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The elastic-gravitational equations in global seismology with low regularity

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Abstract

The elastic-gravitational equations describe the deformation and the gravitational field of an elastic self-gravitating continuum. These second-order evolutionary partial differential equations, boundary conditions, and interface conditions are the governing equations of global seismology. Solutions of this system – as well as its generalization including wave attenuation and viscoelastic rheology – cover a great variety of the Earth's motions, ranging from the high frequency seismic body and surface waves, the longer-period free oscillations and tidal deformations, to the effects of surface loads like glacial rebound. In this thesis we derive and analyze the system of elastic-gravitational equations for a uniformly rotating composite fluid-solid earth model under minimal assumptions concerning the smoothness of material parameters and interface geometry. For this purpose we first establish a consistent mathematical formulation of the low-regularity planetary model within the general framework of nonlinear continuum mechanics. Then we apply calculus of variations in a Sobolev space setting on composite domains and show how the weak form of the linearized elastic-gravitational equations directly arises from Hamilton's principle of stationary action. Finally we prove existence and uniqueness of weak solutions by the method of energy estimates and discuss additional regularity properties. Thereby we obtain a complete low-regularity variational model for seismic waves in a self-gravitating planet. This model allows to further develop the spectral theory of the Earth, may improve numerical modeling of wave propagation, and eventually provides a basis for investigating the inverse problem in global seismology.

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Contents

Introduction

The Earth is a complex physical system in which various mechanical, thermodynamical, chemical, and electromagnetic phenomena are interrelated. Due to the broad range of spatial and temporal scales, different theories will be correct within the different regimes of geodynamic processes. Specifically, increasing the time scale from seconds to millions of years, the description of the mechanical properties of the globe changes from elastic (seismic waves, free oscillations, Earth tides), to viscoelastic (post-seismic deformation, glacial-isostatic adjustment, plate tectonics), and viscous (mantle convection).

The elastic-gravitational equations in global seismology describe the interaction of elastic and gravitational forces in the uniformly rotating Earth. Gravity is clearly negligible in high-frequency seismic wave propagation or is just considered as a passive restoring force in geophysical fluid dynamics of the atmosphere or oceans. However, the Earth's gravest seismic normal modes deform the whole planet and the resulting change in the mass density distribution will induce a detectable temporal variation of the global gravity field, which in turn actively influences the oscillations of the Earth. The elastic-gravitational equations essentially consist of the seismic wave equation, which is generalized to include this effect of self-gravitation. Thus, this second-order evolutionary system of partial differential equations for the displacement and the gravitational potential can be considered as the governing equations of low-frequency global seismology.

In this work, the elastic-gravitational equations are rigorously derived and analyzed within a consistent mathematical model of low regularity, representing a planetary body with a very general three-dimensional structure. Specifically, we only require that the material parameters, that is, mass density and elastic coefficients, are L^∞ (bounded measurable functions) and allow for rough boundary and interface geometry (Lipschitz surfaces). The derivation as well as the analysis of the governing equations are based on variational methods within an appropriate Sobolev space setting: On one hand, Hamilton's principle of stationary action directly leads to the weak formulation of the system, which naturally incorporates the boundary and interface conditions. On the other hand, we employ the variational solution technique of energy estimates to infer existence and uniqueness of solutions of the linearized system of elastic-gravitational equations.

Outline and main results

The thesis consists of two parts: Part I (Chapters 1–3) discusses the prerequisites and Part II (Chapters 4–8) contains the main results of this work.

In view of the needs that readers with a pure geophysical background might have, Part I provides a rather detailed review of the relevant mathematical methods. In Chapter 1 we give an introduction to continuum mechanics and elasticity, with special emphasis on mathematical modeling aspects. After a very brief discussion of global seismology and the gravity field of the Earth, Chapter 2 presents the complete system of elastic-gravitational equations, including all

boundary and interface conditions, both in its full nonlinear version and in its linearized form. Chapter 3 is devoted to variational solution methods for partial differential equations. We discuss calculus of variations including the classical Euler-Lagrange equations as well as the natural boundary and interface conditions. Then we summarize important properties of Sobolev spaces, introduce the concept of weak solutions of partial differential equations, and finally we present the key ingredients of the method of energy estimates for proving well-posedness of second-order evolution equations with time-independent coefficients.

In Part II, a consistent variational model for the elastic-gravitational equations in global seismology with low regularity is developed. In Chapter 4 we introduce a composite fluid-solid earth model describing a terrestrial planet. Since frictional dissipation can be neglected in the seismic regime, the system of governing equations is conservative and may be obtained variationally via Hamilton's principle. The corresponding action integral is defined in Chapter 5 based on energy considerations. In Chapter 6, we set up the linearized variational model by defining small perturbations of the fields and by approximating the action up to second order. This enables us to employ calculus of variations in a Sobolev space setting and rigorously deduce the weak formulation of the linearized elastic-gravitational equations in Chapter 7. In Chapter 8 we finally prove existence and uniqueness of solutions and discuss additional regularity properties.

We summarize the main results of the individual chapters of Part II in more detail:

• **The composite fluid-solid earth model** (Chapter 4): We set up a general planetary model that is composed of fluid and solid interior regions within the framework of nonlinear continuum mechanics. First we define the notions of interior boundaries and Lipschitz composite domains (Definitions 4.1, 4.2) and present an adapted version of the divergence theorem (Lemma 4.11). These are the prerequisites needed to introduce the composite fluid-solid earth model as a disjoint union of Lipschitz domains (Definition 4.12). We define the class of admissible motions φ as piecewise Lipschitz regular maps, additionally satisfying certain kinematical interface conditions on the welded solid-solid interfaces and on the perfectly slipping fluid-solid interfaces (Definition 4.19). The other state variables of the earth model are the spatial mass density ρ^s and the spatial gravitational potential Φ^s.

• **Action** (Chapter 5): We introduce the **nonlinear variational model** by summarizing the consistent regularity conditions on the interface geometry and the state variables, which were identified in the previous chapter (Assumption 1). First-principle energy considerations lead us to the definition of the action, which consists of a volume integral plus a surface integral, taking into account the work done by slip along fluid-solid interfaces. Self-gravitation is modeled by the Poisson equation, which is rigorously included in the variational principle via a Lagrange multiplier argument (Section 5.2.3). In order to apply this method in a Sobolev space setting, the separation of the gravitational monopole term turns out to be a crucial ingredient (Lemma 4.29). Conservation of mass allows us to express ρ^s in terms of its initial value ρ^0 (Section 5.2.4). As a brief interlude, we review an abstract geometric variational formulation of continuum mechanics that also incorporates the kinematic interface conditions via generalized variations (Section 5.3). Finally we show that, at least on a formal level, stationarity of the proposed action indeed yields the Euler-Lagrange equations and natural boundary/interface conditions that coincide with the correct nonlinear system of elastic-gravitational equations (Section 5.4).

• **Linearization** (Chapter 6): We linearize the earth model and decompose the state variables φ and Φ^s into their equilibrium values and small perturbations, representing the displacement u and the incremental gravitational potential Φ^{s1}. Due to static equilibrium between gravitational, centrifugal, and elastic forces, the equilibrium stress (prestress) must not vanish. Consequently the theory of **prestressed linearized elasticity** has to be employed, which in particular results in different incremental stresses (Section 6.4). In view of the regularity conditions of Assumption 1, we find that the linearized fields are naturally defined as piecewise H^1 functions corresponding

to the fluid and solid interior regions. It is straightforward to incorporate the linearization in the variational approach by approximating the action up to second order in the perturbations (Lemmas 6.7, 6.8). However, the occurrence of quadratic terms in the surface action forces us to additionally assume that the fluid-solid interfaces are at least $\mathcal{C}^{1,1}$ (which guarantees bounded curvature) and that the equilibrium pressure p^0 is Lipschitz continuous on the interface; otherwise the interpretation of the surface action in terms of a surface Sobolev duality between $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ seems to be impossible. We conclude with a discussion of the fluid-solid interface energy, explaining why the surface action will vanish in the nonlinear model, but must be nonzero in the linearized model (Section 6.6.3).

• **Variational derivation of the elastic-gravitational equations** (Chapter 7): We introduce the **linearized variational model** which consists of L^∞ material parameters (density and elastic coefficients), piecewise H^1 fields Φ^{s1} and u with tangential slip, and the additional fluid-solid surface regularity conditions identified in the previous chapter (Assumption 2). A general version of calculus of variations for quadratic Lagrangians in a Sobolev space setting is presented, which in particular includes surface integrals and composite domains (Theorem 7.4). In the proof, the unboundedness of the coefficient $\nabla\nabla\Phi^0$ (Hessian of the equilibrium gravitational potential) is treated via Sobolev embedding techniques. By applying Theorem 7.4 to the approximated action, we rigorously obtain the weak formulation of the elastic-gravitational equations (Section 7.3). Stationarity of the first-order terms yields the equilibrium equations (static equilibrium equation (2.19), equilibrium Poisson equation (2.20)); stationarity of the second-order terms implies the dynamical equations (equation of motion (2.26), perturbed Poisson equation (2.27)). Moreover, we identify higher regularity conditions that are sufficient to establish the strong form of the equations, as well as the boundary and interface conditions (Assumption 3, Corollary 7.6). Thereby we recover the complete system of linearized governing equations from Hamilton's principle within a consistent low-regularity setting.

• **Analysis of the elastic-gravitational equations** (Chapter 8): We analyze the linearized system of elastic-gravitational equations and investigate existence, uniqueness, and regularity of solutions within the low-regularity setting defined in Assumption 2. In Propositions 8.1 and 8.2 we establish solvability and regularity properties of the weak equilibrium equations. In particular, solvability implies higher regularity, showing weak-strong uniqueness of the equilibrium equations. The mapping properties of the solution operator of the perturbed Poisson equation (Lemma 8.3) allow us to consider the gravitational contributions as zero-order terms in the equation of motion. Solvability of the dynamical equations is established by the method of **energy estimates**, which requires additional positivity conditions for the material parameters (Assumption 4). We argue that the surface term can be incorporated in the estimates, provided that the fluid-solid interface curvature is sufficiently low compared to the elastic coefficients. In the main result, Theorem 8.9, we prove **existence and uniqueness of weak solutions** of the equation of motion under the Assumptions 2 and 4. We conclude with a discussion of further regularity properties of the solution under more restrictive conditions on the coefficients: Propositions 8.10 and 8.11 show that in case of piecewise Lipschitz material parameters, validity of the dynamical equations in strong form, including the boundary and interface conditions, can be inferred from the weak solution.

In summary, we describe, derive, and analyze the complete system of elastic-gravitational equations (2.19)–(2.35) within a consistent variational Sobolev space framework and under minimal assumptions on interface geometry and regularity of material parameters.

The results of the Chapters 4–7 are also presented in [BdHH17].

Previous work and perspectives

The formal variational derivation of the linearized elastic-gravitational equations and the corresponding second-order action integral is well known in theoretical seismology, e.g. [PJ58, Woo76, WD78, Gil80, WR90, LR90, DT98, WD07]. Without a connection to its variational foundation, the weak formulation of the linearized governing equations is also discussed in the geophysical literature [Val86]. The question of well-posedness of the linearized system was recently addressed in [dHHP15].

A complete solution theory for the general system of nonlinear elastic-gravitational equations presently seems to be out of reach. In fact, the governing equations of nonlinear elastodynamics form a quasilinear system of multidimensional hyperbolic conservation laws, which is hard to analyze [MH83, Bal10, Daf16]. Some results on short-time existence and uniqueness of solutions could be established in $\mathbb{R}^3$ via semigroup methods by [HKM77] and for the displacement problem on a bounded domain via energy estimates by [DH85]. On the contrary, the solution theory for the equations of classical linearized elasticity is well developed, at least in the case of sufficiently smooth coefficients. We mention [HM78], where energy estimates are employed to analyze the equation of motion of classical linearized elasticity on a compact domain with smooth boundary and for smooth material parameters, based on a formulation as a linear first-order symmetric hyperbolic system. However, because of the coupling to gravity and the resulting mixed spatial-material nature of the system, together with the nonzero prestress, the low regularity assumptions, and the presence of fluid-solid interactions, the linearized system of elastic-gravitational equations is situated in between these two extremes.

The mapping properties of the elastic-gravitational operator are related to the spectral problem of the Earth [Val87, Val89a, Val89b]. In particular, questions of completeness of eigenfunctions are crucial, e.g. for the validity of the synthetic seismograms obtained via the mode-summation method [Wah81, LCK98, WD07]. The difficulties are mainly caused by the presence of the fluid outer core [Cro84, SSR84, CHL91, RV09]. Moreover, since the weak formulation is essentially the basis of the finite element method, a consistent function space framework may improve numerical methods for global seismic wave propagation, e.g. the spectral element method [KT02a, KT02b, CV04, MGLD06]. Finally, from a geophysical modeling perspective, the proposed low-regularity variational model for the elastic-gravitational equations solves the forward problem of global seismology and thereby also provides a framework in which the inverse problem can be investigated.

Notation

In order to make the text accessible to both, an audience from geophysics as well as from mathematics, we feel the necessity to briefly discuss the notation. We highlight only those concepts whose notational differences in the literature are most prominent and thus restrict ourselves to mainly collecting our conventions for tensors and derivatives of functions of multiple variables. Other definitions are presented within the main text.

Our basic setting is the classical Newtonian space-time. Thus we generally work in $\mathbb{R}^n$ or consider subsets of $\mathbb{R}^n \times I$, where $I \subseteq \mathbb{R}$ corresponds to a time interval. General subsets of $\mathbb{R}^n$ are often denoted by Ω, V, or B, where the latter is reserved for continuous bodies, in particular for the Earth. However, in accordance with the geophysical literature [DT98, KB08], Ω will also denote the angular velocity vector of the Earth's rotation.

Components of a **vector** (or a tensor) are defined with respect to Cartesian coordinates in $\mathbb{R}^n$, if not stated differently. The ith component of a vector $a \in \mathbb{R}^n$ is denoted by $a_i \in \mathbb{R}$ and $A_{ij} \in \mathbb{R}$ is the entry of an $m \times n$**-matrix** $A \in \mathbb{R}^{m\times n}$ in the ith row and the jth column:

$$a = (a_j)_{j=1}^{n} \in \mathbb{R}^n \qquad \text{and} \qquad A = (A_{ij})_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}.$$

We view vectors as column vectors, i.e. $\mathbb{R}^n \cong \mathbb{R}^{n\times 1}$, and identify $m \times n$-matrices $A \in \mathbb{R}^{m\times n}$ and linear operators $A\colon \mathbb{R}^n \to \mathbb{R}^m$. If $A \in \mathbb{R}^{n\times n}$ is invertible, we denote its inverse by A^{-1} and set $A^{-T} := (A^{-1})^T$, where $(.)^T$ denotes matrix transposition. The unit matrix in $\mathbb{R}^n$ is $1_{n\times n} = (\delta_{ij})_{i,j=1}^{n}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, called the Kronecker delta. In $\mathbb{R}^3$, the Levi-Civita permutation symbol ϵ satisfies $\epsilon_{ijk} = 1$ if the indices i, j, k are an even permutation of $1, 2, 3$, $\epsilon_{ijk} = -1$ if i, j, k are an odd permutation of $1, 2, 3$, and $\epsilon_{ijk} = 0$ if two indices are equal.

Indexed quantities of more general order $p \in \mathbb{N}$ are denoted by $(A_{i_1\dots i_p})_{i_1,\dots,i_p=1}^{n_1,\dots,n_p}$. These higher-order **tensors** correspond to multilinear forms on the respective Cartesian product. In particular, "matrix", "second-order tensor", and "2-tensor" will be used synonymously. Most of the time we will ignore the difference between covariant and contravariant indices. Specifically, we identify tensors of type $\binom{p}{0}$ and $\binom{0}{p}$ and just call them tensors of order p or simply p-tensors.

We often find it useful to employ **summation convention**, i.e. we take the sum over any index that occurs twice (e.g. $a_k b_k = \sum_k a_k b_k$ or $a_k^2 = \sum_k a_k^2$). As an alternative to index notation we adopt the so-called Gibbs notation (see [Wil60, p. 265 and p. 306]): The **tensor product** (dyadic product) AB, the **contraction** (dot product) $A \cdot B$, and the **double contraction** (double dot product) $A : B$ of quantities A, B with multiple indices are denoted by

$$AB := (A_{\dots\, i} B_{j\, \dots})_{\dots\, ij\, \dots}, \qquad A \cdot B := (A_{\dots\, k} B_{k\, \dots})_{\dots}, \qquad \text{and} \qquad A : B := (A_{\dots\, ij} B_{ij\, \dots})_{\dots}.$$

Gibbs notation is widespread in geophysics [BMS81, DT98], but is not standard in mathematics. Specifically, in the mathematical literature one would replace AB by $A \otimes B$. Furthermore, if A, B are linear operators (matrices) the contraction (matrix multiplication) $A \cdot B$ corresponds to the composition of linear operators which is usually written as AB.

The **norm** in a normed vector space E is generally denoted by $\|.\|_E$. If E is equipped with an **inner product**, this is denoted by brackets $\langle .|. \rangle_E$. The bracket $\langle ., . \rangle_{E',E}$ stands for the **duality** between the topological dual $E' = \mathrm{Lin}(E, \mathbb{R}) = \{f\colon E \to \mathbb{R}, \text{ continuous, linear}\}$ and E; if not stated otherwise, $\langle ., . \rangle$ is the distributional duality between the space of distributions $\mathcal{D}'$ and the space of test functions $\mathcal{D}$. In case of Cartesian products we generally do not indicate dimensions of the spaces in the respective norms, products, dualities, e.g. the norm on the space $E^n = E \times \dots \times E$ is denoted by $\|.\|_E$ (instead of $\|.\|_{E^n}$). In particular, in consistence with the notation for contraction introduced above, the Euclidean inner product of $a, b \in \mathbb{R}^n$ is

$\langle a|b\rangle = a \cdot b = a_k b_k$ (dot product) and the Frobenius inner product of matrices $A, B \in \mathbb{R}^{m\times n}$ is $\langle A|B\rangle = A : B = A_{ij}B_{ij}$ (double dot product). Vertical bars $|.|$ denote both, the absolute value $|a|$ of a real (or complex) number $a \in \mathbb{R}$ (or $\mathbb{C}$) as well as the Euclidean norm $|a| := \sqrt{\langle a|a\rangle}$ of a vector $a \in \mathbb{R}^n$.

The **derivative** Df of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator acting from $\mathbb{R}^n$ to $\mathbb{R}^m$; $Df(a)$ is the derivative of f in the direction of $a \in \mathbb{R}^n$. The matrix representation of Df as an element of $\mathbb{R}^{m\times n}$ follows from the identification $Df \cdot a := Df(a)$:

$$Df = (\partial_j f_i)_{i,j=1}^{m,n}.$$

Here, ∂_j denotes the partial derivative with respect to the jth variable. Note that our convention for the derivative Df as $(\partial_j f_i)_{ij}$ is transposed compared to the one in [DT98]. We further use the row-wise definition of the **divergence** of $f\colon \mathbb{R}^n \to \mathbb{R}^{m\times n}$

$$\operatorname{div} f = (\partial_j f_{ij})_{i=1}^{m},$$

that is, the derivative operator is contracted with the last index of f. If F depends on a matrix $A \in \mathbb{R}^{m\times n}$, then the derivative of $A \mapsto F(., A)$ is given by $\partial_A F$ with components $\partial_{A_{ij}} F$. In particular, if $A = Dy$ for $y\colon \mathbb{R}^n \to \mathbb{R}^m$, then $\partial_{Dy} F$ denotes the derivative of $Dy \mapsto F(., Dy)$, in components:

$$(\partial_{Dy} F)_{ij} = \partial_{\partial_j y_i} F.$$

The derivative operator D is replaced by ∇ if only spatial coordinates are involved; the time derivative is denoted by ∂_t or with an overdot. Specifically, the space-time derivative of $\mathbb{R}^n \times \mathbb{R} \ni (x, t) \mapsto q(x, t) \in \mathbb{R}$ is the following row-vector of length $n + 1$,

$$Dq = ((\nabla q)^T, \partial_t q) = (\partial_1 q, \ldots, \partial_n q, \dot{q}).$$

The column vector $\nabla q = (\partial_j q)_{j=1}^n$ is referred to as the **gradient** of q. However, in case of a vector valued function $\mathbb{R}^n \times \mathbb{R} \ni (x, t) \mapsto f(x, t) \in \mathbb{R}^m$ we disregard transposition of the gradient operator ∇ and identify the space-time derivative with the $m \times (n + 1)$-matrix

$$Df = (\nabla f, \partial_t f) = ((\partial_j f_i)_{i,j=1}^{m,n}, (\dot{f}_i)_{i=1}^m).$$

If $\Omega \subseteq \mathbb{R}^n$ is open, we denote the space of (vector-valued and real) continuous functions f by $\mathcal{C}^0(\Omega, \mathbb{R}^m) = \mathcal{C}^0(\Omega)^m = \{f\colon \Omega \to \mathbb{R}^m, \text{ continuous}\}$ for $n, m \in \mathbb{N}$ (or $\mathcal{C}^0(\Omega)$ if $m = 1$). Analogously $\mathcal{C}^k(\Omega)^m$ consists of k-times continuously differentiable functions $f\colon \Omega \to \mathbb{R}^m$. If $X \subseteq \mathbb{R}^n$ is arbitrary but with nonempty interior, then $f \in \mathcal{C}^k(X)^m$ if there exists an open set $Y \supseteq X$ and a function $g \in \mathcal{C}^k(Y)^m$ such that $g|_X = f$.

If $B \subseteq \mathbb{R}^n$, then the identity map $B \ni x \mapsto x \in \mathbb{R}^n$ (embedding of B in $\mathbb{R}^n$) will be denoted by Id_B, that is, $\mathrm{Id}_B(x) = x$ for all $x \in B$. If $B \times I \subseteq \mathbb{R}^n \times \mathbb{R}$, we use the same symbol to denote the map $B \times I \ni (x, t) \mapsto x \in \mathbb{R}^n$, that is we write $\mathrm{Id}_B(x, t) = x$, under a slight abuse of notation. If $f\colon B \times I \to \mathbb{R}^m$ we will frequently write

$$f_t := f(., t)\colon B \to \mathbb{R}^m.$$

This notation is natural in continuum mechanics, since the time variable t can often be considered as a passive parameter.

Part I

Physical principles and mathematical methods

Chapter 1

Continuum mechanics and elasticity

Continuum mechanics is concerned with the motion, the deformation, and the mechanical interaction of matter possessing certain material properties. Solids and fluids are modeled as continuous bodies, neglecting their internal molecular or atomistic structure.

Classical references for continuum mechanics are the monograph [TT60, TN04], more physical introductions are [Mal69, Gur81, GFA09], and [NH81, MH83, Cia88, Šil97, Ant05] deal with more advanced mathematical aspects, in particular of elasticity theory.

The aim of this chapter is twofold: We wish to explain basic physical principles of continuum mechanics and try to identify a mathematical framework which allows to correctly formulate and analyze the governing equations. We first give a brief overview of the mathematical foundations of continuum mechanics and the modeling of continuous bodies, which is still an active branch of research (Section 1.1). Then we discuss the kinematics and dynamics of continuous bodies, that is, their motion, the material and spatial representation of fields, the fundamental balance laws, and the concept of stress (Section 1.2). Finally we present the material properties of elastic media (Section 1.3).

1.1 Mathematical foundations of continuum mechanics

In order to give a concise mathematical formulation of the fundamental concepts of continuum mechanics, one needs to specify the sets or functions for modeling the kinematics and dynamics of continuous bodies, as well as the constitutive relations. The "correct" model is *"mathematically precise, physically natural, conceptually simple"* [Ant05], *"including all that can possibly be imagined by an engineer but excluding all that can be dreamt up only by an ingenious mathematician"* [NV88]. Ideally, the derivation, the formulation, and the analysis of the field equations should be possible within one and the same mathematical framework. Modeling involves expressing physical relations in mathematical terms. However, this often cannot be done in a straightforward way, especially if many different physical processes are involved. This difficulty is not new: As a matter of fact, the *"mathematical treatment of the axioms of physics"* is Hilbert's Sixth Problem [Hil02].

1.1.1 The governing equations of elasticity

As a motivating example we present the field equations of elastodynamics in space dimension three ($n = 3$). In material representation, they consist of a second-order quasilinear system of

PDEs, comprising the **equation of motion**

$$\rho^0(X)\,\ddot{\varphi}(X,t) - \operatorname{div} T^{\mathrm{PK}}(X,t) = f(X,t), \tag{1.1}$$

the **constitutive equation**

$$T^{\mathrm{PK}}(X,t) = \partial_{\nabla\varphi} W(X, \nabla\varphi(X,t)), \tag{1.2}$$

and the (**kinematical** and **dynamical**) **boundary conditions**

$$\varphi\big|_{\Gamma_D\times[t_0,t_1]} = \varphi_D \qquad \text{and} \qquad T^{\mathrm{PK}}\cdot\nu\big|_{\Gamma_N\times[t_0,t_1]} = \tau_N. \tag{1.3}$$

The Cauchy problem for elastodynamics is to determine the solution φ of this system, given the **initial conditions**

$$\varphi(.,t_0) = \varphi_0 \qquad \text{and} \qquad \dot{\varphi}(.,t_0) = v_0. \tag{1.4}$$

Here the continuous body is modeled by a set $B \subseteq \mathbb{R}^3$ that evolves within the time interval $I = [t_0, t_1]$; the pair $(X,t) \in B \times I$ is a point in classical space-time $\mathbb{R}^3 \times \mathbb{R}$ (see the figure below).

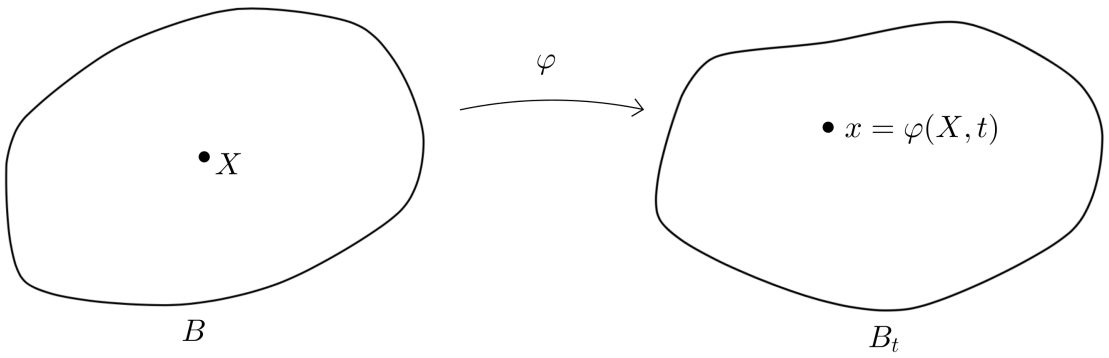

Figure 1.1: The motion $\varphi\colon B\times I \to \mathbb{R}^n$ moves and deforms the original body B to $B_t = \varphi(B,t)$.

The boundary of the body is the disjoint union $\partial B = \Gamma_D \cup \Gamma_N$ which is equipped with the exterior unit normal vector field $\nu\colon \partial B \to \mathbb{R}^3$. Moreover,

$\varphi\colon B\times I \to \mathbb{R}^3$ is the motion of the body (velocity $\dot{\varphi} = \partial_t\varphi$, acceleration $\ddot{\varphi} = \partial_t^2\varphi$),

$\rho^0\colon B \to \mathbb{R}$ is the reference mass density,

$T^{\mathrm{PK}}\colon B\times I \to \mathbb{R}^{3\times 3}$ is the first Piola-Kirchhoff stress tensor,

$f\colon B\times I \to \mathbb{R}^3$ is the external volume force density field,

$W\colon B\times\mathbb{R}^{3\times 3} \to \mathbb{R}$ is the internal energy density per unit volume,

depending on X and the deformation gradient $\nabla\varphi$. Here $\partial_{\nabla\varphi}W$ stands for the derivative of $W(.,F)$ with respect to $F \in \mathbb{R}^{3\times 3}$, in components $(\partial_{\nabla\varphi}W)_{ij} = \partial_{\partial_j\varphi_i}W = \frac{\partial W}{\partial(\partial_j\varphi_i)}$, and div denotes the row-wise divergence: $(\operatorname{div} T^{\mathrm{PK}})_i = (\nabla\cdot T^{\mathrm{PK}})_i = \partial_j T^{\mathrm{PK}}_{ij}$.

The prescribed initial and boundary data are functions

$$\varphi_0\colon B\to\mathbb{R}^3, \quad v_0\colon B\to\mathbb{R}^3 \qquad \text{and} \qquad \varphi_D\colon \Gamma_D\times I\to\mathbb{R}^3, \quad \tau_N\colon \Gamma_N\times I\to\mathbb{R}^3.$$

The elastic-gravitational equations essentially consist of these governing equations of elastodynamics and additionally include the effect of self-gravitation: The volume force is generalized to contain the force of gravity, which in turn depends on the current mass distribution within the body and hence also on its motion.

1.1.2 Modeling continuous bodies

In physics, an n-dimensional **continuous body** is a set that ([Ant05, p. 417], [Sch07]),

(A) can occupy regions in space, and

(B) has volume, mass, can sustain forces, and, in particular, can interact with other bodies.

By requirement (A) it is natural to model a continuous body as a topological space that can be mapped diffeomorphically to suitable subsets of $\mathbb{R}^n$. Then it is convenient to identify the body with the region $B \subseteq \mathbb{R}^n$ which it occupies at some **reference time** $t_0 \in \mathbb{R}$. The evolution of the body within the time interval $I = [t_0, t_1]$ is described via the **motion**, which is defined as a map $\varphi\colon B \times I \to \mathbb{R}^n$ (see Section 1.1.1).

The correct class $\mathcal{B}$ of regions B that can be occupied by continuous bodies and their corresponding motions φ should in particular fulfill the following desiderata [Nol73, GWZ86, NV88]:

(A1) The class $\mathcal{B}$ is closed under finite intersections, finite unions, and complements.

(A2) All regions B in $\mathcal{B}$ have a (hyper-)surface-like boundary ∂B which allows to apply a version of the divergence theorem (Lemma 1.6) for vector fields of suitable regularity.

(A3) The class $\mathcal{B}$ is invariant under differentiable and injective mappings, which correspond to possible motions $\varphi(.,t)\colon B \to \mathbb{R}^n$ for some fixed times $t \in I$.

The first condition (A1) guarantees that if a body is split into finitely many parts, these **subbodies** are still contained in $\mathcal{B}$, and the same is true for a set obtained by merging two or more bodies. In mathematical terms the condition states that the set of bodies must be an algebra of sets (e.g. [Fol99, p. 21]); in [Nol73] it is referred to as the fulfillment of the "axioms of the material universe". The validity of the divergence theorem required by (A2) is crucial, e.g. for the definition of stress tensors or for switching between integral, differential, and weak formulation of the field equations. Injectivity of $\varphi(.,t)$, as imposed in (A3), expresses the requirement of impenetrability of matter, since it prohibits that two initially different points are mapped to one and the same point during the motion. An additional condition such as $\det(\nabla\varphi) > 0$ or a suitable global analog implies that motions also preserve orientation [Cia88].

At first glance the class of sets with piecewise smooth boundary seems to be a reasonable candidate for $\mathcal{B}$. However, intersections might be quite nasty (e.g. the function $x \mapsto e^{-1/x^2}\sin(1/x)$ is smooth on $(0,\infty)$ but has infinitely many oscillations, thus the areas between its graph and the x-axis does not represent a physically reasonable body [NV88]). When bodies are modeled as open or closed sets, then two bodies cannot be in contact without interpenetration. Moreover, one cannot deal with the separation of a body into parts. Thus, open or closed sets violate condition (A1). This suggests to disregard individual points of bodies and to identify bodies whenever they coincide up to a set of measure zero. Yet, the class of all Borel sets is too general: The boundary of a Borel set is in general not surface-like, which is vital for condition (A2) to hold true.

The correct framework to overcome the difficulties described above is provided by geometric measure theory (see Remark 1.1): The sets of finite perimeter in $\mathbb{R}^n$ (or within a given body) form a class $\mathcal{B}$ of continuous bodies (or subbodies) which is closed under intersections, unions, and complements, and allows for the generalized divergence theorem [BF79, BF81], thus satisfying (A1, A2). Admissible motions of bodies $B \in \mathcal{B}$ are the bi-Lipschitz maps $\varphi(.,t)\colon B \to \varphi(B,t)$ for fixed time t (see Definition 1.4), and the class $\mathcal{B}$ is invariant under these mappings (A3).

Thus, a model for geometry and kinematics of continuous bodies fulfilling all desiderata (A1–3) consists of the measure-theoretically open sets of finite perimeter,

$$\mathcal{B} := \{B \subseteq \mathbb{R}^n : \ B \text{ measure-theoretically open, of finite perimeter}\},$$

provided that adapted definitions for unions and complements are employed [Zie83, GWZ86]. The natural general setting to allow for discontinuous processes (for instance fracture, dislocations, incoherent phase transitions, or cavitation) are motions that are functions of bounded variation (BV), or variants thereof [Šil97, Ch. 2.6], [AFP00, Sections 4.6.7-8].

Remark 1.1 (**Geometric measure theory, sets of finite perimeter, and BV functions**). Let $B \subseteq \mathbb{R}^n$ be a Borel set and $\mathscr{L}^n$ denote the n-dimensional Lebesgue measure (see e.g. [Fol99]). Then B is called measure-theoretically open if it coincides with its measure-theoretic interior $B^* := \{x \in \mathbb{R}^n : \lim_{r\to 0+} \frac{\mathscr{L}^n(B\cap B_r(x))}{\mathscr{L}^n(B_r(x))} = 1\}$, i.e. the set of all Lebesgue density points of B, where $B_r(x)$ is the ball centered at x with radius r. The set B is said to be of finite perimeter if the $(n-1)$-dimensional Hausdorff measure $\mathscr{H}^{n-1}$ of its measure-theoretic boundary $\partial^* B := (B^* \cup (B^{\mathrm{c}})^*)^{\mathrm{c}}$ is finite (in this case $\mathscr{H}^{n-1}(\partial B \setminus \partial^* B) = 0$, [Zie83, p. 19, (3)]). Equivalently, B is of finite perimeter if its characteristic function χ_B is of bounded variation $BV(\mathbb{R}^n)$ [AFP00]. If the measure-theoretic boundary and the measure-theoretic unit outer normal are employed, the divergence theorem (1.16) holds in classical form for $\mathcal{C}^1$-functions on sets of finite perimeter. This result can be generalized to Lip- or BV-functions [EG92, CTZ09]. ⌟

Other approaches tried to reduce the amount of measure theory involved (see Remark 1.2). In particular, the concept of measure-theoretically open sets is replaced by employing sets $B \subseteq \mathbb{R}^n$ that are **regularly open**, i.e.

$$B = (\overline{B})^\circ, \tag{1.5}$$

and/or **regularly closed**, i.e.

$$B = \overline{(B^\circ)}. \tag{1.6}$$

Note that regular openness excludes bodies B with missing points, lines, or (hyper-)surfaces, since these features are removed by the closure (compare with the figure p. 15). On the contrary, regular closedness prohibits isolated points, lines, or (hyper-)surfaces attached to a body, because they vanish by taking the open interior.

Remark 1.2 (**Fit regions and structured deformations**). The so-called fit regions, introduced by [NV88], are bounded, regularly open sets of finite perimeter, with negligible boundary with respect to the volume measure (i.e. $\mathscr{L}^n(\partial B) = 0$). A new class of fit regions was proposed by [DP03, DP07], consisting of bounded sets that have regularly open interior (i.e. $B^\circ = (\overline{B})^\circ$), regularly closed closure (i.e. $\overline{B} = \overline{(B^\circ)}$), and a boundary of finite $(n-1)$-dimensional Hausdorff measure (i.e. $\mathscr{H}^{n-1}(\partial B) < \infty$). Admissible motions are $\mathcal{C}^1$-diffeomorphisms. Within the framework of fit regions, fracture of bodies may be described by considering suitable limits of finite unions of fit regions and corresponding piecewise $\mathcal{C}^1$-motions [DPO93]. These structured deformations can be generalized to the variational setting involving special functions of bounded variation, see [CF97] and also [AFP00, Sections 4.6.7-8]. ⌟

By the physical requirement (B) a continuous body should be measurable in various ways:

(B1) A continuous body needs to possess an n-dimensional **volume**:

$$\mathscr{L}^n(\varphi(B,t)) < \infty \qquad \forall\, t \in I, \quad \forall\, B \in \mathcal{B}.$$

(B2) The **mass** of a continuous body is represented by a positive measure $M\colon \mathcal{B} \to \mathbb{R}_0^+$ such that the mass of $B \in \mathcal{B}$ is given by $M(B) = \int_B \rho^0 \,\mathrm{dV} < \infty$ with **mass density** $\rho^0 \in L^1_{\mathrm{loc}}$. The mass of a body must be conserved during its motion (conservation of mass):

$$M(B) = M(\varphi(B,t)) < \infty \qquad \forall\, t \in I, \quad \forall\, B \in \mathcal{B}. \tag{1.7}$$

(B3) There exist measures and corresponding densities describing additional physical properties and forces, modeling various **interactions** with other bodies or fields.

Remark 1.3 (**The continuum hypothesis in physics**)**.** By the continuum hypothesis, the molecular or atomistic structure of matter is neglected. It is assumed that the macroscopic properties of the material can be completely described by density fields. The values of these fields at certain points represent averages of the properties of the real material over small but finite regions. ⌟

We usually write the n-dimensional Lebesgue measure $\mathscr{L}^n$ on $\mathbb{R}^n$ in terms of a volume integral with the n-dimensional **volume element** dV: The volume of a Lebesgue measurable set $A \subseteq \mathbb{R}^n$ is

$$\mathscr{L}^n(A) = \int_A \mathrm{dV}.$$

Thus the volume condition in requirement (B1) reads $\int_{\varphi(B,t)} \mathrm{dV} < \infty$.

Requirement (B2), i.e. the fact that mass is represented by a positive measure as well as the existence of the corresponding density ρ^0, can be deduced from physics by measure theoretical arguments (see e.g. [Ant05, p. 432]): It is physically natural to assume that the empty body has zero mass

$$M(\emptyset) = 0,$$

mass is nonnegative

$$M(B) \geq 0 \qquad \forall\, B \in \mathcal{B},$$

parts of bodies (subbodies) have smaller mass

$$A \subseteq B \qquad \Longrightarrow \qquad M(A) \leq M(B) \qquad \forall\, A, B \in \mathcal{B},$$

and the total mass is obtained by adding masses of disjoint parts (these parts can be countably many, see [Sch07]; in this case the property is referred to as σ-additivity):

$$M\Big(\bigcup_k B_k\Big) = \sum_k M(B_k) \qquad \forall \text{ pairwise disjoint } B_k \in \mathcal{B}.$$

These four conditions are equivalent to saying that M is a positive measure on $\mathcal{B}$. Another physically reasonable assumption is that bodies with zero volume have zero mass, i.e.

$$\mathscr{L}^n(B) = 0 \qquad \Longrightarrow \qquad M(B) = 0 \qquad \forall\, B \in \mathcal{B}. \tag{1.8}$$

Hence it is not possible to consider masses concentrated on surfaces, curves, or points in $\mathbb{R}^3$ (which may be represented in terms of the Dirac delta distribution δ). Condition (1.8) can be rephrased as $M \ll \mathscr{L}^n$, meaning that mass is absolutely continuous with respect to Lebesgue measure. Consequently, by the Radon-Nikodym theorem, there exists a nonnegative function $\rho^0 \in L^1_{\mathrm{loc}}(B)$ such that

$$M(B) = \int_B \rho^0 \,\mathrm{dV} \qquad \forall\, B \in \mathcal{B}. \tag{1.9}$$

Here ρ^0 is the mass density associated to B, whose existence was postulated in (B2). The mass density in spatial representation (Section 1.2), $\rho^s(.,t) \in L^1_{\mathrm{loc}}(\varphi(B,t))$ for $t \in I$, can be introduced along the same lines:

$$M(\varphi(B,t)) = \int_{\varphi(B,t)} \rho^s(.,t) \,\mathrm{dV} \qquad \forall\, t \in I, \quad \forall\, B \in \mathcal{B}.$$

Consequences of conservation of mass will be discussed in Section 1.2.3.

Concerning requirement (B3), the existence of densities associated to other physical properties and forces is inferred in a similar way (see also Remark 1.9).

Given suitable densities in L^1_{loc}, the requirements (B) are fulfilled if $\mathcal{B}$ consists of open and bounded subsets of $\mathbb{R}^n$. Moreover we disregard (A1), which addresses questions of splitting or merging of bodies. Hence we may also assume connectedness of continuous bodies. Thus we may agree on the following intermediate characterization of continuous bodies:

$$\begin{aligned}\mathcal{B} := \{B \subseteq \mathbb{R}^n :\ & B \text{ open, bounded, connected, and with boundary } \partial B \\ & \text{such that (a generalized version of) the divergence theorem holds}\}.\end{aligned} \tag{1.10}$$

However, for our purposes of modeling parts of a general composite fluid-solid Earth, we are particularly interested in bodies with possibly non-smooth boundary.

1.1.3 The divergence theorem for Lipschitz domains

The consideration of sets with non-smooth but sufficiently regular boundaries leads to continuous functions that are not classically differentiable.

Definition 1.4 (Lipschitz and Hölder regularity). Let $\Omega \subseteq \mathbb{R}^n$ be open. A function $f\colon \Omega \to \mathbb{R}$ is called Lipschitz continuous (or Lip) on Ω, if there exists $L > 0$, such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall\, x, y \in \Omega. \tag{1.11}$$

If there exists $H > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(y)| \leq H|x - y|^\alpha \qquad \forall\, x, y \in \Omega, \tag{1.12}$$

the function f is called Hölder continuous with exponent α (or $\mathcal{C}^{0,\alpha}$) on Ω. We say that f is $\mathcal{C}^{k,\alpha}$ if it is $\mathcal{C}^k$ and the kth derivative is $\mathcal{C}^{0,\alpha}$.

On a closed domain $\overline{\Omega}$, Lip and Hölder regular functions form Banach spaces $\text{Lip}(\overline{\Omega})$ and $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ ([Zie89, p. 3], [Eva10, Section 5.1]): Continuous functions $f \in \mathcal{C}^0(\overline{\Omega})$ on a closed domain $\overline{\Omega} \subseteq \mathbb{R}^n$ form a Banach space with respect to the L^∞-norm:

$$\|f\|_{\mathcal{C}^0(\overline{\Omega})} := \sup_{x\in\Omega} |f(x)| = \|f\|_{L^\infty(\Omega)}.$$

The norm of $\mathcal{C}^k(\overline{\Omega})$ is defined as

$$\|f\|_{\mathcal{C}^k(\overline{\Omega})} := \sum_{|\mu|\leq k} \sup_{x\in\Omega} |D^\mu f(x)| = \sum_{|\mu|\leq k} \|D^\mu f\|_{L^\infty(\Omega)} \tag{1.13}$$

(where an equivalent norm is obtained if $\sum_{|\mu|\leq k}$ is replaced by $\max_{|\mu|\leq k}$). Here we have written the partial derivatives in terms of the multi-index notation $D^\mu f$, see (3.23). Based on this, the norm of the Banach space $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ is given by [Cia88, p. 26]

$$\|f\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} := \|f\|_{\mathcal{C}^k(\overline{\Omega})} + \max_{|\mu|=m} \sup_{x,y\in\Omega,\, x\neq y} \frac{|D^\mu f(x) - D^\mu f(y)|}{|x - y|^\alpha}.$$

As is clear from the definition, Lipschitz functions are Hölder continuous with $k = 0$, $\alpha = 1$. Hölder (and thus also Lip) functions are continuous:

$$\text{Lip} = \mathcal{C}^{0,1} \qquad \text{and} \qquad \mathcal{C}^{k,\alpha} \subseteq \mathcal{C}^{0,\alpha} \subseteq \mathcal{C}^0 \qquad (k \in \mathbb{N}, \quad 0 < \alpha \leq 1).$$

Indeed, continuity of $f\colon \Omega \to \mathbb{R}$, that is $\lim_{y\to x} f(y) = f(x)$ for all $x \in \Omega$, directly follows from (1.12) (or (1.11)):

$$|f(x) - f(y)| \le H|x-y|^\alpha \to 0 \qquad \text{as} \qquad x \to y.$$

By Rademacher's theorem [EG92, Section 3.1, p. 79], Lip functions are almost everywhere differentiable with bounded derivative:

$$f \in \mathrm{Lip} \quad \Longrightarrow \quad \partial_j f \in L^\infty \quad (j = 1, \dots, n). \tag{1.14}$$

Consequently, on an open set $\Omega \subseteq \mathbb{R}^n$, the space $\mathrm{Lip}(\Omega)$ may be identified with the Sobolev space (see Definition 3.12 in Section 3.2.1)

$$W^{1,\infty}(\Omega) = \{f \in L^\infty(\Omega) : \, Df \in L^\infty(\Omega)^n\}.$$

The standard example of a Lipschitz continuous function is the absolute value $x \mapsto |x|$ on $\mathbb{R}$ (or the Euclidean norm on $\mathbb{R}^n$). As this example shows, Lip and thus also Hölder functions are not necessarily smooth but may have corners and edges.

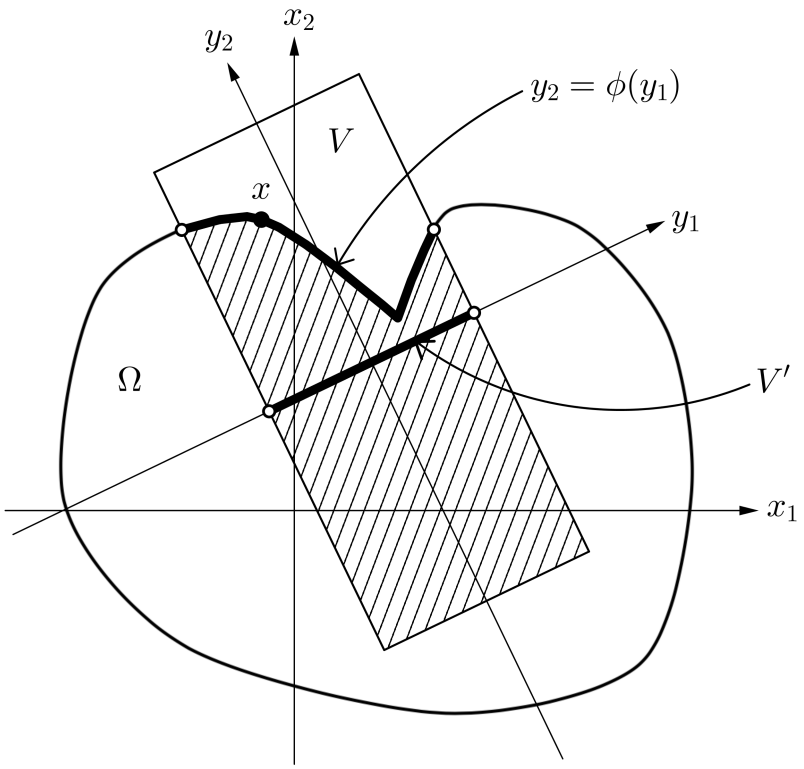

Figure 1.2: Illustration of Definition 1.5 of a Lip-domain.

We are now ready to define Lip-domains and surfaces following [Gri85, Def. 1.2.1.1, p. 5]. Similar formulations can be found in [NH81, p. 17], [Cia88, p. 32], or [Zie89, p. 64].

Definition 1.5 (Lipschitz domains). A bounded open connected set $\Omega \subseteq \mathbb{R}^n$ is called a Lipschitz domain if its boundary $\partial\Omega$ is locally (in a finite cover by open neighborhoods) given as the graph of a Lipschitz continuous function and such that Ω in this local representation is located only on one side of the graph describing the boundary. More precisely, the conditions on the bounded open connected set $\Omega \subseteq \mathbb{R}^n$ are the following: For all $x \in \partial\Omega$ there exists a neighborhood $V \subseteq \mathbb{R}^n$ of x which is an n-rectangle in new orthogonal coordinates $(y_1, \dots, y_n)$, i.e.

$$V := \{(y_1, \dots, y_n) : \, -a_k < y_k < a_k \quad (k = 1, \dots, n)\} \qquad \text{with} \quad a_k > 0 \quad (k = 1, \dots, n),$$

and there exists a Lip function $\phi\colon V' \to \mathbb{R}$ defined in the $(n-1)$-rectangle

$$V' := \{(y_1, \ldots, y_{n-1}) : \ -a_k < y_k < a_k \quad (k = 1, \ldots, n-1)\}$$

with $|\phi(y')| \le a_n/2$ for all $y' = (y_1, \ldots, y_{n-1}) \in V'$, such that

$$\begin{aligned} \Omega \cap V &= \{(y', y_n) \in V : \ y_n < \phi(y')\}, \\ \partial\Omega \cap V &= \{(y', y_n) \in V : \ y_n = \phi(y')\}. \end{aligned}$$

The boundaries of Lip-domains may have corners or edges, whereas sets with cuts or cusps are not allowed [NH81]. In particular, Lip-domains are regularly open $\Omega = (\overline{\Omega})^\circ$ and have regularly closed closure $\overline{\Omega} = \overline{(\Omega^\circ)}$, see (1.5) and (1.6).

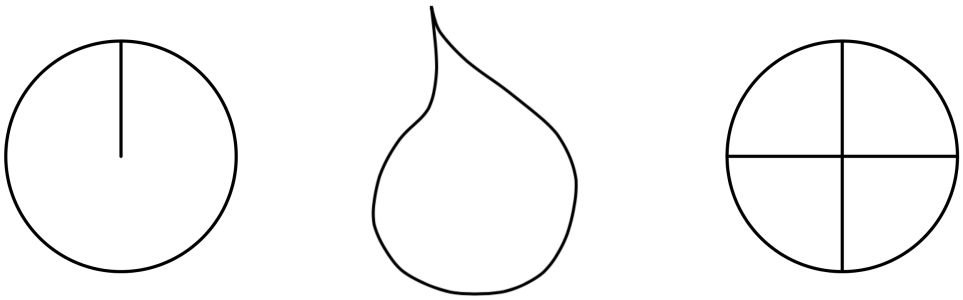

Figure 1.3: Examples of sets in $\mathbb{R}^2$ that are not Lip-domains according to Definition 1.5: Circles with cuts (left & right); a set with a cusp (middle). The cuts are interior boundaries of the circles; only the circle on the right is a Lip-composite domain (Definitions 4.1, 4.2).

The closure $\overline{\Omega}$ of a Lip-domain Ω is an n-dimensional Lip-**submanifold** in $\mathbb{R}^n$ with boundary, see [Gri85, Def. 1.2.1.2, p. 6 and p. 7]. Any connected $(n-1)$-dimensional submanifold $S \subseteq \partial\Omega$ is a Lip-submanifold (possibly with boundary ∂S) and S is called a Lip-**surface** (to be precise, a Lip-hypersurface). In particular, the Lip-surface $\partial\Omega$ is orientable and we choose

$$\nu\colon \partial\Omega \to \mathbb{R}^n$$

as the **exterior unit normal vector**. In the notation of Definition 1.5 we have [Gri85, p. 37]

$$\nu(y', \phi(y')) := \frac{1}{\sqrt{|\nabla\phi(y')|^2 + 1}} \begin{pmatrix} -\nabla\phi(y') \\ 1 \end{pmatrix} = \frac{(-\partial_1\phi(y'), \ldots, -\partial_{n-1}\phi(y'), 1)^T}{\sqrt{(\partial_1\phi(y'))^2 + \ldots + (\partial_{n-1}\phi(y'))^2 + 1}} \tag{1.15}$$

for a.a. $y' = (y_1, \ldots, y_{n-1}) \in V'$. Since ϕ is Lip on V', Rademacher's theorem (1.14) implies that the components of ν are bounded and measurable. By a partition of unity argument, ν may be extended to an L^∞ vector field in a neighborhood of $\overline{\Omega}$, which then is restricted to $\partial\Omega$. Consequently, the unit normal vector of a Lip-domain Ω satisfies $\nu \in L^\infty(\partial\Omega)^n$.

Definition 1.5 of Lip-domains can be generalized to $\mathcal{C}^{k,\alpha}$-**domains**, boundaries, and surfaces upon replacing Lip $= \mathcal{C}^{0,1}$ by $\mathcal{C}^{k,\alpha}$ for $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ [Cia88, p. 34]. We note that $\mathcal{C}^{k,\alpha}$-domains with $0 < \alpha < 1$ may have fractal boundaries, see e.g. [Fed88, Section 6.4], [Hol95, Section 4.1]. However, aiming at modeling the major parts of the Earth without taking into account subtle differences in the fine structure of boundary topography, it appears to be natural to restrict ourselves to the case $\alpha = 1$ and only consider $\mathcal{C}^{k,1}$-domains.

Equation (1.15) implies that the unit normal of a $\mathcal{C}^{k,1}$-domain is a $\mathcal{C}^{k-1,1}$ vector field, if $k \ge 1$. In particular, $\mathcal{C}^{1,1}$-domains have Lip$(= \mathcal{C}^{0,1})$ unit normals, i.e. ν is almost everywhere differentiable.

We recall the classical divergence theorem (Gauß theorem) and the associated formula for integration by parts (a variant of Green's formula), but state them already in the form of their natural extension to H^1 regular matrix-valued fields on Lip-domains:

Lemma 1.6 (**Divergence theorem and integration by parts**). *If $\Omega \subseteq \mathbb{R}^n$ is a* Lip*-domain and $f \in H^1(\Omega)^{m\times n}$, then*

$$\int_\Omega \operatorname{div} f \,\mathrm{dV} = \int_{\partial\Omega} f \cdot \nu \,\mathrm{dS}. \tag{1.16}$$

If, in addition $h \in H^1(\Omega)^m$, then Green's formula for integration by parts holds:

$$\int_\Omega f : Dh \,\mathrm{dV} = -\int_\Omega h \cdot \operatorname{div} f \,\mathrm{dV} + \int_{\partial\Omega} h \cdot f \cdot \nu \,\mathrm{dS}. \tag{1.17}$$

The divergence theorem (1.16) is proved in e.g. [CTZ09, Thm. 2.35] for Ω having finite perimeter, which is true for Lip-domains [Zie89, p. 248]. The surface integrals have to be understood as Sobolev dualities of the traces of the component functions of f in $H^{\frac{1}{2}}(\partial\Omega)$ with corresponding components of ν in $L^\infty(\partial\Omega) \subseteq L^2(\partial\Omega) \subseteq H^{-\frac{1}{2}}(\partial\Omega)$ in the case of (1.16), or, for (1.17), of $h \in H^{\frac{1}{2}}(\partial\Omega)^m$ with $f \cdot \nu$ in $H^{\frac{1}{2}}(\partial\Omega)^{m\times n} \cdot L^\infty(\partial\Omega)^n \subseteq H^{-\frac{1}{2}}(\partial\Omega)^m$ (see Lemma 3.14). Instead of having $f \in H^1(\Omega)^{m\times n}$, the equations (1.16) and (1.17) still hold if one only assumes

$$f \in H_{\operatorname{div}}(\Omega)^{m\times n} := \{f \in L^2(\Omega)^{m\times n} : \operatorname{div} f \in L^2(\Omega)^m\}, \tag{1.18}$$

since $f \in H_{\operatorname{div}}(\Omega)^{m\times n}$ possesses normal traces $f \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)^m$ (see [KO88, p. 93], [DL92, p. 511]). The formula for integration by parts (1.17), also presented in [Gri85, p. 53], is a special case of an abstract Green's formula for linear operators on Hilbert spaces, which is established e.g. in [KO88, Theorem 5.8, p. 90]. If the functions f, h are sufficiently smooth (e.g. $\mathcal{C}^1$), then the identity directly follows from the divergence theorem (1.16) applied to

$$\operatorname{div}(h \cdot f) = \partial_j(h_i f_{ij}) = (\partial_j h_i) f_{ij} + h_i(\partial_j f_{ij}) = f : Dh + h \cdot \operatorname{div} f.$$

However, we note that if the components of f and h are merely H^1, then the calculation above shows that $\operatorname{div}(h \cdot f) \in L^1(\Omega)$ (by Cauchy-Schwarz, $L^2 \cdot L^2 \subseteq L^1$), but, in general, $h \cdot f$ is not an element of $H_{\operatorname{div}}(\Omega)^m$ and thus the divergence theorem cannot be directly applied to derive (1.17).

Definition 1.5 and the validity of the divergence theorem (Lemma 1.6) imply that Lip-domains satisfy all requirements for elements of $\mathcal{B}$ as imposed in (1.10). Moreover, as already indicated in the formulation of the divergence theorem, Lip-domains guarantee good properties of Sobolev spaces, which in turn are important for the weak formulation of partial differential equations, as for instance the equation of motion of a continuous body. This leads us to the following final class of bodies for our purposes:

$$\mathcal{B} := \{B \subseteq \mathbb{R}^n : B^\circ \text{ is a Lip-domain}\}. \tag{1.19}$$

Requiring that only the open interior B° is a Lip-domain allows us to include bodies that may contain parts of their boundaries (in accordance with [Ant05, p. 418]). This slightly generalizes the class (1.10) with respect to openness of bodies (and will be the reason for taking the closure of B in (1.20) compared to the definition given in Section 1.1). As concerns regularity, our basic setting will be the Lipschitz framework and we will consider $\mathcal{C}^{k,1}$-domains with $k \geq 1$ only when necessary.

1.2 Kinematics and balance principles

Continuum mechanics is a branch of classical mechanics and thus splits into **kinematics** and **dynamics**. Kinematics describes the motion and associated quantities, whereas dynamics studies the forces that drive the motion. The dynamical laws of continuum mechanics are formulated in terms of **balance principles** and **constitutive laws**.

1.2.1 The motion of a continuous body

Let B be a body in $\mathbb{R}^n$, that is $B \in \mathcal{B}$. The space $\mathbb{R}^n$ is also referred to as the **ambient space** and we consider the time interval $I = [t_0, t_1]$. We write I° for the open interval (t_0, t_1). The motion of B is a function

$$\varphi\colon \overline{B} \times I \to \mathbb{R}^n \tag{1.20}$$

that represents the change of the shape and position of the body in space as time passes (see Section 1.1). The motion

$$\varphi_t = \varphi(.,t)\colon \overline{B} \to \mathbb{R}^n$$

maps the **reference configuration** $\overline{B}$ of the body to its **current configuration**

$$\varphi_t(\overline{B}) = \{x \in \mathbb{R}^n : \exists\, X \in \overline{B} :\ x = \varphi_t(X)\},$$

representing the actual position and shape of the body. The reference state at time t_0 is chosen such that no motion has yet occurred, i.e.

$$\varphi_{t_0} = \mathrm{Id}_{\overline{B}}. \tag{1.21}$$

In other words, we identify material particles with their initial positions, denoted by

$$X = \varphi_{t_0}(X) \in \overline{B}.$$

We use x to denote points in the ambient space $\mathbb{R}^n$. The point

$$x = \varphi(X,t) = \varphi_t(X) \in \mathbb{R}^n$$

is the position of a material particle of the body at time $t \in I$ that initially, i.e. at time t_0, was at position $X \in \overline{B}$. The situation is illustrated in the figure on p. 9.

Remark 1.7 (**Bundle structure of trajectories**)**.** The collection of all configurations of the body, that is, $\bigcup_{t\in I}(\varphi_t(\overline{B}) \times \{t\}) = \{(x,t) \in \mathbb{R}^n \times I :\ x = \varphi_t(X),\ X \in \overline{B},\ t \in I\}$, has the structure of a bundle on I with fibers $\varphi_t(\overline{B})$ for $t \in I$. This bundle can be considered as the **trajectory** of the body (see also Section 5.3). ⌟

Time derivatives yield the velocity and acceleration field of the body. Also the spatial derivatives (spatial gradient, divergence, and Jacobian determinant) are of physical significance, and in particular give rise to various strain tensors quantifying the change of lengths and angles during the motion of the body. We formally introduce these kinematical quantities derived from the motion. In the classical setting, sufficient regularity for introducing these basic concepts is guaranteed by $\mathcal{C}^2$-regular motions $\varphi\colon \overline{B} \times I \to \mathbb{R}^n$ (see Definition 4.14 or [MH83, Definition 1.4, p. 27]) of bodies $B \in \mathcal{B}$ as in (1.10).

The first (partial) time derivative of the motion defines the **velocity**

$$v\colon \overline{B} \times I \to \mathbb{R}^n, \qquad v := \dot{\varphi} \tag{1.22}$$

and the second time derivative of the motion is the **acceleration**

$$a\colon \overline{B} \times I \to \mathbb{R}^n, \qquad a := \dot{v} = \ddot{\varphi}. \tag{1.23}$$

The derivative of the motion with respect to $X \in B$ is commonly referred to as the **deformation gradient**

$$\nabla\varphi\colon \overline{B} \times I \to \mathbb{R}^{n\times n}, \qquad \nabla\varphi = (\partial_j \varphi_i)_{i,j=1}^n. \tag{1.24}$$

Being the derivative of the motion, the deformation gradient measures the difference of final positions of initially adjacent material particles (actually, the word “gradient” is inappropriate

for the derivative of a map $\overline{B}\times I \to \mathbb{R}^n$; nevertheless, this name is firmly established in continuum mechanics, see also [MH83, p. xii]). The motion φ, taking values in the ambient space $\mathbb{R}^n$, is differentiated with respect to points in the reference body B. Consequently, the deformation gradient $\nabla\varphi$ is a two-point 2-tensor field (see Remark 1.10).

The **Jacobian determinant** of φ is the determinant of the deformation gradient:

$$J\colon \overline{B}\times I \to \mathbb{R}, \qquad J := \det(\nabla\varphi). \tag{1.25}$$

If $J > 0$, the motion preserves the orientation of the material and is called positively oriented. Finally, we introduce the **Green strain tensor** associated to the motion as the symmetric 2-tensor field

$$e\colon \overline{B}\times I \to \mathbb{R}^{n\times n}_{\mathrm{sym}}, \qquad e := \frac{1}{2}\left(\nabla\varphi^T\cdot\nabla\varphi - 1_{n\times n}\right) \quad \text{i.e.} \quad e_{ij} = \frac{1}{2}\left((\partial_i\varphi_k)(\partial_j\varphi_k) - \delta_{ij}\right). \tag{1.26}$$

The Green strain tensor quantifies the deformation of the body, that is, the changes in lengths and angles, but without the effects of rigid body motion or rotation.

1.2.2 Material and spatial representation

Classical physical fields are quantities depending on space and time, that is, functions on space-time taking values in some vector space Q (in most applications $Q = \mathbb{R}^m$ for some $m \in \mathbb{N}$). According to their definition, the motion φ, the velocity v, the acceleration a, the deformation gradient $\nabla\varphi$, the Jacobian J, and the strain tensor e are (scalar, vector, and tensor valued) physical fields that are expressed as functions of space-time points in the reference configuration of the body:

$$(X,t) \in \overline{B}\times I.$$

This is the so-called **material representation** (Lagrangian representation, referential representation). However, physical fields can also be described in the so-called **spatial representation** (Eulerian representation) as functions of space-time points in the current configuration of the body:

$$(x,t) \in \mathbb{R}^n\times I \quad \text{with} \quad x = \varphi_t(X) \ \text{ for } \ X \in \overline{B}.$$

The material and the spatial representation will be denoted by q and q^s respectively, where, for all $t \in I$, the material representation q_t is a function of the material points $X \in \overline{B}$ but the spatial quantity q^s_t depends on the space points $x \in \varphi_t(\overline{B}) \subseteq \mathbb{R}^n$:

$$q\colon \overline{B}\times I \to Q \qquad \text{and} \qquad q^s\colon \bigcup_{t\in I}(\varphi_t(\overline{B})\times\{t\}) \to Q.$$

Thus, in the material picture one describes the quantity as a function q of the reference configuration $\overline{B}\times I$, whereas in the spatial picture the quantity is expressed as a function q^s defined on the trajectory $\bigcup_{t\in I}(\varphi_t(\overline{B})\times\{t\})$ (see Remark 1.7).

Spatial and material representations are related via the motion φ:

$$q_t = q^s_t\circ\varphi_t, \tag{1.27}$$

i.e., for all $(X,t) \in \overline{B}\times I$ the identity $q(X,t) = q^s(\varphi(X,t),t)$ holds. If φ_t is invertible on $\overline{B}$, then this is equivalent to

$$q^s_t = q_t\circ\varphi_t^{-1}, \tag{1.28}$$

i.e., for all $(x,t) \in \bigcup_{t\in I}(\varphi_t(\overline{B})\times\{t\})$ the identity $q^s(x,t) = q(\varphi_t^{-1}(x),t)$ holds.

In other words, the field in material representation is the field in spatial representation composed (pulled-back) by the motion for each fixed time. Provided that motions are admissible, spatial and material picture are completely equivalent (see Section 4.2.2 and Lemma 4.20).

To simplify the notation we will sometimes omit the time-variable in (1.27) and just write

$$q = q^s \circ \varphi.$$

In many situations, spatial quantities (e.g. the gravitational field) are naturally defined on the whole space, that is,

$$q^s \colon \mathbb{R}^n \times I \to Q.$$

However, starting from the material representation q, one only gets and needs information about the restriction of q^s to the trajectory $\bigcup_{t\in I}(\varphi_t(\overline{B}) \times \{t\})$, see (1.28). The diagram

$$\begin{array}{ccc} \varphi_t(\overline{B}) & \hookrightarrow & \mathbb{R}^n \\ {\scriptstyle \varphi_t}\uparrow & & \downarrow{\scriptstyle q_t^s} \\ \overline{B} & \xrightarrow[q_t]{} & Q \end{array}$$

illustrates the situation at fixed time t.

The spatial representation of the motion itself gives the identity map on the current configuration $\varphi_t(\overline{B})$ of the body: $\varphi_t^s = \mathrm{Id}_{\varphi_t(\overline{B})}$. Indeed, (1.28) implies $\varphi_t^s(x) = (\varphi_t \circ \varphi_t^{-1})(x) = x$, that is, $\varphi_t^s(x) = \varphi^s(x,t) = \varphi^s(\varphi(X,t),t) = \varphi(X,t) = x$.

By $\varphi_{t_0} = \mathrm{Id}_{\overline{B}}$ (1.21), spatial and material representation of a physical field coincide at reference time t_0:

$$q_t = q_t^s \circ \varphi_t \qquad \text{and} \qquad \varphi_{t_0} = \mathrm{Id}_{\overline{B}} \qquad \Longrightarrow \qquad q_{t_0} = q_{t_0}^s =: q^0. \tag{1.29}$$

In fluid mechanics one predominantly adopts the spatial representation since there is no distinguished reference configuration. Moreover, instead of the motion, the basic unknown is the spatial velocity, since one usually is not interested in the trajectories of individual fluid particles but on the flow of the material. In the theory of elastic solid continua, however, the material viewpoint is more common, because the equilibrium configuration is a natural reference configuration and the displacements are often very small. In particular, seismometers record the motion (displacement, velocity) in material representation.

We emphasize that the transition from q^s to q defined by $q_t = q_t^s \circ \varphi_t$ (1.27) does not give the differential geometric pull-back in case of vector or tensor fields, since only the base point transformation is involved. We illustrate this issue for the unit normal vector: For a Lip-surface S inside B with unit normal vector ν we denote the corresponding **spatial unit normal vector** of the surface $\varphi_t(S)$ by

$$\nu_t^s \colon \varphi_t(S) \to \mathbb{R}^n. \tag{1.30}$$

Even when φ_t is a diffeomorphism, we typically encounter $\nu_t^s \neq \nu \circ \varphi_t^{-1}$ but the correct (differential geometric) transformation is (compare to (1.34) below and see also [Cia88, p. 41])

$$\nu_t^s = \frac{(J_t(\nabla\varphi_t)^{-T} \cdot \nu)}{|(J_t(\nabla\varphi_t)^{-T} \cdot \nu)|} \circ \varphi_t^{-1}.$$

Another example is the material counterpart $f_t \colon B \to \mathbb{R}^n$ of a spatial force density field $f_t^s \colon \varphi_t(B) \to \mathbb{R}^n$, which is given by $f_t = J_t(f_t^s \circ \varphi_t)$, see (4.47).

The **material time derivative** of a spatial quantity q^s is defined as the spatial representation of the partial time derivative of the corresponding material quantity q:

$$\mathrm{d}_t q^s := (\partial_t q)^s. \tag{1.31}$$

Expressed in purely spatial terms, the material time derivative $d_t q^s$ splits into the local temporal change of q^s plus its change due to the advection by the velocity field v^s: If $(x,t) \mapsto q^s(x,t)$ is a spatial quantity, then its material time derivative is given by

$$\mathrm{d}_t q^s = \partial_t q^s + \nabla q^s \cdot v^s. \tag{1.32}$$

This identity follows from calculating the time derivative of $q_t = q_t^s \circ \varphi_t$ (1.27) by the chain rule:

$$\begin{aligned} \partial_t q_t = \partial_t (q_t^s \circ \varphi_t) &= (\partial_t q_t^s) \circ \varphi_t + ((\nabla q_t^s) \circ \varphi_t) \cdot \dot{\varphi}_t \\ &= (\partial_t q_t^s) \circ \varphi_t + ((\nabla q_t^s) \circ \varphi_t) \cdot v_t \\ &= (\partial_t q_t^s) \circ \varphi_t + ((\nabla q_t^s) \circ \varphi_t) \cdot (v_t^s \circ \varphi_t) = (\partial_t q_t^s + \nabla q_t^s \cdot v_t^s) \circ \varphi_t, \end{aligned}$$

from which we obtain $\partial_t q_t^s + \nabla q_t^s \cdot v_t^s = (\partial_t q_t) \circ \varphi_t^{-1} = (\partial_t q_t)^s = \mathrm{d}_t q^s$, where the last equality is (1.31). For example, the acceleration $a = \dot{v} = \ddot{\varphi}$ (1.23) has the spatial form $a_t^s = \mathrm{d}_t v_t^s = \partial_t v^s + \nabla v^s \cdot v^s$.

Integrals over spatial and material parts of the body are related by the following **volume and surface integral transformation formulas** (the question of sufficient regularity conditions is addressed in Lemma 4.21): If h^s is a volume density in spatial representation and $A \subseteq B$ is an open subset, then

$$\int_{\varphi_t(A)} h_t^s \,\mathrm{dV} = \int_A h_t J_t \,\mathrm{dV}. \tag{1.33}$$

If g^s is a spatial surface density defined on a hypersurface S inside B, then

$$\int_{\varphi_t(S)} g_t^s \,\nu_t^s \mathrm{dS} = \int_S g_t J_t \,(\nabla\varphi)_t^{-T} \cdot \nu \,\mathrm{dS}. \tag{1.34}$$

The identities (1.33) and (1.34) are the classical change of variables formulas for volumes and hypersurfaces under the diffeomorphism $\varphi_t \colon B \to \mathbb{R}^n$ with $J_t = \det \nabla\varphi_t > 0$. These relations are often stated in the form

$$\mathrm{dV}_t = J_t \mathrm{dV} \qquad \text{and} \qquad \nu_t^s \mathrm{dS}_t = J_t (\nabla\varphi)_t^{-T} \cdot \nu \,\mathrm{dS}, \tag{1.35}$$

see e.g. [Cia88, p. 31 and p. 40], [Mal69, (4.5.24) and (4.5.29b)], [DT98, (2.30) and (2.37)]. The surface version is also known as Nanson's formula [Ogd84, Section 2.2.2, formula (2.2.18), p. 88].

Essentially, formula (1.33) can be deduced from the generalization to n dimensions of the three dimensional volume of the parallelepiped spanned by the images of the vectors $a, b, c \in \mathbb{R}^3$ under the linear map $F \in \mathbb{R}^{3\times 3}$: $((F \cdot a) \times (F \cdot b)) \cdot (F \cdot c) = (\det F)\,(a \times b) \cdot c$. Moreover, requiring the validity of this relation for all $c \in \mathbb{R}^3$ implies $F^T \cdot ((F \cdot a) \times (F \cdot b)) = (\det F)\,(a \times b)$ or, if F is invertible, $((F \cdot a) \times (F \cdot b)) = (\det F)(F^{-T}) \cdot (a \times b)$, which explains the specific form (1.34). A detailed proof is given in [Cia88, Thm. 1.7-1, p. 39].

The surface integral transformation law contains the **cofactor matrix** $\operatorname{Cof} F = (\det F)F^{-T}$ of $F = \nabla\varphi$. The representation formula $\det A = A_{ij}(\operatorname{Cof} A)_{ij}$ (summation convention) implies that the cofactor $\operatorname{Cof} A$ of a (possibly noninvertible) quadratic matrix $A \in \mathbb{R}^{n\times n}$ arises as the derivative of its determinant with respect to the matrix components [MH83, p. 10]:

$$\partial_A(\det A) = \operatorname{Cof} A. \tag{1.36}$$

This identity allows to calculate the time derivative of $J = \det(\nabla\varphi)$ (1.25) under $\mathcal{C}^2$-regularity [MH83, Proposition 5.4, p. 86]:

$$\partial_t J_t = J_t \,(\operatorname{div} v^s) \circ \varphi_t. \tag{1.37}$$

Indeed, with $F = \nabla\varphi$ and $\partial_t F = \partial_t(\nabla\varphi) = \nabla(\partial_t\varphi) = \nabla v$, we have

$$\partial_t J = \partial_t(\det F) = (\partial_{F_{ij}}(\det F))(\partial_t F_{ij}) = (\operatorname{Cof} F)_{ij}(\nabla v)_{ij} = \det F (F^{-T})_{ij}(\partial_j v_i),$$

but from $v_t = v_t^s \circ \varphi_t$ we get by the chain rule (and omitting the time variable)

$$\nabla v = \nabla(v^s \circ \varphi) = ((\nabla v^s) \circ \varphi) \cdot \nabla\varphi = ((\nabla v^s) \circ \varphi) \cdot F = F^T \cdot ((\nabla v^s) \circ \varphi)$$

that is, $\partial_j v_i = ((\partial_k v_i^s) \circ \varphi) F_{kj} = ((\partial_k v_i^s) \circ \varphi)(F^T)_{jk} = (F^T)_{jk}(\partial_k v_i^s) \circ \varphi$, which finally leads to

$$\partial_t J = \det F \underbrace{(F^{-T})_{ij}(F^T)_{jk}}_{=\delta_{ik}} (\partial_k v_i^s) \circ \varphi = \det F (\partial_k v_k^s) \circ \varphi = J(\mathrm{div} v^s) \circ \varphi.$$

Remark 1.8 (**Reynold's transport theorem**)**.** Reynold's transport theorem expresses the time derivative of a spatial volume integral in terms of the time derivative of the integrand plus the flux across the boundary [MH83, Chapter 2, Theorem 1.1, p. 121]:

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(A)} f^s \,\mathrm{dV} &= \int_{\varphi_t(A)} (\mathrm{d}_t f^s + f^s(\nabla \cdot v^s)) \,\mathrm{dV} \\ &= \int_{\varphi_t(A)} (\partial_t f^s + \nabla \cdot (f^s v^s)) \,\mathrm{dV} = \int_{\varphi_t(A)} (\partial_t f^s) \,\mathrm{dV} + \int_{\partial\varphi_t(A)} (f^s v^s) \cdot \nu^s \,\mathrm{dS}.\end{aligned}$$

This follows under $\mathcal{C}^1$-regularity from (1.33), (1.37), the identity $\mathrm{d}_t f^s = \partial_t f^s + (\nabla f^s) \cdot v^s$ (1.32), and the divergence theorem. ⌟

1.2.3 Balance principles and stress tensors

The general balance principles in continuum mechanics are

- conservation of mass,
- conservation of linear momentum (balance of forces),
- conservation of angular momentum (balance of torques), and
- conservation of energy (first law of thermodynamics).

In continuum thermomechanics, where thermal effects are included, one has to add an equation for the entropy (second law of thermodynamics).

The principle of **conservation of mass** states that the mass of a body must be constant throughout its motion: $M(B) = M(\varphi(B,t))$ for all $B \in \mathcal{B}$ and $t \in I$, see (1.7). Equivalently,

$$M(\varphi_{t'}(A)) = \int_{\varphi_{t'}(A)} \rho_{t'}^s \,\mathrm{dV} = \int_{\varphi_{t''}(A)} \rho_{t''}^s \,\mathrm{dV} = M(\varphi_{t''}(A)) \tag{1.38}$$

for all open sets $A \subseteq B$ and almost all times t', $t'' \in I$. If ρ^s is differentiable with respect to time, we obtain the formulation

$$\frac{d}{dt}\int_{\varphi_t(A)} \rho_t^s \,\mathrm{dV} = 0. \tag{1.39}$$

The differential analog of conservation of mass in material formulation is the identity

$$\rho^0 = \rho_t J_t, \tag{1.40}$$

which holds on $B \times I$. The proof is based on the volume transformation formula (1.33) (see Lemma 4.23). In spatial representation, the differential form of conservation of mass is the **continuity equation**

$$\partial_t \rho^s + \nabla \cdot (\rho^s v^s) = 0 \tag{1.41}$$

which holds on $\varphi_t(B)$ for all $t \in I$, as can be seen from the transport theorem (Remark 1.8). Under suitable regularity conditions, the different variants (1.39), (1.40), and (1.41)) of conservation of mass are equivalent (see Lemma 4.26). In particular, combining the transport theorem (Remark 1.8) and the continuity equation (1.41) yields

$$\frac{d}{dt}\int_{\varphi_t(A)} q_t^s \rho_t^s \,\mathrm{dV} = \int_{\varphi_t(A)} (\mathrm{d}_t q_t^s)\rho_t^s \,\mathrm{dV} \tag{1.42}$$

for any spatial field q^s and any subbody $A \subseteq B$ (alternatively, this relation may be established by transformation in material formulation, since $\int_{\varphi_t(A)} q_t^s \rho_t^s \,\mathrm{dV} = \int_A q_t \rho^0 \,\mathrm{dV}$, see Lemma 4.24).

The fundamental dynamical equation governing the motion of a continuous body is Newton's second law, which expresses **conservation of linear momentum** or **balance of forces**. The inertial force is instantaneously balanced by the applied forces, which are assumed to decompose into **body forces** and **contact forces**: For an arbitrary spatial region $A_t = \varphi_t(A)$ contained in the current configuration $\varphi_t(B)$ of the body at time $t \in I$,

$$\frac{d}{dt}\int_{A_t} \rho^s\, v^s \,\mathrm{dV} \;=\; \int_{A_t} f^s \,\mathrm{dV} + \int_{\partial A_t} \tau^s \,\mathrm{dS}. \tag{1.43}$$

This is the **integral form of the equation of motion** in spatial representation. The corresponding material version reads

$$\int_A \rho^0\, \ddot{\varphi} \,\mathrm{dV} \;=\; \int_A f \,\mathrm{dV} + \int_{\partial A} \tau \,\mathrm{dS}, \tag{1.44}$$

which holds for each part A of a continuous body B.

Spatial and material body forces are represented by the volume force densities

$$f^s = \rho^s b^s \qquad \text{and} \qquad f = \rho^0 b,$$

which are products of the density and the corresponding spatial and material acceleration field b^s and b (the material form is obtained in consequence of conservation of mass). In particular, the volume force densities are related via the Jacobian: $f = J(f^s \circ \varphi)$ (see (4.47) and Section 4.4 for details). Spatial and material contact forces are described in terms of surface force densities, referred to as **traction** or **stress vectors**, which are assumed to depend on the surface only locally through the unit normal vector:

$$\tau^s = \tau^s(\nu^s) \qquad \text{and} \qquad \tau = \tau(\nu).$$

By **conservation of angular momentum**, the rate of change of angular momentum of any portion of the body must be equal to the sum of the external torques due to the applied forces, see e.g. [MH83, Chapter 2, Def. 2.9, p. 138]: $\frac{d}{dt}\int_{A_t} \rho^s (x \times v^s) \mathrm{dV} = \int_{A_t} (x \times f^s) \mathrm{dV} + \int_{\partial A_t} (x \times \tau^s) \mathrm{dS}$. **Conservation of energy** is discussed in Section 5.2.5.

Remark 1.9 (**General interactions and divergence measure fields**)**.** An early approach towards an axiomatic definition of body and contact forces described as general interactions between bodies is given in [Nol59]. Later the regularity requirements were gradually reduced (e.g. [GM76, Zie83, Šil85, Šil91]). In current research, stress is modeled in terms of divergence measure fields, a setting that allows for concentrated forces [DMM99, Šil08, CTZ09]. ⌟

According to **Cauchy's stress theorem**, balance of forces implies that contact forces must depend linearly on the unit normal vector of the surface on which they act. The matrix corresponding to this linear mapping is the stress tensor.[1] The proof of Cauchy's stress theorem

[1] In the literature, stress tensors are often also defined in a transposed variant with the convention $\tau(\nu) = \nu \cdot T$ instead of $\tau(\nu) = T \cdot \nu$, see e.g. [Mal69, (3.2.8) and (5.3.19)] and also [DT98]). This does not play a role for the symmetric Cauchy stress tensor, but may be a source of confusion in case of the first Piola-Kirchhoff stress tensor.

employs the famous tetrahedron argument (see e.g. [Cia88, Thm. 2.3.1, p. 62]). However, since both, the contact forces as well as the surface may be described in spatial or material representation, there exist different variants of stress tensors.

The spatial **Cauchy stress tensor** T^s relates the spatial traction τ^s to the current surface area element with unit normal ν^s:

$$\tau^s(\nu^s) = T^s \cdot \nu^s \qquad \text{on} \qquad \varphi_t(S). \tag{1.45}$$

Here $\varphi_t(S)$ is the current configuration of a (hyper-)surface S contained in the body B. Since τ^s is assumed to exist for all surfaces A_t in current configuration, Cauchy stress is a spatial 2-tensor field: $T^s\colon \bigcup_{t\in I}(\varphi_t(\overline{B}) \times \{t\}) \to \mathbb{R}^{n\times n}$. Conservation of angular momentum is equivalent to the symmetry of the Cauchy stress tensor: $T^s = (T^s)^T$ (a proof is given in [MH83, Chapter 2, Theorem 2.10] or [Cia88, Theorem 2.3-I]). The material representation of the Cauchy stress tensor (with respect to the base point, i.e. according to (1.27)), that is, $T\colon \overline{B} \times I \to \mathbb{R}^{n\times n}$ given by

$$T_t = T_t^s \circ \varphi_t \tag{1.46}$$

for $t \in I$, is sometimes referred to as the **Lagrangian stress tensor** (e.g. [DT98, p. 35]).

The **first Piola-Kirchhoff stress tensor** T^{PK} relates the spatial traction τ^s (acting on the area element after deformation) to the original undeformed area element (see [TN04, (16.5) and (43 A.1)], [Cia88, p. 71]):

$$\tau^s(\nu^s) = T^{\mathrm{PK}} \cdot \nu \qquad \text{on} \qquad S \tag{1.47}$$

with $T^{\mathrm{PK}}\colon \overline{B} \times I \to \mathbb{R}^{n\times n}$. We note that T^{PK} is a two-point 2-tensor with spatial and material tensor components (see Remark 1.10). Transformation in the purely material formulation gives the second Piola-Kirchhoff stress tensor $T^{\mathrm{SK}}(X,t) = (\nabla\varphi)^{-1}(X,t)\cdot T^{\mathrm{PK}}(X,t)$, which is symmetric [MH83, Def. 2.8, p. 136 and p. 138]. The relation between T^s and T^{PK} follows from investigating the total surface force acting on a deformed surface $\varphi_t(S)$: Definition (1.45) of T^s and (1.34) implies

$$\int_{\varphi_t(S)} \tau^s(\nu^s)\,\mathrm{dS} = \int_{\varphi_t(S)} T^s \cdot \nu^s\,\mathrm{dS} = \int_S J\,T \cdot \nabla\varphi^{-T} \cdot \nu\,\mathrm{dS}, \tag{1.48}$$

whereas by the definition of T^{PK} (1.47) we have $\displaystyle\int_{\varphi_t(S)} \tau^s(\nu^s)\,\mathrm{dS} = \int_S T^{\mathrm{PK}} \cdot \nu\,\mathrm{dS}$.

Since S is arbitrary, we obtain

$$T^{\mathrm{PK}} = J\,(T^s \circ \varphi) \cdot \nabla\varphi^{-T} = J\,T \cdot \nabla\varphi^{-T}. \tag{1.49}$$

In other words, T^{PK} is the **Piola transform** of T^s ([MH83, Section 1.7], [Cia88, pp. 38-39]). In index notation, the Piola transform is $T^{\mathrm{PK}}_{ij} = J\,T_{ik}((\nabla\varphi)^{-T})_{kj} = J\,T_{ik}((\nabla\varphi)^{-1})_{jk}$.

Remark 1.10 (**Two-point tensors**, [MH83, p. 48 and p. 70])**.** The definition $\tau^s = T^{\mathrm{PK}} \cdot \nu$ (1.47) shows that T^{PK} is a two-point 2-tensor, i.e. its two tensor "legs" are defined with respect to different bases, one of the tangent space of the ambient space, and the other one from the tangent space of the body. Specifically, $\tau^s_i = T^{\mathrm{PK}}_{ij}\nu_j$ reveals that the tensor components i of T^{PK}_{ij} are defined with respect to a spatial basis (as those of τ^s), whereas the components j are defined with respect to a material basis (as those of ν). The deformation gradient $\nabla\varphi = (\partial_j\varphi_i)^n_{i,j=1}$ is a two-point 2-tensor as well. However, since we model the body B as an open subset of the ambient space $\mathbb{R}^n$, both tangent spaces may be identified with $\mathbb{R}^n$. It is therefore legitimate to consider the two-point 2-tensor fields $\nabla\varphi$ and T^{PK} as maps taking values in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{n\times n}$, as we did above. ⌟

The notion of symmetry is not defined for two-point tensors. However, as T^{PK} is related to T^s via the Piola transform (1.49), symmetry of T^s implies the identity $\nabla\varphi \cdot T^{\mathrm{PK},T} = T^{\mathrm{PK}} \cdot \nabla\varphi^T$.

An important consequence of Cauchy's stress theorem is that it allows us to localize the equation of motion via the divergence theorem. By (1.47), the material traction takes the form $\tau = T^{\mathrm{PK}} \cdot \nu$ with the first Piola-Kirchhoff stress tensor $T^{\mathrm{PK}} \colon B \times I \to \mathbb{R}^{n \times n}$. Consequently, the divergence theorem (1.16) can be applied to formulate balance of forces (1.44) in terms of volume integrals:

$$\int_A \rho^0 \, \ddot{\varphi} \, \mathrm{dV} \;=\; \int_A f \, \mathrm{dV} + \int_{\partial A} T^{\mathrm{PK}} \cdot \nu \, \mathrm{dS} \;=\; \int_A (f + \nabla \cdot T^{\mathrm{PK}}) \, \mathrm{dV}.$$

This equation has to hold for every subbody $A \subseteq B$. Thus (1.44) reduces to the material **equation of motion in classical differential form** (**local form**), that is (1.1):

$$\rho^0 \ddot{\varphi} - \nabla \cdot T^{\mathrm{PK}} = f.$$

The spatial analog

$$\rho^s \mathrm{d}_t v^s - \nabla \cdot T^s = f^s$$

is deduced in a similar way, where (1.42) is applied to the left-hand side of (1.43).

Clearly, the formulation in terms of differential equations requires more smoothness than the integral form. Equivalence only holds if sufficiently high regularity conditions on the physical fields are imposed.

Another formulation of the equation of motion that holds under low regularity conditions is the weak form (variational form), which formally is obtained in the following way: We take the Euclidean inner product of the equation of motion (1.1) with a suitable test function $h \colon B \times [t_0, t_1] \to \mathbb{R}^3$ and integrate over the space-time domain $B \times [t_0, t_1]$:

$$\int_{t_0}^{t_1} \int_B \left(\rho^0 \ddot{\varphi} - \nabla \cdot T^{\mathrm{PK}}\right) \cdot h \, \mathrm{dV} \mathrm{d}t = \int_{t_0}^{t_1} \int_B f \cdot h \, \mathrm{dV} \mathrm{d}t.$$

Integration by parts in the stress terms and in the acceleration terms via (1.17) and $\frac{d}{dt}(\dot{\varphi} \cdot h) = \ddot{\varphi} \cdot h + \dot{\varphi} \cdot \dot{h}$ yields

$$\int_B \rho^0 \dot{\varphi} \cdot h \, \mathrm{dV} \Big|_{t_0}^{t_1} - \int_B \int_{t_0}^{t_1} \left(\rho^0 \dot{\varphi} \cdot \dot{h} - T^{\mathrm{PK}} : \nabla h\right) \mathrm{dV} \mathrm{d}t = \int_{t_0}^{t_1} \left(\int_B f \cdot h \, \mathrm{dV} + \int_{\partial B} h \cdot T^{\mathrm{PK}} \cdot \nu \, \mathrm{dS} \right) \mathrm{d}t.$$

Restriction to test functions with $h(., t_0) = h(., t_1) = 0$ eliminates the first term (alternatively, we could incorporate the initial condition $\dot{\varphi}(., t_0) = v_0$ of (1.4) and, after yet another integration by parts, also $\varphi(., t_0) = \varphi_0$; final conditions are eliminated in the limit $t_1 \to \infty$). Thus we obtain the **weak form of the equation of motion**

$$-\int_B \int_{t_0}^{t_1} \left(\rho^0 \dot{\varphi} \cdot \dot{h} - T^{\mathrm{PK}} : \nabla h\right) \mathrm{dV} \mathrm{d}t = \int_{t_0}^{t_1} \left(\int_B f \cdot h \, \mathrm{dV} + \int_{\partial B} \tau_N \cdot h \, \mathrm{dS} \right) \mathrm{d}t, \tag{1.50}$$

which has to hold for all suitable test functions h, see e.g. [Ant05, Section 12.9, Equation (9.5), p. 445]. The weak form naturally incorporates the Neumann (dynamical) boundary condition $T^{\mathrm{PK}} \cdot \nu|_{\partial B} = \tau_N$ (1.3), whereas the Dirichlet (kinematical) boundary condition $\varphi|_{\partial B} = \varphi_D$ can be included by restricting the class of test functions (a description of the procedure in a more general context is provided in Section 3.2.1).

If the applied forces are conservative and also the stress possesses a potential (e.g. in a hyperelastic material), then the weak form follows from Hamilton's principle of stationary action (Section 3.1.1): The weak form coincides with the principle of virtual work, see (3.8). The direct equivalence of weak and integral form of the equation of motion in a Sobolev space setting is established in [AO79].

1.3 Elasticity

The mechanical response of a continuous body to an applied force depends on the physical properties of the material. Conversely, the specific response allows to classify materials.

We introduce the following notation for groups of linear transformations $\mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{aligned}
GL(n) &:= \{F \in \mathbb{R}^{n\times n} : \det F \neq 0\} && \text{invertible (general linear group),}\\
GL_+(n) &:= \{F \in \mathbb{R}^{n\times n} : \det F > 0\} && \text{orientation preserving,}\\
SL(n) &:= \{F \in \mathbb{R}^{n\times n} : \det F = 1\} && \text{orientation and volume preserving (special linear group),}\\
UM(n) &:= \{H \in \mathbb{R}^{n\times n} : |\det H| = 1\} && \text{volume preserving (unimodular group),}\\
O(n) &:= \{Q \in \mathbb{R}^{n\times n} : Q^T \cdot Q = 1_{n\times n}\} && \text{rotations and reflections (orthogonal group),}\\
SO(n) &:= \{Q \in \mathbb{R}^{n\times n} : Q^T \cdot Q = 1_{n\times n}, \det Q = 1\} && \text{rotations (special orthogonal group).}
\end{aligned}$$

1.3.1 Elastic solids and fluids

A continuous body $B \subseteq \mathbb{R}^n$ is **elastic** if the first Piola-Kirchhoff stress tensor T^{PK} at $(X,t) \in B \times I$ depends on the motion $\varphi\colon B \times I \to \mathbb{R}^n$ in the following way (e.g. [Cia88, Chapter 3]):

$$T^{\text{PK}}(X,t) = r(X, (\nabla\varphi)(X,t)) \quad \text{with} \quad r\colon B \times GL(n) \to \mathbb{R}^{n\times n}. \tag{1.51}$$

The **response function** (**constitutive relation**) r completely characterizes the elastic behavior of the material (if thermal effects are included, then r also depends on the specific entropy, but this case will not be considered here). An admissible motion φ is orientation preserving, that is, $J = \det(\nabla\varphi) > 0$, see (1.25). Therefore, it suffices to consider only elements from the group of invertible orientation-preserving real $n \times n$-matrices in the matrix argument F (representing $\nabla\varphi_t(X)$) of the response function $r(X,F)$, i.e. (1.51) with $r\colon B \times GL_+(n) \to \mathbb{R}^{n\times n}$.

Independence of the response function r of the motion φ follows from the requirements of differentiability, locality, and energy balance, see [MH83, Chapter 3, Theorem 2.6, p. 192]. Materials that only depend on the deformation gradient $\nabla\varphi$, as in (1.51), are called **simple materials**. It is also possible to allow the dependence on higher-order spatial derivatives of φ (non-simple materials).

A **hyperelastic material** is an elastic material whose response function is given by the derivative of the **internal energy density** W (energy per unit volume) or U (energy per unit mass): $r(.,F) = \partial_F W(.,F) = \rho^0 \partial_F U(.,F)$ with $W = \rho^0 U\colon B \times GL_+(n) \to \mathbb{R}$ (cf. [Cia88, p. 141], [MH83, pp. 210-211]). Thus the constitutive law for hyperelasticity reads

$$T^{\text{PK}} = \frac{\partial W}{\partial(\nabla\varphi)} = \rho^0 \frac{\partial U}{\partial(\nabla\varphi)}. \tag{1.52}$$

Since constitutive laws should characterize the material, they must be independent of changes of the reference frame of the observer, i.e. invariant under rigid rotations or translations of the ambient space. This invariance property is called the principle of **material frame indifference**:

$$U(.,Q\cdot F) = U(.,F) \qquad \forall\, Q \in SO(n),\ \forall\, F \in GL_+(n).$$

Material frame indifference implies that the stored energy function U depends on $F = \nabla\varphi$ only through the so-called right Cauchy-Green tensor $C := \nabla\varphi^T \cdot \nabla\varphi$ (see [Cia88, p. 146] or [MH83, Chapter 3, Theorem 2.10]). Equivalently, U depends on $\nabla\varphi$ only through the material strain tensor $e = (C - 1_{3\times3})/2$ (1.26) (by abuse of notation we again write U for the internal energy as a function of e): $U(.,\nabla\varphi) = U(.,e)$.

Material symmetries are reflected by the invariance of the response function r in (1.51) under certain changes of the reference configuration for an arbitrary deformation gradient. An arbitrary invertible matrix $A \in GL(n)$, representing such a change of reference configuration, can be expressed as the product $A = (|\det A|^{1/n}\, 1_{n\times n}) \cdot (|\det A|^{-1/n} A)$, where the first factor is a pure dilation and the second factor is a unimodular matrix. Thus, it suffices to consider changes of reference configurations associated with the unimodular group $UM(n)$. The **symmetry group** (**isotropy group**) $\mathcal{G}_r$ is the collection of all static density-preserving deformations at $X \in B$ that cannot be detected by experiment. With respect to the reference configuration $\varphi_{t_0} = \mathrm{Id}_B$, it is defined by

$$\mathcal{G}_r := \{H \in UM(n) :\ r(X,F) = r(X, F\cdot H)\ \forall\, F \in GL(n),\ X \in B\}. \tag{1.53}$$

Thus, $\mathcal{G}_r$ encodes the symmetry behavior of the constitutive function r upon volume-preserving (density-preserving) changes of the reference configuration. If the material is hyperelastic, the symmetry group can equivalently be determined from the invariance properties of the internal elastic energy W or U in place of the response function r.

The specific properties of the symmetry group lead to a further classification of elastic continua attributed to Noll [TN04, p. 77 ff.]: A simple elastic material with constitutive function r is an **elastic solid**, if there exists a reference configuration such that

$$\mathcal{G}_r \subseteq O(n)$$

is a proper subgroup. The material is an **elastic fluid** if

$$\mathcal{G}_r = UM(n).$$

In the terminology of fluid dynamics an elastic fluid is called an **inviscid compressible fluid**. Finally, the material is **isotropic**, if

$$\mathcal{G}_r \supseteq O(n).$$

Note that $UM(n) \supseteq O(n)$ implies that elastic fluids are always isotropic. The symmetry classification via $\mathcal{G}_r$ can also be based on orientation preserving groups: $GL(n)$, $UM(n)$, $O(n)$ then have to be replaced by $GL_+(n)$, $SL(n)$, $SO(n)$ respectively (e.g. [Daf16, Chapter 2.5, p. 39]).

By definition, an elastic solid possesses a **natural reference configuration**, such that the constitutive function is not altered by the action of the subgroup $\mathcal{G}_r$ of the orthogonal group. The specific subgroups allow to distinguish solids with different anisotropy properties (e.g. triclinic, monoclinic, cubic, hexagonal, etc.).

An elastic fluid is a material which does not change its response under arbitrary density preserving transformations. Furthermore, the fluid symmetries imply that the response function r of an elastic fluid depends only on its density, i.e. $r(.,\nabla\varphi) = \widetilde{r}(.,\rho)$. Indeed, since in a fluid $r(.,F) = r(.,F\cdot H)$ must hold for all $H \in UM(n)$, we may insert the special matrix $H := J^{1/n}F^{-1} \in UM(n)$ with $F = \nabla\varphi$ and employ conservation of mass $J = \rho^0/\rho$ (1.40) to obtain

$$r(.,F) = r(.,F\cdot H) = r(.,J^{1/n}\,F\cdot F^{-1}) = r(.,J^{1/n}\,1_{n\times n}) = r(.,(\rho^0/\rho)^{1/n}1_{n\times n}) =: \widetilde{r}(\rho).$$

In a hyperelastic fluid, the internal energy U only depends on the density: $U(.,\nabla\varphi) = \widetilde{U}(.,\rho)$.

Since fluids are predominantly described in spatial representation, we are interested in the spatial analog of the material constitutive relation $T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla\varphi)}$ (1.52): First, conservation of mass and the identity (1.36) imply

$$\frac{\partial U}{\partial(\nabla\varphi)} = \frac{\partial\widetilde{U}}{\partial\rho}\frac{\partial\rho}{\partial(\nabla\varphi)} = \frac{\partial\widetilde{U}}{\partial\rho}\frac{\partial(\rho^0/J)}{\partial J}\frac{\partial J}{\partial(\nabla\varphi)} = \frac{\partial\widetilde{U}}{\partial\rho}\left(-\frac{\rho^0}{J^2}\right)J(\nabla\varphi)^{-T} = -\rho\frac{\partial\widetilde{U}}{\partial\rho}(\nabla\varphi)^{-T}.$$

Next, T^{PK} is related to T^s via the Piola transform (1.49):

$$\begin{aligned} T &= \frac{1}{J} T^{\mathrm{PK}} \cdot (\nabla\varphi)^T = \frac{\rho^0}{J} \frac{\partial U}{\partial(\nabla\varphi)} \cdot (\nabla\varphi)^T = \rho \frac{\partial U}{\partial(\nabla\varphi)} \cdot (\nabla\varphi)^T \\ &= \rho\Big(-\rho \frac{\partial \widetilde{U}}{\partial \rho} (\nabla\varphi)^{-T} \Big) \cdot (\nabla\varphi)^T = -\rho^2 \frac{\partial \widetilde{U}}{\partial \rho} (\nabla\varphi)^{-T} \cdot (\nabla\varphi)^T = -\rho^2 \frac{\partial \widetilde{U}}{\partial \rho} 1_{3\times 3}. \end{aligned}$$

Since the transition (1.27) between material and spatial representations involves base point transformations only, we have $U(X, ...) = U^s(\varphi_t(X), ...)$ and get $\widetilde{U}(X, \rho_t(X)) = \widetilde{U}^s(\varphi_t(X), \rho_t(X)) = \widetilde{U}^s(\varphi_t(X), \rho_t^s(\varphi_t(X))) = \widetilde{U}^s(., \rho_t^s) \circ \varphi_t(X)$. This shows $\frac{\partial \widetilde{U}}{\partial \rho}(X) = \frac{\partial \widetilde{U}^s}{\partial \rho^s}(\varphi_t(X))$, i.e. $(\frac{\partial \widetilde{U}}{\partial \rho})^s = \frac{\partial \widetilde{U}^s}{\partial \rho^s}$ and consequently, $T^s = (-\rho^2 \frac{\partial \widetilde{U}}{\partial \rho} 1_{3\times 3})^s = -(\rho^s)^2 \frac{\partial \widetilde{U}^s}{\partial \rho^s} 1_{3\times 3}$. Thus, we obtain

$$T^s = -p^s\, 1_{3\times 3} \qquad \text{with} \qquad p^s = (\rho^s)^2 \frac{\partial \widetilde{U}^s}{\partial \rho^s}. \tag{1.54}$$

The spatial Cauchy stress T^s in elastic fluids is a pure pressure p^s, which is determined from the internal energy density $\widetilde{U}^s$ and the spatial mass density ρ^s.

Remark 1.11 (Gibbs' relation). In thermodynamics of fluids, energy conservation is often expressed as Gibbs' relation: $\mathrm{d}U = \theta \mathrm{d}s - p\,\mathrm{d}(\frac{1}{\rho})$, i.e., $\mathrm{d}U = \theta \mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho$, where θ denotes the absolute temperature and s is the specific entropy (in case of isentropic processes, $\mathrm{d}s = 0$). This equation for differentials implies the constitutive laws $\frac{\partial U}{\partial s} = \theta$ and $\frac{\partial U}{\partial \rho} = \frac{p}{\rho^2}$, i.e. (1.54). ⌟

Generally the (spatial) **pressure** is defined by

$$p^s = -\frac{1}{3} \operatorname{tr} T^s. \tag{1.55}$$

We have seen that the Cauchy stress T^s in elastic fluids reduces to the spatial pressure p^s, which depends on the spatial density ρ^s through U^s. However, with conservation of mass, we can reverse the arguments employed above and express the dependence of pressure on density again by a dependence on the deformation gradient. This allows one to treat hyperelastic solid and fluid regions in a unified way by prescribing the material internal energy density U.

1.3.2 The governing equations of classical linearized elasticity

With a generally nonlinear constitutive law $T^{\mathrm{PK}} = r(., \nabla\varphi)$ ($r = \partial_{\nabla\varphi} W$ for hyperelasticity), the governing equations of elasticity (1.1),

$$\rho^0 \ddot{\varphi} - \nabla \cdot T^{\mathrm{PK}} = f, \tag{1.56}$$

form a nonlinear system of PDEs for the motion φ that is second-order (in space and in time). More precisely, the system is **quasilinear** (in the space variables), that is, the system has a nonlinear principal part that depends linearly on the highest-order derivatives. This can be seen from the chain rule:

$$\nabla \cdot T^{\mathrm{PK}} = \nabla \cdot (r(., \nabla\varphi)) = ((\partial_{\nabla\varphi} r)(., \nabla\varphi)) : \nabla\nabla\varphi + (\nabla \cdot r)\,(., \nabla\varphi).$$

The factor

$$A := (\partial_{\nabla\varphi} r)(., \nabla\varphi) = (\partial^2_{\nabla\varphi} W)(., \nabla\varphi)$$

in front of the Hessian $\nabla\nabla\varphi$ is the generalized elasticity tensor (cf. [MH83, Chapter 3, Def. 4.1, p. 208]), which depends on the deformation gradient $\nabla\varphi$.

The analysis of the governing equations of nonlinear elastodynamics, that is, investigating existence, uniqueness, and regularity of solutions, is still a challenge [MH83, Bal02, Ant05, Daf16]. Because of quasilinearity, the occurrence of shock waves is possible and solvability results can typically only be established on short time intervals [HKM77], [DH85]. Moreover, the requirements of frame-indifference, positive orientation, and global injectivity, are difficult to incorporate and partly incompatible with the convexity assumptions and growth conditions employed in standard solution methods. These problems already arise in the static problem of nonlinear hyperelasticity, where solvability may be formulated in terms of energy minimization and the direct method of calculus of variations can be invoked [Bal76], [BM84].

Classical linearized elasticity results from a linearization of the governing equation (1.56) around a stress-free reference state, called natural state, i.e. with $T^0 = 0$. By this assumption, all stresses coincide in the classical linearized theory, that is, correct to first order, there is no difference between total and incremental stresses (see Section 6.4.2):

$$T^0 = 0 \quad \Longrightarrow \quad T = T^1 = T^s = T^{s1} = T^{\mathrm{PK}} = T^{\mathrm{PK1}}.$$

The approximation of classical linearized elasticity is valid for small **displacement** (cf. (6.4))

$$u\colon B \times I \to \mathbb{R}^n, \qquad u(X,t) := \varphi(X,t) - X, \tag{1.57}$$

small **displacement gradient** $\nabla u = \nabla\varphi - 1_{n\times n}$, and small stress T. The linearized constitutive relation between strain and stress is known as **Hooke's law**: $T = c : \varepsilon$, that is $T_{ij} = c_{ijkl}\varepsilon_{kl}$. Here $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearized strain tensor (cf. (6.39)) and $c\colon B \to \mathbb{R}^{n\times n\times n\times n}$ is the **classical elasticity tensor**. Symmetry of T and ε (i.e. $T_{ij} = T_{ji}$ and $\varepsilon_{kl} = \varepsilon_{lk}$) imply that c_{ijkl} has the minor symmetries

$$c_{ijkl} = c_{jikl} \quad \text{and} \quad c_{ijkl} = c_{ijlk}.$$

Consequently, Hooke's law can be written as

$$T = c : \nabla u, \qquad \text{that is,} \qquad T_{ij} = c_{ijkl}\partial_l u_k. \tag{1.58}$$

The governing equations of classical linearized elasticity consist of the linear second-order system of PDEs for u

$$\rho^0 \ddot{u} - \nabla \cdot (c : \nabla u) = f, \tag{1.59}$$

which is complemented with suitable initial and boundary conditions (see Section 3.2.1). The requirements of frame-indifference, positive orientation, and global injectivity can be dropped in the linearized model. **Linearized hyperelasticity** is obtained with a stored energy W that is a purely quadratic function of the linearized strain tensor ε, i.e.

$$W = \frac{1}{2}\varepsilon : c : \varepsilon = \frac{1}{2}\nabla u : c : \nabla u = \frac{1}{2}c_{ijkl}(\partial_j u_i)(\partial_l u_k). \tag{1.60}$$

Here, the second equality follows from the minor symmetries of the elasticity tensor. Thus c is the Hessian of W with respect to strain:

$$c = \partial_\varepsilon^2 W = \partial_{\nabla u}^2 W, \qquad \text{that is,} \qquad c_{ijkl} = \frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = \frac{\partial^2 W}{\partial(\partial_j u_i)\partial(\partial_l u_k)},$$

which implies the major symmetry

$$c_{ijkl} = c_{klij}.$$

The equation of motion (1.59) is known as the (linearized) **elastic wave equation**. Thus, one expects hyperbolicity, which is indeed true, provided that the elasticity tensor c satisfies some positivity conditions. A typical positivity requirement is strong ellipticity, which is equivalent to positive definiteness of the acoustic tensor $C(\eta) \in \mathbb{R}^{n\times n}$, $C_{ik}(\eta) := c_{ijkl}\eta_j\eta_l$, for all propagation directions $\eta \in \mathbb{R}^n$ (cf. Remark 8.6). This ensures that plane wave solutions of the equation of motion in $\mathbb{R}^n$ will propagate with real wave speeds [MH83, p. 240]. Strong ellipticity suffices to prove existence of solutions in classical linearized elastodynamics [HM78].

Chapter 2

Global seismology and gravity

Global seismology is concerned with earthquakes and the propagation of seismic waves within the Earth. Most of our knowledge on the interior structure of the Earth is obtained by seismological methods [AR80, DT98, BMS81, KB08]. Seismic waves are solutions of the elastic wave equation (Section 2.1). The mass density distribution within the Earth determines its gravitational field (Section 2.2). The elastic-gravitational equations describe the elastic deformation of a rotating planet under the influence of its own gravity (Section 2.3).

2.1 Seismic waves and the free oscillations of the Earth

Seismic waves are elastic deformations that propagate within the Earth. The most prominent sources of seismic waves are earthquakes. Since, except at fault zones, the amplitudes and strains of seismic waves are small, the theory of linearized elasticity is appropriate in first approximation.

The anisotropic **seismic wave equation** is the equation of motion of classical linearized elasticity (1.59), which (for $f = 0$ and writing ρ instead of ρ^0) reads

$$\rho\,\ddot{u} = \nabla \cdot T \qquad \text{with} \qquad T = c : \nabla u.$$

In **isotropic** media, the elasticity tensor c has the components $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $\lambda, \mu \colon B \to \mathbb{R}$ are the Lamé constants (see Remark 4.9). Hooke's law then reads

$$T_{ij} = c_{ijkl}\varepsilon_{kl} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \lambda(\partial_k u_k)\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j),$$

that is,

$$T = \lambda(\mathrm{tr}\varepsilon)1_{3\times 3} + 2\mu\varepsilon = \lambda(\nabla \cdot u)1_{3\times 3} + \mu(\nabla u + \nabla u^T).$$

The material parameter μ is called the **shear modulus** (**rigidity**). In an alternative form of Hooke's law for isotropic media, stress is split into its spherical and its deviatoric part:

$$T = \kappa(\mathrm{tr}\varepsilon)1_{3\times 3} + 2\mu(\varepsilon - \frac{1}{3}(\mathrm{tr}\varepsilon)1_{3\times 3}).$$

Here, $\kappa := \lambda + \frac{2}{3}\mu$ is the **bulk modulus** (**compressibility** or **incompressibility**, as is more common in geophysics).

If the medium is **homogeneous**, λ, μ as well as ρ are constant. Then $\partial_j T_{ij} = \lambda\partial_i(\partial_j u_j) + \mu(\partial_j^2 u_i + \partial_i(\partial_j u_j))$. Consequently, in an isotropic and homogeneous medium, the seismic wave equation reads

$$\rho\,\ddot{u} = (\lambda + \mu)\nabla(\nabla \cdot u) + \mu\triangle u.$$

Taking the divergence then gives $\rho\,\partial_t^2(\nabla\cdot u) = (\lambda+2\mu)\triangle(\nabla\cdot u)$, whereas taking the curl yields $\rho\,\partial_t^2(\nabla\times u) = \mu\triangle(\nabla\times u)$. This shows that both, the divergence $\nabla\cdot u$ and the curl $\nabla\times u$ of the displacement field u, fulfill classical wave equations. A Helmholtz decomposition of u shows that these wave equations also hold for the irrotational and divergence-free part of u respectively. Thus u splits into **P-** and **S-waves**, propagating with velocities α and β given by

$$\alpha := \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{\kappa+\frac{4}{3}\mu}{\rho}} \qquad \text{and} \qquad \beta := \sqrt{\frac{\mu}{\rho}}.$$

P-waves are purely compressional waves, whereas the slower S-waves are pure shear waves. Fluid media have no resistance against shear deformation ($\mu = 0$) and thus Hooke's law reduces to its spherical part: $T = \kappa(\mathrm{tr}\varepsilon)1_{3\times3} = \kappa(\nabla\cdot u)1_{3\times3}$. Therefore, only P-waves but no S-waves can propagate in fluids ($\alpha = \sqrt{\kappa/\rho}$ but $\beta = 0$).

P- and S-waves are body waves that travel through the whole globe. Yet, seismic waves also propagate at the surface of the Earth, as the solution of the seismic wave equation in the half space reveals. These **surface waves** cause the major damage in earthquakes.

Of course, the elasticity of the Earth is not perfect: Seismic waves will lose energy due to friction. These **viscoelastic** effects are typically modeled by generalizing Hooke's law $T = c : \nabla u$ to a time convolution integral $T(x,t) = \int_{-\infty}^{t} c(x,t-t') : \nabla u(x,t')\,\mathrm{d}t'$, see [DT98, (6.3), p. 194].

The seismic **free oscillations** (**normal modes**) are standing elastic waves within the Earth which are observed at periods less than one hour (more precisely, between 200 and 3 000 seconds, [WD07]). They are the eigenfunctions in the **spectral problem** for the seismic wave equation $-\rho^0\omega^2\widehat{u}(\omega) = \nabla\cdot(c : \nabla\widehat{u}(\omega))$ obtained via Fourier transformation in time. Viscoelasticity is easily incorporated by allowing the elastic coefficients c to depend on frequency, since convolution becomes a product in the frequency domain. Then the complex eigenvalues $\omega_n + i\gamma_n$ correspond to the oscillation frequencies and decay rates. Their values constrain average properties of the material parameters within the Earth's deep interior, and in particular, due to the influence of self-gravitation, give valuable information on density [Tro93, IT04].

The normal modes of a spherical, elastic, non-rotating, isotropic earth model can be calculated and they give the spheroidal and toroidal modes ${}_nS_l$ and ${}_nT_l$. Results for more general earth models typically are obtained by perturbing this simple reference model and discussing the splitting and coupling of modes [ZLWSG00, DW01, Rog03]. The modes related to elastic-gravitational motions in the Earth's fluid parts, especially in the outer core, are discussed in [Val89a, SSR84, CHL91, RV09]. Since body and surface waves are superpositions of the free oscillations, one can construct synthetic seismograms from the Earth's free oscillations. The mathematical background for this procedure is the completeness of the normal modes in appropriate function spaces. Yet, the complete description of the elastic-gravitational spectrum for a realistic earth model is an open scientific problem [WD78, Val86, Val89b, RR04b, WD07, WSL07].

Normal modes exist also for the Moon and other planetary objects [BD69]. Information on the oscillations of the Sun or other stars is obtained from radiation variations; elastic-gravitational motions of these completely gaseous objects are studied in the theory of stellar pulsations [Cox80].

2.2 The gravity field of a uniformly rotating planet

Gravity is the mutually attractive force between masses. The **gravitational acceleration** g^s is conservative and thus equal to the negative gradient of the gravitational **potential** Φ^s:

$$g^s = -\nabla\Phi^s. \tag{2.1}$$

The sign convention is the same as in [DT98, (2.107), (2.108)]. The spatial force density field (per unit volume) for a spatial mass density distribution ρ^s in a gravitational potential Φ^s is then given by $\rho^s g^s = -\rho^s \nabla \Phi^s$. Because mass also is the source of gravity, Φ^s is determined as the solution of **Poisson's equation**:

$$\triangle \Phi^s = 4\pi G\, \rho^s. \tag{2.2}$$

Here $\triangle := \nabla \cdot \nabla = \sum_{k=1}^3 \partial_k^2$ is the Laplacian and G denotes the universal gravitational constant. In SI-units, $G = 6.67 \cdot 10^{-11}\ \mathrm{m^3 s^{-2} kg^{-1}}$.

Uniform rotation of the Earth means that the **angular velocity of the Earth's rotation**

$$\Omega \in \mathbb{R}^3 \tag{2.3}$$

is independent of time and space – a valid approximation in the seismic frequency range. Its modulus is $|\Omega| = 2\pi/$"1 sidereal day" $= 2\pi/(23\,\mathrm{h}\ 56\,\mathrm{min}\ 4\,\mathrm{s}) = 2\pi/(86\,164\ \mathrm{s}) = 7.2921 \times 10^{-5}\mathrm{s}^{-1}$.

We establish a co-rotating reference frame of Cartesian basis vectors in $\mathbb{R}^3$, whose origin lies in the Earth's center of mass and whose x_3-basis vector aligns with Ω. Since the rotating coordinate system is not inertial, acceleration contains additional centrifugal and Coriolis components, as will be derived following [TT60, Section 143, p. 437] or [MR94, p. 247]: The relation of the time derivative of a material vector field q in an inertial reference frame to the time derivative calculated in a coordinate system that rotates with (possibly time-dependent) angular velocity Ω is given by

$$\partial_t q = \partial_t^{\mathrm{rot}} q^{\mathrm{rot}} + \Omega \times q^{\mathrm{rot}}, \qquad \text{i.e.} \quad \partial_t = \partial_t^{\mathrm{rot}} (.)^{\mathrm{rot}} + \Omega \times (.)^{\mathrm{rot}}, \tag{2.4}$$

where $(.)^{\mathrm{rot}}$ indicates the coordinates with respect to the rotating reference frame. The material acceleration $a = \ddot{\varphi}$ (1.23) can be written as

$$\begin{aligned}
\ddot{\varphi} &= \partial_t(\partial_t \varphi) = \partial_t(\partial_t^{\mathrm{rot}} \varphi^{\mathrm{rot}} + \Omega \times \varphi^{\mathrm{rot}}) \\
&= \partial_t^{\mathrm{rot}} \left(\partial_t^{\mathrm{rot}} \varphi^{\mathrm{rot}} + \Omega \times \varphi^{\mathrm{rot}}\right)^{\mathrm{rot}} + \Omega \times \left(\partial_t^{\mathrm{rot}} \varphi^{\mathrm{rot}} + \Omega \times \varphi^{\mathrm{rot}}\right)^{\mathrm{rot}} \\
&= (\partial_t^{\mathrm{rot}})^2 \varphi^{\mathrm{rot}} + (\partial_t^{\mathrm{rot}} \Omega) \times \varphi^{\mathrm{rot}} + 2\,\Omega \times (\partial_t^{\mathrm{rot}} \varphi^{\mathrm{rot}}) + \Omega \times \left(\Omega \times \varphi^{\mathrm{rot}}\right) \\
&= \ddot{\varphi}^{\mathrm{rot}} + \dot{\Omega} \times \varphi^{\mathrm{rot}} + 2\,\Omega \times \dot{\varphi}^{\mathrm{rot}} + \Omega \times \left(\Omega \times \varphi^{\mathrm{rot}}\right).
\end{aligned}$$

In a coordinate system rotating at a constant angular velocity (i.e. $\dot{\Omega} = 0$), the acceleration reads (now dropping the label $(.)^{\mathrm{rot}}$)

$$\ddot{\varphi} + 2\,\Omega \times \dot{\varphi} + \Omega \times (\Omega \times \varphi). \tag{2.5}$$

Hence, in spatial representation, $\mathrm{d}_t v^s$ has to be replaced by

$$\mathrm{d}_t v^s + 2\,\Omega \times v_t^s + \Omega \times (\Omega \times x). \tag{2.6}$$

Thus, if φ is the motion in a rotating reference frame, the acceleration field consists of the **inertial** acceleration $a = \ddot{\varphi}$ or $a^s = \mathrm{d}_t v^s$, the **Coriolis** acceleration $2\,\Omega \times \dot{\varphi}$ or $2\,\Omega \times v^s$, and the **centrifugal** acceleration $\Omega \times (\Omega \times \varphi)$ or $\Omega \times (\Omega \times x)$. If Ω depends on time, the calculation above reveals that one must add the Euler acceleration $\dot{\Omega} \times x$ which models the effects of varying rotation periods of the Earth.

The centrifugal force is conservative. The associated **centrifugal potential** is [DT98, (2.116)]

$$\Psi^s \colon \mathbb{R}^3 \to \mathbb{R}, \quad \Psi^s(x) := -\frac{1}{2}\left(\Omega^2 x^2 - (\Omega \cdot x)^2\right) \tag{2.7}$$

and the gradient of Ψ^s gives the centrifugal acceleration:

$$\nabla \Psi^s \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \nabla \Psi^s(x) = \Omega \times (\Omega \times x). \tag{2.8}$$

Geometrically, the centrifugal potential $\Psi^s(x)$ is given by half the squared normal distance of x to the Earth's rotation axis. This can be seen from the following representation:

$$\Psi^s(x) = -\frac{1}{2}(\Omega \times x)^2.$$

Indeed, if α denotes the angle between the vectors Ω and x, the identity follows from

$$(\Omega \times x)^2 = |\Omega|^2|x|^2|\sin\alpha|^2 = |\Omega|^2|x|^2(1-\cos^2\alpha) = |\Omega|^2|x|^2 - (\Omega\cdot x)^2 = \Omega^2 x^2 - (\Omega\cdot x)^2,$$

or alternatively, from

$$(\Omega \times x)^2 = (\epsilon_{ijk}\Omega_j x_k)^2 = \epsilon_{ijk}\Omega_j x_k \epsilon_{ilm}\Omega_l x_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\Omega_j\Omega_l x_k x_m = \Omega_l^2 x_k^2 - (\Omega_k x_k)^2.$$

Let us verify (2.8): The partial derivative of Ψ^s with respect to x_i is

$$\partial_i \Psi^s(x) = -\frac{1}{2}\partial_i\left(\Omega^2 x^2 - (\Omega\cdot x)^2\right) = -\frac{1}{2}\left(2\,\Omega^2 x_i - 2(\Omega\cdot x)\Omega_i\right) = (\Omega\cdot x)\Omega_i - \Omega^2 x_i,$$

which coincides with the ith component of centrifugal acceleration,

$$(\Omega\times(\Omega\times x))_i = \epsilon_{ijk}\Omega_j\epsilon_{klm}\Omega_l x_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_j\Omega_l x_m = \Omega_m x_m \Omega_i - \Omega_l^2 x_i.$$

Since $\Psi^s\colon \mathbb{R}^3 \to \mathbb{R}$ is a pure second-order polynomial, the gradient $\nabla\Psi^s\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, see (2.8), and the Hessian $\nabla\nabla\Psi^s\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is a constant matrix. In particular, the derivative of $\nabla\Psi^s$ in direction of a vector $a \in \mathbb{R}^3$ is given by

$$\nabla\nabla\Psi^s \cdot a = \nabla\Psi^s(a) = \Omega\times(\Omega\times a). \tag{2.9}$$

The components of the Hessian are $\partial_i\partial_j\Psi^s = \partial_i((\Omega_k x_k)\Omega_j - \Omega_k^2 x_j) = (\Omega_k\delta_{ik})\Omega_j - \Omega_k^2(\partial_i x_j) = \Omega_i\Omega_j - \Omega_k^2\delta_{ij}$ and the Laplacian of Ψ^s is $\triangle\Psi^s = \partial_i^2\Psi^s = \Omega_i^2 - 3\Omega_k^2 = -2\,\Omega_k^2 = -2\,\Omega^2$.

Since the centrifugal force has similar properties as the gravitational force, it is customary to combine their potentials: In the equations of motion, both forces will occur together which motivates to introduce the **geopotential** $\Phi^s + \Psi^s$, which is understood as the gravity field of a uniformly rotating planet.

2.3 The system of elastic-gravitational equations

Global seismic wave propagation is significantly influenced by prestress (residual stress) and self-gravitation. Incorporating these effects in the seismic wave equation results in the **elastic-gravitational equations**. This system of equations describes the elastic deformations of a self-gravitating, uniformly rotating, fluid-solid planet, e.g. the Earth. The underlying planetary model is fairly general, in particular it is inhomogeneous, anisotropic, and aspherical. Solutions of the elastic-gravitational equations comprise seismic body and surface waves, free oscillations and tidal deformations [Wah81], including the effects of short-time surface loads as for instance air pressure variations [LCK98]. The various contributions are detectable by superconducting gravimeters [HC00, WS03].

2.3.1 The nonlinear system

We include the effect of uniform rotation and self-gravitation in the general governing equation (1.56). This yields the following **material representation of the equation of motion** [DT98, (2.93)]:

$$\rho^0\left(\ddot{\varphi} + 2\,\Omega\times\dot{\varphi} + \Omega\times(\Omega\times\varphi)\right) \;=\; \nabla\cdot T^{\mathrm{PK}} + \rho^0 g + f, \tag{2.10}$$

that is, for $(X,t) \in B \times I$,

$$\rho^0(X)\Big(\ddot{\varphi}(X,t) + 2\,\Omega \times \dot{\varphi}(X,t) + \Omega \times (\Omega \times \varphi(X,t))\Big) \;=\; \nabla \cdot T^{\mathrm{PK}}(X,t) + \rho^0(X) g(X,t) + f(X,t).$$

The terms on the left-hand side are the inertial, the Coriolis, and the centrifugal force in material representation, with reference density ρ^0. As was discussed in Section 2.2, the appearance of these two forces follows from the adoption of a Cartesian coordinate system which is co-rotating with the Earth with angular velocity Ω. On the right-hand side of (2.10) we have the row-wise divergence of the first Piola-Kirchhoff stress tensor T^{PK} and the material representation of gravity. Compared to [DT98], the system is generalized to contain additional applied force densities f in material representation on the right-hand side. By (2.1) the spatial gravitational acceleration g^s is the negative gradient of the spatial gravitational potential Φ^s, which in turn relates to the spatial mass density ρ^s through Poisson's equation (2.2) [DT98, (2.108)],

$$\triangle \Phi^s = 4\pi G\, \rho^s.$$

Thus, the system of elastic-gravitational equations consists of the equation of motion (2.10), Poisson's equation (2.2), and a constitutive equation, which in case of hyperelasticity takes the form (1.52) [DT98, (2.141)]:

$$T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla \varphi)}.$$

The **spatial representation of the equation of motion** (2.10) reads [DT98, (2.83)]

$$\rho^s(\mathrm{d}_t v^s + 2\,\Omega \times v^s + \Omega \times (\Omega \times x)) = \nabla \cdot T^s + \rho^s g^s + f^s, \tag{2.11}$$

that is, for $(x,t) \in \mathbb{R}^3 \times I$,

$$\rho^s(x,t)(\mathrm{d}_t v^s(X,t) + 2\,\Omega \times v^s(x,t) + \Omega \times (\Omega \times x)) = (\nabla \cdot T^s)(x,t) + \rho^s(x,t) g^s(x,t) + f^s(x,t).$$

Here the material time derivative is given by (1.32), $\mathrm{d}_t v^s = \partial_t v^s + \nabla v^s \cdot v^s$, where the second term is nonlinear in v^s. Moreover, the material and spatial acceleration and force fields are related by $g(X,t) = g^s(\varphi(X,t),t)$ and $f(X,t) = J(X,t) f^s(\varphi(X,t),t)$. Gravitational and centrifugal accelerations are conservative, that is, $g^s(x,t) = -\nabla \Phi^s(x,t)$ and $\Omega \times (\Omega \times x) = \nabla \Psi^s(x)$ with $x = \varphi(X,t)$, see (2.1) and (2.8). Consequently, the material equation of motion (2.10) can be written in terms of the spatial geopotential $\Phi^s + \Psi^s$:

$$\rho^0\,(\ddot{\varphi} + 2\,\Omega \times \dot{\varphi}) \;=\; \nabla \cdot T^{\mathrm{PK}} - \rho^0 (\nabla(\Phi^s + \Psi^s)) \circ \varphi + f, \tag{2.12}$$

or in spatial representation [DT98, (2.117)]

$$\rho^s(\mathrm{d}_t v^s + 2\Omega \times v^s) = \nabla \cdot T^s - \rho^s\, \nabla(\Phi^s + \Psi^s) + f^s.$$

The system is complemented by suitable initial, boundary, interface, and decay conditions:

The **initial conditions** for the second-order in time material formulation (2.10) consist in fixing the values of φ and $\dot{\varphi}$ at the desired initial time t_0:

$$\varphi(.,t_0) \quad \text{and} \quad \dot{\varphi}(.,t_0).$$

Viewing the spatial formulation (2.11) as an equation for v^s, it is only of first order in time and thus it suffices to specify the initial spatial velocity v^s. In principle, the associated motion φ is then determined from solving the nonlinear differential equation

$$\dot{\varphi}(X,t) = v^s(\varphi(X,t),t) \qquad \text{for} \quad (X,t) \in B \times (t_0,t_1).$$

Equivalently, φ has to satisfy the integral equation $\varphi(X,t) = \varphi(X,t_0) + \int_{t_0}^{t} v^s(\varphi(X,t),t)\mathrm{d}t$. This shows that the solution in spatial formulation requires the knowledge of the initial value of the motion $\varphi(.,t_0)$ as well. However, if the initial time t_0 coincides with the reference time, then we already know the initial position $\varphi(X,t_0) = X$ and it remains to specify the initial velocity $\dot{\varphi}(.,t_0) = v^s(.,t_0) = v^0$.

The **boundary conditions** are conditions specified on the Earth's surface. Upon neglecting the interaction with the atmosphere, one has a **free surface** and one only assumes the homogeneous Neumann boundary condition of zero traction:

$$T^s \cdot \nu^s = 0 \quad \text{on the exterior boundary } \partial(\varphi_t(B)). \tag{2.13}$$

Here, ν^s is the unit normal vector field of the exterior boundary surface of the Earth at time $t \in I$, $\partial(\varphi_t(B))$. Actually, the equation should read $T^s_t \cdot \nu^s_t = 0$, but for simplicity we omit indicating the time-dependence, as we will do in the conditions below.

The **interface conditions** (also called **interior boundary conditions**) are jump conditions for fields across interfaces S within the Earth, $S \subseteq B$. We will denote the jump of a quantity across a surface by $[\,.\,]^+_-$, see (4.8).

The **kinematical** conditions state that across **welded** (**solid-solid**) **interfaces** $S = \Sigma^{\mathrm{SS}}$ the motion must be continuous, whereas across **slipping** (**fluid-solid**) **interfaces** $S = \Sigma^{\mathrm{FS}}$ only the normal component of the spatial velocity is continuous. Thereby, tangential slip is allowed, but cavitation or interpenetration is excluded. In spatial representation, these conditions must hold on the current configuration $\varphi_t(S)$ of the interface, [DT98, (2.79), (2.80)]:

$$[v^s]^+_- = 0 \quad \text{on welded interfaces } \varphi_t(\Sigma^{\mathrm{SS}}), \tag{2.14}$$

$$[v^s]^+_- \cdot \nu^s = 0 \quad \text{on slipping interfaces } \varphi_t(\Sigma^{\mathrm{FS}}). \tag{2.15}$$

The **dynamical** conditions express the continuity of the spatial traction vector across any surface $S \subseteq B$ in the current configuration $\varphi_t(S)$, [DT98, (2.81)]:

$$[T^s]^+_- \cdot \nu^s = 0 \qquad \text{on all interfaces } \varphi_t(\Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}}). \tag{2.16}$$

These conditions are a consequence of Newton's third law, as is explained in Section 4.6.2. Specifically, perfectly slipping interfaces are characterized by normality of the traction vector, that is, no tangential stresses can occur, [DT98, (2.82)]:

$$T^s \cdot \nu^s = (\nu^s \cdot T^s \cdot \nu^s)\nu^s \qquad \text{on perfectly slipping interfaces } \varphi_t(\Sigma^{\mathrm{FS}}). \tag{2.17}$$

The spatial gravitational potential as well as the gravitational acceleration must both be continuous across any surface $S \subseteq \mathbb{R}^3$ [DT98, (2.109), (2.110)]:

$$[\Phi^s]^+_- = 0 \quad \text{and} \quad [\nabla\Phi^s]^+_- \cdot \nu^s = 0 \qquad \text{on all surfaces } S \subseteq \mathbb{R}^3. \tag{2.18}$$

Finally, Φ^s must decay to zero at infinity: $\lim\limits_{|x|\to\infty} \Phi^s(x,.) = 0$.

2.3.2 The linearized system

The governing equations are linearized around the equilibrium state of the Earth, which is assumed to hold at reference time t_0 (the definition of the material and spatial perturbations will be given in Section 6.3.1). The linearized system of elastic-gravitational equations (2.19)–(2.35) consists of equilibrium and dynamical equations, boundary and interface conditions:

The equilibrium equations are the **static equilibrium equation**

$$\rho^0 \nabla(\Phi^0 + \Psi) - \nabla \cdot T^0 = 0 \tag{2.19}$$

and the **reference Poisson equation**

$$\triangle \Phi^0 = 4\pi G \rho^0, \tag{2.20}$$

relating the initial density ρ^0, the prestress T^0, and the reference gravitational potential Φ^0 [DT98, (3.8), (3.4)]. The static system is complemented by the boundary and interface conditions

$$\begin{aligned} T^0 \cdot \nu &= 0 \quad \text{on the exterior boundary } \partial B, && (2.21)\\ [T^0]^+_- \cdot \nu &= 0 \quad \text{on all interfaces } \Sigma^{\text{SS}} \cup \Sigma^{\text{FS}}, && (2.22)\\ T^0 \cdot \nu &= (\nu \cdot T^0 \cdot \nu)\nu \quad \text{on perfectly slipping interfaces } \Sigma^{\text{FS}}, && (2.23)\\ [\Phi^0]^+_- &= 0 \ \text{ and} && (2.24)\\ [\nabla \Phi^0]^+_- \cdot \nu &= 0 \quad \text{on all surfaces } S \subseteq \mathbb{R}^3, && (2.25) \end{aligned}$$

with $\lim_{|x|\to\infty} \Phi^0(x) = 0$.

For a given initial density distribution ρ^0, the gravitational potential Φ^0 is defined through the Poisson equation. The static equilibrium equation then constrains T^0 in terms of ρ^0, Φ^0, and Ω.

The dynamical equations, relating the displacement u and the gravitational potential perturbation (mass redistribution potential) Φ^{s1}, consist of the **equation of motion** [DT98, (3.60)]

$$\rho^0 \left(\ddot{u} + 2\,\Omega \times \dot{u} + \nabla\nabla(\Phi^0 + \Psi^s) \cdot u + \nabla \Phi^{s1} \right) - \nabla \cdot (\Lambda^{T^0} : \nabla u) = f \tag{2.26}$$

and the **perturbed Poisson equation** [DT98, (3.96)]

$$\triangle \Phi^{s1} = -4\pi G\, \nabla \cdot (\rho^0 u). \tag{2.27}$$

In the equation of motion, inertia, the Coriolis acceleration, the acceleration due to the initial gravitational and centrifugal potential, the acceleration due to gravitational potential perturbation, and the elastic restoring forces are balanced by the applied force density. The prestressed elasticity tensor Λ^{T^0} allows to generalize Hooke's law to the case of nonzero prestress (see Section 6.4.1):

$$\Lambda^{T^0} : \nabla u = T^{\text{PK1}}.$$

The perturbed Poisson equation states that the source of Φ^{s1} is the divergence of the mass flow due to displacement, where $-\nabla \cdot (\rho^0 u) = \rho^{s1}$ is the incremental spatial density (see (6.44) in Section 6.5.2). If Φ^{s1} is expressed in terms of u by solving the perturbed Poisson equation, the equation of motion assumes the form of a second-order partial integro-differential system for u (see Section 8.2.2).

In index notation, the system (2.19), (2.20), (2.26), and (2.27) reads ($i = 1, 2, 3$)

$$\begin{aligned} \rho^0\, \partial_i(\Phi^0 + \Psi^s) - \partial_j T^0_{ij} &= 0,\\ \partial_k^2 \Phi^0 &= 4\pi G \rho^0,\\ \rho^0(\ddot{u}_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k + u_j \partial_j \partial_i(\Phi^0 + \Psi^s) + \partial_i \Phi^{s1}) - \partial_j(\Lambda^{T^0}_{ijkl}(\partial_l u_k)) &= f_i,\\ \partial_k^2 \Phi^{s1} &= -4\pi G\, \partial_j(\rho^0 u_j). \end{aligned}$$

The dynamical system is complemented by the following boundary and interface conditions:

Kinematical conditions:

$$\begin{aligned} [u]^+_- &= 0 \quad \text{on welded interfaces } \Sigma^{\text{SS}}, && (2.28)\\ [u]^+_- \cdot \nu &= 0 \quad \text{on slipping interfaces } \Sigma^{\text{FS}}. && (2.29) \end{aligned}$$

Dynamical conditions:

$$T^{\mathrm{PK1}} \cdot \nu = 0 \quad \text{on the exterior boundary } \partial B, \tag{2.30}$$
$$[T^{\mathrm{PK1}}]^+_- \cdot \nu = 0 \quad \text{on welded interfaces } \Sigma^{\mathrm{SS}}, \tag{2.31}$$

$$[\tau^{\mathrm{PK1}}]^+_- = 0 \quad \text{and} \tag{2.32}$$
$$\tau^{\mathrm{PK1}} = (\tau^{\mathrm{PK1}} \cdot \nu)\nu \quad \text{on perfectly slipping interfaces } \Sigma^{\mathrm{FS}}, \tag{2.33}$$

$$[\Phi^{s1}]^+_- = 0 \quad \text{and} \tag{2.34}$$
$$[\nabla\Phi^{s1} + 4\pi G\,\rho^0 u]^+_- \cdot \nu = 0 \quad \text{on all surfaces } S \subseteq \mathbb{R}^3, \tag{2.35}$$

with $\lim_{|x|\to\infty} \Phi^{s1}(x,.) = 0$.

The modified surface traction vector on Σ^{FS} is defined by [DT98, eq. (3.80)]:

$$\tau^{\mathrm{PK1}} := T^{\mathrm{PK1}} \cdot \nu + \nu\,\widetilde{\nabla} \cdot (p^0 u) - p^0 \nu \cdot (\widetilde{\nabla} u), \tag{2.36}$$

that is, $\tau_i^{\mathrm{PK1}} = T_{ij}^{\mathrm{PK1}}\nu_j + \nu_i\,\widetilde{\partial}_k(p^0 u_k) - p^0\nu_k(\widetilde{\partial}_k u_i)$, with the surface derivative $\widetilde{\nabla} = (\widetilde{\partial}_j)_{j=1}^3$, see (4.16).

The following table summarizes the set of boundary and interface conditions for the linearized elastic-gravitational equations:

Boundary or interface	Conditions for equilibrium fields	Conditions for perturbations
All surfaces $S \subseteq B$	$[\Phi^0]^+_- = 0$ $[\nabla\Phi^0]^+_- \cdot \nu = 0$	$[\Phi^{s1}]^+_- = 0$ $[\nabla\Phi^{s1} + 4\pi G\rho^0 u]^+_- \cdot \nu = 0$
Solid-solid boundaries Σ^{SS}	– $[T^0]^+_- \cdot \nu = 0$	$[u]^+_- = 0$ $[T^{\mathrm{PK1}}]^+_- \cdot \nu = 0$
Fluid-solid boundaries Σ^{FS}	– $[T^0]^+_- \cdot \nu = 0$ $T^0 \cdot \nu = (\nu \cdot T^0 \cdot \nu)\,\nu$	$[u]^+_- \cdot \nu = 0$ $[\tau^{\mathrm{PK1}}]^+_- = 0$ $\tau^{\mathrm{PK1}} = (\tau^{\mathrm{PK1}} \cdot \nu)\,\nu$
Free surface ∂B	$T^0 \cdot \nu = 0$	$T^{\mathrm{PK1}} \cdot \nu = 0$

In Part II we will establish a low-regularity earth model that allows us to derive the complete system of elastic-gravitational equations from a variational principle. Within this framework, we will eventually solve the Cauchy problem (initial value problem) for the linearized system of elastic-gravitational equations.

Chapter 3

Variational methods for partial differential equations

Variational methods for partial differential equations are a versatile solution technique based on the weak formulation (Section 3.2). The foundations of the method are closely related to calculus of variations, which deals with minimizing, maximizing, or finding stationary points of functionals (Section 3.1).

3.1 Calculus of variations

The objective of calculus of variations is to minimize, maximize, or find critical (stationary) points of a function $f\colon M \to \mathbb{R}$, where M typically consists of numbers or functions, comprising also curves, surfaces or other mathematical objects. Due to the generality of this formulation, calculus of variations possesses a great variety of practical applications, which is reflected in the rich literature [Mor66, Kli88, BB92, Gia83, GH96, Tro96, Sch04, VA04]. The most elementary variational problem is the classical extreme value problem for real functions. More sophisticated examples are the geodesic, the minimal-surface, or the brachystochrone problem. Besides of pure mathematics, calculus of variations has broad applications in natural sciences and economics. Fundamental physical laws as Hamilton's principle of stationary action (Section 3.1.1) or Fermat's principle of geometrical optics are also formulated as variational principles.

Historically, the study of variational problems lead to important theoretical questions which motivated the foundation of whole branches of mathematics. The systematic development of a general theory of calculus of variations started in the middle of the 18th century with the discovery of the Euler-Lagrange equations. These equations are necessary conditions for the stationarity of functionals (that is, functions on function spaces) of a special form (Section 3.1.3). However, a rigorous justification was possible only after the proof of the so-called fundamental lemma of the calculus of variations which was achieved by Du Bois-Reymond based on the concept of continuity introduced by Cauchy. At the end of the 19th century, differential calculus on infinite dimensional spaces and the concepts of Fréchet and Gâteaux derivative were introduced (evolving from the first variation or functional derivative). These mappings generalize the notion of the gradient and the directional derivative of functions on $\mathbb{R}^n$ to the setting of infinite dimensional spaces (Section 3.1.2). Thereby a firm theoretical background of calculus of variations was established and, at the same time, the foundation of functional analysis. Calculus of variations itself thus may be regarded as the oldest part of functional analysis [BB92, p. 6]. The modern theory of calculus of variations combines results from topology and functional analysis.

Since the Euler-Lagrange equations are differential equations, solving a variational problem at the same time provides a solution of the associated differential equations. Conversely, under certain symmetry conditions, the weak formulation of a partial differential equation may be interpreted as a stationarity condition for a functional. The weak formulation is the basic ingredient of the variational solution method (Sections 3.2.1, 3.2.2).

3.1.1 Hamilton's principle of stationary action

Hamilton's principle of stationary action allows a very elegant formulation of classical and relativistic mechanics and field theory [SW68, LL76, FS78, Sal88, MR94, GPS02]:

> *The equations of motion governing a physical system follow from the stationarity of the action, which is a scalar function of the dynamical state variables of the system.*

In many applications the action is not only stationary but also minimal, and therefore Hamilton's principle is also known as the **principle of least action**.

The variational description of a physical system has many advantages: The complete system of governing equations, including natural boundary conditions, follows from the knowledge of a single scalar quantity. Consequently, the formulation does not depend on the choice of coordinates. By Noether's theorem, conservation properties of the system are related to the symmetries of the Lagrangian, that is, its invariance properties under certain transformations. Moreover, Hamilton's principle allows us to study stability of the solution, as well as to systematically incorporate constraint equations by means of the Lagrange multiplier method, and finally to introduce approximations in a transparent way.

We discuss the classical variational description of a general conservative dynamical system. According to Hamilton's principle, the governing equations follow from the variational principle expressing the stationarity of the **action** $\mathscr{A}$ of the system, that is,

$$\delta\mathscr{A} = 0. \tag{3.1}$$

The action is a functional on the **configuration space** which consists of the **state variables** of the system and $\delta\mathscr{A}$ denotes the first derivative (first variation) of $\mathscr{A}$ with respect to these variables. In classical point mechanics, when $q\colon I \to \mathbb{R}^m$ describes the trajectory of a particle in $\mathbb{R}^m$ within the time interval $I = [t_0, t_1]$, the action typically reads

$$\mathscr{A}(q) = \int_I \mathscr{L}(t, q(t), \dot{q}(t))\,\mathrm{d}t$$

where $\mathscr{L}\colon I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is called the **Lagrangian**. In classical field theory, the state of the system is characterized by a function y of space and time, that is $y\colon B \times I \to \mathbb{R}^m$ for an interval $I \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^n$ open and bounded. The Lagrangian $\mathscr{L}$ then is a functional of y which may be expressed as a volume and possibly also a surface integral. Hence, the action is of the form

$$\mathscr{A}(y) = \int_I \mathscr{L}(y)\,\mathrm{d}t = \int_I \left(\int_B L\,\mathrm{d}V + \int_S L_S\,\mathrm{d}S + \int_{\partial B} L_{\partial B}\,\mathrm{d}S \right) \mathrm{d}t. \tag{3.2}$$

The integrand $L\colon B \times I \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \to \mathbb{R}$ is called the **volume Lagrangian density** (or simply volume Lagrangian). Its argument is $(x, t, y(x,t), \nabla y(x,t), \dot{y}(x,t))$. For conservative systems, L is given by the difference of kinetic energy density E_{kin} and potential energy density E_{pot} (see [Syn60, p. 108], [GPS02, p. 22])

$$L = E_{\mathrm{kin}} - E_{\mathrm{pot}}. \tag{3.3}$$

The **surface Lagrangian density** $L_S\colon S \times I \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \to \mathbb{R}$ takes into account the potential interaction energy of different regions within B, separated by the hypersurface $S \subseteq B$. The exterior Lagrangian density $L_{\partial B}\colon \partial B \times I \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \to \mathbb{R}$ corresponds to conditions prescribed on the boundary ∂B. Both surface Lagrangians depend on $(x, t, y(x,t), \widetilde{\nabla} y(x,t), \dot{y}(x,t))$. Here $\widetilde{\nabla}$ denotes the surface gradient introduced in (4.16).

Under suitable regularity conditions, stationarity of $\mathscr{A}$ at y implies that y is a solution of the **Euler-Lagrange equations** (**EL**)

$$\partial_t(\partial_{\dot{y}} L) + \nabla \cdot (\partial_{\nabla y} L) - \partial_y L = 0 \quad \text{in} \quad B \times I \tag{3.4}$$

The strong EL (3.4) coincide with the equation of motion (Newton's second law). Moreover, y satisfies the **natural boundary conditions** (**NBC**), written for the case $L_{\partial B} = 0$,

$$(\partial_{\nabla y} L) \cdot \nu = 0 \quad \text{on} \quad \partial B \times I, \tag{3.5}$$

where $\nu\colon \partial B \to \mathbb{R}^n$ denotes the exterior unit normal vector to ∂B. Furthermore, on sufficiently smooth orientable interior surfaces $\Sigma \subseteq B$ with unit normal $\nu\colon \Sigma \to \mathbb{R}^n$, y satisfies the **natural interior boundary/interface conditions** (**NIBC**)

$$[\partial_{\nabla y} L]^+_- \cdot \nu = 0 \quad \text{on} \quad (\Sigma \setminus S) \times I \tag{3.6}$$

and

$$\partial_y L_S - \partial_t(\partial_{\dot{y}} L_S) - \widetilde{\nabla} \cdot (\partial_{\widetilde{\nabla} y} L_S) - [\partial_{\nabla y} L]^+_- \cdot \nu = 0 \quad \text{on} \quad S \times I \tag{3.7}$$

with $[\,.\,]^+_-$ the jump operator and $\widetilde{\nabla}$ the surface derivative, see (4.8), (4.16). The NBC (3.5) and NIBC (3.6), (3.7) correspond to the dynamical boundary and interface conditions.

The integral formulation of stationarity of the action, $\delta\mathscr{A} = 0$, is referred to as the **weak EL**:

$$\begin{aligned} \delta\mathscr{A}(y,h) \;=\; & \int_I \int_B (\partial_y L \cdot h + \partial_{\dot{y}} L \cdot \dot{h} + \partial_{\nabla y} L : \nabla h)\, \mathrm{dV}\mathrm{d}t \\ & + \int_I \int_S (\partial_y L_S \cdot h + \partial_{\dot{y}} L_S \cdot \dot{h} + \partial_{\widetilde{\nabla} y} L_S : \widetilde{\nabla} h)\, \mathrm{dS}\mathrm{d}t = 0 \end{aligned} \tag{3.8}$$

for all test functions $h\colon B \times I \to \mathbb{R}^m$ (again written for the case $L_{\partial B} = 0$). Here $\delta\mathscr{A}(y,h)$ denotes the first variation of $\mathscr{A}$ at y in direction of h (see Definition 3.2). The weak EL coincide with the **principle of virtual work** (also, though incorrectly, known as the "principle of virtual power"), where the test function h is interpreted as a "virtual displacement".

As an integral formulation of the equations of motion, the weak EL require less regularity of the involved fields than the classical differential form of the strong EL. Hence the weak EL allow us to completely describe the behavior of the system even under very low regularity assumptions. The strong EL (3.4) are formally recovered from the weak EL (3.8) by an integration by parts based on the divergence theorem (Lemma 1.6). The weak EL also include the NBC (3.5) and NIBC (3.6), (3.7), which are obtained by applying the divergence theorem for composite domains as well as the surface divergence theorem (Lemma 4.11, Lemma 4.5). The regularity assumptions that are necessary to make the calculation rigorous are specified in the following sections.

3.1.2 Differential calculus on Banach spaces

Before we consider the stationarity of functionals given in terms of volume and surface integrals as in (3.2), we investigate variational problems in an abstract setting and introduce some basic concepts of differential calculus on Banach spaces (complete normed vector spaces). Examples of Banach spaces that will be of relevance in the application of calculus of variations are spaces of

$\mathcal{C}^k$ functions on closed domains, the Lebesgue spaces L^p, and, most important for our purposes, the Sobolev spaces H^s and $W^{k,p}$ (see Section 3.2.1).

Throughout this section, E_1 and E_2 are Banach spaces and $M \subseteq E_1$ is open with $M \neq \emptyset$. The set of all continuous linear operators $E_1 \to E_2$ is denoted by $\mathrm{Lin}(E_1, E_2)$. We follow [BB92]:

Definition 3.1 (**Fréchet derivative**)**.** The Fréchet derivative (strong derivative) of $f\colon M \to E_2$ at $y \in M$ is an operator $Df(y) \in \mathrm{Lin}(E_1, E_2)$ such that

$$Df(y)(h) = f(y+h) - f(y) + o(y,h)$$

for all $h \in E_1$ with $y + h \in M$ and $o(y, \,.\,)\colon E_1 \to E_2$, $o(y,0) = 0$, and $\lim\limits_{\|h\|_{E_1}\to 0} \dfrac{\|o(y,h)\|_{E_2}}{\|h\|_{E_1}} = 0$.
If $Df(y)$ exists, f is called (Fréchet) **differentiable** at y and if this is true for all $y \in M$, then f is differentiable on M. If $Df\colon M \to \mathrm{Lin}(E_1, E_2)$ is continuous, then f is called continuously differentiable on M.

Definition 3.2 (**Gâteaux derivative**)**.** The Gâteaux derivative (weak derivative, directional derivative, first variation) of $f\colon M \to E_2$ at $y \in M$ in the direction $h \in E_1$ is defined by $\delta f(y,h) := \Phi'(0)$ with $\Phi\colon (-\varepsilon_0, \varepsilon_0) \to E_2$, $\Phi(\varepsilon) = f(y + \varepsilon h)$ for some $\varepsilon_0 > 0$, i.e.,

$$\delta f(y,h) = \frac{d}{d\varepsilon}(f(y+\varepsilon h))\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{f(y+\varepsilon h) - f(y)}{\varepsilon}.$$

If $\delta f(y, .) \in \mathrm{Lin}(E_1, E_2)$, then f is called Gâteaux differentiable on M.

The Fréchet derivative is stronger than the Gâteaux derivative in the sense that the existence of Df implies the existence of δf and that $\delta f = Df$. Conversely, if the Gâteaux derivative enjoys additional regularity, then it is actually a Fréchet derivative: If $\delta f(y, .)$ exists for all y in a neighborhood U of $y_0 \in M$ and if $y \mapsto \delta f(y, .)$ is continuous $U \to \mathrm{Lin}(E_1, E_2)$, then $Df(y_0)$ exists and $Df(y_0) = \delta f(y_0, .)$ [BB92, Lemma 2.3.2, p. 47].

We discuss some examples of Fréchet derivatives:

(i) Every **linear operator** $l\colon E_1 \to E_2$ is Fréchet (and hence Gâteaux) differentiable with constant derivative

$$Dl(y) = l.$$

Indeed, by linearity $l(y+h) = l(y) + l(h)$, which shows that $Dl(y)(h) = l(h)$ for all y and $h \in E_1$.

(ii) Let $Q\colon H \to \mathbb{R}$ be a **quadratic form**,

$$Q(y) = \langle Ay | y \rangle_H$$

with a bounded linear operator $A\colon H \to H$ on a real Hilbert space H with inner product $\langle .|. \rangle_H$ and associated norm $\|.\|_H$. Then Q is Fréchet differentiable on H with

$$DQ(y)(h) = \langle Ay | h \rangle_H + \langle Ah | y \rangle_H.$$

Indeed, $Q(y+h) = \langle A(y+h) | y+h \rangle_H = \langle Ay|y\rangle_H + \langle Ay|h\rangle_H + \langle Ah|y\rangle_H + \langle Ah|h\rangle_H$ with $|\langle Ah|h\rangle_H|/\|h\|_H \leq \|A\|_{\mathrm{op}}\|h\|_H \to 0$ as $\|h\|_H \to 0$ and $h \mapsto \langle Ay|h\rangle_H + \langle Ah|y\rangle_H$ is bounded and linear. In particular, if A is symmetric, then $DQ(y)(h) = 2\langle Ay|h\rangle_H$.

(iii) For every $y_0 \in M$, the Fréchet derivative of a **functional** $\mathcal{J}\colon M \to \mathbb{R}$ is an element of the normed dual space of E_1:

$$D\mathcal{J}(y_0) \in E_1' := \mathrm{Lin}(E_1, \mathbb{R}).$$

(iv) The Fréchet derivative Df of a **vector-valued function** $f\colon M \to \mathbb{R}^m$ on $M \subseteq \mathbb{R}^n$ coincides with the linear operator corresponding to the Jacobi matrix of f (upon identifying linear operators with matrices), i.e.

$$Df(y_0)(h) = Df(y_0) \cdot h \qquad \text{with} \qquad Df = (\partial_j f_i)_{i,j=1}^{m,n}.$$

Moreover, interpreted as Gateaux derivative, $Df \cdot h$ represents the directional derivative of f in the direction of h.

Definition 3.3 (**Regular and stationary points**)**.** Let $f\colon M \to E_2$ be differentiable at $y_0 \in M$. If $Df(y_0)\colon E_1 \to E_2$ is surjective, then y_0 is called a regular point of f. Else, y_0 is called a stationary (or critical) point of f. In particular, a point $y_0 \in M \subseteq E_1$ is a stationary point of a functional $\mathcal{J}\colon M \to \mathbb{R}$ if $D\mathcal{J}(y_0) = 0$ in E_1', that is,

$$D\mathcal{J}(y_0)(h) = 0 \quad \text{for all} \quad h \in E_1. \tag{3.9}$$

Let $g\colon M \to E_2$ be continuous and $y_0 \in M$. Then y_0 is called a **stationary point** of $\mathcal{J}$ **subject to the constraint** $g = 0$ on M, if y_0 is a stationary point for the restricted functional $\mathcal{J}|_{g^{-1}(\{0\})}$. Note that continuity of g implies that $g^{-1}(\{0\})$ is a closed set. The Lagrange multiplier theorem gives necessary conditions for stationary points of a functional $\mathcal{J}\colon M \to \mathbb{R}$ that is restricted to a subset of M by a constraint [BB92, Theorem 4.3.3, p. 74]:

Theorem 3.4 (**Lagrange multiplier theorem**)**.** *Let $\mathcal{J}\colon M \to \mathbb{R}$, $g\colon M \to E_2$ both be (Fréchet) differentiable and let $y_0 \in M$ be a stationary point of $\mathcal{J}$ subject to the constraint*

$$g(y) = 0 \text{ for all } y \in M.$$

If y_0 is a regular point of g and $\ker(Dg(y_0))$ has a topological complement in E_1, then there exists a continuous linear functional $\lambda\colon E_2 \to \mathbb{R}$, called Lagrange multiplier, such that

$$D\mathcal{J}(y_0) = \lambda \circ Dg(y_0).$$

Equivalently, y_0 is a stationary point of the functional $\mathcal{J}^\lambda\colon M \to \mathbb{R}$, $\quad \mathcal{J}^\lambda := \mathcal{J} - \lambda \circ g$.

Remark 3.5 (**Topological complement condition**)**.** By the topological complement condition, $\ker(Dg(y_0))$ should possess a topological complement in E_1. This means that there exists a closed subspace $V \subseteq E_1$ such that $E_1 = \ker(Dg(y_0)) \oplus V$. Note that $\ker(Dg(y_0))$ is closed, since $Dg(y_0) \in \mathrm{Lin}(E_1, E_2)$ by the requirement of Fréchet differentiability. If E_1 is a Hilbert space, then the topological complement condition is automatically satisfied on (the closed subspace) $K := \ker(Dg(y_0))$. Indeed, the orthogonal complement $K^\perp =: V$ is closed and we always have $E_1 = K \oplus K^\perp$. ⌟

3.1.3 Classical calculus of variations

Classical calculus of variations considers functionals given by an integral $\mathcal{J}$ of the form

$$\mathcal{J}(y) := \int_V F(x, y(x), Dy(x)) \,\mathrm{dV}(x) \qquad \text{for} \quad y \in \mathcal{C}^1(\overline{V})^m, \tag{3.10}$$

where $V \subseteq \mathbb{R}^n$ is nonempty, open, and bounded and the Lagrangian $F \in \mathcal{C}^1(U)$ is defined on an open set

$$U \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}.$$

The space $\mathcal{C}^1(\overline{V})^m$ equipped with the norm (1.13), $\|y\|_{\mathcal{C}^1(\overline{V})} := \max_{x\in\overline{V}} (|y(x)| + |Dy(x)|)$, is a Banach space. The set of all **admissible functions** consists of all $y \in \mathcal{C}^1(\overline{V})^m$ whose 1-graph is contained in U:

$$W := \Big\{ y \in \mathcal{C}^1(\overline{V})^m : \{(x, y(x), Dy(x)) : x \in \overline{V}\} \subseteq U \Big\}. \tag{3.11}$$

The admissibility set W is an open subset of $\mathcal{C}^1(\overline{V})^m$: For all $y \in W$ there exists $\delta_0 > 0$ such that $F(x, z(x), Dz(x))$ is defined for all $x \in \overline{V}$ and for all $z \in \mathcal{C}^1(\overline{V})^m$ with $\|z - y\|_{\mathcal{C}^1(\overline{V})} < \delta_0$, cf. [GH96, p. 12]. Consequently

$$\mathcal{J} \colon W \to \mathbb{R}$$

is a functional defined on an open subset of a Banach space. In the language of classical calculus of variations, the Gâteaux derivative (Definition 3.2) of $\mathcal{J}$ at $y_0 \in W$ in the direction $h \in \mathcal{C}^1(\overline{V})^m$ is called the **first variation** of $\mathcal{J}$:

$$\delta\mathcal{J}(y_0, h) = \frac{d}{d\varepsilon}(\mathcal{J}(y_0 + \varepsilon h))\Big|_{\varepsilon=0}.$$

In accordance with Definition 3.3, a point $y_0 \in W$ is a stationary point of $\mathcal{J}$, if

$$\delta\mathcal{J}(y_0, h) = 0 \quad \text{for all} \quad h \in \mathcal{C}^1(\overline{V})^m. \tag{3.12}$$

This stationarity condition represents the **weak EL** for $\mathcal{J}$.

Remark 3.6 (**Variations with fixed and free boundary values**)**.** The weak EL naturally contain some boundary conditions on y_0, which, under higher regularity, are explicitly recovered by integration by parts (see Lemma 3.9 below). In particular, if one considers variations of y with fixed boundary values (fixed endpoint problems), it suffices to require $\delta\mathcal{J}(y_0, h) = 0$ for all $h \in \mathcal{C}^1_0(\overline{V})^m$. By density of the test functions $\mathcal{D}(V)^m$ (3.21) in $\mathcal{C}^1_0(\overline{V})^m$, this is equivalent to $\delta\mathcal{J}(y_0, h) = 0$ for all $h \in \mathcal{D}(V)^m$. In contrast, test functions $h \in \mathcal{C}^1(\overline{V})^m$ allow for more general variations with free boundary values (free endpoint problems), see [Kli88, VA04]. ⌟

Let us derive an explicit formula for the weak EL of $\mathcal{J}$ (3.10) and thus consider its first variation. We introduce the notation

$$D^\alpha F_\varepsilon := (D^\alpha F) \circ \eta_\varepsilon : \overline{V} \to \mathbb{R}$$

for $\alpha \in \mathbb{N}_0^n$, $\varepsilon \in \mathbb{R}$, with $\eta_\varepsilon := (\mathrm{Id}_{\overline{V}}, y_0 + \varepsilon h, Dy_0 + \varepsilon Dh) \colon \overline{V} \to \overline{V} \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$. In particular, for $x \in \overline{V}$,

$$F_0(x) = F(x, y_0(x), Dy_0(x)). \tag{3.13}$$

Lemma 3.7 (**Weak EL**)**.** *The first variation in the weak EL (3.12) is given by*

$$\begin{aligned} \delta\mathcal{J}(y_0, h) &= \int_V \Big(\sum_{i=1}^m (\partial_{y_i} F_0) h_i + \sum_{i=1}^m \sum_{j=1}^n (\partial_{\partial_j y_i} F_0) \partial_j h_i \Big)(x)\, \mathrm{dV}(x) \\ &= \int_V ((\partial_y F_0) \cdot h + (\partial_{Dy} F_0) : Dh)\, \mathrm{dV}. \end{aligned} \tag{3.14}$$

Recall that $\partial_{Dy} F$ denotes the derivative of $A \mapsto F(., A)$ for $A = Dy = (\partial_j y_i)_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}$, in components: $(\partial_{Dy} F)_{ij} = \partial_{\partial_j y_i} F$.

Proof. Following [GH96, p. 12], we have $\delta\mathcal{J}(y_0, h) = \Phi'(0)$ with the continuously differentiable function $\Phi : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $\Phi(\varepsilon) := \mathcal{J}(y_0 + \varepsilon h)$ defined for $\varepsilon_0 < \delta_0 / \|h\|_{\mathcal{C}^1(\overline{V})}$ with δ_0 used to establish openness of the set W of admissible points. Continuity and linearity of the functional

$h \mapsto \delta\mathcal{J}(y_0, h)$ on $\mathcal{C}^1(\overline{V})^m$ follows from the explicit formula (3.14) for $\delta\mathcal{J}(y_0, h)$, which we will now derive: By definition of the first variation as Gâteaux derivative,

$$\delta\mathcal{J}(y_0, h) = \frac{d}{d\varepsilon}\Big(\mathcal{J}(y_0 + \varepsilon h)\Big)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_V F_\varepsilon(x)\,\mathrm{dV}(x)\,\Big|_{\varepsilon=0} = \int_V (\partial_\varepsilon F_\varepsilon)(x)\,\mathrm{dV}(x)\,\Big|_{\varepsilon=0}.$$

In the last equality, interchanging differentiation and integration is justified by dominated convergence (e.g. [Els07, Satz 5.7, p. 147]), since $F_\varepsilon \in L^1(V)$ and the derivative $\partial_\varepsilon F_\varepsilon$ exists a.e. in V and for all ε near zero, and satisfies $|\partial_\varepsilon F_\varepsilon| \leq g$ with $g \in L^1(V)$ independent of ε. Then, applying the chain rule, we obtain

$$\begin{aligned}(\partial_\varepsilon F_\varepsilon)(x) = \partial_\varepsilon(F \circ \eta_\varepsilon)(x) &= (DF)(\eta_\varepsilon(x)) \cdot (\partial_\varepsilon \eta_\varepsilon)(x) \\ &= \sum_{i=1}^m (\partial_{y_i} F)(\eta_\varepsilon(x)) h_i(x) + \sum_{i=1}^m \sum_{j=1}^n (\partial_{\partial_j y_i} F)(\eta_\varepsilon(x))(\partial_j h_i)(x) \\ &= ((\partial_y F)(\eta_\varepsilon(x))) \cdot h(x) + ((\partial_{Dy} F)(\eta_\varepsilon(x))) : Dh(x)\end{aligned}$$

for $x \in \overline{V}$. Setting $\varepsilon = 0$ completes the proof. □

In Section 7.2, Sobolev space techniques will allow us to prove the validity of formula (3.14) for the first variation $\delta\mathcal{J}(y_0, h)$ under substantially lower regularity conditions on y and $F(., y, Dy)$, provided that the integrand F of the functional is a second-order polynomial in y and Dy.

The classical (strong) EL are necessary conditions for stationary points that are twice continuously differentiable. If the (open and bounded) domain V has some additional regularity, then also NBC (natural boundary conditions) can be deduced. Both EL and NBC follow from formula (3.14) using the divergence theorem (Lemma 1.6) and the so-called fundamental lemma of the calculus of variations:

Lemma 3.8 (The fundamental lemma of the calculus of variations).

(i) **Volume integrals:** *Let $V \subseteq \mathbb{R}^n$ be open.*
Let $f \in L^1_{\mathrm{loc}}(V)$. Then, $\int_V f\,\eta = 0$ for all $\eta \in \mathcal{D}(V)$ $\iff$ $f = 0$ a.e. in V.
If $f \in \mathcal{C}^0(V)$, we can conclude $f(x) = 0$ for all $x \in V$.

(ii) **Surface integrals:** *Let S be a* Lip*-submanifold of $\mathbb{R}^n$ of dimension $n-1$ that is contained in the boundary ∂V of a* Lip*-domain $V \subseteq \mathbb{R}^n$.*
Let $g \in H^{-\frac{1}{2}}(S)$. Then, $\int_S g\,\eta = 0$ for all $\eta \in H^1(V)$ $\iff$ $g = 0$ in $H^{-\frac{1}{2}}(S)$.
If $g \in L^2(S)$, we can conclude $g = 0$ a.e. on S; if $g \in \mathcal{C}^0(S)$, we get $g(x) = 0$ for all $x \in S$.

Proof. Part (i) is a standard result in measure theory or distribution theory (e.g. [Hör90, Theorem 1.2.5], [AF03, Chapter 3, Lemma 3.31, p. 74]). We establish (ii): Since the trace operator $T\colon H^1(V) \to H^{\frac{1}{2}}(\partial V)$ is continuous and surjective, see (3.30), $\int_S g\,\eta = 0$ for all $\eta \in H^1(V)$ implies $g = 0$ in $H^{-\frac{1}{2}}(S)$. If $g \in L^2(S)$, we conclude $g = 0$ almost everywhere on S; the argumentation is similar for the continuous case. The converse direction is clear. □

The EL and the NBC follow from the weak EL (3.12) under higher regularity conditions [GH96]:

Lemma 3.9 (Classical EL and NBC). *Let $\mathcal{J}\colon W \to \mathbb{R}$ be as in (3.10) and F, U, V, W be defined as above. In addition, assume that V is a* Lip*-domain with exterior normal $\nu\colon \partial V \to \mathbb{R}^n$, and $F \in \mathcal{C}^2(U)$. If $y_0 \in W \cap \mathcal{C}^2(V)^m$ is a stationary point of $\mathcal{J}$, that is, the weak EL (3.12) hold, then y_0 satisfies the classical EL*

$$\operatorname{div}(\partial_{Dy} F_0) - \partial_y F_0 = 0 \quad \text{in} \quad V \tag{3.15}$$

as well as the classical NBC

$$(\partial_{Dy} F_0) \cdot \nu = 0 \quad \text{on} \quad \partial V. \tag{3.16}$$

In index notation, the EL and NBC read

$$\sum_{j=1}^{n} \partial_j(\partial_{\partial_j y_i} F_0) - \partial_{y_i} F_0 = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} (\partial_{\partial_j y_i} F_0)\, \nu_j = 0 \qquad \text{for} \quad 1 \le i \le m.$$

Proof. The divergence theorem (Green's formula (1.17) for integration by parts, see Lemma 1.6) applied to the first variation (3.14) of $\mathcal{J}$ yields

$$\begin{aligned} \delta\mathcal{J}(y_0, h) &= \int_V ((\partial_y F_0) \cdot h + (\partial_{Dy} F_0) : Dh)\, \mathrm{dV} \\ &= \int_V (\partial_y F_0 - \operatorname{div}(\partial_{Dy} F_0)) \cdot h\, \mathrm{dV} + \int_{\partial V} h \cdot (\partial_{Dy} F_0) \cdot \nu\, \mathrm{dS}, \end{aligned}$$

that is,

$$\delta\mathcal{J}(y_0, h) = \int_V \Big(\sum_{i=1}^{m} \partial_{y_i} F_0 - \sum_{i=1}^{m} \sum_{j=1}^{n} \partial_j(\partial_{\partial_j y_i} F_0) \Big) h_i\, \mathrm{dV} + \int_{\partial V} \sum_{i=1}^{m} \sum_{j=1}^{n} (\partial_{\partial_j y_i} F_0) h_i\, \nu_j\, \mathrm{dS}.$$

We restrict the space of test functions $\mathcal{C}^1(\overline{V})$ to the subset of smooth, compactly supported functions $\mathcal{D}(V)^m$ (3.21) and only require the stationarity $\delta\mathcal{J}(y_0, h) = 0$ for all $h \in \mathcal{D}(V)^m$. Then the fundamental lemma (Lemma 3.8) implies the EL (3.15) in V because by $h|_{\partial V} = 0$ the surface integral vanishes. Then, the validity of the EL together with stationarity for all $h \in \mathcal{C}^1(\overline{V})^m$ leads to the condition

$$\int_{\partial V} h \cdot (\partial_{Dy} F_0) \cdot \nu\, \mathrm{dS} = 0 \quad \text{for all} \quad h \in \mathcal{C}^1(\overline{V})^m$$

and the fundamental lemma applied to ∂V eventually gives the NBC (3.16). □

We emphasize that the classical EL (3.15) and NBC (3.16) hold pointwise. From their derivation it is clear that the equations still hold pointwise if, instead of full $\mathcal{C}^2$-regularity of F, we just assume

$$\partial_y F \in \mathcal{C}^0(U)^m \qquad \text{and} \qquad \partial_{Dy} F \in \mathcal{C}^1(U)^{m \times n},$$

that is, $\partial_{y_i} F \in \mathcal{C}^0(U)$ and $\partial_{\partial_j y_i} F \in \mathcal{C}^1(U)$ for all $1 \le i \le m$ and $1 \le j \le n$ (see [GH96, p. 16 and p. 34] for ∂V piecewise $\mathcal{C}^1$).

Remark 3.10 (Distributional interpretation of the EL). If the conditions

$$\partial_y F_0 \in \mathcal{D}'(V)^m \qquad \text{and} \qquad \partial_{Dy} F_0 \in \mathcal{D}'(V)^{m \times n} \tag{3.17}$$

hold, we may interpret the right-hand side of the explicit formula (3.14) for the weak EL as a distributional duality:

$$\delta\mathcal{J}(y_0, h) = \int_V ((\partial_y F_0) \cdot h + (\partial_{Dy} F_0) : Dh)\, \mathrm{dV} = \langle \partial_y F_0, h \rangle + \langle \partial_{Dy} F_0, Dh \rangle.$$

By the definition of the distributional derivative (3.24), the weak EL (3.12) thus read

$$\delta\mathcal{J}(y_0, h) = \langle \partial_y F_0 - \operatorname{div}(\partial_{Dy} F_0), h \rangle = 0$$

for all $h \in \mathcal{D}(V)^m$ (fixed endpoint variations). Thereby we obtain the EL $\operatorname{div}(\partial_{Dy} F_0) - \partial_y F_0 = 0$ (3.15) in the sense of $\mathcal{D}'(V)^m$. However, the conditions (3.17) severely restrict the possible function spaces for admissible functions y and integrands F. Yet they can be satisfied if, for example, y is a bounded Sobolev function and F is smooth [Hör97, Chapter 8, Theorem 8.5.1, p. 201]: If $y \in L^\infty \cap H^s$ for $s \ge 0$ and $F \in \mathcal{C}^\infty$ with $F(0) = 0$, then $F \circ y \in L^\infty \cap H^s$. ⌟

We generalize the classical calculus of variations to the presence of interior boundaries and functionals with surface integrals (see Section 4.1). The interfaces model surfaces across which the integrand of the functional may be discontinuous or change its form. Specifically, we consider domains V that are composed of finitely many adjacent but non-overlapping subdomains. Moreover, we will also introduce an additional surface integral to the functional $\mathcal{J}$ (3.10). In these cases, stationarity of the functional, in addition to the EL and NBC, also implies certain **interface conditions**, which are the **natural interior boundary conditions** (**NIBC**) on the interior boundaries of the composite domain.

First, we consider the stationarity of volume integrals on **composite domains**. The possible discontinuity of the integrand prevents us from directly applying the divergence theorem (Lemma 1.6). Instead a variant of the divergence theorem for composite domains $V = \bigcup_k V_k$ with interior boundary Σ is used (Lemma 4.11). Hence, assuming the validity of the formula (3.14) for the first variation of $\mathcal{J}$ (this is true e.g. for polynomial integrands in a Sobolev space setting, Section 7.2), and in addition $\partial_{Dy} F_0|_{V_k} \in \mathcal{C}^1(\overline{V_k})^{m\times n}$ for all k, the stationarity of $\mathcal{J}$ for all $h \in \mathcal{C}^1(\overline{V})^m$ can be written in the form

$$\int_V (\partial_y F_0 - \operatorname{div}(\partial_{Dy} F_0)) \cdot h \,\mathrm{dV} + \int_{\partial V} h \cdot (\partial_{Dy} F_0) \cdot \nu \,\mathrm{dS} - \int_\Sigma h \cdot [\partial_{Dy} F_0]_-^+ \cdot \nu \,\mathrm{dS} = 0.$$

Consequently, by the fundamental lemma (Lemma 3.8), a stationary point y_0 fulfills the EL (3.15) in each V_k, the NBC (3.16) on ∂V and, in addition, the NIBC

$$[\partial_{Dy} F_0]_-^+ \cdot \nu = 0 \qquad \text{on} \quad \Sigma. \tag{3.18}$$

Second, we discuss the implications of an additional **surface integral**: Let S be a Lip-hypersurface in $\mathbb{R}^n$ which is contained in the interior boundaries Σ of V. Furthermore, let

$$\mathcal{J}(y) := \int_V F(x, y(x), Dy(x)) \,\mathrm{dV}(x) + \int_S F_S(x, y(x), \widetilde{D}y(x)) \,\mathrm{dS}(x) \tag{3.19}$$

with $\widetilde{D}y = Dy - ((Dy)\cdot\nu)\nu$ denoting the surface derivative (4.16) on S. The surface Lagrangian $F_S \colon S \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is assumed to be suitably regular (e.g. $\mathcal{C}^2$). In this case, (3.14) contains the additional surface integral

$$\int_S \Big((\partial_y F_{S0}) \cdot h + (\partial_{\widetilde{D}y} F_{S0}) : \widetilde{D}h \Big) \,\mathrm{dS},$$

where, analogous to $D^\alpha F_0$ (3.13), we put $D^\alpha F_{S0} := (D^\alpha F_S)(., y_0, Dy_0)$ for all $\alpha \in \mathbb{N}_0^n$. By the surface divergence theorem on S (Lemma 4.5), the surface integral is equal to

$$\int_S h \cdot \Big(\partial_y F_{S0} - \widetilde{\operatorname{div}}(\partial_{\widetilde{D}y} F_{S0}) \Big) \mathrm{dS} + \int_{\partial S} h \cdot (\partial_{\widetilde{D}y} F_{S0}) \cdot \mathrm{d}\lambda.$$

The line integral vanishes if S is closed ($\partial S = \emptyset$) or if the integrand is zero on ∂S. Hence the fundamental lemma (Lemma 3.8) yields additional terms $\partial_y F_{S0} - \widetilde{\operatorname{div}}(\partial_{\widetilde{D}y} F_{S0})$ in the NIBC (3.18). Therefore these conditions have to be replaced by the following modified NIBC:

$$\partial_y F_{S0} - \widetilde{\operatorname{div}}(\partial_{\widetilde{D}y} F_{S0}) - [\partial_{Dy} F_0]_-^+ \cdot \nu = 0 \qquad \text{on} \quad S. \tag{3.20}$$

We note that the presence of the surface integral neither affects the EL (3.15) nor the NBC (3.16) on ∂V, but only the NIBC (3.18) on S. The NIBC on interior boundaries $\Sigma \setminus S$ remain unchanged.

3.2 Solution of partial differential equations with low regularity

Studying **partial differential equations** (**PDEs**) in a low regularity setting means to allow non-smooth coefficients, initial data, and boundary data [LM72, Tré75, DL88, DL92, RR04a, Eva10]. The weak formulation makes it possible to correctly state the equation and search for generalized solutions (Section 3.2.1). Of major importance in this respect are Sobolev spaces, essentially consisting of power integrable functions with power integrable derivatives. These Banach or Hilbert spaces provide a favorable setting for the study of well-posedness of PDEs.

A versatile solution technique for PDEs with low regularity is the variational method, which is also known as the method of energy estimates (Section 3.2.2). The key ingredient is an inequality which bounds the energy of the solution by the energy of the coefficients and data. Thereby the term "energy" refers to a certain norm, which is chosen on the basis of so-called "a priori energy estimates". The validity of energy estimates relies on certain positivity (ellipticity, convexity, coercivity) conditions.

3.2.1 Weak solutions and Sobolev spaces

We introduce some basic terminology for PDEs. Let P be a partial differential operator (PDO). A function $u\colon \Omega \to \mathbb{R}$ defined on an open set $\Omega \subseteq \mathbb{R}^n$ is called a solution of a PDE if it satisfies

$$Pu = f \quad \text{in} \quad \Omega$$

for a given inhomogeneity $f\colon \Omega \to \mathbb{R}$. A PDE is often coupled to **boundary conditions** which the solution must satisfy: For example, Dirichlet conditions are of the form

$$u|_{\partial\Omega} = g \quad \text{on} \quad \partial\Omega$$

for the given boundary data $g\colon \partial\Omega \to \mathbb{R}$. Conditions involving derivatives of u at $\partial\Omega$ are called Neumann conditions. If one of the coordinates is interpreted as time, i.e. $u\colon \Omega \times [t_0, t_1] \to \mathbb{R}$, $(x,t) \mapsto u(x,t)$ and $Pu = f$ is an evolution equation, i.e. contains time derivatives of u, then u typically has to fulfill certain **initial conditions** at initial time t_0, that is, on $\Omega \times \{t_0\}$. In case of PDEs that are of second-order with respect to time, suitable initial conditions are

$$u(.,t_0) = u^0 \qquad \text{and} \qquad \dot{u}(.,t_0) = u^1 \quad \text{in} \quad \Omega$$

for given functions u^0, $u^1\colon \Omega \to \mathbb{R}$. The boundary conditions are then understood as conditions with respect to space coordinates, that is, they hold on $\partial\Omega \times [t_0, t_1]$. The problem $Pu = f$ with initial (and boundary) conditions is called the **Cauchy problem** for P.

In the application of PDEs to problems in physics it is natural to require that the solution exists, is unique, and depends continuously on the data, represented by the source and the model parameters. This is the classical definition of a **well-posed** problem in the sense of Hadamard. More recently, the existence of a convergent numerical scheme (based on a time- or space-discretization method) has been added as a further desirable property of a PDE. Precise statements on existence, uniqueness, continuous dependence on data, and numerical approximation can only be made based on the knowledge of spaces in which the solution and data should lie. The correct choice of these spaces ideally balances the mathematical complexity and the physical reality of the model problem. If the problem is formulated in terms of PDEs, one is lead to the consideration of different solution concepts.

The solution $u\colon \Omega \to \mathbb{R}$ of a PDE $Pu = f$ is a **classical solution**, if $(Pu)(x) = f(x)$ holds pointwise for all $x \in \Omega$. It is obvious that the concept of classical solution is not applicable if f is not continuous. If $f \in L^2(\Omega)$ we may define a **strong solution** by requiring that the

equation $Pu = f$ holds in L^2 almost everywhere in Ω. However, in order to prove existence of solutions it turns out to be beneficial to further relax the solution concept and consider **weak solutions** (**variational solutions**). Their definition is based on reformulating the PDE as a duality relation with respect to a suitable space of test functions. This so-called **weak form** (**variational form**) of the PDE is obtained, if one applies both sides of $Pu = f$ to a test function φ,

$$\langle Pu, \varphi\rangle = \langle f, \varphi\rangle.$$

The derivatives contained in the operator P are then shifted to the test function side as in the definition of the distributional derivative (3.22). Thereby, provided that P is of suitable form, we can gradually reduce the minimal regularity that u must enjoy to represent a "solution".

We briefly summarize some basic concepts and notation of the theory of distributions. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The space of test functions $\mathcal{D}(\Omega)$ consists of smooth, compactly supported functions,

$$\mathcal{D}(\Omega) := \{\varphi \in \mathcal{C}^\infty(\Omega) : \operatorname{supp}(\varphi) \text{ compact in } \Omega\}, \tag{3.21}$$

and is endowed with the topology of uniform convergence of all derivatives with support in a fixed compact set, see e.g. [Tré67, Chapter 21]. The space of **distributions** $\mathcal{D}'(\Omega)$ consists of continuous linear functionals on $\mathcal{D}(\Omega)$. The action of a distribution $u \in \mathcal{D}'(\Omega)$ on a test function $\varphi \in \mathcal{D}(\Omega)$ is written in terms of a duality bracket:

$$u\colon \varphi \mapsto \langle u, \varphi\rangle \in \mathbb{R}.$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\langle u, \varphi\rangle = \int_\Omega u\,\varphi \,\mathrm{dV}.$$

The integration by parts formula

$$\int_\Omega (\partial_j f)\,\varphi \,\mathrm{dV} = -\int_\Omega f\,(\partial_j \varphi)\,\mathrm{dV},$$

which holds classically for $f \in \mathcal{C}^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ (the boundary contribution vanishes because $\operatorname{supp}(\varphi)$ is strictly contained in Ω), motivates the following definition:

Definition 3.11 (Distributional derivative, weak derivative). Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in \mathcal{D}'(\Omega)$, and $\varphi \in \mathcal{D}(\Omega)$. Then, for $1 \le j \le n$, the distributional derivative $\partial_j f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle \partial_j f, \varphi\rangle := -\langle f, \partial_j \varphi\rangle.$$

Higher-order distributional derivatives are then defined by iteration:

$$\langle D^\alpha f, \varphi\rangle := (-1)^{|\alpha|}\langle f, D^\alpha \varphi\rangle. \tag{3.22}$$

In this definition, the higher-order partial derivatives are written using multi-index notation: If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index of order $|\alpha| := \alpha_1 + \cdots + \alpha_n$, then

$$D^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \qquad \text{with} \qquad \partial_j^0 = 1 \qquad (j = 1, \dots, n). \tag{3.23}$$

The symbols ∂_i and D always denote distributional derivatives, which coincide with the classical derivatives for functions of sufficient regularity (see also Remark 4.7). Distributional duality of vector-valued distributions and test functions is defined in terms of scalar duality of the components: If $u \in \mathcal{D}'(\Omega)^m$ and $\varphi \in \mathcal{D}(\Omega)^m$, then $\langle u, \varphi\rangle = \sum_{k=1}^m \langle u_k, \varphi_k\rangle$. In particular, if $f \in \mathcal{D}'(\Omega)^{m\times n}$ is a matrix with distributional components and $h \in \mathcal{D}(\Omega)^m$ a vector of test functions, then $\operatorname{div} f \in \mathcal{D}'(\Omega)^m$, $Dh \in \mathcal{D}(\Omega)^{m\times n}$, and we have $\langle f, Dh\rangle = \langle f_{ij}, \partial_j h_i\rangle = -\langle \partial_j f_{ij}, h_i\rangle =$

$-\langle(\operatorname{div} f)_i, h_i\rangle = -\langle \operatorname{div} f, h\rangle$ (with summation convention for $1 \le i \le m$ and $1 \le j \le n$), that is, we obtain the "distributional integration by parts formula"

$$\langle f, Dh\rangle = -\langle \operatorname{div} f, h\rangle. \tag{3.24}$$

If f is sufficiently regular ($f \in H_{\operatorname{div}}(\Omega)^{m\times n}$ is enough), this coincides with Green's formula (1.17), where the boundary term disappears for $h \in \mathcal{D}(\Omega)^m$.

As a motivation, we formally illustrate the concept of weak solutions for the **Poisson equation**,

$$-\triangle u = f$$

for $u\colon \Omega \to \mathbb{R}$ on an open domain $\Omega \subseteq \mathbb{R}^n$. The PDO $\triangle = \nabla\cdot\nabla = \sum_{k=1}^n \partial_k^2$ is the n-dimensional **Laplace operator** (linear, second-order, constant coefficients). We multiply the Poisson equation with a sufficiently smooth test function $v\colon \Omega \to \mathbb{R}$ that vanishes at $\partial\Omega$, integrate over Ω, apply the product rule $\nabla\cdot((\nabla u)v) = (\triangle u)v + \nabla u\cdot\nabla v$, and the divergence theorem (Lemma 1.6; or directly Green's formula (1.17) for integration by parts). The result is the weak form of the PDE $-\triangle u = f$, namely

$$\int_\Omega \nabla u\cdot\nabla v \,\mathrm{dV} = \int_\Omega fv \,\mathrm{dV} \quad \text{for all test functions } v. \tag{3.25}$$

Moreover, a solution u of (3.25) is also a stationary point of the **Dirichlet integral**

$$u \mapsto \mathcal{J}(u) = \int_\Omega F(u,\nabla u)\mathrm{dV} \qquad \text{with} \qquad F(u,\nabla u) := \tfrac{1}{2}|\nabla u|^2 - fu. \tag{3.26}$$

Indeed, since $\partial_{\nabla u}F = \nabla u$ and $\partial_u F = -f$, the weak EL (3.12), i.e.

$$\delta\mathcal{J}(u,v) = \frac{d}{d\varepsilon}\mathcal{J}(u+\varepsilon v)\big|_{\varepsilon=0} = \int_V \Big((\partial_{\nabla u}F)\cdot\nabla v + (\partial_u F)v\Big)\,\mathrm{dV} = 0,$$

are precisely (3.25). The strong EL $\nabla\cdot(\partial_{\nabla u}F) - \partial_u F = 0$ (3.15) coincide with $-\triangle u = f$.

Thus, the weak solution of the Poisson equation solves a variational problem. This result is not restricted to the Laplacian: Essentially, every weak formulation that involves a symmetric bilinear form represents the weak EL to some functional, explaining why weak solutions are alternatively called variational solutions [LM72, Chapter 2, Section 9.1, p. 201]. With respect to regularity conditions, the Dirichlet integral is defined whenever the distributional gradient of the solution is square integrable, i.e. $\nabla u \in L^2(\Omega)^n$. Hence, the concept of weak solutions naturally leads to the definition of **Sobolev spaces**.

Sobolev functions are Lebesgue measurable and power integrable functions whose distributional derivatives also are elements of a Lebesgue space. The Lebesgue space $L^p(\Omega)$ is the Banach space consisting of (equivalence classes) of Lebesgue measurable functions $f\colon \Omega \to \mathbb{R}$ defined on an open set $\Omega \subseteq \mathbb{R}^n$, with norm $\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \,\mathrm{dV}\right)^{1/p} & 1 \le p < \infty \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & p = \infty. \end{cases} \tag{3.27}$$

Here, $\|f\|_{L^\infty(\Omega)}$ is the essential supremum of $|f(x)|$, that is, is the greatest lower bound (infimum) of all K such that $\sup_{x\in\Omega}|f(x)| \le K$ a.e. on Ω.

Definition 3.12 (Sobolev spaces). Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$ and $1 \le p \le \infty$. Then the L^p-based Sobolev space of order (exponent) k, given by

$$W^{k,p}(\Omega) := \Big\{u \in L^p(\Omega)\colon\ \forall\,\alpha\in\mathbb{N}_0^n,\ |\alpha| \le k :\ D^\alpha u \in L^p(\Omega)\Big\},$$

is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} & 1 \le p < \infty \\ \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty. \end{cases} \tag{3.28}$$

In particular, $W^{0,p}(\Omega) = L^p(\Omega)$. The L^2-based Sobolev spaces, $H^k(\Omega) := W^{k,2}(\Omega)$ are Hilbert spaces with inner product $\langle u|v\rangle_{H^k(\Omega)} := \sum_{|\alpha|\le k}\langle D^\alpha u|D^\alpha v\rangle_{L^2(\Omega)} = \sum_{|\alpha|\le k}\int_\Omega (D^\alpha u)(D^\alpha v)\mathrm{dV}$.

For a general open subset $\Omega \subseteq \mathbb{R}^n$, the **local Sobolev space** $W^{k,p}_{\mathrm{loc}}(\Omega)$ consists of those distributions $u \in \mathcal{D}'(\mathbb{R}^n)$, which, after multiplication with a test function $\varphi \in \mathcal{D}(\Omega)$, are contained in $W^{k,p}(\mathbb{R}^n)$:

$$W^{k,p}_{\mathrm{loc}}(\Omega) := \Big\{u \in \mathcal{D}'(\mathbb{R}^n)\colon\ \forall\, \varphi \in \mathcal{D}(\Omega)\,:\ u\varphi \in W^{k,p}(\mathbb{R}^n)\Big\}.$$

On $\Omega = \mathbb{R}^n$, the L^2-based Sobolev spaces $H^k(\mathbb{R}^n)$ may also be defined as certain subsets of the **temperate distributions** $\mathcal{S}'(\mathbb{R}^n)$ (the dual space to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions, which together with all their derivatives decay at infinity faster than any rational function). Since the Fourier transform $\mathscr{F}$ leaves $\mathcal{S}'(\mathbb{R}^n)$ invariant and higher differentiability of u corresponds to faster decay of $\widehat{u} := \mathscr{F}u$, smoothness can be measured with weighted L^2-norms of $\widehat{u}$:

$$\|(1+|.|^2)^{s/2}\,\widehat{u}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s\,|\widehat{u}(\xi)|^2\,\mathrm{dV}(\xi) < \infty.$$

This allows to extend the definition of Sobolev spaces to arbitrary real Sobolev exponents $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) = \Big\{u \in \mathcal{S}'(\mathbb{R}^n)\colon\ (1+|.|^2)^{s/2}\,\widehat{u} \in L^2(\mathbb{R}^n)\Big\}. \tag{3.29}$$

The equivalence of (3.29) to Definition 3.12 in the case of $\Omega = \mathbb{R}^n$, $p = 2$, and integer exponent $s = k \in \mathbb{N}_0$ is clear from the properties

$$\|u\|_{L^2(\mathbb{R}^n)} = c_n\|\widehat{u}\|_{L^2(\mathbb{R}^n)} \qquad \text{and} \qquad (D^\alpha u)\widehat{\ }(\xi) = (i\xi)^\alpha\,\widehat{u}(\xi)$$

of the Fourier transform for $u \in \mathcal{S}'(\mathbb{R}^n)$. The constant $c_n > 0$ in Plancherel's theorem is a certain power of 2π, determined by the specific choice of factors in $\mathscr{F}$.

Following [DL88, Chapter 5, §4, Prop. 3, p. 118], one may also define Sobolev spaces $H^s(\Omega)$ for $s \in \mathbb{R}$ and on a general open set $\Omega \subseteq \mathbb{R}^n$ by requiring that $u \in \mathcal{D}'(\Omega)$ has an extension to $\widetilde{u} \in \mathcal{S}'(\mathbb{R}^n)$ with $\widetilde{u} \in H^s(\mathbb{R}^n)$ in the sense of (3.29). For integer exponents $s = k \in \mathbb{N}_0$, the definitions coincide, provided that the domain Ω has the **extension property** for H^k, i.e. there exists a bounded linear operator (the extension operator)

$$E\colon H^k(\Omega) \to H^k(\mathbb{R}^n) \qquad \text{with} \qquad (Eu)\big|_\Omega = u \qquad \forall\, u \in H^k(\Omega).$$

It can be shown that Lip-domains (Definition 1.5) possess the extension property for all H^k.

By the **trace theorem**, the restriction of Sobolev spaces to boundaries of Lip-domains Ω in $\mathbb{R}^n$ reduces their exponent by 1/2 (e.g. [Tré75, Theorem 26.2], [Wlo87, Theorem 8.8]): Let $\Omega \subseteq \mathbb{R}^n$ be a Lip-domain and $k \in \mathbb{N}$, then the restriction of a smooth function $u \in \mathcal{C}^\infty(\overline{\Omega})$ to the boundary $\partial\Omega$ uniquely extends to a continuous and surjective linear map, the trace operator

$$T\colon H^k(\Omega) \to H^{k-\frac{1}{2}}(\partial\Omega) \qquad \text{with} \qquad T(u) = u\big|_{\partial\Omega}.$$

In particular, if $u \in H^1(\Omega)$, then there exists $c > 0$ with

$$\|u\|_{H^{1/2}(\partial\Omega)} \le c\|u\|_{H^1(\Omega)}. \tag{3.30}$$

For the definition of the general Sobolev space $W^{k,p}(S)$ with $k \in \mathbb{N}_0$ and the corresponding norm on a $\mathcal{C}^{k,1}$-submanifold S in $\mathbb{R}^n$, see [Gri85, Definition 1.3.3.2].

We briefly discuss **Sobolev duality**. The dual space of a Sobolev space with exponent s is essentially a Sobolev space with exponent $-s$. Specifically we have

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n),$$

i.e., $H^{-s}(\mathbb{R}^n)$ consists of all bounded linear functionals on $H^s(\mathbb{R}^n)$. In general however, test functions $\mathcal{D}(\Omega)$ are not dense in $H^k(\Omega)$ if $\Omega \subsetneq \mathbb{R}^n$ (but $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, e.g. [RR04a, Lemma 7.7, p. 206]). Consequently, we only have

$$(H_0^k(\Omega))' = H^{-k}(\Omega),$$

where $H_0^k(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ with respect to the Sobolev norm $\|.\|_{H^k(\Omega)}$. Similarly, the duality relation for $W^{k,p}(\Omega)$ with $k \in \mathbb{N}_0$ and $1 \le p \le \infty$ reads

$$(W_0^{k,p}(\Omega))' = W^{-k,p'}(\Omega),$$

where the conjugate exponent p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$, including the extreme cases 1 and ∞.

Note that since $H_0^1(\Omega) \subseteq H^1(\Omega)$ and $X \subseteq Y \Rightarrow Y' \subseteq X'$ for normed spaces X, Y, we obtain that the dual space of $H^1(\Omega)$ is, as a set, strictly contained in $H^{-1}(\Omega)$: $(H^1(\Omega))' \subseteq H^{-1}(\Omega)$ (and analogously for exponents $k > 1$). Due to the mentioned lack of density of test functions, the dual $(H^1(\Omega))'$ is not a space of distributions on Ω [LM72, p. 295]. However, the adjoint of the continuous projection $H^1(\Omega) \to H_0^1(\Omega)$ defines a continuous embedding in the converse direction: $H^{-1}(\Omega) \hookrightarrow (H^1(\Omega))'$, see [VP65, Section 3] or [Eva10].

The Sobolev duality on the boundary of a Lip-domain Ω is [DL88, Chapter 4, Appendix, p. 143]

$$(H^s(\partial\Omega))' = H^{-s}(\partial\Omega). \tag{3.31}$$

We will denote the **surface Sobolev duality** between $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ on a closed Lip-surface S by

$$\langle\!\langle ., . \rangle\!\rangle_S := \langle ., . \rangle_{H^{-\frac{1}{2}}(S), H^{\frac{1}{2}}(S)}. \tag{3.32}$$

A function $f \in H_{\mathrm{div}}(\Omega)^{m\times n}$ (1.18) is continuously mapped to its normal trace $f \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)^m$. With $H^1 \subseteq H_{\mathrm{div}}$ we thus have, for some $c > 0$,

$$\|f \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \le c \|f\|_{H^1(\Omega)}. \tag{3.33}$$

By the **Sobolev embedding theorem**, Sobolev functions with sufficiently high exponents are embedded in the space of continuous functions, that is, Sobolev regularity becomes classical regularity. We specify the details. A normed vector subspace $X \subseteq Y$ of a normed space Y is said to be continuously embedded in Y, denoted by $X \hookrightarrow Y$, if the identity mapping $\mathrm{Id}\colon X \to Y$, $\mathrm{Id}(x) := x$ is continuous, see [RR04a, Def. 7.15, p. 209]. By linearity of the identity operator, continuity reduces to the existence of $c > 0$ such that $\|\mathrm{Id}(x)\|_Y \le c\|x\|_X$ for all $x \in X$, that is,

$$\|x\|_Y \le c\|x\|_X. \tag{3.34}$$

For general topological spaces X and Y, we write $X \hookrightarrow Y$ if there exists an injective continuous map $X \to Y$. An example is the embedding of Lebesgue measurable spaces $L^p(\Omega)$ on a bounded open set $\Omega \subseteq \mathbb{R}^n$ [AF03, Theorem 2.14, p. 28]:

$$L^p(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for} \qquad 1 \le q \le p \le \infty. \tag{3.35}$$

Furthermore, $L^p \hookrightarrow L^1_{\mathrm{loc}}$ holds on general domains for all $p \geq 1$.

Embedding results for Sobolev spaces typically require some geometric properties of the domain Ω. An open set $\Omega \subseteq \mathbb{R}^n$ satisfies the **cone condition**, if there exists a fixed cone $C \subseteq \mathbb{R}^n$ of finite height and aperture, such that every $x \in \Omega$ is the vertex of a finite cone $C_x \subseteq \Omega$ congruent to C, that is, C_x coincides with C up to a rigid motion (translation or rotation) [AF03, Chapter 4, 4.6, p. 82]. In particular, the cone condition rules out domains with cusps. We note that the cone condition holds for Lip-domains as well as finite unions of Lip-domains.

We present the Sobolev embedding theorem following [AF03, Chapter 4, Theorem 4.12, p. 85]:

Lemma 3.13 (**Sobolev embedding**). *Let $\Omega \subseteq \mathbb{R}^n$ be open and satisfy the cone condition, $k \in \mathbb{N}_0$, and $1 \leq p \leq \infty$.*

(i) *If $k > \dfrac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^0(\Omega) \cap L^\infty(\Omega)$ (continuous and bounded functions).*

If Ω is also a Lip*-domain, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ with $\alpha = k - \dfrac{n}{p}$.*

(ii) *If $k = \dfrac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q \leq \infty$.*

(iii) *If $k < \dfrac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $k = n\left(\dfrac{1}{p} - \dfrac{1}{q}\right)$, that is for $q = \dfrac{np}{n-kp}$.*

If Ω is bounded, we get $q \leq \frac{np}{n-kp}$ in (iii), since by (3.35) we have $L^q(\Omega) \subseteq L^m(\Omega)$ for $m \leq q$.

As the Sobolev embedding theorem (i) already suggests, $W^{k,p}$ is a Banach algebra if $k > \frac{n}{p}$ [AF03, Chapter 4, p. 106]. Moreover, one obtains a general result on the multiplication of Sobolev functions [Gri85, Theorem 1.4.4.2]:

Lemma 3.14 (**Product in Sobolev spaces**). *Let $\Omega \subseteq \mathbb{R}^n$ be a* Lip*-domain, s, s_1, $s_2 \in \mathbb{R}$ with $s_{1,2} \geq s$, and $1 \leq p$, $p_{1,2} \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p}$. If*

$$s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \qquad \text{and} \qquad s_{1,2} - s \geq n\left(\frac{1}{p_{1,2}} - \frac{1}{p}\right)$$

(or with $>$ and $\geq$ signs interchanged), then the multiplication $(u, v) \mapsto u \cdot v$ is a continuous bilinear map $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \to W^{s,p}(\Omega)$. In particular, there exists a constant $c > 0$ such that for $u \in W^{s_1,p_1}(\Omega)$ and $v \in W^{s_2,p_2}(\Omega)$,

$$\|u \cdot v\|_{W^{s,p}(\Omega)} \leq c\, \|u\|_{W^{s_1,p_1}(\Omega)} \|v\|_{W^{s_2,p_2}(\Omega)}.$$

Finally, we are ready to present some results on **PDEs in Sobolev spaces**. By definition, $f \in W^{k,p}$ implies that its derivative $Df \in W^{k-1,p}$. The definition of Sobolev norms (3.28) implies that the derivative $D\colon W^{k,p} \to W^{k-1,p}$ is continuous:

$$\|Df\|_{W^{k-1,p}} \leq \|f\|_{W^{k,p}}.$$

We restrict our discussion to the L^2-based Sobolev spaces and write $H^s = H^s(\mathbb{R}^n)$. It follows from the mapping properties of the derivative that every **linear PDO** P of order $m \in \mathbb{N}_0$ with constant (or smooth) coefficients a_α,

$$P := \textstyle\sum_{|\alpha| \leq m} a_\alpha D^\alpha,$$

maps H^s to H^{s-m}. In other words, if $u \in H^s$, then $Pu = f \in H^{s-m}$ (this also holds on a general domain Ω). The operator P is called (uniformly) **elliptic** on $\Omega \subseteq \mathbb{R}^n$ open, if its principal symbol

$$p(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x)\, (i\xi)^\alpha$$

fulfills

$$p(x,\xi) \neq 0 \qquad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^n,\ \xi \neq 0. \tag{3.36}$$

The most prominent example of an elliptic operator is the Laplace operator on $\mathbb{R}^n$, which is a bijection $\triangle\colon H^s(\mathbb{R}^n) \to H^{s-2}(\mathbb{R}^n)$ with principal symbol $p(\xi) = -\sum_{k=1}^n \xi_k^2 = -|\xi|^2$.

If P is a linear elliptic operator of order m, then also the Sobolev regularity of the solution u of $Pu = f$ must be m orders higher than the Sobolev regularity of the source f. This converse statement is known as the elliptic regularity theorem (see e.g. [Fol99, Lemma 9.25 and Theorem 9.26, p. 307] or [Hör90, Theorem 7.9.7 p. 246 and Theorem 4.5.13 p. 123]):

Lemma 3.15 (**Elliptic regularity**). *Let P be an elliptic linear PDO of order m with constant coefficients, $s \in \mathbb{R}$, and $\Omega \subseteq \mathbb{R}^n$ be open.*

(i) **Local regularity:** $u \in \mathcal{D}'(\Omega)$ *and* $Pu \in H^s_{\mathrm{loc}}(\Omega) \implies u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.

(ii) **Global regularity:** $u \in H^s(\mathbb{R}^n)$ *and* $Pu \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n)$.

The theorem may be generalized to the case of linear elliptic PDOs with smooth coefficients. A similar result also holds for Sobolev spaces $W^{k,p}$ or for Hölder spaces $\mathcal{C}^{k,\alpha}$ with $0 < \alpha < 1$ (Definition 1.4). Yet, the result is not true for $\mathcal{C}^k$ functions [Fol99, p. 311]: For an mth-order elliptic operator P, the statements $u \in \mathcal{C}^k$ and $Pu \in \mathcal{C}^k$ do not imply $u \in \mathcal{C}^{k+m}$.

The principal symbol of a second-order linear PDO P is a quadratic form on $\mathbb{R}^n$, $p(.,\xi) = \xi \cdot A \cdot \xi$ for a matrix $A\colon \Omega \to \mathbb{R}^{n\times n}$. If A is uniformly **positive definite**, that is, $A(x)$ only has positive eigenvalues for all $x \in \Omega$, then P is called uniformly **strongly elliptic**. Equivalently, there exists $\alpha > 0$ such that

$$p(x,\xi) \geq \alpha|\xi|^2 \qquad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^n.$$

In the context of bilinear forms on Hilbert spaces, strong ellipticity is also known as coercivity (cf. Remark 8.6): A bilinear form $a\colon V \times V \to \mathbb{R}$ on a Hilbert space V is **coercive**, if

$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall\, v \in V. \tag{3.37}$$

To conclude this section, let us illustrate these concepts for the Poisson equation $-\triangle u = f$ on a Lip-domain $\Omega \subseteq \mathbb{R}^n$ and with $f \in L^2(\Omega)$: The weak form (3.25) implies that the bilinear form $a\colon V \times V \to \mathbb{R}$ corresponding to $-\triangle$ is given by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{dV}.$$

Here $V = H^1_0(\Omega)$ in case of Dirichlet boundary conditions or $V = H^1(\Omega)$ in case of Neumann boundary conditions (both are Hilbert spaces). Coercivity of a then follows from Poincaré's inequality [DL88, Chapter 4, §7, Corollary 2, p. 126 and Remark 3, p. 129]. This allows to apply the Lax–Milgram theorem (Remark 3.16), which yields existence and uniqueness (Neumann problem: up to a constant) of solutions $u \in V$ of the weak Poisson equation (3.25):

$$a(u,v) = \langle f, v\rangle \qquad \text{for all} \quad v \in V.$$

3.2.2 Variational solution of second-order evolution equations

We present the **variational solution method** for linear evolution problems of second order, which is also known as the method of **energy estimates**. In this section, we closely follow [DL92, Chapter 8], but consider only time-independent coefficients and real-valued functions.

Remark 3.16 (**Lax-Milgram Theorem**)**.** In the static case of purely spatial PDEs, the variational solution method essentially reduces to the Lax-Milgram Theorem [DL88, Chapter 7, §1.1, Theorem 1, p. 376]: If $a\colon V \times V \to \mathbb{R}$ is a continuous and coercive bilinear form on a Hilbert space V and $L \in V'$ (a continuous linear functional on V), then there exists a unique solution $u \in V$ of the abstract variational problem $a(u, v) = L(v)$ for all $v \in V$. ⌟

Within the context of linear evolutionary PDEs, the most natural Hilbert space framework for the dynamical variable u is $L^2((0,T), H)$ with a Hilbert space H and for finite time $0 < T < \infty$. At each time instant t, some sort of energy of the solution is required to be bounded, which is modeled by the condition $u(t) \in V$ for another Hilbert space $V \subseteq H$.

The Cauchy problem for the evolution equation is formulated in weak form: Find $t \mapsto u(t) \in V$ solving the **weak evolution equation**

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v\rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0,T) \tag{3.38}$$

and satisfying the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$.

Here a, b, c are bilinear forms on V or H and f represents the source. We will see that if a, b, c fulfill to certain symmetry and positivity hypotheses, then the Cauchy problem is well-posed: There exists a unique solution u that depends continuously on the data u^0, u^1, f (see Theorems 3.18 and 3.21).

We specify the **assumptions on the spaces**: Let H and V be separable real Hilbert spaces where V is dense in H and

$$V \hookrightarrow H \hookrightarrow V'.$$

In this so-called variational triple (Gelfand triple), the space V' is the dual of V and the pivot space H is identified with its dual H' (Riesz representation theorem for Hilbert spaces, [RR04a, Theorem 6.52, p. 196]). Let $\langle .|. \rangle_H$ denote the inner product in H with associated norm $\|.\|_H$, $\langle .|. \rangle_V$ and $\|.\|_V$ are those in V, and the norm in V' is $\|.\|_{V'}$. The continuity of the embeddings implies the existence of c, $c' > 0$ such that

$$\frac{1}{c'}\|v\|_{V'} \le \|v\|_H \le c\|v\|_V \qquad \forall\, v \in V. \tag{3.39}$$

Moreover, the duality between V and V', denoted by

$$\langle ., . \rangle := \langle ., . \rangle_{V',V},$$

is a continuous extension of the duality of H' and H, that is,

$$\langle ., . \rangle = \langle .|. \rangle_H \qquad \text{on} \quad H \times V. \tag{3.40}$$

Indeed, the inner product on H may be identified with the duality bracket and explicitly introducing the embedding $\iota\colon V \to H$ and its adjoint $\iota'\colon H' = H \to V'$, where ι and ι' are given by $\iota v = v$ and $\iota' u = u$ for $v \in V$ and $u \in H$ respectively, we obtain

$$\langle u, v\rangle = \langle u, v\rangle_{V',V} = \langle \iota' u, v\rangle_{V',V} = \langle u, \iota v\rangle_{H',H} = \langle u, v\rangle_{H',H} = \langle u|v\rangle_H \quad \forall\, u \in H,\ v \in V.$$

Next we list the **assumptions on the bilinear forms** a, b, c: Let a be a continuous bilinear form on V,

$$a\colon V \times V \to \mathbb{R}, \qquad a = a_0 + a_1, \tag{3.41}$$

where a_0 denotes the principal part of a. By continuity there exists $c_a > 0$ such that

$$|a(u, v)| \le c_a \|u\|_V \|v\|_V \qquad \forall\, u, v \in V.$$

In addition, the principal part a_0 is symmetric and **V-coercive with respect to H**, i.e. there exist $\alpha > 0$, $\lambda \in \mathbb{R}$ such that

$$a_0(u,v) = a_0(v,u) \qquad \text{and} \qquad a_0(u,u) \geq \alpha \|u\|_V^2 - \lambda \|u\|_H^2 \qquad \forall\, u, v \in V. \tag{3.42}$$

The bilinear form a_1 possesses additional regularity: There exist $c_{a_1} > 0$ and $c'_{a_1} > 0$ such that

$$|a_1(u,v)| \leq c_{a_1} \|u\|_V \, \|v\|_H \qquad \forall\, u, v \in V \tag{3.43}$$

and

$$|a_1(u,v)| \leq c'_{a_1} \|u\|_H \, \|v\|_{V'} \qquad \forall\, u, v \in H. \tag{3.44}$$

Let b, c be continuous bilinear forms on H,

$$b,\ c\colon H \times H \to \mathbb{R}. \tag{3.45}$$

By continuity there exist $c_b > 0$ and $c_c > 0$ such that

$$|b(w,v)| \leq c_b \|w\|_H \|v\|_H \qquad \text{and} \qquad |c(w,v)| \leq c_c \|w\|_H \|v\|_H \qquad \forall\, w, v \in H.$$

Finally it is assumed that c is symmetric and **H-coercive**, i.e. there exists $\gamma > 0$ such that

$$c(w,v) = c(v,w) \qquad \text{and} \qquad c(w,w) \geq \gamma \|w\|_H^2 \qquad \forall\, w, v \in H. \tag{3.46}$$

Let us formulate the evolution equation in **operator form**: The continuous bilinear forms a, b, c correspond to continuous linear operators between V, H and V'. In general, if X, Y are Banach spaces, then every bilinear mapping $f\colon X \times Y \to \mathbb{R}$ that is continuous, i.e. $|f(x,y)| \leq c\|x\|_X \|y\|_Y$, corresponds to a continuous linear operator $F\colon X \to Y'$, i.e. $F \in \mathrm{Lin}(X, Y')$ and vice versa: The identification is given by $\langle Fx, y\rangle_{Y',Y} = f(x,y)$ for all $x \in X$ and $y \in Y$.

Specifically, the continuous bilinear form $a = a_0 + a_1$ corresponds to the operator

$$A = A_0 + A_1 \in \mathrm{Lin}(V, V')$$

defined by $\langle Au, v\rangle = a(u,v)$ for all $u, v \in V$. In addition (3.43) and (3.44) imply

$$A_1 \in \mathrm{Lin}(V, H) \cap \mathrm{Lin}(H, V').$$

Similarly, setting $\langle Bw|v\rangle_H = b(w,v)$ and $\langle Cw|v\rangle_H = c(w,v)$ for all $w, v \in H$ yields the following operators corresponding to b and c:

$$B,\ C \in \mathrm{Lin}(H, H).$$

Rewriting the weak evolution equation (3.38),

$$\frac{d}{dt} c(\dot{u}, v) + b(\dot{u}, v) + a(u,v) = \langle f, v\rangle,$$

in terms of the operators A, B, C gives $\frac{d}{dt}\langle C\dot{u}|v\rangle_H + \langle B\dot{u}|v\rangle_H + \langle Au, v\rangle = \langle f, v\rangle$ which by (3.40) is the same as

$$\frac{d}{dt}\langle C\dot{u}, v\rangle + \langle B\dot{u}, v\rangle + \langle Au, v\rangle = \langle f, v\rangle. \tag{3.47}$$

The validity of (3.38) as an equation for all $v \in V$ in the sense of $\mathcal{D}'(0,T)$ means

$$\Big\langle \frac{d}{dt} c(\dot{u}, v) + b(\dot{u}, v) + a(u,v), \psi \Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = \Big\langle \langle f, v\rangle, \psi \Big\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$$

for all $v \in V$ and all test functions $\psi \in \mathcal{D}(0,T)$. We observe that with $f \in L^2((0,T), H)$, the weak formulation (3.38) makes sense as an equation in $\mathcal{D}'(0,T)$, if $u \in \mathcal{C}^0([0,T], V)$ and $\dot{u} \in \mathcal{C}^0([0,T], H)$, that is, $u \in \mathcal{C}^0([0,T], V) \cap \mathcal{C}^1([0,T], H)$.

Remark 3.17 (**Strong form under higher regularity**). Under the higher regularity conditions $u \in \mathcal{C}^2([0,T],V)$ and $f \in \mathcal{C}^0([0,T],H)$, the weak evolution equation (3.47) reads $\langle C\ddot{u} + B\dot{u} + Au, v\rangle = \langle f, v\rangle$, which now holds in $\mathcal{C}^0([0,T])$ for all $v \in V$. The evolution equation can then be written in strong form:

$$Pu = f \qquad \text{where} \qquad P := \frac{d}{dt}C\frac{d}{dt} + B\frac{d}{dt} + A, \qquad P\colon \mathcal{C}^2([0,T],V) \to \mathcal{C}^0([0,T],V'). \tag{3.48}$$

Actually $P = C\frac{d^2}{dt^2} + B\frac{d}{dt} + A$ but we have kept the position of the time-derivatives as in the general case of a time-dependent operator C. Alternatively, $u \in H^2((0,T),V)$ and $f \in L^2((0,T),H)$ implies the validity of the strong form $Pu = f$ a.e. in $L^2((0,T),V')$. ⌟

Having listed all our assumptions, we are ready to present the existence and uniqueness result [DL92, Chapter 8, §5, Problem (P_2), p. 570; Theorems 3 and 4]:

Theorem 3.18 (**Existence and uniqueness of weak solutions**). *Under the hypotheses* (3.41)–(3.46) *on* a, b, c *and given the data*

$$u^0 \in V, \quad u^1 \in H, \quad \text{and} \quad f \in L^2((0,T),H), \tag{3.49}$$

there exists a unique solution

$$u \in \mathcal{C}^0([0,T],V) \cap \mathcal{C}^1([0,T],H) \tag{3.50}$$

of the weak evolution equation (3.38)*, satisfying the initial conditions* $u(0) = u^0$ *and* $\dot{u}(0) = u^1$.

Existence follows from the **Faedo-Galerkin method**, which is based on an approximation of the problem in finite-dimensional subspaces and solvability of ODEs: **Energy estimates** yield boundedness results, which allow to extract a weakly convergent subsequence and thus establishes existence of a solution u (the energy estimates arise similarly as in the proof of Lemma 3.19 and the calculation leading to Theorem 3.21, see below). A direct application of the method to (3.38) yields solutions in $W := \left\{u \in L^2((0,T),V) : \dot{u} \in L^2((0,T),H),\ \frac{d}{dt}(C\dot{u}) \in L^2((0,T),V')\right\}$. Unfortunately, the elements of W are not necessarily continuous or differentiable with respect to time, which is essential for the validity of initial conditions. The higher regularity of u with respect to time is obtained from a **parabolic regularization** (**vanishing viscosity method**): The operator B is replaced by $B_\varepsilon := B + \varepsilon(A_0 + \lambda \operatorname{Id}_V)$, where $\varepsilon(A_0 + \lambda \operatorname{Id}_V)$ is an artificial viscosity term that disappears in the limit $\varepsilon \to 0$. Uniqueness may be established by choosing a special test function v, that consists of a time-integral of u, see [DL92, p. 572].

A solution u of the weak evolution equation necessarily satisfies the following equality [DL92, Chapter 8, §5, Lemma 7, p. 578]:

Lemma 3.19 (**Energy equality**). *Let* $u \in \mathcal{C}^0([0,T],V) \cap \mathcal{C}^1([0,T],H)$ *be the solution of* (3.38) *with* $u(0) = u^0 \in H$ *and* $\dot{u}(0) = u^1 \in V$ *(Theorem 3.18). Then, for* $t \in [0,T]$,

$$\begin{aligned}\frac{1}{2}\big(c(\dot{u}(t),\dot{u}(t)) + a_0(u(t),u(t))\big) + \int_0^t \big(b(\dot{u}(t'),\dot{u}(t')) + a_1(u(t'),\dot{u}(t'))\big)\,\mathrm{d}t' \\ = \frac{1}{2}\left(c(u^1,u^1) + a_0(u^0,u^0)\right) + \int_0^t \langle f(t'),\dot{u}(t')\rangle\,\mathrm{d}t'.\end{aligned} \tag{3.51}$$

Proof. (Heuristic argument) Formally, the energy equality is obtained by evaluating the weak form (3.38), $\frac{d}{dt}c(\dot{u},v) + b(\dot{u},v) + a(u,v) = \langle f, v\rangle$, at a fixed time $t' \in (0,T]$ and for the test

function $v = \dot{u}(t')$ (which corresponds to multiplying $Pu = f$ with $v = \dot{u}$), and time-integration:

$$\begin{aligned}
&\frac{d}{dt}c(\dot{u}(t'), v) + b(\dot{u}(t'), v) + a(u(t'), v) = \langle f(t'), v\rangle \quad \text{for} \quad v = \dot{u}(t'), \quad t' \in (0, T] \\
\implies \quad &\frac{d}{dt}c(\dot{u}, \dot{u}) + b(\dot{u}, \dot{u}) + a(u, \dot{u}) = \langle f, \dot{u}\rangle \quad \text{for} \quad t' \in (0, T] \\
\implies \quad &\frac{1}{2}\, c(\dot{u}, \dot{u})\Big|_0^t + \int_0^t (b(\dot{u}, \dot{u}) + a(u, \dot{u})) = \int_0^t \langle f, \dot{u}\rangle \\
\implies \quad &\frac{1}{2}\, (c(\dot{u}, \dot{u}) + a_0(u, u))\,\Big|_0^t + \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) = \int_0^t \langle f, \dot{u}\rangle.
\end{aligned}$$

The last step follows from $\frac{d}{dt}a_0(u, u) = a_0(u, \dot{u}) + a_0(\dot{u}, u) = 2\, a_0(u, \dot{u})$, which is a consequence of bilinearity and symmetry of a_0. However, the calculation above is only formal since Theorem 3.18 only gives $\dot{u} \in \mathcal{C}^0([0, T], H)$. Hence $v = \dot{u}(t') \in H$ for $t' \in (0, T]$, which generally is not an element of V and thus might not be a valid test function in the weak formulation (3.38). A rigorous proof of the Lemma based on double regularization techniques is provided in [LM72, Chapter 3, Section 8.4, Lemma 8.3, p. 276]. □

The equation in Lemma 3.19 is called energy equality, because the function $E(u)\colon [0, T] \to \mathbb{R}$,

$$E(u)(t) := \frac{1}{2}\big(c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t))\big) \tag{3.52}$$

can be interpreted as the **stored energy** (**kinetic energy** plus **potential energy**) of the solution u as a function of time. In terms of E, the energy equality expresses **energy balance** (i.e. conservation of total energy or the first law of thermodynamics):

$$E(u)(t) + \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u}))\,\mathrm{d}t = E(u)(0) + \int_0^t \langle f, \dot{u}\rangle\,\mathrm{d}t \tag{3.53}$$

or

$$\frac{d}{dt}E(u)(t) + \Big(b(\dot{u}, \dot{u}) + a_1(u, \dot{u})\Big) = \langle f, \dot{u}\rangle, \tag{3.54}$$

where $b(\dot{u}, \dot{u}) + a_1(u, \dot{u})$ corresponds to the **dissipation rate** and $\langle f, \dot{u}\rangle$ is the **external power**.

Together with the coercivity assumptions, the energy equality (Lemma 3.19) allows to estimate the norm of the solution in terms of the data and the initial conditions (the final result will be Theorem 3.21 below): The left-hand side of

$$c(\dot{u}, \dot{u}) + a_0(u, u) = c(u^1, u^1) + a_0(u^0, u^0) + 2\int_0^t \langle f, \dot{u}\rangle - 2\int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u}))$$

is bounded from below thanks to coercivity of c and a_0,

$$c(\dot{u}, \dot{u}) + a_0(u, u) \geq \gamma\|\dot{u}\|_H^2 + \alpha\|u\|_V^2 - \lambda\|u\|_H^2.$$

The right-hand side is bounded from above due to continuity of c and a_0,

$$c(u^1, u^1) + a_0(u^0, u^0) \leq c_c\|u^1\|_H^2 + c_a\|u^0\|_V^2,$$

and by continuity of b and a_1, the Cauchy-Schwarz inequality $\langle f, \dot{u}\rangle \leq \|f\|_H\|\dot{u}\|_H$, as well as the estimate $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$ (applied to the norms):

$$\begin{aligned}
2\int_0^t \langle f, \dot{u}\rangle - 2\int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) &\leq 2\int_0^t \Big(\|f\|_H\|\dot{u}\|_H + c_b\|\dot{u}\|_H^2 + c_{a_1}\|u\|_V\|\dot{u}\|_H\Big) \\
&\leq \int_0^t \Big(\|f\|_H^2 + \|\dot{u}\|_H^2 + 2c_b\|\dot{u}\|_H^2 + c_{a_1}\|u\|_V^2 + c_{a_1}\|\dot{u}\|_H^2\Big) \\
&= \int_0^t \Big(\|f\|_H^2 + (1 + c_{a_1} + 2c_b)\|\dot{u}\|_H^2 + c_{a_1}\|u\|_V^2\Big).
\end{aligned}$$

Combining both estimates yields the inequality

$$\gamma\|\dot u\|_H^2+\alpha\|u\|_V^2\le\lambda\|u\|_H^2+c_c\|u^1\|_H^2+c_a\|u^0\|_V^2+\int_0^t\Big(\|f\|_H^2+(1+c_{a_1}+2c_b)\|\dot u\|_H^2+c_{a_1}\|u\|_V^2\Big).$$

The fundamental theorem of calculus $u(t)=u^0+\int_0^t\dot u$ also holds for $u\in\mathcal{C}^1([0,T],H)$ (cf. [DL92, Chapter 8, §5, p. 561]), which allows to estimate the term $\lambda\|u(t)\|_H^2$ for $t\in[0,T]$:

$$\begin{aligned}\|u(t)\|_H=\|u^0+\textstyle\int_0^t\dot u\|_H\le\|u^0\|_H+\int_0^t\|\dot u\|_H\le\|u^0\|_H+\underbrace{(\textstyle\int_0^t1^2)^{1/2}}_{=\sqrt t}\Big(\int_0^t\|\dot u\|_H^2\Big)^{1/2}\\
\implies\quad\|u(t)\|_H^2\le\Big(\|u^0\|_H+\sqrt t\Big(\int_0^t\|\dot u\|_H^2\Big)^{1/2}\Big)^2\\
\le\|u^0\|_H^2+\underbrace{2\|u^0\|_H\sqrt t\Big(\int_0^t\|\dot u\|_H^2\Big)^{1/2}+t\int_0^t\|\dot u\|_H^2}_{\le\|u^0\|_H^2+t\int_0^t\|\dot u\|_H^2}\le2\|u^0\|_H^2+2t\int_0^t\|\dot u\|_H^2\\
\implies\quad\lambda\|u(t)\|_H^2\le2\lambda\|u^0\|_V^2+2\lambda T\int_0^t\|\dot u\|_H^2.\end{aligned}$$

Thus, the estimate takes the form

$$\begin{aligned}\gamma\|\dot u\|_H^2+\alpha\|u\|_V^2\le c_c\|u^1\|_H^2+(c_a+2\lambda)\|u^0\|_V^2+\int_0^t\|f\|_H^2\\
+\int_0^t\Big((1+c_{a_1}+2c_b+2\lambda T)\|\dot u\|_H^2+c_{a_1}\|u\|_V^2\Big),\end{aligned}$$

from which we deduce

$$\begin{aligned}\min(\alpha,\gamma)\big(\|u\|_V^2+\|\dot u\|_H^2\big)\le c_c\|u^1\|_H^2+(c_a+2\lambda)\|u^0\|_V^2+\int_0^T\|f\|_H^2\\
+(1+c_{a_1}+2c_b+2\lambda T)\int_0^t\big(\|u\|_V^2+\|\dot u\|_H^2\big).\end{aligned}$$

We need the following lemma, proved e.g. in [DL92, Chapter 8, §5, Lemma 1, p. 559]:

Lemma 3.20 (Gronwall's inequality). *Let $\phi\in L^\infty(0,T)$ and $\mu\in L^1(0,T)$ with $\phi,\mu\ge0$ a.e. on $(0,T)$, and $K\in\mathbb{R}$. Then*

$$\phi(t)\le K+\int_0^t\mu(t')\,\phi(t')\,\mathrm{d}t'\quad\implies\quad\phi(t)\le K\,e^{\int_0^t\mu(t')\,\mathrm{d}t'}$$

for a.a. $t\in(0,T)$. In particular, if $K=0$, we deduce $\phi(t)=0$.

In terms of the auxiliary function $\phi\in L^\infty(0,T)$,

$$\phi:=\|u\|_V^2+\|\dot u\|_H^2,$$

the estimate obtained from the energy equality reads

$$\begin{aligned}\min(\alpha,\gamma)\,\phi(t)\ \le\ c_c\|u^1\|_H^2+(c_a+2\lambda)\|u^0\|_V^2+\int_0^T\|f(t')\|_H^2\,\mathrm{d}t'\\
+(1+c_{a_1}+2c_b+2\lambda T)\int_0^t\phi(t')\,\mathrm{d}t'.\end{aligned}$$

Gronwall's inequality leads to the final form of the energy estimate:

Theorem 3.21 (**Energy estimate**). *Let $u \in \mathcal{C}^0([0,T],V) \cap \mathcal{C}^1([0,T],H)$ be the solution of (3.38) with $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$ (Theorem 3.18). Then, for a.a. $t \in (0,T)$,*

$$\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \leq k_1\, e^{k_2 t}, \tag{3.55}$$

with

$$k_1 := \frac{(c_a + 2\lambda)\|u^0\|_V^2 + c_c\|u^1\|_H^2 + \int_0^T \|f\|_H^2}{\min(\alpha,\gamma)} \tag{3.56}$$

and

$$k_2 := \frac{1 + c_{a_1} + 2c_b + 2\lambda T}{\min(\alpha,\gamma)}. \tag{3.57}$$

Remark 3.22 (**Uniqueness via energy estimates**). We note that the energy estimate also allows to establish uniqueness (which we already know from Theorem 3.18): Indeed, let $\tilde{u}$ and $\tilde{\tilde{u}}$ be different solutions of (3.38). Then, by linearity, their difference $u := \tilde{u} - \tilde{\tilde{u}}$ solves the homogeneous equation with zero initial values: $f = 0$, $u^0 = 0$, $u^1 = 0$. But these conditions imply $k_1 = 0$ and the energy estimate gives $\|u\|_V^2 + \|\dot{u}\|_H^2 = 0$ a.e. on $(0,T)$. Consequently we deduce $u = 0$, i.e. $\tilde{u} = \tilde{\tilde{u}}$, in $L^\infty((0,T),V) \cap W^{1,\infty}((0,T),H)$, which proves that solutions are unique. ⌟

The continuity assumptions on a_0 and c imply that the auxiliary function $\phi = \|\dot{u}\|_H^2 + \|u\|_V^2$ bounds the total energy of the solution (5.19):

$$E(u) = \frac{1}{2}\big(c(\dot{u},\dot{u}) + a_0(u,u)\big) \leq \frac{1}{2}\big(c_c\|\dot{u}\|_H^2 + c_a\|u\|_V^2\big) \leq k_0\big(\|\dot{u}\|_H^2 + \|u\|_V^2\big)$$

with

$$k_0 := \frac{1}{2}\max(c_a, c_c). \tag{3.58}$$

Thus we have obtained the **total energy estimate**

$$E(u)(t) \leq k_0 k_1\, e^{k_2 t} \quad \text{for a.a. } t \in (0,T), \tag{3.59}$$

with constants k_0, k_1, k_2 defined in (3.58), (3.56), (3.57). The energy estimate quantifies how the total energy of the solution depends on the data (i.e. the initial data, $\|u^0\|_V$, $\|u^1\|_H$, and the source, $\|f\|_H$), as well as on the coefficients (i.e. the coercivity constants α, λ, γ and the continuity constants c_a, c_{a_1}, c_b, c_c).

Part II

A variational model with low regularity

Chapter 4

The composite fluid-solid earth model

We define a general planetary model that is composed of regions with different physical properties. First we propose the concept of Lip-composite domains, discuss identities and operations on surfaces, and present a generalized version of the divergence theorem for composite domains (Section 4.1). These are the prerequisites needed to define the composite fluid-solid earth model and its admissible motions (Section 4.2). With special emphasis on the consistency of regularity assumptions, we then introduce the relevant physical fields and constitutive properties of the earth model within the framework of nonlinear continuum mechanics: Density, volume forces, gravity, elastic energy, and stress (Sections 4.3, 4.4, 4.5, 4.6).

4.1 Composite domains and surfaces

4.1.1 Lipschitz composite domains with interior boundaries

We introduce an abstract notion of a so-called interior boundary for a general subset of $\mathbb{R}^n$, which represents the union of all interfaces inside a continuous body. For example, the definition describes the cut lines in the interior of the circles in the figure on p. 15.

Definition 4.1 (**Interior boundary**)**.** The interior boundary Σ of a set $V \subseteq \mathbb{R}^n$ is defined by

$$\Sigma := \partial V \setminus \partial \overline{V}. \tag{4.1}$$

According to the definition, interior boundary points of V are elements of the boundary ∂V that are not boundary points of the closure $\overline{V}$. We call $\partial\overline{V}$ the exterior boundary, which by definition is disjoint to the interior boundary: $\Sigma \cap \partial\overline{V} = \emptyset$. One always has $\partial\overline{V} \subseteq \partial V$, because $\partial\overline{V} = \overline{\overline{V}}\setminus\overline{V}^\circ = \overline{V}\setminus\overline{V}^\circ \subseteq \overline{V}\setminus V^\circ = \partial V$. By construction, we have the disjoint union $\partial V = \Sigma \cup \partial\overline{V}$; combined with $\overline{V} = V \cup \partial V$, we obtain the representation

$$\overline{V} = V \cup \Sigma \cup \partial\overline{V} \tag{4.2}$$

which gives a decomposition into disjoint sets, if V is open. Moreover, noting that $\overline{V}^\circ = \overline{V}\setminus\partial\overline{V}$, we may then also deduce the identity

$$\overline{V}^\circ = V \cup \Sigma. \tag{4.3}$$

Examples for interior boundaries are cuts inside V and hypersurfaces $S \subseteq \overline{V}^\circ \setminus V^\circ$ such that V is located on both sides of S. In particular, Lip-domains have no interior boundaries:

$$\Sigma = \emptyset \text{ for all Lip-domains.}$$

Indeed, the requirement of a Lip-domain V being open and located on one side of its boundary (see Definition 1.5) implies that V is regularly open, that is $V = (\overline{V})^\circ$ (1.5), and thus $\partial V \setminus (\partial \overline{V}) = \emptyset$ (taking the closure of a set removes all its interior boundaries, as is clear from Definition 4.1). Therefore, sets $V \subseteq \mathbb{R}^n$ with cuts or other interior boundaries S cannot be Lip-domains, since V is located on both sides of $S \subseteq \partial V$.

Sets which include interior boundaries that are Lip-surfaces may be modeled by considering the following composition of Lip-domains:

Definition 4.2 (Lip-**composite domains**)**.** A subset V of $\mathbb{R}^n$ is called a Lip-composite domain if it can be written as a finite union of pairwise disjoint Lip-domains $V_k \subseteq \mathbb{R}^n$ $(k = 1, \dots, k_0)$, that is,

$$V = \bigcup_{k=1}^{k_0} V_k \qquad \text{with} \qquad V_k \cap V_{k'} = \emptyset \quad \text{for} \quad k \neq k', \tag{4.4}$$

with the additional property that for every subset $M \subseteq \{1, \dots, k_0\}$, the set $V_M := (\bigcup_{k \in M} \overline{V_k})^\circ$ is a finite disjoint union of Lip-domains.

The condition on the sets V_M is required to rule out sets with cusps of the exterior or interior boundary (cf. the middle figure on p. 15). If V is a Lip-composite domain with $\overline{V}$ connected, then $\overline{V}^\circ$ is a Lip-domain.

Consider a Lip-composite domain V with disjoint union $V = \bigcup_{k=1}^{k_0} V_k$ as in the definition. Then the following identity can be derived, where the first equality is a consequence of V being a disjoint union of open sets and the second equality follows from the property $\overline{V_k}^\circ = V_k$ (the Lip-domains V_k have no interior boundary):

$$\partial V = \bigcup_{k=1}^{k_0} \partial V_k = \bigcup_{1 \leq k < k' \leq k_0} \Sigma_{kk'} \cup \partial \overline{V} \qquad \text{with} \qquad \Sigma_{kk'} := \partial V_k \cap \partial V_{k'} \quad \text{for} \quad k \neq k'. \tag{4.5}$$

By Definition 4.1, the interior boundary $\Sigma = \partial V \setminus \partial \overline{V}$ of a Lip-composite domain V is thus given by

$$\Sigma = \bigcup_{1 \leq k < k' \leq k_0} \Sigma_{kk'} \setminus \partial \overline{V}. \tag{4.6}$$

We note that the sets $\Sigma_{kk'}$ are $(n-1)$-dimensional Lip-surfaces in the sense of Definition 1.5 and their boundaries $\partial \Sigma_{kk'}$ (here, boundary in the sense of a submanifold of ∂V with boundary) thus are $(n-2)$-dimensional manifolds. The boundary $\partial \Sigma$ of the interior boundary Σ consists of those parts of $\partial \Sigma_{kk'}$ that lie on the exterior boundary $\partial \overline{V}$ of V, that is,

$$\partial \Sigma = \bigcup_{1 \leq k < k' \leq k_0} \partial \Sigma_{kk'} \cap \partial \overline{V}. \tag{4.7}$$

In particular, in $\mathbb{R}^3$, $\partial \Sigma$ is a finite union of curves in $\partial \overline{V}$.

4.1.2 Identities and differential operators on surfaces

We introduce some basic identities on interfaces. Let S be a hypersurface in $\mathbb{R}^n$ of sufficient regularity (e.g. Lip or $\mathcal{C}^{1,1}$ will be enough). We think of S as an **interface**, i.e. a subset of the interior boundary of a Lip-composite domain, such that the $+$-side and $-$-side of S are unambiguously defined. Let f be any vector- or tensor-valued function, whose traces $f^\pm$ from the $\pm$-sides of S exist (e.g. in some Sobolev sense). Then the **jump** of f across S is defined as

$$[f]^+_- := f^+ - f^-. \tag{4.8}$$

We also introduce the **mean value** of f on S:

$$\{f\} := \frac{1}{2}(f^+ + f^-). \tag{4.9}$$

The jump of a product satisfies a **Leibniz rule**, which we state in two variants: If f, g are functions whose traces $f^\pm, g^\pm$ exist on S, then

$$[fg]^+_- = f^+[g]^+_- + [f]^+_- g^- \tag{4.10}$$

and

$$[fg]^+_- = \{f\}[g]^+_- + [f]^+_-\{g\}. \tag{4.11}$$

Indeed, a direct calculation yields

$$\begin{aligned}
f^+[g]^+_- + [f]^+_- g^- &= f^+(g^+ - g^-) + (f^+ - f^-)g^- = f^+g^+ - f^-g^- = (fg)^+ - (fg)^- = [fg]^+_-, \\
\{f\}[g]^+_- + [f]^+_-\{g\} &= \frac{1}{2}(f^+ + f^-)(g^+ - g^-) + \frac{1}{2}(f^+ - f^-)(g^+ + g^-) \\
&= \frac{1}{2}\left((f^+g^+ - f^+g^- + f^-g^+ - f^-g^-) + (f^+g^+ + f^+g^- - f^-g^+ + f^-g^-)\right) \\
&= f^+g^+ - f^-g^- = (fg)^+ - (fg)^- = [fg]^+_-.
\end{aligned}$$

The jump has a certain similarity to a derivative since it may be written as a limit of a difference (see also (4.13) below). Consequently, the validity of some variant of the Leibniz rule is no surprise.

If f is continuous across S, i.e. $[f]^+_- = 0$, we may identify $f|_S$ with $f^+ = f^- = \{f\}$ and the Leibniz rule (4.10) implies that f can be pulled out of the jump operator:

$$[f]^+_- = 0 \qquad \Longrightarrow \qquad [fg]^+_- = f[g]^+_-.$$

Let S be orientable and let us fix the orientation by defining the **unit normal vector** $\nu\colon S \to \mathbb{R}^n$. In accordance with [DT98, Figure A.1, p. 826], we agree on the convention that ν points in the direction of the jump, that is, from the $-$-side to the $+$-side of S: If $\nu^{(\pm)}$ denotes the (outwards directed) normal vector of the region on the $\pm$-side of the surface, we define

$$\nu := \nu^{(-)} = -\nu^{(+)}. \tag{4.12}$$

Note that this choice of orientation is the reason for the negative sign in front of the integrals over interior boundaries below. The orientation of ν will be fixed from now on. Since ν points to the $+$-side of S, we may also express the traces $f^\pm$ and hence also the jump of f at $x \in S$ as the limit

$$[f]^+_-(x) = f^+(x) - f^-(x) = \lim_{h\to 0}\Big(f(x + h\nu(x)) - f(x - h\nu(x))\Big). \tag{4.13}$$

If the normal vector appears inside a jump as a factor of a scalar product, this abuse of notation (note that the traces $\nu^\pm$ do not make sense) has to be understood as $[f \cdot \nu]^+_- := [f]^+_- \cdot \nu$.

Definition 4.3 (**Normal and tangential parts**). A function $f\colon S \to \mathbb{R}^{m\times n}$ can be uniquely decomposed into its normal part $f^\perp$ and tangential part $f^\parallel$:

$$f = f^\perp + f^\parallel \qquad \text{with} \qquad f^\perp := (f\cdot\nu)\nu \qquad \text{and} \qquad f^\parallel := f\cdot(1_{n\times n} - \nu\nu) = f - (f\cdot\nu)\nu. \tag{4.14}$$

In Cartesian components, where $f = (f_{ij})_{i,j=1}^{m,n}$, we have $f_{ij} = f^\perp_{ij} + f^\parallel_{ij}$ with

$$f^\perp_{ij} = (f_{ik}\nu_k)\nu_j \qquad \text{and} \qquad f^\parallel_{ij} = f_{ik}(\delta_{kj} - \nu_k\nu_j) = f_{ij} - (f_{ik}\nu_k)\nu_j.$$

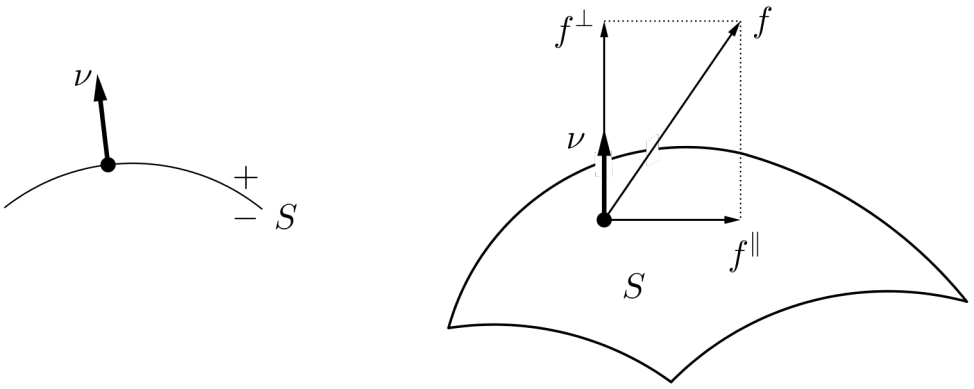

Figure 4.1: Sign convention of ν. Decomposition $f = f^\perp + f^\parallel$

If f takes values in $\mathbb{R}^n$ with $f = (f_j)_{j=1}^n$, the term $f \cdot \nu = f_k \nu_k$ is a scalar called the normal component of f and we have

$$f_j^\perp = (f_k \nu_k)\nu_j \qquad \text{and} \qquad f_j^\parallel = f_k(\delta_{kj} - \nu_k \nu_j) = f_j - (f_k \nu_k)\nu_j.$$

Note that by Definition 4.3 we always multiply with ν from the right.[1] Since $\nu \cdot \nu = 1$, we immediately get

$$f^\perp \cdot \nu = f \cdot \nu \qquad \text{and} \qquad f^\parallel \cdot \nu = 0.$$

These identities justify to call $f^\perp$ the normal and $f^\parallel$ the tangential part of f.

If f is defined in a neighborhood of S, then also the traces $f^\pm$ may be split into tangential and normal parts: $f^\pm = f^{\pm\perp} + f^{\pm\parallel}$. Note that the converse expressions (i.e. $f^{\perp\pm}$ and $f^{\parallel\pm}$) do not make sense, since $f^\perp$ and $f^\parallel$ are defined only on S and thus do not possess traces from different sides. By abuse of notation we set $[f^\parallel]_-^+ := [f]_-^{+\,\parallel}$.

A function f has continuous normal component across the surface, if $f^{+\perp} = f^{-\perp}$ or, equivalently, if $[f]_-^{+\,\perp} = 0$, $f^+ \cdot \nu = f^- \cdot \nu$, or $[f]_-^+ \cdot \nu = 0$. In this case its jump $[f]_-^+ = [f]_-^{+\,\perp} + [f]_-^{+\,\parallel}$ is purely tangential:

$$[f]_-^+ \cdot \nu = 0 \qquad \Longrightarrow \qquad [f]_-^+ = [f]_-^{+\,\parallel}. \tag{4.15}$$

We turn our attention to the surface analogs of gradient and divergence of a $\mathbb{R}^{m\times n}$-valued field f that is differentiable on a neighborhood of S. The **surface gradient** $\widetilde{\nabla}$ is given by the derivative in tangential directions. From (4.14) we obtain

$$\widetilde{\nabla} f := (\nabla f)^\parallel = (\nabla f) \cdot (1_{n\times n} - \nu\nu) = \nabla f - (\nabla f \cdot \nu)\nu. \tag{4.16}$$

Since $\widetilde{\nabla} f$ is the tangential part of ∇f, it follows that

$$\widetilde{\nabla} f \cdot \nu = 0.$$

Consequently, $\widetilde{\nabla} f$ only acts on tangential parts of vector fields, that is, if $f, g \colon S \to \mathbb{R}^n$, then

$$\widetilde{\nabla} f \cdot g = \widetilde{\nabla} f \cdot (g^\parallel + (g \cdot \nu)\nu) = \widetilde{\nabla} f \cdot g^\parallel. \tag{4.17}$$

The surface gradient (4.16) may also be applied to one-sided limits $f^\pm$ and jumps $[f]_-^+$, provided that they are continuously extended to a neighborhood of the surface.

[1] As an alternative, one could take the scalar product with ν from the left and define $f^\perp := \nu(\nu \cdot f)$ and $f^\parallel := (1_{n\times n} - \nu\nu) \cdot f = f - \nu(\nu \cdot f)$ as well as $\nu \cdot f^\perp = \nu \cdot f$, $\nu \cdot f^\parallel = 0$, see e.g. [DT98, A.6, p. 827]. Both conventions coincide if f is vector-valued, upon identification of row vectors with column vectors, but differ if f has multiple indices.

The definition of the surface gradient via formula (4.16) is well-known in the continuum mechanics of surfaces (see e.g. [Gur00, p. 95, 15-8bc]). In application to tangent vectors, the surface gradient $\widetilde{\nabla}$ defined by (4.16) coincides with the covariant derivative on S, see e.g. [O'N06, Chapter 7, Lemma 3.8, p. 343]. Here, S is viewed as a Riemannian manifold, where the metric is induced from the Euclidean metric of $\mathbb{R}^n$ (see also Remark 5.4 with $S = \partial M$).

The **surface divergence** is the (matrix) trace of the surface gradient when the last two indices are contracted:

$$\widetilde{\operatorname{div}} f := \widetilde{\nabla} \cdot f = \operatorname{tr}(\widetilde{\nabla} f) = \nabla \cdot f - (\nabla f \cdot \nu) \cdot \nu. \tag{4.18}$$

In components, $(\widetilde{\operatorname{div}} f)_i = \widetilde{\partial}_j f_{ij} = \partial_j f_{ij} - (\partial_k f_{ij})\nu_k \nu_j = \partial_j f_{ij} - \nu_k (\partial_k f_{ij})\nu_j$. In particular, if f is $\mathbb{R}^n$-valued, we have $(\nabla f \cdot \nu) \cdot \nu = \nu \cdot \nabla f \cdot \nu = \nu_k (\partial_k f_j)\nu_j$.

Application of the surface gradient to the unit normal yields the **Weingarten map** $\widetilde{\nabla}\nu$, which is a symmetric bilinear form on the tangent space of S ($\widetilde{\nabla}\nu$ is related to the curvature of the surface, see Remark 4.4). Symmetry and the projection property (4.17) of $\widetilde{\nabla}\nu$ imply

$$f \cdot \widetilde{\nabla}\nu \cdot g = f \cdot \widetilde{\nabla}\nu \cdot g^{\parallel} = g^{\parallel} \cdot \widetilde{\nabla}\nu \cdot f = g^{\parallel} \cdot \widetilde{\nabla}\nu \cdot f^{\parallel} = g \cdot \widetilde{\nabla}\nu \cdot f$$

for $f, g \colon S \to \mathbb{R}^n$. From $0 = \widetilde{\nabla} 0 = \widetilde{\nabla}(f^{\parallel} \cdot \nu) = \nu \cdot \widetilde{\nabla} f^{\parallel} + f^{\parallel} \cdot \widetilde{\nabla}\nu$ we then also get the identity

$$\nu \cdot \widetilde{\nabla} f^{\parallel} = -f^{\parallel} \cdot \widetilde{\nabla}\nu \quad (= -f \cdot \widetilde{\nabla}\nu = -\widetilde{\nabla}\nu \cdot f = -\widetilde{\nabla}\nu \cdot f^{\parallel}). \tag{4.19}$$

Remark 4.4 (**Shape operator and surface curvature**)**.** In differential geometry of Riemannian surfaces, the negative Weingarten map $-\widetilde{\nabla}\nu$ associated to a manifold M corresponds to the shape operator S: At $p \in M$, $S_p \colon T_pM \to T_pM$, with $S_p(v) = -\widetilde{\nabla}\nu(p) \cdot v$, i.e. $S_p(v)$ is minus the covariant derivative of ν at $p \in M$ in direction $v \in T_pM$ [O'N06, Chapter 5, Def. 1.1, p. 203]. The trace of the Weingarten map, i.e. the surface divergence of the unit normal, $\operatorname{tr}(\widetilde{\nabla}\nu) = \widetilde{\nabla} \cdot \nu$, is the sum of the $n-1$ principal curvatures of M. This term occurs in the following splitting of the volume divergence on a surface: $\operatorname{div} f = \widetilde{\operatorname{div}}(f^{\parallel}) + (\nabla f \cdot \nu) \cdot \nu + (f \cdot \nu)(\widetilde{\nabla} \cdot \nu)$. ⌟

The surface divergence theorem is the classical divergence theorem (1.16) formulated on a surface S as a bounded manifold (see e.g. [Cha06, Thm. III.7.5, p. 152] for a proof in the smooth case).

Lemma 4.5 (**Divergence theorem for surfaces**)**.** *Let S be a* Lip*-hypersurface of $\mathbb{R}^n$ and denote by $\mathrm{d}\lambda$ the line element orthogonal to the surface boundary ∂S and tangential to S. If $f \in \operatorname{Lip}(U)^n$ for $U \subseteq \mathbb{R}^n$ a neighborhood of S, then*

$$\int_S \widetilde{\operatorname{div}} f \, \mathrm{dS} = \int_{\partial S} f \cdot \mathrm{d}\lambda. \tag{4.20}$$

In the smooth case, e.g. if $f \in \mathcal{C}^1(U)^{m \times n}$ and $h \in \mathcal{C}^1(U)^m$ for a neighborhood $U \subseteq \mathbb{R}^n$ of the smooth surface S, the surface divergence theorem (Lemma 4.5) and the product rule

$$\widetilde{\operatorname{div}}(h \cdot f) = \widetilde{\nabla} h : f + h \cdot \widetilde{\operatorname{div}} f$$

imply the following **surface integration by parts formula**, which is the surface analog of (1.17):

$$\int_S \widetilde{\nabla} h : f \mathrm{dS} = -\int_S h \cdot \widetilde{\operatorname{div}} f \, \mathrm{dS} + \int_{\partial S} h \cdot f \cdot \mathrm{d}\lambda. \tag{4.21}$$

For a closed Lip-surface ($\partial S = \emptyset$), the boundary contribution vanishes. In this case (4.21) may be interpreted via surface Sobolev dualities (3.32), if $f \in H^{\frac{1}{2}}(S)^{m \times n}$ and $h \in H^{\frac{1}{2}}(S)^m$:

$$\langle\!\langle \widetilde{\nabla} h, f \rangle\!\rangle_S = -\langle\!\langle \widetilde{\operatorname{div}} f, h \rangle\!\rangle_S. \tag{4.22}$$

Here the negative surface divergence $-\widetilde{\operatorname{div}} \colon H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$ is viewed as the abstract adjoint of the surface gradient $\widetilde{\nabla} \colon H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S) = (H^{\frac{1}{2}}(S))'$, upon noting that reflexivity of $H^{\frac{1}{2}}(S)$ implies $H^{\frac{1}{2}}(S) = (H^{\frac{1}{2}}(S))''$.

Remark 4.6 (**The surface divergence theorem and the classical Stokes' theorem**)**.** In $\mathbb{R}^3$, the surface divergence theorem (4.20) is a variant of the classical Stokes' theorem, which reads $\int_S \operatorname{rot}(\nu \times f)\,\mathrm{dS} = \int_{\partial S}(\nu \times f)\cdot \mathrm{d}l$. Indeed, a straightforward calculation shows $\operatorname{rot}(\nu \times f) = \widetilde{\operatorname{div}} f$, and we have $(\nu \times f)\cdot \mathrm{d}l = f \cdot (\mathrm{d}l \times \nu)$. Since the classical line element $\mathrm{d}l$ is parallel to the boundary curve ∂S, it follows that $\mathrm{d}l \times \nu =: \mathrm{d}\lambda$ is normal to ∂S. ⌟

4.1.3 The divergence theorem for Lipschitz composite domains

We present a variant (Lemma 4.11) of the divergence theorem for Lip-composite domains. Compared to the classical formulation (Lemma 1.6) for Lip-domains, it will contain an additional surface integral over the interior boundary.

As a starting point we discuss some basic properties and notation for piecewise differentiable functions, first in the classical $\mathcal{C}^1$ setting. Let $V = \bigcup_{k=1}^{k_0} V_k$ be a Lip-composite domain as in (4.4). By an $\mathbb{R}^{m\times n}$-valued, bounded piecewise $\mathcal{C}^1$-function f on $\overline{V}$, with possible discontinuities being jumps contained in the interior boundaries Σ, we mean

$$f \in \mathcal{C}^1(\textstyle\bigcup_{k=1}^{k_0} V_k)^{m\times n} \cap L^\infty(\overline{V})^{m\times n}$$

such that every restriction $f|_{V_k}$ $(1 \le k \le k_0)$ possesses a $\mathcal{C}^1$ extension to $\overline{V_k}$, denoted by $f^{\overline{V_k}}$:

$$f^{\overline{V_k}} \in \mathcal{C}^1(\overline{V_k})^{m\times n}.$$

The classical partial derivative of f is continuous on the disjoint union $V = \bigcup_{k=1}^{k_0} V_k$ and may be expressed as a sum involving cutoff functions,

$$(\partial_j f)\chi_V = \sum_{k=1}^{k_0} (\partial_j f^{\overline{V_k}})\chi_{V_k} \in L^\infty(\overline{V})^{m\times n},$$

where the characteristic functions χ_{V_k} have value 1 on V_k and 0 else.

Remark 4.7 (**Derivatives of discontinuous functions: Surface measures**)**.** The distributional partial derivative of a piecewise continuous function may contain additional surface measures at the discontinuity surfaces (cf. [Hör90, Thm. 3.1.9, p. 60]). ⌟

Recalling that $\partial V_k \setminus \Sigma \subseteq \partial\overline{V}$, we define the trace of f on $\partial\overline{V}$ by

$$\overline{f} := \sum_{k=1}^{k_0} f^{\overline{V_k}} \chi_{\partial V_k \setminus \Sigma} \in L^\infty(\partial\overline{V})^{m\times n}. \tag{4.23}$$

We note that there is no contribution of $f^{\overline{V_k}}$ to $\overline{f}$ if $\partial V_k \subseteq \Sigma$, that is, if V_k is a completely interior region. By construction, $\overline{f}$ is $\mathcal{C}^1$ on the finite union $\partial\overline{V} \setminus \partial\Sigma$ of surfaces on the exterior boundary. To simplify the notation we omit the bar introduced in (4.23) in the following, that is, we write f instead of the trace $\overline{f}$ in surface integrals.

In general, an interior boundary Σ will not be a single Lip-surface but a finite union of such, see (4.6). Consequently a surface integral over Σ has to be interpreted as the corresponding sum of surface integrals. More precisely, if $g\colon \Sigma \to \mathbb{R}$ with restrictions $g|_{\Sigma_{kk'}} \in L^1(\Sigma_{kk'})$ for all $1 \le k < k' \le k_0$, then $g \in L^1(\Sigma)$ with

$$\int_\Sigma g\,\mathrm{dS} := \sum_{1\le k<k'\le k_0} \int_{\Sigma_{kk'}} g\,\mathrm{dS}. \tag{4.24}$$

Let us discuss the definition of the jump of f, $[f]^+_- = f^+ - f^-$ (4.8), when passing through Σ. Strictly speaking, the notation $[.]^+_-$ for jumps across the interior boundary is only applicable to a composite domain consisting of two parts, one labeled by $+$, the other one labeled by $-$. Nevertheless it can also be extended to composite domains, which are such that every interior boundary point has a neighborhood containing elements of only an even number of different interior regions. Thereby, points such as triple-junctions where an odd number of different interior regions meet, are not allowed. However, in the later application, we will need to label the interior regions by two different flags (fluid or solid). In addition, if two regions of the same kind share a common boundary, they are glued together by taking the closure of their union, see (4.39). This allows us to consistently use the $[.]^+_-$-notation and choose the normal vector.

After these preparatory observations we are ready to identify the jump across the interior boundary surface $\Sigma = \partial V \setminus \partial\overline{V}$ with

$$[f]^+_-(x) = f^{\overline{V_k}}(x) - f^{\overline{V_{k'}}}(x) \qquad \text{for} \quad x \in \Sigma_{kk'},$$

where V_k corresponds to the region on the $+$-side and $V_{k'}$ to the region on the $-$-side of Σ. Similarly, the identities related to surfaces S in $\mathbb{R}^n$, presented in Section 4.1.2, are applicable to Σ and corresponding piecewise $\mathcal{C}^1$ functions as well. The results also hold for piecewise Lip or H^1 functions f, if $\overline{f}$, $f^\pm$, $[f]^+_- \in H^{\frac{1}{2}}(\Sigma)^{m\times n}$ are interpreted almost everywhere via the trace.

Of particular importance will be the space of **piecewise H^1 vector fields with continuous normal component** across the interior boundary Σ of a Lip-composite domain $V = \bigcup_{k=1}^{k_0} V_k$:

$$H^1_\Sigma(V)^n := \{h \in H^1(V)^n : \ [h]^+_- \cdot \nu = 0 \ \text{ on } \ \Sigma\}. \tag{4.25}$$

This will serve as our space of test functions in the linearized variational framework (see (6.11) and Assumption 2 below). However, to obtain the formula for integration by parts in almost usual form (Lemma 4.11), the restriction of the test functions has to be balanced by requiring some additional symmetry conditions:

Definition 4.8 (Isotropy). Let $V \subseteq \mathbb{R}^n$, then $f\colon V \to \mathbb{R}^{n\times n}$ is called isotropic if there exists $f^0\colon V \to \mathbb{R}$ such that the components of f are given by $f_{ij} = f^0\delta_{ij}$, that is,

$$f = f_0\, 1_{n\times n}. \tag{4.26}$$

Remark 4.9 (Isotropic tensors of higher order). Isotropic 2-tensor fields as above are also known as spherical tensors [Cha76]. Isotropic tensors of arbitrary order are characterized by the invariance of their components under rotations of the coordinate system: A 2-tensor $f \in \mathbb{R}^{n\times n}$ is isotropic if $f = Q^T \cdot f \cdot Q$ for all $Q \in SO(n)$, which is equivalent to the definition above [Ogd84, Section 1.2.5, p. 18]. It can be shown that an isotropic 4-tensor $c \in \mathbb{R}^{n\times n\times n\times n}$ has the coefficients $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk}$ with $\lambda, \mu_1, \mu_2 \in \mathbb{R}$. In case of classical linearized elasticity, where c has major and minor symmetries ($c_{ijkl} = c_{klij}$ and $c_{ijkl} = c_{jikl} = c_{ijlk}$), λ and $\mu := \mu_1 = \mu_2$ are the Lamé coefficients in the isotropic version of Hooke's law. ⌟

Definition 4.10 (Normality condition). Let S be a hypersurface with unit normal ν, then $f\colon S \to \mathbb{R}^{n\times n}$ is said to satisfy the normality condition on S, if

$$f \cdot \nu = (\nu \cdot f \cdot \nu)\nu, \tag{4.27}$$

or equivalently, if $f \cdot \nu$ is purely normal on S or has zero tangential part on S (cf. Definition 4.3):

$$f \cdot \nu = (f \cdot \nu)^\perp \qquad \text{or} \qquad (f \cdot \nu)^\parallel = 0.$$

The term normality condition is also used in [DT98, eq. (3.82)]. Isotropy implies normality in the sense that every isotropic tensor field satisfies the normality condition on any arbitrary

surface S: $f \cdot \nu = (f_0\, 1_{n\times n}) \cdot \nu = f_0\, \nu$ and $\nu \cdot f \cdot \nu = \nu \cdot (f_0\, 1_{n\times n}) \cdot \nu = f_0(\nu \cdot \nu) = f_0$. In our earth model, f will correspond to the stress tensor, therefore normality on a surface means that all tangential stresses vanish. Specifically, if S is an interface and f is the jump of stress tensors across that interface, then the normality condition models the case of perfect slip.

We are ready to present the divergence theorem and different variants of the formula for integration by parts on composite domains. As in the classical versions (1.16) and (1.17) on Lip-domains (without interior boundaries) one has to interpret the surface integrals as Sobolev dualities, cf. (7.18). Moreover it again suffices to assume $f \in H_{\mathrm{div}}$ instead of H^1.

Lemma 4.11 (Divergence theorem and integration by parts for composite domains). *Let $V = \bigcup_{k=1}^{k_0} V_k \subseteq \mathbb{R}^n$ be a* Lip*-composite domain (4.4) with interior boundary $\Sigma = \partial V \setminus \partial\overline{V}$. If $f \in H^1(V)^{m\times n}$, then*

$$\int_V \operatorname{div} f \,\mathrm{dV} = \int_{\partial\overline{V}} f \cdot \nu \,\mathrm{dS} - \int_\Sigma [f]^+_- \cdot \nu \,\mathrm{dS}. \tag{4.28}$$

For $h \in H^1(\overline{V})^m$, this leads to a variant of Green's formula for integration by parts:

$$\int_V f : Dh \,\mathrm{dV} = -\int_V h \cdot \operatorname{div} f \,\mathrm{dV} + \int_{\partial\overline{V}} h \cdot f \cdot \nu \,\mathrm{dS} - \int_\Sigma h \cdot [f]^+_- \cdot \nu \,\mathrm{dS}. \tag{4.29}$$

For $h \in H^1_\Sigma(V)^n$ (4.25) and if $m = n$, we have the following modifications:

(i) *If $f \in H^1(V)^{n\times n}$ is isotropic (4.26), i.e. $f = f^0 1_{n\times n}$ with f^0 an H^1 function on V (or in a neighborhood of Σ), then Green's formula (4.29) holds.*

(ii) *If f satisfies the normality condition (4.27) on the $-$-side of Σ, that is $f^- \cdot \nu = (\nu \cdot f^- \cdot \nu)\nu$ or $(f^- \cdot \nu)^\parallel = 0$, then Green's formula is valid upon replacing h by h^+ in the last term:*

$$\int_V f : Dh \,\mathrm{dV} = -\int_V h \cdot \operatorname{div} f \,\mathrm{dV} + \int_{\partial\overline{V}} h \cdot f \cdot \nu \,\mathrm{dS} - \int_\Sigma h^+ \cdot [f]^+_- \cdot \nu \,\mathrm{dS}. \tag{4.30}$$

A similar statement is true for the mean value: If $\{f\} \cdot \nu = (\nu \cdot \{f\} \cdot \nu)\nu$ on Σ, then

$$\int_V f : Dh \,\mathrm{dV} = -\int_V h \cdot \operatorname{div} f \,\mathrm{dV} + \int_{\partial\overline{V}} h \cdot f \cdot \nu \,\mathrm{dS} - \int_\Sigma \{h\} \cdot [f]^+_- \cdot \nu \,\mathrm{dS}. \tag{4.31}$$

Proof. Equation (4.28) is obtained by using the divergence theorem (1.16) for each region V_k and summing up, using the fact that the two normal vectors on interior boundaries are antiparallel with $\nu^{(\pm)} = \mp\nu$, see (4.12). Similarly, Green's formula for integration by parts (1.17) yields (with summation convention for $i = 1, \dots, m$ and $j = 1, \dots, n$)

$$\int_V f_{ij}(\partial_j h_i) \,\mathrm{dV} = -\int_V h_i(\partial_j f_{ij}) \,\mathrm{dV} + \int_{\partial\overline{V}} h_i\, f_{ij}\, \nu_j \,\mathrm{dS} - \int_\Sigma [h_i\, f_{ij}]^+_- \nu_j \,\mathrm{dS}.$$

Since $h \in H^1(\overline{V})^m$, we have $[h_i\, f_{ij}]^+_- = h_i\, [f_{ij}]^+_-$, which proves (4.29). However, if $h \in H^1_\Sigma(V)^m$, we cannot directly pull out h from $[h \cdot f]^+_- \cdot \nu$ in the surface integral even if $m = n$, i.e. in general $[h \cdot f]^+_- \neq h \cdot [f]^+_-$. A special case where this is possible is when f is isotropic near Σ, $f_{ij} = f^0 \delta_{ij}$ (4.26), which proves (i). Statement (ii) is based on the validity of the normality condition (4.27): The Leibniz rule for the jump (4.10) implies

$$[h \cdot f]^+_- \cdot \nu = h^+ \cdot [f]^+_- \cdot \nu + [h]^+_- \cdot f^- \cdot \nu.$$

We show that the last term vanishes, implying (ii): Indeed, for $h \in H^1_\Sigma(V)^n$ we have $[h]^+_- \cdot \nu = 0$ and thus $[h]^+_-$ is parallel to Σ. But, by normality of f we have $f^- \cdot \nu = (f^- \cdot \nu)^\perp = (\nu \cdot f^- \cdot \nu)\nu$, which is normal to Σ. Hence the product $[h]^+_- \cdot f^- \cdot \nu$ is zero. The claim involving the mean value $\{f\}$ follows from the Leibniz rule (4.11), $[h \cdot f]^+_- \cdot \nu = \{h\} \cdot [f]^+_- \cdot \nu + [h]^+_- \cdot \{f\} \cdot \nu$, where the second term vanishes as before due to normality, $\{f\} \cdot \nu = (\nu \cdot \{f\} \cdot \nu)\nu$. □

We conclude with the observation that also the surface divergence theorem (Lemma 4.5) extends to the case where Σ is the interior boundary of a Lip-composite domain. Indeed, as in (4.24), we just have to interpret both sides of

$$\int_\Sigma \widetilde{\operatorname{div}} f \, \mathrm{dS} = \int_{\partial\Sigma} f \cdot \mathrm{d}\lambda \tag{4.32}$$

via summing up the individual integrals over $\Sigma_{kk'}$ and $\partial\Sigma_{kk'}$. If there are no junctions where an odd number of interior regions meet, the contributions of $\partial\Sigma_{kk'} \setminus \partial\overline{V}$, that is, parts lying in the interior of $\overline{V}$ will cancel and in view of (4.7) we are left with the contribution from $\partial\Sigma$ alone. Thus, under these circumstances, the surface divergence theorem also holds for the interior boundary Σ of a Lip-composite domain.

4.2 Geometry and kinematics

4.2.1 Definition of the composite fluid-solid earth model

We consider a uniformly rotating, elastic and self-gravitating earth model, subdivided into solid and fluid regions. The Earth is described as a bounded continuous body that moves and deforms in space as time passes (see Chapter 1). Specifically, the Earth is modeled as a family $\{B_t\}_{t\in I}$ of bodies, which in accordance with the class (1.19) are Lip-domains $B_t \subseteq \mathbb{R}^3$ and the time interval is $I = [t_0, t_1]$. The open set B_t is the volume occupied by the material of the Earth at time $t \in I$. As a **reference configuration** we identify the Earth with (the closure $\overline{B}$ of) its volume at time t_0,

$$B := B_{t_0} \subseteq \mathbb{R}^3.$$

Thus (the interior of) the Earth is open and bounded with Lip-boundary ∂B. The different configurations B_t are related by the Earth's motion (1.20):

$$\varphi\colon \overline{B} \times I \to \mathbb{R}^3, \qquad \varphi_t(B) = B_t.$$

Due to elasticity of the material of the Earth, there exists a natural equilibrium state which we assume to be adopted in the reference configuration at initial time t_0. That is, B is the equilibrium shape of the Earth, $\varphi(X, t_0) = X$ for every $X \in \overline{B}$, or $\varphi_{t_0} = \mathrm{Id}_{\overline{B}}$ (1.21). Finally, we recall that the point $x = \varphi(X, t) = \varphi_t(X) \in \mathbb{R}^3$ gives the position of the material particle $X \in \overline{B}$ of the Earth at time $t \in I$.

The earth model is subdivided into solid and fluid regions with Lip-continuous interior boundaries. This model is called a composite fluid-solid earth model [DT98, p. 56].

Definition 4.12 (Composite fluid-solid earth model). A composite fluid-solid earth model is a body $B \subseteq \mathbb{R}^3$ according to (1.19), that is, a Lip-domain such that there exists a Lip-composite domain $(\bigcup_{k\in K^{\mathrm{F}}} V_k^{\mathrm{F}}) \cup (\bigcup_{k\in K^{\mathrm{S}}} V_k^{\mathrm{S}})$ as in Definition 4.2 consisting of (disjoint) fluid and solid interior regions V_k^{F} and V_k^{S} respectively, and with interior boundary Σ, such that we have the disjoint union

$$B = \Big(\bigcup_{k\in K^{\mathrm{F}}} V_k^{\mathrm{F}}\Big) \cup \Big(\bigcup_{k\in K^{\mathrm{S}}} V_k^{\mathrm{S}}\Big) \cup \Sigma \tag{4.33}$$

in accordance with the decomposition (4.3). The **exterior boundary** of the earth model is given by $\partial B = \partial\overline{B}$. The interior boundary Σ splits into **solid-solid**, **fluid-solid**, and **fluid-fluid interior boundaries** denoted by Σ^{SS}, Σ^{FS}, and Σ^{FF} respectively, that is, we have the disjoint union

$$\Sigma = \Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}} \cup \Sigma^{\mathrm{FF}}. \tag{4.34}$$

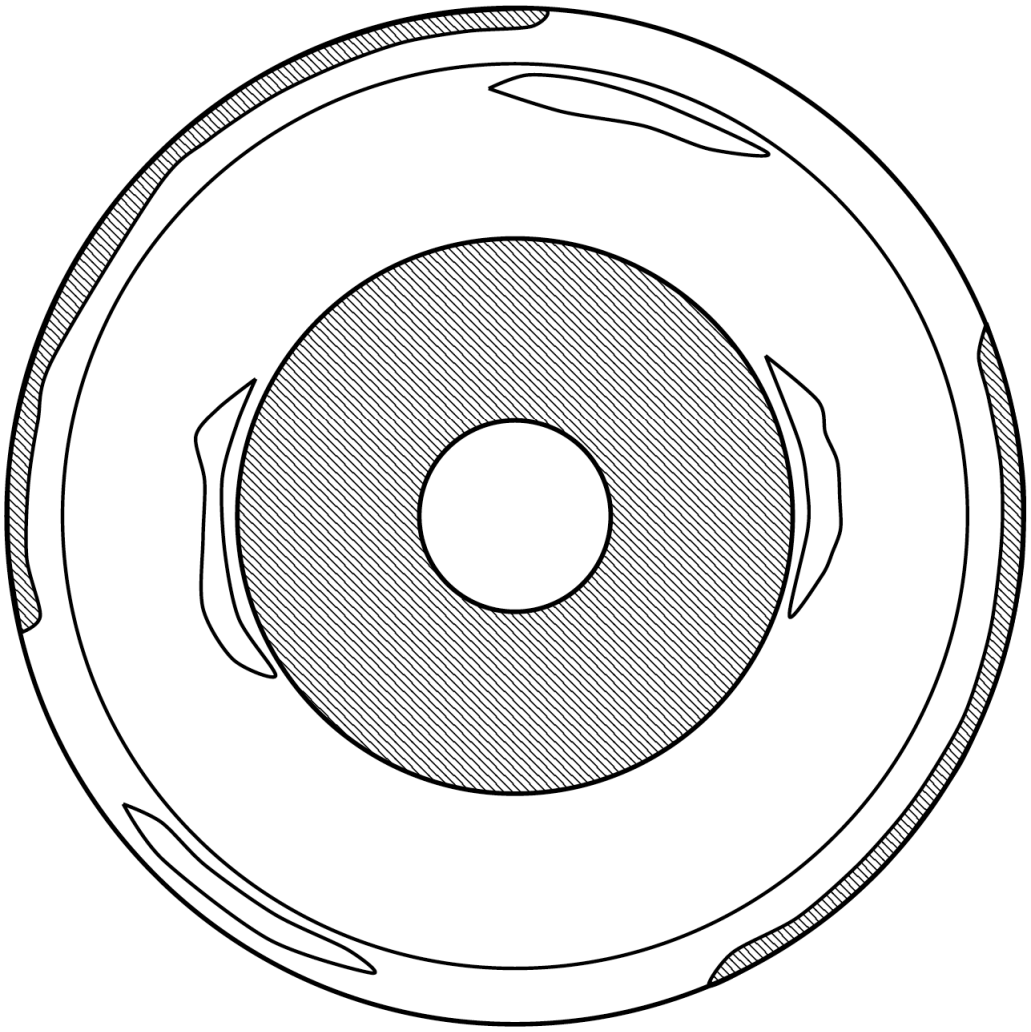

Figure 4.2: An illustration of a composite fluid-solid earth model. Hatched regions correspond to fluid parts which can be interpreted as the outer core and the oceans (depth not to scale).

The interior regions V_k^{F} and V_k^{S} represent parts of the Earth that consist of material whose physical parameters (density, elasticity coefficients) have different values or symmetry properties. On the largest scale, as indicated in the figure above, our planet is divided into almost spherical layers: The crust, the mantle, the outer core and inner core. In the seismic frequency range (that is, frequencies greater than 3 mHz, [LW95, p. 13]), where we can neglect long-time viscosity, the crust, mantle, and inner core consist of solid material, whereas the outer core and the oceans are fluid. The interior regions are further subdivided due to regional to local phase transitions or fluid flow patterns. These smaller scales however are no longer spherically symmetric, which motivates to disregard the assumption of purely radial stratification but consider an earth model with a general heterogeneous structure.

The welded solid-solid interior boundaries Σ^{SS} comprise discontinuities such as the Mohorovičić discontinuity (bottom of the crust), the upper mantle transition zones, and the highly irregular D″ region in the deep mantle on top of the core, which may be explained by buried slabs and partial melt [KB08]. The fluid-solid boundaries Σ^{FS} model the inner core boundary, the core-mantle boundary, and the ocean bottom. Fluids on a smaller scale correspond to molten rock, e.g. in volcanic areas or possibly also at the D″ region. Fluid-fluid interior boundaries Σ^{FF} separate different fluid layers in the inner core or in the oceans, but we will not consider them in the following. Our earth model does not contain fault surfaces, which would correspond

to slipping solid-solid interior boundaries. Interior boundaries thus only divide in welded interior boundaries Σ^{SS} and, neglecting viscosity, perfectly slipping interior boundaries Σ^{FS}. The boundary of the interior boundaries in particular contains the coast lines of oceans $\partial\Sigma^{\mathrm{FS}}$.

Remark 4.13 (No welded Σ^{SS} for L^∞ material parameters). The concept of welded solid-solid boundaries Σ^{SS} is superfluous if the material parameters are modeled as L^∞ functions. Essentially, welded boundaries can only be defined if one assumes the material parameters to be at least piecewise continuous. Indeed, a welded boundary is any surface across which the jump $[.]_-^+$ (4.8) of one of the material parameters is nonzero. Since the evaluation of $[.]_-^+$ involves calculating limits from both sides of the surface, defining a boundary surface amounts to requiring piecewise continuity of the material parameters. However, one may weaken this condition by considering Sobolev spaces with mixed regularity, which are continuous only in one direction [Hör85, Appendix B, Def. B.1.10, p. 477]. ⌟

4.2.2 Regularity conditions for the motion

A reasonable minimal regularity assumption for the Earth's motion $\varphi\colon \overline{B}\times I\to\mathbb{R}^3$ is

$$\varphi\in\mathcal{C}^0(I,L^\infty(B)^3). \tag{4.35}$$

First, continuity with respect to time is required for the initial condition (1.21) to be defined. Second, requiring $\varphi_t\in L^\infty(B)^3$ for all $t\in I$ is natural, since the moving Earth does not disperse to an unbounded set in space. However, to exclude catastrophic phenomena like tearing or interpenetration of the material, we need to introduce additional regularity conditions, which we now define for the motion of a general open and bounded set $V\subseteq\mathbb{R}^n$.

Definition 4.14 (Classes of regular motions Reg, $\mathrm{Reg}_{\mathcal{C}^k}$, $\mathrm{Reg}_{\mathrm{Lip}}$). Let $V\subseteq\mathbb{R}^n$ be open and bounded and consider the motion $\varphi\colon\overline{V}\times I\to\mathbb{R}^n$ with $\varphi\in\mathcal{C}^0(I,L^\infty(V)^n)$ and $\varphi_{t_0}=\mathrm{Id}_{\overline{V}}$.

(i) φ is called **regular**, $\varphi\in\mathrm{Reg}(\overline{V}\times I)$, if for all $t\in I$, $\varphi_t(V)$ is open, $\varphi_t(\overline{V})$ is closed, and $\varphi_t\colon\overline{V}\to\mathbb{R}^n$ is injective.

(ii) φ is called **$\mathcal{C}^k$-regular**, $\varphi\in\mathrm{Reg}_{\mathcal{C}^k}(\overline{V}\times I)$, for $k\in\mathbb{N}_0$, if φ is regular, $\varphi\in\mathcal{C}^k(\overline{V}\times I)^n$, and for all $t\in I$ the inverse φ_t^{-1} is $\mathcal{C}^k$ as a map $\varphi_t(\overline{V})\to\overline{V}$ (that is, φ_t^{-1} is the restriction of a $\mathcal{C}^k$ map defined on an open neighborhood of $\varphi_t(\overline{V})$).

(iii) φ is called **Lip-regular**, $\varphi\in\mathrm{Reg}_{\mathrm{Lip}}(\overline{V}\times I)$, if φ is regular, $\varphi\in\mathrm{Lip}(\overline{V}\times I)^n$, and for all $t\in I$ the inverse φ_t^{-1} is Lipschitz-continuous $\varphi_t(\overline{V})\to\overline{V}$.

By definition, we have the chain of inclusions

$$\bigcup_{1\le k\le\infty}\mathrm{Reg}_{\mathcal{C}^k}\subseteq\mathrm{Reg}_{\mathrm{Lip}}\subseteq\mathrm{Reg}_{\mathcal{C}^0}\subseteq\mathrm{Reg}.$$

Regular and $\mathcal{C}^k$-regular motions are also introduced in [MH83, Definition 1.4, p. 27].

Let us discuss some properties of regular motions $\varphi\in\mathrm{Reg}(\overline{V}\times I)$ and their physical meaning. The injectivity of $\varphi_t\colon\overline{V}\to\mathbb{R}^n$ for $t\in I$ prohibits interpenetration as well as self-contact of material, because two initially different points cannot be mapped to one and the same point by the motion. A $\mathcal{C}^k$-regular motion $\varphi\in\mathrm{Reg}_{\mathcal{C}^k}(\overline{V}\times I)$ enjoys continuity of derivatives of the motion up to order k and guarantees that $\varphi_t\colon V\to\varphi_t(V)$ is a $\mathcal{C}^k$-diffeomorphism. The conditions defining a $\mathcal{C}^0$-regular motion $\varphi\in\mathrm{Reg}_{\mathcal{C}^0}(\overline{V}\times I)$ prevent the material of the body from tearing, since under continuous mappings, connected portions of material stay connected.

As is shown in [Cia88, p. 16], the conditions

$$V\subseteq\mathbb{R}^n\ \text{ open},\quad \overline{V}^{\,\circ}=V,\quad\text{and}\quad\varphi_t\in\mathcal{C}^0(\overline{V})^n\ \text{ injective}$$

imply

$$\varphi_t(\overline{V}) = \overline{\varphi_t(V)}, \quad \varphi_t(V) = (\varphi_t(V))^\circ, \qquad \text{and} \qquad \varphi_t(\partial V) = \partial(\varphi_t(V)) = \partial(\varphi_t(\overline{V})). \tag{4.36}$$

Since $\overline{V}^\circ = V$ is true for Lip-domains V, this shows that $\mathcal{C}^0$-regular motions preserve the interior, the exterior, and the boundary of a Lip-domain.

Lemma 4.15 (**Continuous inverse with respect to time**). *Let $\varphi \in \mathrm{Reg}_{\mathcal{C}^0}(\overline{V} \times I)$. Then, for all $x \in \bigcup_{t\in I} \varphi_t(\overline{V})$, the map $t \mapsto \varphi_t^{-1}(x)$ is continuous $I \to \overline{V}$.*

Proof. By definition, a $\mathcal{C}^0$-regular motion $\varphi \in \mathrm{Reg}_{\mathcal{C}^0}(\overline{V} \times I)$ is continuous up to the boundary ∂V. Consequently, $\varphi_t \colon \overline{V} \to \mathbb{R}^n$ is continuous and injective for $t \in I$ (closed). This implies that $\widetilde{\varphi} \colon \overline{V \times I} \to \widetilde{\varphi}(\overline{V \times I}) = \bigcup_{t\in I}(\varphi_t(\overline{V}) \times \{t\}) \subseteq \mathbb{R}^n \times I$, defined by $\widetilde{\varphi}(x,t) := (\varphi_t(x), t)$, is a continuous and bijective map between compact sets. Therefore, its inverse given by $(x,t) \mapsto \widetilde{\varphi}^{-1}(x,t) := (\varphi_t^{-1}(x), t)$ is continuous, which shows the continuity of $t \mapsto \varphi_t^{-1}(x)$. □

Injectivity of φ is a global requirement, that is, it depends on the motion as a map of the entire body. A related pointwise condition is positive orientation of φ (i.e. φ is orientation preserving), which is related to positivity of the determinant of the deformation gradient $J = \det \nabla\varphi$ (1.25):

Lemma 4.16 (**Positive orientation and local injectivity**). *Let $\varphi \in \mathrm{Reg}_{\mathcal{C}^1}(\overline{V} \times I)$. Then $J > 0$, i.e. φ is positively oriented. Conversely, if $\varphi \colon \overline{V} \times I \to \mathbb{R}^n$ is $\mathcal{C}^1$ and $J > 0$, then φ_t is locally injective on V.*

Proof. Injectivity implies $J(X,t) \neq 0$ for all (X,t) and $\varphi_{t_0} = \mathrm{Id}_{\overline{V}}$ yields $J^0(X) = J(X,t_0) = 1$. Thus, since J is continuous with respect to time, we obtain $J(X,t) = 1 > 0$ for all (X,t). Conversely, by the inverse function theorem, if φ is $\mathcal{C}^1$ and $J > 0$, then φ is locally invertible and thus locally injective. □

However, Lemma 4.16 is not true under mere Lip-regularity, so, in this case, positive orientation has to be imposed as an extra condition:

Definition 4.17 (**Positively oriented** Lip**-regular motions** $\mathrm{Reg}^+_{\mathrm{Lip}}$). The space $\mathrm{Reg}^+_{\mathrm{Lip}}(\overline{V}\times I)$ consists of all motions $\varphi \in \mathrm{Reg}_{\mathrm{Lip}}(\overline{V} \times I)$ that are positively oriented, i.e. such that $J > 0$ holds on I and a.e. in V.

Remark 4.18 (**Global interior injectivity in static problems**). In static problems, a.e. interior injectivity of a deformation $\varphi \colon \overline{V} \to \mathbb{R}^n$ belonging to $W^{1,p}$ and positively oriented ($J = \det \nabla\varphi > 0$) is guaranteed by requiring $\int_V \det \nabla\varphi \,\mathrm{dV} \leq \int_{\varphi(V)} \mathrm{dV}$ [CN87]. This so-called Ciarlet-Nečas condition rules out interpenetration of matter, but allows frictionless self-contact, i.e. parts of the body's boundary may touch. Indeed, the right-hand side represents the volume of the set occupied by the body obtained after the deformation. The left-hand side is the volume of the body expressed in terms of the deformation and thus counts any overlapping regions twice. Consequently, the occurrence of interpenetration violates the inequality. ⌟

4.2.3 Kinematical interface conditions

We have defined regular motions for a general domain $\overline{V} \subseteq \mathbb{R}^n$. For an earth model without interior boundaries we may simply take $V = B$ and consider the motion $\varphi \colon \overline{B}\times I \to \mathbb{R}^3$. However, due to the possible slip along fluid-solid boundaries, the motion of the composite fluid-solid earth cannot be globally injective. Thus, for a composite fluid-solid earth model (Definition 4.12) the regularity properties will be assumed to hold on every interior region, that is, for each of the restrictions $\varphi|_{\overline{V_k^{\mathrm{S}}}\times I}$ and $\varphi|_{\overline{V_k^{\mathrm{F}}}\times I}$ separately, but not necessarily for φ globally.

In order to glue the different parts of the earth model together, additional continuity conditions on the motion have to be imposed on the interior boundaries $\Sigma = \Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}} \cup \Sigma^{\mathrm{FF}}$. These conditions are referred to as **kinematical interface conditions** (**kinematical interior boundary conditions**). In view of the properties of welded solid-solid interfaces of a composite fluid-solid earth model (see also [DT98, pp. 48, 67, and 71]), we must have continuity of the motion across Σ^{SS}, that is

$$[\varphi_t]^+_- = 0 \qquad \text{on} \quad \Sigma^{\mathrm{SS}}. \tag{4.37}$$

On slipping fluid-solid and fluid-fluid interfaces we only have continuity of the normal component of the spatial velocity $v^s_t = v_t \circ \varphi_t^{-1}$ across the current Σ^{FS}-boundaries $\varphi_t(\Sigma^{\mathrm{FS}})$, that is

$$[v^s_t]^+_- \cdot \nu^s_t = 0 \qquad \text{on} \quad \varphi_t(\Sigma^{\mathrm{FS}}), \tag{4.38}$$

and similarly for Σ^{FF}. Here ν^s_t is the spatial normal vector field (1.30) on $\varphi_t(\Sigma^{\mathrm{FS}})$ and the bracket $[.]^+_-$ denotes the jump of the enclosed quantity as defined in (4.8). If the motion is $\mathcal{C}^1$ regular, then the kinematical interface conditions hold for all $t \in I$. If φ is Lip-regular, then v^s as well as ν^s are just L^∞ with respect to time and (4.38) only holds for almost every $t \in I$. In the following we neglect the fluid-fluid interior boundaries Σ^{FF}.

By continuity of the motion across Σ^{SS} (4.37), a function that is Lip on two adjacent solid interior regions $\overline{V^{\mathrm{S}}_k}$ and $\overline{V^{\mathrm{S}}_{k'}}$ is Lip on the closure of their union $\overline{V^{\mathrm{S}}_k \cup V^{\mathrm{S}}_{k'}}$. To define the regularity of the motion it thus suffices to consider the composite earth model as a union of the open solid and fluid interior regions B^{S} and B^{F} obtained by merging adjacent regions:

$$B^{\mathrm{S}} := \Big(\bigcup_{k \in K^{\mathrm{S}}} \overline{V^{\mathrm{S}}_k} \Big)^\circ \qquad \text{and} \qquad B^{\mathrm{F}} := \Big(\bigcup_{k \in K^{\mathrm{F}}} \overline{V^{\mathrm{F}}_k} \Big)^\circ. \tag{4.39}$$

We recall that B, V^{S}_k ($k \in K^{\mathrm{S}}$), and V^{F}_k ($k \in K^{\mathrm{F}}$) are Lip-domains. The sets B^{S} and B^{F} are finite unions of Lip-domains, as they are obtained as finite unions of Lip-domains where possibly resulting interior boundaries are removed by the closure. Note that cusps are ruled out by Definition 4.2 of Lip-composite domains. By construction we have

$$\overline{B} = \overline{B^{\mathrm{S}}} \cup \overline{B^{\mathrm{F}}} \qquad \text{and} \qquad \Sigma^{\mathrm{FS}} \cup \partial\Sigma^{\mathrm{FS}} = \overline{B^{\mathrm{S}}} \cap \overline{B^{\mathrm{F}}} = \partial B^{\mathrm{S}} \cap \partial B^{\mathrm{F}}.$$

For convenience we further introduce the notation

$$\begin{aligned} B^{\mathrm{C}} &:= \mathbb{R}^3 \setminus \overline{B}, \\ B^{\mathrm{FS}} &:= B^{\mathrm{F}} \cup B^{\mathrm{S}} = B \setminus \Sigma^{\mathrm{FS}}, \\ B^{\mathrm{FSC}} &:= B^{\mathrm{F}} \cup B^{\mathrm{S}} \cup B^{\mathrm{C}} = \mathbb{R}^3 \setminus (\Sigma^{\mathrm{FS}} \cup \partial B). \end{aligned} \tag{4.40}$$

Thereby the original decomposition of Definition 4.12,

$$B = \big(\bigcup_{k \in K^{\mathrm{F}}} V^{\mathrm{F}}_k \big) \cup \big(\bigcup_{k \in K^{\mathrm{S}}} V^{\mathrm{S}}_k \big) \cup \Sigma,$$

simplifies to the disjoint union

$$B = B^{\mathrm{F}} \cup B^{\mathrm{S}} \cup \Sigma^{\mathrm{FS}} = B^{\mathrm{FS}} \cup \Sigma^{\mathrm{FS}}. \tag{4.41}$$

The figure on p. 69 illustrates this decomposition.

4.2.4 Admissible motions

Admissible motions for the composite fluid-solid earth model are Lip-regular positively oriented motions on fluid and solid parts separately and have to satisfy additional compatibility conditions, taking into account the possible slip along Σ^{FS}:

Definition 4.19 (Admissible motions). Let $B = B^{\mathrm{F}} \cup B^{\mathrm{S}} \cup \Sigma^{\mathrm{FS}}$ be a composite fluid-solid earth model (Definition 4.12). We define the associated class of admissible motions by

$$\mathcal{A}(\overline{B} \times I) := \Big\{ \varphi \colon \overline{B} \times I \to \mathbb{R}^3 : \ \varphi \in \mathcal{C}^0(I, L^\infty(B)^3),\ \varphi_{t_0} = \mathrm{Id}_{\overline{B}},\ \text{and } \varphi \text{ satisfies (i) to (iv)} \Big\}$$

with

(i) **Global conditions:** $\varphi_t(\overline{B})$ is homeomorphic to $\overline{B}$ and $\bigcup_{t\in I} \varphi_t(B)$ is bounded, $\forall\, t \in I$,

(ii) **Piecewise positively oriented** Lip**-regular motion:**
$\varphi|_{B^{\mathrm{S}}\times I}$ and $\varphi|_{B^{\mathrm{F}}\times I}$ can be extended to $\mathrm{Reg}^+_{\mathrm{Lip}}(\overline{B^{\mathrm{S}}} \times I)$ and $\mathrm{Reg}^+_{\mathrm{Lip}}(\overline{B^{\mathrm{F}}} \times I)$ respectively,

(iii) **No interpenetration:** $\varphi_t(B^{\mathrm{S}}) \cap \varphi_t(B^{\mathrm{F}}) = \emptyset$, $\forall\, t \in I$, and

(iv) **Tangential slip:** (4.38) holds, i.e. $[v^s_t]^+_- \cdot \nu^s_t = 0$ on $\varphi_t(\Sigma^{\mathrm{FS}})$ holds for a.a. $t \in I$.

Actually, instead of the symbol $\overline{B}$, the fluid and solid parts B^{F} and B^{S} should enter the definition of admissible motions, that is, in view of (4.40), $\mathcal{A}(\overline{B} \times I)$ should be understood as $\mathcal{A}(B^{\mathrm{FS}} \times I)$.

By this definition, the class of admissible motions consists of functions $\varphi \in \mathcal{C}^0(I, L^\infty(B))^3$ that preserve the connectedness properties as well as boundedness of the earth model by (i), possess a positively oriented Lip-regular extension to the closure of each interior region by (ii), prohibit interpenetration of different interior regions by (iii), satisfy the slipping interior boundary conditions (4.38) on fluid-solid interior boundaries by (iv), and satisfy the welded interior boundary conditions (4.37) on solid-solid interior boundaries by construction using the domain B^{S} introduced in (4.39) and in view of (ii).

An admissible motion is continuous across Σ^{SS} but possibly discontinuous across Σ^{FS}. We accounted for this discontinuity in (ii) by demanding only the existence of a positively oriented Lip-regular extension for each interior region instead of requiring $\varphi \in \mathrm{Reg}^+_{\mathrm{Lip}}$ on its closure.

Due to slip on interior surfaces, the motion is not globally injective on the closure $\overline{B}$. Therefore it is not a $\mathcal{C}^0$-regular motion of B and the confinement condition (i) needs to be imposed to preserve connectedness and boundedness of $\varphi_t(\overline{B})$. However, φ is injective in the interior of each interior region B^{S} and B^{F} separately by property (ii). Since these sets are finite unions of Lip-domains, an admissible motion satisfies $\varphi_t(\overline{B^{\mathrm{S}}}) = \overline{\varphi_t(B^{\mathrm{S}})}$ and $\varphi_t(\overline{B^{\mathrm{F}}}) = \overline{\varphi_t(B^{\mathrm{F}})}$ for all $t \in I$, see (4.36). Combined with property (i) we thus have

$$\varphi_t(\overline{B}) = \overline{\varphi_t(B)}. \tag{4.42}$$

For a discussion of further regularity properties, let $V = B^{\mathrm{S}}$ or B^{F} for the moment. Then, by property (ii), $\varphi \in \mathcal{A}(\overline{B} \times I)$ implies that the restriction $\varphi|_{V\times I}$ can be extended to yield a motion in $\mathrm{Reg}^+_{\mathrm{Lip}}(\overline{V} \times I) \subseteq \mathrm{Lip}(\overline{V} \times I)^3$. Motions in $\mathcal{A}(\overline{B} \times I)$ are thus Lipschitz continuous functions with respect to time and take values in the space of all piecewise Lipschitz continuous functions of B. Moreover, since $\mathrm{Reg}^+_{\mathrm{Lip}}(\overline{V} \times I) \subseteq H^1(V \times I^\circ)^3$, it follows that

$$\mathcal{A}(\overline{B} \times I) \subseteq H^1(B^{\mathrm{FS}} \times I^\circ)^3. \tag{4.43}$$

The regularity of $\varphi \in \mathcal{A}(\overline{B}\times I)$ implies that the first-order derivatives of the Lip-regular extension of $\varphi|_{V\times I}$ to $\overline{V}$, denoted by $\varphi^{\overline{V}}$, satisfy $\nabla\varphi^{\overline{V}} \in \mathrm{Lip}(I, L^\infty(V))^{3\times 3}$ and $\partial_t\varphi^{\overline{V}} \in L^\infty(I^\circ, \mathrm{Lip}(\overline{V}))^3$. Consequently, the components of v, $\nabla\varphi$, e, and J, defined by (1.22), (1.26), and (1.25) are essentially bounded functions on $B^{\mathrm{FS}} \times I^\circ$. Thus, by $B = B^{\mathrm{FS}} \cup \Sigma^{\mathrm{FS}}$ (4.41) and since Σ^{FS} has zero volume, they can be extended to $L^\infty(B \times I^\circ)$. However, we emphasize that the spatial derivatives occurring in the definitions of $\nabla\varphi$, e, and J need not be identical to the distributional derivatives (see Remark 4.7).

The fluid-solid boundary condition $[v_t^s]_-^+ \cdot \nu_t^s = 0$ (4.38) (or condition (iv) in Definition 4.19) indeed makes sense within the regularity setting specified in (ii), showing consistency of definition of $\mathcal{A}(\overline{B} \times I)$: The Lip-regular extension $\varphi_t^{\overline{V}}$ yields $v_t^{\overline{V}} = \partial_t \varphi_t^{\overline{V}} \in \mathrm{Lip}(\overline{V})^3$ and thus $(v_t^{\overline{V}})^s = v_t^{\overline{V}} \circ (\varphi_t^{\overline{V}})^{-1} \in \mathrm{Lip}(\varphi_t^{\overline{V}}(\overline{V}))^3$ for almost every $t \in I$ with uniform Lipschitz constant. Therefore $(v_t^{\overline{V}})^s$ can be restricted to the moving boundary $\varphi_t(\partial V) = \partial(\varphi_t(V))$ for almost every $t \in I$. By (4.23) this implies that v_t^s can be restricted to moving interior boundaries $\varphi_t(\Sigma^{\mathrm{FS}})$. Hence, its jump $[v_t^s]_-^+$ is a continuous function on $\varphi_t(\Sigma^{\mathrm{FS}})$ which can be multiplied with the bounded spatial unit normal ν_t^s.

Next we consider the validity of material and spatial representation in the composite earth model. The following lemma shows equivalence between boundedness of a spatial quantity q^s and boundedness of the corresponding material quantity q for an admissible motion of a composite earth model.

Lemma 4.20 (**Regularity of material and spatial representations**). *Let the maps*

$$q^s \colon \mathbb{R}^3 \times I \to \mathbb{R} \qquad \text{and} \qquad q \colon \overline{B} \times I \to \mathbb{R}$$

be related by (1.27)*, that is,* $q_t = q_t^s \circ \varphi_t$ *for* $t \in I$ *and* $\varphi \in \mathcal{A}(\overline{B} \times I)$*. Then*

$$q^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)) \quad \Longleftrightarrow \quad q \in \mathcal{C}^0(I, L^\infty(B)).$$

Proof. We first establish the equivalence of the L^∞-condition with respect the spatial variables:

$$q_t^s \in L^\infty(\varphi_t(B)) \quad \Longleftrightarrow \quad q_t \in L^\infty(B).$$

Since φ preserves interior boundaries which are of measure zero it suffices to show

$$q_t^s \in L^\infty(\varphi_t(V)) \quad \Longleftrightarrow \quad q_t \in L^\infty(V)$$

for all interior regions $V = B^{\mathrm{S}}$ or B^{F}. We denote the restrictions of q_t^s and q_t to $\varphi_t(V)$ and V again by q_t^s and q_t. We start with the implication from left to right. First note that if $q_t^s \in L^\infty(\varphi_t(V))$, boundedness of $q_t = q_t^s \circ \varphi_t$ is clear. Measurability is guaranteed by Lip regularity of φ_t^{-1} on the corresponding interior regions $\varphi_t(V)$. More precisely, for a Borel set M in $\mathbb{R}$ with $M \subseteq q_t(V)$, consider the pre-image $q_t^{-1}(M) = \varphi_t^{-1}((q_t^s)^{-1}(M))$. Since $(q_t^s)^{-1}(M)$ is Lebesgue measurable by assumption and Lipschitz-maps preserve Lebesgue measurability (e.g. [Bog07, Lemma 3.6.3 on p. 192]), we have the Lebesgue measurability of $q_t^{-1}(M)$. Hence, $q_t \in L^\infty(V)$. The other implication follows similarly by changing the roles of φ_t and φ_t^{-1}. Finally, continuity of $q_t := q_t^s \circ \varphi_t$ resp. $q_t^s := q_t \circ \varphi_t^{-1}$ with respect to the time variable follows from the continuity of the map $t \mapsto \varphi_t$ resp. $t \mapsto \varphi_t^{-1}$, which was established in Lemma 4.15. □

For $\varphi \in \mathcal{A}(\overline{B} \times I)$, the spatial counterpart of the material velocity field $v_t \in L^\infty(B)^3$ is given by $v_t^s := v_t \circ \varphi_t^{-1} \in L^\infty(\mathbb{R}^3)$ (extended to $\mathbb{R}^3$ by zero) for almost all $t \in I$. Similarly, since J and the components of e are in $L^\infty(B \times I^\circ)$, it follows that J_t^s and the components of e_t^s are in $L^\infty(\mathbb{R}^3)$ (extending again by zero).

The transformation formula between the material and the spatial representation of volume integrals also holds for admissible motions of the composite fluid-solid earth:

Lemma 4.21 (**Spatial and material volume integrals**). *Let* $\varphi \in \mathcal{A}(\overline{B} \times I)$ *and* $f_t \in L^\infty(B)$ *for* $t \in I$*. Then* (1.33) *holds for all open* $A \subseteq B$*:*

$$\int_{\varphi_t(A)} f_t^s \,\mathrm{dV} = \int_A f_t J_t \,\mathrm{dV}.$$

Proof. By Lemma 4.20 we have $f_t^s = f_t \circ \varphi_t^{-1} \in L^\infty(\varphi_t(A))$. Clearly, for $A \subseteq B^{\mathrm{S}}$ or B^{F} we have positivity of J_t and thus (1.33) holds. By the decomposition (4.41) together with the mapping properties of $\varphi \in \mathcal{A}(\overline{B} \times I)$ (properties (ii) and (iii) of Definition 4.19), the result is true for a general open subset $A \subseteq B$. □

Remark 4.22 (**Transformation of surface integrals**)**.** If S is a Lip-surface within the composite fluid-solid earth $B = B^{\mathrm{FS}} \cup \Sigma^{\mathrm{FS}}$, formula (1.34),

$$\int_{\varphi_t(S)} g_t^s \, \nu_t^s \mathrm{dS} = \int_S g_t J_t \, (\nabla\varphi)_t^{-T} \cdot \nu \, \mathrm{dS},$$

holds classically if S avoids slipping surfaces, i.e. $S \cap \Sigma^{\mathrm{FS}} = \emptyset$. The relation is established in [Cia88, Thm. 1.7-1, p. 39] for φ a $\mathcal{C}^1$-regular motion and $g_t \in \mathcal{C}^0(S)$ for $t \in I$. ⌟

4.3 Density and conservation of mass

The mass of a continuous body describes the body's resistance to acceleration when a force acts on it. The composite fluid-solid earth B is a continuous body, thus, as was motivated Section 1.1.2, its mass may be written as a volume integral over an L^1_{loc} density (1.9). Specifically, the spatial mass density of the Earth is a non-negative function

$$\rho^s \colon \mathbb{R}^3 \times I \to \mathbb{R}_0^+$$

which is compactly supported in the closure of the Earth in current configuration at time $t \in I$, that is,

$$\operatorname{supp}(\rho_t^s) \subseteq \overline{\varphi_t(\overline{B})} = \overline{\varphi_t(B)} = \varphi_t(\overline{B}). \tag{4.44}$$

Here we assumed that the motion is admissible, $\varphi \in \mathcal{A}(\overline{B} \times I)$, and invoked (4.42) to obtain the last two equalities. Requiring global boundedness $\rho_t^s \in L^\infty(B_t)$ instead of only $\rho_t^s \in L^1_{\mathrm{loc}}(B_t)$ implies that a body's mass can be estimated by its volume:

$$M(B_t) = \int_{B_t} \rho^s \, \mathrm{dV} \leq \|\rho^s\|_{L^\infty(B_t)} \int_{B_t} \mathrm{dV}.$$

Thus it is natural to assume

$$\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)). \tag{4.45}$$

The requirement of continuity with respect to time will turn out to be consistent with conservation of mass and the class of admissible motions. The corresponding material density

$$\rho \colon \overline{B} \times I \to \mathbb{R}^+$$

is given by $\rho_t := \rho_t^s \circ \varphi_t$ for all $t \in I$, see (1.27). If $\varphi \in \mathcal{A}(\overline{B} \times I)$, then by Lemma 4.20 the condition $\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$ yields $\rho \in \mathcal{C}^0(I, L^\infty(B))$. Extension by zero, i.e. setting $\rho_t(X) := 0$ if $X \notin B$, yields $\rho \colon \mathbb{R}^3 \times I \to \mathbb{R}_0^+$. Thereby we obtain

$$\rho \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$$

with fixed support $\operatorname{supp}(\rho_t) \subseteq \overline{B}$ for all $t \in I$.

The principle of conservation of mass (see Sections 1.1.2 and 1.2.3) allows us to determine the material density ρ directly from the reference density ρ^0 and the motion. The following lemma establishes this assertion under the low regularity conditions in the composite fluid-solid earth model B (see Definitions 4.12 and 4.19).

Lemma 4.23 (Conservation of mass). *Let $\varphi \in \mathcal{A}(\overline{B} \times I)$ and $\rho_t^s \in L^\infty(\varphi_t(B))$ for (almost all) $t \in I$. Then conservation of mass (1.38) is equivalent to the equation (1.40),*

$$\rho^0 = \rho_t J_t,$$

valid almost everywhere on B and for almost all $t \in I$.

Proof. We recall the principle of conservation of mass (1.38) for a continuous body $V \subseteq \mathbb{R}^n$: For all open sets $A \subseteq V$ and almost all times t', $t'' \in I$,

$$\int_{\varphi_{t'}(A)} \rho_{t'}^s \,\mathrm{dV} = \int_{\varphi_{t''}(A)} \rho_{t''}^s \,\mathrm{dV}.$$

We set $t' = t_0$, $t'' = t \in I$ and consider $\varphi|_{V \times I}$ for $V = B^{\mathrm{S}}$ or B^{F}. Then

$$\int_A \rho^0 \,\mathrm{dV} = \int_{\varphi_t(A)} \rho_t^s \,\mathrm{dV}$$

must hold for all $A \subseteq V$ open, where on the left-hand side we used $\varphi_{t_0}(A) = \mathrm{Id}_B(A) = A$ and $\rho_{t_0}^s = \rho_{t_0} = \rho^0$. Lip continuity of φ_t on A for all $t \in I$ allows us to change variables on the right-hand side (see [Fed69, Theorem 3.2.3, p. 243]), which, together with $J > 0$, yields

$$\int_A \rho^0 \,\mathrm{dV} = \int_A \rho_t^s(\varphi_t(X))\, J_t(X) \,\mathrm{dV}(X) = \int_A (\rho_t^s \circ \varphi_t)\, J_t \,\mathrm{dV}.$$

Since $\rho_t^s \circ \varphi_t = \rho_t$ we thus have

$$\int_A (\rho^0 - \rho_t J_t) \,\mathrm{dV} = 0$$

for every open subset $A \subseteq V$. This is equivalent to $\rho^0 = \rho_t J_t$ to holding almost everywhere on B^{FS} and hence almost everywhere on B, which completes the proof. □

As a consequence of Lemma 4.21 and Lemma 4.23 we obtain equations for the integral of a function times mass density in spatial and material representation.

Lemma 4.24 (Spatial and material volume integrals with density). *Let $\varphi \in \mathcal{A}(\overline{B} \times I)$, $b_t \in L^\infty(B)$, $\rho^0 \in L^\infty(B)$, and $t \in I$. Then conservation of mass implies that for all open $A \subseteq B$,*

$$\int_{\varphi_t(A)} b_t^s \rho_t^s \,\mathrm{dV} = \int_A b_t \rho^0 \,\mathrm{dV}. \tag{4.46}$$

Proof. By Lemma 4.20, the condition $b_t \in L^\infty(B)$ gives $b_t^s = b_t \circ \varphi_t \in L^\infty(\varphi_t(B))$. Thus the volume integral transformation formula (Lemma 4.21) and conservation of mass (1.40) (Lemma 4.23) yield

$$\int_{\varphi_t(A)} (b_t^s \rho_t^s)(x) \,\mathrm{dV}(x) = \int_A b_t(X) \rho_t(X)\, J_t(X) \,\mathrm{dV}(X) = \int_A (b_t \rho^0)(X) \,\mathrm{dV}(X).$$

which proves the claim. □

Conservation of mass also relates the regularity of the density to the regularity of the motion: If $\varphi \in \mathcal{A}(\overline{B} \times I)$ and $\rho_t^s \in L^\infty(\varphi_t(B))$, Lemma 4.23 shows that conservation of mass implies $\rho^0 = \rho_t J_t$. Thus, given the initial density ρ^0, this equation expresses the current material density ρ_t in terms of derivatives of the motion φ. In particular, since by Definition 4.19,

$$\varphi \in \mathcal{A}(\overline{B} \times I) \subseteq \mathcal{C}^0(I, L^\infty(B)^3),$$

the assumption $\rho \in \mathcal{C}^0(I, L^\infty(B))$ made in (4.45) (Lemma 4.20 is applied) is in accordance with conservation of mass. Moreover, improving the regularity of φ directly improves the regularity of

$$t \mapsto \rho_t(x) = \rho^0(x)/J_t(x)$$

for almost all x, if J_t is positive:

Lemma 4.25 (Improved time regularity of material density via conservation of mass). *If $\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$ and conservation of mass holds, then*

$$\varphi \in \mathcal{A}(\overline{B} \times I), \quad \rho^0 \in L^\infty(B) \quad \Longrightarrow \quad \rho \in \mathrm{Lip}(I, L^\infty(B))$$

and

$$\varphi \in \mathrm{Reg}_{\mathcal{C}^1}(\overline{B} \times I), \quad \rho^0 \in \mathcal{C}^0(B) \quad \Longrightarrow \quad \rho \in \mathcal{C}^1(I, \mathcal{C}^0(B)).$$

Proof. By (1.40), $\rho = \rho^0/J$. The second claim holds because $\varphi \in \mathrm{Reg}_{\mathcal{C}^1}(\overline{B} \times I) \subseteq \mathcal{C}^1(\overline{B} \times I)^3$ implies $J \in \mathcal{C}^1(I, \mathcal{C}^0(\overline{B}))$ and $J > 0$ (see Lemma 4.16). We prove the first claim: For $\varphi \in \mathcal{A}(\overline{B} \times I)$ and $V = B^{\mathrm{S}}$ or B^{F} we have $\varphi|_{V \times I} \in \mathrm{Lip}(V \times I)^3$ with J positive, bounded away from zero on V, and Lip with respect to time. Consequently, if $\rho^0 \in L^\infty(B)$, then conservation of mass implies $\rho|_{V \times I} \in \mathrm{Lip}(I, L^\infty(V))$ and thus $\rho \in \mathrm{Lip}(I, L^\infty(B))$. □

However, conservation of mass does not necessarily improve the time regularity of the spatial density representation $\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$: Indeed, existence of the time derivative

$$\partial_t(\rho^s(x,t)) = \partial_t(\rho(\varphi_t^{-1}(x), t)) = \partial_t \rho(\varphi_t^{-1}(x), t) + \nabla \rho(\varphi_t^{-1}(x), t) \cdot \partial_t \varphi_t^{-1}(x)$$

would imply boundedness of $\nabla \rho$, which cannot be expected if $\rho_t \in L^\infty(B)$ and $\varphi \in \mathcal{A}(\overline{B} \times I)$.

The local formulation (1.40) of conservation of mass does not require any time-differentiability of the density or the motion. This is a benefit compared to the more prominent formulations, which require $\mathcal{C}^1$ regularity.

Lemma 4.26 (Conservation of mass and continuity equation). *If $\varphi \in \mathrm{Reg}_{\mathcal{C}^1}(\overline{B} \times I)$ and $\rho \in \mathcal{C}^1(\overline{B} \times I)$, then the following are equivalent for all $t \in I$:*

(i) **Integral form:** $\frac{d}{dt} \int_{\varphi_t(A)} \rho_t^s \, \mathrm{dV} = 0$ *for all open $A \subseteq B$.*

(ii) **Local material form:** $\rho^0 = \rho_t J_t$ *on* B.

(iii) **Local spatial form (continuity equation):** $\partial_t \rho^s + \nabla \cdot (\rho^s v^s) = 0$ *on* $\varphi_t(B)$.

Proof. We argue as in [MH83, Theorem 5.7, p. 87]. The equivalence (i)⇔(ii) follows from adapting the proof of Lemma 4.23 to the higher regularity conditions, noting that (i) is equivalent to (1.38). The equivalence (ii)⇔(iii) then follows from differentiating $\rho_t J_t = (\rho_t^s \circ \varphi_t) J_t$ with respect to t, and invoking the material time derivative $\partial_t \rho_t \circ \varphi_t^{-1} = \mathrm{d}_t \rho_t^s = \partial_t \rho_t^s + v_t^s \cdot \nabla \rho_t^s$ (1.32) as well as the identity $\partial_t J_t = J_t \, ((\nabla \cdot v^s) \circ \varphi_t)$ (1.37):

$$\begin{aligned}
\partial_t(\rho_t J_t) &= (\partial_t \rho_t) J_t + \rho_t (\partial_t J_t) \\
&= ((\partial_t \rho_t^s + v_t^s \cdot \nabla \rho_t^s) \circ \varphi_t) J_t + \rho_t J_t ((\nabla \cdot v^s) \circ \varphi_t) \\
&= ((\partial_t \rho_t^s + v_t^s \cdot \nabla \rho_t^s + \rho_t^s \nabla \cdot v^s) \circ \varphi_t) J_t \\
&= ((\partial_t \rho_t^s + \nabla \cdot (\rho_t^s \, v_t^s)) \circ \varphi_t) J_t.
\end{aligned}$$

Since by assumption $J_t > 0$, this equation implies the equivalence of $\partial_t(\rho_t J_t) = 0$ (and hence $\rho_t J_t = \rho^0$, since $J^0 = 1$) and the continuity equation. □

4.4 Conservative volume forces

In classical continuum mechanics, **volume forces** (**body forces**) are modeled by vector fields

$$f^s \colon \mathbb{R}^3 \times I \to \mathbb{R}^3$$

acting on the body. The field f^s is a force density per unit volume, hence we have

$$f^s = \rho^s b^s$$

for the acceleration field $b^s \colon \mathbb{R}^3 \times I \to \mathbb{R}^3$.

The next lemma addresses the definition and regularity of the material representation of volume forces, assuming admissible motions of the composite fluid-solid earth model.

Lemma 4.27 (**Volume forces in material representation**). *Let $\varphi \in \mathcal{A}(\overline{B} \times I)$, $\rho^0 \in L^\infty(B)$. Then the material formulation of a volume force*

$$f^s = \rho^s b^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)^3)$$

is given by

$$f := \rho^0 b \in \mathcal{C}^0(I, L^\infty(B)^3)$$

with $b_t = b_t^s \circ \varphi_t$ for all $t \in I$.

Proof. Lemma 4.24, which results from conservation of mass, allows us to write the total force as the following integral over the reference domain ($t \in I$):

$$\int_{\varphi_t(B)} f_t^s \,\mathrm{dV} = \int_{\varphi_t(B)} \rho_t^s b_t^s \,\mathrm{dV} = \int_B \rho^0 b_t \,\mathrm{dV} = \int_B f_t \,\mathrm{dV}.$$

The integrand on the right-hand side represents the force density per unit volume in material formulation. Finally, Lemma 4.20 ensures that $f \in \mathcal{C}^0(I, L^\infty(B)^3)$. □

Note that f^s and f are not related via (1.27) (which would give $q_t = q_t^s \circ \varphi_t$ for $q = f$). Instead, as a consequence of conservation of mass $\rho^0 = J\rho$ (1.40), the relation merely is

$$f = J(f^s \circ \varphi), \tag{4.47}$$

since $f = \rho^0 b = J\rho\, b = J((\rho^s b^s) \circ \varphi) = J(f^s \circ \varphi)$. The explanation is, that in order to obtain the material formulation, we have to take the change in volume into account, which is represented by the Jacobian determinant $J = \det(\nabla\varphi)$ (1.25) (cf. [MH83, Section 2.1, p. 123]).

We restrict ourselves to the consideration of **conservative volume forces**, since these can be naturally incorporated in the calculus of variation [Cia88, p. 82]. Thus we assume that the body force field f^s can be expressed as density ρ^s times the negative gradient of a scalar potential $F^s \colon \mathbb{R}^3 \times I \to \mathbb{R}$,

$$f^s = -\rho^s \nabla F^s. \tag{4.48}$$

Concerning regularity, we assume

$$F^s \in \mathcal{C}^0(I, \mathrm{Lip}(\mathbb{R}^3)) \tag{4.49}$$

Together with (4.45), i.e. $\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)^3)$ with compact support, this implies that

$$f^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)^3)$$

and f^s is compactly supported as well. By Lemma 4.27,

$$f_t = -\rho^0 (\nabla F_t^s) \circ \varphi_t \tag{4.50}$$

is the corresponding material formulation of the conservative volume force.

4.5 Gravity

At any time instant $t \in I$, the **spatial gravitational potential** Φ^s is determined by the Earth's current density distribution $\rho_t^s \in L_c^\infty(\mathbb{R}^3)$ as the distributional solution of Poisson's equation (2.2)

$$\triangle \Phi^s = 4\pi G \rho^s$$

in $\mathbb{R}^3$ that vanishes at infinity: $\lim_{|x|\to\infty} \Phi_t^s(x) = 0$. The associated material gravitational potential $\Phi\colon \overline{B} \times I \to \mathbb{R}$ is found from (1.27): $\Phi_t := \Phi_t^s \circ \varphi_t$.

4.5.1 Distributional solution of the Poisson equation

We review general properties of distributional solutions of the Poisson equation in $\mathbb{R}^3$. With $\mathcal{E}'(\mathbb{R}^3)$ the set of all distributions in $\mathcal{D}'(\mathbb{R}^3)$ with compact support, we introduce the solution set

$$Y(\mathbb{R}^3) := \{y \in \mathcal{D}'(\mathbb{R}^3) : \triangle y \in \mathcal{E}'(\mathbb{R}^3),\ \lim_{|x|\to\infty} y(x) = 0\} \tag{4.51}$$

and the subset $Y^\infty \subseteq Y$ for bounded compactly supported sources,

$$Y^\infty(\mathbb{R}^3) := \{y \in \mathcal{D}'(\mathbb{R}^3) : \triangle y \in L_c^\infty(\mathbb{R}^3),\ \lim_{|x|\to\infty} y(x) = 0\}. \tag{4.52}$$

Let $E_3 \in \mathcal{D}'(\mathbb{R}^3)$ denote the unique radial **fundamental solution** (**Green's function**) for the Laplacian $\triangle$ in $\mathbb{R}^3$, that is

$$\triangle E_3 = \delta,$$

that vanishes at infinity. The representation $E_3(x) = -\frac{1}{4\pi|x|}$, which holds for $x \neq 0$, implies the local integrability of E_3 and of its partial derivatives $\partial_i E_3(x) = \frac{x_i}{4\pi|x|^3}$:

$$E_3 \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \qquad \text{and} \qquad \nabla E_3 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3. \tag{4.53}$$

Lemma 4.28 (**Properties of distributional solutions of Poisson's equation**).

(i) **Newtonian potential:** *If $\triangle y \in \mathcal{E}'(\mathbb{R}^3)$ is given, then $y \in Y(\mathbb{R}^3)$ is the convolution*

$$y = E_3 * \triangle y.$$

(ii) **Decay conditions:** *If $y \in Y(\mathbb{R}^3)$, then for all multi-indices $\alpha \in \mathbb{N}_0^3$*

$$D^\alpha y(x) = \langle \triangle y, 1\rangle (D^\alpha E_3)(x) + O(1/|x|^{2+|\alpha|}) \quad \text{as } |x| \to \infty.$$

(iii) **Regularity:** *If $y \in Y^\infty(\mathbb{R}^3)$, then $y \in \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3)$.*

Proof. The assertions (i) and (ii) follow by the distributional solution theory for Poisson's equation: For the Newtonian potential (i) and the decay conditions (ii) see [DL90, Ch. II.3 Propositions 2, 3, p. 278, 279] for the case $n = 3$. To prove (iii), note that the regularity properties of $y \in Y^\infty(\mathbb{R}^3)$ are a consequence of the ellipticity of the Laplacian: By definition of Y^∞, we have

$$\triangle y \in L_c^\infty(\mathbb{R}^3) \subseteq L^\infty_{\mathrm{loc}}(\mathbb{R}^3) \subseteq L^p_{\mathrm{loc}}(\mathbb{R}^3) \quad \forall\, p \ge 1.$$

We thus can invoke local elliptic regularity for the L^p-based Sobolev spaces (Lemma 3.15) to obtain $D^\alpha y \in \bigcap_{1\le p<\infty} L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for $|\alpha| = 2$, that is $y \in \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$. Furthermore, by the Sobolev embedding theorem (Lemma 3.13), $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^k(\mathbb{R}^3)$ for $k \in \mathbb{N}_0$ if $0 \le k < 2 - 3/p$. Since this inequality is satisfied for $k = 1$ and $p > 3$, we have $y \in \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3)$, completing the proof. □

We note that the inclusion

$$Y^\infty(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3)$$

in Lemma 4.28 (iii) can also be established as follows: Let $y \in Y^\infty(\mathbb{R}^3)$, then $y = E_3 * \triangle y$ and $\partial_i y = \partial_i E_3 * \triangle y$ by Lemma 4.28 (i). By (4.53), E_3, $\partial_i E_3 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and, since by assumption $\triangle y \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$, the result follows from the inclusion $L^1_{\mathrm{loc}} * L^\infty_{\mathrm{c}} \subseteq \mathcal{C}^0$ (see e.g. [DL90, Ch. II.3 Lemma 3, p. 284]).

4.5.2 Separating the far-field monopole term

We present a decomposition of $y \in Y^\infty(\mathbb{R}^3)$ which will be useful in constructions to follow. The idea is to separate, at large distances to $\operatorname{supp}(\triangle y)$, the monopole term m_y of y from its more rapidly decaying remainder $\widetilde{y} = y - m_y$. We write $B_R(0) := \{x \in \mathbb{R}^3 : |x| < R\}$ for the open ball with radius $R > 0$ around the origin $0 \in \mathbb{R}^3$.

Lemma 4.29 (A decomposition of $Y^\infty(\mathbb{R}^3)$ separating the far-field monopole term). *Let $y \in Y^\infty(\mathbb{R}^3)$ and choose $R > 0$ and sufficiently large to ensure*

$$\operatorname{supp}(\triangle y) \subseteq B_R(0).$$

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^3)$ be a smooth cutoff function with $\chi(x) = \begin{cases} 1, & |x| < R \\ 0, & |x| > 2R. \end{cases}$

Then the function

$$m_y \colon \mathbb{R}^3 \to \mathbb{R}, \quad m_y(x) := \langle \triangle y, 1 \rangle \, E_3(x) \, (1 - \chi(x))$$

satisfies $m_y \in \mathcal{C}^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $\operatorname{supp}(m_y) \subseteq \mathbb{R}^3 \setminus \operatorname{supp}(\triangle y)$, $\triangle m_y \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, and

$$y = m_y + \widetilde{y} \qquad \text{where} \qquad \widetilde{y} \in H^2(\mathbb{R}^3).$$

Proof. The regularity and the support property of m_y is clear from its definition. A direct calculation based on $\triangle E_3 = \delta$ yields $\triangle m_y \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$. We set $\widetilde{y} := y - m_y \in \mathcal{D}'(\mathbb{R}^3)$. Since $\langle \triangle y, 1 \rangle E_3(x) \chi(x)$ vanishes when $|x| > 2R$, the decay conditions in Lemma 4.28 (ii) imply

$$\begin{aligned} \widetilde{y}(x) = y(x) - m_y(x) &= \langle \triangle y, 1 \rangle E_3(x) - \langle \triangle y, 1 \rangle E_3(x)(1 - \chi(x)) + O(1/|x|^2) \\ &= \langle \triangle y, 1 \rangle E_3(x) \chi(x) + O(1/|x|^2) = O(1/|x|^2) \end{aligned}$$

as $|x| \to \infty$, which shows square integrability of $\widetilde{y}$ outside $B_{2R}(0)$, that is, $\widetilde{y} \in L^2(\mathbb{R}^3 \setminus B_{2R}(0))$. The proof of Lemma 4.28 (iii) for $p = 2$ gives $y \in H^2_{\mathrm{loc}}(\mathbb{R}^3) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^3)$. This also follows by applying local elliptic regularity (Lemma 3.15) to $\triangle y \in L^\infty_{\mathrm{c}}(\mathbb{R}^3) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Boundedness of m_y thus yields $\widetilde{y} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Consequently, $\widetilde{y} \in L^2(\mathbb{R}^3)$, which finally allows to use global elliptic regularity to obtain $\widetilde{y} \in H^2(\mathbb{R}^3)$. □

4.5.3 The gravitational potential of the earth model

We apply our findings to the gravitational potential Φ^s of the composite fluid-solid earth model (Definition 4.2) with admissible motion $\varphi \in \mathcal{A}(\overline{B} \times I)$.

If $\rho^s_t \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$, then $\Phi^s_t \in Y^\infty(\mathbb{R}^3)$ for $t \in I$, see (4.52). Lemma 4.28 then yields that Φ^s_t can be expressed as the Newtonian potential

$$\Phi^s_t(x) = (E_3 * 4\pi G \rho^s_t)(x) = -G \int_{\mathbb{R}^3} \frac{\rho^s_t(x')}{|x - x'|} \, \mathrm{dV}(x') \qquad (x \in \mathbb{R}^3), \tag{4.54}$$

for $t \in I$. Moreover (with $i, j \in \{1,2,3\}$),

$$\Phi_t^s \in \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3), \qquad \partial_i \Phi_t^s \in \bigcap_{1\le p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3), \qquad \partial_i\partial_j \Phi_t^s \in \bigcap_{1\le p<\infty} L^{p}_{\mathrm{loc}}(\mathbb{R}^3),$$

and $\langle \triangle \Phi_t^s, 1\rangle (D^\alpha E_3)(x) = \langle 4\pi G \rho_t^s, 1\rangle (D^\alpha E_3)(x) = 4\pi G\, M_B\, (D^\alpha E_3)(x)$ if $x \neq 0$. Here, conservation of mass (1.7) allowed us to replace the deformed Earth's total mass $M(\varphi_t(B))$ by the constant $M_B := M(B)$:

$$\langle \rho_t^s, 1\rangle = \int_{\mathbb{R}^3} \rho_t^s(x) \mathrm{dV}(x) = M(\varphi_t(B)) = M(B) = M_B.$$

Consequently, Φ_t^s satisfies the asymptotic condition

$$D^\alpha \Phi_t^s(x) = -G\, M_B\, D^\alpha(1/|x|) + O(1/|x|^{2+|\alpha|}) \qquad (|x| \to \infty,\ \alpha \in \mathbb{N}_0^3). \tag{4.55}$$

In particular,

$$\Phi_t^s(x) = -\frac{GM_B}{|x|} + O(1/|x|^2) \qquad \text{as} \quad |x| \to \infty,$$

which is consistent with the well-known multipole expansion of the Earth's external gravitational potential (see Remark 4.30).

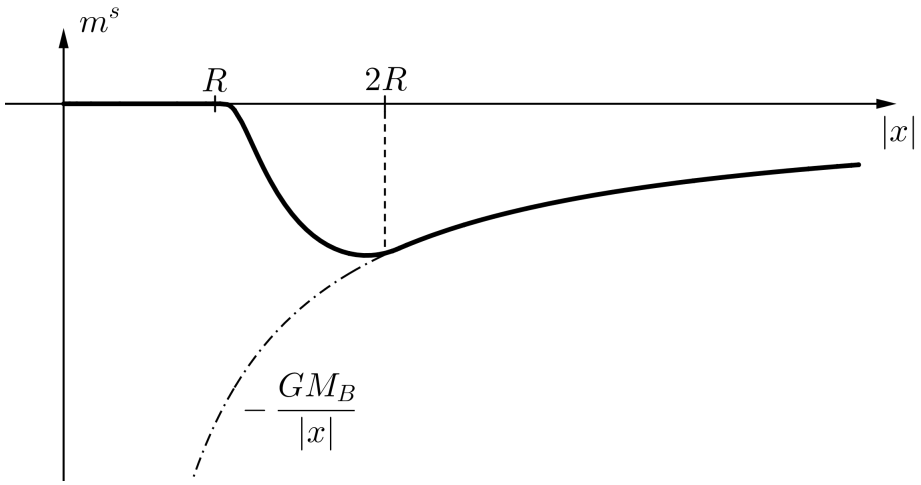

Figure 4.3: The solid line illustrates the far-field monopole term m^s (4.56).

To interpret the result of Lemma 4.29, let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ be a cutoff around the ball

$$B_R(0) \supseteq \bigcup_{t\in I} \varphi_t(B)$$

that vanishes outside $B_{2R}(0)$ (the current Earth is contained in some $B_R(0)$ by condition (i) in Definition 4.19 of admissible motions). As M_B is constant, the function

$$x \mapsto m^s(x) := m_{\Phi_t^s}(x) = -\frac{GM_B}{|x|}\,(1-\chi(x)) \qquad (x \in \mathbb{R}^3) \tag{4.56}$$

neither depends on Φ_t^s nor on ρ_t^s (m^s depends only on the fixed total mass of the earth M_B and the choice of the cutoff χ). Lemma 4.29 thus yields the following decomposition of the gravitational potential:

$$\Phi_t^s = m^s + \widetilde{\Phi}_t^s. \tag{4.57}$$

with $m^s \in \mathcal{C}^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\operatorname{supp}(m^s) \subseteq \mathbb{R}^3 \setminus \bigcup_{t\in I} \varphi_t(B)$. The function m^s represents the **far-field monopole term** of the Earth's gravitational field at large distances ($|x| \geq 2R$) and

$$\widetilde{\Phi}^s_t \in H^2(\mathbb{R}^3)$$

consists of all **higher-order multipole terms** modeling the Earth's near and interior gravitational field.

Remark 4.30 (**Multipole expansion**)**.** We have shown that the gravitational potential $\Phi^s(x,.)$ for large $|x|$ may be approximated by the monopole term $-\frac{GM_B}{|x|}$. More generally, in spherical coordinates (r,ϑ,λ) with $r = |x|$, solving **Laplace's equation** $\triangle\Phi^s = 0$ outside the Earth yields the following multipole expansion (sum of spherical harmonic functions) of the Earth's external gravitational potential [Tor03, p. 62]:

$$\Phi^s(r,\vartheta,\lambda) = -\frac{GM_B}{r}\left(1 + \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{r_B}{r}\right)^l (C_l^m \cos(m\lambda) + S_l^m \sin(m\lambda))\right) P_l^m(\cos\vartheta).$$

for $r > r_B$. Here r_B is the mean radius of the Earth, the functions P_l^m are the associated Legendre polynomials of degree l and order m, and the spherical harmonic coefficients C_l^m and S_l^m are weighted integrals of the Earth's interior density distribution. ⌟

Up to now we have not imposed any regularity condition on the gravitational potential with respect to time. The reason is that via Poisson's equation, the temporal regularity of Φ^s is determined by that of ρ^s. Our basic regularity assumption on ρ^s is (4.45), namely

$$\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$$

with compact support of ρ^s_t in $\mathbb{R}^3$. Thus, in view of Poisson's equation $\triangle\Phi^s = 4\pi G\rho^s$ or the representation of Φ^s as Newtonian potential $\Phi^s = E_3 * (4\pi G\rho^s)$, we obtain

$$\Phi^s \in \mathcal{C}^0(I, Y^\infty(\mathbb{R}^3)). \tag{4.58}$$

We note that $Y^\infty(\mathbb{R}^3)$ is not a Banach space, but a subspace of the topological vector space $\mathcal{D}'(\mathbb{R}^3)$. The topology of $\mathcal{C}^0(I, Y^\infty(\mathbb{R}^3))$ thus is induced from $\mathcal{C}^0(I, \mathcal{D}'(\mathbb{R}^3))$.

The decomposition $\Phi^s_t = m^s + \widetilde{\Phi}^s_t$ (4.57) and $\Phi^s \in \mathcal{C}^0(I, Y^\infty(\mathbb{R}^3))$ imply

$$\widetilde{\Phi}^s \in \mathcal{C}^0(I, H^2(\mathbb{R}^3)).$$

The inclusion $Y^\infty \subseteq \mathcal{C}^1$ shows continuity of Φ^s_t and $\nabla\Phi^s_t$ on $\mathbb{R}^3$. Thus, for any Lip-surface $S \subseteq \mathbb{R}^3$ (Lip regularity guarantees the boundedness of the unit normal ν^s (1.30) if $\varphi \in \mathcal{A}(\overline{B} \times I)$),

$$[\Phi^s_t]^+_- = 0 \qquad \text{and} \qquad [\nabla\Phi^s_t]^+_- \cdot \nu^s_t = 0 \qquad \text{on} \qquad \varphi_t(S).$$

Consequently, all **gravitational interface conditions** (2.18), as well as the decay condition at infinity, already follow from the regularity (4.58) of Φ^s.

4.6 Elastic constitutive model

4.6.1 Elastic energy and stress

The material of the fluid-solid composite earth model is described as a hyperelastic continuum. As discussed in Section 1.3.1, the mechanical response of the earth model is thus encoded in its

internal energy density per unit mass, $U\colon B \times GL_+(3) \to \mathbb{R}$, with first Piola-Kirchhoff stress tensor T^{PK} given by (1.52):

$$T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla\varphi)}.$$

As in [Cia88] we assume that U is of Carathéodory type, that is, bounded (measurable) in its first argument $X \in B$ and $\mathcal{C}^1$ in its second argument $\nabla\varphi(X,t) \in GL_+(3)$ for almost every $X \in B$ and $t \in I$, i.e.

$$U \in L^\infty(B, \mathcal{C}^1(GL_+(3))). \tag{4.59}$$

For admissible motions $\varphi \in \mathcal{A}(\overline{B} \times I)$ this in turn implies that $(X,t) \mapsto U(X,(\nabla\varphi)(X,t))$ is in $\mathcal{C}^0(I, L^\infty(B))$. By abuse of notation, this map will also be denoted by U. Thus we obtain the regularity

$$U \in \mathcal{C}^0(I, L^\infty(B)) \tag{4.60}$$

and

$$T^{\mathrm{PK}} \in \mathcal{C}^0(I, L^\infty(B)^{3\times 3}). \tag{4.61}$$

In fluid regions, which we assume to be elastic (compressible inviscid) fluids, the spatial Cauchy stress is isotropic by (1.54): $T^s = -p^s 1_{3\times 3}$ with $p^s = (\rho^s)^2 \frac{\partial U^s}{\partial \rho^s}$. As was discussed in Section 1.3.1, by conservation of mass one can express the dependence of pressure on density again by a dependence on the deformation gradient. Thereby the elastic response of solid and fluid regions can be modeled in a unified way by prescribing the internal energy density U.

4.6.2 Dynamical interface conditions

The **dynamical interface conditions** (dynamical interior boundary conditions) constrain the traction on Lip-boundaries and interfaces S. In this section, we motivate the specific form of these interface conditions, already presented in Section 2.3.1. As briefly discussed in Section 3.1.1, these conditions can also be obtained from Hamilton's principle.

By **Newton's third law** of action and reaction, the spatial traction vector (see Section 1.2.3) must satisfy

$$\tau^s(-\nu^s) = -\tau^s(\nu^s) \tag{4.62}$$

on any spatial surface $\varphi_t(S)$ (e.g. [Ant05, (7.19)] or [BMS81, (1.2)]). Here the insertion of $\pm\nu^s$ also means evaluating τ^s on the $\pm$-side of the surface $\varphi_t(S)$. With the convention (4.12) that ν^s points from the $-$-side to the $+$-side of the surface, we identify $\nu^s = \nu^{s,-}$. The definition (1.45) of the Cauchy stress then implies

$$-T^{s,-}\cdot\nu^s = T^{s,-}\cdot(-\nu^s) = \tau^s(-\nu^s) = -\tau^s(\nu^s) = -T^{s,+}\cdot\nu^s.$$

Spatial traction is thus continuous across interior boundaries, that is, the spatial jump condition (2.16)

$$[T^s]^+_-\cdot\nu^s = 0 \qquad \text{on} \qquad \varphi_t(S)$$

holds for all $t \in I$. In case of **perfect slip along frictionless surfaces** (i.e. ideal faults or fluid-solid boundaries where viscosity is disregarded), there are no tangential stresses, that is, the traction vector must be purely normal. Thus, $(T^s_t\cdot\nu^s_t)^{\parallel} = 0$ on $\varphi_t(\Sigma^{\mathrm{FS}})$, or equivalently, T^s satisfies the normality condition (4.27) which gives (2.17):

$$T^s\cdot\nu^s = (\nu^s\cdot T^s\cdot\nu^s)\nu^s \quad \text{on} \quad \varphi_t(\Sigma^{\mathrm{FS}}).$$

In particular, if (2.16) is considered on the Earth's exterior boundary and atmospheric stresses are neglected (i.e. the Earth is considered as a body in vacuum), it gives the dynamical boundary condition (2.13):

$$T^s\cdot\nu^s = 0 \qquad \text{on} \qquad \varphi_t(\partial B).$$

In combination, the conditions (2.16) and (2.17) guarantee the absence of the fluid-solid interface and boundary integral contribution to the energy balance in the spatial representation.

Because of tangential slip along fluid-solid interfaces, it is not straightforward to formulate the material counterparts of the dynamical interface condition (2.16) in terms of the first Piola-Kirchhoff stress tensor T^{PK}. We will derive the corresponding condition only in the linearized setting (see Section 6.5.3).

4.6.3 Pressure and deviatoric prestress

The stress at initial time t_0 is called the **prestress** (**residual stress**) T^0. Assumption (1.21), $\varphi_{t_0}(X) = X$ for all $X \in \overline{B}$, implies that the spatial Cauchy stress tensor T^s, the material Cauchy stress tensor T, and the first Piola-Kirchhoff stress tensor T^{PK} all coincide at initial time:

$$T^0 := T^s_{t_0} = T_{t_0} = T^{\mathrm{PK}}_{t_0}. \tag{4.63}$$

Evaluating $p^s = -\frac{1}{3}\operatorname{tr} T^s$ (1.55) at time t_0 gives the **hydrostatic equilibrium pressure**:

$$p^0 = -\frac{1}{3}\operatorname{tr} T^0. \tag{4.64}$$

The **deviatoric prestress** is defined as the trace-free difference

$$T^0_{\mathrm{dev}} := T^0 - \frac{1}{3}(\operatorname{tr} T^0)1_{3\times 3}.$$

Thus, prestress decomposes into hydrostatic and deviatoric prestress:

$$T^0 = -p^0 1_{3\times 3} + T^0_{\mathrm{dev}}. \tag{4.65}$$

As T^0_{dev} is symmetric (due to the symmetry of T^0) and trace-free, only three of its components are independent. In the solid parts of the Earth, these are generally non-zero (see also Remark 6.3). In fluid regions however, $T^0_{\mathrm{dev}} = 0$ and, in accordance with $T^s = -p^s 1_{3\times 3}$ (1.54), prestress reduces to a pure pressure:

$$T^0 = -p^0 1_{3\times 3}. \tag{4.66}$$

Evaluating $[T^s]^+_- \cdot \nu^s = 0$ (2.16) at t_0 immediately gives (2.22):

$$[T^0]^+_- \cdot \nu = 0 \qquad \text{on} \qquad \Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}}.$$

Similarly, (2.13) leads to the equilibrium zero-traction condition (2.21):

$$T^0 \cdot \nu = 0 \qquad \text{on} \qquad \partial B.$$

Combining (2.22) and (4.66) shows that the initial traction $T^0 \cdot \nu$ must be purely normal on a fluid-solid boundary (T^0 satisfies the normality condition (4.27) on Σ^{FS}), i.e. (2.23):

$$T^0 \cdot \nu = (\nu \cdot T^0 \cdot \nu)\nu = -p^0 \nu \qquad \text{with} \qquad p^0 = -\nu \cdot T^0 \cdot \nu \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}.$$

With (2.22) this further implies

$$[p^0]^+_- = 0 \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}, \tag{4.67}$$

and (2.21) yields

$$p^0 = 0 \qquad \text{on} \qquad \partial B. \tag{4.68}$$

Condition (4.67) expresses the continuity of the hydrostatic pressure (4.64) across fluid-solid interfaces and (4.68) states that the pressure vanishes at the Earth's surface, which corresponds to ignoring any atmospheric stresses.

Chapter 5

Action

We introduce the variational framework for the composite fluid-solid earth model and determine the action corresponding to the nonlinear system of elastic-gravitational equations from energy considerations (Sections 5.1, 5.2). Based on this, we show how the nonlinear system of elastic-gravitational equations, including the dynamical interface conditions, formally arises from Hamilton's principle (Section 5.4). As an interlude, we briefly review an abstract geometric variational formulation of continuum mechanics that also incorporates the kinematic interface conditions via generalized variations (Section 5.3).

5.1 The variational model

The state of the uniformly rotating, elastic, and self-gravitating earth model is characterized by specifying its motion φ, its gravitational potential Φ^s, and its density ρ^s for given elastic properties which are encoded in the internal energy density U, and for a given force potential F^s. We summarize the basic regularity assumptions on model geometry and on the state variables developed in the previous chapter, in particular Definitions 4.12, 4.19, (4.58), and (4.45). In order to unify the material and spatial conditions, the motion is extended to $\mathbb{R}^3$.

Assumption 1 (Variational model).

(i) **Geometry:** $B = B^{\mathrm{FS}} \cup \Sigma^{\mathrm{FS}}$ *is a composite fluid-solid earth model (Definition 4.12).*

(ii) **Configuration spaces:** $(\varphi, \Phi^s, \rho^s) \in W_{\mathrm{motion}} \times W_{\mathrm{gravity}} \times W_{\mathrm{density}}$ *for*

$$\begin{aligned} W_{\mathrm{motion}} &:= \{\varphi \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))^3 : \varphi|_{\overline{B}\times I} \in \mathcal{A}(\overline{B} \times I)\}, \\ W_{\mathrm{gravity}} &:= \mathcal{C}^0(I, Y^\infty(\mathbb{R}^3)), \\ W_{\mathrm{density}} &:= \{\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)) : \operatorname{supp}(\rho^s_t) \subseteq \varphi_t(\overline{B})\ \forall\, t \in I\}. \end{aligned}$$

(iii) **Constitutive assumptions:** *The material is hyperelastic, i.e.* $T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla\varphi)}$ (1.52)*, where the stored energy density satisfies* (4.59)*:* $U \in L^\infty(B, \mathcal{C}^1(GL_+(3)))$.

(iv) **External force:** f^s *is conservative, i.e.* $f^s = -\rho^s \nabla F^s$ (4.48)*, where the potential satisfies* (4.49)*:* $F^s \in \mathcal{C}^0(I, \mathrm{Lip}(\mathbb{R}^3))$.

Under these assumptions on φ and U, Lemma 4.20 yields $U^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$ with $\operatorname{supp}(U^s_t) \subseteq \varphi_t(\overline{B})$ for all $t \in I$. The assumption on F^s combined with $\rho^s \in W_{\mathrm{density}}$ implies that the force satisfies $f^s = -\rho^s \nabla F^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))^3$ with compact support contained in $\varphi_t(\overline{B})$ at time $t \in I$.

5.2 Definition of the action integral

5.2.1 The basic structure of the action

The elastic-gravitational deformations of the Earth do not dissipate energy. Thus, Hamilton's principle applies: The configuration $(\varphi, \Phi^s, \rho^s) \in W_{\text{motion}} \times W_{\text{gravity}} \times W_{\text{density}}$ is a stationary point for the action functional

$$\mathscr{A} : W_{\text{motion}} \times W_{\text{gravity}} \times W_{\text{density}} \to \mathbb{R}. \tag{5.1}$$

The action has the following basic structure (this will be justified below):

$$\mathscr{A}(\varphi, \Phi^s, \rho^s) = \int_I \left(\int_{\varphi_t(B)} L^s(x,t)\, \mathrm{dV}(x) + \int_{\varphi_t(\Sigma^{\text{FS}})} L^s_{\varphi_t(\Sigma^{\text{FS}})}(x,t)\, \mathrm{dS}(x) \right) \mathrm{d}t. \tag{5.2}$$

Here, L^s is an abbreviated notation for the volume Lagrangian density, which may depend explicitly on (x,t) besides being a function of the space- and time-derivatives of the state-variables $(\varphi, \Phi^s, \rho^s)$. In detail, the integrand L^s is to be understood as a function

$$(x,t) \mapsto L^s \left(x, t, \varphi(x,t), \Phi^s(x,t), \rho^s(x,t), \nabla\varphi(x,t), \nabla\Phi^s(x,t), \nabla\rho^s(x,t), \dot{\varphi}(x,t), \dot{\Phi}^s(x,t), \dot{\rho}^s(x,t) \right).$$

The part of $\mathscr{A}$ representing the surface action consists of a temporally integrated surface integral over all fluid-solid boundaries within the earth model:

$$\mathscr{A}_{\Sigma^{\text{FS}}}(\varphi) := \int_I \mathscr{A}_{\Sigma^{\text{FS}},t}(\varphi)\, \mathrm{d}t := \int_I \int_{\varphi_t(\Sigma^{\text{FS}})} L^s_{\varphi_t(\Sigma^{\text{FS}})}(x,t)\, \mathrm{dS}(x)\, \mathrm{d}t = \int_I \int_{\Sigma^{\text{FS}}} L_{\Sigma^{\text{FS}}}(X,t)\, \mathrm{dS}(X)\, \mathrm{d}t.$$

Accounting for the mutual interaction of fluid and solid regions, $\mathscr{A}_{\Sigma^{\text{FS}}}(\varphi)$ only occurs if fluid regions are present in the earth model. We will see later that $\mathscr{A}_{\Sigma^{\text{FS}}}$ is, in fact, independent of Φ^s and ρ^s, that is, $L^s_{\varphi_t(\Sigma^{\text{FS}})}$ is a function

$$(x,t) \mapsto L^s_{\varphi_t(\Sigma^{\text{FS}})}\left(x, t, \varphi(x,t), \nabla\varphi(x,t), \dot{\varphi}(x,t)\right).$$

The explicit form (5.11) of the material surface Lagrangian $L_{\Sigma^{\text{FS}}}$ will be obtained as a consequence of energy balance (see Section 5.2.5). Actually, since there is no work done by perfect slip, the surface action will be zero in the full nonlinear model: $\mathscr{A}_{\Sigma^{\text{FS}}} = 0$, see (6.64). However, in the linearized model the purely second-order surface action does not vanish, even in the frictionless case: Its Lagrangian density (6.54) accounts for the work done by slip of material at fluid-solid boundaries against the initial traction due to prestress. This interpretation, which is consistent with [DT98, p. 96, (3.232)], will be discussed in more detail in Section 6.6.3.

The action does not contain an integral over the exterior boundary ∂B, which will correspond to the zero-traction (homogeneous Neumann) boundary condition (5.36): $T^{\text{PK}} \cdot \nu = 0$ on ∂B.

5.2.2 Kinetic and potential energy densities in spatial formulation

Let us further specify the volume Lagrangian density by expressing it in terms of physical quantities based on a first-principles analysis [DT98, (3.251)]. Since we consider elastic, gravitational, and also other (internal or external) conservative forces, according to Hamilton's principle we have

$$L^s = E^s_{\text{kin}} - E^s_{\text{pot}} \qquad \text{with} \qquad E^s_{\text{pot}} = E^s_{\text{elast}} + E^s_{\text{gravity}} + E^s_{\text{ext}}.$$

Here E^s_{kin}, E^s_{elast}, E^s_{gravity}, and E^s_{ext} are the **kinetic**, **elastic**, **gravitational energy densities** of the Earth and of the other (internal or external) conservative forces respectively, given by

$$E^s_{\text{kin}} = \frac{1}{2}(v^s + \Omega \times x)^2 \rho^s, \qquad E^s_{\text{elast}} = U^s \rho^s, \qquad E^s_{\text{gravity}} = \frac{1}{2}\Phi^s \rho^s, \qquad \text{and} \qquad E^s_{\text{ext}} = F^s \rho^s.$$

The factor $\frac{1}{2}$ in E^s_{gravity} is due to self-gravitation (cf. [DT98], [WD07]). Thus

$$L^s = \Big(\frac{1}{2}(v^s + \Omega \times x)^2 - (U^s + \frac{1}{2}\,\Phi^s + F^s)\Big)\rho^s. \tag{5.3}$$

Note that the additional term $\Omega \times x$ in the formula of E^s_{kin} is due to the adoption of a co-rotating coordinate system (see Section 2.2). We have

$$\frac{1}{2}(v^s + \Omega \times x)^2 = \frac{1}{2}(v^s)^2 + v^s \cdot (\Omega \times x) + \frac{1}{2}(\Omega \times x)^2.$$

By abuse of notation, we will frequently keep the argument of $x \mapsto \Omega \times x$; actually, $\Omega \times x$ should read $\Omega \times \mathrm{Id}_{\mathbb{R}^3}$. Alternatively, since the ith component of $\Omega \times x$ is $\epsilon_{ijk}\Omega_j x_k$,

$$\begin{aligned}(v^s + \Omega \times x)^2 &= (v^s_i + \epsilon_{ijk}\Omega_j x_k)^2 \\ &= (v^s_i)^2 + 2\, v^s_i \epsilon_{ijk}\Omega_j x_k + \epsilon_{ijk}\Omega_j x_k \epsilon_{ilm}\Omega_l x_m \\ &= (v^s_i)^2 + 2\, v^s_i \epsilon_{ijk}\Omega_j x_k + (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\Omega_j \Omega_l x_k x_m \\ &= (v^s_i)^2 + 2\, v^s_i \epsilon_{ijk}\Omega_j x_k + \Omega_l^2 x_k^2 - (\Omega_k x_k)^2.\end{aligned}$$

The expression $2v^s_i \epsilon_{ijk}\Omega_j x_k = 2v^s \cdot (\Omega \times x) = 2x \cdot (v^s \times \Omega)$ represents the Coriolis term (the Coriolis acceleration is given by $2\, v^s \times \Omega$). We recall from (2.7) that the centrifugal acceleration reads

$$\Omega \times (\Omega \times x) = \nabla \Psi^s(x),$$

where the centrifugal potential is the second-order polynomial $\Psi^s \colon \mathbb{R}^3 \to \mathbb{R}$,

$$\Psi^s(x) = -\frac{1}{2}(\Omega_l^2 x_k^2 - (\Omega_k x_k)^2) = -\frac{1}{2}(\Omega^2 x^2 - (\Omega \cdot x)^2) = -\frac{1}{2}(\Omega \times x)^2.$$

Therefore, we may write L^s in the form

$$L^s = \Big(\frac{1}{2}\,(v^s)^2 + v^s \cdot (\Omega \times x) - (U^s + \frac{1}{2}\,\Phi^s + \Psi^s + F^s)\Big)\rho^s. \tag{5.4}$$

By Assumption 1, the functions v^s, U^s, ρ^s, Φ^s, F^s are elements of $\mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$, where, for t in the bounded time interval I, ρ^s_t is compactly supported in $\varphi_t(\overline{B})$. Therefore,

$$L^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)) \subseteq L^\infty(\mathbb{R}^3 \times I)$$

with compact support $\operatorname{supp}(L^s_t) \subseteq \varphi_t(\overline{B})$. The regularity conditions on φ and U imply integrability of the surface Lagrangian, as will be seen later. Consequently, the action functional $(\varphi, \Phi^s, \rho^s) \mapsto \mathscr{A}(\varphi, \Phi^s, \rho^s)$ is indeed defined on $W_{\text{motion}} \times W_{\text{gravity}} \times W_{\text{density}}$, which justifies (5.1).

In addition to the stationarity of the action with respect to variations in the state variables φ, Φ^s and ρ^s, the fields Φ^s and ρ^s are linked via Poisson's equation. Furthermore, φ and ρ^s are coupled through conservation of mass. Therefore we have to interpret the stationarity of $\mathscr{A}$ to hold under these constraints. In the following sections we will modify $\mathscr{A}$ in order to incorporate self-gravitation and eliminate the dependence on spatial density via conservation of mass. We thereby arrive at the material formulation of the action (5.9):

$$\mathscr{A}'' \colon W_{\text{motion}} \times W_{\text{gravity}} \to \mathbb{R}.$$

By Hamilton's principle, stationarity of $\mathscr{A}''$ then gives the nonlinear dynamical equations (the nonlinear system of elastic-gravitational equations, Section 2.3.1) describing the motion of a uniformly rotating, nonlinear elastic, self-gravitating composite fluid-solid continuum with initial density $\rho^0 \in L^\infty(B)$ and internal elastic energy function $U \in L^\infty(B, \mathcal{C}^1(\mathbb{R}^{3\times 3}))$. The derivation, on a formal level, is presented in Section 5.4.

However, due to the nonlinearity of U, the interrelation of spatial and material quantities, and the fact that $W_{\text{motion}} \times W_{\text{gravity}}$ is not a Banach space (more precisely, not normable), a rigorous mathematical framework for a calculus of variations (as well as for a linearization) lies beyond the basic notion of Fréchet differentiability or related concepts. The derivation of EL, NBC, and NIBC (3.4) to (3.7) for $\mathscr{A}''$ with rigorous proofs remains an open problem. In Section 6.6, we will introduce a physically reasonable approximation of the action integral, yielding a linearization, which enables us to apply the calculus of variations in a Sobolev framework (see Section 7.2), eventually leading to linear governing equations.

5.2.3 A variational problem constrained by self-gravitation

The Poisson equation (2.2),

$$\triangle \Phi^s = 4\pi G \rho^s \qquad \text{in} \qquad \mathbb{R}^3 \times I,$$

is incorporated in the variational principle via a Lagrange multiplier method. This approach, on a formal level, is also used in [WD07, p. 34] or, in the linearized setting, in [DT98, p. 88]. In order to apply the Lagrange multiplier theorem for constrained variational problems in a Hilbert space setting (Theorem 3.4), the constrained variational problem must be reformulated in a suitable way.

We begin with some preparatory observations. Let $(\varphi, \Phi^s, \rho^s) \in W_{\text{motion}} \times W_{\text{gravity}} \times W_{\text{density}}$ (see Assumption 1). First, note that for $t \in I$, $L^s_t = L^s(.,t)$ is proportional to ρ^s_t and thus supported in $\varphi_t(B)$, that is $L^s_t \in L^\infty_{\text{c}}(\mathbb{R}^3)$ with $\operatorname{supp}(L^s_t) \subseteq \varphi_t(\overline{B})$, and clearly also $L^s_{\varphi_t(\Sigma^{\text{FS}})}$ vanishes outside $\varphi_t(\overline{B})$. Hence, we may integrate L^s_t over $\mathbb{R}^3$ without changing the definition of $\mathscr{A}$ (5.2). Second, we may neglect $L^s_{\Sigma^{\text{FS}}}$ for the moment, because Φ^s and ρ^s do not directly contribute to the surface action, and consider

$$\mathscr{A}(\varphi, \Phi^s, \rho^s) = \int_I \int_{\mathbb{R}^3} L^s(x,t)\, \mathrm{dV}(x) \mathrm{d}t.$$

We now have $\mathbb{R}^3 \times I$ as both, the integration domain and the domain of the constraint equation. Third, we observe that if we replace Φ^s_t by $m^s + \widetilde{\Phi}^s_t$ according to equation (4.57), m^s does not contribute to $\mathscr{A}$, since by construction (which implicitly uses conservation of mass) its support is disjoint from $\operatorname{supp}(\rho^s_t) \supseteq \operatorname{supp}(L^s_t)$. Therefore, $\mathscr{A}(\varphi, \Phi^s, \rho^s) = \mathscr{A}(\varphi, \widetilde{\Phi}^s, \rho^s)$. Fourth, since there is no explicit dependence on time in Poisson's equation, it suffices to investigate the constrained variational problem for the spatially integrated part of the action integral

$$\mathscr{A}_t(\varphi_t, \Phi^s_t, \rho^s_t) := \int_{\mathbb{R}^3} L^s(x,t)\, \mathrm{dV}(x),$$

considered as a functional of φ_t, ρ^s_t, and Φ^s_t for fixed $t \in I$. Since Poisson's equation does not involve φ, we can consider the constrained functional $\mathscr{A}_t$ as a functional of ρ^s_t and Φ^s_t only and keep φ_t fixed (by abuse of notation we use the same symbol $\mathscr{A}_t$).

The observations above thus show that for $t \in I$ fixed, $\mathscr{A}_t$ can be written in the form

$$\mathscr{A}_t(\Phi^s_t, \rho^s_t) = \mathscr{A}_t(\widetilde{\Phi}^s_t, \rho^s_t) = \int_{\mathbb{R}^3} \Big(a^s_t(x) - \frac{1}{2}\widetilde{\Phi}^s_t(x)\Big)\rho^s_t(x)\, \mathrm{dV}(x) = \langle a^s_t - \tfrac{1}{2}\,\widetilde{\Phi}^s_t | \rho^s_t \rangle_{L^2(\mathbb{R}^3)}.$$

Here we use the abbreviation

$$a^s_t(x) := \frac{1}{2}(v^s_t(x))^2 + v^s_t(x) \cdot (\Omega \times x) - (U^s_t(x) + \Psi^s(x) + F^s_t(x))\,.$$

Note that $a_t^s \in L_c^\infty(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3)$ (upon extending Ψ^s by zero) and does not depend on Φ_t^s and ρ_t^s. The bracket $\langle .|. \rangle_{L^2(\mathbb{R}^3)}$ denotes the L^2 inner product. By Lemma 4.29 we have $\widetilde{\Phi}^s \in H^2(\mathbb{R}^3)$. Consequently, we may identify $\mathscr{A}_t$ with the functional

$$\mathcal{I}\colon H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}\,, \quad \mathcal{I}(\widetilde{\Phi}^s, \rho^s) \;:=\; \langle a^s - \tfrac{1}{2}\,\widetilde{\Phi}^s | \rho^s \rangle_{L^2(\mathbb{R}^3)}, \tag{5.5}$$

with $a^s \in L^2(\mathbb{R}^3)$. Poisson's equation can be stated in the form $g = 0$ with

$$g\colon H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)\,, \quad g(\widetilde{\Phi}^s, \rho^s) \;:=\; \triangle \widetilde{\Phi}^s + \triangle m^s - 4\pi G \rho^s. \tag{5.6}$$

Note that we have omitted the explicit time dependence to simplify the notation. The functions $\widetilde{\Phi}^s$, ρ^s, and a^s in $\mathcal{I}$ correspond to $\widetilde{\Phi}_t^s$, ρ_t^s, and a_t^s in $\mathscr{A}_t$; $g = 0$ corresponds to Poisson's equation $\triangle(\widetilde{\Phi}_t^s + m^s) = 4\pi G \rho_t^s$.

Consequently, the constrained variational problem for $\mathscr{A}$ (5.2) and Poisson's equation (2.2) can be formulated as follows: For $\mathcal{I}$ given by (5.5) and g given by (5.6), find $(\widetilde{\Phi}_*^s, \rho_*^s) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that $\mathcal{I}$ is stationary under the constraint $g = 0$.

The Lagrange multiplier method (Theorem 3.4) yields a necessary condition for $(\widetilde{\Phi}_*^s, \rho_*^s)$:

Lemma 5.1 (Unconstrained variational problem via Lagrange multiplier method). *Let $\mathcal{I}$ defined in (5.5) be stationary at $(\widetilde{\Phi}_*^s, \rho_*^s) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ under the constraint $g = 0$ with g defined in (5.6). Then there exists a Lagrange multiplier $\lambda \in L^2(\mathbb{R}^3)$ such that the modified functional $\mathcal{I}^\lambda := \mathcal{I} + \langle \lambda | g(.) \rangle_{L^2(\mathbb{R}^3)}\colon H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}$, that is*

$$\mathcal{I}^\lambda(\widetilde{\Phi}^s, \rho^s) = \mathcal{I}(\widetilde{\Phi}^s, \rho^s) + \langle \lambda | \triangle(\widetilde{\Phi}^s + m^s) - 4\pi G \rho^s \rangle_{L^2(\mathbb{R}^3)},$$

is stationary at $(\widetilde{\Phi}_*^s, \rho_*^s)$. Moreover, λ satisfies the equation $\triangle \lambda = \frac{1}{2}\,\rho_*^s$ in $\mathcal{D}'(\mathbb{R}^3)$.

Proof. To apply the Lagrange multiplier theorem we just have to show differentiability of $\mathcal{I}$ and g and verify that $(\widetilde{\Phi}_*^s, \rho_*^s)$ is a regular point of g (note that $\ker Dg(\widetilde{\Phi}_*^s, \rho_*^s)$ automatically has a topological complement in the Hilbert space $H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$). The functional $\mathcal{I}$ is differentiable on $H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, because it can be written as a sum of a linear functional and a quadratic form associated to a bounded linear operator on $H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$:

$$\mathcal{I}(\widetilde{\Phi}^s, \rho^s) = \left\langle \begin{pmatrix} 0 \\ a \end{pmatrix} \middle| \begin{pmatrix} \widetilde{\Phi}^s \\ \rho^s \end{pmatrix} \right\rangle_{L^2 \times L^2} - \frac{1}{2} \left\langle \begin{pmatrix} 0 & \mathrm{Id}_{L^2} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\Phi}^s \\ \rho^s \end{pmatrix} \middle| \begin{pmatrix} \widetilde{\Phi}^s \\ \rho^s \end{pmatrix} \right\rangle_{L^2 \times L^2}.$$

Here, $\langle .|. \rangle_{L^2 \times L^2}$ denotes the $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$-inner product and boundedness of $\mathcal{I}$ follows from the estimate $\|.\|_{L^2 \times L^2} \leq \|.\|_{H^2 \times L^2}$. Since g is a continuous (affine) linear operator, its differentiability is clear. The derivative $h := Dg(\widetilde{\Phi}^s, \rho^s)\colon H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ at $(\widetilde{\Phi}^s, \rho^s) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ reads

$$h(\widetilde{\Phi}^s, \rho^s) = \triangle \widetilde{\Phi}^s - 4\pi G \rho^s.$$

We show that h is surjective: Let $r \in L^2(\mathbb{R}^3)$ and define $f := -\mathscr{F}^{-1}(\frac{(\mathscr{F} r)(\xi)}{|\xi|^2 + 4\pi G}) \in H^2(\mathbb{R}^3)$, where $\mathscr{F}$ denotes the Fourier transform with spectral variable $\xi \in \mathbb{R}^3$. Then

$$(\mathscr{F} h(f, f))(\xi) = \mathscr{F}(\triangle f - 4\pi G f)(\xi) = -(|\xi|^2 + 4\pi G)(\mathscr{F} f)(\xi) = (\mathscr{F} r)(\xi)$$

on $\mathcal{S}'(\mathbb{R}^3)$, that is $h(f, f) = r$, which proves the surjectivity of h. Thus, every point in $H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is a regular point of g and we can apply the Lagrange multiplier theorem (Theorem 3.4): If $(\widetilde{\Phi}_*^s, \rho_*^s) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is a stationary point for $\mathcal{I}$ under the constraint $g = 0$, then there exists a linear functional $\lambda'\colon L^2(\mathbb{R}^3) \to \mathbb{R}$ such that $(\widetilde{\Phi}_*^s, \rho_*^s)$ is stationary for the modified functional

$$\mathcal{I}^{\lambda'}\colon H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}, \quad \mathcal{I}^{\lambda'} := \mathcal{I} - \lambda' \circ g.$$

Using L^2-duality we may replace the action of λ' by an inner product with $\lambda \in L^2(\mathbb{R}^3)$ such that we can rewrite $\mathcal{I}^{\lambda'}$ in the desired form:

$$\mathcal{I}^\lambda := \mathcal{I} + \langle \lambda | g(.) \rangle_{L^2(\mathbb{R}^3)}.$$

To establish $\triangle\lambda = \frac{1}{2}\,\rho^s_*$ in the sense of distributions, observe that stationarity of $\mathcal{I}^\lambda$ at $(\widetilde{\Phi}^s_*, \rho^s_*) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is equivalent to $D\mathcal{I}^\lambda(\widetilde{\Phi}^s_*, \rho^s_*)(y,z) = 0$ for all $(y,z) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and differentiation yields

$$\begin{aligned} 0 = D\mathcal{I}^\lambda(\widetilde{\Phi}^s_*, \rho^s_*)(y,z) &= \langle a^s - \frac{1}{2}\,\widetilde{\Phi}^s_* | z \rangle_{L^2(\mathbb{R}^3)} - \langle \frac{1}{2}\rho^s_* | y \rangle_{L^2(\mathbb{R}^3)} + \langle \lambda | \triangle y - 4\pi G z \rangle_{L^2(\mathbb{R}^3)} \\ &= \langle a^s - \frac{1}{2}\,\widetilde{\Phi}^s_* - 4\pi G\lambda | z \rangle_{L^2(\mathbb{R}^3)} - \langle \frac{1}{2}\rho^s_* | y \rangle_{L^2(\mathbb{R}^3)} + \langle \lambda | \triangle y \rangle_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Setting $z = 0$, which corresponds to considering only the stationarity of $\mathcal{I}^\lambda$ with respect to its first variable $\widetilde{\Phi}^s$, it follows that $\langle \frac{1}{2}\rho^s_* | y \rangle_{L^2(\mathbb{R}^3)} = \langle \lambda | \triangle y \rangle_{L^2(\mathbb{R}^3)}$ holds for every $y \in H^2(\mathbb{R}^3)$. Since $\mathcal{D}(\mathbb{R}^3) \subseteq H^2(\mathbb{R}^3)$, this implies the $\langle \mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3) \rangle$-duality $\langle \frac{1}{2}\rho^s_*, y \rangle = \langle \lambda, \triangle y \rangle = \langle \triangle\lambda, y \rangle$, hence $\triangle\lambda = \frac{1}{2}\,\rho^s_*$ holds in $\mathcal{D}'(\mathbb{R}^3)$. □

The modified functional $\mathcal{I}^\lambda$ in Lemma 5.1 corresponds to the modified action integral

$$\mathscr{A}^{\lambda_t}_t(\varphi_t, \Phi^s_t, \rho^s_t) := \int_{\mathbb{R}^3} \left(L^s_t + \lambda_t(\triangle\Phi^s_t - 4\pi G\rho^s_t)\right)(x)\,\mathrm{dV}(x)$$

for $t \in I$. Here, the time-dependence of the Lagrange multiplier $\lambda \in L^2(\mathbb{R}^3)$ is indicated by λ_t. As the proof shows, the equation $\triangle\lambda = \frac{1}{2}\,\rho^s$ in $\mathcal{D}'(\mathbb{R}^3)$ is a consequence of the stationarity of the modified action integral $\mathcal{I}^\lambda$ with respect to variations solely in $\widetilde{\Phi}^s$. The Fréchet derivatives of $\mathcal{I}^{\lambda_t}$ and of $\mathscr{A}^{\lambda_t}_t$ coincide (although the functionals act on different spaces). This suggests to consider the equation

$$\triangle\lambda_t = \frac{1}{2}\,\rho^s_t \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3) \tag{5.7}$$

as a necessary condition for the stationarity of the original action integral $\mathscr{A}$ constrained by Poisson's equation $\triangle\Phi^s_t = 4\pi G\rho^s_t$. The next lemma shows that we can solve Equation (5.7) in the regularity setting of the basic variational model (Assumption 1), which allows to identify the Lagrange multiplier λ as a multiple of the gravitational potential:

Lemma 5.2 (Lagrange multiplier). *If $\rho^s_t \in L^\infty_c(\mathbb{R}^3)$, then the unique solution to $\triangle\lambda_t = \frac{1}{2}\rho^s_t$ (5.7) in $Y(\mathbb{R}^3)$ is given by*

$$\lambda_t = \frac{1}{2}\,\rho^s_t * E_3 = \frac{1}{8\pi G}\,\Phi^s_t.$$

Proof. Since $\rho^s_t \in L^\infty_c(\mathbb{R}^3)$ and $\triangle\lambda_t = \frac{1}{2}\,\rho^s_t$ holds by construction, we have $\lambda_t \in Y(\mathbb{R}^3)$ (even in $Y^\infty(\mathbb{R}^3)$). It only remains to observe that $y \in Y(\mathbb{R}^3)$ and $\triangle y = 0$ implies $y = 0$ (by analyticity of y and the decay condition in $Y(\mathbb{R}^3)$). Thus, $\lambda_t = \frac{1}{2}\,\rho^s_t * E_3$. The second equality follows by comparison with $\Phi^s_t = 4\pi G\,\rho^s_t * E_3$. □

Hence, for deriving the equations governing the state of our earth model, it is sufficient to study the unconstrained stationarity of the action integral

$$\mathscr{A}'(\varphi, \Phi^s, \rho^s) := \int_I \mathscr{A}'_t(\varphi_t, \Phi^s_t, \rho^s_t)\,\mathrm{d}t$$

with, still omitting $\mathscr{A}_{\Sigma^{\mathrm{FS}}}(\varphi)$,

$$\mathscr{A}'_t(\varphi_t, \Phi^s_t, \rho^s_t) := \mathscr{A}^{\Phi^s_t/(8\pi G)}_t(\varphi_t, \Phi^s_t, \rho^s_t) = \int_{\mathbb{R}^3} \Big(L^s_t + \frac{1}{8\pi G}\Phi^s_t\triangle\Phi^s_t - \frac{1}{2}\,\Phi^s_t\rho^s_t\Big)(x)\,\mathrm{dV}(x).$$

We now rewrite the term involving the Laplacian via integration by parts. Since $\Phi_t^s \in Y^\infty(\mathbb{R}^3)$, Lemma 4.28 yields $\Phi_t^s \in L^\infty(\mathbb{R}^3)$, $\Phi_t^s(x) = O(1/|x|)$ as $|x| \to \infty$, $\partial_{x_l}^2 \Phi_t^s \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and $\partial_{x_l}^2 \Phi_t^s(x) = O(1/|x|^3)$ as $|x| \to \infty$, hence $\Phi_t^s\, \partial_{x_l}^2 \Phi_t^s \in L^1(\mathbb{R}^3)$. Therefore, when studying the term

$$\int_{\mathbb{R}^3} (\Phi_t^s \triangle \Phi_t^s)(x)\, \mathrm{dV}(x) = \sum_{l=1}^{3} \int_{\mathbb{R}^3} (\Phi_t^s\, \partial_{x_l}^2 \Phi_t^s)(x)\, \mathrm{dV}(x),$$

by Fubini's theorem, it is sufficient to consider the one-dimensional integrals $\int_{\mathbb{R}} (\Phi_t^s\, \partial_{x_l}^2 \Phi_t^s)(x)\, \mathrm{d}x_l$ for $1 \le l \le 3$. By Lemma 4.28, Φ_t^s and $\partial_{x_l} \Phi_t^s$ belong to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$, which consists of functions with a representative that is absolutely continuous on (bounded intervals of) almost all lines parallel to the coordinate axes [Zie89, Theorem 2.1.4, p. 44]. Consequently, integration by parts may be applied [Fol99, Theorem 3.35, p. 106]. Together with the decay conditions of Φ_t^s and $\partial_{x_l} \Phi_t^s$, this leads to

$$\int_{\mathbb{R}} (\Phi_t^s\, \partial_{x_l}^2 \Phi_t^s)(x)\, \mathrm{d}x_l = -\int_{\mathbb{R}} (\partial_{x_l} \Phi_t^s)^2(x)\, \mathrm{d}x_l$$

for $1 \le l \le 3$. Here, $(\partial_{x_l} \Phi_t^s)^2 = (\nabla \Phi^s)^2$ also is in $L^1(\mathbb{R}^3)$ and, again by Fubini, we finally obtain

$$\mathscr{A}_t'(\varphi_t, \Phi_t^s, \rho_t^s) = \int_{\mathbb{R}^3} \left(L_t^s - \frac{1}{2}\, \Phi_t^s \rho_t^s - \frac{1}{8\pi G} (\nabla \Phi_t^s)^2 \right)(x)\, \mathrm{dV}(x).$$

Thus, integrating over the time interval gives the modified action for unconstrained stationarity,

$$\mathscr{A}'(\varphi, \Phi^s, \rho^s) = \int_I \int_{\mathbb{R}^3} \left(\left(\frac{1}{2}\, (v^s)^2 + v^s \cdot (\Omega \times x) - (U^s + \Phi^s + \Psi^s + F^s) \right) \rho^s - \frac{(\nabla \Phi^s)^2}{8\pi G} \right) \mathrm{dV}\mathrm{d}t, \tag{5.8}$$

where the surface contribution $\mathscr{A}_{\Sigma^{\mathrm{FS}}}(\varphi)$ still has to be added.

5.2.4 Incorporating conservation of mass in the material formulation

We use conservation of mass and formulate $\mathscr{A}'$ as an integral over material quantities. This step is crucial, as the material description is the natural one for the variational formulation of field equations based on Hamilton's principle. Moreover, this substitution will eliminate the dependence of the action integral on the variation of density with respect to time.

By Assumption 1 (ii), $\mathrm{supp}(\rho_t^s) \subseteq \varphi_t(\overline{B})$, which implies that $\mathscr{A}_t'$ for $t \in I$ can be decomposed as follows:

$$\begin{aligned} \mathscr{A}_t'(\varphi_t, \Phi_t^s, \rho_t^s) = \int_{\varphi_t(B)} \Big(\frac{1}{2}\, (v^s)^2 + v^s \cdot (\Omega \times x) - (U^s + \Phi^s + \Psi^s + F^s) \Big)(x,t) \rho^s(x,t)\, \mathrm{dV}(x) \\ - \frac{1}{8\pi G} \int_{\mathbb{R}^3} (\nabla \Phi^s)^2(x,t)\, \mathrm{dV}(x) + \mathscr{A}_{\Sigma^{\mathrm{FS}},t}(\varphi). \end{aligned}$$

In the first integral we substitute $x = \varphi(X,t)$ and make the transitions from the spatial to the material representations $v = \dot{\varphi}$, U, Φ, Ψ, and ρ according to the general rule $q_t = q_t^s \circ \varphi_t$ (1.27). By Assumption 1 and Lemma 4.20, the material quantities U, ρ, and Φ are functions in $\mathcal{C}^0(I, L^\infty(B))$ (in fact, $\varphi \in \mathcal{A}(\overline{B} \times I)$ and $\rho^0 \in L^\infty(B)$ give $\rho \in \mathrm{Lip}(I, L^\infty(B))$ by Lemma 4.25).

The assumption of conservation of mass allows us to apply the relation (Lemma 4.24)

$$\int_{\varphi_t(B)} (h_t^s \rho_t^s)(x)\, \mathrm{dV}(x) = \int_B (h_t \rho^0)(X)\, \mathrm{dV}(X)$$

to the function

$$h^s = \frac{1}{2}\, (v^s)^2 + v^s \cdot (\Omega \times x) - (U^s + \Phi^s + \Psi^s + F^s)$$

and thereby reduce the dependence on ρ^s to one on the initial density ρ^0. In particular, the requirement $\rho^s \in W_{\text{density}} = \{\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3)) : \operatorname{supp}(\rho_t^s) \subseteq \varphi_t(\overline{B})\, \forall\, t \in I\}$ in Assumption 1 (ii) can be replaced by the condition $\rho^0 \in L^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(\rho^0) \subseteq \overline{B}$.

Since $\operatorname{supp}(\rho^0) \subseteq \overline{B}$, we may extend the domain of integration to $\mathbb{R}^3$, which allows us to combine both integrals to one (upon renaming the variable in the second integral). To summarize, we may rewrite the action $\mathscr{A}'(\varphi, \Phi^s, \rho^s)$ as $\mathscr{A}''(\varphi, \Phi^s)$, where

$$\mathscr{A}'' \colon W_{\text{motion}} \times W_{\text{gravity}} \to \mathbb{R}$$

is given by

$$\mathscr{A}''(\varphi, \Phi^s) = \int_I \Big(\underbrace{\int_{\mathbb{R}^3} L''(X,t)\, \mathrm{dV}(X)}_{=:\mathscr{A}''_{\text{vol}}(t)} + \underbrace{\int_{\Sigma^{\text{FS}}} L''_{\Sigma^{\text{FS}}}(X,t)\, \mathrm{dS}(X)}_{=:\mathscr{A}''_{\Sigma^{\text{FS}}}(t)} \Big) \mathrm{d}t. \tag{5.9}$$

The material volume Lagrangian density reads

$$L''(X,t) := \Big(\frac{1}{2}\dot{\varphi}^2 + \dot{\varphi} \cdot (\Omega \times \varphi) - (U + \Phi + \Psi + F) \Big)(X,t) \rho^0(X) - \frac{1}{8\pi G} (\nabla \Phi^s)^2(X,t). \tag{5.10}$$

We also announce the explicit form of the material surface Lagrangian density $L_{\Sigma^{\text{FS}}} = L''_{\Sigma^{\text{FS}}}$ (which will be obtained from the energy balance in the next section):

$$L''_{\Sigma^{\text{FS}}}(X,t) = -\int_{t_0}^{t} [\dot{\varphi}_{t'}(X) \cdot T^{\text{PK}}_{t'}(X)]^+_- \cdot \nu(X)\, \mathrm{d}t', \tag{5.11}$$

therefore the surface action $\mathscr{A}_{\Sigma^{\text{FS}}}(\varphi) = \mathscr{A}''_{\Sigma^{\text{FS}}}(\varphi) = \int_I \mathscr{A}''_{\Sigma^{\text{FS}}}(t) \mathrm{d}t$ reads

$$\mathscr{A}''_{\Sigma^{\text{FS}}}(t) = -\int_{t_0}^{t} \int_{\Sigma^{\text{FS}}} [\dot{\varphi}_{t'} \cdot T^{\text{PK}}_{t'}]^+_- \cdot \nu\, \mathrm{dS}\, \mathrm{d}t'. \tag{5.12}$$

We note that if Σ^{FS} is a Lip-surface, if $\dot{\varphi}_t \in H^1(B^{\text{FS}})^3$ (which by (4.43) is true for admissible motions), and if $T^{\text{PK}}_t \in H_{\text{div}}(B^{\text{FS}})^{3\times 3}$ (which is a necessary condition if the nonlinear equation of motion (2.10) holds in $L^2(B^{\text{FS}})$ a.e.), then $\mathscr{A}''_{\Sigma^{\text{FS}}}$ can be interpreted as the time integrated jump of the surface Sobolev duality (3.32) between $T^{\text{PK}} \cdot \nu$ in $H^{-\frac{1}{2}}(\Sigma^{\text{FS}})$ and $\dot{\varphi}$ in $H^{\frac{1}{2}}(\Sigma^{\text{FS}})$.

5.2.5 Surface action from energy balance

The specific form of the surface action is motivated by energy considerations. In general, the volume energy density E of a conservative dynamical system with state variable $(x,t) \mapsto q(x,t)$ is the **Legendre transform** of the volume Lagrangian density L:

$$L \mapsto E := \dot{q}\, \frac{\partial L}{\partial \dot{q}} - L. \tag{5.13}$$

In case of the variational earth model (action (5.9) and Assumption 1), the state variables are $q = (\varphi, \Phi^s)$ and the Lagrangian $L = L''$ is given by (5.10):

$$L'' \colon \mathbb{R}^3 \times I \to \mathbb{R}, \qquad L'' = \Big(\frac{1}{2}\dot{\varphi}^2 + \dot{\varphi} \cdot (\Omega \times \varphi) - (U + \Phi + \Psi + F) \Big) \rho^0 - \frac{1}{8\pi G} (\nabla \Phi^s)^2.$$

Application of the Legendre transform (5.13) yields the corresponding **volume energy density**

$$E'' = \dot{\varphi} \frac{\partial L''}{\partial \dot{\varphi}} + \dot{\Phi}^s \frac{\partial L''}{\partial \dot{\Phi}^s} - L'',$$

that is,

$$E'' \colon \mathbb{R}^3 \to \mathbb{R}, \qquad E'' = \Big(\frac{1}{2}\,\dot\varphi^2 + U + \Phi + \Psi + F\Big)\rho^0 + \frac{1}{8\pi G}(\nabla\Phi^s)^2. \tag{5.14}$$

We identify the material kinetic and (elastic, gravitational, external force) potential energy densities (the corresponding spatial formulation was presented in Section 5.2.2):

$$E'' = E''_{\mathrm{kin}} + E''_{\mathrm{pot}} \qquad \text{with} \qquad E''_{\mathrm{pot}} = E_{\mathrm{elast}} + E''_{\mathrm{gravity}} + E_{\mathrm{ext}},$$

where

$$E''_{\mathrm{kin}} = \frac{1}{2}\rho^0\dot\varphi^2, \qquad E_{\mathrm{elast}} = \rho^0 U, \qquad E''_{\mathrm{gravity}} = \rho^0(\Phi{+}\Psi) + \frac{1}{8\pi G}(\nabla\Phi^s)^2, \qquad \text{and} \qquad E_{\mathrm{ext}} = \rho^0 F.$$

As was demonstrated in Section 5.2.3, self-gravitation (Poisson's equation) is incorporated into the variational model upon replacing the gravitational potential energy $E_{\mathrm{gravity}} = \frac{1}{2}\rho^0\Phi$ by

$$E''_{\mathrm{gravity}} = \frac{1}{2}\rho^0\Phi + \frac{1}{8\pi G}(\nabla\Phi^s)^2.$$

The centrifugal potential $\Psi = \Psi^s \circ \varphi = -\frac{1}{2}(\Omega \times \varphi)^2$ actually is a part of the kinetic energy in the inertial frame:

$$E_{\mathrm{kin}} = \frac{1}{2}\rho^0(\dot\varphi + \Omega \times \varphi)^2.$$

However, in the rotating frame, it is natural to add Ψ to the gravitational potential ($\Phi + \Psi$ is the geopotential, introduced in Section 2.2) and reduce the kinetic energy to inertia, $E''_{\mathrm{kin}} = \frac{1}{2}\rho^0\dot\varphi^2$.

It follows from Noether's theorem that if the Lagrangian density L does not explicitly depend on time, the volume energy density E (related to L via the Legendre transform) satisfies the **local energy balance equation** [MH83, p. 283]

$$\dot E + \operatorname{div}\left(\dot q \cdot \frac{\partial L}{\partial(\nabla q)}\right) = 0.$$

In case of the variational earth model, the energy balance equation for E'' (5.14) reads

$$\dot E'' + \operatorname{div}\Big(\dot\varphi \cdot \frac{\partial L''}{\partial(\nabla\varphi)} + \dot\Phi^s \frac{\partial L''}{\partial(\nabla\Phi^s)}\Big) = 0,$$

that is,

$$\dot E'' - \operatorname{div}\Big(\dot\varphi \cdot T^{\mathrm{PK}} + \frac{1}{4\pi G}\,\dot\Phi^s\,\nabla\Phi^s\Big) = 0. \tag{5.15}$$

We integrate over B^{FS} and apply the divergence theorem for composite domains (Lemma 4.11). The regularity properties of the gravitational potential, $\Phi^s \in W_{\mathrm{gravity}} \subseteq \mathcal{C}^0(I, \mathcal{C}^1(\mathbb{R}^3))$, guarantee that gravity does not cause any jump terms. Thus we obtain the **integral energy balance equation**, expressing **conservation of energy** (cf. [MH83, p. 143; without Σ^{FS}-term] or [DT98, (3.201), p. 91; linearized setting]):

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3} E''_t \,\mathrm{dV}\right) = \int_{\partial B} \dot\varphi \cdot T^{\mathrm{PK}} \cdot \nu \,\mathrm{dS} - \int_{\Sigma^{\mathrm{FS}}} [\dot\varphi \cdot T^{\mathrm{PK}}]^+_- \cdot \nu \,\mathrm{dS}. \tag{5.16}$$

The left-hand side is the time derivative at time t of the total volume energy density. The right-hand side collects all boundary and interface energies. The equation has the same structure as the abstract formulation (3.54). With E'' given by (5.14) and $\operatorname{supp}(\rho^0) \subseteq \overline{B}$, the integrated energy balance equation reads

$$\begin{aligned}\frac{d}{dt}\left(\int_B \Big(\frac{1}{2}\,\dot\varphi^2 + U + \Phi + \Psi + F\Big)\rho^0 \mathrm{dV} + \int_{\mathbb{R}^3} \frac{1}{8\pi G}(\nabla\Phi^s)^2 \mathrm{dV}\right) \\ = \int_{\partial B} \dot\varphi \cdot T^{\mathrm{PK}} \cdot \nu \,\mathrm{dS} - \int_{\Sigma^{\mathrm{FS}}} [\dot\varphi \cdot T^{\mathrm{PK}}]^+_- \cdot \nu \,\mathrm{dS}.\end{aligned} \tag{5.17}$$

By the zero traction natural boundary condition (5.36) at the free surface, the integral over the exterior boundary ∂B vanishes. Rewriting (5.16) as

$$\frac{d}{dt}\left(\int_B E''_t \,\mathrm{dV} + \int_{t_0}^{t}\int_{\Sigma^{\mathrm{FS}}} [\dot{\varphi}_{t'} \cdot T^{\mathrm{PK}}_{t'}]^+_- \cdot \nu \,\mathrm{dS}\,\mathrm{d}t'\right) = 0$$

upon interchanging time and surface integration, energy conservation suggests to introduce a **surface energy density** defined by

$$E''_{\Sigma^{\mathrm{FS}}}(X,t) := \int_{t_0}^{t} \left[\dot{\varphi}_{t'}(X) \cdot T^{\mathrm{PK}}_{t'}(X)\right]^+_- \mathrm{d}t' \cdot \nu(X) \tag{5.18}$$

as an additional part of the **total energy** of the composite fluid-solid earth model:

$$\mathscr{E}'' = \int_I \left(\int_{\mathbb{R}^3} E''\mathrm{dV} + \int_{\Sigma^{\mathrm{FS}}} E''_{\Sigma^{\mathrm{FS}}}\mathrm{dS}\right)\mathrm{d}t. \tag{5.19}$$

The instantaneous surface energy

$$\mathscr{E}''_{\Sigma^{\mathrm{FS}}}(t) := \int_{\Sigma^{\mathrm{FS}}} E''_{\Sigma^{\mathrm{FS}}}(.,t)\mathrm{dS} \tag{5.20}$$

corresponds to the work done by slip against the traction at fluid-solid boundaries Σ^{FS} within the time interval $[t_0,t] \subseteq I$. This result is consistent with the interpretation of [DT98, p. 96] in the linearized setting. The complete Lagrangian density of the earth model is given as kinetic minus potential energy density (see Section 5.2.1). Since we do not consider dissipative forces, the surface energy density obtained above can be viewed as a part of the potential energy (it clearly does not have the form of a kinetic energy). Therefore, as was announced in (5.11), the complete Lagrangian density has to include an additional surface density given by

$$L''_{\Sigma^{\mathrm{FS}}} = -E''_{\Sigma^{\mathrm{FS}}}. \tag{5.21}$$

However, as will be discussed in Section 6.6.3, the surface contribution vanishes in case of frictionless slip along Σ^{FS}.

Remark 5.3 (**Validity of the Legendre transform for surface densities**)**.** If a conservative dynamical system also involves a Lagrangian density L_S that has the form of a normal jump on a discontinuity surface S that only allows for tangential slip, its Legendre transform is given by $L_S \mapsto E_S := -L_S$. This form of the surface Legendre transform is clear from the usual definition $L \mapsto E := \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L$ (5.13), if $L = L_S$ does not depend on $\dot{q}$ (which is true at least in the linear case, cf. the second-order surface Lagrangian density (6.54) or see [DT98, p. 90]). In contrast, we note that (5.11) depends linearly on $\dot{\varphi}$, making the validity of (5.21) questionable at first glance. However, as is discussed in a nonsmooth geometric setting in [FMW03, Lemma 3.1], in case of pure tangential slip at the discontinuity surface S, the elements in the tangent space of the configuration q have zero normal jump across S, see also (5.30). Consequently, the derivative of the surface Lagrangian density with respect to $\dot{q}$ vanishes and the Legendre transform (5.13) reduces to $E_S = -L_S$ when applied to a surface density. ⌟

5.3 Interlude: Geometric variational formulation

As an interlude, we briefly review some results by [MPS98, FMW03] on a geometric variational approach to continuum mechanics, enabling us to infer also the kinematic interface conditions from stationarity of the action.

5.3.1 Geometric formulation of continuum mechanics

The basic field variable in continuum mechanics in the motion, $\varphi\colon X = B \times I \to \mathbb{R}^n = S$. In order to solve the governing equations, one singles out particular elements of the infinite dimensional manifold of all admissible fields

$$W := \{\varphi\colon X \to S\},$$

e.g. by requiring stationarity of an action integral $\mathscr{A}\colon W \to \mathbb{R}$. Alternatively, in the so-called **multisymplectic** (**covariant**) formulation of continuum mechanics [MH83, AMR01], one considers the finite dimensional **configuration bundle**

$$\pi\colon Y \to X \quad \text{with typical fiber } S \tag{5.22}$$

and describes fields via **sections**

$$\phi\colon X \to Y, \quad \pi \circ \phi = \mathrm{Id}_X. \tag{5.23}$$

Here, a **fiber bundle** $\pi\colon Y \to X$ (that is, the bundle Y over X) with typical fiber S consists of manifolds X, Y, S (X base, Y bundle, S fiber) and a surjective map π (projection) such that $\pi^{-1}(x) = S$ for all $x \in X$ and locally $Y = X \times S$. A special case is a **vector bundle**, where X is a submanifold of Y and S is a vector space. The most prominent example is the **tangent bundle** $TM = \bigcup_{m\in M} T_m M$ of a manifold M. If M is n-dimensional, then locally $TM = M \times \mathbb{R}^n$.

We have the following correspondence with the conventional setting of continuum mechanics:

$$\phi\colon X = B \times I \;\to\; (B \times I) \times \mathbb{R}^n = Y, \quad \phi(x,t) = ((x,t), \varphi(x,t))\,.$$

It is straightforward to incorporate additional physical fields in the geometric approach.

Let the manifold X in the bundle (5.22) have dimension $\dim(X) = n + 1$, equipped with coordinates $\{x^\mu\}_{\mu=0,\dots,n}$, and let $\dim(S) = N$ with fiber coordinates $\{y^A\}_{A=1,\dots,N}$ $(n, N \in \mathbb{N})$. Essentially, x^μ can be viewed as space-time coordinates and y^A may be interpreted as the vector components of the fields.

In the geometric variational formulation [MH83, AMR01] one defines the **action** integral as the following functional of sections (5.23)

$$\mathscr{A}(\phi) = \int_X \mathscr{L}(j^1\phi), \tag{5.24}$$

where the **Lagrangian density** is the bundle map

$$\mathscr{L}\colon J^1Y \to \Lambda^{n+1}(X), \qquad \mathscr{L} = L((x^\mu), (y^A), (v^A_\mu))\,\mathrm{d}^{n+1}x.$$

Here $\mathrm{d}^{n+1}x = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^n$ denotes the volume element on X, $\Lambda^{n+1}(X)$ is the set of all $(n+1)$-forms on X, and J^1Y denotes the **first jet bundle** of the bundle Y, which is the bundle analog of the tangent bundle of a manifold. It is defined as the affine bundle over Y whose fiber over $y \in \pi^{-1}(x)$ for $x \in X$ consists of linear maps $\gamma\colon T_xX \to T_yY$ such that $T\pi \circ \gamma = \mathrm{Id}_{TX}$. In coordinates, the first jet $j^1\phi \in J^1Y$ for a section ϕ is given by

$$j^1\phi = ((x^\mu), (\phi^A), (\partial_\mu \phi^A)).$$

In this definition, sections ϕ and the bundle map $\mathscr{L}$ are assumed to be smooth (at least $\mathcal{C}^1$).

Remark 5.4 (**Integration in geometric formulation**). Let (M, g) be an n-dimensional Riemannian manifold with boundary ∂M, embedded as a submanifold via $\iota_{\partial M}\colon \partial M \hookrightarrow M$. The volume measure in M is the Riemannian volume form, given by $\mathrm{dV} = \sqrt{\det g}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ in coordinates. The surface measure on ∂M is the volume form of the induced Riemannian metric $\tilde{g} = \iota_{\partial M}^*(g)$ on ∂M. By [Lee13, Corollary 15.34], we have $\mathrm{dS} = \iota_{\partial M}^*(\nu \lrcorner \mathrm{dV})$, where ν is the outward unit normal vector field on ∂M and $\nu \lrcorner \mathrm{dV}$ denotes the contraction, that is, $\nu \lrcorner \mathrm{dV}(Y_1, \ldots, Y_{n-1}) = \mathrm{dV}(\nu, Y_1, \ldots, Y_{n-1})$. Existence and uniqueness of ν is guaranteed by [Lee13, Propositions 15.33], where the construction of ν is based on a function $f\colon M \to \mathbb{R}$ with $\mathrm{d}f \neq 0$ on ∂M, defining the boundary as the level set $\partial M = f^{-1}(0)$. The result is analogous to (1.15), namely $\nu = -\nabla f / \sqrt{g(\nabla f, \nabla f)}$, but with ∇ denoting the metric gradient (cf. [Lee13, Propositions 5.43]). ⌟

5.3.2 Vertical and horizontal variations

In the following, we will investigate the stationarity of $\mathscr{A}$ under **general variations**. These do not only consist of fiber-preserving variations, called **vertical variations**, but also of base-space variations, called **horizontal variations**. The terminology vertical/horizontal is found e.g. in [MPS98]. Other names for these two types of variations are outer/inner variations [GH96], or simultaneous/nonsimultaneous (generalized) variations [VA04, (3.2.4), p. 133].

In order to define general variations, we follow [MPS98] and introduce another smooth manifold U and generalize the space of smooth sections

$$\{\phi\colon X \to Y : \ \phi \ \text{ is } \ \mathcal{C}^\infty,\ \pi \circ \phi = \mathrm{Id}_X\}$$

to the **configuration space**

$$\mathcal{C} := \{\widetilde{\phi}\colon U \to Y : \ \widetilde{\phi} \ \text{ is } \ \mathcal{C}^\infty,\ \pi \circ \widetilde{\phi}\colon U \to X \ \text{ is an embedding}\}. \tag{5.25}$$

If $\widetilde{\phi} \in \mathcal{C}$, then $\pi \circ \widetilde{\phi}\colon U \to (\pi \circ \widetilde{\phi})(U) \subseteq X$ is a diffeomorphism. With the section ϕ defined by $\phi := \widetilde{\phi} \circ (\pi \circ \widetilde{\phi})^{-1}\colon (\pi \circ \widetilde{\phi})(U) \to Y$, the diagram

$$\begin{array}{ccc} & & Y \\ & {\scriptstyle\widetilde{\phi}}\nearrow & \downarrow{\scriptstyle\pi}\uparrow{\scriptstyle\phi} \\ U & \longrightarrow & X \\ & {\scriptstyle\pi\circ\widetilde{\phi}} & \end{array}$$

is commutative. We redefine the action integral (5.24) as a functional on $\mathcal{C}$:

$$\widetilde{\mathscr{A}}(\widetilde{\phi}) := \mathscr{A}(\phi) = \int_{(\pi\circ\widetilde{\phi})(U)} \mathscr{L}\left(j^1\left(\widetilde{\phi} \circ (\pi \circ \widetilde{\phi})^{-1}\right)\right). \tag{5.26}$$

To define variations of $\widetilde{\phi} \in \mathcal{C}$, [MPS98] consider a Lie group $\mathcal{G}$ with unit e acting on the bundle Y. Let the corresponding representation be given by $\eta\colon \mathcal{G} \to \mathrm{Diff}(Y)$ with $\eta(e) = \mathrm{Id}_Y$, covering $\chi\colon \mathcal{G} \to \mathrm{Diff}(X)$ with $\chi(e) = \mathrm{Id}_X$ (i.e. $\pi \circ \eta(g) = \chi(g) \circ \pi$ for all $g \in \mathcal{G}$). Then a path $g\colon (-1, 1) \to \mathcal{G}$, $\lambda \mapsto g(\lambda)$ with $g(0) = e$ in $\mathcal{G}$ induces a **general variation** $\widetilde{\phi}_\lambda$ of $\widetilde{\phi} \in \mathcal{C}$ via

$$\widetilde{\phi}_\lambda\colon (-1, 1) \to \mathcal{C}, \quad \lambda \mapsto \widetilde{\phi}_\lambda := \eta(g(\lambda)) \circ \widetilde{\phi}.$$

The diagram

$$\begin{array}{ccccccc} & & & {\scriptstyle\eta(g(\lambda))} & & & \\ & & Y & \longrightarrow & Y & \longrightarrow & TY \\ & {\scriptstyle\widetilde{\phi}}\nearrow & \downarrow{\scriptstyle\pi}\uparrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle T\pi}\uparrow{\scriptstyle T\phi} \\ U & \longrightarrow & X & \longrightarrow & X & \longrightarrow & TX \\ & {\scriptstyle\pi\circ\widetilde{\phi}} & & {\scriptstyle\chi(g(\lambda))} & & & \end{array}$$

illustrates these definitions. A **stationary point** of $\widetilde{\mathscr{A}}$ is any section $\widetilde{\phi} \in \mathcal{C}$ such that

$$(\mathrm{d}\widetilde{\mathscr{A}} \cdot V)(\widetilde{\phi}) := \left.\frac{d}{d\lambda}\right|_{\lambda=0} \widetilde{\mathscr{A}}(\widetilde{\phi}_\lambda) = 0,$$

where the **general variation vector** $V \in T_{\widetilde{\phi}}\mathcal{C}$ is defined by

$$V := \left.\frac{d}{d\lambda}\right|_{\lambda=0} \widetilde{\phi}_\lambda = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \eta(g(\lambda)) \circ \widetilde{\phi} : U \to TY. \tag{5.27}$$

With the projection V_X of V on TX, given by

$$V_X := T\pi \circ V = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \pi \circ \eta(g(\lambda)) \circ \widetilde{\phi} = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \chi(g(\lambda)) \circ \pi \circ \widetilde{\phi} : U \to TX, \tag{5.28}$$

the general variation vector V can be decomposed into the **horizontal variation vector** V^{h} and the **vertical variation vector** V^{v}:

$$V^{\mathrm{h}} := T\phi \circ V_X = T\left(\widetilde{\phi} \circ (\pi \circ \widetilde{\phi})^{-1}\right) \circ V_X \qquad \text{and} \qquad V^{\mathrm{v}} := V - V^{\mathrm{h}}.$$

As is shown in [MPS98],

$$(\mathrm{d}\widetilde{\mathscr{A}} \cdot V)(\widetilde{\phi}) = \underbrace{\int_{(\pi\circ\widetilde{\phi})(U)} \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathscr{L}\left(j^1\phi_\lambda\right)}_{=(\mathrm{d}\widetilde{\mathscr{A}}\cdot V^{\mathrm{v}})(\widetilde{\phi})} + \underbrace{\int_{(\pi\circ\widetilde{\phi})(U)} \mathcal{L}_{V_X}\left(\mathscr{L}\left(j^1\phi\right)\right)}_{=(\mathrm{d}\widetilde{\mathscr{A}}\cdot V^{\mathrm{h}})(\widetilde{\phi})}.$$

Here $\mathcal{L}_{V_X}$ denotes the Lie-derivative (e.g. [MH83, Definition 6.8, p. 96]) with respect to V_X (5.28) and $\phi_\lambda := \widetilde{\phi}_\lambda \circ (\pi \circ \widetilde{\phi}_\lambda)^{-1}$. The proof is based on the transport theorem for differential forms (see Remark 5.5).

Remark 5.5 (**Transport theorem for differential forms**)**.** A differential form ω dependent on a parameter t satisfies the following transport theorem, see [Fla73]:

$$\frac{d}{dt}\int_{D_t} \omega = \int_{D_t} (\dot{\omega} + \mathcal{L}_v\,\omega) = \int_{D_t} (\dot{\omega} + v \lrcorner\, \mathrm{d}\omega) + \int_{\partial D_t} v \lrcorner\, \omega.$$

Here $D_t = \varphi(D,t)$ is the image of an initial domain D under the "motion" φ at "time" t and v denotes the corresponding "spatial velocity" (defined on D_t by $v(x,t) := \dot{\varphi}((\varphi(.,t))^{-1}(x),t)$ with $\dot{\varphi} := \partial_t\varphi$). The second equality follows from Cartan's identity for the Lie derivative,

$$\mathcal{L}_v\,\omega = v \lrcorner\, \mathrm{d}\omega + \mathrm{d}(v \lrcorner\, \omega),$$

together with Stokes' theorem, $\int_A \mathrm{d}\alpha = \int_{\partial A} \alpha$. If $\omega = \mathrm{d}V$, then $\mathcal{L}_v\,\omega = \mathcal{L}_v\,\mathrm{d}V = (\operatorname{div} v)\,\mathrm{d}V$ and the classical Reynold's transport theorem (given in Remark 1.8) is recovered. ⌟

Stationarity of $\widetilde{\mathscr{A}}$ with respect to vertical variations yields the classical EL (3.4) and NBC (3.5)

$$\mathrm{d}\widetilde{\mathscr{A}} \cdot V^{\mathrm{v}} = 0 \quad \Longrightarrow \quad \partial_{y^A} L - \partial_{x^\mu}(\partial_{v^A_\mu} L) = 0 \quad \text{and} \quad (\partial_{v^A_\mu} L) N_\mu = 0,$$

whereas stationarity with respect to horizontal variations results in an additional boundary term. In the presence of interfaces, which will be discussed next, this additional term will correspond to the kinematical jump conditions.

5.3.3 Discontinuity surfaces

In [FMW03] the variational formulation of continuum mechanics is extended to the nonsmooth case. The space-time **singularity surface** (e.g. $\Sigma \times I$ for a material discontinuity) is modeled by a codimension-one submanifold D without boundary,

$$D \subseteq U \qquad \text{such that} \qquad U^\circ = U^+ \cup U^- \cup D$$

is a disjoint union, with open sets $U^\pm \subseteq U$. The **nonsmooth configuration space** generalizing (5.25) to possible tangential slip along D (without interpenetration or cavitation) is given by

$$\begin{aligned} \mathcal{C}_{\text{slip}} := \Big\{ &\widetilde{\phi}\colon U \to Y : \ \pi \circ \widetilde{\phi}\colon U \to X \ \text{ is an embedding,} \\ &\widetilde{\phi}\big|_{U^+\cup U^-} \text{ is } \mathcal{C}^2, \ \text{and } \overline{\widetilde{\phi}\big|_{U^+}}(D) = \overline{\widetilde{\phi}\big|_{U^-}}(D) = \widetilde{\phi}(D) \Big\}, \end{aligned} \tag{5.29}$$

where $\overline{\widetilde{\phi}\big|_A}$ denotes the continuous extension of $\widetilde{\phi}\big|_A$ to the closure $\overline{A}$. Then [FMW03, Lemma 3.1] show that a corresponding **nonsmooth general variation vector**

$$V = ((V^\mu), (V^A)) \in T_{\widetilde{\phi}}\mathcal{C}_{\text{slip}}$$

satisfies the continuity condition

$$\big[V^A\big]^+_- N_A = 0 \quad \text{on} \quad D, \tag{5.30}$$

with N the unit normal of D. By taking into account the singularity manifold D in the calculation of [MPS98] (neglecting the contribution of the boundary of U), [FMW03, (3.31)] arrive at the following coordinate expression for the first variation of $\widetilde{\mathscr{A}}$ at $\widetilde{\phi} \in \mathcal{C}_{\text{slip}}$ in direction $V \in T_{\widetilde{\phi}}\mathcal{C}_{\text{slip}}$:

$$\begin{aligned} (\mathrm{d}\widetilde{\mathscr{A}} \cdot V)(\widetilde{\phi}) = &\int_{(\pi\circ\widetilde{\phi})(U^+\cup U^-)} \Big(\partial_{y^A} L - \partial_{x^\mu}(\partial_{v^A_\mu} L)\Big) V^A \, \mathrm{d}^{n+1}x \\ + &\int_{(\pi\circ\widetilde{\phi})(D)} \Big(\Big[(\partial_{v^A_\mu} L) V^A\Big]^+_- + \Big[L V^\mu - (\partial_{v^A_\mu} L)(\partial_{x^\nu}\phi^A) V^\nu\Big]^+_- \Big) \, \mathrm{d}^n x_\mu. \end{aligned} \tag{5.31}$$

Here L and its derivatives are evaluated at $j^1\phi$ for $\phi = \widetilde{\phi} \circ (\pi \circ \widetilde{\phi})^{-1}$ as above. Moreover we employ the notation (cf. Remark 5.4)

$$\mathrm{d}^n x_\mu := \partial_{x^\mu} \lrcorner \, \mathrm{d}^{n+1}x. \tag{5.32}$$

For a spatial coordinate x^i $(i = 1, \ldots, n)$ we have

$$\mathrm{d}^n x_i = N_i \, \mathrm{d}S\mathrm{d}t,$$

where N is the unit normal of the codimension-one hypersurface $x^i = 0$, see [MH83].

Remark 5.6 (**Natural jump conditions for piecewise-smooth real functions**)**.** Formula (5.31) is the space-time analog of the variation $\Delta\mathscr{A}$ of one-dimensional action integrals

$$\mathscr{A}(q) = \int_I L(t, q(t), \dot{q}(t))\mathrm{d}t$$

with piecewise smooth functions: Let $q\colon I \to \mathbb{R}$ have a possible discontinuity at $\tau \in I$, but $q \in \mathcal{C}^1(I^- \cup I^+)$ for $I = (t_0, t_1) = I^+ \cup I^- \cup \{\tau\}$ with $I^- = (t_0, \tau)$ and $I^+ = (\tau, t_1)$. Then (under sufficient regularity of L), stationarity of $\mathscr{A}$ with respect to generalized variations (nonsimultaneous variations)

$$\Delta q = \delta q + \dot{q}\Delta t,$$

which combine vertical variations δq and horizontal variations Δt [VA04, (3.2.5), p. 133], reads

$$\Delta\mathscr{A} = \int_{I^+\cup I^-} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q\,\mathrm{d}t + \left[\frac{\partial L}{\partial \dot q}\right]_-^+ \Delta q + \left[L - \frac{\partial L}{\partial \dot q}\dot q\right]_-^+ \Delta t = 0,$$

where $[f]_-^+ = f(\tau+) - f(\tau-) = \lim_{h\to 0}(f(\tau+h) - f(\tau-h))$ is the jump of f at $\tau \in I$. The jump conditions

$$\left[\frac{\partial L}{\partial \dot q}\right]_-^+ \Delta q = 0 \quad \text{and} \quad \left[L - \frac{\partial L}{\partial \dot q}\dot q\right]_-^+ = 0$$

are known as the **Weierstraß-Erdmann corner conditions** [VA04, (6.6.4), p. 237]. ⌟

In the stationarity equations $\mathrm{d}\widetilde{\mathscr{A}}\cdot V = 0$ (5.31), the first integral and the first term in the second integral are the classical EL (3.4) and NIBC (3.6), arising from purely vertical variations $\mathrm{d}\widetilde{\mathscr{A}}\cdot V^{\mathrm{v}}$. The second term in the second integral results from purely horizontal variations $\mathrm{d}\widetilde{\mathscr{A}}\cdot V^{\mathrm{h}}$ and gives additional IBC. In particular, [FMW03] show that vertical variations imply the nonlinear equation of motion (1.1) and the Cauchy traction condition (2.16) at welded and slipping interfaces, which reads

$$[\tau^s]_-^+ = 0,$$

where $\tau^s = T^s\cdot\nu^s$. For purely temporal variation vectors ($V^j = 0$ for $j = 1,\dots,n$ but $V^0 \neq 0$), the additional IBC due to the horizontal variations precisely yield the spatial **kinematic slip condition** (4.38), expressing continuity of the normal component of the spatial velocity:

$$[v^s]_-^+\cdot\nu^s = 0.$$

Purely spatial variations ($V^j \neq 0$ for $j = 1,\dots,n$ but $V^0 = 0$) are reported to give no new result.

Remark 5.7 (**Geometric variational framework in the linearized setting**)**.** In view of the interface conditions associated to the linearized system of elastic-gravitational equations, it seems to be reasonable to apply the geometric variational framework also to the corresponding approximated action integral (6.48), which depends on the displacement u (and the incremental gravitational potential). The generalized stationarity (5.31) then implies the additional IBC $[\dot u]_-^+\cdot\nu = 0$ and the correct kinematical slip condition $[u]_-^+\cdot\nu = 0$ (2.29) is recovered after time integration. However, in case of the dynamical slip condition $[\tau^{\mathrm{PK1}}]_-^+ = 0$ (2.32), that is $[T^{\mathrm{PK1}}\cdot\nu + \nu\,\widetilde{\nabla}\cdot(p^0 u) - p^0\nu\cdot(\widetilde{\nabla}u)]_-^+ = 0$, the theory only yields the contribution related to T^{PK1}. The other terms follow from the additional surface action in the linear theory (see also the interpretation in Section 6.6.3). ⌟

5.4 Variational derivation of the nonlinear elastic-gravitational equations

We derive the nonlinear governing equations from Hamilton's principle with the action (5.9),

$$\mathscr{A}'' = \mathscr{A}''_{\mathrm{vol}} + \mathscr{A}''_{\Sigma^{\mathrm{FS}}} : W_{\mathrm{motion}}\times W_{\mathrm{gravity}} \to \mathbb{R}.$$

The derivation has the character of a formal calculation, in the sense that we assume that the differentiability of the action on suitable function spaces corresponding to $W_{\mathrm{motion}}\times W_{\mathrm{gravity}}$ has been established (see the comments at the end of Section 5.2.2).

By considering two classes of variations separately, one class with support avoiding the fluid-solid boundaries and the other class with support near these boundaries, we may consider the contributions of $\mathscr{A}''_{\mathrm{vol}}$ and $\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ independently in deriving the EL from stationarity of the total action. While the investigation of stationarity of $\mathscr{A}''_{\mathrm{vol}}$ is classical, in case of $\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ we have to use a concept of weak (or distributional) stationarity described in detail in Section 5.4.2 below.

5.4.1 Stationarity of the volume action

The nonlinear dynamical elastic-gravitational equations are the classical Euler-Lagrange equations (EL) (3.4)

$$\partial_t(\partial_{\dot y}L) + \nabla\cdot(\partial_{\nabla y}L) - \partial_y L = 0$$

for the state variable $y = (\varphi, \Phi^s)$ and the Lagrangian $L = L''$ (5.10) corresponding to the volume action $\mathscr{A}''_{\rm vol}$ in (5.9):

$$L'' = \Big(\frac{1}{2}\,\dot\varphi^2 + \dot\varphi\cdot(\Omega\times\varphi) - U - (\Phi^s + \Psi^s + F^s)\circ\varphi\Big)\,\rho^0 - \frac{1}{8\pi G}(\nabla\Phi^s)^2.$$

We have

$$\begin{aligned}
\partial_{\dot\varphi}L'' &= \rho^0\,(\dot\varphi + \Omega\times\varphi)\,,\\
\partial_{\nabla\varphi}L'' &= -\rho^0\partial_{\nabla\varphi}U = -T^{\rm PK},\\
\partial_{\varphi}L'' &= \rho^0\,(\dot\varphi\times\Omega + (\nabla(\Phi^s + \Psi^s + F^s))\circ\varphi)\,,
\end{aligned}$$

where we applied the constitutive relation (1.52) in the second equation. Variations of $\mathscr{A}''_{\rm vol}$ with respect to φ thus result in the EL

$$\begin{aligned}
&\partial_t(\partial_{\dot\varphi}L'') + \nabla\cdot(\partial_{\nabla\varphi}L'') - \partial_\varphi L''\\
&= \rho^0\,(\ddot\varphi + \Omega\times\dot\varphi) - \nabla\cdot T^{\rm PK} - \rho^0\,(\dot\varphi\times\Omega + (\nabla(\Phi^s + \Psi^s + F^s))\circ\varphi) = 0.
\end{aligned}$$

The terms $g = -(\nabla\Phi^s)\circ\varphi$ and $(\nabla\Psi^s)\circ\varphi = \Omega\times(\Omega\times\varphi)$ are the material gravitational acceleration and centrifugal acceleration respectively. By (4.50), the conservative body force is $f = -\rho^0(\nabla F^s)\circ\varphi$. Consequently, we have obtained the full nonlinear material momentum equation (2.10),

$$\rho^0\,(\ddot\varphi + 2\,\Omega\times\dot\varphi + \Omega\times(\Omega\times\varphi)) = \nabla\cdot T^{\rm PK} + \rho^0 g + f.$$

Variations of $\mathscr{A}''_{\rm vol}$ with respect to Φ^s will give Poisson's equation (2.2): $\triangle\Phi^s = 4\pi G\rho^s$. To obtain this result, we need to write the action $\mathscr{A}''$ (5.9) in **spatial representation**, i.e. as an integral over the current configuration, exclusively involving spatial fields:

$$\mathscr{A}''(\varphi,\Phi^s) = \int_I \Big(\int_{\mathbb{R}^3} L''^s(x,t)\,{\rm dV}(x) \;+\; \mathscr{A}''_{\Sigma^{\rm FS}}(t)\Big){\rm d}t. \tag{5.33}$$

The spatial volume Lagrangian density corresponding to $\mathscr{A}''$ is

$$\begin{aligned}
L''^s &= \Big(\frac{1}{2}\,(v^s + \Omega\times x)^2 - (U^s + \Phi^s + F^s)\Big)\rho^s - \frac{1}{8\pi G}(\nabla\Phi^s)^2\\
&= \Big(\frac{1}{2}\,(v^s)^2 + v^s\cdot(\Omega\times x) - (U^s + \Phi^s + \Psi^s + F^s)\Big)\rho^s - \frac{1}{8\pi G}(\nabla\Phi^s)^2,
\end{aligned} \tag{5.34}$$

which coincides with the integrand in (5.8). The surface action (5.12) is independent of Φ^s and thus does not contribute to the EL (as already mentioned in Section 5.2.3). As a result, we get

$$\partial_{\dot\Phi^s}L''^s = 0, \qquad \partial_{\nabla\Phi^s}L''^s = -\frac{1}{4\pi G}(\nabla\Phi^s), \qquad \partial_{\Phi^s}L''^s = -\rho^s$$

and the corresponding EL reduce to Poisson's equation (2.2):

$$\partial_t(\partial_{\dot\Phi^s}L''^s) + \nabla\cdot(\partial_{\nabla\Phi^s}L''^s) - \partial_{\Phi^s}L''^s = -\frac{1}{4\pi G}\triangle\Phi^s + \rho^s = 0.$$

Remark 5.8 (Spatial variational principle). The spatial equation of motion (2.11),

$$\rho^s({\rm d}_t v^s + 2\Omega\times v^s) = \nabla\cdot T^s - \rho^s\,\nabla(\Phi^s + \Psi^s) + f^s,$$

may also be derived from a completely spatial variational principle, i.e. for the state variables v^s and ρ^s. This approach is discussed in [SW68] based on Lagrange multiplier arguments, as well as in [AOS11], where also self-gravitation Φ^s is incorporated. In [GBMR12] the spatial variational principle is rigorously formulated in a geometric framework. ⌟

The external and welded solid-solid dynamical boundary conditions are consequences of stationarity as well. The NIBC (3.6) obtained from variations of $\mathscr{A}''$ with respect to Φ^s reduce to the condition

$$[\nabla\Phi^s]^+_- \cdot \nu = 0 \qquad \text{on all surfaces } S \subseteq \mathbb{R}^3, \tag{5.35}$$

see (2.18). This is equivalent to continuity of the normal component of the spatial gravitational acceleration $g^s = -\nabla\Phi^s$ and thus is no new constraint: In Section 4.5 we even found $\Phi^s_t \in \mathcal{C}^1(\mathbb{R}^3)$, implying $g^s_t \in \mathcal{C}^0(\mathbb{R}^3)^3$ for a.a. $t \in I$.

As is clear from (3.5) and (3.6), stationarity of $\mathscr{A}''_{\text{vol}}$ with respect to φ directly implies the zero-traction natural boundary condition

$$T^{\text{PK}} \cdot \nu = 0 \quad \text{on the exterior boundary } \partial B, \tag{5.36}$$

and the dynamical interface condition of continuity of normal traction

$$[T^{\text{PK}}]^+_- \cdot \nu = 0 \quad \text{on welded solid-solid interfaces } \Sigma^{\text{SS}}. \tag{5.37}$$

These boundary and interface conditions correspond to $T^s \cdot \nu^s = 0$ (2.13) and $[T^s]^+_- \cdot \nu^s = 0$ (2.16) (on Σ^{SS}) respectively. This can be seen from (1.48), which in turn is a consequence of the Piola transform (1.49) and the surface transformation formula (1.34):

$$\int_{\varphi_t(\partial B)} T^s \cdot \nu^s \mathrm{dS} = \int_{\partial B} T^{\text{PK}} \cdot \nu \,\mathrm{dS} \qquad \text{and} \qquad \int_{\varphi_t(\Sigma^{\text{SS}})} [T^s]^+_- \cdot \nu^s \mathrm{dS} = \int_{\Sigma^{\text{SS}}} [T^{\text{PK}}]^+_- \cdot \nu \,\mathrm{dS}.$$

The dynamical interface condition (2.16) on the slipping fluid-solid boundaries Σ^{FS} follow from the surface action as will be discussed next.

5.4.2 Weak stationarity of the surface action

We investigate the implications of the fluid-solid surface contribution to the action (5.9). At fixed time t, we interpret $\mathscr{A}''_{\Sigma^{\text{FS}}}(t) = \int_{\Sigma^{\text{FS}}} L''_{\Sigma^{\text{FS}}}(X,t)\,\mathrm{dS}(X)$ defined by (5.12) as the action of a distribution on $\mathbb{R}^3$, with support on a two-dimensional surface, that is, the surface integral is considered as distributional action $\langle \mathscr{A}''_{\Sigma^{\text{FS}}}(t), h\rangle$ on a smooth compactly supported test function h on $\mathbb{R}^3$ satisfying $h|_{\Sigma^{\text{FS}}} = 1$ (which bears some similarity with the concept of mass of a 2-current as in [KP08, Subsection 7.2]). With (5.11), the distributional action is given by integration over Σ^{FS} in the form

$$\begin{aligned}
\langle \mathscr{A}''_{\Sigma^{\text{FS}}}(t), h\rangle &= -\int_{\Sigma^{\text{FS}}} h(X) \left(\int_{t_0}^t [\dot\varphi_{t'}(X) \cdot T^{\text{PK}}_{t'}(X)]^+_- \,\mathrm{d}t' \right) \cdot \nu(X)\,\mathrm{dS}(X) \\
&= -\int_{t_0}^t \underbrace{\int_{\Sigma^{\text{FS}}} h(X)\, [\dot\varphi_{t'}(X) \cdot T^{\text{PK}}_{t'}(X)]^+_- \cdot \nu(X)\,\mathrm{dS}(X)}_{=:\left\langle [\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'} \cdot \nu\,\mathrm{dS}]^+_-, h \right\rangle} \mathrm{d}t' = -\int_{t_0}^t \left\langle [\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'} \cdot \nu\,\mathrm{dS}]^+_-, h \right\rangle \mathrm{d}t',
\end{aligned}$$

where the distribution $[\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'} \cdot \nu\,\mathrm{dS}]^+_-$ on $\mathbb{R}^3$ with support in Σ^{FS} coincides with $[\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'}]^+_- \cdot \nu\,\mathrm{dS}$ as a distributional density on the surface (alternatively denoted by $[\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'}]^+_- \cdot \nu\,\delta_{\Sigma^{\text{FS}}}$, e.g., [DL88, Appendix, §1, 4.4, pp. 487-488]; see also [Hör90, Equation (3.1.5) and the comment about extension to Lipschitz surfaces in the second paragraph on p. 61, as well as Theorem 8.1.5 and Example 8.2.5]). Our notation here is chosen in a way that makes the results better comparable with geophysics literature as [DT98].

Note that $t' \mapsto [\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'} \cdot \nu\,\mathrm{dS}]^+_-$ is a weakly measurable and bounded map, hence weakly integrable over any bounded interval of time. Therefore,

$$t \mapsto \int_{t_0}^t [\dot\varphi_{t'} \cdot T^{\text{PK}}_{t'} \cdot \nu\,\mathrm{dS}]^+_- \,\mathrm{d}t'$$

is weakly absolutely continuous and, in particular, almost everywhere weakly differentiable.

Applying the Piola transform (1.49) yields

$$\langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), h\rangle = -\int_{t_0}^{t} \Big\langle [\dot\varphi_{t'} \cdot T^{\mathrm{PK}}_{t'} \cdot \nu \,\mathrm{dS}]^+_-, h \Big\rangle \,\mathrm{d}t' = -\int_{t_0}^{t} \Big\langle [\dot\varphi_{t'} \cdot J_{t'} T_{t'} \cdot (\nabla\varphi)^{-T}_{t'} \cdot \nu \,\mathrm{dS}]^+_-, h \Big\rangle \,\mathrm{d}t'$$

and, by interpreting the surface transformation formula (1.34) in the sense of a distributional pull-back, we obtain, with $h^s \circ \varphi_{t'} = h$ and the spatial velocity $v^s = \dot\varphi^s$,

$$\langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), h\rangle = -\int_{t_0}^{t} \Big\langle [v^s_{t'} \cdot T^s_{t'} \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}]^+_-, h^s \Big\rangle \,\mathrm{d}t'. \tag{5.38}$$

Here, $[v^s_{t'} \cdot T^s_{t'} \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}]^+_-$ is a spatial distributional density with support on $\Sigma^{\mathrm{FS}}_{t'} = \varphi_{t'}(\Sigma^{\mathrm{FS}})$, which again should be understood in the sense $[v^s_{t'} \cdot T^s_{t'}]^+_- \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}$ or $[v^s_{t'} \cdot T^s_{t'}]^+_- \cdot \nu^s_{t'} \,\delta_{\Sigma^{\mathrm{FS}}_{t'}}$ (one may read the explanation following Equation (3.68) in [DT98] also in that way), and is weakly integrable with respect to $t' \in I$. Thus, we may consider (5.38) as action functional with surface Lagrangian in spatial representation at fixed t.

As we will establish next, the variation of $\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ with respect to the spatial velocity v^s reproduces the classical fluid-solid boundary conditions. However, in this derivation, we view the surface Lagrangian as a function of the spatial velocity v^s only; in particular, the dependence of T on $\nabla\varphi$ is neglected. For an arbitrary test function $\tilde h$ on $\mathbb{R}^3 \times I$, we consider stationarity of the surface action as a map

$$v^s \mapsto \langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}, \tilde h\rangle = \int_I \langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), \tilde h(t,.)\rangle \,\mathrm{d}t.$$

In this sense, we obtain a **weak stationarity condition** on $\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ playing the role of the EL. By density of tensor products, it suffices to consider test functions $\tilde h = h_0 \otimes h \colon (X,t) \mapsto h(X)h_0(t)$ and

$$\langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}, h_0 \otimes h\rangle \colon v^s \mapsto \int_I h_0(t) \langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), h\rangle \,\mathrm{d}t = -\int_I h_0(t) \int_{t_0}^{t} \Big\langle [v^s_{t'} \cdot T^s_{t'} \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}]^+_-, h^s \Big\rangle \,\mathrm{d}t' \,\mathrm{d}t.$$

As noted above, $t \mapsto \int_{t_0}^{t} \langle [v^s_{t'} \cdot T^s_{t'} \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}]^+_-, h^s\rangle \,\mathrm{d}t'$ is differentiable almost everywhere and has an integrable derivative. Moreover, $v^s \mapsto \langle [v^s_{t'} \cdot T^s_{t'} \cdot \nu^s_{t'} \,\mathrm{dS}_{t'}]^+_-, h^s\rangle$ defines a family of linear continuous maps $L^\infty(B \times I^\circ)^3 \to \mathbb{R}$ depending measurably on t'. Therefore,

$$v^s \mapsto \langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), h\rangle$$

given in (5.38) is Fréchet differentiable with derivative (independent of v^s)

$$(D\langle \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(t), h\rangle) \cdot w^s_t = -\Big\langle [w^s_t \cdot T^s_t \cdot \nu^s_t \,\mathrm{dS}_t]^+_-, h^s \Big\rangle$$

and the weak stationarity of $\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ means

$$\int_I h_0(t) \Big\langle [w^s_t \cdot T^s_t \cdot \nu^s_t \,\mathrm{dS}_t]^+_-, h^s \Big\rangle \,\mathrm{d}t = 0$$

for all test functions $h_0 \otimes h$ on $\mathbb{R}^3 \times I$ and $w \in L^\infty(B \times I^\circ)^3$. Thus, we arrive at the condition

$$[T^s_t \cdot \nu^s_t \,\mathrm{dS}_t]^+_- = 0 \tag{5.39}$$

to hold across the fluid-solid boundary $\Sigma^{\mathrm{FS}}_t = \varphi_t(\Sigma^{\mathrm{FS}})$ (in agreement with [DT98, (3.68)]). This condition corresponds to (2.16) obtained by Newton's third law (see Section 4.6.2).

Chapter 6

Linearization

In this chapter we develop the linearized model. After some generalities about linearization of physical models around a reference model (Section 6.1), we discuss the prestressed equilibrium state around which the earth model will be linearized (Section 6.2). We then deduce the linearization of kinematical and dynamical fields, as well as prestressed linearized elasticity, as approximations of the nonlinear theories (Sections 6.3, 6.4). On the way, we also provide regularity conditions for the linearized fields, which are obtained in accordance with the nonlinear setting. We derive the second-order approximation of the action, which is the basic ingredient of the linearized variational earth model, and conclude with the interpretation of the interface energy (Section 6.6). In addition, the formal direct linearization of the governing equations is presented (Section 6.5). However, the complete linearized system of the elastic-gravitational equations, including the boundary and interface conditions, will be obtained in Chapter 7 from the linearized variational principle.

6.1 Linearization via perturbation around a reference state

Linearization of a field q means decomposition into a **reference part** and a **perturbation**:

$$q = q^0 + q^1.$$

Formally, the linearization of an equation involving various fields q is obtained by replacing the fields q by their corresponding decompositions $q^0 + q^1$ and neglecting all terms that are of order two or higher in the perturbations. Higher-order approximations are deduced analogously. The omission of terms of order $k+1$ (or higher) in the perturbed quantities or in their derivatives will be indicated by $\approx_k$, where

$$a \approx_k b \tag{6.1}$$

means "a is equal to b up to k-th order" or "correct to order k". This procedure of linearization essentially is based on Taylor expansion and thus one has to assume that the operators P in the equations depend smoothly on the fields q. Consequently, a rigorous systematic linearization procedure is based on formulating the problem in suitable spaces. We restrict ourselves to the Banach space setting [MH83, Section 4.1, Definition 1.1, p. 227]:

Definition 6.1 (**Linearization in Banach spaces**). Let X and Y be Banach spaces and $M \subseteq X$ be open with $M \neq \emptyset$. The linearization of a differentiable map $P\colon M \to Y$, $q \mapsto P(q)$ around $q^0 \in M$ is given by

$$P_{\mathrm{lin}}\colon X \to Y, \quad q^1 \mapsto P_{\mathrm{lin}}(q^1) := P(q^0) + DP(q^0)(q^1).$$

Here $DP(q^0) \in \mathrm{Lin}(X, Y)$ is the Fréchet derivative of P at q^0 (see Definition 3.1). Thus the linearized map P_{lin} coincides with the first-order Taylor expansion

$$P(q^0 + q^1) \approx_1 P(q^0) + DP(q^0)(q^1) = P_{\mathrm{lin}}(q^1).$$

More generally, if X, Y are manifolds, then the first perturbation q^1 is an element of the tangent space $T_{q^0}X$, see [MH83, Section 4.1, p. 232].

The linearization of an equation $P(q) = 0$ may also be obtained by writing $q = q^0 + \varepsilon q^1$ with a scalar parameter ε that is assumed to be small. Expanding $\varepsilon \mapsto f(\varepsilon) := P(q^0 + \varepsilon q^1)$ around $\varepsilon = 0$ up to first order with respect to ε yields

$$P(q^0 + \varepsilon q^1) = f(\varepsilon) = f(0) + \varepsilon f'(0) + O(\varepsilon^2) = P(q^0) + \varepsilon\, DP(q^0)(q^1) + O(\varepsilon^2), \qquad \varepsilon \to 0.$$

If P is differentiable, this procedure is consistent with Definition 6.1: Terms that are $O(1)$, i.e. independent of ε, give the equations in the reference regime $P^0(q^0) = 0$; terms of $O(\varepsilon)$ produce the equations in the first-order regime $P^1(q^1) = 0$ for $P^1 := DP(q^0)$.

Remark 6.2 (**Scale analysis**)**.** The method of scale analysis allows to systematically deduce different approximation regimes of a general nonlinear PDE. It is a common approach in geophysical fluid dynamics [Ped87, p. 345]. Formally it proceeds as follows: One writes the equation $P(q) = 0$ in its dimensionless form by dividing all variables q through their typical scales. Thereby an equation in dimensionless variables q_* is obtained, where the individual terms are weighted by dimensionless factors. One defines a dimensionless ratio ε, whose smallness characterizes a typical regime (in case of fluid flow, e.g., the Reynold's number, comparing advection to friction, or the Rossby number, comparing advection to Coriolis acceleration). Next the unknowns are Taylor-expanded: $q_* = q_*^0 + \varepsilon\, q_*^1 + O(\varepsilon^2)$ as $\varepsilon \to 0$. The approximation level is chosen in terms of a certain power of ε and by neglecting all contributions with different power. The so-obtained system is finally rewritten in dimensional form. ⌟

The physical fields $q = q^0 + q^1$ in continuum mechanics depend on space and time. If the reference state q^0 is defined as the field evaluated at the reference time t_0, the perturbation q^1 must vanish at t_0:

$$q = q^0 + q^1 \quad \text{and} \quad q^0 := q(., t_0) \qquad \Longrightarrow \qquad q^1(., t_0) = 0. \tag{6.2}$$

The **linearization of the motion** $\varphi\colon \overline{B} \times I \to \mathbb{R}^n$ (1.20) of a continuous body $B \subseteq \mathbb{R}^n$ reads

$$\varphi(X, t) = X + u(X, t) \qquad \text{for} \quad (X, t) \in \overline{B} \times I. \tag{6.3}$$

Here the identity initial configuration $\varphi_{t_0} = \mathrm{Id}_{\overline{B}}$ (1.21), where no motion has yet occurred, is used as the reference configuration: $\varphi^0(X) := \varphi(X, t_0) = X$. The difference

$$u\colon \overline{B} \times I \to \mathbb{R}^n, \qquad u := \varphi - \mathrm{Id}_{\overline{B}} \tag{6.4}$$

is referred to as the **displacement** (the perturbation of the motion), already introduced in (1.57). As we have seen in (6.2), our choice of reference configuration implies that the displacement u must vanish at initial time:

$$u(X, t_0) = 0.$$

We note that with the notation of scale analysis (Remark 6.2), the linearization (6.3) of the motion would read $\varphi(X, t) = X + \varepsilon u(X, t)$, where the parameter ε compares the magnitude of the displacement $|u|$ to the characteristic length scale of the body.

In application to the self-gravitating elastic earth model, the dynamical variables φ and Φ^s are decomposed into reference values (Section 6.2) and perturbations (Sections 6.3.2 and 6.3.4):

$$\varphi = \mathrm{Id}_{\overline{B}} + u \qquad \text{and} \qquad \Phi^s = \Phi^0 + \Phi^{s1}.$$

Linearization is indeed justified within the seismic regime, in which the elastic motion of the Earth results only in a slight departure from equilibrium (e.g. [DT98, p. 56]).

Upon linearization, the nonlinear governing equations (2.10) and (2.2) will split into the equilibrium equations (2.19) and (2.20), constraining the reference fields, and the linear governing equations (2.26) and (2.27), determining the evolution of the first-order perturbations. Formally, the equations are obtained by neglecting quadratic or higher-order contributions of the perturbations or of their derivatives; the boundary and interface conditions are treated in a similar way (Section 6.5). The linear equations and boundary/interface conditions derived in this way will be said to hold "correct to first order" in the perturbations. However, as will be established in Chapter 7, the governing equations as well as the dynamical interface and boundary conditions are rigorously obtained from the first variation of a corresponding action (as Euler-Lagrange equations EL and associated natural boundary/interface conditions NBC/NIBC respectively). In the linearized setting, action, Lagrangian, and energy require an approximation correct to second order in the perturbations (Section 6.6).

6.2 The prestressed equilibrium reference state of the Earth

We assume that the Earth is in equilibrium at initial time t_0, which we use as a reference configuration. Thus the equilibrium state is characterized by specifying Earth's rotation Ω, initial density ρ^0 and prestress T^0. Since we use some properties of the equilibrium state when discussing perturbed quantities, we anticipate the equilibrium results. These equilibrium governing equations will follow in Chapter 7 as EL, NBC, and NIBC from stationarity of the action in the first-order approximation (the governing equations for perturbations arise from stationarity of the purely second-order terms of the approximated action).

The assumption of equilibrium at initial time, $\varphi_{t_0} = \mathrm{Id}_{\overline{B}}$, implies that initial acceleration $\ddot{\varphi}_{t_0}$ and initial velocity $\dot{\varphi}_{t_0}$ vanish (and that there is no external force in equilibrium: $f_{t_0} = 0$). Therefore, the equilibrium equations follow from the dynamical equations evaluated at time $t = t_0$, specifically, the spatial Poisson equation (2.2),

$$\triangle \Phi^s = 4\pi G \rho^s,$$

and the nonlinear equation of motion (2.10),

$$\rho^0 \left(\ddot{\varphi} + 2\,\Omega \times \dot{\varphi} + \Omega \times (\Omega \times \varphi) \right) - \nabla \cdot T^{\mathrm{PK}} = \rho^0 g + f.$$

These equations, introduced in Section 2.3.1, were formally variationally derived in Section 5.4.

The **equilibrium earth model** is set up in three steps:

1. Specify ρ^0, whose compact support gives $\overline{B}$. Specify $\Omega \in \mathbb{R}^3$, which determines the centrifugal potential by (2.7), $\Psi^s(x) = -\frac{1}{2}\left(\Omega^2 x^2 - (\Omega \cdot x)^2\right)$, for $x \in \mathbb{R}^3$.

2. Obtain Φ^0 as solution of the equilibrium Poisson equation (2.20),

$$\triangle \Phi^0 = 4\pi G \rho^0.$$

3. Specify T^0 such that the static equilibrium equation (2.19) holds,

$$\rho^0 \nabla (\Phi^0 + \Psi^s) - \nabla \cdot T^0 = 0.$$

We note that the static equilibrium equation (2.19) only constrains the divergence $\nabla \cdot T^0$.

Remark 6.3 (**Constraining the components of prestress**). Note that T^0 is not fully determined by the governing equations. Formally, given ρ^0 and Ω, the three components of the static equilibrium equation (2.19) (plus the interface and boundary conditions) constrain only three out of the six independent components of T^0. The remaining three components, specifically those of the initial deviatoric prestress T^0_{dev} (4.65), need to be treated as additional material parameters that have to be specified independently [DT98, p. 100]. A method of parametrization of the equilibrium stress is discussed in [AAW10]. In particular, it is shown there that the equilibrium stress field with minimum deviatoric component in terms of a given norm corresponds to the solution of a steady-state incompressible viscous flow problem. ⌟

We briefly comment on the consistency of the regularity conditions underlying the equilibrium earth model (a more detailed argumentation is provided in Section 8.1). The regularity conditions are precisely those of the basic variational model (Assumption 1) for the time-dependent density ρ^s, gravitational potential Φ^s, and stress tensor T^{PK} evaluated at the initial time t_0. This restriction to initial time is indeed possible, because all fields in Assumption 1 are continuous with respect to time. In particular, evaluating $\rho^s \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$ with $\mathrm{supp}(\rho^s_t) \subseteq \varphi_t(\overline{B})$ at time t_0 gives

$$\rho^0 \in L^\infty(\mathbb{R}^3) \qquad \text{with} \qquad \mathrm{supp}(\rho^0) = \overline{B}.$$

With these properties of ρ^0, Lemma 4.28 shows that the solution Φ^0 of the equilibrium Poisson equation $\triangle\Phi^0 = -4\pi G\,\rho^0$ (2.20) satisfies

$$\Phi^0 \in \mathcal{C}^1(\mathbb{R}^3).$$

The smoothness of Φ^0 guarantees the validity of the jump conditions (2.25). Furthermore, the right-hand side of the static equilibrium equation $\nabla \cdot T^0 = \rho^0 \nabla(\Phi^0 + \Psi^s)$ (2.19) is bounded, which implies that $\nabla \cdot T^0 \in L^\infty(\mathbb{R}^3)^3$. As T^0 is bounded and supported in $\overline{B}$, this yields $\nabla \cdot T^0 \in L^2(B^{\mathrm{FSC}})^3$, i.e.

$$T^0 \in H_{\mathrm{div}}(B^{\mathrm{FSC}})^{3\times 3}.$$

In particular, since H_{div} (1.18) has normal traces $T^0 \cdot \nu$ in $H^{-1/2}$ on Lip-surfaces, the equilibrium traction boundary and interface conditions (2.21), (2.22), (2.23) make sense in Sobolev spaces.

6.3 Linearized kinematics and dynamical fields

6.3.1 Linearization in material and spatial representation

Choosing the initial configuration as a reference implies that the material representation q and the spatial representation q^s both reduce to the **reference field** q^0 defined on B, cf. (1.29):

$$q^0(X) = q(X, t_0) = q^s(X, t_0) \qquad \text{for} \quad X \in \overline{B}.$$

However, linearization in material or in spatial representation will result in different first-order perturbations:

$$q(X,t) = q^0(X) + q^1(X,t) \qquad \text{for} \quad (X,t) \in \overline{B} \times I \tag{6.5}$$

and

$$q^s(x,t) = q^0(x) + q^{s1}(x,t) \qquad \text{for} \quad (x,t) \in \bigcup_{t\in I} (\varphi_t(\overline{B}) \times \{t\}) \subseteq \mathbb{R}^3 \times I. \tag{6.6}$$

By definition of q^0, both perturbations satisfy $q^1(X, t_0) = 0$ and $q^{s1}(x, t_0) = 0$. Since x and X are related by

$$x = \varphi(X,t) = X + u(X,t),$$

with the displacement u introduced in (6.4), Taylor expansion of q^0 and q^{s1} around X gives

$$\begin{aligned} q(X,t) &= q^s(\varphi(X,t),t) \\ &= q^0(\varphi(X,t)) + q^{s1}(\varphi(X,t),t) \\ &= q^0(X+u(X,t)) + q^{s1}(X+u(X,t),t) \\ &= q^0(X) + \nabla q^0(X)\cdot u(X,t) + q^{s1}(X,t) + \nabla q^{s1}(X,t)\cdot u(X,t) + O(|u(X,t)|^2) \end{aligned}$$

as $|u(X,t)| \to 0$, i.e. for small displacement. The product $\nabla q^{s1} \cdot u$ is a term of second order and we obtain

$$q \approx_1 q^0 + q^{s1} + \nabla q^0 \cdot u \tag{6.7}$$

in the regime of small perturbations. Comparing with $q = q^0 + q^1$, the material and spatial perturbations are thus related by [DT98, Equation (3.16)]

$$q^1 \approx_1 q^{s1} + \nabla q^0 \cdot u. \tag{6.8}$$

Here, $\approx_1$ indicates the omission of terms of higher than first order in u as well as the omission of products of u with derivatives of q^{s1} (or q^1, which can be derived analogously), cf. (6.1).

6.3.2 Displacement and kinematical interface conditions

We recall from (6.4) that the displacement associated to the motion φ of the earth model is given by

$$u = \varphi - \mathrm{Id}_{\overline{B}} \colon \overline{B} \times I \to \mathbb{R}^3 \qquad \text{with} \qquad u(.,t_0) = 0.$$

This definition implies that u and φ have the same differentiability properties. We are free to set u equal to zero outside $\overline{B}$:

$$u \colon \mathbb{R}^3 \times I \to \mathbb{R}^3 \quad \text{with} \quad u(X,.) := 0 \quad \text{for} \quad X \in B^{\mathrm{C}}.$$

By Assumption 1, $\varphi|_{\overline{B}\times I} \in \mathcal{A}(\overline{B}\times I)$ and thus we obtain $u \in \mathrm{Lip}(I, L^\infty(\mathbb{R}^3))^3$ with $\mathrm{supp}(u_t) \subseteq \overline{B}$ for all $t \in I$. Moreover $u|_{B^{\mathrm{S}}\times I}$, $u|_{B^{\mathrm{F}}\times I}$, and $u|_{B^{\mathrm{C}}\times I}(=0)$ have Lipschitz continuous extensions to $\overline{B^{\mathrm{S}}}\times I$, $\overline{B^{\mathrm{F}}}\times I$, and $\overline{B^{\mathrm{C}}}\times I$ respectively. In accordance with the inclusion $\mathcal{A}(\overline{B}\times I) \subseteq H^1(B^{\mathrm{FS}}\times I^\circ)^3$ (4.43), we obtain

$$u|_{B^{\mathrm{FSC}}\times I^\circ} \in H^1(B^{\mathrm{FSC}} \times I^\circ)^3. \tag{6.9}$$

Next we consider the kinematical interface conditions. Continuity (4.37) of φ across welded solid-solid interior boundaries directly implies continuity of $u = \varphi - \mathrm{Id}_{\overline{B}}$, that is, (2.28):

$$[u]^+_- = 0 \qquad \text{on} \qquad \Sigma^{\mathrm{SS}}.$$

As we have seen, the slipping kinematical interface condition across fluid-solid interior boundaries, (4.38), expresses the continuity of the normal component of the spatial velocity $v^s_t \cdot \nu^s_t$ across $\varphi_t(\Sigma^{\mathrm{FS}})$. However, its material counterpart is nonlinear in u. As is established in Lemma 6.4 below, correct to first order in u, we have the **linear tangential slip condition** (2.29)

$$[u]^+_- \cdot \nu \approx_1 0 \qquad \text{on} \qquad \Sigma^{\mathrm{FS}} \tag{6.10}$$

whose structure is similar to the exact slip condition in spatial form (4.38). Thus, the final space for the variational formulation on linear level is

$$H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}} \times I^\circ)^3 := \{u \in H^1(B^{\mathrm{FSC}} \times I^\circ)^3 : \ [u]^+_- \cdot \nu = 0 \ \text{ on } \Sigma^{\mathrm{FS}}\}. \tag{6.11}$$

The time-independent analog of this space was already introduced in (4.25).

In the remainder of this section we present the derivation of the slip condition, correct up to second order in u:

Lemma 6.4 (Tangential slip condition). *Correct to second order in u, the kinematical interface condition for tangential slip (4.38), $[v^s]^+_- \cdot \nu^s = 0$ on $\varphi_t(\Sigma^{\mathrm{FS}})$, reduces to*

$$\left[u \cdot \nu - u \cdot \widetilde{\nabla}(u \cdot \nu) + \frac{1}{2}\, u \cdot \widetilde{\nabla}\nu \cdot u\right]^+_- \approx_2 0 \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}. \tag{6.12}$$

Equation (6.12) is also given in [DT98, (3.95)]. We note that $u \cdot \widetilde{\nabla}(u \cdot \nu) = u \cdot \widetilde{\nabla}\nu \cdot u + \nu \cdot \widetilde{\nabla} u \cdot u$ yields the equivalent condition

$$\left[u \cdot \nu - \nu \cdot \widetilde{\nabla} u \cdot u - \frac{1}{2}\, u \cdot \widetilde{\nabla}\nu \cdot u\right]^+_- \approx_2 0 \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}. \tag{6.13}$$

Proof. We give a proof of (6.12) along the lines of [DT98, p. 72]: In the first step we establish the separate first-order expansions of the spatial surface element $\nu^s\, \mathrm{dS}^s$ [DT98, (3.31) and (3.30)]:

$$\nu^s \approx_1 \nu - (\widetilde{\nabla} u)^T \cdot \nu = \nu - \nu \cdot \widetilde{\nabla} u \qquad \text{and} \qquad \mathrm{dS}^s \approx_1 (1 + \widetilde{\nabla} \cdot u)\, \mathrm{dS}. \tag{6.14}$$

They are deduced from the relation for material and spatial surface elements (Lemma 4.21), the first-order expansions

$$J = \det(\nabla\varphi) = \det(1_{3\times 3} + \nabla u) \approx_1 1 + \nabla \cdot u \qquad \text{and} \qquad (\nabla\varphi)^{-T} \approx_1 1_{3\times 3} - (\nabla u)^{-T}, \tag{6.15}$$

and the identities $\nabla \cdot u = \widetilde{\nabla} \cdot u + \nu \cdot \nabla u \cdot \nu$ and $\nabla u = \widetilde{\nabla} u + (\nabla u \cdot \nu)\nu$ (yielding cancellation of terms in the calculation to follow):

$$\begin{aligned}
\nu^s\, \mathrm{dS}^s = J\, (\nabla\varphi)^{-T} \cdot \nu\, \mathrm{dS} \quad &\approx_1 \quad (1 + \nabla \cdot u)(1_{3\times 3} - (\nabla u)^T) \cdot \nu\, \mathrm{dS} \\
&\approx_1 \quad (\nu + (\nabla \cdot u)\nu - (\nabla u)^T \cdot \nu)\mathrm{dS} \\
&= \quad (\nu + (\nabla \cdot u)\nu - \nu \cdot \nabla u)\mathrm{dS} \\
&= \quad (\nu + (\widetilde{\nabla} \cdot u)\nu - \nu \cdot \widetilde{\nabla} u)\mathrm{dS} \\
&= \quad (\nu + (\widetilde{\nabla} \cdot u)\nu - (\widetilde{\nabla} u)^T \cdot \nu)\mathrm{dS} \\
&\approx_1 \quad (\nu - (\widetilde{\nabla} u)^T \cdot \nu)(1 + \widetilde{\nabla} \cdot u)\mathrm{dS}.
\end{aligned}$$

In the second step we consider a slipping surface $S = \Sigma^{\mathrm{FS}}$. The situation is illustrated in the figure below.

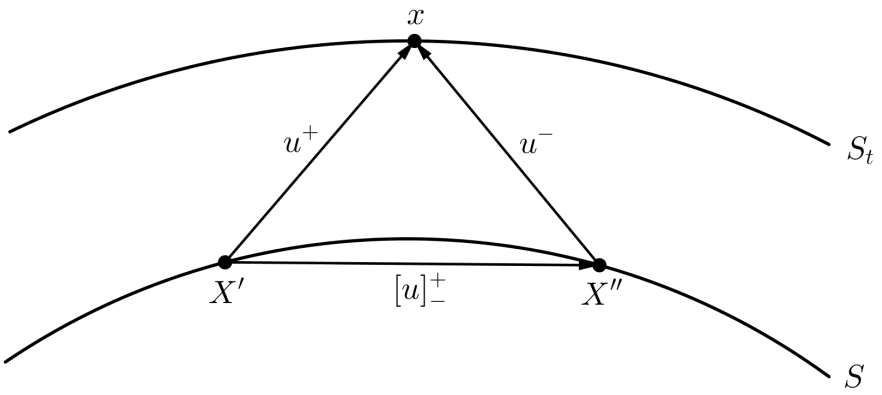

Figure 6.1: The initial surface S deforms to the surface $S_t = \varphi_t(S)$ at time t. Due to slip, the points $X' \neq X''$ on S will be moved to the same point $x \in S_t$ (cf. [DT98, Figure 3.3, p. 68]). Actually, the jump $[u]^+_-$ should be read as $u^+(X') - u^-(X'') = X'' - X'$, see (6.16).

We observe that for all $x \in \varphi_t(\Sigma^{\mathrm{FS}})$ there exist $X', X'' \in \Sigma^{\mathrm{FS}}$ such that

$$X' + u_t^+(X') = \varphi_t^+(X') = x = \varphi_t^-(X'') = X'' + u_t^-(X''), \tag{6.16}$$

where $\varphi_t^+(X')$ denotes the limit from the $+$-side and $\varphi_t^-(X'')$ the limit from the $-$-side of $\varphi_t(\Sigma^{\mathrm{FS}})$, analogoulsy for $u_t^\pm$. (The existence of these limits is guaranteed by the piecewise Lip-regularity of the motion, see (ii) of Definition 4.19.) Omitting the dependence on t for the moment, $x = \varphi(X)$ implies $v^s(x) = \dot{u}(X)$. Together with (6.14), this leads to

$$(v^s \cdot \nu^s)(x) \approx_2 \dot{u}(X) \cdot (\nu - (\widetilde{\nabla} u)^T \cdot \nu)(X) = (\dot{u} \cdot \nu - \dot{u} \cdot (\widetilde{\nabla} u)^T \cdot \nu)(X). \tag{6.17}$$

From (6.16) and the exact spatial slip condition $[v^s]_-^+ \cdot \nu^s = [v^s \cdot \nu^s]_-^+ = 0$ we then get

$$(\dot{u}^+ \cdot \nu - \dot{u}^+ \cdot (\widetilde{\nabla} u^+)^T \cdot \nu)(X') \approx_2 (v^s \cdot \nu^s)^+(x) = (v^s \cdot \nu^s)^-(x) \approx_2 (\dot{u}^- \cdot \nu - \dot{u}^- \cdot (\widetilde{\nabla} u^-)^T \cdot \nu)(X''),$$

that is,

$$(\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \approx_2 (\dot{u}^+ \cdot (\widetilde{\nabla} u^+)^T \cdot \nu)(X') - (\dot{u}^- \cdot (\widetilde{\nabla} u^-)^T \cdot \nu)(X''). \tag{6.18}$$

In particular, up to first order we obtain

$$(\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \approx_1 0,$$

which upon substituting $X'' = X' + u^+(X') - u^-(X'')$ from (6.16) implies

$$[\dot{u} \cdot \nu]_-^+(X') = (\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X') \approx_1 (\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \approx_1 0. \tag{6.19}$$

We observe that integrating with respect to time yields the first-order tangential slip condition (6.10).

To prove the second-order condition (6.12), we use again $X'' = X' + u^+(X') - u^-(X'')$ as well as Taylor expansion of the second term on the right-hand side in (6.18):

$$\begin{aligned}
&(\dot{u}^+ \cdot (\widetilde{\nabla} u^+)^T \cdot \nu)(X') - (\dot{u}^- \cdot (\widetilde{\nabla} u^-)^T \cdot \nu)(X'') \\
&\quad = (\dot{u}^+ \cdot (\widetilde{\nabla} u^+)^T \cdot \nu)(X') - (\dot{u}^- \cdot (\widetilde{\nabla} u^-)^T \cdot \nu)(X' + u^+(X') - u^-(X'')) \\
&\quad \approx_2 (\dot{u}^+ \cdot (\widetilde{\nabla} u^+)^T \cdot \nu)(X') - (\dot{u}^- \cdot (\widetilde{\nabla} u^-)^T \cdot \nu)(X') = [\dot{u} \cdot (\widetilde{\nabla} u)^T \cdot \nu]_-^+(X').
\end{aligned}$$

For the second-order approximation of the left-hand side of (6.18), also the first-order terms of the Taylor expansion have to be taken into account:

$$\begin{aligned}
&(\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \\
&\quad = (\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X' + u^+(X') - u^-(X'')) \\
&\quad \approx_2 (\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X') - \nabla(\dot{u}^- \cdot \nu)(X') \cdot \big(u^+(X') - u^-(X'')\big) \\
&\quad = [\dot{u} \cdot \nu]_-^+(X') - \nabla(\dot{u}^- \cdot \nu)(X') \cdot u^+(X') + \nabla(\dot{u}^- \cdot \nu)(X') \cdot u^-(X'').
\end{aligned}$$

We rewrite the second term and observe that $\nabla[\dot{u} \cdot \nu]_-^+(X') \cdot u^+(X') \approx_2 0$ holds, since $[\dot{u} \cdot \nu]_-^+ \approx_1 0$ by (6.19):

$$\begin{aligned}
&-\nabla(\dot{u}^- \cdot \nu)(X') \cdot u^+(X') \\
&\quad = -\nabla(\dot{u}^- \cdot \nu)(X') \cdot u^+(X') + \nabla(\dot{u}^+ \cdot \nu)(X') \cdot u^+(X') - \nabla(\dot{u}^+ \cdot \nu)(X') \cdot u^+(X') \\
&\quad = \nabla[\dot{u} \cdot \nu]_-^+(X') \cdot u^+(X') - \nabla(\dot{u}^+ \cdot \nu)(X') \cdot u^+(X') \\
&\quad \approx_2 -\nabla(\dot{u}^+ \cdot \nu)(X') \cdot u^+(X').
\end{aligned}$$

The third term, again upon Taylor expansion, may be written in the form

$$\nabla(\dot{u}^- \cdot \nu)(X') \cdot u^-(X'') \approx_2 \nabla(\dot{u}^- \cdot \nu)(X') \cdot u^-(X').$$

The left-hand side of (6.18) can thus be approximated as

$$(\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \approx_2 [\dot{u} \cdot \nu]_-^+(X') - \nabla(\dot{u}^+ \cdot \nu)(X') \cdot u^+(X') + \nabla(\dot{u}^- \cdot \nu)(X') \cdot u^-(X'),$$

that is

$$(\dot{u}^+ \cdot \nu)(X') - (\dot{u}^- \cdot \nu)(X'') \approx_2 \left[\dot{u} \cdot \nu - u \cdot \nabla(\dot{u} \cdot \nu)\right]_-^+(X') = \left[\dot{u} \cdot \nu - u \cdot \widetilde{\nabla}(\dot{u} \cdot \nu)\right]_-^+(X').$$

Combining these results, (6.18) implies the identity [DT98, (3.93)]

$$\left[\dot{u} \cdot \nu - u \cdot \widetilde{\nabla}(\dot{u} \cdot \nu)\right]_-^+ \approx_2 \left[\dot{u} \cdot (\widetilde{\nabla} u)^T \cdot \nu\right]_-^+.$$

With

$$\dot{u} \cdot (\widetilde{\nabla} u)^T \cdot \nu = \nu \cdot \widetilde{\nabla} u \cdot \dot{u} = \widetilde{\nabla}(u \cdot \nu) \cdot \dot{u} - u \cdot \widetilde{\nabla}\nu \cdot \dot{u} = \dot{u} \cdot \widetilde{\nabla}(u \cdot \nu) - u \cdot \widetilde{\nabla}\nu \cdot \dot{u}$$

and by symmetry of $\widetilde{\nabla}\nu$, we can modify the right-hand side and pull out the time derivative:

$$0 \approx_2 \left[\dot{u} \cdot \nu - u \cdot \widetilde{\nabla}(\dot{u} \cdot \nu) - \dot{u} \cdot \widetilde{\nabla}(u \cdot \nu) + u \cdot \widetilde{\nabla}\nu \cdot \dot{u}\right]_-^+ = \partial_t \left[u \cdot \nu - u \cdot \widetilde{\nabla}(u \cdot \nu) + \frac{1}{2}\, u \cdot \widetilde{\nabla}\nu \cdot u\right]_-^+.$$

Since $u_{t_0} = 0$, integration from t_0 to t finally leads to the tangential slip condition (6.12). □

6.3.3 Perturbative force potential

By Assumption 1, the force potential introduced in Section 4.4 satisfies $F^s \in \mathcal{C}^0(I, \mathrm{Lip}(\mathbb{R}^3))$ (4.49). Since external forces are absent in the equilibrium state, F^s may be considered as a perturbative term in the Lagrangian (5.10), similarly as u is viewed as a perturbation of the spatial position in $x = X + u_t(X)$. This approach is also discussed in [MH83, Chapter 4, p. 237]. We observe that

$$F_t = F_t^s \circ \varphi_t = F_t^s \circ (\mathrm{Id}_{\mathbb{R}^3} + u_t)$$

implies the Taylor approximation

$$F \approx_2 F^s + u \cdot \nabla F^s, \tag{6.20}$$

where the first term corresponds to the first-order level, while the second term, being a product of two first-order perturbations, corresponds to second-order approximation. In particular, we obtain

$$f \approx_1 -\rho^0(\nabla F^s) =: f^1. \tag{6.21}$$

Earthquake sources can be modeled by a particular choice of the perturbative force. This so-called equivalent body force distribution can be expressed using the concept of stress glut, assuming that earthquakes occur as a result of a localized and transient failure of the linearized elastic constitutive relation, see [DT98, Chapter 5].

6.3.4 Gravitational potential perturbation

The linearization of the gravitational potential Φ^s takes the form

$$\Phi^s = \Phi^0 + \Phi^{s1} \tag{6.22}$$

with the equilibrium field $\Phi^0 = \Phi^s_{t_0}$. The **gravitational potential perturbation** Φ^{s1} (also called the incremental gravitational potential or the mass-redistribution potential) depends on space and time and, by definition, vanishes in the equilibrium state at time t_0, that is $\Phi^{s1}_{t_0} = 0$.

Since the equilibrium gravitational potential Φ^0 is time-independent, Φ^s and $\Phi^{s1} = \Phi^s - \Phi^0$ have the same regularity with respect to time. Moreover, $x = \varphi(X,t) = X + u(X,t)$ (6.4) and the mean value theorem give

$$\begin{aligned}\Phi^{s1}(x,t) &= \Phi^s(x,t) - \Phi^0(x) = \Phi^s(X+u(X,t),t) - \Phi^0(X+u(X,t)) \\ &= \Phi^s(X,t) - \Phi^0(X) + \int_0^1 \big(\nabla\Phi^s(X+ru(X,t),t) - \nabla\Phi^0(X+ru(X,t))\big)\cdot u(X,t)\,\mathrm{d}r,\end{aligned}$$

which shows that the spatial regularity of Φ^{s1} will be one order lower compared to that of Φ^s or Φ^0. According to Assumption 1, we have $\Phi^s \in \mathcal{C}^0(I, Y^\infty(\mathbb{R}^3))$. Recalling the decay and regularity results in Lemma 4.28 (ii) and (iii), we obtain

$$\Phi^0 \in Y^\infty(\mathbb{R}^3) \subseteq \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3)$$

with $\Phi^0(x) = O(1/|x|)$ as $|x| \to \infty$ and

$$\Phi^{s1} \in \mathcal{C}^0(I, DY^\infty(\mathbb{R}^3)) \subseteq \mathcal{C}^0(I, \bigcap_{1\le p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3))$$

with $\Phi^{s1}_t(x) = O(1/|x|^2)$ as $|x| \to \infty$, where

$$DY^\infty(\mathbb{R}^3) := \{f \in \mathcal{D}'(\mathbb{R}^3) : \exists\, g \in Y^\infty(\mathbb{R}^3),\ \exists\, \alpha \in \mathbb{N}_0^3,\ |\alpha| = 1 : f = D^\alpha g\}.$$

It follows that

$$\Phi^{s1} \in \mathcal{C}^0(I, H^1(\mathbb{R}^3)). \tag{6.23}$$

By the Sobolev embedding theorem (Lemma 3.13), $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^k(\mathbb{R}^3)$ if $0 \le k < 1 - 3/p$. Since this is satisfied for $k = 0$ and $p > 3$, we conclude that Φ^{s1}_t must be continuous and hence

$$\Phi^{s1} \in \mathcal{C}^0(\mathbb{R}^3 \times I).$$

Continuity of Φ^{s1}_t and continuous differentiability of Φ^0 yield the conditions (2.24), (2.25), and (2.34), i.e.

$$[\Phi^0]^+_- = 0, \quad [\nabla\Phi^0]^+_- \cdot \nu = 0, \quad \text{and} \quad [\Phi^{s1}]^+_- = 0 \tag{6.24}$$

on every surface S in $\mathbb{R}^3$ and in particular on the interior and exterior boundaries $\Sigma \cup \partial B$. Moreover, both Φ^0 and Φ^{s1}_t decay to zero at infinity. We announce that the remaining interface condition in (2.35), $[\nabla\Phi^{s1} + 4\pi G\,\rho^0 u]^+_- \cdot \nu = 0$, will follow from the variational principle.

6.4 Linearized prestressed elasticity

6.4.1 Prestressed Hooke's law

In the presence of large ambient stresses encoded in the prestress T^0 (4.63), the purely quadratic energy (1.60) $W = \rho^0 U$ of classical linearized elasticity is generalized to the following second-order approximation of the internal elastic energy density [DT98, p. 76 eq. (3.115)]:

$$\rho^0 U \approx_2 T^0 : e + \frac{1}{2} e : \Xi : e = T^0_{ij} e_{ij} + \frac{1}{2}\,\Xi_{ijkl} e_{ij} e_{kl}. \tag{6.25}$$

The expansion is given in terms of the full (nonlinear) material strain tensor e (1.26), because by material frame indifference, U depends on $\nabla\varphi$ only through e (as was mentioned in Section 1.3.1).

Since e is symmetric, the time-independent tensor fields T^0 and Ξ without loss of generality possess the symmetries (here and everywhere below, $i,j,k,l \in \{1,2,3\}$)

$$T^0_{ij} = T^0_{ji} \qquad \text{and} \qquad \Xi_{ijkl} = \Xi_{klij} = \Xi_{jikl} = \Xi_{ijlk}. \tag{6.26}$$

Recall that by Assumption 1, $U(.,(\nabla\varphi)(.)) \in \mathcal{C}^0(I, L^\infty(\mathbb{R}^3))$, see (4.60). Therefore, the components of T^0 and Ξ need to be bounded (with support contained in $\overline{B}$):

$$T^0_{ij} \quad \text{and} \quad \Xi_{ijkl} \in L^\infty(\mathbb{R}^3).$$

Next we discuss the associated linear stress-strain relation. We introduce

$$\widetilde{U}(.,\nabla u) := U(.,1_{3\times 3} + \nabla u) = U(.,\nabla\varphi),$$

satisfying $\frac{\partial \widetilde{U}}{\partial(\nabla u)} = \frac{\partial U}{\partial(\nabla\varphi)}$. Thus, by common abuse of notation, $T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla\varphi)}$ (1.52) takes the form

$$T^{\mathrm{PK}} = \rho^0 \frac{\partial U}{\partial(\nabla u)} \approx_1 T^0 + T^{\mathrm{PK1}}. \tag{6.27}$$

We express the perturbation T^{PK1} of the first Piola-Kirchhoff stress tensor in terms of u. For the material strain tensor (1.26) we have $e = \frac{1}{2}(\nabla\varphi^T \cdot \nabla\varphi - 1_{3\times 3}) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \cdot \nabla u)$ or in components $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) + \frac{1}{2}(\partial_i u_k)(\partial_j u_k)$. Hence, the second-order expansion of the elastic energy density reads

$$\begin{aligned}
\rho^0 U &\approx_2 T^0_{ij} e_{ij} + \frac{1}{2}\,\Xi_{ijkl} e_{ij} e_{kl} \\
&\approx_2 T^0_{ij}\partial_j u_i + \frac{1}{2}\left(T^0_{ij}(\partial_i u_k)(\partial_j u_k) + \Xi_{ijkl}(\partial_j u_i)(\partial_l u_k)\right) \\
&= T^0_{ij}\partial_j u_i + \frac{1}{2}\left(T^0_{jl}(\partial_j u_k)(\partial_l u_k) + \Xi_{ijkl}(\partial_j u_i)(\partial_l u_k)\right) \\
&= T^0_{ij}\partial_j u_i + \frac{1}{2}\left(\delta_{ik} T^0_{jl} + \Xi_{ijkl}\right)(\partial_j u_i)(\partial_l u_k).
\end{aligned} \tag{6.28}$$

We introduce the **prestressed elasticity tensor** Λ^{T^0} as

$$\Lambda^{T^0}_{ijkl} := \delta_{ik} T^0_{jl} + \Xi_{ijkl}. \tag{6.29}$$

This coincides with prestressed elasticity tensor derived in [MH83, Chapter 3, Proposition 4.15 (a), p. 214] and the formula may alternatively be obtained from a direct linearization of the constitutive law $T^{\mathrm{PK}} = r(.,\nabla\varphi)$, see [MH83, Chapter 4, p. 235].

The definition (6.29) of Λ^{T^0} implies that $\Lambda^{T^0}_{ijkl} \in L^\infty(\mathbb{R}^3)$, has support contained in $\overline{B}$, and satisfies the symmetry relations of hyperelasticity, $\Lambda^{T^0}_{ijkl} = \Lambda^{T^0}_{klij}$. Thus we found

$$\rho^0 U \approx_2 T^0 : \nabla u + \frac{1}{2}\nabla u : \Lambda^{T^0} : \nabla u \tag{6.30}$$

as second-order approximation of the internal energy density in terms of the displacement gradient. Consequently the linearization of the first Piola-Kirchhoff stress tensor is $T^{\mathrm{PK}} \approx_1 T^0 + T^{\mathrm{PK1}}$ with

$$T^{\mathrm{PK1}} := \Lambda^{T^0} : \nabla u, \tag{6.31}$$

or $T^{\mathrm{PK1}}_{ij} := \Lambda^{T^0}_{ijkl}\partial_l u_k$ in components [DT98, (3.120)[1]]. The linear stress-strain relation (6.31) generalizes **Hooke's law** to a prestressed elastic material.

[1]Note that [DT98] employ transposed conventions for ∇u and T^{PK}: Instead of $\Lambda^{T^0}_{ijkl} = \delta_{ik} T^0_{jl} + \Xi_{ijkl}$ (6.29), they replace $i \leftrightarrow j$, $k \leftrightarrow l$ and write $\Lambda^{T^0}_{ijkl} = \delta_{jl} T^0_{ik} + \Xi_{jilk} = \delta_{jl} T^0_{ik} + \Xi_{ijkl}$ [DT98, (3.122)].

Yet, the elastic tensor Ξ may also be written in the more general form [DT98, (3.135)]

$$\Xi^{T^0}_{ijkl} := \Gamma_{ijkl} + a(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij}) + b(T^0_{ik}\delta_{jl} + T^0_{jk}\delta_{il} + T^0_{il}\delta_{jk} + T^0_{jl}\delta_{ik}), \tag{6.32}$$

with parameters $a, b \in \mathbb{R}$ and Γ satisfying the classical symmetries $\Gamma_{ijkl} = \Gamma_{klij} = \Gamma_{jikl} = \Gamma_{ijlk}$. Upon replacing $i \leftrightarrow j$, $k \leftrightarrow l$, $(i,j) \leftrightarrow (k,l)$ and by employing the symmetry of T^0, one easily verifies that Ξ^{T^0} also possesses the desired symmetries (6.26).

In the absence of prestress, Γ is equal to the **classical elasticity tensor** (typically denoted by c, see Sections 1.3.2 and 2.1): In a solid medium, Γ takes the form

$$\Gamma_{ijkl} = (\kappa - \tfrac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \Gamma^{\mathrm{a}}_{ijkl}, \tag{6.33}$$

with bulk modulus κ, shear modulus μ, and the purely anisotropic part Γ^{a}. Thus, the generalized prestressed elasticity tensor (6.29) reads

$$\begin{aligned} \Lambda^{T^0}_{ijkl} &= T^0_{jl}\delta_{ik} + \Xi^{T^0}_{ijkl} \\ &= \Gamma_{ijkl} + a(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij}) + (1+b)T^0_{jl}\delta_{ik} + b(T^0_{ik}\delta_{jl} + T^0_{jk}\delta_{il} + T^0_{il}\delta_{jk}). \end{aligned} \tag{6.34}$$

A suitable choice for the parameters a, b will be discussed in the next section.

6.4.2 Different incremental stress tensors

The **incremental Lagrangian stress tensor** (i.e. the first perturbation of the Cauchy stress tensor in material representation (1.46), $T_t = T^s_t \circ \varphi_t$) reads

$$T^1 \approx_1 T - T^0. \tag{6.35}$$

Its relation to T^{PK1} is obtained based on the Piola transform (1.49) and the expansions (6.15):

$$\begin{aligned} T^0 + T^{\mathrm{PK1}} \approx_1 T^{\mathrm{PK}} = JT \cdot (\nabla\varphi)^{-T} &\approx_1 (1 + \nabla\cdot u)(T^0 + T^1)\cdot(1_{3\times 3} - (\nabla u)^{-T}) \\ &\approx_1 T^0 + T^0(\nabla\cdot u) - T^0\cdot(\nabla u)^T + T^1. \end{aligned} \tag{6.36}$$

Thus we established [DT98, (3.36)]

$$T^1 \approx_1 T^{\mathrm{PK1}} + T^0\cdot(\nabla u)^T - T^0(\nabla\cdot u) = \Upsilon^{T^0} : \nabla u, \tag{6.37}$$

or in index notation,

$$T^1_{ij} \approx_1 T^{\mathrm{PK1}}_{ij} + T^0_{il}\partial_l u_j - T^0_{ij}\partial_k u_k = \Upsilon^{T^0}_{ijkl}\partial_l u_k,$$

where (cf. [DT98, (3.123) with $i \leftrightarrow j$ and $k \leftrightarrow l$])

$$\begin{aligned} \Upsilon^{T^0}_{ijkl} &= \Lambda^{T^0}_{ijkl} + T^0_{il}\delta_{kj} - T^0_{ij}\delta_{kl} \\ &= \Xi^{T^0}_{ijkl} + T^0_{jl}\delta_{ik} + T^0_{il}\delta_{kj} - T^0_{ij}\delta_{kl} \\ &= \Gamma_{ijkl} + (a-1)T^0_{ij}\delta_{kl} + aT^0_{kl}\delta_{ij} + (b+1)(T^0_{il}\delta_{jk} + T^0_{jl}\delta_{ik}) + b(T^0_{ik}\delta_{jl} + T^0_{jk}\delta_{il}). \end{aligned} \tag{6.38}$$

Symmetry of T^1 corresponds to the minor symmetry $\Upsilon^{T^0}_{ijkl} = \Upsilon^{T^0}_{jikl}$.

In accordance with [DT98, p. 80] we will choose $a = -b = 1/2$, because this convention renders T^1 independent of the initial pressure:

Lemma 6.5 (Incremental Lagrangian stress tensor independent of initial pressure). *With the choice of parameters $a = -b = 1/2$ in the definition (6.32) of the generalized prestressed elasticity tensor Ξ^{T^0}, the incremental Lagrangian stress tensor T^1 will be independent of p^0.*

Proof. If $a = -b = 1/2$ then

$$\Upsilon^{T^0}_{ijkl} = \Gamma_{ijkl} + \frac{1}{2}\left(-T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij} + T^0_{il}\delta_{jk} + T^0_{jl}\delta_{ik} - T^0_{ik}\delta_{jl} - T^0_{jk}\delta_{il}\right)$$

and thus (now writing $=$ instead of $\approx_1$)

$$T^1_{ij} = \Gamma_{ijkl}\partial_l u_k + \frac{1}{2}\left(-T^0_{ij}\partial_k u_k + T^0_{kl}\partial_l u_k \delta_{ij} + T^0_{il}\partial_l u_j + T^0_{jl}\partial_l u_i - T^0_{ik}\partial_j u_k - T^0_{jk}\partial_i u_k\right).$$

Applying the decompositions $T^0 = -p^0 1_{3\times 3} + T^0_{\mathrm{dev}}$ (4.65) for initial stress and

$$\nabla u = \varepsilon + \omega \qquad \text{with} \qquad \varepsilon := \frac{1}{2}(\nabla u + \nabla u^T) = \varepsilon^T \quad \text{and} \quad \omega := \frac{1}{2}(\nabla u - \nabla u^T) = -\omega^T \tag{6.39}$$

for strain (cf. Section 1.3.2), as well as employing the symmetry of Γ and T^0, we deduce

$$\begin{aligned} T^1 &= \Gamma : \nabla u + \frac{1}{2}(\underbrace{-T^0(\nabla\cdot u) + (T^0 : \nabla u)1_{3\times 3}}_{=-T^0(\operatorname{tr}\varepsilon)+(T^0:\varepsilon)1_{3\times 3}} + \underbrace{T^0\cdot(\nabla u)^T + \nabla u\cdot T^0 - T^0\cdot\nabla u - (\nabla u)^T\cdot T^0}_{=T^0\cdot((\nabla u)^T-\nabla u)+(\nabla u-(\nabla u)^T)\cdot T^0 = 2(-T^0\cdot\omega+\omega\cdot T^0)}) \\ &= \Gamma : \varepsilon + \frac{1}{2}\left(-T^0(\operatorname{tr}\varepsilon) + (T^0 : \varepsilon)1_{3\times 3}\right) - T^0\cdot\omega + \omega\cdot T^0 \\ &= \Gamma : \varepsilon + \frac{1}{2}\left(p^0(\operatorname{tr}\varepsilon)1_{3\times 3} - T^0_{\mathrm{dev}}(\operatorname{tr}\varepsilon) - p^0(\operatorname{tr}\varepsilon)1_{3\times 3} + (T^0_{\mathrm{dev}} : \varepsilon)1_{3\times 3}\right) - T^0_{\mathrm{dev}}\cdot\omega + \omega\cdot T^0_{\mathrm{dev}}, \end{aligned}$$

that is,

$$T^1 = \Gamma : \varepsilon + \frac{1}{2}\left(-T^0_{\mathrm{dev}}(\operatorname{tr}\varepsilon) + (T^0_{\mathrm{dev}} : \varepsilon)1_{3\times 3}\right) - T^0_{\mathrm{dev}}\cdot\omega + \omega\cdot T^0_{\mathrm{dev}}, \tag{6.40}$$

which does not depend on the initial pressure p^0. □

We note that the choice $a = -b = 1/2$ yields

$$\Lambda^{T^0}_{ijkl} = \Gamma_{ijkl} + \frac{1}{2}\left(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij} + T^0_{jl}\delta_{ik} - T^0_{ik}\delta_{jl} - T^0_{jk}\delta_{il} - T^0_{il}\delta_{jk}\right)$$

and

$$\Xi^{T^0}_{ijkl} = \Gamma_{ijkl} + \frac{1}{2}\left(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij} - T^0_{jl}\delta_{ik} - T^0_{ik}\delta_{jl} - T^0_{jk}\delta_{il} - T^0_{il}\delta_{jk}\right).$$

However, the choice $a = -b = 1/2$ has a major advantage in the case of a hydrostatic earth model (see Section 7.5): The deviatoric prestress T^0_{dev} vanishes by definition and consequently (6.40) implies that

$$T^1 = \Gamma : \varepsilon,$$

showing that T^1 is even independent of T^0. In particular, inserting (6.33) with the Lamé constant $\lambda = \kappa - \frac{2}{3}\mu$, we obtain

$$\begin{aligned} T^1_{ij} = \Gamma_{ijkl}\varepsilon_{kl} &= \left(\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right)\partial_l u_k + \Gamma^{\mathrm{a}}_{ijkl}\varepsilon_{kl} \\ &= \lambda\delta_{ij}\partial_k u_k + \mu\partial_j u_i + \mu\partial_i u_j + \Gamma^{\mathrm{a}}_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + \Gamma^{\mathrm{a}}_{ijkl}\varepsilon_{kl}, \end{aligned}$$

that is, classical Hooke's law including an additional anisotropic part:

$$T^1 = \lambda \operatorname{tr}\varepsilon\, 1_{3\times 3} + 2\mu\,\varepsilon + \Gamma^{\mathrm{a}} : \varepsilon.$$

Hence, in the hydrostatic case, the incremental elastic behavior is independent of prestress. In other words, formulated in terms of the Lagrangian incremental stress T^1, hydrostatically prestressed linearized elasticity reduces to classical linearized elasticity (cf. Section 2.1).

6.4.3 Elasticity tensors in prestressed elastic fluids

In an elastic fluid, shear resistance μ and anisotropy Γ^{a} vanish and Γ (6.33) reduces to

$$\Gamma_{ijkl} = \kappa\, \delta_{ij}\delta_{kl}.$$

In fluid regions of the earth model, prestress is hydrostatic: $T^0 = -p^0 1_{3\times 3}$, that is, $T^0_{\mathrm{dev}} = 0$. Thus (6.40) leads to the following simple form of the incremental Lagrangian stress tensor in prestressed elastic fluids:

$$T^1_{ij} = \kappa\, \varepsilon_{kk}\delta_{ij} = \kappa(\partial_k u_k)\delta_{ij} \qquad \text{i.e.} \qquad T^1 = \kappa(\operatorname{tr}\varepsilon)1_{3\times 3} = \kappa(\nabla\cdot u)1_{3\times 3}. \tag{6.41}$$

Furthermore, we have

$$\begin{aligned}
\Lambda^{T^0}_{ijkl} &= \kappa\delta_{ij}\delta_{kl} - \frac{1}{2}p^0(\delta_{ij}\delta_{kl} + \delta_{kl}\delta_{ij} + \delta_{jl}\delta_{ik} - \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} - \delta_{il}\delta_{jk}) \\
&= \kappa\delta_{ij}\delta_{kl} - p^0(\delta_{ij}\delta_{kl} - \delta_{jk}\delta_{il}) \\
&= p^0(\gamma-1)\delta_{ij}\delta_{kl} + p^0\delta_{jk}\delta_{il}
\end{aligned} \tag{6.42}$$

where

$$\gamma := \kappa/p^0$$

is the fluid's adiabatic index, cf. [Val86]. Hence, the first perturbation of the first Piola-Kirchhoff stress tensor in elastic prestressed fluids is given by

$$T^{\mathrm{PK1}}_{ij} = \Lambda^{T^0}_{ijkl}\partial_l u_k = p^0(\gamma-1)(\partial_k u_k)\delta_{ij} + p^0\partial_i u_j,$$

i.e.

$$T^{\mathrm{PK1}} = p^0(\gamma-1)(\nabla\cdot u)1_{3\times 3} + p^0(\nabla u)^T. \tag{6.43}$$

Similarly,

$$\begin{aligned}
\Xi^{T^0}_{ijkl} &= \kappa\delta_{ij}\delta_{kl} - \frac{1}{2}p^0(\delta_{ij}\delta_{kl} + \delta_{kl}\delta_{ij} - \delta_{jl}\delta_{ik} - \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} - \delta_{il}\delta_{jk}) \\
&= \kappa\delta_{ij}\delta_{kl} - p^0(\delta_{ij}\delta_{kl} - \delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}) \\
&= p^0(\gamma-1)\delta_{ij}\delta_{kl} + p^0(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}).
\end{aligned}$$

Inserting Ξ^{T^0} and $T^0 = -p^0 1_{3\times 3}$ in (6.28) yields the following second-order approximation for the elastic energy density in fluid regions of the earth model

$$\rho^0 U \approx_2 -p^0(\nabla\cdot u) + \frac{p^0}{2}\left((\gamma-1)(\nabla\cdot u)^2 + \nabla u : (\nabla u)^T\right).$$

The latter is consistent with the above equation for T^{PK1} and (6.27) in fluids, since a direct computation shows

$$\rho^0\frac{\partial U}{\partial(\nabla u)} \approx_1 -p^0 1_{3\times 3} + p^0(\gamma-1)(\nabla\cdot u)1_{3\times 3} + p^0(\nabla u)^T = T^0 + T^{\mathrm{PK1}}.$$

6.5 Direct linearization of the governing equations

We formally linearize the nonlinear elastic-gravitational equations directly, that is, we show how the linearized system of Section 2.3.2 arises from the nonlinear system of Section 2.3.1. Essentially, we simply replace the terms occurring in the nonlinear system by the corresponding linearizations, dropping any resulting product terms of higher order. However, we anticipate that the complete linearized system will be obtained independently and in a rigorous way in Section 7.3 from the variational principle applied to the first- and second-order approximation (6.48) of the action integral.

6.5.1 Linearization of the Poisson equation

In order to linearize Poisson's equation (2.2),

$$\triangle \Phi^s = 4\pi G \rho^s,$$

one writes $\Phi^s = \Phi^0 + \Phi^{s1}$ and approximates the spatial density ρ^s correct to first order in u. Linearizing conservation of mass (1.40) in material representation,

$$\rho = \frac{\rho^0}{J} = \frac{\rho^0}{\det \nabla \varphi} = \frac{\rho^0}{\det(1_{3\times 3} + \nabla u)} \approx_1 \frac{\rho^0}{1 + \nabla \cdot u} \approx_1 \rho^0(1 - \nabla \cdot u) = \rho^0 - \rho^0 \nabla \cdot u,$$

and approximating the material density ρ as in (6.7),

$$\rho = \rho^s \circ \varphi \approx_1 \rho^0 + \rho^{s1} + \nabla \rho^0 \cdot u,$$

gives $\rho^0 - \rho^0 \nabla \cdot u \approx_1 \rho^0 + \rho^{s1} + \nabla \rho^0 \cdot u$, that is,

$$\rho^{s1} = -\nabla \cdot (\rho^0 u). \tag{6.44}$$

The identity

$$\rho^s \approx_1 \rho^0 - \nabla \cdot (\rho^0 u)$$

has a clear physical interpretation: A positive value $\nabla \cdot (\rho^0 u) > 0$ represents a diverging mass flux $\rho^0 u$, which must reduce the spatial density ρ^s, whereas $\nabla \cdot (\rho^0 u) < 0$ represents a converging mass flux, increasing the density.

Hence, the **linearized Poisson equation** reads

$$\triangle(\Phi^0 + \Phi^{s1}) = 4\pi G \left(\rho^0 - \nabla \cdot (\rho^0 u)\right). \tag{6.45}$$

The terms of order zero give Poisson's equation in equilibrium, $\triangle \Phi^0 = 4\pi G \rho^0$ (2.20), and first-order terms yield the perturbed Poisson's equation $\triangle \Phi^{s1} = -4\pi G \nabla \cdot (\rho^0 u)$ (2.27).

6.5.2 Linearization of the equation of motion

Our starting point is the nonlinear material equation of motion (2.10),

$$\rho^0 \left(\ddot{\varphi} + 2\,\Omega \times \dot{\varphi} + \Omega \times (\Omega \times \varphi)\right) \;=\; \nabla \cdot T^{\mathrm{PK}} + \rho^0 g + f,$$

in the version (2.12):

$$\rho^0 \left(\ddot{\varphi} + 2\,\Omega \times \dot{\varphi}\right) \;=\; \nabla \cdot T^{\mathrm{PK}} - \rho^0 (\nabla(\Phi^s + \Psi^s)) \circ \varphi + f.$$

We have $\dot{\varphi} = \dot{u}$, $\ddot{\varphi} = \ddot{u}$, and $T^{\mathrm{PK}} \approx_1 T^0 + T^{\mathrm{PK}1}$ (6.27). The geopotential gradient $(\nabla(\Phi^s + \Psi^s)) \circ \varphi$ is approximated analogously to the expansion $q \approx_1 q^0 + q^{s1} + \nabla q^0 \cdot u$ (6.7): Correct to first order in the perturbations u and Φ^{s1},

$$q = (\nabla(\Phi^s + \Psi^s)) \circ \varphi, \quad \text{that is,} \quad q(X,t) = (\nabla(\Phi^s + \Psi^s))(\varphi(X,t),t) \quad \text{for} \;\; (X,t) \in B \times I,$$

is given by

$$\begin{aligned} q(X,t) &= (\nabla((\Phi^0 + \Psi^s) + \Phi^{s1}))(X + u(X,t),t) \\ &\approx_1 \nabla(\Phi^0 + \Psi^s)(X,t) + \nabla \Phi^{s1}(X,t) + (\nabla\nabla(\Phi^0 + \Psi^s) \cdot u)(X,t), \end{aligned}$$

that is,

$$(\nabla(\Phi^s + \Psi^s)) \circ \varphi \approx_1 \nabla(\Phi^0 + \Psi^s) + \nabla \Phi^{s1} + \nabla\nabla(\Phi^0 + \Psi^s) \cdot u.$$

We recognize the material gravitational acceleration which, correct to first order, is given by

$$(\nabla\Phi^s)\circ\varphi = g \approx_1 g^0 + g^1 = g^0 + g^{s1} + \nabla g^0 \cdot u = -\nabla(\Phi^0 + \Phi^{s1}) - \nabla\nabla\Phi^0 \cdot u.$$

The remaining terms correspond to the material centrifugal acceleration, which by (2.8) is linear in u and thus coincides with its linearization:

$$(\nabla\Psi^s)\circ\varphi = \Omega\times(\Omega\times\varphi) = \Omega\times(\Omega\times(.+u)) = \nabla\Psi^s + \nabla\nabla\Psi^s\cdot u.$$

We then obtain the **linearized equation of motion**

$$\rho^0\left(\ddot{u} + 2\,\Omega\times\dot{u} \;+\; \nabla(\Phi^0+\Psi^s) \;+\; \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla\Phi^{s1}\right) - \nabla\cdot(T^0 + T^{\mathrm{PK1}}) = f^1. \quad (6.46)$$

Here $T^{\mathrm{PK1}} = \Lambda^{T^0} : \nabla u$ by (6.31) and $f^1 \approx_1 f$ by (6.21). Equation (6.46) comprises the static equilibrium equation (2.19) (terms of order zero),

$$\rho^0\nabla(\Phi^0+\Psi^s) - \nabla\cdot T^0 = 0,$$

and the linear equation of motion (2.26) (first-order terms),

$$\rho^0\left(\ddot{u} + 2\,\Omega\times\dot{u} \;+\; \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla\Phi^{s1}\right) - \nabla\cdot T^{\mathrm{PK1}} = f.$$

Due to nonzero prestress T^0, expressing the linearized equation of motion not in terms of the perturbation of the first Piola-Kirchhoff stress T^{PK} (but instead in terms of the perturbation of Cauchy stress T^s, Lagrangian stress T, or second Piola-Kirchhoff stress T^{SK}) will result in a different form of the linearized equation of motion, see [DT98, (3.51), (3.56), (3.62) respectively]:

$$\begin{aligned}
\rho^0\left(\ddot{u} + 2\,\Omega\times\dot{u}\right) + \rho^0\nabla\Phi^{s1} + \rho^{s1}\nabla(\Phi^0+\Psi^s) - \nabla\cdot T^{s1} &= f,\\
\rho^0\left(\ddot{u} + 2\,\Omega\times\dot{u}\right) + \rho^0\nabla\Phi^{s1} + \rho^{s1}\nabla(\Phi^0+\Psi^s) - \nabla\cdot(\nabla T^0\cdot u + T^1) &= f,\\
\rho^0\left(\ddot{u} + 2\,\Omega\times\dot{u} + \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla\Phi^{s1}\right) - \nabla\cdot(\nabla u\cdot T^0 + T^{\mathrm{SK1}}) &= f.
\end{aligned}$$

All these variants coincide with (2.26) up to first order in u, Φ^{s1}, and the stress perturbations.

In Section 7.4.2, the linearized elastic-gravitational equations (6.45) and (6.46) will be obtained from the variational principle as Euler-Lagrange equations (3.4).

6.5.3 Linearized dynamical interface condition for traction

We directly linearize the dynamical interface condition asserting that the spatial traction $T^s\cdot\nu^s$ must be continuous across interior surfaces, welded or perfectly slipping.

Lemma 6.6 (**Linearized dynamical interface condition**). *Let S be a sufficiently regular (e.g. $\mathcal{C}^{1,1}$) surface S in B; S may be welded or perfectly slipping. Then the exact dynamical interface condition* (2.16)*, $[T^s]^+_-\cdot\nu^s = 0$ on $\varphi_t(S)$, takes the linearized form*

$$[T^{\mathrm{PK1}}\cdot\nu - \widetilde{\nabla}\cdot((T^0\cdot\nu)u)]^+_- = 0 \quad (6.47)$$

on S, that is, $[T^{\mathrm{PK1}}_{ij}\nu_j - \widetilde{\partial}_k(T^0_{ij}\nu_j u_k)]^+_- = 0$.

Formula (6.47) for the linearized dynamical interface condition coincides with [DT98, (3.73)]. By the product rule, we may write the condition as

$$[T^{\mathrm{PK1}}\cdot\nu - \widetilde{\nabla}(T^0\cdot\nu)\cdot u - (T^0\cdot\nu)(\widetilde{\nabla}\cdot u)]^+_- = 0.$$

Proof. We argue along the lines of [DT98, 3.4.2], based on techniques similar to the proof of Lemma 6.4 and introduce the points X' and X'' as in (6.16), see also the figure on p. 108. We rewrite (6.14), $\mathrm{dS}^s \approx_1 (1+\widetilde{\nabla}\cdot u)\,\mathrm{dS}$, in the form

$$\mathrm{dS} \approx_1 (1+\widetilde{\nabla}\cdot u)^{-1}\,\mathrm{dS}^s \approx_1 (1-\widetilde{\nabla}\cdot u)\,\mathrm{dS}^s.$$

The material counterpart of (2.16), $\big((T^0+T^{\mathrm{PK1}})\cdot\nu\,\mathrm{d}S\big)^+(X') = \big((T^0+T^{\mathrm{PK1}})\cdot\nu\,\mathrm{d}S\big)^-(X'')$, thus reads

$$\Big((T^0+T^{\mathrm{PK1}})\cdot\nu\,(1-\widetilde{\nabla}\cdot u)\Big)^+(X')\,\mathrm{d}S^s(x) \approx_1 \Big((T^0+T^{\mathrm{PK1}})\cdot\nu\,(1-\widetilde{\nabla}\cdot u)\Big)^-(X'')\,\mathrm{d}S^s(x),$$

which implies

$$\underbrace{\big(T^{\mathrm{PK1}}\cdot\nu\big)^+(X') - \big(T^{\mathrm{PK1}}\cdot\nu\big)^-(X'')}_{=:A} + \underbrace{\big(T^0\cdot\nu\big)^+(X') - \big(T^0\cdot\nu\big)^-(X'')}_{=:B}$$
$$\underbrace{-\Big((T^0\cdot\nu)(\widetilde{\nabla}\cdot u)\Big)^+(X') + \Big((T^0\cdot\nu)(\widetilde{\nabla}\cdot u)\Big)^-(X'')}_{=:C} \approx_1 0.$$

A Taylor approximation yields $A \approx_1 [T^{\mathrm{PK1}}\cdot\nu]^+_-(X')$ and $C \approx_1 -[(T^0\cdot\nu)(\widetilde{\nabla}\cdot u)]^+_-(X')$. We find $B \approx_1 -[\widetilde{\nabla}(T^0\cdot\nu)\cdot u]^+_-(X')$ by employing, in addition, $[T^0]^+_-\cdot\nu = 0$ (cf. (2.22)) and the kinematical slip condition $[u]^+_-\cdot\nu = 0$ (6.10). Combining these terms we get $[T^{\mathrm{PK1}}\cdot\nu - \widetilde{\nabla}\cdot((T^0\cdot\nu)u)]^+_- \approx_1 0$, proving (6.47). □

If the surface is welded, as a solid-solid boundary $S = \Sigma^{\mathrm{SS}}$, the conditions $[T^0]^+_-\cdot\nu = 0$ (2.22) and $[u]^+_- = 0$ (2.28) imply continuity of $\widetilde{\nabla}\cdot((T^0\cdot\nu)u)$, and therefore (6.47) reduces to (2.31):

$$\big[T^{\mathrm{PK1}}\big]^+_-\cdot\nu = 0 \quad \text{on} \quad \Sigma^{\mathrm{SS}}.$$

Considering the case of perfectly slipping fluid-solid boundaries $S = \Sigma^{\mathrm{FS}}$, we may substitute $T^0\cdot\nu = -p^0\nu$ (2.23) in (6.47):

$$[T^{\mathrm{PK1}}\cdot\nu - \widetilde{\nabla}\cdot((T^0\cdot\nu)u)]^+_- = [T^{\mathrm{PK1}}\cdot\nu + \widetilde{\nabla}\cdot(p^0\nu\,u)]^+_- = [T^{\mathrm{PK1}}\cdot\nu + \nu\widetilde{\nabla}\cdot(p^0u) + p^0\widetilde{\nabla}\nu\cdot u]^+_-.$$

Since $\widetilde{\nabla}\nu\cdot f^{\parallel} = -\nu\cdot\widetilde{\nabla}f^{\parallel}$ (4.19), $[p^0]^+_- = 0$ (4.67), and $[u]^+_-\cdot\nu = 0$ (6.10), the jump of the last term is

$$[p^0\widetilde{\nabla}\nu\cdot u]^+_- = p^0\widetilde{\nabla}\nu\cdot[u]^+_- = -p^0\nu\cdot\widetilde{\nabla}[u]^+_- = [-p^0\nu\cdot\widetilde{\nabla}u]^+_-.$$

Thus (6.47) reduces to (2.32), i.e. $[\tau^{\mathrm{PK1}}]^+_- = 0$:

$$\Big[T^{\mathrm{PK1}}\cdot\nu + \nu\widetilde{\nabla}\cdot(p^0u) - p^0\nu\cdot\widetilde{\nabla}u\Big]^+_- = 0 \quad \text{on} \quad \Sigma^{\mathrm{FS}}.$$

In Section 7.4.3, the linearized dynamical slip conditions (2.31) and (2.32) will be obtained from the variational principle as natural interior boundary conditions (3.7).

6.6 Second-order approximation of the action

Aiming at dynamical equations which are linear in the perturbations (u, Φ^{s1}), we approximate the action integral by a quadratic expression in these variables and their first-order derivatives. Inserting $\varphi = \mathrm{Id}_{\mathbb{R}^3} + u$, $\Phi^s = \Phi^0 + \Phi^{s1}$, and the second-order relation (6.30) for prestressed linear elasticity in the action integral $\mathscr{A}''(\varphi, \Phi^s)$ (5.9) leads to the approximation

$$\begin{aligned}\mathscr{A}''(\varphi,\Phi^s) &= \mathscr{A}''(\mathrm{Id}_{\mathbb{R}^3}+u, \Phi^0+\Phi^{s1})\\ &\approx_2 \mathscr{A}''_{[0]}(u,\Phi^{s1}) + \mathscr{A}''_{[1]}(u,\Phi^{s1}) + \mathscr{A}''_{[2]}(u,\Phi^{s1})\end{aligned} \tag{6.48}$$

where $\approx_2$ indicates the omission of terms of third order in $(u, \nabla u, \dot{u}, \Phi^{s1}, \nabla\Phi^{s1}, \dot{\Phi}^{s1})$, with

$$\mathscr{A}''_{[i]}(u, \Phi^{s1}) := \int_I \left(\int_{\mathbb{R}^3} L''_{[i]} \, \mathrm{dV} + \int_{\Sigma^{\mathrm{FS}}} L''_{\Sigma^{\mathrm{FS}},[i]} \, \mathrm{dS} \right) \mathrm{d}t \qquad (i = 0, 1, 2). \tag{6.49}$$

As a general principle, we always assume the validity of all first-order results when we modify second-order terms. In particular, we will frequently employ the kinematical slip condition $[u]^+_- \cdot \nu \approx_1 0$ (6.10) as well as the equilibrium stress interface condition $[T^0]^+_- \cdot \nu = 0$ (2.22), which reduces to continuity of $p^0 = -\nu \cdot T^0 \cdot \nu$ across fluid-solid boundaries.

6.6.1 Quadratic volume Lagrangians

The zero-, first-, and second-order volume actions of (6.49) with the approximated volume Lagrangian densities $L''_{[0]}$, $L''_{[1]}$, and $L''_{[2]}$ are defined within the regularity setting of the linearized fields on the composite fluid-solid earth model (Assumption 2). This setting is in complete consistence with the regularity conditions making the action integral (5.9) defined in the nonlinear case (Assumption 1).

The approximated volume Lagrangian densities are functions of x, t, $u(x,t)$, $\Phi^{s1}(x,t)$, $\nabla u(x,t)$, $\dot{u}(x,t)$, and $\nabla\Phi^{s1}(x,t)$ (note their independence of $\dot{\Phi}^{s1}(x,t)$). Their explicit form is as follows:

Lemma 6.7 (**Approximated volume Lagrangian densities**). *The Lagrangian densities $L''_{[0]}$, $L''_{[1]}$, and $L''_{[2]}$, approximating L'' (5.10) up to order two in (u, Φ^{s1}), read*

$$L''_{[0]} = -\rho^0(\Phi^0 + \Psi^s) - \frac{1}{8\pi G}(\nabla\Phi^0)^2, \tag{6.50}$$

$$\begin{aligned} L''_{[1]} &= \rho^0 \dot{u} \cdot (\Omega \times x) - T^0 : \nabla u - \rho^0 u \cdot \nabla(\Phi^0 + \Psi^s) \\ &\quad -\rho^0\Phi^{s1} - \frac{1}{4\pi G}\nabla\Phi^0 \cdot \nabla\Phi^{s1} - \rho^0 F^s, \end{aligned} \tag{6.51}$$

$$\begin{aligned} L''_{[2]} &= \frac{1}{2}\rho^0\dot{u}^2 + \rho^0\dot{u} \cdot (\Omega \times u) - \frac{1}{2}\nabla u : \Lambda^{T^0} : \nabla u - \frac{1}{2}\rho^0 u \cdot (\nabla\nabla(\Phi^0 + \Psi^s)) \cdot u \\ &\quad -\rho^0 u \cdot \nabla\Phi^{s1} - \frac{1}{8\pi G}(\nabla\Phi^{s1})^2 - \rho^0 u \cdot \nabla F^s. \end{aligned} \tag{6.52}$$

The Lagrangian density $L''_{[2]}$ coincides with the volume Lagrangian in the "displacement-potential variational principle" of [DT98, (3.190), p. 89].

Proof. We recall Formula (5.10) for the material volume Lagrangian density of the nonlinear model:

$$L''(X,t) = \Big(\frac{1}{2}\dot{\varphi}^2 + \dot{\varphi} \cdot (\Omega \times \varphi) - U - (\Phi + \Psi) - F\Big)(X,t)\rho^0(X) - \frac{1}{8\pi G}(\nabla\Phi^s)^2(X,t).$$

With $\varphi = \mathrm{Id}_{\mathbb{R}^3} + u$, the kinetic energy density, the Coriolis term, and the centrifugal potential read

$$\begin{aligned} \dot{\varphi}^2 &= \dot{u}^2, \\ \dot{\varphi} \cdot (\Omega \times \varphi) &= \dot{u} \cdot (\Omega \times x) + \dot{u} \cdot (\Omega \times u), \\ \Psi &= \Psi^s + u \cdot \nabla\Psi^s + \frac{1}{2}u \cdot \nabla\nabla\Psi^s \cdot u, \end{aligned}$$

where the last equality holds since $\Psi(X,t) = \Psi^s \circ \varphi_t(X) = \Psi^s(X + u_t(X))$ and Ψ^s is a second-order polynomial, see (2.7). Upon inserting $\Phi^s = \Phi^0 + \Phi^{s1}$, the squared norm of the spatial gradient is

$$(\nabla\Phi^s)^2 = \big(\nabla\Phi^0 + \nabla\Phi^{s1}\big)^2 = (\nabla\Phi^0)^2 + 2\nabla\Phi^0 \cdot \nabla\Phi^{s1} + (\nabla\Phi^{s1})^2.$$

The second-order approximation for the material gravitational potential $\Phi_t = (\Phi^0 + \Phi^{s1}) \circ \varphi_t$ is given by (compare to the calculation in Section 6.5.2)

$$\Phi \approx_2 \Phi^0 + (u \cdot \nabla \Phi^0 + \Phi^{s1}) + \left(\frac{1}{2}\, u \cdot \nabla \nabla \Phi^0 \cdot u + u \cdot \nabla \Phi^{s1} \right)$$

(where $\approx_2$ indicates the omission of terms of third order involving (u, Φ^{s1}) or their derivatives). Here, we have grouped the linear and quadratic terms with brackets.

Inserting these approximations and the second-order elastic energy density (6.30) into the Lagrangian density (5.10) and comparing with the powers of (u, Φ^{s1}) and their derivatives in (6.48) leads to equations (6.50) to (6.52). □

We note that the Lagrangian densities $L''_{[0]}$, $L''_{[1]}$, and $L''_{[2]}$ do not depend on $\dot{\Phi}^{s1}(x,t)$. Moreover, $L''_{[0]}$ is independent of the variables (u, Φ^{s1}), their derivatives, as well as of time. Furthermore, $L''_{[1]}$ and $L''_{[2]}$ are first- and second-order polynomials in u, Φ^{s1}, ∇u, $\dot{u}$, $\nabla \Phi^{s1}$, the force potential F^s, and its gradient ∇F^s with time-independent coefficients (the appearance of the force terms in the different approximation levels is explained in Section 6.3.3).

6.6.2 Quadratic surface Lagrangians

The second-order surface action is obtained as an approximation of the exact surface action

$$\mathscr{A}_{\Sigma^{\mathrm{FS}}}(\varphi) = \mathscr{A}''_{\Sigma^{\mathrm{FS}}}(\varphi) = \int_I \mathscr{A}''_{\Sigma^{\mathrm{FS}},t}(\varphi)\, \mathrm{d}t$$

(5.12) with surface Lagrangian (5.11), that is

$$\mathscr{A}''_{\Sigma^{\mathrm{FS}},t}(\mathrm{Id}_{\mathbb{R}^3} + u) = \int_{\Sigma^{\mathrm{FS}}} L''_{\Sigma^{\mathrm{FS}},t} \mathrm{dS} = -\int_{t_0}^{t} \int_{\Sigma^{\mathrm{FS}}} [\dot{u}_{t'} \cdot T^{\mathrm{PK}}_{t'}]^+_- \cdot \nu \, \mathrm{dS}\, \mathrm{d}t'.$$

The derivation of the second-order surface action is more involved than the derivation of the corresponding approximated volume action. It requires additional surface conditions on the smoothness of the fluid-solid interfaces Σ^{FS} and the interface reference pressure $p^0|_{\Sigma^{\mathrm{FS}}}$ (see (iii) of Assumption 2 below).

We present two variants to derive the approximation of the action up to second order: First, by a direct expansion of the surface Lagrangian density (which is straightforward but rather technical) and then, alternatively, from energy conservation and the second-order tangential slip condition (6.13).

Lemma 6.8 (Approximated surface Lagrangian densities). *Let Σ^{FS} locally be a $\mathcal{C}^{1,1}$-surface and $p^0|_{\Sigma^{\mathrm{FS}}} \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})$. Then the surface Lagrangian densities $L''_{\Sigma^{\mathrm{FS}},[0]}$, $L''_{\Sigma^{\mathrm{FS}},[1]}$, $L''_{\Sigma^{\mathrm{FS}},[2]}$, approximating $L''_{\Sigma^{\mathrm{FS}}}$ (5.11) up to order two in (u, Φ^{s1}), are given by*

$$L''_{\Sigma^{\mathrm{FS}},[0]} = L''_{\Sigma^{\mathrm{FS}},[1]} = 0, \tag{6.53}$$

$$L''_{\Sigma^{\mathrm{FS}},[2]} = -p^0 \left[\nu \cdot \widetilde{\nabla} u \cdot u + \frac{1}{2} u \cdot \widetilde{\nabla} \nu \cdot u \right]^+_-. \tag{6.54}$$

Remark 6.9 lists several equivalent formulas for $L''_{\Sigma^{\mathrm{FS}},[2]}$. The surface Lagrangian $L''_{\Sigma^{\mathrm{FS}},[2]}$ in the equivalent form (6.57) coincides with the surface density proposed in [DT98, (3.165), p. 89].

Remark 6.9 (**Equivalent representations of the surface Lagrangian density**). The identity $\nu \cdot \widetilde{\nabla} u \cdot u = u \cdot \widetilde{\nabla}(u \cdot \nu) - u \cdot \widetilde{\nabla} \nu \cdot u$ allows us to write (6.54) as

$$L''_{\Sigma^{\mathrm{FS}},[2]} = -p^0 \Big[u \cdot \widetilde{\nabla}(\nu \cdot u) - \frac{1}{2} u \cdot \widetilde{\nabla} \nu \cdot u \Big]_-^+. \tag{6.55}$$

Upon integration one also obtains

$$\begin{aligned} \int_{\Sigma^{\mathrm{FS}}} L''_{\Sigma^{\mathrm{FS}},[2]} \, \mathrm{d}S &= \int_{\Sigma^{\mathrm{FS}}} \Big[(\nu \cdot u) \widetilde{\nabla} \cdot (p^0 u) + \frac{1}{2} p^0 u \cdot \widetilde{\nabla} \nu \cdot u \Big]_-^+ \mathrm{d}S & (6.56) \\ &= \frac{1}{2} \int_{\Sigma^{\mathrm{FS}}} \Big[(\nu \cdot u) \widetilde{\nabla} \cdot (p^0 u) - p^0 \nu \cdot \widetilde{\nabla} u \cdot u \Big]_-^+ \mathrm{d}S. & (6.57) \end{aligned}$$

Here, we used the identities

$$\begin{aligned} p^0 u \cdot \widetilde{\nabla}(\nu \cdot u) &= \widetilde{\nabla} \cdot \big(p^0 u (\nu \cdot u) \big) - (\nu \cdot u) \widetilde{\nabla} \cdot (p^0 u) \\ p^0 u \cdot \widetilde{\nabla} \nu \cdot u &= \widetilde{\nabla} \cdot (p^0 u (\nu \cdot u)) - (\nu \cdot u) \widetilde{\nabla} \cdot (p^0 u) - p^0 \nu \cdot \widetilde{\nabla} u \cdot u \end{aligned}$$

and applied the surface divergence theorem (Lemma 4.5) to the corresponding surface action, where the line integrals vanish as $p^0 = 0$ on $\partial \Sigma^{\mathrm{FS}} \subseteq \partial B$. ⌟

Next we present the two variants to prove Lemma 6.8:

Proof. (**Direct expansion**) We first apply the Piola transform (1.49) $T^{\mathrm{PK}} = JT \cdot (\nabla \varphi)^{-T}$ and linearize each factor:

$$T \approx_1 T^0 + T^1, \qquad J \approx_1 1 + \nabla \cdot u \qquad \text{and} \qquad (\nabla \varphi)^{-T} \approx_1 1_{3\times 3} - (\nabla u)^T.$$

Consequently,

$$\begin{aligned} T^{\mathrm{PK}} = JT \cdot (\nabla \varphi)^{-T} &\approx_1 (1 + \nabla \cdot u)(T^0 + T^1) \cdot (1_{3\times 3} - (\nabla u)^T) \\ &\approx_1 T^0 + (\nabla \cdot u) T^0 + T^1 - T^0 \cdot (\nabla u)^T. \end{aligned}$$

Thus, to second order and noticing that $T^0 \cdot (\nabla u)^T = (\nabla u \cdot T^0)^T$ by symmetry of T^0,

$$\begin{aligned} \dot{u} \cdot T^{\mathrm{PK}} \cdot \nu &\approx_2 \dot{u} \cdot T^0 \cdot \nu + (\nabla \cdot u) \dot{u} \cdot T^0 \cdot \nu + \dot{u} \cdot T^1 \cdot \nu - \dot{u} \cdot T^0 \cdot (\nabla u)^T \cdot \nu \\ &= \dot{u} \cdot T^0 \cdot \nu + (\nabla \cdot u) \dot{u} \cdot T^0 \cdot \nu + \dot{u} \cdot T^1 \cdot \nu - \nu \cdot (\nabla u \cdot T^0) \cdot \dot{u}. \end{aligned}$$

Rewriting the second and the last term in terms of surface derivatives yields

$$\begin{aligned} &(\nabla \cdot u) \dot{u} \cdot T^0 \cdot \nu - \nu \cdot (\nabla u \cdot T^0) \cdot \dot{u} \\ &\quad = (\widetilde{\nabla} \cdot u) \dot{u} \cdot T^0 \cdot \nu - \nu \cdot (\widetilde{\nabla} u \cdot T^0) \cdot \dot{u} + (\nu \cdot \nabla u \cdot \nu) \dot{u} \cdot T^0 \cdot \nu - (\nu \cdot \nabla u \cdot \nu) \nu \cdot T^0 \cdot \dot{u} \\ &\qquad\qquad = (\widetilde{\nabla} \cdot u) \dot{u} \cdot T^0 \cdot \nu - \nu \cdot (\widetilde{\nabla} u \cdot T^0) \cdot \dot{u}, \end{aligned}$$

since T^0 is symmetric. As we will consider the jump across Σ^{FS} where the normal component of u is continuous, we argue that we may approximate the material Cauchy stress perturbation T^1 by the surface derivative of T^0 in direction of u: Indeed, by (6.8)

$$T^1 \approx_1 T^{s1} + \nabla T^0 \cdot u = T^s - T^0 + \nabla T^0 \cdot u,$$

and hence

$$[\dot{u} \cdot T^1 \cdot \nu]_-^+ \approx_2 [\dot{u} \cdot T^s \cdot \nu]_-^+ - [\dot{u} \cdot T^0 \cdot \nu]_-^+ + [\dot{u} \cdot (\nabla T^0 \cdot u) \cdot \nu]_-^+ = [\dot{u} \cdot (\widetilde{\nabla} T^0 \cdot u) \cdot \nu]_-^+,$$

since T^s and T^0 satisfy the normality condition (4.27) and $[u\cdot\nu]^+_- = 0$. Therefore we obtain

$$\begin{aligned}[\dot u\cdot T^{\mathrm{PK}}\cdot\nu]^+_- &\approx_2 [\dot u\cdot T^0\cdot\nu + (\widetilde\nabla\cdot u)\dot u\cdot T^0\cdot\nu + \dot u\cdot T^1\cdot\nu - \nu\cdot(\widetilde\nabla u\cdot T^0)\cdot\dot u]^+_-\\ &= [\dot u\cdot T^0\cdot\nu + (\widetilde\nabla\cdot u)\dot u\cdot T^0\cdot\nu + \dot u\cdot(\widetilde\nabla T^0\cdot u)\cdot\nu - \nu\cdot(\widetilde\nabla u\cdot T^0)\cdot\dot u]^+_-.\end{aligned}$$

Successive application of the product rule and symmetry of T^0 and $\widetilde\nabla\nu$ enables us to write the sum of the second and the third term in the form

$$\begin{aligned}\dot u\cdot((\widetilde\nabla\cdot u)T^0 + (\widetilde\nabla T^0\cdot u))\cdot\nu &= \dot u_i((\widetilde\partial_k u_k)T^0_{ij} + (\widetilde\partial_k T^0_{ij})u_k))\nu_j\\ &= \dot u_i\widetilde\partial_k(T^0_{ij}u_k)\nu_j\\ &= \widetilde\partial_k(\dot u_i T^0_{ij}u_k\nu_j) - (\widetilde\partial_k\dot u_i)T^0_{ij}u_k\nu_j - \dot u_i T^0_{ij}u_k(\widetilde\partial_k\nu_j)\\ &= \widetilde\nabla\cdot((\dot u\cdot T^0\cdot\nu)u) - (T^0\cdot\nu)\cdot\widetilde\nabla\dot u\cdot u - \dot u\cdot(T^0\cdot\widetilde\nabla\nu)\cdot u\\ &= \widetilde\nabla\cdot((\dot u\cdot T^0\cdot\nu)u) - \nu\cdot(T^0\cdot\widetilde\nabla\dot u)\cdot u - u\cdot(T^0\cdot\widetilde\nabla\nu)\cdot\dot u.\end{aligned}$$

By the surface divergence theorem (Lemma 4.5) and the zero traction boundary condition (2.21), $T^0\cdot\nu = 0$ on $\partial\Sigma^{\mathrm{FS}}\subseteq\partial B$, the integral of the surface divergence term over Σ^{FS} vanishes. Thus,

$$\begin{aligned}\dot u\cdot T^{\mathrm{PK}}\cdot\nu &\approx_2 \dot u\cdot T^0\cdot\nu - u\cdot(T^0\cdot\widetilde\nabla\nu)\cdot\dot u - \nu\cdot(T^0\cdot\widetilde\nabla\dot u)\cdot u - \nu\cdot(\widetilde\nabla u\cdot T^0)\cdot\dot u\\ &= \frac{\partial}{\partial t}\left(u\cdot T^0\cdot\nu - \frac12\,u\cdot(T^0\cdot\widetilde\nabla\nu)\cdot u\right) - \nu\cdot(\widetilde\nabla u\cdot T^0\cdot\dot u + T^0\cdot\widetilde\nabla\dot u\cdot u).\end{aligned}$$

Considering the deviatoric initial stress T^0_{dev} (4.65) as an effect of first order (see [DT98, p. 102]), we have

$$T^0\approx_0 -p^0 1_{3\times3}. \tag{6.58}$$

Therefore, up to second order, also the last term is a time derivative,

$$-\nu\cdot(\widetilde\nabla u\cdot T^0\cdot\dot u + T^0\cdot\widetilde\nabla\dot u\cdot u)\approx_2 \nu\cdot p^0(\widetilde\nabla u\cdot\dot u + \widetilde\nabla\dot u\cdot u) = p^0\frac{\partial}{\partial t}(\nu\cdot\widetilde\nabla u\cdot u)$$

and with $u\cdot T^0\cdot\nu - \frac12\,u\cdot(T^0\cdot\widetilde\nabla\nu)\cdot u\approx_2 -p^0(u\cdot\nu - \frac12\,u\cdot\widetilde\nabla\nu\cdot u)$ we arrive at

$$\dot u\cdot T^{\mathrm{PK}}\cdot\nu\approx_2 -p^0\frac{\partial}{\partial t}\left(u\cdot\nu - \nu\cdot\widetilde\nabla u\cdot u - \frac12\,u\cdot\widetilde\nabla\nu\cdot u\right).$$

Thus, on a fluid-solid interface, the up-to second-order expansion of the surface Lagrangian (5.11) is given by

$$\begin{aligned}L''_{\Sigma^{\mathrm{FS}}} &\approx_2 \left[p^0(u\cdot\nu - \nu\cdot\widetilde\nabla u\cdot u - \frac12\,u\cdot\widetilde\nabla\nu\cdot u)\right]^+_-\\ &= [p^0(u\cdot\nu)]^+_- - \left[p^0(\nu\cdot\widetilde\nabla u\cdot u + \frac12\,u\cdot\widetilde\nabla\nu\cdot u)\right]^+_- = L''_{\Sigma^{\mathrm{FS}},[0]} + L''_{\Sigma^{\mathrm{FS}},[1]} + L''_{\Sigma^{\mathrm{FS}},[2]}.\end{aligned} \tag{6.59}$$

We observe that there is no constant part: $L''_{\Sigma^{\mathrm{FS}},[0]} = 0$. The linear part vanishes to first order, due to continuity of p^0 (4.67) and the first-order tangential slip condition $[u]^+_-\cdot\nu\approx_1 0$ (6.10):

$$L''_{\Sigma^{\mathrm{FS}},[1]} := [p^0(u\cdot\nu)]^+_- = p^0[u]^+_-\cdot\nu\approx_1 0.$$

Finally, the second-order part $L''_{\Sigma^{\mathrm{FS}},[2]}$ also coincides with (6.54). □

As an additional result, (6.59) incorporates the linear plus quadratic approximation of the surface energy density on fluid-solid interfaces, since $E''_{\Sigma^{\mathrm{FS}}} = -L''_{\Sigma^{\mathrm{FS}}}$ by (5.21):

$$E''_{\Sigma^{\mathrm{FS}}}\approx_2 -p^0\left[u\cdot\nu - \nu\cdot\widetilde\nabla u\cdot u - \frac12\,u\cdot\widetilde\nabla\nu\cdot u\right]^+_- = E''_{\Sigma^{\mathrm{FS}},[0]} + E''_{\Sigma^{\mathrm{FS}},[1]} + E''_{\Sigma^{\mathrm{FS}},[2]}. \tag{6.60}$$

Proof. (**Energy considerations**) As we will show in Section 6.6.3, in the linearized setting, energy conservation implies that the second-order surface energy must be equal to the work due to slip against initial stress on Σ^{FS}, whereas the zero- and first-order contribution to the surface energy vanish:

$$\begin{aligned} E''_{\Sigma^{\mathrm{FS}},[0]} &= E''_{\Sigma^{\mathrm{FS}},[1]} = 0, \\ E''_{\Sigma^{\mathrm{FS}},[2]} &= -[u \cdot T^0]^+_- \cdot \nu. \end{aligned} \tag{6.61}$$

With the second-order tangential slip condition $[u \cdot \nu - \nu \cdot \widetilde{\nabla} u \cdot u - \frac{1}{2}\, u \cdot \widetilde{\nabla} \nu \cdot u]^+_- \approx_2 0$ (6.13), we obtain

$$\begin{aligned} \int_{\Sigma^{\mathrm{FS}}} E''_{\Sigma^{\mathrm{FS}},[2]}\, \mathrm{d}S = -\int_{\Sigma^{\mathrm{FS}}} [u \cdot T^0]^+_- \cdot \nu \, \mathrm{d}S \quad &= \quad \int_{\Sigma^{\mathrm{FS}}} p^0 [u]^+_- \cdot \nu \mathrm{d}S \\ &\approx_2 \quad \int_{\Sigma^{\mathrm{FS}}} p^0 \Big[\nu \cdot \widetilde{\nabla} u \cdot u + \frac{1}{2}\, u \cdot \widetilde{\nabla} \nu \cdot u\Big]^+_- \mathrm{d}S. \end{aligned}$$

Consequently, in agreement with the second-order terms of (6.60), we have

$$E''_{\Sigma^{\mathrm{FS}},[2]} = p^0 \Big[\nu \cdot \widetilde{\nabla} u \cdot u + \frac{1}{2}\, u \cdot \widetilde{\nabla} \nu \cdot u\Big]^+_- \tag{6.62}$$

and the corresponding second-order surface Lagrangian $L''_{\Sigma^{\mathrm{FS}},[2]} = -E''_{\Sigma^{\mathrm{FS}},[2]}$ coincides with (6.54). □

6.6.3 Interpretation of the interface energy

Energy conservation in the nonlinear setting was discussed in Section 5.2.5. In particular, the exact interface energy (5.20), which is minus the full surface action $\mathscr{E}''_{\Sigma^{\mathrm{FS}}} = -\mathscr{A}''_{\Sigma^{\mathrm{FS}}}$ (5.12), was derived from conservation of energy:

$$\mathscr{E}''_{\Sigma^{\mathrm{FS}}}(t) = \int_{t_0}^{t} \int_{\Sigma^{\mathrm{FS}}} [\dot{u}_{t'} \cdot T^{\mathrm{PK}}_{t'}]^+_- \cdot \nu \, \mathrm{dS}\, \mathrm{d}t' = \int_{t_0}^{t} \int_{\varphi_{t'}(\Sigma^{\mathrm{FS}})} [v^s_{t'} \cdot T^s_{t'}]^+_- \cdot \nu^s_{t'} \, \mathrm{d}S_{t'}\, \mathrm{d}t'. \tag{6.63}$$

The spatial version is valid at least in the sense of surface densities on $\varphi_{t'}(\Sigma^{\mathrm{FS}})$ (see (5.38) of Section 5.4.2). However, by the Leibniz rule, by the constitutive law $T^s = -p^s 1_{3\times 3}$ in fluids (on the $-$-side of $\varphi_t(\Sigma^{\mathrm{FS}})$), which in particular implies the characterization of perfect slip via zero tangential stress $T^s \cdot \nu^s = (\nu^s \cdot T^s \cdot \nu^s)\nu^s = -p^s \nu^s$ (2.17), as well as by the exact kinematic and dynamic slip conditions $[v^s]^+_- \cdot \nu^s$ (4.38) and $[T^s]^+_- \cdot \nu^s = 0$ (2.16), the integrand vanishes:

$$[v^s \cdot T^s]^+_- \cdot \nu^s = v^{s,+} \cdot \underbrace{[T^s]^+_- \cdot \nu^s}_{=0} + [v^s]^+_- \cdot T^{s,-} \cdot \nu^s = -p^{s,-} \underbrace{[v^s]^+_- \cdot \nu^s}_{=0} = 0.$$

This shows that the interface energy density $E''_{\Sigma^{\mathrm{FS}}}$ is zero if slip is assumed to be frictionless. Since the interface energy $\mathscr{E}''_{\Sigma^{\mathrm{FS}}} = \int_{\Sigma^{\mathrm{FS}}} E''_{\Sigma^{\mathrm{FS}}} \mathrm{dS}$ corresponds to the work due to slip,

$$E''_{\Sigma^{\mathrm{FS}}} = 0$$

indeed is clear from a physical viewpoint. Consequently, as already mentioned in Section 5.2.1, the full nonlinear variational model of the elastic-gravitational equations possesses zero surface action (5.12):

$$\mathscr{A}''_{\Sigma^{\mathrm{FS}}} = 0. \tag{6.64}$$

In case of the linearized model, where the energy is quadratic, (6.60) then yields

$$0 = E''_{\Sigma^{\mathrm{FS}}} \approx_2 E''_{\Sigma^{\mathrm{FS}},[0]} + E''_{\Sigma^{\mathrm{FS}},[1]} + E''_{\Sigma^{\mathrm{FS}},[2]} = -p^0 \Big[u \cdot \nu - \nu \cdot \widetilde{\nabla} u \cdot u - \frac{1}{2}\, u \cdot \widetilde{\nabla} \nu \cdot u\Big]^+_-.$$

With $E''_{\Sigma^{\mathrm{FS}},[0]} = 0$ we see that the second-order interface energy must balance the first-order interface energy:

$$E''_{\Sigma^{\mathrm{FS}},[2]} \approx_2 -E''_{\Sigma^{\mathrm{FS}},[1]} = p^0[u]^+_- \cdot \nu = -[u \cdot T^0]^+_- \cdot \nu,$$

where we employed the constitutive relation $T^0 = -p^0 1_{3\times 3}$ in fluids. Hence, in order to guarantee energy conservation in the linearized variational model, we have to include a nonzero second-order interface energy $E''_{\Sigma^{\mathrm{FS}},[2]}$ as in (6.61). The interface energy represents the work against the initial stress on slipping fluid-solid boundaries.

Substituting $T^{\mathrm{PK}}_{t'} \approx_1 T^0 + T^{\mathrm{PK1}}_{t'}$ in the exact interface energy $\mathscr{E}''_{\Sigma^{\mathrm{FS}}}(t) = 0$ (6.63) and integrating $\int_{t_0}^t \dot{u}_{t'}\,\mathrm{d}t' = u_t - u_{t_0} = u_t$ yields

$$0 = \int_{t_0}^t \int_{\Sigma^{\mathrm{FS}}} [\dot{u}_{t'} \cdot T^{\mathrm{PK}}_{t'}]^+_- \cdot \nu \,\mathrm{dS}\,\mathrm{d}t' \approx_2 \int_{\Sigma^{\mathrm{FS}}} \left[u_t \cdot T^0\right]^+_- \cdot \nu \,\mathrm{dS} + \int_{t_0}^t \int_{\Sigma^{\mathrm{FS}}} \left[\dot{u}_{t'} \cdot T^{\mathrm{PK1}}_{t'}\right]^+_- \cdot \nu \,\mathrm{dS}\,\mathrm{d}t'.$$

With (6.61), which actually is a second-order approximation as well, we then arrive at

$$\mathscr{E}''_{\Sigma^{\mathrm{FS}},[2]} = \int_{\Sigma^{\mathrm{FS}}} E''_{\Sigma^{\mathrm{FS}},[2]} \,\mathrm{dS} \approx_2 -\int_{\Sigma^{\mathrm{FS}}} \left[u \cdot T^0\right]^+_- \cdot \nu \,\mathrm{dS} = \int_{t_0}^t \int_{\Sigma^{\mathrm{FS}}} \left[\dot{u} \cdot T^{\mathrm{PK1}}\right]^+_- \cdot \nu \,\mathrm{dS}\,\mathrm{d}t'.$$

This gives the following representation of the second-order interface **energy dissipation rate**, which is analogous to the abstract energy balance equation (3.54):

$$\frac{d}{dt}\mathscr{E}''_{\Sigma^{\mathrm{FS}},[2]} \approx_2 \int_{\Sigma^{\mathrm{FS}}} \left[\dot{u} \cdot T^{\mathrm{PK1}}\right]^+_- \cdot \nu \,\mathrm{dS}. \tag{6.65}$$

Remark 6.10 (**Energy release due to friction on earthquake fault surfaces**)**.** On an earthquake fault surface S_{f}, friction is nonzero. This results in a nonzero interface energy similar to (5.20):

$$\mathscr{E}_{S_{\mathrm{f}}}(t) = \int_{S_{\mathrm{f}}} \int_{t_0}^t [\dot{u}_{t'} \cdot T^{\mathrm{PK}}_{t'}]^+_- \cdot \nu \,\mathrm{d}t' \,\mathrm{dS} = \int_{S_{\mathrm{f}}} E_{S_{\mathrm{f}}} \,\mathrm{dS}.$$

In the linearized setting, the second-order approximation of the energy dissipation rate density would be the following modification of the density in (6.65), where τ_{f} represents the **frictional force** which acts tangentially on S_{f}, cf. [BdHH17, Section 6]:

$$\frac{d}{dt} E_{S_{\mathrm{f}}} \approx_2 [\dot{u} \cdot T^{\mathrm{PK1}}]^+_- \cdot \nu + [\dot{u}]^+_- \cdot \tau_{\mathrm{f}}.$$

Following [DT98, Chapter 5.5] one may alternatively obtain an approximation by replacing the first Piola-Kirchhoff stress $T^{\mathrm{PK}} = T^0 + T^{\mathrm{PK1}}$ by the average of its initial and final value on the fault surface, which both are time-independent, i.e. $T^{\mathrm{PK}} = \frac{1}{2}(T^0 + (T^0 + T^{\mathrm{PK1}}_{\mathrm{final}})) = \frac{1}{2}(2T^0 + T^{\mathrm{PK1}}_{\mathrm{final}})$. With $u(.,t_0) = 0$ and $u(.,t_{\mathrm{final}}) =: u_{\mathrm{final}}$, the final displacement field after the earthquake has ceased, the total elastic-gravitational **energy released during earthquake rupture** is [DT98, p. 184, (5.160)]

$$\mathscr{E}_{S_{\mathrm{f}}}(t_{\mathrm{final}}) = \frac{1}{2}\int_{S_{\mathrm{f}}} \left[\left(\int_{t_0}^{t_{\mathrm{final}}} \dot{u}\,\mathrm{d}t'\right) \cdot (2T^0 + T^{\mathrm{PK1}}_{\mathrm{final}})\right]^+_- \cdot \nu \,\mathrm{dS} = \frac{1}{2}\int_{S_{\mathrm{f}}} [u_{\mathrm{final}} \cdot (2T^0 + T^{\mathrm{PK1}}_{\mathrm{final}})]^+_- \cdot \nu \,\mathrm{dS}.$$

Note that in this approach one assumes that the fault surface S_{f} is already fixed. Advanced fracture models cover opening cracks and predict the rupture zone from material weaknesses. ⌟

Chapter 7

Variational derivation of the elastic-gravitational equations

The linear system of elastic-gravitational equations is derived from Hamilton's principle of stationary action in a rigorous way. Based on the regularity assumptions of the linearized fields, we first introduce the linearized variational framework for the composite fluid-solid earth model (Section 7.1). Then, a general low-regularity version of calculus of variations for quadratic Lagrangians in a Sobolev space setting is presented, which in particular includes surface integrals and composite domains (Section 7.2). By applying these results to the second-order approximation of the action integral of the composite fluid-solid earth model, we rigorously obtain the weak formulation of the elastic-gravitational equations under low regularity conditions (Section 7.3). Stationarity of the first-order terms yields the equilibrium equations and the dynamical equations follow from the second-order terms. Moreover, sufficient conditions to establish the strong form of the equations, as well as the boundary and interface conditions, are presented (Section 7.4). Thereby we recover the complete system of linearized governing equations from the variational principle within a consistent low-regularity setting. Finally, we briefly discuss the elastic-gravitational equations under hydrostatic prestress, which is a well established modeling assumption in global seismology (Section 7.5).

7.1 The linearized variational model

In the previous chapter, we have derived a second-order approximation of the full action integral $\mathscr{A}''$ (5.9) by replacing the configuration variables (φ, Φ^s) by reference fields $(\mathrm{Id}_{\mathbb{R}^3}, \Phi^0)$ plus perturbations (u, Φ^{s1}). Stationarity of the resulting quadratic action integral (6.48),

$$(u, \Phi^{s1}) \mapsto (\mathscr{A}''_{[0]} + \mathscr{A}''_{[1]} + \mathscr{A}''_{[2]})(u, \Phi^{s1}),$$

is understood as stationarity of the actions of order zero, one, and two separately. As will be shown in this chapter, the equilibrium equations (2.19) and (2.20) coupling ρ^0, Φ^0, and T^0 will follow from the variation of $\mathscr{A}''_{[1]}$, whereas the variation of $\mathscr{A}''_{[2]}$ will yield the dynamical equations (2.26) and (2.27) for u and Φ^{s1} (Section 7.3).

In order to study stationarity on a rigorous basis, we need to define the linearized variational model. Specifically, our purpose is to establish a suitable function space setting for the different approximations of the action in which we can calculate the Fréchet derivatives.

The quadratic action (6.48) consists of integrals of the form

$$\mathscr{A}''_{[i]}(u, \Phi^{s1}) = \int_I \left(\int_{\mathbb{R}^3} L''_{[i]} \,\mathrm{dV} + \int_{\Sigma^{\mathrm{FS}}} L''_{\Sigma^{\mathrm{FS}},[i]} \,\mathrm{dS} \right) \mathrm{d}t \qquad (i = 0, 1, 2).$$

We observe that, instead of $\mathbb{R}^3$, we may integrate over the unbounded Lip-composite domain B^{FSC} (4.40), since its interior boundaries $\Sigma^{\mathrm{FS}} \cup \partial B$ have measure zero. This modification allows us to benefit from the higher regularity inside the interior regions of the earth model. Indeed, the displacement associated to a piecewise Lip-continuous motion is just L^∞ on the whole volume B, but H^1 on fluid and solid interior regions B^{FS} separately. Furthermore, spatial derivatives of the motion have been defined as L^∞-functions on B by simply neglecting the possible discontinuity surfaces Σ^{FS}, and not globally as distributional derivatives (see Section 4.2.4). Finally we note that since $\mathrm{supp}(\rho^0) \subseteq \overline{B}$, the only non-zero contributions to the volume integral exterior to B are integrands corresponding to the squared gradient of the gravitational potential, namely $-\frac{1}{8\pi G}(\nabla\Phi^0)^2$, $-\frac{1}{4\pi G}(\nabla\Phi^0)\cdot(\nabla\Phi^{s1})$, and $-\frac{1}{8\pi G}(\nabla\Phi^{s1})^2$ appearing in $L''_{[0]}$, $L''_{[1]}$, and $L''_{[2]}$ respectively (cf. Section 6.6.1).

Assumption 2 formulated below collects sufficient regularity conditions on the material parameters c, ρ^0, the reference fields T^0, Φ^0, the perturbations u, Φ^{s1} of the linearized setting, and of the force potential F^s, enabling us to define the approximated action integrals (6.49) and study their Fréchet differentiability.

Assumption 2 (**Linearized variational model**).

(i) **Geometry, material parameters, and equilibrium fields:**
$B = B^{\mathrm{FS}} \cup \Sigma^{\mathrm{FS}}$ *is a composite fluid-solid earth model in the sense of Definition 4.12,*
ρ^0, T^0_{ij}, $c_{ijkl} \in L^\infty(\mathbb{R}^3)$ *for* $i,j,k,l \in \{1,2,3\}$, *with compact support contained in* $\overline{B}$,
$\Phi^0 \in Y^\infty(\mathbb{R}^3)$.

(ii) **Dynamical fields:**
$\Phi^{s1} \in H^1(\mathbb{R}^3 \times I^\circ)$,
$u \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}} \times I^\circ)^3$ *with* $\mathrm{supp}(u_t) \subseteq \overline{B}$ *for* $t \in I$,
$F^s \in L^2(I^\circ, H^1(\mathbb{R}^3))$.

(iii) **Surface conditions:** Σ^{FS} *locally is a* $\mathcal{C}^{1,1}$*-surface and* $p^0|_{\Sigma^{\mathrm{FS}}} \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})$.

Let us emphasize that the conditions (i) and (ii) in Assumption 2 are essentially deduced from the conditions on φ, Φ^s, ρ^0, U, and F^s which are sufficient to define the full action in the nonlinear setting (Assumption 1). More precisely, the regularity conditions on Φ^0, T^0, ρ^0, c, and F^s directly follow from the regularity conditions (4.58), (4.45), (4.59), and (4.49) of Φ^s, ρ^s, U, and F^s respectively. According to (6.11), $u \in H^1_{\Sigma^{\mathrm{FS}}}$ states that u is piecewise H^1 with continuous normal component. This is a consequence of φ being a piecewise Lip-regular motion of the composite fluid-solid earth, as we have already observed in (6.9). We note that the prestressed elasticity tensor Λ^{T^0} also satisfies the conditions in (i): By definition (6.34),

$$\Lambda^{T^0}_{ijkl} = \Gamma_{ijkl} + a(T^0_{ij}\delta_{kl} + T^0_{kl}\delta_{ij}) + (1+b)T^0_{jl}\delta_{ik} + b(T^0_{ik}\delta_{jl} + T^0_{jk}\delta_{il} + T^0_{il}\delta_{jk})$$

for $a, b \in \mathbb{R}$, where the classical elasticity tensor c is identified with Γ, as introduced in (6.33). As concerns the regularity of Φ^0, we recall that Lemma 4.28 shows that

$$Y^\infty(\mathbb{R}^3) = \{y \in \mathcal{D}'(\mathbb{R}^3) : \triangle y \in L^\infty_{\mathrm{c}}(\mathbb{R}^3),\ \lim_{|x|\to\infty} y(x) = 0\} \subseteq \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3).$$

For technical reasons we make a slightly stronger additional assumption for the regularity of Φ^{s1} with respect to time, namely H^1 in place of $\mathcal{C}^0$. This is more than in the nonlinear case, because the regularity (4.58) of Φ^s only translates to (6.23): $\Phi^{s1} \in \mathcal{C}^0(I^\circ, H^1(\mathbb{R}^3))$. However, the time derivative of Φ^{s1} neither appears in the action integral nor in other calculations below.

Moreover, the higher temporal regularity for Φ^{s1} will be justified in Proposition 8.10 from the existence of weak solutions of the dynamical equations.

Condition (iii) is necessary for the definition and differentiability of the surface action in the linearized model (see also Sections 6.6.2 and 7.2.3): The $\mathcal{C}^{1,1}$ regularity of Σ^{FS} renders the curvature of fluid-solid boundaries finite, thereby excluding sharp corners, where the curvature is measured in terms of the Weingarten operator $\widetilde{\nabla}\nu$ (cf. Remark 4.4):

$$\nu \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})^n \qquad \text{and} \qquad \widetilde{\nabla}\nu \in L^\infty(\Sigma^{\mathrm{FS}})^{n\times n}.$$

By Lemma 3.14 on products of Sobolev spaces and with $\mathrm{Lip} = W^{1,\infty}$ we have the inclusion

$$\mathrm{Lip}\cdot H^{\frac{1}{2}} \subseteq H^{\frac{1}{2}} \qquad \text{and, by duality,} \qquad \mathrm{Lip}\cdot H^{-\frac{1}{2}} \subseteq H^{-\frac{1}{2}} \qquad \text{on} \quad \Sigma^{\mathrm{FS}}.$$

Concerning the condition on p^0, we anticipate from Proposition 8.2 that solvability of the equilibrium equations yields $T^0 \in H_{\mathrm{div}}(B^{\mathrm{FS}})^{3\times 3}$, which has normal traces in $T^0\cdot\nu \in H^{-\frac{1}{2}}(\Sigma^{\mathrm{FS}})^3$, see (1.18) and recall from (2.23) that

$$T^0\cdot\nu = -p^0\nu \qquad \text{with} \qquad p^0 = -\nu\cdot T^0\cdot\nu \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}.$$

These identities, with $\nu \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})^n$ which is true for $\mathcal{C}^{1,1}$-surfaces, suggest that $p^0 = -\nu\cdot T^0\cdot\nu$ is an element of $H^{-\frac{1}{2}}(\Sigma^{\mathrm{FS}})$. However, in the quadratic surface Lagrangian for the composite fluid-solid earth model (6.54), p^0 acts as a multiplier on products of the traces of u, $\widetilde{\nabla}u$, ν, and $\widetilde{\nabla}\nu$. Consequently, if the surface action is interpreted as duality of $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ on Σ^{FS}, we need to assume $p^0|_{\Sigma^{\mathrm{FS}}} \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})$. Thus, condition (iii) of Assumption 2 has the status of an additional regularity requirement, which originates from the quadratic approximation of the surface action (compare to the comments after (5.12) for the nonlinear situation).

The conditions summarized in Assumption 2 allow us to study the variational problem for the approximated action (6.48) in a Sobolev space setting. To simplify the notation and draw the attention to the structure of the action, we combine u and Φ^{s1} to a single variable and set

$$y := (u, \Phi^{s1})^T \in H \tag{7.1}$$

for

$$H := H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}}\times I^\circ)^3 \times H^1(\mathbb{R}^3\times I^\circ) = H^1(I^\circ, H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3 \times H^1(\mathbb{R}^3)). \tag{7.2}$$

Upon recalling

$$H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}}\times I^\circ)^3 = \{u \in H^1(B^{\mathrm{FSC}}\times I^\circ)^3 : \ [u]^+_-\cdot\nu = 0 \ \text{ on } \Sigma^{\mathrm{FS}}\}$$

from (6.11), it is clear that $H \subseteq H^1(B^{\mathrm{FSC}}\times I^\circ)^4$. Then the first (space-time) derivative of y reads

$$Dy = \begin{pmatrix} \nabla u & \dot{u} \\ (\nabla\Phi^{s1})^T & \dot{\Phi}^{s1} \end{pmatrix} \in L^2(B^{\mathrm{FSC}}\times I^\circ)^{4\times 4}. \tag{7.3}$$

Finally, we note that due to the specific form of Λ^{T^0} in fluid regions, both, the divergence $\nabla\cdot u$ and the displacement gradient ∇u occur in $T^{\mathrm{PK1}} = p^0(\gamma-1)(\nabla\cdot u)1_{3\times 3} + p^0(\nabla u)^T$, see (6.43) (by contrast, T^1 (6.41) only depends on the divergence). Therefore it is legitimate to test u with H^1-functions not only in the solid parts but also in fluid regions of the earth model.

7.2 Calculus of variations with low regularity

We generalize the classical theory of calculus of variations (presented in Section 3.1) to a low-regularity setting. Based on Sobolev space techniques we will show that for action functionals (3.10),

$$\mathcal{J}(y) = \int_V F(x, y(x), Dy(x))\,\mathrm{dV}(x),$$

but with a Lagrangian F in the form of a second-order polynomial in y and Dy, the first variation $\delta\mathcal{J}(y_0, h)$ may still be determined using formula (3.14) for the weak EL, even with regularity lower than the classical $\mathcal{C}^2$-condition. This will allow us to extend the validity of the classical results to functions of lower regularity, in particular, the EL (3.4) and NBC (3.5), as well as the NIBC (3.6) for composite domains and their modification (3.7) in presence of a surface term.

7.2.1 Quadratic functionals in a Sobolev-space setting

We consider a functional $\mathcal{J}$ (3.10) of $y\colon V \to \mathbb{R}^m$ for $V \subseteq \mathbb{R}^n$ open (and possibly unbounded). The Lagrangian is a quadratic polynomial of y and Dy, which is split into constant, first-order and second-order terms: $F = F_{[0]} + F_{[1]} + F_{[2]}$. We thus consider

$$\mathcal{J}(y) = (\mathcal{J}_{[0]} + \mathcal{J}_{[1]} + \mathcal{J}_{[2]})(y) \qquad \text{with} \qquad \mathcal{J}_{[i]}(y) := \int_V F_{[i]}(x, y(x), Dy(x))\,\mathrm{dV}(x) \tag{7.4}$$

for $i = 0, 1, 2$, where

$$\begin{aligned} F_{[0]}(., y, Dy) &:= f_0 \\ F_{[1]}(., y, Dy) &:= \langle f_1 | (y, Dy)\rangle \\ F_{[2]}(., y, Dy) &:= \langle f_2^0 \cdot y | y\rangle + \langle f_2^1 \cdot y | Dy\rangle + \langle f_2^2 \cdot Dy | Dy\rangle. \end{aligned} \tag{7.5}$$

By abuse of notation, we suppress explicit switching between vector and matrix notation and simply write $\langle .|.\rangle$ for inner products in $\mathbb{R}^p$ or $\mathbb{R}^{q\times r}$, for $p, q, r \in \mathbb{N}$. We will also identify $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$. Then the values of the coefficients at $x \in V$ are as follows:

$$f_0(x) \in \mathbb{R}, \quad f_1(x) \in \mathbb{R}^{m+mn}, \quad f_2^0(x) \in \mathbb{R}^{m\times m}_{\mathrm{sym}}, \quad f_2^1(x) \in \mathbb{R}^{(mn)\times m}, \quad f_2^2(x) \in \mathbb{R}^{(mn)\times(mn)}_{\mathrm{sym}}.$$

The structure of Lagrangian densities in (7.5) is the same as of the volume Lagrangian densities (6.50)–(6.52) of the fluid-solid earth model for $y = (u, \Phi^{s1})^T$ (7.1) and $m = n = 3$: $f_0 = L''_{[0]}$ and the coefficients of f_1 as well as of f_2^j for $j = 0, 1, 2$ correspond to coefficients of $L''_{[1]}$ and $L''_{[2]}$ respectively (except of the force terms $\rho^0 F^s$ and $\rho^0 u \cdot \nabla F^s$, which are of lower order in u),

$$\begin{aligned} L''_{[1]} &= \rho^0 \dot{u} \cdot (\Omega \times x) - T^0 : \nabla u - \rho^0 u \cdot \nabla(\Phi^0 + \Psi^s) - \rho^0 \Phi^{s1} - \frac{1}{4\pi G}\,\nabla\Phi^0 \cdot \nabla\Phi^{s1} - \rho^0 F^s, \\ L''_{[2]} &= \frac{1}{2}\,\rho^0 \dot{u}^2 + \rho^0 \dot{u} \cdot (\Omega \times u) - \frac{1}{2}\,\nabla u : \Lambda^{T^0} : \nabla u - \frac{1}{2}\,\rho^0 u \cdot (\nabla\nabla(\Phi^0 + \Psi^s)) \cdot u \\ &\quad - \rho^0 u \cdot \nabla\Phi^{s1} - \frac{1}{8\pi G}(\nabla\Phi^{s1})^2 - \rho^0 u \cdot \nabla F^s. \end{aligned}$$

Under the regularity conditions

$$f_0 \in L^1(V),\ f_1 \in L^2(V)^{m+mn},\ f_2^0 \in L^\infty(V)^{m\times m}_{\mathrm{sym}},\ f_2^1 \in L^\infty(V)^{(mn)\times m},\ f_2^2 \in L^\infty(V)^{(mn)\times(mn)}_{\mathrm{sym}} \tag{7.6}$$

we easily verify

$$\mathcal{J}\colon H^1(V)^m \to \mathbb{R}.$$

Indeed, integrability of $F_{[i]}(., y, Dy)$ $(i = 0, 1, 2)$ on V is a consequence of the inclusions $H^1 \subseteq L^2$, $L^\infty \cdot L^2 \subseteq L^2$, $L^2 \cdot L^2 \subseteq L^1$, and $L^\infty \cdot L^1 \subseteq L^1$.

However, since f_2^0 will contain $\nabla\nabla\Phi^0$ which by Lemma 4.28 is not L^∞, applicability of the calculus to the composite fluid-solid earth model requires the condition

$$f_2^0 \in \bigcap_{1\le p<\infty} L^p(V)^{m\times m}_{\mathrm{sym}}.$$

(We note that multiplication of $\nabla\nabla\Phi^0$ by ρ^0, which has compact support, allows replacing L^p_{loc} by L^p.) Therefore, proving integrability of this term is more subtle, since the components of f_2^0 are in L^p for $1 \le p < \infty$ but not for $p = \infty$. This case is settled by Lemma 7.1 below, which shows that $\langle f_2^0 y | y \rangle$ is integrable at least on $V \subseteq \mathbb{R}^4$ or $V \subseteq \mathbb{R}^3$. Based on the Sobolev embedding theorem (Lemma 3.13), it provides a result on L^p-multipliers that map H^1 to L^2, if V is open and satisfies the cone property (see Section 3.2.1). Note that this is true if V is a Lip-domain or a Lip-composite domain (Definition 4.2).

Lemma 7.1 (Products of L^p and H^1-functions via Sobolev embeddings). *Let $V \subseteq \mathbb{R}^n$ be open and have the cone property. Then $H^1(V)$ is embedded in $L^q(V)$ for $q = \frac{2n}{n-2}$ with*

$$\|y\|_{L^q(V)} \le c\|y\|_{H^1(V)} \qquad \text{for} \qquad y \in H^1(V)$$

and a constant $c > 0$. The estimate holds for $q = 4$ if $n = 4$ and for $q = 6$ if $n = 3$. Moreover:

(i) *If $n = 4$, $y \in H^1(V)$, and $f \in L^4(V)$, then $fy \in L^2(V)$ with*

$$\|fy\|_{L^2(V)} \le c\|f\|_{L^4(V)}\|y\|_{H^1(V)} \tag{7.7}$$

and $fy^2 \in L^1(V)$ with $\|fy^2\|_{L^1(V)} \le c\|f\|_{L^4(V)}\|y\|^2_{H^1(V)}$.

(ii) *If $n = 3$, $y \in H^1(V)$, and $f \in L^6(V) \cap L^2(V)$, then $fy \in L^2(V)$ with*

$$\|fy\|_{L^2(V)} \le c^{\frac{3}{4}}\Big(\|f\|^3_{L^6(V)}\|f\|_{L^2(V)}\Big)^{\frac{1}{4}}\|y\|_{H^1(V)} \tag{7.8}$$

and $fy^2 \in L^1(V)$ with $\|fy^2\|_{L^1(V)} \le c^{\frac{3}{4}}\Big(\|f\|^3_{L^6(V)}\|f\|_{L^2(V)}\Big)^{\frac{1}{4}}\|y\|^2_{H^1(V)}$.

Proof. To simplify the notation in the proof, we omit the domain V and just write, e.g. H^1 instead of $H^1(V)$. The first claim directly follows from Sobolev embedding, Lemma 3.13 (iii): $H^1 = W^{1,2}$ ($k = 1$, $p = 2$) is embedded in L^q with $\|y\|_{L^q} \le c\|y\|_{H^1}$ for $q = \frac{2n}{n-2}$.

Proof of (i): For $n = 4$, note that $h, g \in L^4$ implies $hg \in L^2$ by Cauchy-Schwarz:

$$\|hg\|^2_{L^2} = \int_V |h|^2|g|^2 = \||h|^2|g|^2\|_{L^1} \le \||h|^2\|_{L^2}\||g|^2\|_{L^2} = \|h\|^2_{L^4}\|g\|^2_{L^4}.$$

Therefore, $y \in H^1$ and $f \in L^4$ give $fy \in L^2$ and by Sobolev embedding of H^1 in L^4, the inequality $\|fy\|_{L^2} \le c\|f\|_{L^4}\|y\|_{H^1}$ holds. Furthermore, using again Cauchy-Schwarz and the inequality above, we find

$$\|fy^2\|_{L^1} = \int_V |fy||y| = \||fy||y|\|_{L^1} \le \|fy\|_{L^2}\|y\|_{L^2} \le c\|f\|_{L^4}\|y\|_{H^1}\|y\|_{L^2} \le c\|f\|_{L^4}\|y\|^2_{H^1},$$

completing the proof of (i).

Proof of (ii): The statement for $n = 3$ is a consequence of Cauchy-Schwarz in combination with the (trivial) embedding $H^1 \subseteq L^2$ and the Sobolev embedding $H^1 \subseteq L^6$. Indeed, if $f, g \in L^6 \cap L^2$, then $fg \in L^2$ follows from

$$\begin{aligned}\|fg\|^2_{L^2} = \int_V |f|^2|g|^2 &= \||f|^2|g|^2\|_{L^1} \\ &\le \||f|^2\|_{L^2}\||g|^2\|_{L^2} = (\||f|^3|f|\|_{L^1}\||g|^3|g|\|_{L^1})^{\frac{1}{2}} \\ &\le (\||f|^3\|_{L^2}\||f|\|_{L^2}\||g|^3\|_{L^2}\||g|\|_{L^2})^{\frac{1}{2}} = \|f\|^{\frac{3}{2}}_{L^6}\|f\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{3}{2}}_{L^6}\|g\|^{\frac{1}{2}}_{L^2}\end{aligned}$$

by applying Cauchy-Schwarz twice. Combining this result with the embedding inequalities $\|y\|_{L^2} \le \|y\|_{H^1}$ and $\|y\|_{L^6} \le c\|y\|_{H^1}$ for $y \in H^1(V)$ justifies to set $g = y$ and similar arguments as for (i) complete the proof of (ii). □

7.2.2 Stationarity of volume integrals

We are ready to prove Fréchet differentiability of the functional (7.4).

Proposition 7.2 (**Fréchet derivatives for volume integrals**). *Let $V \subseteq \mathbb{R}^n$ be open and $F_{[i]}$ $(i = 0, 1, 2)$ be given by (7.5) for coefficients f_0, f_1, f_2^0, f_2^1, and f_2^2 with regularity (7.6). Then the functionals $\mathcal{J}_{[i]}$ $(i = 0, 1, 2)$ in (7.4) are Fréchet-differentiable on $H^1(V)^m$ with derivatives*

$$\begin{aligned} D\mathcal{J}_{[0]}(y)(h) &= 0, \\ D\mathcal{J}_{[1]}(y)(h) &= \int_V \langle f_1 | (h, Dh) \rangle \, \mathrm{dV} \;=\; \mathcal{J}_{[1]}(h), \\ D\mathcal{J}_{[2]}(y)(h) &= \int_V \Big(\langle 2f_2^0 \cdot y + (f_2^1)^T \cdot Dy | h \rangle + \langle f_2^1 \cdot y + 2f_2^2 \cdot Dy | Dh \rangle \Big) \, \mathrm{dV} \end{aligned}$$

for $y, h \in H^1(V)^m$. Furthermore, instead of assuming boundedness of f_2^0, the result also holds if $f_2^0 \in L^4(V)^{m\times m}_{\mathrm{sym}}$ for $V \subseteq \mathbb{R}^4$ with cone property or if $f_2^0 \in L^2(V)^{m\times m}_{\mathrm{sym}} \cap L^6(V)^{m\times m}_{\mathrm{sym}}$ for $V \subseteq \mathbb{R}^3$ with cone property.

Proof. The functional $\mathcal{J}_{[0]}$ is constant and $\mathcal{J}_{[1]}$ linear and bounded, because by Cauchy-Schwarz

$$|\mathcal{J}_{[1]}(y)| \le \|f_1^0\|_{L^2} \|y\|_{L^2} + \|f_1^1\|_{L^2} \|Dy\|_{L^2} \le c \|y\|_{H^1},$$

where $f_1 = (f_1^0, f_1^1)$. Thus their Fréchet derivatives are well-defined. For $\mathcal{J}_{[2]}$ we have

$$\begin{aligned} \mathcal{J}_{[2]}(y+h) &= \int_V \Big(\langle f_2^0 \cdot (y+h) | y+h \rangle + \langle f_2^1 \cdot (y+h) | (Dy + Dh) \rangle \\ &\qquad + \langle f_2^2 \cdot (Dy + Dh) | (Dy + Dh) \rangle \Big) \, \mathrm{dV} = \mathcal{J}_{[2]}(y) + D\mathcal{J}_{[2]}(y)(h) + \mathcal{J}_{[2]}(h) \end{aligned}$$

with the Fréchet derivative $D\mathcal{J}_{[2]}(y)(h)$ as claimed above. Thus $D\mathcal{J}_{[2]}(y)$ is linear and continuity follows from the estimate

$$|D\mathcal{J}_{[2]}(y)(h)| \le c(y) \|h\|_{H^1}^2,$$

obtained from the Cauchy-Schwarz inequality (and using the inequalities (7.7), (7.8) of Lemma 7.1 to estimate the term involving f_2^0 if it is not L^∞). Similarly, the remainder satisfies

$$|\mathcal{J}_{[2]}(h)| \le c \|h\|_{H^1}^2$$

implying $\frac{|\mathcal{J}_{[2]}(h)|}{\|h\|_{H^1}} \to 0$ as $h \to 0$ in H^1, completing the proof of Fréchet-differentiability of $\mathcal{J}_{[2]}$. □

Thus the Fréchet derivative of $\mathcal{J} = \mathcal{J}_{[0]} + \mathcal{J}_{[1]} + \mathcal{J}_{[2]}$ at $y_0 \in H^1(V)^m$ takes the classical form (3.14) and the **weak EL** read

$$D\mathcal{J}(y_0)(h) = \int_V \left(\langle \partial_y F_0 | h \rangle + \langle \partial_{Dy} F_0 | Dh \rangle \right) \, \mathrm{dV} = 0$$

for all $h \in H^1(V)^m$. Here F_0 denotes the evaluation of F at y_0 as introduced in (3.13) and we recall that $\langle . | . \rangle$ is the Euclidean inner product in $\mathbb{R}^m$ or $\mathbb{R}^{m\times n}$. The assumptions of Proposition 7.2 ($y \in H^1(V)^m$ and the coefficients of the Lagrangian are bounded) allow us to formulate the weak EL in terms of L^2 inner products:

$$\langle \partial_y F_0 | h \rangle_{L^2(V)} + \langle \partial_{Dy} F_0 | Dh \rangle_{L^2(V)} = 0. \tag{7.9}$$

If V is a Lip-domain (Definition 1.5) and the quadratic Lagrangian $F = F_{[0]} + F_{[1]} + F_{[2]}$ satisfies the additional conditions

$$\partial_y F_0 \in L^2(V)^m \qquad \text{and} \qquad \partial_{Dy} F_0 \in H_{\mathrm{div}}(V)^{m\times n}, \tag{7.10}$$

we may integrate by parts via the Sobolev version of Green's formula (1.17) and obtain

$$\langle \partial_y F_0 - \operatorname{div}(\partial_{Dy} F_0) | h \rangle_{L^2(V)} + \langle\!\langle \partial_{Dy} F_0 \cdot \nu, h \rangle\!\rangle_{\partial V} = 0$$

where $\langle\!\langle ., . \rangle\!\rangle_{\partial V}$ denotes the duality of $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ on ∂V, see (3.32). Hence we arrive at the **strong EL** (3.15), $\partial_y F_0 - \operatorname{div}(\partial_{Dy} F_0) = 0$, interpreted as equations in $L^2(V)$, together with the **NBC** (3.16), $\partial_{Dy} F_0 \cdot \nu = 0$, valid in $H^{-\frac{1}{2}}(\partial V)$. Specifically, in case of the quadratic volume Lagrangian we have

$$\partial_y F_0 = 2 f_2^0 \cdot y + (f_2^1)^T \cdot Dy \qquad \text{and} \qquad \partial_{Dy} F_0 = f_2^1 \cdot y + 2 f_2^2 \cdot Dy.$$

Under the assumptions of Proposition 7.2, the first expression also satisfies the additional conditions (7.10) required to obtain the strong form. However, the second expression needs higher regularity assumptions: Specifically, L^2 measurability of $\operatorname{div}(\partial_{Dy} F_0)$ is guaranteed, if $y \in H^2(V)^m$ and the coefficients f_2^1 and f_2^2 are Lipschitz regular.

7.2.3 Stationarity of surface integrals

We discuss Fréchet differentiability of a quadratic functional in the form of a surface integral

$$\mathcal{J}_S(y) := \int_S F_S(x, y(x), \widetilde{D} y(x)) \, \mathrm{dS}(x) \tag{7.11}$$

for a closed Lip-hypersurface S in the interior of V and with the following special surface Lagrangian

$$F_S = [G_S]_-^+ \qquad \text{where} \qquad G_S(., y, \widetilde{D} y) := \langle g^0 \cdot y | y \rangle + \langle g^1 \cdot y | \widetilde{D} y \rangle. \tag{7.12}$$

Here y has to be understood in the sense of the trace of functions in $H^1(V)^m$, which possibly consists of different values y^+ and y^- when S is approached from different sides. The surface gradient operator $\widetilde{D}$ (4.16) acts on the different traces of y. The values of the coefficients at $x \in S$ are $g^0(x) \in \mathbb{R}^{m \times m}_{\mathrm{sym}}$ and $g^1(x) \in \mathbb{R}^{(mn) \times m}$. As before, $\langle . | . \rangle$ denotes the Euclidean inner product in $\mathbb{R}^m$ or $\mathbb{R}^{m \times n}$. Linearity of the jump implies

$$\mathcal{J}_S(y) = \int_S F_S \, \mathrm{dS} = \int_S [G_S]_-^+ \, \mathrm{dS} = \left[\int_S G_S \, \mathrm{dS} \right]_-^+.$$

Let us specify the assumptions on the coefficients if we want to keep $y \in H^1(V)^m$. We have the traces $y \in H^{\frac{1}{2}}(S)^m$ and $\widetilde{D} y \in H^{-\frac{1}{2}}(S)^m$. By the properties of products of Sobolev spaces (Lemma 3.14), $L^\infty \cdot H^{\frac{1}{2}} \subseteq H^{-\frac{1}{2}}$ and $\mathrm{Lip} \cdot H^{-\frac{1}{2}} \subseteq H^{-\frac{1}{2}}$. Thus, if

$$g^0 \in L^\infty(S)^{m \times m}_{\mathrm{sym}} \qquad \text{and} \qquad g^1 \in \mathrm{Lip}(S)^{(mn) \times m}, \tag{7.13}$$

then the surface integral makes sense as $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ Sobolev duality on the Lip-surface S,

$$\mathcal{J}_S(y) = [\langle\!\langle g^0 y, y \rangle\!\rangle_S + \langle\!\langle \widetilde{D} y, g^1 y \rangle\!\rangle_S]_-^+,$$

showing

$$\mathcal{J}_S \colon H^1(V)^m \to \mathbb{R}.$$

Moreover, by continuity of the trace operator (3.30),

$$\|y\|_{H^{1/2}(S)} \le c' \|y\|_{H^1(V)} \qquad \text{and} \qquad \|\widetilde{D} y\|_{H^{-1/2}(S)} \le c'' \|y\|_{H^1(V)}.$$

Consequently, Fréchet differentiability of $\mathcal{J}_S$ on $H^1(V)^m$ follows from the same estimates employed in the proof of Proposition 7.2 for the volume integral $\mathcal{J}$.

However, comparing F_S (7.12) with the surface Lagrangian for the composite fluid-solid earth model (6.54), where $S = \Sigma^{\mathrm{FS}}$ and $y = (u, \Phi^{s1})^T$,

$$L''_{\Sigma^{\mathrm{FS}},[2]} = -p^0 \left[\frac{1}{2} u \cdot \widetilde{\nabla}\nu \cdot u + \nu \cdot \widetilde{\nabla} u \cdot u\right]_-^+,$$

it turns out that the nonzero components of g^0 and g^1 must be proportional to $-\frac{1}{2}p^0\widetilde{\nabla}\nu$ and $-p^0\nu$ respectively. If S is a Lip-surface, then $\nu \in L^\infty(S)^n$; yet $\widetilde{\nabla}\nu$ is not even measurable. Nevertheless, the required regularity conditions (7.13), i.e. $g^0 \in L^\infty$ and $g^1 \in \mathrm{Lip}$, are recovered under the additional assumptions that $S = \Sigma^{\mathrm{FS}}$ locally is a $\mathcal{C}^{1,1}$-surface and $p^0|_{\Sigma^{\mathrm{FS}}} \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})$, which are precisely the conditions (iii) of Assumption 2.

We summarize our findings about Fréchet differentiability of the surface integral $\mathcal{J}_S$ in the following proposition.

Proposition 7.3 (Fréchet derivatives for surface integrals). *Let $V \subseteq \mathbb{R}^n$ be open, S a* Lip*-surface in V, and F_S be given by (7.12) with $g^0 \in L^\infty(S)^{m\times m}_{\mathrm{sym}}$ and $g^1 \in \mathrm{Lip}(S)^{(mn)\times m}$. Then $\mathcal{J}_S$ is Fréchet-differentiable on $H^1(V)^m$ with derivative*

$$D\mathcal{J}_S(y)(h) = \int_S [\langle 2g^0 \cdot y + (g^1)^T \cdot \widetilde{D}y | h\rangle + \langle g^1 \cdot y | \widetilde{D}h\rangle]_-^+ \,\mathrm{dS}$$

for $y, h \in H^1(V)^m$.

The corresponding **weak surface EL** at $y_0 \in H^1(V)^m$ take the form

$$D\mathcal{J}_S(y_0)(h) = \int_S \left[\langle \partial_y G_{S0} | h\rangle + \langle \partial_{\widetilde{D}y} G_{S0} | \widetilde{D}h\rangle\right]_-^+ \mathrm{dS} = 0$$

for all $h \in H^1(V)^m$. The regularity conditions on g^0 and g^1 of Proposition 7.3 imply

$$\partial_y G_{S0} = 2g^0 \cdot y + (g^1)^T \cdot \widetilde{D}y \in H^{-\frac{1}{2}}(S)^m \qquad \text{and} \qquad \partial_{\widetilde{D}y} G_{S0} = g^1 \cdot y \in H^{\frac{1}{2}}(S)^{m\times n}, \quad (7.14)$$

allowing us to interpret the Fréchet derivative $D\mathcal{J}_S(y_0)(h)$ in terms of surface Sobolev dualities and to apply the surface divergence theorem (4.22):

$$D\mathcal{J}_S(y)(h) = [\langle\!\langle \partial_y G_{S0}, h\rangle\!\rangle_S + \langle\!\langle \widetilde{D}h, \partial_{\widetilde{D}y} G_{S0}\rangle\!\rangle_S]_-^+ = [\langle\!\langle \underbrace{\partial_y G_{S0} - \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} G_{S0})}_{=:\, g}, h\rangle\!\rangle_S]_-^+. \quad (7.15)$$

The Leibniz rule for the jump gives $[\langle\!\langle g, h\rangle\!\rangle_S]_-^+ = \langle\!\langle [g]_-^+, h^+\rangle\!\rangle_S + \langle\!\langle g^-, [h]_-^+\rangle\!\rangle_S$. Upon restricting to test functions $h \in H^1(V)^m$ that are a.e. continuous across S (that is, in the sense of the trace, $[h]_-^+ = 0$ a.e. on S, and thus $h := h^+ = h^-$), the last term disappears: $[\langle\!\langle g, h\rangle\!\rangle_S]_-^+ = \langle\!\langle [g]_-^+, h\rangle\!\rangle_S$. Under these assumptions, the weak surface EL in Proposition 7.3 take the classical form:

$$\begin{aligned} D\mathcal{J}_S(y)(h) &= \langle\!\langle [\partial_y G_{S0} - \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} G_{S0})]_-^+, h\rangle\!\rangle_S \\ &= \langle\!\langle \partial_y [G_{S0}]_-^+ - \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} [G_{S0}]_-^+), h\rangle\!\rangle_S \\ &= \langle\!\langle \partial_y F_{S0} - \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} F_{S0}), h\rangle\!\rangle_S = \langle\!\langle \partial_y F_{S0}, h\rangle\!\rangle_S + \langle\!\langle \widetilde{D}h, \partial_{\widetilde{D}y} F_{S0}\rangle\!\rangle_S = 0. \end{aligned}$$

In total, we have thus established Fréchet differentiability of a functional $\mathcal{J} + \mathcal{J}_S \colon H^1(V)^m \to \mathbb{R}$, comprising a volume and a surface integral, with integrands that are quadratic polynomials in y and Dy or $\widetilde{D}y$.

7.2.4 Composite domains

The findings of the two previous sections extend to Lagrangians on composite domains. We consider a Lip-composite domain V with interior boundary Σ and restrict to the case $m = n$ (in the application, $m = n = 3$). In view of Assumption 2 (ii), stationarity is only required on the space

$$H^1_S(V)^n = \{h \in H^1(V)^n : \ [h]^+_- \cdot \nu = 0 \ \text{ on } S \subseteq \Sigma\}$$

introduced in (4.25). To compensate this restriction of test functions, we need to assume that a modified normality condition (4.27) holds.

The following theorem collects our main result about calculus of variations with low regularity for quadratic Lagrangians on composite domains:

Theorem 7.4 (**Weak and strong EL, NBC, and NIBC for** Lip**-composite domains**). *Let V be a* Lip*-composite domain with interior boundary Σ and with a* Lip*-surface $S \subseteq \Sigma$. Consider the functionals $\mathcal{J} = \int_V F \,\mathrm{dV}$ (3.10) and $\mathcal{J}_S = \int_S F_S \,\mathrm{dS} = \int_S [G_S]^+_- \,\mathrm{dS}$ (7.11).*

(i) **Weak EL:** *The results of Proposition 7.2 and Proposition 7.3 hold, if one considers the restricted functionals $\mathcal{J}\colon H^1_S(V)^n \to \mathbb{R}$ and $\mathcal{J}_S\colon H^1_S(V)^n \to \mathbb{R}$ respectively. In particular, the weak EL for the combined functional*

$$\mathcal{J} + \mathcal{J}_S = \int_V F \,\mathrm{dV} + \int_S [G_S]^+_- \,\mathrm{dS}, \qquad \mathcal{J} + \mathcal{J}_S\colon H^1_S(V)^n \to \mathbb{R}$$

read

$$\langle \partial_y F_0 | h \rangle_{L^2(V)} + \langle \partial_{Dy} F_0 | Dh \rangle_{L^2(V)} + [\langle\!\langle \partial_y G_{S0}, h \rangle\!\rangle_S + \langle\!\langle \widetilde{D} h, \partial_{\widetilde{D}y} G_{S0} \rangle\!\rangle_S]^+_- = 0$$

for all $h \in H^1_S(V)^n$.

(ii) **Strong EL, NBC, and NIBC under higher regularity:** *If*

$$\mathcal{T} := \partial_{Dy} F_0 \in H_{\mathrm{div}}(V)^{n\times n}, \tag{7.16}$$

then stationarity of $\mathcal{J} + \mathcal{J}_S$ on $H^1_S(V)^n$ implies the strong EL (3.15),

$$\mathrm{div}(\partial_{Dy} F_0) - \partial_y F_0 = 0 \qquad \text{in} \qquad L^2(V)^n,$$

the NBC (3.16),

$$(\partial_{Dy} F_0) \cdot \nu = 0 \qquad \text{in} \qquad H^{-\frac{1}{2}}(\partial V)^n,$$

and the NIBC (3.18),

$$[\partial_{Dy} F_0]^+_- \cdot \nu = 0 \qquad \text{in} \qquad H^{-\frac{1}{2}}(\Sigma \setminus S)^n.$$

(iii) **Modified NIBC under normality:** *If, in addition to the higher regularity (7.16), $\mathcal{T}$ satisfies the normality condition (4.27)*

$$\mathcal{T} \cdot \nu = (\nu \cdot \mathcal{T} \cdot \nu)\nu$$

on S (that is, $\mathcal{T} \cdot \nu$ is purely normal: $(\mathcal{T} \cdot \nu)^{\parallel} = 0$), then stationarity of $\mathcal{J}$ on $H^1_S(V)^n$ implies the NIBC (3.18) in $H^{-\frac{1}{2}}(S)^n$. Under the modified normality condition that

$$\tau = (\tau \cdot \nu)\nu \qquad \text{for} \qquad \tau := \partial_{Dy} F_0 \cdot \nu + \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} G_{S0}) - \partial_y G_{S0} \tag{7.17}$$

on S (that is, τ is purely normal: $\tau^{\parallel} = 0$), stationarity of the combined functional $\mathcal{J} + \mathcal{J}_S$ on $H^1_S(V)^n$ leads to the modified NIBC (3.20),

$$\partial_y F_{S0} - \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y} F_{S0}) - [\partial_{Dy} F_0]^+_- \cdot \nu = 0 \qquad \text{in} \qquad H^{-\frac{1}{2}}(S)^n.$$

In the application of Theorem 7.4 to the composite fluid-solid earth model, the surface S (more precisely, the spatial part of the space-time hypersurface S) will play the role of the fluid-solid boundary Σ^{FS}, whereas Σ also comprises solid-solid interfaces. The 2-tensor $\mathcal{T}$ will correspond to the negative stress tensor ($-T^0$ or $-T^{\mathrm{PK1}}$), the vector $\mathcal{T}\cdot\nu$ to the associated traction vector on the interface ($-T^0\cdot\nu$ or $-T^{\mathrm{PK1}}\cdot\nu$), and $\tau=\mathcal{T}\cdot\nu-g$ will be identified with the reduced traction vector ($-\tau^0$ or $-\tau^{\mathrm{PK1}}$) on perfectly slipping fluid-solid boundaries, see (2.23) and (2.32) respectively.

Proof. (i): Restricting $\mathcal{J}$ to $H^1_S\subseteq H^1$ formally does not alter its derivative, so the Fréchet derivatives (the weak EL) take the same form. Linearity of the derivative implies that the weak EL of the combined functional $\mathcal{J}+\mathcal{J}_S$ are obtained as the sum of the individual weak EL (7.9) and (7.15):

$$\begin{aligned}D(\mathcal{J}+\mathcal{J}_S)(y)(h)&=D\mathcal{J}(y)(h)+D\mathcal{J}_S(y)(h)\\&=\langle\partial_yF_0|h\rangle_{L^2(V)}+\langle\partial_{Dy}F_0|Dh\rangle_{L^2(V)}+[\langle\!\langle\partial_yG_{S0},h\rangle\!\rangle_S+\langle\!\langle\widetilde{D}h,\partial_{\widetilde{D}y}G_{S0}\rangle\!\rangle_S]^+_-.\end{aligned}$$

(ii): We apply formula (4.30) of Lemma 4.11,

$$\int_V f:Dh\,\mathrm{dV}=-\int_V h\cdot\mathrm{div}f\,\mathrm{dV}+\int_{\partial\overline{V}}h\cdot f\cdot\nu\,\mathrm{dS}-\int_\Sigma h^+\cdot[f]^+_-\cdot\nu\,\mathrm{dS},$$

that is

$$\langle f|Dh\rangle_{L^2(V)}=-\langle\mathrm{div}f|h\rangle_{L^2(V)}+\langle\!\langle f\cdot\nu,h\rangle\!\rangle_{\partial\overline{V}}-\langle\!\langle[f]^+_-\cdot\nu,h^+\rangle\!\rangle_S,\tag{7.18}$$

to $f=\mathcal{T}=\partial_{Dy}F_0\in H_{\mathrm{div}}(V)^{n\times n}$ which by assumption satisfies the normality condition (4.27) on S and $\mathcal{T}|_S\cdot\nu\in H^{-\frac12}(S)^n$. This gives

$$\langle\partial_yF_0-\mathrm{div}(\partial_{Dy}F_0)|h\rangle_{L^2(V)}+\langle\!\langle\partial_{Dy}F_0\cdot\nu,h\rangle\!\rangle_{\partial\overline{V}}-\langle\!\langle[\partial_{Dy}F_0]^+_-\cdot\nu,h^+\rangle\!\rangle_S=0$$

for all $h\in H^1_S(V)^n$. First we restrict to test functions that vanish on S and deduce the strong EL (3.15), valid as equality in $L^2(V)^n$. Second we consider the full class of test functions. But by the just established strong EL, only the surface dualities are left, proving the NBC (3.16) valid in $H^{-\frac12}(\partial\overline{V})^n$ and the NIBC (3.18) valid in $H^{-\frac12}(S)^n$. On interior boundaries $\Sigma\setminus S$, the NIBC (3.18) hold in $H^{-\frac12}(\Sigma\setminus S)^n$ without the normality assumption.

(iii): Let us discuss the modifications in presence of $\mathcal{J}_S$. We first observe that the reduced traction vector satisfies $\tau\in H^{-\frac12}(S)^n$. We may argue similarly as in the proof of Lemma 4.11: The conditions $[h]^+_-\cdot\nu=0$, the Leibniz rule applied to surface Sobolev dualities, as well as the modified normality condition (7.17), implying in particular $\tau^-=(\tau^-\cdot\nu)\nu$, yield

$$[\langle\!\langle\tau,h\rangle\!\rangle_S]^+_-=\langle\!\langle[\tau]^+_-,h^+\rangle\!\rangle_S+\langle\!\langle\tau^-,[h]^+_-\rangle\!\rangle_S=\langle\!\langle[\tau]^+_-,h^+\rangle\!\rangle_S$$

and thus

$$\begin{aligned}[\langle\!\langle\partial_yG_{S0}-\widetilde{\mathrm{div}}(\partial_{\widetilde{D}y}G_{S0})-\partial_{Dy}F_0\cdot\nu,h\rangle\!\rangle_S]^+_-&=-[\langle\!\langle\tau,h\rangle\!\rangle_S]^+_-=-\langle\!\langle[\tau]^+_-,h^+\rangle\!\rangle_S\\&=\langle\!\langle[\partial_yG_{S0}-\widetilde{\mathrm{div}}(\partial_{\widetilde{D}y}G_{S0})-\partial_{Dy}F_0\cdot\nu]^+_-,h^+\rangle\!\rangle_S\\&=\langle\!\langle\partial_yF_{S0}-\widetilde{\mathrm{div}}(\partial_{\widetilde{D}y}F_{S0})-[\partial_{Dy}F_0]^+_-\cdot\nu,h^+\rangle\!\rangle_S.\end{aligned}$$

Consequently we obtain

$$\langle\partial_yF_0-\mathrm{div}(\partial_{Dy}F_0)|h\rangle_{L^2(V)}+\langle\!\langle\partial_{Dy}F_0\cdot\nu,h\rangle\!\rangle_{\partial\overline{V}}+\langle\!\langle\partial_yF_{S0}-\widetilde{\mathrm{div}}(\partial_{\widetilde{D}y}F_{S0})-[\partial_{Dy}F_0]^+_-\cdot\nu,h^+\rangle\!\rangle_S=0$$

for all $h\in H^1_S(V)^n$, which implies that the modified NIBC (3.20) are valid in $H^{-\frac12}(S)^n$. □

7.3 The weak Euler-Lagrange equations

We apply the results of Theorem 7.4 (i) and derive the weak EL for the composite fluid-solid earth model from Hamilton's principle of stationary action.

7.3.1 Stationarity of the first- and second-order action

Let the regularity conditions for the linearized variational model listed in Assumption 2 hold. Then the Lagrangians $L''_{[i]}$ in (6.50)–(6.52), that is,

$$\begin{aligned}
L''_{[0]} &= -\rho^0(\Phi^0+\Psi^s) - \frac{1}{8\pi G}(\nabla\Phi^0)^2,\\
L''_{[1]} &= \rho^0\dot{u}\cdot(\Omega\times x) - T^0 : \nabla u - \rho^0 u\cdot\nabla(\Phi^0+\Psi^s)\\
&\quad -\rho^0\Phi^{s1} - \frac{1}{4\pi G}\nabla\Phi^0\cdot\nabla\Phi^{s1} - \rho^0 F^s,\\
L''_{[2]} &= \frac{1}{2}\rho^0\dot{u}^2 + \rho^0\dot{u}\cdot(\Omega\times u) - \frac{1}{2}\nabla u : \Lambda^{T^0} : \nabla u - \frac{1}{2}\rho^0 u\cdot(\nabla\nabla(\Phi^0+\Psi^s))\cdot u\\
&\quad -\rho^0 u\cdot\nabla\Phi^{s1} - \frac{1}{8\pi G}(\nabla\Phi^{s1})^2 - \rho^0 u\cdot\nabla F^s,
\end{aligned}$$

and $L''_{\Sigma^{\mathrm{FS}},[i]}$ in (6.53)–(6.54), that is,

$$\begin{aligned}
L''_{\Sigma^{\mathrm{FS}},[0]} &= L''_{\Sigma^{\mathrm{FS}},[1]} = 0,\\
L''_{\Sigma^{\mathrm{FS}},[2]} &= -p^0[\nu\cdot\widetilde{\nabla}u\cdot u + \frac{1}{2}u\cdot\widetilde{\nabla}\nu\cdot u]^+_-,
\end{aligned}$$

have the same structure and regularity of coefficients as the functionals with quadratic Lagrangians F and F_S discussed in Section 7.2, upon identifying

$$V = B^{\mathrm{FSC}}\times I^\circ \qquad \text{and} \qquad S = \Sigma^{\mathrm{FS}}\times I^\circ.$$

Therefore we have integrability on $B^{\mathrm{FSC}}\times I^\circ$ of the Lagrangians in the approximated action integrals (6.49),

$$\mathscr{A}''_{[i]}(y) = \int_I\left(\int_{\mathbb{R}^3} L''_{[i]}\,\mathrm{dV} + \int_{\Sigma^{\mathrm{FS}}} L''_{\Sigma^{\mathrm{FS}},[i]}\,\mathrm{dS}\right)\mathrm{d}t$$

for $i=0,1,2$ and $y=(u,\Phi^{s1})^T$. Fréchet differentiability of the action integrals on $H^1(B^{\mathrm{FSC}}\times I^\circ)^4$ is established by Propositions 7.2 and 7.3. By Theorem 7.4 (i) the result also holds on the smaller space

$$H = H^1(I^\circ, H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3\times H^1(\mathbb{R}^3))$$

introduced in (7.2). Hence, for $y,h\in H$ the Fréchet derivatives are given by the formulas

$$\begin{aligned}
D\mathscr{A}''_{[0]}(y)(h) &= 0,\\
D\mathscr{A}''_{[1]}(y)(h) &= \int_I\int_{B^{\mathrm{FSC}}}\Big(\langle\partial_y L''_{[1]}|h\rangle + \langle\partial_{Dy}L''_{[1]}|Dh\rangle\Big)\,\mathrm{dV}\,\mathrm{d}t, \qquad (7.19)\\
D\mathscr{A}''_{[2]}(y)(h) &= \int_I\int_{B^{\mathrm{FSC}}}\Big(\langle\partial_y L''_{[2]}|h\rangle + \langle\partial_{Dy}L''_{[2]}|Dh\rangle\Big)\,\mathrm{dV}\,\mathrm{d}t\\
&\quad + \int_I\int_{\Sigma^{\mathrm{FS}}}\Big(\langle\partial_y L''_{\Sigma^{\mathrm{FS}},[2]}|h\rangle + \langle\partial_{\widetilde{D}y}L''_{\Sigma^{\mathrm{FS}},[2]}|\widetilde{D}h\rangle\Big)\mathrm{dS}\,\mathrm{d}t. \qquad (7.20)
\end{aligned}$$

The derivatives of the Lagrangians read, recalling from (7.3) the convention $Dy = \begin{pmatrix} \nabla u & \dot{u} \\ (\nabla \Phi^{s1})^T & \dot{\Phi}^{s1} \end{pmatrix}$,

$$\partial_y L''_{[1]} = \begin{pmatrix} -\rho^0 \nabla(\Phi^0 + \Psi^s) \\ -\rho^0 \end{pmatrix}, \qquad \partial_{Dy} L''_{[1]} = \begin{pmatrix} -T^0 & \rho^0 \Omega \times x \\ -\frac{1}{4\pi G}(\nabla \Phi^0)^T & 0 \end{pmatrix}, \tag{7.21}$$

$$\begin{aligned} \partial_y L''_{[2]} &= \begin{pmatrix} -\rho^0 \nabla F^s - \rho^0 \nabla \nabla(\Phi^0 + \Psi^s) \cdot u - \rho^0 \Omega \times \dot{u} - \rho^0 \nabla \Phi^{s1} \\ 0 \end{pmatrix}, \\ \partial_{Dy} L''_{[2]} &= \begin{pmatrix} -\Lambda^{T^0} : \nabla u & \rho^0(\Omega \times u) + \rho^0 \dot{u} \\ (-\rho^0 u - \frac{1}{4\pi G} \nabla \Phi^{s1})^T & 0 \end{pmatrix}, \end{aligned} \tag{7.22}$$

$$\begin{aligned} \partial_y L''_{\Sigma^{\mathrm{FS}},[2]} &= \begin{pmatrix} -p^0 [\widetilde{\nabla} \nu \cdot u + \nu \cdot \widetilde{\nabla} u]^+_- \\ 0 \end{pmatrix}, \\ \partial_{\widetilde{Dy}} L''_{\Sigma^{\mathrm{FS}},[2]} &= \begin{pmatrix} -p^0 \, [\nu u]^+_- & 0_{3\times 1} \\ 0_{1\times 3} & 0 \end{pmatrix}. \end{aligned} \tag{7.23}$$

According to Hamilton's principle, the system of dynamical equations for $y \in H$ follows from the stationarity of the regional action integrals, expressed by the conditions

$$D\mathscr{A}''_{[1]}(y)(h) = 0 \qquad \text{and} \qquad D\mathscr{A}''_{[2]}(y)(h) = 0 \tag{7.24}$$

for all $h \in H$. These are the linearized EL for the composite fluid-solid earth model in their general weak form, coinciding with the principle of virtual work (3.8).

7.3.2 Weak formulation of the elastic-gravitational equations

We separate space and time components of derivatives by inserting a test function

$$h \in \mathcal{D}(I^\circ, H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3 \times H^1(B^{\mathrm{FSC}})) \subseteq H, \qquad h(x,t) := z(x)\psi(t)$$

for $(x,t) \in B^{\mathrm{FSC}} \times I^\circ$, where

$$z \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3 \times H^1(B^{\mathrm{FSC}}) \qquad \text{and} \qquad \psi \in \mathcal{D}(I^\circ).$$

Then the (temporally distributional) spatially weak EL (7.24) for $\mathscr{A}''_{[1]}$, $\mathscr{A}''_{[2]}$ are given by the equations

$$\int_{B^{\mathrm{FSC}}} \Big(\langle \partial_y L''_{[1]} | z \rangle + \langle \partial_{\nabla y} L''_{[1]} | \nabla z \rangle \Big) \, \mathrm{dV} = 0 \tag{7.25}$$

and

$$\begin{aligned} \frac{d}{dt} \left(\int_{B^{\mathrm{FSC}}} \langle \partial_{\dot{y}} L''_{[2]} | z \rangle \, \mathrm{dV} + \int_{\Sigma^{\mathrm{FS}}} \langle \partial_{\dot{y}} L''_{\Sigma^{\mathrm{FS}},[2]} | z \rangle \, \mathrm{dS} \right) - \int_{B^{\mathrm{FSC}}} \Big(\langle \partial_y L''_{[2]} | z \rangle + \langle \partial_{\nabla y} L''_{[2]} | \nabla z \rangle \Big) \, \mathrm{dV} \\ - \int_{\Sigma^{\mathrm{FS}}} \Big(\langle \partial_y L''_{\Sigma^{\mathrm{FS}},[2]} | z \rangle + \langle \partial_{\widetilde{\nabla} y} L''_{\Sigma^{\mathrm{FS}},[2]} | \widetilde{\nabla} z \rangle \Big) \, \mathrm{dS} = 0. \end{aligned} \tag{7.26}$$

These equations must hold as equations in $\mathcal{D}'(I^\circ)$, for all $z \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3 \times H^1(B^{\mathrm{FSC}})$.

Remark 7.5 (Initial and final conditions in Hamilton's principle). By choosing the temporal part of the test function with compact support, i.e. $\psi \in \mathcal{D}(t_0, t_1)$, we restrict to fixed endpoints of $(x,t) \mapsto y(x,t)$ with respect to time, that is, fixed initial conditions at t_0 and final conditions at t_1. Since final conditions seem unnatural, one may let $t_1 \to \infty$ and consider the problem on the unbounded interval (t_0, ∞). This approach is taken e.g. in [AO79]. ⌟

In order to separate inner products we further decompose the spatial test function z as

$$z = (v, w) \qquad \text{with} \qquad v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3, \quad w \in H^1(B^{\mathrm{FSC}}),$$

in accordance with the components u and Φ^{s1} of y (7.1). Consequently, (7.25) and (7.26) reduce to the weak formulation of the elastic-gravitational equations: Stationarity of $\mathscr{A}''_{[1]}$ implies the **weak equilibrium elastic-gravitational equations**

$$\int_{B^{\mathrm{FSC}}} \left(-T^0 : \nabla v - \rho^0 \nabla(\Phi^0 + \Psi^s) \cdot v - \frac{1}{4\pi G} \nabla\Phi^0 \cdot \nabla w - \rho^0 w \right) \mathrm{dV} = 0 \tag{7.27}$$

$\forall\, v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3, \quad \forall\, w \in H^1(B^{\mathrm{FSC}})$.

Stationarity of $\mathscr{A}''_{[2]}$ implies the **weak dynamical elastic-gravitational equations**

$$\begin{aligned} &\frac{d}{dt} \left(\int_{B^{\mathrm{FSC}}} \rho^0 \dot{u} \cdot v \,\mathrm{dV} \right) + \int_{B^{\mathrm{FSC}}} \rho^0 \Big(2\,\Omega \times \dot{u} + \nabla\nabla(\Phi^0 + \Psi^s) \cdot u + \nabla\Phi^{s1} + \nabla F^s \Big) \cdot v \,\mathrm{dV} \\ &\quad + \int_{B^{\mathrm{FSC}}} \left(\Lambda^{T^0} : \nabla u \right) : \nabla v \,\mathrm{dV} + \int_{B^{\mathrm{FSC}}} \left(\rho^0 u + \frac{1}{4\pi G} \nabla\Phi^{s1} \right) \cdot \nabla w \,\mathrm{dV} \\ &\quad + \int_{\Sigma^{\mathrm{FS}}} p^0 \Big[u \cdot (\widetilde{\nabla}\nu) \cdot v + \nu \cdot (\widetilde{\nabla} u) \cdot v + \nu \cdot (\widetilde{\nabla} v) \cdot u \Big]^+_- \mathrm{dS} = 0 \end{aligned} \tag{7.28}$$

in $\mathcal{D}'(I^\circ), \quad \forall\, v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3, \quad \forall\, w \in H^1(B^{\mathrm{FSC}})$.

In particular, the surface term $s(u, v)$ is recovered from (7.26) and (7.23) with the help of the symmetry of $\widetilde{\nabla}\nu$ and continuity (4.67) of p^0 across Σ^{FS}. Actually, in view of Theorem 7.4 (i), this term is understood as a surface Sobolev duality, see (3.32):

$$\begin{aligned} s(u, v) &:= \int_{\Sigma^{\mathrm{FS}}} p^0 \Big[u \cdot (\widetilde{\nabla}\nu) \cdot v + \nu \cdot (\widetilde{\nabla} u) \cdot v + \nu \cdot (\widetilde{\nabla} v) \cdot u \Big]^+_- \mathrm{dS} \\ &= \Big[\Big\langle\!\Big\langle p^0 u \cdot (\widetilde{\nabla}\nu) + p^0 \nu \cdot (\widetilde{\nabla} u),\, v \Big\rangle\!\Big\rangle_{\Sigma^{\mathrm{FS}}} + \Big\langle\!\Big\langle p^0 \widetilde{\nabla} v,\, \nu u \Big\rangle\!\Big\rangle_{\Sigma^{\mathrm{FS}}} \Big]^+_-. \end{aligned} \tag{7.29}$$

The product rule and the tangential slip conditions $[u]^+_- \cdot \nu = 0$ and $[v]^+_- \cdot \nu = 0$ (2.29), which imply $[\widetilde{\nabla}(\nu \cdot u)]^+_- = 0$ and $[\widetilde{\nabla}(\nu \cdot v)]^+_- = 0$, yield the alternative representation

$$\begin{aligned} s(u, v) &= \int_{\Sigma^{\mathrm{FS}}} p^0 \Big[u \cdot \widetilde{\nabla}(\nu \cdot v) + v \cdot \widetilde{\nabla}(\nu \cdot u) - u \cdot (\widetilde{\nabla}\nu) \cdot v \Big]^+_- \mathrm{dS} \\ &= \int_{\Sigma^{\mathrm{FS}}} p^0 \left([u]^+_- \cdot \widetilde{\nabla}(\nu \cdot v) + [v]^+_- \cdot \widetilde{\nabla}(\nu \cdot u) - [u \cdot (\widetilde{\nabla}\nu) \cdot v]^+_- \right) \mathrm{dS}, \end{aligned}$$

showing that the weak dynamical EL (7.28) are in complete consistence with [Val86, eq. (36)].

We thus have seen that under Assumption 2, Hamilton's principle of stationary action applied to the second-order approximation of the action (6.49) yields the weak EL (7.27) and (7.28).

7.4 Strong form with natural boundary and interface conditions

The strong EL, the NBC, and the NIBC may be derived from the weak EL under higher regularity conditions on the material parameters and equilibrium fields ρ^0, Φ^0, T^0, c than those listed in Assumption 2. Specifically, Theorem 7.4 (ii), (iii) give conditions that will enable us to deduce the strong EL, the NBC, and the NIBC from the weak EL by integration by parts. Rather than applying these results directly (which is of course possible), we will repeat key elements of the proof for the composite fluid-solid earth model and in particular translate the H_{div}-condition (7.16) to corresponding higher regularity assumptions on the specific fields in the linearized earth model (see Assumption 3).

7.4.1 Higher regularity assumptions

In case of the weak elastic-gravitational equations, we consider an interior Lip-regular surface Σ in B^{FSC}, further separating it into connected components V_k. Then the unbounded composite integration domain $V := \bigcup_k V_k$ has interior boundary $\Sigma = \partial V \setminus \partial\overline{V}$ (cf. Definition 4.1). Formula (4.29) of Lemma 4.11, that is

$$\int_V f : \nabla v \,\mathrm{dV} = -\int_V v \cdot (\nabla \cdot f)\,\mathrm{dV} + \int_{\partial\overline{V}} v \cdot f \cdot \nu \,\mathrm{dS} - \int_\Sigma v \cdot [f]^+_- \cdot \nu \,\mathrm{dS}$$

or

$$\int_V g \cdot \nabla w \,\mathrm{dV} = -\int_V w(\nabla \cdot g)\,\mathrm{dV} + \int_{\partial\overline{V}} w\, g \cdot \nu \,\mathrm{dS} - \int_\Sigma w\, [g]^+_- \cdot \nu \,\mathrm{dS},$$

needs to be applied to $f = T^0$ and $f = T^{\mathrm{PK1}}$ or $g = \nabla\Phi^0$ and $g = \rho^0 u + \frac{1}{4\pi G}\nabla\Phi^{s1}$ respectively.

Let us investigate the conditions that, in addition to Assumption 2, are sufficient for integrating the weak elastic-gravitational equations by parts. By definition of the weak equations, the test functions v and w are in H^1 (or at least in $H^1_{\Sigma^{\mathrm{FS}}}$) on B^{FSC}, which is enough for integration by parts. As we will see in Section 7.4.3, the modified normality condition (7.17), which is necessary to handle the more general case where test functions are restricted to $v \in H^1_{\Sigma^{\mathrm{FS}}}$ (4.25), will correspond to the dynamical conditions (2.23) and (2.33), characterizing frictionless tangential slip. Thus we are left with the requirement that f and g must satisfy the H_{div}-condition (7.16), that is, we have to assume the following:

Assumption 3 (**Higher regularity**). *In addition to Assumption 2, let $V = \bigcup_k V_k \subseteq B^{\mathrm{FSC}}$ be a* Lip*-composite domain that is possibly unbounded (see Definition 4.2).*

(i) $T^0 \in H_{\mathrm{div}}(V)^{3\times 3}, \quad \nabla\Phi^0 \in H_{\mathrm{div}}(V)^3.$

(ii) $\dot{u} \in H^1(I^\circ, L^2(V)^3), \quad T^{\mathrm{PK1}} \in L^2(I^\circ, H_{\mathrm{div}}(V)^{3\times 3}), \quad \rho^0 u + \frac{1}{4\pi G}\nabla\Phi^{s1} \in L^2(I^\circ, H_{\mathrm{div}}(V)^3).$

In Assumption 3 (ii), $T^{\mathrm{PK1}} = \Lambda^{T^0} : \nabla u$. Moreover, we have separated the H_{div}-condition with respect to space and time, upon noting that $H_{\mathrm{div}}(I^\circ) = H^1(I^\circ)$.

7.4.2 The elastic-gravitational equations in strong form

By Theorem 7.4 (ii), the strong EL (3.4) hold in L^2 under the higher regularity conditions (7.16). If $y = (u, \Phi^{s1})^T$ as in (7.1), we thus need to assume that $\partial_{Dy} L''_{[i]}$ lies in $H_{\mathrm{div}}(B^{\mathrm{FSC}} \times I^\circ)$. These are the additional conditions of Assumption 3, as (7.21) and (7.22) reveal in view of (7.3):

$$\partial_{Dy} L''_{[1]} = \begin{pmatrix} -T^0 & \rho^0\Omega\times x \\ -\frac{1}{4\pi G}(\nabla\Phi^0)^T & 0 \end{pmatrix} \qquad \text{and} \qquad \partial_{Dy} L''_{[2]} = \begin{pmatrix} -\Lambda^{T^0}:\nabla u & \rho^0(\Omega\times u)+\rho^0\dot{u} \\ (-\rho^0 u - \frac{1}{4\pi G}\nabla\Phi^{s1})^T & 0 \end{pmatrix}.$$

Under these conditions, the strong EL hold in $L^2(B^{\mathrm{FSC}}\times I^\circ)$. Alternatively, one may also keep the time-dependence distributional and only assume that $\partial_{Dy} L$ has components in $\mathcal{D}'(I^\circ, H_{\mathrm{div}}(B^{\mathrm{FSC}}))$. This leads to strong EL that are valid in $\mathcal{D}'(I^\circ, L^2(B^{\mathrm{FSC}}))$.

Specifically, the strong EL for $L''_{[1]}$ (6.51) consist of the equilibrium equations

$$\nabla \cdot (\partial_{\nabla u} L''_{[1]}) - \partial_u L''_{[1]} = 0 \qquad \text{and} \qquad \nabla \cdot (\partial_{\nabla\Phi^0} L''_{[1]}) = 0. \tag{7.30}$$

Thus the component of the equations $D\mathscr{A}''_{[1]}(u, \Phi^{s1}) = 0$ that corresponds to variations with respect to u is the static equilibrium equation (2.19)

$$\rho^0 \nabla(\Phi^0 + \Psi^s) - \nabla \cdot T^0 = 0$$

and variations with respect to Φ^{s1} give the equilibrium Poisson equation (2.20)

$$\triangle \Phi^0 = 4\pi G \rho^0.$$

The strong EL for $L''_{[2]}$ (6.52) consist of the dynamical equations [DT98, (3.169) and (3.192)]

$$\partial_t(\partial_{\dot{u}} L''_{[2]}) + \nabla \cdot (\partial_{\nabla u} L''_{[2]}) - \partial_u L''_{[2]} = 0 \quad \text{and} \quad \nabla \cdot (\partial_{\nabla \Phi^{s1}} L''_{[2]}) = 0. \tag{7.31}$$

The stationarity equations $D\mathscr{A}''_{[2]}(u, \Phi^{s1}) = 0$ yield the linear equation of motion (2.26),

$$\rho^0 \left(\ddot{u} + 2\,\Omega \times \dot{u} \; + \; \nabla\nabla(\Phi^0 + \Psi^s) \cdot u + \nabla \Phi^{s1} \right) - \nabla \cdot (\Lambda^{T^0} : \nabla u) = \rho^0 \nabla F^s,$$

when varying with respect to u, and the perturbed Poisson equation (2.27),

$$\triangle \Phi^{s1} = -4\pi G \, \nabla \cdot (\rho^0 u),$$

follows from stationarity with respect to variations of Φ^{s1}.

We observe that outside the Earth, the only nontrivial equations are the two Laplace equations

$$\triangle \Phi^0 = 0 \qquad \text{and} \qquad \triangle \Phi^{s1} = 0. \tag{7.32}$$

7.4.3 Boundary and interface conditions

With Assumption 3, we consider the weak EL (7.27) and (7.28) and restrict B^{FSC} to the possibly unbounded Lip-composite domain $V = \bigcup_k V_k \subseteq B^{\mathrm{FSC}}$ with interior boundary

$$\Sigma = \Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}} \cup \Sigma^{\mathrm{FF}} \cup \partial B.$$

Then Theorem 7.4 (ii), (iii) allow to deduce $H^{-\frac{1}{2}}$-validity of the NBC (3.5), the NIBC (3.6), and the modified the NIBC (3.7). These conditions comprise the dynamical exterior boundary and interface conditions.

The weak equilibrium EL (7.27) imply the equilibrium NIBC (3.6),

$$[\partial_{\nabla y} L''_{[1]}]^+_- \cdot \nu = 0 \qquad \text{in} \qquad H^{-\frac{1}{2}}(\Sigma).$$

Assumption 3 (i) ensures that the components of T^0 and $\nabla \Phi^0$ are in $H_{\mathrm{div}}(V)$ and we obtain

$$[T^0]^+_- \cdot \nu = 0 \qquad \text{and} \qquad [\nabla \Phi^0]^+_- \cdot \nu = 0 \qquad \text{on} \quad \Sigma.$$

These are (2.22) and part of (2.25), which also contain (2.21), $T^0 \cdot \nu = 0$ on ∂B, as a special case. Since there is no first-order contribution from the surface action, slip at fluid-solid boundaries does not affect the equilibrium equations.

The weak dynamical EL (7.28) gives the dynamical NIBC (3.6),

$$[\partial_{\nabla y} L''_{[2]}]^+_- \cdot \nu = 0 \qquad \text{in} \qquad H^{-\frac{1}{2}}((\Sigma \setminus \Sigma^{\mathrm{FS}}) \times I^\circ).$$

Assumption 3 (ii) ensures that $T^{\mathrm{PK1}} = \Lambda^{T^0} : \nabla u$ and $\rho^0 u + \frac{1}{4\pi G} \nabla \Phi^{s1}$ have components in $H_{\mathrm{div}}(V)$ and we obtain

$$[T^{\mathrm{PK1}}]^+_- \cdot \nu = 0 \qquad \text{on} \quad \Sigma \setminus \Sigma^{\mathrm{FS}} \qquad \text{and} \qquad [4\pi G \rho^0 u + \nabla \Phi^{s1}]^+_- \cdot \nu = 0 \qquad \text{on} \quad \Sigma.$$

These are (2.31) and (2.35), which also contain (2.30), $T^{\mathrm{PK1}} \cdot \nu = 0$ on ∂B, as a special case.

On Σ^{FS} the surface term in (7.28) gives an additional contribution to the dynamical NIBC, as is already clear from (7.20) or (7.26). Since the surface Lagrangian in (6.54) only depends on u, it suffices to evaluate the modified NIBC (3.7),

$$\partial_t(\partial_{\dot u} L''_{\Sigma^{\mathrm{FS}},[2]}) + \widetilde{\nabla}\cdot(\partial_{\widetilde{\nabla} u} L''_{\Sigma^{\mathrm{FS}},[2]}) - \partial_u L''_{\Sigma^{\mathrm{FS}},[2]} + [\partial_{\nabla u} L''_{[2]}]^+_- \cdot \nu = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\Sigma^{\mathrm{FS}} \times I^\circ).$$

With

$$\begin{aligned}
\partial_{\nabla u} L''_{[2]} &= -T^{\mathrm{PK1}}, \quad \partial_{\dot u} L''_{\Sigma^{\mathrm{FS}},[2]} = 0,\\
\partial_u L''_{\Sigma^{\mathrm{FS}},[2]} &= -p^0[\nu\cdot\widetilde{\nabla}u + u\cdot\widetilde{\nabla}\nu]^+_- = -p^0\nu\cdot[\widetilde{\nabla}u]^+_- - p^0\widetilde{\nabla}\nu\cdot[u]^+_-,\\
\widetilde{\nabla}\cdot(\partial_{\widetilde{\nabla}u} L''_{\Sigma^{\mathrm{FS}},[2]}) &= \widetilde{\nabla}\cdot(-p^0\nu[u]^+_-) = -\nu\widetilde{\nabla}\cdot(p^0[u]^+_-) - p^0\widetilde{\nabla}\nu\cdot[u]^+_-,
\end{aligned}$$

noticing that the terms $p^0\widetilde{\nabla}\nu\cdot[u]^+_-$ cancel, the interface condition reads

$$-\nu\widetilde{\nabla}\cdot(p^0[u]^+_-) + p^0\nu\cdot[\widetilde{\nabla}u]^+_- - [T^{\mathrm{PK1}}]^+_-\cdot\nu = 0,$$

that is,

$$[T^{\mathrm{PK1}}]^+_-\cdot\nu + \nu\widetilde{\nabla}\cdot(p^0[u]^+_-) - p^0\nu\cdot[\widetilde{\nabla}u]^+_- = 0.$$

Finally, as $[u]^+_-\cdot\nu = 0$, (4.15) allows us to pull out the jump operator:

$$\Big[T^{\mathrm{PK1}}\cdot\nu + \nu\widetilde{\nabla}\cdot(p^0u) - p^0\nu\cdot\widetilde{\nabla}u\Big]^+_- = 0. \tag{7.33}$$

Upon recalling the definition of the modified surface traction vector (2.36),

$$\tau^{\mathrm{PK1}} = T^{\mathrm{PK1}}\cdot\nu + \nu\widetilde{\nabla}\cdot(p^0u) - p^0\nu\cdot(\widetilde{\nabla}u),$$

the result (7.33) shows that the dynamical traction NIBC on Σ^{FS} reduce to (2.32):

$$[\tau^{\mathrm{PK1}}]^+_- = 0 \qquad \text{on} \qquad \Sigma^{\mathrm{FS}}.$$

As we have seen in Section 6.5.3, this is a special case of (6.47),

$$[T^{\mathrm{PK1}}\cdot\nu - \widetilde{\nabla}\cdot((T^0\cdot\nu)u)]^+_- = 0.$$

We discuss the fluid-solid normality condition (4.27), $f\cdot\nu = (\nu\cdot f\cdot\nu)\nu$ on Σ^{FS}, which by Theorem 7.4 (iii) is required to obtain the NIBC (3.6) or (3.7). For a volume action of the form $\mathcal{J} = \int_V F\mathrm{dV}$ we have $f = \partial_{Dy}F$. Consequently, normality corresponds to the dynamical interface condition (2.23) for prestress,

$$T^0\cdot\nu = (\nu\cdot T^0\cdot\nu)\nu \quad \text{on} \quad \Sigma^{\mathrm{FS}}.$$

In the presence of a surface action, that is, if the action integral is of the form $\int_V F\mathrm{dV} + \int_S F_S\mathrm{dS}$ with $S = \Sigma^{\mathrm{FS}}$, we have $f\cdot\nu = [\partial_{Dy}F]^+_-\cdot\nu + \widetilde{\mathrm{div}}(\partial_{\widetilde{D}y}F_S) - \partial_y F_S$. For the composite fluid-solid earth model only the second-order action contains a surface term. Thus, as already announced, normality corresponds to the dynamical interface condition (2.33),

$$\tau^{\mathrm{PK1}} = (\tau^{\mathrm{PK1}}\cdot\nu)\nu \quad \text{on} \quad \Sigma^{\mathrm{FS}}.$$

The sum of the slip conditions (2.23) and (2.33) coincides with the linearized version of the exact normality condition (2.17) on slipping fluid-solid interfaces:

$$T^s\cdot\nu = (\nu\cdot T^s\cdot\nu)\nu.$$

We note that both slip conditions (2.23) and (2.33) on Σ^{FS} follow from the constitutive laws for T^0 and T^{PK1} in elastic fluids, which is encoded in the prestressed elasticity tensor Λ^{T^0} (see Sections 4.6.3, 6.4.3, and [DT98, Section 3.4.2]).

We summarize our results.

Corollary 7.6 (**Variational derivation of the strong elastic-gravitational equations, including boundary and interface conditions**). *Let Assumption 2 hold and*

$$\Sigma = \Sigma^{\mathrm{SS}} \cup \Sigma^{\mathrm{FS}} \cup \Sigma^{\mathrm{FF}} \cup \partial B$$

be the interior boundary of a possibly unbounded Lip*-composite domain* $V \subseteq B^{\mathrm{FSC}}$.

(i) **Strong equilibrium equations:** *Under Assumption 3* (i)*, the weak equilibrium EL* (7.27) *imply strong EL, NBC, and NIBC, that coincide with*

– *the equilibrium equations* (2.19), (2.20) *a.e. in* $L^2(B^{\mathrm{FSC}})$,

– *the gravity interface condition* (2.25) *in* $H^{-\frac{1}{2}}(\Sigma)$*, and*

– *the traction boundary and interface conditions* (2.21), (2.22) *in* $H^{-\frac{1}{2}}(\Sigma \setminus \Sigma^{\mathrm{FS}})$.

If the frictionless slip condition (2.23) *holds on* Σ^{FS}*, then the NIBC also include the traction interface condition* (2.22) *in* $H^{-\frac{1}{2}}(\Sigma^{\mathrm{FS}})$.

(ii) **Strong dynamical equations:** *Under Assumption 3* (ii)*, the weak dynamical EL* (7.28) *imply strong EL, NBC, and NIBC, that coincide with*

– *the dynamical equations* (2.26), (2.27) *a.e. in* $L^2(B^{\mathrm{FSC}} \times I^\circ)$,

– *the gravity interface condition* (2.35) *in* $H^{-\frac{1}{2}}(\Sigma \times I^\circ)$*, and*

– *the traction boundary and interface conditions* (2.30), (2.31) *in* $H^{-\frac{1}{2}}((\Sigma \setminus \Sigma^{\mathrm{FS}}) \times I^\circ)$.

If the frictionless slip condition (2.33) *holds on* Σ^{FS}*, then the NIBC also include the traction interface condition* (2.32) *in* $H^{-\frac{1}{2}}(\Sigma^{\mathrm{FS}} \times I^\circ)$.

The equilibrium gravity interface condition (2.25) actually holds from the solution of the weak equilibrium Poisson equation alone, which also yields (2.24) and the decay conditions (see Proposition 8.1). Moreover, the analysis in Chapter 8 will reveal that the conditions (i) and (ii) in Assumption 3 (and thus the validity of the strong EL, the NBC, and the NIBC) essentially follow from the existence of solutions of the weak EL (7.27) and (7.28) (see Propositions 8.2, 8.10, and 8.11; in the dynamical case the material parameters must be piecewise Lip). Eventually, as Section 5.3 suggests, a geometric formulation involving generalized variations would further allow to variationally derive the linearized kinematic slip condition (2.29), characterizing Σ^{FS} (see Remark 5.7). Thereby, the complete system of equations, boundary and interface conditions of the linearized elastic-gravitational equations arises from a variational principle.

7.5 Hydrostatic approximation

In a **hydrostatic equilibrium earth model** (which is also, but incorrectly, is referred to as a **hydrostatic earth model**), prestress is a pure pressure, that is, given by (4.66),

$$T^0 = -p^0 1_{3\times 3}.$$

Consequently, by (4.65) the deviatoric prestress vanishes,

$$T^0_{\mathrm{dev}} = 0,$$

and the static equilibrium equation (2.19) reduces to **hydrostatic balance**

$$\nabla p^0 = -\rho^0 \nabla(\Phi^0 + \Psi^s). \tag{7.34}$$

In a hydrostatic equilibrium earth model, pressure gradients must be parallel to the gravitational plus centrifugal acceleration terms: Taking the cross product of (7.34) with ∇p^0 (which with Corollary 7.6 is in L^2) results in

$$\nabla p^0 \times \nabla(\Phi^0 + \Psi^s) = 0.$$

Let us consider regions where ρ^0 is sufficiently smooth such that all terms in (7.34) are at least $\mathcal{C}^1$. This is the case, e.g., if ρ^0 is $\mathcal{C}^{1,\alpha}$ for $0 < \alpha < 1$, because solutions Φ^0 of the Poisson equation (2.20) then are $\mathcal{C}^{3,\alpha}$ by elliptic regularity on Hölder-Zygmund spaces [Hör97, notes at the end in Section 8.6, referring to Corollary 8.4.7]. The properties of these spaces with respect to multiplication (see [Hör97, Proposition 8.6.8]) guarantee that ∇p^0 is $\mathcal{C}^{1,\alpha}$.

Taking the curl of (7.34) then yields

$$\nabla \rho^0 \times \nabla(\Phi^0 + \Psi^s) = 0.$$

Consequently, in a hydrostatic earth model,

$$\nabla \rho^0 \parallel \nabla p^0 \parallel \nabla(\Phi^0 + \Psi^s). \tag{7.35}$$

In other words, the level surfaces of the density ρ^0, the pressure p^0 and the geopotential $\Phi^0 + \Psi^s$ are parallel. Due to the zero-traction condition (2.21) we have $p^0 = 0$ on the exterior boundary ∂B, that is, ∂B is a level set for p^0. Therefore also ρ^0 and $\Phi^0 + \Psi^s$ must be constant on ∂B. Furthermore, in the hydrostatic model one assumes that all interior boundaries Σ are level surfaces of ρ^0, p^0, and $\Phi^0 + \Psi^s$. On $\mathcal{C}^1$-discontinuity surfaces of ρ^0, which are contained in $\Sigma \cup \partial B$, the $\pm$-traces, $(\nabla \rho^0)_\pm$, $(\nabla p^0)_\pm$, and $(\nabla(\Phi^0 + \Psi^s))_\pm$, must also be parallel and the respective surface gradients vanish [DT98, (3.258)]:

$$\widetilde{\nabla} \rho^0 = 0, \quad \widetilde{\nabla} p^0 = 0, \quad \widetilde{\nabla}(\Phi^0 + \Psi^s) = 0 \quad \text{on} \quad \Sigma \cup \partial B. \tag{7.36}$$

The hydrostatic assumption $T^0 = -p^0 1_{3\times 3}$ severely restricts the possible equilibrium earth models. In absence of rotation, hydrostatic equilibrium implies spherical symmetry of the planet, that is, its material parameters are functions of the radial coordinate only. Uniform rotation leads to rotational ellipsoidal symmetry (up to first order in the centrifugal-to-gravitational-force ratio, which is sufficiently accurate in case of the relatively slowly rotating Earth [DT98, p. 597]). Thus, all level sets and interior boundaries must be oblate ellipsoids. In particular, the fields ρ^0, p^0, and $\Phi^0 + \Psi^s$ depend only on the ellipsoidal radial distance from the center of the Earth but are constant in lateral directions (varying latitude or longitude). Consequently, the presence of any lateral heterogeneity in ρ^0 requires a non-zero initial deviatoric stress T^0_{dev} for its support, so that the hydrostatic assumption $T^0 = -p^0 1_{3\times 3}$ does no longer hold. In practice this discrepancy is often ignored and one assumes validity of the hydrostatic equations even if ρ^0 is laterally heterogeneous. However, this so-called **quasi-hydrostatic assumption** is justified, as T^0_{dev} generally is small compared to the Earth's rigidity [DT98, p. 102].

The advantage of the hydrostatic assumption is that it simplifies the elastic-gravitational equations and interface conditions, especially if the Lagrangian stress perturbation T^1 is employed. By (6.40), if $T^0_{\text{dev}} = 0$, then

$$T^1 = \Gamma : \nabla u,$$

which is Hooke's law of classical linearized elasticity (Section 1.3.2 for $\Gamma = c$; recalling also the definition of the linearized strain tensor $\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$).

In particular, this shows that T^1 is independent of initial pressure p^0. The equation of motion (2.26) then can also be written without explicit occurrence of p^0 [DT98, (3.259)] (cf. Section 6.5.2):

$$\rho^0 \left(\ddot{u} + 2\,\Omega \times \dot{u} + \nabla \Phi^{s1}\right) - \left(\nabla \cdot (\rho^0 u)\right) \nabla(\Phi^0 + \Psi^s) + \nabla \left(\rho^0 \nabla(\Phi^0 + \Psi^s) \cdot u\right) - \nabla \cdot (\Gamma : \nabla u) = f.$$

Here we inserted $\rho^{s1} = -\nabla \cdot (\rho^0 u)$ from (6.44) and employed the hydrostatic equilibrium (7.34). The linearized exterior boundary and interface conditions simplify to [DT98, (3.265)–(3.267)]

$$p^0 = 0, \quad T^1 \cdot \nu = 0 \qquad \text{on} \qquad \partial B,$$

and

$$[p^0]^+_- = 0, \quad \left[T^1\right]^+_- \cdot \nu = 0 \qquad \text{on} \qquad \Sigma = \Sigma^{\text{SS}} \cup \Sigma^{\text{FS}}, \tag{7.37}$$

as well as the additional condition for frictionless slip (normality condition)

$$T^1 \cdot \nu = (\nu \cdot T^1 \cdot \nu)\nu \qquad \text{on} \qquad \Sigma^{\text{FS}}. \tag{7.38}$$

These identities follow from the relation (6.37) between T^{PK1} and T^1, $T^0 = -p^0 1_{3\times 3}$, and the isobaric condition $\widetilde{\nabla} p^0 = 0$ in (7.36). In consistence with the variational derivation of the dynamical interface conditions as NIBC (3.7), the second-order surface energy vanishes if prestress is hydrostatic.

Chapter 8

Analysis of the elastic-gravitational equations

We analyze the linearized system of elastic-gravitational equations and investigate existence, uniqueness, and regularity of solutions within the low-regularity setting. First we show solvability of the weak equilibrium equations, comprising the static equilibrium equation and the equilibrium Poisson equation (Section 8.1). Then solvability of the weak equation of motion is discussed by employing the method of energy estimates, where the incremental gravitational potential is expressed in terms of displacement via the solution operator for the perturbed Poisson equation (Section 8.2). Thereby, existence and uniqueness of weak solutions of the linearized system of elastic-gravitational equations can be established. We conclude with a discussion of additional regularity results for the dynamical equations (Section 8.3).

8.1 Solution of the weak equilibrium equations

The weak equilibrium equations (7.27),

$$\int_{B^{\mathrm{FSC}}} \left(T^0 : \nabla v + \rho^0 \nabla(\Phi^0 + \Psi^s) \cdot v + \frac{1}{4\pi G} \nabla\Phi^0 \cdot \nabla w + \rho^0 w \right) \mathrm{dV} = 0,$$

must hold for all test functions $v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3$ and $w \in H^1(B^{\mathrm{FSC}})$, where $B^{\mathrm{FSC}} \subseteq \mathbb{R}^3$ is the composite domain defined in (4.40).

This system constrains the possible reference configurations ρ^0, Φ^0, T^0, Ω of the variational earth model, as was already mentioned in Section 6.2. In this section, we investigate the regularity properties of the weak equilibrium equations more thoroughly. In Propositions 8.1 and 8.2 we will find that, under Assumption 2 (i), validity of the weak equilibrium equations (7.27) implies the hypotheses of Assumption 3 (i). By Corollary 7.6, this shows **weak-strong uniqueness** of the weak equilibrium equations, that is, solutions Φ^0 and T^0 of the weak equations are actually solutions of the equations in strong form.

8.1.1 Equilibrium Poisson's equation

Setting $v = 0$ in (7.27) gives the **weak equilibrium Poisson equation**

$$\int_{B^{\mathrm{FSC}}} \left(\frac{1}{4\pi G} \nabla\Phi^0 \cdot \nabla w + \rho^0 w \right) \mathrm{dV} = 0 \qquad \text{for all} \qquad w \in H^1(B^{\mathrm{FSC}}). \tag{8.1}$$

Its solution determines the equilibrium gravitational potential Φ^0 resulting from a given equilibrium density model ρ^0.

Proposition 8.1 (**Solution and regularity for the equilibrium Poisson equation**). *As in Assumption 2* (i)*, let* $\rho^0 \in L^\infty(\mathbb{R}^3)$ *with* $\operatorname{supp}(\rho^0) \subseteq \overline{B}$ *be given. Then there exists* $\Phi^0 \in Y^\infty(\mathbb{R}^3)$ *that is the unique solution of the weak EL* (8.1)*. Moreover,* Φ^0 *also satisfies the equilibrium Poisson equation* (2.20)*,*

$$\triangle \Phi^0 = 4\pi G \rho^0 \qquad \text{in} \qquad L^2(\mathbb{R}^3),$$

as well as the IBC (2.24) *and* (2.25)*, which hold in a pointwise sense. Furthermore, the solution is explicitly given by*

$$\Phi^0 = 4\pi G E_3 * \rho^0, \qquad \text{that is,} \qquad \Phi^0(x) = -G \int_{\mathbb{R}^3} \frac{\rho^0(x')}{|x - x'|} \,\mathrm{dV}' \quad \text{for} \quad x \in \mathbb{R}^3.$$

Proof. By the hypotheses, $\nabla\Phi^0$ and ρ^0 are in $L^2(\mathbb{R}^3)$. Therefore the integration domain B^{FSC} in the weak EL (8.1) can be replaced by $\mathbb{R}^3$ and one may consider test functions $w \in H^1(\mathbb{R}^3)$. Restricting to $w \in \mathcal{D}(\mathbb{R}^3)$, the weak EL can be interpreted as the distributional duality:

$$\frac{1}{4\pi G} \Big\langle \nabla\Phi^0, \nabla w \Big\rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} + \Big\langle \rho^0, w \Big\rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} = 0.$$

This gives $\langle \triangle\Phi^0, w\rangle = 4\pi G \langle \rho^0, w\rangle$, showing that $\Phi^0 \in Y^\infty(\mathbb{R}^3)$ is the distributional solution of the equilibrium Poisson equation $\triangle\Phi^0 = 4\pi G\rho^0$ (2.20). By Lemma 4.28, $\Phi^0 = 4\pi G E_3 * \rho^0 \in \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$, implying $\nabla\Phi^0 \in \bigcap_{1\le p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^3)^3$. With the decay $\nabla\Phi^0(x) = O(1/|x|^2)$ as $|x| \to \infty$, we obtain

$$\nabla\Phi^0 \in H^1(\mathbb{R}^3)^3 \subseteq H_{\mathrm{div}}(\mathbb{R}^3)^3 \subseteq H_{\mathrm{div}}(B^{\mathrm{FSC}})^3.$$

This gives the relevant condition of Assumption 3 (i) which by Corollary 7.6 suffices to establish the strong EL and IBC. □

Let us note again that all IBC for Φ^0 already follow from the inclusion $\Phi^0 \in Y^\infty(\mathbb{R}^3) \subseteq \mathcal{C}^1(\mathbb{R}^3)$. Moreover, the H_{div}-condition for $\nabla\Phi^0$ can also directly be inferred from the validity of Poisson's equation in distributional sense: $\nabla \cdot \nabla\Phi^0 = \triangle\Phi^0 = 4\pi G\rho^0 \in L^\infty_{\mathrm{c}}(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3)$.

8.1.2 Static equilibrium equation

With $w = 0$, (7.27) reduces to

$$\int_{B^{\mathrm{FSC}}} \left(T^0 : \nabla v + \rho^0 \nabla(\Phi^0 + \Psi^s) \cdot v\right) \mathrm{dV} = 0 \qquad \text{for all} \qquad v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3. \tag{8.2}$$

This equation constrains the possible choice of prestress tensors T^0, for given Ω, ρ^0, and Φ^0 defined as the solution of Poisson's equation (see Proposition 8.1).

Proposition 8.2 (**Regularity for the static equilibrium equation**). *In accordance with Assumption 2* (i)*, let* $T^0 \in L^\infty(\mathbb{R}^3)^{3\times 3}$ *and* $\rho^0 \in L^\infty(\mathbb{R}^3)$ *with support of* T^0 *and* ρ^0 *contained in* $\overline{B}$ *be given and assume* $\Phi^0 \in Y^\infty(\mathbb{R}^3)$*. If the weak EL* (8.2) *hold, then*

$$T^0 \in H_{\mathrm{div}}(B^{\mathrm{FSC}})^{3\times 3}$$

and T^0 *satisfies the static equilibrium equation* (2.19)*,*

$$\nabla \cdot T^0 = \rho^0 \nabla(\Phi^0 + \Psi^s) \qquad \text{in} \qquad L^2(B^{\mathrm{FSC}}),$$

as well as the conditions (2.21) *and* (2.22) *which hold in the sense of* $H^{-1/2}$*.*

Proof. By Assumption 2 (i), the integrands T^0 and $\rho^0\nabla(\Phi^0+\Psi^s)$ are in $L^2(B^{\mathrm{FSC}})$, which together with $v \in \mathcal{D}(B^{\mathrm{FSC}})^3$ allows us to interpret (8.2) as the distributional duality

$$\Big\langle T^0, \nabla v \Big\rangle_{\mathcal{D}'(B^{\mathrm{FSC}}),\mathcal{D}(B^{\mathrm{FSC}})} + \Big\langle \rho^0\nabla(\Phi^0+\Psi^s), v \Big\rangle_{\mathcal{D}'(B^{\mathrm{FSC}}),\mathcal{D}(B^{\mathrm{FSC}})} = 0.$$

Equivalently, $\langle \nabla\cdot T^0, v\rangle = \langle \rho^0\nabla(\Phi^0+\Psi^s), v\rangle$, that is, (2.19) holds in $\mathcal{D}'(B^{\mathrm{FSC}})$. But since $\rho^0\nabla(\Phi^0+\Psi^s) \in L^2(B^{\mathrm{FSC}})^3$, we obtain $\nabla\cdot T^0 \in L^2(B^{\mathrm{FSC}})^3$. This proves $T^0 \in H_{\mathrm{div}}(B^{\mathrm{FSC}})^{3\times 3}$, which by Corollary 7.6 is the relevant condition of Assumption 3 (i) to deduce the strong EL, NBC, and NIBC. □

8.2 Solution of the weak dynamical equations

The weak dynamical equations (7.28) split into the **weak perturbed Poisson equation**

$$\int_{B^{\mathrm{FSC}}} \Big(\rho^0 u + \frac{1}{4\pi G}\nabla\Phi^{s1}\Big)\cdot\nabla w \,\mathrm{dV} = 0, \tag{8.3}$$

valid in $\mathcal{D}'(I^\circ)$ for all $w \in H^1(B^{\mathrm{FSC}})$, and the **weak equation of motion**

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{B^{\mathrm{FSC}}} \rho^0 \dot{u}\cdot v \,\mathrm{dV}\Big) + \int_{B^{\mathrm{FSC}}} \Big(\Lambda^{T^0} : \nabla u\Big) : \nabla v \,\mathrm{dV} \\
&\quad + \int_{B^{\mathrm{FSC}}} \rho^0 \left(2\,\Omega\times\dot{u} + \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla\Phi^{s1}\right)\cdot v \,\mathrm{dV} \\
&\quad + \int_{\Sigma^{\mathrm{FS}}} p^0 \Big[u\cdot(\widetilde{\nabla}\nu)\cdot v + \nu\cdot(\widetilde{\nabla}u)\cdot v + \nu\cdot(\widetilde{\nabla}v)\cdot u\Big]_-^+ \mathrm{dS} \;=\; \int_{B^{\mathrm{FSC}}} f\cdot v \,\mathrm{dV},
\end{aligned} \tag{8.4}$$

valid in $\mathcal{D}'(I^\circ)$ for all $v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3$.

Recall that, correct to first order, the force was given by $f = -\rho^0\nabla F^s$, see (6.21). However, the hypothesis of a conservative force was only necessary to derive the governing equations from Hamilton's principle of stationary action and thus can be dropped in the present discussion of solvability. Therefore, in Assumption 2 (ii), the condition $F^s \in L^2(I^\circ, H^1(\mathbb{R}^3))$ for the force potential can be replaced by the following condition on a **general body force** f:

$$f \in L^2(I^\circ, L^2(\mathbb{R}^3))^3 \qquad \text{with} \qquad \mathrm{supp}(f_t) \subseteq \overline{B} \qquad \text{for a.a. } t \in I^\circ. \tag{8.5}$$

In order to write the dynamical elastic-gravitational equations as a system for the displacement u only, we need to express Φ^{s1} in terms of u by solving the weak perturbed Poisson equation (8.3).

8.2.1 The solution operator for the perturbed Poisson equation

Under Assumption 2 (and by arguments similar as those in the proof of Proposition 8.1), the weak perturbed Poisson equation (8.3) may be interpreted as distributional duality

$$\Big\langle \rho^0 u + \frac{1}{4\pi G}\nabla\Phi^{s1}, \nabla w\Big\rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}.$$

Thus Φ^{s1} solves $\triangle\Phi^{s1} = -4\pi G\nabla\cdot(\rho^0 u)$ (2.27) in the sense of $\mathcal{D}'(\mathbb{R}^3)$. Since $\rho^0 u \in \mathcal{E}'(\mathbb{R}^3)^3$, Lemma 4.28 yields the solution operator $S\colon \mathcal{D}'(\mathbb{R}^3)^3 \to \mathcal{D}'(\mathbb{R}^3)$,

$$\Phi^{s1} = Su := -4\pi G\, E_3 * (\nabla\cdot(\rho^0 u)) = -4\pi G(\nabla E_3) * (\rho^0 u). \tag{8.6}$$

In order to also take the geopotential terms in the weak equation of motion (8.4) into account, we further introduce the operator $Q\colon L^2(\mathbb{R}^3)^3 \to \mathcal{D}'(\mathbb{R}^3)^3$

$$Qu := \rho^0(\nabla\nabla(\Phi^0 + \Psi^s)\cdot u + \nabla(Su)). \tag{8.7}$$

The following lemma states that the restricted operators (by abuse of notation denoted by the same symbols)

$$S\colon L^2(B^{\mathrm{FS}})^3 \to H^1(B^{\mathrm{FS}}) \qquad \text{and} \qquad Q\colon H^1(B^{\mathrm{FS}})^3 \to L^2(B^{\mathrm{FS}})^3, \quad Q\colon L^2(B^{\mathrm{FS}})^3 \to H^{-1}(B^{\mathrm{FS}})^3$$

are continuous within the regularity framework of the linearized variational model. To abbreviate the notation, we temporarily set

$$g := -4\pi G\rho^0 \qquad \text{and} \qquad h := \rho^0\nabla\nabla\Phi^0.$$

Assumption 2 implies that $g \in L^\infty(\mathbb{R}^3)$, with support contained in $\overline{B}$. Lemma 7.1 then shows that the components of h are in $\bigcap_{1\le p<\infty} L^p_{\mathrm{loc}}(\mathbb{R}^3) \subseteq \bigcap_{1\le p<\infty} L^p(B^{\mathrm{FS}}) \subseteq L^2(B^{\mathrm{FS}}) \cap L^6(B^{\mathrm{FS}})$, where we benefit from the boundedness of the restricted domain B^{FS}.

Lemma 8.3 (Mapping properties of the operators S and Q).

(i) *Let $g \in L^\infty_{\mathrm{c}}(\mathbb{R}^3)$ and define the map $\widetilde{S}\colon \mathcal{D}'(\mathbb{R}^3)^3 \to \mathcal{D}'(\mathbb{R}^3)$ by $\widetilde{S}y := E_3 * (\nabla\cdot(gy))$. Then $\widetilde{S}\colon L^2(\mathbb{R}^3)^3 \to H^1(\mathbb{R}^3)$ and its restriction $S\colon L^2(B^{\mathrm{FS}})^3 \to H^1(B^{\mathrm{FS}})$ are continuous.*

(ii) *Let $h \in L^\infty(\mathbb{R}^3)^{3\times 3}$ and $\widetilde{Q}\colon L^2(\mathbb{R}^3)^3 \to \mathcal{D}'(\mathbb{R}^3)^3$ be defined by $\widetilde{Q}y := y\cdot h + \nabla(\widetilde{S}y)$. Then $\widetilde{Q}\colon L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ and its restriction $Q\colon L^2(B^{\mathrm{FS}})^3 \to L^2(B^{\mathrm{FS}})^3$ are continuous.*

(iii) *If the components of h are elements of $L^2(B^{\mathrm{FS}}) \cap L^6(B^{\mathrm{FS}})$, then we obtain continuity of Q as a map $H^1(B^{\mathrm{FS}})^3 \to L^2(B^{\mathrm{FS}})$ and also as a map $L^2(B^{\mathrm{FS}})^3 \to H^{-1}(B^{\mathrm{FS}})$.*

Proof. (i) Let $y \in L^2(\mathbb{R}^3)^3$. Since $gy \in L^2_{\mathrm{c}}(\mathbb{R}^3)^3$, we have $\nabla\cdot(gy) \in H^{-1}_{\mathrm{c}}(\mathbb{R}^3) \subseteq \mathcal{E}'(\mathbb{R}^3)$ and

$$\widetilde{S}y := E_3 * (\nabla\cdot(gy))$$

can be defined in $\mathcal{D}'(\mathbb{R}^3)$, because $E_3 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Young's inequality for convolution products [AF03, Corollary 2.25, p. 34], $\partial_i E_3 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and $gy_i \in L^2_{\mathrm{c}}(\mathbb{R}^3)$ $(i = 1,2,3)$ give

$$\widetilde{S}y = E_3 * \textstyle\sum_{i=1}^3 \partial_i(gy_i) = \sum_{i=1}^3 (\partial_i E_3) * (gy_i) \in L^2(\mathbb{R}^3).$$

Furthermore, we have

$$\triangle\widetilde{S}y = (\triangle E_3) * (\nabla\cdot(gy)) = \delta * (\nabla\cdot(gy)) = \nabla\cdot(gy) \in H^{-1}(\mathbb{R}^3).$$

Elliptic regularity (Lemma 3.15) of $\triangle$ thus yields $\widetilde{S}y \in H^1(\mathbb{R}^3)$, hence $\widetilde{S}\colon L^2(\mathbb{R}^3)^3 \to H^1(\mathbb{R}^3)$. For continuity of $\widetilde{S}$, let $y_k \to y$ in $L^2(\mathbb{R}^3)^3$ and $\widetilde{S}y_k \to z$ in $H^1(\mathbb{R}^3)$ $(k\to\infty)$. Then

$$\widetilde{S}y_k = E_3 * (\nabla\cdot(gy_k)) \to E_3 * (\nabla\cdot(gy))$$

as $k\to\infty$ in $\mathcal{D}'(\mathbb{R}^3)$, implying that $z = \widetilde{S}y$ in $\mathcal{D}'(\mathbb{R}^3)$. However, since $y \in L^2(\mathbb{R}^3)^3$ it follows from above that $\widetilde{S}y \in H^1(\mathbb{R}^3)$ and thus we have $z = \widetilde{S}y$ in $H^1(\mathbb{R}^3)$. Therefore, the closed graph theorem applies, establishing continuity of $\widetilde{S}\colon L^2(\mathbb{R}^3)^3 \to H^1(\mathbb{R}^3)$. Finally, define the operator

$$S\colon L^2(B^{\mathrm{FS}})^3 \to H^1(B^{\mathrm{FS}}), \qquad S := R_{B^{\mathrm{FS}}} \circ \widetilde{S} \circ E_{B^{\mathrm{FS}}},$$

where $E_{B^{\mathrm{FS}}}\colon L^2(B^{\mathrm{FS}})^3 \to L^2(\mathbb{R}^3)^3$ denotes the extension operator to $\mathbb{R}^3$, extending by zero outside B^{FS}, and $R_{B^{\mathrm{FS}}}\colon H^1(\mathbb{R}^3) \to H^1(B^{\mathrm{FS}})$ stands for the restriction operator from $\mathbb{R}^3$ to

B^{FS}. Since $\|E_{B^{\mathrm{FS}}}u\|_{L^2(\mathbb{R}^3)} = \|u\|_{L^2(B^{\mathrm{FS}})}$ for $u \in L^2(B^{\mathrm{FS}})^3$ and $\|R_{B^{\mathrm{FS}}}v\|_{H^1(B^{\mathrm{FS}})} \leq \|v\|_{H^1(\mathbb{R}^3)}$ for $v \in H^1(\mathbb{R}^3)$, all operators are continuous and so is their composition S.

(ii) Continuity of $\widetilde{Q}\colon L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ follows from $\|h \cdot y\|_{L^2(\mathbb{R}^3)} \leq \|h\|_{L^\infty(\mathbb{R}^3)}\|y\|_{L^2(\mathbb{R}^3)}$ for $y \in L^2(\mathbb{R}^3)^3$, $\|\nabla a\|_{L^2(\mathbb{R}^3)} \leq \|a\|_{H^1(\mathbb{R}^3)}$ for $a = \widetilde{S}y \in H^1(\mathbb{R}^3)$, and the continuity of the operator $\widetilde{S}\colon L^2(\mathbb{R}^3)^3 \to H^1(\mathbb{R}^3)$. Continuity of the restricted operator $Q\colon L^2(B^{\mathrm{FS}})^3 \to L^2(B^{\mathrm{FS}})^3$ follows as in (i).

(iii) If $h \in L^2 \cap L^6$ on B^{FS}, then $\|h \cdot y\|_{L^2} \leq c_h\|y\|_{H^1}$ for $y \in H^1$ by Lemma 7.1 for $n = 3$. Thus we obtain $Q\colon H^1 \to L^2$. The conditions $h \in L^6$ and $L^6 \cdot L^2 \subseteq H^{-1}$ (in general, by Lemma 3.14, $L^{2n} \cdot L^2 \subseteq L^{\frac{2n}{n+1}} \subseteq H^{-1}$ on $B^{\mathrm{FS}} \subseteq \mathbb{R}^n$) imply $Q\colon L^2 \to H^{-1}$. Continuity of these maps follows from the continuity of the embeddings. □

8.2.2 The equation of motion as an abstract evolution equation

We eliminate the explicit dependence on the incremental gravitational potential in the weak evolution equation (8.4) via the solution operator $\Phi^{s1} = Su$ (8.6). Then the weak EL for u is written as a sum of bilinear forms,

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a_0(u, v) + a_1(u, v) + s(u, v) = \langle f|v\rangle, \tag{8.8}$$

which holds for all $v \in H^1(B^{\mathrm{FS}})^3$ in the sense of $\mathcal{D}'(I^\circ)$. The framework will be analogous to Section 3.2.2, with variational triple $V \hookrightarrow H \hookrightarrow V'$ given by

$$H^1(B^{\mathrm{FS}})^3 \;\hookrightarrow\; L^2(B^{\mathrm{FS}})^3 \;\hookrightarrow\; (H^1(B^{\mathrm{FS}})^3)'.$$

The continuous embedding $H^{-1} \hookrightarrow (H^1)'$ mentioned in Section 3.2.1 allows to replace $(H^1(B^{\mathrm{FS}})^3)'$ by $H^{-1}(B^{\mathrm{FS}})^3$ in the estimates below.

Remark 8.4 (Restricting the domain from B^{FSC} to B^{FS}). With $\rho^0 = 0$ in B^{C} by Assumption 2, the perturbed Poisson equation (8.3) becomes $\int_{B^{\mathrm{C}}} \nabla\Phi^{s1} \cdot \nabla w \,\mathrm{dV} = 0$ for all $w \in H^1(B^{\mathrm{C}})$, that is, the Laplace equation $\triangle\Phi^{s1} = 0$, whereas the weak equation of motion (8.4) vanishes identically outside B. Consequently, it henceforth suffices to consider the elastic-gravitational equations restricted to the bounded domain B^{FS} instead of B^{FSC}. In Theorem 8.9 below, we establish existence and uniqueness of the weak solution u in $\mathcal{C}^0(I, H^1(B^{\mathrm{FS}}))^3 \cap \mathcal{C}^1(I, L^2(B^{\mathrm{FS}}))^3$. But since B^{FS} is a union of Lip-domains and thus possesses the Sobolev-extension property, this space is contained in $\mathcal{C}^0(I, H^1(B^{\mathrm{FSC}}))^3 \cap \mathcal{C}^1(I, L^2(B^{\mathrm{FSC}}))^3$. ⌟

To abbreviate the notation, we denote the spaces $L^2(B^{\mathrm{FS}})^3$, $H^1(B^{\mathrm{FS}})^3$, $H^{-1}(B^{\mathrm{FS}})^3$ simply by L^2, H^1, H^{-1} and write $\langle w|v\rangle_{L^2} = \int_{B^{\mathrm{FS}}} w\, v \,\mathrm{dV}$ for the standard L^2-inner product on $B^{\mathrm{FS}} \subseteq \mathbb{R}^n$. Moreover, $\langle\!\langle .,. \rangle\!\rangle := \langle\!\langle .,. \rangle\!\rangle_{\Sigma^{\mathrm{FS}}}$ denotes the surface Sobolev duality (3.32) on the $\mathcal{C}^{1,1}$-surface Σ^{FS}.

In accordance with (8.4), the bilinear forms a_0, a_1, b, c, and s are the maps

$$a_0,\, a_1,\, s\colon H^1 \times H^1 \to \mathbb{R} \qquad \text{and} \qquad b,\, c\colon L^2 \times L^2 \to \mathbb{R}, \tag{8.9}$$

given by

$$\begin{aligned}
a_0(w, v) &:= \langle \Lambda^{T^0} : \nabla w|\nabla v\rangle_{L^2}, && (8.10)\\
a_1(w, v) &:= \langle Qw|v\rangle_{L^2}, && (8.11)\\
b(w, v) &:= \langle \rho^0 2\Omega \times w|v\rangle_{L^2}, && (8.12)\\
c(w, v) &:= \langle \rho^0 w|v\rangle_{L^2}, && (8.13)\\
s(w, v) &:= [\langle\!\langle p^0 w \cdot (\widetilde{\nabla}\nu) + p^0\nu \cdot (\widetilde{\nabla}w),\, v\rangle\!\rangle + \langle\!\langle p^0\widetilde{\nabla}v,\, \nu w\rangle\!\rangle]_-^+. && (8.14)
\end{aligned}$$

Here, the operator Q is given by (8.7) and (8.6), that is,

$$Qw = \rho^0(\nabla\nabla(\Phi^0 + \Psi^s)\cdot w + \nabla(Sw)) \qquad \text{with} \qquad Sw = -4\pi G\, E_3 * (\nabla\cdot(\rho^0 w)).$$

Due to the mapping properties of S (Lemma 8.3), the integral operator ∇S is also just a contribution of order zero in the weak formulation (8.8).

Remark 8.5 (**Including the tangential slip condition**). The weak elastic-gravitational equations (7.28) were variationally derived for test functions $v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FS}})^3 := \{v \in H^1(B^{\mathrm{FS}})^3 : [v]^+_- \cdot \nu = 0 \text{ on } \Sigma^{\mathrm{FS}}\}$, see (6.11). In order to establish solvability of (8.8), we enlarge this space to $V := H^1(B^{\mathrm{FS}})^3$, which simplifies the analysis. Unfortunately, the weak solution found will generally not satisfy the tangential slip condition $[u]^+_- \cdot \nu = 0$ on Σ^{FS}. A Sobolev space framework incorporating the slip condition is proposed in [Val87, Val89a, Val89b], where the spectral problem associated to the weak elastic-gravitational equations is studied. ⌟

8.2.3 Conditions on material parameters and fluid-solid interfaces

In addition to Assumption 2, proving existence and uniqueness of weak solutions of the elastic-gravitational equation of motion requires the following conditions on the prestressed elasticity tensor Λ^{T^0}, the initial density ρ^0, and the curvature $\|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})}$ of the fluid-solid interfaces:

Assumption 4 (**Positivity of material parameters and low interface curvature**).

(i) Λ^{T^0} *is uniformly positive definite, in the sense that there exists* $\lambda > 0$ *such that for a.a.* $x \in B^{\mathrm{FS}}$ *and for all matrices* $X \in \mathbb{R}^{3\times 3}$, *the inequality* $X : \Lambda^{T^0}(x) : X \geq \lambda\, X : X$, *i.e.* $\Lambda^{T^0}_{ijkl}(x)X_{ij}X_{kl} \geq \lambda X_{ij}X_{ij}$, *holds.*

(ii) ρ^0 *is uniformly bounded from below on* B^{FS}, *i.e. there exists a constant* $\rho^0_- > 0$ *such that for a.a.* $x \in B^{\mathrm{FS}}$, *the inequality* $\rho^0_- < \rho^0(x)$ *holds.*

(iii) $\|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})}$ *is sufficiently low to ensure* $c_{\Sigma^{\mathrm{FS}}} < \lambda \,/\, \|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})}$, *where* λ *is the lower bound in* (i) *and* $c_{\Sigma^{\mathrm{FS}}} > 0$ *is the constant in the surface estimate*

$$|s(u,u)| \leq c_{\Sigma^{\mathrm{FS}}}\|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})}\|u\|^2_{H^1}. \tag{8.15}$$

As Lemma 8.8 (v) reveals, $c_{\Sigma^{\mathrm{FS}}} = 2(c')^2\|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})} + 4c''c'$ for Sobolev trace constants c', c''.

Condition (i) in Assumption 4 expresses the **strict convexity** of $\Lambda^{T^0}(x)$ as a bilinear map on $\mathbb{R}^{3\times 3}$. This implies strict convexity of the quadratic form

$$u \mapsto a_0(u,u) = \int_{B^{\mathrm{FS}}} (\Lambda^{T^0} : \nabla u) : \nabla u \,\mathrm{dV}$$

on H^1, which in turn corresponds to **coercivity** of a_0 (8.10), i.e. $a_0(u,u) \geq \lambda\|u\|^2_{H^1}$, see (3.37). The slightly weaker condition $a_0(u,u) \geq \alpha\|u\|^2_{H^1} - \beta\|u\|^2_{L^2}$ required in the proof of Theorem 3.18 is known as **Gårding's inequality** [MH83, Box 1.1, p. 323].

Remark 8.6 (**Convexity conditions**). In classical linearized elasticity, existence of weak solutions of the equation of motion $\rho^0\ddot{u} - \nabla\cdot(c : \nabla u) = f$ (1.59) is still guaranteed, if strict convexity is replaced by rank-one convexity (Legendre-Hadamard condition, strong ellipticity): There exists $\lambda > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$, the estimate $(\xi\,\eta) : c : (\xi\,\eta) \geq \lambda\,|\xi|^2|\eta|^2$ (that is, $\xi_i\eta_j\, c_{ijkl}\,\xi_k\eta_l \geq \lambda\,\xi_i^2\eta_j^2$) holds. In nonlinear elasticity, also the intermediate conditions of polyconvexity and quasiconvexity play an important role, see [MH83, Bal02, Ant05]. ⌟

Condition (iii) in Assumption 4 will provide H^1-coercivity of the combined bilinear form $a_0 + s$.

Remark 8.7 (**Validity of the curvature condition in the Earth**). In a very crude first approximation, the constant λ in Assumption 4 (i) may be interpreted as the bulk modulus κ, which in the real Earth is around four times larger than p^0, e.g. [KB08, p. 99]. Consequently, the curvature condition holds if $\|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})}$ is small enough to obtain $c_{\Sigma^{\mathrm{FS}}} < 4$. Combined with the fact that the formula (8.18) for $c_{\Sigma^{\mathrm{FS}}}$ seems not to be the optimal estimate for (8.15), the validity of the curvature condition is indeed realistic. ⌟

8.2.4 A priori energy estimates

A priori estimates are inequalities that a solution must necessarily satisfy if it exists. Thereby one can identify the regularity conditions that constrain the possible function spaces in which a solution can be found. The a priori estimates are usually obtained from the energy equality (or inequality), by employing positivity conditions of the material parameters.

Since the general theory of variational solutions of second-order evolutionary PDEs was already presented in Section 3.2.2, for our purposes it actually suffices to check whether the bilinear forms (8.9)–(8.14) satisfy all relevant conditions. This will be done in Lemma 8.8 below. However, we feel that it is illustrative to briefly sketch the derivation of the a priori estimates for the linearized elastic-gravitational equations.

The **energy equality for the elastic-gravitational equations** formally can be deduced by multiplying the equation of motion (2.26) by $\dot{u}$, the perturbed Poisson equation (2.27) by $\dot{\Phi}^{s1}$, adding the results, integrating over the space domain, and finally incorporating all interface and boundary conditions via the divergence theorem on composite domains (Lemma 4.11):

$$\begin{aligned}\frac{d}{dt}\Bigg(\int_{B^{\mathrm{FS}}}\Big(\frac{1}{2}\,\rho^0\dot{u}^2+\frac{1}{2}\,\nabla u:\Lambda^{T^0}:\nabla u+\frac{1}{2}\,\rho^0 u\cdot(\nabla\nabla(\Phi^0+\Psi^s))\cdot u+\rho^0 u\cdot\nabla\Phi^{s1}\Big)\,\mathrm{dV}\\ +\int_{\mathbb{R}^3}\frac{1}{8\pi G}(\nabla\Phi^{s1})^2\,\mathrm{dV}\Bigg)+\int_{\Sigma^{\mathrm{FS}}}[\dot{u}\cdot T^{\mathrm{PK1}}]^+_-\cdot\nu\,\mathrm{dS}=\int_{B^{\mathrm{FS}}}f\cdot\dot{u}\,\mathrm{dV}.\end{aligned}\tag{8.16}$$

Alternatively, the energy equality may also be obtained based on the weak formulation, by a procedure analogous to the proof of Lemma 3.19: Inserting the test functions $v=\dot{u}$ and $w=\dot{\Phi}^{s1}$ in the weak dynamical equations (7.28) directly yields twice the energy equality (8.16).

The volume integrand is the total second-order energy density $(E''_{\mathrm{kin}}+E''_{\mathrm{pot}})_{[2]}$, whose constituents are in one-to-one correspondence with the terms in the second-order volume Lagrangian density $L''_{[2]}=(E''_{\mathrm{kin}}-E''_{\mathrm{pot}})_{[2]}$ of (6.52), but without the Coriolis term (since Coriolis force is orthogonal to the velocity, $\rho^0(\Omega\times\dot{u})\cdot\dot{u}=0$, it cannot do any work).

As the second-order identity (6.65) for the surface energy dissipation reveals, the surface integral in (8.16) is a total time derivative as well (this can alternatively be deduced directly from the dynamical slip condition (2.32) and the specific form of the constitutive law for T^{PK1} in fluids (6.43), cf. [dHHP15, Lemma 3.1]):

$$\begin{aligned}\int_{\Sigma^{\mathrm{FS}}}[\dot{u}\cdot T^{\mathrm{PK1}}]^+_-\cdot\nu\,\mathrm{dS}&=\frac{d}{dt}\int_{\Sigma^{\mathrm{FS}}}E''_{\Sigma^{\mathrm{FS}},[2]}\,\mathrm{dS}\\ &=\frac{d}{dt}\int_{\Sigma^{\mathrm{FS}}}p^0\Big[\nu\cdot\widetilde{\nabla}u\cdot u+\frac{1}{2}\,u\cdot\widetilde{\nabla}\nu\cdot u\Big]^+_-\mathrm{dS}=\frac{1}{2}\,\frac{d}{dt}\,s(u,u).\end{aligned}\tag{8.17}$$

In terms of the bilinear forms (8.9)–(8.14), the energy equality (8.16) thus reads

$$\frac{d}{dt}\,\frac{1}{2}\,\Big(c(\dot{u},\dot{u})+(a_0+\widetilde{a}_1+s)(u,u)\Big)=\langle f,\dot{u}\rangle.$$

(Here, a_1 is generalized to $\widetilde{a}_1$ by including the gravitational term $\frac{1}{8\pi G}(\nabla\Phi^{s1})^2$, which is not present in the weak equation for u alone.)

In comparison with the abstract version $\frac{d}{dt}\frac{1}{2}\big(c(\dot{u},\dot{u}) + a_0(u,u)\big) + \big(b(\dot{u},\dot{u}) + a_1(u,\dot{u})\big) = \langle f, \dot{u}\rangle$ (3.54), this specific form of the energy equality reflects the fact that the system of elastic-gravitational equations does not include any dissipative effects, except if they are contained in the force f.

With the positivity conditions of Assumption 4 and the mapping properties of the lower-order bilinear forms, one may estimate the terms in the time-integrated energy equality (8.16) from above and below as in Section 3.2.2 and employ Gronwall's inequality (Lemma 3.20). This results in the following **a priori estimates for the elastic-gravitational equations**:

$$\|\dot{u}(t)\|^2_{L^2(B^{\mathrm{FS}})} + \|u(t)\|^2_{H^1(B^{\mathrm{FS}})} \leq c$$

for a.a. $t \in I$ and some constant $c > 0$, dependent on the material parameters, the initial conditions, and the force. These estimates imply that the solution must satisfy $\dot{u} \in L^\infty(I^\circ, L^2(B^{\mathrm{FS}})^3)$ and $u \in L^\infty(I^\circ, H^1(B^{\mathrm{FS}})^3)$, that is,

$$u \in L^\infty(I^\circ, H^1(B^{\mathrm{FS}})^3) \cap W^{1,\infty}(I^\circ, L^2(B^{\mathrm{FS}})^3).$$

Together with existence and uniqueness of solutions, a refined version of these conditions and energy estimates will be rigorously established in Theorem 8.9 and (8.20) below.

8.2.5 Existence and uniqueness of weak solutions

We verify the conditions necessary to directly apply the method of energy estimates for second-order linear evolution equations, presented in Section 3.2.2.

Lemma 8.8 (Properties of the bilinear forms). *Let the Assumptions 2 and 4 hold. Then a_0, a_1, b, c, s defined by (8.9)–(8.14) satisfy the following conditions:*

(i) *a_0 is a continuous, symmetric, bilinear form on H^1.*
For all w, $v \in H^1$, $|a_0(w,v)| \leq c_0\|w\|_{H^1}\|v\|_{H^1}$ with $c_0 = \|\Lambda^{T^0}\|_{L^\infty(B^{\mathrm{FS}})}$.
For all $w \in H^1$, $a_0(w,w) \geq \alpha\|w\|^2_{H^1} - \beta\|w\|^2_{L^2}$ with $\alpha = \beta = \lambda$.

(ii) *a_1 is a continuous bilinear form on H^1.*
For all w, $v \in H^1$, $|a_1(w,v)| \leq c_{a_1}\|w\|_{H^1}\|v\|_{L^2}$ with $c_{a_1} = \|Q\|_{\mathrm{op}}$.
For all w, $v \in L^2$, $|a_1(w,v)| \leq c'_{a_1}\|w\|_{L^2}\|v\|_{H^{-1}}$ with $c'_{a_1} > 0$.

(iii) *b is a continuous bilinear form on L^2.*
For all y, $z \in L^2$, $|b(y,z)| \leq c_b\|y\|_{L^2}\|z\|_{L^2}$ with $c_b = 2|\Omega|\|\rho^0\|_{L^\infty(B^{\mathrm{FS}})}$.

(iv) *c is a continuous, symmetric, bilinear form on L^2.*
For all y, $z \in L^2$, $|c(y,z)| \leq c_c\|y\|_{L^2}\|z\|_{L^2}$ with $c_c = \|\rho^0\|_{L^\infty(B^{\mathrm{FS}})}$.
For all $y \in L^2$, $c(y,y) \geq \gamma\|y\|^2_{L^2}$ with $\gamma = \rho^0_-$.

(v) *s is a continuous, symmetric, bilinear form on H^1.*
For all w, $v \in H^1$, $|s(v,w)| \leq c_s\|v\|_{H^1}\|w\|_{H^1}$ with

$$c_s = c_{\Sigma^{\mathrm{FS}}}\|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})} \qquad \text{where} \qquad c_{\Sigma^{\mathrm{FS}}} := 2(c')^2\|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})} + 4c''c'. \tag{8.18}$$

Proof. Bilinearity of the forms in clear by their definition. Let w, $v \in H^1$ and y, $z \in L^2$.

(i) Continuity of a_0 follows from Cauchy-Schwarz and from boundedness of Λ^{T^0}:

$$|a_0(w,v)| = |\langle \Lambda^{T^0} : \nabla w | \nabla v\rangle_{L^2}| \leq \|\Lambda^{T^0}\|_{L^\infty}\|\nabla w\|_{L^2}\|\nabla v\|_{L^2} \leq \|\Lambda^{T^0}\|_{L^\infty}\|w\|_{H^1}\|v\|_{H^1}.$$

Symmetry of a_0 is a consequence of the major symmetry $\Lambda^{T^0}_{ijkl} = \Lambda^{T^0}_{klij}$:

$$a_0(w,v) = \langle \Lambda^{T^0} : \nabla w | \nabla v \rangle_{L^2} = \textstyle\int_{B^{\mathrm{FS}}} (\nabla v : \Lambda^{T^0} : \nabla w)\, \mathrm{dV} = \langle \Lambda^{T^0} : \nabla v | \nabla w \rangle_{L^2} = a_0(v,w).$$

Positive definiteness of Λ^{T^0} from Assumption 4 (i) yields coercivity:

$$a_0(w,w) = \langle \Lambda^{T^0} : \nabla w | \nabla w \rangle_{L^2} \geq \lambda \langle \nabla w | \nabla w \rangle_{L^2} = \lambda \|w\|^2_{H^1} - \lambda \|w\|^2_{L^2}.$$

(ii) For a_1, Cauchy-Schwarz and continuity of $Q\colon H^1 \to L^2$, established in Lemma 8.3, give

$$|a_1(w,v)| = |\langle Qw | v \rangle_{L^2}| \leq \|Q\|_{\mathrm{op}} \|w\|_{L^2} \|v\|_{L^2} \leq \|Q\|_{\mathrm{op}} \|w\|_{H^1} \|v\|_{L^2} \leq \|Q\|_{\mathrm{op}} \|w\|_{H^1} \|v\|_{H^1},$$

which also implies continuity of a_1. The second estimate claimed is equivalent to continuity of $Q\colon L^2 \to H^{-1}$ (Lemma 8.3 again).

(iii) We have $|b(y,z)| = |\langle \rho^0 2\Omega \times y | z \rangle_{L^2}| \leq 2|\Omega| \|\rho^0\|_{L^\infty} \|y\|_{L^2} \|z\|_{L^2}$, so b is continuous on L^2.

(iv) Boundedness of ρ^0 gives continuity of c: $|c(y,z)| = |\langle \rho^0 y | z \rangle_{L^2}| \leq \|\rho^0\|_{L^\infty} \|y\|_{L^2} \|z\|_{L^2}$. Symmetry of c is clear. Finally, thanks to the uniform lower bound $\rho^0_- > 0$ of ρ^0 from Assumption 4 (ii), we have $c(y,y) = \langle \rho^0 y | y \rangle_{L^2} \geq \rho^0_- \|y\|^2_{L^2}$.

(v) Symmetry of s follows from the symmetry of $\widetilde{\nabla}\nu$. Assumption 2 (iii) guarantees that $\nu \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})^3$ and $p^0 \in \mathrm{Lip}(\Sigma^{\mathrm{FS}})$. Hence, estimating $[.]^+_- \leq 2|.|$ and successively employing the embeddings of Sobolev space products $\mathrm{Lip} \cdot H^{\frac{1}{2}} \subseteq H^{\frac{1}{2}}$ and $\mathrm{Lip} \cdot H^{-\frac{1}{2}} \subseteq H^{-\frac{1}{2}}$ (Lemma 3.14), as well as the Sobolev trace properties

$$\|v\|_{H^{1/2}(\Sigma^{\mathrm{FS}})} \leq c' \|v\|_{H^1} \qquad \text{and} \qquad \|\widetilde{\nabla} v\|_{H^{-1/2}(\Sigma^{\mathrm{FS}})} \leq c'' \|v\|_{H^1},$$

(3.30), and finally $\|\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})} = 1$ results in the estimate

$$\begin{aligned}
|s(w,v)| &= \left| \left[\langle\!\langle p^0 w \cdot (\widetilde{\nabla}\nu) + p^0 \nu \cdot (\widetilde{\nabla} w),\, v \rangle\!\rangle + \langle\!\langle p^0 \widetilde{\nabla} v,\, \nu w \rangle\!\rangle \right]^+_- \right| \\
&\leq 2 |\langle\!\langle p^0 w \cdot (\widetilde{\nabla}\nu),\, v \rangle\!\rangle| + 2 |\langle\!\langle p^0 \nu \cdot (\widetilde{\nabla} w),\, v \rangle\!\rangle| + 2 |\langle\!\langle p^0 \widetilde{\nabla} v,\, \nu w \rangle\!\rangle| \\
&\leq \|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})} \left(2(c')^2 \|\widetilde{\nabla}\nu\|_{L^\infty(\Sigma^{\mathrm{FS}})} + 4 c'' c' \right) \|w\|_{H^1} \|v\|_{H^1} = c_s \|w\|_{H^1} \|v\|_{H^1}.
\end{aligned}$$

We thus identify $c_s = c_{\Sigma^{\mathrm{FS}}} \|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})}$, which establishes (8.18) and also (8.15). □

As announced earlier, combining the estimates (i) and (v) of Lemma 8.8 with Assumption 4 (iii), which states that $\lambda > c_{\Sigma^{\mathrm{FS}}} \|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})} = c_s$, shows that $a_0 + s$ is a continuous, symmetric, coercive, bilinear form on H^1 with

$$(a_0 + s)(w,w) = a_0(w,w) + s(w,w) \geq a_0(w,w) - |s(w,w)| \geq (\lambda - c_s) \|w\|^2_{H^1} - \lambda \|w\|^2_{L^2}.$$

Thus, $a_0 + s$ satisfies a similar coercivity estimate as a_0, but with constants $\alpha, \beta > 0$ given by $\alpha := \lambda - c_s = \lambda - c_{\Sigma^{\mathrm{FS}}} \|p^0\|_{\mathrm{Lip}(\Sigma^{\mathrm{FS}})}$ and $\beta := \lambda$. Consequently, Lemma 8.8 assembles all the conditions that are necessary to infer the existence and uniqueness of weak solutions of the linearized elastic-gravitational equations from Theorem 3.18:

Theorem 8.9 (Existence and uniqueness of weak solutions of the equation of motion). *Let the Assumptions 2 and 4 hold and $I = [t_0, t_1]$. Then, for given data*

$$u^0 \in H^1(B^{\mathrm{FS}})^3, \quad u^1 \in L^2(B^{\mathrm{FS}})^3, \quad \text{and} \quad f \in L^2(I^\circ, L^2(B^{\mathrm{FS}})^3),$$

there exists a unique solution

$$u \in \mathcal{C}^0(I, H^1(B^{\mathrm{FS}})^3) \cap \mathcal{C}^1(I, L^2(B^{\mathrm{FS}})^3) \tag{8.19}$$

of the Cauchy problem for the weak equation of motion of the elastic-gravitational equations (8.4), satisfying the initial conditions $u(.,t_0) = u^0$ and $\dot{u}(.,t_0) = u^1$.

By Theorem 3.21 the weak solution fulfills the following **energy estimate**: For $t \in I$,

$$\|u(.,t)\|^2_{H^1(B^{\mathrm{FS}})} + \|\dot{u}(.,t)\|^2_{L^2(B^{\mathrm{FS}})} \leq k_1(u_0, u_1, f)\, e^{\,k_2(t)\,t} \tag{8.20}$$

with

$$k_1(u_0, u_1, f) = \frac{(\|\Lambda^{T^0}\|_{L^\infty} + \|Q\|_{\mathrm{op}} + 2\lambda)\|u^0\|^2_{H^1} + \|\rho^0\|_{L^\infty}\|u^1\|^2_{L^2} + \int_I \|f\|^2_{L^2}}{\min(\rho^0_-, \lambda - c_s)}$$

and

$$k_2(t) = \frac{1 + \|Q\|_{\mathrm{op}} + 4|\Omega|\|\rho^0\|_{L^\infty} + 2\lambda(t - t_0)}{\min(\rho^0_-, \lambda - c_s)}.$$

We recall from (8.7) and Lemma 8.3 that $\|Q\|_{\mathrm{op}}$ contains norms of the Hessian of the geopotential $\nabla\nabla(\Phi^0 + \Psi^s)$ and norms of the initial density ρ^0. The energy estimate thus indicates the dependence of u on the material parameters, the equilibrium fields, the initial conditions, and the source.

8.3 Additional regularity results

We investigate the validity of the strong form of the dynamical equations, including the boundary and interface conditions.

8.3.1 Perturbed Poisson equation

The results of Theorem 8.9 are obtained independently of the assumption that Φ^{s1} possesses H^1-regularity with respect to time, as an inspection of the proof of Lemma 8.3 reveals. It turns out that this condition rather holds as a consequence of solvability of the weak equations. Moreover, higher regularity conditions on ρ^0, namely piecewise Lip, allow to deduce the strong EL and IBC from the weak EL for Φ^{s1}.

Proposition 8.10 (**Solution and regularity for the perturbed Poisson equation**). *Let u be the solution of (8.4) obtained in Theorem 8.9 and let $\Phi^{s1} \in \mathcal{C}^0(I, H^1(B^{\mathrm{FSC}}))$ satisfy the weak EL (8.3). Then*

$$\Phi^{s1} \in \mathcal{C}^1(I, H^1(B^{\mathrm{FSC}})) \subseteq H^1(B^{\mathrm{FSC}} \times I^\circ). \tag{8.21}$$

Moreover, if in addition

$$\rho^0 \in \mathrm{Lip}(B^{\mathrm{FSC}}),$$

then Φ^{s1} also satisfies the strong EL (2.27) in $L^2(B^{\mathrm{FSC}} \times I^\circ)$ and the IBC (2.35) in $H^{-\frac{1}{2}}$.

Proof. Below, all spaces are defined on the composite domain B^{FSC}, which we do not indicate. From the weak solution $u \in \mathcal{C}^0(I, H^1)^3 \cap \mathcal{C}^1(I, L^2)^3$ and $\rho^0 \in L^\infty$ with compact support, we infer $\rho^0 u \in \mathcal{C}^1(I, L^2)^3$ which gives $\nabla \cdot (\rho^0 u)) \in \mathcal{C}^1(I, H^{-1})$. As we have seen in Section 8.2.1, the perturbed Poisson equation $\triangle \Phi^{s1} = -4\pi G\, \nabla\cdot(\rho^0 u)$ (2.27) holds in $\mathcal{D}'(\mathbb{R}^3)$, therefore elliptic regularity of $\triangle$ yields $\Phi^{s1} \in \mathcal{C}^1(I, H^1)$, proving (8.21). For ρ^0 piecewise Lip, the same argument, starting from $\rho^0 u \in \mathcal{C}^0(I, H^1)^3$, gives $\Phi^{s1} \in \mathcal{C}^0(I, H^2)$. Consequently, $\rho^0 u + \frac{1}{4\pi G}\nabla\Phi^{s1} \in L^2(I^\circ, H_{\mathrm{div}})^3$, which is the relevant condition of Assumption 3 that leads to the strong EL and NBC via Corollary 7.6 (ii). □

We note that the regularity result (8.21) justifies studying stationarity of the approximated action integral for $\Phi^{s1} \in H^1(B^{\mathrm{FSC}} \times I^\circ)$ instead of $\mathcal{C}^0(I, H^1(B^{\mathrm{FSC}}))$ (see Assumption 2).

As concerns the second statement, we observe that if ρ^0 is piecewise Lip, then by $[u]^+_- \cdot \nu = 0$ on Σ^{FS} (and also Σ^{SS}), the NIBC $[\nabla\Phi^{s1}]^+_- \cdot \nu = -4\pi G[\rho^0 u]^+_- \cdot \nu$ (2.35) on surfaces $S \subseteq \mathbb{R}^3$ reduce to

$$[\nabla\Phi^{s1}]^+_- \cdot \nu = -4\pi G[\rho^0]^+_- u \cdot \nu.$$

These interface conditions are satisfied for

$$\Phi^{s1} = -4\pi G(\nabla E_3) * (\rho^0 u),$$

which on the level of convolution of distributions coincides with (8.6). Thus, in case of a piecewise Lip density model, we obtain the following explicit representation of Φ^{s1} [DT98, (3.99), (3.98)]:

$$\begin{aligned}\Phi^{s1}(x,t) &= -G\int_{\mathbb{R}^3} \frac{\left(\rho^0 u\right)(x',t)\cdot(x-x')}{|x-x'|^3}\,\mathrm{dV}(x') \\ &= G\int_{\mathbb{R}^3} \frac{\left(\nabla\cdot(\rho^0 u)\right)(x',t)}{|x-x'|}\,\mathrm{dV}(x') + G\int_S \frac{[\rho^0]^+_-(x')\,(u(x',t)\cdot\nu(x'))}{|x-x'|}\,\mathrm{dS}(x').\end{aligned}$$

The volume integral is the incremental gravitational potential related to the perturbations of spatial volume density, $\rho^{s1} = -\nabla\cdot(\rho^0 u)$ (6.44), whereas the surface integral represents the effects of the apparent surface mass density due to normal displacement, $-[\rho^0]^+_- u\cdot\nu$, cf. [DT98, p. 73].

8.3.2 Equation of motion

In contrast to the static equilibrium equation or the two Poisson equations, the equation of motion is a PDE with variable leading coefficients ρ^0 and Λ^{T^0}. Consequently, we do not expect to obtain similar weak-strong uniqueness results as in Propositions 8.2, 8.1, or 8.10, at least in the case of low-regularity coefficients.

Ideally, in order to obtain weak-strong uniqueness also for the equation of motion, we would like to argue as follows: Assume that the weak solution u obtained in Theorem 8.9 also satisfies the equation of motion (2.26) in $\mathcal{D}'(B^{\mathrm{FS}}\times I^\circ)$. Then we may proceed as in [DL92, p. 585] and arrive at the following equality for the space-time divergence of $(-\Lambda^{T^0} : \nabla u, \rho^0\dot{u})$ in $\mathcal{D}'(B^{\mathrm{FS}}\times I^\circ)$:

$$\begin{aligned}\mathrm{div}_{x,t}(-\Lambda^{T^0} : \nabla u, \rho^0\dot{u}) &= -\nabla\cdot(\Lambda^{T^0} : \nabla u) + \rho^0\ddot{u} \\ &= f - \rho^0\Big(2\,\Omega\times\dot{u} + \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla(Su)\Big).\end{aligned}$$

By the regularity of the weak solution $u \in \mathcal{C}^0(I, H^1(B^{\mathrm{FS}})^3)\cap\mathcal{C}^1(I, L^2(B^{\mathrm{FS}})^3)$, the right-hand side is an element of $L^2(I^\circ, L^2(B^{\mathrm{FS}}))^3 \subseteq L^2(B^{\mathrm{FS}}\times I^\circ)^3$. Therefore, also employing strict positivity of ρ^0 (Assumption 4 (ii)), we obtain the relevant conditions of Assumption 3 (ii), that is

$$\dot{u} \in H^1(I^\circ, L^2(B^{\mathrm{FS}}))^3 \qquad \text{and} \qquad \Lambda^{T^0} : \nabla u \in L^2(I^\circ, H_{\mathrm{div}}(B^{\mathrm{FS}}))^{3\times 3},$$

implying the validity of the equation of motion (2.26) in strong form in $L^2(B^{\mathrm{FSC}}\times I^\circ)$. However, these are essentially the same conditions that are required to obtain the distributional interpretation of the weak equation of motion in the first place.

Let us investigate the situation in more detail: By Theorem 8.9, there exists $u \in \mathcal{C}^0(I, H^1(B^{\mathrm{FS}})^3)\cap\mathcal{C}^1(I, L^2(B^{\mathrm{FS}})^3)$ satisfying the weak equation of motion (8.4),

$$\begin{aligned}&\frac{d}{dt}\left(\int_{B^{\mathrm{FSC}}}\rho^0\dot{u}\cdot v\,\mathrm{dV}\right) + \int_{B^{\mathrm{FSC}}}\Big(\Lambda^{T^0} : \nabla u\Big) : \nabla v\,\mathrm{dV} \\ &\quad + \int_{B^{\mathrm{FSC}}}\rho^0\left(2\,\Omega\times\dot{u} + \nabla\nabla(\Phi^0+\Psi^s)\cdot u + \nabla\Phi^{s1}\right)\cdot v\,\mathrm{dV} \\ &\quad + \int_{\Sigma^{\mathrm{FS}}} p^0\Big[u\cdot(\widetilde{\nabla}\nu)\cdot v + \nu\cdot(\widetilde{\nabla}u)\cdot v + \nu\cdot(\widetilde{\nabla}v)\cdot u\Big]^+_-\,\mathrm{dS} \;=\; \int_{B^{\mathrm{FSC}}} f\cdot v\,\mathrm{dV},\end{aligned}$$

valid in $\mathcal{D}'(I^\circ)$ for all $v \in H^1_{\Sigma^{\mathrm{FS}}}(B^{\mathrm{FSC}})^3$. We evaluate the $\mathcal{D}'(I^\circ)$-duality at a test function $\psi \in \mathcal{D}(I^\circ)$ and only consider spatial test functions $v \in \mathcal{D}(B^{\mathrm{FSC}})^3$. This restriction will eliminate the surface term. However, in order to interpret the equation in $\mathcal{D}'(B^{\mathrm{FS}} \times I^\circ)^3$, we additionally need **higher regularity assumptions** to avoid the occurrence of ill-defined products of distributions. Specifically, in addition to Assumption 2, which gives $\rho^0 \in L^\infty$, we make the hypothesis that

$$u \in H^2(B^{\mathrm{FSC}} \times I^\circ)^3 \qquad \text{and} \qquad \Lambda^{T^0} \in \mathrm{Lip}(B^{\mathrm{FSC}})^{3\times3\times3\times3} \tag{8.22}$$

(that is, Λ^{T^0} is piecewise Lip). Then, by density of tensor products $v(x)\psi(t)$, the weak equation can be written as the following distributional duality for all $h \in \mathcal{D}(B^{\mathrm{FSC}} \times I^\circ)^3$:

$$\left\langle \rho^0 \left(\ddot{u} + 2\,\Omega \times \dot{u} + \nabla\nabla(\Phi^0 + \Psi^s) \cdot u + \nabla\Phi^{s1} \right) - \nabla \cdot (\Lambda^{T^0} : \nabla u), h \right\rangle = \langle f, h \rangle.$$

The distributional interpretation is indeed possible, because the conditions (8.22) ensure that all terms are in $L^2(B^{\mathrm{FSC}} \times I^\circ)$ (else, the terms $\rho^0\ddot{u}$ and $\nabla \cdot (\Lambda^{T^0} : \nabla u)$ would be ill-defined). We note that the conditions (8.22) also directly imply Assumption 3 (ii). Anyway, the new result obtained from these higher regularity assumptions is the validity of the interface and boundary conditions via Corollary 7.6. Let us summarize:

Proposition 8.11 (**Regularity for the equation of motion**). *Let u be the solution of the weak EL (8.4) obtained from Theorem 8.9 and assume that the higher regularity assumptions (8.22) hold, that is*

$$u \in H^2(B^{\mathrm{FSC}} \times I^\circ)^3 \qquad \textit{and} \qquad \Lambda^{T^0} \in \mathrm{Lip}(B^{\mathrm{FSC}})^{3\times3\times3\times3}.$$

Then u satisfies the strong EL (2.26) in $L^2(B^{\mathrm{FSC}} \times I^\circ)$, as well as the NBC (2.30) and the (solid-solid) NIBC (2.31) in $H^{-\frac{1}{2}}$.

This regularity result as well as Assumption 4 (i), (ii) are in accordance with the analysis of classical linearized elastodynamics based on semigroup theory (see Remark 8.12). The lower-order terms due to prestress and gravity can be incorporated in the semigroup framework as bounded perturbations according to Kato, see [MH83, Chapter 6, 2.17, 1., p. 341].

Remark 8.12 (**Strong solution via semigroup theory**). The equation of motion of classical linearized elasticity, $\rho\ddot{u} = \nabla \cdot (c : \nabla u)$ (1.59) on $\mathbb{R}^n$, can be written in the form of a linear first-order symmetric hyperbolic system, see also [HM78]:

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = A' \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \qquad \text{with} \qquad A' = \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix} \qquad \text{and} \qquad Au := \frac{1}{\rho}\nabla \cdot (c : \nabla u).$$

If ρ is $\mathcal{C}^0$, uniformly bounded from below, and c is $\mathcal{C}^1$, strongly elliptic (see Remark 8.6), then $A\colon H^2 \to L^2$ and A' generates a quasi-contractive semigroup on $H^2 \times H^1$, which implies the existence and uniqueness of solutions, see also [MH83, Chapter 6, Theorem 3.1, p. 346]. This result can be generalized to include a nonzero body force f, see [MH83, Chapter 6, 2.17, 5., p. 342]. The semigroup framework allows to deduce further refined regularity results. In particular, if the coefficients and data are smooth, i.e. $\mathcal{C}^\infty$, then the equation of motion has classical smooth solutions. ⌟

The results obtained in this chapter may be further refined and extended by employing generalized functions techniques [Obe92]. Moreover, this framework may also provide the means to analyze the full nonlinear system of elastic-gravitational equations in a low-regularity setting.

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Zusammenfassung

Die elastisch-gravitativen Gleichungen, d.h. die Gleichungen der Elastizitätstheorie unter Berücksichtigung der Eigengravitation, beschreiben die Bewegung und das daran gekoppelte zeitlich veränderliche Gravitationsfeld eines elastischen Kontinuums. Dieses System von partiellen Differentialgleichungen zweiter Ordnung stellt zusammen mit den entsprechenden Rand- und Grenzflächenbedingungen die Grundgleichungen der globalen Seismologie dar. Die Lösungen dieses Systems – sowie seiner Verallgemeinerungen, die auch Wellendämpfung bzw. viskoelastische Rheologie beinhalten – umfassen mannigfaltige Deformationen unseres Planeten; von den hochfrequenten seismischen Raum- und Oberflächenwellen, über die längerperiodischen freien Schwingungen und Gezeitendeformationen, bis hin zu auf großen Zeitskalen ablaufenden Lithosphärenbewegungen, beispielsweise aufgrund glazialer Auflasteffekte. Die vorliegende Arbeit untersucht die elastisch-gravitativen Gleichungen für ein allgemeines, gleichförmig rotierendes, heterogenes Erdmodell, welches neben festem Material auch aus flüssigen Bereichen besteht, die dem äußeren Erdkern oder den Ozeanen entsprechen. Dabei sollen nur minimale Bedingungen hinsichtlich der Glattheit der Materialparameter sowie der Grenzflächengeometrie vorausgesetzt werden. Zu diesem Zweck wird zunächst ein konsistentes mathematisches Erdmodell mit geringer Regularität innerhalb des allgemeinen Rahmens der nichtlinearen Kontinuumsmechanik entwickelt. Dann wird eine Variante der Variationsrechnung in Sobolew-Räumen auf zusammengesetzten Gebieten präsentiert, die es erlaubt, die schwache Formulierung der linearisierten elastisch-gravitativen Gleichungen direkt aus dem Hamiltonschen Prinzip der kleinsten Wirkung abzuleiten. Zum Abschluss werden Existenz und Eindeutigkeit schwacher Lösungen mithilfe der Methode der Energieabschätzungen gezeigt, sowie weitere Regularitätseigenschften diskutiert. Das Ergebnis dieser Arbeit ist ein vollständiges mathematisches Modell für die globale seismische Wellenausbreitung unter dem Einfluss der Eigengravitation und unter geringen Regularitätsannahmen. Dieses Modell kann als theoretische Grundlage für die Weiterentwicklung der Spektraltheorie der Erde sowie für eine Verbesserung der numerischen Modellierung seismischer Wellen dienen. Schließlich stellt es einen mathematisch konsistenten Ausgangspunkt dar, um das inverse Problem der globalen Seismologie unter realistischen Regularitätsannahmen zu untersuchen.