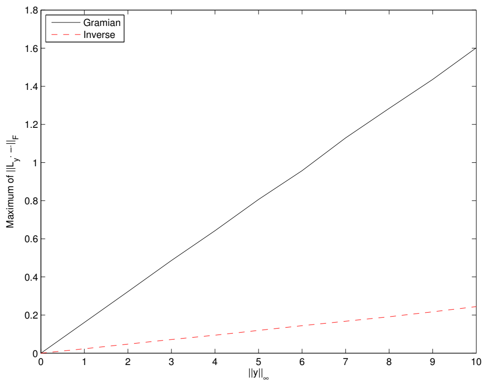
Figure 3.8: Deformation from approximated biorthogonality

In Figure 3.8, we have plotted the distance in norm 2 between the recontruction operator $T_{\phi_{deformed}} T^*_{\tilde{\phi}}$ from identity matrix (the expected $T_\phi T^*_{\tilde{\phi}}$ reconstruction operator). The deformation seems very smooth but, of course, the error becomes big for significant deformation: around 10% for a deformation of 1/3 of the frequency step and 1/10 of probability of deformation.

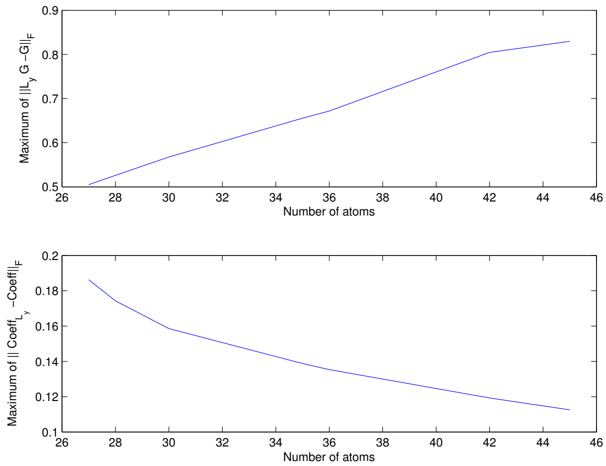
After having *deformed* a given generating set substituting to each element a modulated version $\phi_i \to \phi_{i,m} := \langle \cdot, \widehat{y} \rangle \, \phi$, being $\widehat{y}$ uniformly randomly chosen in a neighbourhood of the unity, we have considered the related biorthogonal system Ψ_m. Then a signal was analysed through the $V_{\Phi,H}$ and the obtained coefficient synthesized an approximation through $U_{\Psi_m,H}$. The approximation error display linear dependency with the magnitude of the deformation.
We employ a *dual* numerical test, computing the approximation of a signal f through

$$f_a = U_{\Psi,H} V_{\Phi_m,H} f$$

This numerical test can simulate the behaviour of an ideal multi-pass filter which fails with uniform probability to use the proper frequency. Even with the precomputing of the expected biorthogonal system, we obtain the same linear error.
For a signal of length L=3780, we consider the Riesz basis generated by the time-frequency shifts of a bump function over the separable lattice having constant gap$_t$=21 and gap$_f$=90. We have then fixed a magnitude ω and considered for every atom a shift in the frequency domain of value $\hat{y}_i \in \{-\omega, \omega\}$, in such a way that the element $\hat{y}$ in formula (3.10) is a vertex of the hypercube centred in 0 and radius of inscribed sphere ω.
The result is shown in Figure 3.9b. It display the linear dependence stated in Theorem 9 between the deformation and ϵ.
To test the linear dependence between the deformation on the number of atoms we have used a system of parameters L=3780, gap$_t$=21 and number of channel #chan=27, 28, 30, 35, 36, 42, 45 and magnitude of deformation $\omega = 5$.
The thesis is validated as it appears in (3.11), since the norm increase in a linear way,



(a) Increasing step size



(b) Increasing number of atoms
Top: Gramian. Bottom: Inverse

Figure 3.9: Deformation as in Theorem 9

while the surprising fact is that the error in the coefficients decrease with the number of atoms. This outcomes is not yet explained, since Theorem 9 focus on the Gramian rather than the coefficients. We think it is connected to the approximation power of the ST space since more windows, hence better localized information, are added. This aspect will be explored in future works. If we select as deformation parameters the magnitude and the probability of deformation as in Figure 3.8, and measure the deformation in Frobenius norm of the Gramian and the related coefficients, we obtain outcomes as in Figure 3.10. We would like to stress that all the outcomes does not depend on the chosen Riesz basis. The example here displayed are just a way to ensure reproducibility of the experiment.

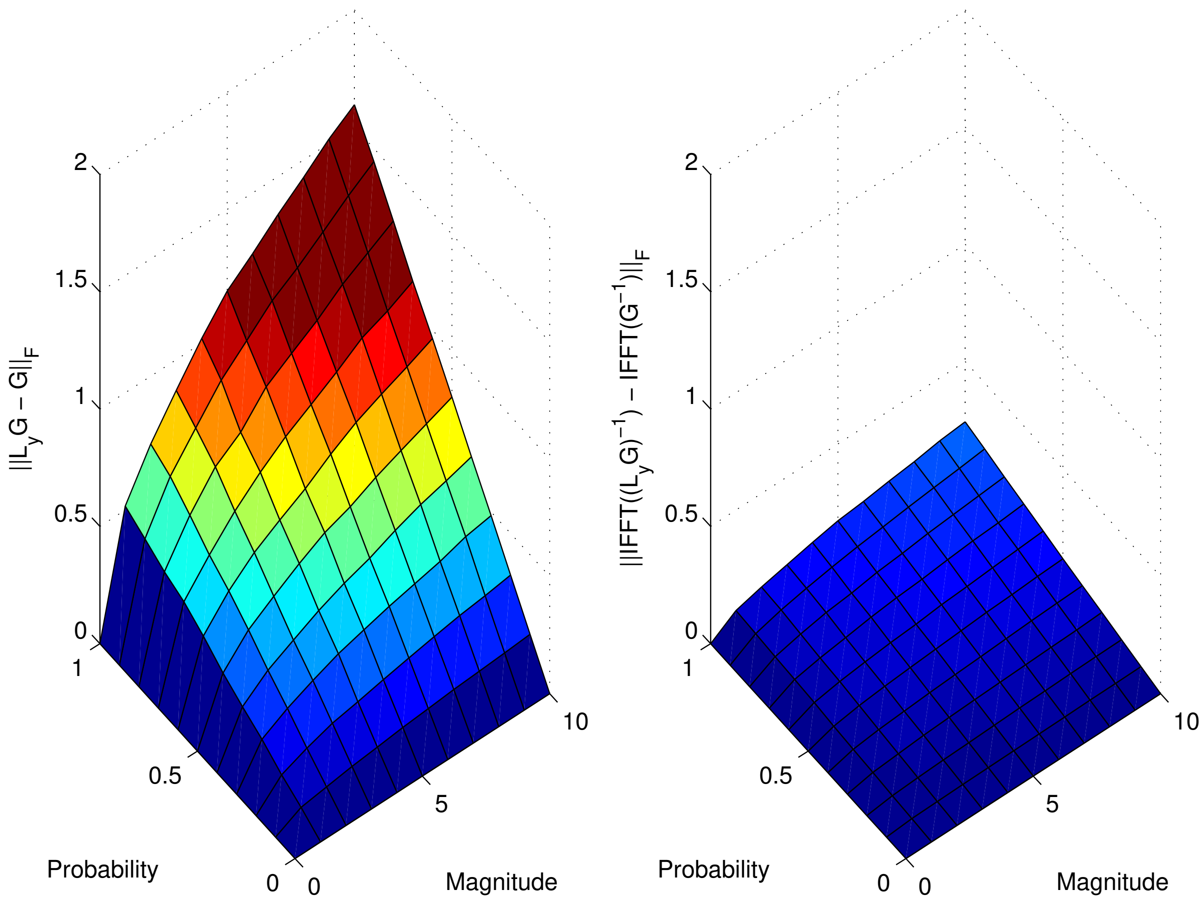


Figure 3.10: Deformation in Frobenius norm of the Gramian and the coefficients

Chapter 4

Gabor-filtered anisotropic generalized Hough transform

In this chapter, we explore the possibility of applying a Gabor system to the outcome of one of the most known transform in image processing, the Hough Transform (HT). Using the Anisotropic Generalized Hough Transform (AGHT), enhanced with a Gabor based time-frequency filtering (GTF), the mandibular canal will be detected in digital dental panoramic radiographs, a support not so easy to deal with due to high presence of noise. The proposed method is based on template matching using the fact that the shape of the mandibular canal is similar in different patients, followed by a filtering of the accumulator space in the Gabor domain for a precise detection of the position. The proposed procedure consists of a detailed description of the shape of the canal in its canonical form and on preserving Gabor filtering information for sorting the hierarchy of location candidates after applying anisotropically the extraction algorithm. The experimental results show that the proposed procedure is robust to recognition under occlusion and under the presence of additional structures e.g. teeth and projection errors.
Out of the scope of this thesis, I would like to mention that after this work which focus on the combination of pattern recognition and harmonic analysis, I continued working on the Hough transform considering in [64] a cascade algorithm which first search for the mandible, then for the mandibular canal inside of it.

4.1 Problem setting

The estimation of the mandibular canal is useful in detecting the nerve for inferior teeth called inferior dental nerve which is found inside it. While there are many research studies trying to visually identify the mandibular canal, eg. [4, 56, 49], the aim of this work is to propose an automatic method based on the AGHT filtered with a Gabor-based transform for marking the mandibular canal in dental panoramic radiography. The clinical motivation relies in the ground fact that these types of images are relatively cheap and easy to acquire compared with other types of imagistic investigations e.g. computer-tomography and they are still widely spread within the community of oromaxillofacial practitioners.
On one side, the HT is a popular technique to extract features from an image. The method was patented in 1962 [90] for the detection of lines in photographs. The functioning of the algorithm lies in a proper choice of the parameters space for the set of lines on the plane.

Consider the implicit equation of a line as expressed in [48]

$$f\left((\hat{m},\hat{c}),(x,y)\right) = y - \hat{m}x - \hat{c} = 0 \tag{4.1}$$

where $\hat{m}$ and $\hat{c}$ are the slope and the intercept that characterize the line. Equation (4.1) maps every pair of parameters $(\hat{m},\hat{c})$ to a specific line, a shape.
By swapping the roles of the parameters and the points and fixing one point, the equation defines all the lines that pass through the fixed point.
So, if we consider some points in an image, they will be collinear if we can find a parameter $(\hat{m},\hat{c})$ which satisfy (4.1) for all the theese fixed points.
Following [8], consider now a generic curve in analytic form

$$f\left(\boldsymbol{x},\boldsymbol{a}\right) = 0 \tag{4.2}$$

where $\boldsymbol{x}$ is a point and $\boldsymbol{a}$ is a parameter vector. For every point we will find a hypersurface of parameters that satisfy (4.2). The curve will be detected in the intersection of these hypersurface. For example if we consider a circle

$$(x - x_0) + (y - y_0) = r_0^2,$$

due to the relation among x_0, y_0, and r_0, the number of free parameters is two.
Every point (x, y) will produce a cone in the parameter space (x_0, y_0, r_0); if we consider three point on the plane, we can find the related circle as the intersection of these 3 cones.
Now we can add further information that can be extracted an image: the gradient and the tangent on the points of the edge. We can use the equation of the curve together with its derivative to find all the parameters which satisfy

$$\frac{\partial f}{\partial \boldsymbol{x}}\left(\boldsymbol{x},\boldsymbol{a}\right) = 0 \tag{4.3}$$

or similarly

$$\frac{\partial y}{\partial x} = \tan\left(\phi(x) - \frac{\pi}{2}\right),$$

which is known since $\phi(x)$ is the gradient direction.
Going back to the example of the circle, fixing the gradient with the point means that the center must lie r_0 units along the direction of the gradient, so the number of free parameters reduces to one.

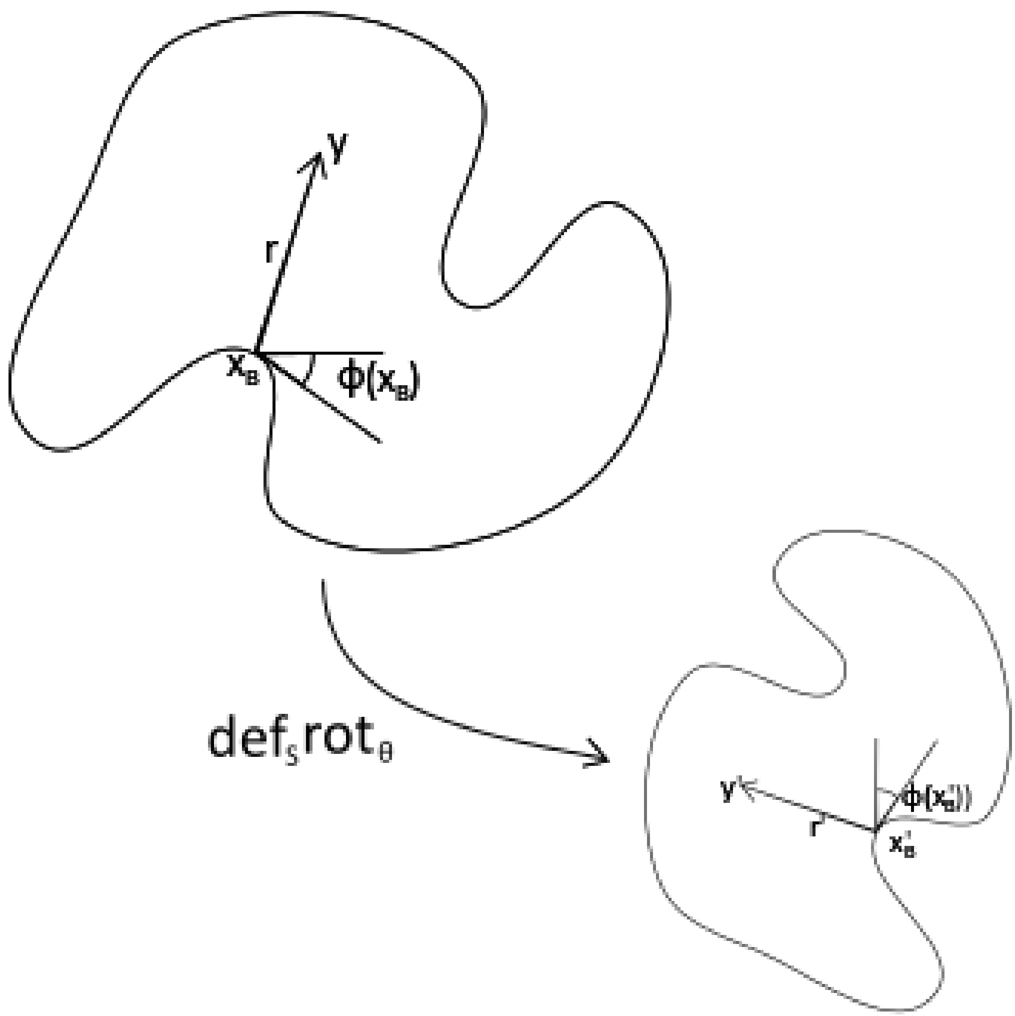


Figure 4.1: Explanation for the notation used in the Hough transform method

In order to develop a method for the recognition of a generic template in an image, in [8] the parameters

$$\boldsymbol{a} = \{\boldsymbol{y}, \boldsymbol{S}, \theta\} \tag{4.4}$$

are used, being

- $\boldsymbol{y} = (x_r, y_r)$ the reference point to represent the translations
- $\boldsymbol{S} = (S_x, S_y)$ scale values for the orthogonal shearing deformations
- θ the angle that represents the rotations

The reference point $\boldsymbol{y}$ is described in terms of a table, called the **R-table** of the template, of possible edge pixel orientations. The other parameters are described in terms of transformations of the aforementioned table.
The key for generalizing the Hough transform is to use the directional information. Given a template, i.e. a set of boundary points $\{\boldsymbol{x}_B\}$, a reference point $\boldsymbol{y}$ is chosen. After the discretization of the straight angle through a uniform partition $\{0, \Delta t, 2\Delta t, \ldots, N\Delta t\}$, for every boundary point the tangent direction $\phi(\boldsymbol{x}_B)$ is computed, then $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}_B$ is stored

in the nth bin of the R-Table if $\mod(\phi(\boldsymbol{x}_B), \pi) \in [(n-1)\Delta t, n\Delta t)$. This choice of using the angle modulus π will be made clear later.
Usually, to control the error in the computation of $\boldsymbol{r}$, the barycenter of the boundary point is chosen as reference point and $\boldsymbol{r}$ is stored in polar coordinates.
We obtain a structure in the form

BIN	*POINTS*
1	$\{\boldsymbol{r} \mid \exists\, \overline{\boldsymbol{x}} \in \{\boldsymbol{x}_B\}\ s.t.\ \mod(\phi(\overline{\boldsymbol{x}}), \pi) \in [0, \Delta t[\}$
2	$\{\boldsymbol{r} \mid \exists\, \overline{\boldsymbol{x}} \in \{\boldsymbol{x}_B\}\ s.t.\ \mod(\phi(\overline{\boldsymbol{x}}), \pi) \in [\Delta t, 2\Delta t[\}$
$\vdots$	$\vdots$
N	$\{\boldsymbol{r} \mid \exists\, \overline{\boldsymbol{x}} \in \{\boldsymbol{x}_B\}\ s.t.\ \mod(\phi(\overline{\boldsymbol{x}}), \pi) \in [(N-1)\Delta t, N\Delta t]\}$

Table 4.1: R-Table

through the following procedure

Algorithm 3 Construction of the R-Table

Precondition: $\begin{cases} \texttt{TMP}\text{: Set of boundary point,} \\ N\text{: Discretization Parameter for Gradients} \end{cases}$

```
function R-TABLE_BUILD(TMP)
    R ← EMPTY_CELL_VECTOR(N) Initialize the R-table as an array of empty cell
    y ← BARYCENTER(TMP)
    for x_B ∈ TMP do
        r_B ← y − x_i
        φ_i ← φ(x_B)
        n ← ⌈φ_i·N / π⌉
        R{n} ← R{n} ∪ {r_B}
    end for
    return R
end function
```

Given this simple structure, if we define the rotation and shearing operators, eg. rot_θ and $def_{\boldsymbol{S}}$, we can build the following procedure to detect the template in the set of edge points of any image

1. Fix a deformation $\boldsymbol{S}$ and a rotation θ
2. Compute the gradient of the edge points $\phi(\boldsymbol{x}_e)$
3. Select the Bin in the R-table corresponding to $def_{\boldsymbol{S}}(rot_\theta(\phi(\boldsymbol{x}_e)))$
4. For every $\boldsymbol{r}$ in the selected Bin, report an occurrence of $\boldsymbol{y}' = \boldsymbol{x}_e + def_{\boldsymbol{S}}(rot_\theta(\boldsymbol{r}))$ as possible reference point.

In this way we find the edge point that satisy the non-analytic version of (4.2) and (4.3). The space of occurrences for the reference point is called **Accumulator Space**.

Algorithm 4 Detection of Pattern in an image

Precondition:
$$\begin{cases} \texttt{IMG}\text{: Image} \\ \texttt{R}\text{: R-Table of a template} \\ \{(S,\theta)\}\text{: Set of scale and rotation parameters} \\ \texttt{WHERE}\text{: subregion of } \texttt{IMG} \end{cases}$$

function HOUGH_RECOGNITION(IMG,R-T,$\{(S,\theta)\}$,WHERE)
    $(\boldsymbol{x}_e, \phi(\boldsymbol{x}_e)) \leftarrow$ EDGE(IMG)
    ACCUMULATOR $\leftarrow$ empty occurrence matrix with the same dimension of WHERE
    for (S,θ) **do**
        for $(\boldsymbol{x}_e, \phi(\boldsymbol{x}_e))$ **do**
            $n \leftarrow \lceil \frac{\phi_i \cdot N}{\pi} \rceil$
            for $\boldsymbol{r} \in R\{n\}$ **do**
                $\boldsymbol{y}' \leftarrow$ DEF(ROT($\boldsymbol{r}, \theta$),S)
                if $\boldsymbol{y}' \in$WHERE **then**
                    ACCUMULATOR$\{\boldsymbol{y}'\} \leftarrow$ ACCUMULATOR$(\boldsymbol{y}') + 1$
                end if
            end for
        end for
    end for
    return ACCUMULATOR
end function

4.2 Recognition of the pattern: the case of mandibular canal

In previous section, we described a common way to detect a template using the AGTH. Problems arise when we have to deal with real images which could be corrupted by noise. That is the case of radiography where the structure of a bone is not well defined and where some part of the bone which would be detected can be disguised - even from the human eye - with unwanted details.
The position of the mandibular canal is described by medical indications as follows: the canal starts at the mandibular foramen in the middle part of the vertical ramus. It continues through the mandibular bone and ends in the menton foramen between apexes of the two inferior premolars.
Any surgical intervention in the mandibular area must prevent any nerve injury. The injury of the nerve would result in prolonged local and lower lip anesthesia for a minimum period of six weeks. Estimating the position of the mandibular canal means knowing the position of the nerve and by this the surgeon can estimate the risks and to adapt the surgical procedure to the individual case.

Figure 4.2: Typical panoramic radiography

In our test we used figure 4.2 to extract the template of the canal. The recognition performs well on the same panoramic as it can be seen in image 4.5, but the aforementioned image is in a critical situation: the patient has only one molar on the right part of the mandibular.
When the mandible is edentated, without any further restriction, the canal could not be recognized anymore because the AGHT algorithm matches the template in image 4.3 with the top part of the alveolar process, the part of the bone where the teeth should be. It happens because the process is detected with a thicker edge than the canal, but has the same gradient direction, so in the recognition process it will have more importance. This undesired, unavoidable, matching has been soften by using the modulus-π direction of the gradient: the canal, as modeled in figure 4.3, looks like the empty space in a edentated mouth; by using the modulus we remove every information about the inside-outside of the model. This seems the best choice for the case of the mandibular canal, a poorly defined region of the mandible enclosed by a slightly brighter contour.
This problem is related to how the AGHT is implemented: one of the weakness of this transform is that the accumulation space does not carry any information about the position in space of the template nor the mutual relation of different shape in the image. We identify three ways to overcome this problem:

- Manual solution: user should restrict the area to be investigated manually through anatomical information given beforehand.
- Semi-automatic solution: after a first, coarse and less accurate search for the mandible template through AGHT, the area to be investigated for the mandibular canal is automatically restricted.
- Automatic ranking of the accumulator space using the Gabor transform: as described in Section 4.3.1.

4.3 Proposed method

The canal template has the following characteristics: it is a connected, compact and simply connected region of the plane. Therefore, the first step is to compute the edges. After the binarization of the template, we can run a contour-following algorithm to detect the boundary points. This way, one could reach the first purpose for the proposed pattern recognition: to obtain an easy manipulable set of data samples.

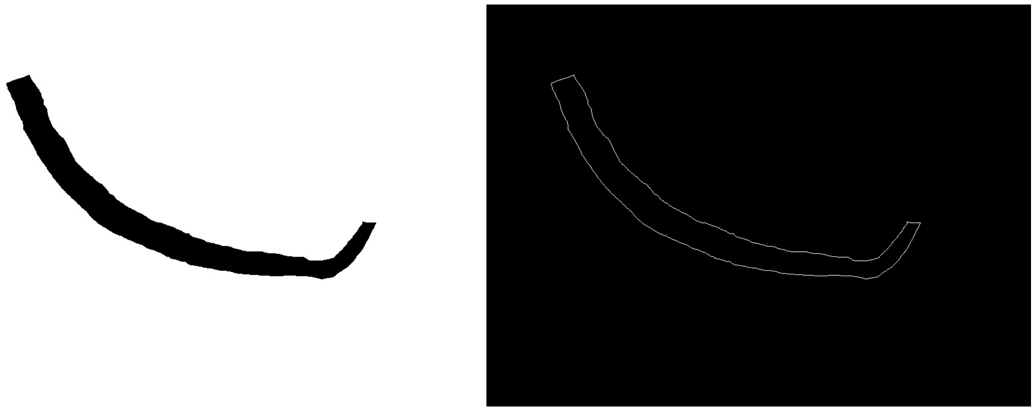


Figure 4.3: Canal and its contour

Nevertheless, the same strategy can not be applied directly to a panoramic radiograph; also common edge detector filters such as Canny, Sobel, etc., fail. Radiographs are spurious images which contain a great amount of unwanted details. So, we concentrated our work on the choice of the good parameters for the detection of the edges of the teeth (for a good survey see [41]). This means the proper choice of the variance in the Gaussian filter and the threshold parameter.
The processes of low pass filtering and thresholding, cancel the mandibular canal from the image; so we have to create an *ad hoc* method for detecting the edges. It can be seen from Figure 4.2 that mandibular canals are drawn by two bright gray curved lines in a darker gray background; the good point is that the orientation of the canal is steady. Therefore, the natural idea is to use a high pass anisotropic filter mask adjusted on the shape of the mandibular canal.
Pursuing this idea, after a histogram equalization, we used the following mask as anisotropic high pass filter (AHPF) to detect the right mandibular canal

$$\begin{aligned}
A_0 &= \begin{pmatrix} -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 2 & 2 & 2 & 2 & 2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} & -\frac{5}{4} \end{pmatrix} \\
A_1 &= \begin{pmatrix} 2 & \frac{1}{4} & -1 & -1 & -1 \\ \frac{1}{4} & 2 & \frac{1}{4} & -1 & -1 \\ -1 & \frac{1}{4} & 2 & \frac{1}{4} & -1 \\ -1 & -1 & \frac{1}{4} & 2 & \frac{1}{4} \\ -1 & -1 & -1 & \frac{1}{4} & 2 \end{pmatrix} \\
A_2 &= \begin{pmatrix} [1 & 2 & -\frac{1}{4} & -1 & -2 \\ 0 & 2 & 1 & -\frac{1}{2} & -2 \\ -1 & -\frac{1}{4} & 2 & -\frac{1}{4} & -1 \\ -2 & -\frac{1}{2} & 1 & 2 & 0 \\ -2 & -1 & -\frac{1}{4} & 2 & 1 \end{pmatrix} \\
A_3 &= A_2^T
\end{aligned} \tag{4.5}$$

and their flipped left-to-right version for the left mandibular canal.

To calculate the gradient needed in the implementation of the AGHT we used a Sobel mask for both the boundary and the edge points to have consistency in the calculation. After the calculation of the barycenter of the template, for the construction of the R-table we chose to store $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}_B$ in Cartesian coordinates to follow the natural discretization introduced by an image. This also helps us to understand the worst case scenarios: after the recognition process the accumulator space will be a matrix with the same dimension of the image so we could print it on screen in grayscale to understand how the error spread and which other shape can be disguised as a mandibular canal.
There are many ways to increment the accumulator space. The general command is

$$A(\boldsymbol{a}) := A(\boldsymbol{a}) + g(\boldsymbol{x})$$

for all the different edge point $\boldsymbol{x}$.
By looking at the regularity of the mandibular canal we chose

$$A(\boldsymbol{a}) = \begin{cases} A(\boldsymbol{a}) + 5 & \text{if } \phi(\boldsymbol{x}) \in \left]\frac{1}{12}\pi, \frac{5}{12}\pi\right[\\ A(\boldsymbol{a}) + 1 & \text{elsewhere} \end{cases}$$

for the right canal and an analogous function for the left one.

Figure 4.4: Accumulator Space for Canal recognition

The last remark should be about the parameters expressed in (4.4). We have not used the rotational parameter θ inasmuch as every panoramic radiography is taken with the patient's head fixed. The important parameter is $\boldsymbol{S} = (S_x, S_y)$ which helps us to reconstruct the anatomical difference among human beings. In this way, it is possible to find the best deformed version of the template in Figure 4.3 that matchs the canal in the panoramic radiograph under analysis.

4.3.1 Gabor ranking of the accumulator space

In order to propose a ranking for the best match in the accumulator space, we will use the Gabor transform with the following interpretation: instead of seeing $\widehat{x}_1$ and $\widehat{x}_2$ as the components of a 2D frequency, one might prefer to see $\widehat{x}$ as the parameter for the pattern density and define an orientation similar to the argument of complex numbers.

In order to detect the position of the best match in the acumulator space of the GHT for the reconstruction of the mandibular canal, we will perfom a Gabor transform to obtain local characteristics of the waves' intensities.

The localization of the image is done by computing the point wise multiplication $Im \cdot T_{\mathbf{l}}Atom$ for a set of different translation parameters $(T_{\mathbf{l}}Atom)_{\mathbf{k}} = Atom_{k-l}$. The last step is the computation of the FFT for all these localized blocks. The drawback of this representation is that it considers a continuous domain. When discretized directly it results a highly redundant representation. Therefore, the proper sampling of position space and frequency space with step widths $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, chosen such that we obtain a Gabor frame [39] is necessary. The result is a set of Gabor coefficients which represent local waves contained in the image.

As a localized and sampled 2D STFT, the Gabor representation, using discrete frames will give us, at each point of the lattice, a two-dimensional spectral representation of the image ordered either with modulation or with translation priority. Using this localized time-frequency representation, the harmonic filtering in phase-space will identify the pattern in the phase arguments.

In order to detect the best match, we first chop the accumulator space into small overlapping hexagonal regions in order to sample using the quincunx lattice. These regions are determined by their geometric coordinates in the accumulator space. Then we compute the Gabor representation on these patches for the first ranking. In the last step, we check if any energy variation happen in similar patches and detect the position of the best match by narrowing the analyzing window. This task is computationally expensive and high performance machines are recommended.

The procedure is presented in detail in Algorithm 5.

Algorithm 5 [DO&SZ] Gabor Ranking of image

Precondition: $\begin{cases} A\text{: Accumulator space} \\ (g_i, \Delta_i)_{i=1}^{N}\text{: Gabor systems in subregions} \end{cases}$

```
function GABOR_RANKING(𝒜,(g_i, Δ_i))
    𝒜 := {A_i}_{i=1}^N ← PARTITION(𝒜, N)
    for x_B ∈ TMP do
        G_i ← V_(g, Δ)A_i
        M_i ← FIND_Max(G_i)
    end for
    MAX ← SORT(M_i)
    return MAX
end function
```

4.3.2 The Algorithm

We sum up in Algorithm 6 the complete procedure for the recognition of the mandibular canal, incorporating the AGTH detection and the GTF ranking.

Algorithm 6 [DO&SZ] Anisotropic Gabor-Hough Transform

Precondition: $\begin{cases} \texttt{IMG}\text{: Image} \\ \texttt{TMP}\text{: Template } (g_i, \Delta_i)_{i=1}^{N}\text{: Gabor systems in subregions} \end{cases}$

function GABOR_RANKING($\mathcal{A}$,N)
  `R` $\leftarrow$ `R-Table_Build(TMP)`
  `ACC` $\leftarrow$ `HOUGH_RECOGNITION(IMG,R)`
  `POS` $\leftarrow$ `GABOR_RANKING`(ACC,$(g_i, \Delta_i)_{i=1}^{N}$)
  return `POS`
end function

4.4 Results

In this section, we present the experimental results of the proposed algorithm. The first test is to search for the template extracted from image 4.2 in the same image. The overlapping is perfect even without any technique of region restriction. The same argument can be brought forward for the flipped template. Since there is no significant anatomical difference between the left and the right side of the same patient, the overlapping is almost perfect.

As presented in Section 4.2, one of the biggest problems we encountered is the mismatch of the canal template with an edentated mandible where we had restricted the edge processing to the left-bottom part of the radiography.

We already identified in Section 4.2 three ways to overcome the problem of mismatching namely manual, semi-automatic, and automatic solution. A comparison could be summarized as follows: the first approach is as expected the worst solution because it is necessary to adjust the region of interest image by image.

Figure 4.5: Perfect matching of the template on the original image. Optimal matching of the flipped-left template

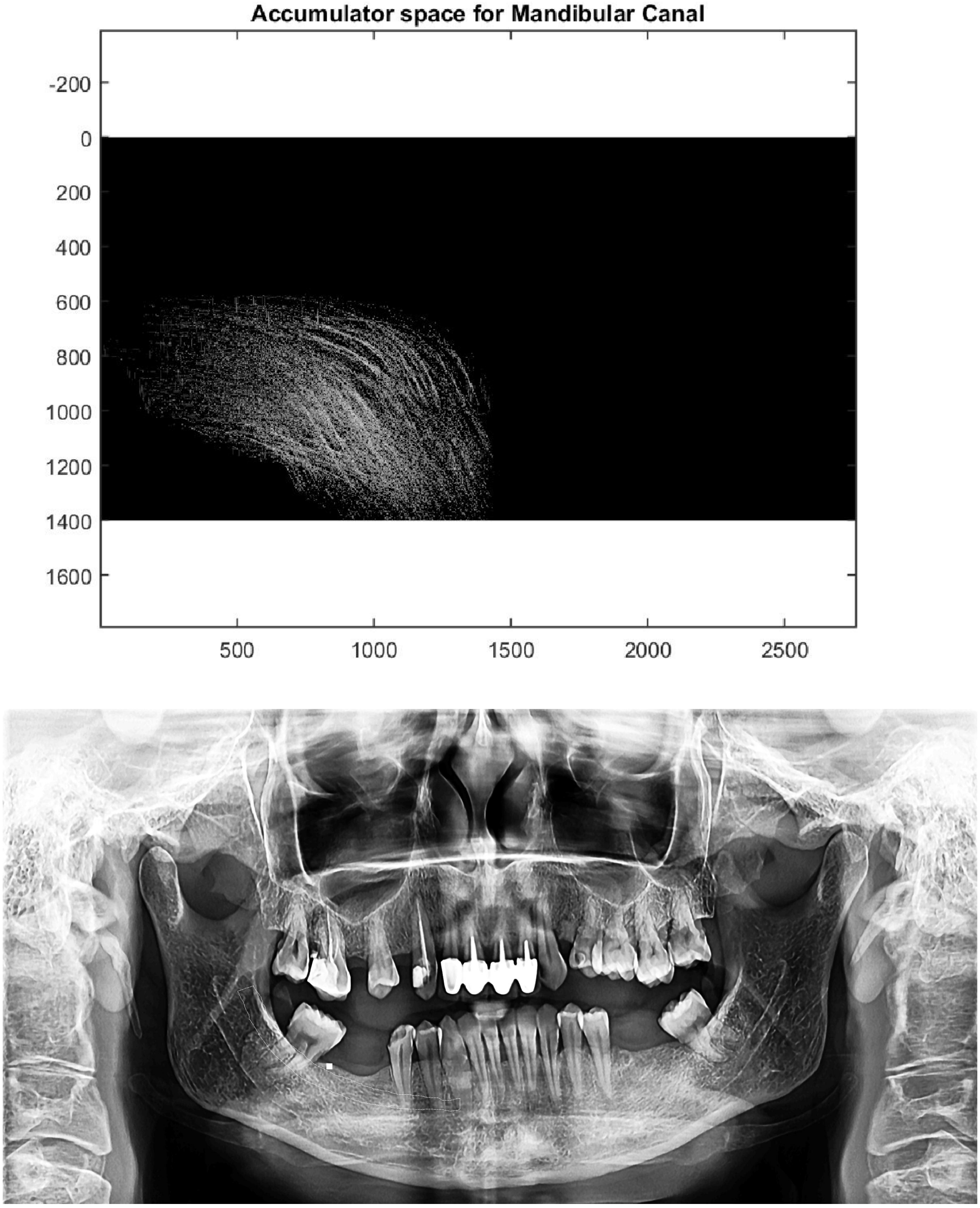


Figure 4.6: **[DO&SZ]** Semi-automatic solution without aria restriction.

The second approach is the *semi-automatic* solution, named in this way because it searches for the whole mandible to find the area of interest, then the AGHT recognition process for the canal is performed.
This process is possible because the canal lies in the center of the mandible, so their reference points (their barycenters) are really close. After the detection of reference points for the mandible $\boldsymbol{y}_M = (x_M, y_M)$, we force the reference points of the canal to be in the square window $[x_M - 50, x_M + 50] \times [x_M - 50, x_M + 50]$, with underlining unit of measure the pixel (Figure 4.6 and 4.7).

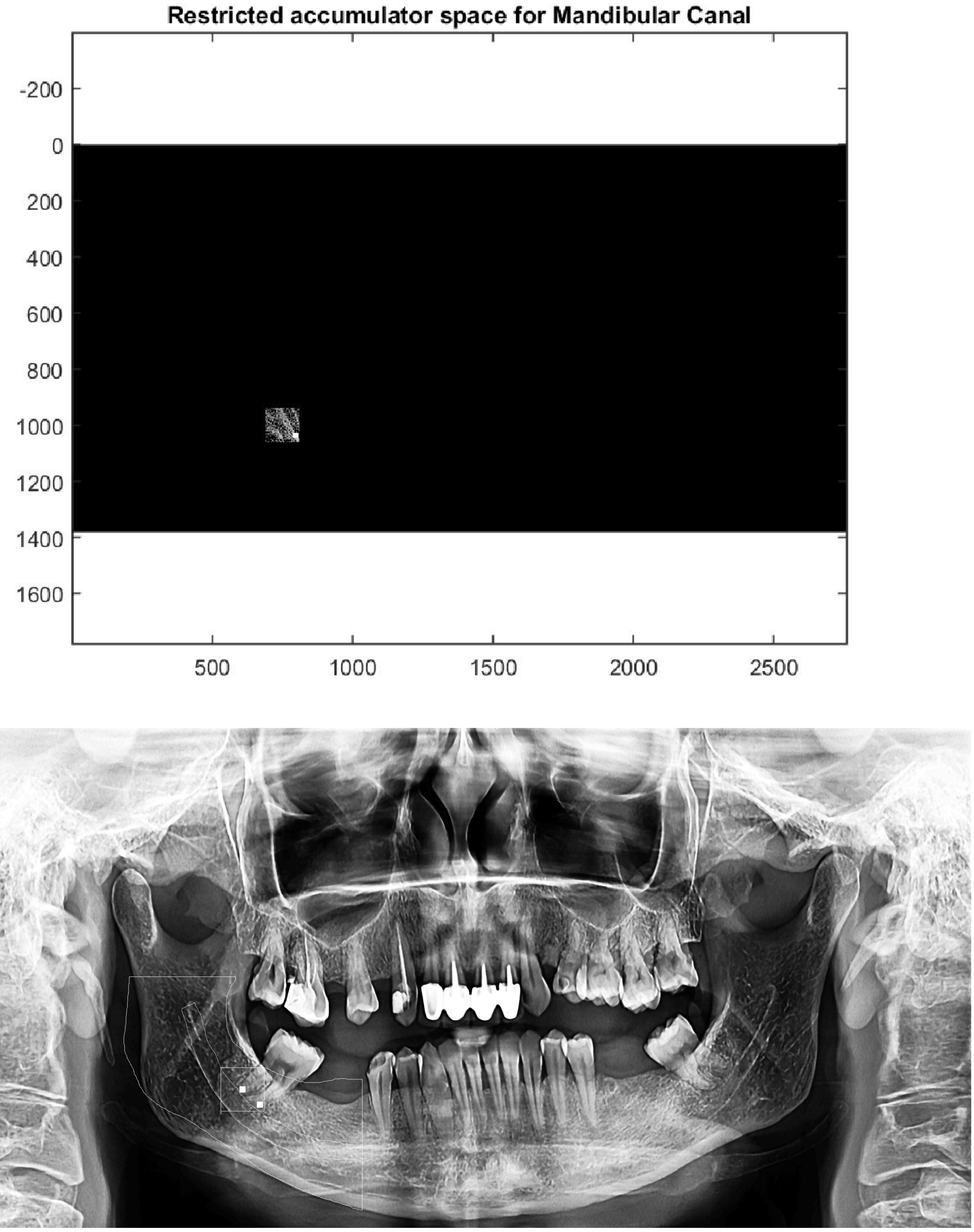


Figure 4.7: Semi-automatic solution with area restriction.

It is easy observe in the image 4.8 that if the patient has all his teeth, the area-restriction process is not required for the canal recognition.

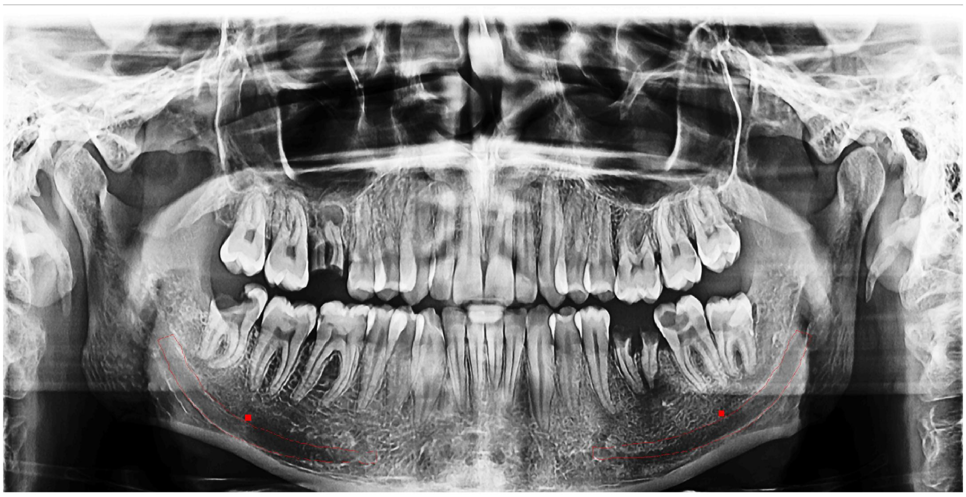


Figure 4.8: Recognition of the template in figure 4.3 with scale parameters $(S_x, S_y) = (1.1, 1.15)$

The *automatic solution* is to run completely the algorithm 6 including the AGTH and the GTF ranking. In figure 4.9, one can see after applying the GTF ranking, the restriction area is automatically detected such that the AGTH reconstruction could be performed for the best match.

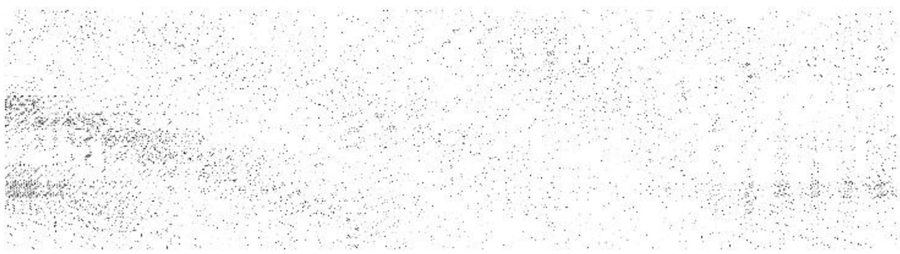


Figure 4.9: GTF ranking (automatic area delimitation on the right side of the image).

4.5 Chapter conclusions

In this work, we used the AGHT for the detection of the mandibular canal in a panoramic radiograph, followed by a GTF filtering of the acumulator space. We have also indicated the problems we encountered in this particular case of pattern recognition:

- choice of parameter space
- edge detection
- zonal restriction

The first issue was solved through the choice of Cartesian coordinates for a better understanding of the local maximas. The second issue was resolved through a proper choice of the anisotropic high pass filter.

We had to deal with different shapes which can be mistaken with each other, such as the top and the bottom of the mandible.
The initial strategy was to restrict the area to be analyzed using medical information. This is not an optimal strategy for the application of the AGHT recognition process in general situations.
Therefore, we have used an automatic selection based on GTF ranking of the accumulator space for finding the best template match.

Chapter 5

Wavelet-like systems based on multi-windows spline-type spaces

Wavelet analysis has shown since its early discovery prominent application in signal analysis [22]. The possibility to employ a multiscale analysis became a milestone for compression and learning of a signal. This work was my introduction to the world of multiresolution analysis over locally compact groups. The exposition is rather informal since we are interested into two practical aspects: establishing the stability of designed algorithms and overcoming standard orthogonal expansion in multiresolution analysis[58].
The structure of this chapter is as follows: in Section 5.1 we present the spline-type reformulation of multiresolution-like analysis including a detailed ST multiresolution Algorithm 5.2 and its complexity analysis in Section 5.2.1 while in the Section 5.3 we present the numerical experiments. The last section is dedicated to conclusions and the outline of future research.
This work was presented in [63].

5.1 Modulations and dilations

The ST analysis measure the correlation of a signal with different shifts of a fixed set of atoms. Together with time-localization, signal and image analysis is concerned with scale and frequency localization.Togeter with the mudulation operator (1.15)we have to define the contraction (or dilation) operator. This is a much more arduous work, since its construction is not as straightforward as for modulation. To mimic the case of the real group, we would need to give the LC group an external product with scalar e.g. a vectorial structure, such that standard dilation

$$D_a f := f(a\cdot)$$

can be employed.
In Chapter 3 we have shown that the Gabor systems can be exactly reformulated in a biorthogonal construction problem; this construction becomes a powerful and easy to use tool in signal analysis. This is due to the favourable case for which it is possible to periodize both time and frequency domain. The same is not possible for wavelet systems since the collection of generating functions is infinite, not periodizable, and defined over different lattices. Wavelet can indeed be defined as a collection of function $\{\phi_i\}_{i\in I}$, which are characterized by being the contraction of a unique mother function ϕ and their shifts over a related lattice H_i such that $H_i \subset H_{i+1}$.

As a Gabor systems can be defined as the collection of shifts over the same lattice of different modulations

$$\{L_x (M_{\widehat{x}}\phi) \ : \ x \in H, \ \widehat{x} \in K\},$$

being K a discrete subset of $\hat{G}$, the wavelet system can be defined as

$$\left\{L_x \left(D^{(i)}\phi\right) \ : \ i \in I, \ x \in H_i\right\},$$

For a union of ST spaces $\cup_i (\phi_i, H_i)$ we get the system operator

$$Sf := \sum_i U_{\phi_i,H_i} V_{\phi_i,H_i} f,$$

With this reformulation Gabor systems can be rewritten as ST spaces being ϕ_i different modulations and H_i a constant sequence, while for Wavelets one uses contractions and an ascending monotone sequence of lattices. The Gabor system operator commutes with both shifts and modulations; this is not the case for wavelets for which dilation do not commutes. If the system operator is invertible, it is possible to build the functions $S^{-1}\phi_i$, called the canonical duals, which behaves as the biorthogonal in the reproduction formula (2.7).
For Gabor systems built on a uniform rational sampling in both space and modulation the inversion procedure is made computationally feasible through the Walnut representation which partition the system operator into a block circulant matrix. Thanks to its multi-window feature, the spline type reformulation gives an analogous partitioning, hence the construction of biorthogonal systems do not suffer from uniform frequency partitioning while maintaining the same algorithmic complexity.

To solve the wavelet construction in a sampling theory, generalized shift invariant systems (GSI) were introduced[76, 18], allowing the construction of ST space as infinite unions of ST spaces, each of them generated by a unique function (in the wavelet case a different dilation) and a different subgroup. This reformulation leads to good result for theoretical purpose [76] but the complexity of the related algorithm raises [18].
The nimbleness of Theorem 4 is lost, so the complexity of the algorithm raise up to usual operator inversion.
Here we use the wavelet scheme in its typical multiresolution reformulation. In its early formulation [58], the mathematical justification of multiresolution analysis (MRA) relies in it an ascending chain of ST spaces in $L^2(\mathbb{R}^d)$

$$V_{-\infty} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_\infty = L^2(\mathbb{R}^d)$$

each of them generated by a scaling function $D_{a^j}\phi$ over the lattice $a^j\mathbb{Z}^d$.
At each level j is then considered W_j the orthogonal complement of V_j in V_{j+1}. Therefore, it is possible to consider any element of $L^2(\mathbb{R}^d)$ as expansion

$$L^2(\mathbb{R}^d) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \ldots$$

for an arbitrary fixed index j_0.
At each scale only one generating function is considered and the orthogonality of the W_j spaces is used to express subsequent projections in the ST spaces $\mathcal{S}(D_{a_i}\phi, H_i)$. Our interest is focussed in biorthogonal expansions which can express different features of a signal. In this particular case

- Time = through the ST structure
- Scale = through the multiresolution iteration
- Frequency = through the multiwindow nature.

Using multiwindow spline type spaces, we can improve the accuracy with a small drawback in complexity. Fixed a scale, we chose to modify the window obtaining different atoms, having the same scale. This modification will be obtained starting from the modulation operator.

5.2 The ST multiresolution algorithm

During the first step of a MRA the low frequency content of the signal is analysed. For this purpose in Step 2 of Algorithm 7 we use few modulations $\Phi := \{M_{\hat{x}_i}\phi\}$ of a Gaussian function ϕ, being $\hat{x}_i$ close to 0. In the function `BIORTH` of Step 3 we compute the related biorthogonal system and compute the low resolution approximation in Step 4.
To handle the three different behaviours that we stated above, we have to modify the multiresolution algorithm building at each scale a stable generating set for which the corresponding ST well represent the given signal, with the less possible effort. As shown in Chapter 3, it is possible to approximate a signal in ST spaces considering only those generators that better intersect its spectrum. Hence we cover the whole spectrum through a modified $\bar{M}$. modulation operator, which will be exploited in Section 5.3. After Algorithm 7 generates the whole set of AVAILABLE atoms, it ranks the outcome of the multiwindow-analysis operator in Steps 9 and 8. The best $\#at$ atoms which ensure stability are consider.
Algorithm 8, which is the procedure to check linear independence, is nothing but the psuedocode for condition (ii) in Theorem 4. A row echelon reduction is used to track the windows which give instability. It is important to notice that we have to check the linear independence of the FT of the atoms for only one representative of each element of $\hat{G}/H^{\perp}$.
Finally the the biorthogonal system is built and the approximation computed. The procedure is iterated on the reminder error.

5.2.1 Complexity analysis

The input signal XX in Algorithm 7 is a sampled function, e.g. an element of $\mathbb{C}^L$ for some $L \in \mathbb{N}$. We consider as LCA group G the set $\overline{L} := \{0, 1, \dots, L-1\}$ with operation

$$x \cdot_G y \equiv x + y, \quad \text{mod } L \qquad x, y\overline{L}$$

Its subgroup are $H_a = \{0, a, \dots, L-a \;:\; L \equiv 0 \mod a\}$. The dual of $\overline{L}$ is $\overline{L}$ itself, and the perpendicular subgroup of H_a is the subgroup $\hat{H}_{\frac{L}{a}}$, pointing out through the *hat* superscript that the related object belong to the dual structure.
For our multiresolution analysis, we consider a sequence of lattices

$$\boldsymbol{H} := \{H_{a_i} \;:\; a_i < a_{i+1}\}_{I=1,\dots,r}$$

So we do not impose the constrain that $H_i \subset H_{i+1}$, but we just assume that levels with higher index have higher resolution.

Algorithm 7 [DO&SZ] Multiresolution ST approximation

Precondition: $\begin{cases} \text{Signal } XX, \\ \text{Scaling function } \phi, \\ \text{Windows } \Gamma = \left\{ D^{(i)}\psi : i \in I \right\}, \\ \text{subgroups } \boldsymbol{H} := \{H_i\}, \\ \text{Fixed number of atorms } \#at \end{cases}$

function `ST_MULTIRESOLUTION`$(XX, \phi, \Psi, \boldsymbol{H}, \#at)$

$\Phi \leftarrow \{M_{\hat{x}_i}\hat{\phi} \;:\; \hat{x}_i \in \widehat{G}\}$
$\tilde{\Phi} \leftarrow$ `BIORTH`(Φ, H_0)
$XXa \leftarrow U_{\Phi,H_0} V_{\tilde{\Phi},H_0} XX$
RR $\leftarrow XX - XXa$
for $i \in I$ **do**
$\Psi \leftarrow \{\bar{M}_{\hat{x}_i} D^{(i)}\psi \;:\; \hat{x}_i \in \widehat{G}\}$
$A \leftarrow V_{\phi_i,H_i}(RR)$
$P_I \leftarrow$ `SORT`(A)
AVAILABLE $\leftarrow \Phi(P_I)$
$\Psi^{(i)} \leftarrow \emptyset$
while $card(\Psi^{(i)}) < \#at$ **do**
GOOD $\leftarrow$ AVAILABLE$(1, \ldots, \#at)$
$(\Psi^{(i)}$,BAD$) \leftarrow$ `ST_STABLE`(GOOD,H_i)
AVAILABLE $\leftarrow$ AVAILABLE$\backslash$BAD
end while
$\tilde{\Psi}^{(i)} \leftarrow$ `BIORTH`$(\Psi^{(i)}, H_i)$
$XXa \leftarrow XXa + U_{\tilde{\Psi}^{(i)},H_i} V_{\Psi^{(i)},H_i} RR$
RR $\leftarrow RR - XXa$
end for
return XXa
end function

Algorithm 8 [DO&SZ] Stability Analysis

Precondition: $\begin{cases} \text{Windows } \Gamma = \{\gamma_i : i \in I\}, \\ \text{subgroup } H \end{cases}$

1: **function** ST_STABLE(Γ, H)

2: $H^\perp \leftarrow$ PERPENDICULAR(H)
3: **for** $\dot{\hat{x}} \in \hat{G}/H^\perp$ **do**
4: $C_{i,j} \leftarrow (\widehat{\gamma}_i(j\hat{x})) \qquad j \in H^\perp$
5: $A \leftarrow$ ROW_ENCHELOR_FORM(C)
6: **for** $j = 1, \ldots, card(\Gamma)$ **do**
7: **if** $a_{i,i} = 0$ **then**
8: INDEPENDENTS $\leftarrow$ INDEPENDENTS $\cup\{\gamma_j\}$
9: **else**
10: DISCARDED $\leftarrow$ DISCARDED $\cup\{\gamma_j\}$
11: **end if**
12: **end for**
13: **end for**
14: **return** (INDEPENDENTS,DISCARDED)
15: **end function**

Since we want to have an $\mathcal{O}(1)$ redundancy at each level, the number of available atoms will be an $\mathcal{O}(\frac{L}{a_i})$ at level H_{a_i}. In this way we showed [61] that the complexity of the biorthogonality construction take an $\mathcal{O}(\mathcal{L})$ in time.
We finish our complexity analysis considering the stability check. Since

$$card\left((H)^\perp\right) = card\left(H_{\frac{L}{a_i}}\right) = a_i,$$

the matrix computed in Step 4 is a $a_i \times \#at$ matrix. Its row echelon form reduction, computed through Gauss-Jordan elimination, has a time cost of $\mathcal{O}(\min(\#at, a_i)^2 \max(\#at, a_i))$ in time. Considering the number of atoms as a constant, Algorithm 8 has $\mathcal{O}(a_i)$ time complexity. Finally, since the number of AVAILABLE atoms is a $\mathcal{O}(\frac{L}{a_i})$, building a stable ST space takes a $\mathcal{O}(L)$ at each level. This proves

Proposition 11. *Let Γ be a set of windows having different scale features suitable for the analysis of signal in $\mathbb{C}^L$.*
The multiresolution ST procedure in Algorithm 7 has time complexity

$$\mathcal{O}(card(\Gamma) \cdot L)$$

5.3 Numerical experiments

A function that is often used in wavelet scheme is the following radial function

$$\varphi(x) = (1 - \parallel x \parallel^2) e^{-\frac{\|x\|^2}{2}} \qquad x \in \mathbb{R} \tag{5.1}$$

In Figure 5.1 two shifts of this window over the periodized domain $\overline{L}$ are shown The high frequency region of $\widehat{\overline{L}}$ is located in the middle of it, so we understand how modulation

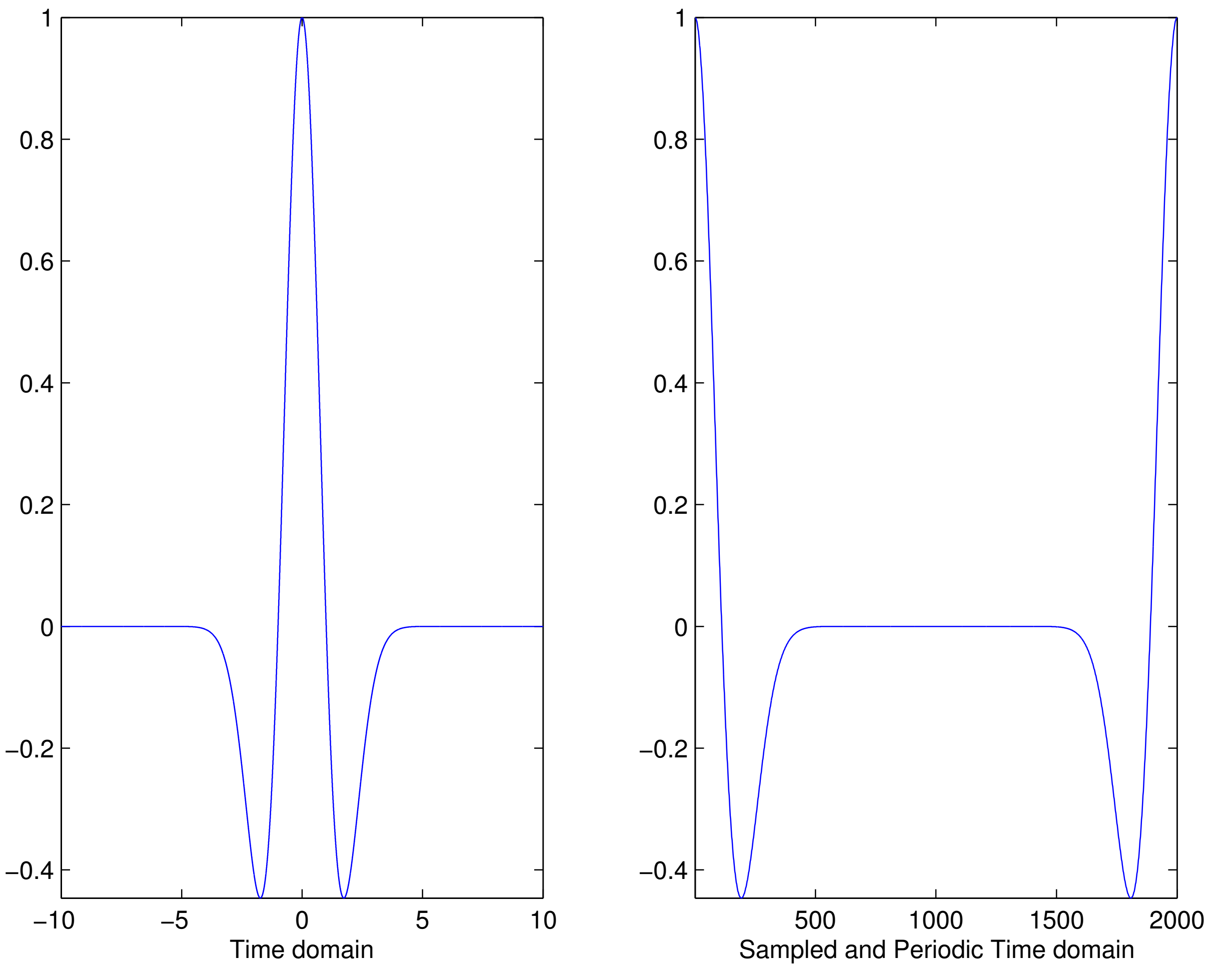


Figure 5.1: Wavelet (5.1) in time and periodized time domain

operator does not handle well the multi-frequency analysis in a wavelet scheme. Mother wavelet is traditionally chosen[22] such that

$$\int |\hat{x}|^{-1}|\hat{\phi}(\hat{x})|^2 d\hat{x} < \infty$$

this imply that $\int \hat{\phi}(\hat{0}) = \int \phi(x)dx = 0$, so we cannot have good frequency localization. This typical behaviour is shown in Figure 5.2.

The ST scheme has the great flexibility to work with modulation as different generating atoms, so we can build a generating set that does not rely on the modulation operator. We build a modified modulation operator

$$\overline{M}_{\hat{x}} = \left(\hat{L}_{\hat{x}} \widehat{f} \left(0, \ldots, \frac{L}{2}\right) + \hat{L}_{\hat{x}} \widehat{f} \left(\frac{L}{2} + 1, \ldots, L\right)\right)^{\vee}$$

which shifts the positive and negative parts of the FT in opposite direction. It is easy to check that this modified modulation build a new window sharing the same decay of the original one, so same scale parameter. In this way we can consider different frequency features of a signal at the same resolution level. An exemplification can be seen in Figure 5.3. As stated in 5.2 we complete the scheme with some low frequency shift of a Gaussian, covering the low region of the frequency domain.

As an example, consider the periodized domain $\overline{1024}$. The low frequency region is analysed by $\mathcal{S}(\Phi, H_{128})$, being Φ a set of 7 modulated Gaussians $\Phi := \{M_{\hat{x}} e^{-\frac{x^2}{2}} \ : \ \hat{x} =$

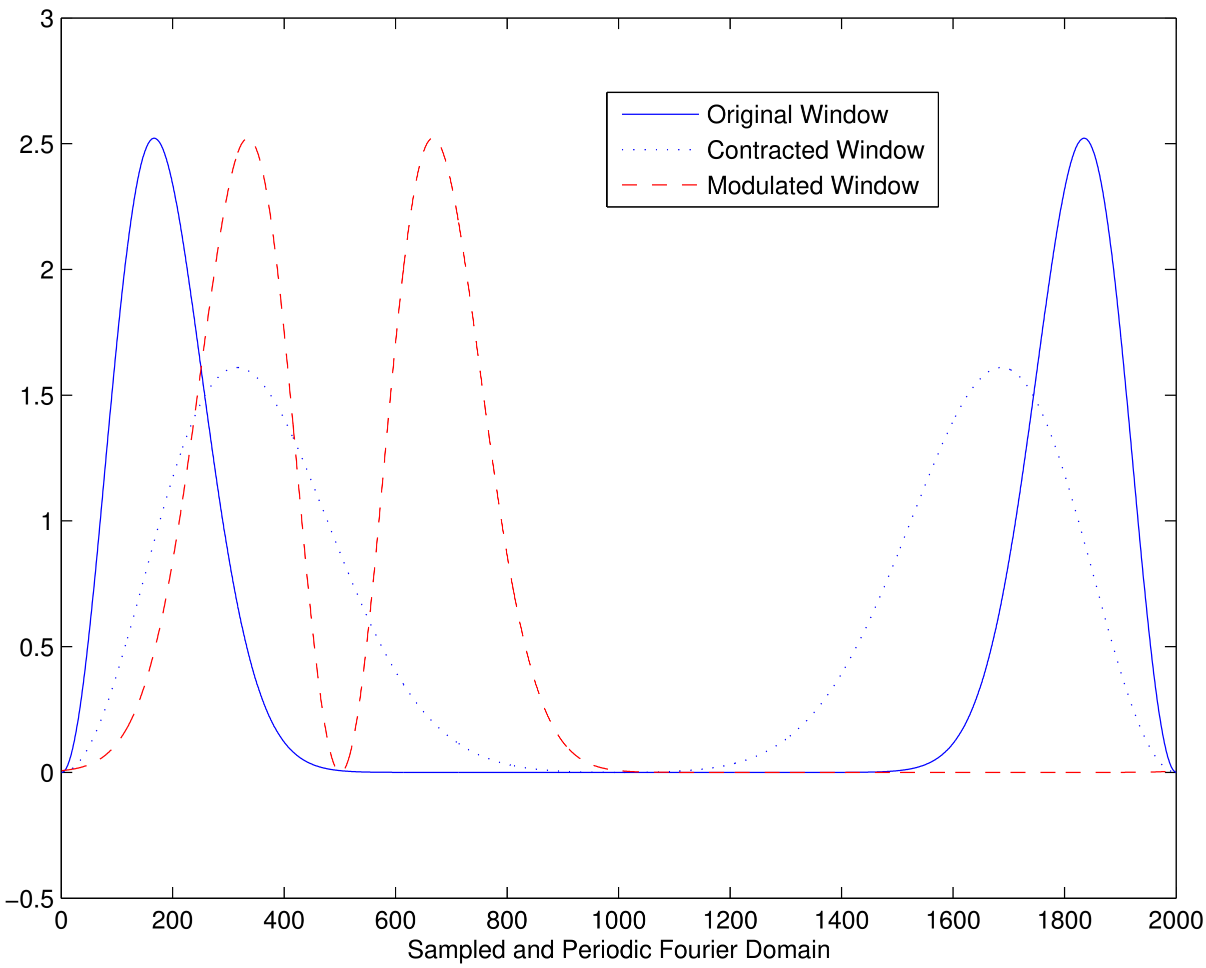


Figure 5.2: Fourier transform of Wavelet (5.1), a modulation and the Fourier transform of a contraction

$0, \pm 8, \pm 16, \pm 24\}$. We then considered for each $a = 64, 32, 16, 8$ the ST space generated over H_a by a mother wavelet φ_a of the type (5.1) having support of radius $3a$ and by the modified modulations $\bar{M}_{\hat{x}}\varphi_a$, $x = \frac{2}{3}\frac{L}{a}k$, $k = 0, \ldots, \lfloor \frac{3}{4}a \rfloor - 1$, with $\lfloor \cdot \rfloor$ the usual round toward negative infinity operation; in this way the spectrum of the ST space covers the whole Fourier domain and the modified modulations show good overlap.
We have performed 3 separate runs of the code:

- Generating only the $\overline{M}_0$ modified modulation to reproduce standard wavelet system
- Generating all the modulations and selecting the best correlated one
- Generating all the modulations and selecting the 4 best windows

The outcome of this test is displayed in Figure 5.4. At each modification of the original multiresolution algorithm, the approximation improves.

5.4 Chapter conclusions

We presented in this chapter a numerical procedure to construct wavelet-like systems based on multiwindow spline-type spaces under the stability argumentation. The next

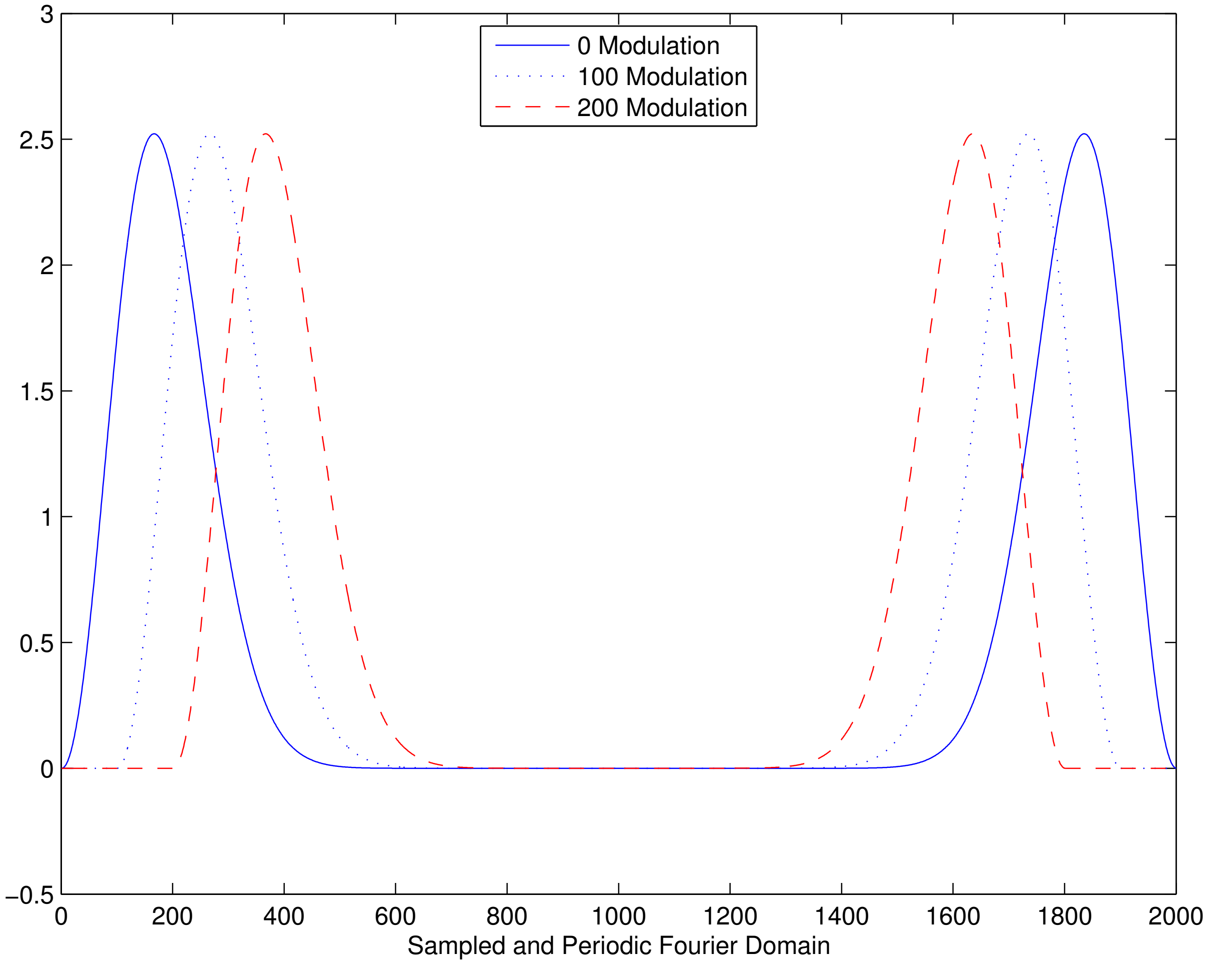


Figure 5.3: Fourier transform of Wavelet (5.1), a modulation and the Fourier transform on another scale

step is to reproduce standard multiresolution wavelet constructions, without the use of generalized spline type spaces, and considering instead only the recurrency structure that comes from the refinement equation upon which these ascending sequences of shift invariant spaces are built.

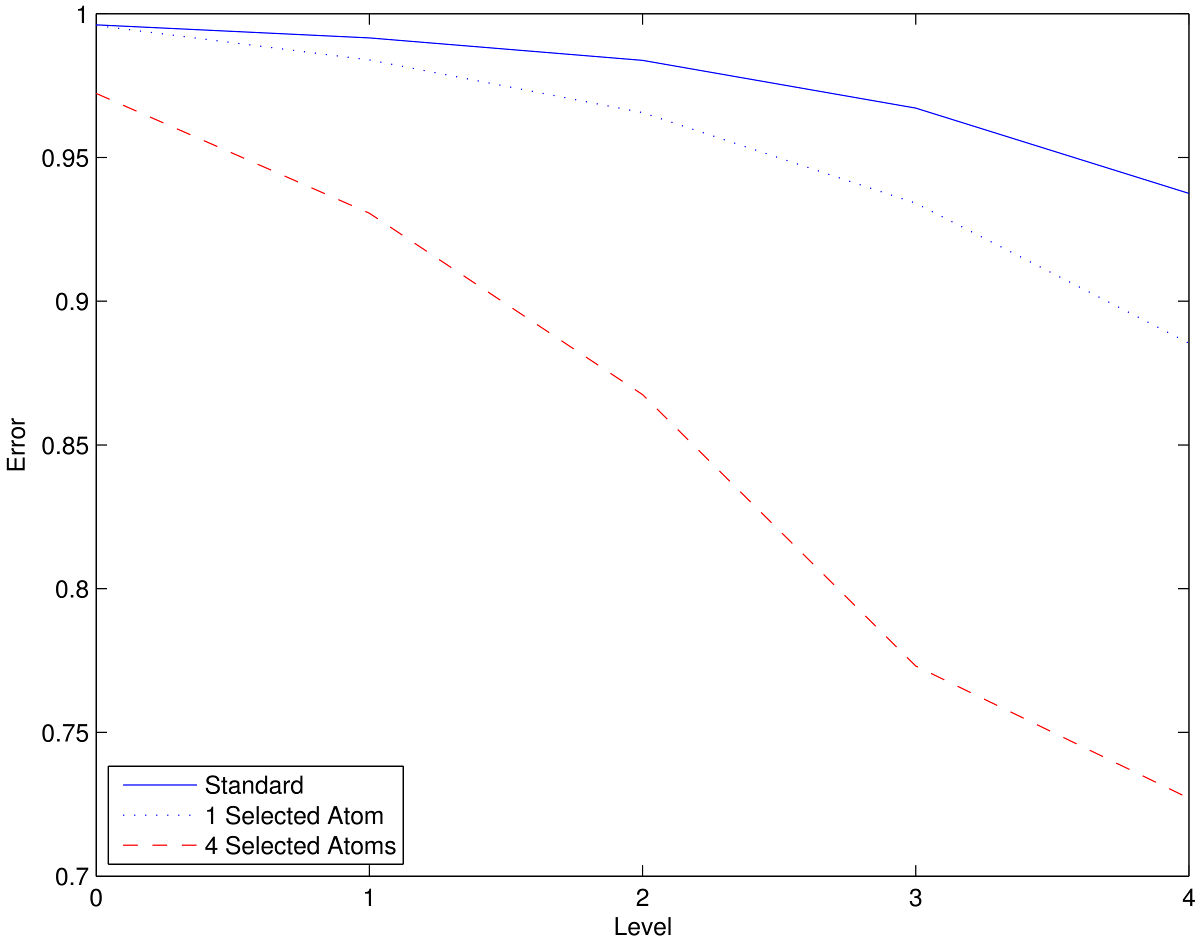


Figure 5.4: Comparison of different runs of algorithm 7: 1 window to reproduce standard multilevel approach, 1 window having maximum correlation, 4 best correlated windows

Part III

Pseudodifferential operator and Generalized Gabor multipliers

Introduction to Part III

In this last Part of the thesis, we will explore the approximation of pseudodifferential operator through spline-type spaces. The bridge between this two, apparently different, mathematical tools will be the use of a particular pseudodifferential operator called Generalized Gabor Multiplier (GGM).
Without any other formal definition let's look at the name of this type of operator:

- **multiplier**: operators on a Banach space $\gamma \otimes \overline{g} : X \mapsto X^*$ in the form

$$g \otimes \overline{\gamma}(f) := \langle f, \gamma \rangle \, g_{X \times X^*} \quad f \in X$$

  being $\gamma, g \in X^*$.

- **Gabor**: use of the time-frequency shift

$$\pi(\lambda) f(t) := M_{\widehat{x}} L_x f(t) = \langle x, \widehat{x} \rangle \, f(x^{-1} t).$$

- **generalized**: multi-window nature of spline-type spaces.

The (multiple) multipliers will be the building blocks of the GGM dictionary, and the time-frequency shift will be the operator that generate all the words.
We will then study the stability of the approximated transcription of a pseudodifferential operator in GGM dictionary as in Theorem 5. This will give us the possibility to characterize the automorphism that build up a sequence of GGM and to connect the uniform stability with traditional stability in the Lax-Richtmeyer sense. Chapter 6 is still unpublished but promising for its possible further developments.
We will then present the approximation of Hilbert-Schmidt operators as already exposed in [62]. The same modulations' selection of Chapter 3 will be employed in the ST approximation of the symbol of a pseudodifferential operator. Interpretation for the related multiplier will be given. Also, the Hough transform of Chapter 4 will be used to overcome the inability of GGM to reproduce discontinuities along lines.

Chapter 6

Pseudodifferential operators on LCA groups and Generalized Gabor multipliers

6.1 Problem setting

Pseudodifferential operators were originally introduced to generalize differential operators. The standard approach [43, Chapter 14] starts from considering a differential operator for the group $\mathbb{R}^d$ of order N

$$Af(x) = \sum_{|\alpha|\leq N} \sigma_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} f(x)$$

having non-constant coefficients σ_α. Since

$$\widehat{\tfrac{\partial^\alpha}{\partial x^\alpha} f}(\omega) = (2\pi i \omega)^\alpha \widehat{f}(\omega) \qquad \omega \in \mathbb{R}^d$$
$$\widehat{(-2\pi i \omega)^\alpha f}(\omega) = \frac{\partial^\alpha}{\partial x^\alpha} \widehat{f}(\omega) \qquad \omega \in \mathbb{R}^d$$

we get that the operator A can be rewritten as

$$\begin{aligned} Af(x) &= \int_{\mathbb{R}^d} \left(\sum_{|\alpha|\leq N} \sigma_\alpha(x)(2\pi i\omega)^\alpha \right) \widehat{f}(\omega) e^{2\pi i x\cdot\omega} d\omega \\ &= \int_{\mathbb{R}^d} \sigma_A(x,\omega) \widehat{f}(\omega) e^{2\pi i x\cdot\omega} d\omega \end{aligned}$$

being

$$\sigma_A(x,\omega) = \sum_{|\alpha|\leq N} \sigma_\alpha(x)(2\pi i\omega)^\alpha$$

the so called *symbol* of the differential operator A.
Another interesting application of pseudodifferential operators can be found in the representation of a Time-varying filter (TVF). They are described by linear operators B of the form

$$Bu(t) = \int_{\mathbb{R}} k_A(t, t\tau) u(\tau) d\tau, \qquad t \in \mathbb{R}.$$

There is not explicit formula for the dependence of the input from the output signal. Thus, it is both necessary and natural to consider numerical methods to determine the input approximately[72]. It is important for applications that such operators can be rewritten as

$$Bf(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma_B(t, \omega) f(\tau) e^{2\pi i (t-\tau)\cdot\omega} d\tau d\omega$$

being

$$\sigma_B(t, \omega) = \int_{\mathbb{R}} k_a(t, \tau) e^{-i\tau\cdot\omega} d\tau$$

Notable application in engineering of time-varying linear operators is the modeling of time-varying linear system. The same reformulation, and related symbolic calculus, can be employed[54]. To understand a process it is important to be able to decompose the system into simple and easy-to-understand blocks. Once again, spline type space helps in this tasf.

6.2 Pseudodifferential operators

Pseudodifferential operators are particularly interesting cases of linear operators acting on Banach spaces of functions. In the previous section we have showed some application, but they can be defined for function on locally compact groups. For an arbitrary locally compact G group and a tempered distribution $\sigma \in S'(G \times \widehat{G})$ we define the operator the operator $Q_\sigma : S(G) \mapsto S^*(G)$

$$Q_\sigma f(x) := \int_{\widehat{G}} \sigma(x, \widehat{x}) \left(\int_G f(t) \overline{\langle t, \widehat{x} \rangle} dt \right) \langle x, \widehat{x} \rangle \, d\widehat{x} \tag{6.1}$$

which is called the Pseudo Differential Operator (ΨDO) with Kohn-Nirenberg (K-N) symbol σ.

The nature of differential operators with non-constant coefficients and time-varying filters is preserved since Formula (6.1) says that the operator Q_σ takes the Fourier transform of the input Schwartz function f, applies the *mask* σ, also called the Kohn-Nirenberg symbol of the operator, and finally the inverse Fourier transform. From this if easy to show that pseudodifferential operator are shift invariant

$$\begin{aligned} Q_\sigma (L_x f) &= \mathcal{F}^{-1} (\sigma \cdot \mathcal{F} (L_x f)) = \mathcal{F}^{-1} (M_{x^{-1}} \sigma \cdot \mathcal{F} (f)) \\ &= L_x \mathcal{F}^{-1} (\sigma \cdot \mathcal{F} (f)) = L_x Q_\sigma (f) \end{aligned}$$

The same does not apply to the modulation operator, since

$$\begin{aligned} Q_\sigma (M_{\widehat{x}} f) &= \mathcal{F}^{-1} (\sigma \cdot \mathcal{F} (M_{\widehat{x}} f)) = \mathcal{F}^{-1} (\sigma \cdot L_{\widehat{x}} \mathcal{F} (f)) \\ &= \mathcal{F}^{-1} (L_{\widehat{x}} ((L_{\widehat{x}^{-1}} \sigma) \cdot f)) = M_{\widehat{x}} \mathcal{F}^{-1} ((L_{\widehat{x}^{-1}} \sigma) \cdot f) \\ &= M_x Q_{L_{\widehat{x}^{-1}} \sigma} (f) \end{aligned}$$

i.e. a shift in the Fourier variable of the symbol $\sigma(x, \widehat{x})$.

Analogous outcome for the operator induced by an automorphism:

$$\begin{aligned} Q_\sigma (D_\tau f) &= \mathcal{F}^{-1} (\sigma \cdot \mathcal{F} (D_\tau f)) = \mathcal{F}^{-1} \left(\sigma \cdot \Delta(\tau) D_{(\tau^*)^{-1}} \mathcal{F} (f) \right) \\ &= \Delta(\tau) \mathcal{F}^{-1} \left(D_{(\tau^*)^{-1}} ((D_{\tau^*} \sigma) \cdot f) \right) = \Delta(\tau) \Delta((\tau^*)^{-1}) D_\tau \mathcal{F}^{-1} ((D_{\tau^*} \sigma) \cdot f) \\ &= D_\tau Q_{D_{\tau^*} \sigma} (f) \end{aligned}$$

In ΨDO literature authors often assume that the symbol of the ΨDO belongs to a *properly chosen* subspace of the tempered distribution; this is due the fact that the symbol class defines the property of the related ΨDO. A particular case is the class of Hilbert-Schmidt (H-S) integral operators. Any $K \in L^2(G \times G)$ defines sesquilinear form on $L^2(G)$

$$Q(v,u) = \int_G \int_G \overline{v(x)} K(x,y) u(y) dx dy$$

which can be seen as the weak definition of an operator $A_K : L^2(G) \mapsto L^2(G)$

$$Q(v,u) = \langle v, A_K u \rangle_{L^2(G)}$$

The operator A_K is a ΨDO [38] which can be extended to the whole $L^2(G)$ space and having symbol

$$\sigma(x,\widehat{x}) = \int_G K(x, y^{-1}x) \langle y, \widehat{x} \rangle \, dy$$

in $L^2(G \times \widehat{G})$ due to the Plancherel theorem.

6.3 Generalized Gabor multipliers

An attempt of discretizing a ΨDO can be seen considering the example of a *real-discrete pseudodifferential operator* acting on $L^2(\mathbb{R}^d)$ in the form

$$\int_{\mathbb{Z}^d} \sigma(x,\widehat{x}) \left(\int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot \widehat{x}} dt \right) e^{2\pi i x \cdot \widehat{x}} d\widehat{x}$$

being $\sigma \in L^2(\mathbb{Z}^d \times \mathbb{Z}^d)$.
This linear system can also have a simple informal interpretation: it consider the coefficients $\left\langle f, e^{-2\pi i (\cdot)\cdot \widehat{x}} \right\rangle_{L^2(\mathbb{R}^d)}$ of the function $f \in L^2(\mathbb{R}^d)$ with respect to the orthonormal Fourier basis $\{e^{-2\pi i t \cdot \widehat{x}} : x \in \mathbb{Z}^d\}$ and, after a modification of these coefficient through the mask σ, consider the inverse reconstruction formula related to the same basis.
A generalization of this concept can be found in a Multiplier. The basic tool for its definition are the rank one operators $\gamma \otimes \overline{g} : S(G) \mapsto S(G)^*$ weakly defined as

$$g \otimes \overline{\gamma}(f) := \langle f, \gamma \rangle \, g \quad f \in S(G) \tag{6.2}$$

being γ and g two distributions.
Multipliers are linear combination of a countable family $\left\langle g_j \otimes \overline{\gamma}_j \right\rangle_{i \in I}$ of operators as in (6.2):

$$\sum_{j \in I} \lambda_j g_j \otimes \overline{\gamma}_j \tag{6.3}$$

Multipliers were originally studied [80] in the case of countable orthonormal basis $\{\gamma_\lambda\}, \{g_\lambda\} \subset L^2(G)$ in context of Hilbert-Schmidt and trace class operators. These operators are particularly useful due to spectral theory for compact operators. Also in case of complete orthonormal families the inversion of these operators is particularly simple: the operator (6.3) is invertible iff

$$0 < |\lambda_j| < \infty$$

and its inverse is

$$\sum_{j\in I}(1/\lambda_j) g \otimes \overline{\gamma_j}_j$$

Similar result is given for discrete Riesz basis in Hilbert spaces [84] and discrete p-Riesz basis in Banach spaces [73].
The operator (6.2) is a ΨDO having symbol

$$R(g,\gamma)(x,\widehat{x}) = \overline{\langle x,\widehat{x}\rangle}\,\overline{\widehat{\gamma}(\widehat{x})} g \tag{6.4}$$

which is called (cross) Rihaczek distribution of g and γ. The Kohn-Nieremberg quantizatation rule

$$S'(G\times\widehat{G}) \ni \sigma \longmapsto Q_\sigma \in \Psi\mathrm{DO} \tag{6.5}$$

expressed in (6.1) says that a the operator Q_σ have an equivalent reformulation in terms of the cross Rihaczek distribution:

$$\langle f_1, Q_\sigma f_2\rangle_{S(G)\times S'(G)} = \langle R(f_1,f_2),\sigma\rangle_{S(G\times\widehat{G})\times S'(G\times\widehat{G})} \tag{6.6}$$

for $f_1, f_2 \in S(G)$. So, for the case of Schatten operators we get

$$\begin{aligned}\langle f_1, g\otimes\overline{\gamma} f_2\rangle_{S(G)\times S'(G)} &= \langle R(f_1,f_2), R(g,\gamma)\rangle_{S(G\times\widehat{G})\times S'(G\times\widehat{G})}\\ &= \langle f_1, g\rangle_{S(G)\times S'(G)}\ \langle f_2,\gamma\rangle_{S(G)\times S'(G)}\end{aligned}$$

Since we are considering operators identified by a symbol defined on the space $G\times\widehat{G}$ it is natural to consider the time-frequency shift operator

$$\pi(x,\widehat{x})f = \langle\cdot,\widehat{x}\rangle\, L_x f \qquad (x,\widehat{x}) \in G\times\widehat{G}$$

used in Chapter 3 for the construction of Gabor systems.
If the Shatten operator (6.2) is shifted along the the time-frequency plane, we obtain operators $\pi(\lambda)g\otimes\overline{\pi(\lambda)\gamma}$ having K-N symbol

$$R(\pi(\lambda)g,\pi(\lambda)\gamma) = L_\lambda^{G\times\widehat{G}} R(g,\gamma), \tag{6.7}$$

being $L_\cdot^{G\times\widehat{G}}$ the shift that act componentwise [46]. More general shifted operator $\pi(\lambda)g\otimes\pi(\lambda')\gamma$, for $\lambda\neq\lambda'$, cannot avoid the symplectic nature of the phase space [44].
Thanks to the time-frequency shift (1.19) and the Schatten operators (6.2) we can define the following class of multipliers

Definition 11. *Let $\boldsymbol{g} := \{g_i : i = 1,\ldots,r\}$ and $\boldsymbol{\gamma} := \{\gamma_i : i = 1,\ldots,r\}$ be two finite subset of $S^*(G)$ and H a subgroup of $G\times\widehat{G}$ we define the class ofGGM $\mathcal{M}^p_{g,\gamma,H}$ as operators in the form*

$$M_{\boldsymbol{g},\boldsymbol{\gamma},H,\boldsymbol{m}} f := \sum_{i=1}^{r}\int_H m_i(\lambda)\,\langle f,\pi(\lambda)g_i\rangle\,\pi(\lambda)\gamma_i \tag{6.8}$$

being $\boldsymbol{m} := \{m_i \in L^p(H) : i = 1,\ldots,r\}$.

By linearity of the K-N representation (6.5) ant the time-frequency shift property (6.7) of the Shatten operators, the GGM is a pseudodifferential operator having symbol

$$\sum_{i=1}^{r}\int_H m_i(\lambda)L_\lambda^{G\times\widehat{G}}R(g_i,\gamma_i)d\lambda = U_{\Phi,H}m \tag{6.9}$$

being

$$\Phi = \{R(g_i,\gamma_i),\ i=1,\dots,r\}. \tag{6.10}$$

If g and $\widehat{\gamma}$ are compactly supported, Theorem 4 can be applied due to property (1.12) which states that $G\times\widehat{G}\in LCA$.

Proposition 12. *Let G be a LCA group, H be a subgroup of $G\times\widehat{G}$, $\boldsymbol{g}$ be compactly supported distributions on G, $\boldsymbol{\gamma}$ having compactly supported FT in $\widehat{G}$, and $M\in\mathbb{M}^p_{g,\gamma,H}$. If the the ST system $S(\Phi,H)$ system satisfy one of the equivalent condition in Theorem 4 and have dual $S(\tilde{\Phi},H)$, then the mask of M in of the form:*

$$m(\lambda) = \left\langle L_\lambda\tilde{\Phi},\sigma_M\right\rangle \quad \lambda\in H$$

A favourable situation is given for the case of a separable lattice, since the dual system can be obtained in "$1D$"

Proposition 13. ***[DO&SZ]*** *Let H a separable subgroup of $G\times\widehat{G}$, i.e. $H=K_1\times\widehat{K}_2$ for K_1 subgroup of G and $\widehat{K}_2$ subgroup of $\widehat{G}$, and let*

- *$\boldsymbol{g}\subset S'(G)$ such that $(\boldsymbol{g},K_1)$ is a Riesz basis of compactly supported distribution with biorthogonal $(\tilde{\boldsymbol{g}},K_1)$*
- *$\boldsymbol{\gamma}\subset S'(G)$ such that $(\widehat{\boldsymbol{\gamma}},\widehat{K}_2)$ is a Riesz basis of compactly supported distribution with biorthogonal $(\tilde{\widehat{\boldsymbol{\gamma}}},\widehat{K}_2)$*

The mask of any element $M_m\in\mathcal{M}(g,\gamma,H)$ can be recovered through

$$m(\lambda) = \left\langle L_\lambda\tilde{\Phi},\sigma_{M_m}\right\rangle \quad \lambda\in H \tag{6.11}$$

being

$$\tilde{\Phi} = \left\{R\left(\tilde{g}_i,\left(\tilde{\widehat{\gamma}}_i\right)^\vee\right),\ i=1,\dots,r\right\}$$

Proof. By looking at the definition of the Rihaczek distribution we get that for each $(x,\widehat{x})\in K_1\times\widehat{K}_2$

$$\begin{aligned}\left\langle R(g_i,\gamma_i),L_{(x,\widehat{x})}R\left(\tilde{g}_i,\left(\tilde{\widehat{\gamma}}_i\right)^\vee\right)\right\rangle_{S(G\times\widehat{G})\times S'(G\times\widehat{G})} &= \langle g_i,L_x\tilde{g}_j\rangle_{S(G)\times S'(G)}\left\langle\widehat{\gamma}_i,L_{\widehat{x}}\tilde{\widehat{\gamma}}_i\right\rangle_{S(G)\times S'(G)}\\ &= \delta_{i,j}\delta_{K_1}\delta_{\widehat{K}_2}\end{aligned}$$

and so we obtain a biorthogonal system ∎

6.4 Automorphism and Generalized Gabor Multipliers

We finally consider the projection of a ΦDO having symbol in $L^p(g \times \widehat{G})$ onto a sequence of GGM built through sequence of projection into spline type spaces of symbols which are related each othe through automorphism.
First of all we see that proposition 13 can also be read in term of reformulation 6.6 of a pseudodifferential operator and using the property 1.19 of the time-frequency shift:

$$m_i(\lambda) = \left\langle L_\lambda R\left(\tilde{g}_i, \left(\hat{\tilde{\gamma}}_i\right)^\vee\right), \sigma\right\rangle = \left\langle \pi(\lambda)\tilde{g}_i, Q_\sigma \pi(\lambda)\left(\hat{\tilde{\gamma}}_i\right)^\vee\right\rangle \tag{6.12}$$

We want to characterize that automorphism that, applied to a Rihaczek distribution, give another Rihaczek distribution that will define the subsequent GGM. Let τ_1 and τ_2 be two automorphism of G, and consider the operator $D_{\tau_1\times\tau_2}$ over defined on $S(G \times \widehat{G})$ or $S'(G \times \widehat{G})$ as

$$D_{\tau_1\times\tau_2}\sigma := D^G_{\tau_1} D^{\widehat{G}}_{\tau_2^*}\sigma$$

i.e. a componentwise morphing.
If we apply such an operator to the Riahczek distribution of two distributions we get

$$\begin{aligned} D_{\tau_1\times\tau_2}R(g,\gamma) &= \overline{\left\langle \tau_1^{-1}(x), (\tau_2^*)^{-1}(\widehat{x})\right\rangle}\, D^{\widehat{G}}_{\tau_2^*}\overline{\widehat{\gamma}(\widehat{x})} D^G_{\tau_1} g \\ &= \overline{\left\langle (\tau_1\tau_2)^{-1}(x), \widehat{x}\right\rangle}\, \overline{\widehat{D^G_{\tau_2^{-1}}\gamma}(\widehat{x})} D^G_{\tau_1} g \end{aligned}$$

The modified Rihaczek distribtion is hence the Rihaczek distribution of a multiplier iff

$$\tau_2 = \tau_1^{-1}$$

In the case $\tau := \tau_2 = \tau_1^{-1}$ we have that

$$D_{\tau\times\tau^{-1}}R(g,\gamma) = R(D_\tau g, D_\tau\gamma)$$

this leads to the following proposition

Proposition 14. ***[DO&SZ]*** *Let H, g and γ as in proposition 13, and let τ be an automorphism.*
The ST space $S(D_{\tau\times\tau^{-1}}R(g,\gamma), \tau K_1 \times \tau^{-1}K_2)$ is the space of symbols for the Gabor class

$$\mathcal{M}(D_\tau g, D_\tau\gamma, \tau K_1 \times \tau^{-1}K_2)$$

Applying the powers of the automorphism τ as in section 2.3.1, the bi-infinite sequence of projections into the ST spaces

$$\left\{S(D_{\tau^i\times\tau^{-i}}R(g,\gamma), \tau^i K_1 \times \tau^{-i}K_2) : \ i \in \mathbb{Z}\right\}$$

is obtained, which correspond to projection into Gabor multipliers spaces

$$\left\{\mathcal{M}(D_{\tau^i}\boldsymbol{g}, D_{\tau^i}\boldsymbol{\gamma}, \tau^i K_1 \times \tau^{-i}K_2) : \ i \in \mathbb{Z}\right\}$$

Chapter 7

Numerical implementation for Hilbert-Schmidt operators

7.1 Spreading function

As said before, the K-N quantization rule establishes a correspondence between $L^2(G\times\widehat{G})$ and HS operator defined on $L^2(G)$ [83].
The problem of approximating Hilbert-Schmidt (HS) via Gabor multipliers was approached by several authors [27, 35, 42] due to its important applications in computational sciences and signal processing [21]. Another important element to analyze a ΨDO is its spreading function.
The spreading function of the operator Q_σ is simply defined as the FT (for the LCA group $G\times\widehat{G}$) of the K-N symbol σ. It is hence a function defined over $\widehat{G\times\widehat{G}}$ which is, thanks to the Pontryagin duality, isomorphic to $\widehat{G}\times G$. The spreading function was mainly used in pseudodifferential operators theory[85, 26] for the possibility of expressing an operator Q_σ as

$$\begin{aligned} Q_\sigma f \int_{\widehat{G}}\int_G \widehat{\sigma}(\widehat{x},x)\,\langle\cdot,\widehat{x}\rangle\, L_{x^{-1}} f d\widehat{x}\,dx \\ = \int_{\widehat{G}}\int_G \widehat{\sigma}(\widehat{x},x)\pi(x^{-1},\widehat{x}) f d\widehat{x}\,dx \end{aligned}$$

This formula says us the the spreading function can be used as a multiplication mask against the time-frequency shifts of the input function f to obtain the image of Q_σ.
In [27], the authors have improved the approximation of HS operators using GGM in they spreading function formulation. In [35], the authors present the connection between the approximation of the HS operators and GGM via multi-window spline-type spaces that we exposed in the previous chapter. The connection was presented only theoretically in that paper and no implementation was given. In this chapter we perform numerical implementation of the theory proposed in [35] and we have designed a realizable algorithm that improves the overall numerical approximation of the HS operators in three ways: in speed by transferring the problems in splines-spaces domain, in complexity by lowering the number of atoms through a proper selection procedure similar to the one performed in Chapters 3 and 5, and geometrically via a feature detection technique i.e. the Hough transform.
Our use of spreading function is instead closer to classification of operators. A ΨDO having spreading function well localized in 0 is called underspread, otherwise it is called

overspread. The same distinction is made in the study of nonstationary random processes [55, 51]. Overspread processes are related to high-lag time-frequency correlation; in Subsection 7.2 we will find the same lag behaviour analysing GGM through their spreading functions.
In Section 3.4.3 of Chapter 3 we have used the reformulation of Gabor system in spline-type spaces to produce a sparse-frequency representation of a signal. We use in here a GGM for the same purpose: selection of proper *channels* for the approximation of over-spread or underspread operators.
This general theory will lead to usual L^2-projection once restricted to square integrable functions. As always, the channels will not be orthogonal, but biorthogonal. For the class of HS operators we will obtain L^2-projection, hence it can be seen as optimal design of biorthogonal frequency division multiplex (BFDM) [79] in a Gaborian dictionary.
The outcome is similar to the one developed in [34] but it handles the multi-window formulation of GGM. We show the robustness of the generators selection procedure for operators having smooth symbol and the inability of GGM of handling discontinuities. We overcome this problem by looking at the FT of the reminder as an image and using the Hough transform introduced in Chapter 4.

7.2 Generalized Gabor multipliers in the Fourier domain: selection procedure

We want to produce an expansion of the symbol σ through well localized *blocks* in the phase space, as it happens for the approximation of a function through Gabor systems, to approximate Hilbert Schmidt Operator.
Theorem 4 is particularly appealing in this case for two main reasons:

(I) The invertibility of the synthesis says that it is possible to identify $S(\Phi, H)$ with ideals of $(L^1(H), *)$.

(II) For symbols defined in $L^2(G \times \widehat{G})$ the projection (2.15) is the L^2-projection[23]. We have best approximation of HS operators in GGM class $\mathbb{M}_{g,\gamma,H}$.

One of the strength-weakness of a ST space is that the spectra of its generators defines most of its features. As pointed out in (I) we reproduce ideals of the convolution algebra $(L^1(H), *)$, which are in one-to-one correspondence with the ideals of the multiplication algebra $(\mathcal{F}L^1(H), \cdot)$ through FT [74]. Informally, given a function f and a ST space generated by an atom ϕ, if the spectra of f and g do not intersect, $P_{\phi,H}f = 0$. Hence, to build a good ST space for the L^2 approximation of a function, it is enough to cover in a *patch-like fashion* its FT fulfilling requirement (ii) of Theorem 4. Straightforward question in our investigation is: *What does each of this patch represent in the case of GGM approximation?*
This quest will lead us in understanding the decomposition of a ΨD overspread operator by the particular GGM dictionary built through an analysis-synthesis couple and some *lagged versions.* By only using properties (1.16) and (1.17) we obtain the forllowing proposition.

Proposition 15. ***[DO&SZ]*** *The spreading function (SF) of the operator* $\langle \cdot, \langle \cdot, \xi' \rangle \, g \rangle \, L_{\eta'}\gamma$ *is* $L^{\widehat{G}\times G}_{(\xi',\eta')}\widehat{\sigma}_{g,\gamma}$

So, from $\widehat{\sigma}_{g,\gamma}$ well localized in (0,0) we have that

- $L^{\widehat{G}\times G}_{(\xi',0)}\widehat{\sigma}_{g,\gamma}$ is a model for a Doppler effect
- $L^{\widehat{G}\times G}_{(0,\eta')}\widehat{\sigma}_{g,\gamma}$ is a model for an impedance effect

Straightforward drawback from the GGM construction is the inability of approximating symbols possessing important Fourier feature badly intersecting these patches. This will be shown in Section 7.3, when time varying filters will be discussed. We summarize the developed theory in the Algorithm 9.

Algorithm 9 [DO&SZ] Selection of the generating set through Proposition 15 and computation of the GGM masks as in (6.11)

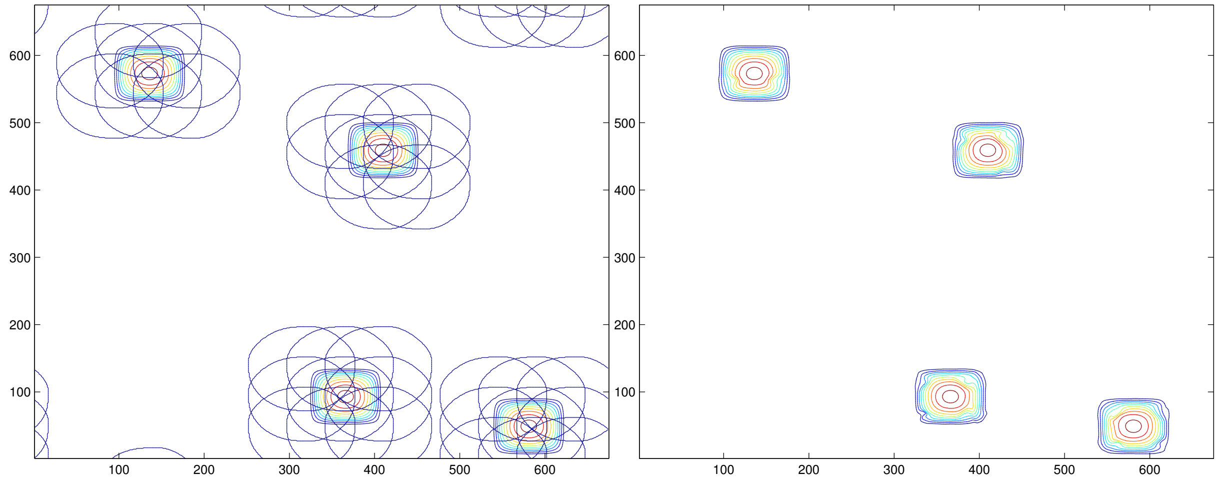
Precondition: $\begin{cases} \text{Symbol } \sigma_P, \\ \text{Analysis window } g, \\ \text{Synthesis windows } \gamma, \\ \text{subgroup } H, \\ \text{Shifts } \widehat{K} \subset \widehat{G}\times G \end{cases}$

function GGM_APPROX(σ_P, g, γ, H, $\widehat{K}$)
  $\widehat{\sigma}_P \leftarrow \mathrm{FFT}(\sigma_P)$
  $\sigma_{g,\gamma} \leftarrow g(x)\overline{\widehat{\gamma}(\widehat{x})}\,\overline{\langle x,\widehat{x}\rangle}$
  $\widehat{\sigma}_{g,\gamma} \leftarrow \mathrm{FFT}(\sigma_{g,\gamma})$
  $\Gamma \leftarrow \emptyset$
  for $(\widehat{x},x)\in\widehat{K}$ **do**
    if INTERSECT($\widehat{\sigma}_P, L^{\widehat{G}\times G}_{(\widehat{x},x)}\widehat{\sigma}_{g,\gamma}$) **and** ST_STABLE($\Gamma\cup\{\langle\cdot,\langle\cdot,\widehat{x}\rangle\, g\rangle\, L_x\gamma\}$, H) **then**
      $\Gamma \leftarrow \Gamma\cup\{\langle\cdot,\langle\cdot,\widehat{x}\rangle\, g\rangle\, L_x\gamma\}$
    end if
  end for
  $\tilde{\Gamma} \leftarrow$ BIORTHOGONAL(Γ, H)
  $\boldsymbol{m} \leftarrow \emptyset$
  for $\widehat{\sigma}\in\tilde{\Gamma}$ **do**
    $\boldsymbol{m} \leftarrow \boldsymbol{m}\cup\{$IFFT(FFT($\sigma_P$)FFT($\tilde{\sigma}$))$(H)\}$
  end for
  return $(\Gamma,\boldsymbol{m})$
end function

7.3 Numerical tests

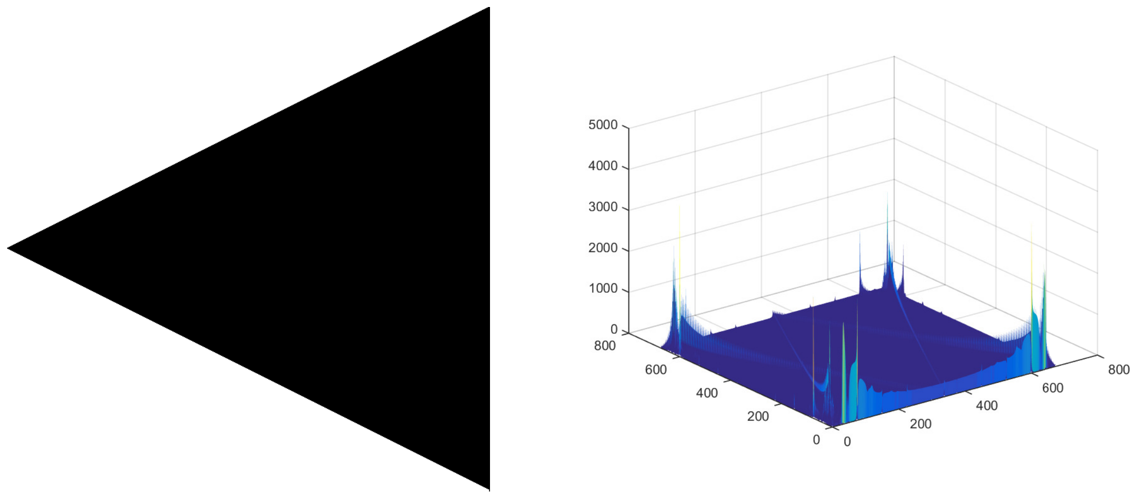
In numerical tests we consider HS operators defined over the class of discrete signals of finite length L as function over a periodic time domain $\mathbb{Z}/L\mathbb{Z}$. As analysis and synthesis we have considered Gaussians or bump function as analysis and synthesis being the IFT of a bump function, in such a way that the related Rihaczek distribution is compactly supported, well localized in 0, and the SF is localized in 0 and quickly decaying. With M_σ we will denote the obtained GGM approximation of the HS operator having symbol σ. We approach two types PD of operators: overspread operators and TVF.

The first shows the outcome of the dual shifts selection expressed in Subsection 7.2, while the latter exploit intrinsic limitation of the GGM approximation procedure. To build an overspread operator we have synthesized its SF $\widehat{\sigma}_{over}$ as sum of tensor products 1D bumps. After fixing two Gaussian functions as analysis and synthesis window, we have selected a



(a) (b)

Figure 7.1: 7.1a SF of overspread operator and 7.1b GGM approximation



(a) (b)

Figure 7.2: 7.2a TVF and 7.2b SF reminder in GGM approximation

tolerance under which the SF of the GGM is considered null and selected the shifts over a rectangular lattice that intersect $\widehat{\sigma}_{over}$ for at least of $1/5$ of its support; we call this set Φ. The result is shown in Figure 7.1a. The SF $\widehat{P_{\Phi,H}\sigma_{over}}$ of the GGM approximation is shown in Figure 7.1b. Computing the error in Frobenius norm, hence in HS norm

$$E_{\sigma_{over}} := \frac{\| \sigma_{over} - P_{\Phi,H}\sigma_{over} \|_{\mathcal{F}}}{\| \sigma_{over} \|_{\mathcal{F}}} \approx 0.0822$$

and applying this GGM approximation $M_{\sigma_{over}}$, for a mean of 1000 random signal f we get

$$E_{PDO_{\sigma_{over}}} := \frac{\| PDO_{\sigma_{over}} f - M_{\sigma_{over}} f \|_2}{\| PDO_{\sigma_{over}} \|_2} \approx 0.1040$$

Our interest in acoustics leads us in studying approximation of time-varying filters through GGM. In Figure 7.2a a simple example of non-stationary frequency filter is shown: its symbol σ_{filt} takes only two values, 0 and 1, so it annihilates the frequency's content of the input signal starting from the high frequencies (the middle point of $\widehat{\mathbb{Z}_L}$). Since

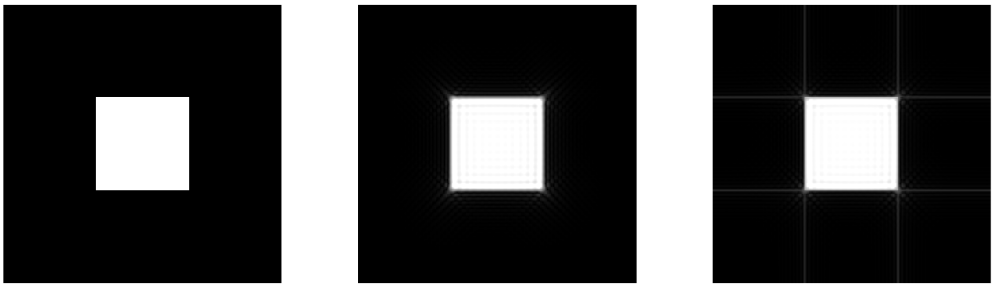


Figure 7.3: Square mask approximation
Left: Original Center: GM approximation Right: GM+HF approximation

Figure 7.4: Detail of Figure 7.3:
Comparison between GM and GM+HT

the symbol is characterized by a flat region, the most relevant harmonic is the 0th. For this reason we have considered a GM (i.e. mono-window case) generated by a Rihaczek distribution ϕ. We obtained $E_{\sigma_{filt}} \approx 0.0604$, a quite small error predictable result in this 1-window case, but for a mean of 1000 randomly generated signal f we got $E_{PDO_{\sigma_{filt}}} \approx 0.1224$ with variance $5 \cdot 10^{-5}$, which means a global relative error of the 12%. We need to analyse the reason for such a discrepacy between the two errors. As usual the answer is encapsulated in Fourier domain. In Figure7.2b the magnitude of $\widehat{\sigma_{filt}} - \widehat{P_{\phi,H}\sigma_{filt}}$ is shown in the Gaussian case. Low frequencies are not participating any longer, meaning that the GM approximates well the flat region of 1s, but lines running through 0 and having slope perpendicular to the jumps between 0 and 1 regions still persist. Using shifts selection does not help, and the error slightly increases. For this reason we decided to approximate these missed lines in 1D. We first employ a Hough transform to detect discontinuity in the symbol of the operator. From this, rotating by 90 we can have the lines l_k on which the contribution in Fourier domain is relevant. This contribution can be then extracted and approximated by a favourable dictionary. We decide to use a reduced Gabor transform for fixed 1D Gaussian or bump atom, as done in [61]. After this correction we get $E^c_{\sigma_{filt}} \approx 0.0245$ and $E^c_{PDO_{\sigma_{filt}}} \approx 0.0695$, which is a quite satisfactory outcome.

Another example of HT post-processing can be see in in the approximation of a square mask as in Figure 7.3. By looking at the zoom in Figure 7.4 we can appreciate a much more sharpened bound between the original 1/0 regions of the mask.

7.4 Chapter conclusions

In this chapter, we have introduced a realizable algorithm for HS operators approximation based on Gabor multipliers and multi-window spline-type spaces. We have used the spreading function of an operator in an operational way, getting to a novel interpretation of lag-behaviour of a channel through a Gabor dictionary. After the error analysis, we understood the limitation of this approach and come with a new use of the Hough transform for feature extraction over lines.

Documentation of the CREAM toolbox

The CREAM toolbox can be downloaded at the page:
http://homepage.univie.ac.at/darian.onchis/

```
%BUMP build a normalised and periodized bump function centered in 0
%  Input:  L    =  length of the signal
%          time =  basic radius of the support
%          mult =  scaling parameter
%  Output: b    =  normalized bump of length L with radius time*mult
%  Usage:  [ b ] = bump( L,time,mult ) return a bump function centered in 0
%  of length L and support time*mult, such that it is possible to generate
%  quickly Spline-type wavelet-like spaces
%          [ b ] = bump( L,time,mult ) return a bump function centered in 0
%  of length L and support time


%CHEB_NODE computes Chebichev-like rational disposition of nodes
%  Input:  a = left extreme of the interval
%          b = right extreme of the interval
%          L = length of the discretized domain
%          n = order
%  Output: x   =  Cheb-like node
%  Usage:  [ x ] = cheb_node( a,b,L,n ) select the breackpoints of the
%  uniform partition of size (b-a)/L of the interval [a,b] the nodes
%  closer to the nth order Chebichev zeroes


%FIND_MODULATIONS find the modulations of an atom g which intersect a signal X
%  Input:  X = bandlimited signal (it can be a complex signal)
%          g = bandlimited window
%          fgap = stepsize in modulation for the Gabor system
%          TOLL = a tolerance
%  Output: MODULATIONS is a set of integer , the modulations
```

```
%   Usage:   [ MODULATIONS ] = find_modulations( X,g,fgap,TOLL ) approximate
%   the spectrum of the XX given signal and the fixed atom g considering
%   null frequency content of absolute value smaller than TOLL; then finds
%   the modulations with phase defined by fgap of the atom g whose spectrum
%   intersect the spectrum of the signal XX


%FROBENIUS_NORM Compute the Frobenius norm of a matrix valued
%function over a sampling set
%   Input:  MM         = #shifts X n^2 matrix containing the function
%                        or
%                        n X n X #shifts matrix containing the function
%   Output: fro_norm   = vector containing the Frobenius norm of each matrix
%           max_fro_norm   = maximum of the norms
%   Usage:   [ fro_norm,max_fro_norm ] = Frobenius_norm( MM ) computes the
%   Frobenius normes of the matrices collected in the stack MM and stores
%   them in the vector fro_norm. It gives also the maximum of this vector
%   through the scalar fro_norm_max

% GAUSSSZ
% Simone Zappala'Âă modification of gaussnk allowing contraction
%   Usage:  g=gausssz(n,q) scales the variance of the normalized gaussian
%   by a factor q
% GAUSSNK (Norbert Kaiblinger's Gauss function, modified by HGFei)
%
% GAUSSFIX  - canonical discrete gauss
%
% Usage:       g = gaussnk(n);
%      or      g = gaussnk(xx);   % where xx is some signal
%
% Input:       n   dimension
% Output:      g   a canonically discretized gaussian,
%                  an eigenvector with eigenvalue 1
%                  of the unitary DFT (that is g=fft(g)/sqrt(n))
%                  (normed with respect to 2-norm)
% Author:      N.Kaiblinger, 20.Feb.1996 (kaibling@tyche.mat.univie.ac.at)
```

```
%
% Copyright:  (c) NUHAG, Dept. Math., University of Vienna, Austria
%             Permission is granted to modify and re-distribute this
%             code in any manner as long as this notice is preserved.
%             All standard disclaimers apply.



%GENERATE_SPARSE_FREQ_SIGNAL generate a random signal having sum of square
%function as Fourier transform
%   Input:  L        =  length of the signal
%           n_band   =  number of band in the fourier domain
%           dim_band =  radius of the band
%   Output: XX the signal
%   Usage:  [ XX ] = generate_sparse_freq_signal( L,n_band, dim_band )
%   generates a random periodic signal of length L having Fourier transform
%   equal to 1 over some intervals [a_i,b_i] i=1,...,n_band with
%   b_i-a_i = dim_band.
%   Type 'generate_sparse_freq_signal;' on the Command Window to have a
%   graphical demonstration


%LATTICE_GENERATION Generate a logical lattice
%   Input:  L = scalar length of the domain
%           gap = amplitude of the lattice
%   Output: ll = the lattice
%   Usage:
%           [ ll ] = lattice_generation( L ,gap ) generate a boolean index
%   set
%   ll(k) = true iff k belongs to 1:gap:(L-gap+1)


%LIN_DEP_TEST test linear dependence of input spline type system
%   Input:  ATOM  = matrix having on its column the atoms
%           time  =  a vector containing the time lattice
%                 or
%                 a scalar which measure the norm of a uniform partition
%   Output: flag  =  boolean variable saying whether the system is
%                  consistent
```

```
%MOD_COMPLETION Completion of a generating set of delay through some
%modulation
%   Input:  L = length of the signal
%           ATOM =  string of char containing the type of the atom
%           time =  a vector containing the sample location in time
%                   or
%                   a scalar which measure the norm of a uniform partition
%           del =  set of delay
%   Output: freq = modulation of the most peaked delay to be used


%MOD_GENERATION Generate the desired modulation for a given atom
%   Input:  ATOM =  a vector containing the generating function
%           freq =  a vector containing the sample location in frequency
%   Output: GG   =  a matrix having on its column the modulations


%NUGABDUAL Build the biorthogonal system of Gabor system through ST space
%NUGABDUAL consider different modulations in a Gabor system as different
%generating atoms of a ST system, so they can be non-uniform
%NUGABDUAL works to for general spine-type systems too
%   Input:  ATOM =  a vector containing the generating function of Gabor
%                   or
%                   a matrix with column containg the generating functions
%                   (the case of NON-GABORIAN spline-type space)
%           time =  a vector containing the time lattice
%                   or
%                   a scalar which measure the norm of a uniform partition
%               (for Gabor only!)
%           freq =  a vector containing the sample location in frequency
%                   or
%                   a scalar which measure the norm of a uniform partition
%   Output: GG   =  Set of modulations
```

```
%           GGD  =  Biothogonal system
%           FGG  =  FT of the system for further application of the
%            analysis operator
%           CC   =  Synthesis coefficients for the biorthogonal system (for
%            test only)
%           GRAM =  Gramian of the ST system (for test only)
%   Usage:
%   [ GG,GGD ] = NUgabdual( ATOM,time,freq ), if ATOM is a vector and time
%   and freq are scalars, build the uniform Gabor system
%       S({M_{h*freq} ATOM) : h integer} , 0:time:length(ATOM)-time)
%   and it's biorthogonal
%   [ GG,GGD ] = NUgabdual( ATOM,time,freq ), if ATOM is a vector and time
%   a scalars and freq a set, build the uniform Gabor system
%        S({M_{freq_i} ATOM) :  i=1:length(freq) } , 0:time:length(ATOM)-time)
%   and it's biorthogonal
%   [ GG,GGD ] = NUgabdual( ATOM,time ), if ATOM is a matrix and time
%   is a scalars, build the uniform ST system
%        S(ATOM,  0:time:length(ATOM)-time)
%   and it's biorthogonal


%IRLMW For a shift invariant system multiwindow system, build the dual
%system starting from the precomputed coefficients.
%   Input:  CC = matrix having on its column the the coefficient the shift of every atoms
%           FGG = matrix having on its column the  FFT of some atoms GG
%           time = a vector containing the shifts location
%   Output: GGD = Matrix having on its column the dual atoms of GG


%NUIRLMWAP approximate a signal in ST space
%   Input:  xx   =  a vector containing the signal to be approximated
%           FGG  =  a matrix with column containing the Fourier transform of
%            the analysis ST space
%           GGD  =  a matrix with column containing the the analysis ST,
%            space Biothogonal system to ifft(FGG)
```

```
%           time =  a vector containing the time lattice
%  Output: xxa  =  approximation of the signal
%          cxx  =  Synthesis coefficients for the biorthogonal synthesis
%          coming from the analysis related to the (GG,time) system
%  Usage:
%  [xxa,cxx] = NUtrlMWap(xx,FGG,GGD,time), if time is a scalar approximate
%  the signal xx analyzing it through the system
%      S(ifft(FGG) , 0:time:length(ATOM)-time)
%  and use these coefficients for the synthesis space
%      S( GGD , 0:time:length(ATOM)-time)
%  [xxa,cxx] = NUtrlMWap(xx,FGG,GGD,time), if time is a lattice
%  approximate the signal xx analyzing it through the system
%      S(ifft(FGG) , time)
%  and use these coefficients for the synthesis space
%      S( GGD , time)


%NUTRLMWAP computes the analysis of a signal related to a ST space
%  Input:  xx   =  a vector containing the signal to be approximated
%          FGG  =  a matrix with column containing the Fourier transform of
%          the generating set
%          time =  a vector containing the time lattice
%                  or
%                  a scalar which measure the norm of a uniform partition
%  Output: coef =  analyzed signal
%  Usage:
%  [coef] = NUtrlMWcof(xx,FGG,time), if time is a scalar, returns the
%  analysis coefficients related to the ST system
%      S(ifft(FGG)  , 0:time:length(ATOM)-time)
%  [coef] = NUtrlMWcof(xx,FGG,time), if time is a lattice, returns the
%  analysis coefficients related to the ST system
%      S(ifft(FGG)  , time)


%NUTRLMWGr return the coefficients of the dual system of the input ST space
% correlations of the atoms, to obtain the coefficients of the dual atoms
%  Input:  GG = matrix having on its column the atoms
%          time = data location in time
```

```
%   Output: FGG = Fourier transform of the atoms in GG (useful in
%             subsequent reconstruction precess)
%             CIFab = matrix of dimension length(time) X length(ATOM) having on
%             its column the coefficient on the data location


%IRLMW For a shift invariant system multiwindow system, synthesis operator
%   Input:  CC = matrix having on its column the the coefficient the shift of every atoms
%            FGG = matrix having on its column the  FFT of some atoms GG
%             time = Shift gap: even if we have the coefficient in every sample
%             point, we can choose to use only the ones that belongs to a
%             regular lattice.
%   Output: CCGD = synthesized signal


%RANDI_MODUL generate random uniform deformation of modulations
%   Input:  L    =  length of the discretized domain
%             freq =  a vector containing the sample location in frequency
%                       or
%                       a scalar which measure the norm of a uniform partition
%             prob =  Probability of deformation
%             magn =  Step size in the deformation
%   Output: def_freq    =  vector containing the new modulations


%SAMPLE_CONTROL create or control an input index set as lattice
%   Input:  sample   =  a vector containing the lattice
%                       or
%                       a logical vector containing the lattice
%                       or
%                       a scalar which measure the norm of a uniform partition
%             L        = a scalar: length of the domain
%             domain   = a string of character that identify the domain
%   Output: sample   = set index which represent the sample points
%   Usage:
%             [ lattice ] = sample_control( sample,L,'time' ) for sample a
```

```
%   scalar, generate the lattice lattice=1:sample:(L-sample+1), suitable
%   for indexing
%            [ lattice ] = sample_control( sample,L,'freq' ) for sample a
%   scalar, generate the lattice lattice=0:sample:(L-sample) suitable for
%   generating modulations
%            [ lattice ] = sample_control( sample,L,'time' ) for sample a
%   an index set check whether their set is a lattice


%CORRELATIONS Compute the L^2 products between two ST system
%defined over the same time lattice
%   Input:  GG   =  a matrix with column containing one generating set
%           DD   =  a matrix with column containing one generating set
%           time =  a vector containing the time lattice
%                   or
%                   a scalar which measure the norm of a uniform partition
%   Output: BB   =  a size(GG,2) X size(DD,2) X (size(GG,1)/time) stack of
%                   matrices having BB(i,j,kk) = <GG(:,i),L_{kk} GG(:,j)>


%APPLY_OP Apply a pseudifferential operator
%   Input:  f    =  Vector of length L containing the discretized function
%   variable
%           KN   =  LXL matrix containing the Kohn-Nirenberg symbol
%   Output: Pf   =  Image under the Pseudodifferential operator
%   Usage:  [ Pf ] = apply_op( f,KN ) return the image under the
%   pseudodifferential operator induced by the symbol KN of the function f.
%   The output is such that size(Pf)==size(f)


%AS_CONTROL control the dimension of the input matrices
%   Input:  SS = matrix function on the time-frequency plane that represent the KN symbol.
%           GG = matrix having on its column the synthesis atoms
%           HH = matrix having on its column the analysis atoms
%   Output: [ L,n_atom ] = size(GG);
```

```
%   Usage:   [ L,n_atom ] = AS_control( GG,HH ) check the consistency
%   whether the input matrix representing the synthesis and the analysis in
%   a Generalized Gabor multiplier are coherent, checking the sizes


%DEC_FREQ generate a time varying filter as pseudodifferential operator
%   Input:  L  = length of the domain
%   Output: KN = Kohn Nirenberg symbol
%   Usage:   [ KN ] = dec_freq( L ) return a LxL matrix containing the
%   discrete Kohn Nirenberg symbol of the filter
%   Example:
%   [ KN ] = dec_freq( 10 )
%   produces the time-frequency mask
% KN =
%
%      1     1     1     1     1     1     1     1     1     1
%      1     1     1     1     1     1     1     1     1     1
%      1     1     1     1     0     0     1     1     1     1
%      1     1     1     1     0     0     1     1     1     1
%      1     1     1     0     0     0     0     1     1     1
%      1     1     1     0     0     0     0     1     1     1
%      1     1     0     0     0     0     0     0     1     1
%      1     1     0     0     0     0     0     0     1     1
%      1     0     0     0     0     0     0     0     0     1
%      1     0     0     0     0     0     0     0     0     1


%GGM_STAP Spline type approximation of Generalized Gabor multiplier
%   Input:  OP = matrix KN representation of the operator
%           GG = matrix having on its column the synthesis atoms (time)
%           HH = matrix having on its column the analysis atoms (freq)
%           time =  a vector containing the sample location in time
%                   or
%                   a scalar which measure the norm of a uniform partition
%           freq =  a vector containing the sample location in frequency
%                   or
%                   a scalar which measure the norm of a uniform partition
```

```
%   Output: gab_mult = Stack of Matrices of Gabor multiplier.


%GGMW Application of generalized Gabor multiplier
%   Input:  f = Vector containing the function
%           gab_mult = Stack of Matrices of Gabor multiplier
%           GG = matrix having on its column the synthesis atoms (time)
%           HH = matrix having on its column the analysis atoms (freq)
%   Output: Gf = Image of f through gab_mult


%KNMcof Compute the coefficient in multi-window setting

%   Input:  SS = matrix function on the time-frequency plane that represent the .
%           GG = matrix having on its column the synthesis atoms (time)
%           HH = matrix having on its column the analysis atoms (freq)
%           time =  a vector containing the sample location in time
%                   or
%                   a scalar which measure the norm of a uniform partition
%           freq =  a vector containing the sample location in frequency
%                   or
%                   a scalar which measure the norm of a uniform partition
%   Output: DUAL = Stack of Matrices of duals.
%           RONE = Stack of Matrices of atoms.


%NUTRLMWAP approximate a signal in ST space in R^2
%   Input:  xx   =  a vector containing the signal to be approximated
%           FGG  =  a stack of matrix with column containing the Fourier
%           transform of the analysis ST space
%           GGD  =  a stack of matrix with column containing the analysis ST,
%           space Biothogonal system to ifft(FGG)
%           a =  a scalar containing the time lattice in first dimension
%           b =  a scalar containing the time lattice in second dimension
%   Output: xxa  =  approximation of the signal
```

```
%TRL2MWCOF compute the analysis of a ST space in 2-dimensional domain
%   Input:  xx   =  a matrix containing the signal
%           time =  a stack of matrix containing the Fourier transform of
%           the atoms
%           a    =  scalar containing step in second dimension
%           b    =  scalar containing step in first dimension
%   Output: coef =  a stack of matrix containing the analysis coefficients


%TRL2synFWW Synthesis operator of bivariate ST space
%   Input:  CC   = stack of matrix having containing the synthesis
%           coefficients
%           FGG  = stack of matrix containing the FFT the generating set
%           a    = Shift gap in first dimension
%           b    = Shift gap in second dimension
%   Output: CCGD = synthesized signal
```

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