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Contents

1 Introduction

The total energy, momentum and similar global quantities of any physical theory play a key role in that theory. But, the energy and mass in General Relativity (GR) have no unique definition, compared against the Special Relativity (SR) where the mass is well-defined. It turns out that there is no local notion of mass and energy in GR. However, we can attribute global quantities, defined at large distances, to some space-times.The Arnowitt-Deser-Misner (ADM) formalism [1] provides a way to calculate such quantities. There is also the notion of Komar quantities [19], which are defined in space-times with some symmetries.

Based on the geometric Hamiltonian formalism of Kijowski, Tulczyjew and Chruściel [18], which is discussed in section 2, we derive new expressions for the total Hamiltonian energy of gravitating systems in higher dimensions in terms of Riemann tensor. One can also obtain in this way the ADM and the Komar quantities for the corresponding space-times. This analysis is independent of the asymptotic behavior of the space-times. In this context, we derive for the first time an expression for the Komar mass in higher dimensional space-times with various asymptotics known to us, which satisfy certain conditions.

We then apply these results to the cases of asymptotic flat (section 3), asymptotic Kaluza-Klein (section 4) and asymptotic anti de-Sitter space-times (section 5). This analysis was carried out carefully for the case of Rasheed's solutions [21] by myself and Piotr T. Chruściel together with Michael Hörzinger in a paper which is accepted for publication in Physical Review D [6]. As it turns out, for the stationary asymptotically flat space-time the Komar mass coincides with the ADM mass in all dimensions.This provides a generalization of the result of Beig in four-dimensional space-times [7]. Above all we show that, in general, the Hamiltonian mass, the ADM mass and the Komar mass defer from each other in the non-asymptotically flat setting. In line with our analysis, we derive formulae for the mass and momentum associated with asymptotically Anti-de Sitter space-times, which have been already obtained by Ashtekar and Das [2] earlier, with stronger conditions required in comparison to ours.

By the courtesy of the co-authors of [6], I heavily used the parts, to which I mainly contributed, in this thesis.

2 Preliminaries

We start with the Hamiltonian analysis of Kijowski, Tulczyjew and Chruściel [10, 13, 18]. Consider a Lagrangian field theory with the Lagrangian $L(\varphi^A, \partial_\mu \varphi^A)$. The field equations then read

$$\partial_\mu p^\mu{}_A = \frac{\partial L}{\partial \varphi^A}\,, \tag{2.1}$$

where

$$p^\mu{}_A := \frac{\partial L}{\partial(\partial_\mu \varphi^A)} \tag{2.2}$$

and A labels the fields on the given space-time.

The definition of the energy is associated with the time derivative. Hence, the Lie derivative (denoted by $\mathcal{L}$) along a space-time vector field X defining the evolution which we want to describe, is a natural replacement of time derivative in field theories, since away from zeros of X we can always find find adapted coordinates to this vector such that we have

$X = \partial_0$, and the corresponding Lie derivative reduces to the time derivative.
Given a spacelike hypersurface $\mathscr{S}$ and a vector field X, there is a natural notion of Hamiltonian, with respect to a suitable symplectic structure, generating the flow of X which is given by the formula [18]:

$$H(X, \mathscr{S}) = \int_{\mathscr{S}} H^{\mu}(X) d\Sigma_{\mu} \,, \tag{2.3}$$

where

$$H^{\mu}(X) = p^{\mu}{}_{A} \mathcal{L}_X \varphi^{A} - X^{\mu} L \tag{2.4}$$

and

$$\Sigma_{\mu} = \partial_{\mu} \lrcorner \, (dx^0 \wedge \cdots \wedge dx^{D-1}) \tag{2.5}$$

is the surface element one-form of a D-dimensional space-time, with $\lrcorner$ denoting the contraction.
Analogous to the classical mechanics, where the Hamiltonian is conserved (the time derivative of Hamiltonian vanishes), the divergence of $H^{\mu}(X)$ is zero, provided that the theory is invariant under changes of the coordinates.[1]

Let us apply this formulation to the gravitational field. Let (M, g) be a D-dimensional space-time. Then, the gravitational field is described by a metric tensor $g_{\mu\nu}$, i.e., $\varphi = g_{\mu\nu}$. Moreover, we use a background metric $\overline{g}_{\mu\nu}$ to which the metric $g_{\mu\nu}$ is supposed to asymptote. Then (2.3), in vacuum, becomes

$$H(X, \mathscr{S}) = \int_{\mathscr{S}} \left(p^{\mu}{}_{\alpha\beta} \mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^{\mu} L \right) d\Sigma_{\mu} \,, \tag{2.6}$$

where

$$\begin{aligned} L \;&:=\; \mathfrak{g}^{\mu\nu} \left[\left(\Gamma^{\alpha}{}_{\sigma\mu} - \overline{\Gamma}^{\alpha}{}_{\sigma\mu} \right) \left(\Gamma^{\sigma}{}_{\alpha\nu} - \overline{\Gamma}^{\sigma}{}_{\alpha\nu} \right) - \left(\Gamma^{\alpha}{}_{\mu\nu} - \overline{\Gamma}^{\alpha}{}_{\mu\nu} \right) \left(\Gamma^{\sigma}{}_{\alpha\sigma} - \overline{\Gamma}^{\sigma}{}_{\alpha\sigma} \right) + \overline{\mathbf{R}}_{\mu\nu} - \frac{2}{D} \Lambda g_{\mu\nu} \right] \\ &\quad - \frac{1}{16\pi} \sqrt{-\det \overline{g}}\, \overline{g}^{\mu\nu} \left(\overline{\mathbf{R}}_{\mu\nu} - \frac{2}{D} \Lambda \overline{g}_{\mu\nu} \right) , \end{aligned} \tag{2.7}$$

with $\overline{\mathbf{R}}_{\mu\nu}$ being the Ricci tensor of the background metric $\overline{g}_{\mu\nu}$, Λ the cosmological constant, D the dimension of the space-time, $\det g \equiv \det(g_{\mu\nu})$, $\det \overline{g} \equiv \det(\overline{g}_{\mu\nu})$ and

$$\mathfrak{g}^{\mu\nu} := \frac{1}{16\pi} \sqrt{-\det g}\, g^{\mu\nu} \,, \quad p^{\lambda}{}_{\mu\nu} := \frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}{}_{,\lambda}} = \left(\overline{\Gamma}^{\lambda}_{\mu\nu} - \delta^{\lambda}_{(\mu} \overline{\Gamma}^{\kappa}_{\nu)\kappa} \right) - \left(\Gamma^{\lambda}_{\mu\nu} - \delta^{\lambda}_{(\mu} \Gamma^{\kappa}_{\nu)\kappa} \right) . \tag{2.8}$$

The last two terms in (2.7), which are independent of the metric $g_{\mu\nu}$, have been added by hand to ensure the convergence of the integral.

A further point needs to be made here is that the formalism in [18] is quite general, independent of the asymptotic conditions and of dimension of the space-time. However, it is not guaranteed that the integrals are convergent and the results are well-posed for every asymptotic conditions. Thus, one has to investigate this question for different sets of asymptotic conditions.

If X is a Killing vector field of $\overline{g}_{\mu\nu}$ and if the Einstein equations with cosmological constant Λ are satisfied,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,, \tag{2.9}$$

[1] Recall that we can always find adapted coordinates in which $X^{\mu}\partial_{\mu} = \partial_0$.

the integrand (2.3), in this case, can be rewritten as the divergence of a "Freud-type super-potential", up to source and renormalization terms:

$$\begin{aligned} H^\mu &= p^\mu_{\alpha\beta}\mathcal{L}_X\mathfrak{g}^{\alpha\beta} - X^\mu L \\ &= \partial_\alpha \mathbb{U}^{\mu\alpha} - \sqrt{-\det g}\, T^\mu{}_\alpha X^\alpha + \frac{1}{16\pi}\sqrt{-\det \overline{g}}\,\overline{g}^{\alpha\beta}\big(\overline{\mathbf{R}}_{\alpha\beta} - \frac{2}{D}\Lambda\overline{g}_{\alpha\beta}\big)X^\mu, \end{aligned} \tag{2.10}$$

with

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_\beta X^\beta - \frac{1}{8\pi}\sqrt{|\det g|}\, g^{\alpha[\nu}\delta^{\lambda]}_\beta \overline{\nabla}_\alpha X^\beta\,, \tag{2.11}$$

$$\mathbb{U}^{\nu\lambda}{}_\beta = \frac{2|\det \overline{g}|}{16\pi\sqrt{|\det g|}} g_{\beta\gamma}\overline{\nabla}_\kappa\left(e^2 g^{\gamma[\lambda}g^{\nu]\kappa}\right)\,, \tag{2.12}$$

where $\overline{\nabla}$ denotes the covariant derivative of the background metric $\overline{g}_{\mu\nu}$ and

$$e^2 \equiv \frac{\det g}{\det \overline{g}}\,. \tag{2.13}$$

In vacuum this leads to the formula

$$H(X,\mathscr{S}) = H_b(X,\mathscr{S}) := \frac{1}{2}\int_{\partial\mathscr{S}}\big(\mathbb{U}^{\nu\lambda} - \mathbb{U}^{\nu\lambda}\big|_{g=\overline{g}}\big)d\Sigma_{\nu\lambda}\,, \tag{2.14}$$

where

$$d\Sigma_{\nu\lambda} = \partial_\lambda \rfloor d\Sigma_\nu \tag{2.15}$$

and the subscript "b" on H_b stands for "boundary" and again we have added the second term in the integrand for convergence of the integral by hand.

3 Flat Asymptotics

Consider the initial data surfaces in a d-dimensional space-time with $d = n+1$, where n denotes the dimension of the space, containing asymptotic ends of the form

$$\mathscr{S}_{\rm ext} := \mathbb{R}^n \setminus B(0,R)\,. \tag{3.1}$$

A space-time is asymptotically flat if g has the following asymptotic conditions [2] along $\mathscr{S}_{\rm ext} \equiv \{x^0 = 0\}$

$$g_{\mu\nu} = \eta_{\mu\nu} + o\left(r^{-\alpha}\right)\,,\quad \partial_\sigma g_{\mu\nu} = o\left(r^{-\alpha-1}\right)\,,\quad \overline{g}_{\mu\nu} = \eta_{\mu\nu} + o\left(r^{-\alpha}\right)\,,\quad \partial_\sigma \overline{g}_{\mu\nu} = o\left(r^{-\alpha-1}\right)\,. \tag{3.2}$$

with $\alpha = (n-2)/2$ which is an optimal choice to have a well-posed definition of the energy, and

$$\overline{\Gamma}^\alpha{}_{\beta\gamma} = o(r^{-\alpha-1})\,, \tag{3.3}$$

[2]Note that these conditions are weaker than

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}\left(r^{-\alpha}\right)\,,\qquad \partial_\rho g_{\mu\nu} = \mathcal{O}\left(r^{-\alpha-1}\right)\,,\qquad T_{\mu\nu} = \mathcal{O}\left(r^{-2\alpha-2}\right)\,,$$

with $\alpha > (n-2)/2$.

with

$$r := \sqrt{(x^1)^2 + \ldots + (x^n)^2}\,. \tag{3.4}$$

We first assume that X is $\overline{g}$-covariantly constant (hence, a Killing vector of the background metric $\overline{g}_{\mu\nu}$). One then can check that in the coordinates of (3.2) the vector field X has to be of the form

$$X^\mu = X^\mu_\infty + o(r^{-\alpha})\,, \quad \partial_\nu X^\mu_\infty = 0\,. \tag{3.5}$$

As $\Lambda = 0$ in the current case, convergence of the boundary integrals in vacuum will be guaranteed if one assumes $\partial_\rho g_{\mu\nu} \in L^2(\mathscr{S}_{\rm ext})$, i.e.,

$$\sum_{\mu\alpha\beta} \int_{\mathscr{S}\cap\{r\geq R\}} |\partial_\mu g_{\alpha\beta}|^2 d^n x < \infty\,. \tag{3.6}$$

This follows immediately from Stokes' theorem together with (2.6)-(2.8) and (2.10), keeping in mind that $\Lambda = 0 = \overline{\mathbf{R}}_{\mu\nu}$ in the current context.

While we are mostly interested in vacuum solutions, the analysis below applies to non-vacuum ones, provided that one also has

$$T_{\mu\nu} = o(r^{-n}) \qquad \text{and} \qquad \sum_{\alpha\beta} \int_{\mathscr{S}\cap\{r\geq R\}} |T_{\alpha\beta}| d^n x < \infty\,. \tag{3.7}$$

The last term in (2.11) drops out when $\overline{\nabla}_\beta X_\alpha = 0$. Hence, one obtains

$$\begin{aligned}
\mathbb{U}^{\nu\lambda} &= \mathbb{U}^{\nu\lambda}{}_\beta X^\beta \\
&= -\frac{1}{16\pi}\left(1+o(r^{-\alpha})\right)\left(\eta_{\beta\gamma}+o(r^{-\alpha})\right) X^\beta \Big[\left(\eta^{\gamma\nu}\eta^{\lambda\kappa}\eta_{\rho\sigma} - \eta^{\gamma\lambda}\eta^{\nu\kappa}\eta_{\rho\sigma}\right) g^{\rho\sigma}{}_{,\kappa} \\
&\quad + g^{\gamma\lambda}{}_{,\kappa}\eta^{\nu\kappa} - g^{\gamma\nu}{}_{,\kappa}\eta^{\lambda\kappa} + \eta^{\gamma\lambda} g^{\nu\kappa}{}_{,\kappa} - \eta^{\gamma\nu} g^{\lambda\kappa}{}_{,\kappa} + o\left(r^{-2\alpha-1}\right)\Big] \\
&= -\frac{1}{16\pi}\Big[\left(\eta^{\lambda\kappa}X^\nu - \eta^{\nu\kappa}X^\lambda\right)\eta_{\rho\sigma} g^{\rho\sigma}{}_{,\kappa} + \eta^{\nu\kappa}\eta_{\beta\gamma} g^{\gamma\lambda}{}_{,\kappa} X^\beta - \eta^{\lambda\kappa}\eta_{\beta\gamma} g^{\gamma\nu}{}_{,\kappa} X^\beta \\
&\quad + g^{\nu\kappa}{}_{,\kappa} X^\lambda - g^{\lambda\kappa}{}_{,\kappa} X^\nu \Big] + o\left(r^{-2\alpha-1}\right) \\
&= -\frac{1}{16\pi}\eta^{\delta\kappa}\eta_{\beta\gamma} g^{\gamma\tau}{}_{,\kappa} X^\xi \Big(\delta^\lambda_\delta \delta^\nu_\xi \delta^\beta_\tau - \delta^\lambda_\xi \delta^\nu_\delta \delta^\beta_\tau \\
&\quad + \delta^\lambda_\tau \delta^\nu_\delta \delta^\beta_\xi - \delta^\lambda_\delta \delta^\nu_\tau \delta^\beta_\xi + \delta^\lambda_\xi \delta^\nu_\tau \delta^\beta_\delta - \delta^\lambda_\tau \delta^\nu_\xi \delta^\beta_\delta \Big) + o\left(r^{-2\alpha-1}\right) \\
&= \frac{3}{8\pi}\eta^{\delta\kappa}\eta_{\beta\gamma} g^{\gamma\tau}{}_{,\kappa} X^\xi\, \delta^{\nu\lambda\beta}_{\tau\delta\xi} + o\left(r^{-2\alpha-1}\right)\,.
\end{aligned} \tag{3.8}$$

Plugging the result into (2.14) and renaming indices, in the limit $r \to \infty$, we obtain the following form of (2.14), which will be a key formula for further calculations

$$H_b\left(X, \mathscr{S}\right) = \frac{3}{16\pi} \lim_{R\to\infty} \int_{S(R)} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho}\eta_{\gamma\sigma}\partial_\rho g^{\sigma\mu} dS_{\alpha\beta}\,, \tag{3.9}$$

where $S(R)$ denotes a sphere of radius R in the $\mathbb{R}^n$ factor of $\mathscr{S}_{\rm ext}$, the surface element two-forms $dS_{\alpha\beta}$ is given by

$$dS_{\alpha\beta} = \frac{1}{(n-1)!}\epsilon_{\alpha\beta\xi_1\cdots\xi_{n-1}}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-1}} \equiv \partial_\beta \lrcorner\, \partial_\alpha \lrcorner \sqrt{\det g}\, dx^0 \wedge\cdots\wedge dx^n\,. \tag{3.10}$$

and

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} := \delta^\alpha_{[\lambda}\delta^\beta_\mu\delta^\gamma_{\nu]}\,. \tag{3.11}$$

3.1 Hamiltonian Charges and the ADM Quantities

We see from (3.5) that the Hamiltonian charge $H_b\,(X,\mathscr{S})$ can be written as

$$H_b\,(X,\mathscr{S}) =: p_\mu X^\mu_\infty\,, \tag{3.12}$$

where the coefficients p_μ are called the ADM four-momentum of $\mathscr{S}$ [1]. It means that in this case Hamiltonian charges coincide with the ADM four-momentum.

If $X=\partial_0$ we find the ADM mass formula in any dimension ($n\geq 3$):

$$\begin{aligned} p_0 \;&:=\; H_b\,(\partial_t,\mathscr{S}) = \frac{1}{16\pi}\lim_{R\to\infty}\int_{S(R)}\sum_{j=1}^{n}(\partial_j g_{ij}-\partial_i g_{jj})\frac{x^i}{R}\,d^n x \\ &=\; m_{\mathrm{ADM}}\,. \end{aligned} \tag{3.13}$$

Next, when $X^0=0$, after using Stokes theorem for the following integral

$$\int_{S(R)} \partial_j(g_{j0}\delta^\ell_i - g_{\ell 0}\delta^j_i)\partial_\ell \lrcorner\,(dx^1\wedge\cdots\wedge dx^n) = 0\,, \tag{3.14}$$

we obtain the formula for ADM momentum

$$p_i := H_b\,(\partial_i,\mathscr{S}) = \frac{1}{8\pi R}\lim_{R\to\infty}\int_{S(R)} P_{ij}x^j\,d^n x\,, \tag{3.15}$$

where

$$P_{ij} := g^{\ell m}\mathcal{K}_{\ell m}g_{ij} - \mathcal{K}_{ij}\,, \tag{3.16}$$

and $\mathcal{K}_{ij}$ is the second fundamental form given by

$$\mathcal{K}_{ij} := \frac{1}{2}\mathcal{L}_T g_{ij} = \frac{1}{2}(\partial_0 g_{ij} - \partial_i g_{0j} - \partial_j g_{0i}) + o(r^{-2\alpha-1})\,, \tag{3.17}$$

where $\mathcal{L}_T$ denotes the Lie derivative in the direction of the unit-timelike future directed field T of normals to the level sets of x^0.

3.2 The Ashtekar-Hansen Equation

We start with the key equation (3.9) and try to rewrite it in terms of Christoffel symbols. To this end, we note that

$$\epsilon_{\lambda\mu\nu\rho}\epsilon^{\alpha\beta\gamma\delta} = -4!\delta^\alpha_{[\lambda}\delta^\beta_\mu\delta^\gamma_\nu\delta^\delta_{\rho]} \equiv -4!\delta^{\alpha\beta\gamma\delta}_{\lambda\mu\nu\rho} \tag{3.18}$$

and consequently

$$\epsilon_{\lambda\mu\nu\kappa}\epsilon^{\alpha\beta\gamma\kappa} = -3!\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\,, \tag{3.19}$$

and so on. We then have

$$\begin{aligned} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\,dS_{\alpha\beta} \;&=\; -\frac{1}{3!(n-2)!}\cdot\frac{1}{(n-1)!}\epsilon_{\lambda\mu\nu\kappa_1\cdots\kappa_{n-2}}\epsilon^{\alpha\beta\gamma\kappa_1\cdots\kappa_{n-2}}\epsilon_{\alpha\beta\xi_1\cdots\xi_{n-1}}\,dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-1}} \\ &=\; \frac{2}{3!(n-2)!}\,\epsilon_{\lambda\mu\nu\kappa_1\cdots\kappa_{n-2}}\,\delta^{\gamma\kappa_1\cdots\kappa_{n-2}}_{\alpha\beta\xi_1\cdots\xi_{n-1}}\,dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-1}} \\ &=\; \frac{1}{3}\cdot\frac{1}{(n-2)!}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}}\,dx^\gamma\wedge dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-2}}\,. \end{aligned} \tag{3.20}$$

On the other hand, from $\nabla_\rho g^{\sigma\mu}=0$ we get

$$\partial_\rho g^{\sigma\mu} = -\Gamma^\sigma{}_{\alpha\rho} g^{\alpha\mu} - \Gamma^\mu{}_{\alpha\rho} g^{\alpha\sigma} \,. \tag{3.21}$$

Plugging (3.20) and (3.21) into (3.9), one obtains

$$p_\mu X^\mu_\infty = \frac{(-1)^{n-1}}{16\pi(n-2)!} \lim_{R\to\infty} \int_{S(R)} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} \Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma \,. \tag{3.22}$$

Now, we can rewrite the integrand of (3.22) in form of the exterior derivative as follows

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho}\Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma \\
&\quad = d\left(\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu x^{\xi_1} g^{\lambda\rho}\Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\right) \\
&\qquad -(-1)^{n-3}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} x^{\xi_1} dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge \underbrace{\left(\partial_\sigma\Gamma^\mu{}_{\gamma\rho} dx^\sigma\wedge dx^\gamma\right)}_{=\frac{1}{2}R^\mu{}_{\rho\sigma\gamma}dx^\sigma\wedge dx^\gamma} \\
&\qquad +o(r^{-2\alpha-1})\,,
\end{aligned} \tag{3.23}$$

where we used

$$d\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} = \frac{1}{2}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} g^{\alpha\beta} dg_{\alpha\beta} \tag{3.24}$$

and

$$R^\mu{}_{\rho\sigma\gamma} = \partial_\sigma\Gamma^\mu{}_{\rho\gamma} - \partial_\gamma\Gamma^\mu{}_{\rho\sigma} + o(r^{-2\alpha-2})\,. \tag{3.25}$$

Inserting these results into (3.22) and applying Stokes's theorem, we finally get a generalization of a similar formula derived by Ashtekar and Hansen in 4 dimensions [3] (compare [11]) to $n+1$ dimensions

$$p_\mu X^\mu_\infty = \frac{1}{32\pi(n-2)!} \lim_{R\to\infty} \int_{S(R)} \epsilon_{\nu\xi_1\cdots\xi_{n-2}\lambda\mu} X^\nu x^{\xi_1} R^{\lambda\mu}{}_{\delta\gamma}\, dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\delta\wedge dx^\gamma \,. \tag{3.26}$$

This expression looks somehow more geometric than the familiar ADM formula. However, we note that the coordinate functions x^μ appearing in (3.26) do not transform as a vector field under coordinate changes. Nevertheless, it turns out that (3.26) is invariant under coordinate transformation (cf. [12]).

We can also rewrite (3.26) in a $n+1$ form. For this purpose, we consider $X=\partial_0$ and look at the hypersurface $\mathscr{S}=\{x^0=0\}$. We then obtain

$$m := p_0 = \frac{1}{32\pi(n-2)!} \lim_{R\to\infty} \int_{S(R)} \epsilon_{il_1\cdots l_{n-1}jk} x^i\, R^{jk}{}_{mn}\, dx^{l_1}\wedge\cdots\wedge dx^{l_{n-2}}\wedge dx^m\wedge dx^n \,. \tag{3.27}$$

Note that $R^{jk}{}_{mn}$ is the Riemann tensor of the $n+1$-dimensional metric $g_{\mu\nu}dx^\mu dx^\nu$. We can replace $R^{jk}{}_{mn}$ with the Riemann tensor of the n-dimensional metric $g_{\mu\nu}dx^\mu dx^\nu$, denoted by $\mathcal{R}^{jk}{}_{mn}$, by means of the Gauss-Codazzi-Mainardi equation

$$R^{jk}{}_{mn} = \mathcal{R}^{jk}{}_{mn} + o(r^{-2\alpha-2})\,, \tag{3.28}$$

where we used the fact that the second fundamental form falls of as $o(r^{-\alpha-1})$. Since the error term does not contribute in the limit, (3.27) becomes

$$m = \frac{1}{32\pi(n-2)!} \lim_{R\to\infty} \int_{S(R)} \epsilon_{il_1\cdots l_{n-1}jk}\, x^i \mathcal{R}^{jk}{}_{mn}\, dx^{l_1}\wedge\cdots\wedge dx^{l_{n-2}}\wedge dx^m\wedge dx^n \,. \tag{3.29}$$

3.3 The Komar Integral

In order to achieve an expression for the Komar integral, we rewrite (3.26) as follows. First, from (3.10) we get

$$dx^{\xi_2} \wedge \cdots \wedge dx^{\xi_{n-2}} \wedge dx^{\delta} \wedge dx^{\gamma} = -\frac{1}{2}\epsilon^{\xi_2\cdots\xi_{n-2}\delta\gamma\rho\sigma}\, dS_{\rho\sigma}\,. \tag{3.30}$$

Therefore, using the identity (A.1) from Appendix A the integrand of (3.26) becomes

$$\begin{aligned}
&-\frac{1}{2}\epsilon_{\nu\xi_1\xi_2\cdots\xi_{n-2}\lambda\mu}X^{\nu}x^{\xi_1}R^{\lambda\mu}{}_{\delta\gamma}\epsilon^{\xi_2\cdots\xi_{n-2}\delta\gamma\rho\sigma}\, dS_{\rho\sigma}\\
&= \frac{1}{2}(n-3)!4!\, X^{\nu}x^{\xi_1}\, \delta^{\delta\gamma\rho\sigma}_{\lambda\mu\nu\xi_1}R^{\lambda\mu}{}_{\delta\gamma}\, dS_{\rho\sigma}\\
&= 2(n-3)!X^{\nu}x^{\xi_1}\left(R_{\nu\xi_1}{}^{\rho\sigma} + \delta^{\rho\sigma}_{\nu\xi_1}R - 4\delta^{[\rho}_{[\nu}R^{\sigma]}{}_{\xi_1]}\right) dS_{\rho\sigma}\\
&= 2(n-3)!\, X^{\nu}x^{\xi_1}\, R_{\nu\xi_1\rho\sigma}dS^{\rho\sigma} + o\left(1\right)\,,
\end{aligned} \tag{3.31}$$

where we used $R_{\mu\nu} \sim T_{\mu\nu} = o(r^{-n})$ and $dS_{\rho\sigma} \sim \left(r^{n-1}d^{n-1}\Omega\right)_{\rho\sigma}$ with $d^{n-1}\Omega$ being the round metric on a unit $(n-1)$-sphere. Inserting (3.31) into (3.26) and renaming some indices, we finally arrive at

$$p_{\mu}X^{\mu}_{\infty} = \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{S(R)} X^{\mu}x^{\nu}\, R_{\mu\nu\rho\sigma}dS^{\rho\sigma}\,. \tag{3.32}$$

Now, if X is also a Killing vector field of metric g then we have the following identity

$$R_{\mu\nu\alpha\beta}X^{\mu} = \nabla_{\nu}\nabla_{\alpha}X_{\beta}\,. \tag{3.33}$$

Hence, the equation (3.32) can be written as

$$p_{\mu}X^{\mu}_{\infty} = \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{S(R)} X^{[\beta;\alpha]}{}_{;\gamma}x^{\gamma}dS_{\alpha\beta}\,. \tag{3.34}$$

In order to be able to use the Gauss's theorem, we should have a total divergence of an antisymmetric tensor in integrand. Thus, the integrand of (3.34) can be rewritten as a total divergence as follows (for future purposes, we do the calculation in d space-time dimensions, i.e., $\delta^{\mu}{}_{\mu} = d$)

$$\begin{aligned}
\left(X^{[\alpha;\beta}x^{\gamma]}\right)_{;\gamma} &= X^{[\alpha;\beta}{}_{;\gamma}x^{\gamma]} + X^{[\alpha;\beta}x^{\gamma]}_{;\gamma}\\
&= \frac{1}{3!}\Big(X^{\alpha;\beta}{}_{;\gamma}x^{\gamma} + X^{\beta;\gamma}{}_{;\gamma}x^{\alpha} + X^{\gamma;\alpha}{}_{;\gamma}x^{\beta} - X^{\beta;\alpha}{}_{;\gamma}x^{\gamma} - X^{\alpha;\gamma}{}_{;\gamma}x^{\beta} - X^{\gamma;\beta}{}_{;\gamma}x^{\alpha}\\
&\quad + X^{\alpha;\beta}\underbrace{x^{\gamma}_{;\gamma}}_{=d} + X^{\beta;\gamma}\underbrace{x^{\alpha}_{;\gamma}}_{=\delta^{\alpha}_{\gamma}} + X^{\gamma;\alpha}\underbrace{x^{\beta}_{;\gamma}}_{=\delta^{\beta}_{\gamma}} - X^{\beta;\alpha}\underbrace{x^{\gamma}_{;\gamma}}_{=d} - X^{\alpha;\gamma}\underbrace{x^{\beta}_{;\gamma}}_{=\delta^{\beta}_{\gamma}} - X^{\gamma;\beta}\underbrace{x^{\alpha}_{;\gamma}}_{=\delta^{\alpha}_{\gamma}}\Big)\\
&= \frac{1}{3!}\left(2X^{[\alpha;\beta]}{}_{;\gamma}x^{\gamma} + 2X^{[\beta;\gamma]}{}_{;\gamma}x^{\alpha} + 2X^{[\gamma;\alpha]}{}_{;\gamma}x^{\beta} + 2(d-2)X^{[\alpha;\beta]}\right).
\end{aligned} \tag{3.35}$$

Using again the identity (3.33) for the second and third therm in the last line of (3.35), on gets

$$\begin{aligned}
\left(X^{[\alpha;\beta}x^{\gamma]}\right)_{;\gamma} &= \frac{1}{3}\left(X^{[\alpha;\beta]}{}_{;\gamma}x^{\gamma} + 2R_{\mu}{}^{[\alpha}x^{\beta]}X^{\mu} + (d-2)X^{[\alpha;\beta]}\right)\\
&= \frac{1}{3}\left(X^{[\alpha;\beta]}{}_{;\gamma}x^{\gamma} + (d-2)X^{[\alpha;\beta]} + o\left(r^{-2\alpha-1}\right)\right)\,,
\end{aligned} \tag{3.36}$$

or equivalently

$$X^{[\beta;\alpha]}{}_{;\gamma}x^\gamma = (d-2)X^{[\alpha;\beta]} - 3\left(X^{[\alpha;\beta}x^{\gamma]}\right)_{;\gamma} + o\left(r^{-2\alpha-1}\right) . \tag{3.37}$$

In the current context we have $d = n+1$. Inserting these results into (3.34) and applying the Gauss's theorem, we obtain our final expression

$$p_\mu X^\mu_\infty = \frac{n-1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} X^{[\alpha;\beta]} dS_{\alpha\beta} . \tag{3.38}$$

This is the Komar integral in $(n+1)$-dimensional space-time ($n \geq 3$). However, one should note that this expression represents the Komar integral only if the vector field X satisfies (3.5). For instance, if $X = \partial_0$ then (3.38) is the Komar mass in $(n+1)$-dimensional space-time, but, (3.38) *cannot* represents the Komar angular momentum for $X = \partial_\varphi$.

Moreover, we showed that in the case of stationary asymptotically flat space-times, i.e., for the Killing vector field $X = \partial_0$, the Komar mass coincides with the ADM mass in all dimensions ($x \geq 3$). This generalizes the result of Beig in four-dimensional space-times [7].

As we have seen, in the simplest case, namely in the case of asymptotic flat space-times, all three notions of energy, the Hamiltonian charge, ADM mass and Komar mass, coincide with each other.

4 Kaluza-Klein Asymptotics

In this section, we are going to generalize the results of section 3 to the asymptotically Kaluza-Klein (KK) space-times which are defined as follows: We say that a metric g is KK-asymptotically flat if along hypersurfaces $\mathscr{S}_{\rm ext} \equiv \{x^0 = 0\}$ which are initial data surfaces in an D-dimensional space-time, with $D = d+K$, containing asymptotic ends of the form

$$\mathscr{S}_{\rm ext} := (\mathbb{R}^n \setminus B(0,R)) \times \underbrace{S^1 \times \cdots \times S^1}_{K \text{ factors}} =: (\mathbb{R}^n \setminus B(0,R)) \times \mathbb{T}^K \tag{4.1}$$

the metric g has the following asymptotic form

$$g = \underbrace{\eta_{ab}dx^a dx^b + \delta_{AB}dx^A dx^B}_{=:\eta_{\mu\nu}dx^\mu dx^\nu} + o(r^{-\alpha}) , \quad \partial_\mu g_{\nu\rho} = o(r^{-\alpha-1}) , \tag{4.2}$$

where S^1 is the unit circle, $\mathbb{T}^K$ is a K-torus and Greek indices run from 0 to $n+K$, upper case Latin indices from the beginning of alphabet run from $n+1$ to $n+K$, lower case Latin indices from the beginning of alphabet running from 0 to n, and lower case Latin indices from the middle of alphabet running from 1 to n, and finally upper case Latin indices from the middle of alphabet run from 1 to $n+K$. That is,

$$\begin{aligned} \alpha, \beta, \gamma, \ldots &\in \{0, \ldots, n+K\} , \\ a, b, c, \ldots, h &\in \{0, \ldots, n\} , \\ i, j, k, \ldots &\in \{1, \ldots, n\} , \\ A, B, C, \ldots, H &\in \{n+1, \ldots, n+K\} , \\ I, J, K, \ldots, &\in \{1, \ldots, n+K\} \end{aligned} \tag{4.3}$$

or equivalently

$$(x^\mu) \equiv (x^0, x^i, x^A) \equiv (x^a, x^A) \equiv (x^0, x^I)\,. \tag{4.4}$$

As in section 3, d is the physical space-time dimension, n is the physical space-dimension and K denotes the number of Kaluza-Klein directions. We also use the same definitions for r and α from section 3. In the KK case (3.2)-(3.3) remains valid with the above definition of $\eta_{\mu\nu}$ and (3.6)-(3.7) generalize to

$$\sum_{\mu\alpha\beta} \int_{\mathscr{S}\cap\{r\geq R\}} |\partial_\mu g_{\alpha\beta}|^2 d^{n+K}x < \infty \tag{4.5}$$

and

$$\sum_{\alpha\beta} \int_{\mathscr{S}\cap\{r\geq R\}} |T_{\alpha\beta}| d^{n+K}x < \infty\,, \tag{4.6}$$

where $T_{\mu\nu} = o(r^{-n})$. Also in this case we assume that X is $\overline{g}$-covariantly constant and therefore (3.5) holds. Obviously, we could assume that $\overline{\nabla}X$ falls off fast enough so that it does not contribute to the integrals in the limit.

4.1 Hamiltonian Charges

Under the conditions mentioned above, the calculations of (3.8) remains also in the KK case true and we arrive at the same key equation (3.9)

$$H_b\,(X, \mathscr{S}) = \frac{3}{16\pi} \lim_{R\to\infty} \int_{S(R)} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho} \eta_{\gamma\sigma} \partial_\rho g^{\sigma\mu} dS_{\alpha\beta}\,, \tag{4.7}$$

where

$$dS_{\alpha\beta} = \frac{1}{(n+K-1)!} \epsilon_{\alpha\beta\xi_1\cdots\xi_{n+K-1}}\, dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n+K-1}}\,. \tag{4.8}$$

Again we have because of (3.5) we have

$$H_b\,(X, \mathscr{S}) =: p_\mu X^\mu_\infty\,, \tag{4.9}$$

with the difference that in this case p_μ does not necessarily coincide with the ADM four-momentum of $\mathscr{S}$. Obviously, if $K = 0$ then p_μ will represent the ADM four-momentum.

If $X = \partial_0$ we find the generalized form of (3.13)

$$\begin{aligned} p_0 \quad &:= \quad H_b\,(\partial_t, \mathscr{S}) = \frac{1}{16\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \sum_{I=1}^{n+K} (\partial_I g_{iI} - \partial_i g_{II}) \frac{x^i}{R}\, d^{n+K-1}\mu \\ &= \quad |\mathbb{T}^K|\, p_{0,\mathrm{ADM}} + \frac{1}{16\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \sum_{A=n+1}^{n+K} (\partial_A g_{iA} - \partial_i g_{AA}) \frac{x^i}{R}\, d^{n+K-1}\mu\,, \end{aligned} \tag{4.10}$$

where $d^{n+K-1}\mu$ is the measure induced on $S(R) \times \mathbb{T}^K$ by the flat metric, $|\mathbb{T}^K|$ denotes the volume of $\mathbb{T}^K$, and $p_{0,\mathrm{ADM}}$ is the usual ADM mass of the physical-space metric $g_{ij}dx^i dx^j$. This shows that the ADM mass does not coincide with the Hamiltonian generating time translations in general.

Again, if $X^0 = 0$, using the generalized form of (3.14)

$$\int_{S(R)\times\mathbb{T}^K} \partial_J (g_{J0}\delta^L_I - g_{L0}\delta^J_I) \partial_L \lrcorner\, (dx^1 \wedge \cdots \wedge dx^{K+n}) = 0\,, \tag{4.11}$$

one obtains the formula

$$p_I := H_b(\partial_I, \mathscr{S}) = \frac{1}{8\pi R} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} P_{Ii} x^i \, d^{n+K-1}\mu \,. \tag{4.12}$$

Here P_{IJ} is the generalized form of (3.16)

$$P_{IJ} := g^{LM} \mathcal{K}_{LM} g_{IJ} - \mathcal{K}_{IJ} \,, \tag{4.13}$$

where

$$\mathcal{K}_{IJ} := \frac{1}{2}\mathcal{L}_T g_{IJ} = \frac{1}{2}(\partial_0 g_{IJ} - \partial_I g_{0J} - \partial_J g_{0I}) + o(r^{-2\alpha-1}) \,. \tag{4.14}$$

As an example, we compute the above integrals for the Rasheed metrics, described in Appendix C, with $P = 0$:

$$p_0 = 2\pi M \,, \quad p_i = 0 \,, \quad p_4 = 2\pi Q \,. \tag{4.15}$$

Equation (4.15) includes a 2π factor arising from a normalization in which the Kaluza-Klein coordinate x^4 in the Rasheed solutions runs over a circle of length 2π.

This should be compared with the ADM four-momentum $p_{\mu,\mathrm{ADM}}$ of the n-dimensional space metric $g_{ij}dx^i dx^j$, which reads

$$p_{0,\mathrm{ADM}} = M - \frac{\Sigma}{\sqrt{3}} \,, \quad p_{i,\mathrm{ADM}} = 0 \,. \tag{4.16}$$

4.2 Towards the Komar Integral

In order to achieve the Komar integral in KK case we should repeat the calculations of section 3.2. It turns out that we *cannot* easily generalize all results in that section. There are some subtleties that should be taken into account. For example, the coordinate functions x^A are periodic and therefore are *not* defined uniquely. Hence, these functions are *not* allowed to appear in our calculations, whereas expressions such as dx^A are well-defined.

Before we proceed, let us introduce some notations which should be kept in mind throughout this work:

- We will use the following symbols to denote the Riemann tensor
  1. $\mathbf{R}^\alpha{}_{\beta\gamma\delta}$ of the $(d+K)$-dimensional metric $g_{\mu\nu}dx^\mu dx^\nu$,
  2. $R^a{}_{bcd}$ of the d-dimensional metric $g_{ab}dx^a dx^b$,
  3. $\mathsf{R}^I{}_{JKL}$ of the $(n+K)$-dimensional metric $g_{IJ}dx^I dx^J$, and
  4. $\mathcal{R}^i{}_{jk\ell}$ of the n-dimensional metric $g_{ij}dx^i dx^j$.
- We will write "n.c." for the sum of those terms which do not contribute to the integral either because of the integration domain, or by Stokes theorem, or by passage to the limit.

We note that in the case of $K = 0$ there are no differences between $g_{ab}dx^a dx^b$ and $g_{\mu\nu}dx^\mu dx^\nu$ and consequently $\mathbf{R}^\alpha{}_{\beta\gamma\delta}$ and $R^a{}_{bcd}$.

We again start with the equation (4.7). One can readily generalize (3.20) to the KK case

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\, dS_{\alpha\beta} = \frac{1}{3}\cdot\frac{1}{(n+K-2)!}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}}\, dx^\gamma \wedge dx^{\xi_1} \wedge \cdots \wedge dx^{\xi_{n+K-2}} \,. \tag{4.17}$$

Plugging (4.17) into (4.7) provides the generalized form of (3.22)

$$p_\mu X^\mu_\infty = \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!} \lim_{R\to\infty} \int_{S(R)\times\mathbb{T}^n} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}} X^\nu g^{\lambda\rho} {\Gamma^\mu}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n+K-2}}\wedge dx^\gamma\,. \tag{4.18}$$

In this case we will be integrating the integrand of (4.18) over

$$S^{n-1}\times\mathbb{T}^K = S^{d-2}\times\mathbb{T}^K\,.$$

So only those forms in the sum which contain a $dx^d\wedge\cdots\wedge dx^{d+K-1}$ factor will survive integration. Hence, we can manipulate the integrand of (4.18) as follows

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\ldots\xi_{d+K-3}} X^\nu g^{\lambda\rho}{\Gamma^\mu}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{d+K-3}}\wedge dx^\gamma\\
&\quad= \frac{(d+K-3)!}{(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}{\Gamma^c}_{ae}\, dx^{a_1}\wedge\cdots\wedge dx^{A_K}\wedge dx^a\\
&\qquad+\frac{(d+K-3)!}{(d-2)!(N-1)!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}{\Gamma^\mu}_{A\rho}\, dx^{a_1}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A\\
&\quad= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f \eta^{be}{\Gamma^c}_{ae}\, x^{a_1} dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^a)\\
&\qquad-\frac{(d+K-3)!}{(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f \eta^{be} x^{a_1} d{\Gamma^c}_{ae}\wedge\cdots\wedge dx^{A_K}\wedge dx^a\\
&\qquad+\frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu \eta^{\lambda\rho}{\Gamma^\mu}_{A\rho}\, x^{a_1} dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A)\\
&\qquad-\frac{(d+K-3)!}{(d-2)!(N-1)!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu \eta^{\lambda\rho} x^{a_1} d{\Gamma^\mu}_{A\rho}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A + \text{n.c.}\\
&\quad= \frac{(d+K-3)!}{2(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f x^{a_1} {\mathbf{R}^{bc}}_{a_{d-1}a_{d-2}} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^{a_{d-2}}\\
&\qquad+\frac{(d+K-3)!}{(d-2)!(N-1)!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu x^{a_1} {\mathbf{R}^{\lambda\mu}}_{a_{d-1}A} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A\\
&\qquad+\text{n.c.}
\end{aligned} \tag{4.19}$$

Using

$$\begin{aligned}
&\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A\\
&\quad= 3(N-1)!(-1)^{d+K-1}\delta^{Aab}_{\lambda\mu\nu}\epsilon_{aba_1\ldots a_{d-2}} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{a_{d-2}}\wedge dx^{d+1}\wedge\cdots\wedge dx^{d+K}\,,
\end{aligned}$$

after some reordering of indices one obtains

$$\begin{aligned}
p_\mu X^\mu_\infty \;=\; &\frac{(-1)^n}{32\pi(n-1)!}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K} x^{a_1}\Big[(n-1)\epsilon_{a_1a_2\cdots a_{n-2}abc} X^a {\mathbf{R}^{bc}}_{a_{n-1}a_n}\\
&-\underbrace{\epsilon_{a_1a_2\cdots a_{n-1}ab}\left(4X^a{\mathbf{R}^{bA}}_{a_nA}+2X^A{\mathbf{R}^{ab}}_{a_nA}\right)}_{(*)}\Big] dx^{a_2}\wedge\cdots\wedge dx^{a_n}\wedge dx^{d+1}\wedge\cdots\wedge dx^{d+K}\,.
\end{aligned} \tag{4.20}$$

From

$$dx^{a_2}\wedge\cdots\wedge dx^{a_n}\wedge dx^{d+1}\wedge\cdots\wedge dx^{d+K} = -\frac{1}{2}\epsilon^{a_2\cdots a_n ef} dS_{ef} \tag{4.21}$$

and

$$\mathbf{R}^{ad}{}_{bd} \;=\; \mathbf{R}^{a}{}_{b} - \mathbf{R}^{aA}{}_{bA}$$

we get for the first term of the Hamiltonian integral

$$\begin{aligned}
&\epsilon_{a_1a_2\cdots a_{n-2}abc}\, x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dx^{a_2}\wedge\cdots\wedge dx^{a_n}\wedge dx^{d+1}\wedge\cdots\wedge dx^{d+K}\\
&= -\frac{1}{2}\epsilon_{a_1a_2\cdots a_{n-2}abc}\,\epsilon^{a_2\cdots a_n ef} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dS_{ef}\\
&= \frac{1}{2}(-1)^{n-1}(n-3)!\,4!\,\delta^{a_{n-1}a_n ef}_{a_1\,a\,b\,c}\, x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1}a_n} dS_{ef}\\
&= 2(-1)^{n-1}(n-3)! x^{a_1} X^a \left(\mathbf{R}^{ef}{}_{a_1a} + \delta^{ef}_{a_1a}\mathbf{R}^{bc}{}_{bc} - 4\delta^{[e}_{[a_1}\mathbf{R}^{f]c}{}_{a]c}\right) dS_{ef}\\
&= 2(-1)^{n-1}(n-3)! x^{a_1} X^a \left[\mathbf{R}^{ef}{}_{a_1a} + \delta^{ef}_{a_1a}\left(\mathbf{R}^{c}{}_{c} - \mathbf{R}^{cA}{}_{cA}\right) - 4\left(\delta^{[e}_{[a_1}\mathbf{R}^{f]}{}_{a]} - \delta^{[e}_{[a_1}\mathbf{R}^{f]A}{}_{a]A}\right)\right] dS_{ef}\,,
\end{aligned} \tag{4.22}$$

where we used (A.1) in the fourth line.

Now, recall that finiteness of the total energy of matter fields together with the dominant energy condition implies (3.7). This, together with the Einstein equations, takes for granted that the Ricci-tensor contribution to the integrals will vanish in the limit $R \to \infty$. Nevertheless, we will keep the Ricci tensor terms for future reference.

Using

$$-\frac{1}{2}\epsilon_{a_1a_2\cdots a_{n-1}ab}\,\epsilon^{a_2\cdots a_n ef} dS_{ef} = 3(-1)^n(n-2)!\,\delta^{a_n ef}_{a_1ab} dS_{ef}\,,$$

the terms involving $(*)$ in (4.20) can be manipulated as

$$\begin{aligned}
&6(-1)^n(n-2)!\,\delta^{a_n ef}_{a_1ab} x^{a_1}\left(2X^a\mathbf{R}^{bA}{}_{a_nA} + X^A\mathbf{R}^{ab}{}_{a_nA}\right) dS_{ef}\\
&\quad= 2(-1)^n(n-2)!\, x^{a_1}\left(\delta^{a_n e}_{a_1a}\delta^f_b + \delta^{ef}_{a_1a}\delta^{a_n}_b + \delta^{fa_n}_{a_1a}\delta^e_b\right)\left(2X^a\mathbf{R}^{bA}{}_{a_nA} + X^A\mathbf{R}^{ab}{}_{a_nA}\right) dS_{ef}\\
&\quad= 2(-1)^n(n-2)!\Big[2\left(x^{[a_n}X^{e]}\mathbf{R}^{fA}{}_{a_nA} + x^{[e}X^{f]}\mathbf{R}^{a_nA}{}_{a_nA} + x^{[f}X^{a_n]}\mathbf{R}^{eA}{}_{a_nA}\right)\\
&\qquad + X^A\left(x^{[a_n}\mathbf{R}^{e]f}{}_{a_nA} + x^{[e}\mathbf{R}^{f]a_n}{}_{a_nA} + x^{[f}\mathbf{R}^{a_n]e}{}_{a_nA}\right)\Big] dS_{ef}\,.
\end{aligned}$$

Renaming the indices, rearranging terms, and plugging the results into the integral one gets the following final expression

$$\begin{aligned}
p_\mu X^\mu_\infty &= \frac{1}{32\pi(n-2)}\times\\
&\quad \lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K}\Bigg\{-2x^bX^a\left[\mathbf{R}^{ef}{}_{ba} + \delta^{ef}_{ba}\left(\mathbf{R}^c{}_c - \mathbf{R}^{cA}{}_{cA}\right) - 4\left(\delta^{[e}_{[b}\mathbf{R}^{f]}{}_{a]} - \delta^{[e}_{[b}\mathbf{R}^{f]A}{}_{a]A}\right)\right]\\
&\quad -2\frac{n-2}{n-1}\left[2\left(2x^{[b}X^{e]}\mathbf{R}^{fA}{}_{bA} + x^eX^f\mathbf{R}^{bA}{}_{bA}\right) + 3X^Ax^e\mathbf{R}^{fb}{}_{bA}\right]\Bigg\} dS_{ef}\\
&= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^K}\Bigg(X^ax^b\mathbf{R}_{ab}{}^{ef} + 4x^{[e}X^{a]}\mathbf{R}^f{}_a - x^eX^f\mathbf{R}^c{}_c\\
&\quad -\frac{1}{n-1}\left[(n-3)x^eX^f\mathbf{R}^{bA}{}_{bA} + 4x^{[e}X^{a]}\mathbf{R}^{fA}{}_{aA} + 3(n-2)X^Ax^e\mathbf{R}^{fb}{}_{bA}\right]\Bigg) dS_{ef}\,.
\end{aligned} \tag{4.23}$$

Some special cases are of interest:

1. Suppose that $X^\mu = \delta^\mu_0$, thus X has just time component. At $x^0 = 0$ we have

$$\begin{aligned} p_0 &= \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big[x^j \mathbf{R}_{0j}{}^{0i} + \frac{1}{2} x^i \left(\mathbf{R}^j{}_j - \mathbf{R}^0{}_0 \right) - x^j \mathbf{R}^i{}_j \\ &\qquad - \frac{1}{n-1} \left(\frac{1}{2} x^i \mathbf{R}^{0A}{}_{0A} - x^j \mathbf{R}^{iA}{}_{jA} \right) \Big] dS_{0i} \,. \end{aligned} \tag{4.24}$$

where the terms involving the Ricci tensor give a vanishing contribution because of (3.7); similarly for (4.25) below.

2. Suppose that $X^A = 0$, thus X has only space-time components. Then

$$\begin{aligned} p_a X^a_\infty &= \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \Big(X^a x^b \mathbf{R}_{ab}{}^{ef} + 4 x^{[e} X^{a]} \mathbf{R}^f{}_a - x^e X^f \mathbf{R}^c{}_c \\ &\qquad - \frac{1}{n-1} \left[(n-3) x^e X^f \mathbf{R}^{bA}{}_{bA} + 4 x^{[e} X^{a]} \mathbf{R}^{fA}{}_{aA} \right] \Big) dS_{ef} \,. \end{aligned} \tag{4.25}$$

We will see below that the first term in the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when $X^0 = 0$, at $x^0 = 0$ the third term in the integrand gives a vanishing contribution, so that the generators of space-translations read

$$p_i X^i_\infty = \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} \left[X^i x^k \mathbf{R}_{ik}{}^{0j} + 2 x^{[i} X^{j]} \left(\frac{2}{n-1} \mathbf{R}^{0A}{}_{iA} + \mathbf{R}^0{}_i \right) \right] dS_{0j} \,. \tag{4.26}$$

We also note that when $K = 1$ the contribution of the fourth term in the integrand in (4.25) always vanishes because, denoting the Kaluza-Klein coordinate by x^4, we have

$$\mathbf{R}^{bA}{}_{bA} = \mathbf{R}^{b4}{}_{b4} = \mathbf{R}^{\mu 4}{}_{\mu 4} = \mathbf{R}^4{}_4 = o(r^{-n}) \,,$$

which gives a zero contribution in the limit.

3. Suppose instead that $X^a = 0$, thus X has only components tangential to the Kaluza-Klein fibers. Then, again at $x^0 = 0$,

$$\begin{aligned} p_A X^A_\infty &= \frac{3}{16(n-1)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^e \mathbf{R}_{Ab}{}^{fb} dS_{ef} \\ &= \frac{3}{16(n-1)\pi} \lim_{R\to\infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^i \mathbf{R}_{AB}{}^{0B} dS_{0i} \,, \end{aligned} \tag{4.27}$$

where the decay $o(r^{-n})$ of the Ricci tensor of the $(n+K+1)$-dimensional metric has been used.

If X^μ is also a Killing vector field of g, we can rewrite (4.23) using the identity (3.33). Indeed, the first integral of (4.23) can be manipulated as (keeping in mind that dS_{0A} and

dS_{AB} do not contribute)

$$\begin{aligned}
&\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} X^a x^b \mathbf{R}_{abef} dS^{ef} \\
&= \lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} \left(X^{[f;e]}{}_{;b} x^b - X^A x^b \mathbf{R}_{Ab}{}^{ef} \right) dS_{ef} \\
&= \lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} \left\{ (n-1) X^{[e;f]} - 3\left(X^{[e;f} x^{b]} \right)_{;b} + 2\mathbf{R}_{\mu b}{}^{b[f} x^{e]} X^\mu - X^A x^b \mathbf{R}_{Ab}{}^{ef} \right\} dS_{ef} \\
&= \lim_{R\to\infty}\left[(n-1)\int_{S(R)}\int_{\mathbb{T}^N} X^{\alpha;\beta} dS_{\alpha\beta} + \int_{S(R)}\int_{\mathbb{T}^N} \left(2X^\mu x^e \mathbf{R}^{fb}{}_{b\mu} - X^A x^b \mathbf{R}_{Ab}{}^{ef} \right) dS_{ef} \right] ,
\end{aligned} \tag{4.28}$$

where we used the identity (3.37) in the third line. Hence, we have

$$\begin{aligned}
p_\mu X^\mu_\infty &= \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \Bigg\{ (n-1)\int_{S(R)}\int_{\mathbb{T}^N} X^{\alpha;\beta} dS_{\alpha\beta} \\
&\quad + \int_{S(R)}\int_{\mathbb{T}^N} \Bigg(2X^\mu x^e \mathbf{R}^{fb}{}_{b\mu} - X^A x^b \mathbf{R}_{Ab}{}^{ef} - \frac{1}{n-1}\Big[(n-3) x^e X^f \mathbf{R}^{bA}{}_{bA} \\
&\quad + 4x^{[e} X^{a]} \mathbf{R}^{fA}{}_{aA} + 3(n-2) X^A x^e \mathbf{R}^{fb}{}_{bA} \Big] \Bigg) dS_{ef} \Bigg\} .
\end{aligned} \tag{4.29}$$

The first integrand is the Komar integrand. In asymptotically flat case, $K=0$, (4.29) reduces to (3.38). Thus, it appears that, in general, the Komar-type integral does not coincide with the Hamiltonian generators. This is really the case, as can be seen for the Rasheed solutions. Using (C.12) one readily finds for $X=\partial_t$, keeping in mind that $n=3$:

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} X^{\alpha;\beta} dS_{\alpha\beta} = 2\pi\big(M + \frac{\Sigma}{\sqrt{3}}\big) , \tag{4.30}$$

which does *not* coincide with p_0 (cf.). On the other hand, for $X=\partial_4$ we obtain

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N} X^{\alpha;\beta} dS_{\alpha\beta} = 2\pi Q , \tag{4.31}$$

which is twice the Hamiltonian charge p_4.

The equation (4.29) represents also an alternative expression for energy in terms of Riemann tensor.

5 General Asymptotics

In this section we will generalize the results of previous sections to a more general class of space-times which are *not* asymptotically flat. The motivation of doing this was a special case of Rasheed solutions which are not KK asymptotically flat, see Appendix C.

Consider an asymptotic region $\mathscr{S}_{\rm ext} \subset \mathscr{S}$ diffeomorphic to

$$\mathscr{S}_{\rm ext} \approx E(R_0) , \text{ where } E(R) := \big(\mathbb{R}^n \setminus B(R)\big) \times {}^K\mathscr{N} , \tag{5.1}$$

with some K-dimensional compact manifold ${}^K\mathscr{N}$, for some $R_0 \geq 0$. We therefore have an associated global coordinate system x^i on $\mathbb{R}^n \setminus B(R_0)$. We also will use the dilation vector field $Z = x^i \partial_i \equiv r\partial_r$ which will play a key role in some calculation below.

Somewhat more generally, in order to be able to include general "Birmingham-Kottler-Schwarzschild anti-de Sitter" metrics, we will consider ends $E(R)$ equipped with a radial function r so that

$$\mathscr{S}_{\rm ext} \approx E(R_0)\,, \text{ with } E(R) := \{r \geq R\} \equiv [R,\infty) \times \mathscr{K}\,, \tag{5.2}$$

where $\mathscr{K}$ is a compact manifold. Here r is a coordinate running along the $[R_0,\infty)$ factor of $\mathscr{S}_{\rm ext}$, and the dilation vector Z is defined as $Z := r\partial_r$.

For the usual $(n+1)$-dimensional Schwarzschild-anti-de Sitter metric the manifold $\mathscr{K}$ will be an $(n-1)$-dimensional sphere, but it can be an arbitrary compact manifold admitting Einstein metrics in the case of metrics (B.1) in Appendix B.1.

Along $\mathscr{S}_{\rm ext}$ we are given two Lorentzian metrics g and $\overline{g}$, with g asymptotic to the background $\overline{g}$ in a sense which we make precise now. Denoting by $\overline{\nabla}$ the Levi-Civita connection associated with $\overline{g}$, we assume the existence of a $\overline{g}$-orthonormal frame $\{\overline{e}_{\hat\mu}\}$ defined along $\mathscr{S}_{\rm ext}$ such that, decorating frame-indices with hats,

$$g_{\hat\mu\hat\nu} := g(\overline{e}_{\hat\mu}, \overline{e}_{\hat\nu}) = \overline{g}_{\hat\mu\hat\nu} + o\left(r^{-\alpha}\right)\,, \qquad \overline{\nabla}_{\hat\lambda} g_{\hat\mu\hat\nu} = o\left(r^{-\beta}\right)\,. \tag{5.3}$$

It seems that the specific values of α and β as needed for our mass formulae can only be chosen after a case-by-case study of the background metric $\overline{g}$; compare (5.7)-(5.8) below.

In what follows we will use the following convention: given two tensor fields u and v, we will write

$$u = v + o(r^{-\alpha}) \tag{5.4}$$

if the frame components of $u - v$, within the class of $\overline{g}$-ON frames chosen, decay as $o(r^{-\alpha})$. If $\overline{e}_{\hat 0}$ is orthogonal to $\mathscr{S}_{\rm ext}$ (which will often be assumed) then, if we denote by $\overline{g}_{\mathscr{S}} := \overline{g}_{IJ} dx^I dx^J$ the Riemannian metric induced by $\overline{g}$ on $\mathscr{S}_{\rm ext}$, and by $|\cdot|_{\overline{g}_{\mathscr{S}}}$ the associated norm, we have e.g.

$$u_{\mu\nu} = o(r^{-\alpha}) \quad \Longleftrightarrow \quad |u_{\hat0\hat0}| + |u_{\hat0 I} dx^I|_{\overline{g}_{\mathscr{S}}} + |u_{IJ} dx^I dx^J|_{\overline{g}_{\mathscr{S}}} = o(r^{-\alpha})\,.$$

We first assume that X is $\overline{g}$-covariantly constant and therefore the second term of (2.11) vanishes and for the first term we have the same expression as in the KK-asymptotically flat case, with the difference that instead of $\eta_{\mu\nu}$ we have $\overline{g}_{\mu\nu}$ and instead of partial derivatives we have covariant derivatives of the background metric, i.e.,

$$\begin{aligned}
\mathbb{U}^{\nu\lambda} &= \mathbb{U}^{\nu\lambda}{}_{\xi} X^{\xi} \\
&= \left(\frac{3}{8\pi}\delta^{\nu\lambda\sigma}_{\tau\delta\xi}\overline{g}^{\delta\kappa}\overline{g}_{\sigma\gamma}X^{\xi}\overline{\nabla}_{\kappa}g^{\gamma\tau} + o\left(|X|r^{-\alpha-\beta}\right)\right)\sqrt{|\det g|}\,,
\end{aligned} \tag{5.5}$$

where

$$|X|^2 := \sum_{\mu} (X^{\hat\mu})^2\,. \tag{5.6}$$

In order to control the error terms appearing in (5.5) we will assume that

> α and β are such that the subleading terms $o\left(|X|r^{-\alpha-\beta}\right)$ in (5.5) give vanishing contribution to the boundary integrals after passing to the limit. (5.7)

This will e.g. be the case for all Rasheed metrics when $\alpha = (n-2)/2$, $\beta = \alpha + 1$, with X asymptotic to ∂_μ in coordinates as in (C.12).

Similarly (5.7) will be satisfied for asymptotically anti-de Sitter metrics with

$$\alpha=\beta=n/2\,, \tag{5.8}$$

where r is the area coordinate for the anti-de Sitter metric. Note that in this case we have $|X|=O(r)$.

Now, instead of (3.9) one gets in view of the conditions mentioned above

$$H_b\left(X,\mathscr{S}\right)=\frac{3}{16\pi}\lim_{R\to\infty}\int_{\partial E(R)}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\mu}dS_{\alpha\beta}\,. \tag{5.9}$$

We can now compute the Hamiltonian charges for this general case. We have

$$\begin{aligned}
&\frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\mu}dS_{\alpha\beta}\\
&\quad=\frac{1}{16\pi}\left(\delta^{\alpha\beta}_{\lambda\mu}\delta^\gamma_\nu+\delta^{\alpha\beta}_{\mu\nu}\delta^\gamma_\lambda+\delta^{\alpha\beta}_{\nu\lambda}\delta^\gamma_\mu\right)X^\nu\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\mu}dS_{\alpha\beta}\\
&\quad=\frac{1}{16\pi}\left(X^\gamma\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\mu}dS_{\lambda\mu}+X^\nu\overline{\nabla}_\sigma g^{\sigma\mu}dS_{\mu\nu}+X^\nu\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\gamma}dS_{\nu\lambda}\right)\,.
\end{aligned} \tag{5.10}$$

Now, in order to simplify the calculations, we use an $\overline{g}$-orthonormal frame $\{\overline{e}_{\hat\mu}\}$ with $\overline{e}_{\hat 0}$ orthogonal to $\mathscr{S}$ and $\overline{e}_{\hat A}$ tangent to $\partial E(R)$. Then, only the forms $dS_{\hat 0\hat i}$ give a non-vanishing contribution to the boundary integral.

If $X=\partial_0$, and assuming that

$$\partial_0=X^{\hat 0}\overline{e}_{\hat 0} \tag{5.11}$$

one finds, using frame indices throughout the calculation,

$$\begin{aligned}
\frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^\nu\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_\rho g^{\sigma\mu}dS_{\alpha\beta}&=\frac{3}{8\pi}X^{\hat 0}\delta^{\hat 0\hat k\hat J}_{\hat\lambda\hat\mu\hat 0}\overline{g}^{\hat\lambda\hat\rho}\overline{g}_{\hat J\hat\nu}\overline{\nabla}_{\hat\rho}g^{\hat\nu\hat\mu}dS_{\hat 0\hat k}\\
&=\frac{1}{16\pi}X^{\hat 0}\left[\overline{\nabla}^{\hat k}\left(\overline{g}_{\hat J\hat L}g^{\hat J\hat L}\right)-\overline{\nabla}_{\hat J}g^{\hat J\hat k}\right]d\sigma_{\hat k}+\text{n.c.}\,,
\end{aligned} \tag{5.12}$$

where

$$d\sigma_{\hat i}:=dS_{\hat 0\hat i}\equiv\overline{e}_{\hat i}\rfloor\overline{e}_{\hat 0}\rfloor(\sqrt{|\det g|}dx^0\wedge\cdots\wedge dx^{n+K})=\overline{e}_{\hat i}\rfloor(\sqrt{|\det g_{IJ}|}dx^1\wedge\cdots\wedge dx^{n+K})+\text{n.c.} \tag{5.13}$$

Hence, we obtain the following generalization of the ADM mass:

$$p_0=H_b\left(\partial_t,\mathscr{S}\right)=\frac{1}{16\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^{\hat 0}\left[\overline{\nabla}^i\left(\overline{g}_{JK}g^{JK}\right)-\overline{\nabla}_J g^{Ji}\right]d\sigma_i\,. \tag{5.14}$$

Existence of the limit in (5.14) will be guaranteed if, instead of (4.5)-(4.6), one assumes now, e.g.,

$$\int_{\mathscr{S}\cap\{r\ge R\}}|X|\Big(\sum_{\hat\mu\hat\alpha\hat\beta}|\overline{\nabla}_{\hat\mu}g_{\hat\alpha\hat\beta}|^2+\sum_{\hat\alpha\hat\beta}|T_{\hat\alpha\hat\beta}|+|\Lambda||g^{\mu\nu}(g_{\mu\nu}-\overline{g}_{\mu\nu})|\Big)d\mu_{\overline{g}_{\mathscr{S}}}<\infty\,, \tag{5.15}$$

where $d\mu_{\overline{g}_{\mathscr{S}}}$ is the $(n+K)$-dimensional Riemannian measure induced on $\mathscr{S}$ by $\overline{g}$. A condition on the metric and the energy-momentum tensor of matter fields naturally associated with (5.15) is

$$\lim_{R\to\infty}|X||\partial E(R)|T_{\hat\mu\hat\nu}=0\,,\qquad\lim_{R\to\infty}|X||\partial E(R)||\Lambda||g^{\mu\nu}(g_{\mu\nu}-\overline{g}_{\mu\nu})|=0\,, \tag{5.16}$$

where $|\partial E(R)|$ denotes the area of $\partial E(R)$; compare (4.6).

As an example, we consider the Rasheed metrics of Appendix C with $P \neq 0$, which are vacuum. The $\overline{g}$-Killing vector $X = \partial_t$ is $\overline{g}$-covariantly constant so that (5.14) applies. The asymptotic behaviour of the metric coefficients in the frame (C.18) coincides with the asymptotic behaviour of the metric coefficients in manifestly asymptotically Minkowskian coordinates when $P = 0$, and is given by (C.12). One obtains

$$p_0 = 4\pi P M \,, \tag{5.17}$$

where the extra factor $4P$, as compared to (4.15), is due to the $8P\pi$-periodicity of the coordinate x^4 (cf. (C.16)), as enforced by the requirement of smoothness of the metric. Note that the formulae (5.17) for the ADM four-momentum remain unchanged.

We now generalize our results to the case where X is not $\overline{g}$-covariantly constant. Thus, the second term of (2.11) does not vanish. Disregarding those terms which do *not* involve the forms $d\Sigma_{0i}$, we obtain, keeping in mind that X is a Killing vector field of $\overline{g}$,

$$\begin{aligned}
\frac{1}{2}(U^{\alpha\beta} - U^{\alpha\beta}\big|_{g=\overline{g}})d\Sigma_{\alpha\beta} &= \frac{1}{2}\big(\mathbb{U}^{\alpha\beta}{}_{\lambda}X^{\lambda} - \frac{1}{8\pi}\sqrt{|\det g|}g^{\mu[\alpha}\delta^{\beta]}_{\nu}\overline{\nabla}_{\mu}X^{\nu} - U^{\alpha\beta}\big|_{g=\overline{g}}\big)d\Sigma_{\alpha\beta} \\
&= \frac{3}{16\pi}\delta^{\alpha\beta\gamma}_{\lambda\mu\nu}X^{\nu}\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_{\rho}g^{\sigma\mu}dS_{\alpha\beta} - \frac{1}{16\pi}\big(\sqrt{|\det g|}g^{\mu[\alpha} - \sqrt{|\det \overline{g}|}\overline{g}^{\mu[\alpha}\big)\overline{\nabla}_{\mu}X^{\beta]}d\Sigma_{\alpha\beta} + \text{n.c.} \\
&= \frac{1}{8\pi}\Big(3\delta^{0i\gamma}_{\lambda\mu\nu}X^{\nu}\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_{\rho}g^{\sigma\mu} - \big(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\big)\overline{\nabla}_{\mu}X^{i]}\Big)N d\sigma_i + \text{n.c.}
\end{aligned}$$

Here we have used $dS_{0i} = N d\sigma_i$, where N is the lapse function of the foliation by the level sets of t, defined by writing the metric as $g = -N^2dt^2 + g_{IJ}(dx^I + N^I dt)(dx^J + N^J dt)$. We conclude that

$$H_b\left(X, \mathscr{S}\right) = \frac{1}{8\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big(3\delta^{0i\gamma}_{\lambda\mu\nu}X^{\nu}\overline{g}^{\lambda\rho}\overline{g}_{\gamma\sigma}\overline{\nabla}_{\rho}g^{\sigma\mu} - \big(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\big)\overline{\nabla}_{\mu}X^{i]}\Big)N d\sigma_i \,. \tag{5.18}$$

We can apply the last formula to the background Killing vectors ∂_i and ∂_4 for Rasheed metrics with $P \neq 0$. A calculation gives

$$p_i = 0\,, \quad p_4 = 4\pi P Q\,. \tag{5.19}$$

Here one can note that ∂_z is $\overline{g}$-covariantly constant so that the last term in (5.18) does certainly not contribute; while $p_x = p_y = 0$ follows from the axi-symmetry of the Rasheed metrics. (In fact $\overline{\nabla}X = O(r^{-2})$ or better for these Killing vectors so that the last term never contributes in the current case.)

Equation (5.18) applies for completely general background metrics $\overline{g}$, assuming that (5.15) and (5.7) hold, for a large class of field equations. In particular, it applies to asymptotically Kottler ("anti-de Sitter") metrics, compare [14, 15, 16, 22].

5.1 Hamiltonian Charges

Assuming that X is a $\overline{g}$-Killing vector field, (5.9) becomes for general background metrics

$$H_b(X,\mathscr{S}) = \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!}\lim_{R\to\infty}\int_{\partial E(R)} \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n+K-2}}X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\,dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n+K-2}}\wedge dx^\gamma\,, \tag{5.20}$$

where

$$\delta\Gamma^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \overline{\Gamma}^\alpha{}_{\beta\gamma} = o(r^{-\beta})\,, \tag{5.21}$$

with the last equality following from (5.3).

In order to obtain a version of (3.23) suitable to the current setting we will assume that there exists a vector field Z with $Z^A=0$ and a real number $\gamma>0$ such that

$$\overline{\nabla}_a Z^b = \delta^b_a + O(r^{-\gamma}) \quad \mathrm{mod}\ (\delta^r_\mu, \delta^t_\mu)\,, \tag{5.22}$$

where we used "mod $(\delta^r_\mu,\delta^t_\mu)$" for a tensor which has the form $\delta^r_\mu\mathring{\alpha}_{\dots}+\delta^t_\mu\mathring{\beta}_{\dots}$ for some tensor fields $\mathring{\alpha}$, and $\mathring{\beta}$. In other words, if X is a vector field tangent to the submanifolds of constant t, r, and if "$u_{\mu\dots}=0\ \ \mathrm{mod}\ (\delta^r_\mu,\delta^t_\mu)$", then $X^\mu u_{\mu\dots}=0$.

As shown in Appendix B, the vector field defined in appropriate coordinates as

$$Z = r\partial_r \tag{5.23}$$

satisfies a) (5.22) for asymptotically anti-de Sitter metrics, and b) for general Rasheed metrics, in both cases without the error term $O(r^{-\gamma})$; equivalently, the exponent γ can be taken as large as desired. We have introduced the $O(r^{-\gamma})$ term for possible future generalizations.

We further assume that

$$\overline{\nabla}_\mu X^\nu = O(|X|r^{-\beta}) \quad \mathrm{mod}\ (\delta^r_\mu,\delta^t_\mu)\,, \tag{5.24}$$

which will certainly be the case if X is $\overline{g}$-covariantly constant. And finally, we replace (5.7) by the requirement that

> terms $o\left(|X|r^{-\alpha-\beta}\right)$, $o\left(|Z||X|r^{-2\beta}\right)$ and $o\left(|X|r^{-\beta-\gamma}\right)$ give vanishing contribution to boundary integrals at fixed r and t, after passing to the limit $r\to\infty$. (5.25)

We note that under the condition (5.25) one gets (5.20) from (5.9) for a vector field X which is not $\overline{g}$-covariantly constant if it satisfies (5.24).

We study two different cases in details.

5.1.1 $\Lambda\neq 0$ and $K=0$

In the first case we assume that $\Lambda\neq0$ and $K=0$. We first derive an identity which will be integrated on submanifolds of fixed r and t, so that any forms containing a factor dr or dt will give zero contribution.

$$
\begin{aligned}
& d\left(\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\right)\\
&= \overline{\nabla}_\sigma\left(\sqrt{|\det g|}\mathring{\epsilon}_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\right) dx^\sigma\wedge dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\\
&= Z^{\xi_1}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\underbrace{\overline{\nabla}_\sigma X^\nu dx^\sigma}_{\text{n.c.}}\wedge dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\\
&\quad +\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\underbrace{\overline{\nabla}_\sigma Z^{\xi_1}\, dx^\sigma}_{dx^{\xi_1}+\text{n.c.}}\wedge dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\\
&\quad +(-1)^{n-3}\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho} Z^{\xi_1} dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge \underbrace{\left(\overline{\nabla}_\sigma\delta\Gamma^\mu{}_{\gamma\rho} dx^\sigma\wedge dx^\gamma\right)}_{=\left(\frac12\delta R^\mu{}_{\rho\sigma\gamma}+o(r^{-2\beta})\right)\ dx^\sigma\wedge dx^\gamma} +\text{n.c.}\\
&= \epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\gamma\\
&\quad +(-1)^{n-3}\frac12\epsilon_{\lambda\mu\nu\xi_1\cdots\xi_{n-2}} X^\nu g^{\lambda\rho}\delta R^\mu{}_{\rho\sigma\gamma}\, Z^{\xi_1} dx^{\xi_2}\wedge\cdots\wedge dx^{\xi_{n-2}}\wedge dx^\sigma\wedge dx^\gamma+\text{n.c.} \qquad (5.26)
\end{aligned}
$$

This identity replaces (3.23) in the current setting. One can now repeat the remaining calculations of section 4 by replacing every occurrence of the Christoffel symbols by the difference of those of g and $\overline{g}$, every appearance of the Riemann tensor by the difference of the Riemann tensors of g and $\overline{g}$, and every occurrence of an undifferentiated x^α by Z^α. Some care must be taken when generalizing (4.23) when passing from the background Riemann tensor to the background Ricci tensor because in (4.23) all indices are lowered and raised with g. Thus, (A.1) is replaced now by

$$
\begin{aligned}
3!\delta^{\sigma\gamma\alpha\beta}_{\lambda\mu\nu\xi}\left(\mathbf{R}^\mu{}_{\rho\sigma\gamma}-\overline{\mathbf{R}}^\mu{}_{\rho\sigma\gamma}\right)g^{\lambda\rho} &= \left(\mathbf{R}^{\alpha\beta}{}_{\xi\nu}-\overline{\mathbf{R}}^{[\alpha}{}_{\rho\xi\nu}g^{\beta]\rho}\right)+\left(\mathbf{R}-\overline{\mathbf{R}}_{\rho\lambda}g^{\rho\lambda}\right)\delta^{\alpha\beta}_{\xi\nu}\\
&\quad -4\delta^{[\alpha}_{[\xi}\mathbf{R}^{\beta]}{}_{\nu]}+2\delta^{[\alpha}_{[\xi}g^{\beta]\rho}\overline{\mathbf{R}}_{\nu]\rho}-2\overline{\mathbf{R}}^{[\alpha}{}_{\rho\lambda[\xi}\delta^{\beta]}_{\nu]}g^{\rho\lambda}\,. \qquad (5.27)
\end{aligned}
$$

The simplest situation is obtained when $K=0$ so that ${}^K\mathscr{N}$ is reduced to a point, and (5.18) becomes

$$
\begin{aligned}
H_b(X,\mathscr{S}) &= \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{S(R)}\Big\{X^\nu Z^\xi\left(R^{0i}{}_{\nu\xi}-\overline{R}^{[0}{}_{\rho\nu\xi}g^{i]\rho}\right)\\
&\quad +X^{[0}Z^{i]}\left(R-\overline{R}_{\rho\lambda}g^{\rho\lambda}\right)+2X^\nu Z^{[0}R^{i]}{}_\nu-2Z^\nu X^{[0}R^{i]}{}_\nu+(Z^\nu X^{[0}g^{i]\rho}-X^\nu Z^{[0}g^{i]\rho})\overline{R}_{\nu\rho}\\
&\quad -X^{[\nu}Z^{i]}\overline{R}^0{}_{\rho\lambda\nu}g^{\rho\lambda}+X^{[\nu}Z^{0]}\overline{R}^i{}_{\rho\lambda\nu}g^{\rho\lambda}-(n-2)\left(g^{\mu[0}-e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]}\Big)\Big\}N d\sigma_i\,. \qquad (5.28)
\end{aligned}
$$

We want to analyze (5.28) for metrics g which asymptote a maximally symmetric background $\overline{g}$ with $\Lambda\neq 0$. This case requires separate attention as then the background curvature tensor does not approach zero as we recede to infinity. We note that the calculations in this section are formally correct independently of the sign of Λ, but to the best of our knowledge they are only relevant in the case $\Lambda<0$.

It is useful to decompose the Riemann tensor into its irreducible components,

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= W_{\alpha\beta\gamma\delta}+\frac{1}{d-2}(R_{\alpha\gamma}g_{\beta\delta}-R_{\alpha\delta}g_{\beta\gamma}+R_{\beta\delta}g_{\alpha\gamma}-R_{\beta\gamma}g_{\alpha\delta})\\
&\quad -\frac{R}{(d-1)(d-2)}(g_{\beta\delta}g_{\alpha\gamma}-g_{\beta\gamma}g_{\alpha\delta})\\
&= W_{\alpha\beta\gamma\delta}+\frac{1}{d-2}(P_{\alpha\gamma}g_{\beta\delta}-P_{\alpha\delta}g_{\beta\gamma}+P_{\beta\delta}g_{\alpha\gamma}-P_{\beta\gamma}g_{\alpha\delta})+\frac{R}{d(d-1)}(g_{\beta\delta}g_{\alpha\gamma}-g_{\beta\gamma}g_{\alpha\delta})\,,
\end{aligned}
$$

where $W_{\alpha\beta\gamma\delta}$ is the Weyl tensor and $P_{\alpha\beta}$ the trace-free part of the Ricci tensor,

$$
P_{\mu\nu}=R_{\mu\nu}-\frac{R}{d}g_{\mu\nu}\,.
$$

This leads to the following rewriting of (5.28)

$$\begin{aligned}H_b(X,\mathscr{S}) &= \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi\Big(W^{0i}{}_{\nu\xi} - \overline{W}^{[0}{}_{\rho\nu\xi}g^{i]\rho} + \frac{2R}{n(n+1)}\delta^{[0}_{[\nu}\delta^{i]}_{\xi]} \\ &\quad -\frac{2\overline{R}}{n(n+1)}\delta^{[0}_{[\nu}\overline{g}_{\xi]\rho}g^{i]\rho}\Big) + X^{[0}Z^{i]}\left(R-\overline{R}_{\rho\lambda}g^{\rho\lambda}\right) + 2X^\nu Z^{[0}R^{i]}{}_\nu \\ &\quad -2Z^\nu X^{[0}R^{i]}{}_\nu + (Z^\nu X^{[0}g^{i]\rho} - X^\nu Z^{[0}g^{i]\rho})\overline{R}_{\nu\rho} \\ &\quad -\frac{2\overline{R}}{n(n+1)}\big(X^{[\nu}Z^{i]}\delta^0_{[\lambda}\overline{g}_{\nu]\rho} - X^{[\nu}Z^{0]}\delta^i_{[\lambda}\overline{g}_{\nu]\rho}\big)g^{\rho\lambda} \\ &\quad -(n-2)\big(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\big)\overline{\nabla}_\mu X^{i]}\Big\}Nd\sigma_i\,.\end{aligned} \tag{5.29}$$

Assuming that the background Weyl tensor falls-off sufficiently fast so that it does not contribute to the integrals (e.g., vanishes, when the background is a space-form such as the anti-de Sitter metric), and that both the energy-momentum tensor of matter and $e-1$ decay fast enough (cf. (5.16)), and setting

$$\Delta^{\mu\nu} := g^{\mu\nu} - \overline{g}^{\mu\nu}\,,$$

we obtain

$$\begin{aligned}H_b(X,\mathscr{S}) = \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi\Big(W^{0i}{}_{\nu\xi} - \frac{2\overline{R}}{n(n+1)}\delta^{[0}_{[\nu}\overline{g}_{\xi]\rho}\Delta^{i]\rho}\Big) \\ -X^{[0}Z^{i]}\overline{R}_{\rho\lambda}\Delta^{\rho\lambda} + (Z^\nu X^{[0}\Delta^{i]\rho} - X^\nu Z^{[0}\Delta^{i]\rho})\overline{R}_{\nu\rho} \\ -\frac{2\overline{R}}{n(n+1)}\big(X^{[\nu}Z^{i]}\delta^0_{[\lambda}\overline{g}_{\nu]\rho} - X^{[\nu}Z^{0]}\delta^i_{[\lambda}\overline{g}_{\nu]\rho}\big)\Delta^{\rho\lambda} - (n-2)\Delta^{\mu[0}\overline{\nabla}_\mu X^{i]}\Big\}Nd\sigma_i\,.\end{aligned} \tag{5.30}$$

where we have also used the hypothesis (5.7) that terms such as $|X||Z|\Delta^{\mu\nu}\Delta_{\rho\sigma}$ and $|X||Z|g_{\mu\nu}\Delta^{\mu\nu}$ fall off fast enough so that they give no contribution to the integral in the limit. With some further work one gets

$$\begin{aligned}H_b(X,\mathscr{S}) &= \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}\Big\{X^\nu Z^\xi W^{0i}{}_{\nu\xi} \\ &\quad +(n-2)\Delta^{\mu[0}\left[\frac{\overline{R}}{n(n+1)}\left(X_\mu Z^{i]} - Z_\mu X^{i]}\right) - \overline{\nabla}_\mu X^{i]}\right]\Big\}Nd\sigma_i\,.\end{aligned} \tag{5.31}$$

To continue, we assume the Birmingham-Kottler form (B.1)-(B.3) of the background metric $\overline{g}$. If X is the $\overline{g}$-Killing vector field ∂_t then, writing momentarily X_ν for $\overline{g}_{\nu\mu}X^\nu$

$$\begin{aligned}\overline{\nabla}_\sigma X_\nu dx^\sigma\otimes dx^\nu &= \overline{\nabla}_{[\sigma}X_{\nu]}dx^\sigma\otimes dx^\nu \\ &= \partial_{[\sigma}X_{\nu]}dx^\sigma\otimes dx^\nu \\ &= \partial_{[\sigma}\overline{g}_{\nu]0}dx^\sigma\otimes dx^\nu \\ &= \frac{1}{2}\partial_r\overline{g}_{00}dx^r\wedge dx^0 \\ &= \frac{1}{2}\partial_r\overline{g}_{00}\overline{\Theta}^1\wedge\overline{\Theta}^0\,.\end{aligned} \tag{5.32}$$

Using this one checks that all terms linear in Δ in (5.31) cancel out, leading to the elegant formulae

$$\begin{aligned}H_b(X,\mathscr{S}) &= \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^\nu Z^\xi W^{0i}{}_{\nu\xi}\,Nd\sigma_i \\ &= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)}X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi}\,dS_{\alpha\beta}\,,\end{aligned} \tag{5.33}$$

which, at this stage, hold for all X belonging to the $(n+1)$-dimensional family of Killing vectors of the anti-de Sitter background which are normal to $\{t=0\}$.

If $X=\partial_\varphi$, then we have

$$\begin{aligned}\overline{\nabla}_\sigma X_\nu dx^\sigma\otimes dx^\nu &= \overline{\nabla}_{[\sigma}X_{\nu]}dx^\sigma\otimes dx^\nu\\ &= \frac{1}{2}\partial_\sigma\overline{g}_{\nu\varphi}dx^\sigma\wedge dx^\nu\\ &= \frac{1}{2}\partial_r\overline{g}_{\varphi\varphi}dr\wedge d\varphi+\frac{1}{2}\partial_\theta\overline{g}_{\varphi\varphi}d\theta\wedge d\varphi\\ &= \sqrt{V}\sin\theta\,\overline{\Theta}^1\wedge\overline{\Theta}^3+\cos\theta\,\overline{\Theta}^2\wedge\overline{\Theta}^3\,,\end{aligned}$$

where we used the co-frame of the background metric (B.1) with the following co-basis

$$\overline{\Theta}^0=\sqrt{V}dt\,,\quad \overline{\Theta}^1=\frac{1}{\sqrt{V}}dr\,,\quad \overline{\Theta}^2=rd\theta\,,\quad \overline{\Theta}^3=r\sin\theta d\varphi\,. \tag{5.34}$$

Hence, in this co-frame one obtains

$$\overline{\nabla}_{\hat{1}}X_{\hat{3}}=\sqrt{V}\sin\theta=-\overline{\nabla}_{\hat{3}}X_{\hat{1}}\,,\qquad \overline{\nabla}_{\hat{2}}X_{\hat{3}}=\cos\theta=-\overline{\nabla}_{\hat{3}}X_{\hat{2}}\,.$$

Therefore, the second term of the integrand in (5.31) vanishes for $r\longrightarrow\infty$, since (keeping in mind that $dS_{\hat{0}\hat{i}}$ for $i\neq 1$ gives zero contribution to the integrals)

$$\begin{aligned}&\frac{\overline{R}}{n(n+1)}\Delta^{\mu[0}\left(X_\mu Z^{1]}-Z_\mu X^{1]}\right)-\Delta^{\mu[0}\overline{\nabla}_\mu X^{1]}=-\frac{\lambda}{2}\Delta^{\hat{\mu}\hat{0}}X_{\hat{\mu}}Z^{\hat{1}}-\frac{1}{2}\Delta^{\hat{\mu}\hat{0}}\overline{\nabla}_{\hat{\mu}}X^{\hat{1}}\\ &=-\frac{\lambda}{2}\Delta^{\hat{3}\hat{0}}\underbrace{X_{\hat{3}}}_{=(\overline{g}_{\hat{3}\hat{3}}+\Delta_{\hat{3}\hat{3}})X^{\hat{3}}}Z^{\hat{1}}-\frac{1}{2}\Delta^{\hat{3}\hat{0}}\overline{\nabla}_{\hat{3}}X_{\hat{1}}=\left(-\frac{\lambda}{2}\cdot\frac{r^2}{\sqrt{V}}+\frac{1}{2}\sqrt{V}\right)\sin\theta\Delta^{\hat{3}\hat{0}}\xrightarrow{r\to\infty}0\,.\end{aligned}$$

Hence (5.33) also holds for $X=\partial_\varphi$. Since all Killing vectors of AdS space-time can be obtained as linear combinations of images of these two vectors by isometries preserving $\{t=0\}$, we conclude that (5.33) holds for all Killing vectors of the AdS metric.

It bears mentioning that (5.33) has already been observed in [5], following-up on the pioneering definitions in [2, 4]. We note that our conditions for the equality in (5.33) are quite weaker than those in [5].

5.1.2 $\Lambda=0$ and $K\neq 0$

In the second case we assume that $\Lambda=0$ and $K\neq 0$. We will impose conditions which guarantee that all terms which are quadratic or higher in $g_{\mu\nu}-\overline{g}_{\mu\nu}$ give zero contribution to the integrals in the limit $R\to\infty$. Without these assumptions the final formulae become unreasonably long. Hence we assume (5.21)-(5.22), (5.24) and (5.25).

In the current context, the calculation (4.19) is replaced by

$$
\begin{aligned}
&\epsilon_{\lambda\mu\nu\xi_1\ldots\xi_{d+K-3}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{\gamma\rho}\, dx^{\xi_1}\wedge\cdots\wedge dx^{\xi_{d+K-3}}\wedge dx^\gamma \\
&\quad= \frac{(d+K-3)!}{(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}\delta\Gamma^c{}_{ae} \underbrace{dx^{a_1}}_{\overline{\nabla}_h Z^{a_1}dx^h+\text{n.c.}=\delta^{a_1}_h dx^h+\text{n.c.}} \wedge\cdots\wedge dx^{A_K}\wedge dx^a \\
&\qquad+\frac{(d+K-3)!}{(d-2)!(N-1)!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho}\, dx^{a_1}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\quad= \frac{(d+K-3)!}{(d-3)!N!}\Big[\overline{\nabla}_h(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}\delta\Gamma^c{}_{ae} Z^{a_1}) dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^a \\
&\qquad-\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} \underbrace{\overline{\nabla}_h X^f}_{\text{n.c.}} g^{be}\delta\Gamma^c{}_{ae} Z^{a_1} dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^a \\
&\qquad-\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be} Z^{a_1}\overline{\nabla}_h\delta\Gamma^c{}_{ae}\, dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^a+\text{n.c.}\Big] \\
&\qquad+\frac{(d+K-3)!}{(d-2)!(N-1)!}\Big[\overline{\nabla}_h(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} Z^{a_1}) dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\qquad-\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} \underbrace{\overline{\nabla}_h X^\nu}_{\text{n.c.}} g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} Z^{a_1} dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\qquad-\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho} Z^{a_1}\overline{\nabla}_h\delta\Gamma^\mu{}_{A\rho} dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A+\text{n.c.}\Big] \\
&\quad= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f g^{be}\delta\Gamma^c{}_{ae} Z^{a_1} dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^a) \\
&\qquad \frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu g^{\lambda\rho}\delta\Gamma^\mu{}_{A\rho} Z^{a_1} dx^h\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A) \\
&\qquad-\frac{(d+K-3)!}{(d-3)!N!}\epsilon_{bcfa_1\ldots a_{d-3}A_1\ldots A_K} X^f x^{a_1} g^{be}\delta\mathbf{R}^c{}_{ea_{d-1}a_{d-2}} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_K}\wedge dx^{a_{d-2}} \\
&\qquad-\frac{(d+K-3)!}{2(d-3)!N!}\epsilon_{\lambda\mu\nu a_1\ldots a_{d-2}A_1\ldots A_{K-1}} X^\nu x^{a_1} g^{\lambda\rho}\delta\mathbf{R}^\mu{}_{\rho a_{d-1}A} dx^{a_{d-1}}\wedge dx^{a_2}\wedge\cdots\wedge dx^{A_{K-1}}\wedge dx^A \\
&\qquad+\text{n.c.}
\end{aligned}
$$

As before, in the last equality we have used the fact that the first $\overline{\nabla}_h$-terms in the first expression in each of the square brackets can be replaced by $\overline{\nabla}_\mu$, because each form appearing in the first line above must already contain K-factors of the KK differentials dx^A, otherwise it will give zero contribution to the integral.

In addition to all the hypotheses so far we will also assume that the Riemann tensor decays at a rate $o(r^{-\beta_R})$:

$$
\mathbf{R}^\alpha{}_{\beta\gamma\delta}=o(r^{-\beta_R})\,,\quad \overline{\mathbf{R}}^\alpha{}_{\beta\gamma\delta}=o(r^{-\beta_R})\,, \tag{5.35}
$$

with β_R satisfying

$$
\text{terms } |X||Z|o(r^{-\alpha-\beta_R}) \text{ give no contribution to the integral in the limit } R\to\infty. \tag{5.36}
$$

All these conditions are satisfied by the five-dimensional Rasheed metrics, with $\alpha>0$ as close to one as one wishes, $\beta=1+\alpha$, $\beta_R=3$, with γ as large as desired.

In line with our previous notation, we will write $\mathbf{R}^\alpha{}_{\beta\gamma\delta}-\overline{\mathbf{R}}^\alpha{}_{\beta\gamma\delta}$ for the difference of Riemann tensors of the $(d+K)$-dimensional metrics $g_{\mu\nu}dx^\mu dx^\nu$ and $\overline{g}_{\mu\nu}dx^\mu dx^\nu$, $R^a{}_{bcd}-\overline{R}^a{}_{bcd}$

for that of the d-dimensional metrics $g_{ab}dx^a dx^b$ and $\overline{g}_{ab}dx^a dx^b$, $\mathsf{R}^I{}_{JKL} - \overline{\mathsf{R}}^I{}_{JKL}$ for the $(n+K)$-dimensional metrics $g_{IJ}dx^I dx^J$ and $\overline{g}_{IJ}dx^I dx^J$, and $\mathcal{R}^i{}_{jk\ell} - \overline{\mathcal{R}}^i{}_{jk\ell}$ for that of the n-dimensional metrics $g_{ij}dx^i dx^j$ and $\overline{g}_{ij}dx^i dx^j$.

With the above hypotheses, the derivation of the key formula (4.23) follows closely the remaining calculations in section 4, and leads to

$$\begin{aligned} H_b\left(X, \mathscr{S}\right) = & \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \Bigg\{ \int_{\partial E(R)} \Bigg(X^a Z^b (\mathbf{R}_{ab}{}^{ef} - \overline{\mathbf{R}}_{ab}{}^{ef}) + 4Z^{[e}X^{a]} \left(\mathbf{R}^f{}_a - \overline{\mathbf{R}}^f{}_a\right) \\ & - Z^e X^f \left(\mathbf{R}^c{}_c - \overline{\mathbf{R}}^c{}_c\right) - \frac{1}{n-1}\Big[(n-3)Z^e X^f (\mathbf{R}^{bA}{}_{bA} - \overline{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e}X^{a]}(\mathbf{R}^{fA}{}_{aA} - \overline{\mathbf{R}}^{fA}{}_{aA}) \\ & + 3(n-2)X^A Z^e (\mathbf{R}^{fb}{}_{bA} - \overline{\mathbf{R}}^{fb}{}_{bA})\Big]\Bigg) dS_{ef} - (n-2)\int_{\partial E(R)} \left(g^{\mu[a} - e^{-1}\overline{g}^{\mu[a}\right)\overline{\nabla}_\mu X^{b]} dS_{ab} \Bigg\} . \end{aligned} \tag{5.37}$$

For Rasheed solutions, or more generally for solutions which asymptote to the Rasheed backgrounds $\overline{g}$ given by (C.13) with the usual decay $o(r^{-(n-2)/2})$, with $T_{\mu\nu} = o(r^{-3})$, one has (cf. (C.22)-(C.23)) $\overline{\mathbf{R}}^0{}_{\lambda\mu\nu} = 0$, $\overline{\mathbf{R}}^{\hat\mu}{}_{\hat 4\hat\alpha\hat 4} = O(r^{-4})$, $\overline{\mathbf{R}}_{\mu\nu} = O(r^{-4})$ and we obtain, for $X = \partial_\mu$ and after passing to the limit $R \to \infty$, an integrand which is formally identical to that for metrics which are KK-asymptotically flat:

$$\begin{aligned} H_b\left(X, \mathscr{S}\right) &= \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{S(R)\times S^1} X^\nu Z^\mu \mathbf{R}^{\alpha\beta}{}_{\nu\mu} dS_{\alpha\beta} \\ &= \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \int_{S(R)\times S^1} X^\nu x^j \mathbf{R}^{0i}{}_{\nu j} N d\sigma_i . \end{aligned}$$

Some special cases, without necessarily assuming that g asymptotes to the Rasheed background, are of interest:

1. Suppose that $X^\mu = \delta^\mu_0$, thus X has just a time component. Keeping in mind that $Z^0 = 0$ and $\partial E(R) \subset \{x^0 = 0\}$ we have

$$\begin{aligned} H_b\left(\partial_0, \mathscr{S}\right) &= \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \Bigg\{ \int_{\partial E(R)} \Bigg(Z^j \left(\mathbf{R}_{0j}{}^{0i} - \overline{\mathbf{R}}_{0j}{}^{0i}\right) \\ & + \frac{1}{2} Z^i \left(\mathbf{R}^j{}_j - \mathbf{R}^0{}_0 + \overline{\mathbf{R}}^j{}_j - \overline{\mathbf{R}}^0{}_0\right) - Z^j \left(\mathbf{R}^i{}_j - \overline{\mathbf{R}}^i{}_j\right) \\ & - \frac{1}{2(n-1)}\Big[Z^i(\mathbf{R}^{0A}{}_{0A} - \overline{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j(\mathbf{R}^{iA}{}_{jA} - \overline{\mathbf{R}}^{iA}{}_{jA})\Big]\Bigg) dS_{0i} \\ & - (n-2)\int_{\partial E(R)} \left(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]} dS_{0i} \Bigg\} \\ &= \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \Bigg\{ \int_{\partial E(R)} \Bigg(\frac{1}{2} Z^i \left(\mathbf{R}^j{}_j - \overline{\mathbf{R}}^j{}_j\right) - Z^j \left(\mathbf{R}_{Ij}{}^{Ii} - \overline{\mathbf{R}}_{Ij}{}^{Ii}\right) \\ & - \frac{1}{2(n-1)}\Big[Z^i(\mathbf{R}^{0A}{}_{0A} - \overline{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j(\mathbf{R}^{iA}{}_{jA} - \overline{\mathbf{R}}^{iA}{}_{jA})\Big]\Bigg) dS_{0i} \\ & - (n-2)\int_{\partial E(R)} \left(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]} dS_{0i} \Bigg\} . \end{aligned} \tag{5.38}$$

2. Suppose that $X^A=0$, thus X has only space-time components. Then

$$
\begin{aligned}
H_b\left(X,\mathscr{S}\right) &= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\left\{\int_{\partial E(R)}\left(X^aZ^b({\mathbf{R}_{ab}}^{ef}-{\overline{\mathbf{R}}_{ab}}^{ef})\right.\right.\\
&\quad +4Z^{[e}X^{a]}\left({\mathbf{R}^f}_a-{\overline{\mathbf{R}}^f}_a\right)-Z^eX^f\left({\mathbf{R}^c}_c-{\overline{\mathbf{R}}^c}_c\right)\\
&\quad \left.-\frac{1}{n-1}\Big[(n-3)Z^eX^f({\mathbf{R}^{bA}}_{bA}-{\overline{\mathbf{R}}^{bA}}_{bA})+4Z^{[e}X^{a]}({\mathbf{R}^{fA}}_{aA}-{\overline{\mathbf{R}}^{fA}}_{aA})\Big]\right)dS_{ef}\\
&\quad \left.-2(n-2)\int_{\partial E(R)}\left(g^{\mu[0}-e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]}dS_{0i}\right\}. \qquad (5.39)
\end{aligned}
$$

We will see below that the first term in the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when $X^0=0$, at $t=0$ the third and fourth terms in the integrand in (5.39) give a vanishing contribution so that the generators of space-translations read

$$
\begin{aligned}
H_b\left(X,\mathscr{S}\right) &= \frac{1}{8(n-2)\pi}\lim_{R\to\infty}\left\{\int_{\partial E(R)}\left(X^iZ^k\left({\mathbf{R}_{ik}}^{0j}-{\overline{\mathbf{R}}_{ik}}^{0j}\right)\right.\right.\\
&\quad \left.+Z^{[i}X^{j]}\left[\frac{2}{n-1}\left({\mathbf{R}^{0A}}_{iA}-{\overline{\mathbf{R}}^{0A}}_{iA}\right)+{\mathbf{R}^0}_i-{\overline{\mathbf{R}}^0}_i\right]\right)dS_{0j}\\
&\quad \left.-(n-2)\int_{\partial E(R)}\left(g^{\mu[0}-e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]}dS_{0i}\right\}. \qquad (5.40)
\end{aligned}
$$

3. Suppose instead that $X^a=0$, thus X has only components tangential to the Kaluza-Klein fibers. Then, again at $x^0=0$,

$$
\begin{aligned}
H_b\left(X,\mathscr{S}\right) &= \lim_{R\to\infty}\left\{\frac{3}{16(n-1)\pi}\int_{\partial E(R)}X^AZ^e({\mathbf{R}_{Ab}}^{fb}-{\overline{\mathbf{R}}_{Ab}}^{fb})dS_{ef}\right.\\
&\quad \left.-\frac{1}{8\pi}\int_{\partial E(R)}\left(g^{\mu[0}-e^{-1}\overline{g}^{\mu[0}\right)\overline{\nabla}_\mu X^{i]}dS_{0i}\right\}. \qquad (5.41)
\end{aligned}
$$

5.2 $(n+K)+1$–decomposition

In a Cauchy-data context it is convenient to express the global charges explicitly in terms of Cauchy data. Here one can use the Gauss-Codazzi-Mainardi embedding equations to reexpress our space-time-Riemann-tensor integrals in terms of the Riemann tensor of the initial-data metric and of the extrinsic curvature tensor. For this we consider $X^\mu=\delta^\mu_0$ and $x^0=0$, i.e., we consider (5.38).

We start with the case of KK asymptotically flat initial data sets. Keeping in mind our convention that $(x^I)=(x^i,x^A)$, we can replace ${\mathbf{R}^I}_{JKL}$ with the $(n+K)$-dimensional Riemann tensor, which we denote by ${\mathsf{R}^I}_{JKL}$, by means of the Gauss-Codazzi relation

$$
{\mathbf{R}^I}_{JKL}={\mathsf{R}^I}_{JKL}+o(r^{-2\alpha-2})\,. \qquad (5.42)
$$

In this case (5.38) reduces to (4.24). Hence, we obtain

$$
p_0 = -\frac{1}{8(n-2)\pi}\lim_{R\to\infty}\int_{S(R)}\int_{\mathbb{T}^N}x^j\left({\mathsf{R}^i}_j+\frac{1}{2(n-1)}{\mathsf{R}^k}_k\delta^i_j+\frac{1}{n-1}{\mathsf{R}^{iA}}_{jA}\right)N d\sigma_i\,. \qquad (5.43)
$$

We note that in the usual asymptotically flat case, $K = 0$, the last integral is not present. Further, $\mathsf{R}^k{}_k$ becomes then the Ricci scalar of the initial data metric, with $\mathsf{R}^k{}_k = o(r^{-2\alpha-2})$ because of the scalar constraint equation, and hence does not contribute to the integral. The above reproduces thus the well-known-by-now formula for the ADM mass in terms of the Ricci tensor of the initial data metric [9, 17, 20] when the Ricci scalar decays fast enough, as we assumed here.

We pass now to the case covered in section 5.1.1, namely $K = 0$ but $\Lambda < 0$, with the background metric $\overline{g}$ is as in (B.1)-(B.3). Let $\mathcal{K}_{IJ}$ be the extrinsic curvature tensor of the slices $\{x^0 = \mathrm{const}\}$. If we assume that $\mathcal{K}_{IJ}$ satisfies

$$|k| := \sqrt{g^{IJ} g^{LM} \mathcal{K}_{IL} \mathcal{K}_{JM}} = o(r^{-n/2}) \,, \tag{5.44}$$

from (5.33) we obtain a formula first observed in [17]:

$$H_b(X, \mathscr{S}) = -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{\partial E(R)} X^0 Z^j (\mathsf{R}^i{}_j - \frac{\mathsf{R}}{n}\delta^i_j) N d\sigma_i \,, \tag{5.45}$$

where in (5.45) we have assumed that X is a Killing vector of the anti-de Sitter background which is normal to the hypersurface $\{t = 0\}$.

Finally, consider general configurations as in section 5.1.2. Under the hypothesis that

$$|\mathcal{K}|^2 |Z| |\partial E(R)| \xrightarrow{R\to\infty} 0 \,, \tag{5.46}$$

from (5.38) we find

$$\begin{aligned} H_b(\partial_0, \mathscr{S}) = \frac{1}{8(n-2)\pi} \lim_{R\to\infty} \Bigg\{ \int_{\partial E(R)} \Bigg(& \left[\frac{1}{2} Z^i \left(\mathbf{R}^j{}_j - \overline{\mathbf{R}}^j{}_j\right) - Z^j \left(\mathsf{R}^i{}_j - \overline{\mathsf{R}}^i{}_j\right)\right] \\ & - \frac{1}{2(n-1)} \Bigg[Z^i \left(\left(\mathbf{R}^A{}_A - \overline{\mathbf{R}}^A{}_A\right) - \left(\mathsf{R}^A{}_A - \overline{\mathsf{R}}^A{}_A\right)\right) \\ & - 2Z^j \left(\left(\mathsf{R}^i{}_j - \overline{\mathsf{R}}^i{}_j\right) - \left(\mathsf{R}^{ik}{}_{jk} - \overline{\mathsf{R}}^{ik}{}_{jk}\right)\right) \Bigg] \Bigg) N d\sigma_i \\ & - (n-2) \lim_{R\to\infty} \int_{\partial E(R)} \left(g^{\mu[0} - e^{-1}\overline{g}^{\mu[0}\right) \overline{\nabla}_\mu X^{i]} N d\sigma_i \Bigg\} \,. \end{aligned} \tag{5.47}$$

5.3 Komar Integrals

If X^α is a Killing vector field of both g and $\overline{g}$, we have

$$X^\alpha \mathbf{R}_{\alpha bcd} = \nabla_b \nabla_c X_d \,, \qquad \text{and} \qquad X^\alpha \overline{\mathbf{R}}_{\alpha bcd} = \overline{\nabla}_b \overline{\nabla}_c X_d \,. \tag{5.48}$$

This allows us to express some of the integrals above as Komar-type integrals.

We start with the set-up of section 5.1.2, the KK-asymptotically flat case can be obtained directly from the calculations here by setting $\overline{\mathbf{R}}_{\alpha\beta\gamma\delta} = 0$. To make things clear: we assume (5.21)-(5.22), (5.24)-(5.25), together with (5.35)-(5.36), and recall that all these hypotheses are satisfied under the corresponding hypotheses made in the KK-asymptotically flat case.

The contribution from the first integrand in (5.37) can be manipulated as [3]

$$
\begin{aligned}
&\lim_{R\to\infty}\int_{\partial E(R)} X^a Z^b \left(\mathbf{R}_{ab}{}^{ef} - \overline{\mathbf{R}}_{ab}{}^{ef}\right) dS_{ef} \\
&= \lim_{R\to\infty}\int_{\partial E(R)} \left[\left(X^{[f;e]}{}_{;b} - X^{[f\|e]}{}_{\|b}\right) Z^b - X^A Z^b \left(\mathbf{R}_{Ab}{}^{ef} - \overline{\mathbf{R}}_{Ab}{}^{ef}\right)\right] dS_{ef} \\
&= \lim_{R\to\infty}\int_{\partial E(R)} \Bigg\{(n-1)\left(X^{[e;f]} - X^{[e\|f]}\right) - 3\left(X^{[e;f}Z^{b]}\right)_{;b} + 3\left(X^{[e\|f}Z^{b]}\right)_{\|b} \\
&\quad +2\left(\mathbf{R}_{\mu b}{}^{b[f}Z^{e]} - \overline{\mathbf{R}}_{\mu b}{}^{b[f}Z^{e]}\right) X^\mu dS_{ef} - X^A Z^b \left(\mathbf{R}_{Ab}{}^{ef} - \overline{\mathbf{R}}_{Ab}{}^{ef}\right)\Bigg\} dS_{ef} \\
&= \lim_{R\to\infty}\Bigg\{(n-1)\int_{\partial E(R)}\left(X^{[\alpha;\beta]} - X^{[\alpha\|\beta]}\right) dS_{\alpha\beta} + \int_{\partial E(R)}\Big[2X^\mu Z^e\left(\mathbf{R}^{fb}{}_{b\mu} - \overline{\mathbf{R}}^{fb}{}_{b\mu}\right) \\
&\quad - X^A Z^b\left(\mathbf{R}_{Ab}{}^{ef} - \overline{\mathbf{R}}_{Ab}{}^{ef}\right)\Big] dS_{ef}\Bigg\},
\end{aligned}
\tag{5.49}
$$

where the semicolon (;) denotes the covariant derivative of the metric g and the double bar ($\|$) denotes the covariant derivative of the background metric $\overline{g}$. Moreover, we used the identity (3.37) and the Gauss's theorem [4] , e.g.

$$
\lim_{R\to\infty}\int_{\partial E(R)}\left(X^{[e\|f}Z^{b]}\right)_{\|b}\sqrt{|\det g|}\,d\Sigma_{ef} = \lim_{R\to\infty}\int_{\partial E(R)}\left(X^{[e\|f}Z^{b]}\right)_{\|b}\sqrt{|\det \overline{g}|}\,d\Sigma_{ef} = 0\,. \tag{5.50}
$$

Hence, under the hypotheses used in the derivation of (5.37), we can rewrite (5.37) as

$$
\begin{aligned}
H_b(X,\mathscr{S}) &= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\Bigg\{\int_{\partial E(R)}(n-1)\left(X^{[\alpha;\beta]} - X^{[\alpha\|\beta]}\right) dS_{\alpha\beta} \\
&+\int_{\partial E(R)}\Bigg(2X^\mu Z^e\left(\mathbf{R}^{fb}{}_{b\mu} - \overline{\mathbf{R}}^{fb}{}_{b\mu}\right) - X^A Z^b\left(\mathbf{R}_{Ab}{}^{ef} - \overline{\mathbf{R}}_{Ab}{}^{ef}\right) + 4Z^{[e}X^{a]}\left(\mathbf{R}^f{}_a - \overline{\mathbf{R}}^f{}_a\right) \\
&-Z^eX^f\left(\mathbf{R}^c{}_c - \overline{\mathbf{R}}^c{}_c\right) - \frac{1}{n-1}\Big[(n-3)Z^eX^f(\mathbf{R}^{bA}{}_{bA} - \overline{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e}X^{a]}\left(\mathbf{R}^{fA}{}_{aA} - \overline{\mathbf{R}}^{fA}{}_{aA}\right) \\
&+3(n-2)X^AZ^e\left(\mathbf{R}^{fb}{}_{bA} - \overline{\mathbf{R}}^{fb}{}_{bA}\right)\Big]\Bigg) dS_{ef} - (n-2)\lim_{R\to\infty}\int_{\partial E(R)}\left(g^{\mu[a} - e^{-1}\overline{g}^{\mu[a}\right)\overline{\nabla}_\mu X^{b]} dS_{ab}\Bigg\}.
\end{aligned}
\tag{5.51}
$$

The first integrand is the difference of Komar integrands of g and $\overline{g}$.

Specializing to the KK-asymptotically flat case for background-covariantly constant Killing vectors, one finds (4.29).

Now, we can calculate Koamr integral for Rasheed solutions with $P \neq 0$ which are not KK asymptotically flat, compare (4.30)-(4.31). We summarize the results:

Again using (C.12) one gets for $X = \partial_t$ (recall that $n = 3$):

$$
\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 2\pi\big(M + \frac{\Sigma}{\sqrt{3}}\big), & P = 0; \\ 8\pi P\big(M + \frac{\Sigma}{\sqrt{3}}\big), & P \neq 0, \end{cases} \tag{5.52}
$$

[3] Strictly speaking, under our asymptotic conditions each individual integrand might fail to have a finite limit as $R \to \infty$, it is only the integral of the sum of all terms which is guaranteed to have a limit. A careful reader will make the calculation below with the remaining terms from (5.37) added to each integral.

[4] One can also show that we do not necessary need the application of the Gauss's theorem.

which does *neither* coincide with p_0, cf. (4.15), *nor* with the ADM mass of the space metric $g_{ij}dx^i dx^j$. Note that the Komar integral of the space-time metric $g_{ab}dx^a dx^b$ will equal $M+\frac{\Sigma}{\sqrt{3}}$ regardless of the value of P.

Next, for $X = \partial_4$ we obtain

$$\frac{1}{8\pi}\lim_{R\to\infty}\int_{S(R)}\int_{S^1} X^{\alpha;\beta}dS_{\alpha\beta} = \begin{cases} 4\pi Q, & P=0; \\ 16\pi PQ, & P\neq 0, \end{cases} \tag{5.53}$$

which is again twice the Hamiltonian charge p_4.

As a simple application of (5.52), suppose that there exists a Rasheed metric without a black hole region. Since the divergence of the Komar integrand is zero, we obtain $M = -\Sigma/\sqrt{3}$. But this is precisely one of the parameter values excluded in the Rasheed metrics, cf. (C.4) below. We conclude that the regular metrics in the Rasheed family must be black-hole solutions.

For the case of metrics which asymptote to a maximally symmetric background $\overline{g}$ with $\Lambda \neq 0$, as in section 5.1.1, the Komar integral resulting from (5.33) reads

$$\begin{aligned} H_b\,(X,\mathscr{S}) &= \frac{1}{16(n-2)\pi}\lim_{R\to\infty}\int_{\partial E(R)} X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi}\,dS_{\alpha\beta} \\ &= \lim_{R\to\infty}\left\{\frac{n-1}{16(n-2)\pi}\int_{\partial E(R)} X^{[\alpha;\beta]}dS_{\alpha\beta} - \frac{\Lambda}{4(n-2)(n-1)n\pi}\int_{\partial E(R)} X^\alpha Z^\beta dS_{\alpha\beta}\right\}. \end{aligned} \tag{5.54}$$

Appendix A An identity for the Riemann tensor

For completeness we prove the following identity satisfied by the Riemann tensor, which is valid in any dimension, is clear in dimension two and three, implies the double-dual identity for the Weyl tensor in dimension four, and is probably well known in higher dimensions as well, keeping in mind that we write $\delta^{\alpha\beta}_{\gamma\delta}$ for $\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta} \equiv \frac{1}{2}(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta})$, etc.:

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\gamma\delta} = \frac{1}{3!}\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu} R - 4\delta^{[\alpha}_{[\mu} R^{\beta]}{}_{\nu]}\right) . \tag{A.1}$$

The above holds for any tensor field satisfying

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\delta\gamma} . \tag{A.2}$$

To prove (A.1) one can calculate as follows:

$$\begin{aligned}
4!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\gamma\delta} &= 2\big[\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho}\right) \\
&\quad -\delta^{\alpha}_{\nu}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho}\right) \\
&\quad +\delta^{\alpha}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right) \\
&\quad -\delta^{\alpha}_{\sigma}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho} + \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right)\big] R^{\rho\sigma}{}_{\gamma\delta} \\
&= 2\left(2\delta^{\alpha\beta}_{\mu\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - 4\delta^{\alpha\gamma}_{\mu\nu}\delta^{\beta\delta}_{\rho\sigma} + 4\delta^{\beta\gamma}_{\mu\nu}\delta^{\alpha\delta}_{\rho\sigma} + 2\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right) R^{\rho\sigma}{}_{\gamma\delta} \\
&= 4\left(\delta^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}{}_{\gamma\delta} - 2\delta^{\alpha\gamma}_{\mu\nu} R^{\beta\sigma}{}_{\gamma\sigma} + 2\delta^{\beta\gamma}_{\mu\nu} R^{\alpha\sigma}{}_{\gamma\sigma} + R^{\alpha\beta}{}_{\mu\nu}\right) \\
&= 4\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}{}_{\gamma\delta} - 4\delta^{[\alpha}_{[\mu} R^{\beta]\gamma}{}_{\nu]\gamma}\right) .
\end{aligned} \tag{A.3}$$

If the sums are over all indices we obtain (A.1). The reader is warned, however, that in some of our calculations the sums will be only over a subset of all possible indices, in which case the last equation remains valid but the last two terms in (A.3) *cannot* be replaced by the Ricci scalar and the Ricci tensor.

Appendix B The Dilation Vector Field Z

Let $Z = r\partial_r$. We want to calculate $\overline{\nabla}_\mu Z_\nu$ for the Kottler metrics and the Rasheed metrics appearing in the calculations.

B.1 Kottler Background Metric

Let $\overline{g}$ be the $(n+1)$-dimensional anti-de Sitter (Kottler) metric,

$$\overline{g} = -V dt^2 + V^{-1} dr^2 + r^2 \overline{h}\,, \tag{B.1}$$

where

$$V = \lambda r^2 + \kappa\,, \tag{B.2}$$

with $\kappa \in \{0, \pm 1\}$ and

$$\lambda = -\frac{2\Lambda}{n(n-1)}\,, \tag{B.3}$$

where $\overline{h}$ is an (r-independent) Einstein metric on an $(n-1)$-dimensional compact manifold $\mathscr{K}$, with scalar curvature $(n-1)(n-2)\kappa$. It holds that (cf., e.g., [8])

$$\overline{R} = -n(n+1)\lambda\,. \tag{B.4}$$

We have

$$\begin{aligned}
\overline{\nabla}_{(\mu} Z_{\nu)} dx^\mu \otimes dx^\nu &= \frac{1}{2}\mathcal{L}_Z \overline{g} = \frac{1}{2}\left(Z^\alpha \partial_\alpha \overline{g}_{\mu\nu} + \partial_\mu Z^\alpha \overline{g}_{\alpha\nu} + \partial_\nu Z^\alpha \overline{g}_{\alpha\mu}\right) dx^\mu dx^\nu \\
&= \frac{1}{2}\Big[r\partial_r(-V) dt^2 + r\partial_r(V^{-1}) dr^2 + r\partial_r(r^2) d\Omega + 2V^{-1} dr^2 \Big] \\
&= \frac{1}{2}\left[\frac{r\partial_r V}{V}(-V dt^2) + (2 - rV^{-1}\partial_r V) V^{-1} dr^2 + 2r^2 d\Omega^2 \right],
\end{aligned} \tag{B.5}$$

and the antisymmetric part vanishes

$$\overline{\nabla}_{[\mu} Z_{\nu]} dx^\mu \otimes dx^\nu = \partial_{[\mu} Z_{\nu]} dx^\mu \otimes dx^\nu = 0 \tag{B.6}$$

Thus, we have

$$\begin{aligned}
\overline{\nabla}_\mu Z_\nu \, dx^\mu \otimes dx^\nu &= \overline{\nabla}_{(\mu} Z_{\nu)} dx^\mu \otimes dx^\nu + \overline{\nabla}_{[\mu} Z_{\nu]} dx^\mu \otimes dx^\nu \\
&= \overline{g} \mod (\delta^t_\mu, \delta^r_\mu)
\end{aligned} \tag{B.7}$$

which gives (5.22).

B.2 Rasheed Background Metrics

Next, for the Rasheed background metrics (C.13) one finds

$$\mathcal{L}_Z \overline{g} = 2(dr^2 + r^2 d\Omega^2)\,, \quad d(\overline{g}_{\alpha\beta} Z^\alpha dx^\beta) = d(r dr) = 0\,, \tag{B.8}$$

and (5.22) without the $o(r^{-\gamma})$ term readily follows.

Appendix C Rasheed's Solutions

D. Rasheed [21] has constructed a family of stationary axi-symmetric five-dimensional solutions of the Einstein-Maxwell equations which take the form

$$ds^2_{(5)} = \frac{B}{A}\left(dx^4 + 2A_\mu dx^\mu\right)^2 + \sqrt{\frac{A}{B}} ds^2_{(4)}\,, \tag{C.1}$$

where a, M, P, Q and Σ are real numbers satisfying

$$\tfrac{Q^2}{\Sigma+M\sqrt{3}} + \tfrac{P^2}{\Sigma-M\sqrt{3}} = \tfrac{2\Sigma}{3}\,, \tag{C.2}$$

$$M^2+\Sigma^2-P^2-Q^2 \neq 0\,, \quad \left(M+\Sigma/\sqrt{3}\right)^2 - Q^2 \neq 0\,, \quad \left(M-\Sigma/\sqrt{3}\right)^2 - P^2 \neq 0\,, \tag{C.3}$$

$$M \pm \tfrac{\Sigma}{\sqrt{3}} \neq 0\,, \quad F^2 := \tfrac{\left[\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right]\left[\left(M-\Sigma/\sqrt{3}\right)^2 - P^2\right]}{M^2+\Sigma^2-P^2-Q^2} > 0\,, \tag{C.4}$$

and where

$$ds^2_{(4)} = -\frac{G}{\sqrt{AB}}\left(dt + {\omega^0}_\phi d\phi\right)^2 + \frac{\sqrt{AB}}{\Delta}dr^2 + \sqrt{AB}d\theta^2 + \frac{\Delta\sqrt{AB}}{G}\sin^2(\theta)d\phi^2\,, \tag{C.5}$$

with

$$\begin{aligned}
A &= \left(r-\Sigma/\sqrt{3}\right)^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + a^2\cos^2(\theta) + \frac{2JPQ\cos(\theta)}{\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2}\,, \\
B &= \left(r+\Sigma/\sqrt{3}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2\cos^2(\theta) - \frac{2JPQ\cos(\theta)}{\left(M-\Sigma/\sqrt{3}\right)^2 - P^2}\,, \\
G &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2(\theta)\,, \\
\Delta &= r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\,, \\
{\omega^0}_\phi &= \frac{2J\sin^2(\theta)}{G}\left[r+E\right]\,, \\
J^2 &= a^2F^2\,,
\end{aligned} \tag{C.6}$$

whereas E is given by

$$E = -M + \frac{\left(M^2+\Sigma^2-P^2-Q^2\right)\left(M+\Sigma/\sqrt{3}\right)}{\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2}\,. \tag{C.7}$$

The Maxwell field is given by

$$2A_\mu dx^\mu = \frac{C}{B}dt + \left({\omega^5}_\phi + \frac{C}{B}{\omega^0}_\phi\right)d\phi\,, \tag{C.8}$$

where

$$C = 2Q\left(r-\Sigma/\sqrt{3}\right) - \frac{2PJ\cos(\theta)\left(M+\Sigma/\sqrt{3}\right)}{\left(M-\Sigma/\sqrt{3}\right)^2 - P^2}\,, \tag{C.9}$$

$${\omega^5}_\phi = \frac{H}{G}\,, \tag{C.10}$$

and

$$H := 2P\Delta\cos(\theta) - \frac{2QJ\sin^2(\theta)\left[r\left(M-\Sigma/\sqrt{3}\right) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2\right]}{\left[\left(M+\Sigma/\sqrt{3}\right)^2 - Q^2\right]}\,. \tag{C.11}$$

C.1 Asymptotic Behaviour

When $P = 0$ the Rasheed metrics satisfy the KK-asymptotic flatness conditions. This can be seen by introducing manifestly-asymptotically-flat coordinates (t, x, y, z) in the usual way. With some work one finds that the metric takes the form

$$\begin{pmatrix} \frac{2M}{r}+\frac{2\Sigma}{\sqrt{3}r}-1 & 0 & 0 & 0 & \frac{2Q}{r} \\ 0 & \frac{2Mx^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r}+1 & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} & 0 \\ 0 & \frac{2Mxy}{r^3} & \frac{2My^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r}+1 & \frac{2Myz}{r^3} & 0 \\ 0 & \frac{2Mxz}{r^3} & \frac{2Myz}{r^3} & \frac{2Mz^2}{r^3}-\frac{2\Sigma}{\sqrt{3}r}+1 & 0 \\ \frac{2Q}{r} & 0 & 0 & 0 & \frac{4\Sigma}{\sqrt{3}r}+1 \end{pmatrix} + O(r^{-2})\,. \tag{C.12}$$

It turns out that when $P \neq 0$, the Rasheed metrics do *not* satisfy the KK-asymptotic flatness requirements anymore: Indeed, the phase space decomposes into sectors, labeled by $P \in \mathbb{R}$, in which the metrics g asymptote to the background metric

$$\overline{g} := \big(dx^4 + 2P\cos(\theta)d\varphi\big)^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\,. \tag{C.13}$$

The metrics (C.1) and (C.13) are singular at $\sin(\theta) = 0$. This can be resolved by replacing x^4 by $\overline{x}^4$, respectively by $\tilde{x}^4$, on the following coordinate patches:

$$\begin{cases} \overline{x}^4 := x^4 + 2P\varphi\,, & \theta \in [0,\pi)\,, \\ \tilde{x}^4 := x^4 - 2P\varphi\,, & \theta \in (0,\pi]\,. \end{cases} \tag{C.14}$$

Indeed, the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\overline{x}^4 + 2P(\cos(\theta)-1)d\varphi = d\overline{x}^4 - \frac{2P}{r(r+z)}(xdy - ydx)$$

is smooth for $r > 0$ on $\{\theta \in [0,\pi)\}$. Similarly the one-form

$$dx^4 + 2P\cos(\theta)d\varphi = d\tilde{x}^4 + 2P(\cos(\theta)+1)d\varphi = d\tilde{x}^4 + \frac{2P}{r(r-z)}(xdy - ydx)$$

is smooth on $\{\theta \in (0,\pi]\,, r > 0\}$. Smoothness of both g and $\overline{g}$ in the d.o.c., under the constraints discussed above, readily follows.

We note the relation

$$\overline{x}^4 = \tilde{x}^4 + 4P\varphi\,, \tag{C.15}$$

which implies a smooth geometry with periodic coordinates $\overline{x}^4$ and $\tilde{x}^4$ if and only if

$$\text{both } \overline{x}^4 \text{ and } \tilde{x}^4 \text{ are periodic with period } 8P\pi. \tag{C.16}$$

From this perspective x^4 is *not a coordinate* anymore: instead the basic coordinates are $\overline{x}^4$ for $\theta \in [0,\pi)$ and $\tilde{x}^4$ for $\theta \in (0,\pi]$, with dx^4 (but *not* x^4) well defined away from the axes of rotation $\{\sin(\theta) = 0\}$ as

$$dx^4 = \begin{cases} d\overline{x}^4 - 2Pd\varphi\,, & \theta \in [0,\pi)\,, \\ d\tilde{x}^4 + 2Pd\varphi\,, & \theta \in (0,\pi]\,. \end{cases} \tag{C.17}$$

C.1.1 Curvature of The Asymptotic Background

We continue with a calculation of the curvature tensor of the asymptotic background. It is convenient to work in the coframe

$$\overline{\Theta}^{\hat{0}} = dt\,, \qquad \overline{\Theta}^{\hat{1}} = dx\,, \qquad \overline{\Theta}^{\hat{2}} = dy\,, \qquad \overline{\Theta}^{\hat{3}} = dz\,, \qquad \overline{\Theta}^{\hat{4}} = dx^4 + 2P\cos(\theta)d\varphi\,, \tag{C.18}$$

which is manifestly smooth after replacing dx^4 as in (C.17). Using

$$d\overline{\Theta}^{\hat{4}} = -2P\sin(\theta)\,d\theta\wedge d\varphi = -2P\frac{x^i}{r^3}\partial_i\rfloor(dx\wedge dy\wedge dz) = -\frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^i dx^{\hat{j}}\wedge dx^{\hat{k}}\,, \tag{C.19}$$

where $\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}} \in \{0, \pm 1\}$ denotes the usual epsilon symbol, one finds the following non-vanishing connection coefficients

$$\overline{\omega}^{\hat{4}}{}_{\hat{i}} = \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^{\hat{j}}\overline{\Theta}^{\hat{k}}\,, \quad \overline{\omega}^{\hat{i}}{}_{\hat{j}} = \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}x^{\hat{k}}\overline{\Theta}^{\hat{4}}\,, \tag{C.20}$$

where $x^{\hat{i}} \equiv x^i$. This leads to the following curvature forms

$$\begin{aligned}
\overline{\Omega}^{\hat{i}}{}_{\hat{j}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{k}}x^{\hat{\ell}} + \delta^{\hat{k}}_{\hat{\ell}}\right)\overline{\Theta}^{\hat{\ell}}\wedge\overline{\Theta}^{\hat{4}} - \frac{2P^2}{r^6}\mathring{\epsilon}_{\hat{i}\hat{m}(\hat{k}}\epsilon_{\hat{j})\hat{n}\hat{\ell}}x^{\hat{m}}x^{\hat{n}}\overline{\Theta}^{\hat{k}}\wedge\overline{\Theta}^{\hat{\ell}}\,,\\
\overline{\Omega}^{\hat{4}}{}_{\hat{i}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{k}}\left(-\frac{3}{r^2}x^{\hat{j}}x^{\hat{\ell}} + \delta^{\hat{j}}_{\hat{\ell}}\right)\overline{\Theta}^{\hat{\ell}}\wedge\overline{\Theta}^{\hat{k}} + \frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{j}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\overline{\Theta}^{\hat{j}}\wedge\overline{\Theta}^{\hat{4}}\,,
\end{aligned} \tag{C.21}$$

hence the following non-vanishing curvature tensor components

$$\begin{aligned}
\overline{\mathbf{R}}^{\hat{i}}{}_{\hat{j}\hat{k}\hat{4}} &= \frac{P}{r^3}\mathring{\epsilon}_{\hat{i}\hat{j}\hat{\ell}}\left(-\frac{3}{r^2}x^{\hat{\ell}}x^{\hat{k}} + \delta^{\hat{\ell}}_{\hat{k}}\right)\,,\\
\overline{\mathbf{R}}^{\hat{4}}{}_{\hat{i}\hat{j}\hat{4}} &= \frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{j}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\,, \quad \overline{\mathbf{R}}_{\hat{i}\hat{j}\hat{k}\hat{\ell}} = -\frac{2P^2}{r^6}(\mathring{\epsilon}_{\hat{i}\hat{j}\hat{n}}\mathring{\epsilon}_{\hat{k}\hat{\ell}\hat{m}} + \mathring{\epsilon}_{\hat{i}\hat{m}[\hat{k}}\mathring{\epsilon}_{\hat{\ell}]\hat{j}\hat{n}})x^{\hat{m}}x^{\hat{n}}\,.
\end{aligned} \tag{C.22}$$

The non-vanishing components of the Ricci tensor read

$$\overline{\mathbf{R}}_{\hat{i}\hat{j}} = -\frac{2P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{i}}\mathring{\epsilon}_{\hat{k}\hat{n}\hat{j}}x^{\hat{m}}x^{\hat{n}}\,, \qquad \overline{\mathbf{R}}_{\hat{4}\hat{4}} = -\frac{P^2}{r^6}\mathring{\epsilon}_{\hat{k}\hat{m}\hat{i}}\mathring{\epsilon}_{\hat{k}\hat{i}\hat{\ell}}x^{\hat{m}}x^{\hat{\ell}}\,. \tag{C.23}$$

Subsequently the Ricci scalar is $\overline{\mathbf{R}} = -2P^2/r^4$.

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Zusammenfassung

Ausgehend von dem geometrischen Hamilton-Formalismus leiten wir in dieser Arbeit neue Ausdrücke für die Hamiltonsche Gesamtenergie von gravitativen Systemen in höheren Dimensionen in Termen des Riemann-Tensors her, welche auch Raumzeiten mit einer kosmologischen Konstante $\Lambda \in \mathbb{R}$ einschließen. Daraus lassen sich in weiterer Folge die entsprechenden ADM- und Komar-Größen der betreffenden Raumzeiten herleiten. Wir wenden diese gewonnenen Ausdrücke auf eine weite Klasse von höherdimensionalen Raumzeiten mit verschiedenen asymptotischen Verhalten an, welche bestimmten Bedingungen unterliegen. Insbesondere deckt unsere Analyse asymptotisch anti-de-Sitter und asymptotisch flache sowie Kaluza-Klein asymptotisch flache Raumzeiten ab. Es zeigt sich, dass im Fall von stationären asymptotisch flachen Raumzeiten die Komar-Masse mit der ADM-Masse übereinstimmt. Dies gilt jedoch nicht allgemein, wie z.B. im Fall von Kaluza-Klein asymptotisch flachen Raumzeiten. Wir zeigen des Weiteren, dass die Hamiltonsche Masse, ADM-Masse und Komar-Masse im Allgemeinen nicht miteinander übereinstimmen.

Abstract

Based on the geometric Hamiltonian formalism we derive new expressions for the total Hamiltonian energy of gravitating systems in higher dimensions in terms of the Riemann tensor, allowing a cosmological constant $\Lambda \in \mathbb{R}$. In this way, we also obtain the ADM and the Komar expressions of the corresponding space-times. We then apply this analysis to a large class of higher dimensional space-times with various asymptotics known to us, which satisfy certain conditions. In particular, our analysis covers asymptotically anti-de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. As it turns out, the Komar mass equals the ADM mass in stationary asymptotically flat space-times in all dimensions, generalizing the four-dimensional result of Beig. This is not true for arbitrary space-times in general, e.g. for those with Kaluza-Klein asymptotics. We show that the Hamiltonian mass, the ADM mass and the Komar mass do not, in general, coincide with each other in the non-asymptotically flat setting.

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