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„Modeling Technological Progress

and the Steady-State Growth Theorem"

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Abstract

This thesis explores the necessity of steady-state technological progress to be purely labor-augmenting. Apart from the findings of Uzawa (1961) and Irmen (2016), which indicate that this restriction applies irrespective of whether technological progress is treated as an exogenous or as an endogenous variable, this thesis also presents two models that find such a constraint unnecessary. The first one extends the neoclassical growth model by including adjustment costs for the capital investment which create inequality between the growth of gross capital investment and the growth of capital stock. Steady-state capital-augmenting technological progress is then needed to balance out this inequality. The second model addresses the declining relative price of investment goods as an indicator of ongoing embodied technological progress. Balanced growth can be achieved by introducing endogenous schooling into the model, and assuming a sufficient level of capital-skill complementarity.

Keywords: Steady-state growth theorem; Technological progress; Uzawa; Harrod-neutral; Labor-augmenting; Capital-augmenting

Contents

1 Introduction

In 1961 Hirofumi Uzawa formulated a theorem claiming that in order for a neoclassical growth model to exhibit steady-state growth (balanced growth), technological progress must have a representation as purely labor-augmenting (Harrod-neutral). Since it demands only a representation in such a form, one way how to always satisfy this requirement is to use the the Cobb-Douglas production function, in which the distinction between capital-augmenting and labor-augmenting technology becomes irrelevant and technological progress can be always expressed as purely labor-augmenting. In fact, some economists interpret Uzawa's theorem as a requirement that production function must have the Cobb-Douglas functional form. But one way or the other, this is a very strong restriction and it is natural to ask whether there is any economic reason for steady-state technological progress to be purely labor-augmenting.

Years before Uzawa published his theorem, in 1932, Hicks suggested (as cited in Acemoglu (2003, p. 6)) that direction of technological change is determined by the relative factor prices, and that it is directed at the factor which has become relatively expensive. The so-called induced innovations literature of the 1960s, which includes the work of Fellner (1961), von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966), attempted to develop a framework which would formalize this notion. However, this work was met with criticism for not specifying the development of new technologies – who developed them, what was the cost and resources that financed them.

Decades later, Acemoglu (2002, 2003, 2009) returned to this topic with his models of directed technological change. In these models, profit-maximizing firms can invest into both capital-augmenting and labor-augmenting innovations. Acemoglu finds that as long as the elasticity of substitution between capital and labor satisfies $\sigma < 1$, firms' profit-maximizing incentives lead to a purely labor-augmenting technological progress along the steady-state path.

Irmen and Tabaković (2015) contributed to the discussion with their extended Ramsey-Cass-Koopmans model, which allows for endogenous capital- and labor-augmenting technological change, and obtained the same result. But in comparison to Acemoglu (2002, 2003, 2009), their result holds independent of the value of elasticity of substitution between capital and labor.

These results suggest, despite the fact that technological change is endogenous, that Uzawa's theorem somehow applies to these models too. To confirm this, Irmen (2016) extended Uzawa's theorem and applied it to four models of endogenous technological change (including the models of Acemoglu (2009) and Irmen and Tabaković (2015)) that differ in the way technological progress is created. Irmen shows that all four models can be simplified into a form where either the original steady-state growth theorem or the generalized one apply. This might be interpreted as that purely labor-augmenting steady-state technological progress is necessary.

However, in recent years, economists return to Uzawa's theorem and attempt to find conditions that would allow for capital-augmenting technological progress to be present along the steady-state path. One way how to achieve this is to include adjustment costs for capital investment, which create gap between gross and net investment. Capital-augmenting technological progress then closes this gap.

The aim of this thesis is to identify the reasons why steady-state technological progress should be purely labor-augmenting, and to give an overview of conditions that relax this restriction.

The structure of this thesis is the following: Chapter 2 starts with a description of neoclassical growth model, defines three different types of neutral technological progress and closes with the statement and the proof of Uzawa's steady-state growth theorem. Chapter 3 shows how Acemoglu's (2009) model of directed technological change can be used to identify the reasons behind purely labor-augmenting steady-state technological progress. In Chapter 4, the generalized steady-state growth theorem is presented with an example how it applies to the previously introduced model of Acemoglu (2009). Finally, to show that it is possible to reconcile the capital-augmenting technological change with the requirements of balanced growth, Chapter 5 contains two models that attempt to do so by including adjustment costs for capital investment, or endogenous schooling into the model. Chapter 6 provides a conclusion.

2 The Steady-State Growth Theorem

This chapter presents both the statement and the proof of Uzawa's steady-state growth theorem which applies to the basic one-sector neoclassical growth model with exogenous technological progress. The content of this chapter is based mostly on Acemoglu (2009, Chapter 2) and Barro and Sala-i-Martin (2004, Chapter 1).

2.1 The Economic Environment

Consider a closed economy in continuous time which admits a representative firm producing the final good with the aggregate production function

$$Y(t) = \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right], \tag{2.1}$$

where $Y(t)$ is aggregate output of the final good, $K(t)$ is capital stock, $L(t)$ is the labor endowment, and $\tilde{A}(t)$ is the state of technology at time t.

The aggregate production function $\tilde{F} : \mathbb{R}^3_+ \mapsto \mathbb{R}_+$ is a *neoclassical production function*, meaning that it satisfies the following standard assumptions:

1. $\tilde{F}$ is twice continuously differentiable in K and L.
2. $\tilde{F}$ has positive marginal products of capital and labor:

   $$\tilde{F}_K(K, L, \tilde{A}) \equiv \frac{\partial \tilde{F}(K, L, \tilde{A})}{\partial K} > 0 \quad \text{and} \quad \tilde{F}_L(K, L, \tilde{A}) \equiv \frac{\partial \tilde{F}(K, L, \tilde{A})}{\partial L} > 0$$

   for all $(K, L, \tilde{A}) \in \mathbb{R}^3_+$.
3. $\tilde{F}$ has diminishing marginal products (diminishing returns) of capital and labor:

   $$\tilde{F}_{KK}(K, L, \tilde{A}) \equiv \frac{\partial^2 \tilde{F}(K, L, \tilde{A})}{\partial K^2} < 0 \quad \text{and} \quad \tilde{F}_{LL}(K, L, \tilde{A}) \equiv \frac{\partial^2 \tilde{F}(K, L, \tilde{A})}{\partial L^2} < 0$$

for all $(K, L, \tilde{A}) \in \mathbb{R}^3_+$.

4. $\tilde{F}$ exhibits constant returns to scale (linear homogeneity) with respect to capital and labor: For all $\lambda > 0$ and all $(K, L, \tilde{A}) \in \mathbb{R}^3_+$ it holds that

$$\tilde{F}(\lambda K, \lambda L, \tilde{A}) = \lambda \tilde{F}(K, L, \tilde{A}).$$

5. $\bar{F}$ satisfies the Inada conditions:

$$\lim_{K \to 0} \tilde{F}_K(K, L, \tilde{A}) = \infty \quad \text{and} \quad \lim_{K \to \infty} \tilde{F}_K(K, L, \tilde{A}) = 0 \quad \text{for all } L > 0 \text{ and all } \tilde{A},$$

$$\lim_{L \to 0} \tilde{F}_L(K, L, \tilde{A}) = \infty \quad \text{and} \quad \lim_{L \to \infty} \tilde{F}_L(K, L, \tilde{A}) = 0 \quad \text{for all } K > 0 \text{ and all } \tilde{A}.$$

6. Both inputs are essential for production:

$$\tilde{F}(0, L, \tilde{A}) = \tilde{F}(K, 0, \tilde{A}) = 0 \quad \text{for all } K, L \text{ and } \tilde{A}.$$

Both production factors $K(t)$ and $L(t)$ are owned by a representative household. Labor is supplied inelastically for a time t rental price (wage rate) $w(t)$. The rental price of capital is denoted by $R(t)$. Both these prices are expressed in units of final output. Capital goods are assumed to depreciate at rate $\delta_K > 0$, therefore the rate of return (interest rate) on capital is $r(t) = R(t) - \delta_K$.

A representative firm seeks to maximize its profits, given the factor prices $R(t)$ and $w(t)$, and the technology $\tilde{A}(t)$, which can be written as

$$\max \quad \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right] - R(t)K(t) - w(t)L(t).$$

This implies for the factor prices the following conditions

$$R(t) = \tilde{F}_K\left[K(t), L(t), \tilde{A}(t)\right] \quad \text{and} \quad w(t) = \tilde{F}_L\left[K(t), L(t), \tilde{A}(t)\right].$$

Euler's theorem then implies that with these prices the representative firm makes zero profits,

$$Y(t) = w(t)L(t) + R(t)K(t).$$

Next, the income shares of capital and labor are defined as

$$\alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)} \quad \text{and} \quad \alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)},$$

and by Euler's theorem $\alpha_K(t) + \alpha_L(t) = 1$.

The representative household decides whether the final good output is to be consumed or invested

$$Y(t) = C(t) + I(t). \tag{2.2}$$

Investing means replacing the worn-out capital and increasing the capital stock

$$I(t) = \delta_K K(t) + \dot{K}(t). \tag{2.3}$$

Combining the last two equations with the aggregate production function (2.1) implies

$$\dot{K}(t) = Y(t) - C(t) - \delta_K K(t) = \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right] - C(t) - \delta K(t), \quad K(0) > 0. \tag{2.4}$$

Finally, the evolution of the labor force is given by

$$L(t) = L(0)e^{\gamma_L t}, \quad L(0) > 0, \quad \gamma_L > 0. \tag{2.5}$$

2.2 Technological Progress

Exogenous technological progress, represented by $\tilde{A}(t)$, can have various forms. It may enable to produce the same amount of output by saving relatively more of capital or labor input. Such a technological progress can be called *capital-saving*, or *labor-saving*, respectively. Eventually, if it is neither capital-saving nor labor-saving, it is called *neutral* or *unbiased*. Usually three different types of neutral technological progress are defined.

2.2.1 Hicks-Neutral Technological Progress

First definition was formulated by John Hicks, and it states that technological progress is neutral if the ratio of marginal product of capital to marginal product of labor, $\tilde{F}_K/\tilde{F}_L$, remains unchanged for a given capital-labor ratio, K/L. Production function with Hicks-neutral technological progress can be written as

$$Y(t) = \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right] = A(t) \cdot F\left[K(t), L(t)\right],$$

where F is some linearly homogeneous function.

2.2.2 Harrod-Neutral Technological Progress

Alternative definition was proposed by Roy Harrod, who defined technological progress as neutral if for a given capital-output ratio, K/Y, the relative input shares $(K \cdot \tilde{F}_K)/(L \cdot \tilde{F}_L)$

remain unchanged. This definition implies a production function in the form

$$Y(t) = \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right] = F\left[K(t), A(t) \cdot L(t)\right].$$

Technological progress of this type is called *labor-augmenting* because an increase in technology $A(t)$ increases output in the same way as if there was an increase in the labor input.

2.2.3 Solow-Neutral Technological Progress

The third definition is by Robert Solow, according to whom technological progress classifies as neutral if the relative input shares, $(L \cdot \tilde{F}_L)/(K \cdot \tilde{F}_K)$, remain unchanged for a given labor-output ratio, L/Y. This implies that the production function takes the form

$$Y(t) = \tilde{F}\left[K(t), L(t), \tilde{A}(t)\right] = F\left[A(t) \cdot K(t), L(t)\right].$$

Technological progress of this form is also referred to as *capital-augmenting* because an increase in technology $A(t)$ raises output in the same way as an increase in the capital input.

2.3 Uzawa's Steady-State Growth Theorem

Definition 2.1. *A balanced growth path (BGP) in the neoclassical growth model is a path along which all variables grow at constant exponential rates.*

Alternatively, such a path is also referred to as a *steady-state path*. Throughout this work these terms are used interchangeably depending on which term is used by the author(s) of the model that is being presented.

In the following, $\gamma_x(t) \in \mathbb{R}$ stands for the instantaneous growth rate of a variable $x(t)$ at time t. In steady-state $\gamma_x(t) = \gamma_x$ for all variables.

Uzawa's (1961) original proof is quite complicated. Both the statement and the proof presented here are the variants originally formulated by Schlicht (2006), which were then adopted by Jones and Scrimgeour (2008) and Acemoglu (2009, Chapter 2).

Theorem 2.1 (The Steady-State Growth Theorem, Uzawa, 1961)**.** *Suppose that the neoclassical growth model with production function given by* (2.1)*, capital accumulation given by* (2.4) *and labor input evolving according to* (2.5)*, exhibits a steady-state beginning at some* $\tau < \infty$ *along which* $\forall t \geq \tau : \frac{\dot{Y}(t)}{Y(t)} = \gamma_Y > 0$, $\frac{\dot{K}(t)}{K(t)} = \gamma_K > 0$ *and* $\frac{\dot{C}(t)}{C(t)} = \gamma_C > 0$. *Then for all* $t \geq \tau$ *holds:*

1. $\gamma_Y = \gamma_K = \gamma_C$ *and*

2. the aggregate production function has a representation

$$Y(t) = F\left(K(t), A(t)L(t)\right),$$

where $A(t) \in \mathbb{R}_+$ *and*

$$\frac{\dot{A}(t)}{A(t)} = \gamma = \gamma_Y - \gamma_L.$$

Proof. Following the proof in Acemoglu (2009, p. 60-61):

Claim I.: By assumption that there is a steady-state path starting at some $\tau < \infty$ along which all the variables grow at some positive rates, for $\forall t \geq \tau$, time t quantities can be expressed as $Y(t) = Y(\tau)e^{\gamma_Y(t-\tau)}$, $K(t) = K(\tau)e^{\gamma_K(t-\tau)}$, and $C(t) = C(\tau)e^{\gamma_C(t-\tau)}$. Substituting these values into the capital accumulation equation (2.4) in the form

$$\underbrace{\dot{K}(t)}_{=\gamma_K \cdot K(t)} = Y(t) - C(t) - \delta_K K(t) \quad \longrightarrow \quad \left(\gamma_K + \delta_K\right) K(t) = Y(t) - C(t),$$

gives

$$K(\tau)e^{\gamma_K(t-\tau)}\left(\gamma_K + \delta_K\right) = Y(\tau)e^{\gamma_Y(t-\tau)} - C(\tau)e^{\gamma_C(t-\tau)}.$$

Dividing both sides by $e^{\gamma_K(t-\tau)}$ and then differentiating it with respect to time t gives

$$0 = \left(\gamma_Y - \gamma_K\right) Y(\tau)e^{(\gamma_Y - \gamma_K)(t-\tau)} - \left(\gamma_C - \gamma_K\right) C(\tau)e^{(\gamma_C - \gamma_K)(t-\tau)}.$$

In order for this to hold for all t, any of these four conditions must be fulfilled:

$$\begin{aligned}
i) \quad & \gamma_Y = \gamma_K \quad \wedge \quad \gamma_C = \gamma_K \quad \Longrightarrow \quad \gamma_Y = \gamma_K = \gamma_C, \\
ii) \quad & \gamma_Y = \gamma_K \quad \wedge \quad C(\tau) = 0, \\
iii) \quad & \gamma_C = \gamma_K \quad \wedge \quad Y(\tau) = 0, \\
iv) \quad & \gamma_Y = \gamma_C \quad \wedge \quad Y(\tau) = C(\tau).
\end{aligned}$$

Assumption that the growth rates γ_Y, γ_K and γ_C are positive implies $Y(\tau) > C(\tau) > 0$. However, this requirement is contradicted in variants *ii)*–*iv)*, therefore there is only one possible solution and that is $\gamma_Y = \gamma_K = \gamma_C$, as stated in the theorem.

Claim II.: Substituting $Y(\tau) = e^{-\gamma_Y(t-\tau)} \cdot Y(t)$, $K(\tau) = e^{-\gamma_K(t-\tau)} \cdot K(t)$ and $L(\tau) = e^{-\gamma_L(t-\tau)} \cdot L(t)$ into the aggregate production function (2.1) gives for any $t \geq \tau$

$$e^{-\gamma_Y(t-\tau)} \cdot Y(t) = \tilde{F}\left[e^{-\gamma_K(t-\tau)} \cdot K(t), e^{-\gamma_L(t-\tau)} \cdot L(t), \tilde{A}(\tau)\right].$$

Because $\tilde{F}$ exhibits constant returns to scale in $K(t)$ and $L(t)$, multiplying both sides by $e^{\gamma_Y(t-\tau)}$ gives

$$Y(t) = \tilde{F}\left[e^{(\gamma_Y-\gamma_K)(t-\tau)} \cdot K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} \cdot L(t), \tilde{A}(\tau)\right].$$

The previous part proved that $\gamma_Y = \gamma_K$, applying this result here gives for any $t \geq \tau$

$$Y(t) = \tilde{F}\left[K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} \cdot L(t), \tilde{A}(\tau)\right].$$

The fact that the last equality holds for all $t \geq \tau$ and that $\tilde{F}$ exhibits constant returns to scale in $K(t)$ and $L(t)$, implies existence of a function $F : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$, with constant returns to scale in $K(t)$ and $L(t)$, such that

$$Y(t) = F\left[K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} \cdot L(t)\right] = F\left[K(t), A(t)L(t)\right],$$

with $A(t) = e^{(\gamma_Y-\gamma_L)(t-\tau)} \in \mathbb{R}_{++}$.

□

The message of the theorem is that constant growth of $Y(t)$, $C(t)$, and $K(t)$ after some finite time τ is possible only if technological progress has a representation as purely labor-augmenting (Harrod-neutral). The reason is that capital grows faster than labor, which is inconsistent with the requirement of constant growth. Labor-augmenting technological progress balances out this inequality.

However, the theorem does not provide any economic reason why such a strict assumption should hold. One attempt to justify this restriction is presented in the next chapter.

Finally, as was said in the introductory part, one way how to avoid the discussion about the required form of technological progress is to use only Cobb-Douglas production functions. Starting with

$$Y(t) = \left(B(t)K(t)\right)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha},$$

where $B(t)$ is capital-augmenting and $A(t)$ is labor-augmenting technological progress with some constant growth rates, after defining $C(t) = B(t)^{\alpha/(1-\alpha)}A(t)$, this function becomes

$$Y(t) = K(t)^{\alpha}\left(C(t)L(t)\right)^{1-\alpha},$$

that is a production function with purely labor-augmenting technological progress $C(t)$.

3 Directed Technological Change

This chapter presents a model of directed technological change from Acemoglu (2009, Chapter 15) which attempts to find a reason for steady-state technological progress to be purely labor-augmenting. In this model, technological progress is endogenous and results from development of new factor-complementing machines. So the question is what determines the direction of technological progress. An intuitive assumption is that profit-maximizing firms prefer to invest into inventing that factor-augmenting technological change, which has a higher profitability. In fact, the relative profitability of different factor-augmenting technologies depends on two effects:

1. *The price effect*: firms tend to invent technologies that produce goods commanding higher prices.
2. *The market size effect*: a higher profit is gained by inventing technologies that have a larger market.

3.1 The Model

Consider an economy with a representative household whose preferences are given by

$$\int_0^\infty e^{-\rho t}\frac{C(t)^{1-\theta}-1}{1-\theta}\mathrm{d}t, \tag{3.1}$$

where ρ denotes the time preference, and θ represents the risk aversion. This household holds a balanced portfolio of all firms that operate in this economy.

In the baseline model of directed technological change, supply of both input factors is constant, i.e., $\forall t : K(t) = K$ and $L(t) = L$.

Final good $Y(t)$ is assembled from a labor-intensive intermediate good $Y_L(t)$ and a capital-intensive intermediate good $Y_K(t)$, according to a constant-elasticity-of-substitution (CES) production function

$$Y(t) = \left[\lambda_L Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda_K Y_K(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{3.2}$$

where λ_L and λ_K are share parameters determining the importance of a particular intermediate good in the aggregate production such that $\lambda_K + \lambda_K = 1$, and ε is the elasticity of substitution between the two intermediate goods, $0 \leq \varepsilon < \infty$.

- When $\varepsilon = 0 \implies \frac{\varepsilon-1}{\varepsilon} \longrightarrow -\infty$, there is no substitution between $Y_K(t)$ and $Y_L(t)$ and production function is Leontieff.
- When $\varepsilon = 1 \implies \frac{\varepsilon-1}{\varepsilon} \longrightarrow 0$, then the production function is Cobb-Douglas.
- When $\varepsilon = \infty \implies \frac{\varepsilon-1}{\varepsilon} = 1$, then $Y_K(t)$ and $Y_L(t)$ are perfect substitutes and the production function is linear.
- With $\varepsilon < 1$, $Y_K(t)$ and $Y_L(t)$ are gross complements, for $\varepsilon > 1$ they are gross substitutes.

The intermediate goods $Y_L(t)$ and $Y_K(t)$ are produced competitively by using machines and inputs of labor or capital:

$$\begin{aligned} Y_L(t) &= \frac{1}{1-\beta}\left(\int_0^{N_L(t)} x_L(j,t)^{1-\beta}\,\mathrm{d}j\right) L^\beta, \\ Y_K(t) &= \frac{1}{1-\beta}\left(\int_0^{N_K(t)} x_K(j,t)^{1-\beta}\,\mathrm{d}j\right) K^\beta, \end{aligned} \tag{3.3}$$

where $x_L(j,t)$ is the quantity of type j machine complementing labor, $x_K(j,t)$ is the quantity of type j machine complementing capital. Sets $[0, N_L(t)]$ and $[0, N_K(t)]$ denote the varieties of different machines $x_L(j,t)$ and $x_K(j,t)$ used in the production process at time t. These machines fully depreciate after they are used. Terms K^β and L^β are scaling terms to reflect the size of a market for each intermediate good. Important feature is that for a given values of $N_L(t)$ and $N_K(t)$ these production functions have constant returns to scale.

Economy's resource constraint is

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{3.4}$$

where $X(t)$ is total spending on machines and $Z(t)$ is aggregate R&D expenditure.

In Acemoglu's (2009) baseline model, the innovation possibilities frontier, which describes invention of new types of machines in R&D sector, has a so-called "*lab-equipment specification*". Investment is directed into laboratories and research equipment, and not into researches. It takes the form

$$\dot{N}_L(t) = \eta_L Z_L(t) \quad \text{and} \quad \dot{N}_K(t) = \eta_K Z_K(t), \tag{3.5}$$

where η_K, $\eta_L > 0$, $Z_L(t)$ is the amount of resources spend on inventing new labor-augmenting machines, and $Z_K(t)$ is spending on invention of new capital-augmenting machines, such that

$$Z_L(t) + Z_K(t) = Z(t).$$

The parameters η_L and η_K determine how many new types of machines are created with one unit of final good. The more is invested into R&D, the more machines are invented. Expanding the set of machines by increasing $N_L(t)$ or $N_K(t)$ corresponds to labor-augmenting or capital-augmenting technological progress.

Once a new type of machine is developed, the research company that developed it obtains a patent on this machine, which makes the company this machine's only supplier. Each machine is produced at a fixed marginal cost and fully depreciates after being used in the production of intermediate goods. Since the machine producers act under monopolistic competition, they choose the rental prices for their machines as a fixed mark-up over the marginal cost.

Inventing a machine brings to its inventor a value which is a net present discounted value of future profits. This value depends on instantaneous profits from renting the machine to the intermediate good producers, and also on the market interest rate $r(t)$.

After formulating the maximization problems of the intermediate goods producers, from which the demands for factor-complementing machines are obtained, the monopolists' profits, and therefore also their net present discounted values, are found to be independent of the concrete machine type j, but dependent solely on which factor their machines complement. In other words, whether the monopolist supplies the sector producing the intermediate good $Y_K(t)$, or the sector producing $Y_L(t)$.

Assuming that the model exhibits balanced growth, the relative profitability of inventing new capital-complementing machines is derived [see Acemoglu (2009, pp. 505–508)] as

$$\frac{V_K}{V_L} = \underbrace{\left(\frac{p_K}{p_L}\right)^{1/\beta}}_{\text{price effect}} \cdot \underbrace{\frac{K}{L}}_{\text{market size effect}}, \tag{3.6}$$

where p_K and p_L are the equilibrium prices of intermediate goods $Y_K(t)$ and $Y_L(t)$, respectively. This equation shows the two effects that influence the direction of technological change:

- The *price effect*: V_K/V_L is increasing in p_K/p_L, therefore, there are stronger incentives to develop machines that produce the more expensive intermediate good.
- The *market size effect*: V_K/V_L is also increasing in K/L. The larger is the supply of a production factor, the larger is the market for machines that complement this factor, thus there are stronger incentives for innovations directed at this factor.

After eliminating relative prices of intermediate goods, the relative profitability is given by

$$\frac{V_K}{V_L} = \left(\frac{\lambda_K}{\lambda_L}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K}{N_L}\right)^{-\frac{1}{\sigma}} \left(\frac{K}{L}\right)^{\frac{\sigma-1}{\sigma}}, \tag{3.7}$$

where $\sigma \equiv \varepsilon - (\varepsilon - 1)(\beta - 1) = 1 + (\varepsilon - 1)\beta$ is the derived elasticity of substitution between capital and labor. This parameter determines which of the two effects will dominate, because with $\sigma > 1$, V_K/V_L is increasing in K/L, and for $\sigma < 1$, V_K/V_L decreases in K/L.

Crucial for the analysis is the following equation for BGP technological augmentation

$$\frac{N_K}{N_L} = \left(\frac{\eta_K}{\eta_L}\right)^{\sigma} \left(\frac{\lambda_K}{\lambda_L}\right)^{\varepsilon} \left(\frac{K}{L}\right)^{\sigma-1}, \tag{3.8}$$

where the relationship between K/L and N_K/N_L again depends on the value of σ. Acemoglu (2009) favours the empirical evidence suggesting that the elasticity of substitution between capital and labor is smaller than 1. Chirinko (2008) analysed the results of various studies that estimated the value of σ and concluded that $\sigma \in (0.40, 0.60)$. Therefore, assuming that $\sigma < 1$ in eq. (3.8), an increase in the relative supply K/L increases N_L/N_K. Since growth models usually assume that capital is not constant but accumulates, increasing$K(t)/L$ increases $N_L(t)/N_K(t)$. Acemoglu (2009) interprets this as a natural reason for technology to be more labor-augmenting than capital-augmenting. But is there a reason why technology should be purely labor-augmenting? Unfortunately, the answer is the same as the one provided by Uzawa (1961) — it is the only technological form that allows the model to exhibit balanced growth once the capital deepening is included. To make the analysis as simple as possible, Acemoglu (2009) assumes that capital stock grows at exogenous rate

$$\frac{\dot{K}(t)}{K(t)} = b_K > 0. \tag{3.9}$$

Also capital depreciation is left out, and so the price of capital equals the interest rate $r(t)$. Along the balanced growth path consumption grows at a constant rate. Household optimization implies

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left(r(t) - \rho\right). \tag{3.10}$$

Obviously, $\dot{C}(t)/C(t)$ is constant if and only if the interest rate $r(t)$ is constant. In this model, the equilibrium interest rate is given by

$$r(t) = \beta\lambda_K N_K(t) \left[\lambda_L \left(\frac{N_L(t)L}{N_K(t)K(t)}\right)^{\frac{\sigma-1}{\sigma}} + \lambda_K\right]^{\frac{1}{\sigma-1}}. \tag{3.11}$$

Equation (3.11) implies that constant interest rate requires constancy of $N_K(t)$, i.e., no capital-augmenting technological change, and also constancy of the term $(N_L(t)L)\,/\,(N_K(t)K(t))$.

For this, it is necessary to introduce the innovation possibilities frontier "*specification with knowledge spillovers*", where the research is carried out by scientists who benefit from past inventions. There is a constant supply of S scientist, and the creation of new machines is given by

$$\begin{aligned} \dot{N}_L(t) &= \eta_L N_L(t)^{(1+\delta)/2} N_K(t)^{(1-\delta)/2} S_L(t), \\ \dot{N}_K(t) &= \eta_K N_L(t)^{(1-\delta)/2} N_K(t)^{(1+\delta)/2} S_K(t), \end{aligned} \tag{3.12}$$

where $S_L(t)$ and $S_K(t)$ are the numbers of scientists working on development of labor- and capital-augmenting machines, respectively. For all t: $S_L(t) + S_K(t) \leq S$. Parameter $\delta \leq 1$ represents the innovations' state dependence.

- $\delta = 0$ (no state dependence): Technologies in both sectors create knowledge spillovers to the other sector.
- $\delta = 1$ (extreme state dependence): Knowledge spillovers affect only the corresponding sector where they were created.

However, it turns out that that balanced growth is possible only in the case of extreme state dependence, that is equation (3.12) with $\delta = 1$:

$$\frac{\dot{N}_L(t)}{N_L(t)} = \eta_L S_L(t) \quad \text{and} \quad \frac{\dot{N}_K(t)}{N_K(t)} = \eta_K S_K(t). \tag{3.13}$$

The following calculations are a replication of the ones in Acemoglu (2009, Section 15.6). This specification implies that relative factor shares are constant, i.e.,

$$\frac{r(t)K(t)}{w(t)L} = \frac{\eta_L}{\eta_K}. \tag{3.14}$$

In this economy, the relative factor prices are equal to

$$\frac{r(t)}{w(t)} = \left(\frac{\lambda_K}{\lambda_L}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K(t)}{N_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{K(t)}{L}\right)^{-\frac{1}{\sigma}}. \tag{3.15}$$

The relative factor income shares can then be obtained from equation (3.15) as

$$\underbrace{\frac{r(t)K(t)}{w(t)L}}_{=\,\eta_L/\eta_K} = \left(\frac{\lambda_K}{\lambda_L}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K(t)}{N_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{K(t)}{L}\right)^{\frac{\sigma-1}{\sigma}} = \left(\frac{\lambda_K}{\lambda_L}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K(t)K(t)}{N_L(t)L}\right)^{\frac{\sigma-1}{\sigma}}. \tag{3.16}$$

Since the term on the left-hand side of the equation is constant, $(N_K(t)K(t)) / (N_L(t)L)$ must also be constant, which means that $N_K(t)K(t)$ and $N_L(t)$ grow at equal rates, i.e.,

$$\frac{\dot{N}_K(t)}{N_K(t)} + b_K = \frac{\dot{N}_L(t)}{N_L(t)}. \tag{3.17}$$

Equation (3.17) then implies that $(N_L(t)L) / (N_K(t)K(t))$ in (3.11) is constant.

Then the balanced growth is achieved when $N_K(t)$ remains constant, and so the technological progress is purely labor-augmenting. This result is summed up in the following proposition.

Proposition 3.1 (Acemoglu, 2009, Proposition 15.12)**.** *In the baseline model of directed technological change where capital grows at exogenous rate according to* (3.9)*, and the innovation possibilities frontier is given by* (3.13)*, there exists a unique balanced growth path along which technological progress takes purely labor-augmenting (Harrod-neutral) form, output and consumption grow at constant rates, and the interest rate remains constant.*

Capital-augmenting technological progress will therefore appear only along the transition path, if the ratio of initial values $N_K(0) > 0$ and $N_L(0) > 0$ is below the BGP equilibrium value, i.e., $N_K(0)/N_L(0) < N_K/N_L$. But in the long run technology becomes purely labor-augmenting to balance out the effect of capital deepening.

3.2 Remarks

Acemoglu's (2009) model is supported by the empirical study by Klump *et al.* (2004). In their study, which estimates a supply-side system of US economy between the years 1953 and 1998 by using a normalized CES function with factor-augmenting technological progress, Klump *et al.* find that the elasticity of substitution is $\sigma \in (0.5, 0.7)$, that is a value well below one. Furthermore, the growth rates of factor-augmenting technological progress are found to be asymmetric. Labor-augmenting technological progress is found to be growing exponentially and dominating, while capital-augmenting technological progress is hyperbolic (or logarithmic). This is in line with Acemoglu's theory that because of firms' profit-maximizing motivations, technological progress is relatively more labor-augmenting, and in the long run capital-augmenting technological progress vanishes. Similar paths of technological progress are found by Arpaia *et al.* (2009) in nine countries of the EU-15 member states in period 1970-2004.

However, Acemoglu's model also meets some criticism. For example Brugger and Gehrke (2017) criticise that Acemoglu builds on the assumption that the elasticity of substitution between capital and labor is below 1. That means that all the studies that find $\sigma > 1$, or approximately equal to 1, are overlooked. But when Li and Stewart (2014) applied the framework of Klump *et al.* (2004) to data from Canada 1961-2010, the resulting elasticity of substitution was not significantly below 1, and technological progress was purely labor-augmenting in both short term and long term.

Another point of their criticism is the focus on profit-maximizing motivation, which sure is an important motivation for many companies, but other motivations like a development of general purpose technologies or research aimed to extend corporate power cannot be explained by Acemoglu's models.

Finally, their biggest concern is how strongly the result about existence of balanced growth path relies on the form of innovation possibility frontier which is given exogenously. Supporting this as a strong shortcoming is the finding of Li (2016), who points out that Proposition 3.1 holds only if $S = \frac{b_K}{\eta_L}$. Acemoglu (2009) omits to mention what the requirement of purely labor-augmenting technological progress implies for the number of scientist working in the research sector. Assuming the full employment of scientist, i.e., $S_L(t) + S_K(t) = S$, equation (3.13) implies that absence of capital-augmenting technological progress along the BGP means that $\dot{N}_K(t)/N_K(t) = \eta_K S_K(t) = 0$. Since $\eta_K > 0$, it must hold that $S_K(t) = 0$, i.e., no scientist works on development of new capital-complementary machines, but all scientists work on invention of labor-complementary machines, and so $S_L(t) = S$. Then to have $(N_L(t)L) \,/\, (N_K(\tau)K(t))$ constant for all $t \geq \tau$ requires

$$\frac{\dot{N}_L(t)}{N_L(t)} = \eta_L S_L(t) = \eta_L S = b_K = \frac{\dot{K}(t)}{K(t)}.$$

Expressing the number of scientist gives $S = \frac{b_K}{\eta_L}$. The same condition holds in case that some scientist are unemployed. Therefore, according to Li (2016), Acemoglu's (2009) model is not suitable as an explanation of why technological change must be purely labor-augmenting along the balanced growth path, because the Proposition 3.1 holds only under "extremely restrictive requirement".

4 The Generalized Steady-State Growth Theorem

The fact that growth models with endogenous capital- and labor-augmenting technological progress, like the one by Acemoglu (2009), exhibit balanced growth only when technological progress takes purely labor-augmenting form, suggests that Uzawa's steady-state growth theorem applies to them too, despite their more complex structure. To confirm this intuition, Irmen (2016) extended Uzawa's theorem and showed that the requirement for technological progress to be purely labor-augmenting is the result of models' analytical structure.

4.1 The Model

Consider a closed economy in continuous time with a production sector consisting of two parts – final good production and intermediate good production. Final output is produced according to the aggregate production function given by

$$Y(t) = \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right], \tag{4.1}$$

where $\tilde{F} : \mathbb{R}_+^2 \times \mathfrak{A}_F \mapsto \mathbb{R}_+$ is assumed to be increasing in $K(t)$ and $L(t)$ and exhibits constant returns to scale in both $K(t)$ and $L(t)$. $\mathbf{A}_F(t) \in \mathfrak{A}_F$ stands for the technology which affects the final good production.

In each period t, a fraction of the aggregate production is used as an input elsewhere in the economy, for example, as in the previous model, to finance the R&D research, or to produce intermediate goods. These resources are therefore not used for capital accumulation or consumption. They are referred to as *aggregate intermediate expenses*

$$E(t) = \tilde{E}\left[K(t), L(t), \mathbf{A}_E(t)\right], \tag{4.2}$$

where $\tilde{E} : \mathbb{R}^2_+ \times \mathfrak{A}_E \mapsto \mathbb{R}_+$ is the *aggregate intermediate expenses function* and $\mathbf{A}_E(t) \in \mathfrak{A}_E$ is the technology affecting the quantity of expended final output. Also $\tilde{E}$ is assumed to be increasing in $K(t)$ and $L(t)$ with constant returns to scale in these arguments.

The difference between aggregate production $Y(t)$ and aggregate expenses $E(t)$ determines the amount of resources which are available for consumption and capital accumulation. These resources are referred to as a net output $V(t)$.

$$\begin{aligned} V(t) &= \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right] - \tilde{E}\left[K(t), L(t), \mathbf{A}_E(t)\right], \\ &\equiv \tilde{V}\left[K(t), L(t), \mathbf{A}_F(t), \mathbf{A}_E(t)\right], \end{aligned} \tag{4.3}$$

where $\tilde{V} : \mathbb{R}^2_+ \times \mathfrak{A}_F \times \mathfrak{A}_E \mapsto \mathbb{R}_+$ is the *net output function* with constant returns to scale in capital and labor. Capital accumulation is given by

$$\dot{K}(t) = V(t) - C(t) - \delta_K K(t), \quad \delta_K \in \mathbb{R}_+, \tag{4.4}$$

where $C(t)$ represents aggregate consumption and δ_K denotes the instantaneous depreciation rate of capital. Labor supply evolves according to

$$L(t) = L(0)e^{\gamma_L t}, \quad L(0) > 0, \quad \gamma_L \in \mathbb{R}. \tag{4.5}$$

Compared to equation (2.5), here γ_L is allowed to be also negative or zero, as in some models of endogenous technical change (like the one by Acemoglu (2009)) the labor force remains constant over time.

4.2 The Generalized Steady-State Growth Theorem

The following theorem extends Uzawa's steady-state growth theorem to the neoclassical economy with aggregate intermediate expenses.

Theorem 4.1 (The Generalized Steady-State Growth Theorem; Irmen, 2016, Theorem 1)**.** *Suppose that in the economy where net output is produced according to* (4.3)*, capital accumulates according to* (4.4)*, and the evolution of labor input is given by* (4.5)*, there exists a steady-state path beginning at some* $\tau < \infty$ *along which* $\forall t \geq \tau : Y(t) > V(t) > C(t) > 0$*. Then for all* $t \geq \tau$ *holds:*

I. $\gamma_V = \gamma_Y = \gamma_E = \gamma_K = \gamma_C$*.*

II. Net output takes the form

$$V(t) = V\left[K(t), A(t)L(t)\right],$$

where $A(t) = e^{(\gamma_V - \gamma_L)(t-\tau)} \in \mathbb{R}_{++}$, *and*

$$\gamma = \gamma_V - \gamma_L,$$

is the growth rate of per-worker variables.

The generalized steady-state growth theorem states that technological progress in the net output function takes purely labor-augmenting form in the steady-state, and that all per-worker variables $V(t)/L(t)$, $Y(t)/L(t)$, $I(t)/L(t)$, $K(t)/L(t)$ and $C(t)/L(t)$ grow at the growth rate of technology.

Proof. Following the proof in Irmen (2016, p. 23-24, part A.1.):

Claim I.: Proof of the first claim comes in two parts.

1. part: Along the steady-state path for all $t \geq \tau$, it holds that $V(t) = V(\tau)e^{\gamma_V(t-\tau)}$, $Y(t) = Y(\tau)e^{\gamma_Y(t-\tau)}$ and $E(t) = E(\tau)e^{\gamma_E(t-\tau)}$. Then, from the definition $V(t) = Y(t) - E(t)$, it holds that

$$V(\tau)e^{\gamma_V(t-\tau)} = Y(\tau)e^{\gamma_Y(t-\tau)} - E(\tau)e^{\gamma_E(t-\tau)}, \quad \text{for all } t \geq \tau.$$

Dividing both sides of equation by $e^{\gamma_V(t-\tau)}$ and then differentiating it with respect to time t gives

$$0 = (\gamma_Y - \gamma_V)\,Y(\tau)e^{(\gamma_Y-\gamma_V)(t-\tau)} - (\gamma_E - \gamma_V)\,E(\tau)e^{(\gamma_E-\gamma_V)(t-\tau)}.$$

In order for this equation to hold for all t, any of these four conditions must be fulfilled:

$$\begin{aligned}
&i) && \gamma_Y = \gamma_V && \wedge && \gamma_E = \gamma_V && \Longrightarrow && \gamma_Y = \gamma_V = \gamma_E,\\
&ii) && \gamma_Y = \gamma_V && \wedge && E(\tau) = 0,\\
&iii) && \gamma_E = \gamma_V && \wedge && Y(\tau) = 0,\\
&iv) && \gamma_Y = \gamma_E && \wedge && Y(\tau) = E(\tau).
\end{aligned}$$

Theorem 4.1 assumes that $Y(t) > V(t) > C(t) > 0$ for all $t \geq \tau$. Together with the definition of net output $V(t) = Y(t) - E(t)$, it follows that $Y(t) > E(t) > 0$ for all $t \geq \tau$. Out of the four possible solutions, the only one which does not violate this restriction is the variant *i*), with the growth rates being identical to each other, i.e. $\gamma_Y = \gamma_V = \gamma_E$.

2. part: Similar approach is used for the capital accumulation equation (4.4):

$$\dot{K}(t) = V(t) - C(t) - \delta_K K(t).$$

This part of the proof is identical to the proof of Uzawa's theorem 2.1 when $Y(t)$ is replaced by the net output function $V(t)$. There are again four possible solutions: *i)* $\gamma_V = \gamma_K = \gamma_C$, or *ii)* $\gamma_V = \gamma_K$ and $C(\tau) = 0$, or *iii)* $\gamma_C = \gamma_K$ and $V(\tau) = 0$, or finally *iv)* $\gamma_V = \gamma_C$ and $V(\tau) = C(\tau)$. The only solution that satisfies the assumption $V(\tau) > C(\tau) > 0$ is the first one, in which the growth rates γ_V, γ_K and γ_C coincide. Combining this with the result from the first part gives the first statement of the theorem: $\gamma_V = \gamma_Y = \gamma_E = \gamma_K = \gamma_C$.

Claim II.: Proof of the second claim starts with the equation (4.3) for net output at time $t = \tau$:

$$V(\tau) = \tilde{V}\left[K(\tau), L(\tau), \mathbf{A}_F(\tau), \mathbf{A}_E(\tau)\right].$$

Applying $V(\tau) = e^{-\gamma_V(t-\tau)} \cdot V(t)$, $K(\tau) = e^{-\gamma_K(t-\tau)} \cdot K(t)$ and $L(\tau) = e^{-\gamma_L(t-\tau)} \cdot L(t)$ gives for any $t \geq \tau$

$$e^{-\gamma_V(t-\tau)} \cdot V(t) = \tilde{V}\left[e^{-\gamma_K(t-\tau)} \cdot K(t), e^{-\gamma_L(t-\tau)} \cdot L(t), \mathbf{A}_F(\tau), \mathbf{A}_E(\tau)\right].$$

Because $\tilde{V}$ exhibits constant returns to scale in $K(t)$ and $L(t)$, multiplying both sides by $e^{\gamma_V(t-\tau)}$ gives

$$V(t) = \tilde{V}\left[e^{(\gamma_V-\gamma_K)(t-\tau)} \cdot K(t), e^{(\gamma_V-\gamma_L)(t-\tau)} \cdot L(t), \mathbf{A}_F(\tau), \mathbf{A}_E(\tau)\right].$$

Applying the result from the previous part that $\gamma_V = \gamma_K$, gives for any $t \geq \tau$

$$V(t) = \tilde{V}\left[K(t), e^{(\gamma_V-\gamma_L)(t-\tau)} \cdot L(t), \mathbf{A}_F(\tau), \mathbf{A}_E(\tau)\right].$$

The fact that the last equality holds for all $t \geq \tau$ and that $\tilde{V}$ exhibits constant returns to scale in $K(t)$ and $L(t)$, implies existence of a function $V : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$, with constant returns to scale in $K(t)$ and $L(t)$, such that

$$V(t) = V\left[K(t), e^{(\gamma_V-\gamma_L)(t-\tau)} \cdot L(t)\right] = V\left[K(t), A(t)L(t)\right],$$

with $A(t) = e^{(\gamma_V-\gamma_L)(t-\tau)} \in \mathbb{R}_{++}$.

In addition, since $\gamma_V = \gamma_Y = \gamma_E = \gamma_K = \gamma_C$, growth rate of all per-worker variables $V(t)/L(t)$, $Y(t)/L(t)$, $E(t)/L(t)$, $K(t)/L(t)$ and $C(t)/L(t)$ can be expressed as $\gamma = \gamma_V - \gamma_L$, which is also the growth rate of labor-augmenting technology $A(t)$. This clarifies the second statement of the theorem.

$\square$

4.3 Special Cases

4.3.1 The Absence of Aggregate Intermediate Expenses Function $E(t)$

As mentioned above, assumptions of Theorem 4.1 imply that $E(t) > 0$ for all $t \geq \tau$. If there were no aggregate intermediate expenses, i.e., if $\forall t \geq \tau$: $E(t) = 0$, then the net output would be equal to the aggregate output, because $V(t) = Y(t) - E(t) = Y(t) - 0 = Y(t)$. In that case, Irmen's model is reduced to one-sector production model and Irmen's theorem is the same as Uzawa's original theorem.

4.3.2 A Model Without Capital Accumulation

This point highlights the importance of capital accumulation in the model. $V(t) = C(t)$ means that whole net output is consumed and nothing is left for the capital accumulation. How this assumption changes Theorem 4.1 is reflected in the following corollary from Irmen (2016, p. 7).

Corollary 4.1 (Irmen, 2016, Corollary 1)**.** *Suppose that in the economy where net output is produced according to* (4.3)*, capital accumulates according to* (4.4)*, and the evolution of labor input is given by* (4.5)*, exists a steady-state path beginning at some $\tau < \infty$ along which $\forall t \geq \tau : Y(t) > V(t) = C(t) > 0$. Then for all $t \geq \tau$ holds:*

I. $\gamma_V = \gamma_Y = \gamma_E = \gamma_C$ *and* $\gamma_K = -\delta_K$.

II. Net output takes the form

$$V(t) = V\left[B(t)K(t), A(t)L(t)\right],$$

where $B(t) = e^{(\gamma_V + \delta_K)(t-\tau)} \in \mathbb{R}_{++}$ *and* $A(t) = e^{(\gamma_V - \gamma_L)(t-\tau)} \in \mathbb{R}_{++}$.
Capital per worker grows at rate $-(\delta_K + \gamma_L)$*. All the other per-worker variables grow at rate* $\gamma = \gamma_V - \gamma_L$.

Proof. See Irmen (2016, p. 24, part A.2.). □

First result is quite intuitive since $Y(t) > E(t) > 0$ still holds as in Theorem 4.1, therefore $\gamma_V = \gamma_Y = \gamma_E$ as was proved above. $V(t) = C(t)$ implies that $\gamma_V = \gamma_C$, and no capital accumulation means that capital only depreciates at its depreciation rate, so $\gamma_K = -\delta_K$.

Second claim shows that in the steady-state the capital-augmenting technological progress $B(t)$ does not vanish as it compensates for the missing capital accumulation to match the growth of capital to the growth of net output. $B(t)$ vanishes only when the growth rates of $V(t)$ and $K(t)$ coincide, i.e., only if $\gamma_V = -\delta_K$.

The impact of capital accumulation equation on the steady-state technological progress will be further discussed in Chapter 5.

4.3.3 Aggregate Functions With Factor-Augmenting Technological Change

Aggregate production function and aggregate intermediate expenses function with factor-augmenting technological change can be expressed as

$$Y(t) = F\left[B_F(t)K(t), A_F(t)L(t)\right] \text{ and } E(t) = E\left[B_E(t)K(t), A_E(t)L(t)\right],$$

where $B_F(t), B_E(t), A_F(t)$ and $A_E(t) \in \mathbb{R}_{++}$. Capital-augmenting technology is represented by $B_F(t)$ and $B_E(t)$, labor-augmenting technology by $A_F(t)$ and $A_E(t)$. The last point of the Discussion part 2.3 in Irmen (2016) aims to provide conditions which allow for $\gamma_{B_F} \neq 0$ and/or $\gamma_{B_E} \neq 0$ in steady-state. These conditions are described in the following corollary.

Corollary 4.2 (Irmen, 2016, Corollary 2)**.** *Suppose that in the economy where net output is is given by*

$$V(t) = F\left[B_F(t)K(t), A_F(t)L(t)\right] - E\left[B_E(t)K(t), A_E(t)L(t)\right], \tag{4.6}$$

capital accumulates according to (4.4)*, and the evolution of labor input is given by* (4.5)*, exists a steady-state path beginning at some $\tau < \infty$ along which $\forall t \geq \tau : Y(t) > V(t) > C(t) > 0$. Then for all $t \geq \tau$ holds:*

1. *If $\gamma_{B_F} = \gamma_{B_E} = 0$, then $\gamma = \gamma_{A_F} = \gamma_{A_E}$.*
2. *If $\gamma_{B_F} = 0$ and $\gamma_{B_E} \neq 0$, then net output is given by*

$$V(t) = F\left[B_F(\tau)K(t), A_F(t)L(t)\right] - \beta_E K(t)^{\alpha_E}\left(e^{\gamma(t-\tau)}L(t)\right)^{1-\alpha_E},$$

   where $0 < \alpha_E < 1$, $\beta_E = c_E B_E(\tau)^{\alpha_E} A_E(\tau)^{(1-\alpha_E)} > 0$, and

$$\gamma = \frac{\alpha_E \gamma_{B_E}}{1-\alpha_E} + \gamma_{A_E} = \gamma_{A_F}. \tag{4.7}$$

3. *If $\gamma_{B_F} \neq 0$ and $\gamma_{B_E} = 0$, then net output is given by*

$$V(t) = \beta_F K(t)^{\alpha_F}\left(e^{\gamma(t-\tau)}L(t)\right)^{1-\alpha_F} - E\left[B_E(\tau)K(t), A_E(t)L(t)\right],$$

   where $0 < \alpha_F < 1$, $\beta_F = c_F B_F(\tau)^{\alpha_F} A_F(\tau)^{(1-\alpha_F)} > 0$, and

$$\gamma = \frac{\alpha_F \gamma_{B_F}}{1-\alpha_F} + \gamma_{A_F} = \gamma_{A_E}. \tag{4.8}$$

4. If $\gamma_{B_F} \neq 0$ and $\gamma_{B_E} \neq 0$, then net output is given by

$$V(t) = \beta_F K(t)^{\alpha_F} \left(e^{\gamma(t-\tau)} L(t)\right)^{1-\alpha_F} - \beta_E K(t)^{\alpha_E} \left(e^{\gamma(t-\tau)} L(t)\right)^{1-\alpha_E},$$

and

$$\gamma = \frac{\alpha_E \gamma_{B_E}}{1-\alpha_E} + \gamma_{A_E} = \frac{\alpha_F \gamma_{B_F}}{1-\alpha_F} + \gamma_{A_F}. \tag{4.9}$$

Proof. See Irmen (2016, pp. 24-26, part A.3.). □

Corollary 4.2 shows that capital-augmenting technological progress may persist in the steady-state if the corresponding aggregate function is Cobb-Douglas. This means that, unfortunately, Irmen (2016) does not derive anything new, but simply uses the fact that technological progress in Cobb-Douglas function can be always expressed as purely labor-augmenting.

For example, in the case described in Claim 2., $\gamma_{B_F} = 0$ and $\gamma_{B_E} \neq 0$ imply that technology in $Y(t)$ is purely labor-augmenting, while it is factor-augmenting in $E(t)$. Proof in Irmen (2016, p. 24-25, part A.3.) shows that existence of steady-state implies for $E(t)$ to take the form

$$E(t) = c_E \left(B_E(t)K(t)\right)^{\alpha_E} \left(A_E(t)L(t)\right)^{1-\alpha_E},$$

where $c_E > 0$ is an integration constant. Then substituting $B_E(t) = B_E(\tau)e^{\gamma_{B_E}(t-\tau)}$ and $A_E(t) = A_E(\tau)e^{\gamma_{A_E}(t-\tau)}$ gives

$$\begin{aligned} E(t) &= c_E \left(B_E(\tau)e^{\gamma_{B_E}(t-\tau)} \cdot K(t)\right)^{\alpha_E} \left(A_E(\tau)e^{\gamma_{A_E}(t-\tau)} \cdot L(t)\right)^{1-\alpha_E} \\ &= c_E \left(B_E(\tau)K(t)\right)^{\alpha_E} \left(A_E(\tau)e^{\left(\frac{\alpha_E\gamma_{B_E}}{1-\alpha_E}+\gamma_{A_E}\right)(t-\tau)} \cdot L(t)\right)^{1-\alpha_E}. \end{aligned}$$

Denoting $\gamma = \frac{\alpha_E\gamma_{B_E}}{1-\alpha_E} + \gamma_{A_E}$ and combining all the constant terms into $\beta_E = c_E B_E(\tau)^{\alpha_E} A_E(\tau)^{(1-\alpha_E)}$ simplifies the term above into

$$E(t) = \beta_E K(t)^{\alpha_E} \left(e^{\gamma(t-\tau)} \cdot L(t)\right)^{1-\alpha_E},$$

which is a function with purely labor-augmenting technological change at rate γ. Net output function then takes the form stated in the corollary. Because $Y(t)$ and $E(t)$ are required to grow at the same rate, γ_{A_F} and γ must align as described by condition (4.7). An analogous conditions hold for the other cases.

However, if Cobb-Douglas function is excluded from the discussion because of the empirical evidence suggesting that the elasticity of substitution between capital and labor is less than 1, then this corollary becomes irrelevant.

4.4 Example of Application

This section demonstrates how the generalized steady-state growth theorem can be applied to Acemoglu's (2009) model of directed technological change, which was introduced in the previous chapter. The description here follows Irmen (2016, Section 3.3).

First of all, Acemoglu (2009) states that having a CES production function (3.2) is not necessary for his results. Thus without loss of generality this production function can be replaced by

$$Y(t) = F\left[Y_K(t), Y_L(t)\right], \tag{4.10}$$

where F is increasing and has constant returns to scale in $Y_K(t)$ and $Y_L(t)$.

Next simplification is setting $\beta = 1/2$ in the production functions of intermediate goods (3.3), which implies

$$\begin{aligned} Y_K(t) &= 2\sqrt{K(t)} \int_0^{N_K(t)} \sqrt{x_K(j,t)}\,\mathrm{d}j, \\ Y_L(t) &= 2\sqrt{L(t)} \int_0^{N_L(t)} \sqrt{x_L(j,t)}\,\mathrm{d}j. \end{aligned} \tag{4.11}$$

The final output serves an in input in the production of factor-complementing machines. These resources are what Irmen (2016) referred to as intermediate expenses. Since all the machines are assumed to have an equal marginal production cost, the amount of final output that is required for producing the whole range of capital- or labor-complementary machines, respectively, can be expressed as

$$N_K(t)x_K(t) = \zeta_K(t)Y(t) \quad \text{and} \quad N_L(t)x_L(t) = \zeta_L(t)Y(t), \tag{4.12}$$

where $\zeta_K(t) > 0$ and $\zeta_L(t) > 0$. To ensure that not all amount of final output is used on production of machines, it holds that $0 < \zeta_L(t) + \zeta_K(t) < 1$. Then aggregate intermediate expenses can be written as a sum of these two quantities,

$$E(t) = \left[\zeta_K(t) + \zeta_L(t)\right] Y(t). \tag{4.13}$$

Expressing $x_K(t)$ and $x_L(t)$ from equation (4.12) and substituting them into the production functions of intermediate goods (4.11) gives these production function in form

$$Y_K(t) = 2\sqrt{\zeta_K(t)N_K(t)K(t)Y(t)} \quad \text{and} \quad Y_L(t) = 2\sqrt{\zeta_L(t)N_L(t)LY(t)}. \tag{4.14}$$

Substituting these functions into (4.10) gives

$$Y(t) = F\left[2\sqrt{\zeta_K(t)N_K(t)K(t)Y(t)}, 2\sqrt{\zeta_L(t)N_L(t)LY(t)}\right]. \tag{4.15}$$

Because F exhibits constant returns to scale in both arguments, this equation can be solved for $Y(t)$ and the aggregate output function can be written as

$$Y(t) = 4\left(F\left[\sqrt{\zeta_K(t)N_K(t)K(t)}, \sqrt{\zeta_L(t)N_L(t)L}\right]\right)^2 = \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right], \tag{4.16}$$

which corresponds to equation (4.1).

Substituting this into equation (4.13) implies that the aggregate intermediate expenses function takes form

$$\tilde{E}\left[K(t), L(t), \mathbf{A}_E(t)\right] = \left[\zeta_K(t) + \zeta_L(t)\right]\tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right], \tag{4.17}$$

which corresponds to equation (4.2).

Finally, the specification of R&D sector remains the same as in (3.13). Without loss of generality, η_K and η_L can be set equal to 1.

Given the equations (4.16) and (4.17), the net output function $V(t) = Y(t) - E(t)$ has the form

$$\tilde{V}\left[K(t), L(t), \mathbf{A}_F(t), \mathbf{A}_E(t)\right] = \left(1 - \zeta_K(t) - \zeta_L(t)\right)\tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right]. \tag{4.18}$$

The following proposition describes the conditions for existence of steady-state path in this model. For simplification of notation: $B_F(t) \equiv \zeta_K(t)N_K(t)$ and $A_F(t) \equiv \zeta_L(t)N_L(t)$.

Proposition 4.1 (Irmen, 2016, Proposition 3)**.** *Suppose that in the economy where net output is given by* (4.18)*, capital accumulates according to* (4.4)*, and the evolution of labor input is given by* (4.5) *with* $\gamma_L = 0$*, there exists a steady-state path beginning at some* $\tau < \infty$ *along which* $\forall t \geq \tau : Y(t) > V(t) > C(t) > 0$*. Then for all* $t \geq \tau$ *holds:*

1. *With* $\gamma_{N_K} \neq 0$*, net output must be given by*

$$V(t) = \left(1 - \zeta_K(\tau) - \zeta_L(\tau)\right)\beta_F\left[K(t)\right]^{\alpha_F}\left[e^{\gamma(t-\tau)}L\right]^{1-\alpha_F},$$

where

$$\gamma = \frac{\alpha_F\gamma_{B_F}}{1-\alpha_F} + \gamma_{A_F}.$$

2. *In case that the production function* F *is not Cobb-Douglas, then*

$$V(t) = \left(1 - \zeta_K(\tau) - \zeta_L(\tau)\right)F\left(N_K(\tau)K(t), N_L(t)L\right),$$

$\gamma_{N_K} = 0$ *and* $\gamma = \gamma_{N_L}$*.*

The simplified model satisfies the assumptions of Theorem 4.1 with $\gamma_L = 0$. Then from the Theorem's first claim, in steady-state $\gamma_V = \gamma_Y = \gamma_E$, which holds only when both $\zeta_K(t)$ and $\zeta_L(t)$ remain constant along the steady-state path. That justifies the term $(1 - \zeta_K(\tau) - \zeta_L(\tau))$ in the net output function. According to the second claim of Theorem 4.1, steady-state technological change must have a representation as purely labor-augmenting. This condition, according to Corollary 4.2, gives the two options presented in Proposition 4.1. If γ_{N_K} is to be non-zero in the steady-state, then the function F must be Cobb-Douglas. If F is not Cobb-Douglas, then the capital-augmenting technology must be constant in the steady-state, i.e., $\gamma_{N_K} = 0$.

The other examples presented in Irmen (2016) are made for the models of Irmen and Tabaković (2015), Acemoglu (2003), and the 1960s literature on induced innovations. In the first one, final output is used to generate technological progress. This qualifies as intermediate expenses and Theorem 4.1 characterizes the steady-state path of the model in a similar way as for the model of Acemoglu (2009). In the model of Acemoglu (2003), neither the intermediate goods nor the technological progress require resources in the form of final good output, so there are no intermediate expenses, i.e., $E(t) = 0$. Then $V(t) = Y(t)$ and Uzawa's theorem applies to the model. The same holds for the model of induced innovations, where technological progress is cost-free.

5 Steady-State With Capital-Augmenting Technological Change

The statements of Theorems 2.1 and 4.1 indicate that steady-state technological progress must be represented as purely labor-augmenting irrespective of whether it is treated as an exogenous or as an endogenous variable. However, models presented in this chapter argue that such a constraint is unnecessary, and that there are conditions that allow for a steady-state technological progress to be capital-augmenting as well.

5.1 A Model with Adjustment Costs

This section presents neoclassical growth model with adjustment costs, as described in Irmen (2013) and Li *et al.* (2013, 2014, 2016).

Consider an economy in which production function of final output and all its properties is identical to the function (4.1), that is

$$Y(t) = \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right]. \tag{5.1}$$

The resource constraint also remains the same as before

$$Y(t) = C(t) + I(t). \tag{5.2}$$

Replacing old capital stock with a new one usually generates all sorts of costs that come, for example, from the installation of new machines, employees' training about manipulation with these machines, etc. Therefore, the capital investment $I(t)$ can be expressed as a sum of newly purchased capital goods, $G(I(t))$, and the additional adjustment costs, $AC(I(t))$, that came up with this capital purchase

$$I(t) = G(I(t)) + AC(I(t)). \tag{5.3}$$

$G(I(t))$ is such that

$$G : \mathbb{R}_+ \mapsto \mathbb{R}_+, \quad G(0) = 0. \tag{5.4}$$

Irmen (2013) calls $G(I(t))$ the "*installation function of capital*", Li *et al.* (2013) use the term "*efficiency function of capital*". Including adjustment costs into the model creates inequality between gross and net capital investment

$$I(t) > G(I(t)) > 0. \tag{5.5}$$

For all $I(t) > 0$ the adjustment costs function $AC : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing and strictly convex, which imposes a restriction

$$1 > \frac{dG(I(t))}{dI(t)} > 0 > \frac{d^2G(I(t))}{dI(t)^2}. \tag{5.6}$$

Capital accumulation equation in this economy is given by

$$\dot{K}(t) = G(I(t)) - \delta_K K(t), \quad \delta \in \mathbb{R}_+, \tag{5.7}$$

and the labor force evolves according to

$$L(t) = L(0)e^{\gamma_L t}, \quad L(0) > 0, \quad \gamma_L \in \mathbb{R}. \tag{5.8}$$

Definition 5.1 (Irmen, 2013, Definition 1)**.** *In the economy described by equations* (5.1), (5.2), (5.7), *and* (5.8), *a steady-state is a path along which* $\forall t \geq \tau \geq 0$: $Y(t)$, $I(t)$, $C(t)$, $K(t)$, $L(t)$ *and* $G(I(t))$ *grow at constant rates.*

Li *et al.* (2013) claim that presence of capital-augmenting technological change $B(t)$ in the steady state is restricted by the condition

$$\frac{\dot{G}_I}{G_I} + \frac{\dot{B}}{B} = 0, \tag{5.9}$$

where $G_I \equiv dG(I(t))/dI(t)$ is the *marginal efficiency of capital investment.*

Irmen (2013) goes one step further and demonstrates that, apart from other conditions, $G(I(t))$ must have a form of a power function. His results about the steady-state of this model are summarized in the following theorem.

Theorem 5.1 (Irmen, 2013, Theorem 1)**.** *Suppose that in the economy described by equations* (5.1)*,* (5.2)*,* (5.7)*, and* (5.8)*, exists a steady-state path beginning at some* $\tau < \infty$ *along which* $\forall t \geq \tau : Y(t) > C(t) > 0$*. Then for all* $t \geq \tau$ *holds:*

I. $\gamma_Y = \gamma_C = \gamma_I$.

II. Either

$$\gamma_K = \gamma_G = \phi\gamma_Y \neq 0,$$

together with

$$G(I(t)) = c \cdot I(t)^{\phi}, \quad c > 0, \quad 0 < \phi < 1,$$

or

$$\gamma_K = \gamma_G = \phi\gamma_Y = 0.$$

III. Total output takes the form

$$Y(t) = F\left[B(t)K(t), A(t)L(t)\right]$$

with $B(t) = e^{\gamma_Y(1-\phi)(t-\tau)} \in \mathbb{R}_{++}$ *and* $A(t) = e^{(\gamma_Y - \gamma_L)(t-\tau)} \in \mathbb{R}_{++}$*.*
Growth rate of capital per worker is $\gamma_K - \gamma_L = \phi\gamma_Y - \gamma_L$*. All the other per worker variables grow at rate* $\gamma = \gamma_Y - \gamma_L$*.*

Proof. This proof once again builds on and extends the variant by Schlicht (2006), and can be found in Irmen (2013, pp. 2870–2871).

Claim I.: Theorem's assumption $Y(t) > C(t) > 0$ implies $I(t) > 0$ for all $t \geq \tau$. Substituting $Y(t) = Y(\tau)e^{\gamma_Y(t-\tau)}$, $I(t) = I(\tau)e^{\gamma_I(t-\tau)}$ and $C(t) = C(\tau)e^{\gamma_C(t-\tau)}$ into the resource constraint (5.2) gives

$$I(\tau)e^{\gamma_I(t-\tau)} = Y(\tau)e^{\gamma_Y(t-\tau)} - C(\tau)e^{\gamma_C(t-\tau)}, \quad \text{for all } t \geq \tau.$$

Dividing both sides of the equation by $e^{\gamma_I(t-\tau)}$ and then differentiating it with respect to time gives the familiar equation

$$0 = (\gamma_Y - \gamma_I)\, Y(\tau)e^{(\gamma_Y - \gamma_I)(t-\tau)} - (\gamma_C - \gamma_I)\, C(\tau)e^{(\gamma_C - \gamma_I)(t-\tau)},$$

which has four possible solutions:

$$\begin{aligned}
&i) && \gamma_Y = \gamma_I && \wedge && \gamma_C = \gamma_I && \Longrightarrow && \gamma_Y = \gamma_I = \gamma_C,\\
&ii) && \gamma_Y = \gamma_I && \wedge && C(\tau) = 0, &&&&\\
&iii) && \gamma_C = \gamma_I && \wedge && Y(\tau) = 0, &&&&\\
&iv) && \gamma_Y = \gamma_C && \wedge && Y(\tau) = C(\tau). &&&&
\end{aligned}$$

Only the first one does not violate the assumption $Y(\tau) > C(\tau) > 0$, and so $\gamma_Y = \gamma_I = \gamma_C$.

Claim II.: Capital accumulation equation (5.7) can be written as

$$(\gamma_K + \delta_K)\, K(t) = G(I(t)).$$

Term $(\gamma_K + \delta_K)$ is constant in the steady, so $\frac{G(I(t))}{K(t)}$ must also be constant, implying that $\gamma_K = \gamma_G$. Derivation with respect to time gives

$$(\gamma_K + \delta_K)\, \dot{K}(t) = \frac{dG(I(t))}{dI(t)} \dot{I}(t).$$

Dividing the last equation by the previous one delivers a differential equation

$$\begin{aligned} \frac{\dot{K}(t)}{K(t)} &= \frac{dG(I(t))}{dI(t)} \frac{\dot{I}(t)}{G(I(t))} \\ \gamma_K &= \frac{dG(I(t))}{dI(t)} \frac{I(t)}{G(I(t))} \gamma_I. \end{aligned}$$

If the economy is stationary, i.e., $\gamma_K = \gamma_I = 0$, then the equation holds for any function $G(I(t))$. If $\gamma_K \neq 0$ and $\gamma_I \neq 0$, then the installation function of capital must have a form of a power function

$$G(I(t)) = c \cdot I(t)^{\phi},$$

where $c > 0$ is an integration constant and the exponent $\phi = \gamma_K/\gamma_I$. Requirements on $G(I(t))$ imposed in (5.6) imply that $\phi \in (0, 1)$. Since $\gamma_Y = \gamma_I$, $\phi = \gamma_K/\gamma_I$ implies that $\gamma_K = \phi\gamma_Y$.

Claim III.: Using time τ quantities of $Y(t)$, $K(t)$ and $L(t)$ gives for any $t \geq \tau$ total output at time τ as

$$e^{-\gamma_Y(t-\tau)} \cdot Y(t) = \tilde{F}\left[e^{-\gamma_K(t-\tau)} \cdot K(t), e^{-\gamma_L(t-\tau)} \cdot L(t), \mathbf{A}_F(\tau)\right].$$

Since $\tilde{F}$ exhibits constant returns to scale in $K(t)$ and $L(t)$, multiplying both sides by $e^{\gamma_Y(t-\tau)}$ delivers

$$Y(t) = \tilde{F}\left[e^{(\gamma_Y-\gamma_K)(t-\tau)} K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} L(t), \mathbf{A}_F(\tau)\right]. \tag{5.10}$$

Substituting $\gamma_K = \phi\gamma_Y \neq 0$ yields

$$Y(t) = \tilde{F}\left[e^{\gamma_Y(1-\phi)(t-\tau)} K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} L(t), \mathbf{A}_F(\tau)\right].$$

Because this holds for all $t \geq \tau$ and because $\tilde{F}$ exhibits constant returns to scale in the first two arguments, there must be constant returns to scale function $F : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ such that

$$Y(t) = F\left[e^{\gamma_Y(1-\phi)(t-\tau)} K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)} L(t)\right] = F\left[B(t)K(t), A(t)L(t)\right]$$

with $B(t) = e^{\gamma_Y(1-\phi)(t-\tau)} \in \mathbb{R}_{++}$ and $A(t) = e^{(\gamma_Y-\gamma_L)(t-\tau)} \in \mathbb{R}_{++}$.

Stationary economy, in which $\gamma_K = \gamma_I = \gamma_Y = 0$, implies

$$Y(t) = F\left[K(t), e^{\,-\gamma_L(t-\tau)}L(t)\right] = F\left[K(t), A(t)L(t)\right]$$

with $A(t) = e^{-\gamma_L(t-\tau)} \in \mathbb{R}_{++}$.

As $\gamma_K = \phi\gamma_Y$, capital per worker, $K(t)/L(t)$, grows at rate $\phi\gamma_Y - \gamma_L$. And because $\gamma_Y = \gamma_C = \gamma_I$, all the other per-worker variables $Y(t)/L(t)$, $C(t)/L(t)$ and $I(t)/L(t)$ grow at rate $\gamma = \gamma_Y - \gamma_L$. $\square$

Theorem 5.1 shows that due to adjustment cost for capital investment, capital-augmenting technological change does not vanish in the steady-state but it persist to balance the growth rates of gross and net capital investment. However, including adjustment costs into the model by itself does not guarantee appearance of steady-state capital-augmenting technological change. It requires fulfilment of two independent conditions – function $G(I(t))$ must exhibit diminishing returns, and the economy cannot be stationary.

The condition (5.9) can be derived in the following way. First, Li *et al.* (2013) prove that along the steady state path $\gamma_Y = \gamma_C = \gamma_K - \gamma_{G_I}$, where $\gamma_{G_I} \equiv \dot{G}_I/G_I \neq 0$ [see Li *et al.* (2013, pp. 10-12)]. This is an expected result since, because of the adjustment costs, one unit of final output generates less than one unit of new capital and so the capital stock grows at a slower rate than final output. Substituting $-\gamma_{G_I} = \gamma_Y - \gamma_K$ into equation (5.10) gives

$$Y(t) = F\left[e^{\left(-\gamma_{G_I}\right)(t-\tau)}K(t), e^{(\gamma_Y-\gamma_L)(t-\tau)}L(t)\right] = F\left[B(t)K(t), A(t)L(t)\right]$$

with $B(t) \equiv e^{\left(-\gamma_{G_I}\right)(t-\tau)}$ and $A(t) \equiv e^{(\gamma_Y-\gamma_L)(t-\tau)}$. Then for the growth rate of capital-augmenting technology $B(t)$ holds that $\gamma_B = -\gamma_{G_I}$, which can be rewritten into

$$\frac{\dot{G}_I}{G_I} + \frac{\dot{B}}{B} = 0.$$

This condition implies that when the change in marginal efficiency of capital is not zero, $\dot{G}_I/G_I \neq 0$, steady-state requires presence of capital-augmenting technological change. It also includes the result of Uzawa's theorem. Among the assumptions of Theorem 2.1 is that capital accumulates according to $\dot{K}(t) = I(t) - \delta_K K(t)$, which means that $G(I(t)) = I(t)$. Then the marginal efficiency of capital remains constant, because in this case $G_I = dG(I(t))/dI(t) = 1$, and so $\dot{G}_I/G_I = 0$. Condition (5.9) then implies $\dot{B}/B = 0$, i.e., steady-state technological progress must be purely labor-augmenting (Harrod-neutral). Li *et al.* (2013) claim that adding adjustment costs to the model makes it more realistic, and suggest that Uzawa's theorem should be considered a special case. Therefore the theorem should be modified into the following: "*If the marginal efficiency of capital accumulation remains constant, then the steady-state technological progress must be Harrod-neutral.*"[Li and Huang (2016, pp. 11-12)].

5.2 A Model with Endogenous Schooling

Since the 1950s, the relative price of quality-adjusted investment goods, and in particular the price of capital equipment, has been decreasing, as depicted by Figure 5.1. Greenwood *et al.* (1997) interpret this decline as an indicator of significant capital-augmenting technological progress, which enabled the producers to be relatively more efficient in production of capital equipment. In their paper they investigate how much of the output growth is due to *investment-specific technological progress*, which is a form of *embodied* technological progress. In this case it is embodied in new capital equipment.

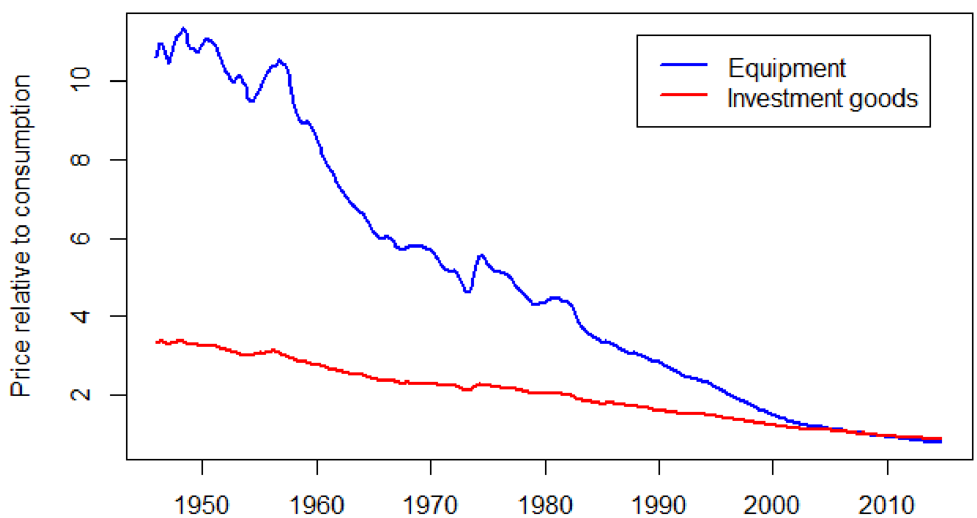

Figure 5.1: Relative prices of equipment and investment goods in U.S., 1947–2015
(Data source: FRED, Federal Reserve Bank of St. Louis; series PERIC and PIRIC)

In contrast to the *disembodied* technological progress, where higher level of technology implies higher productivity of all input factors, no matter how old the equipment involved in the production is, to make use of the latest technological knowledge available, the *embodied* technological progress constantly requires investment in new capital goods containing (*embodying*) this new technology. Once the technology advances again, new investment is needed to increase the productivity. Newer the capital stock, higher the productivity. Groth and Wendner (2011, p. 6) illustrate this nicely with an example that "*only the most recent vintage of a computer series incorporates the most recent advance in information technology*".

In their analysis, Greenwood *et al.* (1997) find that embodied technological progress accounts for almost 60% of the growth in output per hours worked.

Grossman *et al.* (2017) incorporate this type of technological progress into their model which attempts to reconcile capital-augmenting technological change with a steady-state growth path. As before, final output $Y(t)$ is produced according to

$$Y(t) = F\left(B(t)K(t), A(t)L(t), s(t)\right), \tag{5.11}$$

where $K(t)$ is physical capital, $L(t)$ is raw labor, and $B(t)$ and $A(t)$ represent disembodied capital- and labor-augmenting technological progress, respectively.[1] F is a standard neoclassical production function with constant returns to scale in capital and labor. $s(t)$ here is a scalar variable that represents the level of schooling. The labor input $L(t)$ is assumed to grow at some constant rate γ_L. This growth rate can be either positive, zero, or negative.

Each unit of final output can be either consumed or it can be invested to create $q(t)$ units of capital,

$$Y(t) = C(t) + I(t)/q(t). \tag{5.12}$$

In this equation, $I(t)$ is the quantity of newly installed capital in units of final output. The concept of Greenwood's investment-specific technological progress is captured by changes in $q(t)$.[2] Greenwood *et al.* (1997) note that the growth in $q(t)$ can be explained in two different ways:

- Since one unit of output can be invested to create $q(t)$ units of capital, the cost of producing one unit capital in terms of final output is $p(t) = 1/q(t)$. As $q(t)$ increases over time, the production cost decreases.

[1] To maintain the notation from the previous part, $B(t)$ refers to capital-augmenting and $A(t)$ to labor-augmenting technological progress. In Grossman *et al.* (2017) the notation is the other way around.

[2] An alternative way to formulate a model with embodied technological progress is to write $Y(t) = C(t)+I(t)$; $\dot{K} = q(t)I(t) - \delta K(t)$. These two formulations are equivalent.

- Or, assuming that the price of producing new unit of capital remains unchanged over time, and that this price is one unit of final output, $q(t)$ can be seen as an indicator of productivity of new unit of capital, where $q(t)$ increases over time. For example, theoretically, if $q(t)$ doubled every period of time, then the newest capital goods would be twice as productive as the capital goods produced in the previous period.

That means that technological progress either makes new capital goods less expensive, or improves their quality when compared to the older ones.

Finally, again as before, capital depreciates at rate δ_K, that is

$$\dot{K}(t) = I(t) - \delta_K K(t). \tag{5.13}$$

Grossman *et al.* (2017) define a balanced growth path as a path along which $Y(t)$, $C(t)$, and $K(t)$ grow at constant and proportional rates.

Lemma 5.1 (Grossman *et al.*, 2017, Lemma 1)**.** *Suppose that $q(t)$ grows at constant rate γ_q, and suppose that there exists a balanced growth path beginning at some $\tau < \infty$ along which $\forall t \geq \tau : Y(t) > C(t) > 0$. Then, for the growth rates it holds that $\gamma_Y = \gamma_C = \gamma_K - \gamma_q$.*

Proof. Substituting the capital accumulation equation (5.13) into the resource constraint (5.12), differentiating it with respect to time and then rearranging it gives

$$(\gamma_K - \gamma_q - \gamma_C)\, C(t) = (\gamma_K - \gamma_q - \gamma_Y)\, Y(t).$$

This equation has four possible solutions:

$$\begin{aligned}
i)&\quad \gamma_K - \gamma_q = \gamma_C &\wedge&\quad \gamma_K - \gamma_q = \gamma_Y,\\
ii)&\quad \gamma_K - \gamma_q = \gamma_C &\wedge&\quad Y(t) = 0,\\
iii)&\quad \gamma_K - \gamma_q = \gamma_Y &\wedge&\quad C(t) = 0,\\
iv)&\quad \gamma_C = \gamma_Y &\wedge&\quad C(t) = Y(t).
\end{aligned}$$

The last three options violate assumption that $\forall t : Y(t) > C(t) > 0$, hence $\gamma_Y = \gamma_C = \gamma_K - \gamma_q$ must hold. □

But obviously, with these growth rates the capital-output ration, $K(t)/Y(t)$, cannot be constant. The thing is, as Grossman *et al.* (2017) note, once the price of each capital good in units of final goods is determined by the cost of required investment, i.e. $p(t) = 1/q(t)$, it is not the amount of capital units but the value of capital in units of final output that is growing at the same rate as final output.

In this setting, capital stock is affected by two types of technological progress – disembodied progress $B(t)$ and embodied progress $q(t)$. Then the *total rate of capital-augmenting technological progress* can be defined as $\tilde{\gamma}_K \equiv \gamma_B + \gamma_q$.

So far the index of schooling did not receive any attention. The following Proposition specifies the relation between capital-augmenting technological change, $\tilde{\gamma}_K$, and the change in the level of schooling, $\dot{s}$, that must hold in order to have constant factor shares along the steady-state path. As before, σ denotes the elasticity of substitution between inputs $K(t)$ and $L(t)$.

Proposition 5.1 (Grossman *et al.*, 2017, Proposition 1)**.** *Suppose that the growth rate of embodied technological progress, γ_q, is constant. Further assume that factors are paid the value of their marginal products. In order for a balanced growth path with constant and strictly positive factor income shares to exist, it has to hold that*

$$(1-\sigma)\,\tilde{\gamma}_K = \sigma \frac{F_L}{F_K} \frac{\partial\,(F_s/F_L)}{\partial K}\dot{s}. \tag{5.14}$$

Proof. See the online appendix of Grossman *et al.* (2017), pp. 1-2. □

To see how this relates to the steady-state growth theorem, $\dot{s}$ has to be set equal to zero, since education does not play any role in Uzawa (1961). Then the right-hand side of equation (5.14) is equal to zero, and

$$(1-\sigma)\,\tilde{\gamma}_K = 0 \quad \Longleftrightarrow \quad \sigma = 1 \quad \vee \quad \tilde{\gamma}_K = 0,$$

which is the usual interpretation of Uzawa's theorem — either the elasticity of substitution between capital and labor has to be equal to one, and therefore the production function takes Cobb-Douglas form, where the distinction between capital- and labor-augmenting type of technology is irrelevant, or there is no capital-augmenting technological progress involved along the steady-state path.

However, interpreting $\tilde{\gamma}_K = 0$ as that "there is no capital-augmenting technological progress" is slightly incorrect. As $\tilde{\gamma}_K = \gamma_B + \gamma_q$, having $\tilde{\gamma}_K = 0$ is indeed satisfied if both γ_B and γ_q equal to 0, i.e., if there is neither disembodied nor embodied capital-augmenting technological change. But, assuming that the level of technological knowledge can decrease over time, this condition also holds when there is embodied technological progress with some positive growth rate γ_q, and disembodied technological decline with negative growth rate $\gamma_B = -\gamma_q$. These two types offset one another, but technically there is some type of capital-augmenting technological progress involved.

For σ not to be forced to equal to 1, or $\tilde{\gamma}_K$ not to be 0, it is required that

$$\frac{\partial \left(F_s/F_L\right)}{\partial K} \neq 0 \quad \wedge \quad \dot{s} \neq 0 \tag{5.15}$$

Empirical evidence suggests that $\sigma < 1$. Then $(1-\sigma) > 0$, and together with assumption that the index of schooling is increasing over time, i.e. $\dot{s} > 0$, equation (5.14) implies that $\frac{\partial \left(F_s/F_L\right)}{\partial K} > 0$. This is an important assumption of the model – "*capital accumulation must increase the marginal product of schooling proportionally more than it increases the marginal product of labor*"(Grossman *et al.*, 2017, p. 1300). The assumption of increasing index of schooling is supported by empirical evidence presented in Figure 5.2. For over a century the number of years devoted to education by native U.S. workers born between the years 1876 and 1982 has been increasing.

5.2.1 A Model with a Reduced-Form Education Function

As was said in the beginning of this section, $s(t)$ is a scalar index that captures the level of education in the economy. For simplicity, the first model of Grossman *et al.* (2017) deals with an unspecified education function $D[s(t)]$, such that $D'[s(t)] < 0$ for all values of $s(t)$. Some concrete examples of this education function are presented in the next part.

Suppose that the economy is populated by a continuum of identical family dynasties, which each has a continuum of $N(t)$ family members. Size of the family grows at exogenous rate n. A social planner will seek to maximize dynastic utility

$$u = \int_0^\infty N(t) e^{-\rho t} \, \frac{c(t)^{1-\theta} - 1}{1-\theta} \mathrm{d}t, \tag{5.16}$$

where $c(t)$ denotes consumption of individual member of the family.

There is an inverse relationship between the level of education and the amount of labor used in the production process described by

$$L(t) = D[s(t)]N(t), \qquad \forall s(t): \quad D'[s(t)] < 0. \tag{5.17}$$

The following Assumption is a crucial one because it restricts the functional form for the production function to imply that $\frac{\partial \left(F_s/F_L\right)}{\partial K} > 0$, which is a requirement for the existence

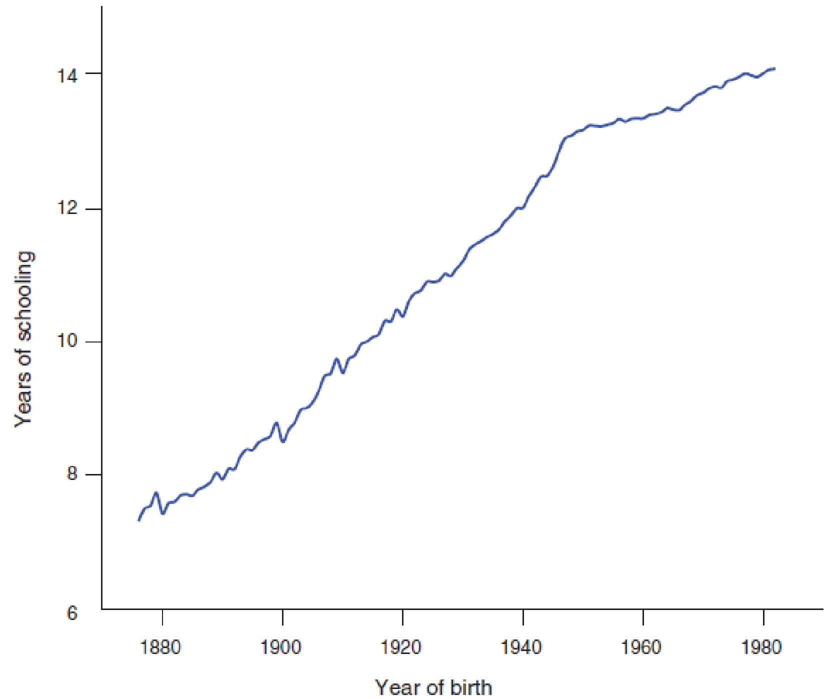

Figure 5.2: Average years of schooling in U.S. by birth cohorts, 1876-1982
(Source: Grossman et al.: *Balanced Growth Despite Uzawa* (2017), Figure 2)

of a balanced growth path with non-zero capital-augmenting technological progress under the assumption that that the level of education $s(t)$ increases over time.[3]

Assumption 5.1 (Grossman *et al.*, 2017, Assumption 1)**.** *The production function can be expressed as* $F(BK, AL, s) = \tilde{F}\left[D(s)^b BK, D(s)^{-a} AL\right]$*,* $a, b > 0$*, where*

1. $h(z) \equiv \tilde{F}(z, 1)$ *is twice differentiable, strictly increasing and strictly concave for* $\forall z$*;*

2. $\sigma \equiv \dfrac{F_L F_K}{F F_{LK}} < 1.$

Also a certain parameter restrictions are needed to guarantee that $F_s \geq 0$, the optimal $s > 0$, and that the utility of a dynasty is finite.

[3]For the proof of this implication see the online Appendix of Grossman *et al.* (2017), pp. 4-5.

Assumption 5.2 (Grossman *et al.*, 2017, Assumption 2)**.**

1. $\lim_{z\to\infty} \dfrac{zh'(z)}{h(z)} \quad < \quad \dfrac{a-1}{a+b-1} \quad < \quad \lim_{z\to 0} \dfrac{zh'(z)}{h(z)} \quad < \quad \dfrac{a}{a+b};$
2. $\rho > n + (1-\eta)\left[\gamma_A + \dfrac{a-1}{b}\tilde{\gamma}_K\right].$

Focus of the paper is on the relationship between the level of schooling and the technological progress. The social planner has to choose the optimal balance between the level of schooling $s(t)$ and the amount of raw labor $L(t)$ to achieve the maximal possible output $Y(t)$. That is,

$$Y(t) = \max_{s(t)} \tilde{F}\left[D(s(t))^b B(t)K(t), D(s(t))^{-a} A(t)L(t)\right]$$

subject to

$$L(t) = D[s(t)]N(t).$$

This implies the maximization problem in the form of

$$\begin{aligned} Y(t) &= \max_{s(t)} \tilde{F}\left[D(s(t))^b B(t)K(t), D(s(t))^{1-a} A(t)N(t)\right] \\ &= \max_{s(t)} D(s(t))^{1-a} A(t)N(t)\tilde{F}\left[\frac{D(s(t))^{a+b-1}B(t)K(t)}{A(t)N(t)}, 1\right]. \end{aligned}$$

Substituting $z(t) \equiv D(s(t))^{a+b-1}B(t)K(t)/A(t)N(t)$, and applying $h(z(t)) \equiv \tilde{F}(z(t), 1)$ yields

$$Y(t) = \max_{z(t)} (A(t)N(t))^{1-\alpha}(B(t)K(t))^{\alpha} z(t)^{-\alpha} h(z(t)),$$

where $\alpha \equiv (a-1)/(a+b-1)$. FOC with respect to $z(t)$ gives

$$\frac{z(t)h'(z(t))}{h(z(t))} = \alpha.$$

This implies that the planner opts for such a level of education $s(t)$, so that $z(t) \equiv D(s(t))^{a+b-1}B(t)K(t)/A(t)N(t)$ is constant, meaning that increased schooling balances out the effect of capital deepening. Once the optimal value of $z(t)$ is used, $Y(t)$ becomes a Cobb-Douglas function of $A(t)N(t)$ and $B(t)K(t)$, therefore the growth rate of $B(t)$ is not restricted to be zero along the balanced growth path.

Second, the planner has to find the optimal consumption path given the capital accumulation constrain, that is

$$\max_{c(t)} \int_0^{\infty} N(t)e^{-\rho t} \frac{c(t)^{1-\theta}-1}{1-\theta} \mathrm{d}t$$

subject to

$$\dot{K}(t) = I(t) - \delta K(t) = q(t)\left(Y(t) - N(t)c(t)\right) - \delta K(t).$$

Results of the first social planner's problem are summarized in the following proposition.

Proposition 5.2 (Grossman *et al.*, 2017, Proposition 2)**.** *Let Assumptions 5.1 and 5.2 hold. Further suppose that there is an inverse relationship between the level of schooling and stock of labor given by $L(t) = D[s(t)]N(t)$. Then the economy converges to a balanced growth path along which*

1. *aggregate output $Y(t)$ and aggregate consumption $C(t)$ have identical growth rates*

$$\gamma_Y = \gamma_C = n + \gamma_A + \frac{a-1}{b}\tilde{\gamma}_K; \tag{5.18}$$

2. *the index of education evolves according to*

$$\dot{s} = -\frac{\tilde{\gamma}_K D(s)}{bD'(s)}, \tag{5.19}$$

so that

$$\gamma_D = -\frac{\tilde{\gamma}_K}{b}; \tag{5.20}$$

3. *the capital share α_K remains constant over time*

$$\alpha_K = \frac{a-1}{a+b-1}. \tag{5.21}$$

Important implication of Assumptions 5.1 and 5.2 is that $a > 1$. Therefore, since $b > 0$, the fraction $\frac{a-1}{b}$ in (5.18) is positive. As a result, both the labor-augmenting technology and the combined capital-augmenting technology contribute to the growth of per capita income. So, contrary to Uzawa, it is theoretically possible to have a balanced growth path with non-zero capital-augmenting technological progress, thanks to the optimal choice of schooling which "*neutralizes the effect of capital-augmenting progress and the declining investment-good prices on the growth of the effective capital stock*" (Grossman *et al.*, 2016, p. 15).

Following this result, the authors examine whether their model could do without the functional form restriction from Assumption 5.1. The answer is no, it could not, implying that Assumption 5.1 is a necessary condition to achieve a balanced growth path with $\tilde{\gamma}_K > 0$.

5.2.2 Specifications of the Education Function

Following the reduced-form model in Grossman *et al.* (2017) are three examples of market economies that can be simplified into the model presented in the previous subsection. The description here omits most of the information about these economies and focuses solely on the specifications of the education function.

The setting with family dynasties with $N(t)$ members of the family alive at time t remains unchanged in all these models.

5.2.2.1 "Time-in-School" Model

All of the $N(t)$ family members spend first part of their life, denoted by the variable $s(t)$, studying and then part $1 - s(t)$ working. The value of $s(t)$ is decided by a representative family member of the generation alive at time t. Here $D[s(t)] = 1 - s(t)$, and so the labor supply is given by the number of family members and the years they spend working, that is $L(t) = N(t)\,(1 - s(t))$. All members of the family attempt to maximize dynastic utility described in equation (5.16) by maximizing their wage income, which depends on the level of schooling the representative person chose to gain.

5.2.2.2 "Manager-Worker" Model

In the second example firms employ teams consisting of "managers" and "production workers". The ratio of managers to workers determines team's productivity. To be employed as a manager, a fraction m of person's life must be dedicated to education. Those who do not pursue this education work as production workers.

Since the population consists of managers and production workers, denoting their respective numbers by $N^M(t)$ and $N^W(t)$ gives $N(t) = N^M(t) + N^W(t)$. Let $M(t)$ and $L(t)$ be the time units provided by managers and production workers to be spent in the production process at time t. Production workers spend their whole life performing their job, so $L(t) = N^W(t)$. Managers have only the faction $1-m$ of their life to perform their duties, which gives $M(t) = (1-m)N^M(t)$. From that the number of managers can be expressed as $N^M(t) = M(t)/(1-m)$. And so

$$N(t) = N^M(t) + N^W(t) = \frac{M(t)}{1-m} + L(t). \tag{5.22}$$

The index of schooling is here defined as $s(t) = M(t)/L(t)$. Substituting $M(t) = s(t)L(t)$ into equation (5.22) and expressing $L(t)$ yields $L(t) = N(t)/\left(1 + \frac{s(t)}{1-m}\right)$. Since $L(t) = D[s(t)]N(t)$, it must be that $D[s(t)] = [1 + s(t)/(1 - m)]^{-1}$.

When deciding which career to pursue, the representative person compares the lifetime incomes that would be earned in the two professions.

In both the "time-in-school" model and the "manager-worker" model the production function takes the form described in Assumption 5.1. Also the parameter restrictions are identical to those made in Assumption 5.2. And so both models converge to a steady-state with the properties listed in Proposition 5.2.

5.2.2.3 Overlapping Generations Model

The last example is the most realistic one. The "time-in-school" model is extended to allow the cohorts to overlap instead of just replacing the preceding one. So far the level of schooling was a scalar variable whose value could jump between periods t and $t + 1$. In this model $s(t)$ evolves slowly over time.

New member of a dynasty is born with an instantaneous probability λ, and dies with an instantaneous probability ν. Therefore the population grows at rate $n = \lambda - \nu$, and the size of a dynasty is $N(t) = e^{(\lambda-\nu)t}N(0)$.

Each person attends school for s years, then works for $\bar{u}$ years and then retires. Firms employ workers from various cohorts who differ in their education and experience. Qualification and working experience are reflected in their wages. At first worker's productivity increases, but then drops to zero as the number of working years reaches $\bar{u}$. Conditional on survival, a representative person of a cohort born at time τ decides how long part of her life, s_τ, dedicate to schooling. The choice is such as to maximize the expected present discounted value of lifetime earnings. However, therein lies a potential problem. Assumptions made in this model imply that the age distribution in the workforce must remain stationary over time. As authors of the paper admit, if some cohorts decide to gain more (or less) education than they are supposed to, and therefore start their working life later (or earlier), then the stationarity of the age distribution in the workforce will be disrupted.

This model's assumptions are not identical, but analogous to Assumptions 5.1 and 5.2 in the simplified model. The results about the balanced growth path are similar to those in Proposition 5.2. For example, the BGP growth rate of output is $\gamma_Y = n + \gamma_A + \frac{a-\lambda}{b}\tilde{\gamma}_K$, which differs from (5.18) only by the presence of λ. Parameter restrictions imply that $\lambda < a$, and so again both capital- and labor-augmenting technological progress determine the growth rate of aggregate output. The level of schooling by birth cohort must evolve according to

$$\dot{s}_\tau = \frac{\tilde{\gamma}_K}{b - \tilde{\gamma}_K}, \tag{5.23}$$

which is said to be roughly in agreement with the data presented in Figure 5.2, especially for the cohorts born between the years 1876 and 1955, and then for the later ones.

In contrast to the previous models, where equation (5.21) shows that the capital share is constant and independent of technological growth rates, in the OLG model factor shares are affected by changes in $\tilde{\gamma}_K$.

5.2.3 Remarks

In all fairness to Grossman *et al.* (2017), they clearly say that schooling *can* offset the impact of capital-augmenting technological change on capital stock, not that it necessarily *will*. Their model relies on quite a lot of restrictions. Production function must belong to a specific class of functions described in Assumption 5.1 to imply a certain capital-skill complementarity. It it different from restricting the functional form to be Cobb-Douglas, which avoids the discussion about capital-augmenting technological progress, and certainly an improvement in a sense that it allows $\sigma < 1$, but it is a restriction nonetheless.

Next problem is that each individual must receive just the right amount of education. The level of schooling must evolve according to equations (5.19) or (5.23), where s_τ increases linearly. It is unlikely that empirical data about education of country's workers would match with this demand in more than a few developed countries, and if yes, then not for longer than a few decades in a row.

Finally, when Greenwood *et al.* (1997) studied the role of investment specific-technological progress, they distinguished between two types of capital input – capital equipment $K_e(t)$, and structures $K_s(t)$. Apart from consumption, final output could then be used as an investment into either equipment or structures. As the fall of relatives prices in Figure 5.1 is more apparent for the capital equipment, factor $q(t)$ representing the technological knowledge was included only with equipment investment term. In Grossman *et al.* (2017) there is only one type of capital and so the factor $q(t)$ in equation (5.12) affects all the investment. This can be seen as a more recent development, as Figure 5.1 shows that since the year 2003 the relative price of equipment more or less coincides with the relative price of investment goods. On the other hand, the model might not be so well historically applicable.

6 Conclusion

Is there any economic reason why steady-state technological change should be purely labor-augmenting? Literature dedicated to this topic fails to provide any reasonable justification.

The explanation provided by Acemoglu's (2009) directed technological change model seems promising, but it only shows that under the assumption that elasticity of substitution between capital and labor is below one, there are natural incentives for technology to be relatively labor-augmenting, not purely labor-augmenting. The later is stated to be a necessary condition for achieving balanced growth in the model. However, the desired outcome of a stable long-run equilibrium is shown to be obtained only with a very specific number of scientists in the R&D sector.

Since Irmen (2016) extends Uzawa's steady-state growth theorem and shows that it applies to multiple growth models with endogenous technological change, it might be interpreted as that it is not possible to obtain steady-state growth without restricting technological change to be purely labor-augmenting. However, the framework of Uzawa (1961) and Irmen (2016) confirms that the shape of capital accumulation equation is the key to this result. Once the relationship between capital investment and capital accumulation is not linear, for example due to adjustment costs for capital investment, capital-augmenting technological progress may arise in steady state.

Interestingly, growth models that attempt to find the reasons behind steady-state purely labor-augmenting technological progress did not give much attention to the embodied capital-augmenting technological progress, despite its strong empirical evidence. The model by Grossman *et al.* (2017) presented in Section 5.2 shows that it is possible to reconcile such a technological progress with the requirements for balanced growth, but only under some restrictive assumptions which can be seen as highly unrealistic.

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Zusammenfassung

Diese Masterthese untersucht, ob stationärer technischer Fortschritt ausschließlich Arbeit vermehrt. Neben den Ergebnissen von Uzawa (1961) und Irmen (2016), die darauf hinweisen, dass diese Einschränkung unabhängig davon, ob technischer Fortschritt als exogene oder endogene Variable behandelt wird, zutrifft, werden in dieser Masterthese auch zwei Modelle vorgestellt, die aufzeigen, dass eine solche Einschränkung nicht notwendig ist. Das erste Modell erweitert das neoklassische Wachstumsmodell, indem Kosten für die Anpassung von Kapitalinvestitionen inkludiert werden, welche Ungleichheit zwischen dem Wachstum der Bruttokapitalinvestitionen und dem Wachstum des Kapitalstocks erzeugen. Stationärer technologischer Fortschritt, welcher das Kapital erhöht, ist daraufhin eine Notwendigkeit, um diese Ungleichheit auszugleichen. Das zweite Modell beschäftigt sich mit dem abnehmenden relativen Preis von Investitionsgütern als Indikator für laufenden gebundenen technologischen Fortschritt. Ausgewogenes Wachstum kann durch die Einführung endogener Ausbildung und unter der Annahme, dass eine ausreichende Komplementarität von Kapital und (Arbeits-)Qualifikation gegeben ist, erreicht werden.

Stichwörter: stationärer Zustand; technischer Fortschritt; Uzawa's theorem; Harrod-Neutralität; Arbeitsvermehrung; Kapitalvermehrung