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„Factorization of second-order strictly hyperbolic operators with non-smooth coefficients and generalized microlocal approximations“

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Abstract

In the first part we give a factorization procedure for a strictly hyperbolic partial differential operator of second order with logarithmic slow scale coefficients. From this we can microlocally diagonalize the full wave operator which results in a coupled system of two first-order pseudodifferential equations in a microlocal sense. Under the assumption that the full wave equation is microlocal regular in a fixed domain of the phase space, we can approximate the problem by two one-way wave equations where a dissipative term is added to suppress singularities outside the given domain. We obtain well-posedness of the corresponding Cauchy problems for the approximated one-way wave equations with a dissipative term.

In the second part of this thesis we study strictly hyperbolic partial differential operators of second order with less regular coefficients as in the first part. After modeling them as semiclassical Colombeau equations of log-type we provide a factorization procedure on some time-space-frequency domain. As a result the operator is written as a product of two semiclassical first-order constituents of log-type which approximates the modeled operator microlocally at infinite points. We then present a diagonalization method so that microlocally at infinity the governing equation is equal to a coupled system of two semiclassical first-order strictly hyperbolic pseudodifferential equations. Furthermore we compute the coupling effect.

Zusammenfassung

Im ersten Teil dieser Arbeit befassen wir uns mit der Faktorisierung von strikt hyperbolischen partiellen Differentialoperatoren zweiter Ordnung wobei die Koeffizienten durch gewisse verallgemeinerte logarithmische slow-scale Funktionen gegeben sind. Davon ausgehend zeigen wir eine mikrolokale Diagonalisierung des verallgemeinerten partiellen Differentialoperators. Wir erhalten daher ein gekoppeltes System von zwei Pseudodifferentialgleichungen erster Ordnung in einem mikrolokalen Setting. Unter der Annahme, dass die volle Wellengleichung mikrolokal regulär auf einem hinreichend schönen Gebiet des Phasenraums ist, kann das Wellengleichungenproblem durch zwei One-Way Gleichungen approximativ beschrieben werden. Im Speziellen enthalten diese approximierenden One-Way Operatoren einen zusätzlichen dissipativen Term, der es erlaubt, ungewollte Singularitäten zu unterdrücken. Im Anschluss daran zeigen wir die Wohldefiniertheit der entsprechenden Cauchy Probleme für die approximierenden One-Way Gleichungen mit dissipativem Term.

Als Motivation für den zweiten Teil dieser Arbeit dient der Versuch die Bedingungen an die Koeffizienten aus dem ersten Teil abzuschwächen. Dazu betrachten wir einerseits einen, im Vergleich zu oben, leicht modifizieren strikt hyperbolischen Differentialoperator aber andererseits sind die Koeffizienten nunmehr durch logarithmisch skalierte Funktionen gegeben. Um die verlorene Regularität der Koeffizienten auszugleichen modellieren wir die Gleichungen nun in einem semiklassischen Sinne. Als erstes Resultat erhalten wir eine Faktorisierung des nun semiklassischen Operators in ein Produkt von zwei Operatoren erster Ordnung auf einem geeigneten Gebiet des Phasenraums. Anschließend zeigen wir eine mikrolokale Diagonalisierung des semiklassischen Operators und der damit verbundenen Möglichkeit eines approximierenden gekoppelten Systems von zwei mikrolokalen semiklassischen Pseudodifferentialgleichungen erster Ordnung - nun aber nur mehr auf sogenannten unendlichen Punkten des entsprechenden Gebiets des Phasenraums.

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Chapter 0

Introduction

0.1 Motivation and the smooth background case

A particular example of a strictly hyperbolic partial differential equation of second order is the wave equation describing phenomena such as propagation of elastic waves and vibrations. However, in this section of the introduction we give a motivation for the present survey and is devoted to one-way wave propagation in inhomogeneous acoustic media in the case of smooth background coefficients. For more information on the relevance in seismic imaging and migration models we refer the reader to [7, 4, 53, 54].

To start with a real life problem we may ask whether and how one can establish a tap-proof communication link between a source and a receiver in an underwater environment. In the following we analyze how the correlation between the source and the receiver location can help to detect a desired information in downward continuation problems. However, in this real life problem one is mainly interested in the extrapolation of a wave field along the depth-direction. The mathematical reformulation of such problems corresponds to initial value problems of second-order partial differential equations with a space-like direction as evolution parameter and is in general not well-defined (cf. [48], [38]).

To overcome the ill-posedness the concept of wave-field decomposition enables us to rewrite the full wave equation into a coupled system of approximative one-way wave equations. Using microlocal techniques the system of one-way equations can then be decoupled assuming that the wave field propagates in one direction within certain propagation angles and prohibits propagation in the opposite direction. Therefore, the approximation allows to determine the high-frequency components of the solution as they propagate along curved trajectories in the phase space.

In the following we recall the crucial statements given by Stolk for the directional wave field decomposition in the present of smooth background data [51, 52]. For a discussion on the inverse scattering problem with simultaneous consideration of possible reflection data see for example [50, 54, 11] and the references therein. Also, we refer to [46, 47] for investigations of the Cauchy problem of first-order pseudodifferential equations and to [58] for numerical implementations.

As in [51] our basic type of model will be the wave equation for inhomogeneous acoustic media in n-dimensions, $n \geqslant 2$. Since we are interested in approximative one-way equations we allocate the vertical direction z, which we call the depth, the lateral directions are denoted by x. The medium itself is described by the wave speed $c = c(x,z)$ and the fluid density $\rho = \rho(x,z)$. Also we let $U = U(t,x,z)$ denote the acoustic wave field and $F = F(t,x,z)$ a source which usually describes the initiation of an acoustic wave. Then the acoustic wave equation is given by

$$PU := \Big(-\frac{1}{\rho}\frac{1}{c^2}\partial_t^2 + \sum_{j=1}^{n-1} \partial_{x_j}\frac{1}{\rho}\partial_{x_j} + \partial_z \frac{1}{\rho}\partial_z \Big) U = F \tag{0.1.1}$$

where U and F are in the distribution space $\mathcal{D}'(\mathbb{R}^{n+1})$. Further we assume smooth background data, i.e. the wave speed and the density are functions in $\mathcal{C}^\infty(\mathbb{R}^n)$, which shall also satisfy the boundedness conditions $0 < c(x,z), \rho(x,z) < \infty$ for all $(x,z) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

In the case of slow varying media and the presence of a source function we follow Stolk's approach and restrict the analysis to a microlocal region I_{Θ_2} that is given by

$$\begin{aligned} I_{\Theta_2} &:= \{(t,x,z,\tau,\xi,\zeta) \mid (x,z,\tau,\xi) \in I'_{\Theta_2},\ |\zeta| \leqslant C|\tau|\} \\ I'_{\Theta_2} &:= \{(x,z,\tau,\xi) \mid \tau \neq 0,\ |\xi| \leqslant \sin(\Theta_2)|c(x,z)^{-1}\tau|\} \end{aligned}$$

for some angle $\Theta_2 \in (0, \pi/2)$. To give an impression of the first of these domains we give a picture below in figure 1 (see [46]).

Then, under the assumption that the wave front set of U is contained in I_{Θ_2}, one can rewrite a microlocal equivalent model to (0.1.1) in terms of a coupled system of one-way wave equations with the depth as evolution parameter.

The main result is then the following. The equation

$$PU = F \qquad \text{microlocally on } I_{\Theta_2}$$

is equivalent to the system of two first-order strictly hyperbolic partial differential equations of the form

$$P_{0,\pm}u_\pm := \big(\partial_z - iB_\pm(x,z,D_t,D_x)\big)u_\pm = f_\pm \quad \text{microlocally on } I_{\Theta_2}, \tag{0.1.2}$$

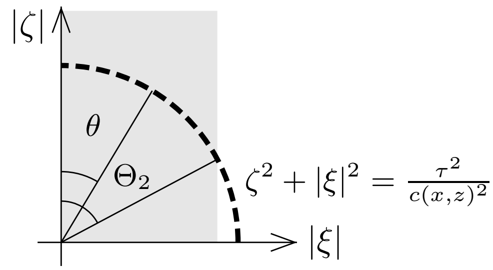

Figure 1: Here the shaded area corresponds to I_{Θ_2} at given (t, x, z) and a given frequency τ. The bold dotted line designates the characteristic set $\mathrm{Char}(P)$ and θ is the propagation angle of the singularities.

where the plus-minus sign refers to downward and upward migration. Here $u_\pm, f_\pm$ are distributions and $B_\pm$ are pseudodifferential operators of order 1 that can be chosen selfadjoint. Furthermore the coupling effect of the counterpropagating constituents $u_\pm, f_\pm$ and the original data U, F can be computed by a Douglis Nirenberg elliptic pseudodifferential operator transfer matrix Q. In detail, we have

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = Q^{-1} \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix}, \qquad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ F \end{pmatrix}. \tag{0.1.3}$$

To model wave propagation in the downward direction in the source-free case one decouples (0.1.2) microlocally and we are interested in the solutions to the problem

$$\begin{aligned} &PU = 0 \qquad z > z_0 \\ &\mathrm{WF}(U) \cap \{z = z_0\ ,\ \zeta/\tau > 0\} = \varnothing \end{aligned}$$

for some initial depth $z_0 \in \mathbb{R}$ such that $U|_{z_0}$ is well-defined.

Using the geometry of the microlocal region we let the set $J_{\Theta,+}(z_0) \subset T^*\mathbb{R}^{n+1}\backslash 0$ consist of points $(t, x, z, \tau, \xi, \zeta)$ so that the bicharacteristics $(t(z), x(z), \tau(z), \xi(z))$ corresponding to B_+, parametrized by z, and with propagation angle $\theta(z)$, pass through $(t_0, x_0, \tau_0, \xi_0)$ at initial depth $z = z_0$ and the points $(x(z), z, \tau(z), \xi(z))$ remain in I'_Θ for all $z \in [0, Z]$. So $J_{\Theta,+}(z_0)$ is the set that can be reached from the initial depth $z = z_0$ while staying in $I_{\Theta,+}$ and the propagation angle $\theta(z)$ along the bicharacteristic is always smaller than Θ (cf. Figure 1).

Since the equation for downward migration in (0.1.2) holds only microlocally, one obtains approximative solutions when studying a perturbation P_+ of the operator $P_{0,+}$ including an additional damping term $C = C(x, z, D_t, D_x)$ which vanishes in I_{Θ_1} for some fixed positive angle $0 < \Theta_1 < \Theta_2$ and suppresses singularities outside I_{Θ_2}. Also note that u_- is vanishing on $I_{\Theta_2} \cap \{z = z_0\}$ so that the perturbed initial value problem for downward propagation now reads

$$P_+ u_+ = (\partial_z - iB_+ + C)\, u_+ = 0 \qquad \mathbb{R}^n \times (z_0, Z)$$
$$u_+|_{z_0} = Q_+^{-1}(z_0) U|_{z_0}$$

where Q_+^{-1} is the essential component of the transfer matrix mentioned above. As a result the solution to the initial value problem for the perturbed first-order pseudodifferential equation can be related to that of $P_{0,+}u_+ = 0$ using the geometry in the region $J_{\Theta,+}$. In detail one can recover the high frequency part of the original wave field on some subset $J_{\Theta_1,+}$ of the phase space that can be reached from an initial given depth z_0 while staying in I_{Θ_1} for propagation angles $\theta(z) < \Theta_1 < \Theta_2$.

As a result, we get the following relations

$$\begin{cases} Q_+ u_+ \equiv U & \text{on } J_+(z_0, \Theta_1), \\ Q_+ u_+ \equiv 0 & \text{outside } J_+(z_0, \Theta_2) \end{cases}$$

and we refer to [51] and [52] for a deeper discussion.

0.2 The case of logarithmic slow scale background coefficients

When studying strictly hyperbolic partial differential equations with generalized coefficients the results commonly depend on the appropriate choice of the asymptotic scale of the regularizing parameter. In the first part of this work we will consider operators of the form (0.1.1) where the coefficients are logarithmic slow scale and where U and F belong to $\mathcal{G}_{L^2}$. We continue to write

$$LU = F \tag{0.2.1}$$

for the generalized version of (0.1.1).

The aim then is to provide a symbolic calculus in order to explain a diagonalization with respect to the parameter z of the Colombeau generalized version of equation (0.2.1). To do so, we will reduce the diagonalization problem to a factorization theorem which in the case of smooth coefficients can be found in [37, Appendix II], [31, Chapter 23]. We note that in these references the results are only valid for operators with simple characteristics on the phase space with the zero section excluded. This is different to our approach as we are interested in factorizations with respect to the parameter z.

For recent contributions to a related topic we refer the reader to the work by Garetto and Oberguggenberger [24]. There, the authors established existence and regularity results for solutions to strictly hyperbolic systems with Colombeau coefficients using symmetrisation up to regularizing errors in the time evolution of the wave field. In detail, they proved existence and uniqueness of generalized solutions in the case of logarithmic slow scale coefficients. Note that a net $r_\varepsilon \in \mathbb{R}^{(0,1]}$ is a logarithmic slow scale net if $|r_\varepsilon| = \mathcal{O}(\log^p(1/\varepsilon))$ for all $p > 0$ as $\varepsilon \to 0$ and a generalized coefficient $(a_\varepsilon)_\varepsilon$ is said to be logarithmic slow scale regular if

$$\forall \alpha \in \mathbb{N}^n \; \exists \text{ a logarithmic slow scale net } r_\varepsilon : \quad \|\partial^\alpha a_\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(r_\varepsilon) \qquad \varepsilon \to 0.$$

Again, with view to (0.2.1), the roots in ζ, the dual variable of z, of the principal symbol of the operator are assumed to be simple on the phase space without the zero section in this case.

So, when studying an equation of the form (0.2.1) with coefficients that satisfy strongly positive logarithmic slow scale estimates one can adapt the results in [21, 37] to obtain a diagonalization in some microlocal subregion of the phase space and microlocal regularity has to be understood in a $\mathcal{G}^\infty_{L^2}$ sense. Recall that $\mathcal{G}^\infty_{L^2}$ is the space of regular generalized functions in $\mathcal{G}^\infty_{L^2}$ and is characterized by uniform ε-growth in all derivatives. This is a generalization of the results in the smooth coefficient case which was briefly explained in the beginning of Section 0.1.

In detail, we obtain a generalized version of the equations (0.1.2) and (0.1.3).

Since in the Colombeau framework the evolution behavior of propagating singularities is not yet sufficiently understood we lose the geometrical interpretations from the smooth setting. But, we will show in the Sections 1.8 and 1.9 how one can derive well-posed approximated Cauchy problems from the resulting microlocal first-order equations.

In detail, we will show the well-posedness of the Cauchy problems for two first-order equations of the form

$$\begin{aligned} L_\pm u_\pm &= 0 && \text{on } (z_0, Z) \times \mathbb{R}^n \\ u_\pm(z_0) &= u_{\pm,0} && \text{on } \mathbb{R}^n \end{aligned}$$

such that the solutions $u_\pm \in \mathcal{G}_{L^2}$ allow for an approximation of the problem

$$LU \equiv 0 \qquad \text{microlocally on } I_{\theta_1},\; z \in (z_0, Z).$$

where $U := Q_+u_+ + Q_-u_- + \widetilde{f}$ for some $\widetilde{f} \equiv 0$ microlocally on I_{θ_1}, $z \in (z_0, Z)$ and $Q_+ := Q_{11}$, $Q_- := Q_{12}$ are suitable entries of a generalized transfer matrix (cf. (0.1.3)). The appropriate choice for the set I_{θ_1} is defined in (1.5.2). It depends on the limit of the wave speed function, i.e. $c^*(x,z) = \lim_{\varepsilon\to 0} c_\varepsilon(x,z)$ (i.e. $c^*(x,z)$ is the limit of the generalized coefficient $c_\varepsilon(x,z)$) which will be assumed to be a Hölder continuous function and is independent of the regularization parameter. More details about the set I_{θ_1} will be given in the Sections 1.5 and 1.7 and in Subsection 1.3.3.

We note that this first chapter goes back to the previous work done [26].

0.3 A semiclassical approach for less regular background coefficients

In the second part we will ask ourselves if it is possible to relax the logarithmic slow scale conditions on the coefficients to coefficients of log-type when using a different kind of quantization and we refer the reader to [25]. As this is a first step we will consider simpler operators as in Chapter 1 but with less regular coefficients. In particular, we will study of the following form:

$$L_\varepsilon(x, z, D_t, D_x, D_z) := \partial_z^2 + \sum_{j=1}^{n-1} b_{j,\varepsilon}(x, z)\partial_{x_j}^2 - c_\varepsilon(x, z)\partial_t^2. \tag{0.3.1}$$

Here, the coefficient $c_\varepsilon(x, z)$ is of log-type up to order $r \in \mathbb{N}$, i.e. for some constant $C > 0$ we have $\|\partial^\alpha c_\varepsilon\|_{L^\infty} = \mathcal{O}(\log^{|\alpha|/r}(C/\varepsilon))$ as $\varepsilon \to 0$, and strictly non-zero in the sense that there exists $\varepsilon_1 \in (0, 1]$ such that $\inf_{(x,z)\in\mathbb{R}^n} |c_\varepsilon(x, z)| \geqslant C$ for all $\varepsilon \in (0, \varepsilon_1]$ and some constant $C > 0$ independent of ε. Also, $b_{j,\varepsilon}$ is of log-type up to order r and strictly non-zero. Moreover we assume that $c_\varepsilon \to c$ and $b_{j,\varepsilon} \to b_j$ in the Hölder space $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$ with exponent $\mu \in (0, 1)$ as $\varepsilon \to 0$.

The observations of Hörmann made in [32] provides an indication for studying operators with coefficients that are of log-type. In detail, the author showed well-posedness for certain first-order hyperbolic pseudodifferential equations where the occurring symbols are of log-type up to fixed specified order. As we will slightly manipulate the second-order operator (i.e. we will consider certain quantizations of the operator L) we note that after a factorization procedure we will obtain first-order operators where the appearing symbols are of log-type.

So, as just mentioned, to analyze operators as in (0.3.1) with less regular coefficients as in Chapter 1 we try a different approach. In order to overcome the necessity of the logarithmic slow scale assumption (e.g. construction of an approximative inverse, well-posedness of certain first-order equations) we associate to the operator L_ε in (0.3.1) a semiclassical modification $L_{\psi,\varepsilon}$ such that $L_{\psi,\varepsilon}(x, z, D_t, D_x, D_z) := L_\varepsilon(x, z, \varepsilon D_t, \varepsilon D_x, \varepsilon D_z)$. Here, the particular choice $\psi = \psi(\varepsilon) = \varepsilon$ refers to the phase function in which the semiclassical scaling is kept and depends on the regularizing parameter $\varepsilon \in (0, 1]$. The detailed explanation of the correspondence between the operators L and L_ψ is given in Section 2.3. Then, instead of working with the equation $LU = F$, we consider the corresponding semiclassical problem

$$L_\psi U = F \tag{0.3.2}$$

where U, F are generalized functions in $\mathcal{G}_{L^2}$ and the operator $L_\psi : \mathcal{G}_{L^2} \to \mathcal{G}_{L^2}$ acts on the level of representatives as

$$(u_\varepsilon)_\varepsilon \mapsto (L_{\psi,\varepsilon}(x, z, D_t, D_x, D_z)u_\varepsilon)_\varepsilon \qquad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}.$$

Due to the additional ε-dependent semiclassical scaling parameter we then apply the theory of generalized semiclassical pseudodifferential operators which we call ψ-pseudodifferential operators in the following for short. This enables us to state a factorization theorem for L_ψ in terms of two first-order ψ-pseudodifferential operators with respect to the parameter z modulo two different types of error operators. As already mentioned above the factorization is only valid when imposing certain restrictions on the underlying time-space-frequency domain which we left ε-independent. In detail, the error operators are characterized by a semiclassical negligibility condition outside a compact set. Since global semiclassical negligible operators are easy to handle as they map moderate nets to negligible ones the interesting case is to understand the compactly supported error operators. But, due to the semiclassical scaling, worse ε-asymptotics are introduced when studying boundedness of such compactly supported operators. The basic notions and a general calculus of ψ-pseudodifferential operators can be found in Sections 2.2 and 2.4, the factorization procedure is explained in Section 2.5.

To eliminate the remainder terms produced in the factorization procedure we then proceed in Section 2.6 by presenting an adapted notion of microlocal regularity which essentially corresponds to the semiclassical wave front set at infinite points within the Colombeau framework. This allows us to overcome the difficulties that arise due to the compactly supported error operators occurring in the factorization. Concepts of the wave front set in Colombeau's theory have been explored in [40, 21, 22, 18]. For an introduction to semiclassical analysis and microlocalization we refer to [39, 14, 29, 1]. Again, we remark that unlike to the case of smooth coefficients we lose the property that singularities are propagating on geometrically determined phase space trajectories.

In Section 2.7 we then combine the factorization with our notion of microlocal regularity in order to describe the desired microlocal diagonalization method. As a result, we obtain a coupled system of two first-order ψ-pseudodifferential equations that approximate equation (0.3.2) microlocally at infinite points on an adequate subdomain of the phase space.

Chapter 1

Factorization of second-order strictly hyperbolic operators with logarithmic slow scale coefficients and well-posedness of generalized microlocal approximations

1.1 Basic notions

In this section we specify the basic notions that will be needed for our constructions. As the problem is treated within the framework of Colombeau algebras we refer to the literature [8, 41, 40, 28, 20, 19] for a systematic treatment in this field.

One of the main objects in our setting are Colombeau generalized functions based on $\mathcal{G}_{L^2}$ which were first introduced in [3]. The elements in this algebra are given by equivalence classes $u := [(u_\varepsilon)_{\varepsilon\in(0,1]}]$ of nets of regularizing functions u_ε in the Sobolev space $H^\infty = \bigcap_{k\in\mathbb{Z}} H^k$ corresponding to certain asymptotic seminorm estimates. More precisely, we denote by $\mathcal{M}_{H^\infty}$ the nets of moderate growth whose elements are characterized by the property

$$\forall\alpha \in \mathbb{N}^n\ \exists N \in \mathbb{N} : \|\partial^\alpha u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0.$$

Negligible nets will be denoted by $\mathcal{N}_{H^\infty}$ and are nets in $\mathcal{M}_{H^\infty}$ whose elements satisfy the following additional condition

$$\forall q \in \mathbb{N} : \|u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \text{ as } \varepsilon \to 0.$$

Then the algebra of generalized functions based on L^2-norm estimates is defined as the factor space $\mathcal{G}_{H^\infty} = \mathcal{M}_{H^\infty}/\mathcal{N}_{H^\infty}$. By abuse of notation, we continue to write $\mathcal{G}_{L^2}(\mathbb{R}^n)$ for $\mathcal{G}_{H^\infty}$. For simplicity, we shall also use the notation $(u_\varepsilon)_\varepsilon$ instead of $(u_\varepsilon)_{\varepsilon\in(0,1]}$ throughout the paper.

Using [3, Theorem 2.7], we first note that the distributions $H^{-\infty} = \bigcup_{k\in\mathbb{Z}} H^k$ are linearly embedded in $\mathcal{G}_{L^2}(\mathbb{R}^n)$ by convolution with a mollifier $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function such that

$$\int \varphi(x)\,dx = 1, \qquad \int x^\alpha \varphi(x)\,dx = 0 \quad \text{for all } \alpha \in \mathbb{N}^n,\ |\alpha| \geqslant 1. \tag{1.1.1}$$

Further, by the same result, $H^\infty(\mathbb{R}^n)$ is embedded as a subalgebra of $\mathcal{G}_{L^2}(\mathbb{R}^n)$.

More generally, we introduce Colombeau algebras based on a locally convex vector space E topologized through a family of seminorms $\{p_i\}_{i\in I}$ as in [24, Section 1]. To continue, the elements

$$\begin{aligned}
\mathcal{M}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I\ \exists N \in \mathbb{N} : p_i(u_\varepsilon) = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}\\
\mathcal{M}_E^\infty &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \exists N \in \mathbb{N}\ \forall i \in I : p_i(u_\varepsilon) = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}\\
\mathcal{M}_E^{lsc} &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I\ \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc} : p_i(u_\varepsilon) = \mathcal{O}(\omega_\varepsilon) \text{ as } \varepsilon \to 0\}\\
\mathcal{N}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I\ \forall q \in \mathbb{N} : p_i(u_\varepsilon) = \mathcal{O}(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}
\end{aligned} \tag{1.1.2}$$

are said to be E-moderate, E-regular, E-moderate of logarithmic slow scale type and E-negligible, respectively.

Then, $\mathcal{N}_E$ is an ideal in $\mathcal{M}_E$ and the space of Colombeau algebra based on E is defined by the factor space $\mathcal{G}_E = \mathcal{M}_E/\mathcal{N}_E$ and possesses the structure of a $\widetilde{\mathbb{C}}$-module. The space of the regular Colombeau algebra $\mathcal{G}_E^\infty = \mathcal{M}_E^\infty/\mathcal{N}_E$ is again a $\widetilde{\mathbb{C}}$-module whereas the space of the logarithmic slow scale algebra $\mathcal{G}_E^{lsc} = \mathcal{M}_E^{lsc}/\mathcal{N}_E$ is a $\mathbb{C}$-module.

Example 1.1.1. Setting $E = \mathbb{C}$ one gets the ring of complex generalized numbers $\widetilde{\mathbb{C}} = \mathcal{G}_{\mathbb{C}}$ with the absolute value as the corresponding seminorm. Furthermore, we denote by $\widetilde{\mathbb{R}} = \mathcal{G}_{\mathbb{R}}$ the ring of real generalized numbers.

Further, let Ω be an open subset of $\mathbb{R}^n$. Then the Colombeau algebra $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ is obtained by taking $E = \mathcal{C}^\infty(\Omega)$ endowed with the topology induced by the family of seminorms $p_i(f) := p_{K_i,i}(f) = \sup\{|\partial^\alpha f(x)| : x \in K_i, |\alpha| \leqslant i\}$ for some fixed $f \in \mathcal{C}^\infty(\Omega)$ and where $\{K_i\}_{i\in\mathbb{N}}$ is an increasing sequence of relatively compact open subsets of Ω exhausting

Ω.

Another example is the Colombeau algebra $\mathcal{G}_{p,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, when E is set to $W^{\infty,p}(\Omega)$ and the topology is determined by the collection of seminorms $p_i(f) = \sup\{||\partial^\alpha f||_p : |\alpha| \leqslant i\}$, $f \in W^{\infty,p}(\Omega)$, as i varies over $\mathbb{N}$. We note that $\mathcal{G}_{L^2} = \mathcal{G}_{2,2}$. For more general Colombeau algebras based on Sobolev spaces we refer to [3, Section 2]. Using the notation there, we have the following equivalences: $\mathcal{E}_{M,p}(\Omega) = \mathcal{M}_{W^{\infty,p}(\Omega)}$ and $\mathcal{N}_{p,p}(\Omega) = \mathcal{N}_{W^{\infty,p}(\Omega)}$.

As different growth types are crucial in regularity theory we introduce the set of logarithmic slow scale nets by

$$\Pi_{lsc} := \Big\{ (\omega_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \;\big|\; \exists \eta \in (0,1]\ \exists c > 0\ \forall \varepsilon \in (0,\eta] : c \leqslant \omega_\varepsilon,$$
$$\exists \eta \in (0,1]\ \forall p \geqslant 0\ \exists c_p > 0 : |\omega_\varepsilon|^p \leqslant c_p \log\left(\frac{1}{\varepsilon}\right) \quad \varepsilon \in (0,\eta] \Big\}.$$

Further, we call a net $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$ logarithmic slow scale strictly nonzero if there exists $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and an $\eta \in (0,1]$ such that $|\omega_\varepsilon| \geqslant 1/r_\varepsilon$ for $\varepsilon \in (0,\eta]$.

1.2 Pseudodifferential calculus

In this section we introduce a general calculus for pseudodifferential operators which are standard quantizations of generalized symbols. Since most of the techniques are similar to the usual theory of pseudodifferential operators we also want to refer to [31, 55]. A detailed discussion on pseudodifferential operators with Colombeau generalized symbols can be found in [20, 19]. As usual, we will write $D_{x_j} = -i\partial_{x_j} = -i\frac{\partial}{\partial x_j}$.

We shall initially discuss the main notions of generalized symbols. As already indicated above, we will study symbols which satisfy asymptotic growth conditions with respect to $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$. In what follows, we use the notation $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$.

1.2.1 Generalized symbol classes

We let $\rho, \delta \in [0,1]$ and $m \in \mathbb{R}$. Moreover, we denote by $S^m_{\rho,\delta} = S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ the set of symbols of order m and type (ρ,δ) as first introduced by Hörmander in [30, Definition 2.1]. Since the symbol class $S^m_{\rho,\delta}$ satisfies global estimates we remark that our space $\mathcal{M}_{S^m_{\rho,\delta}}$ is different to that in [23, Section 1.4].

As in the following we will typically encounter subspaces of $\mathcal{M}_{S^m_{\rho,\delta}}$ subjected to logarithmic slow scale asymptotics we can choose in (1.1.2) $E = S^m_{\rho,\delta}$ and obtain symbols of the form:

Definition 1.2.1. *Let $m \in \mathbb{R}$. Then the set of moderate logarithmic slow scale symbols $S^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ of order m and type (ρ, δ) consists of all $(a_\varepsilon)_\varepsilon \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)^{(0,1]}$ such that*

$$\forall \alpha, \beta \in \mathbb{N}^n \ \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc} :$$
$$q^{(m)}_{\alpha,\beta}(a_\varepsilon) := \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial^\alpha_\xi \partial^\beta_x a_\varepsilon(x,\xi)| \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} = \mathcal{O}(\omega_\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

An element of $N^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is said to be negligible in $S^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ if it fulfills the following condition:

$$\forall \alpha, \beta \in \mathbb{N}^n \ \forall q \in \mathbb{N} : \quad q^{(m)}_{\alpha,\beta}(a_\varepsilon) = \mathcal{O}(\varepsilon^q) \qquad \text{as } \varepsilon \to 0.$$

Then, logarithmically slow scaled symbols of order m are defined as the factor space

$$\widetilde{S}^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n) := S^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n)/N^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n).$$

and in the following we will assume that $\delta < \rho$. In the case that $\rho = 1$ and $\delta = 0$ we use the abbreviations S^m_{lsc}, N^m_{lsc} and $\widetilde{S}^m_{lsc}$ for $S^m_{1,0,lsc}, N^m_{1,0,lsc}$ and $\widetilde{S}^m_{1,0,lsc}$, respectively.

Remark 1.2.2. Moreover, the space $\widetilde{S}^{-\infty}_{lsc}$ of logarithmic slow scale symbols of order $-\infty$ consists of equivalence classes a whose representatives $(a_\varepsilon)_\varepsilon$ have the property that

$$\forall m \in \mathbb{R} \ \forall \alpha, \beta \in \mathbb{N}^n \ \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc} : \quad q^{(m)}_{\alpha,\beta}(a_\varepsilon) = \mathcal{O}(\omega_\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Since $(\omega_\varepsilon)_\varepsilon$ is a logarithmic slow scale net, we note that the net of a symbol $(a_\varepsilon)_\varepsilon$ in S^m_{lsc} can always be estimated as follows

$$\forall \alpha, \beta \in \mathbb{N}^n : \quad q^{(m)}_{\alpha,\beta}(a_\varepsilon) = \mathcal{O}\Big(\log\Big(\frac{1}{\varepsilon}\Big)\Big) \qquad \text{as } \varepsilon \to 0.$$

To give an example, let $P(x, D_x) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha_x$ be a partial differential operator with bounded and measurable coefficients. Then the logarithmically slow scaled regularization of the symbol is given by $p_\varepsilon(x,\xi) := (p(.,\xi) * \varphi_{\omega_\varepsilon^{-1}})(x)$ and $(p_\varepsilon)_\varepsilon$ is contained in $S^m_{lsc}(\mathbb{R}^n \times \mathbb{R}^n)$.

Also, we will make use of the following symbol class:

Definition 1.2.3. *Let $U \subset \mathbb{R}^n \times \mathbb{R}^n$ be open and conic with respect to the second variable. We say that the generalized symbol a is in $\widetilde{S}^m_{\rho,\delta,lsc}(U)$ if it has a representative $(a_\varepsilon)_\varepsilon$ with $a_\varepsilon \in \mathcal{C}^\infty(U)$ for fixed $\varepsilon \in (0, 1]$ and for any compact set $K \subset pr_2(U)$ (independent of ε) and $V_K := \{(x,\xi) \in U \mid \xi \in K\}$ we have*

$$\forall \alpha, \beta \in \mathbb{N}^n \ \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc} \ \exists C > 0 :$$
$$|\partial^\alpha_\xi \partial^\beta_x a_\varepsilon(x,\xi)| \leqslant C \omega_\varepsilon \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \qquad (x,\xi) \in V^c_K$$

for ε sufficiently small and where $V_K^c := \{(x, \lambda\xi) \mid (x,\xi) \in V_K, \lambda \geqslant 1\}$.

Note that Definition 1.2.1 is equivalent to Definition 1.2.3 in the case that $U = \mathbb{R}^n \times \mathbb{R}^n$. For completeness, we recall the definition for global symbols $S^m_{\rho,\delta}(U)$ which is similar to that for local symbols, see [6, Definition 2.3, page 141]: For U being an open subset of $\mathbb{R}^n \times \mathbb{R}^n$, which is conic in the second variable, we say that $a \in S^m_{\rho,\delta}(U)$ if $a \in \mathcal{C}^\infty(U)$ and for each compact $K \subset \mathrm{pr}_2(U)$ and $V_K := \{(x,\xi) \in U \mid \xi \in K\}$ we have

$$\forall \alpha, \beta \in \mathbb{N}^n : \quad \sup_{(x,\xi)\in V_K^c} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} < \infty$$

where $V_K^c := \{(x, \lambda\xi) \mid (x,\xi) \in V_K, \lambda \geqslant 1\}$.

We choose the following convention for defining the Fourier transform $\mathcal{F}$ of a function $u \in L^2(\mathbb{R}^n)$:

$$\mathcal{F}u(\xi) := \hat{u}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx := \lim_{\sigma \to 0_+} \int_{\mathbb{R}^n} e^{-ix\xi - \sigma\langle x \rangle} u(x)\, dx.$$

Then, the Fourier transform is an isomorphism on $L^2(\mathbb{R}^n)$ and the inverse Fourier transform of $u \in L^2(\mathbb{R}^n)$ is given by the formula

$$\mathcal{F}^{-1}u(x) := \int_{\mathbb{R}^n} e^{ix\xi} u(\xi)\ đ\xi := \lim_{\sigma \to 0_+} \int_{\mathbb{R}^n} e^{ix\xi - \sigma\langle \xi \rangle} u(\xi)\ đ\xi.$$

where we have set $đ\xi := (2\pi)^{-n}\, d\xi$. More information about the Fourier transform acting on $\mathcal{G}_{L^2}$ can be found in [2]. As already mentioned above, we will focus on generalized pseudodifferential operators having the following phase-amplitude representation:

Definition 1.2.4. *Let $(a_\varepsilon)_\varepsilon \in a \in \widetilde{S}^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ and let $(u_\varepsilon)_\varepsilon$ be a representative of $u \in \mathcal{G}_{L^2}$. We define the corresponding linear operator $A : \mathcal{G}_{L^2} \to \mathcal{G}_{L^2}$ by*

$$A(x,D)u(x) := \int e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\ đ\xi := (A_\varepsilon(x,D)u_\varepsilon(x))_\varepsilon + \mathcal{N}_{H^\infty}(\mathbb{R}^n)$$

where

$$\begin{aligned} A_\varepsilon(x,D_x)u_\varepsilon(x) &:= \int e^{i(x-y)\xi} a_\varepsilon(x,\xi) u_\varepsilon(y)\, dy\ đ\xi = \int e^{ix\xi} a_\varepsilon(x,\xi)\hat{u}_\varepsilon(\xi)\ đ\xi = \\ &= \mathcal{F}^{-1}_{\xi\to x}\Big(a_\varepsilon(x,\xi)\hat{u}_\varepsilon(\xi)\Big) \end{aligned}$$

where the last integral is interpreted as an oscillatory integral. The operator A is called the generalized pseudodifferential operator with generalized symbol a. Later on, we will sometimes write $A \in \Psi^m_{\rho,\delta,lsc}(\mathbb{R}^n)$ to denote that A is a generalized pseudodifferential operator *with symbol in* $\widetilde{S}^m_{\rho,\delta,lsc}(\mathbb{R}^n \times \mathbb{R}^n)$*. In the case that* $(\rho,\delta) = (1,0)$*, we will write* $A \in \Psi^m_{lsc}(\mathbb{R}^n)$ *for short.*

1.2.2 Generalized point values of a generalized symbol

As above, let $U \times \Gamma \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be open and conic with respect to the second variable and $(a_\varepsilon)_\varepsilon$ a generalized symbol in $S^m_{\rho,\delta,lsc}(U \times \Gamma)$.

Definition 1.2.5. *Let* $\Gamma \subseteq \mathbb{R}^n$ *be an open cone. On*

$$\Gamma_M := \{(\zeta_\varepsilon)_\varepsilon \in \Gamma^{(0,1]} \mid \exists N \in \mathbb{N} : |\zeta_\varepsilon| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

we introduce the equivalence relation

$$(\zeta^1_\varepsilon)_\varepsilon \sim (\zeta^2_\varepsilon)_\varepsilon \Leftrightarrow \forall m \in \mathbb{N} : |\zeta^1_\varepsilon - \zeta^2_\varepsilon| = \mathcal{O}(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by $\widetilde{\Gamma} := \Gamma_M/\sim$ *the generalized cone with respect to* Γ*.*

Lemma 1.2.6. *Let* $a \in \widetilde{S}^m_{\rho,\delta,lsc}(U \times \Gamma)$ *and* $\widetilde{\zeta} \in \widetilde{\Gamma}$*. Then the generalized point value of* a *at* $\widetilde{\zeta} = [(\zeta_\varepsilon)_\varepsilon]$*,*

$$a(z, \widetilde{\zeta}) := [(a_\varepsilon(z, \zeta_\varepsilon))_\varepsilon]$$

is a well-defined element in $\widetilde{\mathbb{C}}$*.*

Proof. The proof is similar to the proof of [28, Proposition 1.2.45]. We let $(z, \zeta_\varepsilon)_\varepsilon \in U \times \Gamma_M$ and $(a_\varepsilon)_\varepsilon \in S^m_{\rho,\delta,lsc}(U \times \Gamma)$. For any index $i \in \mathbb{N}$, C_i denotes a positive constant, $(\omega_{i,\varepsilon})_\varepsilon$ a logarithmic slow scale net and N_i a natural number. Then

$$|a_\varepsilon(z, \zeta_\varepsilon)| \leqslant C_1\omega_{1,\varepsilon}(1 + |\zeta_\varepsilon|)^m \leqslant C_1\omega_{1,\varepsilon}(1 + C_2\varepsilon^{-N_2})^{\max(m,0)} \leqslant C_3\varepsilon^{-N_3},$$

so $(a_\varepsilon(z, \zeta_\varepsilon))_\varepsilon \in \mathbb{R}_M = \mathcal{M}_{\mathbb{R}}$.

We now show that $\widetilde{\zeta}^1 \sim \widetilde{\zeta}^2$ implies $a(z, \widetilde{\zeta}^1) \sim a(z, \widetilde{\zeta}^2)$. So let $\widetilde{\zeta}^1 \sim \widetilde{\zeta}^2$. Then

$$\begin{aligned} |a_\varepsilon(z, \zeta^1_\varepsilon) - a_\varepsilon(z, \zeta^2_\varepsilon)| &\leqslant |\zeta^1_\varepsilon - \zeta^2_\varepsilon| \int_0^1 |D_\zeta a_\varepsilon(z, \zeta^1_\varepsilon + \sigma(\zeta^2_\varepsilon - \zeta^1_\varepsilon))|\, d\sigma \leqslant \\ &\leqslant C_4\omega_{4,\varepsilon}(1 + C_5\varepsilon^{-N_5})^{\max(m,0)}|\zeta^1_\varepsilon - \zeta^2_\varepsilon| \leqslant C_6\varepsilon^{-N_6}\varepsilon^p \end{aligned}$$

for all $p \geqslant 0$ and ε small enough. Hence $(a_\varepsilon(z, \zeta^1_\varepsilon))_\varepsilon - (a_\varepsilon(z, \zeta^2_\varepsilon))_\varepsilon \sim 0$.

Finally, if $(a_\varepsilon)_\varepsilon \in \mathcal{N}_{S^m}(U \times \Gamma)$ we have

$$|a_\varepsilon(z, \zeta_\varepsilon)| \leqslant C\varepsilon^p(1 + |\zeta_\varepsilon|)^m \leqslant C\varepsilon^p(1 + C_7\varepsilon^{-N_7})^{\max(m,0)}$$

for any $p \geqslant 0$ as $\varepsilon \to 0$. So $(a_\varepsilon(z, \zeta_\varepsilon))_\varepsilon \sim 0$. $\square$

In the case of generalized logarithmic slow scale cones we get a similar result.

Definition 1.2.7. *Let $\Gamma \subseteq \mathbb{R}^n$ be an open cone. On*

$$\Gamma_{M,lsc} := \{(\zeta_\varepsilon)_\varepsilon \in \Gamma^{(0,1]} \mid \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc} : |\zeta_\varepsilon| = \mathcal{O}(\omega_\varepsilon) \text{ as } \varepsilon \to 0\}$$

we introduce the equivalence relation

$$(\zeta^1_\varepsilon)_\varepsilon \sim (\zeta^2_\varepsilon)_\varepsilon \Leftrightarrow \forall m \in \mathbb{N} : |\zeta^1_\varepsilon - \zeta^2_\varepsilon| = \mathcal{O}(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by $\widetilde{\Gamma}_{lsc} := \Gamma_{M,lsc}/\sim$ the generalized logarithmic slow scale cone of Γ.

Lemma 1.2.8. *Let $a \in \widetilde{S}^m_{\rho,\delta,lsc}(U \times \Gamma)$ and $\widetilde{\zeta} \in \widetilde{\Gamma}_{lsc}$. Then the generalized point value of a at $\widetilde{\zeta} = [(\zeta_\varepsilon)_\varepsilon]$,*

$$a(z, \widetilde{\zeta}) := [(a_\varepsilon(z, \zeta_\varepsilon))_\varepsilon]$$

is a well-defined element of $\widetilde{\mathbb{C}}_{lsc}$.

1.2.3 Asymptotic expansion

To prepare the factorization theorem of Section 1.5 we will have to consider products of pseudodifferential operators. In the sequel, we will construct a complete symbolic calculus for generalized pseudodifferential operators.

We therefore start with some general observations concerning the notion of asymptotic expansion of a generalized symbol in $\widetilde{S}^m_{\rho,\delta,lsc}$. The presentation of the results are inspired by the results given in [19, Section 2.5] and [20]. There, the techniques are described by means of generalized symbols as well as for slow scale symbols. The modification to logarithmic slow scale symbols is evident.

The definition of the asymptotic expansion is now the following:

Definition 1.2.9. *Let $\{m_j\}_j$ be a strictly decreasing sequence of real numbers such that $m_j \to -\infty$ as $j \to \infty$, $m_0 = m$.*

(i) Further, let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{S^{m_j}_{\rho,\delta}}$. We say that the formal series $\sum_{j=0}^{\infty} (a_{j,\varepsilon})_\varepsilon$ is the asymptotic expansion for $(a_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)^{(0,1]}$, denoted by

$(a_\varepsilon)_\varepsilon \sim \sum_j (a_{j,\varepsilon})_\varepsilon$, if and only if

$$(a_\varepsilon - \sum_{j=0}^{N-1} a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{S^{m_N}_{\rho,\delta}} \qquad \forall N \geqslant 1.$$

(ii) Also, let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in S^{m_j}_{\rho,\delta,lsc}$. We say that the formal series $\sum_{j=0}^{\infty}(a_{j,\varepsilon})_\varepsilon$ is the asymptotic expansion for $(a_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)^{(0,1]}$, denoted by $(a_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (a_{j,\varepsilon})_\varepsilon$, if and only if

$$(a_\varepsilon - \sum_{j=0}^{N-1} a_{j,\varepsilon})_\varepsilon \in S^{m_N}_{\rho,\delta,lsc} \qquad \forall N \geqslant 1.$$

In both cases, $(a_{0,\varepsilon})_\varepsilon$ is said to be the principal symbol of $(a_\varepsilon)_\varepsilon$.

Moreover, we introduce the restriction $\mathcal{M}_{S^m_{\rho,\delta,cl}}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \mathcal{M}_{S^m_{\rho,\delta}}$ of classical generalized symbols.

Definition 1.2.10. *We say that a generalized symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^m_{\rho,\delta}}$ is classical, denoted by $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^m_{\rho,\delta,cl}}$, if there exists a sequence $\{(a_{j,\varepsilon})_\varepsilon\}_j$ with symbols $(a_{j,\varepsilon})_\varepsilon$ in $\mathcal{M}_{S^{m-j}_{\rho,\delta}}(\mathbb{R}^n \times (\mathbb{R}^n \backslash 0))$ homogeneous of degree $m-j$ in $|\xi| \geqslant 1$, $j \in \mathbb{N}$, such that for any cut-off function $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ equal to 1 near the origin we have*

$$\Big(a_\varepsilon(x,\xi) - \sum_{j=0}^{N-1}(1-\varphi(\xi))a_{j,\varepsilon}(x,\xi)\Big)_\varepsilon \in \mathcal{M}_{S^{m-N}_{\rho,\delta}} \qquad \forall N \geqslant 1. \tag{1.2.1}$$

As above, we will write $(a_\varepsilon)_\varepsilon \sim \sum_j (a_{j,\varepsilon})_\varepsilon$ if (1.2.1) holds and we call $(a_{0,\varepsilon})_\varepsilon$ the principal symbol of $(a_\varepsilon)_\varepsilon$.

Replacing $\mathcal{M}_{S^m_{\rho,\delta}}$ by $\mathcal{M}^{lsc}_{S^m_{\rho,\delta}}$ and $\sim$ by $\sim_{lsc}$ in the definition above, one obtains classical symbols of logarithmic slow scale type.

As a result, we obtain that any infinite sum of symbols of strictly decreasing orders can be summarized as follows.

Theorem 1.2.11.

(i) Let $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{S^{m_j}_{\rho,\delta}}$ with $m_j \searrow -\infty$ as $j \to \infty$, $m_0 = m$ as in (i) of Definition 1.2.9. Then there exists $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^m_{\rho,\delta}}$ such that $(a_\varepsilon)_\varepsilon \sim \sum_j (a_{j,\varepsilon})_\varepsilon$. Moreover, if $(a'_\varepsilon)_\varepsilon \sim \sum_j (a_{j,\varepsilon})_\varepsilon$, then $(a_\varepsilon - a'_\varepsilon)_\varepsilon \in \mathcal{M}_{S^{-\infty}}$.

(ii) *Let* $(a_{j,\varepsilon})_\varepsilon \in S^{m_j}_{\rho,\delta,lsc}$ *with* $m_j \searrow -\infty$ *as* $j \to \infty$, $m_0 = m$ *as in (ii) of Definition 1.2.9. Then there exists* $(a_\varepsilon)_\varepsilon \in S^m_{\rho,\delta,lsc}$ *such that* $(a_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (a_{j,\varepsilon})_\varepsilon$. *Moreover, if* $(a'_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (a_{j,\varepsilon})_\varepsilon$, *then* $(a_\varepsilon - a'_\varepsilon)_\varepsilon \in S^{-\infty}_{lsc}$.

Remark 1.2.12. The proof of this theorem can be found in [19, Theorem 2.2]. The same proof remains valid for classical symbol classes. We recall the construction of the symbol $(a_\varepsilon)_\varepsilon$ of Theorem 1.2.11:

Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \leqslant \psi(\xi) \leqslant 1$ such that $\psi(\xi) = 0$ for $|\xi| \leqslant 1$ and $\psi(\xi) = 1$ for $|\xi| \geqslant 2$. As in [19, Theorem 2.2] one can define

$$a_\varepsilon(x,\xi) := \sum_{j\in\mathbb{N}} \psi(\lambda_{j,\varepsilon}\xi) a_{j,\varepsilon}(x,\xi)$$

where $\lambda_{j,\varepsilon}$ are appropriate positive constants with $\lambda_{j+1,\varepsilon} < \lambda_{j,\varepsilon} < 1$, $\lambda_{j,\varepsilon} \to 0$ as $j \to \infty$. In case (i) of Theorem 1.2.11, $\lambda_{j,\varepsilon}$ are taken to be the inverse of an appropriate strict positive net. In Theorem 1.2.11 (ii), it turns out that $(\lambda_{j,\varepsilon})$ can be chosen to be an inverse of a logarithmic slow scale net.

We also remark that Theorem 1.2.11 can be carried over to classical symbols in a corresponding way.

Noting that in Theorem 1.2.11 (i) one can replace the moderateness by negligibility and we introduce as in [19, Definition 2.6 (ii)]:

Definition 1.2.13. *Let* $\{m_j\}_j$ *be as in Definition 1.2.9. Further let* $\{a_j\}_j$ *be a sequence in* $\widetilde{S}^{m_j}_{\rho,\delta,lsc}$. *We say that the formal series* $\sum_{j=0}^\infty a_j$ *is the asymptotic expansion of* $a \in \widetilde{S}^m_{\rho,\delta,lsc}$, *denoted by* $a \sim_{lsc} \sum_j a_j$, *if and only if there exists a representative* $(a_\varepsilon)_\varepsilon$ *of* a *and, for every* j *representatives* $(a_{j,\varepsilon})_\varepsilon$ *of* a_j, *such that* $(a_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (a_{j,\varepsilon})_\varepsilon$.

Similarly, we introduce the following definition for classical symbols.

Definition 1.2.14. *Let* $\{a_j\}_{j\in\mathbb{N}}$ *be a sequence with* $a_j \in \widetilde{S}^{m-j}_{\rho,\delta,lsc}(\mathbb{R}^n \times (\mathbb{R}^n\backslash 0))$ *homogeneous of degree* $m-j$. *We say that the formal series* $\sum_{j=0}^\infty a_j$ *is the asymptotic expansion of* $a \in \widetilde{S}^m_{\rho,\delta,cl,lsc}$, *denoted by* $a \sim_{lsc} \sum_j a_j$, *if and only if there exists a representative* $(a_\varepsilon)_\varepsilon$ *of* a *and, for every* j *representatives* $(a_{j,\varepsilon})_\varepsilon$ *of* a_j, *such that* $(a_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (a_{j,\varepsilon})_\varepsilon$.

1.2.4 Composition and adjoint of pseudodifferential operators

In this subsection we briefly recall the composition law of two pseudodifferential operators and adjoint operators. A detailed discussion for slow scale regular generalized symbols can be found in [20, Section 5]. Again, the adaptation to logarithmic slow scale symbols is evident.

Therefore, for logarithmic slow scale symbols, we get the following result:

Theorem 1.2.15. *Let $A(x, D_x)$ and $B(x, D_x)$ be two pseudodifferential operators with generalized symbols $a \in \widetilde{S}^{m_1}_{\rho,\delta,lsc}$ and $b \in \widetilde{S}^{m_2}_{\rho,\delta,lsc}$ respectively. Then the product AB is well-defined and maps $\mathcal{G}_{L^2}$ into itself. Moreover AB is a pseudodifferential operator with generalized symbol $a\#b$ in $\widetilde{S}^{m_1+m_2}_{\rho,\delta,lsc}$ having the representation*

$$a\#b(x,\xi) \sim_{lsc} \sum_{|\alpha|\geqslant 0} \frac{1}{\alpha!} D^\alpha_\xi a(x,\xi)\partial^\alpha_x b(x,\xi).$$

For the proof we refer to [20, Theorem 5.15]. We note that Theorem 1.2.15 can also be stated for classical symbols. In detail, if $a \in \widetilde{S}^{m_1}_{\rho,\delta,cl,lsc}$ and $b \in \widetilde{S}^{m_2}_{\rho,\delta,cl,lsc}$, then $a\#b$ in $\widetilde{S}^{m_1+m_2}_{\rho,\delta,cl,lsc}$.

Also useful is the following theorem:

Theorem 1.2.16. *Let $A(x, D_x)$ be a pseudodifferential operators with generalized symbol $a \in \widetilde{S}^{m}_{\rho,\delta,lsc}$. Then the adjoint operator A^* is in $\Psi^m_{\rho,\delta,lsc}$ and its symbol a^* is given by the asymptotic expansion*

$$a^*(x,\xi) \sim_{lsc} \sum_{|\alpha|\geqslant 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi \overline{a(x,\xi)}.$$

For the proof see [20, Theorem 5.12].

1.3 The governing equation in the Colombeau setting

The present survey is devoted to wave propagation in inhomogeneous acoustic media in the case of non-smooth background data. The operator structure is motivated from wave propagation phenomena occurring in a number of physical applications, such as underwater acoustic or seismology. Here, the evolution direction is the depth coordinate.

1.3.1 Derivation of the inhomogeneous wave equation

The acoustic wave equation characterizes fluid motions and can be derived from the conservation laws of mass and momentum together with the equation of state of thermodynamical equilibrium. This system of acoustic equations can be linearized and therefore describe small perturbations from a state of rest of the pressure $U(t,x) \in \mathbb{R}$, the density $\rho'(t,x) \in \mathbb{R}$ and the velocity $\vec{v'}(t,x) \in \mathbb{R}^n$ that moves in a fluid with given wave speed $c(x)$ and density $\rho(x)$, $(x \in \mathbb{R}^n, t \in \mathbb{R})$. To explain how these waves are generated or added to the fluid one can introduce so-called sources by adding source terms in the equation of mass, momentum and energy. Assuming an isentropic process the linearized acoustic system can be written as (cf. [35, 36, 42, 5])

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} &= \operatorname{div}(\rho \vec{v'}) + m && \text{(mass conservation)}\\
\frac{\partial \vec{v'}}{\partial t} &= \frac{1}{\rho}\nabla U + \vec{f} && \text{(momentum conservation)}\\
\frac{\partial U}{\partial t} &= c^2\Big(\frac{\partial \rho'}{\partial t} + \vec{v'}\nabla\rho\Big) && \text{(equation of state)}
\end{aligned}$$

where the mass source term m and the force source term $\vec{f}$ are supposed to vanish in the undisturbed state. Here, m represents a volume injection of a source such as, for example, bodies whose volume is oscillating. An example for a body force $\vec{f}$ of a source is an oscillating rigid body of constant volume. Note that these equations also hold for the fluid at rest.

Substituting the state equation into the equation of mass and using the conservation of momentum to eliminate the velocity term gives

$$\operatorname{div}(\frac{1}{\rho}\nabla U) - \frac{1}{c^2\rho}\frac{\partial^2}{\partial t^2}U = F$$

where $F := -\frac{\partial m/\rho}{\partial t} + \operatorname{div}(\vec{f}/\rho)$ denotes the source function. We remark that in the absence of sources one derives the homogeneous wave equation.

In the following we will study the Colombeau generalized partial differential equation of the form

$$LU := \big(\partial_z \frac{1}{\rho}\partial_z + \sum_{j=1}^{n-1}\partial_{x_j}\frac{1}{\rho}\partial_{x_j} - \frac{1}{\rho}\frac{1}{c^2}\partial_t^2\big)U = F \tag{1.3.1}$$

with the (pressure) wave field $U \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ and $F \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ a source term. For the space coordinates we will allocate the vertical direction z, which we call the depth, the lateral directions are denoted by x.

We assume that the coefficients $\frac{1}{\rho} = \frac{1}{\rho}(x,z)$ and $\frac{1}{c} = \frac{1}{c}(x,z)$ are Colombeau generalized functions in $\mathcal{G}^{lsc}_{\infty,\infty} := \mathcal{M}^{lsc}_{W^{\infty,\infty}}/\mathcal{N}_{W^{\infty,\infty}}$ and meet the following requirements: there are representatives $\left(\frac{1}{\rho_\varepsilon}\right)_\varepsilon \in \frac{1}{\rho}$, $\left(\frac{1}{c_\varepsilon^2}\right)_\varepsilon \in \frac{1}{c^2}$ such that

(i) there exist Hölder continuous functions $\frac{1}{\rho^*}, \frac{1}{c^*} \in \mathcal{C}^{0,\mu}(\mathbb{R}^n)$ for some $\mu \in (0,1)$ such that for all $(x,z) \in \mathbb{R}^n$ we have $\frac{1}{\rho_\varepsilon} = \frac{1}{\rho^*} * \varphi_{\omega_\varepsilon^{-1}}, \frac{1}{c_\varepsilon} = \frac{1}{c^*} * \varphi_{\omega_\varepsilon^{-1}}$ where φ is as in (1.1.1), $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and the convolution is taken with respect to x and z

(ii) $\exists \eta \in (0,1], \exists$ constants $c_0,\ c_1,\ \rho_0$ and ρ_1 such that $0 < c_0 \leqslant c_\varepsilon(x,z) \leqslant c_1 < \infty$ and $0 < \rho_0 \leqslant \rho_\varepsilon(x,z) \leqslant \rho_1 < \infty$ for all $(x,z) \in \mathbb{R}^n$ and $\varepsilon \in (0,\eta]$.

The lower bound assumption is sometimes referred to as strong positivity (cf. [21, Section 1] or [18, Section 2]). The upper bound means uniform boundedness of the 0-th derivative of the representatives. For more details on the assumption (i) we refer to the remark given below.

Given a regularized operator as above, we carry out all transformations within algebras of generalized functions from now on. More explicitly, we will study the action of the linear operator L from $\mathcal{G}_{L^2}$ into itself in the following sense: on the level of representatives L acts as

$$(u_\varepsilon)_\varepsilon \mapsto \Big(\big(\partial_z \frac{1}{\rho_\varepsilon} \partial_z + \sum_{j=1}^{n-1} \partial_{x_j} \frac{1}{\rho_\varepsilon} \partial_{x_j} - \frac{1}{\rho_\varepsilon} \frac{1}{c_\varepsilon^2} \partial_t^2\big) u_\varepsilon\Big)_\varepsilon \qquad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}.$$

This explains our governing equation in (1.3.1) where $U = [(u_\varepsilon)_\varepsilon]$.

Remark 1.3.1. In order to realize the logarithmic slow scale type conditions on the coefficients of a partial differential operator, one usually uses a rescaling in the mollification. In our case, the regularization is obtained by convolution with the logarithmically scaled mollifier $\varphi_{\omega_\varepsilon^{-1}}(.) := \omega_\varepsilon^n \varphi(\omega_\varepsilon .)$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as in (1.1.1) and $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$.

For completeness, we give the following implications for our type of coefficients: Let $u \in \mathcal{C}^{0,\mu}(\mathbb{R}^n)$ be a real-valued Hölder continuous function with exponent $\mu \in (0,1)$ and such that $\inf(u) \geqslant c$ for some positive constant c. Further, φ is a mollifier as above and $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$. One may think of $\omega_\varepsilon = \log(\log(\frac{1}{\varepsilon}))$ as an explicit example. Then $\exists c > 0 : \omega_\varepsilon \geqslant c$ for all $\varepsilon \in (0, \frac{1}{11}]$ and $\forall p \geqslant 0, \exists c_p > 0 : \omega_\varepsilon^p \leqslant c_p \log(\frac{1}{\varepsilon})$ for all $\varepsilon \in (0,1]$. So, given some $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$ the logarithmic slow scale regularization of u is given by $u_\varepsilon(x) = u * \varphi_{\omega_\varepsilon^{-1}}(x)$ and satisfies the following estimates (see [33, Theorem 7], or [45, Section 5]):

$$\|\partial^\alpha u_\varepsilon\|_{L^\infty} = \begin{cases} \mathcal{O}(1) & |\alpha| = 0 \\ \mathcal{O}\big(\omega_\varepsilon^{|\alpha|-\mu}\big) & |\alpha| > 0 \end{cases} \qquad (\varepsilon \to 0)$$

and

$$\exists u_0 > 0\ \exists \eta \in (0,1] : \quad |u_\varepsilon(x)| \geqslant u_0 \qquad x \in \mathbb{R}^n,\ \varepsilon \in (0,\eta].$$

Here, the latter inequality is the strong positivity and the first estimate guarantees that the net $(u_\varepsilon)_\varepsilon$ is in $\mathcal{M}^{lsc}_{W^{\infty,\infty}}$. Moreover, note that $u \in \mathcal{C}^{0,\mu}$ implies that for every $x \in \mathbb{R}^n$ we have

$$|u_\varepsilon(x) - u(x)| \leqslant \int |u(x - \omega_\varepsilon^{-1} y) - u(x)|\, |\varphi(y)|\, dy = \mathcal{O}(\omega_\varepsilon^{-\mu}) \qquad \varepsilon \to 0.$$

Again, with a slight abuse of notation, we write u for the equivalence class of the Colombeau regularization of $u \in \mathcal{C}^{0,\mu}$, i.e.

$$u := (u_\varepsilon)_\varepsilon + \mathcal{N}_{W^{\infty,\infty}} \in \mathcal{G}^{lsc}_{\infty,\infty}.$$

1.3.2 Time extrapolation of the wave field

In a paper of Garetto and Oberguggenberger, see [24], the authors studied well-posedness of Cauchy problems with respect to the time variable for strictly hyperbolic systems and higher order equations in the Colombeau setting using symmetrisers and the theory of generalized pseudodifferential operators. There, they imposed conditions concerning the asymptotic scale of the regularization parameter of the operator. Concretely, the authors proved existence, uniqueness and regularity of generalized solutions in the case that the regularization parameter is chosen logarithmic slow scale. More details can be found in [24, Theorem 4.2].

1.3.3 Depth evolution processes

The concept of one-way wave equations, also known as paraxial wave equations, was first introduced by [7] and has become a standard tool in depth migration processes due to ill-posedness of the full wave equation. In fact, they are expressions for the first depth derivative of a wave field and thus of the form

$$\partial_z + iB_\pm(x, z, D_t, D_x). \tag{1.3.2}$$

where $B_\pm$ are (microlocal) pseudodifferential operators. Our derivation of one-way wave equations starts with a factorization of the operator L in (1.3.1) into terms of the form (1.3.2).

Remark 1.3.2. As already noted in Subsection 1.3.2, one obtains well-posed problems in the model of time migration for strictly hyperbolic operators (cf. [24]). We therefore give the following link.

Note that the operator L in (1.3.1) is strictly hyperbolic in the following sense: Let $\theta \in (0, \pi/2)$ be a fixed angle and

$$I'_\theta := \Big\{(x, z, \tau, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \tau \neq 0,\ |c^*(x,z)\tau^{-1}\xi| < \sin\theta\Big\}$$

with $c^*(x,z)$ as in (i) of Subsection 1.3.1.

For completeness we recall the following definition.

Definition 1.3.3. *The operator L from (1.3.1) is called generalized strictly hyperbolic in I'_θ, if there exists a choice of representatives of the coefficients in $\mathcal{G}^{lsc}_{\infty,\infty}(\mathbb{R}^n)$ such that the corresponding principal symbol of $L_\varepsilon(\zeta;x,z,\tau,\xi)$ has two distinct real-valued roots $(\zeta_{j,\varepsilon})_\varepsilon$, $j=1,2$ such that*

$$|\zeta_{1,\varepsilon}(x,z,\tau,\xi)-\zeta_{2,\varepsilon}(x,z,\tau,\xi)| \geqslant \zeta_\varepsilon\langle(\tau,\xi)\rangle \tag{1.3.3}$$

holds for some strictly nonzero net $(\zeta_\varepsilon)_\varepsilon$, for all $(x,z,\tau,\xi)\in I'_\theta$ and ε sufficiently small.

1.4 Ellipticity of pseudodifferential operators

In this section we introduce the notion of elliptic operators which enables us to construct a parametrix for such operators. Furthermore, such operators will play an important role in the characterization of the generalized wave front set (cf. Section 1.6).

1.4.1 Ellipticity

In [21, Definition 1.2] and [19, Proposition 2.7] the authors introduced the concept of slow scale ellipticity which can be carried over to logarithmic slow scale ellipticity easily. With this in mind, we define the property of logarithmic slow scale ellipticity for our class of operators as follows.

Definition 1.4.1. *We say that a generalized symbol $a\in\widetilde{S}^m_{lsc}$ is logarithmic slow scale micro-elliptic at $(x_0,\xi_0)\in T^*\mathbb{R}^n\backslash 0$ if it has a representative $(a_\varepsilon)_\varepsilon$ such that the following is satisfied: $\exists$ relatively compact open neighborhood U of x_0, $\exists$ conic neighborhood Γ of ξ_0, $\exists$ nets $(r_\varepsilon)_\varepsilon\in\Pi_{lsc}$, $(s_\varepsilon)_\varepsilon\in\Pi_{lsc}$ and a constant $\eta\in(0,1]$ such that*

$$|a_\varepsilon(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m \quad (x,\xi)\in U\times\Gamma,\ |\xi|\geqslant r_\varepsilon,\ \varepsilon\in(0,\eta]. \tag{1.4.1}$$

We denote by $Ell_{lsc}(a)$ the set of points $(x_0,\xi_0)\in T^*\mathbb{R}^n\backslash 0$ where a is logarithmic slow scale micro-elliptic. If there exists a representative $(a_\varepsilon)_\varepsilon\in a$ such that (1.4.1) holds for any $(x_0,\xi_0)\in T^*\mathbb{R}^n\backslash 0$ then the symbol a is called logarithmic slow scale elliptic.

Definition 1.4.2. *More generally, we say that a generalized symbol $a \in \widetilde{S}^m_{sc}$ is slow scale micro-elliptic at $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ if (1.4.1) holds for some nets $(r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon \in \Pi_{sc}$.*

Note that a logarithmic slow scale elliptic symbol is also slow scale elliptic but not vice versa.

Remark 1.4.3. We first observe that condition (1.4.1) is independent of the choice of representative. Indeed, let $(a_\varepsilon)_\varepsilon \in a$ as in Definition 1.4.1 and $(a'_\varepsilon)_\varepsilon$ another representative of a. Then, for some arbitrary but fixed $p > 0$ there $\exists c_1 > 0,\ \exists \eta_1 \in (0,1]$ such that on $U \times \Gamma$ we have

$$|a'_\varepsilon(x,\xi)| \geqslant |a_\varepsilon(x,\xi)| - |(a'_\varepsilon - a_\varepsilon)(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m \big(1 - c_1 s_\varepsilon \varepsilon^{2p}\big) \qquad |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta_1].$$

Since $\exists \eta_2 \in (0,1],\ \exists c_2 > 0$ so that $s_\varepsilon \leqslant c_2 \varepsilon^{-p}$ for $\varepsilon \in (0,\eta_2]$ we get

$$1 - c_1 s_\varepsilon \varepsilon^{2p} \geqslant 1 - c_1 c_2 \varepsilon^p \geqslant 1/2 \qquad \forall \varepsilon \in (0, \min(\eta_2, \big(2c_1c_2\big)^{-1/p})].$$

Redefining $\eta := \min(\eta_1, \eta_2, \big(2c_1c_2\big)^{-1/p})$ we obtain

$$|a'_\varepsilon(x,\xi)| \geqslant \frac{1}{2s_\varepsilon}\langle\xi\rangle^m \quad (x,\xi) \in U \times \Gamma,\ |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta].$$

Moreover, we notice that the notion of (logarithmic) slow scale ellipticity is stable under lower order (logarithmic) slow scale perturbations. A proof for the slow scale case can be found in [21, Proposition 1.3]. For the proof in the logarithmic slow scale case one essentially reworks the same lines. For completeness, we give the proof of the following proposition (cf. [21, Proposition 1.3]).

Proposition 1.4.4. *Let $(a_\varepsilon)_\varepsilon \in S^m_{lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ be logarithmic slow scale micro-elliptic at the point (x_0, ξ_0). Then*

(i) $\forall \alpha, \beta \in \mathbb{N}^n\ \exists (\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}\ \exists \eta \in (0,1]:$
$$|\partial^\alpha_\xi \partial^\beta_x a_\varepsilon(x,\xi)| \leqslant \lambda_\varepsilon |a_\varepsilon(x,\xi)| \langle\xi\rangle^{-|\alpha|} \quad U \times \Gamma,\ |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta]$$

(ii) if $(b_\varepsilon)_\varepsilon \in S^{m'}_{lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m' < m$, then (1.4.1) is valid for the net $(a_\varepsilon + b_\varepsilon)_\varepsilon$.

Proof. We first show (i). Since $(a_\varepsilon)_\varepsilon \in S^m_{lsc}$ is logarithmic slow scale elliptic we obtain

$$\forall \alpha, \beta \in \mathbb{N}^n\ \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}\ \exists c > 0\ \exists \eta \in (0,1] \text{ such that}$$
$$|\partial^\alpha_\xi \partial^\beta_x a_\varepsilon(x,\xi)| \leqslant c\omega_\varepsilon \langle\xi\rangle^{m-|\alpha|} \leqslant c\omega_\varepsilon s_\varepsilon \langle\xi\rangle^{-|\alpha|} |a_\varepsilon(x,\xi)|$$

on $U \times \Gamma$ for $|\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0, \eta]$. Setting $\lambda_\varepsilon := \omega_\varepsilon s_\varepsilon$ then $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$ as desired.

To show the second part, let $(b_\varepsilon)_\varepsilon \in S^{m'}_{lsc}$. Then

$$|a_\varepsilon(x,\xi) + b_\varepsilon(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m - c\omega_\varepsilon\langle\xi\rangle^{m'} = \frac{1}{s_\varepsilon}\langle\xi\rangle^m\Big(1 - cs_\varepsilon\omega_\varepsilon\langle\xi\rangle^{m'-m}\Big) \geqslant \frac{1}{2s_\varepsilon}\langle\xi\rangle^m$$

on $U \times \Gamma$, $|\xi| \geqslant r'_\varepsilon := \max(r_\varepsilon, (2cs_\varepsilon\omega_\varepsilon)^{1/(m-m')})$ and $(r'_\varepsilon)_\varepsilon \in \Pi_{lsc}$. ◻

Before we proceed, we will present some general facts of (logarithmic) slow scale elliptic symbols.

Remark 1.4.5. In case of smooth symbols the notion of logarithmic slow scale ellipticity reduces to the classical one and is therefore equivalent to the complement of the characteristic set. Indeed, given $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and a representative $(a_\varepsilon)_\varepsilon$ of a logarithmic slow scale elliptic symbol $a \in S^m_{lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ then there exist nets $(s_\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and constants $c > 0$, $\eta \in (0,1]$ such that for every $q \in \mathbb{N}$ we have:

$$|a(x,\xi)| \geqslant |a_\varepsilon(x,\xi)| - |a(x,\xi) - a_\varepsilon(x,\xi)| \geqslant \langle\xi\rangle^m\Big(\frac{1}{s_\varepsilon} - c\varepsilon^q\Big) \qquad \text{for } |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta].$$

As we are allowed to fix an ε small enough the last expression is bounded away from 0. In particular $\mathrm{Ell}_{lsc}(a)^c$ is equal to the characteristic set $\mathrm{Char}(a(x,D))$. Note that this result remains valid if one replaces lsc by sc (cf. [21, Remark 1.4]).

Moreover, we give an equivalent characterization of the notion of logarithmic slow scale ellipticity concerning the principal symbol. In [34, Proposition 3.3] it is proved that a generalized partial differential operator with regular coefficients in $\mathcal{G}^\infty = \mathcal{G}^\infty_{C^\infty}$ is W-elliptic (weak elliptic) if its principal symbol is S-elliptic (strong elliptic). For the precise meaning of weak and strong ellipticity consult [34]. Referring to this, we want to give the following proposition:

Proposition 1.4.6. *Given a generalized symbol $(a_\varepsilon)_\varepsilon \in S^m_{cl,lsc}$, then the following properties are equivalent:*

(i) $(a_\varepsilon)_\varepsilon$ is logarithmic slow scale elliptic

(ii) the principal symbol $(a_{m,\varepsilon})_\varepsilon$ satisfies the following condition: $\exists (s_\varepsilon)_\varepsilon \in \Pi_{lsc}$, $\exists \eta \in (0,1]$ and $\exists r > 0$ such that

$$|a_{m,\varepsilon}(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m \qquad \forall (x,\xi) \in T^*\mathbb{R}^n \text{ with } |\xi| \geqslant r \text{ and } \varepsilon \in (0,\eta].$$

Proof. The proof follows similar arguments to those used in [34, Proposition 3.3]. We first assume that condition (ii) holds. Then there exist nets $(s_\varepsilon)_\varepsilon \in \Pi_{lsc}, (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and constants $c, r > 0$ such that

$$|a_\varepsilon(x,\xi)| = |a_{m,\varepsilon}(x,\xi) + \sum_{j\geqslant 1} a_{m-j,\varepsilon}(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m - c\omega_\varepsilon\langle\xi\rangle^{m-1} \geqslant$$
$$\geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m\Big(1 - c\omega_\varepsilon s_\varepsilon\langle\xi\rangle^{-1}\Big)$$

for $|\xi| \geqslant r$ and ε small enough. Setting $r_\varepsilon = \max(r, 2c\omega_\varepsilon s_\varepsilon)$ then $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and we obtain

$$|a_\varepsilon(x,\xi)| \geqslant \frac{1}{2s_\varepsilon}\langle\xi\rangle^m \qquad |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta]$$

for some $\eta \in (0,1]$, showing (i).

To prove that (i) implies (ii), suppose that $(a_\varepsilon)_\varepsilon$ satisfies (1.4.1) on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\zeta \in \mathbb{R}^n$, $|\zeta| = 1$ and choose $\xi \in \mathbb{R}^n$ such that $|\xi| \geqslant r_\varepsilon$ and $\zeta = \xi/|\xi|$ with $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$ as in (1.4.1). Then $\exists (s'_\varepsilon)_\varepsilon \in \Pi_{lsc},\ \exists (r'_\varepsilon)_\varepsilon \in \Pi_{lsc},\ \exists \eta \in (0,1]$ so that

$$|\sum_{j\geqslant 0} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j}| = |a_\varepsilon(x,\xi)||\xi|^{-m} \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m\frac{1}{|\xi|^m} = \frac{1}{s_\varepsilon}\langle|\xi|^{-1}\rangle^m \geqslant$$
$$\geqslant \frac{\min(2^{m/2},1)}{s_\varepsilon} =: \frac{1}{s'_\varepsilon}$$

for $|\xi| \geqslant r'_\varepsilon := \max(1, r_\varepsilon),\ \varepsilon \in (0,\eta]$.

Now fix $N \in \mathbb{N},\ N \geqslant 1$. Since $(a_\varepsilon)_\varepsilon \in S^m_{lsc}$ we obtain $\forall (t_\varepsilon)_\varepsilon \in \Pi_{lsc},\ \forall 1 \leqslant j \leqslant N,\ \exists (\omega_{j,\varepsilon})_\varepsilon \in \Pi_{lsc},\ \exists \eta \in (0,1]$ that

$$|a_{m-j,\varepsilon}(x,\zeta)|\frac{1}{|\xi|^j} \leqslant \omega_{j,\varepsilon}\frac{1}{|\xi|^j} \leqslant \frac{1}{t_\varepsilon} \qquad \text{for } |\xi| \geqslant \max_{1\leqslant j\leqslant N}(r_\varepsilon, (t_\varepsilon\omega_{j,\varepsilon})^{1/j})$$

and $\varepsilon \in (0,\eta]$. At this point we are free to choose $(t_\varepsilon)_\varepsilon \in \Pi_{lsc}$ such that $\frac{N}{t_\varepsilon} = \frac{1}{4s'_\varepsilon}$.

On the other hand, we can use Theorem 1.2.11 for classical symbols to show that $\exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}, \exists \eta \in (0,1]$ such that

$$|\sum_{j\geqslant N+1} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j}| \leqslant \frac{1}{|\xi|^{N+1}}|\sum_{j\geqslant N+1} a_{m-j,\varepsilon}(x,\zeta)| \leqslant \frac{1}{|\xi|^{N+1}}\omega_\varepsilon\langle\zeta\rangle^{m-N-1} \leqslant$$
$$\leqslant \frac{2^{m-N-1}\omega_\varepsilon}{|\xi|^{N+1}} \leqslant \frac{1}{4s'_\varepsilon}$$

for any $|\xi| \geqslant \max(1, r_\varepsilon, (2^{m-N-1}\omega_\varepsilon s'_\varepsilon)^{\frac{1}{N+1}})$ and $\varepsilon \in (0,\eta]$. At this point we reset r'_ε by defining $r'_\varepsilon := \max_{1\leqslant j\leqslant N}(1, r_\varepsilon, (2^{m-N-1}\omega_\varepsilon s'_\varepsilon)^{\frac{1}{N+1}}, (t_\varepsilon\omega_{j,\varepsilon})^{1/j})$. Then $(r'_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and we obtain: $\exists\eta \in (0,1]$ such that

$$|a_{m,\varepsilon}(x,\zeta)| \geqslant |\sum_{j=0}^{\infty} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j}| - |\sum_{j=1}^{\infty} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j}| \geqslant$$

$$\geqslant \frac{1}{s'_\varepsilon} - |\sum_{j=1}^{N} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j} + \sum_{j=N+1}^{\infty} a_{m-j,\varepsilon}(x,\zeta)|\xi|^{-j}| \geqslant \frac{1}{s'_\varepsilon} - \frac{1}{2s'_\varepsilon} = \frac{1}{2s'_\varepsilon}$$

for $|\xi| \geqslant r'_\varepsilon$, $\varepsilon \in (0,\eta]$ by the above. Thus, since $a_{m,\varepsilon}$ is homogeneous of degree m in the second variable

$$|a_{m,\varepsilon}(x,\xi)| \geqslant \frac{1}{2s'_\varepsilon}|\xi|^m \geqslant \frac{1}{2s'_\varepsilon}\min\Big(\frac{1}{2^m},1\Big)\langle\xi\rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n,\ |\xi| \geqslant 1$$

for ε sufficiently small. □

Remark 1.4.7. Again, Proposition 1.4.6 remains valid if we replace logarithmic slow scale by slow scale.

1.4.2 Strong ellipticity

As already announced in Section 1.3 we are interested in the depth extrapolation of the wave field U in (1.3.1). We will therefore introduce a notion of (logarithmic) slow scale ellipticity which in addition has a certain behavior with respect to ζ, the dual variable of the depth.

We first introduce some notation. We denote by $(\tau,\xi,\zeta) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$ the dual variables corresponding to $(t,x,z) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$. Thus, we will work on a $2(n+1)$-dimensional phase space. We already mentioned in Subsection 1.2.2 that, given a symbol $a \in \widetilde{S}^m_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ and a generalized point $(t,x,z,\tau,\xi,\widetilde{\zeta})$ in $\mathbb{R}^{n+1}\times\mathbb{R}^n\times\widetilde{\mathbb{R}}$ then the generalized point value of the symbol a at $(t,x,z,\tau,\xi,\widetilde{\zeta})$ is well-defined in $\widetilde{\mathbb{C}}$.

Moreover, let $\Gamma_s \subseteq \mathbb{R}^{n+1}$ be a (classical) conic neighborhood of (τ_0,ξ_0,ζ_0) of the following form

$$\Gamma_s = \{\lambda(\tau,\xi;\zeta) \in \mathbb{R}^{n+1}\backslash 0 \mid (\tau,\xi;\zeta) \in B_s\Big(\frac{(\tau_0,\xi_0;\zeta_0)}{|(\tau_0,\xi_0;\zeta_0)|}\Big) \cap S^1(0),\ \lambda > 0\}$$

for some $s > 0$ small enough, where $B_s(\eta)$ denotes the open ball with radius s around $\eta \in \mathbb{R}^{n+1}$ and $S^1(0) := \{\eta \in \mathbb{R}^{n+1} : |\eta| = 1\}$.

We set

$$\Gamma_{s,M} := \{(\lambda(\tau,\xi;\zeta_\varepsilon))_\varepsilon \in \Gamma_s^{(0,1]} \mid \exists N \in \mathbb{N} : |\zeta_\varepsilon| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0,\ \lambda > 0\}$$

and introduce an equivalence relation on $\Gamma_{s,M}$ by

$$(\zeta_\varepsilon^1)_\varepsilon \sim (\zeta_\varepsilon^2)_\varepsilon \Leftrightarrow \forall m > 0 : |\zeta_\varepsilon^1 - \zeta_\varepsilon^2| = \mathcal{O}(\varepsilon^m) \quad \varepsilon \to 0.$$

We then denote by $\widetilde{\Gamma}_s := \Gamma_{s,M}/\sim$ the generalized cone with respect to ζ generated from Γ_s. So $\widetilde{\Gamma}_s$ is a generalized conic neighborhood of (τ_0,ξ_0,ζ_0).

Moreover we recall that $\Gamma_s \hookrightarrow \widetilde{\Gamma}_s$. The canonical embedding of Γ_s into $\widetilde{\Gamma}_s$ is given by $(\tau,\xi,\zeta) \mapsto (\tau,\xi,(\zeta)_\varepsilon + \mathcal{N}_\mathbb{R})$.

Also note that in the definition of $\Gamma_{s,M}$, $(\tau,\xi;\zeta_\varepsilon)_\varepsilon \in \Gamma_s^{(0,1]}$ means that

$$(\tau,\xi;\zeta_\varepsilon) \in B_s\Big(\frac{(\tau_0,\xi_0;\zeta_0)}{|(\tau_0,\xi_0;\zeta_0)|}\Big) \cap S^1(0) \qquad \text{for every fixed } \varepsilon \in (0,1].$$

With this notation we now give the definition of strong logarithmic slow scale micro-ellipticity.

Definition 1.4.8. *Let a be a generalized symbol in $\widetilde{S}^m_{lsc}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. We say that the symbol a is strong logarithmic slow scale micro-elliptic at a point $(t_0,x_0,z_0,\tau_0,\xi_0;\zeta_0) \in T^*\mathbb{R}^{n+1}\backslash 0$, and we write $(t_0,x_0,z_0,\tau_0,\xi_0;\zeta_0) \in Ell^*_{lsc}(a)$, if and only if there exist a relatively compact open neighborhood U of (t_0,x_0,z_0), a (classical) conic neighborhood Γ of (τ_0,ξ_0,ζ_0) such that $\Gamma \hookrightarrow \widetilde{\Gamma}$ where $\widetilde{\Gamma} = \Gamma_M/\sim$ is the generalized conic neighborhood of (τ_0,ξ_0,ζ_0) generated from Γ as above, nets $(r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and an $\eta \in (0,1]$ such that*

$$|a_\varepsilon(t,x,z,\tau,\xi;\zeta_\varepsilon)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau,\xi;\zeta_\varepsilon)\rangle^m \qquad U \times \Gamma_M,\ |(\tau,\xi;\zeta_\varepsilon)| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta]. \tag{1.4.2}$$

Figure 1.1 below shall illustrate the relation of the sets Γ_s and $\Gamma_{s,M}$ in the following special situation. We fix a point $(x_0,z_0) \in \mathbb{R}^n$, let $\zeta_0^2 := \frac{1}{c^{*2}}(x_0,z_0)\tau_0^2 - |\xi_0|^2$ and let Γ_s be a conic neighborhood around (τ_0,ξ_0,ζ_0). Then for any $s > 0$, however small, we can define

$$\zeta_{0,\varepsilon}^{s2} := \begin{cases} \frac{1}{c^{*2}}(x_0,z_0)\tau_0^2 - |\xi_0|^2 & \varepsilon \in (C_s,1] \\ \frac{1}{c_\varepsilon^2}(x_0,z_0)\tau_0^2 - |\xi_0|^2 & \varepsilon \in (0,C_s] \end{cases}$$

for some positive constant C_s depending on $s > 0$ and will be specified below. Then

$$|\zeta_0^2 - \zeta_{0,\varepsilon}^{s\ 2}| = \Big|\Big(\frac{1}{c^{*2}} - \frac{1}{c_\varepsilon^2}\Big)(x_0,z_0)\tau_0^2\Big| \leqslant C(\tau_0)\omega_\varepsilon^{-\mu} \qquad \forall \varepsilon \in (0,1].$$

Now let $r(s)$ be a function depending on s such that $r(s) \to 0$ as $s \to 0$. Then, since $C(\tau_0)\omega_\varepsilon^{-\mu} \leqslant r(s)/2$ for all ε small enough, i.e. for ε less or equal a $C_s > 0$ (with $C_s \to 0$ as $s \to 0$) we get

$$|\zeta_0^2 - \zeta_{0,\varepsilon}^{s\ 2}| \leqslant r(s)/2 \quad \forall \varepsilon \in (0, C_s].$$

By the definition of $\zeta_{0,\varepsilon}^{s\ 2}$ we even get

$$|\zeta_0^2 - \zeta_{0,\varepsilon}^{s\ 2}| \leqslant r(s)/2 \quad \forall \varepsilon \in (0, 1].$$

So in particular we have $(\tau_0, \xi_0, \zeta_{0,\varepsilon}^{s\ 2}) \in \Gamma_s$ for all $\varepsilon \in (0, 1]$ and $\zeta_{0,\varepsilon}^{s\ 2} \to \zeta_0^2$ as $\varepsilon \to 0$.

Therefore, if we are working in an open ball around (τ_0, ξ_0, ζ_0) with radius $r(s) > 0$ we can find a generalized point $(\tau_0, \xi_0, \zeta_{0,\varepsilon}^s)_\varepsilon$ such that $(\tau_0, \xi_0, \zeta_{0,\varepsilon}^s)$ is contained in $B_{r(s)}\big((\tau_0, \xi_0, \zeta_0)\big)$ for all $\varepsilon \in (0, 1]$.

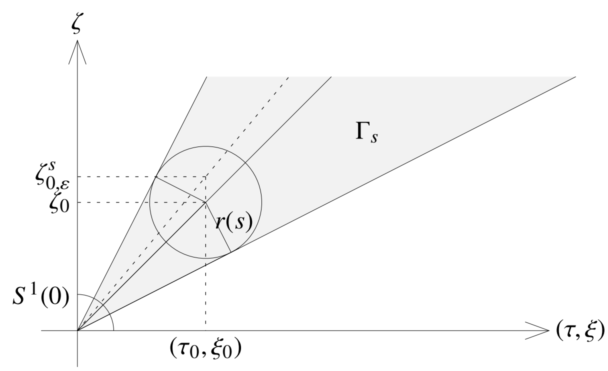

Figure 1.1: Here the shaded area corresponds to the set Γ_s. The dashed line through $(\tau_0, \xi_0, \zeta_{0,\varepsilon}^s)$ and the origin moves to to the line through (τ_0, ξ_0, ζ_0) and the origin as $\varepsilon \to 0$.

Remark 1.4.9. In particular, since $\Gamma \hookrightarrow \widetilde{\Gamma}$, condition (1.4.2) implies

$$|a_\varepsilon(t, x, z, \tau, \xi; \zeta)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau, \xi; \zeta)\rangle^m \qquad U \times \Gamma,\ |(\tau, \xi; \zeta)| \geqslant r_\varepsilon,\ \varepsilon \in (0, \eta]. \tag{1.4.3}$$

So, the symbol a is logarithmic slow scale micro-elliptic at $(t_0, x_0, z_0, \tau_0, \xi_0; \zeta_0)$.

Remark 1.4.10. The definition of strong slow scale micro-ellipticity is straightforward. Again, one simply replaces lsc by sc in Definition 1.4.8.

Lemma 1.4.11. *Let $a \in \widetilde{S}^m_{lsc}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Then a is strong logarithmic slow scale micro-elliptic at $(t_0, x_0, z_0, \tau_0, \xi_0; \zeta_0)$ if and only if it is logarithmic slow scale micro-elliptic there. So, if a satisfies (1.4.3), then it also satisfies (1.4.2) and vice versa.*

Proof. The first direction is already shown by Remark 1.4.9.

On the other hand, suppose that the symbol a is logarithmic slow scale micro-elliptic at the point $(t_0, x_0, z_0, \tau_0, \xi_0; \zeta_0) \in T^*\mathbb{R}^{n+1}\backslash 0$. Then there exists a representative $(a_\varepsilon)_\varepsilon$ of a such that the following is satisfied: $\exists$ relatively compact open neighborhood U of (t_0, x_0, z_0), $\exists$ (classical) conic neighborhood Γ of $(\tau_0, \xi_0; \zeta_0)$, $\exists$ nets $(r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon \in \Pi_{lsc}$, $\exists \eta \in (0,1]$ such that

$$|a_\varepsilon(t,x,z,\tau,\xi;\zeta)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau,\xi;\zeta)\rangle^m \quad \text{ on } U \times \Gamma,\ |(\tau,\xi;\zeta)| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta].$$

In particular, since Γ is conic neighborhood of $(\tau_0, \xi_0; \zeta_0)$, it is of the form

$$\Gamma = \{\lambda(\tau,\xi;\zeta) \in \mathbb{R}^{n+1}\backslash 0 \mid (\tau,\xi;\zeta) \in B_s\Big(\frac{(\tau_0,\xi_0;\zeta_0)}{|(\tau_0,\xi_0;\zeta_0)|}\Big) \cap S^1(0),\ \lambda > 0\}$$

for some $s > 0$ small enough.

We define $\widetilde{\Gamma} := \Gamma_M / \sim$ the generalized cone with respect to ζ constructed from Γ as above.

Take $(\tau,\xi;\zeta_\varepsilon)_\varepsilon \in \Gamma_M$. Then $(\tau,\xi;\zeta_\varepsilon) \in \Gamma$ for any fixed $\varepsilon \in (0,1]$ and $\exists N \in \mathbb{N}$ such that $|\zeta_\varepsilon| = \mathcal{O}(\varepsilon^{-N})$ as $\varepsilon \to 0$. Hence $\exists \eta \in (0,1]$:

$$|a_\varepsilon(t,x,z,\tau,\xi;\zeta_\varepsilon)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau,\xi;\zeta_\varepsilon)\rangle^m \quad \text{ on } U \times \Gamma_M,\ |(\tau,\xi;\zeta_\varepsilon)| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta].$$

As a conclusion we obtain that $(t_0, x_0, z_0, \tau_0, \xi_0; \zeta_0) \in \mathrm{Ell}^*_{lsc}(a)$, i.e., the symbol a is strong logarithmic slow scale micro-elliptic at the point $(t_0, x_0, z_0, \tau_0, \xi_0; \zeta_0)$.

We have therefore shown that logarithmic slow scale micro-ellipticity at a particular point is equivalent to strong logarithmic slow scale micro-ellipticity at that point. $\square$

An inspection of the proof of Lemma 1.4.11 shows the following result.

Corollary 1.4.12. *Given $a \in \widetilde{S}^m_{sc}$, then a is strong slow scale elliptic at a phase space point is equivalent to saying that a is slow scale elliptic at that point.*

Lemma 1.4.13. *Let L be the generalized differential operator given in (1.3.1). Then $Ell_{sc}(L)$ is a subset of the complement of the set Σ, where*

$$\Sigma := \{(t,x,z,\tau,\xi,\zeta) \in T^*\mathbb{R}^{n+1}\backslash 0 \mid \zeta^2 - \frac{1}{c^{*2}}(x,z)\tau^2 + |\xi|^2 = 0\}$$

and $\frac{1}{c^{*2}} \in \mathcal{C}^{0,\mu}(\mathbb{R}^n)$ is as in the beginning of Section 1.3. Note that Σ is a conic set and independent of the regularization parameter $\varepsilon \in (0,1]$. Moreover, we have the following inclusion relations:

$$\Sigma \subseteq Ell_{sc}(L)^c \subseteq Ell_{lsc}(L)^c.$$

Proof. Let $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0) \in T^*\mathbb{R}^{n+1}\backslash 0$ be an arbitrary but fixed point of Σ. Hence, $\zeta_0^2 = \frac{1}{c^{*2}}(x_0, z_0)\tau_0^2 - |\xi_0|^2$.

Assume that the operator L with generalized symbol in $\widetilde{S}^2_{lsc}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ is slow scale micro-elliptic at the point $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0) \in T^*\mathbb{R}^{n+1}\backslash 0$. Then, by Corollary 1.4.12, there is a relatively compact open neighborhood U of (t_0, x_0, z_0), a conic neighborhood Γ of (τ_0, ξ_0, ζ_0) such that for some $(r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon \in \Pi_{sc}$ and some $\eta \in (0, 1]$ we have

$$|L_\varepsilon(x, z, \tau, \xi, \zeta_\varepsilon)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau, \xi, \zeta_\varepsilon)\rangle^2$$

for all $(t, x, z, \tau, \xi, \zeta_\varepsilon)_\varepsilon \in U \times \Gamma_M$, $|(\tau, \xi, \zeta_\varepsilon)| \geqslant r_\varepsilon$, $\varepsilon \in (0, \eta]$ and $\widetilde{\Gamma} = \Gamma_M/\sim$ is the generalized conic neighborhood of (τ_0, ξ_0, ζ_0) from above such that $\Gamma \hookrightarrow \widetilde{\Gamma}$.

For ε small enough we now define $\zeta_{0,\varepsilon}^2 := \frac{1}{c_\varepsilon^2}(x_0, z_0)\tau_0^2 - |\xi_0|^2$ which tends to ζ_0^2 (for fixed $\tau_0 \in \mathbb{R}$) as ε tends to 0.

Then one can choose $(\tau_0, \xi_0, \zeta_{0,\varepsilon})_\varepsilon \in \Gamma_M$, i.e. there is a positive constant s such that

$$(\tau_0, \xi_0; \zeta_{0,\varepsilon}) \in B_s\Big(\frac{(\tau_0, \xi_0; \zeta_0)}{|(\tau_0, \xi_0; \zeta_0)|}\Big) \cap S^1(0) \qquad \text{for every fixed } \varepsilon \in (0, 1]$$

since

$$(\tau_0, \xi_0; \zeta_{0,\varepsilon}) \to (\tau_0, \xi_0; \zeta_0) \qquad \text{as } \varepsilon \to 0.$$

On the other hand, we have that the principal symbol of L_ε vanishes on $(x_0, z_0, \tau_0, \xi_0, \zeta_{0,\varepsilon})$, i.e.

$$\sigma(L_\varepsilon)(x_0, z_0, \tau_0, \xi_0, \zeta_{0,\varepsilon}) = 0$$

for all ε small enough. So, by Proposition 1.4.6 and Remark 1.4.7, the symbol L is not strong slow scale micro-elliptic and therefore not slow scale micro-elliptic at the point $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0)$ - a contradiction. ∎

Lemma 1.4.14. *Let L and Σ be as in Lemma 1.4.13. Then $\Sigma = Ell^c_{lsc}(L)$.*

Proof. By Lemma 1.4.13 it suffices to show the inclusion $\Sigma^c \subseteq \mathrm{Ell}_{lsc}(L)$. To do this, we introduce the continuous function

$$f(x, z, \tau, \xi, \zeta) := \zeta^2 - \frac{1}{c^{*2}}(x, z)\tau^2 + |\xi|^2,$$

which coincides with the limit of the principal symbol of $\rho_\varepsilon L_\varepsilon$ as $\varepsilon \to 0$. So,

$$\Sigma^c = \{(t, x, z, \tau, \xi, \zeta) \in T^*\mathbb{R}^{n+1}\backslash 0 \mid |f(x, z, \tau, \xi, \zeta)| > 0\}$$

is an open subset of the phase space.

Now given a point $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0) \in \Sigma^c$, there exist a relatively compact open neighborhood U of (t_0, x_0, z_0), a conic neighborhood Γ of (τ_0, ξ_0, ζ_0) and a (small) $\delta > 0$ such that

$$|f(x,z,\tau,\xi,\zeta)| = \big|\zeta^2 - \frac{1}{c^{*2}}(x,z)\tau^2 + |\xi|^2\big| \geqslant \delta|(\tau,\xi,\zeta)|^2 \qquad U \times \big(\Gamma \cap S^1(0)\big).$$

Since f is homogeneous of degree 2 in (τ, ξ, ζ) it follows that

$$|f(x,z,\tau,\xi,\zeta)| = \big|\zeta^2 - \frac{1}{c^{*2}}(x,z)\tau^2 + |\xi|^2\big| \geqslant \delta|(\tau,\xi,\zeta)|^2 \qquad U \times \Gamma,\ |(\tau,\xi,\zeta)| \geqslant 1.$$

Hence, there exists a constant $C_\delta \in (0,1]$, depending on $\delta > 0$, such that

$$\begin{aligned}|\sigma(L_\varepsilon(t,x,z,\tau,\xi,\zeta))| &\geqslant \big|\zeta^2 - \frac{1}{c^{*2}}(x,z)\tau^2 + |\xi|^2\big| - \big|\Big(\frac{1}{c_\varepsilon^2} - \frac{1}{c^{*2}}\Big)(x,z)\tau^2\big| \geqslant \\ &\geqslant \delta|(\tau,\xi,\zeta)|^2 - C\omega_\varepsilon^{-\mu}|(\tau,\xi,\zeta)|^2 \geqslant \\ &\geqslant \frac{\delta}{2}|(\tau,\xi,\zeta)|^2 \qquad U \times \Gamma,\ |(\tau,\xi,\zeta)| \geqslant 1,\ \varepsilon \in (0, C_\delta].\end{aligned}$$

Therefore, L_ε is logarithmic slow scale elliptic at $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0)$, which is the desired conclusion. □

Remark 1.4.15. We define the following set:

$$\begin{aligned}\Lambda_M := \{&(t,x,z,\tau,\xi;\zeta_\varepsilon)_\varepsilon \in \mathbb{R}^{n+1} \times (\mathbb{R}^n \times \mathbb{R}^{(0,1]})\backslash 0 \mid \\ &\zeta_\varepsilon^2 = \frac{1}{c_\varepsilon^2}(x,z)\tau^2 - |\xi|^2, \exists N \in \mathbb{N} : |\zeta_\varepsilon| = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}.\end{aligned}$$

If a is a symbol on Σ, then the evaluation on $\Lambda_M/\sim$ is in general not defined. If a is a symbol on Γ, then the generalized point value of a at a point of $\Gamma_M/\sim$ is a well-defined element on $\widetilde{\mathbb{C}}$.

A fixed generalized point $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_{0,\varepsilon})_\varepsilon \in \Lambda_M$ satisfies $\zeta_{0,\varepsilon}^2 = \frac{1}{c_\varepsilon^2}(x_0,z_0)\tau_0^2 - |\xi_0|^2$. Since $\tau_0 \in \mathbb{R}$ is fixed we get

$$\zeta_{0,\varepsilon}^2 = \frac{1}{c_\varepsilon^2}(x_0,z_0)\tau_0^2 - |\xi_0|^2 \to \zeta_0^2 = \frac{1}{c^{*2}}(x_0,z_0)\tau_0^2 - |\xi_0|^2$$

as $\varepsilon \to 0$ and where $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0) \in \Sigma$. Moreover,

$$|\zeta_{0,\varepsilon}^2 - \zeta_0^2| = |\Big(\frac{1}{c_\varepsilon^2} - \frac{1}{c^{*2}}\Big)(x_0,z_0)\tau_0^2| \leqslant C\omega_\varepsilon^{-\mu}\tau_0^2.$$

In the next subsection we will construct an approximated inverse for logarithmic slow scale elliptic pseudodifferential operators.

1.4.3 Construction of a parametrix

In this subsection we will discuss uniqueness and existence results concerning the invertibility of logarithmic slow scale elliptic pseudodifferential operators. Recall that the asymptotic expansion is unique modulo operators of order $-\infty$. As a consequence we will show that one obtains uniqueness of a parametrix modulo operators of infinite order.

Note that given $(a_\varepsilon)_\varepsilon$ a symbol of class $S^{-\infty}_{lsc}$ the operator defined by $(u_\varepsilon)_\varepsilon \mapsto (a_\varepsilon(x,D)u_\varepsilon)_\varepsilon$, $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}$ has a regular kernel representation in the sense that moderate nets are mapped into regular nets.

In the following theorem we shall work on the level of representatives.

Theorem 1.4.16. *Let $(a_\varepsilon)_\varepsilon \in S^m_{lsc}$ be logarithmic slow scale elliptic, i.e.*

$$\exists (r_\varepsilon)_\varepsilon \in \Pi_{lsc},\ \exists (s_\varepsilon)_\varepsilon \in \Pi_{lsc},\ \exists \eta \in (0,1]:$$
$$|a_\varepsilon(x,\xi)| \geqslant \frac{1}{s_\varepsilon}\langle\xi\rangle^m \quad |\xi| \geqslant r_\varepsilon,\ \varepsilon \in (0,\eta].$$

Then there exists $(p_\varepsilon)_\varepsilon \in S^{-m}_{lsc}$ such that $(p_\varepsilon \# a_\varepsilon)_\varepsilon = 1$ modulo $S^{-\infty}_{lsc}$.

Proof. Step 1: As in [19, Proposition 2.8] we show that there exists $(p_{-m,\varepsilon})_\varepsilon \in S^{-m}_{lsc}$ such that

$$(p_{-m,\varepsilon}a_\varepsilon - 1)_\varepsilon \in S^{-\infty}_{lsc}. \tag{1.4.4}$$

For the proof we let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \leqslant \psi \leqslant 1$, $\psi(\xi) = 0$ for $|\xi| \leqslant 1$ and $\psi(\xi) = 1$ for $|\xi| \geqslant 2$. Then

$$p_{-m,\varepsilon}(x,\xi) := a_\varepsilon^{-1}(x,\xi)\psi(\xi/r_\varepsilon)$$

with $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$ as in the theorem does the job. Indeed, one can show that

$$|\partial_\xi^\alpha \partial_x^\beta a_\varepsilon^{-1}(x,\xi)| \leqslant \omega_\varepsilon(\alpha,\beta)\langle\xi\rangle^{-|\alpha|}|a_\varepsilon^{-1}(x,\xi)| \qquad |\xi| \geqslant r_\varepsilon$$

for some $\omega_\varepsilon(\alpha,\beta) \in \Pi_{lsc}$ (cf. [20, Lemma 6.3]). Then, by the properties of the function ψ, we get $(p_{-m,\varepsilon})_\varepsilon \in S^{-m}_{lsc}$ for $|\xi| \geqslant 2r_\varepsilon$ and $|\xi| \leqslant r_\varepsilon$. On the set $r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon$ we observe that

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta p_{-m,\varepsilon}(x,\xi)| &= |\sum_{\alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} \partial_\xi^{\alpha'} \partial_x^\beta a_\varepsilon^{-1}(x,\xi) \partial_\xi^{\alpha-\alpha'} \psi(\xi/r_\varepsilon)| \leqslant \\
&\leqslant \sum_{\alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} \omega_\varepsilon(\alpha',\beta) |a_\varepsilon^{-1}(x,\xi)| \langle\xi\rangle^{-|\alpha'|} r_\varepsilon^{-|\alpha-\alpha'|} \chi_{[r_\varepsilon,2r_\varepsilon]}(\xi) \sup |(\partial^{\alpha-\alpha'}\psi)(\xi/r_\varepsilon)| \leqslant \\
&\leqslant \sum_{\alpha' \leqslant \alpha} \binom{\alpha}{\alpha'} \omega_\varepsilon(\alpha',\beta) r_\varepsilon^{-|\alpha-\alpha'|} s_\varepsilon \langle 2r_\varepsilon\rangle^{|\alpha-\alpha'|} \langle\xi\rangle^{-m-|\alpha|} \sup |(\partial^{\alpha-\alpha'}\psi)(\xi)|
\end{aligned}$$

and therefore, $(p_{m,\varepsilon})_\varepsilon$ is in S^{-m}_{lsc}. Note that χ denotes the characteristic function. In order show (1.4.4) we write

$$p_{-m,\varepsilon} a_\varepsilon(x,\xi) = a_\varepsilon^{-1} a_\varepsilon(x,\xi) + a_\varepsilon^{-1} a_\varepsilon(x,\xi)(\psi(\xi/r_\varepsilon) - 1) = 1 + (\psi(\xi/r_\varepsilon) - 1).$$

Since for any $l > 0$ and $\alpha \in \mathbb{N}^n \backslash 0$

$$\begin{aligned}
&\sup_{r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon} |\langle\xi\rangle^l (\psi(\xi/r_\varepsilon) - 1)| \leqslant c(\psi) \langle 2r_\varepsilon\rangle^l \\
&\sup_{r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon} |\langle\xi\rangle^l \partial_\xi^\alpha (\psi(\xi/r_\varepsilon) - 1)| = \sup_{r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon} |\langle\xi\rangle^l r_\varepsilon^{-|\alpha|} (\partial^\alpha\psi)(\xi/r_\varepsilon)| \leqslant c(\psi) r_\varepsilon^{-|\alpha|} \langle 2r_\varepsilon\rangle^l
\end{aligned}$$

we have shown (1.4.4).

Step 2: We recursively define for $k \geqslant 1$:

$$p_{-m-k,\varepsilon}(x,\xi) := -\Big\{ \sum_{\substack{|\gamma|+j=k \\ j<k}} \frac{(-i)^{|\gamma|}}{\gamma!} \partial_x^\gamma a_\varepsilon(x,\xi) \partial_\xi^\gamma p_{-m-j,\varepsilon}(x,\xi) \Big\} a_\varepsilon^{-1}(x,\xi) \psi(\xi/r_\varepsilon).$$

By the same arguments as in [20, Propositions 6.5 and 6.6], one can show that each $(p_{-m-k,\varepsilon})_\varepsilon$ is in S^{-m-k}_{lsc} and by Theorem 1.2.11 there exists a net $(p_\varepsilon)_\varepsilon \in S^{-m}_{lsc}$ with $(p_\varepsilon)_\varepsilon \sim_{lsc} \sum_j (p_{-m-j,\varepsilon})_\varepsilon$.

Step 3: The aim is to show that $(p_\varepsilon \# a_\varepsilon - 1)_\varepsilon \in S^{-\infty}_{lsc}$. Therefore, let $(\sigma_\varepsilon)_\varepsilon$ be the generalized symbol with the asymptotic expansion

$$\sigma_\varepsilon(x,\xi) \sim_{lsc} \sum_{|\gamma| \geqslant 0} \frac{1}{\gamma!} \big(\partial_\xi^\gamma p_\varepsilon D_x^\gamma a_\varepsilon\big)(x,\xi).$$

and we will show that $(\sigma_\varepsilon - 1)_\varepsilon$ in $S^{-\infty}_{lsc}$.

We write

$$\sigma_\varepsilon - \sum_{|\gamma|<N} \frac{1}{\gamma!} D_x^\gamma a_\varepsilon \partial_\xi^\gamma p_\varepsilon = \sigma_\varepsilon - \sum_{|\gamma|<N} \frac{1}{\gamma!} D_x^\gamma a_\varepsilon \sum_{l<N} \partial_\xi^\gamma p_{-m-l,\varepsilon} + \\ + \sum_{|\gamma|<N} \frac{1}{\gamma!} D_x^\gamma a_\varepsilon \sum_{l\geqslant N} \partial_\xi^\gamma p_{-m-l,\varepsilon}.$$

Since the left-hand side and the last sum on the right-hand side are in S^{-N}_{lsc} we obtain

$$\big(\sigma_\varepsilon - \sum_{|\gamma|<N} \frac{1}{\gamma!} D_x^\gamma a_\varepsilon \sum_{l<N} \partial_\xi^\gamma p_{-m-l,\varepsilon}\big)_\varepsilon \in S^{-N}_{lsc}.$$

We now write

$$\sum_{|\gamma|<N} \frac{1}{\gamma!} D_x^\gamma a_\varepsilon \sum_{l<N} \partial_\xi^\gamma p_{-m-l,\varepsilon} = p_{-m,\varepsilon} a_\varepsilon + \sum_{k=1}^{N-1} \Big\{ p_{-m-k,\varepsilon} a_\varepsilon + \sum_{\substack{|\gamma|+l=k \\ l<k}} \frac{1}{\gamma!} \partial_\xi^\gamma p_{-m-l,\varepsilon} D_x^\gamma a_\varepsilon \Big\} + \\ + \sum_{\substack{|\gamma|+l\geqslant N \\ |\gamma|<N, l<N}} \frac{1}{\gamma!} \partial_\xi^\gamma p_{-m-l,\varepsilon} D_x^\gamma a_\varepsilon,$$

where the last expression is in S^{-N}_{lsc}. The second sum on the right-hand side vanishes for $|\xi| \geqslant 2r_\varepsilon$ and $|\xi| \leqslant r_\varepsilon$ by construction. So, it remains to estimate this expression on the set $r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon$. We observe for any $l > 0$ and $\alpha \in \mathbb{N}^n$ that

$$\sup_{r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon} |\langle\xi\rangle^l \partial_\xi^\alpha \psi(\xi/r_\varepsilon)| = \sup_{r_\varepsilon \leqslant |\xi| \leqslant 2r_\varepsilon} |\langle\xi\rangle^l r_\varepsilon^{-|\alpha|} (\partial^\alpha \psi)(\xi/r_\varepsilon)| \leqslant c(\psi) r_\varepsilon^{-|\alpha|} \langle 2r_\varepsilon\rangle^l$$

and we conclude that this term is contained in S^{-N}_{lsc}.

Using Step 1, i.e. $(p_{-m,\varepsilon} a_\varepsilon)_\varepsilon = 1 \mod S^{-\infty}_{lsc}$, we get for every $N \geqslant 1$

$$\big(\sigma_\varepsilon - \sum_{|\gamma|<N} \frac{1}{\gamma!} (\partial_\xi^\gamma p_\varepsilon D_x^\gamma a_\varepsilon)\big)_\varepsilon = (\sigma_\varepsilon - 1)_\varepsilon \qquad \mod S^{-N}_{lsc}$$

and the proof is complete. □

Recall that $a \in \widetilde{S}^m_{lsc}$ is called logarithmic slow scale elliptic if one of its representatives $(a_\varepsilon)_\varepsilon$ is logarithmic slow scale elliptic. Then, the operator $p(x, D)$ has the symbol $(p_\varepsilon)_\varepsilon + \mathcal{N}_{S^{-m}} \in \widetilde{S}^{-m}_{lsc}$ and $(p_\varepsilon)_\varepsilon$ is as in Theorem 1.4.16. Furthermore, by Step 3 of the same theorem we get

$$p(x,D)\circ a(x,D)u(x) = \Big(\int e^{i(x-y)\xi}u_\varepsilon(y)\,dy\,đ\xi\Big)_\varepsilon + \mathcal{N}_{H^\infty}(\mathbb{R}^n).$$

Therefore, we obtain the following result.

Theorem 1.4.17. *Let $a \in \widetilde{S}^m_{lsc}$ be a logarithmic slow scale elliptic symbol of order m. Then there exists a logarithmic slow scale elliptic pseudodifferential operator of order $-m$ with symbol $p \in \widetilde{S}^{-m}_{lsc}$ such that for all $u \in \mathcal{G}_{L^2}$*

$$\begin{aligned} a(x,D)\circ p(x,D)u &= u + r(x,D)u \\ p(x,D)\circ a(x,D)u &= u + s(x,D)u \end{aligned}$$

where $r(x,D)$ and $s(x,D)$ are regularizing operators, that is they map $\mathcal{G}_{L^2}$ into $\mathcal{G}^\infty_{L^2}$ (see Section 1.6).

1.5 A factorization procedure for the generalized strictly hyperbolic wave operator

Concerning products of logarithmic slow scale pseudodifferential operators that approximate the generalized operator L from (1.3.1), we will follow the ideas in [37, Appendix II], [31, Chapter 23]. First we recall from (1.3.1)

$$\begin{aligned} L(x,z,D_t,D_x,D_z) &= \partial_z\frac{1}{\rho}\partial_z + \sum_{j=1}^{n-1}\partial_{x_j}\frac{1}{\rho}\partial_{x_j} - \frac{1}{\rho}\frac{1}{c^2}\partial_t^2 = \\ &=: \partial_z\frac{1}{\rho}\partial_z + A_\rho(x,z,D_t,D_x) \end{aligned} \tag{1.5.1}$$

where the coefficients $1/\rho$ and $1/c^2$ are in $\mathcal{G}^{lsc}_{\infty,\infty}(\mathbb{R}^n)$ in the variables (x,z). This implies that L is an operator of class $\Psi^2_{lsc}(\mathbb{R}^n\times\mathbb{R}^{n+1})$ and a representative $(l_\varepsilon)_\varepsilon$ of the symbol of L is in $S^2_{lsc}(\mathbb{R}^n\times\mathbb{R}^{n+1})$. Furthermore, the symbol of the operator A_ρ is in $S^2_{lsc}(\mathbb{R}^n\times\mathbb{R}^n)$ and is given by

$$A_\rho(x,z,D_t,D_x) = -\frac{1}{\rho}\Big(\frac{1}{c^2}\partial_t^2 - \sum_{j=1}^{n-1}\partial^2_{x_j}\Big) + \sum_{j=1}^{n-1}\Big(\partial_{x_j}\frac{1}{\rho}\Big)\partial_{x_j}.$$

Note that this operator is only a pseudodifferential operator in (x,t) depending smoothly on the parameter z on the level of representatives for any fixed $\varepsilon\in(0,1]$. We denote by $a_\rho \in \widetilde{S}^2_{lsc}(\mathbb{R}^n\times\mathbb{R}^n)$ the principal symbol of A_ρ.

Moreover, we set

$$A(x,z,D_t,D_x) := -\frac{1}{c^2}(x,z)\partial_t^2 + \sum_{j=1}^{n-1}\partial_{x_j}^2.$$

As already mentioned in Section 1.3, the operator L is not globally strictly hyperbolic but we can restrict the analysis to an appropriate space on which the operator becomes strictly hyperbolic. In particular, we follow Stolk in [51] and introduce the following set of points on which we will then establish the factorization. Therefore we recall from Section 1.3:

$$I'_\theta := \Big\{(x,z,\tau,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \tau \neq 0,\ |c^*(x,z)\tau^{-1}\xi| \leqslant \sin\theta\Big\} \tag{1.5.2}$$

for some fixed $\theta \in (0,\pi/2)$ and $c^*(x,z) = \lim_{\varepsilon\to 0} c_\varepsilon(x,z)$ in $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$. Then I'_θ is a subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash 0)$, independent of the regularization parameter $\varepsilon \in (0,1]$ and conic with respect to (τ,ξ).

Fix $\theta \in (0,\pi/2)$ and denote by a the (principal) symbol of A, i.e.

$$a_\varepsilon(x,z,\tau,\xi) := \frac{\tau^2}{c_\varepsilon^2(x,z)} - |\xi|^2 \qquad \varepsilon \in (0,1].$$

Then a is logarithmic slow scale elliptic on I'_θ and the generalized roots $[(\zeta_{j,\varepsilon})_\varepsilon]$ of the principal symbol of L are given by $\zeta_{j,\varepsilon} := \mp i\sqrt{a_\varepsilon(x,z,\tau,\xi)}$ which satisfy (1.3.3) on I'_θ, i.e.

$$|\zeta_{1,\varepsilon}(x,z,\tau,\xi) - \zeta_{2,\varepsilon}(x,z,\tau,\xi)| \geqslant C|(\tau,\xi)| \qquad \text{on } I'_\theta,\ \text{for } |(\tau,\xi)| \geqslant M$$

for some constants $C > 0, M > 0$ and all ε sufficiently small.

Indeed, we have on I'_θ

$$\begin{aligned}|a_\varepsilon(x,z,\tau,\xi)| = \Big|\frac{\tau^2}{c_\varepsilon^2(x,z)} - |\xi|^2\Big| &\geqslant \left|\Big|\frac{\tau^2}{c^{*2}(x,z)} - |\xi|^2\Big| - \Big|\Big(\frac{1}{c_\varepsilon^2(x,z)} - \frac{1}{c^{*2}(x,z)}\Big)\tau^2\Big|\right| \geqslant \\ &\geqslant \frac{\tau^2}{c^{*2}(x,z)}(1-\sin^2\theta) - C\omega_\varepsilon^{-\mu}\tau^2 \geqslant C(\tau^2 + |\xi|^2)\end{aligned}$$

for some generic constant $C > 0$ and ε small enough. So, in particular a_ρ is logarithmic slow scale elliptic, because the generalized function $1/\rho$ is strongly positive (see assumption (ii) in Section 1.3).

Remark 1.5.1. In the classical theory of pseudodifferential operators the characteristic set plays an important role in microlocal analysis. We recall that the characteristic set is defined as the set of points where the homogeneous principal symbol of the operator vanishes. Concerning vanishing properties of the homogeneous principal symbol in the generalized setting, we make the following remarks. We denote by $(l_\varepsilon)_\varepsilon$ the principal symbol of L.

(1) First note that the equation

$$\begin{aligned} l_\varepsilon(\zeta;x,z,\tau,\xi) &= -\frac{1}{\rho_\varepsilon(x,z)}\zeta^2 + a_{\rho,\varepsilon}(x,z,\tau,\xi) = \\ &= \left(i\zeta - i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right)\frac{1}{\rho_\varepsilon(x,z)}\left(i\zeta + i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right) = 0 \quad \text{on } I'_\theta,\ |(\tau,\xi)| \geqslant M \end{aligned}$$

for some constant $M > 0$ is not (classically, i.e. independent of ε) solvable with respect to ζ in $\mathbb{R}$ as ε tends to 0.

But for any $(x,z,\tau,\xi) \in I'_\theta$ there exists two distinct $\zeta_\varepsilon(x,z,\tau,\xi)$ such that

$$\begin{aligned} l_\varepsilon(\zeta_\varepsilon;x,z,\tau,\xi) &= -\frac{1}{\rho_\varepsilon(x,z)}\zeta_\varepsilon^2 + a_{\rho,\varepsilon}(x,z,\tau,\xi) = \\ &= \left(i\zeta_\varepsilon - i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right)\frac{1}{\rho_\varepsilon(x,z)}\left(i\zeta_\varepsilon + i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right) = 0 \end{aligned}$$

for all $|(\tau,\xi)| \geqslant M > 0$ and $\varepsilon \in (0,1]$. In particular, this is satisfied if we set for fixed $(x,z,\tau,\xi) \in I'_\theta$

$$\zeta_\varepsilon(x,z,\tau,\xi) := \pm\sqrt{\frac{\tau^2}{c_\varepsilon^2(x,z)} - |\xi|^2} \to \zeta(x,z,\tau,\xi) := \pm\sqrt{\frac{\tau^2}{c^{*2}(x,z)} - |\xi|^2} \quad \text{as } \varepsilon \to 0.$$

So, the equation $l_\varepsilon = 0$ has two generalized solutions $\zeta_{j,\varepsilon}$, $j = 1,2$, for fixed $(x,z,\tau,\xi) \in I'_\theta$.

Note that the generalized principal symbol can always be factorized, i.e.

$$\begin{aligned} l_\varepsilon(\zeta;x,z,\tau,\xi) &= -\frac{1}{\rho_\varepsilon(x,z)}\zeta^2 + a_{\rho,\varepsilon}(x,z,\tau,\xi) = \\ &= \left(i\zeta + i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right)\frac{1}{\rho_\varepsilon(x,z)}\left(i\zeta - i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\right) \quad \text{on } I'_\theta,\ |(\tau,\xi)| \geqslant M, \end{aligned}$$

$M > 0$, regardless of whether the symbol vanishes or not.

(2) The roots $\zeta_{1,\varepsilon}(x,z,\tau,\xi)$, $\zeta_{2,\varepsilon}(x,z,\tau,\xi)$ of the equation

$$-\frac{1}{\rho_\varepsilon(x,z)}\zeta_\varepsilon^2 + a_{\rho,\varepsilon}(x,z,\tau,\xi) = 0$$

are called the generalized characteristic roots of L.

Before we state the main theorem of this section, we give a few more details about notation.

In the following we will study operators of the form

$$S = \sum_{j=0}^{2} S_j(x, z, D_t, D_x)\partial_z^{2-j}$$

on $\mathbb{R}^{n+1}$ where the coefficients are operators with symbols $S_j \in S_{lsc}^{k_j}$ for some real numbers k_j, $j = 0, 1, 2$. Further, we write

$$S = \sum_{j=0}^{2} S_j(x, z, D_t, D_x)\partial_z^{2-j} \qquad \text{on } I'_\theta$$

when the symbols of the coefficients S_j are restricted to the set I'_θ.

We will now establish the factorization procedure in order to write the operator L in terms of two first-order pseudodifferential operators of the form $L_j = \partial_z + A_j$ on I'_θ where A_j are pseudodifferential operators with generalized symbols in $\widetilde{S}^1_{lsc}$, $j = 1, 2$.

We are now in the position to show the following result:

Theorem 1.5.2. *Let $L = \partial_z \frac{1}{\rho}\partial_z + A_\rho(x, z, D_t, D_x)$ and I'_θ be as in (1.5.1) and (1.5.2) respectively. Then, on the set I'_θ there are generalized roots $\{(\zeta_{1,\varepsilon})_\varepsilon, (\zeta_{2,\varepsilon})_\varepsilon\}$ of the principal symbol of $(L_\varepsilon)_\varepsilon$ that fulfill the separation property*

$$|\zeta_{1,\varepsilon}(x, z, \tau, \xi) - \zeta_{2,\varepsilon}(x, z, \tau, \xi)| \geqslant C|(\tau, \xi)| \qquad \text{on } I'_\theta, \text{ for } |(\tau, \xi)| \geqslant M$$

for some constants $C > 0, M > 0, \eta \in (0, 1]$ and all $\varepsilon \in (0, \eta]$. Furthermore, the operator L can be factorized into

$$L = L_1 \frac{1}{\rho} L_2 + R \qquad \text{on } I'_\theta \tag{1.5.3}$$

where L_j is written as $L_j = \partial_z + A_j$ and $A_j = A_j(x, z, D_t, D_x)$ is a parameter-dependent logarithmic slow scale (classical) generalized pseudodifferential operator of order one in (t, x). The principal symbol of A_j can be chosen either equal to $(-i\zeta_{j,\varepsilon})_\varepsilon = (\mp i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$, or equal to $(i\zeta_{j,\varepsilon})_\varepsilon = (\mp i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$, on I'_θ. Furthermore, the remainder is given by $R = \Gamma_1 + \Gamma_2\partial_z$ for some pseudodifferential operators $\Gamma_j = \Gamma_j(x, z, D_t, D_x)$ with parameter z and generalized symbol γ_j in $\widetilde{S}^{-\infty}_{lsc}$ on I'_θ, $j = 1, 2$.

Remark 1.5.3. Note that the product $L_1 \frac{1}{\rho} L_2$ in (1.5.3) is not a pseudodifferential operator on $\mathbb{R}^{n+1}$. But one can overcome this by introducing a generalized cut-off $\psi(x, z, D_t, D_x, D_z)$ such that the difference $\psi L_1 \frac{1}{\rho} L_2 - L_1 \frac{1}{\rho} L_2$ is insignificant on some adequate subdomain of the phase space $T^*\mathbb{R}^{n+1}$ under a microlocal point of view. This will be specified in the next section where we introduce a generalized version of microlocal analysis.

1.5.1 Technical preliminaries

As already mentioned above, the symbol $(a_\varepsilon)_\varepsilon$ is logarithmic slow scale elliptic on I'_θ, $\theta \in (0, \pi/2)$, i.e. for any $(x_0, z_0, \tau_0, \xi_0) \in I'_\theta$ there exist a relatively compact open neighborhood U of (x_0, z_0) and a conic neighborhood Γ of (τ_0, ξ_0) such that for some $(r_\varepsilon)_\varepsilon \in \Pi_{lsc}$, $(s_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and a constant $\eta \in (0, 1]$ we have

$$|a_\varepsilon(x, z, \tau, \xi)| \geqslant \frac{1}{s_\varepsilon}\langle(\tau, \xi)\rangle^2 \qquad \text{for } (x, z, \tau, \xi) \in U \times \Gamma,\ |(\tau, \xi)| \geqslant r_\varepsilon,\ \varepsilon \in (0, \eta].$$

As demonstrated in Proposition 1.4.4, we have stability under lower order logarithmic slow scale perturbations, and therefore the total symbol of the operator $(A_{\rho,\varepsilon})_\varepsilon$ itself is logarithmic slow scale elliptic on I'_θ.

In order to describe a factorization for the operator L, we have to give a meaning to the square root of the symbol of A on the set I'_θ.

Note that the square root of $(a_\varepsilon)_\varepsilon$ is prescribed only on the set I'_θ. Outside of I'_θ it is in general not defined but we choose it without the singularity of the square root. We remark that such an extension is equal to $(\sqrt{a_\varepsilon})_\varepsilon$ when restricted to I'_θ.

In order to cut off singularities of the square root of $(a_\varepsilon)_\varepsilon$, we define a generalized symbol $(\chi_\varepsilon)_\varepsilon \in S^0_{lsc}(\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1})$ homogeneous of degree 0 for $|(\tau, \xi)| \geqslant 1$ and such that

$$\chi_\varepsilon(x, z, \tau, \xi) = \begin{cases} 1 & \text{on } I'_{\theta_1} \\ 0 & \text{outside } I'_{\theta_2} \end{cases}$$

for angles $\theta_1, \theta_2 \in (0, \pi/2)$, $\theta_1 < \theta_2$. Then, $(\chi_\varepsilon \sqrt{a_\varepsilon})_\varepsilon \in S^0_{lsc}(\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1})$ and its restriction to I'_{θ_1} is equal to $(\sqrt{a_\varepsilon})_\varepsilon$. Moreover, $(\chi_\varepsilon \sqrt{a_\varepsilon})_\varepsilon$ vanishes outside I'_{θ_2}.

Remark 1.5.4. For the construction of χ_ε we define for every fixed $\varepsilon \in (0, 1]$ the function

$$f_\varepsilon(x, z, \tau, \xi) := c_\varepsilon(x, z)\xi\tau^{-1} \qquad \text{on } \mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1}.$$

Then, for every fixed $\varepsilon \in (0, 1]$ we let χ_ε be the smooth function defined on $\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1}$, $0 \leqslant \chi_\varepsilon \leqslant 1$, which is given by

$$\chi_\varepsilon(x, z, \tau, \xi) := \begin{cases} 0, & |f_\varepsilon| \geqslant \sin\gamma_2 \\ 1, & |f_\varepsilon| \leqslant \sin\gamma_1 \\ 1 - \frac{1}{1 + e^{\frac{1}{|f_\varepsilon| - \sin\gamma_1} - \frac{1}{\sin\gamma_2 - |f_\varepsilon|}}}, & \sin\gamma_1 < |f_\varepsilon| < \sin\gamma_2 \end{cases} \tag{1.5.4}$$

for some fixed angles γ_1, γ_2 with $0 < \theta_1 < \gamma_1 < \gamma_2 < \theta_2 < \pi/2$.

We therefore get $\chi_\varepsilon = 1$ on I'_{θ_1}, since $\exists \eta \in (0,1]$ such that

$$I'_{\theta_1} \subseteq \{(x,z,\tau,\xi) \mid \tau \neq 0,\ |c_\varepsilon(x,z)\xi/\tau| \leqslant \sin\gamma_1\} \qquad \varepsilon \in (0,\eta].$$

To show the inclusion, let $(x,z,\tau,\xi) \in I'_{\theta_1}$ and $f(x,z,\tau,\xi) := \lim_{\varepsilon\to 0} f_\varepsilon(x,z,\tau,\xi)$. Then there exists a constant $C > 0$ such that

$$\begin{aligned}|f_\varepsilon(x,z,\tau,\xi)| &\leqslant |(f_\varepsilon - f)(x,z,\tau,\xi)| + |f(x,z,\tau,\xi)| \leqslant \\ &\leqslant |(c_\varepsilon(x,z) - c^*(x,z))\xi/\tau| + |c^*(x,z)\xi/\tau| \leqslant \\ &\leqslant \sin\theta_1(1 - C\omega_\varepsilon^{-\mu})\end{aligned}$$

Since $\omega_\varepsilon^{-\mu}$ tends to 0 as $\varepsilon \to 0$ and $\theta_1 < \gamma_1$, there exists $\eta \in (0,1]$ such that the right-hand side of the last equation is bounded from above by $\sin\gamma_1$.

Similarly, one can show that $\exists \eta \in (0,1]$ such that for all $\varepsilon \in (0,\eta]$:

$$\{(x,z,\tau,\xi) \mid \tau \neq 0,\ |c^*(x,z)\xi/\tau| \geqslant \sin\theta_2\} \subseteq \{(x,z,\tau,\xi) \mid \tau \neq 0,\ |c_\varepsilon(x,z)\xi/\tau| \geqslant \sin\gamma_2\}$$

and hence $\chi_\varepsilon = 0$ outside I'_{θ_2} (note χ_ε is defined only for $\tau \neq 0$). Indeed

$$\begin{aligned}|f_\varepsilon(x,z,\tau,\xi)| &\geqslant |f(x,z,\tau,\xi)| - |(f_\varepsilon - f)(x,z,\tau,\xi)| \geqslant \\ &\geqslant |c^*(x,z)\xi/\tau| \left(1 - \left|\frac{c_\varepsilon(x,z) - c^*(x,z)}{c^*(x,z)}\right|\right) \geqslant \\ &\geqslant \sin\theta_2(1 - C\omega_\varepsilon^{-\mu}) \geqslant \sin\gamma_2\end{aligned}$$

as $\varepsilon \to 0$. Now, since $\frac{1}{c^2} \in \mathcal{G}^{lsc}_{\infty,\infty}$ and for every $\varepsilon \in (0,1]$ the function χ_ε is smooth on $\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1}$ and homogeneous of order 0 for $|(\tau,\xi)| \geqslant 1$, we can conclude that $\chi_\varepsilon \in S^0_{lsc}(\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1})$ (cf. [27, Example 1.2]).

1.5.2 Factorization procedure

The aim is to decompose the operator L as announced in (1.5.3) in Theorem 1.5.2. Therefore, we give a construction scheme for the generalized symbols $(a_{j,\varepsilon})_\varepsilon$ of the operators A_j, $j = 1,2$, by means of their asymptotic expansions (in the sense of Definition 1.2.14), that is

$$a_j(x,z,\tau,\xi) \ \sim_{lsc} \sum_{\mu=0}^{\infty} b_j^{(\mu)}(x,z,\tau,\xi) \qquad \text{in } \widetilde{S}^1_{cl,lsc}(I'_\theta) \tag{1.5.5}$$

where the sequence $\{b_j^{(\mu)}\}_{\mu\in\mathbb{N}}$ consists of appropriate elements $b_j^{(\mu)} \in S^{1-\mu}_{lsc}(I'_\theta)$, $j = 1,2$, and will be constructed recursively. Recall that (1.5.5) means that there are representatives $(a_{j,\varepsilon})_\varepsilon$ of a_j and $(b_{j,\varepsilon}^{(\mu)})_\varepsilon$ of $b_j^{(\mu)}$ such that for any cut-off φ equal to 1 near the origin we have

$$\Big(a_{j,\varepsilon} - \sum_{\mu=0}^{N-1} (1-\varphi) b_{j,\varepsilon}^{(\mu)}\Big)_\varepsilon \in S_{lsc}^{1-N}(I'_\theta). \tag{1.5.6}$$

Proof of Theorem 1.5.2. For the proof we apply a decomposition method similar to [37, Appendix II] and [31, Chapter 23]. In the following, we will show the desired factorization in the case that the principal symbol of A_j is equal to $(-i\zeta_{j,\varepsilon})_\varepsilon = (\mp i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1,2$. The proof for the second possible choice of the principal symbol of A_j is essentially the same.

We will now give a construction scheme for the symbols a_j of the operator A_j, $j = 1,2$, by means of their asymptotic expansions. To begin with, let $\zeta_{j,\varepsilon} = \pm\sqrt{a_\varepsilon}$, $j = 1,2$ and $\varepsilon \in (0,1]$. For $j = 1,2$ we set on I'_θ

$$b_{j,\varepsilon}^{(0)} := -i\zeta_{j,\varepsilon}, \qquad a_{j,\varepsilon}^{(1)} := b_{j,\varepsilon}^{(0)}, \qquad A_{j,\varepsilon}^{(1)} := \mathrm{OP}(a_{j,\varepsilon}^{(1)})$$

where $A_{j,\varepsilon}^{(1)}$ should be thought as the operator whose symbol has the classical asymptotic expansion $a_{j,\varepsilon}^{(1)}$ on I'_θ. We emphasize that $A_{j,\varepsilon}^{(1)}$ is intrinsically the restriction of a globally defined operator restricted to the set I'_θ as the symbol can always be multiplied by a generalized cut-off function $(\chi_\varepsilon)_\varepsilon$ of the form as indicated in (1.5.4) which is identically 1 on I'_θ (with θ_1 equal θ). Moreover, the function φ as in (1.5.6) cuts off the singularities near $\tau = 0$.

We define $L_{j,\varepsilon}^{(1)} := \partial_z + A_{j,\varepsilon}^{(1)}$, $j = 1,2$, $\varepsilon \in (0,1]$. Using $L_{1,\varepsilon}^{(1)} \frac{1}{\rho_\varepsilon} L_{2,\varepsilon}^{(1)}$ as a first approximation to L_ε we obtain an error of the following form

$$\begin{aligned} L_{1,\varepsilon}^{(1)} \frac{1}{\rho_\varepsilon} L_{2,\varepsilon}^{(1)} - L_\varepsilon &= \Big(\partial_z + A_{1,\varepsilon}^{(1)}\Big)\frac{1}{\rho_\varepsilon}\Big(\partial_z + A_{2,\varepsilon}^{(1)}\Big) - \partial_z \frac{1}{\rho_\varepsilon}\partial_z - A_{\rho,\varepsilon} = \\ &= \Big(A_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}A_{2,\varepsilon}^{(1)}\Big)\partial_z + \partial_z\Big(\frac{1}{\rho_\varepsilon}A_{2,\varepsilon}^{(1)}\Big) + A_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon}A_{2,\varepsilon}^{(1)} - A_{\rho,\varepsilon} = \\ &=: \sum_{j=1}^{2} \Gamma_{j,\varepsilon}^{(1)} \partial_z^{2-j} \qquad \text{on } I'_\theta. \end{aligned} \tag{1.5.7}$$

Since $A_1^{(1)}$ and $-A_2^{(1)}$ have the same principal symbol on I'_θ it follows that the symbols of the operators $\Gamma_j^{(1)} = \Gamma_j^{(1)}(x,z,D_t,D_x)$ are in $S_{cl,lsc}^{j-1}$ on I'_θ, $j = 1,2$.

To improve this approximation we proceed by induction. For convenience of the reader, we also compute the second order approximation of L. Therefore let $\varepsilon \in (0,1]$ be fixed and define

$$a_{j,\varepsilon}^{(2)} := a_{j,\varepsilon}^{(1)} + b_{j,\varepsilon}^{(1)}, \qquad A_{j,\varepsilon}^{(2)} := \mathrm{OP}(a_{j,\varepsilon}^{(2)}), \qquad L_{j,\varepsilon}^{(2)} := \partial_z + A_{j,\varepsilon}^{(2)}.$$

Here $b^{(1)}_{j,\varepsilon}$ will be specified immediately and in such a way that the symbol of $A^{(2)}_{j,\varepsilon}$ with classical asymptotic expansion $\left(a^{(2)}_{j,\varepsilon}\right)_\varepsilon$ is in $S^1_{cl,lsc}$. From the above we already know that

$$L^{(1)}_1 \frac{1}{\rho} L^{(1)}_2 - L = \sum_{j=1}^{2} \Gamma^{(1)}_j \partial_z^{2-j}.$$

We can formally write on I'_θ

$$\begin{aligned} L^{(2)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} L^{(2)}_{2,\varepsilon} - L_\varepsilon &= \left(L^{(1)}_{1,\varepsilon} + B^{(1)}_{1,\varepsilon}\right) \frac{1}{\rho_\varepsilon} \left(L^{(1)}_{2,\varepsilon} + B^{(1)}_{2,\varepsilon}\right) - L_\varepsilon = \\ &= \sum_{j=1}^{2} \Gamma^{(1)}_{j,\varepsilon} \partial_z^{2-j} + \left(\partial_z + A^{(1)}_{1,\varepsilon}\right) \frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon} + B^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} \left(\partial_z + A^{(1)}_{2,\varepsilon}\right) + B^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon} = \\ &= \sum_{j=1}^{2} \Gamma^{(1)}_{j,\varepsilon} \partial_z^{2-j} + \left(B^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon}\right) \partial_z + \partial_z \left(\frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon}\right) + \\ &\qquad + B^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} A^{(1)}_{2,\varepsilon} + A^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon} + B^{(1)}_{1,\varepsilon} \frac{1}{\rho_\varepsilon} B^{(1)}_{2,\varepsilon}. \end{aligned}$$

We now specify $\left(b^{(1)}_{1,\varepsilon}\right)_\varepsilon$, $\left(b^{(1)}_{2,\varepsilon}\right)_\varepsilon$ as follows: as $\left(a^{(1)}_{j,\varepsilon}\right)_\varepsilon = \mp\left(i\sqrt{a_\varepsilon}\right)_\varepsilon$, $j = 1, 2$, are non-vanishing on I'_θ the matrix

$$\frac{1}{\rho_\varepsilon} \begin{pmatrix} 1 & 1 \\ a^{(1)}_{2,\varepsilon} & a^{(1)}_{1,\varepsilon} \end{pmatrix}$$

is invertible on I'_θ.

Then, on I'_θ

$$-\rho_\varepsilon \frac{1}{a^{(1)}_{1,\varepsilon} - a^{(1)}_{2,\varepsilon}} \begin{pmatrix} a^{(1)}_{1,\varepsilon} & -1 \\ -a^{(1)}_{2,\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} \gamma^{(1)}_{1,\varepsilon} \\ \gamma^{(1)}_{2,\varepsilon} \end{pmatrix} = \begin{pmatrix} b^{(1)}_{1,\varepsilon} \\ b^{(1)}_{2,\varepsilon} \end{pmatrix} \tag{1.5.8}$$

where the net $(\rho_\varepsilon)_\varepsilon$ has the same properties as the net $(1/\rho_\varepsilon)_\varepsilon$.

Recall that $\exists C > 0, \exists \eta \in (0,1]$ such that

$$|a^{(1)}_{1,\varepsilon}(x,z,\tau,\xi)| \geqslant C|(\tau,\xi)| \quad \text{ on } I'_\theta,\ \varepsilon \in (0,\eta]$$

and $\forall \alpha, \beta \in \mathbb{N}^n, \exists (\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}, \forall (x,z,\tau,\xi) \in I'_\theta$

$$|\partial^\alpha_{(\tau,\xi)} \partial^\beta_{(x,z)} a^{(1)}_{1,\varepsilon}(x,z,\tau,\xi)| = \mathcal{O}(\omega_\varepsilon)|(\tau,\xi)|^{1-|\alpha|} \qquad \text{for } |(\tau,\xi)| \geqslant 1 \text{ as } \varepsilon \to 0.$$

So, $\left(a^{(1)-1}_{1,\varepsilon}\right)_\varepsilon \in S^{-1}_{cl,lsc}$ on I'_θ, $|(\tau,\xi)| \geqslant 1$. Then, modulo $S^{-\infty}_{lsc}(I'_\theta)$ there are uniquely determined symbols $\left(b_{1,\varepsilon}^{(1)}\right)_\varepsilon$, $\left(b_{2,\varepsilon}^{(1)}\right)_\varepsilon \in S_{lsc}^0$ on I'_θ homogeneous for $|(\tau,\xi)| \geqslant 1$ solving the system

$$\begin{aligned} -\gamma_{1,\varepsilon}^{(1)} &= b_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}b_{2,\varepsilon}^{(1)} \\ -\gamma_{2,\varepsilon}^{(1)} &= b_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon}a_{2,\varepsilon}^{(1)} + a_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon}b_{2,\varepsilon}^{(1)} \end{aligned}$$

on the set I'_θ where $\left(\gamma_{j,\varepsilon}^{(1)}\right)_\varepsilon$ denotes the principal symbol of $\Gamma_j^{(1)}$, $j = 1,2$. Note that $\left(b_{j,\varepsilon}^{(1)}\right)_\varepsilon \in S_{lsc}^0(I'_\theta)$ and is homogeneous for $|(\tau,\xi)| \geqslant 1$. Since $(a_{j,\varepsilon}^{(1)})_\varepsilon$ is logarithmic slow scale elliptic and $(b_{j,\varepsilon}^{(1)})_\varepsilon$ is of lower order, we conclude that $(a_{j,\varepsilon}^{(2)})_\varepsilon$ is also logarithmic slow scale elliptic.

We write $B_{j,\varepsilon}^{(1)} := \mathrm{OP}(b_{j,\varepsilon}^{(1)})$ and obtain

$$L_1^{(2)}\frac{1}{\rho}L_2^{(2)} - L = \sum_{j=1}^{2}\Gamma_j^{(2)}\partial_z^{2-j}$$

with $\Gamma_j^{(2)} = \Gamma_j^{(2)}(x,z,D_t,D_x)$ having symbols in $S_{cl,lsc}^{j-2}$ on I'_θ, $j = 1,2$.

For $N \geqslant 1$ assume $\left(b_{j,\varepsilon}^{(\mu)}\right)_\varepsilon \in S_{lsc}^{1-\mu}(I'_\theta)$ is homogeneous of order $1-\mu$ for $|(\tau,\xi)| \geqslant 1$ and determined for all $\mu < N$, $j = 1,2$. For $\varepsilon \in (0,1]$ we set

$$a_{j,\varepsilon}^{(N)} := \textstyle\sum_{\mu=0}^{N-1} b_{j,\varepsilon}^{(\mu)}, \qquad A_{j,\varepsilon}^{(N)} := \mathrm{OP}(a_{j,\varepsilon}^{(N)}), \qquad L_{j,\varepsilon}^{(N)} := \partial_z + A_{j,\varepsilon}^{(N)}.$$

Again, here $A_j^{(N)}$ is the polyhomogeneous generalized pseudodifferential operator with symbol $\left(a_{j,\varepsilon}^{(N)}\right)_\varepsilon$ in $S_{cl,lsc}^1$. Furthermore, we assume

$$L_1^{(N)}\frac{1}{\rho}L_2^{(N)} - L = \sum_{j=1}^{2}\Gamma_j^{(N)}\partial_z^{2-j}$$

with $\Gamma_j^{(N)} = \Gamma_j^{(N)}(x,z,D_t,D_x)$ having symbols in $S_{cl,lsc}^{j-N}$ on I'_θ, $j = 1,2$.

For the induction step we specify $b_1^{(N)}, b_2^{(N)}$ as follows: We denote by $\left(\gamma_{j,\varepsilon}^{(N)}\right)_\varepsilon$ the top order symbol of $\Gamma_j^{(N)}$. Since $\left(a_{j,\varepsilon}^{(1)}\right)_\varepsilon$ is logarithmic slow scale elliptic on I'_θ, $j = 1, 2$, the matrix

$$\frac{1}{\rho_\varepsilon}\begin{pmatrix} 1 & 1 \\ a_{2,\varepsilon}^{(1)} & a_{1,\varepsilon}^{(1)} \end{pmatrix}$$

is invertible on I'_θ and the system

$$\begin{aligned} -\gamma_{1,\varepsilon}^{(N)} &= b_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}b_{2,\varepsilon}^{(N)} \\ -\gamma_{2,\varepsilon}^{(N)} &= b_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon}a_{2,\varepsilon}^{(1)} + a_{1,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon}b_{2,\varepsilon}^{(N)} \end{aligned}$$

is uniquely solvable for $\left(b_{1,\varepsilon}^{(N)}\right)_\varepsilon, \left(b_{2,\varepsilon}^{(N)}\right)_\varepsilon \in S_{lsc}^{1-N}$ on I'_θ. Moreover, $b_{2,\varepsilon}^{(N)}$ are homogeneous for $|(\tau,\xi)| \geqslant 1$. We write $B_{j,\varepsilon}^{(N)} := \mathrm{OP}(b_{j,\varepsilon}^{N})$ and indeed we get

$$\begin{aligned} L_{1,\varepsilon}^{(N+1)}\frac{1}{\rho_\varepsilon}L_{2,\varepsilon}^{(N+1)} - L_\varepsilon &= \left(L_{1,\varepsilon}^{(N)} + B_{1,\varepsilon}^{(N)}\right)\frac{1}{\rho_\varepsilon}\left(L_{2,\varepsilon}^{(N)} + B_{2,\varepsilon}^{(N)}\right) - L_\varepsilon = \\ &= \sum_{j=1}^{2}\Gamma_{j,\varepsilon}^{(N)}\partial_z^{2-j} + \left(\partial_z + A_{1,\varepsilon}^{(N)}\right)\frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)} + B_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon}\left(\partial_z + A_{2,\varepsilon}^{(N)}\right) + B_{N,\varepsilon}^{(1)}\frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)} = \\ &= \sum_{j=1}^{2}\Gamma_{j,\varepsilon}^{(N)}\partial_z^{2-j} + \left(B_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)}\right)\partial_z + \partial_z\left(\frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)}\right) + \\ &\qquad + B_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon}A_{2,\varepsilon}^{(N)} + A_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)} + B_{1,\varepsilon}^{(N)}\frac{1}{\rho_\varepsilon}B_{2,\varepsilon}^{(N)} = \\ &=: \sum_{j=1}^{2}\Gamma_{j,\varepsilon}^{(N+1)}\partial_z^{2-j} \end{aligned}$$

where $\Gamma_j^{(N+1)} = \Gamma_j^{(N+1)}(x,z,D_t,D_x)$ have symbols in $S_{cl,lsc}^{j-(N+1)}$ on I'_θ.

Then with a_j, $j = 1, 2$, such that

$$a_j \sim_{lsc} \sum_{\mu=0}^{\infty} b_j^{(\mu)} \quad \text{on } I'_\theta$$

we found the desired operator A_j that solves the problem. □

Remark 1.5.5. Theorem 1.5.2 allows for two distinct factorizations, i.e.

$$L = (\partial_z + A_{11})\frac{1}{\rho}(\partial_z + A_{21}) + R_1 = (\partial_z + A_{12})\frac{1}{\rho}(\partial_z + A_{22}) + R_2 \quad \text{on } I'_\theta$$

where the principal symbol of $(A_{j1,\varepsilon})_\varepsilon$ is equal to $(\mp i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$, and that of $(A_{j2,\varepsilon})_\varepsilon$ is equal to $(\pm i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$. R_1 and R_2 denote the corresponding remainders of the factorizations.

For completeness, we will compute the zeroth-order term for the symbols of $(A_{11,\varepsilon})_\varepsilon$ and $(A_{12,\varepsilon})_\varepsilon$ explicitly as they will be used later in Section 1.7. First, we calculate $(A_{11,\varepsilon})_\varepsilon$. Recall from (1.5.7) that

$$\gamma^{(1)}_{1,\varepsilon} = -i\sum_{j=1}^{n-1}\frac{\partial a^{(1)}_{1,\varepsilon}}{\partial \xi_j}\frac{\partial \rho_\varepsilon^{-1}}{\partial x_j} = -a_\varepsilon^{-1/2}\frac{1}{\rho_\varepsilon^2}\sum_{j=1}^{n-1}\frac{\partial \rho_\varepsilon}{\partial x_j}\xi_j$$

and

$$\gamma^{(1)}_{2,\varepsilon} \equiv \partial_z\Big(\frac{1}{\rho_\varepsilon}a^{(1)}_{2,\varepsilon}\Big) - a^{(1)}_{1,\varepsilon}\#(\frac{1}{\rho_\varepsilon}a^{(1)}_{2,\varepsilon}) - a_{\rho,\varepsilon} - i\sum_{j=1}^{n-1}\Big(\partial_{x_j}\frac{1}{\rho_\varepsilon}\Big)\xi_j \qquad \mathrm{mod}\ S^0_{lsc}.$$

Hence

$$\begin{aligned}\gamma^{(1)}_{2,\varepsilon} &= \partial_z\Big(\frac{1}{\rho_\varepsilon}a^{(1)}_{2,\varepsilon}\Big) - \frac{i}{\rho_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial a^{(1)}_{1,\varepsilon}}{\partial \xi_j}\frac{\partial a^{(1)}_{2,\varepsilon}}{\partial x_j} = \\ &= \frac{i}{2}\frac{a_{\rho,\varepsilon}^{1/2}}{\rho_\varepsilon^{1/2}}\Big(\frac{\partial a_{\rho,\varepsilon}}{\partial z}\frac{1}{a_{\rho,\varepsilon}} - \frac{\partial \rho_\varepsilon}{\partial z}\frac{1}{\rho_\varepsilon}\Big) + \frac{i}{\rho_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial a_\varepsilon^{1/2}}{\partial \xi_j}\frac{\partial a_\varepsilon^{1/2}}{\partial x_j}.\end{aligned}$$

We set $b_\varepsilon = a_\varepsilon^{1/2}$ and obtain by (1.5.8)

$$\begin{aligned}a_{11,\varepsilon} = -ib_\varepsilon &+ \frac{1}{2b_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial b_\varepsilon}{\partial \xi_j}\frac{\partial b_\varepsilon}{\partial x_j} - \frac{1}{4}\Big(\frac{\partial a_{\rho,\varepsilon}}{\partial z}\frac{1}{a_{\rho,\varepsilon}} - \frac{\partial \rho_\varepsilon}{\partial z}\frac{1}{\rho_\varepsilon}\Big) + \\ &+ \frac{1}{2}\frac{1}{\rho_\varepsilon b_\varepsilon}\sum_{j=1}^{n-1}\xi_j\frac{\partial \rho_\varepsilon}{\partial x_j} + \mathrm{order}(-1).\end{aligned} \tag{1.5.9}$$

Using the same arguments, we obtain for the operator A_{12} the following expression.

$$\begin{aligned}a_{12,\varepsilon} = \quad ib_\varepsilon &- \frac{1}{2b_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial b_\varepsilon}{\partial \xi_j}\frac{\partial b_\varepsilon}{\partial x_j} - \frac{1}{4}\Big(\frac{\partial a_{\rho,\varepsilon}}{\partial z}\frac{1}{a_{\rho,\varepsilon}} - \frac{\partial \rho_\varepsilon}{\partial z}\frac{1}{\rho_\varepsilon}\Big) - \\ &- \frac{1}{2}\frac{1}{\rho_\varepsilon b_\varepsilon}\sum_{j=1}^{n-1}\xi_j\frac{\partial \rho_\varepsilon}{\partial x_j} + \mathrm{order}(-1).\end{aligned} \tag{1.5.10}$$

1.6 The generalized wave front set

Given a pseudodifferential equation with smooth symbols, regularity properties can be used to describe the behavior of the solution. To study propagation of singularities the notion of the wave front set was introduced. We recall that the complement of the classical wave front set of a distribution u is the set $\mathrm{WF}(u)^c$ and measures smoothness near a point in the sense that the Fourier transform of a localized piece of u is rapidly decreasing in an open cone.

Another version is the Sobolev-based wave front set, where one studies subsets $\mathrm{WF}^m(u)$ of $T^*\mathbb{R}^n \backslash 0$, $m \in \mathbb{N}$, such that

$$(x_0, \xi_0) \notin \mathrm{WF}^m(u)$$

if there exists a $P \in \Psi^m$ elliptic at (x_0, ξ_0) such that $Pu \in L^2(\mathbb{R}^n)$. In this sense, wave front sets give a description of local smoothness of a distribution since $\mathrm{WF}^m(u) = \varnothing$ iff $u \in H^m_{loc}(\mathbb{R}^n)$. As pointed out in [56, Section 4], there is the following relation between these two versions:

Theorem 1.6.1. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then*

$$WF(u) = \overline{\bigcup_k WF^k(u)}.$$

Relating to the notion of wave front sets to pseudodifferential operators with smooth symbols, it is well known that for any $P \in \Psi^m$ with homogeneous principal symbol P_m one has the following inclusion (on $T^*\mathbb{R}^n \backslash 0$)

$$\mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \mathrm{Char}(P)$$

where $\mathrm{Char}(P) = P_m^{-1}(0) \cap T^*\mathbb{R}^n \backslash 0$ is the characteristic set depending on the principal symbol of the operator. Moreover, pseudodifferential operators do not increase the wave front sets. In particular one can show $\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u) \cap \mu\mathrm{supp}(p)$, where $\mu\mathrm{supp}(p)$ denotes the microsupport of P. For more information, we refer the reader to [31, Section 18].

In the framework of Colombeau generalized functions in $\mathcal{G}_{L^2}(\mathbb{R}^n)$, we follow this idea and measure regularity by considering rapid decay on cones in the frequency domain after localization in space. We refer to [40, 21, 22, 34, 49, 16, 10] for more details on the commonly used notion of a generalized wave front set based on $\mathcal{G}^\infty$-regularity.

1.6.1 Generalized microlocal analysis

Recall that a function $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ is regular, denoted by $u \in \mathcal{G}^\infty_{L^2}(\mathbb{R}^n)$, if and only if there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that

$$\exists N \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^n : \quad \|D^\alpha u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

As already indicated above, we introduce the following definition which is similar to [21]:

Definition 1.6.2. *A generalized function $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ is said to be microlocally regular at $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ if there exist $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(x_0) = 1$ and a conic neighborhood $\Gamma \subseteq \mathbb{R}^n \backslash 0$ of ξ_0 such that*

$$\exists N \in \mathbb{N}\ \forall l \in \mathbb{N} : \quad \|\langle \xi \rangle^l \mathcal{F}(\phi u_\varepsilon)(\xi)\|_{L^2(\Gamma)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0. \tag{1.6.1}$$

For $u \in \mathcal{G}_{L^2}$ we denote by $WF_g(u) \subseteq T^*\mathbb{R}^n \backslash 0$ the generalized wave front set of u which is defined as the complement of all points in phase space where u is microlocally regular.

Moreover, we say that two generalized functions $u, v \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ are microlocally equivalent at $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ if and only if there are $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(x_0) = 1$ and a conic neighborhood $\Gamma \subseteq \mathbb{R}^n \backslash 0$ of ξ_0 such that

$$\exists N \in \mathbb{N}\ \forall l \in \mathbb{N} : \quad \|\langle \xi \rangle^l \mathcal{F}\big(\phi(u_\varepsilon - v_\varepsilon)\big)(\xi)\|_{L^2(\Gamma)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

As we pointed out in Subsection 1.4.3, one can construct a generalized parametrix for a logarithmic slow scale elliptic operator $A(x, D) \in \Psi^m_{lsc}$ modulo some regularizing error in $\Psi^{-\infty}_{lsc}$ that maps $\mathcal{G}_{L^2}$ into $\mathcal{G}^\infty_{L^2}$. This fact can be deduced by the L^2-boundedness theorem for operators with symbol of class S^0, see [37, Theorem 4.1].

Remark 1.6.3. Let $P \in \Psi^{-\infty}_{rg}(\mathbb{R}^n)$, i.e. $p \in \widetilde{S}^{-\infty}_{rg} = \mathcal{M}^\infty_{S^{-\infty}}/\mathcal{N}_{S^{-\infty}}$, and $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$. Then Pu is in $\mathcal{G}^\infty_{L^2}(\mathbb{R}^n)$ and hence microlocally regular on $T^*\mathbb{R}^n \backslash 0$. This follows from the fact that the symbol of $D^\gamma P(x, D)$ is in $S^{-\infty}_{rg} = \mathcal{M}^\infty_{S^{-\infty}} \subseteq S^0_{rg}$ uniformly in $\gamma \in \mathbb{N}^n$. More precisely we have the estimate:

$$\exists N \in \mathbb{N}\ \forall \gamma \in \mathbb{N}^n\ \exists\ l \in \mathbb{N}\ \exists\ c > 0 :$$
$$\|D^\gamma P_\varepsilon(x, D) u_\varepsilon\|_{L^2} \leqslant c \max_{|\alpha+\beta| \leqslant l} \sup_{(x,\xi) \in \mathbb{R}^{2n}} |\partial^\alpha_\xi \partial^\beta_x \Big(\xi^\gamma \# p_\varepsilon(x, \xi)\Big)| \langle \xi \rangle^{|\alpha|} \|u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^{-N})$$

as $\varepsilon \to 0$. In particular, an operator $P \in \Psi^{-\infty}_{lsc}$ maps $\mathcal{G}_{L^2}(\mathbb{R}^n)$ into $\mathcal{G}^\infty_{L^2}(\mathbb{R}^n)$.

We now introduce the generalized microsupport of a symbol as follows: Let $p \in S^m_{rg}$ and $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$. Then the symbol p is $\mathcal{G}^\infty_{L^2}$-smoothing at (x_0, ξ_0) if there exist a representative $(p_\varepsilon)_\varepsilon \in p$, a relatively compact open neighborhood U of x_0 and a conic neighborhood

Γ of ξ_0 such that

$$\exists N \in \mathbb{N}\ \forall m \in \mathbb{R}\ \forall \alpha, \beta \in \mathbb{N}^n\ \forall (x,\xi) \in U \times \Gamma :$$
$$|\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x,\xi)| = \mathcal{O}(\varepsilon^{-N})\langle\xi\rangle^{m-|\alpha|} \quad \text{as } \varepsilon \to 0.$$

The generalized microsupport of p, denoted by $\mu\mathrm{supp}_g(p)$, is defined as the complement of the set of points (x_0, ξ_0) where p is $\mathcal{G}^\infty_{L^2}$-smoothing. In the sequel we will follow the ideas made in [21] where the authors measured regularity of a generalized function in $u \in \mathcal{G}(\Omega)$ and Ω is some open subset of $\mathbb{R}^n$ when acting on generalized pseudodifferential operators with slow scale symbols. Our definition (for $u \in \mathcal{G}_{L^2}$) is now the following:

Definition 1.6.4. *For any $u \in \mathcal{G}_{L^2}$ we define*

$$W_{sc}(u) := \bigcap_{\substack{p(x,D) \in {}_{pr}\Psi^0_{sc} \\ p(x,D)u \in \mathcal{G}^\infty_{L^2}}} Ell_{sc}(p)^c$$

where $Ell_{sc}(p)$ denotes the set of points where p is slow-scale elliptic and ${}_{pr}\Psi^0_{sc}$ is the set of properly supported slow scale pseudodifferential operators of order zero. Note that in the manner described in [21] we have that for any slow scale elliptic symbol $p \in \widetilde{S}^m_{sc}$ there exists a parametrix $q \in \widetilde{S}^{-m}_{rg}$ such that $p\#q = q\#p = 1$ in $\widetilde{S}^0_{rg}$.

As in [21, Proposition 2.8] one can show the following result: Let $\pi : T^*(\mathbb{R}^n)\backslash 0 \to \mathbb{R}^n : (x,\xi) \to x$. Then for any $u \in \mathcal{G}_{L^2}$ we have

$$\pi(W_{sc}(u)) = \mathrm{singsupp}_g(u)$$

where $\mathbb{R}^n \backslash \mathrm{singsupp}_g(u) := \{x \in \mathbb{R}^n : \exists U_x \subseteq \mathbb{R}^n \text{ open such that } u|_{U_x} \in \mathcal{G}^\infty_{L^2}\}$.

Before we proceed, let us briefly recall the three main theorems in [21, Theorem 3.6, Theorem 3.11, Theorem 4.1] but applied to the symbol classes and function spaces we use. We renounce to give the proofs as they can be obtained by minor changes in the arguments.

Theorem 1.6.5. *[21] For any $P = p(x,D) \in {}_{pr}\Psi^m_{rg}$ and $u \in \mathcal{G}_{L^2}$ we have*

$$W_{sc}(p(x,D)u) \subseteq W_{sc}(u) \cap \mu supp_g(p).$$

Moreover, we have the following theorem.

Theorem 1.6.6. *[21] For $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ we have*

$$WF_g(u) = W_{sc}(u) = \bigcap_{\substack{P \in {}_{pr}\Psi^0 \\ Pu \in \mathcal{G}^\infty_{L^2}}} Char(P).$$

The proof of this theorem follows the same lines as in [21, Theorem 3.9] but with slight changes. For completeness we give the proof.

Proof. We first show that if $(x_0,\xi_0) \notin \mathrm{WF}_g(u)$ then $(x_0,\xi_0) \notin \mathrm{W}_{sc}(u)$. So, suppose that (1.6.1) holds. By [21, Remark 3.5] there exists $p(\xi) \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\mathrm{supp}(p) \subseteq \Gamma$, $p = 1$ in a conic neighborhood Γ' of ξ_0, $|\xi| \geqslant 1$. Taking a typical proper cut-off χ one can write the properly supported pseudodifferential operator with amplitude $\chi(x,y)p(\xi)\phi(y)$ in the form $\sigma(x,D) \in {}_{pr}\Psi^0_{sc}(\mathbb{R}^n)$ such that $\sigma(x,D)v - p(D)(\phi v) \in \mathcal{G}^\infty_{L^2}(\mathbb{R}^n)$ for all $v \in \mathcal{G}_{L^2}$. Then

$$p(D)\phi(x,D)u = \left[\left(\int_\Gamma e^{ix\xi}p(\xi)\widehat{\phi u_\varepsilon}(\xi)đ\xi\right)_\varepsilon\right] \in \mathcal{G}^\infty_{L^2}$$

Since $\sigma(x,D)u - p(D)\phi u \in \mathcal{G}^\infty_{L^2}$ it follows that $\sigma(x,D)u \in \mathcal{G}^\infty_{L^2}$ and hence $(x_0,\xi_0) \notin \mathrm{W}_{sc}(u)$.

To show the converse assume that $(x_0,\xi_0) \notin \mathrm{W}_{sc}(u)$. Then there exists an open neighborhood U of x_0 such that $(x,\xi_0) \in \mathrm{W}_{sc}(u)^c$ for all $x \in U$. We choose $\phi \in \mathcal{C}^\infty_c(U)$ and define the closed conic set Σ' by

$$\Sigma' := \{\xi \in \mathbb{R}^n \backslash 0 \mid \exists x \in \mathbb{R}^n : (x,\xi) \in \mathrm{W}_{sc}(\phi u)\}.$$

By Theorem 1.6.5 we obtain $\mathrm{W}_{sc}(\phi u) \subseteq \mathrm{W}_{sc}(u) \cap (\mathrm{supp}(\phi) \times \mathbb{R}^n)$. Since there is a $p \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leqslant p \leqslant 1$ such that $p = 1$ in a conic neighborhood Γ' of ξ_0, $|\xi| \geqslant 1$ and $p(\xi) = 0$ in a conic neighborhood Σ'_0 of Σ' we get that $\mu\mathrm{supp}(p) \subseteq \mathbb{R}^n \times \Sigma'_0$ and $\mathrm{W}_{sc}(\phi u) \subseteq \mathbb{R}^n \times \Sigma'$. Therefore $\mathrm{W}_{sc}(p(D)\phi u) \subseteq \mathrm{W}_{sc}(\phi u) \cap \mu\mathrm{supp}(p) = \varnothing$. Hence $p(D)\phi u \in \mathcal{G}^\infty_{L^2}$ and

$$\int e^{ix\xi}p(\xi)\widehat{\phi u_\varepsilon}(\xi)đ\xi = (\phi u_\varepsilon * \check{p})(x) \in \mathcal{M}^\infty_{H^\infty}.$$

Since $\check{p}$ is a Schwartz function outside the origin we distinguish between two cases. First, assume that $\mathrm{dist}(x,\mathrm{supp}(\phi)) > \delta$ for some $\delta > 0$. Then we have that $\partial^\alpha(\phi u_\varepsilon * \check{p})(x) = \int_{|y|>\delta} \phi u_\varepsilon(x-y)\partial^\alpha\check{p}(y)\,dy$ and therefore for every $l \geqslant 0$:

$$\begin{aligned}\langle x\rangle^l|\partial^\alpha(\phi u_\varepsilon * \check{p})(x)| &\leqslant c\int_{|y|\geqslant\delta} |\phi u_\varepsilon(x-y)|\langle x-y\rangle^l\langle y\rangle^l|\partial^\alpha\check{p}(y)|\,dy \leqslant \\ &\leqslant c\int_{\mathrm{supp}(\phi)} |\phi u_\varepsilon(z)|\langle z\rangle^l \leqslant \\ &\leqslant c\|\phi(z)\langle z\rangle^l\|_{L^2(\mathrm{supp}(\phi))}\|u_\varepsilon\|_{L^2(\mathrm{supp}(\phi))}\end{aligned}$$

where in the last inequality we used the Hölder inequality. Note that there is an $N \in \mathbb{N}$ such that for every $l \geqslant 0$ and every $\alpha \in \mathbb{N}^n$ the last expression is uniformly bounded by $\mathcal{O}(\varepsilon^{-N})$ as $\varepsilon \to 0$.

Therefore we have: $\exists N \in \mathbb{N}, \forall l \geqslant 0, \forall \alpha \in \mathbb{N}^n$

$$\begin{aligned}
&\|\langle x\rangle^l \partial^\alpha(\phi u_\varepsilon * \check{p})(x)\|_{L^2(\{x:\mathrm{dist}(x,\mathrm{supp}(\phi))>\delta\})} = \\
&\quad = \mathcal{O}(\varepsilon^{-N})\|\langle x\rangle^l \partial^\alpha(\phi u_\varepsilon * \check{p})(x)\|_{L^\infty(\{x:\mathrm{dist}(x,\mathrm{supp}(\phi))>\delta\})} \int \langle x\rangle^{-n-1}\, dx = \\
&\quad = \mathcal{O}(\varepsilon^{-2N}) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Next, consider the compact set $\mathrm{dist}(x, \mathrm{supp}(\phi)) \leqslant \delta$. Since $(\phi u_\varepsilon * \check{p})(x) \in \mathcal{M}^{\infty}_{H^\infty}$ we obtain $\exists N \in \mathbb{N}, \forall l \geqslant 0, \forall \alpha \in \mathbb{N}^n$:

$$\|\langle x\rangle^l \partial^\alpha(\phi u_\varepsilon * \check{p})(x)\|_{L^2(\{x:\mathrm{dist}(x,\mathrm{supp}(\phi))\leqslant\delta\})} = \mathcal{O}(\varepsilon^{-2N})$$

as $\varepsilon \to 0$.

Putting this together we get $\exists N \in \mathbb{N}, \forall \alpha \in \mathbb{N}^n$, so that

$$\|\partial^\alpha(\phi u_\varepsilon * \check{p})(x)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

Taking the Fourier transform we obtain that $\exists N \in \mathbb{N}, \forall \alpha \in \mathbb{N}^n$

$$\|\langle \xi\rangle^{|\alpha|}(\phi u_\varepsilon * \check{p})\hat{}\,(\xi)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

Hence

$$\|\langle \xi\rangle^{|\alpha|} p(\xi)\widehat{\phi u_\varepsilon}(\xi)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

In particular, since $p(\xi) = 1$ in a neighborhood Γ of ξ_0, $|\xi| \geqslant 1$, we get: $\exists N \in \mathbb{N}, \forall \alpha \in \mathbb{N}^n$

$$\|\langle \xi\rangle^{|\alpha|}\widehat{\phi u_\varepsilon}(\xi)\|_{L^2(\Gamma)} = \mathcal{O}(\varepsilon^{-N}) \text{ as } \varepsilon \to 0,$$

as desired. □

Theorem 1.6.7. *[21] Let $P = p(x, D) \in {}_{pr}\Psi^m_{sc}$ and $u \in \mathcal{G}_{L^2}$. Then*

$$WF_g(Pu) \subseteq WF_g(u) \subseteq WF_g(Pu) \cup Ell_{sc}(p)^c.$$

In particular, this relation holds for any $P \in {}_{pr}\Psi^m_{lsc}$.

1.7 Microlocal decomposition of the wave equation

The purpose of this section is to establish a microlocal diagonalization of the operator L. In what follows, we will restrict our analysis to the following subset I_{θ_2} of the phase space

$$I_{\theta_2} := \{(t, x, z, \tau, \xi; \zeta) \in T^*\mathbb{R}^{n+1}\backslash 0 \mid (x, z, \tau, \xi) \in I'_{\theta_2},\ |\zeta| < C|\tau|\},$$

assuming that $\mathrm{WF}_g(U) \subseteq I_{\theta_2}$ (cf. [51]). Recall that if $LU = 0$, then by Sections 1.4 and 1.6 we have that $\mathrm{WF}_g(U) \subseteq \mathrm{Ell}_{\mathrm{sc}}(L)^c = \mathrm{Ell}_{\mathrm{lsc}}(L)^c = \Sigma$. The inequality $|\zeta| \leqslant \frac{1}{c^*}(x, z)|\tau|$ on Σ implies $|\zeta| \leqslant c_0^{-1}|\tau|$ on Σ and explains the inequality $|\zeta| < C|\tau|$ in the definition of I_{θ_2}. The microlocal diagonalization will then be stated on the set I_{θ_2}.

In order to decompose $LU = F$ microlocally on I_{θ_2} into a system of two first-order components, we will consider (u_+, u_-) that are obtained from $(U, \frac{1}{\rho}\partial_z U)$ by a logarithmic slow scale elliptic 2×2 pseudodifferential operator matrix $Q = Q(x, z, D_t, D_x)$, i.e. there exists $P = P(x, z, D_t, D_x)$ the generalized parametrix such that $PQ = QP = I$ modulo an operator with symbol in $S^{-\infty}_{lsc}$. Then, with the change of variables

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} := Q^{-1} \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix}, \qquad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := Q^{-1} \begin{pmatrix} 0 \\ F \end{pmatrix} \tag{1.7.1}$$

we search for an equivalent model to the equation

$$LU = F \qquad \text{microlocally on } I_{\theta_2}$$

with $U, F \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ in terms of two first-order generalized pseudodifferential equations of the form

$$\begin{aligned} \big(\partial_z - iB_+(x, z, D_t, D_x)\big)u_+ &= f_+ \qquad \text{microlocally on } I_{\theta_2} \\ \big(\partial_z - iB_-(x, z, D_t, D_x)\big)u_- &= f_- \qquad \text{microlocally on } I_{\theta_2} \end{aligned} \tag{1.7.2}$$

where $u_\pm, f_\pm \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ and $B_\pm = B_\pm(x, z, D_t, D_x)$ are logarithmic slow scale pseudodifferential operators of order one. Note that the operators $B_\pm$ are acting in (t, x) and depend on the parameter z. Moreover, we will show that there is a choice of the normalization Q of the wave field such that the operators $B_\pm$ become self-adjoint.

We start by rewriting the homogeneous equation $LU = 0$ in $\mathcal{G}_{L^2}$ into a first-order system with evolution parameter z:

$$\left[I\partial_z - \begin{pmatrix} 0 & \rho \\ -A_\rho & 0 \end{pmatrix} \right] \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} = \vec{0} \tag{1.7.3}$$

with $U \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$, I the $2{\times}2$ identity matrix and

$$A_\rho = A_\rho(x, z, D_t, D_x) = -\frac{1}{\rho}\frac{1}{c^2}\partial_t^2 + \sum_{j=1}^{n-1} \partial_{x_j} \frac{1}{\rho} \partial_{x_j}.$$

Notice that A_ρ is a logarithmic slow scale elliptic pseudodifferential operator of order two on I'_{θ_2}. For brevity, we will drop the identity matrix I in equation (1.7.3) in the following.

Lemma 1.7.1. *For suitable chosen Q and $B_\pm$ we can write*

$$P\left[\partial_z - \begin{pmatrix} 0 & \rho \\ -A_\rho & 0 \end{pmatrix} \right] Q = \partial_z - \begin{pmatrix} iB_+ & 0 \\ 0 & iB_- \end{pmatrix} + R \tag{1.7.4}$$

where $R = R(x, z, D_t, D_x)$ is a $2{\times}2$ pseudodifferential operator matrix with entries in $\widetilde{S}^1_{lsc}$ and that are of order $-\infty$ on I'_{θ_2}.

For the proof we introduce the following notation. Let R be a pseudodifferential operator valued $2{\times}2$ error-matrix with entries of the form

$$\sum_{j=1}^{2} R_j(x, z, D_t, D_x)\partial_z^{2-j} \tag{1.7.5}$$

and the symbol of $R_j = R_j(x, z, D_t, D_x)$ is in $\widetilde{S}^{-\infty}_{lsc}$, $j = 1, 2$. In the following we will sometimes write $\Psi^{-\infty}_{1,lsc}$ to denote an operator of the form (1.7.5). Similarly we will write $\Psi^{-\infty}_{2,lsc}$ for an operator of the form

$$\sum_{j=0}^{2} R_j(x, z, D_t, D_x)\partial_z^{2-j}$$

with $R_j = R_j(x, z, D_t, D_x)$ having symbols in $\widetilde{S}^{-\infty}_{lsc}$ for $j = 0, 1, 2$.

Proof. We will now search for appropriate choices of the operators P, Q, R and $B_\pm$, where Q, P, R are as above and $B_\pm$ as already indicated in (1.7.1) and (1.7.2) such that equation (1.7.4) holds on I'_{θ_2}, i.e. R is of order $-\infty$ on I'_{θ_2}.

To start with, we set $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$. We first apply the left-hand side of (1.7.4) to $P\begin{pmatrix} 1 \\ \frac{1}{\rho}\partial_z \end{pmatrix}$ and obtain

$$P\begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} QP \begin{pmatrix} 1 \\ \frac{1}{\rho}\partial_z \end{pmatrix} = \begin{pmatrix} P_{12}(\partial_z \frac{1}{\rho}\partial_z + A_\rho) + S^{(1,+)} \\ P_{22}(\partial_z \frac{1}{\rho}\partial_z + A_\rho) + S^{(1,-)} \end{pmatrix} \tag{1.7.6}$$

on I'_{θ_2}, with $S^{(1,\pm)}$ in $\Psi^{-\infty}_{2,lsc}$ on I'_{θ_2}. Similarly, we compute for the right-hand side

$$\left[\begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} + R\right] P \begin{pmatrix} 1 \\ \frac{1}{\rho}\partial_z \end{pmatrix} = \begin{pmatrix} (\partial_z - iB_+)(P_{12}\frac{1}{\rho}\partial_z + P_{11}) + S^{(2,+)} \\ (\partial_z - iB_-)(P_{22}\frac{1}{\rho}\partial_z + P_{21}) + S^{(2,-)} \end{pmatrix} \quad \text{on } I'_{\theta_2} \tag{1.7.7}$$

where $S^{(2,\pm)} \in \Psi^{-\infty}_{2,lsc}$ on I'_{θ_2}. Summarizing this, allows us to write

$$\begin{aligned} P_{12}(\partial_z \tfrac{1}{\rho}\partial_z + A_\rho) &= (\partial_z - iB_+)(P_{12}\tfrac{1}{\rho}\partial_z + P_{11}) + \sum_{j=0}^{2} R_j^{(+)}\partial_z^{2-j} \\ P_{22}(\partial_z \tfrac{1}{\rho}\partial_z + A_\rho) &= (\partial_z - iB_-)(P_{22}\tfrac{1}{\rho}\partial_z + P_{21}) + \sum_{j=0}^{2} R_j^{(-)}\partial_z^{2-j} \end{aligned}$$

on I'_{θ_2} where $R_j^{(\pm)} = R_j^{(\pm)}(x, z, D_t, D_x)$ have symbols in $\widetilde{S}^{-\infty}_{lsc}$ on I'_{θ_2}, $j = 0, 1, 2$.

Otherwise, in view of Theorem 1.5.2, the operator $L = \partial_z \frac{1}{\rho}\partial_z + A_\rho$ can be written in the following form

$$L = \big(\partial_z + A_{11}\big)\frac{1}{\rho}\big(\partial_z + A_{21}\big) + \sum_{j=1}^{2} R_{j,1}^{(\infty)}\partial_z^{2-j} \quad \text{on } I'_{\theta_2} \tag{1.7.8}$$

with the symbols of $A_{j1} = A_{j1}(x, z, D_t, D_x)$ in $\widetilde{S}^1_{lsc}$ on I'_{θ_2}, $j = 1, 2$. Here, the representatives of A_{11} and $-A_{21}$ have the same principal symbol equal to $-i(\sqrt{a_\varepsilon})_\varepsilon$, where $a_\varepsilon := \frac{\tau^2}{c_\varepsilon^2(x,z)} - |\xi|^2$ is the principal symbol of $\rho_\varepsilon A_{\rho,\varepsilon}$. Also, we obtain

$$L = \big(\partial_z + A_{12}\big)\frac{1}{\rho}\big(\partial_z + A_{22}\big) + \sum_{j=1}^{2} R_{j,2}^{(\infty)}\partial_z^{2-j} \quad \text{on } I'_{\theta_2} \tag{1.7.9}$$

where the symbols of $A_{j2} = A_{j2}(x, z, D_t, D_x)$ are in $\widetilde{S}^1_{lsc}$ on I'_{θ_2}, $j = 1, 2$. At this point, the representatives of A_{12} and $-A_{22}$ have the same principal symbol equal to $i(\sqrt{a_\varepsilon})_\varepsilon$. Expansion of the right-hand side of (1.7.8), respectively (1.7.9), results in

$$\partial_z\frac{1}{\rho}\partial_z + A_\rho = \partial_z\frac{1}{\rho}\partial_z + \big(A_{1k}\frac{1}{\rho} + \frac{1}{\rho}A_{2k}\big)\partial_z + \partial_z\big(\frac{1}{\rho}A_{2k}\big) + A_{1k}\frac{1}{\rho}A_{2k} + \sum_{j=1}^{2} R^{(\infty)}_{j,k}\partial_z^{2-j}$$

on I'_{θ_2}, $k = 1,2$. Equating coefficients gives $A_{2k} = -\rho A_{1k}\frac{1}{\rho}$ modulo an operator $\widetilde{S}^{-\infty}_{lsc}$ on I'_{θ_2}, $k = 1,2$. Using this, equations (1.7.8) and (1.7.9) now read

$$\begin{aligned} \partial_z\frac{1}{\rho}\partial_z + A_\rho &= \big(\partial_z + A_{11}\big)\big(\frac{1}{\rho}\partial_z - A_{11}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{1,lsc}) &&\text{(1.7.8')}\\ &= \big(\partial_z + A_{12}\big)\big(\frac{1}{\rho}\partial_z - A_{12}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{1,lsc}) &&\text{(1.7.9')}\end{aligned}$$

on I'_{θ_2}. At this point we choose P_{12} and P_{22} such that the requirements of Theorem 1.4.16 are satisfied on I'_{θ_2} and we denote by P_{12}^{-1} and P_{22}^{-1} the corresponding parametrices. Inserting $P_{12}^{-1}P_{12}$ into (1.7.8') and $P_{22}^{-1}P_{22}$ into (1.7.9') yields

$$\begin{aligned} \partial_z\frac{1}{\rho}\partial_z + A_\rho &= \big(\partial_z + A_{11}\big)P_{12}^{-1}\big(P_{12}\frac{1}{\rho}\partial_z - P_{12}A_{11}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{2,lsc})\\ &= \big(\partial_z + A_{12}\big)P_{22}^{-1}\big(P_{22}\frac{1}{\rho}\partial_z - P_{22}A_{12}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{2,lsc}).\end{aligned}$$

on I'_{θ_2}. We define

$$\begin{aligned} -iB_+ &:= P_{12}A_{11}P_{12}^{-1} + P_{12}\partial_z\big(P_{12}^{-1}\big) \qquad \text{on } I'_{\theta_2}\\ -iB_- &:= P_{22}A_{12}P_{22}^{-1} + P_{22}\partial_z\big(P_{22}^{-1}\big) \qquad \text{on } I'_{\theta_2}\end{aligned} \tag{1.7.10}$$

to denote the uniquely determined solutions to

$$\begin{aligned} \big(\partial_z + A_{11}\big)P_{12}^{-1} &= P_{12}^{-1}\big(\partial_z - iB_+\big)\\ \big(\partial_z + A_{12}\big)P_{22}^{-1} &= P_{22}^{-1}\big(\partial_z - iB_-\big),\end{aligned}$$

modulo logarithmic slow scale smoothing operators on I'_{θ_2}. Consequently, $B_\pm$ are operators with symbols in $\widetilde{S}^1_{lsc}$ on I'_{θ_2} with real principal symbols. Note that $B_\pm$ are only prescribed on I'_{θ_2}. Thus

$$\begin{aligned} \partial_z\frac{1}{\rho}\partial_z + A_\rho &= P_{12}^{-1}\big(\partial_z - iB_+\big)\big(P_{12}\frac{1}{\rho}\partial_z - P_{12}A_{11}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{2,lsc})\\ &= P_{22}^{-1}\big(\partial_z - iB_-\big)\big(P_{22}\frac{1}{\rho}\partial_z - P_{22}A_{12}\frac{1}{\rho}\big) \qquad (\mathrm{mod}\ \Psi^{-\infty}_{2,lsc})\end{aligned}$$

on I'_{θ_2}. Further, if we choose for P from above $P_{11} := -P_{12}A_{11}\frac{1}{\rho}$ and $P_{21} := -P_{22}A_{12}\frac{1}{\rho}$ we obtain

$$P = \begin{pmatrix} -P_{12}A_{11}\frac{1}{\rho} & P_{12} \\ -P_{22}A_{12}\frac{1}{\rho} & P_{22} \end{pmatrix} = \begin{pmatrix} P_{12} & 0 \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} -A_{11}\frac{1}{\rho} & 1 \\ -A_{12}\frac{1}{\rho} & 1 \end{pmatrix} \qquad \text{on } I'_{\theta_2} \tag{1.7.11}$$

where P_{12} and P_{22} are logarithmic slow scale elliptic pseudodifferential operators (in the sense of Theorem 1.4.16) on I'_{θ_2}.

Since the principal symbol of A_{11} is $-i(\sqrt{a_\varepsilon})_\varepsilon$ and the principal symbol of A_{12} is $i(\sqrt{a_\varepsilon})_\varepsilon$, it follows that P is logarithmic slow scale elliptic (in the above sense) whenever P_{12} and P_{22} satisfy the requirements of Theorem 1.4.16. Furthermore, the approximative inverse matrix Q of P is given by

$$Q = \begin{pmatrix} -C & C \\ -A_{12}\frac{1}{\rho}C & 1 + A_{12}\frac{1}{\rho}C \end{pmatrix} \begin{pmatrix} P_{12}^{-1} & 0 \\ 0 & P_{22}^{-1} \end{pmatrix} \qquad \text{on } I'_{\theta_2}$$

where C is the generalized parametrix of $A_{11}\frac{1}{\rho} - A_{12}\frac{1}{\rho}$ in the sense of Theorem 1.4.16. We have therefore found appropriate operator-valued matrices P, Q, R and operators $B_\pm$ solving (1.7.4) on I'_{θ_2}.

Outside I'_{θ_2}, we choose P logarithmic slow scale elliptic and of the same order as in (1.7.11) on I'_{θ_2}. Then Lemma 1.7.1 is satisfied also on the complement of I'_{θ_2}. □

We note that the derived one-way wave equations are not unique. Only the principal symbol remains the same for any choice of P_{12} and P_{22}. Recall that the principal symbol of the operator $B_\pm$ is directly related to the generalized wave front set of the full wave equation. In the smooth setting it turns out, that in order to also describe the wave amplitudes, the subprincipal symbols of $B_\pm$ are needed (cf. [58], [44]).

Moreover, by (1.5.9), (1.5.10) and (1.7.10) the zeroth-order terms of $B_\pm$ also depend on the partial derivatives with respect to z of the coefficients, i.e. $\partial_z c_\varepsilon(x,z)$ and $\partial_z \rho_\varepsilon(x,z)$. In the following, we will show that for an appropriate choice of the operator Q such zeroth-order terms can be eliminated. In particular, we have the following result.

Lemma 1.7.2. *Let Q, P, $B_\pm$ and R as in Lemma 1.7.1. Then there is a choice of Q such that the operators $B_\pm$ become self-adjoint on I'_{θ_2}. Moreover, the symbols of the operators $B_\pm$ are of the form*

$$b_\pm(x,z,\tau,\xi) = \pm \left(b + \frac{i}{2b} \sum_{j=1}^{n-1} \frac{\partial b}{\partial \xi_j} \frac{\partial b}{\partial x_j} \right) + \mathit{order}(-1) \qquad \text{on } I'_{\theta_2}. \tag{1.7.12}$$

Proof. In the following, we will restrict ourselves to the set I'_{θ_2} and show that the operators $B_\pm$ can be chosen self-adjoint there. The result follows by an argument used in [51].

We will tackle the problem by choosing generalized classical pseudodifferential operators P_{12}, P_{22} such that

$$Q^{-1} = P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.7.13}$$

is valid modulo a smoothing operator matrix and where Q^* denotes the adjoint of Q. We therefore explicitly compute the right hand side and obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i(P_{12}^{-1})^* C^* \frac{1}{\rho} A_{12}^* & i(P_{12}^{-1})^* C^* \\ -i(P_{22}^{-1})^* - i(P_{22}^{-1})^* C^* \frac{1}{\rho} A_{12}^* & i(P_{22}^{-1})^* C^* \end{pmatrix}.$$

Then (1.7.13) is equivalent to the following two systems of equations

$$\begin{cases} -P_{12} A_{11} \frac{1}{\rho} = -i(P_{12}^{-1})^* C^* \frac{1}{\rho} A_{12}^* \\ \quad P_{12} = i(P_{12}^{-1})^* C^* \end{cases}$$

and

$$\begin{cases} -P_{22} A_{12} \frac{1}{\rho} = -i(P_{22}^{-1})^* - i(P_{22}^{-1})^* C^* \frac{1}{\rho} A_{12}^* \\ \quad P_{22} = i(P_{22}^{-1})^* C^*. \end{cases}$$

These systems are equivalent to solving the following equations

$$i(A_{11}\frac{1}{\rho} - \frac{1}{\rho} A_{11}^*) = P_{12}^{-1}(P_{12}^{-1})^* \qquad -i(A_{12}\frac{1}{\rho} - \frac{1}{\rho} A_{12}^*) = P_{22}^{-1}(P_{22}^{-1})^*$$

where we used the fact that $\frac{1}{\rho} A_{11}^* - \frac{1}{\rho} A_{12}^* = (C^{-1})^* = (C^*)^{-1}$.

As a next step, we will choose P_{12}^{-1} and P_{22}^{-1} equal to the self-adjoint square root of $i(A_{11}\frac{1}{\rho} - \frac{1}{\rho} A_{11}^*)$ and $-i(A_{12}\frac{1}{\rho} - \frac{1}{\rho} A_{12}^*)$ respectively. We show that P_{12}^{-1} can be chosen in such a way. The result for P_{22}^{-1} follows by the same method. We first note that the operator $T := i(A_{11}\frac{1}{\rho} - \frac{1}{\rho} A_{11}^*)$ is self-adjoint. As we are searching for the square root, we are interested in an asymptotic expansion $X = \sum_{j=0}^{\infty} X^{(j)}$ with $X^{(j)} \in \Psi_{lsc}^{1/2-j}$ with $X \sim \pm T^{1/2}$ in $\widetilde{S}^1_{cl,lsc}$.

Therefore, let $X_\varepsilon^{(0)} := \big((b_\varepsilon \frac{1}{\rho_\varepsilon})^{1/2} + (b_\varepsilon \frac{1}{\rho_\varepsilon})^{1/2})^*\big)/2$ be self-adjoint in $S_{lsc}^{1/2}$ where b_ε is the principal symbol of $iA_{11,\varepsilon}$. Then

$$X_\varepsilon^{(0)2} = T_\varepsilon + R_\varepsilon^{(0)} \qquad \text{with } R_\varepsilon^{(0)} \in S^0_{lsc}.$$

Moreover, the remainder $R_\varepsilon^{(0)}$ is self-adjoint. Suppose we have $X_\varepsilon^{(j)}$, $j = 0, \ldots, N$, such that

$$\sum_{j=0}^{N} X_\varepsilon^{(j)} = T_\varepsilon + R_\varepsilon^{(N)} \tag{1.7.14}$$

with $\left(R_\varepsilon^{(N)}\right)_\varepsilon \in S^{-N}_{lsc}$ self-adjoint. Then with $X_\varepsilon^{(N+1)} := -\frac{1}{2\sqrt{b_\varepsilon/\rho_\varepsilon}} R_\varepsilon^{(N)}$, $X_\varepsilon^{(N+1)}$ is self-adjoint since $b_\varepsilon/\rho_\varepsilon$ is real-valued and (1.7.14) holds with $N+1$ instead of N.

We recall from (1.7.4) that

$$P \begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} Q = \partial_z + P \begin{pmatrix} 0 & -\rho \\ A_\rho & 0 \end{pmatrix} Q + P\frac{\partial Q}{\partial z} = \partial_z - i \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} + R.$$

Using (1.7.13) we obtain for the second term in the middle expression

$$P \begin{pmatrix} 0 & -\rho \\ A_\rho & 0 \end{pmatrix} Q = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[P \begin{pmatrix} 0 & -\rho \\ A_\rho & 0 \end{pmatrix} Q \right]^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now this term is the sum of an anti-self-adjoint diagonal matrix and a self-adjoint off-diagonal matrix. Also we have

$$P\frac{\partial Q}{\partial z} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\; P\frac{\partial Q}{\partial z} \; \right]^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So this expression is again the sum of a self-adjoint off-diagonal part and an anti-self-adjoint diagonal part.

Hence the operators $-iB_\pm$ are anti-self-adjoint and therefore $B_\pm$ are self-adjoint.

It remains to show that $B_\pm$ have the form (1.7.12). In order to compute the zeroth order term of B_+, we need to incorporate the principal and the subprincipal symbol of P_{12}^{-1}. We define the operator $P_{12}^{-1} = \mathrm{OP}(\sqrt{2} a_\rho^{1/4} \rho^{-1/4}) + R_1$ of order $1/2$ and the symbol of R_1 is in

$\widetilde{S}_{lsc}^{-1/2}$. Then the parametrix P_{12} has the following asymptotic expansion

$$\begin{aligned}P_{12,\varepsilon} = \psi\Big[&\frac{1}{\sqrt{2}} a_{\rho,\varepsilon}^{-1/4}\rho_\varepsilon^{1/4} - \frac{1}{2} R_{1,\varepsilon} a_{\rho,\varepsilon}^{-1/2}\rho_\varepsilon^{1/2} + \\ &+ \frac{1}{\sqrt{2}}\Big(\frac{i}{8} a_\varepsilon^{-3/2}\sum_{j=1}^{n-1}\frac{\partial a_\varepsilon}{\partial x_j}\xi_j - \frac{i}{4}\frac{1}{\rho_\varepsilon} a_\varepsilon^{-1/2}\sum_{j=1}^{n-1}\frac{\partial \rho_\varepsilon}{\partial x_j}\xi_j\Big)\rho_\varepsilon^{1/2} a_\varepsilon^{-3/4}\Big] + \\ &+ \mathrm{order}(-5/2).\end{aligned}$$

where ψ is a cut-off function as in the proof of Theorem 1.4.16. We obtain

$$P_{12,\varepsilon}\frac{\partial P_{12,\varepsilon}^{-1}}{\partial z} = \frac{1}{4}\Big(\frac{\partial a_{\rho,\varepsilon}}{\partial z}\frac{1}{a_{\rho,\varepsilon}} - \frac{\partial \rho_\varepsilon}{\partial z}\frac{1}{\rho_\varepsilon}\Big) + \mathrm{order}(-1)$$

and after some calculation

$$P_{12,\varepsilon}A_{11,\varepsilon}P_{12,\varepsilon}^{-1} = A_{11,\varepsilon} - \frac{1}{2}\frac{1}{\rho_\varepsilon} a_\varepsilon^{-1/2}\sum_{j=1}^{n-1}\frac{\partial \rho_\varepsilon}{\partial x_j}\xi_j + \mathrm{order}(-1).$$

Using (1.5.9) and (1.5.10) from Section 1.5 we get

$$-ib_{+,\varepsilon} = -ib_\varepsilon + \frac{1}{2b_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial b_\varepsilon}{\partial \xi_j}\frac{\partial b_\varepsilon}{\partial x_j} + \mathrm{order}(-1)$$

where $b_\varepsilon := \sqrt{a_\varepsilon}$. Similarly, if we write $P_{22}^{-1} := \mathrm{OP}(-\sqrt{2}a_\rho^{1/4}\rho^{-1/4}) + R_2$, $R_2 \in \widetilde{S}_{lsc}^{-1/2}$, then one can show that

$$-ib_{-,\varepsilon} = ib_\varepsilon - \frac{1}{2b_\varepsilon}\sum_{j=1}^{n-1}\frac{\partial b_\varepsilon}{\partial \xi_j}\frac{\partial b_\varepsilon}{\partial x_j} + \mathrm{order}(-1).$$

So the proof is complete. □

Remark 1.7.3. We derived the decoupled system (1.7.2) by using the two different exact factorizations of the full wave equation. Once factorized the equation, we were able to write down the normalization matrix explicitly. As a result, we obtained the decomposition in two first-order hyperbolic wave equations. Here, the factorizations were shown by induction and we got a direct connection between the exact factorizations of the wave equation and the decoupled first-order system.

This method is different to the one used in [51]. There, the first-order wave equations were obtained by recursively defining a normalization matrix which diagonalizes the matrix in (1.7.3) in the sense of (1.7.4). Moreover, no exact factorization of the wave equation is directly used as input data.

With this in mind, we are now in the position to state the following main theorem.

Theorem 1.7.4. *Let $L = \partial_z \frac{1}{\rho}\partial_z + A_\rho(x,z,D_t,D_x)$ and $U, F \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$. Then there are first-order pseudodifferential operators $B_\pm$ and a globally elliptic pseudodifferential operator matrix Q as in Lemma 1.7.1 such that the equation*

$$LU \equiv F \quad \textit{microlocally on } I_{\theta_2} \tag{1.7.15}$$

holds if and only if

$$L_{0,\pm}u_\pm := \big(\partial_z - iB_\pm(x,z,D_t,D_x)\big)u_\pm \equiv f_\pm \;\; \textit{microlocally on } I_{\theta_2} \tag{1.7.16}$$

where

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} := Q^{-1}\begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix}, \qquad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := Q^{-1}\begin{pmatrix} 0 \\ F \end{pmatrix}.$$

Here the operators $B_\pm$ are as in (1.7.10) on I'_{θ_2} and Q is as in (1.7.11) on I'_{θ_2}. Moreover, by Lemma 1.7.2 the matrix Q can be chosen in such a way that $B_\pm$ become self-adjoint on I'_{θ_2}.

Proof. We define a microlocal cut-off $\psi = \psi(x,z,D_t,D_x,D_z)$ around $\xi = \tau = 0$. In particular let $\psi(x,z,\tau,\xi,\zeta)$ be a homogeneous symbol of degree 0 in (τ,ξ,ζ) outside a compact set that satisfies

$$\psi(x,z,\tau,\xi,\zeta) = \begin{cases} 0 & |\zeta| > 3C|\tau| \\ 1 & |\zeta| < 2C|\tau|, \; |\zeta| > 1. \end{cases}$$

We note that ψ satisfies the requirements of [16, Theorem 18.1.35] since it vanishes if $|\zeta| > 3C|\tau|$.

So, given an operator $W \in \Psi^m_{lsc}$ on I'_{θ_2} the operator given by ψW is in Ψ^m_{lsc} on I_{θ_2} by [31, Theorem 18.1.35]. Also, the symbol of ψW is equal to the symbol of W modulo $\Psi^{-\infty}_{lsc}$ on I_{θ_2}.

As already announced in Section 1.5, the operators L_j, $j = 1,2$ in (1.5.3) are not pseudodifferential operators on $\mathbb{R}^{n+1}$ (only in (t,x)). But, when applied to the cut-off function $\psi = \psi(x,z,D_t,D_x,D_z)$, we see that ψL_j equals $L_j\psi$ and $L_j\psi$ equals L_j modulo $\Psi^{-\infty}_{lsc}$ on I_{θ_2}, $j = 1,2$.

We may regard (1.7.6) and (1.7.7) as microlocal equations on I_{θ_2}.

Then $LU \equiv F$ microlocally on I_{θ_2} if and only if

$$\psi P \begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} QP \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \equiv P \begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \qquad \text{microlocally on } I_{\theta_2},$$

$U \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$. On the other hand we achieve

$$\psi \Big[\begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} + R \Big] P \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \equiv \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} P \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix}$$

microlocally on I_{θ_2} for $U \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$. Thus

$$P \begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \equiv \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} P \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix}$$

holds microlocally on I_{θ_2}. Therefore equation (1.7.15) is equivalent to

$$P \begin{pmatrix} 0 \\ F \end{pmatrix} \equiv \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} P \begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \qquad \text{microlocally on } I_{\theta_2}.$$

□

1.8 Approximated first-order wave equations

One particular technique for wave modeling is the one-way wave equation which is often used since it requires only one boundary condition, as it is a first-order evolution equation. In the case that the background data is smooth, it also provides a way to propagate in a predetermined direction ([43]). Our background data involves generalized functions and due to the lack of an available theory connecting propagation of singularities with a generalized Hamiltonian flow we need a substitute.

With view to Theorem 1.7.4, we now introduce approximated first-order equations to (1.7.16) of the form

$$L_\pm u_\pm := (\partial_z - iB_\pm(x,z,D_t,D_x) + C(x,z,D_t,D_x))u_\pm = 0 \qquad z > z_0 \tag{1.8.1}$$

where the operator C has a symbol $(C_\varepsilon)_\varepsilon$ in S^1_{lsc} and serves as a correction term which should suppress certain singularities of the solutions. In detail, we choose the principal symbol $c = [(c_\varepsilon)_\varepsilon]$ of C such that

$$\begin{aligned} c_\varepsilon(x,z,\tau,\xi) &= 0 && \text{on } I'_{\theta_1} \\ c_\varepsilon(x,z,\tau,\xi) &\geqslant \eta(\tau^2+|\xi|^2)^{\frac{1}{2}} && \text{outside } I'_{\theta_2}. \end{aligned}$$

for some positive constant $\eta \in \mathbb{R}$, all $\varepsilon \in (0,1]$ and where $0 < \theta_1 < \theta_2 < \pi/2$. The precise definition of the damping operator will be given in the subsection below. In this case, the operator C is chosen logarithmic slow scale elliptic outside I'_{θ_2} and therefore C is also slow scale elliptic there. Without the operator C in (1.8.1) the operators $L_\pm$ reduce to $L_{0,\pm}$ and are standard first-order hyperbolic operators on I'_{θ_2}. In the region where c is logarithmic slow scale elliptic the principal symbols of $L_\pm$ are logarithmic slow scale elliptic and therefore c leads to a suppression of singularities of the solutions to $L_\pm u_\pm = 0$, $z > z_0$. In detail, we have for the singularities of the solutions to $L_\pm u_\pm = 0$ $z > z_0$ that

$$\begin{aligned} \mathrm{WF}_g(u_\pm) &\subseteq \mathrm{WF}_g(L_\pm u_\pm) \cup \mathrm{Ell}^c_{lsc}(L_\pm) \subseteq \\ &\subseteq \mathrm{Ell}^c_{lsc}(L_{0,\pm}) \cap \mathrm{Ell}^c_{lsc}(C) \subseteq \\ &\subseteq \{(t,x,z,\tau,\xi;\zeta) \in T^*\mathbb{R}^{n+1}\backslash 0 \mid (x,z,\tau,\xi) \in I'_{\theta_2}, |\zeta| < C|\tau|\} \qquad z > z_0 \end{aligned}$$

since $L_\pm u_\pm = 0$ for $z > z_0$. So, C suppresses singularities outside I'_{θ_2}. Moreover, having solutions $u_\pm$ to $L_\pm u_\pm = 0$, $z > z_0$ with initial conditions $u_\pm(z_0) = u_{\pm,0}$ then they are also approximations to $L_{0,\pm} u_\pm \equiv 0$ microlocally on I_{θ_1} for $z > z_0$ with the same initial conditions. This can be seen by

$$\mathrm{WF}_g(L_\pm u_\pm - L_{0,\pm} u_\pm) = \mathrm{WF}_g(C u_\pm) \subseteq \mathrm{WF}_g(u_\pm) \cap \mu\mathrm{supp}_g(C) \qquad z > z_0$$

where $\mu\mathrm{supp}_g(C) \subseteq I'^c_{\theta_1}$ and therefore

$$\mathrm{WF}_g(L_\pm u_\pm - L_{0,\pm} u_\pm) \cap I_{\theta_1} = \varnothing \qquad z > z_0.$$

We conclude that $L_\pm u_\pm \equiv L_{0,\pm} u_\pm$ microlocally on I_{θ_1}, $z > z_0$. Then

$$\psi \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \equiv \vec{0} \qquad \text{microlocally on } I_{\theta_1},\ z > z_0$$

and by the proof of Theorem 1.7.4 this is equivalent to

$$\psi \left[P \begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix} Q - R \right] \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \equiv \vec{0} \qquad \text{microlocally on } I_{\theta_1},\ z > z_0. \tag{1.8.2}$$

Since the entries of the pseudodifferential matrix R are operators with symbol in $S^{-\infty}_{lsc}$ on I'_{θ_2}, we have

$$R\begin{pmatrix} u_+ \\ u_- \end{pmatrix} \equiv \vec{0} \qquad \text{microlocally on } I_{\theta_1},\ z > z_0.$$

We now define U being microlocally equivalent to $Q_+u_+ + Q_-u_-$ on $I_{\theta_1}, z > z_0$ where we have set $Q_+ := Q_{11}$ and $Q_- := Q_{12}$, i.e.

$$U := Q_+u_+ + Q_-u_- + \widetilde{f}$$

for some $\widetilde{f} \equiv 0$ microlocally on $I_{\theta_1},\ z > z_0$. Then

$$\begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \equiv Q\begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \text{microlocally on } I_{\theta_1},\ z > z_0$$

since $u_+ \equiv Q_+u_+ \equiv Q_{21}u_+$ and $u_- \equiv Q_-u_- \equiv Q_{22}u_-$ microlocally on $I_{\theta_1},\ z > z_0$. Hence (1.8.2) holds if and only if

$$\psi P\begin{pmatrix} \partial_z & -\rho \\ A_\rho & \partial_z \end{pmatrix}\begin{pmatrix} U \\ \frac{1}{\rho}\partial_z U \end{pmatrix} \equiv \vec{0} \qquad \text{microlocally on } I_{\theta_1},\ z > z_0$$

which means that $LU \equiv 0$ microlocally on $I_{\theta_1},\ z > z_0$.

In summary, we get the following. If the functions $u_\pm$ solve the problems

$$L_\pm u_\pm = 0 \qquad \text{on } (z_0, Z) \times \mathbb{R}^n \tag{1.8.3}$$
$$u_\pm(z_0) = u_{\pm,0} \qquad \text{on } \mathbb{R}^n \tag{1.8.4}$$

then $U := Q_+u_+ + Q_-u_- + \widetilde{f}$ for some $\widetilde{f} \equiv 0$ microlocally on $I_{\theta_1},\ z \in (z_0, Z)$ solves

$$LU \equiv 0 \qquad \text{microlocally on } I_{\theta_1},\ z \in (z_0, Z).$$

1.8.1 Requirements on the damping operator

We now give sufficient conditions on the damping operator C for the operators $L_\pm$ such that the Cauchy problems in (1.8.3)-(1.8.4) are well-posed (cf. [51, 52]). The well-posedness will be the content of the next section.

We define the principal symbol of c of C by

$$c_\varepsilon(x,z,\tau,\xi) := \omega(x,z,\tau,\xi)(1-\chi_\varepsilon(x,z,\tau,\xi))$$

with χ_ε as in Subsection 1.5.1 and ω is a smooth symbol homogeneous of degree 1 in (τ,ξ) and bounded below by some constant times $(\tau^2+|\xi|^2)^{1/2}$. Then $(c_\varepsilon)_\varepsilon \in S^1_{lsc}(\mathbb{R}^n\times\mathbb{R}^n)$. Now, since $c_\varepsilon(x,z,\tau,\xi)$ is real-valued and homogeneous of degree one, there exists a self-adjoint symbol $(C_\varepsilon)_\varepsilon \in S^1_{lsc}(\mathbb{R}^n\times\mathbb{R}^n)$ with principal symbol equal to c_ε. This can easily seen by the following.

Defining

$$C_\varepsilon := \frac{c_\varepsilon + c_\varepsilon^*}{2},$$

then C_ε is the self-adjoint symbol as desired (cf. next section).

In the following, we will work with such a self-adjoint damping operator $C_\varepsilon = C_\varepsilon^{(0)} + C_\varepsilon^{(1)}$ of order one with parameter z and $(C_\varepsilon^{(k)})_\varepsilon \in S^{1-k}_{cl,lsc}$ $(k=0,1)$. We set $C_\varepsilon^{(0)} := c_\varepsilon$ and get

$$\begin{aligned}|\partial^\beta_{(x,z)}\partial^\alpha_{(\tau,\xi)}C_\varepsilon^{(k)}(x,z,\tau,\xi)| &\leqslant C\omega_\varepsilon(1+|(\tau,\xi)|)^{1-k-|\alpha|} \leqslant \\ &\leqslant C\omega_\varepsilon(1+|(\tau,\xi)|)^{-|\alpha|-k+\frac{|\alpha|+|\beta|}{L}}(1+c_\varepsilon(x,z,\tau,\xi))^{1-\frac{|\alpha|+|\beta|}{L}}\end{aligned}$$

on the support of $C_\varepsilon^{(0)} = c_\varepsilon$ since we have $1+c_\varepsilon(x,z,\tau,\xi) \geqslant 1+\eta(\tau^2+|\xi|^2)^{1/2}$ there. We thus get,

$$\begin{gathered}\forall\alpha,\beta\in\mathbb{N}^n\ \forall L>|\alpha|+|\beta|+2k\ \exists\omega_\varepsilon\in\Pi_{lsc}\ \exists C>0\ \exists\eta\in(0,1]: \\ |\partial^\beta_{(x,z)}\partial^\alpha_{(\tau,\xi)}C_\varepsilon^{(k)}(x,z,\tau,\xi)| \leqslant C\omega_\varepsilon(1+|(\tau,\xi)|)^{-|\alpha|-k+\frac{|\alpha|+|\beta|}{L}}(1+c_\varepsilon(x,z,\tau,\xi))^{1-\frac{|\alpha|+|\beta|}{L}}\end{gathered}$$

for all $\varepsilon\in(0,\eta]$, $k=0,1$.

1.9 Well-posedness of the approximated first-order equations

In this section we will mainly follow the ideas made in [52] and show the well-posedness of the Cauchy problems (1.8.3)-(1.8.4). To avoid an overkill through parameters, we will reduce the problem (1.8.3)-(1.8.4) and show well-posedness for the following generalized

Cauchy problem

$$\begin{aligned} Pu &:= (\partial_z - iA(z,x,D_x) + B(z,x,D_x))u = f && \text{on } (0,Z)\times\mathbb{R}^n && (1.9.1)\\ u(0,.) &= u_0 && \text{on } \mathbb{R}^n. && (1.9.2)\end{aligned}$$

where $u = u(z,x) \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ and $f = f(z,x) \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$ under the following assumptions:

(i) A is a generalized pseudodifferential operator with symbol in $\widetilde{S}^1_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$.

(ii) B is a generalized pseudodifferential operator of order $\gamma > 0$ with non-negative real homogeneous principal symbol $b = B^{(0)} = B^{(0)}(z,x,\xi)$, i.e. b is homogeneous for $|\xi| \geqslant 1$ and in $\widetilde{S}^{-\infty}_{lsc}$ for $|\xi| \leqslant 1$.

(iii) Moreover, there are representatives $(B^{(0)}_\varepsilon)_\varepsilon$ and $(B^{(1)}_\varepsilon)_\varepsilon$ of $B^{(0)}$ and $B^{(1)} := B - B^{(0)}$ respectively such that the derivatives of $(B^{(0)}_\varepsilon)_\varepsilon$ and $(B^{(1)}_\varepsilon)_\varepsilon$ can be estimated as follows: $\forall\alpha\in\mathbb{N}^n$, $\forall\beta\in\mathbb{N}^{n+1}$, $\exists\omega_{0,\varepsilon},\omega_{1,\varepsilon}\in\Pi_{lsc}$ such that

$$|\partial^\beta_{(x,z)}\partial^\alpha_\xi B^{(0)}_\varepsilon(x,z,\xi)| = \mathcal{O}(\omega_{0,\varepsilon})(1+|\xi|)^{-|\alpha|+\frac{|\alpha|+|\beta|}{L}\gamma}(1+B^{(0)}_\varepsilon(x,z,\xi))^{1-\frac{|\alpha|+|\beta|}{L}}$$

for $|\alpha|+|\beta| < L$ as $\varepsilon\to 0$ and

$$|\partial^\beta_{(x,z)}\partial^\alpha_\xi B^{(1)}_\varepsilon(x,z,\xi)| = \mathcal{O}(\omega_{1,\varepsilon})(1+|\xi|)^{-|\alpha|-1+\frac{|\alpha|+|\beta|+2}{L}\gamma}(1+B^{(0)}_\varepsilon(x,z,\xi))^{1-\frac{|\alpha|+|\beta|+2}{L}}$$

for $2+|\alpha|+|\beta| < L$ as $\varepsilon\to 0$.

Remark 1.9.1. In this section we construct a square root for the operator $1+B$ modulo a smoothing operator with symbol in $\widetilde{S}^{-\infty}_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$. Using this square root we show that the Cauchy problem (1.9.1)-(1.9.2) has a unique solution that satisfies an energy estimate.

Lemma 1.9.2. *Assume that $B(z,x,D_x)$ is a self-adjoint generalized pseudodifferential operator with symbol in $\widetilde{S}^\gamma_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$, $\gamma>0$, that satisfies (ii), (iii) and $2\gamma < L$. Then, there exists $Q(z,x,D_x)\in\Psi^{\gamma/2}_{1-\frac{\gamma}{L},\frac{\gamma}{L},lsc}(\mathbb{R}^n)$ with $\partial^j_z Q(z,x,D_x)\in\Psi^{\gamma/2-j\gamma/L}_{1-\frac{\gamma}{L},\frac{\gamma}{L},lsc}(\mathbb{R}^n)$ such that Q is self-adjoint and $Q^2 = 1+B+R$ where R is a generalized pseudodifferential operator with symbol in $\widetilde{S}^{-\infty}_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$.*

Remark 1.9.3. In the proof we will make use of the following fact.

Let $f:[0,\infty]\to\mathbb{R}$ be a classical symbol of order δ, i.e. $\forall k\in\mathbb{N}$, $\exists C_k>0$ such that $|f^{(k)}(y)|\leqslant C_k(1+y)^{\delta-k}$, then $f\circ B^{(0)}$ is a symbol in $\widetilde{S}^{\gamma\max(\delta,0)}_{1-\frac{\gamma}{L},\frac{\gamma}{L},lsc}$.

First note that $|f(B^{(0)})| \leqslant C(1+B^{(0)})^{\delta} \leqslant C\omega_\varepsilon(1+|\xi|)^{\gamma \max(\delta,0)}$.

So let $(\alpha,\beta) \neq \vec{0}$. Using Faà di Bruno's formula, we get that the expression $\partial^{\beta}_{(x,z)}\partial^{\alpha}_{\xi} f(B^{(0)})$ is a sum of terms of the form

$$c f^{(k)}(B^{(0)}) \prod_{j=1}^{k} \partial^{\beta_j}_{(x,z)}\partial^{\alpha_j}_{\xi} B^{(0)} \tag{1.9.3}$$

for some constant c and such that $\sum_j \alpha_j = \alpha$, $\sum_j \beta_j = \beta$ and $(\alpha_j,\beta_j) \neq 0$. Since $B^{(0)}$ satisfies (iii) one can estimate the expression in (1.9.3) by

$$C\omega_\varepsilon(1+|\xi|)^{-|\alpha|+\frac{|\alpha|+|\beta|}{L}\gamma}(1+B^{(0)})^{\delta-\frac{|\alpha|+|\beta|}{L}} \leqslant C\omega_\varepsilon(1+|\xi|)^{\gamma\max(\delta,0)-|\alpha|+\frac{|\alpha|+|\beta|}{L}\gamma}$$

where the constant $C>0$ and the logarithmic slow scale net ω_ε depend on α and β.

Proof. The proof of Lemma 1.9.2 follows the same lines as in [52, Lemma 3]. □

In order to get the energy estimates, we will adapt a version of the sharp Gårding inequality of Hörmander as stated in [31, Theorem 18.1.14].

Lemma 1.9.4 (Sharp Gårding inequality)**.** *Suppose that $a \in \widetilde{S}^{1}_{lsc}(\mathbb{R}^n \times \mathbb{R}^n)$ has a representative $(a_\varepsilon)_\varepsilon$ such that $\exists \eta \in (0,1]$*

$$\operatorname{Re} a_\varepsilon(x,\xi) \geqslant 0 \qquad \forall \varepsilon \in (0,\eta].$$

Then there exists a logarithmic slow scale net $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$ and a constant $C>0$ such that

$$\operatorname{Re}(a_\varepsilon(x,D_x)u,u) \geqslant -C\omega_\varepsilon \|u\|^2_{L^2} \tag{1.9.4}$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $\varepsilon \in (0,\eta]$.

The proof follows the same lines as in [31, Theorem 18.1.14]. We only give some supplementary explanations.

Proof. Let $r \in \widetilde{S}^{0}_{lsc}$ and $(r_\varepsilon)_\varepsilon \in r$ be a representative. Then, there exist a logarithmic slow scale net $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$, a constant $C>0$ and an $\eta \in (0,1]$ such that

$$|(r_\varepsilon(x,D)u,u)| \leqslant C\omega_\varepsilon\|u\|^2_{L^2} \qquad \varepsilon \in (0,\eta]$$

and

$$\operatorname{Re}(r_\varepsilon(x,D)u,u) \geqslant -C\omega_\varepsilon\|u\|^2_{L^2} \qquad \varepsilon \in (0,\eta]$$

Now let $(a_\varepsilon)_\varepsilon \in S^1_{lsc}$ and write

$$a_\varepsilon = \operatorname{Re} a_\varepsilon + i \operatorname{Im} a_\varepsilon = \frac{a_\varepsilon + a_\varepsilon^*}{2} + \frac{a_\varepsilon - a_\varepsilon^*}{2}.$$

Then, since the second term on the right side is anti-self-adjoint it follows that

$$\operatorname{Re}\Big(\frac{a_\varepsilon - a_\varepsilon^*}{2} u, u\Big) = 0.$$

Moreover, since

$$\operatorname{Re}(a_\varepsilon) - \frac{a_\varepsilon + a_\varepsilon^*}{2} \in S^0_{lsc}$$

it suffices to show (1.9.4) with a replaced by $\operatorname{Re} a$.

As a next step, we let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n})$ be an even function with L^2-norm equal to 1. Also we define $\psi \in \mathscr{S}$ by $\psi(x, D) := \phi(x, D)^* \phi(x, D)$. Then, ψ is even and $\iint \psi(y, \eta)\, dy\, d\eta = 1$.

The aim is to write a_ε as a superposition of two terms $a_{1,\varepsilon}$ and $a_{0,\varepsilon}$ where $a_{1,\varepsilon}$ is of the form

$$a_{1,\varepsilon}(x, \xi) := \iint \psi((x - y)q(\eta), (\xi - \eta)/q(\eta)) a_\varepsilon(y, \eta)\, dy\, d\eta \qquad \varepsilon \in (0, 1]$$

with $q(\eta) = (1 + |\eta|^2)^{1/4}$. Then, one can show that there exists an $\eta \in (0, 1]$ such that

$$(a_{1,\varepsilon}(x, D)u, u) \geqslant 0 \qquad u \in \mathscr{S},\ \ \varepsilon \in (0, \eta]$$

and $D_\xi^\alpha D_x^\beta a_{0,\varepsilon}(x, \xi) = D_\xi^\alpha D_x^\beta (a_\varepsilon(x, \xi) - a_{1,\varepsilon}(x, \xi))$ is a finite sum of terms

$$\begin{aligned} b_{0,\varepsilon}(x, \xi) = \iint \psi_1((x - y)q(\eta), (\xi - \eta)/q(\eta)) b_\varepsilon(y, \eta)\, dy\, d\eta\, - \\ - b_\varepsilon(x, \xi) \iint \psi_1(y, \eta)\, dy\, d\eta \qquad \varepsilon \in (0, 1] \end{aligned}$$

where $\psi_1 \in \mathscr{S}$ is even and $(b_\varepsilon)_\varepsilon \in S^{1-|\alpha|}_{lsc}$. Then, by the same arguments as in [31, Theorem 18.1.14], one can show that if $(b_\varepsilon)_\varepsilon \in S^\mu_{lsc}$ then there exists a logarithmic slow scale net $(\omega_\varepsilon)_\varepsilon$ such that $|b_{0,\varepsilon}(x, \xi)| = \mathcal{O}(\omega_\varepsilon)(1 + |\xi|)^{\mu - 1}$ as $\varepsilon \to 0$. This finishes the proof. □

Remark 1.9.5. Suppose that $r \in \widetilde{S}^0_{lsc}$. Then there exist a representative $(r_\varepsilon)_\varepsilon \in r$, a logarithmic slow scale net $(\omega_{1,\varepsilon})_\varepsilon \in \Pi_{lsc}$, a constant $C_1 > 0$ and an $\eta_1 \in (0, 1]$ such that $|r_\varepsilon(x, \xi)| \leqslant C_1 \omega_{1,\varepsilon}$ for all $\varepsilon \in (0, \eta_1]$. Hence, the symbol $(r_\varepsilon(x, \xi) + C_1 \omega_{1,\varepsilon})_\varepsilon$ satisfies the requirements of Theorem 1.9.4, i.e.

$$\operatorname{Re}(r_\varepsilon(x, \xi) + C_1 \omega_{1,\varepsilon}) \geqslant 0 \qquad \varepsilon \in (0, \eta_1].$$

We therefore can conclude that there exist $(\omega_{2,\varepsilon})_\varepsilon \in \Pi_{lsc}$, a constant $C_2 > 0$ and an $\eta_2 \in (0,1]$ such that for any $u \in \mathscr{S}$ and $\varepsilon \in (0,\eta_2]$ we have

$$\mathrm{Re}((r_\varepsilon(x,D_x) + C_1\omega_{1,\varepsilon})u,u) \geqslant -C_2\omega_{2,\varepsilon}\|u\|^2_{L^2}$$

which is equivalent to the following

$$\exists C > 0\ \exists(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}: \quad \mathrm{Re}(r_\varepsilon(x,D_x)u,u) \geqslant -C\omega_\varepsilon\|u\|^2_{L^2}$$

for all $u \in \mathscr{S}$ and ε small enough.

Concerning the well-posedness of the Cauchy problem (1.9.1)-(1.9.2), we will first show the following energy estimate as shown in [52] and [31]. With obvious slight changes in the proofs one can show that there is also an energy estimate when the symbols depend on ε.

Lemma 1.9.6. *Let A and B satisfy $(i),(ii),(iii)$ with $2\gamma < L$. If $s \in \mathbb{R}$ and $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$ is generalized real and larger than some logarithmic slow scale net depending on s, then for every $u \in \mathcal{C}^1([0,Z];H^s) \cap \mathcal{C}^0([0,Z];H^{s+\hat{\gamma}})$ with $\hat{\gamma} := \max(\gamma,1)$ and every $p \in [1,\infty]$ we have:*

$$\Big(\frac{1}{2}\int_0^Z \|e^{-\lambda_\varepsilon z}u(z,.)\|^p_{H^s}\lambda_\varepsilon\, dz\Big)^{\frac{1}{p}} \leqslant \|u(0,.)\|_{H^s} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|P_\varepsilon u(z,.)\|_{H^s}\, dz \tag{1.9.5}$$

with the interpretation of the maximum when $p = \infty$.

Proof. Let $(A_\varepsilon)_\varepsilon$ be a representative of A. Since the principal symbol of A is real-valued, we have that

$$\mathrm{Re}(-iA_\varepsilon(x,z,\xi)) \geqslant -C_1\omega_{1,\varepsilon}$$

for some constant $C_1 > 0$ and some $(\omega_{1,\varepsilon})_\varepsilon \in \Pi_{lsc}$, for all ε sufficiently small. Using the sharp Gårding inequality we obtain

$$\begin{aligned} &\exists(\omega_{2,\varepsilon})_\varepsilon \in \Pi_{lsc}\ \exists C_2 > 0\ \exists \eta_2 \in (0,1]: \\ &\qquad \mathrm{Re}(-iA_\varepsilon(x,z,D_x)v,v) \geqslant -C_2\omega_{2,\varepsilon}\|v\|^2_{L^2} \quad v \in H^1 \end{aligned} \tag{1.9.6}$$

for $z \in [0,Z]$, $\varepsilon \in (0,\eta_2]$.

Note that by Remark 1.9.5 the property (1.9.6) is invariant under zeroth-order perturbations of $-iA_\varepsilon(x,z,D_x)$. Now using Lemma 1.9.2 we can write $B_\varepsilon = Q_\varepsilon^2 - 1 - R_\varepsilon$ for some self-adjoint $(Q_\varepsilon)_\varepsilon \in S^{\gamma/2}_{1-\frac{\gamma}{L},\frac{\gamma}{L},lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$ and some $(R_\varepsilon)_\varepsilon \in S^{-\infty}_{lsc}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$. We therefore obtain that there exist $(\omega_\varepsilon)_\varepsilon \in \Pi_{lsc}$, $C > 0$ and $\eta \in (0,1]$ such that

$$\begin{aligned} \mathrm{Re}((-iA_\varepsilon + B_\varepsilon)(x,z,D_x)v,v) &= \mathrm{Re}((-iA_\varepsilon - 1 - R_\varepsilon)(x,z,D_x)v,v) + (Q_\varepsilon^2 v,v) = \\ &= \mathrm{Re}((-iA_\varepsilon - 1 - R_\varepsilon)(x,z,D_x)v,v) + (Q_\varepsilon v, Q_\varepsilon v) \geqslant \\ &\geqslant -C\omega_\varepsilon \|v\|_{L^2}^2 \qquad v \in H^{\hat{\gamma}} \end{aligned} \tag{1.9.7}$$

for any $z \in [0,Z]$ and $\varepsilon \in (0,\eta]$.

We first consider the corresponding L^2-energy estimates, i.e. $s = 0$. Therefore, set $f_\varepsilon := \partial_z u - (iA_\varepsilon - B_\varepsilon)(x,z,D_x)u$. Then, taking the scalar products with respect to z we

$$\frac{\partial}{\partial z} e^{-2\lambda_\varepsilon z}\|u(z)\|_{L^2}^2 = -2\lambda_\varepsilon e^{-2\lambda_\varepsilon z}\|u(z)\|_{L^2}^2 + e^{-2\lambda_\varepsilon z} 2\,\mathrm{Re}\big(\frac{\partial}{\partial z}u(z), u(z)\big)$$

and hence

$$\begin{aligned} &2\,\mathrm{Re}(f_\varepsilon(z), u(z))e^{-2\lambda_\varepsilon z} = \\ &\qquad = \frac{\partial}{\partial z} e^{-2\lambda_\varepsilon z}\|u(z)\|_{L^2}^2 + 2\,\mathrm{Re}\Big(\big((-iA_\varepsilon + B_\varepsilon)(x,z,D_x) + \lambda_\varepsilon\big)u(z), u(z)\Big)e^{-2\lambda_\varepsilon z} \\ &\qquad \geqslant \frac{\partial}{\partial z} e^{-2\lambda_\varepsilon z}\|u(z)\|_{L^2}^2 \end{aligned}$$

under the requirement that $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$ with $\lambda_\varepsilon \geqslant C\omega_\varepsilon$, where C and ω_ε are as in (1.9.7). Integration from 0 to z, $\bar{z} \leqslant z \leqslant Z$ then gives

$$e^{-2\lambda_\varepsilon z}\|u(z)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 \leqslant 2\int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{-\lambda_\varepsilon \bar{z}}\|u(\bar{z})\|_{L^2}\, d\bar{z}.$$

Setting

$$M_\varepsilon(z) := \sup_{0\leqslant \bar{z}\leqslant z} e^{-\lambda_\varepsilon \bar{z}}\|u(\bar{z})\|_{L^2}$$

we get

$$M_\varepsilon(z)^2 \leqslant \|u(0)\|_{L^2}^2 + 2M_\varepsilon(z)\int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2}\, d\bar{z}.$$

Hence

$$\Big(M_\varepsilon(z) - \int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2}\, d\bar{z}\Big)^2 \leqslant \Big(\|u(0)\|_{L^2} + \int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2}\, d\bar{z}\Big)^2$$

which yields

$$e^{-\lambda_\varepsilon z}\|u(z)\|_{L^2} \leqslant \|u(0)\|_{L^2} + 2\int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2}\, d\bar{z} \tag{1.9.8}$$

for all $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$ with $\lambda_\varepsilon \geqslant C\omega_\varepsilon$. In (1.9.8) we set $\lambda_\varepsilon = C\omega_\varepsilon$ and multiply the obtained result by the factor $e^{(C\omega_\varepsilon-\lambda_\varepsilon)z}$. Then

$$e^{-\lambda_\varepsilon z}\|u(z)\|_{L^2} \leqslant e^{(C\omega_\varepsilon-\lambda_\varepsilon)z}\|u(0)\|_{L^2} + 2\int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{(C\omega_\varepsilon-\lambda_\varepsilon)(z-\bar{z})}\,d\bar{z}.$$

If we choose $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$ with $\lambda_\varepsilon > 2C\omega_\varepsilon$, then $C\omega_\varepsilon - \lambda_\varepsilon \leqslant -\frac{\lambda_\varepsilon}{2}$ and hence

$$e^{-\lambda_\varepsilon z}\|u(z)\|_{L^2} \leqslant e^{-\frac{\lambda_\varepsilon}{2}z}\|u(0)\|_{L^2} + 2\int_0^z e^{-\lambda_\varepsilon \bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{-\frac{\lambda_\varepsilon}{2}(z-\bar{z})}\,d\bar{z}. \tag{1.9.9}$$

This shows the desired energy estimate in the case that $p = \infty$. For $1 \leqslant p < \infty$ we observe for the L^p-norm of the first term on the right-hand side of (1.9.9):

$$\int_0^Z e^{-\frac{\lambda_\varepsilon}{2}pz}\,dz = \frac{2}{\lambda_\varepsilon p}\Big(1 - e^{-\frac{\lambda_\varepsilon}{2}pZ}\Big) \leqslant \frac{2}{\lambda_\varepsilon p}.$$

So,

$$\|e^{-\frac{\lambda_\varepsilon}{2}z}\|u(0)\|_{L^2}\|_{L^p} \leqslant \Big(\frac{2}{\lambda_\varepsilon}\Big)^{\frac{1}{p}}\|u(0)\|_{L^2}.$$

Concerning the L^p-norm of the second term on the right-hand side of (1.9.9), we compute

$$\begin{aligned}
2^p &\int_0^Z e^{-\frac{\lambda_\varepsilon}{2}pz}\Big(\int_0^z e^{-\lambda_\varepsilon\bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{\frac{\lambda_\varepsilon}{2}\bar{z}}\,d\bar{z}\Big)^p dz = \\
&= -2^p\frac{2}{\lambda_\varepsilon p}e^{-\frac{\lambda_\varepsilon}{2}pz}\Big(\int_0^z e^{-\lambda_\varepsilon\bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{\frac{\lambda_\varepsilon}{2}\bar{z}}\,d\bar{z}\Big)^p\Big|_0^Z + \\
&\quad + 2^p\frac{2}{\lambda_\varepsilon}\int_0^Z e^{-\frac{\lambda_\varepsilon}{2}pz}\Big(\int_0^z e^{-\lambda_\varepsilon\bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{\frac{\lambda_\varepsilon}{2}\bar{z}}\,d\bar{z}\Big)^{p-1} e^{-\lambda_\varepsilon z}\|f_\varepsilon(z)\|_{L^2} e^{\frac{\lambda_\varepsilon}{2}z}\,dz \leqslant \\
&\leqslant 2^p\frac{2}{\lambda_\varepsilon}\int_0^Z \Big(\int_0^z e^{-\lambda_\varepsilon\bar{z}}\|f_\varepsilon(\bar{z})\|_{L^2} e^{-\frac{\lambda_\varepsilon}{2}(z-\bar{z})}\,d\bar{z}\Big)^{p-1} e^{-\lambda_\varepsilon z}\|f_\varepsilon(z)\|_{L^2}\,dz \leqslant \\
&\leqslant \frac{2}{\lambda_\varepsilon}\Big(2\int_0^Z e^{-\lambda_\varepsilon z}\|f_\varepsilon(z)\|_{L^2}\,dz\Big)^p.
\end{aligned}$$

Summarizing and using the Minkowski inequality, we obtain for every $p \in [1,\infty]$

$$\Big(\int_0^Z \|e^{-\lambda_\varepsilon z}u(z,.)\|_{L^2}^p\,dz\Big)^{\frac{1}{p}} \leqslant \Big(\frac{2}{\lambda_\varepsilon}\Big)^{\frac{1}{p}}\Big(\|u(0,.)\|_{L^2} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|P_\varepsilon u(z,.)\|_{L^2}\,dz\Big)$$

which shows the theorem in the case that $s = 0$.

The L^2-energy estimate can now be used to obtain the H^s-energy estimates for $s \in \mathbb{R}$, $s \neq 0$. Therefore, let $u \in \mathcal{C}^1([0,Z];H^s(\mathbb{R}^n)) \cap \mathcal{C}^0([0,Z];H^{s+\hat{\gamma}}(\mathbb{R}^n))$. Then, $\langle D_x\rangle^s P_\varepsilon \langle D_x\rangle^{-s}$ satisfies the same assumptions as P_ε and $\langle D_x\rangle^s u \in \mathcal{C}^1([0,Z];L^2(\mathbb{R}^n)) \cap \mathcal{C}^0([0,Z];H^{\hat{\gamma}}(\mathbb{R}^n))$.

Note that in particular we have an estimate of the form (1.9.7) if we replace $-iA_\varepsilon + B_\varepsilon$ by $\langle D_x\rangle^s(-iA_\varepsilon + B_\varepsilon)\langle D_x\rangle^{-s}$ and v by $\langle D_x\rangle^s u$ but now with a lower bound $C(s)\omega_\varepsilon(s)$ depending on s. Using the L^2-energy estimate we obtain

$$\Big(\frac{1}{2}\int_0^Z \|e^{-\lambda_\varepsilon z}u(z,.)\|_{H^s}^p\lambda_\varepsilon\,dz\Big)^{\frac{1}{p}} = \Big(\frac{1}{2}\int_0^Z \|e^{-\lambda_\varepsilon z}\langle D_x\rangle^s u(z,.)\|_{L^2})^p\lambda_\varepsilon\,dz\Big)^{\frac{1}{p}} \leqslant$$
$$\leqslant \|\langle D_x\rangle^s u(0,.)\|_{L^2} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|\langle D_x\rangle^s P_\varepsilon\langle D_x\rangle^{-s}\langle D_x\rangle^s u(z,.)\|_{L^2}\,dz =$$
$$= \|u(0,.)\|_{H^s} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|P_\varepsilon u(z,.)\|_{H^s}\,dz$$

since $(P_\varepsilon\langle D_x\rangle^{-s}\langle D_x\rangle^s)_\varepsilon - (P_\varepsilon)_\varepsilon \in S^{-\infty}_{lsc}$. This completes the proof. □

Theorem 1.9.7. *Let A and B be generalized symbols with parameter $z \in [0,Z]$ that satisfy $(i),(ii)$ and (iii) with $2\gamma < L$. Then, for every $f \in \mathcal{G}_{L^2}((0,Z)\times\mathbb{R}^n)$ and $u_0 \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ the Cauchy problem (1.9.1)-(1.9.2) has a unique solution $u \in \mathcal{G}_{L^2}((0,Z)\times\mathbb{R}^n)$ and the energy estimate (1.9.5) remains valid for this solution.*

Proof. Let $(g_\varepsilon)_\varepsilon \in g$, $(f_\varepsilon)_\varepsilon \in f$ be representatives. We fix $\varepsilon \in (0,1]$ and consider the smooth Cauchy problem

$$\begin{aligned} P_\varepsilon u_\varepsilon &= f_\varepsilon \qquad \text{on } (0,Z)\times\mathbb{R}^n \\ u_\varepsilon(0,.) &= u_{0,\varepsilon} \qquad \text{on } \mathbb{R}^n. \end{aligned} \tag{1.9.10}$$

Then, by [31, Theorem 23.1.2] and [52, Theorem 5], we get the existence of a solution $u_\varepsilon \in \mathcal{C}^\infty([0,Z];H^\infty(\mathbb{R}^n))$. Note that the additional regularity with respect to z variable follows from (1.9.10).

Concerning uniqueness, assume that $f_\varepsilon = 0$ and $u_{0,\varepsilon} = 0$. Furthermore, suppose that moderateness of the solutions is already shown. We apply the energy estimate (1.9.5) and get

$$e^{-\lambda_\varepsilon Z}\|u_\varepsilon(z,.)\|_{H^s} \leqslant \max_{z\in[0,Z]}\|e^{-\lambda_\varepsilon z}u_\varepsilon(z,.)\|_{H^s} = 0 \qquad \forall s \geqslant 0$$

Therefore, $u_\varepsilon = 0$.

It remains to show that $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{M,L^2}((0,Z)\times\mathbb{R}^n)$. For moderateness with respect to $z \in [0,Z]$ we note that $\|u_\varepsilon\|_{L^2((0,Z)\times\mathbb{R}^n)} \leqslant Z\sup_{z\in[0,Z]}\|u_\varepsilon\|_{L^2(\mathbb{R}^n)}$.

Since (1.9.5) is applicable for $u_\varepsilon \in \mathcal{C}^\infty([0,Z]; H^\infty(\mathbb{R}^n))$, we obtain

$$e^{-\lambda_\varepsilon Z}\|u_\varepsilon(z,.)\|_{H^s} \leqslant \max_{z\in[0,Z]}\|e^{-\lambda_\varepsilon z}u_\varepsilon(z,.)\|_{H^s} \leqslant$$
$$\leqslant \|u_\varepsilon(0,.)\|_{H^s} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|P_\varepsilon u_\varepsilon(z,.)\|_{H^s}\,dz.$$

Hence

$$\|u_\varepsilon(z,.)\|_{H^s} \leqslant e^{\lambda_\varepsilon Z}\Big(\|u_\varepsilon(0,.)\|_{H^s} + 2\int_0^Z e^{-\lambda_\varepsilon z}\|P_\varepsilon u_\varepsilon(z,.)\|_{H^s}\,dz\Big).$$

and therefore u_ε is moderate with respect to x if and only if λ_ε is of log-type, which is indeed satisfied because of the assumption $(\lambda_\varepsilon)_\varepsilon \in \Pi_{lsc}$. Recall that a net r_ε is of log-type if $|r_\varepsilon| = \mathcal{O}\big(\log(\frac{1}{\varepsilon})\big)$ as $\varepsilon \to 0$.

Concerning the z-derivatives, we write

$$\partial_z\langle D_x\rangle^s u_\varepsilon = \langle D_x\rangle^s(iA_\varepsilon - B_\varepsilon)(x,z,D_x)u_\varepsilon + \langle D_x\rangle^s f_\varepsilon$$

and obtain that for every $s \geqslant 0$ there exists an $N \in \mathbb{N}$ such that $\|\partial_z\langle D_x\rangle^s u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^{-N})$ uniformly in z. For the higher order z-derivatives one uses an induction argument when differentiating the equation (1.9.10). □

Chapter 2

A semiclassical approach

2.1 The setting

In Chapter 1 we showed the well-posedness of approximating first-order equations to the problem

$$LU \equiv 0 \qquad \text{microlocally on } I_{\theta_1},\ z \in (z_0, Z).$$

Recall that in this special case we used the standard or Kohn-Nirenberg correspondence between operators and symbols. The precise relation was given in Definition 1.2.4.

In this second part we will ask ourselves if it is possible to relax the logarithmic slow scale conditions on the coefficients to coefficients of log-type when using a different kind of quantization. In particular, we will consider the semiclassical Kohn-Nirenberg quantization (cf. Section 2.3 or [15] by introducing Plancks's constant and study operators of the form

$$A_\varepsilon(x, \varepsilon D)u_\varepsilon(x) = (2\pi\varepsilon)^{-n} \iint e^{i(x-y)\xi/\varepsilon} a_\varepsilon(x,\xi) u_\varepsilon(y)\, dy\, d\xi \tag{2.1.1}$$

as $\varepsilon \to 0$. We note that the semiclassical asymptotic expansion of the symbol a of order m is of the form

$$a_\varepsilon(x,\xi) \sim \sum_{j \geqslant 0} \varepsilon^j a_{j,\varepsilon}(x,\xi) \tag{2.1.2}$$

valid for $|\xi| \geqslant \frac{1}{C} > 0$ with $a_{j,\varepsilon} \in S^{m-j}(T^*\mathbb{R}^n)$ are classical (Hörmander) symbols of order $m-j$ for every fixed $\varepsilon \in (0,1]$ on this set for some $C > 0$, see [57, Section 1.10.1], [13, Chapters 7 and 11] or [14, Section 8.6]. As we will see later, this will cause, that the principal symbol $a_{0,\varepsilon}$ gives the highest contribution with respect to ε as the lower order terms appear with an additional factor ε^j, $j \geqslant 1$.

As this is a first approach we will consider operators $L_\varepsilon(x, z, hD_t, hD_x, hD_z)$ (in particular we will study the case where $h = \varepsilon$) with L_ε having less regular coefficients as in Chapter 1. In detail, we will work with

$$L_\varepsilon(x, z, D_t, D_x, D_z) := \partial_z^2 + \sum_{j=1}^{n-1} b_{j,\varepsilon}(x, z)\partial_{x_j}^2 - c_\varepsilon(x, z)\partial_t^2. \tag{2.1.3}$$

Here, the coefficients are supposed to be of log-type up to some fixed order $r \in \mathbb{N}$ and strictly non-zero. As in Chapter 1 we assume that the coefficients are certain regularizations of Hölder continuous functions, i.e. $c_\varepsilon \to c$ and $b_{j,\varepsilon} \to b_j$ in the Hölder space $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$ with exponent $\mu \in (0, 1)$ as $\varepsilon \to 0$.

In this chapter we will first provide a factorization into two first-order operators of the form (2.1.1) of the operator $L_\varepsilon(x, z, hD_t, hD_x, hD_z)$ with $h = \varepsilon$. With the asymptotic expansion as in (2.1.2) the highest contribution comes from the principal symbol $a_{0,\varepsilon}$ which is of log-type in our case.

As already mentioned in the introduction Hörmann showed in [32] well-posedness for certain first-order hyperbolic pseudodifferential equations when the occurring symbols are of log-type up to a fixed specified order. The question now is: When using a different quantization, is it possible to relax the logarithmic slow scale condition on the coefficients to coefficients that are of log-type? We will see that working with the semiclassical quantization will change the view of notions such as the asymptotic expansion, ellipticity and microlocality. As a result we will be able to state a factorization theorem as well as a microlocal diagonalization of the governing equation on a sufficiently nice subset of the phase space. The question, whether the approximating first-order equations are well-posed remains open.

Furthermore I want to mention a paper of Colin de Verdière (see [12]) who combined semiclassical analysis with wave equations and more general with elastic wave equations as used in seismology. As there, we won't discuss the physical interpretations of our modeling techniques.

2.2 Basic notions

In this section we specify the basic notions concerning the symbol classes and the semiclassical quantization of the governing equation given as already indicated in the previous section.

In order to realize the log-type up to order $r \in \mathbb{N}$ conditions on the coefficients of the model operator (2.1.3) we will follow the idea in Remark 1.3.1 and consider a regularization obtained by convolution with a logarithmically scaled mollifier $\varphi_{\omega_\varepsilon^{-1}}(.) := \omega_\varepsilon^n \varphi(\omega_\varepsilon .)$ with $\varphi \in \mathscr{S}(\mathbb{R}^n)$ as in (1.1.1) where $\omega_\varepsilon := (\log(C/\varepsilon))^{1/r}$ for some $r \in \mathbb{N}$ and a constant $C > 0$ such that $(\omega_\varepsilon)_\varepsilon$ is strongly positive.

In the following we will typically encounter symbol classes that are subspaces of $\mathcal{M}_{S^m}$ and subjected to two different asymptotic scales. This extends the definition of $\mathcal{M}_{S^m}$ and $\mathcal{N}_{S^m}$ in the following way:

Definition 2.2.1. *Let ν be a non-negative real number and $l, k \in \mathbb{R}$. For $m \in \mathbb{R}$ we let $(a_\varepsilon)_\varepsilon$ be a family of symbols $a_\varepsilon \in S^m$. We then say that $(a_\varepsilon)_\varepsilon$ is in the generalized symbol class $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ if and only if*

$$\begin{aligned} &\exists \eta \in (0,1]\ \forall \alpha, \beta \in \mathbb{N}^n\ \exists C > 0\ \forall \varepsilon \in (0,\eta] : \\ &q_{\alpha,\beta}^{(m)}(a_\varepsilon) := \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_x^\beta a_\varepsilon(x,\xi)| \langle \xi \rangle^{-m+|\alpha|} \leqslant C\varepsilon^k \omega_\varepsilon^{\nu|\beta|+l}. \end{aligned} \tag{2.2.1}$$

We call m, k and (ν, l) the order, the growth type and the log-type respectively of $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$.

This definition is similar to that for generalized symbols $\mathcal{S}_{\rho,\delta,\omega}^{m,\mu}$ in [20, Definition 4.1]. Also, since $(\omega_\varepsilon)_\varepsilon$ is a strongly positive slow scale net we note that a generalized symbol $(a_\varepsilon)_\varepsilon$ in $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ can always be estimated in the following way:

$$\exists \eta \in (0,1]\ \forall \alpha, \beta \in \mathbb{N}^n\ \exists C > 0 :\ q_{\alpha,\beta}^{(m)}(a_\varepsilon) \leqslant C\varepsilon^{k-1} \quad \forall \varepsilon \in (0,\eta]. \tag{2.2.2}$$

Later, in Lemma 2.5.3 we give the construction scheme for approximative inverse operators and the advantage of using (2.2.1) rather than (2.2.2) will be clear then. Hereafter we typically work in spaces $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ with $\nu \in \{0,1\}$. Analogously to Definition 2.2.1 one introduces the space $\mathcal{M}_{S_{\mathrm{hg}}^{m,k}}^{(\nu,l)}(\mathbb{R}^n \times (\mathbb{R}^n \backslash 0))$.

Furthermore, the notion of negligibility in $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ is defined as follows:

Definition 2.2.2. *An element of $\mathcal{M}_{S^{m,k}}^{(\nu,l)}$ is said to be negligible, denoted by $\mathcal{N}_{S^m}$, if the following condition is fulfilled:*

$$\exists \eta \in (0,1]\ \forall q \in \mathbb{N}\ \forall \alpha, \beta \in \mathbb{N}^n\ \exists C > 0 :\ q_{\alpha,\beta}^{(m)}(a_\varepsilon) \leqslant C\varepsilon^q \quad \forall \varepsilon \in (0,\eta].$$

So, if $P(x, D_x) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D_x^\alpha$ with bounded and measurable coefficients, then the logarithmically scaled regularized coefficient $a_{\alpha,\varepsilon} := a_\alpha * \varphi_{\omega_\varepsilon^{-1}}$ satisfy the estimate

$$\|\partial^\alpha a_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\omega_\varepsilon^{|\alpha|}) = \mathcal{O}(\log^{|\alpha|/r}(C/\varepsilon)).$$

Therefore $(a_{\alpha,\varepsilon})_\varepsilon$ is of log-type up to order r, that is $\|\partial^\alpha a_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\log(C/\varepsilon))$ for all $|\alpha| \leqslant r$, but it is not logarithmic slow scale regular. Recall that the asymptotic norm estimate (of any order) of a logarithmic slow scale coefficient has a bound $\mathcal{O}(r_\varepsilon)$ where $|r_\varepsilon| = \mathcal{O}(\log^p(1/\varepsilon))$ for all $p > 0$ as ε tends to 0. In the same manner we obtain for the logarithmically scaled regularization of the operator a symbol $(p_\varepsilon)_\varepsilon$ where $p_\varepsilon(x,\xi) := (p(.,\xi) * \varphi_{\omega_\varepsilon^{-1}})(x)$ which is of class $\mathcal{M}^{(1,0)}_{S^{m,0}}$ and of log-type up to order (∞, r), cf. [32, Remark 3.2]. Recall that a symbol $p_\varepsilon(x,\xi)$ is of log-type up to order (k,l) if

$$q^{(m)}_{k,l}(p_\varepsilon) := \max_{|\alpha|\leqslant k, |\beta|\leqslant l} \sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-m+|\alpha|} |\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x,\xi)| = \mathcal{O}(\log(1/\varepsilon)) \quad \text{as } \varepsilon \to 0.$$

Also, we will use the following symbol classes:

Definition 2.2.3. *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be open and conic with respect to the second variable. We say that a generalized symbol $(a_\varepsilon)_\varepsilon$ is in $\mathcal{M}^{(\nu,l)}_{S^{m,k}}(U)$ if $a_\varepsilon \in \mathcal{C}^\infty(U)$ for fixed $\varepsilon \in (0,1]$ and there is a constant $K > 0$ independent of ε such that:*

$$\exists \eta \in (0,1]\ \forall \alpha, \beta \in \mathbb{N}^n\ \exists C > 0 \text{ for which}$$
$$|\partial_\xi^\alpha \partial_x^\beta a_\varepsilon(x,\xi)| \leqslant C\varepsilon^k \omega_\varepsilon^{\nu|\beta|+l} \langle\xi\rangle^{m-|\alpha|} \quad \text{for } (x,\xi) \in V_K,\ \varepsilon \in (0,\eta],$$

where $V_K := \{(x,\xi) \in U \mid d(\xi,\zeta) \geqslant K,\ \zeta \in \partial pr_2(U) \neq \varnothing\}$. We observe that, if $(x,\xi) \in V_K$ then $(x,\lambda\xi) \in V_K$ for all $\lambda \geqslant 1$.

Note that Definition 2.2.1 is equivalent to Definition 2.2.3 in the case that $U = \mathbb{R}^n \times \mathbb{R}^n$.

2.3 Semiclassical quantization

In the theory of pseudodifferential operators the idea is to assign to each operator A of a suitable class a function σ_A (i.e. a symbol). This is closely related to integrated representations of the Heisenberg group $\mathbb{H}_n$. We will mainly follow Folland, so we refer the reader to [15] for a deeper discussion.

In the Kohn-Nirenberg correspondence or standard quantization of pseudodifferential operators one associates to a symbol $a \in S^m$ an operator $A : \mathcal{S} \to \mathcal{S}$ in the following way

$$\begin{aligned} A(x, D_x) &:= (2\pi)^{-n} \int (\mathcal{F}a)(q,p) e^{-ipq/2} e^{i(pD+qX)}\, dp\, dq = \\ &= (2\pi)^{-n} \int (\mathcal{F}(Ta))(q,p) \rho(q,p)\, dp\, dq =: \rho(\widehat{Ta}) \end{aligned}$$

where $\rho(p,q,t) = e^{i(tI+pD+qX)}$ is a unitary representation of the Heisenberg group $\mathbb{H}_n$ and

$$\rho(q,p) := \rho(q,p,0) = e^{i(pD+qX)}.$$

Here, the above integral is an Bochner integral if $\widehat{Ta} \in L^1(\mathbb{R}^{2n})$. Note that $\rho(\widehat{Ta})$ makes also sense as an operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ if $\rho(\widehat{Ta}) \in \mathcal{S}'(\mathbb{R}^{2n})$. In [15, Theorem 2.37] it is shown that the operator T, defined by $\widehat{Ta}(q,p) = e^{-ipq/2}\hat{a}(q,p)$, maps the symbol class $S^m_{\rho,\delta}$ ($m \in \mathbb{R}$, $0 \leqslant \rho, \delta \leqslant 1$, $\delta < 1$) into themselves and is a Fréchet space isomorphism on each of them. With this in mind, we obtain the classical representation which is given by

$$\begin{aligned} A(x, D_x)u(x) &:= (2\pi)^{-n} \int (\mathcal{F}a)(q,p) e^{-ipq/2} e^{i(pD+qX)} u(x)\, dp\, dq = \\ &= (2\pi)^{-n} \int (\mathcal{F}(a))(q,p) e^{i(qX+pD)} u(x)\, dp\, dq = \\ &= (2\pi)^{-n} \int (\mathcal{F}(a))(q,p) e^{iqx} u(x+p)\, dp\, dq = \\ &= (2\pi)^{-n} \int a(x,\xi) e^{-ip\xi} u(x+p)\, dp\, d\xi = \\ &= (2\pi)^{-n} \int e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi. \end{aligned}$$

One can reformulate these results in a more quantum-mechanical spirit by using the Schrödinger representation ρ_h of the Heisenberg group which is given by

$$\rho_h(q,p,t) := \rho(q,hp,ht) = e^{i(htI+hpD+qX)}$$

and its simplification

$$\rho_h(q,p) := \rho_h(q,p,0) = e^{i(hpD+qX)}$$

respectively and where h denotes the Planck constant. In detail we have

$$\big(\rho_h(q,p)u\big)(x) = e^{i(qx+hpq/2)} u(x+hp).$$

Note that for $h = 1$ this representation leads to the usual Kohn-Nirenberg calculus. Then we can introduce the corresponding h-dependent operator A_h by

$$A_h(x, D_x)u(x) := (2\pi)^{-n} \int (\mathcal{F}a)(q,p)e^{-ipqh/2}\rho_h(p,q)u(x)\,dp\,dq = \rho_h(\widehat{T_h a}).$$

Using the same computations as above, but now with h-dependency, we get

$$\begin{aligned} A_h(x, D_x)u(x) &= (2\pi)^{-n} \int (\mathcal{F}a)(q,p)e^{-ipqh/2}\rho_h(p,q)u(x)\,dp\,dq = \\ &= (2\pi h)^{-n} \int e^{i(x-y)\xi/h}a(x,\xi)u(y)\,dy\,d\xi. \end{aligned}$$

2.3.1 The governing equation in a semiclassical Colombeau setting

Once introduced such an additional artificial parameter h gives us now the opportunity to apply the theory of h-pseudodifferential operators used in semiclassical analysis which is a formalism in the asymptotic regime $h \ll 1$. As we want to combine this with the theory of generalized pseudodifferential operators we start writing $h := h(\varepsilon)$ with the restriction $h(\varepsilon) \to 0$ in the case that $\varepsilon \to 0$.

For technical reasons (e.g. construction of an approximative inverse) we also have to prevent other possible values of the function $h(\varepsilon)$. As this is a first approach in this field we have chosen $h(\varepsilon) := \varepsilon$ at this point.

So, we will concentrate on operators as given in (2.1.3) which are characterized by homogeneity. Also note that our model operators have less regular log-type coefficients than logarithmic slow scale. So, when given a family $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^m}$ we define the corresponding semiclassical pseudodifferential operator of Colombeau type to be the linear operator $A_\psi : \mathcal{G}_{L^2} \to \mathcal{G}_{L^2}$ which acts on the level of representatives as

$$(u_\varepsilon)_\varepsilon \mapsto (A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}. \tag{2.3.1}$$

Here, for fixed $\varepsilon \in (0,1]$, the right hand side corresponds to the standard quantization of the symbol $(a_\varepsilon)_\varepsilon$ with parameter $h(\varepsilon) = \varepsilon$, that is,

$$\begin{aligned} A_{\psi,\varepsilon}(x, D_x)u_\varepsilon(x) &:= (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathcal{F}a_\varepsilon)(q,p)e^{-ipq\varepsilon/2}\rho_\varepsilon(q,p)u_\varepsilon(x)\,dq\,dp = \\ &= (2\pi\varepsilon)^{-n} \int e^{i(x-y)\xi/\varepsilon}a_\varepsilon(x,\xi)u_\varepsilon(y)\,dy\,d\xi. \end{aligned}$$

Then, with L_ε as in (2.1.3), the corresponding ψ-pseudodifferential operator L_ψ is essentially given by the oscillatory integral

$$L_{\psi,\varepsilon}(y, D_y)u_\varepsilon(y) = (2\pi\varepsilon)^{-n-1} \int e^{iy\eta/\varepsilon} l_\varepsilon(y,\eta)\hat{u}_\varepsilon(\eta/\varepsilon)\, d\eta$$

for fixed $\varepsilon \in (0,1]$, where we have set $y = (t,x,z) \in \mathbb{R}^{n+1}$ and $\eta = (\tau,\xi,\zeta) \in \mathbb{R}^{n+1}$. The expression $(l_\varepsilon)_\varepsilon$ is called the generalized symbol of L_ψ and is given by

$$l_\varepsilon(x,z,\tau,\xi,\zeta) := -\zeta^2 - \langle b_\varepsilon(x,z)\xi,\xi\rangle + c_\varepsilon(x,z)\tau^2$$

where we have set $b_\varepsilon(x,z) := \operatorname{diag}(b_{1,\varepsilon}(x,z),\ldots,b_{n-1,\varepsilon}(x,z))$. Hence, $(l_\varepsilon)_\varepsilon \in \mathcal{M}^{(1,0)}_{S^{2,0}}$ and the operator L_ε can easily be reconstructed from $L_{\psi,\varepsilon}$, i.e. $L_\varepsilon = \varepsilon^{-2}L_{\psi,\varepsilon}$, because of the homogeneity of the operators.

Given such a scaled generalized operator as above we will carry out all transformations within algebras of generalized functions from now on. More explicitly, we will study the action of the linear operator $L_\psi = \mathrm{OP}_\psi(l_\varepsilon)$ from $\mathcal{G}_{L^2}$ into itself in the following sense: on the level of representatives L_ψ acts as in (2.3.1). This explains our governing equation

$$L_\psi U = F$$

for which we will now present a factorization theorem and a method for the microlocal diagonalization.

2.4 ψ-pseudodifferential calculus

In this section we introduce a general calculus for ψ-pseudodifferential operators which are certain semiclassical standard quantizations of generalized symbols as demonstrated in the previous section. For an introduction in semiclassical analysis we refer the reader to [39, 14, 29]. Since most of the techniques are similar to the classical theory of pseudodifferential operators we also want to give [31, 55] as references. Moreover, a detailed discussion on pseudodifferential operators with Colombeau generalized symbols can be found in [20, 19].

To continue, given a generalized symbol $(a_\varepsilon(x,\xi))_\varepsilon$ one can assign the semiclassical standard quantization which we denote by $\mathrm{OP}_\psi(a_\varepsilon) := a_\varepsilon(x,\varepsilon D_x)$. In this sense $\psi = \psi(\varepsilon)$ can be thought of as a scaled phase function on phase space. Therefore, we let

$$\psi_\varepsilon(x,\xi) := \langle x,\xi\rangle/\varepsilon := x\xi/\varepsilon \quad \varepsilon \in (0,1]$$

throughout this work.

As already mentioned above we will focus on generalized pseudodifferential operators having the following phase-amplitude representation.

Definition 2.4.1. *Let $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}}(\mathbb{R}^n \times \mathbb{R}^n)$. We define the corresponding linear operator $A_\psi : \mathcal{G}_{L^2} \to \mathcal{G}_{L^2}$ such that on the level of representatives we have*

$$(u_\varepsilon)_\varepsilon \mapsto (A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}$$

and

$$\begin{aligned} A_{\psi,\varepsilon}(x, D_x)u_\varepsilon(x) &:= (2\pi\varepsilon)^{-n} \int e^{i(x-y)\xi/\varepsilon} a_\varepsilon(x,\xi) u_\varepsilon(y)\, dy\, d\xi = \\ &= \varepsilon^{-n} \mathcal{F}^{-1}_{\xi\to x/\varepsilon}\Big(a_\varepsilon(x,\xi)\hat{u}_\varepsilon(\xi/\varepsilon)\Big). \end{aligned} \tag{2.4.1}$$

We call A_ψ the ψ-pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon = (a_\varepsilon(x,\xi))_\varepsilon$. Also, we will often write $A_\psi \in \mathrm{OP}_\psi\, \mathcal{M}^{(\nu,l)}_{S^{m,k}}$ when the generalized symbol of A_ψ belongs to the class $\mathcal{M}^{(\nu,l)}_{S^{m,k}}$.

Remark 2.4.2. (a) Note that the map in the above definition preserves moderateness and negligibility, respectively, so that A_ψ is well-defined on equivalence classes and continuous (cf. [32, Section 1.2] and [37, Theorem 2.7]). Further, we remark that $(A_{\psi,\varepsilon}(x, D_x)u_\varepsilon)_\varepsilon = (A_\varepsilon(x, \varepsilon D_x)u_\varepsilon)_\varepsilon$ since formally we can always perform the rescaling. The problem of rescaling will be discussed in the next subsections.

(b) Note that Definition 2.4.1 agrees with the notion of the semiclassical standard quantization of a generalized symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}}(\mathbb{R}^n \times \mathbb{R}^n)$ and can simply written as in (2.4.1) using scaled or semiclassical Fourier transforms. Another commonly used quantization is the Weyl quantization $a^W(x, \varepsilon D)$ which has the nice property that real symbols correspond to self-adjoint Weyl quantizations. Since we will exclusively deal with the standard quantization of generalized symbols we decided to call them ψ-pseudodifferential operators.

To prepare the factorization theorem of Section 2.5 we will have to consider products of generalized ψ-pseudodifferential operators. We therefore start with some general observations concerning the notion of asymptotic expansion of a generalized symbol in $\mathcal{M}^{(\nu,l)}_{S^{m,k}}$.

2.4.1 Asymptotic expansion of the first kind

First, we present the asymptotic expansion of the first kind which is inspired by the results given in [19, Section 2.5] but also uses a relevant technical aspect from the semiclassical approach. Since this first notion of an asymptotic expansion turns out to have no suitable invariant character under rescaling we will further introduce the asymptotic expansion of second kind which behaves slightly different when performing the rescaling. We should emphasize here that this invariance property of the asymptotic expansion of the second kind will be essential from Section 2.6 onwards, where we discuss microlocal regularity at infinite points.

The definition of the asymptotic expansion of the first kind is now the following:

Definition 2.4.3. *For $j \in \mathbb{N}$ let $\{m_j\}_j$ be a strictly decreasing sequence of real numbers with $m_j \searrow -\infty$ as $j \to \infty$, $m_0 = m$ and $\{l_j\}_j$ be a sequence of the form $l_j = \sigma j + l$ for some fixed $\sigma, l \in \mathbb{R}$ and $\sigma \geqslant 0$. Further let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l_j)}_{S^{m_j,k}}$ so that the following uniform growth type condition is satisfied:*

$$\begin{gathered}\exists \eta \in (0,1]\ \forall j \in \mathbb{N}\ \forall \alpha,\beta \in \mathbb{N}^n\ \exists C > 0 \text{ for which} \\ q^{(m_j)}_{\alpha,\beta}(a_{j,\varepsilon}) \leqslant C\varepsilon^k \omega_\varepsilon^{\nu|\beta|+l_j} \quad \forall \varepsilon \in (0,\eta].\end{gathered} \tag{2.4.2}$$

We say that $\sum_{j=0}^\infty (\varepsilon^j a_{j,\varepsilon})_\varepsilon$ is the asymptotic expansion of the first kind for $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}}$, denoted by $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$, if and only if

$$(a_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m_N,N+k-1}} \quad \forall N \geqslant 1.$$

Moreover $(a_{0,\varepsilon})_\varepsilon$ is said to be the principal symbol of $(a_\varepsilon)_\varepsilon$ if $a_{0,\varepsilon}$ is not identically vanishing for any fixed $\varepsilon \in (0,\eta]$.

We are now in a position to state our first result.

Lemma 2.4.4. *Let $\{l_j\}_j$, $\{m_j\}_j$ and $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be as in Definition 2.4.3. Then there exists a generalized symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}}$ such that $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$ in $\mathcal{M}^{(\nu,l)}_{S^{m,k}}$. Moreover the asymptotic expansion of the first kind determines $(a_\varepsilon)_\varepsilon$ uniquely modulo $\mathcal{N}_{S^{-\infty}}$.*

Before presenting the proof of Lemma 2.4.4 we need a technical auxiliary result.

Lemma 2.4.5. *Let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be as in Definition 2.4.3 and χ a smooth function such that $\chi \equiv 0$ on $[0,1]$, $\chi \equiv 1$ on $[2,\infty)$ and $0 \leqslant \chi \leqslant 1$. Then there exists a zero sequence $\{\mu_j\}_j$ such that for some fixed $\eta \in (0,1]$ we have for every $j \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}^n$ with $|\alpha+\beta| \leqslant j$:*

$$|\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) q^{(m_j)}_{\alpha,\beta}(a_{j,\varepsilon})| \leqslant 2^{-j-1}\varepsilon^k \omega_\varepsilon^{\nu|\beta|+l_j}(\varepsilon\omega_\varepsilon^\sigma)^{-1} \quad \forall \varepsilon \in (0,\eta],$$

with the same σ as in Definition 2.4.3.

Proof. Using the uniformity condition (2.4.2) of the sequence $\{(a_{j,\varepsilon})_\varepsilon\}_j$ we obtain that for some $\eta \in (0,1]$ and $\forall j \in \mathbb{N}$, $\forall \alpha, \beta \in \mathbb{N}^n$, $\exists C^{(1)}_{j,\alpha,\beta} > 0$ and for all $\varepsilon \in (0,\eta]$ we have

$$\begin{aligned}\varepsilon^{-k}\omega_\varepsilon^{-\nu|\beta|-l_j}|\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})q^{(m_j)}_{\alpha,\beta}(a_{j,\varepsilon})| &\leqslant C^{(1)}_{j,\alpha,\beta}\ \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) = \\ &= C^{(1)}_{j,\alpha,\beta}\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})\frac{\varepsilon\omega_\varepsilon^\sigma}{\mu_j} \leqslant \\ &\leqslant C^{(1)}_{j,\alpha,\beta}\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\end{aligned}$$

since $\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) = 0$ for $\mu_j \leqslant \varepsilon\omega_\varepsilon^\sigma$ and $0 \leqslant \chi \leqslant 1$. We now choose the sequence $\{\mu_j\}_j$ in the following way: for all $j \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha+\beta| \leqslant j$ we have

$$C^{(1)}_{j,\alpha,\beta}\mu_j \leqslant 2^{-j-1} \tag{2.4.3}$$

and the proof is complete. □

Proof of Lemma 2.4.4. In the following proof we basically combine techniques from the theory of nonlinear generalized functions, [19, Theorem 2.2], and Borel's Theorem from the semiclassical approach as in [14, Theorem 4.11] or alternatively [39, Proposition 2.3.2]. To avoid overload calculations, we may assume without loss of generality that $k = 0$ in Lemma 2.4.4 and therefore also in the uniformity condition (2.4.2).

We let $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \leqslant \phi \leqslant 1$ such that $\phi(\xi) = 0$ for $|\xi| \leqslant 1$ and $\phi(\xi) = 1$ for $|\xi| \geqslant 2$. Further let χ and $\{\mu_j\}_j$ be as in Lemma 2.4.5. We introduce functions

$$\begin{aligned}b^{(1)}_{j,\varepsilon}(x,\xi) &:= \chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})a_{j,\varepsilon}(x,\xi)\\ b^{(2)}_{j,\varepsilon}(x,\xi) &:= \left(1-\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})\right)\phi(\lambda_j\xi)a_{j,\varepsilon}(x,\xi)\end{aligned}$$

where $\{\lambda_j\}_j$ is a strictly decreasing zero sequence such that for all $j : \lambda_j \leqslant 1$ and will be specified later. For fixed $\varepsilon \in (0,1]$ we also define

$$a_\varepsilon(x,\xi) := \sum_{j\geqslant 0}\varepsilon^j b^{(1)}_{j,\varepsilon}(x,\xi) + \sum_{j\geqslant 0}\varepsilon^j b^{(2)}_{j,\varepsilon}(x,\xi).$$

By construction the first sum consists of at most finitely many nonzero terms (depending on $\varepsilon \in (0,1]$ fixed), since $\mu_j \to 0$ as $j \to \infty$. Moreover the second sum is locally finite and we conclude that $(a_\varepsilon)_\varepsilon \in \mathcal{E}(\mathbb{R}^n \times \mathbb{R}^n)^{(0,1]}$.

Step 1: In the first part of the proof we show that $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,0}}$. Since $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l_j)}_{S^{m_j,0}}$ satisfy the uniform growth type condition with $k = 0$ we obtain that there exists $\eta \in (0,1]$

such that for all $j \in \mathbb{N}$, and all $\alpha, \beta \in \mathbb{N}^n$ we have

$$|\partial_\xi^\alpha \partial_x^\beta b_{j,\varepsilon}^{(1)}(x,\xi)| \leqslant C_{j,\alpha,\beta}^{(1)} \omega_\varepsilon^{\nu|\beta|+l_j} \langle \xi \rangle^{m_j - |\alpha|} \quad \forall \varepsilon \in (0,\eta] \tag{2.4.4}$$

and the constant $C_{j,\alpha,\beta}^{(1)} > 0$ is the same as in the proof of Lemma 2.4.5.

Concerning $b_{j,\varepsilon}^{(2)}$, we observe that $\mathrm{supp}(\partial^\alpha \phi)(\lambda_j \xi) \subseteq \{\xi : \lambda_j^{-1} \leqslant |\xi| \leqslant 2\lambda_j^{-1}\}$ for every $|\alpha| \geqslant 1$. Accordingly, if $|\alpha| \geqslant 1$, we can assume $\lambda_j \leqslant 2/|\xi| \leqslant 4\langle \xi \rangle^{-1}$ on $\mathrm{supp}(\partial^\alpha \phi)(\lambda_j \xi)$ and $b_{j,\varepsilon}^{(2)}$ can be estimated as follows: $\exists \eta \in (0,1]$ independent of α, β and j such that

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta b_{j,\varepsilon}^{(2)}(x,\xi)| &\leqslant \langle \xi \rangle^{m_j - |\alpha|} \sum_{\gamma \leqslant \alpha} c(\chi, \phi, \gamma) 4^{|\alpha - \gamma|} q_{\gamma,\beta}^{(m_j)}(a_{j,\varepsilon}) \leqslant \\ &\leqslant C_{j,\alpha,\beta}^{(2)} \omega_\varepsilon^{\nu|\beta|+l_j} \langle \xi \rangle^{m_j - |\alpha|} \end{aligned} \tag{2.4.5}$$

for all $\varepsilon \in (0,\eta]$. At this point we choose the sequence $\{\lambda_j\}_j$ strictly decreasing and so that

$$\forall j \in \mathbb{N} \ \forall \alpha, \beta \in \mathbb{N}^n \text{ with } |\alpha + \beta| \leqslant j \text{ we have: } \quad C_{j,\alpha,\beta}^{(2)} \lambda_j \leqslant 2^{-j-1}. \tag{2.4.6}$$

To show that $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^{m,0}}^{(\nu,l)}$ we note that

$$\forall \alpha, \beta \in \mathbb{N}^n \ \exists j_0 \in \mathbb{N} : \ |\alpha + \beta| \leqslant j_0, \ m_{j_0} + 1 \leqslant m \tag{2.4.7}$$

and decompose $(a_\varepsilon)_\varepsilon$ in an appropriate way, that now is

$$a_\varepsilon(x,\xi) = \sum_{j \leqslant j_0 - 1} \varepsilon^j b_{j,\varepsilon}(x,\xi) + \sum_{j \geqslant j_0} \varepsilon^j b_{j,\varepsilon}(x,\xi) =: f_\varepsilon(x,\xi) + s_\varepsilon(x,\xi) \tag{2.4.8}$$

where we have set $b_{j,\varepsilon}(x,\xi) := b_{j,\varepsilon}^{(1)}(x,\xi) + b_{j,\varepsilon}^{(2)}(x,\xi)$. Then $\exists \eta \in (0,1]$ such that for all $\alpha, \beta \in \mathbb{N}^n$

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta f_\varepsilon(x,\xi)| &= |\sum_{j \leqslant j_0 - 1} \varepsilon^j \partial_\xi^\alpha \partial_x^\beta \{b_{j,\varepsilon}^{(1)}(x,\xi) + b_{j,\varepsilon}^{(2)}(x,\xi)\}| \leqslant \\ &\leqslant \omega_\varepsilon^{\nu|\beta|+l} \langle \xi \rangle^{m - |\alpha|} \sum_{j \leqslant j_0 - 1} (\varepsilon \omega_\varepsilon^\sigma)^j \big(C_{j,\alpha,\beta}^{(1)} + C_{j,\alpha,\beta}^{(2)}\big) \leqslant \\ &\leqslant C_{j_0} \omega_\varepsilon^{\nu|\beta|+l} \langle \xi \rangle^{m - |\alpha|} \end{aligned}$$

for all $\varepsilon \in (0,\eta]$ by (2.4.4) and (2.4.5). We next estimate the remainder s_ε. For this purpose, we apply (2.4.3) from Lemma 2.4.5 and use (2.4.6) to obtain

$$\begin{aligned}
\omega_\varepsilon^{-\nu|\beta|-l}|\partial_\xi^\alpha \partial_x^\beta s_\varepsilon(x,\xi)| &\leqslant \sum_{j\geqslant j_0} (\varepsilon\omega_\varepsilon^\sigma)^j \langle\xi\rangle^{m_j-|\alpha|} 2^{-j-1}\left\{\mu_j^{-1}+\lambda_j^{-1}\right\} \leqslant \\
&\leqslant (\varepsilon\omega_\varepsilon^\sigma)^{j_0-1}\langle\xi\rangle^{m_{j_0}+1-|\alpha|}\sum_{j\geqslant j_0} 2^{-j}(\varepsilon\omega_\varepsilon^\sigma)^{j-j_0} \leqslant \\
&\leqslant (\varepsilon\omega_\varepsilon^\sigma)^{j_0-1}\langle\xi\rangle^{m_{j_0}+1-|\alpha|} \leqslant \langle\xi\rangle^{m-|\alpha|} \qquad \forall\varepsilon\in(0,\eta]
\end{aligned}$$

for some $\eta \in (0,1]$ independent of the order of differentiation. Note that in the second inequality we used that $\mu_j^{-1} \leqslant (\varepsilon\omega_\varepsilon^\sigma)^{-1}$ on $\mathrm{supp}(b_{j,\varepsilon}^{(1)})$ and $\lambda_j^{-1} \leqslant \langle\xi\rangle$ on $\mathrm{supp}(b_{j,\varepsilon}^{(2)})$. And so, $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{S^{m,0}}^{(\nu,l)}$ as required.

Step 2: We complete the proof by showing that for every $N \geqslant 1$ we have

$$(a_\varepsilon - \sum_{j\leqslant N-1}\varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}_{S^{m_N,N}}^{(\nu,l_N)}(\mathbb{R}^n\times\mathbb{R}^n). \tag{2.4.9}$$

Therefore, we let $N \geqslant 1$ be a fixed natural number and write

$$\begin{aligned}
a_\varepsilon(x,\xi) - \sum_{j\leqslant N-1}\varepsilon^j a_{j,\varepsilon}(x,\xi) = \sum_{j\leqslant N-1}&\varepsilon^j\big(1-\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})\big)\big(\phi(\lambda_j\xi)-1\big)a_{j,\varepsilon}(x,\xi) + \\
&+\sum_{j\geqslant N}\varepsilon^j b_{j,\varepsilon}(x,\xi) =: g_\varepsilon(x,\xi)+t_\varepsilon(x,\xi).
\end{aligned}$$

We first calculate the contribution of g_ε. For this note that $|\xi| \leqslant 2\lambda_j^{-1}$ on the support of $(\phi(\lambda_j\xi)-1)$ and we obtain that $\exists\eta\in(0,1]$ such that $\forall j \leqslant N-1$ we have

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta\big(\phi(\lambda_j\xi)-1\big)a_{j,\varepsilon}(x,\xi)| &\leqslant \\
&\leqslant \sum_{\gamma\leqslant\alpha} c(\phi,\gamma)\lambda_j^{|\gamma|} q_{\alpha-\gamma,\beta}^{(m_j)}(a_{j,\varepsilon})\langle\xi\rangle^{m_j-|\alpha-\gamma|} \leqslant \\
&\leqslant \langle\xi\rangle^{m_N-|\alpha|}\sum_{\gamma\leqslant\alpha} c(\phi,\gamma)\lambda_j^{|\gamma|}\langle 2\lambda_{N-1}^{-1}\rangle^{m_j-m_N+|\gamma|} q_{\alpha-\gamma,\beta}^{(m_j)}(a_{j,\varepsilon}) \leqslant \\
&\leqslant C_{N,\alpha,\beta}\,\omega_\varepsilon^{\nu|\beta|+l_j}\langle\xi\rangle^{m_N-|\alpha|} \qquad \forall\varepsilon\in(0,\eta].
\end{aligned}$$

Now taking into account that $1-\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1}) \neq 0$ only if $\mu_j \leqslant 2\varepsilon\omega_\varepsilon^\sigma$ we compute

$$\begin{aligned}
&\omega_\varepsilon^{-\nu|\beta|-l}|\partial_\xi^\alpha\partial_x^\beta g_\varepsilon(x,\xi)| \leqslant \\
&\qquad \leqslant C_{N,\alpha,\beta}\langle\xi\rangle^{m_N-|\alpha|}\sum_{j\leqslant N-1}(\varepsilon\omega_\varepsilon^\sigma)^j\left(1-\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})\right) = \\
&\qquad = C_{N,\alpha,\beta}(\varepsilon\omega_\varepsilon^\sigma)^N\langle\xi\rangle^{m_N-|\alpha|}\sum_{j\leqslant N-1}\mu_j^{j-N}\left(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\right)^{N-j}\left(1-\chi(\mu_j(\varepsilon\omega_\varepsilon^\sigma)^{-1})\right) \leqslant \\
&\qquad \leqslant \widetilde{C}_{N,\alpha,\beta}(\varepsilon\omega_\varepsilon^\sigma)^N\langle\xi\rangle^{m_N-|\alpha|} \qquad \forall\varepsilon\in(0,\eta].
\end{aligned}$$

We proceed with the estimation of t_ε in which we again use a decomposition as in (2.4.7) and (2.4.8). Therefore we let $N\in\mathbb{N}$, $N\geqslant 1$ be fixed. Then

$$\forall\alpha,\beta\in\mathbb{N}^n\ \exists j_0\in\mathbb{N}:\ |\alpha+\beta|\leqslant j_0,\ m_{j_0}+1\leqslant m_N \text{ and } j_0-1\geqslant N;$$

we write

$$t_\varepsilon(x,\xi) = \sum_{j=N}^{j_0-1}\varepsilon^j b_{j,\varepsilon}(x,\xi) + s_\varepsilon(x,\xi) \tag{2.4.10}$$

with s_ε as in (2.4.8). As already shown in the estimate for s_ε we also have

$$|\partial_\xi^\alpha\partial_x^\beta s_\varepsilon(x,\xi)| \leqslant \omega_\varepsilon^{\nu|\beta|+l}(\varepsilon\omega_\varepsilon^\sigma)^{j_0-1}\langle\xi\rangle^{m_{j_0}+1-|\alpha|}.$$

Furthermore we obtain for the first term on the right hand side of (2.4.10)

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta\sum_{j=N}^{j_0-1}\varepsilon^j b_{j,\varepsilon}(x,\xi)| &\leqslant \omega_\varepsilon^{\nu|\beta|+l}\langle\xi\rangle^{m_N-|\alpha|}\sum_{j=N}^{j_0-1}(\varepsilon\omega_\varepsilon^\sigma)^j\left(C^{(1)}_{j,\alpha,\beta}+C^{(2)}_{j,\alpha,\beta}\right) \leqslant \\
&\leqslant C_{j_0}(\varepsilon\omega_\varepsilon^\sigma)^N\omega_\varepsilon^{\nu|\beta|+l}\langle\xi\rangle^{m_N-|\alpha|}.
\end{aligned}$$

Putting this together we get (2.4.9) which completes the proof. □

Remark 2.4.6. Before we proceed we make some observations concerning the construction of the generalized symbol $(a_\varepsilon)_\varepsilon$ in Lemma 2.4.4 which was given by the expression

$$a_\varepsilon(x,\xi) = \sum_{j\geqslant 0}\varepsilon^j\chi\Big(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\Big)a_{j,\varepsilon}(x,\xi) + \sum_{j\geqslant 0}\varepsilon^j\Big\{1-\chi\Big(\frac{\mu_j}{\varepsilon\omega_\varepsilon^\sigma}\Big)\Big\}\phi(\lambda_j\xi)a_{j,\varepsilon}(x,\xi).$$

Here, the first sum is inspired from the semiclassical approach whereas the second sum is also used in the usual Colombeau framework. Then, it is may worth to mention the following facts:

(a) To construct the square root of the homogeneous operator L_ψ we also need to discuss polyhomogeneous symbols. Then the second sum in the definition of a_ε also makes sense for homogeneous symbols $a_{j,\varepsilon}$, but the first sum would induce singularities (at $\xi = 0$) for fixed $\varepsilon \in (0,1]$. To overcome this problem we will introduce generalized symbols that are smoothed off at the zero section so that we can require a symbol to have an asymptotic expansion in homogeneous terms if we allow some additional error terms in the asymptotic expansion. For the precise description of these so-called polyhomogeneous generalized symbols we refer to Definition 2.5.1.

(b) Another important property concerns the effect of rescaling. This will be used in Section 2.6 where we test microlocal regularity of generalized functions using the L^2-boundedness result of zeroth-order ψ-pseudodifferential operators. To exemplify this, let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l_j)}_{S^{-j,k}}$ and l_j as in Definition 2.4.3, $j \geqslant 0$. By Lemma 2.4.4 there exists $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{0,k}}$ such that for all $N \geqslant 1$ we have

$$(r_\varepsilon)_\varepsilon := (a_\varepsilon - \sum_{j\leqslant N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{-N,N+k-1}}. \tag{2.4.11}$$

Now using the rescaling $(x,\xi) \mapsto (r_\varepsilon(x,\varepsilon\xi))_\varepsilon$ we obtain for every $N \geqslant 1$

$$(x,\xi) \mapsto (r_\varepsilon(x,\varepsilon\xi))_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{-N,k-1}}$$

concluding that the asymptotic expansion of the first kind keeps the order of the error operator invariant under rescaling but not its growth type. According to the L^2-boundedness it seems suitable to have an asymptotic expansion which preserves the growth type of the error operator in (2.4.11) under rescaling rather than the order of the same.

Therefore, note that $(r_\varepsilon)_\varepsilon$ in (2.4.11) can also be considered as a symbol of class $\mathcal{N}_{S^0}$. Then the rescaling does not influence the negligibility of the symbol $(r_\varepsilon)_\varepsilon$ since

$$(x,\xi) \mapsto (r_\varepsilon(x,\varepsilon\xi))_\varepsilon \in \mathcal{N}_{S^0}.$$

Using this, we now introduce the asymptotic expansion of the second kind in order to achieve an invariant growth type under rescaling at least when studying zeroth-order operators. In detail, we will study asymptotic expansions of the form $(\sum_{j\geqslant 0} \varepsilon^j a_{j,\varepsilon})_\varepsilon$, which are similar to those in Definition 2.4.3, but we use the fact that the sequence $\{(a_{j,\varepsilon})_\varepsilon\}_j$ is uniformly bounded in the generalized symbol class of order $m = m_0$.

2.4.2 Asymptotic expansion of the second kind

To eliminate the rescaling problem arising in Remark 2.4.6 (b) we now present the asymptotic expansion of the second kind.

Definition 2.4.7. *Let $m, k \in \mathbb{R}$ and $\{l_j\}_j$ be a sequence of real numbers of the form $l_j = \sigma j + l$ for some fixed $\sigma, l \in \mathbb{R}$ and $\sigma \geqslant 0$, $j \in \mathbb{N}$. Further we let $\{(a_{j,\varepsilon})_\varepsilon\}_j$ be a sequence with $(a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l_j)}_{S^{m,k}}$ satisfying the uniformity condition:*

$$\exists \eta \in (0,1] \; \forall j \in \mathbb{N} \; \forall \alpha, \beta \in \mathbb{N}^n \; \exists C > 0 \text{ such that}$$
$$q^{(m)}_{\alpha,\beta}(a_{j,\varepsilon}) \leqslant C \varepsilon^k \omega_\varepsilon^{\nu|\beta|+l_j} \qquad \forall \varepsilon \in (0,\eta].$$

We say that the $\sum_{j=0}^\infty (\varepsilon^j a_{j,\varepsilon})_\varepsilon$ is the asymptotic expansion of the second kind for the symbol $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}}$, denoted by $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$, if

$$(a_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j a_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,l_N)}_{S^{m,N}} \quad \forall N \geqslant 1.$$

Again we call $(a_{0,\varepsilon})_\varepsilon$ the principal symbol of $(a_\varepsilon)_\varepsilon$ if $a_{0,\varepsilon}$ is not identically zero for every fixed $\varepsilon > 0$ sufficiently small.

Comparing Definition 2.4.3 and Definition 2.4.7 we observe that any generalized symbol $(a_\varepsilon)_\varepsilon$ from Lemma 2.4.4 also admits an asymptotic expansion of the second kind. We remark that in both definitions we used the notation $(a_\varepsilon)_\varepsilon \sim \sum_j (\varepsilon^j a_{j,\varepsilon})_\varepsilon$, but in the following the context will always explain which notion is meant.

Furthermore, we note that our notions of an asymptotic expansion are different from the one used in the semiclassical approach. A crucial difference is that in the semiclassical setting one uses the Hörmander symbol class $S^m_{0,0}$ as the underlying classical space and not $S^m := S^m_{1,0}$ as in our case, cf. [30]. As above, the semiclassical asymptotic expansion provides an invariant character of zeroth-order operators under rescaling. For more details on the semiclassical asymptotic expansion we refer to [13, 39, 29].

2.4.3 Composition of ψ-pseudodifferential operators

In this subsection we establish the composition law of two ψ-pseudodifferential operators utilizing the method of stationary phase to determine the asymptotic behavior in the occurring oscillatory integrals. Because of the specific construction of the ψ-pseudodifferential operators it suffices to study a corollary of the stationary phase formula which can be found in [39, Corollary 2.6.3] and [27, Example 2.2].

In detail, this simplified formula says that given a function $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2n})$ then for every $N \geqslant 1$ one has

$$\left(\frac{\lambda}{2\pi}\right)^n \int e^{-i\lambda xy} a(x,y)\,dx\,dy = \sum_{|\alpha| \leqslant N-1} \frac{1}{\alpha!\lambda^{|\alpha|}} \left(D_x^\alpha \partial_y^\alpha a\right)(0,0) + S_N(a;\lambda)$$

where the parameter λ is considered in the limit $\lambda \to \infty$. Moreover the remainder term $S_N(a;\lambda)$ can be estimated as follows

$$|S_N(a;\lambda)| \leqslant \frac{C}{N!\lambda^N} \sum_{|\alpha+\beta| \leqslant 2n+1} \|\partial_x^\alpha \partial_y^\beta (\partial_x \cdot \partial_y)^N a\|_{L^1(\mathbb{R}^{2n})} \quad \lambda \geqslant 1$$

where $\partial_x \cdot \partial_y = \sum_{j=1}^n \partial_{x_j}\partial_{y_j}$ and the constant $C > 0$ is independent of λ. In the application we will work with smooth functions a which are not compactly supported but are well-behaved at infinity and also depend on the parameter λ. For a more general discussion on the method of stationary phase we refer the reader to [39, Section 2.6] or alternatively to [27, Section 2].

Using the above formula, we can show the following result.

Proposition 2.4.8. *Let $A_\psi(x, D_x)$ and $B_\psi(x, D_x)$ be two ψ-pseudodifferential operators with generalized symbols $(a_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l_1)}_{S^{m_1,k_1}}$ and $(b_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l_2)}_{S^{m_2,k_2}}$ respectively. Then the composition $A_\psi B_\psi$ is well-defined and maps $\mathcal{G}_{L^2}$ into itself. Moreover, $A_\psi B_\psi$ is a ψ-pseudodifferential operator with generalized symbol $(a_\varepsilon \# b_\varepsilon)_\varepsilon$ in $\mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2,k_1+k_2}}$ and we have the following representation*

$$(a_\varepsilon \# b_\varepsilon)_\varepsilon \sim \sum_{|\alpha| \geqslant 0} \left(\frac{\varepsilon^{|\alpha|}}{\alpha!} D_\xi^\alpha a_\varepsilon(x,\xi) \partial_x^\alpha b_\varepsilon(x,\xi) \right)_\varepsilon . \tag{2.4.12}$$

Note that in Proposition 2.4.8 the generalized symbol $(a_\varepsilon \# b_\varepsilon)_\varepsilon$ is given by its asymptotic expansion of the first kind. Hence it can also be reinterpreted as a symbol having an asymptotic expansion of the second kind.

The following proof is an adaption of [39, Theorem 2.6.5] and [14, Theorem 9.13].

Proof of Proposition 2.4.8. Without loss of generality we assume that $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are of growth type 0, i.e. $k_1 = k_2 = 0$, as the proof for more general growth type assumptions requires only slight changes in the argumentation. Let $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ having $(u_\varepsilon)_\varepsilon$ as representative and let $\varepsilon \in (0,1]$ be fixed and arbitrary. Then $A_\psi B_\psi u$ makes sense as an oscillatory integral and we write

$$A_{\psi,\varepsilon} B_{\psi,\varepsilon} u_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^n} \lim_{\sigma\to 0_+} \int e^{i(x-y)\eta/\varepsilon - \sigma\langle y\rangle - \sigma\langle\eta\rangle} a_\varepsilon(x,\eta)(B_{\psi,\varepsilon}u_\varepsilon)(y)\, dy\, d\eta =$$

$$= \frac{1}{(2\pi\varepsilon)^n} \lim_{\sigma\to 0_+,\, \tau\to 0_+} \int e^{ix\xi/\varepsilon - \tau\langle\xi\rangle} c_{\sigma,\varepsilon}(x,\xi)\hat{u}_\varepsilon(\xi/\varepsilon)\, d\xi \tag{2.4.13}$$

where we have set

$$c_{\sigma,\varepsilon}(x,\xi) = (2\pi\varepsilon)^{-n} \int e^{i(x-y)(\eta-\xi)/\varepsilon - \sigma\langle y\rangle - \sigma\langle\eta\rangle} a_\varepsilon(x,\eta) b_\varepsilon(y,\xi)\, dy\, d\eta.$$

In what follows we show that $c_{\sigma,\varepsilon}(x,\xi)$ corresponds to a generalized symbol in $\mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2,0}}$ uniformly for all $\sigma > 0$. Here the relevant asymptotic behavior of $c_{\sigma,\varepsilon}$ as $\varepsilon \to 0$ is described by using the method of stationary phase. Finally, by passing to the limit $\sigma \to 0_+$, we obtain by Lebesgue's dominated convergence theorem that the limit of $(c_{\sigma,\varepsilon})_\varepsilon$ exists in $\mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2,0}}$ and admits a representation as stated in (2.4.12).

For that purpose, we first split the integral representation of $c_{\sigma,\varepsilon}(x,\xi)$ into three parts. Therefore, let $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\chi(s) = 1$ for $|s| \leqslant 1/4$ and $\chi(s) = 0$ for $|s| \geqslant 1/2$ and set $\chi_1(x,y) = \chi(|x-y|)$ for $x, y \in \mathbb{R}^n$. For $|\xi| \geqslant 1$ we write

$$c_{\sigma,\varepsilon}(x,\xi) = d_{\sigma,\varepsilon}(x,\xi) + e_{\sigma,\varepsilon}(x,\xi) + f_{\sigma,\varepsilon}(x,\xi), \tag{2.4.14}$$

where $d_{\sigma,\varepsilon}, e_{\sigma,\varepsilon}$ and $f_{\sigma,\varepsilon}$ are defined by the additional factors $1 - \chi(|\xi-\eta|/|\xi|)$, $\chi(|\xi - \eta|/|\xi|)(1-\chi_1(x,y))$ and $\chi(|\xi-\eta|/|\xi|)\chi_1(x,y)$ in the integrand of $c_{\sigma,\varepsilon}$. Also we note that $f_{\sigma,\varepsilon}$ represents the part of $c_{\sigma,\varepsilon}$ that will determine its asymptotic behavior as we meet the critical points of the phase.

In the following we give the proof under the assumption that $|\xi| \geqslant 1$. In the case that $|\xi| \leqslant 1$ a likewise decomposition as in (2.4.14) will lead to the desired result. Again, this is similar as in the proof of [39, Theorem 2.6.5].

As a first step we check that the integrand of $d_{\sigma,\varepsilon}$ is in $L^1(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ uniformly in $\sigma > 0$. Therefore, we introduce the operator

$$L_\varepsilon := \Big(1 + \frac{|\xi-\eta|^2}{\varepsilon^2} + \frac{|x-y|^2}{\varepsilon^2}\Big)^{-1}\Big(1 + \frac{\xi-\eta}{\varepsilon}D_y + \frac{x-y}{\varepsilon}D_\eta\Big)$$

satisfying the equality

$$L_\varepsilon e^{i(x-y)(\eta-\xi)/\varepsilon} = e^{i(x-y)(\eta-\xi)/\varepsilon}.$$

Using integration by parts we obtain

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x,\xi) = \\ = \int e^{i(x-y)(\eta-\xi)/\varepsilon}({}^tL_\varepsilon)^k\Big\{(1-\chi(|\xi-\eta|/|\xi|))e^{-\sigma\langle y\rangle-\sigma\langle\eta\rangle}a_\varepsilon(x,\eta)b_\varepsilon(y,\xi)\Big\}\,dy\,d\eta.$$

Further, for every $k \in \mathbb{N}$ with $k > n + 1/2$ we obtain

$$\begin{aligned}(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x,\xi) &= \\ &= \int_{|\xi-\eta|\geqslant|\xi|/4} \mathcal{O}\Big(\frac{\omega_\varepsilon^{k\nu+l_1+l_2}\langle\eta\rangle^{m_1}\langle\xi\rangle^{m_2}}{(1+|\xi-\eta|/\varepsilon+|x-y|/\varepsilon)^k}\Big)\,dy\,d\eta = \\ &= \int_{|\xi-\eta|\geqslant|\xi|/4} \mathcal{O}\Big(\varepsilon^n\omega_\varepsilon^{k\nu+l_1+l_2}\langle\eta\rangle^{m_1}\langle\xi\rangle^{m_2}\Big\{1+\frac{|\xi-\eta|}{\varepsilon}\Big\}^{n+1/2-k}\Big)\,d\eta = \\ &= \int_{|\xi-\eta|\geqslant|\xi|/4} \mathcal{O}\Big(\varepsilon^{k-1/2}\omega_\varepsilon^{k\nu+l_1+l_2}\frac{\langle\eta\rangle^{m_1}\langle\xi\rangle^{m_2}}{(1+|\xi|+|\eta|)^{k-n-1/2}}\Big)\,d\eta \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Then, by a straightforward calculation one shows that for every $k \geqslant |m_1| + 2n + 1$ we have

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x,\xi) = \mathcal{O}\left(\varepsilon^{k-1/2}\omega_\varepsilon^{k\nu+l_1+l_2}\langle\xi\rangle^{m_1+m_2-k+2n+1+|m_1|}\right)$$

for ε sufficiently small. Since $(\omega_\varepsilon)_\varepsilon$ is a slow scale net we conclude that

$$(2\pi\varepsilon)^n d_{\sigma,\varepsilon}(x,\xi) = \mathcal{O}\left(\varepsilon^N\omega_\varepsilon^{l_1+l_2}\langle\xi\rangle^{m_1+m_2-N}\right) \quad \forall\ N\geqslant 0$$

uniformly for every $(x,\xi) \in \mathbb{R}^{2n}$ with $|\xi| \geqslant 1$ and $\sigma > 0$ as $\varepsilon \to 0$.

We next estimate $e_{\sigma,\varepsilon}(x,\xi)$. Using the coordinate change $z = y - x$ and integration by parts we can write

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x,\xi) = \int_{\substack{|z|\geqslant 1/4\\ |\xi-\eta|\leqslant|\xi|/2}} e^{-iz(\eta-\xi)/\varepsilon}({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x,z,\xi,\eta)\,dz\,d\eta$$

where we have set $L_{1,\varepsilon} := -\varepsilon z/|z|^2 D_\eta$ and

$$r_{\sigma,\varepsilon}(x,z,\xi,\eta) := \chi(|\xi-\eta|/|\xi|)(1-\chi(|z|))e^{-\sigma\langle x+z\rangle-\sigma\langle\eta\rangle}a_\varepsilon(x,\eta)b_\varepsilon(x+z,\xi).$$

Note that the operator $L_{1,\varepsilon}$ satisfies

$${}^tL_{1,\varepsilon} = -L_{1,\varepsilon} \quad \text{and} \quad L_{1,\varepsilon}e^{-iz(\eta-\xi)/\varepsilon} = e^{-iz(\eta-\xi)/\varepsilon}.$$

Also, the integrand of $e_{\sigma,\varepsilon}(x,\xi)$ is integrable with respect to z if we take $k > n$. To check the integrability with respect to the η-variable we partition the domain of integration into the two regions

$$\Omega_1 := \{\eta \in \mathbb{R}^n : |\xi - \eta| \leqslant 1/4\}, \quad \Omega_2 := \{\eta \in \mathbb{R}^n : |\xi - \eta| \geqslant 1/4\}$$

and write

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x,\xi) = \sum_{j=1}^{2} \int_{\Omega_j} e^{-iz(\eta-\xi)/\varepsilon} ({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x,z,\xi,\eta)\, dz\, d\eta =: \sum_{j=1}^{2} I_j.$$

Concerning I_1 we use the coordinate transformation $\zeta = \eta - \xi$ and by Peetre's inequality $\langle \xi + \zeta\rangle^s \leqslant 2^{|s|/2}\langle\xi\rangle^s\langle\zeta\rangle^{|s|}$ for every $s \in \mathbb{R}$ we obtain

$$I_1 = \int_{\substack{|z|\geqslant 1/4 \\ |\zeta|\leqslant 1/4}} \mathcal{O}(\varepsilon^k \omega_\varepsilon^{l_1+l_2} |z|^{-k} \langle\xi\rangle^{m_2} \langle \xi + \zeta\rangle^{m_1-k})\, dz\, d\zeta = \mathcal{O}(\varepsilon^k \omega_\varepsilon^{l_1+l_2} \langle\xi\rangle^{m_1+m_2-k})$$

for every $k > n$ as $\varepsilon \to 0$. For the estimation of the second part I_2 we introduce

$$L_{2,\varepsilon} := \frac{\varepsilon^2}{|\xi-\eta|^2} \frac{\xi - \eta}{\varepsilon} D_z$$

and let $j \in \mathbb{N}$. Again using integration by parts we obtain for $|\xi| \geqslant 1$

$$\begin{aligned} I_2 &= \int e^{-iz(\eta-\xi)/\varepsilon} ({}^tL_{2,\varepsilon})^j ({}^tL_{1,\varepsilon})^k r_{\sigma,\varepsilon}(x,z,\xi,\eta)\, dz\, d\eta = \\ &= \int_{\substack{|z|\geqslant 1/4 \\ 1/4\leqslant|\xi-\eta|\leqslant|\xi|/2}} \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} |z|^{-k} \langle\xi\rangle^{m_2} \langle\eta\rangle^{m_1-k} |\xi-\eta|^{-j})\, dz\, d\eta \end{aligned}$$

and is integrable with respect to z if we take $k > n$ for sufficiently small ε. Then, using the coordinate change $\eta = \xi + \zeta$ and applying Peetre's inequality gives

$$\begin{aligned} I_2 &= \int_{1/4\leqslant|\zeta|\leqslant|\xi|/2} \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} \langle\xi\rangle^{m_1+m_2-k} \langle\zeta\rangle^{|m_1|+k-j})\, d\zeta = \\ &= \mathcal{O}(\varepsilon^{j+k} \omega_\varepsilon^{j\nu+l_1+l_2} \langle\xi\rangle^{m_1+m_2-k}) \end{aligned}$$

for every $j, k \in \mathbb{N}$ with $k > n$ and $j > |m_1| + k + n$ as $\varepsilon \to 0$. Combining the above yields to

$$(2\pi\varepsilon)^n e_{\sigma,\varepsilon}(x,\xi) = \mathcal{O}\left(\varepsilon^N \omega_\varepsilon^{l_1+l_2} \langle\xi\rangle^{m_1+m_2-N}\right) \quad \forall N \geqslant 0$$

uniformly for all $\sigma > 0$ and $(x,\xi) \in \mathbb{R}^{2n}$, $|\xi| \geqslant 1$ as $\varepsilon \to 0$. By the same arguments as above, one can also show that for every $\alpha,\beta\in\mathbb{N}^n$, $|\xi|\geqslant 1$ and $N\in\mathbb{N}$ we have

$$|\partial_\xi^\alpha\partial_x^\beta d_{\sigma,\varepsilon}(x,\xi)|+|\partial_\xi^\alpha\partial_x^\beta e_{\sigma,\varepsilon}(x,\xi)|=\mathcal{O}\left(\varepsilon^N\omega_\varepsilon^{\nu|\beta|+l_1+l_2}\langle\xi\rangle^{m_1+m_2-|\alpha|-N}\right)$$

uniformly for $\sigma>0$ as $\varepsilon\to 0$. So we deduce that $(d_{\sigma,\varepsilon})_\varepsilon$ and $(e_{\sigma,\varepsilon})_\varepsilon$ are contained in $\mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2-N,N}}$ for every $N\in\mathbb{N}$ uniformly with respect to $\sigma>0$ and $(x,\xi)\in\mathbb{R}^{2n}$, $|\xi|\geqslant 1$.

So it remains to study the term $(2\pi\varepsilon)^n f_{\sigma,\varepsilon}(x,\xi)$ which was given by

$$\int e^{i(x-y)(\eta-\xi)/\varepsilon}\chi(|\xi-\eta|/|\xi|)\chi_1(x,y)e^{-\sigma(\langle y\rangle+\langle\eta\rangle)}a_\varepsilon(x,\eta)b_\varepsilon(y,\xi)\,dy\,d\eta.$$

Now writing $\xi=\lambda\nu$ where $\lambda=|\xi|$ and using the coordinate transformations

$$\zeta=(\eta-\xi)/\lambda,\quad z=y-x$$

we obtain

$$f_{\sigma,\varepsilon}(x,\xi)=\frac{\lambda^n}{(2\pi\varepsilon)^n}\int_{|z|\leqslant 1/2,\,|\zeta|\leqslant 1/2}e^{-i\lambda z\zeta/\varepsilon}t_{\sigma,\varepsilon}(z,\zeta;x,\xi)\,dz\,d\zeta$$

where $t_{\sigma,\varepsilon}$ is a function in $\mathcal{C}_c^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_\zeta)$ containing (x,ξ) as parameters and is given by

$$t_{\sigma,\varepsilon}(z,\zeta;x,\xi)=\chi(|\zeta|)\chi(|z|)e^{-\sigma\langle x+z\rangle-\sigma\langle\lambda(\nu+\zeta)\rangle}a_\varepsilon(x,\lambda(\nu+\zeta))b_\varepsilon(x+z,\lambda\nu).$$

Then the method of stationary phase gives for every $N\in\mathbb{N}$, $N\geqslant 1$

$$f_{\sigma,\varepsilon}(x,\xi)=\sum_{|\alpha|\leqslant N-1}\frac{\varepsilon^{|\alpha|}}{\alpha!\lambda^{|\alpha|}}D_\zeta^\alpha\partial_z^\alpha t_{\sigma,\varepsilon}(z,\zeta;x,\xi)\Big|_{\substack{z=0\\ \zeta=0}}+S_N(t_{\sigma,\varepsilon};\lambda/\varepsilon)$$

and the remainder can be estimated as follows

$$\begin{aligned}|S_N(t_{\sigma,\varepsilon};\lambda/\varepsilon)|&\leqslant\frac{C\varepsilon^N}{N!\lambda^N}\sum_{|\alpha+\beta|\leqslant 2n+1}\|\partial_\zeta^\alpha\partial_z^\beta(\partial_\zeta\cdot\partial_z)^N t_{\sigma,\varepsilon}\|_{L^1(\mathbb{R}^n_z\times\mathbb{R}^n_\zeta)}=\\&=\mathcal{O}\left(\varepsilon^N\omega_\varepsilon^{(2n+1+N)\nu+l_1+l_2}\lambda^{m_1+m_2-N}\right)=\\&=\mathcal{O}\left(\varepsilon^{N-1}\omega_\varepsilon^{l_1+l_2}\langle\xi\rangle^{m_1+m_2-N}\right)\end{aligned}$$

for $|\xi|\geqslant 1$ and ε sufficiently small. Finally, since similar estimates hold true for the derivatives, we conclude that $(S_N(t_{\sigma,\varepsilon}))_\varepsilon$ is contained in $\mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2-N,N-1}}$ for every $N\in\mathbb{N}$ uniformly with respect to $\sigma>0$ for every $(x,\xi)\in\mathbb{R}^{2n}$ with $|\xi|\geqslant 1$.

As already mentioned in the beginning of the proof we now pass to the limit $\sigma\to 0_+$. Then by Lebesgue's dominated convergence theorem we have

$$c_{\sigma,\varepsilon}(x,\xi) \to c_\varepsilon(x,\xi) \quad \text{as } \sigma \to 0_+$$

so that equation (2.4.13) reads

$$\begin{aligned} A_{\psi,\varepsilon}B_{\psi,\varepsilon}u_\varepsilon(x) &= (2\pi\varepsilon)^{-n} \lim_{\tau\to 0_+} \int e^{ix\xi/\varepsilon - \tau\langle\xi\rangle} c_\varepsilon(x,\xi)\hat{u}_\varepsilon(\xi/\varepsilon)\, d\xi = \\ &= (2\pi\varepsilon)^{-n} \int e^{ix\xi/\varepsilon} c_\varepsilon(x,\xi)\hat{u}_\varepsilon(\xi/\varepsilon)\, d\xi \end{aligned}$$

where the last equation holds in the sense of oscillatory integrals. Moreover the generalized symbol $(c_\varepsilon)_\varepsilon$ is equal to $(a_\varepsilon \# b_\varepsilon)_\varepsilon$ as described in (2.4.12) and in particular $(c_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l_1+l_2)}_{S^{m_1+m_2,0}}$. This completes the proof. □

2.5 A factorization procedure for L_ψ

In this section we mainly follow the ideas made in Section 1.5. First, we write for $L_\psi = \varepsilon^2 L_\varepsilon$ with L_ε from the beginning

$$L_\psi(x,z,D_t,D_x,D_z) =: (\partial_z^2 + A(x,z,D_t,D_x))_\psi \tag{2.5.1}$$

where $A_\psi = A_\psi(x,z,D_t,D_x)$ is the ψ-pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon$ and

$$a_\varepsilon(x,z,\tau,\xi) = c_\varepsilon(x,z)\tau^2 - \langle b_\varepsilon(x,z)\xi,\xi\rangle \quad \varepsilon \in (0,1]. \tag{2.5.2}$$

We briefly recall the requirements made on the coefficients $c_\varepsilon(x,z)$ and $b_{j,\varepsilon}(x,z), 1 \leqslant j \leqslant n-1$, from Section 2.1. The coefficient $c_\varepsilon(x,z)$ is of log-type up to order $r \in \mathbb{N}$, i.e. for some constant $C > 0$ we have $\|\partial^\alpha c_\varepsilon\|_{L^\infty} = \mathcal{O}(\log^{|\alpha|/r}(C/\varepsilon))$ as $\varepsilon \to 0$, and strictly non-zero in the sense that there exists $\varepsilon_1 \in (0,1]$ such that $\inf_{y\in\mathbb{R}^n} |c_\varepsilon(x,z)| \geqslant C$ for all $\varepsilon \in (0,\varepsilon_1]$ and some constant $C > 0$ independent of ε. The same is assumed to hold for the coefficients $b_{j,\varepsilon}$. Moreover, $c_\varepsilon \to c$ and $b_{j,\varepsilon} \to b_j$ in the Hölder space $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$ with exponent $\mu \in (0,1)$ as $\varepsilon \to 0$.

Also, we will study operators on $\mathbb{R}^{n+1}$ of the form

$$S_\psi = \sum_{j=0}^{2} S_{j,\psi}(x,z,D_t,D_x)(\partial_z^{2-j})_\psi$$

with coefficients $S_{j,\psi} \in \mathrm{OP}_\psi\, \mathcal{M}^{(1,l_j)}_{S^{k_j,0}}$ for some $k_j, l_j \in \mathbb{R}$, $j = 0,1,2$.

Concerning the factorization domain we make similar considerations as in Section 1.5 and introduce the set

$$I'_\theta := \{(x,z,\tau,\xi) \mid \tau \neq 0, \langle b(x,z)\xi,\xi\rangle < \sin^2(\theta)c(x,z)\tau^2\} \tag{2.5.3}$$

for some $\theta \in (0,\pi/2)$. Then I'_θ is an open subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash 0)$ and conic with respect to the second variable. Moreover, we have chosen I'_θ independent of the parameter $\varepsilon \in (0,1]$. Again, the main reason for restricting the analysis to the domain I'_θ is that $(a_\varepsilon)_\varepsilon$ is non-negative there.

Further we write

$$S_\psi = \sum_{j=0}^{2} S_{j,\psi}(x,z,D_t,D_x)(\partial_z^{2-j})_\psi \quad \text{on } I'_\theta$$

when the symbols of the coefficients $S_{j,\psi}$ are restricted to a set I'_θ.

Further, as already mentioned in Section 2.4, it is necessary to introduce the notion of polyhomogeneous generalized symbols that are smoothed off at the origin.

2.5.1 Polyhomogeneous symbols

Definition 2.5.1. *We say that a generalized symbol $(p_\varepsilon)_\varepsilon$ is polyhomogeneous in $\mathcal{M}^{(\nu,l)}_{S^{m,k}}$, denoted by $(p_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}_{\mathrm{phg}}}$, if there exist a sequence of symbols $(p_{m-j,\varepsilon})_\varepsilon$ in $\mathcal{M}^{(\nu,\nu j+l)}_{S^{m-j,k}_{\mathrm{hg}}}$, $j \geqslant 0$, and a cut-off $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 near the origin so that $\forall N \geqslant 1$ we have*

$$\Big(p_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j (1-\varphi)p_{m-j,\varepsilon}\Big)_\varepsilon \in \mathcal{M}^{(\nu,\nu N+l)}_{S^{m-N,N+k}} \quad \textit{for } \xi \textit{ sufficiently large.} \tag{2.5.4}$$

For (2.5.4) we will use the following notation in the sequel

$$(p_\varepsilon)_\varepsilon \sim \sum_{j\geqslant 0} \big(\varepsilon^j p_{m-j,\varepsilon}\big)_\varepsilon \quad \textit{in } \mathcal{M}^{(\nu,l)}_{S^{m,k}_{\mathrm{phg}}}.$$

Here the homogeneous part $(p_{m,\varepsilon})_\varepsilon$ is called the principal symbol of $(p_\varepsilon)_\varepsilon$ if there exist an $\eta \in (0,1]$ and a constant $C > 0$ such that $p_{m,\varepsilon} \not\equiv 0$ for all $|\xi| \geqslant 1/C$ and any fixed $\varepsilon \in (0,\eta]$. In particular, a ψ-pseudodifferential operator is polyhomogeneous if its generalized symbol is polyhomogeneous.

As in Definition 2.4.3 and in Definition 2.4.7 we used the notation $(a_\varepsilon)_\varepsilon \sim \sum(\varepsilon^j a_{j,\varepsilon})_\varepsilon$, but in the following it will be clear from the text which notion we use. Note that in (2.5.4) the function φ cuts off singularities of $p_{m-j,\varepsilon}$ around $\xi = 0$.

Remark 2.5.2. First note that the principal symbol is uniquely determined if the dual variable is sufficiently large. Also (2.5.4) can be interpreted in the following sense: $\forall N \geqslant 1$ we have

$$\big(p_\varepsilon - \sum_{j=0}^{N-1} \varepsilon^j (1-\varphi) p_{m-j,\varepsilon}\big)_\varepsilon = (r_\varepsilon)_\varepsilon + (s_\varepsilon)_\varepsilon \tag{2.5.5}$$

for some $(s_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,\nu N+l)}_{S^{m-N,N+k}}$ and $(r_\varepsilon)_\varepsilon$ is some compactly supported regular generalized symbol of order $-\infty$ in the following sense:

$$\begin{gathered} \exists \eta \in (0,1]\ \exists k \in \mathbb{R}\ \forall m \in \mathbb{R}\ \forall \alpha,\beta \in \mathbb{N}^n\ \exists C > 0 \text{ such that} \\ |\partial_\xi^\alpha \partial_x^\beta r_\varepsilon(x,\xi)| \leqslant C\varepsilon^k \langle\xi\rangle^{m-|\alpha|} \qquad \forall \varepsilon \in (0,\eta]. \end{gathered}$$

We refer to Section 2.6 where it is essential to distinguish between the two different types of errors on the right hand side of (2.5.5).

2.5.2 Construction of a parametrix

In the sequel we will need the following lemma.

Lemma 2.5.3. *Let $m \in \mathbb{R}$ and P_ψ be a polyhomogeneous ψ-pseudodifferential operator whose symbol is given by $(p_\varepsilon)_\varepsilon \sim \sum_{j=0}^{\infty} (\varepsilon^j p_{m-j,\varepsilon})_\varepsilon$ in $\mathcal{M}^{(\nu,0)}_{S^{m,0}_{\mathrm{phg}}}$ for some $(p_{m-j,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,\nu j)}_{S^{m-j,0}_{\mathrm{hg}}}$, $j \geqslant 0$. Furthermore suppose that the principal symbol $(p_{m,\varepsilon})_\varepsilon$ satisfies an estimate of the form*

$$\exists C > 0\ \exists \eta \in (0,1]: \quad |p_{m,\varepsilon}(x,\xi)| \geqslant C|\xi|^m \quad \varepsilon \in (0,\eta]. \tag{2.5.6}$$

Then, there exists a polyhomogeneous ψ-pseudodifferential operator Q_ψ with symbol in $\mathcal{M}^{(\nu,0)}_{S^{-m,0}_{\mathrm{phg}}}$ such that the following is valid

$$Q_\psi P_\psi = I_\psi + R_\psi, \tag{2.5.7}$$

where I is the identity and the symbol of R_ψ is in $\mathcal{N}_{S^0}$ for $|\xi| \geqslant 1/C$ for some $C > 0$. More precisely, the polyhomogeneous generalized symbol $q = (q_\varepsilon)_\varepsilon$ of Q_ψ can be written in terms of its asymptotic expansion, i.e.

$$(q_\varepsilon)_\varepsilon \sim \big(\sum_{k\geqslant 0} \varepsilon^k q_{-m-k,\varepsilon}\big)_\varepsilon \quad \text{in } \mathcal{M}^{(\nu,0)}_{S^{-m,0}_{\mathrm{phg}}} \tag{2.5.8}$$

for some $(q_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,\nu k)}_{S^{-m-k,0}_{\mathrm{hg}}}$, $k \geqslant 0$. Here, note that the asymptotic expansion should be understood as in Definition 2.5.1.

Proof. The proof follows the classical arguments given in [37, Chapter 2] but involves additional ε-dependencies of the symbols. In order to construct Q_ψ as in (2.5.8) that satisfies (2.5.7) we will recursively define $(q_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,\nu k)}_{S^{-m-k,0}_{\mathrm{hg}}}$, $k \geqslant 0$. Because of (2.5.6) we may define for some $\eta \in (0,1]$

$$q_{-m,\varepsilon}(x,\xi) := \begin{cases} p_{m,\varepsilon}(x,\xi)^{-1}, & \varepsilon \in (0,\eta] \\ 0, & \text{else.} \end{cases}$$

Concerning the asymptotic behavior of the derivatives of $q_{-m,\varepsilon}$ we use the Leibniz rule and obtain that $\exists C > 0$ such that for every $\varepsilon > 0$ sufficiently small

$$\begin{aligned} |\partial^\alpha_{(\tau,\xi)} \partial^\beta_y q_{-m,\varepsilon}| &= |\sum_{\substack{\alpha_1+\ldots+\alpha_\mu=\alpha \\ \beta_1+\ldots+\beta_\mu=\beta}} \left(\partial^{\alpha_1}_{(\tau,\xi)} \partial^{\beta_1}_y p_{m,\varepsilon}\right) \ldots \left(\partial^{\alpha_\mu}_{(\tau,\xi)} \partial^{\beta_\mu}_y p_{m,\varepsilon}\right) \frac{1}{p^{1+\mu}_{m,\varepsilon}}| \leqslant \\ &\leqslant C \omega_\varepsilon^{\nu|\beta|} |\xi|^{-m-|\alpha|} \end{aligned}$$

from which we deduce that $(q_{-m,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,0)}_{S^{-m,0}_{\mathrm{hg}}}$. We now proceed by induction. Therefore we define

$$q_{-m-k,\varepsilon} := -\Big\{ \sum_{\substack{|\gamma|+j+l=k \\ l<k}} \frac{1}{\gamma!} D^\gamma_\xi q_{-m-l,\varepsilon} \partial^\gamma_x p_{m-j,\varepsilon} \Big\} \frac{1}{p_{m,\varepsilon}} \qquad k \geqslant 1. \tag{2.5.9}$$

Assuming that $(q_{-m-N,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,\nu N)}_{S^{-m-N,0}_{\mathrm{hg}}}$ for $N < k$ then from the construction given in (2.5.9) one easily verifies that $(q_{-m-k,\varepsilon})_\varepsilon \in \mathcal{M}^{(\nu,\nu k)}_{S^{-m-k,0}_{\mathrm{hg}}}$.

Moreover, by Lemma 2.4.4 we get the existence of a generalized polyhomogeneous symbol $(q_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,0)}_{S^{-m,0}_{\mathrm{phg}}}$ having the following asymptotic expansion

$$(q_\varepsilon)_\varepsilon \sim \sum_{k \geqslant 0} (\varepsilon^k q_{-m-k,\varepsilon})_\varepsilon \qquad \text{in } \mathcal{M}^{(\nu,0)}_{S^{-m,0}}.$$

Let Q_ψ be the ψ-pseudodifferential operator with symbol $(q_\varepsilon)_\varepsilon$. Then (2.5.8) is satisfied and it remains to show equation (2.5.7). Concerning the asymptotic expansion of $Q_\psi P_\psi$

one has for every $N \geqslant 1$

$$\begin{aligned}\sum_{|\gamma|<N} \frac{\varepsilon^{|\gamma|}}{\gamma!} D_\xi^\gamma q_\varepsilon \partial_x^\gamma p_\varepsilon = q_{-m,\varepsilon} p_{m,\varepsilon} + \sum_{k=1}^{N-1} \sum_{|\gamma|+l+j=k} \frac{\varepsilon^k}{\gamma!} D_\xi^\gamma q_{-m-l,\varepsilon} \partial_x^\gamma p_{m-j,\varepsilon} + \\ + \sum_{k\geqslant N} \sum_{\substack{|\gamma|+l+j=k \\ |\gamma|<N}} \frac{\varepsilon^k}{\gamma!} D_\xi^\gamma q_{-m-l,\varepsilon} \partial_x^\gamma p_{m-j,\varepsilon}.\end{aligned} \tag{2.5.10}$$

Here the second term on the right hand side vanishes by (2.5.9). Furthermore the last expression in (2.5.10) is in $\mathcal{N}_{S^0}$ for $|\xi| \geqslant 1/C$ for some $C > 0$ and $q_{-m,\varepsilon} p_{m,\varepsilon} = 1$, which establishes the statement made in (2.5.7). $\square$

Similarly, one can construct a ψ-pseudodifferential operator $\widetilde{Q}_\psi$ with symbol in $\mathcal{M}^{(\nu,0)}_{S^{-m,0}_{\mathrm{phg}}}$ having a representation of the form (2.5.8) and satisfies

$$P_\psi \widetilde{Q}_\psi = I_\psi + \widetilde{R}_\psi$$

where the generalized symbol of $\widetilde{R}_\psi$ is in $\mathcal{N}_{S^0}$ for $|\xi| \geqslant 1/C$ and some $C > 0$. Furthermore Q_ψ is related to $\widetilde{Q}_\psi$ in the following way: $\exists C > 0$ such that

$$Q_\psi = Q_\psi(P_\psi \widetilde{Q}_\psi) = (Q_\psi P_\psi)\widetilde{Q}_\psi = \widetilde{Q}_\psi \qquad \mathrm{mod}\ \mathrm{OP}_\psi \mathcal{N}_{S^m} \text{ on the set } |\xi| \geqslant 1/C.$$

The aim is now to factorize L_ψ on I'_θ in terms of two first-order operators of the form $L_{j,\psi} = (\partial_z + A_j)_\psi$ on I'_θ with $A_{j,\psi}$ a polyhomogeneous ψ-pseudodifferential operator with generalized symbol in $\mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}$ on I'_θ, $j = 1, 2$. Here, a first rough approximation would suggest $A_{j,\psi}$ to be the square root of the operator $A_\psi = \mathrm{OP}_\psi((a_\varepsilon)_\varepsilon)$ given in (2.5.1) on the set I'_θ where $\pm(i\sqrt{a_\varepsilon})_\varepsilon$ is described explicitly, $j = 1, 2$.

2.5.3 Factorization theorem

We are now going to show the following result.

Theorem 2.5.4. *Let $L_\psi = (\partial_z^2 + A)_\psi$ and I'_θ be as in (2.5.1) and (2.5.3) respectively. Then the operator L_ψ can be factorized into*

$$L_\psi = L_{1,\psi} L_{2,\psi} + R_\psi \qquad \text{on } I'_\theta \tag{2.5.11}$$

where $L_{j,\psi} = (\partial_z + A_j)_\psi$ and $A_{j,\psi} \in \mathrm{OP}_\psi \mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}$ on I'_θ whose principal symbol is equal to $\pm(i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$. Moreover, the remainder is given by $R_\psi = \Gamma_{0,\psi} + \Gamma_{1,\psi}(\partial_z)_\psi$ for some

ψ-pseudodifferential operators $\Gamma_{j,\psi}$ with generalized symbol $(\gamma_{j,\varepsilon})_\varepsilon$ in $\mathcal{N}_{S^{2-j}}$ on I'_θ, $j = 0, 1$. On the symbol side (2.5.11) should be understood in the sense of Definition 2.5.1.

Note that the product $L_{1,\psi}L_{2,\psi}$ in (2.5.11) is not a ψ-pseudodifferential operator on $\mathbb{R}^{n+1}$. As the next section will show, one can overcome this by introducing certain microlocal cut-off functions.

As in Subsection 1.5.1 we can define

$$f_\varepsilon(x, z, \tau, \xi) := \frac{\langle b_\varepsilon(x, z)\xi, \xi\rangle}{c_\varepsilon(x, z)\tau^2} \quad \text{on } \mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1}.$$

since $(c_\varepsilon(x, z))_\varepsilon$ is strictly non-zero. Moreover we let χ_ε be the smooth function defined on $\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1}$, $0 \leqslant \chi_\varepsilon \leqslant 1$, which is given by

$$\chi_\varepsilon(x, z, \tau, \xi) := \begin{cases} 0, & |f_\varepsilon| \geqslant \sin^2\gamma_2 \\ 1, & |f_\varepsilon| \leqslant \sin^2\gamma_1 \\ 1 - \left(1 + e^{\frac{1}{|f_\varepsilon| - \sin^2\gamma_1} - \frac{1}{\sin^2\gamma_2 - |f_\varepsilon|}}\right)^{-1}, & \sin^2\gamma_1 < |f_\varepsilon| < \sin^2\gamma_2 \end{cases}$$

for some fixed γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < \pi/2$. Then the symbol $(\chi_\varepsilon)_\varepsilon$ is in $\mathcal{M}^{(1,0)}_{S^{0,0}}(\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1})$. We let θ_1, θ_2 be fixed so that $0 < \theta_1 < \gamma_1 < \gamma_2 < \theta_2 < \pi/2$. Then

$$I'_{\theta_1} \subset \{(x, z, \tau, \xi) \mid \tau \neq 0 \text{ and } |f_\varepsilon| \leqslant \sin^2\gamma_1\}$$

for ε sufficiently small. Furthermore

$$\{(x, z, \tau, \xi) \mid \tau \neq 0, |f| \geqslant \sin^2\theta_2\} \subset \{(x, z, \tau, \xi) \mid \tau \neq 0, |f_\varepsilon| \geqslant \sin^2\gamma_2\}$$

for ε small enough and where we have set $f := \lim_{\varepsilon\to 0} f_\varepsilon$.

Defining

$$\left(c^{(0)}_{j,\varepsilon}\right)_\varepsilon := \left(\pm i\chi_\varepsilon\sqrt{a_\varepsilon}\right)_\varepsilon \in \mathcal{M}^{(1,0)}_{S^{1,0}}(\mathbb{R}^n \times (\mathbb{R}\backslash 0) \times \mathbb{R}^{n-1})$$

we get $(c^{(0)}_{j,\varepsilon} \mp i\sqrt{a_\varepsilon})_\varepsilon \in \mathcal{N}_{S^1}$ on I'_{θ_1} since $\chi_\varepsilon \equiv 1$ on I'_{θ_1}, $j = 1, 2$.

Factorization procedure

In order to factorize the operator L_ψ as in (2.5.11) we will construct generalized symbols $(a_{j,\varepsilon})_\varepsilon$ of the operators $A_{j,\psi}$, $j = 1, 2$ by means of their polyhomogeneous asymptotic expansions (in the sense of Definition 2.5.1) on I'_θ, that is

$$(a_{j,\varepsilon}(x,z,\tau,\xi))_\varepsilon \sim \sum_{\mu\geqslant 0}\big(\varepsilon^\mu c^{(\mu)}_{j,\varepsilon}(x,z,\tau,\xi)\big)_\varepsilon \quad \text{in } \mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}$$

and the sequence $\{(c^{(\mu)}_{j,\varepsilon})_\varepsilon\}_{\mu\in\mathbb{N}}$ consists of elements $(c^{(\mu)}_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(1,\mu)}_{S^{1-\mu,0}}(\mathbb{R}^n\times(\mathbb{R}\backslash 0)\times\mathbb{R}^{n-1})$ and satisfies the uniformity condition:

$$\begin{gathered}\exists\eta\in(0,1]\ \exists K>0\ \forall\mu\in\mathbb{N}\ \forall\alpha,\beta\in\mathbb{N}^n\ \exists C>0 \text{ for which}\\ |\partial^\alpha_{(\tau,\xi)}\partial^\beta_{(x,z)}c^{(\mu)}_{j,\varepsilon}(x,z,\tau,\xi)|\leqslant C\omega_\varepsilon^{|\beta|}\langle\xi\rangle^{1-\mu-|\alpha|}\quad \text{for } |(\tau,\xi)|\geqslant K,\ \varepsilon\in(0,\eta]\end{gathered}$$

and $j=1,2$. More precisely, we will recursively construct a sequence $\{(b^{(\mu)}_{j,\varepsilon})_\varepsilon\}_\mu$ of symbols $(b^{(\mu)}_{j,\varepsilon})_\varepsilon$ in $\mathcal{M}^{(1,\mu)}_{S^{1-\mu,\mu}_{\mathrm{hg}}}$ on I'_θ, $j=1,2$ such that

$$\big(c^{(\mu)}_{j,\varepsilon}(x,z,\tau,\xi)\big)_\varepsilon = \big(\varepsilon^{-\mu}b^{(\mu)}_{j,\varepsilon}(x,z,\tau,\xi)\big)_\varepsilon \quad \text{on } I'_\theta.$$

Proof of Theorem 2.5.4. The proof follows the same lines as in the proof of Theorem 1.5.2 but with additional ε-dependence.

To start, we set for $j=1,2$ and ε small enough

$$a^{(1)}_{j,\varepsilon} := b^{(0)}_{j,\varepsilon},\quad l^{(1)}_{j,\varepsilon} := i\zeta + a^{(1)}_{j,\varepsilon} \qquad \text{on } I'_\theta$$

where

$$b^{(0)}_{j,\varepsilon}(x,z,\tau,\xi) = \pm i\sqrt{a_\varepsilon(x,z,\tau,\xi)}\quad \text{on } I'_\theta,\ j=1,2.$$

Note that in particular $(a^{(1)}_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}$ on I'_θ. Furthermore we define the first-order operator $L^{(1)}_{j,\psi}$ through $(\partial_z + A^{(1)}_j)_\psi$ where $A^{(1)}_{j,\psi} = \mathrm{OP}_\psi((a^{(1)}_{j,\varepsilon})_\varepsilon)$ for $j=1,2$.

Now taking $L^{(1)}_{1,\psi}L^{(1)}_{2,\psi}$ as a first approximation for L_ψ we can compute the remainder as follows:

$$\begin{aligned} L^{(1)}_{1,\psi}L^{(1)}_{2,\psi} - L_\psi &= (\partial_z + A^{(1)}_1)_\psi(\partial_z + A^{(1)}_2)_\psi - (\partial_z^2 + A)_\psi = \\ &= (A^{(1)}_1 + A^{(1)}_2)_\psi(\partial_z)_\psi + \mathrm{OP}_\psi\big(\big(\varepsilon\,\partial_z a^{(1)}_{2,\varepsilon}\big)_\varepsilon\big) + A^{(1)}_{1,\psi}A^{(1)}_{2,\psi} - A_\psi = \\ &=: \Gamma^{(1)}_{0,\psi} + \Gamma^{(1)}_{1,\psi}(\partial_z)_\psi \end{aligned}$$

where we have used the Leibniz composition rule for ψ-pseudodifferential operators. In particular this means $(\partial_z)_\psi A^{(1)}_{2,\psi} = A^{(1)}_{2,\psi}(\partial_z)_\psi + \mathrm{OP}_\psi\big((\varepsilon\partial_z a^{(1)}_{2,\varepsilon})_\varepsilon\big)$. Then $\Gamma^{(1)}_{1,\psi} = (A^{(1)}_1 +$

$A_2^{(1)})_\psi \in \mathrm{OP}_\psi \mathcal{N}_{S^1}$ on I'_θ. Moreover $\Gamma_{0,\psi}^{(1)}$ is equal to $\mathrm{OP}_\psi\big((\varepsilon\,\partial_z a_{2,\varepsilon}^{(1)})_\varepsilon\big) + A_{1,\psi}^{(1)} A_{2,\psi}^{(1)} - A_\psi$ and is a polyhomogeneous ψ-pseudodifferential operator with generalized symbol $(\gamma_{0,\varepsilon}^{(1)})_\varepsilon$ in $\mathcal{M}^{(1,1)}_{S^{1,1}_{\mathrm{phg}}}(I'_\theta)$ by Proposition 2.4.8. In detail $(\gamma_{0,\varepsilon}^{(1)})_\varepsilon$ admits the asymptotic expansion

$$\big(\gamma_{0,\varepsilon}^{(1)}\big)_\varepsilon \sim \Big(\varepsilon\partial_z a_{2,\varepsilon}^{(1)} + \sum_{|\alpha|\geqslant 1} \frac{\varepsilon^{|\alpha|}}{\alpha!}\big(D_\xi^\alpha a_{1,\varepsilon}^{(1)}\big)\big(\partial_x^\alpha a_{2,\varepsilon}^{(1)}\big)\Big)_\varepsilon \quad \text{on } I'_\theta.$$

To improve this argument and will construct recursively the N-th approximation: $L_{1,\psi}^{(N)} L_{2,\psi}^{(N)}$ of L_ψ on the set I'_θ.

Therefore, assume for $N \geqslant 1$ that $(b_{j,\varepsilon}^{(\nu)})_\varepsilon \in \mathcal{M}^{(1,\nu)}_{S^{1-\nu,\nu}_{\mathrm{hg}}}$ on I'_θ is determined for all $\nu \leqslant N-1$ and $j = 1,2$. For fixed $\varepsilon \in (0,\varepsilon_0]$ and $j = 1,2$ we set on I'_θ

$$a_{j,\varepsilon}^{(N)} := \sum_{\nu=0}^{N-1} b_{j,\varepsilon}^{(\nu)}, \quad L_{j,\psi}^{(N)} = (\partial_z + A_j^{(N)})_\psi$$

with polyhomogeneous $A_{j,\psi}^{(N)}(x,z,D_t,D_x) = \mathrm{OP}_\psi\big((a_{j,\varepsilon}^{(N)})_\varepsilon\big)$. Again, for all other $\varepsilon \in (0,1]$ we set $a_{j,\varepsilon}^{(N)}$ and $L_{j,\varepsilon}^{(N)}$ equal to zero, $j = 1,2$. Furthermore, we suppose an N-th order approximation for L_ψ of the form

$$L_{1,\psi}^{(N)} L_{2,\psi}^{(N)} - L_\psi = \Gamma_{0,\psi}^{(N)} + \Gamma_{1,\psi}^{(N)}(\partial_z)_\psi \quad \text{on } I'_\theta$$

where $\Gamma_{0,\psi}^{(N)}$ and $\Gamma_{1,\psi}^{(N)}$ are the ψ-pseudodifferential operators with generalized symbols $(\gamma_{0,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{M}^{(1,N)}_{S^{2-N,N}_{\mathrm{phg}}}$ on I'_θ and $(\gamma_{1,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{N}_{S^1}$ on I'_θ, respectively.

We denote by $(\overline{\gamma}_{0,\varepsilon}^{(N)})_\varepsilon$ and $(\overline{\gamma}_{1,\varepsilon}^{(N)})_\varepsilon$ the top order symbols of $(\gamma_{0,\varepsilon}^{(N)})_\varepsilon$ and $(\gamma_{1,\varepsilon}^{(N)})_\varepsilon$ on I'_θ. For the induction step we specify $(b_{1,\varepsilon}^{(N)})_\varepsilon, (b_{2,\varepsilon}^{(N)})_\varepsilon$ on I'_{θ_2} as follows: since $(a_{j,\varepsilon}^{(1)})_\varepsilon$ satisfies

$$\exists \eta \in (0,1]: \ |a_{j,\varepsilon}^{(1)}(x,z,\tau,\xi)| \geqslant C|(\tau,\xi)| \quad \text{on } I'_\theta,\ \forall \varepsilon \in (0,\eta]$$

for some constant $C > 0$ that is independent of $\varepsilon \in (0,\eta]$, $j = 1,2$, the matrix

$$\begin{pmatrix} 1 & 1 \\ a_{2,\varepsilon}^{(1)} & a_{1,\varepsilon}^{(1)} \end{pmatrix}$$

is invertible on I'_θ and thus the system

$$\begin{aligned} -\overline{\gamma}_{1,\varepsilon}^{(N)} &= b_{1,\varepsilon}^{(N)} + b_{2,\varepsilon}^{(N)} \\ -\overline{\gamma}_{0,\varepsilon}^{(N)} &= b_{1,\varepsilon}^{(N)} a_{2,\varepsilon}^{(1)} + a_{1,\varepsilon}^{(1)} b_{2,\varepsilon}^{(N)} \end{aligned}$$

is uniquely solvable for $(b_{j,\varepsilon}^{(N)})_\varepsilon$ in $\mathcal{M}^{(1,N)}_{S^{1-N,N}_{\mathrm{hg}}}$ on I'_θ, $j = 1, 2$.

We write $B_{j,\psi}^{(N)}$ for the ψ-pseudodifferential operator with polyhomogeneous generalized symbol $(b_{1,\varepsilon}^{(N)})_\varepsilon$ and set $L_{j,\psi}^{(N+1)} := (L_j^{(N)} + B_j^{(N)})_\psi$. Then the following holds

$$\begin{aligned} L_{1,\psi}^{(N+1)} L_{2,\psi}^{(N+1)} - L_\psi &= \big(L_1^{(N)} + B_1^{(N)}\big)_\psi \big(L_2^{(N)} + B_2^{(N)}\big)_\psi - L_\psi = \\ &= \Gamma_{0,\psi}^{(N)} + \Gamma_{1,\psi}^{(N)} (\partial_z)_\psi + \big(B_1^{(N)} + B_2^{(N)}\big)_\psi \big(\partial_z\big)_\psi + \\ &\quad + \mathrm{OP}_\psi\big(\big(\varepsilon \partial_z b_{2,\varepsilon}^{(N)}\big)_\varepsilon\big) + A_{1,\psi}^{(N)} B_{2,\psi}^{(N)} + B_{1,\psi}^{(N)} A_{2,\psi}^{(N)} + B_{1,\psi}^{(N)} B_{2,\psi}^{(N)}. \end{aligned}$$

Indeed, we have an $(N+1)$-th order approximation of the form

$$L_{1,\psi}^{(N+1)} L_{2,\psi}^{(N+1)} - L_\psi = \Gamma_{0,\psi}^{(N+1)} + \Gamma_{1,\psi}^{(N+1)} (\partial_z)_\psi \quad \text{on } I'_\theta$$

and $\Gamma_{0,\psi}^{(N+1)}$ and $\Gamma_{1,\psi}^{(N+1)}$ are the ψ-pseudodifferential operators with generalized symbols $(\gamma_{0,\varepsilon}^{(N+1)})_\varepsilon \in \mathcal{M}^{(1,N+1)}_{S^{1-N,N+1}_{\mathrm{phg}}}$ and $(\gamma_{1,\varepsilon}^{(N+1)})_\varepsilon \in \mathcal{N}_{S^1}$ on I'_θ, respectively. This completes the induction step. □

2.6 The generalized infinite wave front set

In this section we will discuss an alternative description of microlocality and regularity theory in Colombeau algebras of generalized functions.

In the case of classical generalized pseudodifferential operators that satisfy logarithmic slow scale estimates microlocality is a straightforward modification of the regularity results in [21] within the Colombeau theory and was shown in Subsection 1.6 of Chapter 1. But the situation changes dramatically when working with ψ-pseudodifferential operators with symbols of log-type.

For this reason we will introduce the notion of generalized infinite wave front set describing negligibility of a generalized function at infinite points. For further studies about the infinite wave front set and the more refined semiclassical wave front set we refer to [9, 1] and [39, 14, 29].

Remark 2.6.1. Before we start, we will first study properties of the compactly supported operators that appeared in Remark 2.5.2. Therefore we let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ equal to 1 near the origin ($\xi = 0$) and $(p_\varepsilon)_\varepsilon$ in $\mathcal{M}^{(\nu,l)}_{S^{m,k}}$ as in Subsection 2.5.1. We now consider the symbol $r_\varepsilon(x,\xi) := \varphi(\xi)p_\varepsilon(x,\xi) \in \mathcal{M}^{(\nu,l)}_{S^{-\infty,k}}$. We denote by $H^s_\varepsilon(\mathbb{R}^n)$ the L^2-based semiclassical Sobolev space defined by

$$\|u_\varepsilon\|^2_{H^s_\varepsilon(\mathbb{R}^n)} := \frac{1}{(2\pi\varepsilon)^n}\int (1+|\xi|^2)^s|\mathcal{F}_\varepsilon(u_\varepsilon)(\xi)|^2\,d\xi = \frac{1}{(2\pi)^n}\int(1+|\varepsilon\xi|^2)^s|\hat{u}_\varepsilon)(\xi)|^2\,d\xi$$

for any $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$. Here, $\mathcal{F}_\varepsilon(u_\varepsilon)(\xi) := \hat{u}_\varepsilon(\xi/\varepsilon)$ denotes the semiclassical Fourier transform. Then, we get for any $s \in \mathbb{R}$:

$$\|r_{\psi,\varepsilon}(x,D)u_\varepsilon\|^2_{H^s_\varepsilon(\mathbb{R}^n)} = \|\big(\langle\varepsilon D_x\rangle^s \circ r_{\psi,\varepsilon}(x,D_x)\big)u_\varepsilon\|_{L^2(\mathbb{R}^n)}.$$

Since the symbol of the ψ-pseudodifferential operator $\langle\varepsilon D_x\rangle^s \circ r_{\psi,\varepsilon}(x,D_x)$ belongs to $\mathcal{M}^{(\nu,l)}_{S^{-\infty,k}}$ for any $s \in \mathbb{R}$, we obtain by the L^2-boundedness result (see below): $\exists N \in \mathbb{N}$, $\forall s \in \mathbb{R}$:

$$\|r_{\psi,\varepsilon}(x,D)u_\varepsilon\|^2_{H^s_\varepsilon(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

In this sense $r_{\psi,\varepsilon}(x,D)u_\varepsilon$ is regular in a L^2-based semiclassical Sobolev space.

2.6.1 Microlocal behavior at infinity

Here we introduce a suitable notion of asymptotic negligibility of a generalized function $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ with respect to certain regularities on the phase space.

To do so, we use the following notions which are similar to [9]. For a non-zero vector $\xi_0 \in \mathbb{R}^n$ we write for the projection onto the unit sphere $\xi_0/|\xi_0|$. Moreover, given such ξ_0 we say that $\Gamma_{\infty\xi_0} \subseteq \mathbb{R}^n$ is a conic neighborhood of the direction $\xi_0/|\xi_0|$ if $\Gamma_{\infty\xi_0}$ is the intersection of the complement of some open ball centered at the origin with an open cone containing the direction.

Furthermore, for (x_0,ξ_0) in $T^*\mathbb{R}^n\backslash 0$ we say that a generalized symbol $p := (p_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,0)}_{S^{m,0}_{\mathrm{phg}}}$ is elliptic at $(x_0,\infty\xi_0)$, or elliptic at infinity at (x_0,ξ_0), if there are an open neighborhood U of x_0, a conic neighborhood Γ_{ξ_0} of ξ_0 and a constant $C > 0$ such that

$$|p_\varepsilon(x,\xi)| \geqslant C\langle\xi\rangle^m \qquad \forall (x,\xi) \in U \times \Gamma_{\infty\xi_0} \text{ as } \varepsilon \to 0.$$

where $\Gamma_{\infty\xi_0} := \Gamma_{\xi_0} \cap \{\xi \in \mathbb{R}^n \mid |\xi| \geqslant K\}$ for some $K > 0$. We denote by $\mathrm{Ell}^{\mathrm{i}}(p)$ the set of all points $(x_0,\xi_0) \in T^*\mathbb{R}^n\backslash 0$ where p is elliptic at infinity.

Then, the generalized infinite wave front set is defined as follows :

Definition 2.6.2. *For* $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ *we denote by* $\mathrm{WF}^{\mathrm{i}}(u) \subseteq T^*\mathbb{R}^n \backslash 0$ *the generalized infinite wave front set of* u *which is characterized as follows. We say that* $(x_0, \xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$ *at infinity, denoted by* $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$*, if* $\exists\ \chi := (\chi_\varepsilon)_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{\infty,0}_{\mathrm{phg}}}$ *elliptic at* $(x_0, \infty\xi_0)$ *so that*

$$\mathrm{OP}_\psi(\chi)u = 0 \quad \text{in } \mathcal{G}_{L^2}(\mathbb{R}^n).$$

Remark 2.6.3. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{H^\infty}$ and suppose that there exists $\chi_m := (\chi_{m,\varepsilon})_\varepsilon$ in $\mathcal{M}^{(0,0)}_{S^{m,0}_{\mathrm{phg}}}$ for some $m \in \mathbb{R}$ elliptic at $(x_0, \infty\xi_0)$ with

$$\forall q \in \mathbb{N}: \quad \|\mathrm{OP}_{\psi,\varepsilon}(\chi_{m,\varepsilon})u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \qquad \text{as } \varepsilon \to 0.$$

Using [17, Proposition 3.4] then the above line is equivalent to $\mathrm{OP}_\psi(\chi_m)u = 0$ in $\mathcal{G}_{L^2}$.

Furthermore, we introduce the notion of the infinite conic support to describe singularities of ψ-pseudodifferential operators (cf. [1, Definition 1]). Therefore let $p := (p_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}_{\mathrm{phg}}}$ be the generalized symbol of P_ψ. Then the infinite conic support of P_ψ is the set $\mathrm{conesupp}^{\mathrm{i}}(p) \subseteq T^*\mathbb{R}^n \backslash 0$ and is defined as the complement (in $T^*\mathbb{R}^n \backslash 0$) of the set of points $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ such that there exists an open neighborhood U of x_0, a conic open neighborhood Γ_{ξ_0} of the direction ξ_0 and a constant $K > 0$ such that the following is satisfied

$$\forall \alpha, \beta \in \mathbb{N}^n\ \forall N \in \mathbb{N}: \quad |\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x,\xi)|\langle\xi\rangle^{-m+N+|\alpha|} = \mathcal{O}(\varepsilon^N) \qquad \text{as } \varepsilon \to 0 \tag{2.6.1}$$

uniformly in $(x, \xi) \in U \times \Gamma_{\infty\xi_0}$ where $\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{\xi \in \mathbb{R}^n \mid |\xi| \geqslant K\}$. This is again a condition on the behavior for a generalized symbol at infinity and we therefore say that $(x_0, \xi_0) \notin \mathrm{conesupp}^{\mathrm{i}}(p)$ at infinity and write $(x_0, \infty\xi_0) \notin \mathrm{conesupp}^{\mathrm{i}}(p)$ whenever (2.6.1) is fulfilled.

The idea here is that $\mathrm{conesupp}^{\mathrm{i}}(p)^c$ are the directions on the phase space in which P_ψ annihilates singularities as they are contained in $\mathcal{N}_{S^m}(U \times \Gamma)$ for $|\xi| \geqslant K$ and some $K > 0$. To give a connection to these two notions of wave front sets we state the following theorem (cf. [1, Lemma 5]).

Theorem 2.6.4. *Given* $u \in \mathcal{G}_{L^2}$ *and* P_ψ *a* ψ*-pseudodifferential operator with generalized symbol* $p = (p_\varepsilon)_\varepsilon \in \mathcal{M}^{(\nu,l)}_{S^{m,k}_{\mathrm{phg}}}$*. Then the following statement is valid*

$$\mathrm{WF}^{\mathrm{i}}(P_\psi u) \subseteq \mathrm{WF}^{\mathrm{i}}(u) \cap \mathrm{conesupp}^{\mathrm{i}}(p)$$

and we say that ψ*-pseudodifferential operator microlocal at infinity.*

We remark that most of the properties of the infinite wave front set of a generalized function in $\mathcal{G}_{L^2}$ can be derived from the theorem of Calderón-Vaillancourt for the class pseudodifferential operators with symbols in $S^0_{0,0}$.

Proof. We first show the inclusion relation $\mathrm{WF}^{\mathrm{i}}(P_\psi u) \subseteq \mathrm{conesupp}^{\mathrm{i}}(p)$. Therefore we let $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ such that $(x_0, \infty\xi_0) \notin \mathrm{conesupp}^{\mathrm{i}}(p)$ which in turn implies that $(p_\varepsilon)_\varepsilon$ is in $\mathcal{N}_{S^m}(U \times \Gamma_{\infty\xi_0})$ for some open neighborhood U of x_0 and some conic open neighborhood Γ_{ξ_0} of ξ_0. More precisely, we have that $\exists C > 0$:

$$\forall \alpha, \beta \in \mathbb{N}^n \ \forall N \in \mathbb{N}: \quad |\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x,\xi)| \langle \xi \rangle^{-m+N+|\alpha|} = \mathcal{O}(\varepsilon^N) \qquad \text{as } \varepsilon \to 0$$

uniformly in $(x,\xi) \in U \times \Gamma_{\infty\xi_0}$ with $\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{\xi \in \mathbb{R}^n \mid |\xi| \geqslant C\}$.

We will now construct a symbol $\big(\chi^{(-m)}_{J,K}\big)_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{-m,0}_{\mathrm{phg}}}$ elliptic at $(x_0, \infty\xi_0)$ such that

$$\forall q \in \mathbb{N}: \quad \|\mathrm{OP}_{\psi,\varepsilon}(\chi^{(-m)}_{J,K}) P_{\psi,\varepsilon} u_\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^q) \qquad \text{as } \varepsilon \to 0.$$

For that reason let $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\phi(z) = 1$ for $|z| \leqslant 1/2$ and $\phi(z) = 0$ for $|z| \geqslant 1$. Now, given $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$ we define for fixed $J > 0$ the function

$$\lambda^{(-m)}_J(x,\xi) := \phi\Big(\frac{x - x_0}{J}\Big) \phi\left(\Big\{\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\Big\}\frac{1}{J}\right) |\xi|^{-m} \qquad \text{for } \xi \neq 0.$$

Further for some fixed $K > 0$ let

$$\chi^{(-m)}_{J,K}(x,\xi) := (1-\phi)\Big(\frac{\xi}{2K}\Big) \lambda^{(-m)}_J(x,\xi). \tag{2.6.2}$$

Then for $J, K > 0$ fixed $\big(\chi^{(-m)}_{J,K}(x,\xi)\big)_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{-m,0}_{\mathrm{phg}}}$ is elliptic at $(x_0, \infty\xi_0)$ and supported in

$$|x - x_0| \leqslant J, \quad \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \leqslant J, \quad |\xi| \geqslant K.$$

So $\chi^{(-m)}_{J,K}$ is a cut-off in a conic neighborhood of ξ_0 and is supported in a cone of directions near ξ_0. In particular, we have $\chi^{(-m)}_{J,K} \# p_\varepsilon(x,\xi) = 0$ for $|\xi| \leqslant K$, where we used the notation $a\#b(x,D) = a(x,D) \circ b(x,D)$. We now choose $J, K > 0$ such that

$$\mathrm{supp}(\chi^{(-m)}_{J,K}) \subseteq U \times \Gamma_{\infty\xi_0}. \tag{2.6.3}$$

Then, by the L^2-boundedness theorem of Calderón-Vaillancourt for symbols of class $S^0_{0,0}$, see [55, Chapter 13, Theorem 1.3], there are constants $j_0, j_1 \in \mathbb{N}$ and $C_1 > 0$, each of which depends on the dimension n but is independent of $\varepsilon > 0$, such that

$$\|\mathrm{OP}_{\psi,\varepsilon}(\chi_{J,K}^{(-m)})P_{\psi,\varepsilon}u_\varepsilon\|_{L^2} = \|\mathrm{OP}_{\psi,\varepsilon}(\chi_{J,K}^{(-m)}\#p_\varepsilon)u_\varepsilon\|_{L^2} \leqslant$$
$$\leqslant C_1 \sup_{|\alpha|\leqslant j_0, |\beta|\leqslant j_1} \|\partial_\xi^\alpha\partial_x^\beta(\chi_{J,K}^{(-m)}\#p_\varepsilon)\|_{L^\infty(\mathbb{R}^{2n})}\|u_\varepsilon\|_{L^2}$$

where in the last step we performed the rescaling by means of the asymptotic expansion of the second kind for the composition formula of the pseudodifferential operators expressed by using the notation $\#$, see Subsection 2.4.2. Moreover, we used the following estimation

$$|\partial_\xi^\alpha\partial_x^\beta\big(\chi_{J,K}^{(-m)}\#p_\varepsilon(x,\varepsilon\xi)\big)| = \varepsilon^{|\alpha|}|(\partial_\xi^\alpha\partial_x^\beta\ \chi_{J,K}^{(-m)}\#p_\varepsilon)(x,\varepsilon\xi)| \leqslant$$
$$\leqslant \varepsilon^{|\alpha|} \sup_{(x,\xi)\in\mathbb{R}^{2n}} |(\partial_\xi^\alpha\partial_x^\beta\ \chi_{J,K}^{(-m)}\#p_\varepsilon)(x,\varepsilon\xi)| = \varepsilon^{|\alpha|}\|\partial_\xi^\alpha\partial_x^\beta\ \chi_{J,K}^{(-m)}\#p_\varepsilon\|_{L^\infty(\mathbb{R}^{2n})}.$$

Now using the fact that $(p_\varepsilon)_\varepsilon \in \mathcal{N}_{S^m}(U \times \Gamma_{\infty\xi_0})$ and (2.6.3) one has for every $q \in \mathbb{N}$ that

$$\sup_{|\alpha|\leqslant j_0, |\beta|\leqslant j_1} \|\partial_\xi^\alpha\partial_x^\beta(\chi_{J,K}^{(-m)}\#p_\varepsilon)\|_{L^\infty(\mathbb{R}^{2n})} = \mathcal{O}(\varepsilon^q) \qquad \text{as } \varepsilon \to 0$$

by Proposition 2.4.8 and we deduce that $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(P_\psi u)$ which completes the first part of the proof.

We continue the proof by showing the inclusion: $\mathrm{WF}^{\mathrm{i}}(P_\psi u) \subseteq \mathrm{WF}^{\mathrm{i}}(u)$.

We take $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$ and let U be some open neighborhood of x_0 and Γ_{ξ_0} a conic neighborhood of ξ_0. Then, for some $j \in \mathbb{R}$ there exists $\chi_j := (\chi_{j,\varepsilon})_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{j,0}_{\mathrm{phg}}}$ elliptic at $(x_0, \infty\xi_0)$ such that for some constant $K_1 > 0$ we have

$$\exists C > 0\ \exists \eta \in (0,1] : \qquad |\chi_{j,\varepsilon}(x,\xi)| \geqslant C\langle\xi\rangle^j \qquad \forall \varepsilon \in (0,\eta] \tag{2.6.4}$$

uniformly on $U \times \Gamma_{\infty\xi_0}$ with $\Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{(x,\xi) \mid |\xi| \geqslant K_1\}$ and

$$\forall q \in \mathbb{N}: \quad \|\mathrm{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon})u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \to 0.$$

Then the assumptions in Lemma 2.5.3 are satisfied for $\chi_{j,\varepsilon}$ on the set $U \times \Gamma_{\infty\xi_0}$. So, there is a $\widetilde{\chi}_{-j} := (\widetilde{\chi}_{-j,\varepsilon})_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{-j,0}_{\mathrm{phg}}}$ such that

$$\mathrm{OP}_\psi(\widetilde{\chi}_{-j})\mathrm{OP}_\psi(\chi_j) = I_\psi + R_\psi \qquad \text{on } U \times \Gamma_{\infty\xi_0}$$

where the generalized symbol $(r_\varepsilon)_\varepsilon$ of R_ψ is in $\mathcal{M}^{(0,0)}_{S^{-N,N}}$ on $U \times \Gamma_{\infty\xi_0}$ for every $N \in \mathbb{N}$.

As in (2.6.2), one constructs a generalized symbol $(\kappa_{j-m})_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{j-m,0}_{\mathrm{phg}}}$ elliptic at $(x_0, \infty\xi_0)$ such that $\mathrm{supp}(\kappa_{j-m})$ is contained in the set where (2.6.4) is valid. We then write

$$\mathrm{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#p_\varepsilon) = \mathrm{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#p_\varepsilon\#\widetilde{\chi}_{-j,\varepsilon})\mathrm{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon}) - \mathrm{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#p_\varepsilon\#r_\varepsilon).$$

Concerning the first term on the right hand side we rescale and obtain by the Calderón-Vaillancourt theorem that there are constants $j_0, j_1 \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned}\|\mathrm{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#p_\varepsilon\#\widetilde{\chi}_{-j,\varepsilon})\mathrm{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon})u_\varepsilon\|_{L^2} &\leqslant \\ \leqslant C \sup_{|\alpha|\leqslant j_0, |\beta|\leqslant j_1} \|\partial^\alpha_\xi\partial^\beta_x \kappa_{j-m}\#p_\varepsilon\#\widetilde{\chi}_{-j,\varepsilon})\|_{L^\infty(\mathbb{R}^{2n})}&\|\mathrm{OP}_{\psi,\varepsilon}(\chi_{j,\varepsilon})u_\varepsilon\|_{L^2}\end{aligned}$$

and the latter expression is $\mathcal{O}(\varepsilon^q)$ as $\varepsilon \to 0$ for every $q \in \mathbb{N}$ by assumption. Similarly, we obtain for every $q \in \mathbb{N}$:

$$\|\mathrm{OP}_{\psi,\varepsilon}(\kappa_{j-m}\#p_\varepsilon\#r_\varepsilon)u_\varepsilon\|_{L^2} = \mathcal{O}(\varepsilon^q) \quad \text{as } \varepsilon \to 0$$

since $(r_\varepsilon)_\varepsilon \in \mathcal{N}_{S^0}$ on the support of κ_{j-m}. Therefore $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(P_\psi u)$. □

As a next result we reformulate the infinite wave front set in terms of the Fourier transform of a localized function.

For this purpose we introduce the space $\mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$ of compactly supported generalized functions consisting of those $u \in \mathcal{G}(\mathbb{R}^n)$ such that for some $K \Subset \mathbb{R}^n$ the restriction u to $\mathbb{R}^n \backslash K$ is equal to 0 as an element of $\mathcal{G}(\mathbb{R}^n \backslash K)$. Note that for an open subset Ω of $\mathbb{R}^n$ the space $\mathcal{G}(\Omega)$ is defined by the space $\mathcal{G}_E$ by setting $E = \mathcal{E}(\Omega)$ the space of smooth functions on Ω topologized through the family of seminorms $p_{K_n,j}(f) := \sup_{x\in K_n, |\alpha|\leqslant j} |\partial^\alpha f(x)|$ where $(K_n)_n$ is an exhausting sequence of compact subsets of Ω.

Proposition 2.6.5. *Let $u \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$. If $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$ then there exists $\gamma \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\gamma(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that $\forall N, q \in \mathbb{N}$, $\exists C > 0$, $\exists K > 0$ satisfying*

$$|\mathcal{F}(\gamma u_\varepsilon)(\xi)| \leqslant C\varepsilon^q\langle\xi\rangle^{-N} \qquad \text{as } \varepsilon \to 0 \tag{2.6.5}$$

for all $\xi \in \Gamma$ with $|\xi| \geqslant K/\varepsilon$. This is equivalent to saying

$$\begin{aligned}&\forall N, q \in \mathbb{N}\ \exists C > 0\ \exists K > 0 : \\ &\qquad |\mathcal{F}(\gamma u_\varepsilon)(\xi/\varepsilon)| \leqslant C\varepsilon^q\langle\xi\rangle^{-N} \qquad \text{as } \varepsilon \to 0\end{aligned}$$

for all $\xi \in \Gamma$ with $|\xi| \geqslant K$.

Note that our notion of regularity is derived from the infinite wave front set and therefore differs from [34, Section 6] and [21, Sections 2 and 3]. It is also different to the Definitions

3.13 and 3.14 given in [40, Section 3.2.2] if one replaces there the condition of $\mathcal{G}^\infty$ rapid decrease by rapid decrease. We remark that a generalized function $u \in \mathcal{G}(\mathbb{R}^n)$ is of rapid decrease if it has a representative $(u_\varepsilon)_\varepsilon$ with the property:

$$\exists N \in \mathbb{N}\ \forall p \in \mathbb{N}\ \exists C > 0 \exists \eta \in (0,1]: \quad |u_\varepsilon(x)| \leqslant C\varepsilon^{-N}\langle x\rangle^{-p} \quad \forall x \in \mathbb{R}^n, \varepsilon \in (0,\eta].$$

Proof. The following proof is similar to [27, Proposition 7.4]. First note that $\mathcal{G}_c \subseteq \mathcal{G}_{L^2}$. Since $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$ we can find a $\chi := \big(\chi_\varepsilon\big)_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{-m,0}_{\mathrm{phg}}}$ elliptic at $(x_0, \infty\xi_0)$ such that $\mathrm{OP}_\psi(\chi)u = 0$ in $\mathcal{G}_{L^2}$ for some $m \in \mathbb{R}$. Furthermore we let $\gamma \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\gamma(x_0) \neq 0$. Then there exists a symbol $\phi \in S^0(\mathbb{R}^n), \mathrm{supp}(\phi) \subset \Gamma_{\infty\xi_0} = \Gamma_{\xi_0} \cap \{\xi \mid |\xi| \geqslant K\}$ for some $K > 0$ and $\phi(t\xi) = 1$ for $t \geqslant 1$, $\xi \in \Gamma_{\infty\xi_0}$, so that

$$\phi_\psi(D)\gamma(x) = A_\psi \mathrm{OP}_\psi(\chi) + R_\psi$$

where A_ψ and R_ψ is a ψ-pseudodifferential operator with generalized symbol in $\mathcal{M}^{(0,0)}_{S^{m,0}}$ and $\mathcal{N}_{\mathcal{S}^0}$, respectively. Then by assumption we deduce that $\phi_\psi(D)\gamma u = 0$ in $\mathcal{G}_{L^2}$. Note that in particular we have that $\phi_\psi(D)\gamma u = 0$ in $\mathcal{G}_{\mathscr{S}}$ and $\mathcal{G}_{\mathscr{S}} \hookrightarrow \mathcal{G}_{L^2}$, see [17, Proposition 3.5]. Using the fact that the Fourier transform is an isomorphism on $\mathcal{G}_{\mathscr{S}}$ we obtain $\xi \mapsto (\phi(\varepsilon\xi)\widehat{\gamma u}_\varepsilon(\xi))_\varepsilon$ is in $\mathcal{N}_{\mathscr{S}}$ under consideration of the scaling. Hence, we also have

$$\forall N \in \mathbb{N}\ \forall q \in \mathbb{N}: \quad \|\langle \xi/\varepsilon\rangle^N \phi(\xi)\widehat{\gamma u}_\varepsilon(\xi/\varepsilon)\|_{L^\infty} = \mathcal{O}(\varepsilon^q) \quad \varepsilon \to 0,$$

showing (2.6.5). ◻

We also study the behavior of the infinite wave front set of a function under the action of ψ-pseudodifferential operators that are elliptic at infinity.

Theorem 2.6.6. *Let $u \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ and let P_ψ be a ψ-pseudodifferential operator with symbol $p := (p_\varepsilon)_\varepsilon$ in $\mathcal{M}^{(\nu,0)}_{S^{m,0}_{\mathrm{phg}}}$ that is elliptic at $(x_0, \infty\xi_0) \in T^*\mathbb{R}^n \backslash 0$. Then the following holds:*

$$\mathrm{WF}^{\mathrm{i}}(u) \subseteq \mathrm{WF}^{\mathrm{i}}(P_\psi u) \cup \mathrm{Ell}^{\mathrm{i}}(p)^{\mathrm{c}}.$$

Proof. Suppose $(x_0, \infty\xi_0)$ is not contained in the right hand side of the claimed inclusion relation. Since $(x_0, \infty\xi_0) \in \mathrm{Ell}^{\mathrm{i}}(p)$ the symbol $(p_\varepsilon)_\varepsilon$ is elliptic at $(x_0, \infty\xi_0)$ and therefore there is an open neighborhood U of x_0 and conic neighborhood Γ_{ξ_0} containing ξ_0 such that

$$|p_\varepsilon(x,\xi)| \geqslant C\langle\xi\rangle^m \qquad \text{on } U \times \Gamma_{\infty\xi_0} \text{ as } \varepsilon \to 0.$$

By Lemma 2.5.3 there exists an approximative inverse Q_ψ so that

$$I_\psi = Q_\psi P_\psi + R_\psi \qquad \text{on } U \times \Gamma_{\infty\xi_0} \tag{2.6.6}$$

and $(x_0, \infty\xi_0) \notin \operatorname{conesupp}^{\mathrm{i}}(r)$ where $r := (r_\varepsilon)_\varepsilon$ is the generalized symbol of R_ψ. Hence, $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(R_\psi u)$ by Theorem 2.6.4.

Furthermore, since also $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(P_\psi u)$, we obtain

$$(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(Q_\psi P_\psi u) \subseteq \mathrm{WF}^{\mathrm{i}}(P_\psi u).$$

By (2.6.6) we deduce that $(x_0, \infty\xi_0) \notin \mathrm{WF}^{\mathrm{i}}(u)$. □

2.6.2 Microlocal factorization

In this section we use the notion of microlocal behavior at infinite points of a given generalized function in $\mathcal{G}_{L^2}(\mathbb{R}^n)$ to give a microlocal interpretation of Theorem 2.5.4. Therefore, we say that two generalized functions $u, v \in \mathcal{G}_{L^2}$ are microlocally equivalent at $(x_0, \infty\xi_0)$ if and only if $\exists (\chi_\varepsilon)_\varepsilon \in \mathcal{M}^{(0,0)}_{S^{\infty,0}_{\mathrm{phg}}}$ elliptic at $(x_0, \infty\xi_0)$ such that $\chi_\psi(u - v) = 0$ in $\mathcal{G}_{L^2}$.

As in Chapter 1 and [51] we introduce a subset of the phase space associated to I'_θ which is given by

$$I_\theta := \{(t, x, z, \tau, \xi, \zeta) \mid (x, z, \tau, \xi) \in I'_\theta,\ |\zeta| \leqslant C|\tau|\}. \tag{2.6.7}$$

We remark that the set I_θ will serve as the adequate space-frequency domain on which we establish the microlocal factorization theorem at infinite points.

As already mentioned in the preceding section $L_{j,\psi} = (\partial_z + A_j(x, z, D_t, D_x))_\psi$, $j = 1, 2$ is not a ψ-pseudodifferential operator on $\mathbb{R}^{n+1}$. We therefore define a microlocal cut-off φ as in the proof of Theorem 1.7.4. Then, by construction, $\varphi \in S^0(\mathbb{R}^n \times \mathbb{R}^{n+1})$ and the symbol of $\mathrm{OP}_\psi(\varphi) W_\psi$ is in $\mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ for all $W_\psi \in \mathrm{OP}_\psi(\mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}))$.

In particular, we obtain

$$(t, x, z, \infty(\tau, \xi, \zeta)) \notin \operatorname{conesupp}^{i}(\varphi - 1) \quad \forall (t, x, z, \tau, \xi, \zeta) \in I_\theta,$$

and thus

$$\varphi_\psi L_{1,\psi} L_{2,\psi} = L_{1,\psi} L_{2,\psi} \text{ on } I_\theta \text{ with } |(\tau, \xi, \zeta)| \geqslant K$$

for some constant $K > 0$. And we say

$$\varphi_\psi L_{1,\psi} L_{2,\psi} \equiv L_{1,\psi} L_{2,\psi} \text{ microlocally at infinity on } I_\theta. \tag{2.6.8}$$

Furthermore, since

$$\mathrm{WF}^{\mathrm{i}}(\varphi_\psi L_{1,\psi} L_{2,\psi} u - \varphi_\psi L_\psi u) \subseteq \mathrm{WF}^{\mathrm{i}}(u) \cap \operatorname{conesupp}^{\mathrm{i}}\Big(\varphi \#(l_{1,\varepsilon} \# l_{2,\varepsilon} - l_\varepsilon)\Big)$$

and

$$(t,x,z,\infty(\tau,\xi,\zeta)) \notin \operatorname{conesupp}^{\mathrm{i}}\Big(\varphi\#(l_{1,\varepsilon}\#l_{2,\varepsilon}l_\varepsilon)\Big)$$

for $(t,x,z,\tau,\xi,\zeta) \in I_\theta$ by Theorem 2.5.4, we obtain

$$(t,x,z,\infty(\tau,\xi,\zeta)) \notin \mathrm{WF}^{\mathrm{i}}(\varphi_\psi(L_{1,\psi}L_{2,\psi} - L_\psi)u) \quad \forall(t,x,z,\tau,\xi,\zeta) \in I_\theta,$$

which is equivalent to saying

$$L_{1,\psi}L_{2,\psi}u \equiv L_\psi u \quad \text{microlocally at infinity on } I_\theta$$

for any $u \in \mathcal{G}_{L^2}$. Summarizing the observations from above we obtain the following theorem:

Theorem 2.6.7. *Let L_ψ and I_θ be as in (2.5.1) and (2.6.7). Then the operator L_ψ can be factorized into a product of two first-order ψ-pseudodifferential operators as follows*

$$L_\psi = L_{1,\psi}L_{2,\psi} \quad \textit{microlocally at infinity on } I_\theta$$

where $L_{j,\psi} = \big(\partial_z + A_j\big)_\psi$ and $A_{j,\psi}$ is the ψ-pseudodifferential operator is as in Theorem 2.5.4.

2.7 Microlocal diagonalization for L_ψ

The main issue in this section is to diagonalize of the microlocal equation $L_\psi U = F$ using the refined factorization of Theorem 2.6.7. In detail, we will rewrite the equation

$$L_\psi U = F \qquad \text{microlocally at infinity on } I_{\theta_1}$$

into an equivalent system of the form

$$\big(\partial_z - iB_\pm(x,z,D_t,D_x)\big)_\psi u_\pm = f_\pm \quad \text{microlocally at infinity on } I_{\theta_1}.$$

To show this we will discuss a different approach to the one given by Stolk in [51] as it turns out that the factorization theorem already contains all the ingredients for the diagonalization.

As in Chapter 1 we start rewriting the inhomogeneous equation $L_\psi U = F$ into a first-order system with respect to the parameter z:

$$\left[(\mathrm{Id}\,\partial_z)_\psi - \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}_\psi \right] \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix} \tag{2.7.1}$$

with $U, F \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$, Id the 2×2 identity matrix and A_ψ the ψ-pseudodifferential operator with generalized symbol $(a_\varepsilon)_\varepsilon$ as in (2.5.2). For brevity we again drop the identity matrix Id in the equations from now on. Hereafter we are going to reduce the operator in (2.7.1) to diagonal form. To make this notion rigorous we make the following arrangements.

We let

$$Q_\psi = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}_\psi, \qquad P_\psi = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}_\psi$$

be 2×2 matrices of ψ-pseudodifferential operators whose entries satisfy the requirements of Lemma 2.5.3. Also we choose the top order of the symbols of $Q_{11,\psi}$, $Q_{21,\psi}$ equal to $1-m$ and those of $Q_{12,\psi}$, $Q_{22,\psi}$ equal to $-m$ for some fixed $m \in \mathbb{R}$.

Using Lemma 2.5.3 we choose P_ψ to be the approximative inverse of Q_ψ in the following sense: with $P_{11,\psi}$, $P_{12,\psi}$ being operators of order $m-1$ and $P_{21,\psi}$, $P_{22,\psi}$ of order m and one has that there exists a constant $K > 0$ such that $P_\psi Q_\psi = \mathrm{Id}_\psi + E_\psi$ for $|(\tau,\xi)| > K$ for some 2×2 ψ-pseudodifferential operator matrix E_ψ of the form

$$E_\psi = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}_\psi$$

and $E_{11,\psi}$, $E_{22,\psi} \in \mathrm{OP}_\psi\mathcal{N}_{S^0}$, $E_{12,\psi} \in \mathrm{OP}_\psi\mathcal{N}_{S^{-1}}$ and $E_{21,\psi} \in \mathrm{OP}_\psi\mathcal{N}_{S^1}$ at infinite points.

Then by (2.6.8) and (2.7.1) the equation

$$L_\psi U = F \qquad \text{microlocally at infinity on } I_{\theta_1}$$

holds if and only if

$$\varphi_\psi Q_\psi \begin{pmatrix} \partial_z & -1 \\ A & \partial_z \end{pmatrix}_\psi P_\psi Q_\psi \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix} = \varphi_\psi Q_\psi \begin{pmatrix} 0 \\ F \end{pmatrix} \qquad \text{microlocally at infinity on } I_{\theta_1}$$

where φ_ψ is the microlocal cut-off function from the previous section.

Furthermore, we will use the following notation: for $l = 1, 2$ and $k \in \mathbb{N}$ we denote by $\mathcal{E}_\psi^{(k,l)}$ operators of the form

$$\sum_{k \leqslant j \leqslant k+l} R_{2-j,\psi}(x, z, D_t, D_x)(\partial_z^{j-k})_\psi$$

and the generalized symbol of $R_{2-j,\psi} = R_{2-j,\psi}(x,z,D_t,D_x)$ is in $\mathcal{N}_{S^{2-j}}$ for $k \leqslant j \leqslant k+l$.

In the following we assume that R_ψ is a ψ-pseudodifferential operator valued 2×2 error matrix with entries in $\mathcal{E}_\psi^{(1,1)}$ at infinite points of I'_{θ_1}. Also, we let $B_{\pm,\psi} = B_{\pm,\psi}(x,z,D_t,D_x) \in \mathrm{OP}_\psi \mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{phg}}}$ at infinite points of I'_{θ_1}.

In order to obtain a diagonalization for the operator in (2.7.1) we search for operators P_ψ, Q_ψ, R_ψ and $B_{\pm,\psi}$ as above such that the following adapted formulation of the problem is valid in the region of infinite points of I'_{θ_1}:

$$Q_\psi\Big[\,(\partial_z)_\psi - \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}_\psi\Big]P_\psi = \begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix}_\psi + R_\psi. \tag{2.7.2}$$

To show the existence of these operators we will construct them explicitly. Therefore, we presume that (2.7.2) can be solved for some P_ψ, Q_ψ, R_ψ and $B_{\pm,\psi}$ with the above conditions. We remark that the above equality can be extended outside of I'_{θ_1} but then the error matrix may not be negligible (cf. Chapter 1).

Multiplying equation (2.7.2) with $Q_\psi \left(\begin{smallmatrix} 1 \\ \partial_z \end{smallmatrix}\right)_\psi$ from the right we obtain for the left hand side an expression of the form

$$Q_\psi \begin{pmatrix} \partial_z & -1 \\ A & \partial_z \end{pmatrix}_\psi P_\psi Q_\psi \begin{pmatrix} 1 \\ \partial_z \end{pmatrix}_\psi = \begin{pmatrix} Q_{12,\psi}(\partial_z^2 + A)_\psi + S_\psi^{(1,+)} \\ Q_{22,\psi}(\partial_z^2 + A)_\psi + S_\psi^{(1,-)} \end{pmatrix} \quad \text{on infinte points of } I'_{\theta_1} \tag{2.7.3}$$

with $S_\psi^{(1,\pm)} \in \mathcal{E}_\psi^{(m,2)}$ on infinite points of I'_{θ_1}. Similarly, we compute for the right hand side

$$\begin{aligned}\Big[\begin{pmatrix} \partial_z - iB_+ & 0 \\ 0 & \partial_z - iB_- \end{pmatrix}_\psi + R_\psi\Big]Q_\psi \begin{pmatrix} 1 \\ \partial_z \end{pmatrix}_\psi &= \\ = \begin{pmatrix} (\partial_z - iB_+)_\psi(Q_{12,\psi}(\partial_z)_\psi + Q_{11,\psi}) + S_\psi^{(2,+)} \\ (\partial_z - iB_-)_\psi(Q_{22,\psi}(\partial_z)_\psi + Q_{21,\psi}) + S_\psi^{(2,-)} \end{pmatrix} &\quad \text{at infinity of } I'_{\theta_1}\end{aligned} \tag{2.7.4}$$

where $S_\psi^{(2,\pm)} \in \mathcal{E}_\psi^{(m,2)}$ at infinity of I'_{θ_1}. Combining (2.7.3) and (2.7.4) we get an improved formulation of (2.7.2) which reads

$$\begin{aligned} Q_{12,\psi}(\partial_z^2 + A)_\psi &= (\partial_z - iB_+)_\psi \big(Q_{12,\psi}(\partial_z)_\psi + Q_{11,\psi}\big) + R_\psi^{(+)} \quad \text{at infinity on } I'_{\theta_1} \\ Q_{22,\psi}(\partial_z^2 + A)_\psi &= (\partial_z - iB_-)_\psi \big(Q_{22,\psi}(\partial_z)_\psi + Q_{21,\psi}\big) + R_\psi^{(-)} \quad \text{at infinity on } I'_{\theta_1}, \end{aligned} \tag{2.7.5}$$

with $R_\psi^{(\pm)} \in \mathcal{E}_\psi^{(m,2)}$ on I'_{θ_1}, $|(\tau,\xi)| > K$ for some $K > 0$.

We note that (2.7.5) is a coupled system of two equations each of which stating a factorization similar to (2.5.11) in Theorem 2.5.4. In the following we relate (2.5.11) to (2.7.5) which will guarantee existence of the basic approach made in (2.7.2). As Theorem 2.5.4 allows two different factorizations (depending on the sign of the principal symbol) we will modify both of them to deduce the refined reformulation in (2.7.5).

Thus, with a view to Theorem 2.5.4, we first write $L_\psi = (\partial_z^2 + A)_\psi$ in the following form

$$L_\psi = (\partial_z + A_{11})_\psi(\partial_z + A_{12})_\psi + \Gamma_{1,\psi} \quad \text{on } I'_{\theta_1} \tag{2.7.6}$$

and for $j = 1,2$, $A_{1j,\psi} = A_{1j,\psi}(x,z,D_t,D_x)$ is a polyhomogeneous ψ-pseudodifferential operator with generalized symbol $(a_{1j,\varepsilon})_\varepsilon$ as in Theorem 2.5.4. Moreover we choose $A_{11,\psi}$ and $-A_{12,\psi}$ such that their principal symbols are equal to $(-i\sqrt{a_\varepsilon})_\varepsilon$. Furthermore $\Gamma_{1,\psi}$ is in $\mathcal{E}_\psi^{(0,1)}$ at infinite points on I'_{θ_1}. Likewise we obtain

$$L_\psi = (\partial_z + A_{21})_\psi(\partial_z + A_{22})_\psi + \Gamma_{2,\psi} \quad \text{at infinity on } I'_{\theta_1} \tag{2.7.7}$$

where $A_{2j,\psi} = A_{2j,\psi}(x,z,D_t,D_x)$, $j = 1,2$, are polyhomogeneous ψ-pseudodifferential operators but at this point the top order symbols of $A_{21,\psi}$ and $-A_{22,\psi}$ equal to $(i\sqrt{a_\varepsilon})_\varepsilon$. Again $\Gamma_{2,\psi}$ is in $\mathcal{E}_\psi^{(0,1)}$ on I'_{θ_1} with $|(\tau,\xi)| \geqslant K$ for some $K > 0$.

An expansion in (2.7.6) and (2.7.7) then gives the following for $j = 1,2$

$$(\partial_z^2 + A)_\psi = (\partial_z^2)_\psi + (A_{j1} + A_{j2})_\psi(\partial_z)_\psi + \mathrm{OP}_\psi\big((\varepsilon\partial_z a_{j2})_\varepsilon\big) + A_{j1,\psi}A_{j2,\psi} + \Gamma_{j,\psi}$$

on I'_{θ_1} with $|(\tau,\xi)| \geqslant K > 0$ where we used the Leibniz rule. By construction of the generalized symbols of $A_{j1,\psi}$ and $A_{j2,\psi}$ we observe that $(a_{j1,\varepsilon})_\varepsilon = (-a_{j2,\varepsilon})_\varepsilon$ modulo $\mathcal{N}_{S^1}$ at infinity on I'_{θ_1}, $j = 1,2$. Using this (2.7.6) and (2.7.7) then read

$$\begin{aligned} (\partial_z^2 + A)_\psi &= (\partial_z + A_{11})_\psi(\partial_z - A_{11})_\psi \qquad \text{mod } \mathcal{E}_\psi^{(0,1)} \text{ at infinity on } I'_{\theta_1} \\ &= (\partial_z + A_{21})_\psi(\partial_z - A_{21})_\psi \qquad \text{mod } \mathcal{E}_\psi^{(0,1)} \text{ at infinity on } I'_{\theta_1}. \end{aligned} \tag{2.7.8}$$

With m being the fixed real number from the beginning of this section we now choose $\widetilde{Q}_\psi^{(\pm)} \in \mathrm{OP}_\psi\mathcal{M}^{(1,0)}_{S^{m,0}_{\mathrm{phg}}}$ and so that the requirements of Lemma 2.5.3 are fulfilled. Using the same result we obtain the existence of two ψ-pseudodifferential operators $Q_\psi^{(\pm)}$ with generalized symbols in $\mathcal{M}^{(1,0)}_{S^{-m,0}_{\mathrm{phg}}}$ satisfying (2.5.8) and

$$\widetilde{Q}_\psi^{(\pm)} Q_\psi^{(\pm)} = Q_\psi^{(\pm)} \widetilde{Q}_\psi^{(\pm)} = I_\psi \qquad \text{mod } \mathrm{OP}_\psi \mathcal{N}_{S^0} \text{ at infinity on } I'_{\theta_1}.$$

Inserting $\widetilde{Q}_\psi^{(+)} Q_\psi^{(+)}$ into the first line of (2.7.8) and $\widetilde{Q}_\psi^{(-)} Q_\psi^{(-)}$ into the second line yields

$$\begin{aligned}
(\partial_z^2 + A)_\psi &= (\partial_z + A_{11})_\psi \widetilde{Q}_\psi^{(+)} \big(Q_\psi^{(+)} (\partial_z)_\psi - Q_\psi^{(+)} A_{11,\psi}\big) \qquad && \text{mod } \mathcal{E}_\psi^{(0,2)} \text{ on } I'_{\theta_1} \\
&= (\partial_z + A_{21})_\psi \widetilde{Q}_\psi^{(-)} \big(Q_\psi^{(-)} (\partial_z)_\psi - Q_\psi^{(-)} A_{21,\psi}\big) && \text{mod } \mathcal{E}_\psi^{(0,2)} \text{ on } I'_{\theta_1}
\end{aligned}$$

for $|(\tau, \xi)| \geqslant K > 0$. Here, we define

$$\begin{aligned}
-iB_{+,\psi} &:= Q_\psi^{(+)} A_{11,\psi} \widetilde{Q}_\psi^{(+)} - \mathrm{OP}_\psi\big((\varepsilon \partial_z q_\varepsilon^{(+)})_\varepsilon\big) \widetilde{Q}_\psi^{(+)} \\
-iB_{-,\psi} &:= Q_\psi^{(-)} A_{21,\psi} \widetilde{Q}_\psi^{(-)} - \mathrm{OP}_\psi\big((\varepsilon \partial_z q_\varepsilon^{(-)})_\varepsilon\big) \widetilde{Q}_\psi^{(-)}
\end{aligned} \tag{2.7.9}$$

for $|(\tau, \xi)|$ large enough and where $(q_\varepsilon^{(\pm)})_\varepsilon$ denote the principal symbols of $Q_\psi^{(\pm)}$. Consequently, the operators $B_{\pm,\psi}$ are generalized polyhomogeneous ψ-pseudodifferential operators with real-valued top order symbol in $\mathcal{M}^{(1,0)}_{S^{1,0}_{\mathrm{hg}}}$ on I'_{θ_1}. Furthermore, a straightforward calculation shows that with this choices for $B_{\pm,\psi}$ we have

$$\begin{aligned}
\big(\partial_z + A_{11}\big)_\psi \widetilde{Q}_\psi^{(+)} &= \widetilde{Q}_\psi^{(+)} \big(\partial_z - iB_+\big)_\psi \qquad && \text{mod } \mathcal{E}_\psi^{(1-m,0)} \text{ on } I'_{\theta_1} \\
\big(\partial_z + A_{21}\big)_\psi \widetilde{Q}_\psi^{(-)} &= \widetilde{Q}_\psi^{(-)} \big(\partial_z - iB_-\big)_\psi && \text{mod } \mathcal{E}_\psi^{(1-m,0)} \text{ on } I'_{\theta_1}
\end{aligned}$$

for $|(\tau, \xi)| \geqslant K > 0$, leading to the following decomposition on I'_{θ_1}

$$\begin{aligned}
\big(\partial_z^2 + A\big)_\psi &= \widetilde{Q}_\psi^{(+)} \big(\partial_z - iB_+\big)_\psi \big(Q_\psi^{(+)} (\partial_z)_\psi - Q_\psi^{(+)} A_{11,\psi}\big) \qquad && \text{mod } \mathcal{E}_\psi^{(0,2)} \\
&= \widetilde{Q}_\psi^{(-)} \big(\partial_z - iB_-\big)_\psi \big(Q_\psi^{(-)} (\partial_z)_\psi - Q_\psi^{(-)} A_{21,\psi}\big) && \text{mod } \mathcal{E}_\psi^{(0,2)}
\end{aligned}$$

at infinity. Comparing this with the refined approach (2.7.5) we make the following choice for the matrix Q_ψ from the beginning

$$\begin{aligned}
Q_{11,\psi} &:= -Q_\psi^{(+)} A_{11,\psi}, \qquad & Q_{12,\psi} &:= Q_\psi^{(+)}, \\
Q_{21,\psi} &:= -Q_\psi^{(-)} A_{21,\psi}, & Q_{22,\psi} &:= Q_\psi^{(-)}.
\end{aligned}$$

Hence

$$Q_\psi = \begin{pmatrix} -Q_\psi^{(+)} A_{11,\psi} & Q_\psi^{(+)} \\ -Q_\psi^{(-)} A_{21,\psi} & Q_\psi^{(-)} \end{pmatrix} = \begin{pmatrix} Q^{(+)} & 0 \\ 0 & Q^{(-)} \end{pmatrix}_\psi \begin{pmatrix} -A_{11} & 1 \\ -A_{21} & 1 \end{pmatrix}_\psi \tag{2.7.10}$$

and $Q_\psi^{(\pm)}$ are polyhomogeneous ψ-pseudodifferential operators of order $-m$ on I'_{θ_1} and elliptic in the sense of (2.5.6).

Since the top order symbol of $A_{j1,\psi}$ is given by $(\pm i\sqrt{a_\varepsilon})_\varepsilon$, $j = 1, 2$ it follows from (2.7.10) that Q_ψ is elliptic in the sense of (2.5.6). Furthermore, the approximative inverse matrix P_ψ of Q_ψ is given by

$$P_\psi = \begin{pmatrix} -C_\psi & C_\psi \\ -A_{21,\psi} C_\psi & 1_\psi + A_{21,\psi} C_\psi \end{pmatrix} \begin{pmatrix} \tilde{Q}^{(+)} & 0 \\ 0 & \tilde{Q}^{(-)} \end{pmatrix}_\psi \qquad \text{on } I'_{\theta_1}$$

where C_ψ is the generalized parametrix of $(A_{11} - A_{21})_\psi$ in the sense of Lemma 2.5.3. We have therefore found an appropriate operator-valued matrix P_ψ, Q_ψ, R_ψ and operators $B_{\pm,\psi}$ solving (2.7.2). We have therefore proved the following theorem:

Theorem 2.7.1. *Let L_ψ as in (2.5.1) and $U, F \in \mathcal{G}_{L^2}(\mathbb{R}^{n+1})$. Then, there are operators Q_ψ as in (2.7.10) and $B_{\pm,\psi}$ as in (2.7.9), such that the equation*

$$L_\psi U = F \quad \textit{microlocally at infinity on } I_{\theta_1} \tag{2.7.11}$$

holds if and only if

$$\begin{aligned} \big(\partial_z - iB_+(x,z,D_t,D_x)\big)_\psi u_+ &= f_+ \quad \textit{microlocally at infinity on } I_{\theta_1} \textit{ and} \\ \big(\partial_z - iB_-(x,z,D_t,D_x)\big)_\psi u_- &= f_- \quad \textit{microlocally at infinity on } I_{\theta_1}. \end{aligned} \tag{2.7.12}$$

Furthermore the coupling effect is computed as follows

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} := Q_\psi \begin{pmatrix} U \\ (\partial_z)_\psi U \end{pmatrix}, \qquad \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := Q_\psi \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

2.7.1 Closing remarks

Let us briefly summarize the above. First we explained a factorization procedure for the semiclassical operator L_ψ in the non-trivial case of generalized coefficients of log-type when acting on $\mathcal{G}_{L^2}$. To overcome the error made in this factorization we further introduced an adapted notion of microlocal regularity.

Concerning the motivating part of this chapter we want to make the following remarks. Because of the lack of an adequate description of propagation of singularities in this setting it is not clear so far if and how one can derive approximated solutions to (2.7.11) from solutions of a perturbation of the problem (2.7.12) as we have seen in the smooth case in Subsection 0.1. Even if we allow the operator L_ε given in (2.1.3) to have logarithmic slow scale regular coefficients this problem remains unsolved. Once again, we want to mention that in the case of logarithmic slow scale case no semiclassical interpretation of the situation is necessary and microlocal regularity is based on local $\mathcal{G}_{L^2}$-regularity.

Moreover, we note that apart from the microlocal phase space restrictions in the equations of (2.7.12) the operators $L_{j,\psi}$ meet the conditions of Theorem 3.1 of [32], $j = 1, 2$, if the coefficients in (2.1.3) from the beginning are of log-type with an appropriately chosen exponent $r \in \mathbb{N}$ which depends only on the dimension n. More precisely r plays the same role as k does in Remark 3.2 in [32]. It is not clear so far if it is possible to associate to (2.7.12) a global description of the same by introducing suitable damping operators a in the logarithmic slow scale (and smooth) setting (cf. Subsections 1.8 and 1.9).

However, we already pointed out in Subsection 2.3 that so far we only handled the case where the semiclassical asymptotic regime was restricted to $h(\varepsilon) = \varepsilon$ as $\varepsilon \to 0$. This suggests itself to ask for general criteria of other possible choices for the semiclassical scale $h(\varepsilon)$. Under the viewpoint of parameter-dependent representation theory for generalized pseudodifferential operators this will hopefully also give more insight in the notion of microlocal regularity at infinity. Here, a future aim is to obtain a refined notion of microlocalization which is capable of giving a global characterization and hence has to include regularity results for Colombeau generalized objects also for finite points (cf. Remark 2.6.1).

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