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DISSERTATION

Arithmetic groups acting on quaternion hyperbolic spaces - structure and geometric construction of homology

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Introduction

Locally symmetric spaces form a special class of Riemannian manifolds. In this work our attention is focused on locally symmetric spaces of non-compact type (i.e. having non-positive sectional curvature). These are of the form $\Gamma\backslash G/K$ where G is a noncompact, connected semisimple Lie group with finite center, K a maximal compact subgroup of G and Γ a torsion-free discrete subgroup of G. If G is the group of $\mathbb{R}$-points of an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, one can construct discrete subgroups of G as arithmetic subgroups of the group $\mathbf{G}(\mathbb{Q})$. Algebraic properties of Γ and $\mathbf{G}$ are reflected in geometric characteristics of the locally symmetric space $\Gamma\backslash G/K$ and vice versa. In many aspects, the space $\Gamma\backslash G/K$ can be thought of as a geometric analogue of the group Γ. For instance, since G/K is simply connected, it is the universal cover of $\Gamma\backslash G/K$ so the fundamental group of $\Gamma\backslash G/K$ is Γ. Moreover, G/K is contractible, so $\Gamma\backslash G/K$ is a "$K(\Gamma,1)$" Eilenberg-MacLane space and the group cohomology of Γ can be identified with the de Rham cohomology of the manifold $\Gamma\backslash G/K$.
A thorough analysis of these cohomology groups brings together group theory, arithmetic and geometry in a most fruitful way. In addition, the cohomology of an arithmetic group Γ is strongly related to the theory of automorphic forms for the underlying algebraic group $\mathbf{G}$ with respect to Γ. More precisely, for a congruence subgroup Γ the cohomology $H^*(\Gamma\backslash X;\mathbb{C})$ can be interpreted as the relative Lie algebra cohomology $H^*(\mathbf{g},K;\mathcal{A}(G,\Gamma))$ of $\mathbf{g} := Lie(G)$ with respect to the automorphic spectrum (see [Fr]). Hence, any geometric construction of a non-trivial cohomology class has an interpretation in the theory of automorphic forms. For example, this approach led to non-vanishing results of multiplicities with which specific automorphic forms occur in the automorphic spectrum ([Mil], [MR]).
For a compact locally symmetric space $\Gamma\backslash X$ one has an injective homomorphism of the algebra $Inv^*(G)$ of G-invariant forms on X into $H^*(\Gamma\backslash X;\mathbb{R})$ (see [Mat]). These cohomology classes have also a geometric origin, since the algebra $Inv^*(G)$ can be identified with the cohomology $H^*(X_u;\mathbb{R})$ of the compact dual locally symmetric space X_u of X.

In this work we apply geometric methods ([Mil], [MR], [RSchw], [LS]) for construction of homology classes in locally symmetric spaces to treat the case of arithmetically defined compact quaternion hyperbolic manifolds. Such a locally symmetric space arises as a quotient of the quaternion hyperbolic space $H^n(\mathbb{H}) = Sp(n,1)/(Sp(n)\times Sp(1))$ under the action of a cocompact torsion-free arithmetic lattice Γ in $Sp(n,1)$. The lattice Γ is constructed in the following way: let F be a totally real algebraic number field, D a quaternion algebra over F which does not split at any archimedean place of F, B a

Hermitian form[1] on D^{n+1} of signature $(n,1)$ over D, all of whose conjugates are positive definite. Denote by $SU(B,D,\tau_c)$ the group of isometries of B in $SL_{n+1}(D)$. Then, for any O_F-order[2] Λ in D, the group $SU(B,\Lambda,\tau_c) := SU(B,D,\tau_c) \cap SL_{n+1}(\Lambda)$ embeds in $Sp(n,1)$ as a cocompact arithmetic lattice and the torsion-freeness can be settled by passing to suitable congruence subgroups. We describe the way to obtain such torsion-free subgroups of Γ by passing to congruence subgroups modulo ideals of Λ. Then we consider suitable pairs of commuting involutions on $\Gamma\backslash Sp(n,1)/(Sp(n)\times Sp(1))$ and apply the methods of [MR] and [RSchw] to produce pairs of compact oriented submanifolds[3] in $\Gamma\backslash Sp(n,1)/(Sp(n)\times Sp(1))$ of complementary dimensions intersecting with nonzero intersection number. As a consequence, they represent non-trivial homology classes on $\Gamma\backslash Sp(n,1)/(Sp(n)\times Sp(1))$. The dimensions in which these submanifolds occur are $4, 8, \ldots,$ and $4n-4$. Passing to a finite covering it can be arranged that these homology classes are not dual to $Sp(n,1)$-invariant cohomology classes, i.e. they do not come from $Inv^*(Sp(n,1))$.
Geometric constructions mentioned above rely on an application of the Poincaré duality for compact oriented manifolds. Quotients of the quaternion hyperbolic space $Sp(n,1)/(Sp(n)\times Sp(1))$ under the action of a lattice $\Gamma \leq Sp(n,1)$ are oriented since the group $Sp(n,1)$ is connected. This is not the case for the group $SO(n,1)$, and the orientability problem for arithmetically defined real hyperbolic manifolds[4] cannot be settled by imposing certain "low" congruence conditions on the arithmetic group in question, as it is cited in some recent published papers. We exhibit an example contradicting these claims.
Furthermore, we discuss the problem of finding non G-invariant cohomology classes on compact locally symmetric spaces of non-compact type and the way to obtain lower bounds on the dimensions of the corresponding cohomology groups.
As mentioned above, a cocompact arithmetic group Γ acting on a quaternion hyperbolic space arises as the group of $\Lambda-$points of the group of isometries of a certain Hermitian form B over a quaternion skewfield D constructed over a totally real algebraic number field F, where Λ is an O_F-order in D. We present an elementary proof (independent of the one in [Ba]) of the result on the stable rank of an arithmetic order Λ in a (finite dimensional, separable) skewfield over an algebraic number field. This result can be used to analyze the structure of the linear groups over Λ (i.p. of the arithmetic group Γ) and the first "K-group" of Λ. Although there is no apparent connection to the topic of cohomology of locally symmetric spaces, we note that in [Bo2] a connection between the stable cohomology of arithmetic groups and the "K-groups" of arithmetic orders is established.

We now describe the content of the chapters more precisely:

Chapter 1 contains a description of the geometric methods of construction of cohomology classes in locally symmetric spaces initiated by Millson in [Mil] and further generalized by Millson and Raghunathan ([MR]) and Rohlfs and Schw-

[1] B is a Hermitian form with respect to the standard symplectic anti-involution τ_c on D.
[2] An O_F-order in D is a subring Λ containing 1 which is a finitely generated O_F- module s.t. $F\Lambda = D$ holds.
[3] These are again compact locally symmetric quaternion manifolds.
[4] These arise as quotients of the real hyperbolic space $SO(n,1)/S(O(n)\times O(1))$ under the action of an arithmetic group constructed similarly as in the case of quaternion manifolds discussed above.

ermer ([RSchw]). We give a summary of the Millson-Raghunathan method, where the homology classes in compact locally symmetric spaces $\Gamma\backslash X$ arise from projections of two orthogonal subspaces of the given symmetric space X which intersect transversely at the origin of X. Projecting these subspaces to a locally symmetric space their intersection becomes more complicated. Degenerate intersection components may occur and, even in the case of transversal intersection components still the sign (i.e. the multiplicity) of the intersection has to be determined. The idea of the Millson-Raghunathan approach is to pass to a congruence subgroup $\tilde{\Gamma} \leq \Gamma$, i.e., to a finite cover $\tilde{\Gamma}\backslash X$ of $\Gamma\backslash X$, where only transversal intersection components with positive intersection multiplicity occur. As a consequence, the intersection number of the corresponding homology classes does not equal 0, hence, the classes (and their cohomological duals) cannot be trivial in $H_*(\tilde{\Gamma}\backslash X;\mathbb{Z})$ (resp. $H^*(\tilde{\Gamma}\backslash X;\mathbb{Z})$). The Rohlfs-Schwermer method is a generalization of this result: it considers suitable pairs σ, τ of commuting automorphisms of finite order and the connected components $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ of their fixed points sets on the locally symmetric space $\Gamma\backslash X$. Passing to a finite cover (i.e. to some finite index subgroup $\tilde{\Gamma}$ of Γ) it can be arranged that the intersection number $[C_\sigma(\tilde{\Gamma})][C_\tau(\tilde{\Gamma})]$ equals the sum of Euler characteristics of the connected components of the intersection $C_\sigma(\tilde{\Gamma}) \cap C_\tau(\tilde{\Gamma})$, which are all positive, hence the sum is nonzero and as a consequence $[C_\sigma(\tilde{\Gamma})]$ and $[C_\tau(\tilde{\Gamma})]$ are non-trivial homology classes.
In addition, there is a recent result of Lafont and Schmidt ([LS]) which can be applied to show that certain locally symmetric subspaces (which include some of the above-mentioned cycles) of dimensions in which the cohomology of the compact dual does not vanish represent non-trivial homology classes. However, using this method we do not know how to distinguish between the G-invariant and non G-invariant classes constructed by this method.

Having found the corresponding locally symmetric space $\Gamma\backslash X$ where the cycles intersect with nonzero intersection number, we can pass further to a finite covering $\tilde{\Gamma}\backslash X$ of $\Gamma\backslash X$ where the (cohomological duals of the) cycles do not represent G-invariant classes. This is explained in Chapter 2. Moreover, using the fact that the deck transformation group $\Gamma/\tilde{\Gamma}$ acts non-trivially on the homology groups of $\tilde{\Gamma}\backslash X$ the dimension of the smallest non-trivial representation of $\Gamma/\tilde{\Gamma}$ gives a lower bound on the corresponding Betti number. We give a brief overview of the necessary technique for determining the group $\Gamma/\tilde{\Gamma}$.

In Chapter 3 we apply the above-mentioned methods to the case of quaternion hyperbolic spaces. We describe arithmetic groups acting on quaternion hyperbolic spaces. These groups arise from arithmetic orders in quaternion algebras over number fields in the following way: let F be a totally real algebraic number field, D a quaternion algebra over F which does not split at any archimedean place of F, B a hermitian form on D^{n+1} (with respect to the standard symplectic anti-involution τ_c on D) of signature $(n, 1)$ all of whose conjugates are positive definite. Denote by $SU(B, D, \tau_c)$ the group of isometries of B in $SL_{n+1}(D)$. Then, for any order Λ in D, $SU(B, \Lambda, \tau_c) := SU(B, D, \tau_c) \cap SL_{n+1}(\Lambda)$ embeds in $Sp(n, 1)$ as a cocompact arithmetic lattice. In order for a discrete subgroup of isometries $\Gamma \leq G$ to act freely on the symmetric space G/K it has to be torsion-free. We can pass to a finite index torsion-free subgroup of $SU(B, \Lambda, \tau_c)$ using the following observation:
Let D be a (central, simple) skewfield over a (totally real) algebraic number field F. Let O_F be the ring of integers of F and Λ a maximal O_F-order in D.

Let Γ be a subgroup of $SL_n(\Lambda)$. For an ideal β of Λ let $\Gamma(\beta) := \{\gamma \in \Gamma \mid \gamma \equiv I \pmod{\beta}\}$. Let $\beta := \mathbf{p}_1^{k_1}\mathbf{p}_2^{k_2}\dots\mathbf{p}_r^{k_r}$, where $k_i \geq 1$ for all $i = 1, \dots, r$, be the decomposition of β into distinct prime ideals $\mathbf{p}_i$ of Λ. The intersection of $\mathbf{p}_i$ with $\mathbb{Z}$ is a prime ideal, hence we have $\mathbf{p}_i \cap \mathbb{Z} = p_i\mathbb{Z}$ for a corresponding prime number p_i. In this setting we have (see section 3.2.2):

Proposition. *Let Γ be a subgroup of $SL_n(\Lambda)$. Let $\beta := \mathbf{p}_1^{k_1}\mathbf{p}_2^{k_2}\dots\mathbf{p}_r^{k_r}$ be an ideal of Λ. Suppose either that there exist $i, j \in \{1, \dots, r\}$, $i \neq j$ s.t. $p_i \neq p_j$ or that there exists a single $j \in \{1, \dots, r\}$ such that*

$$exp_{\mathbf{p}_j}(p_j) < k_j(p_j - 1),$$

where $exp_{\mathbf{p}_j}(p_j)$ is the exponent of $\mathbf{p}_j$ in the prime ideal decomposition of the two sided ideal $< p_j >_\Lambda = \Lambda p_j$ of Λ generated by p_j. Then $\Gamma(\beta)$ is torsion-free.

If Γ is a torsion-free arithmetic group of the above form, it embeds in $Sp(n,1)$ as a cocompact lattice and we get a compact locally symmetric space $\Gamma\backslash X$, where $X := Sp(n,1)/(Sp(n) \times Sp(1))$. Fix $m \in \{1, \dots, n-1\}$ and consider two commuting F-involutions σ resp. τ of the underlying algebraic group $\mathbf{G}(F) = SU(B, D, \tau_c)$ given by conjugation by the matrix $\begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ resp. $\begin{bmatrix} -I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. These induce involutions on Γ and X, whose fixed point sets we denote by Γ_σ, X_σ, resp. Γ_τ, X_τ. Then, the locally symmetric spaces $\Gamma_\sigma\backslash X_\sigma$ and $\Gamma_\tau\backslash X_\tau$ embed in $\Gamma\backslash X$ as totally geodesic submanifolds. Let $[\Gamma_\sigma\backslash X_\sigma]$, $[\Gamma_\tau\backslash X_\tau]$ denote their homology classes in $H_{4m}(\Gamma\backslash X; \mathbb{Z})$ resp. $H_{4(n-m)}(\Gamma\backslash X; \mathbb{Z})$. Applying the results of [RSchw] and [MR] we get (see section 3.3):

Theorem. *In the setting above, there exists a $< \sigma, \tau >$-stable normal subgroup $\tilde{\Gamma}$ of finite index in Γ s.t. the intersection number $[\tilde{\Gamma}_\sigma\backslash X_\sigma][\tilde{\Gamma}_\tau\backslash X_\tau]$ is not 0. In particular, $[\tilde{\Gamma}_\sigma\backslash X_\sigma]$ resp. $[\tilde{\Gamma}_\tau\backslash X_\tau]$ are non-trivial homology classes in $H_{4m}(\tilde{\Gamma}\backslash X; \mathbb{Z})$ resp. $H_{4(n-m)}(\tilde{\Gamma}\backslash X; \mathbb{Z})$.*

Applying the result of [MR] (described in Ch. 2), we get:

Corollary. *There exists a $< \sigma, \tau >$-stable normal subgroup Γ' of finite index in Γ s.t. the cohomological duals of $[\Gamma'_\sigma\backslash X_\sigma]$ resp. $[\Gamma'_\tau\backslash X_\tau]$ represent non $Sp(n,1)$-invariant cohomology classes in $H^{4(n-m)}(\tilde{\Gamma}\backslash X; \mathbb{R})$ resp. $H^{4m}(\tilde{\Gamma}\backslash X; \mathbb{R})$.*

In Chapter 4 we discuss orientability questions in locally symmetric spaces modelled on the real hyperbolic space. We exhibit an example contradicting claims in some published papers on the topic that the orientability of the locally symmetric space $\Gamma\backslash SO(n,1)/S(O(n) \times O(1))$ can be achieved for certain "low" congruence subgroups. Furthermore, we show how an observation of Xue in [X] can be used to give an alternative proof of the existence of finite coverings of certain locally symmetric spaces where the corresponding cycles (constructed as in Ch. 1) represent non G-invariant classes. We mention the results on the dimensions of the smallest non-trivial representations of some finite Lie groups which can serve as lower bounds on the Betti numbers of locally symmetric spaces in question.

In addition, we present an elementary proof independent of the one in [Ba] of a result concerning the stable range properties [5] of arithmetic orders in skewfields over number fields (see section 4.2):

Proposition. *Let D be a finite dimensional separable skewfield over an algebraic number field K. Let O_K be the ring of integers in K and let Λ be an O_K-order in D. Then $sr(\Lambda) \leq 2$.*

We conclude with an application of the Proposition above for the structure of the stable and general linear groups over Λ.

Finally, in Appendix we give a brief overview of the background theory and results which are frequently used in the thesis.

[5]This concept plays an important role in the structure theory for the general linear group over a ring R in dimensions above the stable rank $sr(R)$, which is defined as the smallest integer k s.t. for all $m > k$ every unimodular element $(x_1, \ldots, x_m) \in R^m$ is reducible, i.e. if $\sum_{i=1}^{m} Rx_i = R$, then there exist $t_1, \ldots, t_{m-1} \in R$ s.t. $\sum_{i=1}^{m-1} R(x_i + t_i x_m) = R$.

Contents

Notation, conventions and definitions

Common abbreviations

We often use the following abbreviations: "ch." meaning "chapter", "f.e." meaning "for example", "i.e." meaning "that is"/"in other words" (from latin "id est"), "i.p." meaning "in particular", "p." meaning "page"/"pages", "prop." meaning "proposition", "resp." meaning "respectively", "s.t."meaning "so that" or "such that", "th." meaning "theorem" and "w.l.o.g." meaning "without loss of generality".

If $f : X \longrightarrow X$ is a map on a set X, we denote by $Fix_X(f) := \{x \in X \mid f(x) = x\}$ the set of fixed points of the map f.

Rings, fields and algebraic groups

For a ring R we denote by $R^* := R \backslash 0$ its group of units.
The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ have the usual meaning: nonnegative integers, integers, rational numbers, real numbers, complex numbers. For an algebraic number field F we denote by O_F its ring of integers. If $\mathbf{p}$ is a prime ideal of F, we denote by $F_{\mathbf{p}}$ the $\mathbf{p}$-adic completion of F and by $O_{\mathbf{p}}$ the ring of integers in $F_{\mathbf{p}}$. The notion of the adele ring and the corresponding notations are introduced and explained in section 5.4.
The algebraic groups we consider are linear. If G is an algebraic group defined over a field F, and L a field extension of F, we write $G(L)$ for the group of L-rational points of G. For the sake of brevity, the symbol G is sometimes used to denote the group of $\mathbb{R}$-rational points of the algebraic group $\underline{G}$ defined over $\mathbb{Q}$, f.e. in section 1.2.
We shall often use the fact that in order to specify an algebraic group G defined over an algebraic number field F, since $G(F)$ is Zariske dense in G (see [PR], Th.2.2.), it suffices to specify $G(F)$.
If G is an algebraic group defined over a field F, for a field extension L we define $G \otimes L$ to be the group obtained by replacing the elements of F with elements of L, i.e. if G is defined over F by the set Ψ of polynomials in $F[x_{11}, \ldots, x_{ll}]$, then let $G \otimes L := \{g \in SL_l(L) \mid p(g) = 0 \text{ for all } p \in \Phi\}$.
The notions connected-, simply connected-, simple-, reductive- and semisimple algebraic group are standard, i.p. correspond to those used in [PR].

Manifolds and group actions

A Riemannian manifold X is called a *homogeneous space* if its isometry group $Isom(X)$ acts transitively, i.e. for all $x, y \in X$ there exists an isometry ϕ of X

such that $\phi(x) = y$.
Let $\phi : X \longrightarrow X$ be a map on a Riemannian manifold X. ϕ is called an *involution* of X if $\phi^2 = Id$. A fixed point x of ϕ is called an *isolated fixed point of* ϕ if there is a neighborhood U of x such that x is the only fixed point of ϕ contained in U.
A Riemannian manifold X is called a *symmetric space* if X is connected and homogeneous and if there exists an involutive isometry ϕ of X such that ϕ has at least one isolated fixed point.
Let Γ be a group acting on a topological space X. We say that Γ acts *freely* on X if

$$\gamma \cdot x = x \Rightarrow \gamma = e.$$

Furthermore, Γ is said to act *properly discontinuously* on X if for every compact subset C of X:

$$|\{\gamma \in \Gamma | C \cap \gamma C \neq \emptyset\}| < \infty.$$

It can be seen that a group Γ acts freely and properly discontinuously on a topological space X, the natural map $X \longrightarrow \Gamma\backslash X$ is a covering map.
A Riemannian manifold M is called a *locally symmetric space* if its universal cover is a symmetric space, i.e. there exist a symmetric space X and a group Γ of isometries of X, such that Γ acts freely and properly discontinuously on X and M is isometric to $\Gamma\backslash X$.
A subgroup Γ of a topological group G is said to be *discrete* if Γ is a discrete subset of a topological group G, i.e. for every $\gamma \in \Gamma$ there exists an open neighborhood U of γ in G such that $\Gamma \cap U = \gamma$.
A discrete subgroup Γ of a Lie group G is called a *lattice* in G if $\Gamma\backslash G$ has finite volume [6]. Note that if Γ is discrete and $\Gamma\backslash G$ compact, then Γ is a lattice since every compact Riemannian manifold has finite volume.

Congruence subgroups

The term *congruence subgroup*, unless otherwise noted, is used in the "classical" sense: let F be a number field, G a linear algebraic group defined over F and $G \subset SL_n$ a fixed embedding. Let $G(O_F) := G \cap SL_n(O_F)$. For a nonzero ideal $\mathbf{q}$ of O_F let $\Gamma(\mathbf{q}) := Ker(G(O_F) \longrightarrow SL_n(O_F/\mathbf{q}))$ denote the kernel of the natural map $G(O_F) \longrightarrow SL_n(O_F/\mathbf{q})$. A subgroup of $G(O_F)$ is called a congruence subgroup if it contains $\Gamma(\mathbf{q})$ for some nonzero ideal $\mathbf{q}$ of O_F. (Usually, the congruence subgroups we shall refer to will be the groups $\Gamma(\mathbf{q})$ themselves.)
Passing to the "adelic setting", it can be seen that the group $\Gamma(\mathbf{q})$, for $\mathbf{q} = \prod_{i=1}^n \mathbf{p}_i^{k_i}$ equals the group $L \cap G(F)$ for

$$L := \prod_{i=1}^n L_i \ \times \prod_{\mathbf{p} \in \mathbb{P} \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}} G(O_{\mathbf{p}}),$$

where $L_i := \{A \in G(O_{\mathbf{p}_i}) \,|\, a_{ij} - \delta_{ij} \in \mathbf{p}_i^{k_i} O_{\mathbf{p}_i}\}$.
Hence, this is a special case of the following more general notion of a congruence subgroup: an arithmetic subgroup $\Gamma \leq G(F)$ is called a congruence subgroup if Γ is of the form $K \cap G(F)$ (the intersection taken in $G(\mathbb{A}_F)$), where K is an

[6] Fix an left-invariant Haar measure μ on G (unique up to a scalar multiple), which induces a measure $\overline{\mu}$ on $\Gamma\backslash G$. By volume of $\Gamma\backslash G$ we mean the volume relative to $\overline{\mu}$.

open compact subgroup of $G(\mathbb{A}_f)$. The term congruence subgroup is used in that meaning in sections 1.3.6 and 4.1.1 (as it is noted there).

Chapter 1

Geometric constructions of homology classes in locally symmetric spaces

This chapter presents some ideas which enable geometric constructions of homology classes in arithmetically defined locally symmetric spaces. We start with the necessary geometric background, such as Poincaré duality and some results in the intersection theory on manifolds. Then we give a summary of the Millson-Raghunathan method, where the homology classes in compact locally symmetric spaces $\Gamma\backslash X$ arise from projections of two orthogonal subspaces of the given symmetric space X. In the symmetric space they intersect transversely at the origin, however, passing to a locally symmetric space the intersection becomes more complicated. Degenerate intersection components may occur and, even in the case of transversal intersection components still the sign (i.e. the multiplicity) of the intersection has to be determined. The idea of the Millson-Raghunathan approach ([MR]) is to pass to a deeper congruence subgroup $\tilde{\Gamma} \leq \Gamma$, i.e. , to a finite cover $\tilde{\Gamma}\backslash X$ of $\Gamma\backslash X$, where only transversal intersection components with positive intersection multiplicity occur. Then the intersection number of the corresponding homology classes is > 0, hence the classes cannot be trivial in $H_*(\tilde{\Gamma}\backslash X;\mathbb{Z})$. The Rohlfs-Schwermer method ([RSchw]) is a generalization of this result: it considers pairs σ, τ of commuting automorphisms of finite order and the connected components $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ of their fixed points sets on the compact locally symmetric space $\Gamma\backslash X$, parametrizes the intersection components of $C_\sigma(\Gamma)\cap C_\tau(\Gamma)$ by certain elements of $H^1(<\sigma,\gamma>;\Gamma)$ and then, passing to a finite cover (i.e. to some finite index subgroup of Γ) it is arranged that all the intersection numbers are positive. Hence, the difference in the geometric approach is that the Rohlfs-Schwermer method analyzes degenerate intersections and finds a cover where all of their intersection numbers are positive, whereas Millson and Raghunathan get rid of the degenerate intersections. Finally, there is a method due to Lafont-Schmidt ([LS]), where locally symmetric subspaces of the compact locally symmetric space of a non-compact type are considered. In some cases it can be shown that the corresponding fundamental classes are non-trivial as a result of the connection with the cohomology of the compact dual symmetric space via the Matsushima map. However, this method cannot be used to distinguish between the non G-invariant classes (i.e., homology classes whose Poincaré dual is not a G-invariant cohomology class) and G-invariant classes, which arise from the cohomology of the compact dual.

1.1 Geometric background

1.1.1 Poincaré duality

We assume that the reader is familiar with some concepts of homology and cohomology theory. The notation we use is standard, i.p. corresponds to the notation used in [Ha]. The following is a short summary of some of the homology theory which can be found in [Ha]:
Given a differentiable n-dimensional manifold M, one can choose an orientation of T_xM for every $x \in M$: Let $\xi_1, \ldots, \xi_n$ be vector fields such that $\xi_1(x), \ldots, \xi_n(x)$ form a basis of T_xM, and let $(U, u : U \longrightarrow \mathbb{R}^n)$ be a chart for a neighborhood U of x. Let ξ_i^j be the mappings ξ_i for the corresponding coordinate maps. Since $\xi_1(x), \ldots, \xi_n(x)$ is a basis of T_xM, we have $det((\xi_i^j)_{i,j}) \neq 0$, i.p. we get a neighborhood V of x such that for every $y \in V$: $\xi_1(y), \ldots, \xi_n(y)$ is a basis of T_yM. Now, choosing the orientation on T_xM, we get the orientation on all of T_yM, $y \in V$ by setting $(\xi_1(y), \ldots, \xi_n(y))$ as a positive (resp. negative) oriented basis if $(\xi_1(x), \ldots, \xi_n(x))$ is positive (resp. negative) oriented basis of T_xM.
Let M now be a smooth n-dimensional manifold. We say M is *orientable* if one can choose orientations for all T_xM, $x \in M$ in such a way, that given an open connected $U \subset M$ and smooth vector fields $\xi_1, \ldots \xi_n$ on U s.t. $(\xi_1(x), \ldots, \xi_n(x))$ is a basis of T_xM for all $x \in U$, then this bases are either all positive oriented or all negative oriented.
The choice of orientation for an oriented n-dimensional manifold gives rise to an n-form ω such that $\omega_x \neq 0$ for all $x \in M$. The converse is also true, hence for a n-dimensional smooth manifold M:

$$M \text{ orientable} \Leftrightarrow \text{there exists a nowhere vanishing } n\text{-form on } M.$$

One can define orientability for manifolds with the use of homology. Recall the definition of the relative homology groups: let $A \subset X$ be a subspace of X, and let $C_n(X, A) := C_n(X)/C_n(A)$ be the quotient group of chains in X modulo the chains in A. Then the boundary map defined for chains on X induces a boundary map on the quotient group, i.e. we have a chain complex

$$\cdots \longrightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \cdots$$

and now the *relative homology groups* $H_n(X, A)$ are defined as the homology groups of this chain complex. Elements of $H_n(X, A)$ are represented by *relative cycles*: n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
One can easily see that for all $x \in \mathbb{R}^n$

$$H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{x\}) \cong \mathbb{Z},$$

and it makes sense to define a *local orientation* of $\mathbb{R}^n$ at a point x as a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{x\}) \cong \mathbb{Z}$. This is naturally generalized on arbitrary n-dimensional manifolds, i.e. the *local orientation* of M at a point x is a choice of the generator μ_x of $H_n(M, M \backslash \{x\}) \cong \mathbb{Z}$.

For abbreviation, we shall write $H_n(X|A) := H_n(X, X\backslash A)$, and call this the *local homology groups* of X at A since it depends only on a neighborhood of $\overline{A}$. An *orientation* of an n-dimensional manifold M is a function $x \mapsto \mu_x$, i.e. a choice of generators of $H_n(M|x)$ such that each $x \in M$ has a neighborhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M|B) \cong H_n(\mathbb{R}^n|B)$ under the natural maps $H_n(M|B) \longrightarrow H_n(M|y)$. If such an orientation exists for M, then M is called *orientable*. This can be generalized to homology with coefficients in a commutative ring R, i.e. an R-orientation would then be a choice of the generator of $H_n(M|x) \cong R$ with the corresponding "local consistency" condition (analogous as above where we have $R = \mathbb{Z}$).
One can see that this definition of orientability with $R = \mathbb{Z}$ is equivalent to the "standard" definition of orientability discussed in the beginning of this section. Orientability of the manifold is reflected in the following:

Proposition 1. *If a closed*[1] *connected n-dimensional manifold M is R-orientable, then*

$$H_n(M; R) \longrightarrow H_n(M|x; R) \cong R$$

is an isomorphism for all $x \in M$.

This enables us to define the *fundamental class* $[M]$ of an oriented closed n-dim. manifold as an element of $H_n(M; R)$ whose image in $H_n(M|x; R)$ is a generator for all $x \in M$. The fundamental class is sometimes referred to as the *orientation class* $[M]$ for M.

Now, having defined the fundamental class of an oriented closed n-dim. manifold, we continue with the Poincaré duality isomorphism which relates the homology and the cohomology of a manifold:

let us recall the notion of the *cup* and *cap product*:
For two cochains $\varphi \in C^k(M; R)$ and $\psi \in C^l(M; R)$ we define $\varphi \cup \psi \in C^{k+l}(M; R)$ in a following way: for a simplex $\sigma \in C_{k+l}(M; R)$, $\sigma : \Delta^{k+l} := [e_0, \dots, e_{k+l}] \mapsto M$ let

$$(\varphi \cup \psi)(\sigma) := \varphi(\sigma_{|[e_0,\dots,e_k]}) \cdot \psi(\sigma_{|[e_k,\dots,e_{k+l}]}),$$

and extend this by R-linearity. This definition is compatible with the coboundary map, thus we get a cohomology pairing:

$$H^k(M; R) \times H^l(M; R) \xrightarrow{\cup} H^{k+l}(M; R)$$

which is associative and distributive, and turns $H^*(M; R) := \oplus_{n=0}^{\infty} H^n(M; R)$ into a graded ring.
The "cap" product of a chain in $C_k(M; R)$ and a cochain in $C^l(M; R)$ is defined as follows: for a simplex $\sigma : \Delta^{k+l} := [e_0, \dots, e_{k+l}] \mapsto M$ and $\psi \in C^l(M; R)$ we set

$$\sigma \cap \psi := \psi(\sigma_{|[e_0,\dots,e_k]}) \cdot \sigma_{|[e_k,\dots,e_{k+l}]} \in C_{k-l}(M; R)$$

and extend this by R-linearity. This again induces a map on the homology/cohomology, and makes $H_*(M; R) := \oplus_{n=0}^{\infty} H_n(M; R)$ a graded module over the R-algebra $H^*(M; R)$.

Now we have the following:

[1] By a closed manifold we mean a compact manifold without boundary.

Theorem 1. *[Ha], Th.3.30.*

If M is a closed R-orientable n-dimensional manifold with fundamental class $[M] \in H_n(M;R)$, then for all $0 \leq k \leq n$:

$$H^k(M;R) \longrightarrow H_{n-k}(M;R)$$

$$[\alpha] \mapsto [M] \cap [\alpha]$$

is an isomorphism.

The following commutative diagram for a compact connected R-oriented n-dimensional manifold M is a consequence of the theorem above:

$$\begin{tikzcd}
H^i(M;R) \times H^{n-i}(M;R) \arrow[r, "\cup"] \arrow[d, "(Poincar\ Duality)\times Id"', "\cong"] & H^n(M;R) \arrow[d, "(Poincar\ Duality)"', "\cong"] \\
H_{n-i}(M;R) \times H^{n-i}(M;R) \arrow[r, "\cap"] & H_0(M;R)
\end{tikzcd}$$

Here we can see that the cup product of two cohomology classes in complementary dimension is translated by the Poincaré duality map into the evaluation homomorphism

$$H_{n-i}(M;R) \times H^{n-i}(M;R) \longrightarrow H_0(M;R) \cong R,$$

and this duality pairing is the basis for the intersection theory which we describe in the following sections.

1.1.2 Cup product and transversal intersections

This section is a summary of chapter VI, section 11. of [Br], the main results being concerned with the transversal intersections:

The intersection product is a product defined on homology classes in a compact oriented manifold. It is the Poincaré dual of the cup product of corresponding cohomology classes, and enables us in some cases to interpret the cup product (defined in a purely algebraic way) in a geometric language. If the homology classes are represented by the fundamental classes of submanifolds, the intersection product can be related to the geometry of the intersection of the corresponding submanifolds.

Let M be a closed, oriented, connected n-dimensional manifold. Let $D : H_i(M) \longrightarrow H^{n-i}(M)$ be (the inverse of) the Poincaré duality isomorphism, i.e.[2]

$$D(\alpha) \cap [M] = \alpha.$$

[2]Greek letters $\alpha, \beta, \ldots$ are now used to denote homology classes (these were previously used to denote the cohomology classes).

Define the *intersection product*:

$$\bullet : H_i(M) \otimes H_j(M) \longrightarrow H_{i+j-n}(M)$$

$$\alpha \bullet \beta :=$$

$$D^{-1}(D(\beta) \cup D(\alpha)) = (D(\beta) \cup D(\alpha)) \cap [M] = D(\beta) \cap (D(\alpha) \cap [M]) = D(\beta) \cap \alpha$$

We have $\alpha \bullet \beta = (-1)^{(n-i)(n-j)} \beta \bullet \alpha$ and $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma$.
In order to explore the geometric interpretation of the intersection product we have to introduce the following:
Let $\pi : W \longrightarrow M$ be a $k-$disk bundle over a connected, oriented, closed n-dimensional manifold M, i.e. each point of M has a neighborhood U such that $\pi^{-1}(U)$ has the structure of a product $U \times \mathbf{D}^k$, where $\mathbf{D}^k := \{x \in \mathbb{R}^k \,|\, |x| \leq 1\}$ is the closed k-disk.
Then W is an $(n+k)$-dimensional manifold with boundary ∂W, identified with the $(k-1)$-sphere bundle over M. Assume that W is oriented.
The origin of each fibre $\mathbf{D}^k$ provides an inclusion of M in W, denoted $i : M \hookrightarrow W$. Thus M can be considered as a subspace of W and the induced map i^* in cohomology is inverse to π^*.
The *Thom class* of the disk bundle π is the class $\tau \in H^k(W, \partial W)$ given by the image under the map D_W (dual to Poincaré duality map on W) of the homology class of $[M]$ inside W, i.e.

$$\tau := D_W(i_*([M])),$$

or equivalently:

$$\tau \cap [W] = i_*([M]).$$

In this case we have the *Thom Isomorphism Theorem*: the maps in

$$H^p(M) \xrightarrow{\pi*} H^p(W) \xrightarrow{\cup\tau} H^{p+k}(W, \partial W)$$

are isomorphisms and the composition of them coincides with the mapping $D_W i_* D_M^{-1}$.

There is another way to define the Thom class for oriented, rank k real vector bundles:

let $\pi : E \longrightarrow B$ be an oriented, rank k real vector bundle over a connected, oriented, closed n-dimensional manifold B. Then there exists a cohomology class τ in $H^k(E|B; \mathbb{Z})$ s.t. for every $x \in B$ the restriction of τ to $H^k((E|B)_x; \mathbb{Z}) \cong \mathbb{Z}$ is the distinguished generator determined by the orientation. This class is called the Thom class of E. Under this assumption, the following is another version of the *Thom Isomorphism Theorem*:

Theorem 2. *[Hut], Th.3.2.*

i) E has a unique Thom class $\tau \in H^k(E|B; \mathbb{Z})$,
ii) the map

$$H^k(B;\mathbb{Z}) \longrightarrow H^{k+n}(E|B;\mathbb{Z})$$

$$\alpha \mapsto \pi^*\alpha \cup \tau$$

is an isomorphism.

The link between the given two versions of a definition for the Thom class is the following:

let U be a *tubular neighborhood*[3] of a closed oriented $(n-k)$-dimensional submanifold B in a closed oriented n-dimensional manifold X. Then U can be regarded as an oriented rank k vector bundle over B (since it is isomorphic to the normal bundle of B in X), and the Thom class of U over B is an element τ_B of $H^k(U|B;\mathbb{Z})$(defined as the unique class which restricts for each $x \in B$ to a generator of $H^k((U|B)_x;\mathbb{Z})$). In this setting, we have:

Proposition 2. *The Poincaré dual $\beta \in H^k(X;\mathbb{Z})$ of $[B] \in H_{n-k}(X;\mathbb{Z})$ is the image of the Thom class τ_B under the composition:*

$$H^k(U|B;\mathbb{Z}) \longrightarrow H^k(X|B;\mathbb{Z}) \longrightarrow H^k(X;\mathbb{Z}).$$

Hence, the Thom class corresponding to a tubular neighborhood (i.e. to a normal bundle) of closed oriented submanifold B can be identified with the Poincaré dual of $[B]$. We shall sometimes call this class the Thom class of a submanifold B.

Now suppose we have two smoothly embedded m- resp. n-dimensional closed, oriented manifolds M and N in a $(m+n)$-dimensional closed oriented manifold W:
let $[M]_W$ and $[N]_W$ be homology classes in $H_*(W)$ induced by the embeddings i_M and i_N, and let $\tau_M^W := D_W([M]_W) \in H^n(W)$, $\tau_N^W := D_W([N]_W) \in H^m(W)$ be the corresponding Thom classes (i.e. Poincaré duals).
Note that

$$[N]_W \bullet [M]_W = (\tau_N^W \cap [W]) \bullet (\tau_M^W \cap [W]) = (\tau_M^W \cup \tau_N^W) \cap [W].$$

Now assume that M and N intersect *transversely*[4] in W. Then the normal bundle of $M \cap N$ in M is the restriction of the normal bundle of N in W to $M \cap N$. In particular, the Thom class of N in W restricts to the Thom class of $M \cap N$ in M, and we have the formula

$$\tau_{M\cap N}^M = i_M^*(\tau_N^W),$$

where i_M denotes the inclusion of M in W.
With these assumptions we have the following:

[3]For the definition of a tubular neighborhood and its properties, see [Br], ch.II, section 11.

[4]Two submanifolds of a given finite dimensional smooth manifold intersect transversely if at every point of the intersection their tangent spaces generate the tangent space of the ambient manifold at that point. Every intersection can be "perturbed" (by shifting these submanifolds with a homotopy) to be transversal - for the detailed explanation see [GP], "Transversality Homotopy Theorem" on p.70. If two submanifolds are of complementary dimension and intersect transversely, then their intersection consists of isolated points.

Theorem 3. *[Br] p.372, Th.11.9*

$$\tau^W_{M\cap N} = \tau^W_N \cup \tau^W_M,$$

equivalently (Poincaré duality):

$$[M \cap N]_W = [M]_W \bullet [N]_W$$

In this way one can compute the cup product of cohomology classes of W which are dual to the orientation classes of two transversely intersecting submanifolds M and N by looking at the intersection $M \cap N$. In that case the cup product is the cohomology class dual to the orientation class of the intersection $M \cap N$. Although this gives the geometric interpretation of the intersection product for classes carried by submanifolds, one can give some general criteria:

Corollary 1. *Let M be an orientable closed manifold and $A, B \subset M$. Let $\alpha \in H_p(A)$ and $\beta \in H_q(B)$, and let α_M and β_M denote their images in $H_*(M)$. If $\alpha_M \bullet \beta_M \neq 0$, then $A \cap B \neq \emptyset$.*

Remark 1. *If $\alpha, \beta \in H_*(M)$ are carried by the subsets A and B of M, then $\alpha \bullet \beta$ is represented by a cycle carried by any given neighborhood of $A \cap B$.*[5]

We finish this section with the description of how to compute the intersection product in a special case, which we present without proof (see [Br], discussion on p.375):

suppose W is a smooth oriented closed $m+n$-dimensional manifold, and that M and N are closed oriented, $m-$, resp. $n-$ dim. smooth submanifolds embedded in W with fundamental classes $[M] \in H_m(W;\mathbb{Z})$ resp. $[N] \in H_n(W;\mathbb{Z})$. Let $\omega_M \in H^n(W;\mathbb{Z})$ resp. $\omega_N \in H^m(W;\mathbb{Z})$ be the Poincaré duals of $[M]$ resp. $[N]$.
If the intersection of M and N in W is transversal, i.e.

$$\text{for all } x \in M \cap N \text{ we have } T_xW = T_xM \oplus T_xN,$$

then the number $[M][N] := [M] \bullet [N] = (\omega_M \cup \omega_N) \cap [W]$ [6] can be computed as follows:

the intersection of M and N consists of isolated points, and since $M \cap N$ is compact, there is only a finite number of them. For $x \in M \cap N$ set $\epsilon(x) := 1$, if the basis of T_xW obtained via $T_xM \oplus T_xN$ is positive oriented, and $\epsilon(x) := -1$ otherwise. Then we have

Lemma 1.
$$[M][N] = \sum_{x \in M\cap N} \epsilon(x).$$

[5] To see that $\alpha \bullet \beta$ is i.g. not represented by a cycle given contained in $A \cap B$, see the example on p.373 in [Br]

[6] We use the notation $[M][N]$ instead of $[M] \bullet [N]$

1.1.3 The Euler class and degenerate intersections

In the last section, we have seen the geometry behind the computation of the cup product of two classes which are Poincaré duals of submanifolds that intersect transversely. In that case the cup product is simply the Poincaré dual of the intersection. However, most of the time the intersection will not be transversal, and the computation of the intersection number, if defined, is considerably more difficult. This is where the notion of the Euler class is needed.

Let $\pi : E \longrightarrow B$ be an oriented[7], rank k real vector bundle over a connected, oriented, closed n-dimensional manifold B. For the inclusion $j : E \longrightarrow (E|B) = (E, E\backslash B)$, there is a corresponding map in the cohomology $j^* : H^i(E|B) \longrightarrow H^i(E)$, and analogously for the inclusion $s : B \hookrightarrow E$ as the zero section of the bundle E. Define the *Euler class* of E, denoted $e(E)$ as the image of the Thom class τ under the composition

$$H^k(E|B;\mathbb{Z}) \xrightarrow{j^*} H^k(E;\mathbb{Z}) \overset{s^*=\pi^{*-1}}{\longrightarrow} H^k(B;\mathbb{Z}),$$

i.e.

$$e(E) := \pi^{*-1} j^*(\tau).$$

As a consequence of Theorem 2, we have the following isomorphisms:

$$\begin{tikzcd}
H^i(B;\mathbb{Z}) \arrow[rr, "\alpha\mapsto\pi^*\alpha\cup\tau"] \arrow[dr, "\pi^*"'] & & H^{i+k}(E|B;\mathbb{Z}) \\
& H^i(E;\mathbb{Z}) \arrow[ur, "\beta\mapsto\beta\cup\tau"'] &
\end{tikzcd}$$

Let ϕ denote the isomorphism $H^k(B;\mathbb{Z}) \longrightarrow H^{2k}(E|B;\mathbb{Z})$, $\alpha \mapsto \pi^*\alpha \cup \tau$. Then it can be seen that

$$e(E) = \phi^{-1}(\tau \cup \tau),$$

in particular, since $\alpha \cup \beta = (-1)^{(dim\alpha)(dim\beta)} \beta \cup \alpha$, for the Euler class of an odd dimensional bundle E we have $2 \cdot e(E) = 0$.

The following are some important properties of the Euler class (for details see [Hus], p.240-242. and Theorem 5.1 in [Hut]):

$*$ the Euler class is functorial, i.e. if E is a vector bundle over B, and $f : \tilde{B} \longrightarrow B$ a smooth map (between two base spaces $\tilde{B}$, B), then for the induced bundle we have $f^*(e(E)) = e(f^*(E))$, i.p. if E is a trivial bundle, then $e(E) = 0$.

$*$ for the Euler class $e(E)$ of an odd dimensional bundle E we have $2(e(E)) = 0$, [8] i.p. for the homology groups with $\mathbb{Z}-$coefficients (or coefficients in any ring

[7] We restrict our attention to oriented vector bundles. In the case of a non-oriented vector bundle, analogous statements can be made using the cohomology groups with $\mathbb{Z}/2\mathbb{Z}-$coefficients.

[8] This follows from the functorial property of the Euler class, since an odd dimensional vector bundle has an orientation reversing automorphism.

with characteristic $\neq 2$), we have $e(E) = 0$

$*$ given two vector bundles E, F over B, the Euler class satisfies the Whitney product formula:

$$e(E \oplus F) = e(E) \cup e(F),$$

i.p. if a vector bundle E has a nowhere-zero section, then $E = E_{tr} \oplus \eta$, where E_{tr} is a trivial bundle, hence $e(E) = e(E_{tr}) \cup e(\eta) = 0$ since $e(E_{tr}) = 0$.

$*$ let ψ be a transversal (i.e. its graph Ψ meets the zero section B of E transversely) section of an oriented vector bundle $\pi : E \longrightarrow B$, and let $Z := \Psi \cap B$ be the zero locus of ψ. Then Z is a submanifold of B, and its normal bundle in B is simply the restriction of E to Z: $N_Z^B \cong E_{|Z}$. I.p. this defines an orientation on Z. In this setting, we have:

$$e(E) = D_B([Z]) \in H^k(B; Z),$$

i.e. the Euler class is the Poincaré dual to the zero locus of a transversal section.

$*$ the following sequence is exact ("The Gysin sequence"):

$$\cdots \longrightarrow H^i(B) \overset{\alpha \mapsto \alpha \cup e(E)}{\longrightarrow} H^{i+k}(B) \overset{\pi^*}{\longrightarrow} H^{i+k}(E) \longrightarrow H^{i+1}(B) \longrightarrow \cdots$$

Now consider the following case:

Let V be a closed oriented manifold of dimension v, M resp. N closed oriented submanifolds of V of dimensions m resp. n, s.t. the intersection $M \cap N$ is compact. We say that M and N *intersect perfectly*[9] if the connected components of the intersection $M \cap N$ are submanifolds of V and if for all such components F of $M \cap N$ one has for the tangent bundles: $TF = TM_{|F} \cap TN_{|F}$. Suppose that M and N intersect perfectly, $m + n = v$, and that the intersection $F := M \cap N$ is a connected oriented compact submanifold of dimension $f \geq 1$. Let η be the *excess bundle*[10] of the intersection, i.e. the vector bundle over F which is the quotient of the tangent bundle of V restricted to F by the sum of the tangent bundles of M and N restricted to F. Then η is given by the following exact sequence:

$$0 \longrightarrow TM_{|F} + TN_{|F} \longrightarrow TV_{|F} \longrightarrow \eta \longrightarrow 0.$$

(Note that if M, N intersect transversely, then $\dim F = 0$ and $\eta = 0$.) Since F is oriented, η becomes an oriented vector bundle. Let $e(\eta)$ be the corresponding *Euler class* of η.
Let D_X denote the inverse of the Poincaré duality map on a manifold X ($X = V; M; N; F$), as in the beginning of the section 1.1.2. For a map $i : Y \longrightarrow X$, let $i_! : H_*(X) \longrightarrow H_*(Y)$ denote the map $[A] \mapsto D_Y^{-1} i^* D_X([A])$; analogously $i^! : H^*(Y) \longrightarrow H^*(X)$, $\alpha \mapsto D_X i_* D_Y^{-1}(\alpha)$, where i_* and i^* denote the canonically induced maps on homology/ cohomology.
Consider the following diagram, with the maps being the obvious inclusions:

[9] Some authors use the term *clean intersection* instead of *perfect intersection*.
[10] For this notion, see [Fu] §6.3.

$$\begin{tikzcd}
F \arrow[r, "i_F^M"] \arrow[d, "i_F^N"'] & M \arrow[d, "i_M^V"] \\
N \arrow[r, "i_N^V"'] & V
\end{tikzcd}$$

Then we have the following ([Qu], Prop. 3.3.):

Lemma 2. *"Quillen's Clean Intersection Formula"*
Under the assumptions above (i.p. M and N intersect perfectly), if $\alpha \in H^*(M)$, we have in $H^*(N)$:

$$i_N^{V\,*} i_M^{V\,!}(\alpha) = i_F^{N\,!}(e(\eta) \cup i_F^{M\,*}(\alpha))$$

and as a consequence we get a description of the intersection product in the case of a degenerate intersection of submanifolds, $\dim M + \dim N = \dim V$, $\dim F \geq 1$:

Corollary 2.

$$[M][N] = e(\eta) \cap [F]$$

Proof. There is some abuse of notation (f.e. we write $[N]$ both for a class in $H^n(N)$ and for a class in $H^n(V)$), however it is always clear from the context what is meant by the symbol:

$$\begin{aligned}
&[M][N] = \\
&\quad = (\omega_M \cup \omega_N) \cap [V] = \omega_M \cap (\omega_N \cap [V]) = \\
&\quad = \omega_M \cap (i^V_{N*}([N])) = i^V_{N*}(i_N^{V\,*}(\omega_M) \cap [N]) = \\
&\quad (\omega_M = i_M^{V\,!}(1), \ \textit{then use the Clean Intersection Formula}) \\
&\quad = i^V_{N*}(i_F^{N\,!}(e(\eta) \cup 1) \cap [N]) = i^V_{N*}(i_F^{N\,!}(e(\eta)) \cap [N]) = \\
&\quad = i^V_{N*}(i^N_{F*}(e(\eta) \cap [F]) = i^V_{F*}(e(\eta) \cap [F]) = \\
&\quad = e(\eta) \cap F,
\end{aligned}$$

the last equation holds in $\mathbb{Z}$. $\square$

Remark 2. *For the direct proof of this fact, which contains in its approach the idea of the Clean Intersection Formula see [RSchw], p.766.*

1.2 The Millson-Raghunathan method

This section is a summary of some of the results in [MR]. The goal is to give a sketch of the construction of non-trivial homology classes in certain locally symmetric spaces:
Let X be a symmetric space, X_1 and X_2 two *totally geodesic*[11] subsymmetric spaces of X which intersect in exactly one point (w.l.o.g. the origin 0 of X). $\{X_1, X_2\}$ is an *orthogonal decomposition of* X if the tangent spaces of X_1 and X_2 at 0 form an orthogonal decomposition of T_0X (i.e. orthogonal and $T_0X_1 \oplus T_0X_2 = T_0X$). We say that the two orthogonal decompositions $\{X_1, X_2\}$ and $\{\tilde{X_1}, \tilde{X_2}\}$ are equivalent if there is an isometry ϕ of X with $\phi(X_1) = \tilde{X_1}$ and $\phi(X_2) = \tilde{X_2}$.
If $V \subset T_0X$ is a sub-vector space, then $expV$ is a totally geodesic subsymmetric space if and only if V is a triple Lie system, i.e. if for all $a, b, c \in V$ we have: $[[a, b], c] \in V$. Thus, the infinitesimal equivalent of an orthogonal decomposition of a symmetric space X would be the orthogonal decomposition $T_0X = V \oplus W$ with V, W both Lie triple systems.
There is a one to one correspondence between equivalence classes of orthogonal decompositions of X and conjugacy classes of involutive isometries of X. Given an involutive isometry σ of X fixing 0, set $\tau := \sigma\theta$; where θ denotes the geodesic symmetry at 0. Then the sets of fixed points of σ resp. τ on X, denoted X_σ resp. X_τ, form an orthogonal decomposition of X; which is sometimes denoted by $\{\sigma, \tau\}$.

The authors now proceed to establish the following description of cycles in the quotients of X:
Let $X := G/K$ with G a connected component of the isometry group of X, which is assumed to be linear and semisimple and K is the isotropy group at the origin 0 of X. Given a discrete subgroup Γ compatible with the decomposition $\{\sigma, \tau\}$ (i.e. $\sigma\Gamma\sigma \subset \Gamma$, $\tau\Gamma\tau \subset \Gamma$) then σ and τ induce involutions of $\Gamma\backslash X$. Let $\Gamma_\sigma := \{\gamma \in \Gamma | \sigma\gamma\sigma = \gamma\}$, and define Γ_τ; G_σ and G_τ analogously. Then we have the following:

Proposition 3. *Let $\hat{K} := G_\sigma \cap G_\tau$. Then $\hat{K} = K \cap G_\sigma = K \cap G_\tau$ is a maximal compact subgroup of G_σ and G_τ and $X_\sigma \cong G_\sigma/\hat{K}$, $X_\tau \cong G_\tau/\hat{K}$.*

Proposition 4. *If Γ is cocompact in G, then so are Γ_σ and Γ_τ in G_σ and G_τ.*

Let $\pi : X \longrightarrow \Gamma\backslash X$ be a natural projection, then:

Proposition 5. *If Γ acts freely on X, then $\Gamma_\sigma\backslash X_\sigma \cong \pi(X_\sigma)$, analogously for X_τ.*

In this way we get a compact locally symmetric space $\Gamma\backslash X$ (oriented, since G is connected) containing two totally geodesic complementary dimensional submanifolds $Y_\sigma \cong \Gamma_\sigma\backslash X_\sigma$ and $Y_\tau \cong \Gamma_\tau\backslash X_\tau$. Under the assumption that the universal cover of G is linear, one can find an arithmetic[12] subgroup $\tilde{\Gamma} \leq \Gamma$ s.t. $\tilde{Y_\sigma} \cong \tilde{\Gamma_\sigma}\backslash X_\sigma$ and $\tilde{Y_\tau} \cong \tilde{\Gamma_\tau}\backslash X_\tau$ are orientable. [13] Since the groups G we

[11] For the definition of a totally geodesic submanifold, see ch I, §14 in [Hel].
[12] Here, the group G is the group of $\mathbb{R}$-rational points of a linear algebraic group $\underline{G}$ defined over $\mathbb{Q}$. Choosing a $\mathbb{Q}$-rational structure on G, by an arithmetic subgroup of G we mean a subgroup of $\underline{G}(\mathbb{Q})$ commensurable with $\underline{G}(\mathbb{Z})$. For details, see section 5.2.
[13] For this, see [MR], p.109-110 Prop.2.3

are interested in will either be simply connected or will satisfy the assumptions mentioned above s.t. the orientability questions do not present a difficulty, and since our interest is in the arithmetic groups, from now on we deal with an arithmetic subgroup $\Gamma \leq G$ s.t. in the corresponding locally symmetric space our "cycles" $Y_\sigma \cong \Gamma_\sigma \backslash X_\sigma$ and $Y_\tau \cong \Gamma_\tau \backslash X_\tau$ are orientable.

In order to analyse the contribution of Y_σ and Y_τ to the cohomology of $\Gamma \backslash X$, it is important to describe their intersection:

Proposition 6. *There is a one to one correspondence between $Y_\sigma \cap Y_\tau$ and the set of $\Gamma_\sigma \times \Gamma_\tau$ equivalence classes of triples*

$$T := \{(x_1, x_2, \gamma) \in X_\sigma \times X_\tau \times \Gamma \mid \gamma x_1 = x_2\}$$

where the action of $\Gamma_\sigma \times \Gamma_\tau$ on T is given by

$$(\gamma_1, \gamma_2) \cdot (x_1, x_2, \gamma) := (\gamma_1 x_1, \gamma_2 x_2, \gamma_2 \gamma \gamma_1^{-1}).$$

Proposition 7. *Let $(x_1, x_2, \gamma) \in T$, then the connected component of Y_σ and Y_τ passing through $y := \pi(x_1)$ is just the image of $\gamma X_\sigma \cap X_\tau$ under π.*

Let $\tilde{\sigma} := \gamma \sigma \gamma^{-1}$. The intersection component containing $y = \pi(x_1)$ is a locally symmetric space $\Lambda \backslash A/B$, where A resp. B are centralizers of $\tilde{\sigma}$ in G_τ resp.$\hat{K}$ and Λ is the centralizer of $\tilde{\sigma}$ in Γ_τ.

The connected components of the intersection Y_σ and Y_τ are indexed by the Γ_τ, Γ_σ double cosets of those γ which appear in the third component of the triples of the set T of Proposition 6 . Choose a set of representatives for this double cosets including $1 \in \Gamma$ (which corresponds to the projection of the origin in X which is precisely $X_\sigma \cap X_\tau$ and thus appears as the intersection point in $\Gamma \backslash X$); say $1, \gamma_2, \ldots, \gamma_r$; and set

$$I(\Gamma) := \{1, \gamma_2, \ldots, \gamma_r\}.$$

Let H_σ resp. H_τ be (any) subgroups of G_σ resp. G_τ which act transitively on X_σ resp. X_τ. Let $\gamma \in I(\Gamma)$, then there exist $x_1 \in X_\sigma$ and $x_2 \in X_\tau$ s.t. $\gamma x_1 = x_2$. Choose $h_1 \in H_\sigma^0$, $h_2 \in H_\tau^0$ s.t. $h_i \cdot 0 = x_i$. Let $\delta(\gamma) := h_2^{-1} \gamma h_1$ Then $\delta(\gamma) \cdot 0 = 0$, hence $\delta(\gamma) \in K$. As a corollary we get:

Lemma 3. *There is a one-to-one correspondence between $I(\Gamma)$ and the $\Gamma_\sigma \times \Gamma_\tau$-orbits of $\Gamma \cap H_\tau^0 K H_\sigma^0$..*

If $f : X \longrightarrow X$ is a smooth map and $v_1, \ldots, v_p \in T_x(X)$, then let $f_*(v_1 \wedge \ldots \wedge v_p) := f_*(v_1) \wedge \ldots \wedge f_*(v_p)$, where $f_*(v_j) \in T_{f(x)}X$ is the image of v_j under the tangential map of f.
By the *intersection multiplicity* we mean the local contribution to the intersection number, as in the discussion on the end of section 1.1.2. (The multiplicity is also defined in the case of a degenerate intersection since a degenerate intersection can be homotopically deformed in a transversal one[14], and there we can read of the multiplicity.)

[14]See [GP], Ch.II, §3. "The Transversality Theorem".

Proposition 8. *Let $e_1, \ldots, e_k$ resp. $e_{k+1}, \ldots, e_n$ be an ordered basis of T_0X_σ resp T_0X_τ. Let $\epsilon(\gamma)$ be a real number which satisfies*

$$\delta(\gamma)_*(e_1 \wedge \ldots \wedge e_k) \wedge e_{k+1} \wedge \ldots \wedge e_n = \epsilon(\gamma)(e_1 \wedge e_2 \wedge \ldots \wedge e_n).$$

Then:
$\epsilon(\gamma) = 0 \Leftrightarrow$ *the intersection corresponding to γ is degenerate*
$\epsilon(\gamma) > 0 \Leftrightarrow$ *the intersection corresponding to γ has multiplicity $+1$*
$\epsilon(\gamma) < 0 \Leftrightarrow$ *the intersection corresponding to γ has multiplicity -1*

This can be seen in a following way: since $h_1 \in H_\sigma^0$, $h_2 \in H_\tau^0$ connected, $\omega := (L_{h_1})_*(e_1 \wedge \ldots \wedge e_k)$ resp. $\tau := (L_{h_2})_*(e_{k+1} \wedge \ldots \wedge e_n)$ is a positive orientation of X_σ at x_1 resp X_τ at x_2. Thus the intersection will be positive, negative or degenerate according to whether $(L_\gamma)_*(\omega) \wedge \tau$ is a positive, negative, or zero multiple of the volume element of X at x_2. The rest is a simple calculation.

Remark 3. *In the case $\epsilon(\gamma) = 0$, i.e. a degenerate intersection, 0 is not necessarily the intersection multiplicity.*

Let $\overline{G_\sigma}$ resp. $\overline{G_\tau}$ be the subgroup of G_σ resp. G_τ preserving the orientation of X_σ resp. X_τ.[15] The following result[16] will be crucial in manipulating the intersection multiplicities:

Lemma 4. *If γ, $\tilde{\gamma}$ are in the same $\overline{G_\tau}$, $\overline{G_\sigma}$ - double coset in G, then $\epsilon(\gamma) = \epsilon(\tilde{\gamma})$.*

Choosing orientations s.t. X_σ and X_τ intersect at 0 with the positive intersection multiplicity, we have:

Corollary 3. *If γ is in the trivial $\overline{G_\tau}$, $\overline{G_\sigma}$-double coset, i.e. $\gamma \in \overline{G_\tau}\ \overline{G_\sigma}$, then $\epsilon(\gamma) = +1$.*

Now suppose E is a totally real number field, the groups G, H_σ and H_τ are the real rational points of groups M, M_σ and M_τ defined over E and $\Gamma \leq GL_N(O_E)$. If $\mathbf{p}$ is a prime ideal of O_E, let $\Gamma(\mathbf{p}) := Ker(\Gamma \longrightarrow GL_N(O_E/\mathbf{p}))$. In general, $I(\Gamma(\mathbf{p}))$ is not contained in $I(\Gamma)$. Let $\delta(I(\Gamma)) := \{\delta(\gamma) | \gamma \in I(\Gamma)\}$. Then we have:

Lemma 5. $\delta(I(\Gamma(\mathbf{p}))) \subseteq \delta(I(\Gamma))$.

The rest of the work is to find a suitable congruence subgroup $\Gamma(\mathbf{p}) \leq \Gamma$ s.t. $\delta(I(\Gamma(\mathbf{p})))$ is contained in the connected component of the normaliser of $\hat{K}$, and it can be summed in the following few results[17]:

Theorem 4. *[MR], Th.3.1.*
For all but finitely many primes $\mathbf{p}$ in O_E we have

$$\gamma \in \delta(I(\Gamma(\mathbf{p}))) \Rightarrow \gamma \in (H_\tau)_{\mathbb{C}}(H_\sigma)_{\mathbb{C}},$$

where $(H_\sigma)_{\mathbb{C}}$, $(H_\tau)_{\mathbb{C}}$ denote the complexification of H_σ, H_τ.

[15] In our case, we can take $\overline{G_\sigma} = H_\sigma^0$ and $\overline{G_\tau} = H_\tau^0$
[16] For the proof, see [MR], Lemma 2.5.
[17] For the proofs see [MR], §3.

According to Corollary 3, if $\gamma \in H_\tau^0 H_\sigma^0$, the intersection multiplicity of the intersection component corresponding to γ is $+1$. Thus, using Theorem 4 and in the case that H_σ and H_τ are connected, it suffices to study the following problem: given $\gamma \in (H_\tau)_{\mathbb{C}}(H_\sigma)_{\mathbb{C}}$, when can we obtain $\gamma \in H_\tau H_\sigma$? This is solved using Galois cohomology theory in the following way:

let $\gamma \in (H_\tau)_{\mathbb{C}}(H_\sigma)_{\mathbb{C}}$, i.e. $\gamma \in M_\tau(\mathbb{C})M_\sigma(\mathbb{C})$. Write $\gamma = g_\tau g_\sigma$ with $g_\sigma \in M_\sigma(\mathbb{C})$, $g_\tau \in M_\tau(\mathbb{C})$. Now to every such γ we associate a Galois cycle $a(\gamma) \in H^1(E, M_\sigma \cap M_\tau)$ in the following way: for $\vartheta \in Gal_{\mathbb{C}/E}$ we have $\vartheta(\gamma) = \gamma$ (since γ has entries in E), and so

$$g_\sigma \vartheta(g_\sigma)^{-1} = g_\tau^{-1}\vartheta(g_\tau) =: a(\gamma)(\vartheta).$$

Note that for $g_\sigma \in M_\sigma(\mathbb{C})$ we have: $g_\sigma \in M_\sigma(E) \Leftrightarrow$ for every $\vartheta \in Gal_{\mathbb{C}/E}$ we have $\vartheta(g_\sigma) = g_\sigma \Leftrightarrow a(\gamma)(\vartheta) = 1$ for every ϑ. Analogously the case $g_\sigma \in M_\sigma(\mathbb{R})$ can be treated, hence:

Lemma 6. *We can write $\gamma = \mu_\tau \mu_\sigma$ with $\mu_\sigma \in M_\sigma(E)$, $\mu_\tau \in M_\tau(E)$ if and only if the class of $a(\gamma)$ in $H^1(E, M_\sigma \cap M_\tau)$ is trivial.*
Since $E \subset \mathbb{R}$, there is a map $H^1(E, M_\sigma \cap M_\tau) \longrightarrow H^1(\mathbb{R}, M_\sigma \cap M_\tau)$, let $\alpha(\gamma)$ be the image of $a(\gamma)$ under this map.
We can write $\gamma = h_\tau h_\sigma$ with $h_\sigma \in M_\sigma(\mathbb{R})$, $h_\tau \in M_\tau(\mathbb{R})$ if and only if the class of $\alpha(\gamma)$ is trivial in $H^1(\mathbb{R}, M_\sigma \cap M_\tau)$.

Note that $Gal_{\mathbb{C}/\mathbb{R}} = \{Id, c\}$ where c is simply the complex conjugation which we write $c(x) = \overline{x}$. Then we have $\alpha(\gamma)(c) = g_\sigma \overline{g_\sigma^{-1}}$, and thus choosing $x = g_\sigma \in M_\sigma(\mathbb{C})$ we have $x \cdot \alpha(\gamma)(c) \cdot c(x) = 1$, implying that the image of $\alpha(\gamma)$ in $H^1(\mathbb{R}, M_\sigma)$ is trivial. Analogously for M_τ, hence:

Lemma 7. *If $H^1(\mathbb{R}, M_\sigma \cap M_\tau) \longrightarrow H^1(\mathbb{R}, M_\sigma) \times H^1(\mathbb{R}, M_\tau)$ is injective, then the class of $\alpha(\gamma)$ is trivial in $H^1(\mathbb{R}, M_\sigma \cap M_\tau)$.*
Consequently, we can can write $\gamma = h_\tau h_\sigma$ with $h_\sigma \in M_\sigma(\mathbb{R})$, $h_\tau \in M_\tau(\mathbb{R})$.

Summing up these results, we have that the following holds for two closed orientable cycles $\Gamma_\sigma(\mathbf{p})\backslash X_\sigma$, $\Gamma_\tau(\mathbf{p})\backslash X_\tau$ in a closed orientable locally symmetric space $\Gamma(\mathbf{p})\backslash X$:

Theorem 5. *With the notation above assume $\mathbf{p}$ is a prime ideal of O_E satisfying the hypothesis of theorem 4. If H_σ and H_τ are connected, and the map $H^1(\mathbb{R}, M_\sigma \cap M_\tau) \longrightarrow H^1(\mathbb{R}, M_\sigma) \times H^1(\mathbb{R}, M_\tau)$ is injective, then all intersections of $\Gamma_\sigma(\mathbf{p})\backslash X_\sigma$ and $\Gamma_\tau(\mathbf{p})\backslash X_\tau$ in $\Gamma(\mathbf{p})\backslash X$ are isolated and have multiplicity $+1$. For the intersection number we have:*

$$[\Gamma_\sigma(\mathbf{p})\backslash X_\sigma][\Gamma_\tau(\mathbf{p})\backslash X_\tau] = \sum_{\gamma \in \delta(I(\Gamma(\mathbf{p})))} 1 > 0.$$

I.p., using the Poincaré duality for closed orientable manifolds, both cycles represent non trivial homology classes in $H_*(\Gamma(\mathbf{p})\backslash X; \mathbb{Z})$.

1.3 The Rohlfs-Schwermer method

This section is a summary of the results in [RSchw], which is a generalization of the Millson-Raghunathan approach to the construction of nontrivial homology classes in locally symmetric spaces. The results are presented for a reductive algebraic group defined over $\mathbb{Q}$, however, using the restriction of scalars (see [PR], §2.1.2.) we can work over arbitrary number fields.

1.3.1 Construction of special cycles

For the purposes of the construction which will be presented we have to be familiar with non-abelian Galois cohomology, elements of which will parametrize certain fixed point sets on arithmetically defined manifolds.
Let Θ be a group acting on a group A as a group of automorphisms. We denote the action of s on a by ${}^s a$. A *cocycle* of Θ in A is a map $\gamma : \Theta \longrightarrow A, s \mapsto \gamma_s$ s.t. $\gamma_{st} = \gamma_s \cdot^s \gamma_t$ for all $s, t \in \Theta$. Two cocycles γ, ξ are said to be equivalent if there exists an element $a \in A$ s.t. $\xi_s = a^{-1} \cdot \gamma_s \cdot^s a$ for all $s \in \Theta$.Then $H^1(\Theta; A)$, defined as the set of all equivalence classes of cocycles under this relation is the *first non-abelian cohomology set* of Θ in A. The class of the trivial cocycle is denoted by 1_Θ.
If the groups Θ and A act on a set M in a compatible way, ${}^s(am) =^s a\ {}^s m$, then, given a cocycle $\gamma \in H^1(\Theta; A)$ there is a *γ-twisted action* of Θ on M given by $m \mapsto \gamma_s\ {}^s m$ for $s \in \Theta$. The fixed point set of this action is denoted by $M(\gamma)$.

Let G be a reductive algebraic group defined over $\mathbb{Q}$ and let Θ be a finite abelian group of $\mathbb{Q}$-automorphisms of G. Choose a Θ-stable maximal compact subgroup K of $G(\mathbb{R})$. Then Θ acts on a symmetric space $X := G(\mathbb{R})/K$. Given a Θ-stable torsion-free arithmetic subgroup Γ of $G(\mathbb{Q})$, there is a natural action of Θ on the locally symmetric space $\Gamma\backslash X$ and given a cocycle $\gamma \in H^1(\Theta; \Gamma)$, there are γ-twisted actions of Θ on G and on Γ given by $g \mapsto \gamma_s \cdot^s g \cdot \gamma_s^{-1}$ with the induced action of Θ on $\Gamma\backslash X$ coinciding with the original one. Denote by $\Gamma(\gamma)$, $X(\gamma)$ the set of fixed points under the γ-twisted action of Θ on Γ and X. Then, using the fact that Γ acts freely on X we have the injective[18] natural map:

$$\pi_\gamma : \Gamma(\gamma)\backslash X(\gamma) \hookrightarrow \Gamma\backslash X.$$

The image $F(\gamma)$ of π_γ lies in the fixed point set $Fix_{\Gamma\backslash X}(\Theta)$, and depends only on the class of γ in $H^1(\Theta; \Gamma)$. It is a connected totally geodesic closed nonempty submanifold of $\Gamma\backslash X$.[19]
Furthermore, one can see that all fixed points of the action of Θ on $\Gamma\backslash X$ arise by this construction, and we have

$$Fix_{\Gamma\backslash X}(\Theta) = \overset{*}{\bigcup_{\gamma \in H^1(\Theta,\Gamma)}} F(\gamma).$$

Now consider $\Theta := < \mu >$ for a $\mathbb{Q}$-rational automorphism μ of G of finite order. Then we have

$$Fix_{\Gamma\backslash X}(< \mu >) = \overset{*}{\bigcup_{\gamma \in H^1(<\mu>,\Gamma)}} F(\gamma).$$

[18]The injectivity of this map is a consequence of the fact that Γ acts freely on X and the fact that we are dealing with fixed points of involutions, for the proof, see [Schw] §2.2.
[19]For the proof of this fact, see [Hel] ch I, §13 and §14

The connected component corresponding to the base point 1_μ will be called a *special cycle*, denoted by $C_\mu(\Gamma)$.
Note that each of the components $F(\gamma)$ of $Fix_{\Gamma\backslash X}(<\mu>)$ can be viewed as a special cycle obtained by the rational automorphism obtained by twisting μ with γ.
By construction, the special cycle $C_\mu(\Gamma)$ is diffeomorphic to the locally symmetric space originating with the group of real points of the fixed point group $G(\mu)$ of G under μ, since we can identify

$$G(\mu)(\mathbb{R})/K(\mu) \cong X(\mu),$$

where $X(\mu) := X(1_\mu)$, and now the natural map

$$\Gamma(\mu)\backslash X(\mu) \cong C_\mu(\Gamma)$$

with $\Gamma(\mu) := \Gamma(1_\mu) = \{\gamma \in \Gamma | \mu(\gamma) = \gamma\}$ provides a diffeomorphism between the locally symmetric space $\Gamma(\mu)\backslash X(\mu)$ and the closed immersed submanifold $C_\mu(\Gamma)$.

1.3.2 Intersection of two special cycles

Proposition 9. *Let σ, τ be $\mathbb{Q}$-rational automorphisms of G with $\sigma\tau = \tau\sigma$. Let $\Theta := <\sigma, \tau>$. Given a Θ-stable arithmetic subgroup Γ of $G(\mathbb{Q})$ let*

$$res_\sigma \times res_\tau : H^1(\Theta; \Gamma) \longrightarrow H^1(<\sigma>; \Gamma) \times H^1(<\tau>; \Gamma)$$

be the map which sends a cocycle γ to the pair of cocycles determined by γ_σ, γ_τ, i.e. by a natural restriction.
Then we have:
1) *If $\gamma \in Ker(res_\sigma \times res_\tau)$, then $F(\gamma) \subset C_\sigma(\Gamma) \cap C_\tau(\Gamma)$ is a connected component of the intersection of $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$.*
2) *There is a bijection*

$$Ker(res_\sigma \times res_\tau) \leftrightarrow \pi_0(C_\sigma(\Gamma) \cap C_\tau(\Gamma)),$$

given by $\gamma \mapsto F(\gamma)$; where $\pi_0(C_\sigma(\Gamma) \cap C_\tau(\Gamma))$ is the set of connected components of $C_\sigma(\Gamma) \cap C_\tau(\Gamma)$.

The proof of this result can be found in [RSchw], and relies on the fact that the γ-twisted Θ-action on X is determined by the γ_σ-twisted σ-action and the γ_τ-twisted τ-action and we have $X(\gamma) = X(\gamma_\sigma) \cap X(\gamma_\tau)$.

If V a C^∞-manifold, M and N two closed immersed submanifolds of V, we say that M and N *intersect perfectly* if the connected components of the intersection $M \cap N$ are immersed submanifolds of V and if for all such components F of $M \cap N$ one has for the tangent bundles: $TF = TM_{|F} \cap TN_{|F}$.
The exponential map $exp : T_xX \longrightarrow X$ is a bijection commuting with the γ-twisted action of Θ, and this fact is the base of the proof of the following:

Proposition 10. *The special cycles $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ intersect perfectly in $\Gamma\backslash X$.*[20]

[20] For the whole proof, see [RSchw], Lemma 1.4.

1.3.3 Orientability of special cycles and the connected components of their intersection

In order to analyze the contribution of the closed submanifolds on $\Gamma\backslash X$ given by special cycles to homology of $\Gamma\backslash X$, we have to deal with the problem of orientation of special cycles and the connected components of the intersection of two special cycles. This is settled in [RSchw]; §2. Here we state just the main result:

Proposition 11. *Let Θ be a finite group of $\mathbb{Q}$-rational automorphisms of a reductive algebraic group G defined over $\mathbb{Q}$. Let Γ be a Θ-stable arithmetic subgroup of $G(\mathbb{Q})$. Then there exists a normal Θ-stable torsion-free arithmetic subgroup $\tilde{\Gamma} \leq \Gamma$ such that $\tilde{\Gamma}(\gamma) \subset G(\gamma)(\mathbb{R})^0$ for all $\gamma \in H^1(\Theta; \tilde{\Gamma})$. I.e. the corresponding components $F(\gamma)$ of $Fix_{\tilde{\Gamma}\backslash X}(\Theta)$ in $\tilde{\Gamma}\backslash X$ are orientable.*

1.3.4 Evaluation of the intersection number

Suppose we have chosen a suitable torsion-free arithmetic subgroup $\Gamma \leq G(\mathbb{Q})$ of a connected reductive algebraic group G defined over $\mathbb{Q}$ s.t. the two complementary dimensional special cycles $C_\sigma(\Gamma)$, $C_\tau(\Gamma)$ in $\Gamma\backslash X$ and all connected components of their intersection are orientable and that $C_\sigma(\Gamma) \cap C_\tau(\Gamma)$ is compact. Then the intersection number is summed over all connected components of the intersection (some of them might be degenerate intersections), thus, using Proposition 10, Corollary 2 and the discussion at the end of the section 1.1.2, we have:

Corollary 4.

$$[C_\sigma(\Gamma)][C_\tau(\Gamma)] = \sum_\gamma e(\eta(\gamma))[F(\gamma)],$$

where the sum ranges over

$$\gamma \in Ker(res_\sigma \times res_\tau : H^1(<\sigma, \tau>; \Gamma) \longrightarrow H^1(<\sigma>; \Gamma) \times H^1(<\tau>; \Gamma)),$$

$\eta(\gamma)$ denotes the excess bundle of the intersection component $F(\gamma)$, and in the case of transversal intersection, where $F(\gamma) = x$ is a point, we define $e(\eta(\gamma))[F(\gamma)] := 1$ if $T_x C_\sigma(\Gamma) \oplus T_x C_\tau(\Gamma)$ equals $T_x \Gamma\backslash X$ as an oriented vector space, and $e(\eta(\gamma))[F(\gamma)] := -1$, otherwise.

1.3.5 Euler number of the excess bundle

Without going into details, we present the main results of §4 in [RSchw], where the formula for $e(\eta(\gamma))[F(\gamma)]$ is derived:
Let $\gamma \in Ker(res_\sigma \times res_\tau : H^1(<\sigma, \tau>; \Gamma) \longrightarrow H^1(<\sigma>; \Gamma) \times H^1(<\tau>; \Gamma))$.
For a chosen maximal compact subgroup $K(\gamma)$ of $G(\gamma)(\mathbb{R})$ we have

$$X(\gamma) \cong G(\gamma)(\mathbb{R})/K(\gamma), \qquad F(\gamma) \cong \Gamma(\gamma)\backslash X(\gamma).$$

The excess bundle $\eta(\gamma)$ determines a bundle η over $X(\gamma)$. Let $G_u(\gamma)$ be a compact real form of $G(\gamma)$. For a suitable realization of $G_u(\gamma)(\mathbb{R})$ we have an embedding of $X(\gamma)$ into the compact dual $X_u(\gamma) := G_u(\gamma)(\mathbb{R})/K(\gamma)$, and the bundle η determines a bundle $\eta_u(\gamma)$ on $X_u(\gamma)$. An invariant positive measure ω on $G(\gamma)(\mathbb{R})$ induces a measure ω_u on $G_u(\gamma)(\mathbb{R})$. In this set-up, we have the following result:

Proposition 12.

$$e(\eta(\gamma))[F(\gamma)] = (-1)^{\frac{f(\gamma)}{2}} \frac{e(\eta_u(\gamma))[X_u(\gamma)]}{vol_{\omega_u}(G_u(\gamma)(\mathbb{R}))^0} \int_{\Gamma(\gamma)\backslash G(\gamma)(\mathbb{R})^0} \omega,$$

where $f(\gamma) := dim\ F(\gamma)$.
Moreover, if $G(\gamma)(\mathbb{R})^0$ *does not contain a compact Cartan subgroup or if* $f(\gamma)$ *is odd, then* $e(\eta(\gamma))[F(\gamma)] = 0$.

Remark 4. *1) For the Euler number* $\chi(F(\gamma))$ *of* $F(\gamma)$ *we get in the same way:*

$$\chi(F(\gamma)) = (-1)^{\frac{f(\gamma)}{2}} \frac{\chi(X_u(\gamma))}{vol_{\omega_u}(G_u(\gamma)(\mathbb{R}))^0} \int_{\Gamma(\gamma)\backslash G(\gamma)(\mathbb{R})^0} \omega,$$

2) The number $|e(\eta_u(\gamma))[X_u(\gamma)]|$ *depends only on the image of* γ *in* $H^1(<\sigma,\tau>, G(\mathbb{R}))$.

Now, having in mind part 2) of Remark4, we analyze what happens in the case that $\gamma, \xi \in Ker(res_\sigma \times res_\tau)$ map to the same class in $H^1(<\sigma,\tau>, G(\mathbb{R}))$,i.e. there exists $g \in G(\mathbb{R})$ s.t. $\gamma_s = g^{-1} \cdot \xi_s \cdot {}^s g$ for all $s \in <\sigma,\tau>$. In this case, the left translation by g induces isomorphisms[21]:

$$X(\xi) \cong X(\gamma),$$

and

$$X(\xi_\sigma) \times X(\xi_\tau) \times X \cong X(\gamma_\sigma) \times X(\gamma_\tau) \times X.$$

Since $\gamma \in Ker(res_\sigma \times res_\tau)$, we have $\gamma_\sigma = a^{-1}\,{}^\sigma a$, $\gamma_\tau = b^{-1}\,{}^\tau b$ with $a, b \in \Gamma$ and unique modulo $\Gamma(\sigma)$ resp. $\Gamma(\tau)$, and we have orientation preserving isomorphisms

$$X(\sigma) \cong X(\gamma_\sigma), \qquad X(\tau) \cong X(\gamma_\tau).$$

Given cocycles γ_s and ξ_s, the equation $\gamma_s = g^{-1} \cdot \xi_s \cdot {}^s g$ determines g up to a factor $a \in G(\xi_s)(\mathbb{R})$. Now define $s(\gamma, \xi)$ in the following way:
$s(\gamma, \xi) := 0$ if there exists an $a \in G(\xi_s)(\mathbb{R})$ s.t. the left translation by a acts orientation reversing on $X(\xi_\sigma) \times X(\xi_\tau) \times X$.
If all $a \in G(\xi_s)(\mathbb{R})$ act orientation preserving on $X(\xi_\sigma) \times X(\xi_\tau) \times X$, then let $s(\gamma, \xi) := 1$ if the left translation by g acts orientation preserving on $X(\xi_\sigma) \times X(\xi_\tau) \times X$, and $s(\gamma, \xi) := -1$ otherwise.

Proposition 13. *If the classes* $\gamma, \xi \in Ker(res_\sigma \times res_\tau)$ *map to the same class in* $H^1(<\sigma,\tau>, G((R)))$, *then*

$$e(\eta_u(\gamma))[X_u(\gamma)] = s(\gamma, \xi) e(\eta_u(\xi))[X_u(\xi)].$$

In fact, in most of the groups in interest, we will have $s(\gamma, \xi) = 1$:

Lemma 8. *If* $G(\mathbb{R})$ *resp.* $G(\sigma)(\mathbb{R})$ *resp.* $G(\tau)(\mathbb{R})$ *act orientation preserving on* X *resp.* $X(\sigma)$ *resp.* $X(\tau)$, *in the setting of Proposition 13, we have* $s(\gamma, \xi) = 1$.

That the assumptions of Lemma 8 are satisfied for many groups can be seen from the following (see 4.7 in [RSchw]):

[21] Here $X(\gamma_\sigma)$ denotes the fixed point set of the γ_σ-twisted action of σ on X, recall that $X(\gamma) = X(\gamma_\sigma) \cap X(\gamma_\tau)$, as seen in section 1.3.2

Proposition 14. *Let G be a connected, simply connected, semisimple algebraic group defined over $\mathbb{R}$. Let σ be an automorphism of finite order of G defined over $\mathbb{R}$. If $G(\mathbb{R})$ is simply connected as a topological group, then $G(\sigma)(\mathbb{R})$ is connected.*

This will be automatically satisfied in the following case: let $G(\mathbb{R}) = G_c \times G_h \times G_r$, where G_c is the product of compact simple factors of $G(\mathbb{R})$ and G_h is the product of factors which give rise to hermitian symmetric spaces. The rest of factors makes up G_r. If the simple factors of G_r are not of the type $SL_n(\mathbb{R})$, $n \geq 3$; $SO_n(q)(\mathbb{R})$ with q a quadratic form of index ≥ 3; split G_2, F_4, E_6, E_7, E_8; $E_{6,2}$, $E_{7,-5}$, $E_{8,-24}$ where the second index is the Cartan index, then $G(\mathbb{R})$ is simply connected and Proposition 14 applies.

Now, using the results so far, and under the assumption that there exists a point $x \in C_\sigma(\Gamma) \cap C_\tau(\Gamma)$ where the special cycles meet transversely, one "shifts" the problem of evaluation of $e(\eta_u(\gamma))[X_u(\gamma)]$ to the problem of evaluation of $e(\eta_u(\xi))[X_u(\xi)]$ for a certain class $\xi \in H^1(<\sigma>, S)$ which maps to the same class as γ in $H^1(<\sigma, \tau>, G(\mathbb{R}))$, where $S := T(\sigma)^0$, for $T \leq K(\tau)^0$ a maximal σ-stable torus contained in $K(\tau)^0$, where K is the maximal compact subgroup of $G(\mathbb{R})$ corresponding to x. In this case, it can be seen that $|e(\eta_u(\xi))[X_u(\xi)]| = |\chi(X_u(\xi))| = \chi(X_u(\xi))$[22], and since $X(\gamma) \cong X(\xi)$ we get:

Proposition 15. *Assume that the special cycles $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ intersect transversely in at least one point. Then we have:*

$$|e(\eta_u(\gamma))[X_u(\gamma)]| = \chi(X_u(\gamma)).$$

We have to use the absolute value $|e(\eta_u(\gamma))[X_u(\gamma)]|$, since it might happen that the element $g \in G(\mathbb{R})^0$ for which we have $\gamma_s = g^{-1} \cdot \xi_s \cdot {}^s g$ gives rise to an orientation reversing isomorphism $X(\xi_\sigma) \times X(\xi_\tau) \times X \cong X(\gamma_\sigma) \times X(\gamma_\tau) \times X$. In the case that this map is orientation preserving, for instance in the case that Proposition 14 holds, we have

$$e(\eta_u(\gamma))[X_u(\gamma)] = \chi(X_u(\gamma)) > 0.$$

In this situation, using Part 2) of Remark 4 we get $[C_\sigma(\Gamma)][C_\tau(\Gamma)] = \sum_\gamma \chi(F(\gamma))$, where the sum is taken over $\gamma \in Ker(res_\sigma \times res_\tau : H^1(<\sigma, \tau>; \Gamma) \longrightarrow H^1(<\sigma>; \Gamma) \times H^1(<\tau>; \Gamma))$, and since the sign of $\chi(F(\gamma))$ is $(-1)^{\frac{f(\gamma)}{2}}$ the next step in achieving $[C_\sigma(\Gamma)][C_\tau(\Gamma)] > 0$ is to find an arithmetic subgroup Γ s.t. all the dimensions $f(\gamma)$ are congruent modulo 4.
The idea how to achieve this can be found in [RSp], and we present a brief sketch in the next section:

1.3.6 Nonvanishing of Euler-characteristics

In this section, let Γ be an arithmetic subgroup of a connected Lie group G, where G is constructed as a subgroup of a semisimple algebraic group $\underline{G}$ defined over $\mathbb{Q}$ s.t. G is a quotient with compact kernel of $\underline{G}(\mathbb{R})^o$, the connected

[22] The last equality $|\chi(X_u(\xi))| = \chi(X_u(\xi)) > 0$ holds, since $X_u(\xi) = G_\xi / K_\xi$ is a representation space of a closed Lie group G_ξ and we have $rk(G_\xi) = rk(K_\xi)$, see f.e. [BN] Theorem 17. and the references therein.

component of the real points of $\underline{G}$. In $\underline{G}$ we work with an involution $\underline{\theta}$ defined over $\mathbb{Q}$ which induces a Cartan involution of G. Such an involution $\underline{\theta}$ is called a Cartan like involution. Now, if $\underline{K}$ is a maximal compact subgroup of $\underline{G}(\mathbb{R})$ with corresponding Cartan involution θ_0 s.t. $\underline{\theta}$ and θ_0 commute on $\underline{G}(\mathbb{R})$. Let $X := \underline{G}(\mathbb{R})/\underline{K}$, and denote by θ the involution on X induced by $\underline{\theta}$ and θ_0.

Let F be one of the field $\mathbb{Q}$,$\mathbb{R}$ or $\mathbb{Q}_p$. Let $t \in \underline{G}(F)$ represent a class in $H^1(\underline{\theta}, \underline{G}(F))$, i.e. $t \cdot \underline{\theta}(t) = 1$. Then we have the t-twisted action $\underline{\theta}_t$ on $\underline{G}(F)$ given by

$$\underline{\theta}_t(g) := t\underline{\theta}(g)t^{-1},$$

which induces corresponding action on $\mathbf{g}(F) := \mathbf{g} \otimes_{\mathbb{Q}} F$, where $\mathbf{g}$ is the $\mathbb{Q}$-Lie algebra of $\underline{G}$, which we again denote by $\underline{\theta}_t$. $\underline{\theta}_t$ acts as an isometry of the Killing form B of $\mathbf{g}(F)$, thus the eigenspaces of $\underline{\theta}_t$ are orthogonal with respect to B. Let $\mathbf{g}(F)(t)$ denote the fixed point set of $\underline{\theta}_t$ on $\mathbf{g}(F)$. Then $B_{|\mathbf{g}(F)(t)}$ is a non degenerate bilinear form. It is easy to see that the isometry class of the quadratic space $B_F(t) := (\mathbf{g}(F)(t), B_{|\mathbf{g}(F)(t)})$ depends only on the class of t in $H^1(\underline{\theta}, \underline{G}(F))$.

A quadratic form q on a $\mathbb{Q}-$ vector space V can be diagonalized over $\mathbb{R}$ with r factors 1 and s factors -1 on the diagonal. Then we write $sgn(q) := r - s$, which depends only on the isometry class of $q \otimes \mathbb{R}$ in $V \otimes \mathbb{R}$.

Lemma 9. *In notation above, let $X(t) := \{x \in X | \underline{\theta}(x)t^{-1} = x\}$ denote the fixed point set of $\underline{\theta}_t$ on X. Then*

$$2dimX(t) = dim\mathbf{g}(\mathbb{R})(t) + sgn(B_{\mathbb{R}}(t))$$

Proof. The inclusion $\underline{K} \hookrightarrow \underline{G}(\mathbb{R})$ induces a bijection $H^1(\underline{\theta}, \underline{K}) \cong H^1(\underline{\theta}, \underline{G}(\mathbb{R}))$,[23] thus we can assume $t \in \underline{K}$; then $\underline{\theta}$ and θ_0 commute and there is an eigenspace decomposition

$$\mathbf{g}(\mathbb{R})(t) = \mathbf{k}_0 + \mathbf{p}_0$$

of $\mathbf{g}(\mathbb{R})(t)$ with respect to the θ_0-action. Since θ_0 is the Cartan involution corresponding to $\underline{K}$, $B_{|\mathbf{k}_0}$ is negative definite and $B_{|\mathbf{p}_0}$ is positive definite, therefore:

$$dim\mathbf{g}(\mathbb{R})(t) + sgn(B_{\mathbb{R}}(t)) = dim\mathbf{k}_0 + dim\mathbf{p}_0 - dim\mathbf{k}_0 + dim\mathbf{p}_0 =$$

$$= 2dim\mathbf{p}_0 = 2dimX(t).$$

□

Let $W(F)$ be the Grothendieck-Witt ring of quadratic forms over F (see [Scha]), and let

$$B_F : H^1(\underline{\theta}, \underline{G}(F)) \longrightarrow W(F),$$

$$t \mapsto (\mathbf{g}(F)(t), B_{|\mathbf{g}(F)(t)}).$$

The inclusions $\mathbb{Q} \hookrightarrow \mathbb{Q}_v$ induce obvious Hasse maps h in cohomology and of the Witt rings, making the following diagram commutative:

[23] see [RSp] 2.2.

$$\begin{array}{ccc} H^1(\underline{\theta}, \underline{G}(\mathbb{Q})) & \xrightarrow{\quad h \quad} & \prod_{v \in V^{\mathbb{Q}}} H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_v)) \\ \Big\downarrow{\scriptstyle B} & & \Big\downarrow{\scriptstyle \prod_{v \in V^{\mathbb{Q}}} B_v} \\ W(\mathbb{Q}) & \xrightarrow{\quad h \quad} & \prod_{v \in V^{\mathbb{Q}}} W(\mathbb{Q}_v) \end{array}$$

If (q, V) is a $\mathbb{Q}$-rational quadratic space, then for every place $v \in V^{\mathbb{Q}}$ of $\mathbb{Q}$ there is defined a Gauss sum $\gamma_v(q)$ with values in the eight's root of unity, and depending only on the class of $(q \otimes \mathbb{Q}_v, V \otimes \mathbb{Q}_v)$. If $v = \infty$, then $\gamma_\infty(q) = \varepsilon^{-sgn(q)}$, where $\varepsilon = \frac{1+i}{\sqrt{2}}$ is the primitive eight's root of unity. Weil's product formula[24] says:

$$\prod_{v \in V^{\mathbb{Q}}} \gamma_v(q) = 1.$$

Now, if we have two quadratic forms $B(t)$ and $B(\tilde{t})$ which are equal on $\mathbb{Q}_v$ for all finite primes v, we have $\gamma_\infty(B(t)) = \gamma_\infty(B(\tilde{t}))$, i.e.

$$\varepsilon^{-sgn(B_{|\mathbb{R}}(t))} = \varepsilon^{-sgn(B_{|\mathbb{R}}(\tilde{t}))},$$

and since ε is the primitive eight's root of unity, $sgn(B_{|\mathbb{R}}(t)) \equiv sgn(B_{|\mathbb{R}}(\tilde{t}))$ mod 8, and now using Lemma 9 we have the following result:

Lemma 10. *Consider the Hasse map*

$$h := \prod_{v \in V^{\mathbb{Q}}} h_v : H^1(\underline{\theta}, \underline{G}(\mathbb{Q})) \longrightarrow \prod_{v \in V^{\mathbb{Q}}} H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_v)).$$

If $t, \tilde{t} \in \underline{G}(\mathbb{Q})$ represent classes in $H^1(\underline{\theta}, \underline{G}(\mathbb{Q}))$ and if $h_v(t) = h_v(\tilde{t})$ for all $v \neq \infty$, then

$$dimX(t) \equiv dimX(\tilde{t}) \ mod\ 4.$$

Now, in [RSp], the authors show how to produce a "small enough" congruence subgroup $\Gamma \leq \underline{G}(\mathbb{Q})$ s.t. all classes of $H^1(\underline{\theta}, \Gamma)$ are trivial at all finite places of $\mathbb{Q}$, meaning that

$$\prod_{p \in V_f^{\mathbb{Q}}} h_p : H^1(\underline{\theta}, \Gamma) \longrightarrow \prod_p H^1(\underline{\theta}, \underline{G}(\mathbb{Q}_p))$$

is trivial.

Congruence subgroups are produced in the following way:
we have an embedding $\underline{G} \hookrightarrow SL_N$ over $\mathbb{Q}$. Let $K_{\mathbf{p}}(j)$ denote the kernel of the canonical map

$$SL_N(\mathbb{Z}_{\mathbf{p}}) \longrightarrow SL_N(\mathbb{Z}_{\mathbf{p}}/\mathbf{p}^j\mathbb{Z}_{\mathbf{p}}).$$

Let $sl_N(\mathbb{Z}_{\mathbf{p}})$ denote the set of $N \times N$ matrices with coefficients in $\mathbb{Z}_{\mathbf{p}}$ and trace zero. Then we have a bijection induced by the usual matrix exponential series:

$$exp : \mathbf{p}^j sl_N(\mathbb{Z}_{\mathbf{p}}) \longrightarrow K_{\mathbf{p}}(j),$$

[24] See [Scha], Ch.5.

for $\mathbf{p} > 2$ or $\mathbf{p} = 2$ and $j \geq 2$. Let $\mathbf{g}(\mathbb{Q}_\mathbf{p})$ denote the Lie algebra of $\underline{G}(\mathbb{Q}_\mathbf{p})$ considered as a subset of $M_N(\mathbb{Q}_\mathbf{p})$. Then we have an induced bijection

$$exp : \mathbf{p}^j sl_N(\mathbb{Z}_\mathbf{p}) \cap \mathbf{g}(\mathbb{Q}_\mathbf{p}) \longrightarrow \Gamma_\mathbf{p}(j),$$

where $\Gamma_\mathbf{p}(j) := K_\mathbf{p}(j) \cap \underline{G}(\mathbb{Q}_\mathbf{p})$.

The following technical Lemma enables the suitable choice of a congruence subgroup:

Lemma 11. *Let $U_\mathbf{p} \subset \underline{G}(\mathbb{Z}_\mathbf{p})$ be an open $\underline{\theta}$-stable subgroup. Then there exists an open $\underline{\Theta}$-stable normal subgroup $V_\mathbf{p}$ of $U_\mathbf{p}$ such that the map*

$$H^1(\underline{\theta}, V_\mathbf{p}) \longrightarrow H^1(\underline{\theta}, U_\mathbf{p})$$

induced by the inclusion $V_\mathbf{p} \hookrightarrow U_\mathbf{p}$ is trivial.

Since our embedding $\underline{G} \hookrightarrow SL_N$ has been chosen without care with respect to the $\underline{\theta}$-action, $\underline{\theta}$ will not preserve all $\underline{G}(\mathbb{Z}_\mathbf{p}) := \underline{G}(\mathbb{Q}_\mathbf{p}) \cap SL_N(\mathbb{Z}_\mathbf{p})$. This happens at a finite set S_0 of primes. For technical reasons, enlarge S_0 by assuming $2 \in S_0$. Then $\underline{\theta}$ preserves $\underline{G}(\mathbb{Z}_\mathbf{p})$ for all $\mathbf{p} \notin S_0$, and for a finite set of primes S containing S_0 define

$$\Gamma(S) := \underline{G}(\mathbb{Q}) \cap (\prod_{\mathbf{p} \in S} V_\mathbf{p} \times \prod_{\mathbf{p} \notin S} \underline{G}(\mathbb{Z}_\mathbf{p})).$$

Now start with a suitable congruence subgroup $\Gamma(S_0)$, then $H^1(\underline{\theta}, \Gamma(S_0))$ is finite[25], and for a $\gamma \in H^1(\underline{\theta}, \Gamma(S_0))$ with a prime v s.t. $h_v(\gamma) \neq 1$, choose $S = S_0 \cup v$. After finitely many steps, by Lemma 11 one arrives at a desired "small enough" congruence subgroup $\Gamma(S)$, for which, using Lemma 10, we have

$$\gamma, \tilde{\gamma} \in H^1(\underline{\theta}, \Gamma(S)) \Rightarrow dim X(\gamma) \equiv dim X(\tilde{\gamma}) \; mod \; 4,$$

i.p. the sign of $\chi(F(\gamma))$ equals the sign of $\chi(F(\tilde{\gamma}))$.

1.3.7 Finite covering with positive intersection number of cycles

Summing up the results in previous sections we can now formulate the main result of [RSchw]:

Let G be a connected reductive algebraic group defined over $\mathbb{Q}$; σ, τ two $\mathbb{Q}$-rational automorphisms of finite order of G which commute with each other, and let Γ be a $< \sigma, \tau >$-stable torsion-free arithmetic subgroup of $G(\mathbb{Q})$ s.t. the special cycles $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ and all connected components of their intersection are orientable. Suppose $\dim C_\sigma(\Gamma) + \dim C_\tau(\Gamma) = \dim \Gamma\backslash X$ and that $C_\sigma(\Gamma) \cap C_\tau(\Gamma)$ is compact. Furthermore, suppose $G(\mathbb{R})$ resp. $G(\sigma)(\mathbb{R})$ resp. $G(\tau)(\mathbb{R})$ act orientation preserving on X resp. $X(\sigma)$ resp. $X(\tau)$.

Theorem 6. *[RSchw], Th.4.11.*

With the assumptions above, assume in addition that G is semisimple and that $X(\sigma)$ and $X(\tau)$ intersect in exactly one point with positive intersection

[25]For this, see [BoS].

number[26]*. Then there exists a* $<\sigma,\tau>$*-stable normal subgroup* $\tilde{\Gamma}$ *of finite index in* Γ *s.t.*

$$[C_\sigma(\Gamma)][C_\tau(\Gamma)] = \sum_\gamma \chi(F(\gamma)),$$

where the sum ranges over $\gamma \in Ker(res_\sigma \times res_\tau : H^1(<\sigma,\tau>;\Gamma) \longrightarrow H^1(<\sigma>;\Gamma) \times H^1(<\tau>;\Gamma))$*, and where all Euler characteristics* $\chi(F(\gamma))$ *are positive.*

Remark 5. *The results of section 1.3.6 can be applied to* $H^1(<\sigma,\theta>;\Gamma)$ *in order to produce a congruence subgroup where all Euler characteristics* $\chi(F(\gamma))$ *for* $\gamma \in Ker(res_\sigma \times res_\tau : H^1(<\sigma,\tau>;\Gamma) \longrightarrow H^1(<\sigma>;\Gamma) \times H^1(<\tau>;\Gamma))$ *are positive.*

[26] This enables the use of Proposition 15. Note that $X(\sigma)$ and $X(\tau)$ intersect in exactly one point with positive intersection number if and only if the group $G(\sigma,\tau)(\mathbb{R})$ of fixed points of $<\sigma,\tau>$ is compact. Positivity of the intersection number can always be easily achieved by rearranging the orientation on X, $X(\sigma)$ and $X(\tau)$.

1.4 Result of Lafont-Schmidt

Here we present the result of [LS]:

1.4.1 Background: CW-structure of real, complex and quaternion projective spaces

The following can be regarded as a summary of Ch.2. in [Vi]:

Let A, X, Y be topological spaces with $A \subset X$, $X \cap Y = \emptyset$, and let $f : A \longrightarrow Y$ be a continuous function. Consider $X \cup Y$ as a topological space in which X and Y are both open and closed, carrying their original topology. Let $\sim_f$ be the least equivalence relation on $X \cup Y$ such that $a \sim_f f(a)$ for all $a \in A$. ($a \sim f(a)$ defines a partial relation, and the least equivalence relation generated by a partial relation can be obtained by letting $x \sim_f y$ if there exists a sequence $x = x_0, x_1, \ldots, y = x_n$ such that $x_i \sim x_{i+1}$, $x_{i+1} \sim x_i$ or $x_i = x_{i+1}$ for all i).
The identification space $(X \cup Y)/\sim_f$ is the *space obtained by attaching X to Y via $f : A \longrightarrow Y$*, denoted $X \cup_f Y$.
Of particular importance is the case $X := \mathbf{D}^n$, $A := \mathbf{S}^{n-1} = \partial\mathbf{D}^n$. The space $\mathbf{D}^n \cup_f Y$ is called the *space obtained by attaching an n-cell to Y via f.*

Example 1. *$X := \boldsymbol{D}^2$, $A := \boldsymbol{S}^1 = \partial\boldsymbol{D}^2$, and Y a copy of $\boldsymbol{S}^1$ disjoint from X. Let $f : A \longrightarrow Y$ be the map given in complex coordinates by $f(e^{i\theta}) := e^{2i\theta}$. The identification space $\boldsymbol{D}^2 \cup_f \boldsymbol{S}^1$ (as we shall later see) is the real projective plane $P^2(\mathbb{R})$.*
The homology groups of this space may be computed by choosing an open "cell" U in the interior of $\boldsymbol{D}^2$, a point $p \in U$, and a set $V := P^2(\mathbb{R})\backslash\{p\}$. Now U, V cover $P^2(\mathbb{R})$ and the Mayer-Vietoris sequence can be applied. As a result we get[27]

$$H_1(P^2(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_2(P^2(\mathbb{R}); \mathbb{Z}) \cong 0.$$

The method used in this example can be adapted to prove the following:

Proposition 16. *If $f : \boldsymbol{S}^{n-1} \longrightarrow Y$ is continuous and Y Hausdorff, then there is an exact sequence:*

$$\cdots \longrightarrow H_m(\boldsymbol{S}^{n-1}) \xrightarrow{f_*} H_m(Y) \xrightarrow{i_*} H_m(\boldsymbol{D}^n \cup_f Y) \longrightarrow H_{m-1}(\boldsymbol{S}^{n-1}) \longrightarrow \cdots$$

$$\longrightarrow H_0(\boldsymbol{S}^{n-1}) \longrightarrow H_0(Y) \oplus \mathbb{Z} \longrightarrow H_0(\boldsymbol{D}^n \cup_f Y).$$

This exact sequence says that the homology groups of Y and $\mathbf{D}^n \cup_f Y$ can only differ in dimensions n and $(n-1)$. The attaching of an n-cell may have created a new n-dimensional "hole" and on the other hand this new n-cell may have filled an existing $(n-1)$-dimensional "hole" in Y. Away from these dimensions, the homology groups do not differ.

[27] For the details see [Vi], example on p.43.

Example 2. *Define the real n-dimensional projective space to be $P^n(\mathbb{R}) := \boldsymbol{S}^n/\sim$, the set of equivalence classes under the equivalence relation $\sim$ defined by $x \sim y :\Leftrightarrow x = -y$. Let $\pi : \boldsymbol{S}^n \longrightarrow P^n(\mathbb{R})$ be the quotient map. We can attach an $(n+1)$-cell to $P^n(\mathbb{R})$ via π in the following way:*
Regard $\boldsymbol{S}^n \subset \boldsymbol{S}^{n+1}$ by identifying $(x_1, \dots, x_n) \in \boldsymbol{S}^n$ with $(x_1, \dots, x_n, 0) \in \boldsymbol{S}^{n+1}$. This induces the inclusion $i : P^n(\mathbb{R}) \hookrightarrow P^{n+1}(\mathbb{R})$ of a closed subset. Write $\boldsymbol{S}^{n+1}$ as a union $E_+ \cup E_-$, where $E_+ := \{(x_1, \dots, x_n, x_{n+1}) \in \boldsymbol{S}^{n+1} \mid x_{n+1} \geq 0\}$ resp. $E_- := (x_1, \dots, x_n, x_{n+1}) \in \boldsymbol{S}^{n+1} \mid x_{n+1} \leq 0)$ is the upper resp. lower closed hemisphere. Then $E_+ \cap E_- = \boldsymbol{S}^n$.
There is a homeomorphism $g : \boldsymbol{D}^{n+1} \longrightarrow E_+$. Let $f_1 : \boldsymbol{D}^{n+1} \longrightarrow P^{n+1}(\mathbb{R})$ be the composition of the maps

$$\boldsymbol{D}^{n+1} \xrightarrow{g} E_+ \subset \boldsymbol{S}^{n+1} \xrightarrow{\tilde{\pi}} P^{n+1}(\mathbb{R})$$

where $\tilde{\pi}$ is the quotient map on $\boldsymbol{S}^{n+1}$ (analogous to π).
This induces:

$$\boldsymbol{D}^{n+1} \cup P^n(\mathbb{R}) \xrightarrow{f_1 \cup i} P^{n+1}(\mathbb{R})$$

which can easily be seen to be surjective (in fact, f_1 is already surjective).
It is not difficult to see that this map induces a homeomorphism

$$P^{n+1}(\mathbb{R}) \simeq \boldsymbol{D}^{n+1} \cup_\pi P^n(\mathbb{R}).$$

Thus, $P^{n+1}(\mathbb{R})$ can be constructed by attaching an $(n+1)$-cell to $P^n(\mathbb{R})$ via the quotient map $\pi : \boldsymbol{S}^n \longrightarrow P^n(\mathbb{R}) = \boldsymbol{S}^n/\sim$.

The complex and the quaternion projective spaces are constructed analogously using the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbb{H}^n \cong \mathbb{R}^{4n}$:

Example 3. *Consider $\boldsymbol{S}^{2n+1}$ as a subspace of $\mathbb{C}^{n+1}$:*

$$\boldsymbol{S}^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} |z_i|^2 = 1\},$$

and let $\sim$ denote the equivalence relation $(z_1, \dots, z_{n+1}) \sim (\tilde{z}_1, \dots, \tilde{z}_{n+1})$ if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ s.t. $z_i = \lambda \tilde{z}_i$ for all $i = 1, \dots, n+1$.
The complex $n-$dimensional projective space $P^n(\mathbb{C})$ is defined as the quotient $\boldsymbol{S}^{n+1}/\sim$. Let π denote the quotient map $\boldsymbol{S}^{2n+1} \longrightarrow P^n(\mathbb{C})$. Then, analogously as in the case of real projective spaces, we have that $P^{n+1}(\mathbb{C})$ can be obtained by attaching a $2(n+1)$-cell to $P^n(\mathbb{C})$ via π.

Example 4. *Define the quaternion $n-$dimensional projective space to be $P^n(\mathbb{H}) := \boldsymbol{S}^{4n+3}/\sim$, where the equivalence relation $\sim$ is obtained in the following way:*
Consider $\boldsymbol{S}^{4n+3}$ as the subspace $\mathbb{H}^{n+1}$:

$$\boldsymbol{S}^{4n+3} = \{(a_1, \dots, a_{n+1}) \in \mathbb{H}^{n+1} \mid \sum_{i=1}^{n+1} |a_i|^2 = 1\},$$

where for $a \in \mathbb{H}$ we have $|a| := \mathbf{n}(a)$ (for definition see section 3.1.1). Let $\sim$ be the equivalence relation given by $(a_1, \dots, a_{n+1}) \sim (\tilde{a}_1, \dots, \tilde{a}_{n+1})$ if and only if there exists $\lambda \in \mathbb{H}$ with $|\lambda| = 1$ s.t. $a_i = \lambda \tilde{a}_i$ for all $i = 1, \dots, n+1$. It can be seen that $P^{n+1}(\mathbb{H})$ is obtained by attaching an $4(n+1)$-cell to $P^n(\mathbb{H})$ via the projection map $\boldsymbol{S}^{4n+3} \longrightarrow P^n(\mathbb{H}) = \boldsymbol{S}^{4n+3}/\sim$.

This description of projective spaces enables us to calculate their homology:

1.4.2 Homology of projective spaces

As opposed to the complex and quaternion case, real projective spaces have cells in each dimension, which does not an enable elegant use of Prop.16 for the homology computations. Since our interest is mainly in the quaternion case, where the computation is analogous to the complex case, we do not go into details of the computation of homology groups of real projective spaces (for the proof, see p. 62-63. of [Vi]).

Proposition 17. *$H_i(P^n(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$ for $i = 0$ and for $i = n$, when n is odd. $H_i(P^n(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for i odd, $0 < i < n$; and $H_i(P^n(\mathbb{R});\mathbb{Z}) \cong 0$ otherwise.*

Proposition 18. *$H_i(P^n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2, 4, \dots, 2n$ and trivial for all other i.*

Proof. $P^0(\mathbb{C})$ is a point, and $P^1(\mathbb{C})$ is obtained by attaching $\mathbf{D}^2$ to a point, thus $P^1(\mathbb{C})$ is homeomorphic to $\mathbf{S}^2$. Thus $H_*(P^0(\mathbb{C});\mathbb{Z}) \cong H_*(\cdot;\mathbb{Z})$ and $H_*(P^1(\mathbb{C});\mathbb{Z}) \cong H_*(\mathbf{S}^2;\mathbb{Z})$, and the result is true for $n = 0, 1$. Now one can use induction:
Start from $P^{n-1}(\mathbb{C})$ by attaching an $2n$-cell as in Ex.3. Using Prop.16 we get for $i > 1$ a collection of short exact sequences

$$0 \longrightarrow H_i(P^{n-1}(\mathbb{C});\mathbb{Z}) \longrightarrow H_i(P^n(\mathbb{C});\mathbb{Z}) \longrightarrow H_{i-1}(\mathbf{S}^{2n-1};\mathbb{Z}) \longrightarrow 0,$$

which gives the desired result. □

Analogously, we get

Proposition 19. *$H_i(P^n(\mathbb{H})) \cong \mathbb{Z}$ for $i = 0, 4, 8, \dots, 4n$ and trivial for all other i.*

1.4.3 Homologically essential submanifolds

The following is a short overview of the main theorem of [LS]:

Let X and Y be compact locally symmetric spaces of non-compact type, $Y \hookrightarrow X$ embedded as a totally geodesic submanifold. Let X resp. Y be locally modelled on a symmetric space G/K resp. $\tilde{G}/\tilde{K}$ (this symmetric space is the universal cover of X resp. Y), with G, $\tilde{G}$ non-compact semisimple Lie groups; K, $\tilde{K}$ maximal compact subgroups.
Since the embedding $Y \hookrightarrow X$ is totally geodesic, one can consider $\tilde{G}$ as a subgroup of G,[28] and hence take $\tilde{K} = \tilde{G} \cap K$.
Let G_u resp. $\tilde{G}_u$ be compact duals [29] of G resp. $\tilde{G}$, and denote by $X_u := G_u/K$ resp. $Y_u := \tilde{G}_u/\tilde{K}$ the dual symmetric space to G/K resp. $\tilde{G}/\tilde{K}$.
Totally geodesic embedding $Y \hookrightarrow X$ induces a totally geodesic embedding $\tilde{G}/\tilde{K} \hookrightarrow G/K$, which further induces a totally geodesic embedding of dual symmetric spaces $Y_u \hookrightarrow X_u$, since the commutative diagram

[28] In the case that $Y \hookrightarrow X$ is totally geodesic, then any isometry of Y extends to an isometry of Y.

[29] Compact dual G_u of a non-compact Lie group G is the maximal compact subgroup of the complexification of G. If K is a (maximal) compact subgroup of G, one can assume $K \leq G_u$ (up to conjugation).

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] & G \arrow[d] \\ \tilde{K} \arrow[r] & K \end{tikzcd}$$

after passing to complexification and descending to the maximal compacts, yields a commutative diagram

$$\begin{tikzcd} \tilde{G}_u \arrow[r] \arrow[d] & G_u \arrow[d] \\ \tilde{K} \arrow[r] & K \end{tikzcd}$$

When X is a compact locally symmetric space modelled on G/K, Matsushima ([Mat]) constructed a map on the cohomology

$$j^* : H^*(G_u/K;\mathbb{R}) \longrightarrow H^*(X;\mathbb{R}),$$

with the following properties:
i) j^* is injective
ii) there exists a constant $\mathbf{m}(\mathbf{g})$ ("the Matsushima constant") depending only on the Lie algebra $\mathbf{g}$ of G, such that j^* is surjective in cohomology in degrees up to dimension $\mathbf{m}(\mathbf{g})$.
As mentioned in the introduction, the classes which are in the image of this map are exactly the G-invariant classes.

In the notation used above, the following is a main result of interest to us in [LS]:

Theorem 7. *[LS], Th.1.1.*
Let $Y \hookrightarrow X$ be a compact totally geodesic $m-$dim. submanifold of non-compact type inside a compact $n-$dim. locally symmetric space of non-compact type [30]*. Let ρ denote the map $H^m(X_u;\mathbb{R}) \longrightarrow H^m(Y_u;\mathbb{R})$ on the cohomology induced by the embedding $Y_u \hookrightarrow X_u$. Then:*
· if $[Y] = 0$ in $H_m(X;\mathbb{R})$, then $\rho \equiv 0$
· if $\rho \equiv 0$ and $m \leq \mathbf{m}(\mathbf{g})$, where $\mathbf{m}(\mathbf{g})$ is the Matsushima constant of $\mathbf{g} := Lie(G)$, then $[Y] = 0$ in $H_m(X;\mathbb{R})$.

Proof. For a group H acting on a manifold M, let $\Omega^H(M)$ denote the complex of H-invariant differential forms on M.
Let $X := \Gamma\backslash G/K$, $Y := \Lambda\backslash\tilde{G}/\tilde{K}$. We consider the following complexes of differential forms: $\Omega^G(G/K)$, $\Omega^{\tilde{G}}(\tilde{G}/\tilde{K})$, $\Omega^\Gamma(G/K)$, $\Omega^\Lambda(\tilde{G}/\tilde{K})$.
We have the following identifications:

$$\Omega^G(G/K) \cong H^*(\mathbf{g},\mathbf{k}) \cong H^*(\mathbf{g_u},\mathbf{k}) \cong \Omega^{G_u}(G_u/K)$$

[30]Note that every totally geodesic submanifold of a locally symmetric space is locally symmetric, see [Hel], p.189.

and the cohomology of the complex $\Omega^{G_u}(G_u/K)$ is the cohomology of X_u[31].
Analogously we can identify $\Omega^{\tilde{G}}(\tilde{G}/\tilde{K})$ with the cohomology of Y_u.
The cohomology of complexes $\Omega^{\Gamma}(G/K)$ resp. $\Omega^{\Lambda}(\tilde{G}/\tilde{K})$ is simply the cohomology of of X resp. Y.
We have the following commutative diagram of chain complexes:

$$\begin{array}{ccc} \Omega^G(G/K) & \xrightarrow{\psi} & \Omega^{\tilde{G}}(\tilde{G}/\tilde{K}) \\ \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j_Y} \\ \Omega^{\Gamma}(G/K) & \xrightarrow{\phi} & \Omega^{\Lambda}(\tilde{G}/\tilde{K}) \end{array}$$

and passing to the homology of the chain complexes, we obtain a commutative diagram in dimension $m = \dim Y$:

$$\begin{array}{ccc} H^m(X_u;\mathbb{R}) & \xrightarrow{\psi^*} & H^m(Y_u;\mathbb{R}) \cong \mathbb{R} \\ \downarrow{\scriptstyle j_X^*} & & \downarrow{\scriptstyle j_Y^*} \\ H^m(X;\mathbb{R}) & \xrightarrow{\phi^*} & H^m(Y;\mathbb{R}) \cong \mathbb{R} \end{array}$$

The two vertical maps are precisely the Matsushima maps for the respective locally symmetric spaces, which are injective, and since $H^m(Y_u;\mathbb{R}) \cong \mathbb{R} \cong H^m(Y;\mathbb{R})$ we have that j_Y is an isomorphism. Furthermore, if $m \leq \mathbf{m}(\mathbf{g})$, then j_X^* is also surjective. Hence, we have:

$$\phi^* \equiv 0 \Rightarrow \psi^* \equiv 0,$$

and if $m \leq \mathbf{m}(\mathbf{g})$, then $\phi^* \equiv 0 \Leftrightarrow \psi^* \equiv 0$.
Both of horizontal maps coincide with the maps induced on the cohomology by the respective inclusions $Y \hookrightarrow X$ and $Y_u \hookrightarrow X_u$, i.p. ψ^* coincides with the map ρ in the statement of the theorem. On the other hand, ϕ^* is not zero precisely when $[Y] \neq 0$ in $H_m(X;\mathbb{R})$.

□

Remark 6. *Using the description of the homology of complex and quaternion projective spaces in section 1.4.2 , we shall see that Theorem 7 can be used to construct nontrivial homology classes on locally symmetric spaces modelled on complex and quaternion hyperbolic space. The quaternion case will be explained later in the discussion before Corollary 11, section 3.3. For the case of complex hyperbolic manifolds, the framework is analogous, see [LS], example on p.10-11.*

[31] The first and third equality are identifications of complex of harmonic forms with the relative Lie algebra cohomology, and the second equality comes via the dual Cartan decompositions $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ and $\mathbf{g_u} = \mathbf{k} \oplus i\mathbf{p}$. The fact that the cohomology of the complex $\Omega^{G_u}(G_u/K)$ equals the cohomology of the X_u is a consequence of the fact that X_u is a compact oriented manifold, for which the Hodge theorem (see f.e. [Mo], p.159.) applies and the fact that for a symmetric space M the space of harmonic forms coincides with the space of the forms invariant under the largest group of isometries of M, which is in our case exactly G_u.

Chapter 2

Bounds on Betti numbers

In the last chapter, it is shown how to produce some nontrivial homology classes in locally symmetric spaces $\Gamma\backslash X$. We now want to distinguish between the classes representing the dual of a G-invariant cohomology class and those dual to a non G-invariant cohomology class. For example, in a compact locally symmetric spaces of non-compact type[1], the cohomological duals of some of the homology classes constructed in Chapter 1 might simply be the image of a cohomology class in the compact dual space under the Matsushima map. Thus, in order to capture a wider range of homology classes, we might have to pass to a deeper congruence subgroup where the cohomological dual of a special cycle does not represent a G-invariant class, i.p. it is not an image of the class in the compact dual.

Once we are able to produce such a class in $H_i(\tilde{\Gamma}\backslash X)$ for some congruence subgroup $\tilde{\Gamma} \leq \Gamma$, we have a nontrivial representation of the group $\Gamma/\tilde{\Gamma}$ on $H_i(\tilde{\Gamma}\backslash X)$. Applying the strong approximation property of the group G (or a suitable subgroup), there is a way to describe the group $\Gamma/\tilde{\Gamma}$, and the dimension of its smallest nontrivial representation gives a bound on the i-th Betti number.

At the end of this chapter we make a brief note on the result of Agol, [Ag] ,where it is shown how to construct arithmetic subgroups $\Gamma_i \leq \Gamma$ with arbitrary large k-th Betti number of the corresponding locally symmetric space, under the assumption that there is a non G-invariant homology class in $H^k(\Gamma\backslash X)$. The subgroups Γ_i are not constructed as congruence subgroups as opposed to those in [MR].

2.1 Existence of non G-invariant homology classes

This section presents Theorem 2.1 of [MR]:
The notation is as in Chapter 1: having fixed an arithmetic subgroup Γ of $G(\mathbb{Q})$, for an involution μ of G we denote the involutions induced by μ on Γ and on the corresponding symmetric (X) and locally symmetric space ($\Gamma\backslash X$) again by μ, and by G_μ, Γ_μ, X_μ its fixed point sets. For a prime $\mathbf{p}$, we denote by $\Gamma(\mathbf{p})$ the corresponding congruence subgroup of Γ. We denote by $C_\mu(\mathbf{p}) :=$

[1] I.e. $\Gamma\backslash X$ is compact but X is a non compact Riemannian symmetric space.

$C_\mu(\Gamma(\mathbf{p}))$ the special cycle obtained as a connected component corresponding to the cohomology class 1_μ in $H^1(<\mu>, \Gamma(\mathbf{p}))$ of the fixed point set of μ on $\Gamma(\mathbf{p})\backslash X$, canonically diffeomorphic to $\Gamma_\mu(\mathbf{p})\backslash X_\mu$.

The idea how to find a non-invariant class is based on the observation of W. Dwyer cited in [MR] that if a homology class $[C_\sigma(\mathbf{q})]$ is dual to a G-invariant class, then i.p. it must be invariant under the *deck transformations*[2] of the covering $\Gamma(\mathbf{q})\backslash X \longrightarrow \Gamma\backslash X$.

In the setting as in the section 1.2, suppose we have found Γ s.t. the special cycles $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ [3] intersect transversely and that the intersection multiplicity of each intersection component (in this case an isolated point) is positive (i.e. $+1$).[4] Then, we have $[C_\sigma(\Gamma)][C_\tau(\Gamma)] > 0$ and the cycles represent nontrivial homology classes. Passing to a subgroup $\Gamma(\mathbf{q}) \leq \Gamma$, we note that every intersection component of $C_\sigma(\Gamma(\mathbf{q})) \cap C_\tau(\Gamma(\mathbf{q}))$ lies over some intersection component of $C_\sigma(\Gamma) \cap C_\tau(\Gamma)$, hence it again has a positive intersection multiplicity[5], and we have $[C_\sigma(\Gamma(\mathbf{q}))][C_\tau(\Gamma(\mathbf{q}))] > 0$.
The multiplication by an element of the group $\Gamma/\Gamma(\mathbf{q})$ on $\Gamma(\mathbf{q})\backslash X$ is a deck transformation of the covering $\Gamma(\mathbf{q})\backslash X \longrightarrow \Gamma\backslash X$. Hence, if we find an element $\alpha \in \Gamma$ s.t.

$$[\alpha C_\sigma(\Gamma(\mathbf{q}))] \neq [C_\sigma(\Gamma(\mathbf{q}))],$$

then both $[C_\sigma(\Gamma(\mathbf{q}))] \neq 0$ and $[\alpha C_\sigma(\Gamma(\mathbf{q}))] \neq 0$ [6] in $H_*(\Gamma(\mathbf{q})\backslash X)$ cannot be dual to a G-invariant cohomology class.
It suffices to find an $\alpha \in \Gamma$ s.t. $\alpha C_\sigma(\Gamma(\mathbf{q})) \cap C_\tau(\Gamma(\mathbf{q})) = \emptyset$: in that case we have $[\alpha C_\sigma(\Gamma(\mathbf{q}))][C_\tau(\Gamma(\mathbf{q}))] = 0$, and since $[C_\sigma(\Gamma(\mathbf{q}))][C_\tau(\Gamma(\mathbf{q}))] \neq 0$ we must have $[\alpha C_\sigma(\Gamma(\mathbf{q}))] \neq [C_\sigma(\Gamma(\mathbf{q}))]$.
Note that $\alpha C_\sigma(\Gamma(\mathbf{q})) \cap C_\tau(\Gamma(\mathbf{q})) = \emptyset$ follows immediately if $\tau(\alpha C_\sigma(\Gamma(\mathbf{q}))) \cap \alpha C_\sigma(\Gamma(\mathbf{q})) = \emptyset$. We have $\tau(\alpha C_\sigma(\Gamma(\mathbf{q}))) = \tau(\alpha) C_\sigma(\Gamma(\mathbf{q}))$, and the cycles $\tau(\alpha) C_\sigma(\Gamma(\mathbf{q}))$ and $\alpha C_\sigma(\Gamma(\mathbf{q}))$ are disjoint if $\tau(\alpha)\alpha^{-1}$ is not congruent to an element of Γ_σ modulo $\mathbf{q}$, so we are searching for elements of Γ whose image in the group

$$(\Gamma/\Gamma(\mathbf{q}))/(\Gamma_\sigma/\Gamma_\sigma(\mathbf{q}))$$

is not fixed under the involution induced by τ.

If σ, τ are as in section 1.2, we have the following:

Lemma 12. *There exists a prime ideal* $\mathbf{q}$ *and* $\alpha \in \Gamma$ *s.t.* $\tau(\alpha)\alpha^{-1}$ *is not congruent to an element of* Γ_σ *modulo* $\mathbf{q}$.

[2] A deck transformation of the covering $\pi : X \longrightarrow Y$ is a map $f : X \longrightarrow X$ s.t. $\pi \circ f = \pi$.

[3] For now, σ is an involution of X and τ is the composition of σ with the geodesic symmetry at the origin of X, as in the section 1.2. Then σ, τ are given by elements $s, t \in G(\mathbb{R})$ of order 2 (i.e. $s^{-1} = s$, $t^{-1} = t$), and we denote again by σ, τ the involutions $g \mapsto sgs$, $g \mapsto tgt$ on $G(\mathbb{R})$. If σ, τ are defined over $\mathbb{Q}$, we have induced involutions on $G(F)$ for every field $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

[4] In order to achieve this, starting with a torsion-free arithmetic subgroup Γ i.g. we have to pass to a suitable congruence subgroup $\Gamma(\mathbf{p}) \leq \Gamma$. Having this in mind, we work with such $\Gamma(\mathbf{p})$ and for the notation simplicity we denote it by Γ.

[5] This is a direct consequence of Lemma 5 and Proposition 8.

[6] $[\alpha C_\sigma(\Gamma(\mathbf{q}))] \neq 0$ in $H_*(\Gamma(\mathbf{q})\backslash X)$ since $[\alpha C_\sigma(\Gamma(\mathbf{q}))][\alpha C_\tau(\Gamma(\mathbf{q}))] = [C_\sigma(\Gamma(\mathbf{q}))][C_\tau(\Gamma(\mathbf{q}))] > 0$.

Proof. First, there exists an element $\alpha \in \Gamma$ s.t. $\tau(\alpha)\alpha^{-1} \notin \Gamma_\sigma$:

define $\phi : G(\mathbb{C}) \longrightarrow G(\mathbb{C})$ by $g \mapsto \tau(g)g^{-1}$. Suppose that for all $\alpha \in \Gamma$ we have $\tau(\alpha)\alpha^{-1} \in \Gamma_\sigma$, i.e. $\phi(\alpha) \in \Gamma_\sigma$. So, under this assumption we have $\phi(\Gamma) \subset \Gamma_\sigma$. Since Γ is Zariski dense in $G(\mathbb{C})$, this implies $\phi(G(\mathbb{C})) \subset G_\sigma(\mathbb{C})$ (the polynomial function $\sigma \circ \phi - \phi$ is 0 on Γ, and thus it is 0 on $G_\mathbb{C}$).
Since τ is given on G by the inner automorphism $g \mapsto tgt^{-1} = tgt$ for an element $t \in G$ (which is exactly the involution τ on X), we have $\phi(g) = R_{g^{-1}} \circ Int_t(g)$ and $d\phi : \mathbf{g}(\mathbb{C}) \longrightarrow \mathbf{g}(\mathbb{C})$ is given by $x \mapsto txt - x$. [7] Under the assumption above, we have $d\phi(\mathbf{g}(\mathbb{C})) \subset \mathbf{g}(\mathbb{C})_\sigma$.
We have $\mathbf{g} = \mathbf{k} \oplus (\mathbf{p}_\sigma \oplus \mathbf{p}_\tau)$. Identify $\mathbf{g}$ with its image in $\mathbf{g}(\mathbb{C})$. Let $x_1 \in \mathbf{p}_\sigma$, $x_2 \in \mathbf{p}_\tau$. Then $x := [x_1, x_2] \in \mathbf{k}$. Ad_t is the Cartan involution on $\mathbf{g}_\sigma$, hence it is the identity $Id_{\mathbf{k}_\sigma}$ on $\mathbf{k}_\sigma$ and $-Id_{\mathbf{p}_\sigma}$ on $\mathbf{p}_\sigma$. We have $Ad_t(x_1) = -x_1$, $Ad_t(x_2) = x_2$, hence $d\phi(x) = -2x$. However, $-2x \notin \mathbf{g}(\mathbb{C})_\sigma$, since otherwise $-2x \in \mathbf{k}_\sigma$ but we have $Ad_t(-2x) = 2x$, a contradiction since as mentioned above Ad_t acts as identity on $\mathbf{k}_\sigma$.
Hence, there exists an element $\alpha \in \Gamma$ s.t. $\tau(\alpha)\alpha^{-1} \notin \Gamma_\sigma$.

Now let $\mu := \tau(\alpha)\alpha^{-1} \notin \Gamma_\sigma$. Then choose a rational representation ρ of G on a vector space V such that G_σ is the stabilizer of a line through $v \in V$. Then v and $\rho(\mu)(v)$ are linearly independent, since $\mu \notin G_\sigma$. Complete $v, \rho(\mu)(v)$ to a $\mathbb{Q}$- basis of V. Let $d = \frac{a}{b} \in \mathbb{Q}$ be the discriminant of this basis. Let Ω be a lattice in V, obtained as the $\mathbb{Z}$-span of this basis. Now if we take a prime ideal $\mathbf{q}$ not dividing a or b, then our basis of Ω projects to a $\mathbb{Z}/\mathbf{q}\mathbb{Z}$- basis of the reduced lattice $\Omega(\mathbf{q})$, and the images of v and $\rho(\mu)(v)$ in $\Omega(\mathbf{q})$ are again linear independent, hence μ is not congruent to an element of Γ_σ modulo $\mathbf{q}$.

□

Note that almost all prime ideals satisfy the assumption of the Lemma above (only finitely many divide the numerator or denominator of a discriminant $\frac{a}{b} \in \mathbb{Q}$ above), hence we have the following:

Corollary 5. *In the setting as in the section 1.2, suppose we have found Γ s.t. the special cycles $C_\sigma(\Gamma)$ and $C_\tau(\Gamma)$ intersect transversely and that the intersection multiplicity of each intersection component is positive. Then, for almost all prime ideals $\mathbf{q}$ we have that the special cycle $C_\sigma(\Gamma(\mathbf{q}))$ is not dual to a G-invariant cohomology class in $H^*(\Gamma(\mathbf{q})\backslash X)$.*

[7] The derivative of the inner automorphism $Int_t : g \mapsto tgt$ is the adjoint map Ad_t, which is given by $x \mapsto txt$ on a Lie algebra of a linear Lie group, and the derivative of the right multiplication (by the multiplicative inverse element) is the right translation (by the additive inverse element), so $d\phi = dR_{g^{-1}}{}_{|tgt} \circ Ad_{t|g}$ is given by $x \mapsto txt - x$

2.2 Description of the deck transformation group using strong approximation

In the last section we have seen that by passing to a deeper congruence subgroup $\tilde{\Gamma}$ we find special cycles which represent non G-invariant homology classes in $H_i(\tilde{\Gamma}\backslash X)$. There is a natural action of $\Gamma/\tilde{\Gamma}$ on classes of $H_i(\tilde{\Gamma}\backslash X)$ represented by submanifolds, given by

$$\gamma\tilde{\Gamma} \circ [M] := [\gamma M],$$

and the choice of $\tilde{\Gamma}$ (from the previous section) guarantees that this action is not trivial. Thus, the structure of the group $\Gamma/\tilde{\Gamma}$ and its representations can be used to get bounds on the dimension of $H_i(\tilde{\Gamma}\backslash X)$. In this section we focus on the description of the group $\Gamma/\tilde{\Gamma}$.

In the paper by Millson, [Mil], the author considers the following example:

let F be a totally real number field, O_F the ring of integers in F; $Id, \sigma_2, \ldots, \sigma_m$ the embeddings of F into $\mathbb{R}$, and f the quadratic form given by

$$f(X_1, \ldots, X_{n+1}) := \sum_{i=1}^{n} a_i X_i^2 - a_{n+1} X_{n+1}^2$$

with $a_i \in O_F, a_i > 0$, s.t. f^{Id} has signature $(n,1)$ and all its conjugates f^{σ_i} are positive definite. Denote by Φ the subgroup of $GL_{n+1}(O_F)$ consisting of matrices which preserve f. Then Φ is a subgroup of a group isomorphic to $O(n,1)$, w.l.o.g. $\Phi \leq O(n,1)$.

The group $O(n,1)$, $n+1 \geq 4$ (defined as the group of matrices in $GL_{n+1}(\mathbb{R})$ which preserve the quadratic form $\sum_{i=1}^{n} X_i^2 - X_{n+1}^2$) has 4 connected components. Denote by $O(n,1)^+$ the subgroup of $O(n,1)$ which preserves the half-space $x_{n+1} > 0$. Then $O(n,1)^+$ has two connected components, and the connected component of the identity equals $SO(n,1)^0$. Let $\Gamma := \Phi \cap O(n,1)^+$. For a collection of prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_r$ of O_F, define

$$\Gamma(\mathbf{p}_1, \ldots, \mathbf{p}_r) := \{\gamma \in \Gamma | \gamma \equiv 1 \bmod \mathbf{p_i}, 1 \leq i \leq r\}.$$

The group $\Gamma(\mathbf{p})$ acts on hyperbolic n-space $O(n,1)^+/O(n)$, and hence we get a compact[8] constant negatively curved manifold

$$\Gamma(\mathbf{p})\backslash O(n,1)^+/O(n),$$

and except for those ideals $\mathbf{p}$ with norm 2, we have $\Gamma(\mathbf{p}) \leq SO(n,1)^0$ (since we have already intersected Γ with $O(n,1)^+$ so the sufficient condition for orientability is that the matrices contained in Γ have the determinant 1. The determinant of a matrix in $O(n,1)^+$ is either 1 or -1 and the congruence condition with respect to primes with norm different from 2 forces the determinant to be 1), so we can work in $SO(n,1)^0/SO(n)$.

In order to apply the method of strong approximation to determine $\Gamma/\Gamma(\mathbf{p}_1, \ldots, \mathbf{p}_r)$, in the case where $\Gamma \leq O(n,1)$ it is necessary to consider the groups $\Phi^{'} := \{\gamma \in$

[8] To see that this is a compact locally symmetric space, see f.e. [Mil], discussion on p.239.

$\Phi \cap SO(n,1)^o | \theta(\gamma) = 1\}$ and $\Gamma^{'} := \{\gamma \in \Gamma | \theta(\gamma) = 1\}$, where θ is defined in the following way:

given $u \in O(f)$, we can write $u = r_{v_1} \cdots r_{v_k}$ as a product of reflections r_{v_i} in certain vectors v_i (see [O'M], Theorem 43.3.), and let

$$\theta : O(f) \longrightarrow K^*/(K^*)^2$$

$$u \mapsto f(v_1) \cdots f(v_k) \bmod (K^*)^2.$$

Denote by $O(f)^{'}$ the kernel of θ. Now, $O(f)^{'}$ is the image of $Spin(f)$, a simply connected group, in $SO(f)$ and has the strong approximation property (see Appendix, section 5.4), which is the key concept in the proof of the following:

Proposition 20. *Suppose $\mathbf{p}_1, \ldots, \mathbf{p}_r$ is a collection of prime ideals of O_F which do not divide the discriminant of f. Then f reduced modulo $\mathbf{p}_i$ gives rise to a nondegenerate form $\overline{f}_i$ taking values in the finite field $O_F/\mathbf{p}_i$, and we have an exact sequence*

$$1 \longrightarrow \Gamma^{'}(\mathbf{p}_1, \ldots, \mathbf{p}_r) \longrightarrow \Phi^{'} \longrightarrow O(\overline{f}_1)^{'} \times \cdots \times O(\overline{f}_r)^{'} \longrightarrow 1$$

Proof. The set of $\mathbb{R}$-points of $O(f)^{'}$ is not compact, hence (see Theorem 12 in Appendix, section 5.4) the set of K-rational points of $O(f)^{'}$ is dense in the finite adeles of $O(f)^{'}$. Intersecting with integral adeles (which are an open subset of the finite adeles) we get that $\Phi^{'}$ is dense in the integral ideals of $O(f)^{'}$. Given $a_i \in O(\overline{f}_i)^{'}$, $1 \leq i \leq r$, the set of finite integral adeles of $O(f)^{'}$ congruent to a_i modulo $\mathbf{p}_i$ for all $1 \leq i \leq r$ is an open subset of the finite integral ideals of $O(f)^{'}$ and as such has a nonzero intesection with (a dense subset) $\Phi^{'}$. This shows the surjectivity of the map $\Phi^{'} \longrightarrow O(\overline{f}_1)^{'} \times \cdots \times O(\overline{f}_r)^{'}$. The rest is obvious. $\square$

As a direct consequence we get

Corollary 6.

$$\Gamma^{'}(\mathbf{p}_1)/\Gamma^{'}(\mathbf{p}_1, \mathbf{p}_2) \cong O(\overline{f}_2)^{'}$$

and

$$O(\overline{f}_2)^{'} \subseteq \Gamma(\mathbf{p}_1)/\Gamma(\mathbf{p}_1, \mathbf{p}_2) \subseteq SO(\overline{f}_2)$$

which basically describes the deck transformation group.

Note that the complications in this case arise from the fact that the group $O(n,1)$ is not a simply connected group, and thus we must consider the group $\Gamma^{'}$ instead of Γ. In the case where Γ is an arithmetic subgroup of a reductive simply connected group G defined over some number field F, then (with respect to a suitable finite set of places of F) G itself already has the strong approximation property (see Appendix, section 5.4) and in order to describe the deck transformation group, we can use the following (see [PR] p.434, Th.7.15):

For a finite set of prime numbers S, let $\mathbb{Z}(S) := \{x \in \mathbb{Q} \mid |x|_p \leq 1$ for all prime numbers $p \notin S\}$, where $|\cdot|_p$ denotes the $p-$adic norm on $\mathbb{Q}$. Let G be a linear algebraic $\mathbb{Q}$-group (i.e. a linear algebraic group defined over $\mathbb{Q}$). The group $G(\mathbb{Z}(S))$ is called the group of S-integral points of G. We have $G(\mathbb{Z}(S)) = G(\mathbb{A}_{\mathbb{Q}}(S)) \cap G(\mathbb{Q})$ (for definitions, see section 5.4).

Theorem 8. *[PR], Th.7.15.*

Let G be a simply connected simple $\mathbb{Q}$-group, S a finite set of prime numbers and $\Gamma \leq G(\mathbb{Z}(S))$ be a Zariski-dense subgroup of G. Let $\pi_p : G(\mathbb{Z}) \longrightarrow G_p(\mathbb{Z}/p\mathbb{Z})$ denote the reduction map modulo p, where G_p denotes the reduction[9] *of G modulo p. Then, for almost all $p \notin S$, we have*

$$\pi_p(\Gamma) = G_p(\mathbb{Z}/p\mathbb{Z}).$$

Corollary 7. *In the setting above, we have for almost all $p \notin S$:*

$$\Gamma/\Gamma(p) \cong G_p(\mathbb{Z}/p\mathbb{Z}).$$

We present a short sketch of the result in [MOV], Ch.4., which can be considered as a special case of the Corollary 7: [10]

let A be a central simple algebra over an algebraic number field F, Λ an O_F-order in A. If $\mathbf{p}$ is a maximal ideal of O_F, let $F_\mathbf{p}$ and $O_\mathbf{p}$ denote the corresponding $\mathbf{p}$-adic field/ring of integers, and let $\Lambda_\mathbf{p} := \Lambda \otimes_{O_F} O_\mathbf{p}$ and $A_\mathbf{p} := A \otimes_F F_\mathbf{p}$.
For a finite subset of maximal(prime) ideals of O_F, $S \subset V_f^F$, let $\tilde{O}_F(S) := \prod_{\mathbf{p}\in V_f^F\setminus S} O_\mathbf{p}$ and $\tilde{\Lambda}(S) := \prod_{\mathbf{p}\in V_f^F\setminus S} \Lambda_\mathbf{p}$, and let $\tilde{A}(S)$ be the restricted product $\prod'_{\mathbf{p}\in V_f^F\setminus S} A_\mathbf{p}$ with respect to $\Lambda_\mathbf{p}$'s. If $S = \emptyset$, then let $\tilde{O}_F := \tilde{O}_F(\emptyset)$, $\tilde{\Lambda} := \tilde{\Lambda}(\emptyset)$ and $\tilde{A} := \tilde{A}(\emptyset)$.
The kernel of the reduced norm defines $SL_1(A)$, the special linear group of A, and $SL_1(\tilde{A})$ is obtained analogously. Endow each $\Lambda_\mathbf{p}$ with the $\mathbf{p}$-adic topology, $\tilde{\Lambda}$ with the product topology and $\tilde{A}$ with the smallest topology in which $\tilde{\Lambda}$ is an open topological subring. Now, if A is noncommutative and Eichler[11], then $SL_1(A)$ is a simple, simply connected algebraic group (see [Re]) and by the strong approximation theorem, $SL_1(A)$ is dense in $SL_1(\tilde{A})$, a fact which is applied in the proof of the following Lemma:

Lemma 13. *In the setting above, choose $c \in O_F$ s.t. for every prime ideal $\mathbf{p}$ of O_F containing c: $\Lambda_\mathbf{p}$ is a maximal $O_\mathbf{p}$-order in $A_\mathbf{p}$*[12] *and $A_\mathbf{p} \cong M_m(F_\mathbf{p})$ for some $m \in \mathbb{N}$. Then:*

a)

$$\Lambda/c\Lambda \cong_{O_F} M_m(O_F/cO_F)$$

b) If $c_1 \in O_F$ s.t. $O_F = cO_F + c_1O_F$, the composite map

$$1 + \Lambda c_1 \hookrightarrow \Lambda \longrightarrow \Lambda/\Lambda c \longrightarrow M_m(O_F/cO_F)$$

induces a surjective homomorphism

[9] This notion is explained in Appendix, section 5.5.

[10] This result considers arithmetic groups similar to those we have to deal with in the case of quaternion hyperbolic spaces, namely, those arising from orders in quaternion algebras.

[11] Such an algebra A is called Eichler algebra if it is unramified at at least one archimedean place

[12] This is always true if Λ is a maximal O_F-order in A, since in O_F prime ideals are maximal ideals, and then use [Re], Th.11.1 on p.132.

$$SL(1+\Lambda c_1) \longrightarrow SL_m(O_F/cO_F).$$

Proof. (sketch)
The part a) of the Lemma above is a technical application of the Chinese Remainder Theorem. For the part b), note that for a prime ideal $\mathbf{p}$ of O_F containing c, and e the exponent of $\mathbf{p}$ in the prime ideal factorization of the ideal cO_F, the reduction modulo the image of $\mathbf{p}^e$ in $O_{\mathbf{p}}$ is a surjective map $SL_m(O_{\mathbf{p}}) \longrightarrow SL_m(O_{\mathbf{p}}/\mathbf{p}^e)$, hence using the Chinese Remainder Theorem the reduction modulo c induces a surjective map φ at the bottom of the following diagram:

$$\begin{tikzcd}
SL_1(1+\Lambda c_1) \arrow[r] \arrow[d, "i"'] & SL_m(O_F/cO_F) \arrow[d, "\cong"] \\
SL_1(1+\tilde{\Lambda} c_1) \arrow[r, "\varphi"] & SL_m(\tilde{O}_F/c\tilde{O}_F)
\end{tikzcd}$$

We have $SL_1(1+\Lambda c_1) = SL_1(A) \cap SL_1(1+\tilde{\Lambda} c_1)$. Since $SL_1(1+\tilde{\Lambda} c_1)$ is an open subset of $SL_1(\tilde{A})$ and $SL_1(A)$ is a dense subset of $SL_1(\tilde{A})$, we have that $SL_1(1+\Lambda c_1)$ is dense in $SL_1(1+\tilde{\Lambda} c_1)$.
Since the group $SL_m(\tilde{O}_F/c\tilde{O}_F)$ is finite, we have that $\mathrm{Ker}\varphi$ is open in $SL_1(1+\tilde{\Lambda} c_1)$ and so the cosets of $SL_1(1+\tilde{\Lambda} c_1)$ modulo $\mathrm{Ker}\varphi$ are open in $SL_1(1+\tilde{\Lambda} c_1)$, and since $SL_1(1+\Lambda c_1)$ is dense in $SL_1(1+\tilde{\Lambda} c_1)$, we have that every such coset contains an element of $SL_1(1+\Lambda c_1)$. Thus, $\varphi \circ i$ is surjective, and so is $SL_1(1+\Lambda c_1) \longrightarrow SL_m(O_F/cO_F)$.

□

2.3 A result of I. Agol on bounds for Betti numbers

Here, we just make a brief note on the result of I. Agol ([Ag]). There, under the assumption that there is a non G-invariant cohomology class in $H^k(\Gamma\backslash X)$, where $\Gamma\backslash X$ is a compact locally symmetric space, it is shown how to produce subgroups of Γ s.t. the corresponding locally symmetric space has arbitrary large k-th Betti number. This is proven as follows:

if $\dim H^k(\Gamma\backslash X) > \dim\ H^k_G(X)$, [13] then one can find a non G-invariant, Γ-invariant harmonic form ω on X (obtained as a pullback of a non G-invariant harmonic k-form on $\Gamma\backslash X$). Then consider the span of the form ω by all the elements in the commensurator of Γ:

$$V := < g_*(\omega) | g \in Comm(\Gamma) > .$$

Now, the author shows that $\dim V = \infty$, hence for all $i \geq 1$ we can find $g_1, \ldots, g_i \in Comm(\Gamma)$ s.t. $g_{1*}(\omega), \ldots, g_{i*}(\omega)$ are linearly independent. Let

$$\Gamma_i := \bigcap_{j=1,\ldots,i} g_j \Gamma {g_j}^{-1}.$$

Then $g_{1*}(\omega), \ldots, g_{i*}(\omega)$ are linearly independent Γ_i-invariant forms, and hence

$$dim H^k(\Gamma_i\backslash X) > i.$$

[13] Here, $H^k_G(X)$ denotes the space of G-invariant harmonic forms on X.

Chapter 3

An Example: Quaternion hyperbolic manifolds

3.1 Description of the quaternion hyperbolic space

In the section 1.4.1 we have seen the CW-structure (i.e. the cell complex structure) of the projective quaternion space. The duality between the real projective and real hyperbolic spaces extends to the complex and quaternion case. Concretely, in the symmetric space description of a quaternion hyperbolic space G/K resp. quaternion projective space G_u/K we have that the group G_u is the compact dual group of G. We have already seen the example of the use of this duality in section 1.4.3. Now, we construct and describe the quaternion hyperbolic space as a Riemannian symmetric manifold.

3.1.1 Standard model of quaternion hyperbolic spaces

Let $\mathbb{H}$ be the skewfield of quaternions over $\mathbb{R}$ with basis $1, i, j, k$ subject to the equations $i^2 = -1$, $j^2 = -1$, $k = ij = -ji$. Let $\mathbb{H}^{n+1}$ be a right $\mathbb{H}$-vector space. Let $x \mapsto \overline{x}$ be the *standard symplectic (anti-)involution* on quaternions given by:

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \overline{x_0 + x_1 i + x_2 j + x_3 k} := x_0 - x_1 i - x_2 j - x_3 k.$$

Define a quadratic form $\mathbf{b} :=< \cdot, \cdot >: \mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \longrightarrow \mathbb{H}$ on $\mathbb{H}^{n+1}$ by:

$$< a, b >:= \sum_{k=1}^{n} \overline{a_k} b_k - \overline{a_{n+1}} b_{n+1},$$

where $a = (a_1, \dots, a_{n+1}), b = (b_1, \dots, b_{n+1}) \in \mathbb{H}^{n+1}$.
Then $< ., . >$ is a Hermitian bilinear form.[1] Note that for $x := x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ we have the reduced norm $\mathbf{n} : \mathbb{H} \longrightarrow \mathbb{R}$, $\mathbf{n}(x) := x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 \in \mathbb{R}$. Now, for $a = (a_1, \dots, a_{n+1})$ we have $< a, a >= \sum_{i=1}^{n} \mathbf{n}(a_i) - \mathbf{n}(a_{n+1}) \in \mathbb{R}$. The *quaternion hyperbolic space of dimension* n is defined as the quotient

$$H^n(\mathbb{H}) := \{a \in \mathbb{H}^{n+1} | < a, a > < 0\}/\mathbb{H}^*,$$

[1] The general notion of a *Hermitian form* on quaternion algebras will be defined in section 3.2.1

where $\lambda \in \mathbb{H}^*$ acts on $a = (a_1, \ldots, a_{n+1}) \in \mathbb{H}^{n+1}$ by $a \cdot \lambda := (a_1\lambda, \ldots, a_{n+1}\lambda)$. Let $[x] = x\mathbb{H}^*$ denote the image of $x \in \mathbb{H}^{n+1}$ under the identification with respect to the above described action of $\mathbb{H}^*$.
Let $||x|| := \sqrt{< x, x >}$ denote the norm of $x \in \mathbb{H}^{n+1}$, and let $|q| := \sqrt{\mathbf{n}(q)}$ for $q \in \mathbb{H}$; then

$$d([x],[y]) := arccosh(\frac{|< x, y >|}{||x|| \cdot ||y||})$$

is a well defined distance function on $H^n(\mathbb{H})$, i.e. defines a metric on $H^n(\mathbb{H})$.

Since we consider $\mathbb{H}^{n+1}$ as a right vector space over $\mathbb{H}$, then the group $GL_{n+1}(\mathbb{H})$ acts on $\mathbb{H}^{n+1}$ in a natural way. Let G be the subgroup of $GL_{n+1}(\mathbb{H})$ preserving the hermitian form $\mathbf{b}$. Then

$$G = \{g \in GL_{n+1}(\mathbb{H}) \mid g^* \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}\},$$

where $g^* := \overline{g}^t$ is the matrix obtained by replacing every entry g_{ij} of g by $\overline{g_{ij}}$ and taking the transpose matrix.
The group G obtained in this way is the quaternion-analogue of the groups $O(n,1)$ and $U(n,1)$ [2] and is denoted by $Sp(n,1)$. It can easily be seen that $G \leq SL_{n+1}(\mathbb{H})$, where $SL_{n+1}(\mathbb{H})$ is defined as the group of matrices in $GL_{n+1}(\mathbb{H})$ having *reduced norm* 1. [3]
Let $Sp(n)$ denote the subgroup of $GL_n(\mathbb{H})$ acting on $\mathbb{H}^n$ as a right vector space over $\mathbb{H}$ which preserves the quadratic form $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \mapsto \sum_{k=1}^n \overline{a_k} b_k$ on $\mathbb{H}^n$. The group $Sp(n)$ is the analogue of the groups $O(n)$ and $U(n)$.

The action of the group $G = Sp(n,1)$ on $\mathbb{H}^{n+1}$ induces the action on $H^n(\mathbb{H})$ via $[x] \mapsto [Ax]$. The latter action is transitive, which is ensured by quotienting $\{a \in \mathbb{H}^{n+1} | < a, a > < 0\}$ modulo $\mathbb{H}^*$.
Let $K \leq G$ be a subgroup of G which leaves the point $[(0, \ldots, 0, 1)] = (0, \ldots, 0, 1)\mathbb{H}^*$ fixed. Then $K \cong Sp(n) \times Sp(1) \leq G$, and the quaternion hyperbolic space $H^n(\mathbb{H})$ can be identified with G/K:

$$H^n(\mathbb{H}) \cong Sp(n,1)/(Sp(n) \times Sp(1)).$$

We shall describe in the next section the geometry of the $H^n(\mathbb{H})$ induced by this description.

Before we proceed with the standard description of the quaternion hyperbolic space, we note that there are some alternative descriptions isometric to the standard one:

Remark 7. ***The upper half-space model in*** $\mathbb{C}^{2n}$.
One can realise the quaternion hyperbolic space as the upper half space in $\mathbb{C}^{2n}$, $\{z = (z_1, \ldots, z_{2n}) || Im(z_1) > 0\}$ *endowed with a Riemannian metric*

[2] For the definition and properties of this groups, see f.e. [Hel].
[3] We have $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$, hence there is an isomorphism $M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2n}(\mathbb{C})$, and the reduced norm of $A \in M_n(\mathbb{H})$ is defined as the determinant of the image of A in $M_{2n}(\mathbb{C})$ under this isomorphism (see f.e. [Re], §9a).
Note that here lies one of the differences between the real/complex case and the quaternion case. Foe example, we have $det(-\mathbb{I}_n) = -1$, but $n_{red}(-\mathbb{I}_n) = 1$.

$$ds^2(z) = \frac{(dy)^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n}(dz_k d\overline{z}_k + dz_{n+k}d\overline{z}_{n+k}) +$$

$$+\frac{1}{y^4}(dx + Im(\sum_{k=2}^{n}(\overline{z}_k dz_k + \overline{z}_{n+k}dz_{n+k})))^2 +$$

$$+\frac{1}{y^4}|dz_{n+1} + \sum_{k=2}^{n}(z_{n+k}dz_k - z_k dz_{n+k})|,$$

where $z_1 = x + i \cdot y$, $z_k = x_k + i \cdot y_k$ for $k \geq 2$. The Riemann volume element is given by

$$dV(z) = y^{-(4n+3)}dydx \prod_{k=2}^{2n} dx_k dy_k$$

For details, see [Ve].

The following description can be found in [Vol]:

Remark 8. ***An analogue of the Poincaré model***
Let $x := x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$. We have $x = (x_0 + x_1 i) + (x_2 + x_3 i)j$, where $x_0 + x_1 i$ and $(x_2 + x_3 i)$ can be considered as elements in $\mathbb{C}$.
This identification induces the identification of $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ if we identify $a := (a_1, \dots a_n) \in \mathbb{H}^n$ with $(\alpha_1, \dots \alpha_{2n}) \in \mathbb{C}^{2n}$ where we set $a_k = \alpha_k + \alpha_{n+k}j$ as above with $\alpha_k, \alpha_{n+k} \in \mathbb{C}$.
This induces the representation of the general linear group $GL_n(\mathbb{H})$ in $GL_{2n}(\mathbb{C})$:

$$A + Bi \mapsto \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}$$

Let $B^{4n} := \{a \in \mathbb{H}^n | ||a||_{\mathbb{H}} < 1\} = \left\{z \in \mathbb{C}^{2n} | ||z||_{\mathbb{C}} < 1\right\}$ with the above identification.
For $w := (w_1, \dots, w_{2n}), z := (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$, we have the distance function

$$d(z,w) = \frac{1}{2}ln(\frac{|1- <z,w>| + \sqrt{|w-z|^2 + |<w,z>|^2 - |z|^2|w|^2}}{|1- <z,w>| - \sqrt{|w-z|^2 + |<w,z>|^2 - |z|^2|w|^2}}),$$

which defines the metric on B^{4n}. With this metric, B^{4n} is isometric to the quaternion hyperbolic space (as described in the next section).

3.1.2 The group $Sp(n,1)$ and quaternion hyperbolic space as a Riemannian symmetric space

The group $Sp(n,1)$ belongs to the class of non-compact *simple real Lie groups* [4] (see [WM], p.23., Ex.3.23).

[4] A Lie group G is called *simple* if it is not abelian and if it contains no nontrivial, connected, closed, proper, normal subgroups. Note that this definition allows the existence of discrete normal subgroups.

The Lie algebra of the Lie group $Sp(n,1)$ is identified with

$$sp(n,1) := \{X \in M_{n+1}(\mathbb{H}) \mid X^* \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} X, Re(tr(X)) = 0\},$$

where $Re(tr(X))$ denotes the real part of the (quaternion) trace of the matrix X. $sp(n,1)$ is a real simple Lie algebra with the Cartan involution $X \mapsto X^* \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} X$. In the corresponding Cartan decomposition $sp(n,1) = \mathbf{k} \oplus \mathbf{p}$ we have $\mathbf{k} \cong sp(n) \oplus sp(1)$, where $sp(n) := \{X \in M_n(\mathbb{H}) \mid X^* = -X, Re(tr(X)) = 0\}$ is the Lie algebra of the Lie group $Sp(n)$. $\mathbf{k}$ is the Lie algebra of $Sp(n) \times Sp(1)$, which is a maximal compact subgroup of $Sp(n,1)$.

Now consider the group $Sp(1)$. It is defined as the subgroup of $GL_1(\mathbb{H}) \cong \mathbb{H}^*$ which preserves the form $\mathbb{H} \longrightarrow \mathbb{H}$, $x \mapsto x\overline{x} = \mathbf{n}(x)$. Since the norm $\mathbf{n}$ is a multiplicative function, we can identify

$$Sp(1) \cong \mathbb{H}^1,$$

where $\mathbb{H}^1$ is the multiplicative group of quaternions of norm 1. We have $\mathbb{H}^1 = \{x = (x_0 + x_1 i + x_2 j + x_3 k) \in \mathbb{H} | x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$, and this gives the identification of $Sp(1)$ with the 3-sphere in $\mathbb{R}^4$:

$$Sp(1) \cong S^3,$$

i.p. $Sp(1)$ is simply connected and compact.
Similarly, $Sp(n)$ acts transitively on a sphere S^{4n-1}, and the isotropy of $(1,0,\ldots,0)$ can be identified with $Sp(n-1)$. Hence, $Sp(n)/Sp(n-1)$ can be identified with the sphere S^{4n-1}. By induction (since $Sp(1)$ is simply connected, and so are all the spheres S^k for $k \geq 2$), this implies the following (similarly, using Hopf fibrations, all of the homotopy groups of orthogonal and unitary groups can be described, see [Ha], p.383-384.) :

Corollary 8. *$Sp(n)$ is simply connected for all $n \in \mathbb{N}$.*

Since $Sp(n,1)$ is diffeomorphic to $(Sp(n) \times Sp(1)) \times \mathbb{R}^{dim\mathbf{p}}$ (see Appendix, Theorem 9), we have that $\pi_1(Sp(n,1)) \cong \pi_1(Sp(n) \times Sp(1)) \times \pi_1(\mathbb{R}^{dim\mathbf{p}}) \cong 0$, thus:

Corollary 9. *$Sp(n,1)$ is a simply connected Lie group for all $n \in \mathbb{N}$.*

We describe briefly how the quaternion hyperbolic space $H^n(\mathbb{H})$, now described as the symmetric space G/K becomes a Riemannian manifold:

let $\mathbf{g} = sp(n,1)$ and $\mathbf{k} \cong sp(n) \oplus sp(1)$ be the Lie algebras of G and K, respectively, and let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the Cartan decomposition of $\mathbf{g}$. Since $G = K \cdot exp(\mathbf{p})$ (again, as in Theorem 9 in Appendix), we have:

$$T_{eK}(G/K) \cong \mathbf{p}.$$

A multiple of the Killing form on $\mathbf{p}$ induces a positive definite scalar product on $T_{eK}(G/K) \cong \mathbf{p}$, which will define a G-invariant Riemannian metric on G/K. This metric is described as follows:

using the standard identification $\mathbb{H} \cong \mathbb{C}^2$ as in Remark 8, we have the following descriptions (see f.e. [Hel2], Ch.X.):

$$sp(n,1) = \{ \begin{bmatrix} A & c & B & d \\ \overline{c}^t & u & d^t & v \\ -\overline{B} & \overline{d} & \overline{A} & -\overline{c} \\ \overline{d}^t & -\overline{v} & -c^t & \overline{u} \end{bmatrix}, A \in u(n), B \in Symm(n,\mathbb{C}), c, d \in \mathbb{C}^n, u \in u(1), v \in \mathbb{C}\},$$

$$\mathbf{k} = \{ \begin{bmatrix} A & o & B & 0 \\ 0 & u & 0 & v \\ -\overline{B} & 0 & \overline{A} & 0 \\ 0 & -\overline{v} & 0 & \overline{u} \end{bmatrix}, A \in u(n), B \in Symm(n,\mathbb{C}), u \in u(1), v \in \mathbb{C}\},$$

$$\mathbf{p} = \{ \begin{bmatrix} 0 & c & 0 & d \\ \overline{c}^t & 0 & d^t & 0 \\ 0 & \overline{d} & 0 & -\overline{c} \\ \overline{d}^t & 0 & -c^t & 0 \end{bmatrix}, c, d \in \mathbb{C}^n\}.$$

The Killing form on $\mathbf{g}$ is given by $\kappa(X,Y) := 2(n+2)tr_{\mathbb{C}}(XY)$; and one further constructs a bilinear form $\tilde{\kappa}(X,Y) := \frac{1}{4}tr_{\mathbb{C}}(XY)$, which is positive definite on $\mathbf{p} \cong T_{eK}(G/K)$ and induces a G-invariant Riemannian metric on $H^n(\mathbb{H})$. For further description of the geometric properties of $H^n(\mathbb{H})$ with the above constructed Riemannian metric see [Pe].
From the description above, one can easily read off that the dimension of $\mathbf{p}$ equals $4n$ and we have that $H^n(\mathbb{H})$ is a simply connected Riemannian symmetric space of non-compact type of dimension $4n$.
We note that the compact group $Sp(n+1)$ is the *compact dual*[5] of $Sp(n,1)$. The corresponding symmetric space $Sp(n+1)/(Sp(n)\times Sp(1))$ associated to $Sp(n+1)$ is the compact symmetric space $P^n(\mathbb{H})$, called the *quaternion projective space*.

Now we want to pass from symmetric space X to locally symmetric spaces. We have to find certain subgroups of the group of isometries of X acting freely and properly discontinuously on the symmetric space X and consider the quotient space:

[5]By a *compact dual* of a non compact real Lie group G having K as a maximal compact subgroup of G, we mean the maximal compact subgroup G_u of the complexification $G \otimes \mathbb{C}$ containing K. On a Lie algebra level, if $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ is the Cartan decomposition of the Lie algebra $\mathbf{g}$ of G, then the Lie algebra of G_u is $\mathbf{k} \oplus i\mathbf{p}$.

3.2 Description of arithmetic groups acting on $H^n(\mathbb{H})$

3.2.1 Arithmetic lattices in $Sp(n,1)$

The theory of arithmetic lattices in classical groups is described in [WM]. In order to avoid repetition, we just state the result which is of importance to us ([WM] p.152., Cor.10.58.):

for a field F; $a,b \in F^*$ let $Q(a,b|F)$ denote the quaternion algebra over F with basis $1,i,j,k$; where the defining equations are $i^2 = a$, $j^2 = b$, $k = ij = -ji$. The *standard symplectic (anti-)involution*[6] τ_c on $Q(a,b|F)$ is given by

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto x_0 - x_1 i - x_2 j - x_3 k.$$

A *τ_c-Hermitian form* on $Q(a,b|F)^n$ is a map $B : Q(a,b|F)^n \times Q(a,b|F)^n \longrightarrow Q(a,b|F)$ s.t.:
* $B(x+y,z) = B(x,z) + B(y,z)$ and $B(x,y+z) = B(x,y) + B(x,z)$ for all $x,y,z \in Q(a,b|F)^n$,
* $B(\lambda x, y) = \lambda B(x,y)$, $B(x,\lambda y) = B(x,y)\tau_c(\lambda)$ for all $\lambda \in Q(a,b|F)$, $x,y \in Q(a,b|F)^n$
* $B(y,x) = \tau_c(B(x,y))$ for all $x,y \in Q(a,b|F)^n$

The Hermitian form B is *non-degenerate* if for each nonzero $x \in Q(a,b|F)^n$ there exists $y \in Q(a,b|F)^n$ s.t. $B(x,y) \neq 0$. For such a form, let $[B] := [(B(e_i,e_j))_{i,j}]$ be the corresponding matrix of B for a (standard) $Q(a,b|F)$-basis $\{e_1, \dots e_n\}$ of $Q(a,b|F)^n$. Then the group of isometries $SU(B; Q(a,b|F), \tau_c)$ of B in $SL_n(Q(a,b|F))$ is given by

$$SU(B; Q(a,b|F), \tau_c) = \{g \in SL_n(Q(a,b|F)) \mid \tau_c(g^t)[B]g = [B]\}.$$

An *O_F-order* Λ in $Q(a,b|F)$ is a subring of $Q(a,b|F)$ with $1 \in \Lambda$ s.t. Λ is a *full O_F- lattice* in $Q(a,b|F)$, i.e. a finitely generated O_F-submodule with $F \cdot \Lambda = Q(a,b|F)$.[7]

Proposition 21. *Let Γ be an arithmetic lattice in $G = Sp(p,q)$,[8] for some $p,q \geq 1$. Then there exist:*

** a totally real algebraic number field F*

** $a,b \in F$ such that for every place σ of F: $\sigma(a), \sigma(b) < 0$*

** $\beta_1, \dots, \beta_{p+q} \in F$ such that $\beta_1, \dots \beta_p$ are positive, $\beta_{p+1}, \dots, \beta_{p+q}$ are negative and for each place $\sigma \neq Id$ of F, the numbers $\sigma(\beta_1), \dots, \sigma(\beta_{p+q}) \in \mathbb{R}$ are either all positive or all negative*

** an isomorphism $\phi : SU(B; D := Q(a,b|F), \tau_c) \otimes_F \mathbb{R} \to Sp(p,q)$, where B is the τ_c-hermitian form on D^{p+q} defined by*

$$B(x,y) := \sum_{i=1}^{p+q} \beta_i x_i \tau_c(y_i)$$

[6] This map is sometimes referred to as the *conjugation* or the *standard antiinvolution* of $Q(a,b|F)$.

[7] For the existence and properties of orders, see [Re].

[8] For the definition of the group $Sp(p,q)$ see f.e. [WM], §3D.

such that the image of $SU(B;\Lambda,\tau_c)$ under ϕ is commensurable with Γ for any O_F- order Λ in D .

We shall construct examples of cocompact arithmetic subgroups of $Sp(n,1)$ in the following way:

let p be a prime, then $F=\mathbb{Q}(\sqrt{p})$ is a totally real number field, having the Galois group $Gal_{F/\mathbb{Q}}=\{Id,\sigma\}$, where σ is given by $\sqrt{p}\mapsto-\sqrt{p}$. Choose $a,b\in F$ s.t. $a,b;\sigma(a),\sigma(b)$ are all negative and such that $D:=Q(a,b|F)$ is a quaternion skewfield over F [9]. Let τ_c be the standard symplectic involution on D, and B=$<\cdot,\cdot>$ a Hermitian form on D^{n+1} given by the matrix $\begin{bmatrix} I_n & 0 \\ 0 & -\sqrt{p}\end{bmatrix}$, i.e.

$$<x,y>:=\sum_{i=1}^{n} x_i\tau_c(y_i)-\sqrt{p}x_{n+1}\tau_c(y_{n+1}).$$

Consider the group[10]

$$G(F):=SU(B;D,\tau_c)=\{g\in SL_{n+1}(D)|\tau_c(g)^t\begin{bmatrix} I_n & 0 \\ 0 & -\sqrt{p}\end{bmatrix}g=\begin{bmatrix} I_n & 0 \\ 0 & -\sqrt{p}\end{bmatrix}\}.$$

Then G is an algebraic group defined over F and since D *does not split*[11] over $\mathbb{R}$, we have (see [WM], Prop.10.55.):

$$G\otimes\mathbb{R}\cong Sp(n,1)$$

Furthermore, the σ- conjugate B^σ of the form B is given by a definite form represented by the matrix $\begin{bmatrix} I_n & 0 \\ 0 & \sqrt{p}\end{bmatrix}$, which (since a,b are chosen so that $D\otimes_F F_\sigma\cong Q(\sigma(a),\sigma(b)|\mathbb{R})\cong\mathbb{H}$) implies that the group G^σ is isomorphic to $Sp(n+1)$.[12]
We have

$$G(\mathbb{R})\cong G^{Id}\times G^\sigma\cong Sp(n,1)\times Sp(n+1),$$

and choosing a maximal O_F- order Λ in D,

$$\Gamma:=SU(B;\Lambda,\tau_c)$$

embeds in $G^{Id}\cong Sp(n,1)$ as a cocompact arithmetic lattice by Proposition 6.49. in [WM].

---

[9] We can choose $a=b=-1$ or any $a,b\in\mathbb{Q}\subset F$ s.t. $a,b<0$. In that case, since $F\subset\mathbb{R}$, the equation $x^2-ay^2-bz^2=0$ has no nontrivial solution in F^3, which implies that $Q(a,b|F)$ is a skewfield (see f.e. §10.7, Lemma on p.86. in [Ke]).

[10] Here, we abuse notation and actually specify the F-points $G(F)$ of the group G. This does not induce complications and inconsistencies, since in order to specify a group G that is defined over F, it suffices to specify $G(F)$.

[11] By this we mean: $D\otimes_F\mathbb{R}\cong\mathbb{H}$.

[12] For a place σ of F, the group G^σ is given the set of solutions to $\tau_c(g^t)[B^\sigma]g=[B^\sigma]$ in $SL_n(D\otimes_F F_\sigma)$, where F_σ is the completion of F with respect to the embedding $\sigma:F\hookrightarrow\mathbb{C}$. In our case, since F is totally real, $Gal_{F/\mathbb{Q}}=\{Id,\sigma\}$ we have $F_{Id}=F_\sigma=\mathbb{R}$ and by the choice of a,b we have for $D=Q(a,b|F)$: $D\otimes_F F_{Id}\cong D\otimes_F F_\sigma\cong\mathbb{H}$.

Remark 9. *Using Proposition 21 above, we see that any arithmetic lattice in* $Sp(n, 1)$ *is commensurable to a lattice constructed similarly as in our example. The choice of* F *and* D *are the only arithmetically relevant informations, not the choice of* Λ *itself. However, we choose to deal with the maximal orders* Λ, *since maximal orders are much more convenient to deal with - the arithmetic of the prime ideals in maximal orders fully resembles the arithmetic of Dedekind rings, and enables some direct elementary calculations.*

3.2.2 Torsion-free criterion for SL_n over a maximal arithmetic order

In order to ensure that the quotient space $\Gamma\backslash X$ of a symmetric space under a discrete subgroup of isometries is a manifold, we need to ensure that there are no nontrivial elements of finite order in Γ. Otherwise, the quotient is an *orbifold*[13].
Since the groups Γ we are dealing with are subgroups of $SL_n(\Lambda)$ for some maximal order Λ in a quaternion skewfield over a number field, the following criterion for groups arising from orders in skewfields applies:

let D be a (central, simple) skewfield over a (totally real) algebraic number field F. Let O_F be the ring of integers of F and Λ a maximal O_F-order in D. According to [Re], p.193., Theorem 22.10. the arithmetic of the two-sided ideals of Λ fully resembles the case of Dedekind rings, in the sense that we have a unique factorization of a two-sided ideal as a product of prime ideals.
By an *ideal* β *of* Λ we mean a *full* O_F*-lattice in* D[14] contained in Λ which is a two-sided ideal in Λ in the usual ring-theoretic sense. By a *prime ideal* $\mathbf{p}$ *of* Λ we mean an ideal of Λ s.t. for all ideals α, β of Λ with $\alpha\beta \subseteq \mathbf{p}$ we have $\alpha \subseteq \mathbf{p}$ or $\beta \subseteq \mathbf{p}$.

Remark 10. *Since D is a skewfield, every two-sided ideal (in a ring-theoretic sense) in Λ is a full O_F-lattice so the notion "ideal of Λ" applies to two-sided ideals in Λ in a ring-theoretic sense. This can be seen in a following way:*

let $\alpha \subset \Lambda$ be a nonzero ideal in Λ in a ring theoretic sense. Then there exists a non-zero $x \in \alpha$. Since D is a skewfield, there exists an inverse x^{-1} of x in D. Now, $D = F \cdot \Lambda$, so there exist finitely many $k_i \in F$, $\lambda_i \in \Lambda$ s.t. $x^{-1} = \sum_i k_i\lambda_i$. Since F is the field of fractions of O_F, we can write $k_i = \frac{r_i}{s_i}$ with $r_i, s_i \in O_F$. Let $s := \prod_i s_i \neq 0$. Then $sx^{-1} = \sum_i r_i\lambda_i \in \Lambda$. I.p. $sx^{-1}x = s \in \alpha \cap O_F$. Hence, in the case that D is a skewfield, every (ring-theoretic) ideal in Λ contains a non-zero element of O_F.
Now let $y \in D$. Since $F \cdot \Lambda = D$, there exist finitely many $f_i \in F$, $\lambda_i \in \Lambda$ s.t. $y = \sum_i f_i\lambda_i$. For $s \neq 0$, $s \in \alpha \cap O_F$ we have

$$y = \sum_i \frac{f_i}{s}(s\lambda_i),$$

with $\frac{f_i}{s} \in F$, $s\lambda_i \in \alpha$, hence $F \cdot \alpha = D$.

Let Γ be a subgroup of $SL_n(\Lambda)$. For an ideal β of Λ let

$$\Gamma(\beta) := \{\gamma \in \Gamma \mid \gamma \equiv I \pmod{\beta}\ \}.$$

Let $\beta := \mathbf{p}_1^{k_1}\mathbf{p}_2^{k_2}\dots\mathbf{p}_r^{k_r}$, where $k_i \geq 1$ for all $i = 1, \dots, r$, be the unique decomposition of β into distinct prime ideals $\mathbf{p}_i$ of Λ (see [Re], Th. 22.10.). The intersection of $\mathbf{p}_i$ with $\mathbb{Z}$ is a prime ideal, hence we have $\mathbf{p}_i \cap \mathbb{Z} = p_i\mathbb{Z}$ for a corresponding prime number p_i. Note that although $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$ we can have $p_i = p_j$.

[13]An *orbifold* is a generalization of a manifold, in the sense that every point has an open neighborhood homeomorphic to the quotient of an Euclidean space under the linear action of a finite group.
[14]I.e. a finitely generated O_F-submodule containing an F-basis of D.

For such a Γ, its congruence torsion-free subgroups can be determined by the following:

Lemma 14. *Let Γ be a subgroup of $SL_n(\Lambda)$. Let $\beta := \mathbf{p}_1^{k_1}\mathbf{p}_2^{k_2}\ldots\mathbf{p}_r^{k_r}$ be an ideal of Λ. Suppose either that there exist $i, j \in \{1, \ldots, r\}$, $i \neq j$ s.t. $p_i \neq p_j$ or that there exists a single $j \in \{1, \ldots, r\}$ such that*

$$exp_{\mathbf{p}_j}(p_j) < k_j(p_j - 1),$$

where $exp_{\mathbf{p}_j}(p_j)$ is the exponent of $\mathbf{p}_j$ in the prime ideal decomposition of the two sided ideal $< p_j >_\Lambda = \Lambda p_j$ of Λ generated by p_j.[15]
Then $\Gamma(\beta)$ is torsion-free.

Proof. We show that given a prime number k and $\gamma \in \Gamma(\beta)$, $\gamma \neq I$ we have $\gamma^k \neq I$, hence $\Gamma(\beta)$ does not contain any nontrivial element of finite order.[16]
Let $\gamma \in \Gamma(\beta)$ and write

$$\gamma = I + B$$

with .

$$B \equiv 0 \pmod{\beta^m}$$
$$B \not\equiv 0 \pmod{\beta^{m+1}}$$

for some $m \geq 1$. I.p., we have for $j \in \{1, \ldots, r\}$:

$$B \equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j}}$$
$$B \not\equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j + 1}}$$

for some $l_j \geq 0$.
Now suppose $\gamma^k = I$ for some prime number $k \in \mathbb{Z}$. If there exist $i, j \in \{1, \ldots, r\}$, $i \neq j$ s.t. $p_i \neq p_j$ then the proof reduces to the case 1^0 below using congruences modulo $\mathbf{p}_i$ resp. $\mathbf{p}_j$ if $k = p_j$ resp. $k = p_i$. For $k \notin \{\mathbf{p}_i, \mathbf{p}_j\}$ we can use congruences modulo any of the prime ideals $\mathbf{p}_i$,$\mathbf{p}_j$. Otherwise, suppose there exists $j \in \{1, \ldots, r\}$ such that $exp_{\mathbf{p}_j}(p_j) < k_j(p_j - 1)$. Fix this j and consider the following two cases:

CASE 1^0: $k \neq p_j$
Now:

$$\gamma^k = (I + B)^k = I + k \cdot B + \binom{k}{2} \cdot B^2 + \ldots + B^k,$$

and since

$$B^2 \equiv 0 \pmod{\mathbf{p}_j^{2(k_j m + l_j)}}$$
$$B^3 \equiv 0 \pmod{\mathbf{p}_j^{3(k_j m + l_j)}}$$
$$\ldots$$

we have $\binom{k}{2} \cdot B^2 + \ldots + B^k \equiv 0 \bmod \mathbf{p}_j^{k_j m + l_j + 1}$, and:

$$\gamma^k \equiv I + k \cdot B \pmod{\mathbf{p}_j^{k_j m + l_j + 1}}.$$

[15] This number is uniquely determined by the following property: $(\mathbf{p}_j)^{exp_{\mathbf{p}_j}(p_j)+1} \subsetneq < p_j >_\Lambda \subseteq (\mathbf{p}_j)^{exp_{\mathbf{p}_j}(p_j)}$.
[16] It is important for the proof that we assume that k is a prime number, since then k divides $\binom{k}{j}$ for all $j = 1, ..., k-1$.

Now if $\gamma^k = I$, we get:

$$k \cdot B \equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j + 1}},$$

so there exists an entry b_{ij} of B such that $b_{ij} \in \mathbf{p}_j^{(k_j m + l_j)}$, $b_{ij} \notin \mathbf{p}_j^{k_j m + l_j + 1}$ but $k \cdot b_{ij} \in \mathbf{p}_j^{k_j m + l_j + 1}$.
Since k is contained in the center of Λ, we have

$$< k \cdot b_{ij} >_\Lambda = < k >_\Lambda \cdot < b_{ij} >_\Lambda,$$

where $< x >_\Lambda$ is the two sided ideal of Λ generated by x. This implies:

$$\mathbf{p}_j^{k_j m + l_j + 1} \mid < k >_\Lambda \cdot < b_{ij} >_\Lambda$$

and since $< b_{ij} >_\Lambda$ is divisible[17] by $\mathbf{p}_j^{(k_j m + l_j)}$, but not by $\mathbf{p}_j^{k_j m + l_j + 1}$, we must have $\mathbf{p}_j \mid < k >_\Lambda$, a CONTRADICTION since k is a prime number other than p_j which implies $k \notin \mathbf{p}_j$.

CASE 2^0: $k = p_j$
Let $e := exp_{\mathbf{p}_j}(p_j)$ be the exponent of $\mathbf{p}_j$ in the prime-ideal factorization of $< p_j >_\Lambda = < k >_\Lambda$, i.e. $\mathbf{p}_j^e \mid < k >_\Lambda$ but $\mathbf{p}_j^{e+1}$ does not divide $< k >_\Lambda$.
Now:

$$\gamma^k = (I + B)^k = I + k \cdot B + \binom{k}{2} \cdot B^2 + \binom{k}{3} \cdot B^3 + \ldots + \binom{k}{k-1} \cdot B^{k-1} + B^k$$

We have $k \mid \binom{k}{r}$ for all $r = 2 \ldots k-1$ since k is a prime number. Now, $B^r \equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j + 1}}$ for all $r = 2 \ldots k-1$, so

$$\binom{k}{j} \cdot B^j \equiv 0 \pmod{\mathbf{p}_j^{m + l_j + 1 + e}} \text{ for all } j = 2 \ldots k-1.$$

We have $B^k \equiv 0 \pmod{\mathbf{p}_j^{k(k_j m + l_j)}}$, and the condition $e < k_j(k-1)$, (here $k = p_j$), implies $k_j m + l_j + 1 + e \leq k(k_j m + l_j)$, since

$$k_j m + l_j + 1 + e \leq k(k_j m + l_j) \Leftrightarrow (k_j m + l_j)(k-1) \geq e + 1,$$

and the minimum of the term $(k_j m + l_j)(k-1)$ is achieved when $m = 1$, $l_j = 0$; in that case the last inequality becomes equivalent to $e < k_j(k-1)$.
Hence we have

$$\binom{k}{2} \cdot B^2 + \binom{k}{3} \cdot B^3 + \ldots + \binom{k}{k-1} \cdot B^{k-1} + B^k \equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j + 1 + e}}$$

which implies that

$$\gamma^k \equiv I + k \cdot B \pmod{\mathbf{p}_j^{k_j m + l_j + 1 + e}}.$$

Now we proceed as in Case **1'**: under the assumption that $\gamma^k = I$ we have $k \cdot B \equiv 0 \pmod{\mathbf{p}_j^{k_j m + l_j + 1 + e}}$, so we find b_{ij}, entry of the matrix B such that $b_{ij} \in \mathbf{p}_j^{k_j m + l_j}$, $b_{ij} \notin \mathbf{p}_j^{k_j m + l_j + 1}$ but $k \cdot b_{ij} \in \mathbf{p}_j^{k_j m + l_j + 1 + e}$ and analogously we get that $\mathbf{p_j}^{e+1} \mid < k >_\Lambda$, a CONTRADICTION.

□

[17] Here we use the term *divisible* in a ideal-theoretic sense, $\alpha \mid \beta$ means that $\beta \subset \alpha$.

The criterion above enables us to deal with congruence subgroups of Γ constructed directly by means of ideals in Λ. Usually, Γ is embedded in $SL_m(O_F)$ for a suitable $m \in \mathbb{N}$ and the congruence subgroups are constructed modulo the ideals in O_F. However, the distinction in using the ideals of O_F and those of Λ is purely technical (although the above criterion captures a wider range of torsion-free subgroups than the one based on the same idea and dealing with prime ideals of O_F or prime numbers in $\mathbb{Z}$), since in our case, for a maximal order Λ there is a $1-1$ correspondence between the ideals of Λ and those of O_F (see [Re], Th.22.4, p.191.).
Another resemblance between the ideals in a maximal order and those of O_F is reflected in the following (see [Re], Ex.13. on p.204.):

Proposition 22. *Let $\wp$ be a prime ideal of O_F, and let $\wp\Lambda = \mathbf{p_1}^{e_1} \cdot \ldots \mathbf{p_l}^{e_l}$ be a prime ideal decomposition of $\wp\Lambda$ in Λ. Let $f_i := dim_{O_F/\wp}\Lambda/\mathbf{p_i}$. Then:*

$$\sum_{i=1}^{l} e_i f_i = dim_F D$$

I.p., we have that $dim_F D$ (which equals 4 for the quaternion skewfield) is an upper bound for the exponents $exp_{\mathbf{p}}(p)$, so for almost all prime ideals we have $exp_{\mathbf{p}}(p) < p-1$. Now, either by enlarging k_i's in the prime ideal decomposition of an ideal or by multiplying the ideal with prime ideals which do not divide the ideal itself (i.e. which are not factors in the prime ideal decomposition) , we will get a torsion-free congruence subgroup after finitely many steps:

Corollary 10. *Let β be an ideal in a maximal order Λ in D. Then, for all but finitely many ideals $\alpha \subset \beta$:*

$$\Gamma(\alpha) := \{\gamma \in \Gamma \ / \ \gamma \equiv I \ (mod\ \alpha)\ \} \text{ is torsion-free.}$$

3.3 Cycles in compact locally symmetric spaces modelled on $H^n(\mathbb{H})$

Let $D := Q(a, b|F)$ be a quaternion skewfield over an algebraic number field $F := \mathbb{Q}(\sqrt{p})$, s.t. D does not split at both archimedean places of F (see section 3.2.1). Equip D^{n+1} with the hermitian form $< \cdot, \cdot >$ with respect to the standard symplectic involution τ_c, given by the matrix $\begin{bmatrix} I_n & 0 \\ 0 & -\sqrt{p} \end{bmatrix}$, i.e.

$$< x, y >:= \sum_{i=1}^{n} x_i \tau_c(y_i) - \sqrt{p} x_{n+1} \tau_c(y_{n+1}).$$

Let $m \in \{1, \ldots, n-1\}$. Write D^{n+1} as a direct sum:

$$D^{n+1} =< e_1, \ldots, e_m > \oplus < e_{m+1}, \ldots, e_n > \oplus < e_{n+1} >$$

with respect to the standard D-basis $(e_1, \ldots e_{n+1})$ of D^{n+1}. Consider the following two commuting involutions σ, τ on D^{n+1} defined by:

$$\sigma_{|<e_1,\ldots,e_m>} \equiv Id,\ \sigma_{|<e_{m+1},\ldots,e_n>} \equiv -Id,\ \sigma_{|<e_{n+1}>} \equiv Id$$

$$\tau_{|<e_1,\ldots,e_m>} \equiv -Id,\ \tau_{|<e_{m+1},\ldots,e_n>} \equiv Id,\ \tau_{|<e_{n+1}>} \equiv Id.$$

Let[18] $G(F) := SU(B; D, \tau_c)$ be the F-points of an algebraic group G defined over F. We have $\sigma = \begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\tau = \begin{bmatrix} -I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We see that $\sigma, \tau \in G(F)$, hence they induce the following involutions on $G(F)$, which we again denote by σ and τ:

$$\sigma :\ g \mapsto \begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix} g \begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tau :\ g \mapsto \begin{bmatrix} -I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix} g \begin{bmatrix} -I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of fixed points $G(\sigma)$ of σ resp. $G(\tau)$ of τ on $G(F)$ are matrices of the form

$$\left[\begin{array}{cccccccc} * & * & * & & & & & * \\ * & * & * & & & \mathbf{0} & & * \\ * & * & * & & & & & * \\ & & & * & * & * & * & \\ & & & * & * & * & * & \\ & \mathbf{0} & & * & * & * & * & \mathbf{0} \\ & & & * & * & * & * & \\ * & * & * & & & \mathbf{0} & & * \end{array}\right] \text{resp.} \left[\begin{array}{cccccccc} * & * & * & & & & & \\ * & * & * & & \mathbf{0} & & & \mathbf{0} \\ * & * & * & & & & & \\ & & & * & * & * & * & * \\ & \mathbf{0} & & * & * & * & * & * \\ & & & * & * & * & * & * \\ & & & * & * & * & * & * \\ & \mathbf{0} & & * & * & * & * & * \end{array}\right]$$

We have:

[18] Again, as in section 3.2.1 we specify the F-points of an algebraic group G defined over F.

$$G(\sigma) \otimes \mathbb{R} \cong Sp(m,1) \times Sp(n-m),$$
$$G(\tau) \otimes \mathbb{R} \cong Sp(m) \times Sp(n-m,1).$$

Remark 11. *In the case that G is a form of the orthogonal group $SO(p,q)$, pursuing the analogous construction of the involution σ we have to be careful about the real points of $G(\sigma)$. In that case we get $S(O(p) \times O(q))$ as set of fixed points. This does not happen in our case since $SL(\mathbb{H})$ is defined via the reduced norm, which has value 1 both on the matrix I and $-I$.*

We have $\sigma = \tau\Theta$, where Θ is the Cartan involution on $Sp(n,1)$ with respect to the maximal compact subgroup $Sp(n) \times Sp(1)$ (see section 3.1.2), and it is easy to see that the involutions induced by σ, τ on a quaternion hyperbolic space $H^n(\mathbb{H})$ form an orthogonal decomposition of $H^n(\mathbb{H})$ satisfying the assumptions of section 1.2.
The arithmetic subgroup $\Gamma \leq G(\mathbb{R})$ arising from an order Λ in D as described in section 3.2.1 is obviously $< \sigma, \tau >$- stable, since the matrix entries of σ, τ are just 1's and -1's on the diagonal.
Using the torsion-free criterion (section 3.2.2) we can easily find a torsion-free (finite-index) congruence subgroup of Γ, hence w.l.o.g. we assume Γ is torsion-free.
Let Γ_σ resp. Γ_τ be the subgroup of Γ fixed by σ resp. τ. Let $X := G(\mathbb{R})/K \cong Sp(n,1)/(Sp(n) \times Sp(1))$. Let X_σ, X_τ denote the sets of fixed points of σ, τ on X. They are the subsymmetric spaces corresponding to the inclusions $G(\sigma)(\mathbb{R}) \hookrightarrow G(\mathbb{R})$ resp. $G(\tau)(\mathbb{R}) \hookrightarrow G(\mathbb{R})$, and we have

$$X_\sigma \cong H^m(\mathbb{H}), \quad X_\tau \cong H^{n-m}(\mathbb{H}).$$

The quotients $\Gamma_\sigma \backslash X_\sigma$ resp. $\Gamma_\tau \backslash X_\tau$ are compact locally symmetric spaces[19], and correspond to the projection of X_σ resp. X_τ under the natural map $X \longrightarrow \Gamma \backslash X$ (see Proposition 5), and we have totally geodesic immersions[20]:

$$\Gamma_\sigma \backslash X_\sigma \hookrightarrow \Gamma \backslash X \text{ resp. } \Gamma_\tau \backslash X_\tau \hookrightarrow \Gamma \backslash X.$$

In this way we have constructed compact totally geodesic locally symmetric subspaces of non-compact type inside a compact locally symmetric space of non-compact type arising from the canonical inclusion $H^m(\mathbb{H}) \hookrightarrow H^n(\mathbb{H})$ resp. $H^{n-m}(\mathbb{H}) \hookrightarrow H^{n-m}(\mathbb{H})$. The latter inclusions induce the corresponding inclusions in the compact dual space:

$$P^m(\mathbb{H}) \hookrightarrow P^n(\mathbb{H}) \text{ resp. } P^{n-m}(\mathbb{H}) \hookrightarrow P^n(\mathbb{H}),$$

and the induced maps on the cohomology (this is the map $\rho : H^m(P^n(\mathbb{H}); \mathbb{R}) \longrightarrow H^m(P^m(\mathbb{H}); \mathbb{R})$ resp. $H^{n-m}(P^n(\mathbb{H}); \mathbb{R}) \longrightarrow H^{n-m}(P^{n-m}(\mathbb{H}))$ as in Theorem 7) are nontrivial, as it can be seen from the cell complex structure of quaternion projective spaces and its homology described in sections 1.4.1, 1.4.2 (and then using the Poincaré duality for compact oriented manifolds). Now, Theorem7 applies, and we have:

[19]This follows from Proposition4.
[20]The injectivity follows as in section 1.3.1.

Corollary 11. *Under the assumptions above, the homology classes $[\Gamma_\sigma\backslash X_\sigma]$ resp. $[\Gamma_\tau\backslash X_\tau]$ represented by the submanifolds $\Gamma_\sigma\backslash X_\sigma$ resp. $\Gamma_\tau\backslash X_\tau$ are not trivial in $H^m(\Gamma\backslash X;\mathbb{R})$ resp. $H^{n-m}(\Gamma\backslash X;\mathbb{R})$.*

However, as mentioned earlier, we do not have any information about the G-invariance of this classes (i.e. if their cohomological duals represent invariant or non-G invariant classes). For this, we use the fact that these two complementary dimensional submanifolds satisfy the conditions needed to apply the Millson-Raghunathan and the Rohlfs-Schwermer method:

Note that the submanifolds $\Gamma_\sigma\backslash X_\sigma$ resp. $\Gamma_\tau\backslash X_\tau$ are special cycles as defined in section 1.3.1.
We can assume w.l.o.g. that the submanifolds represented by $\Gamma_\sigma\backslash X_\sigma$ and $\Gamma_\tau\backslash X_\tau$ and all of their intersection components are oriented, since this can be arranged by passing to a suitable congruence subgroup of Γ (see section 1.2 or 1.3.3). Since the (topological) group $G(\mathbb{R}) = Sp(n,1)\times Sp(n+1)$ is simply connected[21] (see section 3.1.2), and X_σ, X_τ intersect exactly in one point (w.l.o.g. with positive intersection number). $\Gamma_\sigma\backslash X_\sigma$ and $\Gamma_\tau\backslash X_\tau$ are compact, thus their intersection $\Gamma_\sigma\backslash X_\sigma \cap \Gamma_\tau\backslash X_\tau$ is compact[22]. Hence, the special cycles $\Gamma_\sigma\backslash X_\sigma$ and $\Gamma_\tau\backslash X_\tau$ satisfy the assumptions of the Theorem 6:

Corollary 12. *There exists a $<\sigma,\tau>$-stable normal subgroup $\tilde\Gamma$ of finite index in Γ s.t.*

$$[\tilde\Gamma_\sigma\backslash X_\sigma][\tilde\Gamma_\tau\backslash X_\tau] > 0.$$

I.p. both special cycles represent nontrivial homology classes.

This follows also by the Millson-Raghunathan method (but for not necessarily the same subgroup $\tilde\Gamma \le \Gamma$) since the inclusions $G(\sigma)\cap G(\tau) \hookrightarrow G(\sigma)$, $G(\sigma)\cap G(\tau) \hookrightarrow G(\tau)$ induce injective map $H^1(\mathbb{R}, G(\sigma)\cap G(\tau)) \longrightarrow H^1(\mathbb{R}, G(\sigma)) \times H^1(\mathbb{R}, G(\tau))$ on the real cohomology.[23]

Hence, the submanifolds constructed above always represent a non-trivial homology class by Corollary 11, which is a consequence of the result of Lafont-Schmidt; and using the Rohlfs-Schwermer or the Millson-Raghunathan method, one can find a finite index subgroup of Γ s.t. the corresponding cycles intersect nontrivially. Now, using the result of section 2.1, there exists a further congruence subgroup where the corresponding cycles represent non-G-invariant homology classes.

[21] This is the assumption of Proposition 14

[22] The intersection of two compact subsets of a Hausdorff space is compact.

[23] Note that the same holds for orthogonal and unitary groups. For a summary of similar injectivity questions see [Se], p.192.

Chapter 4

Further results

4.1 Remarks concerning locally symmetric spaces

4.1.1 Orientability questions

Contrary to the case of the group $Sp(n,1)$, the special orthogonal group $SO(n,1)$ is not connected, hence a locally symmetric space of the form

$$\Gamma\backslash SO(n,1)/S(O(n)\times O(1))$$

is not necessarily orientable. Namely, the group Γ in question is i.g. not contained in the group $SO(n,1)^o$, which is the connected component of the identity of the group $SO(n,1)$. The group $SO(n,1)^o$ is described in the following way: The group $SO(n,1)$ is defined by matrices in $SL_{n+1}(\mathbb{R})$ which satisfy

$$A^t \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

The *polar decomposition* of a matrix contained in the group $SO(n,1)$ is of one of the following two forms (see [Ga], Prop.2.3.):

$$A = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{I+vv^t} & v \\ v^t & c \end{pmatrix}$$

or

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{I+vv^t} & v \\ v^t & c \end{pmatrix}.$$

where Q is an orthogonal matrix with determinant -1 in the first case and 1 in the second case; v a vector in $\mathbb{R}^n$ and $c = \sqrt{\|v\|^2+1}$. The second case corresponds to matrices contained in $SO(n,1)^o$ (see [Ga], Cor.2.27.). This is exactly the case when the lower right entry $a_{n+1,n+1}$ of a matrix $A \in SO(n,1)$ is positive, hence:

$$A \in SO(n,1)^o \Leftrightarrow a_{n+1,n+1} > 0.$$

Consider the following class of arithmetic subgroups of $SO(n,1)$: let $F = \mathbb{Q}(\sqrt{p})$ for a prime number p, having the Galois group $< Id, \sigma >$, and let f be a quadratic form on $\mathbb{R}^{n+1}$ given by $\sum_{i=1}^n x_i^2 - \sqrt{p}x_{n+1}^2$. Let G be defined as the group of matrices in $SL_{n+1}(F)$ which preserve the form f, i.e.

$$G := \{A \in SL_{n+1}(F)|\ A^t \begin{pmatrix} I_n & 0 \\ 0 & -\sqrt{p} \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -\sqrt{p} \end{pmatrix}\}$$

There is a natural isomorphism

$$G(\mathbb{R}) = G^{Id} \times G^{\sigma} \cong SO(n,1) \times SO(n+1)$$

The isomorphism $\phi : G^{Id} \cong SO(n,1)$ is given by sending the matrix $X \in G^{Id}$ to

$$\begin{pmatrix} I_n & 0 \\ 0 & p^{1/4} \end{pmatrix} X \begin{pmatrix} I_n & 0 \\ 0 & 1/p^{1/4} \end{pmatrix}$$

This isomorphism preserves topological properties as well and since it does not change the lowest right entry of the matrix, we have:

$$X \in (G^{Id})^o \text{ if and only if } x_{n+1,n+1} > 0.$$

Now, the group of units of f, i.e. the intersection $\Gamma := GL_{n+1}(O_F) \cap G$, embeds in $G(\mathbb{R})$ as an arithmetic lattice. (see section 5.2 in Appendix). In this way, we have an arithmetic subgroup $\phi(\Gamma) \leq SO(n,1)$.

As described in section 2.2, Millson ([Mil]) considers the analogous examples of arithmetic groups contained in $O(n,1)$, then he takes the intersection with $O(n,1)^+$ (which is the subgroup of $O(n,1)$ preserving the upper half-space $x_{n+1} > 0$ in $\mathbb{R}^{n+1}$) to obtain a group $\Gamma \leq O(n,1)^+$. Further, choosing an ideal $\mathbf{p}$ of O_F with norm different from 2, for the congruence subgroup $\Gamma(\mathbf{p}) := \{\gamma \in \Gamma \mid \gamma \equiv I \pmod{\mathbf{p}}\ \}$ the desired $det = 1$ condition is met and $\Gamma(\mathbf{p}) \leq SO(n,1)^o$ holds.

In some published papers this is falsely cited: an arithmetic subgroup $\Gamma \leq SO(n,1)$ is considered, arising in the same way as above from an algebraic number field F, and it is claimed that for any ideal $\mathbf{q}$ of O_F with norm different from 2, the group $\Gamma(\mathbf{q})$ is contained in $SO(n,1)^o$ (the difference lies in the fact that Millson already intersected the group $\Gamma \leq O(n,1)$ with $O(n,1)^+$ and further required just the $det = 1$ condition).

In order to see that the latter claim is not correct, we give the following counterexample:

let $F = \mathbb{Q}(\sqrt{3})$. Then $O_F = \mathbb{Z} \oplus \mathbb{Z}\sqrt{3}$. Let f be a quadratic form on $\mathbb{R}^2$ given by

$$x_1^2 - \sqrt{3}x_2^2,$$

and let

$$G := \{A \in SL_2(F)|\ A^t \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{3} \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{3} \end{pmatrix}\},$$

$$\Gamma := G \cap GL_2(O_F).$$

Let

$$\gamma := \begin{pmatrix} -98 - 57\sqrt{3} & (-75 - 43\sqrt{3})\sqrt{3} \\ -75 - 43\sqrt{3} & -98 - 57\sqrt{3} \end{pmatrix}$$

We have

$$\gamma^t \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{3} \end{pmatrix} \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$$

and $det(\gamma) = 1$, hence $\gamma \in \Gamma$. Furthermore:

$$\gamma = \begin{pmatrix} -98-57\sqrt{3} & (-75-43\sqrt{3})\sqrt{3} \\ -75-43\sqrt{3} & -98-57\sqrt{3} \end{pmatrix} =$$

$$= \begin{pmatrix} 1+(-99-57\sqrt{3}) & (-75-43\sqrt{3})\sqrt{3} \\ -75-43\sqrt{3} & 1+(-99-57\sqrt{3}) \end{pmatrix} =$$

$$= \begin{pmatrix} 1+(-15-9\sqrt{3})(3+2\sqrt{3}) & \sqrt{3}(-11-7\sqrt{3})(3+2\sqrt{3}) \\ (-11-7\sqrt{3})(3+2\sqrt{3}) & 1+(-15-9\sqrt{3})(3+2\sqrt{3}) \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod (< (3+2\sqrt{3}) >_{O_F})$$

Thus, for $\mathbf{q} :=< (3+2\sqrt{3}) >_{O_F}$ we have $\gamma \in \Gamma(\mathbf{q})$. However, $N(\mathbf{q}) \neq 2$ and for the lower right entry one has $\gamma_{2,2} = -98-57\sqrt{3} < 0$, so $\gamma \notin (G^{Id})^o$. Using the above mentioned isomorphism $\phi : G^{Id} \cong SO(1,1)$, and the canonical inclusion $SO(1,1) \hookrightarrow SO(n,1)$ for $n > 1$ we get a counterexample to the claim of "orientability properties" for arithmetic groups in $SO(n,1)$. Hence, the orientability condition cannot be settled by such weak congruence requirements.

We note that the orientability question (i.p. for an arithmetic group $\Gamma \leq SO(n,1)$) can be solved (passing to congruence subgroups in the "adelic sense") using the following result of Rohlfs and Schwermer (see [RSchw], Prop. 2.2 and discussion before for the definition of a congruence subgroup):

Proposition 23. *Let G be a linear algebraic group defined over $\mathbb{Q}$, and let $\Gamma \leq G(\mathbb{Q})$ be an arithmetic subgroup. Then there exists a congruence subgroup $\tilde{\Gamma} \leq \Gamma$ s.t. $\tilde{\Gamma}$ is contained in the connected component of the identity $G(\mathbb{R})^o$ of the real Lie group $G(\mathbb{R})$.*

4.1.2 Non-G-invariant classes

In section 2.1 the Millson-Raghunathan idea is presented which enables to pass to a covering where the class carried by the corresponding special cycle is not dual to a G-invariant cohomology class. This is settled by passing to a congruence subgroup $\Gamma(\mathbf{q}) \leq \Gamma$ s.t. we can find $\alpha \in \Gamma$ with $\alpha C_\sigma(\Gamma(\mathbf{q})) \cap C_\tau(\Gamma(\mathbf{q})) = \emptyset$ which is fulfilled by ensuring that $\tau(\alpha C_\sigma(\Gamma(\mathbf{q}))) \cap \alpha C_\sigma(\Gamma(\mathbf{q})) = \emptyset$.

An alternative approach can be applied based on the following idea of X.Xue ([X]):

in the setting of section 1.2 choose a locally symmetric space $\Gamma\backslash X$ where the corresponding cycles intersect in finitely many points with positive multiplicity (see Theorem 5). These intersection points correspond to a finite set $I(\Gamma) \subset \Gamma$ (as in section 1.2). Passing to a congruence subgroup $\Gamma(\mathbf{q}) \leq \Gamma$ we get coverings $\pi : \Gamma(\mathbf{q})\backslash X \longrightarrow \Gamma\backslash X$, $\pi_\sigma : C_\sigma(\Gamma(\mathbf{q})) \longrightarrow C_\sigma(\Gamma)$ and $\pi_\sigma : C_\tau(\Gamma(\mathbf{q})) \longrightarrow C_\tau(\Gamma)$ with the corresponding deck transformation groups $\Gamma/\Gamma(\mathbf{q})$, $\Gamma_\sigma/\Gamma_\sigma(\mathbf{q})$ and $\Gamma_\tau/\Gamma_\tau(\mathbf{q})$.

We denote by $d(\mathbf{q})$, $d_\sigma(\mathbf{q})$ and $d_\tau(\mathbf{q})$ the (finite) orders of these groups. In $\Gamma(\mathbf{q})\backslash X$ we can "shift" the cycle $C_\tau(\sigma)$ by elements of the deck transformation group $\Gamma/\Gamma(\mathbf{q})$. The group $\Gamma_\tau/\Gamma_\tau(\mathbf{q})$ (considered as the subgroup of $\Gamma/\Gamma(\mathbf{q})$) stabilizes $C_\tau(\Gamma(\mathbf{q}))$, and since Γ acts freely on X, by "shifting" $C_\tau(\Gamma(\mathbf{q}))$ we get $d(\mathbf{q})/d_\tau(\mathbf{q})$ disjoint submanifolds $\mu_1 C_\tau(\Gamma(\mathbf{q})), \ldots, \mu_{d(\mathbf{q})/d_\tau(\mathbf{q})} C_\tau(\Gamma(\mathbf{q}))$, for certain elements $\mu_1, \ldots, \mu_{d(\mathbf{q})/d_\tau(\mathbf{q})} \in \Gamma$, $\mu_1 = 1$.

The following observation is Lemma 5 of [X]:
note that every point in the intersection $(C_\sigma(\Gamma(\mathbf{q})) \cap \mu_i C_\tau(\Gamma(\mathbf{q})))$ lies over a point in the intersection $C_\sigma(\Gamma) \cap C_\tau(\Gamma)$, i.p. we have

$$(C_\sigma(\Gamma(\mathbf{q})) \cap \textstyle\bigcup_{i=1}^{d/d_\tau} \mu_i C_\tau(\Gamma(\mathbf{q}))) \subset \pi_\sigma^{-1}(C_\sigma(\Gamma) \cap C_\tau(\Gamma)),$$

hence

$$|(C_\sigma(\Gamma(\mathbf{q})) \cap \textstyle\bigcup_{i=1}^{d(\mathbf{q})/d_\tau(\mathbf{q})} \mu_i C_\tau(\Gamma(\mathbf{q})))| \leq d_\sigma(\mathbf{q})|I(\Gamma)|$$

and we see that $C_\sigma(\Gamma(\mathbf{q}))$ can intersect at most $d_\sigma(\mathbf{q})|I(\Gamma)|$ of the "shifted" cycles $\mu_i C_\tau(\Gamma(\mathbf{q}))$.

Now consider the following:
we have $d(\mathbf{q})/d_\tau(\mathbf{q})$ cycles $\mu_i C_\tau(\Gamma(\mathbf{q}))$ and we know that $C_\sigma(\Gamma(\mathbf{q}))$ can intersect at most $d_\sigma(\mathbf{q})|I(\Gamma)|$ of them, so if $d_\sigma(\mathbf{q})|I(\Gamma)| < d(\mathbf{q})/d_\tau(\mathbf{q})$ we know that there exists μ s.t $C_\sigma(\Gamma(\mathbf{q})) \cap \mu C_\tau(\Gamma(\mathbf{q})) = \emptyset$, and consequently $[(C_\sigma(\Gamma(\mathbf{q}))] \neq [\mu C_\sigma(\Gamma(\mathbf{q})))]$ are non G-invariant classes.
$d_\sigma(\mathbf{q})|I(\Gamma)| < d(\mathbf{q})/d_\tau(\mathbf{q})$ is equivalent to $|I(\Gamma)| < d(\mathbf{q})/d_\tau(\mathbf{q})d_\sigma(\mathbf{q})$ and since $|I(\Gamma)|$ is a fixed number, in order to get the desired result it suffices to show that the right hand side of the inequality can be made large enough by taking large primes $\mathbf{q}$.

Given the above settings, denote by G^σ resp. G^τ the groups of fixed points of σ resp. τ of G and assume in addition that the groups G, G^σ and G^τ are $\mathbb{R}$-rational points of the simply connected simple $\mathbb{Q}$-groups $\underline{G}$, $\underline{G}^\sigma$ and $\underline{G}^\tau$. Suppose (as in Theorem 8) that the group Γ is a Zariski dense subgroup of $\underline{G}(\mathbb{Z}(S))$, where S is a finite set of prime numbers. Using Theorem 8 and the discussion above we have:

Corollary 13. *For almost all prime ideals $\mathbf{q} \notin S$ we have*

$$\frac{d(\mathbf{q})}{d_\tau(\mathbf{q})d_\sigma(\mathbf{q})} = \frac{|G_\mathbf{q}(\mathbb{F}_\mathbf{q})|}{|G_\mathbf{q}^\tau(\mathbb{F}_\mathbf{q})||G_\mathbf{q}^\sigma(\mathbb{F}_\mathbf{q})|}.$$

Furthermore, if

$$I(\Gamma) < \frac{|G_\mathbf{q}(\mathbb{F}_\mathbf{q})|}{|G_\mathbf{q}^\tau(\mathbb{F}_\mathbf{q})||G_\mathbf{q}^\sigma(\mathbb{F}_\mathbf{q})|},$$

then the homology class carried by $C_\sigma(\Gamma(\mathbf{q}))$ and $C_\tau(\Gamma(\mathbf{q}))$ are not dual to G-invariant classes.

In the case where G reduces to a finite classical group this number can be easily calculated and goes to ∞ as $\mathbf{q}$ grows (for the orders of the finite classical groups in question see [W], Ch.3.), for example:

$*$ if $G_\mathbf{q}(\mathbb{F}_\mathbf{q}) \cong Sp_{2n}(\mathbf{q})$, $G_\mathbf{q}^\sigma(\mathbb{F}_\mathbf{q}) \cong Sp_{2m}(\mathbf{q})$ and $G_\mathbf{q}^\tau(\mathbb{F}_\mathbf{q}) \cong Sp_{2(n-m)}(\mathbf{q})$, we have

$$\frac{|Sp_{2n}(\mathbf{q})|}{|Sp_{2m}(\mathbf{q})||Sp_{2(n-m)}(\mathbf{q})|} = \frac{\mathbf{q}^{n^2}\prod_{i=1}^{n}(\mathbf{q}^{2i}-1)}{(\mathbf{q}^{m^2}\prod_{i=1}^{m}(\mathbf{q}^{2i}-1))(\mathbf{q}^{(n-m)^2}\prod_{i=1}^{n-m}(\mathbf{q}^{2i}-1))}$$

$$= \mathbf{q}^{2m(n-m)}\frac{\prod_{i=m+1}^{n}(\mathbf{q}^{2i}-1)}{\prod_{i=1}^{n-m}(\mathbf{q}^{2i}-1)},$$

and this can be made arbitrary large by enlarging $\mathbf{q}$.

4.1.3 Growth of Betti numbers

In the paper of Millson ([Mil]) the author constructs nontrivial 1-dimensional homology classes in compact locally symmetric spaces modelled on real hyperbolic spaces. These locally symmetric spaces correspond to arithmetic groups arising from a quadratic form, as explained in section 2.2. Without going into details, we give a brief sketch of the idea: The first Betti number can be made arbitrarily large by passing to congruence subgroups where in the corresponding locally symmetric space M we can find disjoint homologically non-trivial hypersurfaces[1] $F_1, \dots, F_r$ s.t. $M \setminus \bigcup_{i=1}^{r} F_i$ is connected. For each hypersurface F_i we can find a 1-dimensional compact loop C_i having intersection number 1 with the chosen hypersurface and 0 with other hypersurfaces (for example C_i can be chosen s.t. $C_i \cap F_j =$ for $i \neq j$). Hence for the intersection number we have $[C_i][F_i] = 1$ and $[C_j][F_i] = 0$ for $i \neq j$, i.p. $[C_i] \neq [C_j]$.
Linear independence of these classes follows immediately: suppose $\sum_{i=1}^{r}\lambda_i[C_i] = [0]$ represents a trivial homology class in $H_1(\Gamma\backslash X;\mathbb{R})$, then (for technical purpose we denote by $[f_j]$ the cohomology class dual to the homology class of a submanifold F_j and we use the cap product as explained in section 1.1.1) $0 = [f_j] \cap (\sum_{i=1}^{r}\lambda_i[C_i]) = \lambda_j$ for all $j = 1, \dots, r$; hence the classes $[C_1], \dots, [C_r]$ are linearly independent.

For higher dimensional homology groups it is more difficult to find k-dimensional submanifolds ($k \geq 2$) carrying different homology classes. In the notation and the setting of section 2.1 we can "shift" the submanifold $C_\sigma(\mathbf{q})$ by a deck transformation represented by an element of the group $\Gamma/\Gamma(\mathbf{q})$. It is obvious that elements γ, μ in Γ whose images in $\Gamma/\Gamma(\mathbf{q})$ are not congruent modulo the subgroup $\Gamma_\sigma/\Gamma_\sigma(\mathbf{q})$ produce disjoint submanifolds $\gamma C_\sigma(\mathbf{q}), \mu C_\sigma(\mathbf{q})$ but these can represent the same homology class (this fact is ignored in some published papers on this subject). For example, this happens if the homology class carried by $C_\sigma(\mathbf{q})$ has a G-invariant dual cohomology class. However, the result of Millson-Raghunathan in section 2.1 enables to find a congruence subgroup $\Gamma(\mathbf{q}) \leq \Gamma$ s.t. there exists $\alpha \in \Gamma$ s.t. $[\alpha C_\sigma(\mathbf{q})] \neq [C_\sigma(\mathbf{q})]$. In this case, the deck transformation group acts non-trivially on the $\mathbb{R}$- (resp. $\mathbb{C}$-) vector space $H_k(\Gamma(\mathbf{q})\backslash X;\mathbb{R})$ (resp. $H_k(\Gamma(\mathbf{q})\backslash X;\mathbb{C})$).

If the assumptions of the Theorem 8 are satisfied, we have that $\Gamma/\Gamma(p) \cong G_p(\mathbb{Z}/p\mathbb{Z})$ for almost all primes p, and we get an estimate of the lower bound for Betti numbers by calculating the dimension of the smallest (non-trivial) representation spaces for a finite group $G_p(\mathbb{Z}/p\mathbb{Z})$.

[1] A hypersurface is a submanifold of codimension 1.

For a large class of finite groups the bounds on the minimal degree of a non-trivial irreducible complex character (i.e. a minimal dimension of a nontrivial irreducible complex representation) are already known. Let $\tilde{G}$ be a finite group of Lie type[2], and $r(\tilde{G})$ defined as in [LiSha], p. 327. Consider $\tilde{G}(q)$, a group of Lie type over a finite field $\mathbb{F}_q$, where $q = p^l$ for a prime p and $l \in \mathbb{N}$. Then there is a constant c (independent of q) s.t. the dimension of the smallest nontrivial representation is greater than $c \cdot q^{r(\tilde{G})}$ ([LiSha], Lemma 2.1.).
Bounds of this type should hold for the homology groups of locally symmetric spaces modelled on the quaternion symmetric space, however this topic requires more careful analysis (f.e. to determine the reduction modulo a prime of the underlying algebraic group).

[2]For a definition of a finite group of Lie type see [Cu], §4.1.

4.2 Description of the stable general linear group over an arithmetic order

As mentioned in the introduction, we present a new proof of a result on the stable range of an arithmetic order in a skewfield over a number field:

We shall briefly introduce some ring-theoretic notions, for a general theory of noncommutative rings see [L]. Let R be a ring with 1. Denote by $radR$ the *Jacobson radical* of R, defined as the intersection of all the maximal left ideals of R. A ring R is called *semisimple* if it is isomorphic to a finite product $M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r)$ of matrix rings over skewfields D_i. A ring R is called *semilocal* if $R/radR$ is a semisimple ring. If R is commutative, this is equal to the condition that R has finitely many maximal ideals. If a (noncommutative) ring has only finitely many maximal left ideals, then it is semilocal. I.p. finite rings are semilocal.

Let R be a ring, $k \in \mathbb{Z}, k \geq 1$. An element $(x_1, x_2, \ldots, x_k) \in R^k$ is called *unimodular* in R^k if $\sum_{i=1}^{k} Rx_i = R$. Let $m \in \mathbb{Z}$, $m \geq 2$. We call a unimodular element in R^m *reducible* if there exist elements $t_1, t_2, \ldots, t_{m-1} \in R$ such that

$$(x_1 + t_1 x_m, x_2 + t_2 x_m, \ldots, x_{m-1} + t_{m-1} x_m)$$

is unimodular in R^{m-1}. If every unimodular vector in R^m is reducible, then so is every unimodular vector in R^n, for all $n > m$.

Let $k \in \mathbb{Z}, k \geq 1$. We say: *$k$ defines a stable range for GL(R)*, if for all $m > k$ every unimodular element $(x_1, x_2, \ldots, x_m) \in R^m$ is reducible. The smallest integer n such that for every $k \geq n$, k defines a stable range for $GL(R)$, is called the *stable rank of R*, to be denoted $sr(R)$. To prove that the stable rank of a ring R equals r, it suffices to show that every unimodular vector in R^{r+1} is reducible.

We will make use of the following result often referred to as "Bass' Theorem", which we cite without proof:

Lemma 15. *[Ba], Lemma 6.4.*
Let R be a semilocal ring, $a \in R$ and I a left ideal of R. If $Ra + I = R$, then the coset $a + I$ contains a unit of R.

We present an elementary proof of the result on the stable rank of an arithmetic order in a skewfield (this result is proved in [Ba] using the theorem of Serre on the structure of direct summands of a direct sum of finitely presented modules over a finitely generated algebra over a commutative Noetherian ring):

Lemma 16. *Let D be a finite dimensional separable[3] skewfield over an algebraic number field K, let O_K be the ring of integers in K. Let Λ be an O_K-order in D.*
Then $sr(\Lambda) \leq 2$.

Proof. It suffices to show that every unimodular element in Λ^3 is reducible, that is: given x_1, x_2 and x_3 in Λ such that

$$\Lambda \cdot x_1 + \Lambda \cdot x_2 + \Lambda \cdot x_3 = \Lambda$$

[3] By a *separable* K-algebra D we mean that the center of D is a separable field extension of K.

there exist μ_1, $\mu_2 \in \Lambda$ such that

$$\Lambda \cdot (x_1 + \mu_1 \cdot x_3) + \Lambda \cdot (x_2 + \mu_2 \cdot x_3) = \Lambda$$

W.l.o.g. we may suppose $x_1 \neq 0$. Let I:=$\Lambda \cdot x_1$, a left ideal in Λ. Since K$\cdot\Lambda$=D, we have

$$x_1^{-1} = \sum_{i=1}^{n} k_i \cdot \lambda_i,$$

(x_1^{-1} is the inverse of x_1 in D, this element needs not to be in Λ).
Now $k_i = \frac{r_i}{s_i}$ with r_i, $s_i \in O_K$, so for $s = \prod_{i=1}^{n} s_i$ we have: $sx_1^{-1} \in \Lambda$, with s $\in O_K$, $s \neq 0$. Then

$$s = sx_1^{-1} \cdot x_1 \in I,$$

so $\beta =$ I $\cap\, O_K$ is a nonzero ideal in O_K.
Consider

$$ß = \Lambda \cdot \beta = \{\textstyle\sum_{finite} \lambda_i \cdot b_i \mid \lambda_i \in \Lambda,\, b_i \in \beta\},$$

contained in $\Lambda \cdot x_1 = I$. The definition of ß implies that ß is a left ideal in Λ, and since b_i's are elements of the center of Λ, ß is a two sided ideal. Since Λ is a finitely generated module over O_K, we have that $\Lambda/ß$ is a finitely generated module over O_K/β. O_K/β is finite, hence $\Lambda/ß$ is finitely generated over a finite ring, so $\Lambda/ß$ is finite itself. I.p. $\Lambda/ß$ is a semilocal ring.
The equality $\Lambda \cdot x_1 + \Lambda \cdot x_2 + \Lambda \cdot x_3 = \Lambda$ leads to

$$\Lambda/ß \cdot (x_2 + ß) + \Lambda/\, ß \cdot \langle (x_1 + ß), (x_3 + ß) \rangle = \Lambda/ß$$

and now we can apply Lemma 15 for semilocal rings to conclude that

$$(x_2 + ß) + \Lambda/\, ß \cdot \langle (x_1 + ß), (x_3 + ß) \rangle$$

contains a unit, so there exist ρ,$\tau \in \Lambda$ such that

$$\Lambda/ß \cdot ((x_2 + \rho \cdot x_1 + \tau \cdot x_3) + ß) = \Lambda/ß.$$

That implies

$$ß + \Lambda \cdot (x_2 + \rho \cdot x_1 + \tau \cdot x_3) = \Lambda.$$

Now consider $\Lambda x_1 + \Lambda(x_2 + \tau \cdot x_3)$:

$$\Lambda x_1 \supseteq ß$$
$$\text{and } x_2 + \rho \cdot x_1 + \tau \cdot x_3 \in \Lambda x_1 + \Lambda(x_2 + \tau \cdot x_3).$$

Thus

$$\Lambda x_1 + \Lambda(x_2 + \tau \cdot x_3) = \Lambda$$

and we get the desired reduction. □

Remark 12. *Note that the idea on the proof is based on the fact that the left ideal $\Lambda \cdot x_1$ has a nonzero intersection with O_K. This allows us to factor the ring Λ modulo ß and then at the end capture ß with x_1. However, this can not be done if we leave out the condition "D is a skewfield", since then there exist left ideals of orders having zero intesection with O_K. One can verify this easily for $M_n(\mathbb{Z})$ as a $\mathbb{Z}$-order in the matrix algebra $M_n(\mathbb{Q})$.*

The stable range conditions have applications for the structure of the general linear group over a ring (see [Ba], Ch.1.). We briefly mention one application concerning the structure of the general linear group over an arithmetic order considered in the Lemma above. Identify $X \in GL_n(R)$ with $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$. This induces the embedding $GL_n(R) \longrightarrow GL_m(R)$ for $m \geq n$. Let E_{ij}, $i \neq j$, $1 \leq i, j \leq n$ be the matrix in $GL_n(R)$ having 1 in the (i, j)-th place and zeros elsewhere. A matrix in $GL_n(R)$ is called *elementary* if it is of the form $x_{ij}(r) := I_n + rE_{ij}$ for some $r \in R$ and $i \neq j$. Denote by $E_n(R)$ the group generated by elementary matrices in $GL_n(R)$. Let G be a group. For $x, y \in G$ we define the *commutator* of x and y as $[x, y] := xyx^{-1}y^{-1}$. Define $[G, G]$ to be the subgroup of G generated by $[h_1, h_2]$ with $h_1, h_2 \in G$. Now, using Theorem 4.2. in [Ba] and the Lemma above we have the following description of the general linear group over an arithmetic order:

Corollary 14. *Let D be a finite dimensional separable skewfield over algebraic number field K, O_K ring of integers in K. Let Λ be an O_K-order in D. Then:*

1. *For all $r > 2$:*
$$GL_r(\Lambda) = GL_2(\Lambda) \cdot E_r(\Lambda).$$

2. $$GL_2(\Lambda) \longrightarrow K_1(\Lambda) := lim_{n\to\infty} GL_n(\Lambda)/[GL_n(\Lambda), GL_n(\Lambda)],$$
is surjective.

3. *For all $r \geq 4$*
$$E_r(\Lambda) = [GL_r(\Lambda), GL_r(\Lambda)].$$

4. *For $n \geq 4$ all the maps below are surjective, and all these groups are abelian:*
$$\ldots SL_n(\Lambda)/E_n(\Lambda) \longrightarrow SL_{n+1}(\Lambda)/E_{n+1}(\Lambda) \longrightarrow \ldots \longrightarrow SL(\Lambda)/E(\Lambda).$$

Chapter 5

Appendix

5.1 Decomposition of a semisimple Lie group

The following theorem, quoted from [Hel], Ch.VI, p.214., is frequently used:
Let $\mathbf{g}$ be a noncompact semisimple Lie algebra over $\mathbb{R}$ with the Cartan decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$, and θ the corresponding Cartan involution (i.e. $\theta_{|\mathbf{k}} = Id_{\mathbf{k}}$, $\theta_{|\mathbf{p}} = -Id_{\mathbf{p}}$). Let G be a connected Lie group with Lie algebra $\mathbf{g}$, K a Lie subgroup of G with Lie algebra $\mathbf{k}$.

Theorem 9. *In the notation above, we have:*

1) K is connected, closed and contains the center Z_G of G. Moreover, K is compact if and only if Z_G is finite.

2) there exists an involutive analytic automorphism Θ of G with $Fix_G\Theta = K$, and whose differential at e is θ.

3) the mapping $\mathbf{p} \times K \longrightarrow G$, $(X, k) \mapsto exp(X)k$ is a diffeomorphism, and the mapping $exp : \mathbf{p} \longrightarrow G/K$ is a diffeomorphism onto the symmetric space G/K.

5.2 Arithmetic groups

Suppose $V_{\mathbb{Q}}$ is a $\mathbb{Q}-$ subspace of a finite dimensional vector space V over $\mathbb{C}$ s.t. $dim_{\mathbb{Q}}V_{\mathbb{Q}} = dim_{\mathbb{C}}V$ and $V_{\mathbb{Q}}$ contains a $\mathbb{C}$-basis of V. By a *lattice* of $V_{\mathbb{Q}}$ we mean a free $\mathbb{Z}$-submodule of $V_{\mathbb{Q}}$ of maximal rank.
Let Λ be a lattice in $V_{\mathbb{Q}}$, G an algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$. Denote by G_Λ be the stabilizer of Λ in $G(\mathbb{Q})$, i.e. $G_\Lambda := \{g \in G(\mathbb{Q}) | g(\Lambda) = \Lambda\}$. A subgroup $\Gamma \leq G(\mathbb{Q})$ is called an *arithmetic subgroup of* G if there exists a lattice Λ of $V_{\mathbb{Q}}$ such that Γ is commensurable with G_Λ (i.e. $\Gamma \cap G_\Lambda$ has finite index both in Γ and in G_Λ)
It can be seen that the definition of an arithmetic subgroup of G is independent of the choice of a faithful representation of G on a vector space V and a lattice Λ, i.e. it depends only on the $\mathbb{Q}$-structure of G.
In the above definition of an arithmetic group one can replace $\mathbb{Q}$ by an arbitrary algebraic number field F and $\mathbb{Z}$ by the ring of integers O_F of F. Given an algebraic subgroup $G \leq GL_n(\mathbb{C})$ one can define arithmetic subgroups of G as those subgroups commensurable with $G_{O_F} := G \cap SL_n(O_F)$. However, this does

not produce any new groups since as it can be seen by reducing the latter case via restriction of scalars[1] to the case $F = \mathbb{Q}$.

5.3 Cohomological dimension

As described in the previous section, a torsion-free arithmetic subgroup Γ of G acts freely and properly discontinuously on $X = G/K$. Since X is contractible, $\Gamma\backslash X$ is an "Eilenberg-Maclane space" of type $K(\Gamma, 1)$, i.e. $\pi_1(\Gamma\backslash X) \cong \Gamma$ and $\pi_i(\Gamma\backslash X) \cong 0$ for $i > 1$.
$\Gamma\backslash X$ can be thought of as a geometric analogue of the arithmetic group Γ. The group (Eilenberg-MacLane) cohomology of the group Γ equals the cohomology of the manifold $\Gamma\backslash X$.[2] I.p. the cohomological dimension $cd(\Gamma)$ is at most $n = dimX$, more precisely

$$cd(\Gamma) = n, \text{ if } \Gamma\backslash X \text{ compact}$$

$$cd(\Gamma) < n, \text{ if } \Gamma\backslash X \text{ non-compact.}$$

In the latter case, when X does not have any compact factors, the difference $n - cd(\Gamma)$ equals the $\mathbb{Q}$-rank of Γ.

5.4 Adeles, strong approximation and its applications

The following is a short summary of the theory of adeles and adele groups which can be found in detail in [PR], ch.5.:
Let K be an algebraic number field. For every prime ideal $\mathbf{p}$ of the ring of integers O_K of K we have a corresponding valuation $|\cdot|_v \in V_f^K$, and for every infinite place of K (equivalent embeddings of K in $\mathbb{C}$) we have an archimedean valuation $|\cdot|_v \in V_\infty^K$. Let K_v denote the completion of K with respect to the valuation $|\cdot|_v$ and $O_v := \{x \in K_v | |x|_v \leq 1\}$ the valuation ring (which is the closure of the ring of integers O_K in K_v). Let $V^K := V_f^K \cup V_\infty^K$.

Denote by $\mathbb{A}_K$ the subset of the direct product $\prod_{v \in V^K} K_v$ consisting of those $x = (x_v)$ such that $x_v \in O_v$ for almost all $v \in V_f^K$ ("restricted product"). $\mathbb{A}_K$ is a ring with respect to the addition and multiplication in the direct product.

We can introduce a topology on $\mathbb{A}_K$, letting the base of the open sets consist of the sets of the form

$$\prod_{v \in S} W_v \times \prod_{v \in V^K \backslash S} O_v,$$

where $S \subset V^K$ is a finite subset containing V_∞^K, and $W_v \subset K_v$ are open subsets for each $v \in S$. This is called the adele topology and it makes $\mathbb{A}_K$ a locally compact topological ring.

[1] For this notion, see [PR], p.49, §2.1.2. "Restriction of scalars".
[2] This is explained in detail in [Bo], §1.

For a finite subset $S \subset V^K$ containing V_∞^K, let

$$\mathbb{A}_K(S) := \prod_{v \in S} K_v \times \prod_{v \notin S} O_v$$

be the ring of S-integral adeles. For $S = V_\infty^K$, the corresponding ring is called the ring of integral adeles, denoted by $\mathbb{A}_K(\infty)$.

There exists a diagonal embedding $K \hookrightarrow \mathbb{A}_K$, given by $x \mapsto (x, x, x, \ldots)$, whose image is called the ring of principal adeles, and can be identified with K. Since $O_K = \cap_{v \in V_f^K}(K \cap O_v)$, we have $K \cap \mathbb{A}_K(\infty) = O_K$, and using the fact that O_K is discrete in $\prod_{v \in V_\infty^K} K_v = K \otimes_\mathbb{Q} \mathbb{R}$ we have:

K (identified with its image in $\mathbb{A}_K$) is discrete in $\mathbb{A}_K$.

Analogously, one can define the ideles $\mathbb{I}_K$ as the set consisting of those $x = (x_v)$ in $\prod_{v \in V^K} K_v^*$ s.t. $x_v \in U_v := O_v^*$ for almost all $v \in V_f^K$. Then $\mathbb{I}_K = \mathbb{A}_K^*$. $\mathbb{I}_K$ is not a topological group with respect to the topology induced from $\mathbb{A}_K$, so the idele topology is defined as the topology induced from $\mathbb{A}_K \times \mathbb{A}_K$ with the embedding $\mathbb{I}_K \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$ and it makes $\mathbb{I}_K$ a locally compact topological group. Analogously as in the case of adeles, we have the group of principal ideles obtained as the image of $K^* \hookrightarrow \mathbb{I}_K$, and is discrete in $\mathbb{I}_K$.

Weak and Strong approximation

Let S be a finite subset of V^K. Denote by $\mathbb{A}_{K,S}$ the image of $\mathbb{A}_K$ under the natural projection onto $\prod_{v \notin S} K_v$. A topology on $\mathbb{A}_{K,S}$ is given analogously as in the case of $\mathbb{A}_K$. For T a finite subset of V^K containing S, let $\mathbb{A}_{K,S}(T)$ denote the image of the ring of T-integral adeles $\mathbb{A}_K(T)$ in $\mathbb{A}_{K,S}$. For $S = V_\infty^K$ we write $\mathbb{A}_{K,S} = \mathbb{A}_{K,f}$, called the ring of finite adeles.

We have $\mathbb{A}_K = K_S \cup \mathbb{A}_{K,S}$, where $K_S := \prod_{v \in S} K_v$ with the direct product topology, and the diagonal embedding of K in $\mathbb{A}_K$ is the product of the diagonal embeddings in K_S and $\mathbb{A}_{K,S}$. Although the image of K in $\mathbb{A}_K$ is discrete, each embedding $K \hookrightarrow K_S$ and $K \hookrightarrow \mathbb{A}_{K,S}$ is dense:

Theorem 10. *Weak Approximation*

The image of K under the diagonal embedding is dense in K_S.

Theorem 11. *Strong Approximation*

If $S \neq 0$. the image of K under the diagonal embedding is dense in $\mathbb{A}_{K,S}$.

Remark 13. *The strong approximation theorem can be considered as a generalization of the Chinese Remainder Theorem, in the following sense: Let $K = \mathbb{Q}$ and $S = \{\infty\}$. If $x \in \mathbb{A}_{\mathbb{Q},f}$, there exists an integer m s.t. $mx \in \mathbb{A}_{\mathbb{Q},f}(\infty)$, and the statement of the theorem can be reduced to the question of density of $\mathbb{Z}$ in $\mathbb{A}_{\mathbb{Q},f}(\infty)$. Now, any open set in $\mathbb{A}_{\mathbb{Q},f}(\infty)$ contains a set contained in the base of topology, a set of the form $W = \prod_{i=1}^r (a_i + p_i^{\alpha_i})\mathbb{Z}_{p_i} \times \prod_{p \neq p_i} \mathbb{Z}_p$, for a finite collection of prime numbers $\{p_1, \ldots p_r\}$ and integers $a_i > 0$. Then $\mathbb{Z} \cap W \neq \emptyset$ is equivalent to the solvability of the system of congruences $\{x \equiv a_i($ mod $p_i^{\alpha_i})\}_{i=1,\ldots,r}$.*

Adelized varieties and algebraic groups

Let X be an affine variety defined over an algebraic number field K with presentation as a K-closed subset of some affine space $\mathbb{A}^n$. Then $X(\mathbb{A}_K)$ consists of the points of X over $\mathbb{A}_K$. $X(\mathbb{A}_K)$ is a restricted topological product (analogously as in the case of adeles) of the spaces $X(K_v)$ with respect to the distinguished open spaces $X(O_v)$. As in the case with adeles, $X(K)$ embeds diagonally into $X(\mathbb{A}_K)$ as a discrete subset. It can be seen that this construction is independent of the choice of a geometric realization of X.

Let G be a linear algebraic K-group. G can be realized as a closed K-subgroup of some full linear group GL_n. In order to describe $G(\mathbb{A}_K)$ it suffices to describe $GL_n(\mathbb{A}_K))$ (a set of matrices in $M_n(\mathbb{A}_K)$ with invertible determinant in $\mathbb{A}_K$): $GL_n(\mathbb{A}_K)$ can be viewed as a restricted topological product of the $GL_n(K_v)$ with respect to the distinguished subgroups $GL_n(O_v)$ for $v \in V_f^K$. The topology on $GL_n(\mathbb{A}_K)$ is defined by the base of open sets consisting of sets of the form

$$\prod_{v\in S} U_v \times \prod_{v\notin S} GL_n(O_v),$$

where S is a finite subset of V^K containing V_∞^K, and U_v are open subsets of $GL_n(K_v)$. Let $GL_n(\mathbb{A}_K(S)) := \prod_{v\in S} GL_n(K_v) \times \prod_{v\notin S} GL_n(O_v)$ (S-integral adeles), and $GL_n(\mathbb{A}_K(\infty)) := GL_n(\mathbb{A}_K(V_\infty^K))$.
Now the adele group can be defined for any closed subgroup G of GL_n as the restricted product of the groups $G(K_v)$ with respect to the distinguished subgroups $G(O_v) := G \cap GL_n(O_v)$ [3] for $v \in V_f^K$, with the topology induced from $GL_n(\mathbb{A}_K)$. $G(\mathbb{A}_K)$ is a locally compact topological group, and $G(K)$ embeds into $G(\mathbb{A}_K)$ as a discrete subgroup. The groups $G(\mathbb{A}_K(S))$ are defined analogously as in the case of $GL_n(\mathbb{A}_K)$.
Analogously as in the case of adeles, we can consider the "truncated" adele group: for an arbitrary subset S of V^K, define the group of S-adeles $G(\mathbb{A}_{K,S})$ as the image of $G(\mathbb{A})$ under the natural projection of $\prod_v G(K_v)$ onto $\prod_{v\notin S} G(K_v)$. Then, $G(\mathbb{A}_{K,S})$ is a restricted product of the $G(K_v)$, $v \notin S$ with respect to $G(O_v)$, $v \in V_f^K \backslash (V_f^K \cap S)$.
As in the case of adeles, it is useful to consider the diagonal embedding $G(K) \hookrightarrow G(\mathbb{A}_{K,S})$:
G is said to satisfy the strong approximation property relative to S if the image under the diagonal embedding $G(K) \hookrightarrow G(\mathbb{A}_{K,S})$ is dense.
The following theorem gives a criterion for strong approximation in algebraic groups:

Theorem 12. *Let G be a reductive algebraic group over an algebraic number field K, and S a finite subset of V^K. Then G satisfies the strong approximation property with respect to S if and only if the following two conditions are satisfied:*
** G is simply connected (i.p. G is semisimple)*
** G does not contain a K-simple component G^i with $G_S^i := \prod_{v\in S} G^i(K_v)$ compact*

Corollary 15. *If G is a K-simple, simply connected group G, then G has a strong approximation with respect to S if and only if $G_S := \prod_{v\in S} G(K_v)$ is noncompact.*

[3] One has to be careful, since $G(O_v)$ depends on the choice of matrix realization of G.

5.5 Reduction of algebraic varieties

The summary of the theory of the reduction of the affine algebraic varieties is given in [PR], p.141.-148. Here we briefly mention some definitions and results: Let K be the field of fractions of an integral domain R, and let $\mathbf{p}$ be a maximal ideal in R. Let X be an affine algebraic variety defined over K. By means of reduction, one can associate to X an algebraic variety $X_{\mathbf{p}}$ defined over the field $k := R/\mathbf{p}$.

Let $\mathbf{a} \subset K[x_1, \ldots, x_n]$ be the ideal of polynomial that vanish on X. By reduction of X modulo $\mathbf{p}$ we mean the variety $X_{\mathbf{p}}$ over a universal domain containing the field $k = R/\mathbf{p}$, defined by the ideal $\mathbf{a}_{\mathbf{p}}$ which is obtained by reducing all the polynomials of $\mathbf{a} \cap R[x_1, \ldots, x_n]$ modulo the ideal $\mathbf{p}$.

There is a natural reduction map $\rho : X(R) \longrightarrow X_{\mathbf{p}}(k)$ whose nonempty fibers are the congruence classes modulo $\mathbf{p}$ of points of $X(R)$.

Let K be an algebraic number field and O_K its ring of integers. Then K is the field of fractions of O_K. Let X be an affine algebraic variety defined over K. For a finite valuation $v \in V_f^K$ of K let X_v denote the reduction of X modulo the corresponding maximal ideal $\mathbf{p} = \mathbf{p}(v)$ of O_K. Let K_v denote the completion of K at the place $v \in V_f^K$, O_v the ring of integers in K_v and $\mathbf{p}_v$ the (unique) maximal ideal of O_v. Then X can be considered as an affine algebraic variety over K_v and the reduction $X_{\mathbf{p}_v}$ of X coincides with X_v.

If $G \subset GL_n$ is an algebraic group defined over K, then for all $v \in V_f^K$ the reduction G_v is an algebraic group defined over the residue field $O_K/\mathbf{p}(v)$ and the reduction map $G(O_K) \longrightarrow G_v(O_K/\mathbf{p}(v))$ is a group homomorphism.

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Abstract

In this thesis, we apply geometric methods ([Mil], [MR], [RSchw], [LS]) for construction of (co-)homology classes in locally symmetric spaces to treat the case of arithmetically defined compact quaternion hyperbolic manifolds. These are of the form $\Gamma\backslash Sp(n,1)/(Sp(n) \times Sp(1))$ for a suitable torsion-free arithmetic group Γ arising from an arithmetic order in a quaternion skewfield over a totally real number field. We construct non-trivial homology classes on $\Gamma\backslash Sp(n,1)/(Sp(n) \times Sp(1))$ in dimensions $4, 8, \ldots, 4n-4$ using special cycles (as in [MR] and [RSchw]) and describe the way to obtain non $Sp(n,1)$-invariant classes and to give a lower bound of the dimension of the part of homology generated by certain translates of the cycles in question. Furthermore, we discuss orientability questions arising in the analogous context of arithmetically defined real hyperbolic manifolds. In addition, we present an elementary proof of the stable range property of an arithmetic order Λ in a skewfield over a number field, a result which is used to study the structure of general and special linear groups over such an order (these are related to our choice of the arithmetic group Γ above).

Zusammenfassung

In dieser Arbeit werden geometrische Methoden ([Mil], [MR], [RSchw], [LS]) zur Konstruktion von Kohomologieklassen in lokal symmetrischen Räumen angewandt, um den Fall von arithmetisch definierten kompakten quaternionisch hyperbolischen Mannigfaltigkeiten zu behandeln. Jene sind von der Form $\Gamma\backslash Sp(n,1)/(Sp(n) \times Sp(1))$ für eine geeignete torsionsfreie arithmetische Gruppe Γ entstehend aus einer arithmetischen Ordnung in einem Quaternionenschiefkörper über einem total reellen Zahlkörper. Es werden, unter Verwendung spezieller Zykeln (wie in [MR] and [RSchw]), nicht triviale Homologieklassen auf $\Gamma\backslash Sp(n,1)/(Sp(n) \times Sp(1))$ in den Graden $4, 8, \ldots, 4n-4$ konstruiert. Anschließend wird der Weg beschrieben, nicht $Sp(n,1)$-invarianten Klassen und eine untere Schranke der Dimension der Kohomologiegruppen zu erhalten.
Darüber hinaus diskutieren wir Fragen der Orientierbarkeit wie im analogen Fall einer arithmetisch definierten reellen hyperbolischen Mannigfaltigkeit. Abschließend wird ein elementarer Beweis der stabilen Rang Bedingung einer arithmetischen Ordnung Λ in einem Schiefkörper über einem Zahlkörper gegeben. Dieses Resultat wird in der Strukturanalyse von allgemeinen linearen Gruppen über Λ benutzt. Diese wirkt sich auf die Struktur von Γ aus.