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Abstract

This dissertation is dedicated to the portfolio optimization problems which aim to maximize the expected utility value of the terminal wealth. The main concern is to study properties of optimal dual processes producing the unique dual optimizer by applying duality theorems provided in [63, 13, 19].

It has been pointed out in [63] that the unique optimal dual process is in general not a local martingale, but a supermartingale. Here, provided that the stock price process has continuous paths, we show the local martingale property of the unique optimal dual process. For Brownian filtration, this result is extended to frictionless markets with bounded random endowment. Mathematically, this result means that the countably additive part of the dual optimizer obtained in [13] can be attained by a local martingale, which is a supermartingale deflator defined in [63].

In chapter 3, we consider markets with proportional transaction costs. Under the strict positivity of the liquidation value process of the optimal trading strategy and the continuity of the stock price process, we prove that all optimal dual processes are local martingales. Furthermore, we perform a stability analysis for the utility maximization problem by investigating the small perturbations of the initial investment and the investor's preference. The sequence of optimal dual processes is proved to converge in the sense of [18, Theorem 2.7] to an optimal dual process in the original market under some conditions. This limit process allows us to construct a shadow price for the original market.

In the last part, we focus on the utility maximization problem in markets with random endowment and proportional transaction costs. For such markets equipped with at least one supermartingale-consistent price system, we find the unique solution to the utility maximization problem and verify the existence of a shadow price under the no-short-selling constraint. Finally, we present an example showing that the shadow prices may exist even if the no-short-selling constraint does not hold.

Zusammenfassung

Diese Dissertation widmet sich den Portfolio-Optimierungsproblemen, die darauf abzielen, den erwarteten Nutzen des Endvermögens zu maximieren. Das Hauptinteresse dieser Arbeit besteht darin, unter Zuhilfenahme von Dualitätsergebnissen aus [63, 13, 19], die optimalen dualen Prozesse und ihre Eigenschaften genauer zu charakterisieren. Dies ist von besonderem Interesse, da diese Prozesse die Lösung des dualen Problems eindeutig festlegen.

Es wurde in [63] darauf hingewiesen, dass der eindeutige optimale duale Prozess im Allgemeinen kein lokales Martingal, sondern ein Supermartingal ist. Für Preisprozesse mit stetigen Pfaden zeigen wir, dass dieser optimale Prozess ein lokales Martingal ist. Im Fall einer Brownschen Filtration wird dieses Ergebnis auf Märkte mit beschränktem, zufälligem Kapital erweitert.

In Kapitel 3 untersuchen wir die optimalen dualen Prozesse für Märkte mit Transaktionskosten. Für Preisprozesse mit stetigen Pfaden und unter der Annahme, dass der Wertprozess aller optimalen Handelsstrategien streng positiv ist, zeigen wir, dass alle optimalen dualen Prozesse lokale Martingale sind. Weiterhin führen wir eine Stabilitätsanalyse für das Nutzenmaximierungsproblem durch. Wir beweisen, dass unter bestimmten Bedingungen die Folge der optimalen dualen Prozesse im Sinne von [18, Theorem 2.7] gegen einen optimalen dualen Prozess im Originalmarkt konvergiert. Dieser Grenzprozess ermöglicht es uns, einen Schattenpreis für den Originalmarkt zu konstruieren.

Im letzten Teil konzentrieren wir uns auf das Nutzenmaximierungsproblem in Märkten mit zufälligem Kapital und proportionalen Transaktionskosten. Unter der Zusatzannahme, dass ein solcher Markt mindestens ein Supermartingal-konsistentes Preissystem besitzt, beweisen wir, dass das Nutzenmaximierungsproblem eine eindeutige Lösung hat. Außerdem verifizieren wir die Existenz eines Schattenpreises unter der Annahme, dass Leerverkäufe verboten sind. Schließlich stellen wir ein Beispiel vor, das zeigt, dass Schattenpreise auch dann existieren können, wenn Leerverkäufe zulässig sind.

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Contents

Chapter 1

Introduction

1.1 Background

Portfolio optimization is a classical problem in mathematical finance, which concerns an economic agent who invests his money in different assets in a financial market so as to maximize the expected utility of his terminal wealth. The portfolio optimization problem is most generally specified as a constrained utility maximization problem, which leads to an efficient pricing tool. Such problems can classically be solved by primal methods leading to Hamilton-Jacobi-Bellman equations, which are hard to solve (see e.g., [20, 85]); or by applying the duality methods (martingale approach) where one optimizes a conjugate functional (Legendre Transform). Duality methods enable to deal with models which are not necessarily Markovian.

A complete review of the literature on utility maximization problems is too extensive. We only concentrate on such problems solved by duality methods, where a utility function U is supported on $\mathbb{R}_+$. For the articles in the area of utility maximization in a complete market solved by martingale methodology, see [76, 10, 11, 57]. In incomplete financial markets, after the problem in Itô-processes models was solved in [58], it is generalized to the framework of semimartingale market by Kramkov and Schachermayer (1999) in [63]. They provided a necessary and sufficient condition called "reasonable asymptotic elasticity" of the utility function for the existence of an optimal solution of utility maximization problem, and constructed an optimal trading strategy by solving a dual problem defined on $\mathbf{L}^0_+$. The other important innovation is that the dual domain is a family of supermartingale deflators, which is not necessarily a local martingale as in [58].

The result in [63] is subsequently generalized by Cvitanić, Schachermayer and Wang [13] to allow for receiving a bounded random endowment e_T other than the deterministic initial wealth, which, for example, can be an exogenous non-traded European contingent claim. In this case, the dual domain is no longer in the space $\mathbf{L}^1$, but in $(\mathbf{L}^\infty)^*$, so the authors of [13] employ the duality between $\mathbf{L}^\infty$ and $(\mathbf{L}^\infty)^*$. The obtained dual optimizer $\widehat{Q}$ is unique up to the singular part, and moreover the primal optimizer can be formulated by the regular part of the dual optimizer $\widehat{Q}^r$. Then, in [48], the authors relaxed the boundedness assumption on the random endowment and instead only required an integrability condition. Another extension of [13] is provided by Karatzas and Žitković in [60] by additionally considering intertemporal consumption.

Besides the duality between the primal and dual problem, many authors are concerned with the properties of the dual optimizer in the case when U is defined on $\mathbb{R}_+$. Particularly, the following representation conjecture is studied: *can the dual optimizer be attained by an equivalent local martingale deflator (for short ELMDs, see e.g., [56, 61])?* Actually, this conjecture fails for general semimartingale models, because Kramkov and Schachermayer provided an example in frictionless case showing that the associated process could only be a strict supermartingale (cf. Example 5.1' in [63]). However, once further conditions are imposed on the stock price process S, one could have some positive results. For example, Karatzas and Žitković observed that this conjecture is true for Itô-process models in [60]. Another important result is presented in [65]: the represented conjecture is true for all continuous semimartingale models. Even in the market with bounded random endowment, but in the Brownian framework, [38] shows the countably additive part $\widehat{Q}^r$ of the dual optimizer $\widehat{Q} \in (\mathbf{L}^\infty)^*$ obtained in [13] can be represented by the terminal value of a supermartingale deflator Y defined in [63], which is a local martingale.

However, if we look over the order book in real stock exchanges, there always exists a bid-ask spread in the price of a stock's share, meanwhile transaction fees are always charged for many kinds of reasons. The presence of these factors in real financial markets are crucial in the equilibrium between traders and market makers and also affects the market liquidity. Therefore, many researchers devoted to improve classical models such that these factors can be taken into consideration, in particular, the constant proportional transaction costs. Since the 1990s, many authors have found that essential properties on the frictionless market can be still valid here, merely through defining some concepts in a new language. Precisely, the notion called "consistent price system" (CPS) is introduced, which plays a similar role as the equivalent local martingale measure in the frictionless framework and so that the equivalence between the existence of a such system and the "no arbitrage condition" can be well set up (see e.g., [50, 81, 42, 44, 45, 43]).

On markets with transaction costs, Magill and Constantinides [69] have first discussed this problem in Merton's model (cf. [71, 72]) with the continuous time setting. Since then, many authors have contributed to this topic from different aspects. In fact, the dual approach for solving this problem under transaction costs dates back to Cvitanić and Karatzas [12], yet the authors assume a priori that the dual optimization problem admits a solution so that the primal optimizer can be constructed. This assumption is proved true by Cvitanić and Wang in [14] by suitably enlarging the domain of the dual problem as Kramkov and Schachermayer did in [63] for investigating a frictionless counterpart. For more general market models with transaction costs, Kabanov [52] introduced a more general multi-currency model based on the concept of solvency cone. In this framework, Deelstra, Pham and Touzi gave a dual formulation of the multivariate utility maximization with the non-smooth utility function in [21], when the market is associated with a continuous semimartingale of classical no-arbitrage features. Afterwards, Campi and Owen [6] solved the utility maximization problem by convex duality under existence of CPSs (cf. [7]). In particular, duality results in [6] rely on the enlargement of the dual domain formed by CPSs to a set of finitely additive measures, which is based on the idea of [13, 60, 74]. Notice that this step is altered in the proof of a similar problem in the numéraire-based context (cf. [17, 19]), where the abstract theorem in [63] applies, which is owed to the $\mathbf{L}^0_+$-bipolar property between the dual domain defined in terms of supermartingale deflators and the primal one. The results in [6] has been subsequently generalized by Benedetti and Campi in [4] to the case with bounded random endowment.

It has been observed that the original market with transaction costs can sometimes be replaced by a frictionless "shadow market" that yields the same optimal strategy and utility, though even small transaction costs could dramatically influence the optimal choice of the investor in utility maximization (cf. [66]). The existence of such shadow market was first implied in [12], which shows if the solution of a suitable dual problem could be attained by a CPS which is a (local) martingale, then the optimal solution could be characterized by solving a hedging problem in the "shadow" market associated with this CPS. Then, Kallsen and Muhle-Karbe [55] showed that the shadow prices always exist in finite probability spaces. More recently, for the continuous stock price processes S, a sufficient (but not necessary) condition for the existence of a shadow price is provided in [19], that is, S satisfies the condition of "no unbounded profit with bounded risk" (NUPBR). After that, in [16] a shadow price is still proved to exist by the weaker condition of "two way crossing" (TWC) introduced by Bender [3]. For general càdlàg stock price processes S, Czichowsky and Schachermayer [17] clarified the existence of a shadow price in the sense of "sandwiched" process, which consists of a predictable and an optional strong supermartingale. This idea is afterward generalized by Bayraktar and Yu [2] in order to study a similar problem but with random endowment: they constructed shadow prices in the sandwiched sense as [17], whenever the duality result holds and a sufficient condition on the dual optimizer is assumed.

If such a shadow price exists, then transaction costs do not lead to qualitatively new effects for portfolio choice, as their impact can be replicated by passing to a suitably modified frictionless market. Starting from [54], shadow prices have also proved to be useful for solving optimization problems. With geometric Brownian motion models, the authors of [54] considered the optimal investment and consumption problem with logarithmic utility in an infinite horizon by employing tools from stochastic control and constructed explicitly a shadow price by solving a free boundary problem. In the same framework, this idea was afterward generalized in [8, 46] for power utility to derive a shadow price. By the same token, Gerhold et al. [35, 34] constructed shadow prices for logarithmic and long-run power utility functions when the time horizon becomes finite.

Instead of studying the dual optimizer, Loewenstein [67] constructed shadow markets directly from the derivatives of dynamic primal value functions under no-short-selling constraints. This argument is based on the existence of the constrained primal solution, which was a hypothesis assumed by [67] but has been indeed obtained by Benedetti et al. [5] when they applied Loewenstein's approach to a similar problem with Kabanov's multi-currency model.

What will happen to the optimal investment problem, if there are small changes of input parameters? This question brings us an interesting study on stability. Until now, the investigation of stability only focuses on the frictionless market. There are two different settings for such stability analysis. One is with respect to the small variations of the preference, while the stock price process is fixed. The other one is with respect to the changes of asset price but the utility function is fixed.

One type of stability problem launched by Larsen and Žitković [65] refers to misspecifications in the model. That is to say, the stability of an agent's optimal behavior is studied through the perturbations of the market coefficient processes and fixed utility function. Such problem has been researched later by Frei [33], Bayraktar and Kravitz [1] for the exponential utility function. More recently, Weston [87] investigated the stability with respect to a sequence of financial markets in incomplete Brownian setting.

The first analysis of continuity with respect to the preferences goes back as far as to the article of Jouini and Napp [51]. The authors received the $\mathbf{L}^p$-convergence ($p \geq 1$) of optimal wealth and consumption at each date when they considered the complete Itô-process model, the pointwise convergence of utility functions. Larsen [64] extended the stability study to more general model – continuous semimartingale model. He also pointed out that only the pointwise convergence of utility functions is not enough to have the stability of utility maximization problems by providing a counterexample. Placing a growth condition on the possible choice utility functions, that is $U(\cdot) \leq \overline{U}(\cdot)$, where $\overline{U}(\cdot)$ is the universal upper bounded utility function, [64] shows the continuity of value function as well as the continuity of the optimizer in the sense of $\mathbf{L}^0(\mathbf{P})$. Afterwards, Kardaras and Žitković [62] proved the same result as in [64], when they considered not only the changes of the utility function, but also variations of the subjective probability measure in a more general model additionally with random endowments. More recently, Mocha and Westray [73] discussed the stability with respect to the relative risk aversion of power utility function, market price of risk, statistical probability measure and extra investment constraints that the set of strategies is a convex cone. Xing [88] focused on the stability of the exponential utility maximization when there are small variations on the agent's utility defined both on $\mathbb{R}$ and $\mathbb{R}_+$.

1.2 Outline

This dissertation is based on a published paper [38], a submitted paper [39], and a working paper [40] written with collaborators Yiqing Lin and Junjian Yang. Focusing on the properties of optimal dual processes which induce the unique dual optimizer, we begin our study on incomplete markets introduced in [63], then we investigate such problem in markets with bounded endowment. Further, financial markets with proportional transaction costs is also concerned, for example, the stability problem and the optimization problem are observed.

We pay attention to the dual problem of utility maximization in incomplete markets with and without bounded random endowment in Chapter 2, which is based on the published paper [38]. Firstly, we show that under the continuity assumption of the stock price process, the optimal dual process $\widehat{Y} \in \mathcal{Y}(\widehat{y})$ obtained in [63, Theorem 2.2], is a local martingale, where $\mathcal{Y}(\widehat{y})$ is the family of supermartingale deflators. Then we extend this result to the case with bounded random endowment e_T introduced in [13]. That is, under the Brownian filtration, the regular part $\widehat{Q}^r$ of any dual optimizer $\widehat{Q}$ obtained in [13, Theorem 3.1] can be attained by a local martingale, which belongs to $\mathcal{Y}(1)$.

In chapter 3, which is based on the working paper [40], we consider the stability problem of utility maximization in markets with transaction costs. According to the duality results in [17, Theorem 3.2], where the stock price is càdlàg, we perform the static stability analysis by perturbing initial capitals, utility functions and physical probability measures. We have the continuity of the value function and the convergence in $\mathbf{L}^0(\mathbf{P})$ of primal and dual optimizers by imposing an additional assumption of uniform integrability of the family of V_n^+, where V_n is the conjugate of U_n. In order to discuss the dynamic stability, we assume the stock price process is continuous. The local martingale property of all optimal dual processes still holds true, when the liquidation value process of each optimal trading strategy is strictly positive. In line with such property, which guarantees the existence of a shadow price process, we prove the convergence in the sense of [18, Theorem 2.7] for optimal dual processes in terms of the changes of initial capital and utility function. Mathematically, for the sequence of optimal dual processes $\{\widehat{Y}^n\}_{n\in\mathbb{N}} := \left\{ \left(\widehat{Y}^0_t(x_n; U_n, \mathbf{P}), \widehat{Y}^1_t(x_n; U_n, \mathbf{P}) \right)_{0\leq t\leq T} \right\}_{n\in\mathbb{N}}$ with respect to the perturbed market parameterized by $(x_n; U_n, \mathbf{P})$, there is a local martingale $\widehat{Y}$ and a sequence of convex combinations of $\{\widehat{Y}^n\}_{n\in\mathbb{N}}$, still denoted by $\{\widehat{Y}^n\}_{n\in\mathbb{N}}$, then for every $[0, T]$-valued stopping time τ,

$$\widehat{Y}^n_\tau \xrightarrow{\mathbf{P}} \widehat{Y}_\tau.$$

Moreover, the limit process $\widehat{Y}$ is also an optimal dual process in the original market parameterized by $(x; U, \mathbf{P})$. This convergence offers a possibility to construct a shadow price in both perturbed and original markets according to the ensured existence of a shadow price.

Considering the optimal portfolio in financial markets with transaction costs and random endowment, we find the optimal terminal wealth and the maximal expected utility in Chapter 4. It is based on a preprint [39]. In this case, under the existence of supermartingale-consistent price systems, we assume the agent cannot short sell assets and is endowed with a strictly positive contingent claim rather than a deterministic initial wealth. We first show that a primal optimizer of this utility maximization problem exists and is unique. Then, we follow the lines of [67, 5] to construct a shadow price directly from the primal solution. Further, we provide an example in the Black-Scholes framework with a constructive random endowment, which shows that even without the no-short-selling constraint shadow prices can be explicitly defined.

Chapter 2

Dual optimizer in utility maximization problems in incomplete markets

A dual optimizer of utility maximization problem is obtained by duality methods, where a conjugate functional (Legendre Transform) should be optimized. We discuss the property of dual optimizer in the cases with and without random endowment.

2.1 Dual optimizer in the classical case

We start with a brief introduction of settings and notation described by Kramkov and Schachermayer (1999) in [63], where the authors are concerned with optimal solutions of utility maximization problems solved by martingale approach in incomplete semimartingale markets, then we discuss the property of the obtained optimal dual process inducing the unique dual optimizer.

2.1.1 Financial model and duality results

We consider a security market consisting of $d+1$ assets, one bond and d stocks. The time horizon T is assumed to be finite $T < \infty$. We suppose without loss of generality that the price of the bond is constant. The price process of d stocks denoted by $S = (S^i)_{1\leq i\leq d}$ is a strictly positive semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ satisfying the usual conditions, i.e., the filtration is $\mathbf{P}$-complete and right-continuous ($\mathcal{F}_t = \mathcal{F}_{t^+} := \cap_{s>t}\mathcal{F}_s$), where $\mathcal{F}_0$ is assumed to be trivial.

Assume the agent is endowed with initial capital x and the amount of each asset held in his portfolio is described by $H = (H^i)_{0\leq i\leq d}$, which is a predictable S-integrable process. The value process $X = (X_t)_{0\leq t\leq T}$ of such a portfolio (x, H) is given by

$$X_t = X_0 + (H \bullet S)_t, \qquad 0 \leq t \leq T, \tag{2.1.1}$$

where $(H \bullet S)_t = \int_0^t H_u dS_u$ denotes the stochastic integral with respect to S.

Definition 2.1.1. *The predictable process (trading strategy) H is called **admissible**, if it is S-integrable with $H_0 = 0$, and the process $(H \bullet S)$ is uniformly bounded from below by some constant.*

For $x \in \mathbb{R}_+$, the family of wealth processes with nonnegative capital at any moment is denoted by $\mathcal{X}(x)$, that is, $X_t \geq 0$ for all $t \in [0, T]$, and with initial value X_0 equal to x,

$$\mathcal{X}(x) = \{X \geq 0 : X \text{ is defined by (2.1.1) with } X_0 = x\}.$$

Definition 2.1.2 (ELMM)**.** *A probability measure $\mathbf{Q} \sim \mathbf{P}$ is called an **equivalent local martingale measure (ELMM)**, if any $X \in \mathcal{X}(1)$ is a local martingale under $\mathbf{Q}$. The family of ELMMs is denoted by $\mathcal{M}^e(S)$ or shortly by $\mathcal{M}$.*

Remark 2.1.3. if S is bounded (resp. locally bounded), under an ELMM $\mathbf{Q}$, S is a $\mathbf{Q}$ martingale (resp. $\mathbf{Q}$-local martingale).

Assumption 2.1.4. *We assume throughout this chapter that*

$$\mathcal{M} \neq \emptyset. \tag{2.1.2}$$

Remark 2.1.5. This assumption is equivalent to the absence of arbitrage opportunities on the security market. For the general semimartingale S, the existence of equivalent σ-martingale measure is equivalent to "no free lunch vanishing risk" (NFLVR). For precise statements and references please see [22] and [26]. Moreover, if S is locally bounded (resp. bounded), then the existence of ELMM (resp. EMM) is equivalent to (NFLVR).

Assumption 2.1.6. *Throughout the paper, suppose the agent's preferences over terminal wealth are modeled by a utility function $U : (0, \infty) \to \mathbb{R}$, which is strictly increasing, strictly concave, continuously differentiable and satisfies the **Inada conditions**:*

$$U'(0) := \lim_{x \to 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

*as well as the condition of "**reasonable asymptotic elasticity**", that is,*

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

Remark 2.1.7. $AE(U) < 1$ is the necessary and sufficient condition on utility function to guarantee the existence of the optimal solution to the following primal problem (2.1.3). For more financial interpretation and results about the condition of "reasonable asymptotic elasticity", we refer to [63].

For a given initial capital $x > 0$, the goal of the agent is to maximize the expected value from terminal wealth:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right]. \tag{2.1.3}$$

To avoid the trivial case, we assume

Assumption 2.1.8.

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right] < \infty, \qquad \textit{for some } x > 0.$$

To obtain the dual optimizer, we should dualize the optimization problem (2.1.3). Let us denote $V : \mathbb{R}_+ \to \mathbb{R}$ the convex conjugate function of U defined by

$$V(y) := \sup_{x>0}\{U(x) - xy\}, \quad y > 0.$$

From classical results of convex analysis, we know that V is strictly decreasing, strictly convex and continuously differentiable and satisfies

$$V'(0) = -\infty, \quad V'(\infty) = 0, \quad V(0) = U(\infty), \quad V(\infty) = U(0).$$

We also define $I : (0,\infty) \to (0,\infty)$ the inverse function of U' on $(0,\infty)$, which is strictly decreasing, and satisfies

$$I(0) = \infty, \quad I(\infty) = 0 \quad \text{and} \quad I = -V' = (U')^{-1}.$$

The dual problem is to minimize the expectation of V over the dual domain $\mathcal{Y}(y)$:

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V(Y_T)\right], \tag{2.1.4}$$

where $\mathcal{Y}(y)$ is the family of nonnegative semimartingales Y with $Y_0 = y$ and such that, for any $X \in \mathcal{X}(1)$, the product XY is a supermartingale,

$$\mathcal{Y}(y) = \left\{Y \geq 0 : Y_0 = y, XY = (X_tY_t)_{0\leq t\leq T} \text{ is a supermartingale, } \forall X \in \mathcal{X}(1)\right\}. \tag{2.1.5}$$

Theorem 2.1.9 ([63, Theorem 2.2]). *Let Assumptions 2.1.4, 2.1.6 and 2.1.8 hold true. Then:*

(i) The primal value function u is finitely valued, i.e., $u(x) < \infty$, for all $x > 0$, and is continuously differentiable, strictly concave on $(0,\infty)$.

(ii) The dual value function v is finitely valued, i.e., $v(y) < \infty$, for all $y > 0$, and is continuously differentiable, strictly convex on $(0,\infty)$.

(iii) The value functions u and v are conjugate,

$$v(y) = \sup_{x>0}\left\{u(x) - xy\right\}, \quad y > 0,$$
$$u(x) = \inf_{y>0}\left\{v(y) + xy\right\}, \quad x > 0.$$

The functions u' and $-v'$ are strictly decreasing and satisfy

$$u'(0) = \lim_{x\to 0} u'(x) = \infty, \quad v'(\infty) = \lim_{y\to\infty} v'(y) = 0.$$
$$u'(\infty) = \lim_{x\to\infty} u'(x) = 0, \quad -v'(0) = \lim_{y\to 0} -v'(y) = \infty.$$

The asymptotic elasticity $AE(u)$ of u is also less then 1, precisely,

$$AE(u)^+ \leq AE(U)^+ < 1,$$

where x^+ denotes $\max\{x, 0\}$.

(iv) *The optimal solution $\widehat{X}(x) \in \mathcal{X}(x)$ to (2.1.3) exists and is unique. The optimal solution $\widehat{Y}(\widehat{y}) \in \mathcal{Y}(\widehat{y})$ to (2.1.4) exists and is unique, where $\widehat{y} = u'(x)$, and we have the dual relation*

$$\widehat{X}_T(x) = I(\widehat{Y}_T(\widehat{y})), \quad \widehat{Y}_T(\widehat{y}) = U'(\widehat{X}_T(x)).$$

Moreover, the process $\widehat{X}(x)\widehat{Y}(\widehat{y})$ is a uniformly integrable martingale on $[0, T]$.

(v) *We have the following relations between u', v' and $\widehat{X}, \widehat{Y}$, respectively,*

$$u'(x) = \mathbb{E}\left[\frac{\widehat{X}_T(x)U'(\widehat{X}_T(x))}{x}\right] \quad v'(\widehat{y}) = \mathbb{E}\left[\frac{\widehat{Y}_T(\widehat{y})V'(\widehat{Y}_T(\widehat{y}))}{\widehat{y}}\right].$$

(vi)

$$v(y) = \inf_{\mathbf{Q}\in\mathcal{M}} \mathbb{E}\left[V(y\frac{d\mathbf{Q}}{d\mathbf{P}})\right], \tag{2.1.6}$$

where $\frac{d\mathbf{Q}}{d\mathbf{P}}$ denotes the Radon-Nikodým derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$ on $(\Omega, \mathcal{F}_T)$.

Remark 2.1.10. The value function u can also be regarded as a kind of utility function, namely the expected utility of the agent at time T. For more properties of u we refer to [63].

Actually, Kramkov and Schachermayer also gave a useful "abstract version" of above mentioned Theorem 2.1.9, which is also copied here. Define

$$\begin{aligned} \widetilde{\mathcal{C}}(x) &:= \left\{g \in \mathbf{L}^0_+(\Omega, \mathcal{F}, \mathbf{P}) : 0 \leq g \leq X_T, \text{ for some } X \in \mathcal{X}(x)\right\}, \\ \widetilde{\mathcal{D}}(y) &:= \left\{h \in \mathbf{L}^0_+(\Omega, \mathcal{F}, \mathbf{P}) : 0 \leq h \leq Y_T, \text{ for some } Y \in \mathcal{Y}(y)\right\}. \end{aligned} \tag{2.1.7}$$

For simplicity, write $\widetilde{\mathcal{C}},\ \widetilde{\mathcal{D}},\ \mathcal{X},\ \mathcal{Y}$ for $\widetilde{\mathcal{C}}(1),\ \widetilde{\mathcal{D}}(1),\ \mathcal{X}(1),\ \mathcal{Y}(1)$ and observe that

$$\widetilde{\mathcal{C}}(x) = x\widetilde{\mathcal{C}} = \left\{xg : g \in \widetilde{\mathcal{C}}\right\}, \text{ for } x > 0,$$

and the analogous relations for $\widetilde{\mathcal{D}}(y)$, $\mathcal{X}(x)$ and $\mathcal{Y}(y)$. The sets $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are polar to each other in $\mathbf{L}^0_+$. The details are showed in [63, Proposition 3.1].

Proposition 2.1.11 ([63, Proposition 3.1])**.** *Suppose that the $\mathbb{R}^d$-valued semimartingale S satisfies (2.1.2). Then the sets $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{D}}$ have the following properties:*

(a) $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are subsets of $\mathbf{L}^0_+(\Omega, \mathcal{F}, \mathbf{P})$ which are convex, solid, and closed in the topology of convergence in measure.

(b)

$$g \in \widetilde{\mathcal{C}} \quad \textit{iff} \quad \mathbb{E}\left[gh\right] \leq 1, \quad \textit{for all} \;\; h \in \widetilde{\mathcal{D}} \quad \textit{and}$$
$$h \in \widetilde{\mathcal{D}} \quad \textit{iff} \quad \mathbb{E}\left[gh\right] \leq 1, \quad \textit{for all} \;\; g \in \widetilde{\mathcal{C}}.$$

(c) $\widetilde{\mathcal{C}}$ is a bounded subset of $\mathbf{L}^0(\mathbf{P})$ and contains the constant function $\mathbf{1}$.

Remark 2.1.12. The first assertion of (b) in Proposition 2.1.11 is in the spirit of Super-replication theorem.

The "abstract versions" of primal problem (2.1.3) and the dual problem (2.1.4) can be denoted by

$$u(x) = \sup_{g \in \widetilde{\mathcal{C}}(x)} \mathbb{E}\left[U(g)\right] = \sup_{g \in \widetilde{\mathcal{C}}(1)} \mathbb{E}\left[U(xg)\right], \tag{2.1.8}$$
$$v(y) = \inf_{h \in \widetilde{\mathcal{D}}(y)} \mathbb{E}\left[V(h)\right] = \inf_{h \in \widetilde{\mathcal{D}}(1)} \mathbb{E}\left[V(yh)\right]. \tag{2.1.9}$$

We denote by u' and v' the right-continuous versions of the derivatives of u and v.

Theorem 2.1.13 ([63, Theorem 3.2])**.** *Assume that the sets $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ satisfy the assertions of Proposition 2.1.11. Let Assumptions 2.1.4, 2.1.6 and 2.1.8 hold true. Then in addition to assertions (i)-(iii) of Theorem 2.1.9, we have*

(1) The optimal solution $\widehat{g}(x) \in \widetilde{\mathcal{C}}(x)$ to (2.1.8) exists and is unique. The optimal solution $\widehat{h}(\widehat{y}) \in \widetilde{\mathcal{D}}(\widehat{y})$ to (2.1.9) exists and is unique, where $\widehat{y} = u'(x)$, we then have the dual relation

$$\widehat{g}(x) = I(\widehat{h}(\widehat{y})), \quad \widehat{h}(\widehat{y}) = U'(\widehat{g}(x)).$$

Moreover,

$$\mathbb{E}\left[\widehat{g}(x)\widehat{h}(\widehat{y})\right] = x\widehat{y}. \tag{2.1.10}$$

Obviously, $\widehat{X}_T(x) = \widehat{g}(x)$, and $\widehat{Y}_T(\widehat{y}) = \widehat{h}(\widehat{y})$.

(2) We have the following relations between u', v' and $\widehat{g}, \widehat{h}$, respectively,

$$u'(x) = \mathbb{E}\left[\frac{\widehat{g}(x)U'(\widehat{g}(x))}{x}\right] \quad v'(\widehat{y}) = \mathbb{E}\left[\frac{\widehat{h}(\widehat{y})V'(\widehat{h}(\widehat{y}))}{\widehat{y}}\right].$$

The proof of Theorem 2.1.13 we refer to [63]. By the uniqueness of the solution $\widehat{g}(x), \widehat{h}(\widehat{y})$ to (2.1.8), (2.1.9) respectively, we can show the uniqueness of the optimal processes $\widehat{Y}(\widehat{y})$ and $\widehat{X}(x)$ from assertion (iv) of Theorem 2.1.13. For simplicity, we give the following definition.

Definition 2.1.14. *Let $\widehat{y} = u'(x)$. The optimal solution $\widehat{X}(x) \in \mathcal{X}(x)$ (resp. $\widehat{X}_T(x)$) to the primal problem* (2.1.3) *is called* ***optimal wealth process*** *(resp.* ***primal optimizer****). The optimal solution $\widehat{Y}(\widehat{y}) \in \mathcal{Y}(\widehat{y})$ (resp. $\widehat{Y}_T(\widehat{y})$) to the dual problem (2.1.4) is called* ***optimal dual process (ODP)*** *(resp.* ***dual optimizer****). The strategy $\widehat{H}$ which induces the optimal wealth process, i.e., $\widehat{X}(x) = x + \widehat{H} \bullet S$ is called* ***optimal trading strategy****.*

Proof of the uniqueness of the optimal processes. Let $\widehat{X}^1, \widehat{X}^2 \in \mathcal{X}(x)$ be two different optimal wealth processes with $\widehat{X}^1_T = \widehat{X}^2_T$ a.s., which are produced by the optimal trading strategies $\widehat{H}^1$ and $\widehat{H}^2$, respectively. Thus, the terminal values $\widehat{X}^1_T$ and $\widehat{X}^2_T$ are maximal, then by [25, Remark 2.7] which shows if a contingent claim f produced by the strategy H is maximal admissible, then the strategy is uniquely determined in the sense that any other admissible strategy K that produces f necessarily satisfies $H \bullet S = K \bullet S$, we have $\widehat{H}^1 \bullet S = \widehat{H}^2 \bullet S$, which shows $\widehat{X}^1$ and $\widehat{X}^2$ are indistinguishable.

For the uniqueness of the optimal dual process, we suppose both $\widehat{Y}^1, \widehat{Y}^2 \in \mathcal{Y}(\widehat{y})$ are optimal solutions to (2.1.4) with $\widehat{Y}^1_T = \widehat{Y}^2_T$ a.s., and such that there exists a time $t \in [0, T)$ and a set $A \in \mathcal{F}_t$ with $\mathbf{P}(A) > 0$, such that $\widehat{Y}^1_t > \widehat{Y}^2_t$ on A. The value function V is strictly decreasing, then,

$$\mathbb{E}\left[V(\frac{\widehat{Y}^1_t}{\widehat{Y}^2_t}\widehat{Y}^2_T) \middle| \mathcal{F}_t \right] < \mathbb{E}\left[V(\widehat{Y}^2_T) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[V(\widehat{Y}^1_T) \middle| \mathcal{F}_t \right].$$

The optimizer $Y_T > 0$ a.s. implies $Y > 0$ a.s., since Y is supermartingale. Then define

$$Y := \widehat{Y}^1 \mathbf{1}_{[0,t]} + \frac{\widehat{Y}^1_t}{\widehat{Y}^2_t} \widehat{Y}^2 \mathbf{1}_{(t,T]} \mathbf{1}_{\{A\}} + \widehat{Y}^1 \mathbf{1}_{(t,T]} \mathbf{1}_{\{A^c\}}$$

which is right continuous, and also an element in the set $\mathcal{Y}(\widehat{y})$. Indeed, for any admissible wealth process $X \in \mathcal{X}(1)$, XY is a supermartingale.

$$\begin{aligned}
\text{If } 0 \le u < s \le t, \quad & \mathbb{E}\left[X_s Y_s \middle| \mathcal{F}_u \right] = \mathbb{E}\left[X_s \widehat{Y}^1_s \middle| \mathcal{F}_u \right] \le X_u \widehat{Y}^1_u; \\
\text{if } 0 \le t < u < s \le T, \quad & \mathbb{E}\left[X_s Y_s \middle| \mathcal{F}_u \right] = \mathbb{E}\left[X_s \frac{\widehat{Y}^1_t}{\widehat{Y}^2_t} \widehat{Y}^2_s \mathbf{1}_{\{A\}} \middle| \mathcal{F}_u \right] + \mathbb{E}\left[X_s \widehat{Y}^1_s \mathbf{1}_{\{A^c\}} \middle| \mathcal{F}_u \right] \\
& = \frac{\widehat{Y}^1_t}{\widehat{Y}^2_t} \mathbf{1}_{\{A\}} \mathbb{E}\left[X_s \widehat{Y}^2_s \middle| \mathcal{F}_u \right] + \mathbf{1}_{\{A^c\}} \mathbb{E}\left[X_s \widehat{Y}^1_s \middle| \mathcal{F}_u \right] \\
& \le \frac{\widehat{Y}^1_t}{\widehat{Y}^2_t} \mathbf{1}_{\{A\}} X_u \widehat{Y}^2_u + \mathbf{1}_{\{A^c\}} X_u \widehat{Y}^1_u \\
& = X_u Y_u;
\end{aligned}$$

if $0 \le u \le t < s \le T$, then

$$\begin{aligned}
\mathbb{E}\left[X_s Y_s \middle| \mathcal{F}_u\right] &= \mathbb{E}\left[X_s \frac{\widehat{Y}_t^1}{\widehat{Y}_t^2} \widehat{Y}_s^2 \mathbf{1}_{\{A\}} \middle| \mathcal{F}_u\right] + \mathbb{E}\left[X_s \widehat{Y}_s^1 \mathbf{1}_{\{A^c\}} \middle| \mathcal{F}_u\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\widehat{Y}_t^1}{\widehat{Y}_t^2} X_s \widehat{Y}_s^2 \mathbf{1}_{\{A\}} \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_u\right] + \mathbb{E}\left[\mathbb{E}\left[X_s \widehat{Y}_s^1 \mathbf{1}_{\{A^c\}} \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_u\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{A\}} \frac{\widehat{Y}_t^1}{\widehat{Y}_t^2} \mathbb{E}\left[X_s \widehat{Y}_s^2 \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_u\right] + \mathbb{E}\left[\mathbf{1}_{\{A^c\}} \mathbb{E}\left[X_s \widehat{Y}_s^1 \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_u\right] \\
&\le \mathbb{E}\left[\mathbf{1}_{\{A\}} \frac{X_t \widehat{Y}_t^1}{\widehat{Y}_t^2} \widehat{Y}_t^2 \middle| \mathcal{F}_u\right] + \mathbb{E}\left[\mathbf{1}_{\{A^c\}} X_t \widehat{Y}_t^1 \middle| \mathcal{F}_u\right] \\
&= \mathbb{E}\left[X_t \widehat{Y}_t^1 \middle| \mathcal{F}_u\right] \\
&\le X_u \widehat{Y}_u^1 = X_u Y_u.
\end{aligned}$$

It is clear that

$$\mathbb{E}\left[V(Y_T)\right] < \mathbb{E}\left[V(\widehat{Y}_T^2)\right] = \mathbb{E}\left[V(\widehat{Y}_T^1)\right],$$

which is in contradiction to the optimal solution of $\widehat{Y}^1, \widehat{Y}^2$. Thus, $\widehat{Y}_t^1 = \widehat{Y}_t^2$ a.s. for each $t \in [0, T]$, which implies they are indistinguishable, as they are càdlàg. □

2.1.2 Property of the dual optimizer

Let us recall Example 5.1' of one period process defined on a countable probability space Ω in [63]. The obtained optimal dual process $\widehat{Y}(1) \in \mathcal{Y}(1)$ fails to be a local martingale, but only a supermartingale. One may ask an interesting question: may the dual optimizer $\widehat{Y} \in \mathcal{Y}(\widehat{y})$ be a (local) martingale? In which setting could it be a local martingale?

Theorem 2.1.15. *Let Assumptions 2.1.4, 2.1.6 and 2.1.8 hold true. In addition, assume the price process S is continuous. Then the optimal dual process $\widehat{Y} \in \mathcal{Y}(\widehat{y})$ obtained in Theorem 2.1.9 is a local martingale.*

Remark 2.1.16. Without the continuity of S, a counterexample for continuous time model can be constructed based on Example 5.1' in [63]. Let S_0, S_1 be the same one period stock price process as defined in [63]. Set $S_t = S_0$ for all $t \in [0, T)$ and $S_T = S_1$, which means that the price process jumps suddenly at time T. The dual optimizer is associated with a process $\widehat{Y}$ that is constant 1 at all time $t \in [0, T)$. However, jumps to S_T^{-1} at the time T, i.e., $\widehat{Y} = S_T^{-1}$. Its expectation is strictly less than 1.

Remark 2.1.17. Larsen and Žitković showed in [65, §3.2] that if the stock-price process S is a continuous semimartingale in the problem in [63], then the dual optimizer is associated with a local martingale living in the set of supermartingale deflators. Their method is based on the representation theorem in [23] for continuous arbitrage-free price processes and the multiplicative decomposition of supermartingale deflators. They examined the decomposition of the dual optimal process, and found that the decreasing part vanishes and only the local martingale part left due to the maximality. Here, we give alternative proofs for the same assertion based on the dynamics of the primal solution $\widehat{X}$ and the dual one $\widehat{Y}$.

We recall the definition of optional strong supermartingales. These processes are introduced by Mertens [70] as a generalization of càdlàg supermartingales. We also refer to Appendix I of [28] for more properties of these processes.

Definition 2.1.18. *A real-valued stochastic process $Y = (Y_t)_{0 \le t \le T}$ is called an* ***optional strong supermartingale (OSS)*** *, if*

1. *Y is optional;*
2. *Y_τ is integrable for every $[0, T]$-valued stopping time τ;*
3. *For all stopping times σ and τ with $0 \le \sigma \le \tau \le T$, we have*

$$Y_\sigma \ge \mathbb{E}\left[Y_\tau | \mathcal{F}_\sigma\right].$$

Proof of Theorem 2.1.15. Let $\widehat{X} \in \mathcal{X}(x)$ be the optimal wealth process with primal optimizer $\widehat{X}_T(x) > 0$, $\mathbf{P}$-a.s. obtained in Theorem 2.1.9, then

(1) $\widehat{X}(x)$ is continuous.

(2) $\widehat{X}_T > 0$, $\mathbf{Q}$-a.s., as $\mathbf{Q} \sim \mathbf{P}$.

(3) $\widehat{X}$ is $\mathbf{Q}$-supermartingale with $\widehat{X}_T > 0$, $\mathbf{Q}$-a.s., by Theorem VI-17 in [28], we have $\inf_{0 \le t \le T} \widehat{X}_t > 0$, $\mathbf{Q}$-a.s. Equivalently,

$$\inf_{0 \le t \le T} \widehat{X}_t > 0, \ \mathbf{P}\text{-a.s.}$$

By the continuity of $\widehat{X}$, define a sequence of stopping times

$$\sigma_k := \left\{ t > 0 : \widehat{X}_t < \frac{1}{k} \right\}, \text{ for } k \in \mathbb{N}, \tag{2.1.11}$$

which goes to infinity.

Since the assertion (vi) in Theorem 2.1.9 (see also [63, Proposition 3.2]) shows

$$v(y) = \inf_{h \in \widehat{\mathcal{D}}} \mathbb{E}\left[V(yh)\right] = \inf_{\mathbf{Q} \in \mathcal{M}} \mathbb{E}\left[V(y\frac{d\mathbf{Q}}{d\mathbf{P}})\right],$$

the value $v(y)$ of the dual problem can be approximated by choosing a minimizing sequence $\{\mathbf{Q}^n\}_{n=1}^\infty \subseteq \mathcal{M}$ associated with its density process $\{Y^n\}_{n=1}^\infty$ such that Y_T^n converges almost surely to the dual optimizer $\widehat{h} \in \widetilde{\mathcal{D}}(1)$ to (2.1.9), where

$$Y_t^n := \mathbb{E}\left[\frac{d\mathbf{Q}^n}{d\mathbf{P}}\middle|\mathcal{F}_t\right] > 0 \tag{2.1.12}$$

is a strictly positive martingale. By [18, Theorem 2.7], there exists a subsequence $\widetilde{Y}^n \in$ conv $(Y^n, Y^{n+1}, \cdots)$ and an optional strong supermartingale (not necessarily càdlàg) $\widehat{Y}$ s.t. for each $[0,T]$-valued stopping time τ,

$$\widetilde{Y}_\tau^n \xrightarrow{\mathbf{P}} \widehat{Y}_\tau \quad \text{and} \quad \widetilde{Y}_T^n \xrightarrow{\mathbf{P}} \widehat{Y}_T.$$

In particular, for a fixed stopping time, passing to a subsequence, still denoted by $\widetilde{Y}^n$, we have

$$\widetilde{Y}^n_{\sigma_k\wedge T} \quad \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \quad \widehat{Y}_{\sigma_k\wedge T} \quad \text{and} \quad \widetilde{Y}^n_T \quad \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \quad \widehat{Y}_T = \widehat{h},$$

so that $\widehat{Y}_T$ is the dual optimizer.

Then we will prove the sequence $\left\{\widetilde{Y}^n_{\sigma_k\wedge T}\right\}_{n=1}^\infty$ is uniformly integrable. Applying once again Theorem 2.7 in [18] and passing to a subsequence, we can show for each stopping time σ_k,

$$\widetilde{Y}^n_{\sigma_k\wedge T}\widehat{X}_{\sigma_k\wedge T} \quad \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \quad \widehat{Y}_{\sigma_k\wedge T}\widehat{X}_{\sigma_k\wedge T} \quad \text{and} \quad \widetilde{Y}^n_T\widehat{X}_T \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \quad \widehat{Y}_T\widehat{X}_T,$$

where $\widehat{Y}\widehat{X}$ is an optional strong supermartingale, then we can deduce that $\widehat{Y}\widehat{X}$ is a martingale from (2.1.10). Consequently, we arrive at

$$\begin{cases} \mathbb{E}^{\mathbf{P}}[\widetilde{Y}^n_{\sigma_k\wedge T}\widehat{X}_{\sigma_k\wedge T}] & \leq x; \\ \mathbb{E}^{\mathbf{P}}[\widehat{Y}_{\sigma_k\wedge T}\widehat{X}_{\sigma_k\wedge T}] & = x. \end{cases} \tag{2.1.13}$$

It follows immediately by Lemma 5.1.1 that $\{\widetilde{Y}^n_{\sigma_k\wedge T}\}_{n=1}^\infty$ is uniformly integrable. Then

$$\widetilde{Y}^n_{\sigma_k\wedge T} \xrightarrow{\mathbf{L}^1} \widehat{Y}_{\sigma_k\wedge T}, \quad \text{i.e.,} \quad \mathbb{E}\left[\left|\widetilde{Y}^n_{\sigma_k\wedge T} - \widehat{Y}_{\sigma_k\wedge T}\right|\right] \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Fix $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\left[\left|\mathbb{E}\left[\widetilde{Y}^n_{\sigma_k\wedge T}\middle|\mathcal{F}_t\right] - \mathbb{E}\left[\widehat{Y}_{\sigma_k\wedge T}\middle|\mathcal{F}_t\right]\right|\right] &= \mathbb{E}\left[\left|\mathbb{E}\left[\widetilde{Y}^n_{\sigma_k\wedge T} - \widehat{Y}_{\sigma_k\wedge T}\middle|\mathcal{F}_t\right]\right|\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\left|\widetilde{Y}^n_{\sigma_k\wedge T} - \widehat{Y}_{\sigma_k\wedge T}\right|\middle|\mathcal{F}_t\right]\right] \\ &= \mathbb{E}\left[\left|\widetilde{Y}^n_{\sigma_k\wedge T} - \widehat{Y}_{\sigma_k\wedge T}\right|\right] \longrightarrow 0. \end{aligned}$$

Therefore we have

$$\widetilde{Y}^n_{t\wedge\sigma_k\wedge T} \xrightarrow{\mathbf{L}^1} \widehat{Y}_{t\wedge\sigma_k\wedge T}, \text{ as } n \longrightarrow \infty.$$

Moreover, $\widetilde{Y}^n_{t\wedge\sigma_k\wedge T} = \mathbb{E}\left[\widetilde{Y}^n_{\sigma_k\wedge T}\middle|\mathcal{F}_t\right]$ is a martingale, and $\widetilde{Y}^n_{t\wedge\sigma_k\wedge T} \xrightarrow{\mathbf{P}} \widehat{Y}_{t\wedge\sigma_k\wedge T}$, which imply

$\widehat{Y}_{t\wedge\sigma_k\wedge T} = \mathbb{E}\left[\widehat{Y}_{\sigma_k\wedge T}\middle|\mathcal{F}_t\right]$, so $\widehat{Y}_{\cdot\wedge\sigma_k\wedge T}$ is a true martingale. We now can conclude that $\widehat{Y}$ is a local martingale, thus is càdlàg.

For any $X \in \mathcal{X}(1)$, again applying Theorem 2.7 in [18], $X\widehat{Y}$ is an optional strong supermartingale. But X is continuous and $\widehat{Y}$ is càdlàg, one can see that $X\widehat{Y}$ is still a (càdlàg) supermartingale, which implies $\widehat{Y} \in \mathcal{Y}(1)$. □

Remark 2.1.19. Actually, we can prove the uniform integrability of the sequence $\left\{\widetilde{Y}^n_{\sigma_k\wedge T}\right\}_{n=1}^{\infty}$ directly, where σ_k is defined in (2.1.11).

Proof. If the sequence $\left\{\widetilde{Y}^n_{\sigma_k\wedge T}\right\}_{n=1}^{\infty}$ is not uniformly integrable, by the definition, there exists $\epsilon > 0$, for all $n \in \mathbb{N}$, there is a sequence of sets A_n s.t. $\mathbf{P}(A_n) < 1/2^n$ and $\mathbb{E}\left[\widetilde{Y}_{\sigma_k\wedge T}\mathbf{1}_{A_n}\right] \geq \epsilon$. Define

$$\eta^n := \widetilde{Y}^n_{\sigma_k\wedge T}\mathbf{1}_{A_n}; \quad \xi^n := \widetilde{Y}^n_{\sigma_k\wedge T}\mathbf{1}_{A_n^c}.$$

Obviously, $\widetilde{Y}^n_{\sigma_k\wedge T} := \eta^n + \xi^n$ and $\xi^n \longrightarrow \widehat{Y}_{\sigma_k\wedge T}$ a.s., as $n \longrightarrow \infty$.

By Fatou's Lemma,

$$\begin{aligned}
x &= \mathbb{E}\left[\widehat{X}_{\sigma_k\wedge T}\widehat{Y}_{\sigma_k\wedge T}\right] \\
&\leq \liminf_n \mathbb{E}\left[\widehat{X}_{\sigma_k\wedge T}\xi^n\right] \\
&= \liminf_n \mathbb{E}\left[\widehat{X}_{\sigma_k\wedge T}\widehat{Y}_{\sigma_k\wedge T} - \widehat{X}_{\sigma_k\wedge T}\eta^n\right] \\
&= \liminf_n \left\{\mathbb{E}\left[\widehat{X}_{\sigma_k\wedge T}\widehat{Y}_{\sigma_k\wedge T}\right] - \mathbb{E}\left[\widehat{X}_{\sigma_k\wedge T}\eta^n\right]\right\} \\
&\leq x - \frac{\epsilon}{k}
\end{aligned}$$

which is a contradiction. □

Remark 2.1.20. According to [56], the inspection of the proofs in [63] reveals that the usual assumption (NFLVR) can be replaced by a weaker one (NUPBR), which is equivalent to that $\mathcal{Y}(1) \neq \emptyset$ or the existence of ELMDs (see e.g. [61]). In this case, to deduce that $\widehat{Y}$ is a local martingale, we could proceed the same as above with a minimizing sequence of ELMDs and a classical localization argument if necessary.

2.2 Dual optimizer of utility maximization with random endowment

In this section, we extend the above mentioned classical financial model to a little complexer case with bounded random endowment. That is, we discuss the dual optimizer in utility maximization problem in incomplete markets with bounded random endowment.

The model and duality results are described by Cvitanić, Schachermayer and Wang (2001) in [13]. Clearly, if there is no random endowment, then this complexer model goes back to the classical case introduced in [63].

2.2.1 Convex duality in the case with random endowment

With the same settings introduced in subsection §2.1.1, we additionally assume the agent receives an exogenous endowment e_T at time T, for example, a non-traded European contingent claim. It is $\mathcal{F}_T$-measurable and satisfies $\rho := \|e_T\|_\infty < \infty$. Then, the corresponding value of his portfolio at time T can be written as

$$W_T = x + (H \bullet S)_T + e_T = X_T + e_T.$$

Let $\widetilde{\mathscr{C}_0}$ be the convex cone of $\mathcal{F}_T$-measurable random variables dominated by admissible stochastic integrals, i.e.,

$$\widetilde{\mathscr{C}_0} := \{g \in \mathcal{F}_T : g \leq (H \bullet S)_T, \text{ for some admissible strategy } H\}.$$

Moreover, we define $\widetilde{\mathscr{C}} := \widetilde{\mathscr{C}_0} \cap \mathbf{L}^\infty$.

Remark 2.2.1. In markets with bounded random endowment, the $\mathcal{F}_T$-measurable random variable $g \in \widetilde{\mathscr{C}_0} \subset \mathbf{L}^0$ can be negative.

Employing a utility function in Assumption 2.1.6, we assume $U(\infty) > 0$ and define $U(x) = -\infty$, if $x \leq 0$. The primal problem in this case with random endowment is to maximize the expected utility from terminal wealth

$$u(x) = \sup_{g \in \widetilde{\mathscr{C}_0}} \mathbb{E}\left[U(x + g + e_T)\right]. \tag{2.2.1}$$

To build the dual problem, we should first define the dual domain. However, in this case with bounded random endowment, the usual dual domain $\mathcal{Y}(y) \subset \mathbf{L}^1$ is not large enough, so we extend it to $(\mathbf{L}^\infty)^*$, the dual space of $\mathbf{L}^\infty$. Define the subset of $(\mathbf{L}^\infty)^*_+$,

$$\widetilde{\mathscr{D}} := \left\{Q \in (\mathbf{L}^\infty)^*_+ : \|Q\|_{(\mathbf{L}^\infty)^*} = 1, \langle Q, g\rangle \leq 0, \text{ for all } g \in \widetilde{\mathscr{C}}\right\}, \tag{2.2.2}$$

which is not empty, convex, closed and compact with respect to the weak-star topology $\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)$. For details please see Appendix §5.2.2.

In fact, the space $(\mathbf{L}^\infty)^*$ can be identified with the space of bounded additive measures. For each element in $(\mathbf{L}^\infty)^*_+$, it admits a unique Yosida-Hewitt decomposition $Q = Q^r + Q^s$, where the regular part $Q^r \in \mathbf{L}^1$ is countably additive and the singular part Q^s is purely finitely additive (see e.g. §5.2.1 or [89]). Then the dual problem can be formulated as

$$v(y) := \inf_{Q \in \widetilde{\mathscr{D}}} \left\{\mathbb{E}^{\mathbf{P}}\left[V(y\frac{dQ^r}{d\mathbf{P}})\right] + y\,\langle Q, e_T\rangle\right\}, \tag{2.2.3}$$

where V is the conjugate of U.

Similarly, as Assumption 2.1.8, we need the following assumption.

Assumption 2.2.2. *In this case, we assume $|u(x)| < \infty$ holds for some $x > \rho$.*

Now we summarize the results obtained in [13] as the following theorem:

Theorem 2.2.3 (Theorem 3.1 and Lemma 4.1 in [13])**.** *Under Assumptions 2.1.4, 2.1.6, and 2.2.2, we have*

(1) The primal value function u is finitely valued for all $x > x_0$, and $u(x) = -\infty$ for all $x < x_0$, where $x_0 := \sup_{Q \in \widetilde{\mathscr{D}}} \langle Q, -e_T \rangle$.

(2) The dual value function v is finitely valued and continuously differentiable on $(0, \infty)$.

(3) The functions u and v are conjugate in the sense that

$$\begin{aligned} v(y) &= \sup_{x > x_0} \{u(x) - xy\}, \ y > 0, \\ u(x) &= \inf_{y > 0} \{v(y) + xy\}, \ x > x_0. \end{aligned}$$

(4) For all $y > 0$, there exists a solution $\widehat{Q} \in \widetilde{\mathscr{D}}$ to the dual problem (2.2.3), which is unique up to the singular part. For all $x > x_0$, $\widehat{g} := I(\widehat{y} \frac{d\widehat{Q}^r_{\widehat{y}}}{d\mathbf{P}}) - x - e_T$ is the solution to the primal problem (2.2.1), where $I = -V'$ and $\widehat{y}$ attains the infimum of $\{v(y) + xy\}$ satisfying $v'(\widehat{y}) = -x$. There is an admissible trading strategy $\widehat{H}$ s.t. $\widehat{g} = (\widehat{H} \bullet S)_T$.

(5) The following equality is verified for the solutions of the primal and the dual problems:

$$\left\langle \widehat{Q}^r_{\widehat{y}}, x + (\widehat{H} \bullet S)_T + e_T \right\rangle = \left\langle \widehat{Q}_{\widehat{y}}, x + (\widehat{H} \bullet S)_T + e_T \right\rangle = x + \left\langle \widehat{Q}_{\widehat{y}}, e_T \right\rangle \qquad (2.2.4)$$

Remark 2.2.4.

1. Since the random variable $x + \left(\widehat{H} \bullet S\right)_T + e_T$ in Theorem 2.2.3 is uniformly bounded from below, then $\langle \widehat{Q}_{\widehat{y}}, x + \left(\widehat{H} \bullet S\right)_T + e_T \rangle$ is well defined by

$$\langle \widehat{Q}_{\widehat{y}}, x + \left(\widehat{H} \bullet S\right)_T + e_T \rangle := \lim_{M \to \infty} \langle \widehat{Q}_{\widehat{y}}, \left(x + \left(\widehat{H} \bullet S\right)_T + e_T\right) \wedge M \rangle,$$

although it is not necessarily an element in $\mathbf{L}^\infty$.

2. From the construction of the primal solution, one can see that $\frac{d\widehat{Q}^r_{\widehat{y}}}{d\mathbf{P}} > 0$, $\mathbf{P}$-a.s., so that $\widehat{Q}^r_{\widehat{y}} \sim \mathbf{P}$.

3. The equality of optimality (2.2.4) shows that the purely finitely additive part $\widehat{Q}^s_{\widehat{y}}$ "concentrates" its mass on the sets,

$$\left\{x + \left(\widehat{H} \bullet S\right)_T + e_T < \tfrac{1}{n}\right\}, \text{ for any } n \in \mathbb{N}.$$

2.2.2 Property of the dual optimizer in the case with random endowment

We discuss here the properties of the dual optimizer obtained in Theorem 2.2.3. Especially, we are interested in the property that the dual optimizer is associated with a local martingale, as we know that the optimal dual process is not always the local martingale, even in the classical case. Actually, our main result in this section shows that in the Brownian filtration, the regular part Q^r of the dual optimizer $\widehat{Q} \in \widetilde{\mathscr{D}} \subset (\mathbf{L}^\infty)^*_+$ obtained in Theorem 2.2.3 can be attained by the terminal value of a local martingale $\widehat{Y} \in \mathcal{Y}(1)$ defined in (2.1.5), which is the set of all supermartingale deflators.

Proposition 2.2.5. *Under Assumptions 2.1.4, 2.1.6 and 2.2.2, let $\widehat{y}$ satisfy $v'(\widehat{y}) = -x$. If $\widehat{Q}_{\widehat{y}}$ is a dual optimizer (denote by $\widehat{Q}$ for short) for the problem (2.2.3), then there exists a sequence $\{\mathbf{Q}^n\}_{n=1}^\infty \subseteq \mathcal{M}$, s.t.*

$$\frac{d\mathbf{Q}^n}{d\mathbf{P}} \to \frac{d\widehat{Q}^r}{d\mathbf{P}},\ \mathbf{P} - a.s. \text{ and } \langle \mathbf{Q}^n, e_T\rangle \longrightarrow \langle \widehat{Q}, e_T\rangle,\ \text{ as } n \to \infty. \tag{2.2.5}$$

Proof. As Proposition 5.2.11 shows that the dual domain $\widetilde{\mathscr{D}}$ is $\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)$-closure of $\mathcal{M}$, then it follows from Corollary 5.2.4 that we can find a sequence $\{\mathbf{Q}^n\}_{n=1}^\infty \subset \mathcal{M}$, such that (2.2.5) holds. □

Remark 2.2.6. Since the dual solution $\widehat{Q} \in \widetilde{\mathscr{D}}$ is unique up to the singular part, one question arises: is any cluster point $\mathcal{Q}$ of the sequence chosen in Proposition 2.2.5 the dual solution to the dual problem (2.2.3)? Actually, the answer is affirmative. It is guaranteed by [13, Proposition A.1], which is also presented in Proposition 5.2.2.

Similarly as in the proof of Theorem 2.1.15, we choose a sequence $\{\mathbf{Q}^n\}_{n=1}^\infty \subseteq \mathcal{M}$, s.t. it converges to the dual optimizer $\widehat{Q}$ in the sense of (2.2.5). For each $n \in \mathbb{N}$, define the density process $(Y_t^n)_{0\le t\le T}$ of $\mathbf{Q}^n$ as (2.1.12). By Theorem 2.7 in [18], there exists a sequence $\{\widetilde{Y}^n\}_{n=1}^\infty$ of convex combinations $\widetilde{Y}^n \in \mathrm{conv}(Y^n, Y^{n+1}, \ldots)$, and a non-negative optional strong supermartingale $\widehat{Y}$ (not necessarily càdlàg), such that for every $[0,T]$-valued stopping time σ, we have

$$\widetilde{Y}^n_\sigma \xrightarrow{\mathbf{P}} \widehat{Y}_\sigma,\ \text{as } n \to \infty. \tag{2.2.6}$$

Then we have the following properties of the optional strong supermartingale $\widehat{Y}$:

(p1) Obviously,

$$\frac{d\widetilde{\mathbf{Q}}^n}{d\mathbf{P}} = \widetilde{Y}^n_T \longrightarrow \widehat{Y}_T = \frac{d\widehat{Q}^r}{d\mathbf{P}},\ \mathbf{P} - a.s.,\ \text{as } n \to \infty,$$

where $d\widetilde{\mathbf{Q}}^n = \widetilde{Y}^n_T d\mathbf{P}$.

(p2) As stated in Remark 2.2.4, $\widehat{Y}_T = \frac{d\widehat{Q}^r}{d\mathbf{P}} > 0$, $\mathbf{P}$-a.s.

(p3) For the nonnegative supermartingale with $\widehat{Y}_T > 0$, $\mathbf{P}$-a.s., Theorem VI-17 in [28] points out that

$$\inf_{0 \le t \le T} \widehat{Y}_t > 0, \ \mathbf{P}\text{-a.s.} \tag{2.2.7}$$

Moreover, although $\widehat{Y}$ is only an optional strong supermartingale, we can also apply martingale inequality to show that (see P395, Appendice I-3 in [28])

$$\sup_{0 \le t \le T} \widehat{Y}_t < \infty, \ \mathbf{P}\text{-a.s.} \tag{2.2.8}$$

Our aim is to prove that $\widehat{Y}$ *is a local martingale* under the following assumption:

Assumption 2.2.7. *The underlying filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by a Brownian motion.*

Once the local martingale property is verified, then $\widehat{Y}$ is continuous and thus a (càdlàg) supermartingale. By the similar argument as in the proof of Theorem 2.1.15, for any $X \in \mathcal{X}(1)$, applying Theorem 2.7 in [18] again, one can see that $X\widehat{Y}$ is still a supermartingale, which implies $\widehat{Y} \in \mathcal{Y}(1)$.

Remark 2.2.8. One can also apply the well-known Fatou-convergence result (see Lemma 5.2 in [31]) to construct $\widehat{Y}'$ as the Fatou limit of $\{\widetilde{Y}^n\}_{n \in \mathbb{N}}$, whose terminal value is exactly the density $\frac{d\widehat{Q}^r}{d\mathbf{P}}$. Although the process $\widehat{Y}'$ constructed in this way is certainly càdlàg, yet (2.2.6) may fail. Therefore, the advantage of the result in [18] is that we could find a unified sequence which is not only the limit of $\widetilde{Y}^n$ at the terminal time T but also at any intermediate time. Particularly, one can pick a subsequence such that the convergence holds $\mathbf{P}$-a.s. at countably many times. Note that the difference between the two kinds of limit is only on the graph of countably many stopping times (see Section A.1 in [19]).

Remark 2.2.9. Under Assumption 2.2.7, every local martingale has a continuous modification, which is equivalent that every $\mathcal{F}$-stopping time is predictable (cf. P.30 in [9]). In particular, from Assumption 2.1.4, the stock-price process S in our setting is indeed continuous. It is not clear to us whether this assumption is really necessary for the following theorem or it could be weakened. We leave this as an open question.

Now, we are ready to state our main result. Its proof is postponed to the next subsection.

Theorem 2.2.10. *Under Assumptions 2.1.4, 2.1.6, 2.2.2 and 2.2.7, the process $\widehat{Y}$ defined in (2.2.6) is a local martingale and thus, the regular part $\widehat{Q}^r$ of any dual optimizer obtained in Theorem 2.2.3 ([13, Theorem 3.1]) can be attained by a local martingale $\widehat{Y} \in \mathcal{Y}(1)$.*

2.2.3 Proof of the local martingale property

We present the proof of Theorem 2.2.10 in this subsection. The main idea is to apply Lemma 5.1.1 and then to verify that the singular part of the dual optimizer only works in a small set as states in Remark 2.2.4. We split the proof into three main steps. In the sequel, each part stands for a step.

Fictitious optimal wealth process

Firstly, we should make the optimal wealth process clear, since we only know the random endowment e_T at the terminal time T, but not the process e of random endowment. The terminal value of an optimal wealth process is $\widehat{W}_T = x + (\widehat{H} \bullet S)_T + e_T$, so we can construct a fictitious optimal wealth process $\widehat{W}$ with the optimal $\widehat{W}_T$ by the sequence $\{\widetilde{Y}^n\}_{n=1}^{\infty}$ associated with $\{\widetilde{\mathbf{Q}}^n\}_{n=1}^{\infty} \subset \mathcal{M}$. Then, we look for a sequence of stopping times, such that at each stopping time, the process $\widehat{W}$ is bounded away from 0.

Define

$$\widetilde{W}_t^n := x + (\widehat{H} \bullet S)_t + \mathbb{E}^{\widetilde{\mathbf{Q}}^n}\left[e_T \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad \text{for all } n \in \mathbb{N}.$$

Then we have the following properties of $\widetilde{W}_t^n$:

(P1) It follows from the optimality of $\widehat{H}$ that $\widetilde{W}_T^n = x + (\widehat{H} \bullet S)_T + e_T = \widehat{W}_T > 0$, $\mathbf{P}$-a.s., equivalently, $\widetilde{\mathbf{Q}}^n$-a.s.

(P2) Since $\mathcal{M}$ is closed with respect to convex combination, $\widetilde{\mathbf{Q}}^n \in \mathcal{M}$ is still an ELMM, so that $\widetilde{W}^n$ is a $\widetilde{\mathbf{Q}}^n$-supermartingale. Indeed, $(\mathbb{E}^{\widetilde{\mathbf{Q}}^n}\left[e_T \middle| \mathcal{F}_\cdot \right])$ is $\widetilde{\mathbf{Q}}^n$-martingale, $(\widehat{H} \bullet S)$ is $\widetilde{\mathbf{Q}}^n$-supermartingale.

(P3) For the nonnegative supermartingale, we can employ the same argument as (2.2.7) to deduce that

$$\inf_{0 \leq t \leq T} \widetilde{W}_t^n > 0, \quad \widetilde{\mathbf{Q}}^n\text{-a.s., even } \mathbf{P}\text{-a.s., as } \widetilde{\mathbf{Q}}^n \sim \mathbf{P}. \tag{2.2.9}$$

Consider the process

$$\begin{aligned} \widetilde{Y}_t^n \widetilde{W}_t^n &= \widetilde{Y}_t^n (x + (\widehat{H} \bullet S)_t + \mathbb{E}^{\widetilde{\mathbf{Q}}^n}\left[e_T \middle| \mathcal{F}_t \right]) \\ &= \widetilde{Y}_t^n (x + (\widehat{H} \bullet S)_t) + \mathbb{E}^{\mathbf{P}}\left[\widetilde{Y}_T^n e_T \middle| \mathcal{F}_t \right] > 0, \ 0 \leq t \leq T, \end{aligned}$$

which is obviously a strictly positive $\mathbf{P}$-supermartingale. Again applying Theorem 2.7 in [18], there exists a subsequence of convex combinations of $\{\widetilde{Y}^n\}_{n=1}^{\infty}$, denoted still by $\{\widetilde{Y}^n\}_{n=1}^{\infty}$, and a nonnegative optional strong supermartingale $\widehat{Z}$, such that for every $[0, T]$-valued stopping time σ we have

$$\widetilde{Y}_\sigma^n \widetilde{W}_\sigma^n \xrightarrow{\mathbf{P}} \widehat{Z}_\sigma, \ \text{as } n \to \infty. \tag{2.2.10}$$

Recall (2.2.6) still holds for the subsequence $\{\widetilde{Y}^n\}_{n=1}^{\infty}$, i.e.,

$$\widetilde{Y}_{\sigma}^{n} \xrightarrow{\mathbf{P}} \widehat{Y}_{\sigma}, \text{ as } n \to \infty. \tag{2.2.6}$$

Hence the fictitious optimal wealth process $\widehat{W}$ can be defined as follows.

Proposition 2.2.11. *The process $\widehat{W} := \widehat{Z}/\widehat{Y}$ is well defined.*

Proof. The strict positivity of $\widehat{Y}$ (2.2.7) guarantees that $\widehat{W}$ is well-defined. □

Proposition 2.2.12. *The process $\widehat{Z}$ is a martingale and thus, it has a continuous modification as the underlying filtration $\mathcal{F}$ is Brownian filtration. Moreover,*

$$\inf_{0 \le t \le T} \widehat{Z}_t > 0, \ \mathbf{P}\text{-}a.s., \quad \text{and} \tag{2.2.11}$$

$$\sup_{0 \le t \le T} \widehat{Z}_t < \infty, \ \mathbf{P}\text{-}a.s. \tag{2.2.12}$$

Proof. From (2.2.5) in Proposition 2.2.5 and the optimal equality (2.2.4) in Theorem 2.2.3, we have

$$\begin{aligned}\widehat{Z}_0 &= \lim_{n\to\infty} \widetilde{Y}_0^n \widetilde{W}_0^n = \lim_{n\to\infty} \widetilde{W}_0^n = x + \lim_{n\to\infty} \mathbb{E}^{\widetilde{\mathbf{Q}}^n}\left[e_T\right] = x + \lim_{n\to\infty} \left\langle \widetilde{\mathbf{Q}}^n, e_T \right\rangle \\ &= x + \left\langle \widehat{Q}, e_T \right\rangle = \left\langle \widehat{Q}^r, x + (\widehat{H} \bullet S)_T + e_T \right\rangle = \mathbb{E}^{\mathbf{P}}\left[\widehat{Y}_T \widehat{W}_T\right] = \mathbb{E}^{\mathbf{P}}\left[\widehat{Z}_T\right].\end{aligned}$$

It is known that the process $\widehat{Z}$ is an optional strong supermartingale, so it is a martingale.

Note that

$$\widehat{W}_T = \widetilde{W}_T^n = x + (\widehat{H} \bullet S) + e_T, \quad \text{for all} \quad n \in \mathbb{N},$$

so $\widehat{Z}_T = \widehat{Y}_T \widehat{W}_T > 0, \mathbf{P}$-a.s. Together with the martingale property of $\widehat{Z}$, we have (2.2.11) by [28, Theorem VI-17]. Proceeding with the same argument as (2.2.8), we obtain (2.2.12). □

Therefore, we can find a sequence of stopping times to keep the stopped optimal wealth process bounded away from 0.

Proposition 2.2.13. *There exists a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$, s.t.*

$$\widehat{W}_{t\wedge\tau_k} \ge \frac{1}{k}, \quad \text{for all} \ \ 0 \le t \le T,$$

and $\mathbf{P}(\tau_k = T) \nearrow 1$, as $k \to \infty$.

Proof. Without loss of generality, we assume that $\widehat{Z}_t = \widehat{Z}_T$, for all $t \ge T$. From (2.2.11) and (2.2.8), it is easy to see that

$$\inf_{0 \le t \le T} \widehat{W}_t > 0, \ \mathbf{P}\text{-a.s.} \tag{2.2.13}$$

Like (2.1.11), define

$$\sigma_k := \left\{ t > 0 : \widehat{W}_t < \frac{1}{k} \right\}, \text{ for all } \ k \in \mathbb{N}, \tag{2.2.14}$$

which goes to infinity. Under the Assumption 2.2.7, all stopping times defined in (2.2.14) are predictable. Thus, for every k, we can choose a sequence $\sigma_{k,m} \to \sigma_k$ and $\sigma_{k,m} < \sigma_k$ on the set $\{\sigma_k > 0\}$. Define $\tau_k := \sigma_{k,m_k} \wedge T$, where

$$\mathbf{P}(|\sigma_{k,m_k} - \sigma_k| > 2^{-k}) < 2^{-k}.$$

The sequence $\{\tau_k\}_{k=1}^{\infty}$ is as desired. □

Fictitious process for the random endowment

The fictitious optimal wealth process $\widehat{W} = \frac{\widehat{Z}}{\widehat{Y}}$ has been obtained, but the process of e for random endowment e_T is still unknown. We shall construct a fictitious process $(e_t)_{0 \le t \le T}$ and a dual optimizer $\mathcal{Q}$. Then we discuss how and where the dual optimizer works on the fictitious optimal wealth process. In fact, the random variable e_τ is the conditional expectation of e_T under a dual optimizer $\mathcal{Q}$.

From now on, fix $k \in \mathbb{N}$ and denote $\tau = \tau_k$. Then according to (2.2.14), $\widehat{W}_\tau \ge 1/k > 0$, $\mathbf{P}$-a.s. It follows from (2.2.6) and (2.2.10) that for every stopping time $\tau \in [0, T]$,

$$\widetilde{W}^n_\tau \xrightarrow{\mathbf{P}} \widehat{W}_\tau, \text{ as } \ n \to \infty.$$

Then, we can rewrite the process $\widehat{W}$ as

$$\widehat{W}_t = x + (\widehat{H} \bullet S)_t + e_t, \quad 0 \le t \le T,$$

where

$$e_t := \mathbf{P} - \lim_{n\to\infty} \ \mathbb{E}^{\widetilde{\mathbf{Q}}^n}[e_T|\mathcal{F}_t] \tag{2.2.15}$$

with

$$\mathbb{E}^{\widetilde{\mathbf{Q}}^n}[e_T|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathbf{P}}\left[\widetilde{Y}^n_T e_T|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbf{P}}\left[\widetilde{Y}^n_T|\mathcal{F}_t\right]}.$$

Remark 2.2.14. In [13], e_T is indeed associated with a cumulative process $e := (e_t)_{0 \le t \le T}$ with $e_0 = 0$, however, only the terminal value e_T influences the choice of the agent. Here, $e := (e_t)_{0 \le t \le T}$ is the fictitious value process with the terminal value e_T, which is constructed by (2.2.15) and should be differed from the one in [13].

With the stopping time τ, both sequences $\left\{ \mathbb{E}^{\mathbf{P}} \left[\widetilde{Y}^n_T e_T \middle| \mathcal{F}_\tau \right] \right\}_{n=1}^{\infty}$ and $\left\{ \mathbb{E}^{\mathbf{P}} \left[\widetilde{Y}^n_T \middle| \mathcal{F}_\tau \right] \right\}_{n=1}^{\infty}$ are bounded in $\mathbf{L}^1(\mathbf{P})$. Recalling Komlós' lemma, we can find a subsequence $\{\overline{Y}^n\}_{n=1}^{\infty}$ of convex combinations $\overline{Y}^n \in \text{conv}(\widetilde{Y}^n, \widetilde{Y}^{n+1}, \cdots)$ associated with $\overline{\mathbf{Q}}^n \in \text{conv}(\widetilde{\mathbf{Q}}^n, \widetilde{\mathbf{Q}}^{n+1}, \cdots)$

s.t. for some $g \in \mathbf{L}^1(\Omega, \mathcal{F}_\tau, \mathbf{P})$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Y}^n_T e_T \middle| \mathcal{F}_\tau\right] = g, \quad \mathbf{P}\text{-a.s.},\\
&\lim_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Y}^n_T \middle| \mathcal{F}_\tau\right] = \lim_{n\to\infty} \overline{Y}^n_\tau = \widehat{Y}_\tau, \quad \mathbf{P}\text{-a.s., and}\\
&e_\tau = \lim_{n\to\infty} \mathbb{E}^{\overline{\mathbf{Q}}^n}\left[e_T\middle| \mathcal{F}_\tau\right] = \lim_{n\to\infty} \frac{\mathbb{E}^{\mathbf{P}}\left[\overline{Y}^n_T e_T \middle| \mathcal{F}_t\right]}{\mathbb{E}^{\mathbf{P}}\left[\overline{Y}^n_T \middle| \mathcal{F}_t\right]} = \frac{g}{\widehat{Y}_\tau}, \quad \mathbf{P}\text{-a.s..}
\end{aligned}$$

Remark 2.2.15. The random variables $\{\mathbb{E}^{\overline{\mathbf{Q}}^n}[e_T|\mathcal{F}_\tau]\}_{n=1}^\infty$ and e_τ are elements in $\mathbf{L}^\infty(\Omega, \mathcal{F}_\tau, \mathbf{P})$, and

$$\mathbb{E}^{\overline{\mathbf{Q}}^n}[e_T|\mathcal{F}_\tau] \overset{\sigma(\mathbf{L}^\infty, \mathbf{L}^1)}{\longrightarrow} e_\tau, \text{ as } n \to \infty.$$

Indeed, for each step function $\xi^m \in \mathcal{F}_\tau$, $m \in \mathbb{N}$, it follows from the bounded convergence theorem that $\langle \mathbb{E}^{\overline{\mathbf{Q}}^n}[e_T|\mathcal{F}_\tau], \xi^m\rangle \to \langle e_\tau, \xi^m\rangle$. Suppose $\xi^m \to \xi$ in $\mathbf{L}^1(\Omega, \mathcal{F}_\tau, \mathbf{P})$, from the uniform integrability of

$$\left\{\mathbb{E}^{\overline{\mathbf{Q}}^n}[e_T|\mathcal{F}_\tau]\xi^m\right\}_{m,n=1}^\infty,$$

we have $\left\langle \mathbb{E}^{\overline{\mathbf{Q}}^n}[e_T|\mathcal{F}_\tau], \xi\right\rangle \to \langle e_\tau, \xi\rangle$.

Further, by Egorov's Theorem, there exists an increasing sequence of sets $\{\Omega_m\}_{m=1}^\infty$, s.t. for each m, $\mathbf{P}(\Omega_m) > 1 - 2^{-m}$, and $\{\overline{Y}^n_\tau\}_{n=1}^\infty$ converges uniformly to $\widehat{Y}_\tau$ on Ω_m. Observing that for each $n \in \mathbb{N}$, $\overline{Y}^n_\tau \in \mathbf{L}^1_+(\Omega, \mathcal{F}_\tau, \mathbf{P})$, from Fatou's Lemma that $\mathbb{E}^{\mathbf{P}}\left[\left|\widehat{Y}_\tau\right|\right] \leq 1$, thus

$$\{\overline{Y}^n_\tau\}_{n=1}^\infty \text{ is uniformly integrable on } \Omega_m.$$

We need to verify that the cluster point of the sequence $\{\overline{\mathbf{Q}}^n\}_{n=1}^\infty$ associated with $\{\overline{Y}^n_\tau\}_{n=1}^\infty$ is a dual optimizer as required.

Proposition 2.2.16. *The sequence $\{\overline{\mathbf{Q}}^n\}_{n=1}^\infty \subset \mathcal{M}$ associated with $\{\overline{Y}^n_\tau\}_{n=1}^\infty$ admits a cluster point $\mathcal{Q} \in \widetilde{\mathscr{D}}$, s.t.*

(1) $\overline{Y}^n_T \longrightarrow \frac{d\mathcal{Q}^r}{d\mathbf{P}}$, $\mathbf{P}$*-a.s.;*

(2) $\langle \widehat{Q}, e_T\rangle = \langle \mathcal{Q}, e_T\rangle$*;*

(3) $\mathcal{Q}$ is a dual optimizer to the problem (2.2.3)*.*

Proof. Note $\{\overline{\mathbf{Q}}^n\}_{n=1}^\infty \subset \mathcal{M} \subset \mathcal{D}$. Since $\mathcal{D}$ is a weak-star compact subset of $(\mathbf{L}^\infty)^*$, the sequence $\{\overline{\mathbf{Q}}^n\}_{n=1}^\infty$ admits a cluster point $\mathcal{Q} \in \mathcal{D}$. (1) follows from Proposition A.1 in [13]; (2) holds because of (2.2.5); (1) and (2) imply (3). □

Then we have the following corollary:

Corollary 2.2.17. *The finitely additive measure $\mathcal{Q}|_{\mathcal{F}_\tau}$ is countably additive on Ω_m, for each $m \in \mathbb{N}$, where $\mathcal{Q}|_{\mathcal{F}_\tau}$ denotes the restriction of $\mathcal{Q}$ on $\mathcal{F}_\tau$. In other words, $(\mathcal{Q}|_{\mathcal{F}_\tau})^s$ vanishes on Ω_m, i.e., for each $A \subset \Omega_m$, $A \in \mathcal{F}_\tau$, $\langle(\mathcal{Q}|_{\mathcal{F}_\tau})^s, \mathbf{1_A}\rangle = 0$.*

Proof. $\mathcal{Q}$ is the cluster point of the sequence $\{\overline{\mathbf{Q}}^n\}_{n=1}^\infty$, according to the procedure in the proof of Corollary 5.2 in [75], $\mathcal{Q}|_{\mathcal{F}_\tau}$ is the cluster point of the sequence $\{\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau}\}_{n=1}^\infty$. By Proposition A.1 in [13]

$$\frac{d(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau})}{d\mathbf{P}} \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}},$$

then for each $A \subset \Omega_m,\ A \in \mathcal{F}_\tau$,

$$\frac{d(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau})}{d\mathbf{P}}\mathbf{1}_A \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}\mathbf{1}_A.$$

It follows from the uniform integrability of $\overline{Y}_\tau^n$ on Ω_m and the fact $\overline{Y}_\tau^n = \mathbb{E}^{\mathbf{P}}\left[\left.\frac{d\overline{\mathbf{Q}}^n}{d\mathbf{P}}\right| \mathcal{F}_\tau\right] = \frac{d(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau})}{d\mathbf{P}}$, that

$$\mathbb{E}^{\mathbf{P}}\left[\left|\frac{d(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau})}{d\mathbf{P}}\mathbf{1}_A - \frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}\mathbf{1}_A\right|\right] \longrightarrow 0,$$

which means

$$\left\langle(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau}), \mathbf{1}_A\right\rangle \longrightarrow \left\langle(\mathcal{Q}|_{\mathcal{F}_\tau})^r, \mathbf{1}_A\right\rangle.$$

On the other hand, we have $\left\langle\overline{\mathbf{Q}}^n, \mathbf{1}_A\right\rangle \longrightarrow \langle\mathcal{Q}, \mathbf{1}_A\rangle$, then $\left\langle(\overline{\mathbf{Q}}^n|_{\mathcal{F}_\tau}), \mathbf{1}_A\right\rangle \longrightarrow \langle(\mathcal{Q}|_{\mathcal{F}_\tau}), \mathbf{1}_A\rangle$, which gives

$$\left\langle(\mathcal{Q}|_{\mathcal{F}_\tau})^r, \mathbf{1}_A\right\rangle = \left\langle(\mathcal{Q}|_{\mathcal{F}_\tau}), \mathbf{1}_A\right\rangle.$$

□

Again from [13, Proposition A.1] we have

Proposition 2.2.18.

$$\widehat{Y}_\tau = \lim_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\left.\overline{Y}_T^n\right| \mathcal{F}_\tau\right] = \frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}},\ \ \mathbf{P}\text{-}a.s.$$

Remark 2.2.19. We remark that the choice of the finitely additive measure $\mathcal{Q}$ depends on the stopping time τ.

The following corollary is straightforward from (2.2.7), which implies $\widehat{Y}_\tau > 0,\ \mathbf{P}$-a.s.

Corollary 2.2.20.

$$(\mathcal{Q}|_{\mathcal{F}_\tau})^r \sim \mathbf{P}.$$

Proceeding with a similar argument as Proposition 2.2.18, we obtain

Proposition 2.2.21.

$$g = \lim_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Y}_T^n e_T|\mathcal{F}_\tau\right] = \frac{d\big((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\big)^r}{d\mathbf{P}},\ \ \mathbf{P}\text{-}a.s.,$$

where $\mathcal{Q}e_T$ denotes the linear operator $\langle\mathcal{Q}, e_T\cdot\rangle \in (\mathbf{L}^\infty)^*$ and

$$\begin{aligned}\big((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\big)^r &:= \big((\mathcal{Q}e_T^+)|_{\mathcal{F}_\tau}\big)^r + \big((\mathcal{Q}e_T^-)|_{\mathcal{F}_\tau}\big)^r;\\ \big((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\big)^s &:= \big((\mathcal{Q}e_T^+)|_{\mathcal{F}_\tau}\big)^s + \big((\mathcal{Q}e_T^-)|_{\mathcal{F}_\tau}\big)^s.\end{aligned}$$

The lemma below is the core of this part, even is the core of the proof to Theorem 2.2.10:

Lemma 2.2.22. *The random variable e_τ is the conditional expectation of e_T under $\mathcal{Q}$ with respect to $\mathcal{F}_\tau$. In particular,*

$$\langle \mathcal{Q}|_{\mathcal{F}_\tau}, e_\tau \rangle = \langle \mathcal{Q}, e_T \rangle.$$

Proof. It follows from the boundedness of e_T, there exists a unique random variable $\eta \in \mathcal{F}_\tau$, (see Definition 7.1 and Theorem 7.2 in [68]), such that for each $A \in \mathcal{F}_\tau$,

$$\langle (\mathcal{Q}|_{\mathcal{F}_\tau})\eta, \mathbf{1}_A \rangle = \langle \mathcal{Q}e_T, \mathbf{1}_A \rangle .$$

Our aim is to prove

$$\eta = e_\tau = \frac{\frac{d((\mathcal{Q}e_T)|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}}{\frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}}, \ \mathbf{P}\text{-a.s.}$$

It is evident that $\mathcal{Q}e_T$ is a cluster point of the sequence $\{\overline{\mathbf{Q}}^n e_T\}_{n=1}^\infty$, and similar to Corollary 2.2.17, we know that for each $m \in \mathbb{N}$, $((\mathcal{Q}e_T)|_{\mathcal{F}_\tau})^s$ vanishes on Ω_m. Then, for any $A \subset \Omega_m$, $A \in \mathcal{F}_\tau$,

$$\begin{aligned}
\left\langle (\mathcal{Q}|_{\mathcal{F}_\tau})^r, \frac{\frac{d\left((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\right)^r}{d\mathbf{P}}}{\frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}}\mathbf{1}_A \right\rangle &= \mathbb{E}^{\mathbf{P}}\left[\frac{d\left((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\right)^r}{d\mathbf{P}}\mathbf{1}_A\right] = \left\langle \left((\mathcal{Q}e_T)|_{\mathcal{F}_\tau}\right)^r, \mathbf{1}_A \right\rangle \\
&= \langle (\mathcal{Q}e_T)|_{\mathcal{F}_\tau}, \mathbf{1}_A \rangle = \langle \mathcal{Q}e_T, \mathbf{1}_A \rangle = \langle \mathcal{Q}|_{\mathcal{F}_\tau}\eta, \mathbf{1}_A \rangle \\
&= \langle (\mathcal{Q}|_{\mathcal{F}_\tau})^r, \eta \mathbf{1}_A \rangle,
\end{aligned}$$

which implies $\eta = e_\tau$, $(\mathcal{Q}|_{\mathcal{F}_\tau})^r$-a.s. on Ω_m. Thanks to Corollary 2.2.20, $\eta = e_\tau$, $\mathbf{P}$-a.s. on Ω_m. Letting $m \to \infty$, we end the proof. □

Now let consider the dynamic version of the optimal equality (2.2.4) in Theorem 2.2.3:

Proposition 2.2.23.

$$\begin{aligned}
&\langle \mathcal{Q}|_{\mathcal{F}_\tau}, \widehat{W}_\tau \rangle = \langle \mathcal{Q}|_{\mathcal{F}_\tau}, x + (\widehat{H} \bullet S)_\tau + e_\tau \rangle = x + \langle \mathcal{Q}, e_\tau \rangle, \ \textit{especially,} \\
&\langle \mathcal{Q}|_{\mathcal{F}_\tau}, \widehat{W}_\tau \rangle = \langle (\mathcal{Q}|_{\mathcal{F}_\tau})^r, \widehat{W}_\tau \rangle.
\end{aligned}$$

Proof. Since $\mathcal{Q} \in \widetilde{\mathscr{D}}$, by definition, $\langle \mathcal{Q}|_{\mathcal{F}_\tau}, (\widehat{H} \bullet S)_\tau \rangle = \langle \mathcal{Q}, (\widehat{H} \bullet S)_\tau \rangle \le 0$. Thus, from the positivity of $(\mathcal{Q}|_{\mathcal{F}_\tau})^s$, the martingale property of $\widehat{Y}\widehat{W}$ together with Lemma 2.2.22, we obtain

$$\langle \mathcal{Q}|_{\mathcal{F}_\tau}, \widehat{W}_\tau \rangle = \langle \mathcal{Q}|_{\mathcal{F}_\tau}, x + (\widehat{H} \bullet S)_\tau \rangle + \langle \mathcal{Q}|_{\mathcal{F}_\tau}, e_\tau \rangle \le x + \langle \mathcal{Q}, e_T \rangle,$$

and

$$\langle \mathcal{Q}|_{\mathcal{F}_\tau}, \widehat{W}_\tau \rangle \ge \langle (\mathcal{Q}|_{\mathcal{F}_\tau})^r, \widehat{W}_\tau \rangle = x + \langle \mathcal{Q}, e_T \rangle.$$

As all inequalities are equalities, we have

$$\langle (\mathcal{Q}|_{\mathcal{F}_\tau})^s, \widehat{W}_\tau \rangle = 0. \tag{2.2.16}$$

$\square$

Proof of the local martingale property

We show that $\{\overline{Y}^n_\tau\}_{n=1}^\infty$ is uniformly integrable here, so that $\widehat{Y}$ is a martingale up to τ. Then, substituting τ by $\tau_k, k \in \mathbb{N}$, we can conclude $\widehat{Y}$ is a local martingale.

Proposition 2.2.24. *The sequence of random variables* $\{\overline{Y}^n_\tau\}_{n=1}^\infty = \left\{\mathbb{E}^{\mathbf{P}}\left[\frac{d\overline{\mathbf{Q}}^n}{d\mathbf{P}}\middle| \mathcal{F}_\tau\right]\right\}_{n=1}^\infty$ *is uniformly integrable and*

$$\overline{Y}^n_\tau \xrightarrow{\mathbf{L}^1} \widehat{Y}_\tau, \text{ as } n \to \infty.$$

Proof. According to Proposition 2.2.13, $\widehat{W}_\tau > 0$, $\mathbf{P}$-a.s. Then together with (2.2.16), we derive that $(\mathcal{Q}|_{\mathcal{F}_\tau})^s \equiv 0$. Recalling that $\|\mathcal{Q}\|_{(\mathbf{L}^\infty)^*} = 1$, we have

$$\mathbb{E}^{\mathbf{P}}\left[\widehat{Y}_\tau\right] = \mathbb{E}^{\mathbf{P}}\left[\frac{d(\mathcal{Q}|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}\right] = 1.$$

On the other hand,

$$\mathbb{E}^{\mathbf{P}}\left[\overline{Y}^n_\tau\right] = 1 \text{ and } \overline{Y}^n_\tau \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \widehat{Y}_\tau, \text{ as } n \to \infty.$$

By Lemma 5.1.1, we obtain that $\{\overline{Y}^n_\tau\}_{n=1}^\infty$ is uniformly integrable. (Actually, the desired result can also follow by Scheffé's lemma.) $\square$

Proof of Theorem 2.2.10 For each $\tau_k, k \in \mathbb{N}$ defined in Proposition 2.2.13, by Proposition 2.2.24, $\{\overline{Y}^n_{\tau_k}\}_{n=1}^\infty$ is also uniformly integrable. Thus,

$$\overline{Y}^n_{\tau_k} \xrightarrow{\mathbf{L}^1} \widehat{Y}_{\tau_k}, \text{ as } n \to \infty.$$

Similarly as in the proof to Theorem 2.1.15, from the martingale property and (2.2.6), we also have

$$\overline{Y}^n_{t\wedge\tau_k} \xrightarrow{\mathbf{L}^1} \widehat{Y}_{t\wedge\tau_k}, \text{ as } n \to \infty, \quad \text{for all } t \in [0,T],$$

which implies $\widehat{Y}$ a martingale up to τ_k.

By the definition of $\{\tau_k\}_{k=1}^\infty$, we see that $\widehat{Y}$ is a local martingale, thus, $\widehat{Y}$ has a càdlàg modification. Applying the similar argument in the proof of Theorem 2.1.15 in the previous section, for each $X \in \mathcal{X}(1)$, $X\widehat{Y}$ is a supermartingale. Therefore, $\widehat{Y}$ is a supermartingale deflator, i.e., $\widehat{Y}$ belongs to $\mathcal{Y}(1)$. $\square$.

Remark 2.2.25. Indeed, for each k, the dual optimizer $\mathcal{Q}$ we constructed generates an ELMM on $[\![0,\tau_k]\!]$ such that the pricing of the fictitious random endowment e_{τ_k} under this measure is exact $\langle \widehat{Q}, e_T\rangle$, in particular, $\langle \widehat{Q}, e_T\rangle$ is a no-arbitrage price for e_{τ_k}.

Remark 2.2.26. We would like to explain a little about the dynamics of $\mathcal{Q}|_{\mathcal{F}_{\cdot}}$. Under Assumption 2.2.7 that $\mathcal{F}$ is the Brownian filtration, the underlying price process S and the local martingale $\widehat{Y}$ are continuous, so is the wealth process $\widehat{W}$. Consider a set $A \in \mathcal{F}_\tau$, where τ is a $[0,T]$-valued stopping time, and suppose $\widehat{W}_\tau$ is strictly positive on A. For an infinitesimal δt such that $\widehat{W}_{\tau+\delta t}$ cannot suddenly jump to 0, we can show that if $(\mathcal{Q}|_{\mathcal{F}_\tau})^s \equiv 0$ on A, then $(\mathcal{Q}|_{\mathcal{F}_{\tau+\delta t}})^s \equiv 0$ on A. Otherwise,

$$\begin{aligned}\langle \mathcal{Q}|_{\mathcal{F}_{\tau+\delta}}, \widehat{W}_{\tau+\delta}\mathbf{1}_A\rangle &= \langle (\mathcal{Q}|_{\mathcal{F}_{\tau+\delta}})^r, \widehat{W}_{\tau+\delta}\mathbf{1}_A\rangle + \langle (\mathcal{Q}|_{\mathcal{F}_{\tau+\delta}})^s, \widehat{W}_{\tau+\delta}\mathbf{1}_A\rangle \\ &> \langle (\mathcal{Q}|_{\mathcal{F}_{\tau+\delta}})^r, \widehat{W}_{\tau+\delta}\mathbf{1}_A\rangle = \langle (\mathcal{Q}|_{\mathcal{F}_{\tau}})^r, \widehat{W}_{\tau}\mathbf{1}_A\rangle = \langle \mathcal{Q}|_{\mathcal{F}_{\tau}}, \widehat{W}_{\tau}\mathbf{1}_A\rangle,\end{aligned}$$

which implies $\left\langle \mathcal{Q}, \big((\widehat{H} \bullet S)_{\tau+\delta t} - (\widehat{H} \bullet S)_\tau\big)\mathbf{1}_A \right\rangle > 0$ in contradiction to the definition of $\widetilde{\mathscr{D}}$.

Remark 2.2.27. We compare with the proof for the case without random endowment in the last subsection § 2.1.2.

1. Obviously, in the case without random endowment, the proof is much simpler, since we do not care where the mass loses by the "optimal" supermartingale deflator, we could avoid settling in the much more complicated $(\mathbf{L}^\infty)^*$ space. Just as showed in [13], the singular part of the dual optimizer defined in $(\mathbf{L}^\infty)^*$ only works on the bounded random endowment e_T. Thus, without random endowment, the dual domain of supermartingale deflators $\widetilde{\mathcal{D}}(y)$ works enough.

2. Without random endowment, we only need the continuity of the stock-price process rather than the assumption on the underlying filtration. We explain as follows. Firstly, the wealth process $\widehat{X}$ in both lines of (2.1.13) is unified, in contrast, for the case with non-trivial e_T, our sequence of fictitious wealth processes depends on n, and it is not easy to find a sequence of stopping times that stop the fictitious wealth processes simultaneously to let all of them stay above 0. Secondly, by the continuity of $(\widehat{H} \bullet S)$, we can easily stop the process by some τ and let it be bounded away from 0, while in the other case, we lack the continuity of $\widehat{W}$.

Chapter 3

Stability of utility maximization problems under transaction costs

In general, there are different sorts of transaction costs in financial markets. We are interested in portfolio optimization problems in markets with proportional transaction costs, for instance, the property of dual optimizer and the stability of the problem.

3.1 Formulation of the financial model

We introduce the market model and the convex duality results obtained in [17]. The financial market consists of only one bond which has constant price 1 and one stock under transaction costs. Fix a finite time horizon $T > 0$. The market involves proportional transaction costs $0 < \lambda < 1$. The process $(S_t, (1-\lambda)S_t)$ models the bid and ask price, in other words, a financial agent has to buy stock shares at a higher ask price S_t, and only receive a lower bid price $(1-\lambda)S_t$ when selling them.

Assumption 3.1.1. *The stock price process $S = (S_t)_{0 \leq t \leq T}$ is strictly positive, càdlàg and adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ satisfying the usual conditions of right continuity and saturatedness. Additionally, we assume $\mathcal{F}_{T-} = \mathcal{F}_T$ and $S_{T-} = S_T$.*

Remark 3.1.2. The additional assumptions $\mathcal{F}_{T-} = \mathcal{F}_T$ and $S_{T-} = S_T$ are to avoid special notation for possible trading at the terminal time T. Without loss of generality, we can assume that the price S cannot jump at the terminal time T, while the investor can still liquidate her position in the stock shares, i.e., we can assume $\varphi_T^1 = 0$. For more details on these assumptions we refer to, e.g., [7, Remark 4.2] or [17, p. 1895].

Definition 3.1.3. *A **trading strategy**, which models the holdings in units of the bond and the stock, is an $\mathbb{R}^2$-valued, predictable, finite variation process $\varphi = (\varphi_t^0, \varphi_t^1)_{0\leq t\leq T}$ such that the following λ-**self-financing** constraint is satisfied:*

$$\int_s^t d\varphi_u^0 \leq -\int_s^t S_u d\varphi_u^{1,\uparrow} + \int_s^t (1-\lambda) S_u d\varphi_u^{1,\downarrow}, \tag{3.1.1}$$

for all $0 \le s < t \le T$, where $\varphi^1 = \varphi^1_0 + \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ denotes Jordan-Hahn decompositions of φ^1 into the difference of two nondecreasing processes both null at zero, and the integrals are defined via

$$\int_s^t S_u d\varphi_u^{1,\uparrow} := \int_s^t S_u d\varphi_u^{1,\uparrow,c} + \sum_{s<u\le t} S_{u-}\Delta\varphi_u^{1,\uparrow} + \sum_{s\le u<t} S_u \Delta_+ \varphi_u^{1,\uparrow},$$
$$\int_s^t S_u d\varphi_u^{1,\downarrow} := \int_s^t S_u d\varphi_u^{1,\downarrow,c} + \sum_{s<u\le t} S_{u-}\Delta\varphi_u^{1,\downarrow} + \sum_{s\le u<t} S_u \Delta_+ \varphi_u^{1,\downarrow}.$$

The self-financing condition (3.1.1) states that purchases and sales of the risky asset are accounted for in the riskless position:

$$\begin{aligned} d\varphi_t^{0,c} &\le -S_t d\varphi_t^{1,\uparrow,c} + (1-\lambda) S_t d\varphi_t^{1,\downarrow,c}, \\ \Delta\varphi_t^0 &\le -S_{t-}\Delta\varphi_t^{1,\uparrow} + (1-\lambda) S_{t-}\Delta\varphi_t^{1,\downarrow}, \\ \Delta_+\varphi_t^0 &\le -S_t\Delta_+\varphi_t^{1,\uparrow} + (1-\lambda) S_t\Delta_+\varphi_t^{1,\downarrow}, \end{aligned}$$

for $0 \le t \le T$.

Remark 3.1.4. Any finite variation process φ is làdlàg. As pointed out in [19], we may assume trading strategies to be càdlàg, if the price process is continuous.

Definition 3.1.5. *Fix the level $0 < \lambda < 1$ of transaction costs. For a trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$, we define the **liquidation value** $V_t^{liq}(\varphi)$ at time $t \in [0,T]$ by*

$$V_t^{liq}(\varphi) := \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- S_t.$$

*If $V_t^{liq}(\varphi) \ge 0$, a.s., for all $0 \le t \le T$, we call the trading strategy **0-admissible**.*

For $x > 0$, we denote by $\mathcal{A}(x)$ the set of all 0-admissible, self-financing trading strategies $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ under transaction costs λ, starting with the initial endowment $(\varphi_0^0, \varphi_0^1) = (x, 0)$. Denote by $\mathcal{C}(x)$ the convex subset of terminal liquidation values

$$\mathcal{C}(x) := \left\{ V_T^{liq}(\varphi) : \ \varphi \in \mathcal{A}(x) \right\} \subseteq L_+^0(\mathbf{P}), \tag{3.1.2}$$

which equals the set $\{\varphi_T^0 : \ \varphi = (\varphi^0, \varphi^1) \in \mathcal{A}(x), \ \varphi_T^1 = 0\}$ as defined in [17].

We accept the same utility function from Assumption 2.1.6 to model the agent's preferences. Given the initial endowment $x > 0$, the agent wants to maximize his expected utility at the terminal time T, i.e.,

$$\mathbb{E}[U(g)] \to \max!, \qquad g \in \mathcal{C}(x). \tag{3.1.3}$$

In order to avoid the trivial case, we have the following assumption.

Assumption 3.1.6. *Suppose that*

$$\sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)] < \infty,$$

for some $x > 0$.

Similarly to the assumption of the existence of equivalent local martingale measures or some similar weaker conditions in the frictionless setting, the duality theorem for the utility maximization in [17] requires the existence of consistent price systems, which are defined as follows.

Definition 3.1.7. *Fix* $0 < \lambda < 1$. *A* ***λ-consistent price system (CPS$^\lambda$)*** *is a two dimensional strictly positive process* $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ *with* $Z_0^0 = 1$, *that consists of a martingale* Z^0 *and a local martingale* Z^1 *under* $\mathbf{P}$ *such that*

$$\widetilde{S}_t := \frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \qquad 0 \le t \le T, \quad a.s.$$

We denote by $\mathcal{Z}^\lambda(S)$ *the set of* λ*-consistent price systems and we say that* S *satisfies the condition* (CPS^λ) *of admitting a* λ*-consistent price system, if* $\mathcal{Z}^\lambda(S)$ *is nonempty.*

Sometime, we define the λ-consistent price system by using the other notation.

Definition 3.1.8. *Fix* $0 \le \lambda < 1$. *A process* $S = (S_t)_{0 \le t \le T}$ *satisfies the condition* (CPS^λ) *of having a consistent price system under transaction costs* λ*, if there is a process* $\widetilde{S} = (\widetilde{S}_t)_{0 \le t \le T}$ *such that*

$$(1-\lambda)S_t \le \widetilde{S}_t \le S_t, \quad 0 \le t \le T,$$

as well as a probability measure $\mathbf{Q}$ *on* $\mathcal{F}$ *equivalent to* $\mathbf{P}$, *such that* $(\widetilde{S}_t)_{0 \le t \le T}$ *is a local martingale under* $\mathbf{Q}$.

We say that S *admits consistent price systems for arbitrarily small transaction costs if* (CPS^λ) *is satisfied, for all* $0 < \lambda < 1$.

Remark 3.1.9. In the above definitions, Z^0 defines a density process of an equivalent local martingale measure $\mathbf{Q} \sim \mathbf{P}$ for a price process $\widetilde{S}$ evolving in the bid-ask spread $[(1-\lambda)S, S]$, and $Z^1 = Z^0 \widetilde{S}$.

Definition 3.1.10. *We say that* S *satisfies the condition* (CPS^λ) locally*, if there exists a strictly positive process* Z *and a sequence* $(\tau_n)_{n=1}^\infty$ *of* $[0, T] \cup \{\infty\}$*-valued stopping times, increasing to infinity, such that each stopped process* Z^{τ_n} *defines a* λ*-consistent price system for the stopped process* S^{τ_n}. *Denote* $\mathcal{Z}^{loc,\lambda}$ *the set of all such processes* Z*, which are called* ***local λ-consistent price systems (LCPS$^\lambda$)****.*

Assumption 3.1.11. *The stock price process* S *satisfies* (CPS^μ) *locally, for all* $0 < \mu < \lambda$.

We consider all optional strong supermartingale deflators as dual variables. The set of such processes is defined in the following.

Definition 3.1.12. *We denote by $\mathcal{B}(y)$ the set of all* ***optional strong supermartingale deflators (OSSD)****, which are pairs of nonnegative optional strong supermartingales $Y = (Y_t^0, Y_t^1)_{0\le t\le T}$ such that $Y_0^0 = y$ and*

$$\widetilde{S}_t := \frac{Y_t^1}{Y_t^0} \in [(1-\lambda)S_t, S_t],$$

for all $0 \le t \le T$, and such that $Y^0(\varphi^0 + \varphi^1\widetilde{S}) = Y^0\varphi^0 + Y^1\varphi^1$ is a nonnegative optional strong supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}(1)$. Accordingly, define

$$\mathcal{D}(y) = \Big\{ Y_T^0 : \, (Y^0, Y^1) \in \mathcal{B}(y) \Big\}, \quad \textit{for } y > 0.$$

Remark 3.1.13. In the frictionless case, we do not need to pass to the optional version of supermartingale deflator, since the wealth process associated with a trading strategy is always càdlàg as a stochastic integral. However, in this new context, we do not have this property.

The polar properties of the sets $\mathcal{C} := \mathcal{C}(1)$ and $\mathcal{D} := \mathcal{D}(1) \subseteq \mathbf{L}_+^0(\mathbf{P})$ are described in [17, Lemma A.1], which means that $\mathcal{C}$ and $\mathcal{D}$ in this new context satisfy verbatim the conditions listed in [63, Proposition 3.1]. As a result, the following duality result is obtained in [17] by following the lines of [63].

Theorem 3.1.14 (Duality Theorem, [17, Theorem 3.2]). *Under Assumptions 2.1.6, 3.1.1, 3.1.6, 3.1.11, define the primal and dual value functions as*

$$\begin{aligned} u(x) &:= \sup_{g\in\mathcal{C}(x)} \mathbb{E}[U(g)], \\ v(y) &:= \inf_{h\in\mathcal{D}(y)} \mathbb{E}[V(h)], \end{aligned} \tag{3.1.4}$$

where

$$V(y) := \sup_{x>0}\{U(x) - xy\}, \quad y > 0,$$

is the conjugate function of U.

Then, the following statements hold true.

(i) The functions $u(x)$ and $v(y)$ are finitely valued, for all x, $y > 0$, and mutually conjugate

$$v(y) = \sup_{x>0}[u(x) - xy], \quad u(x) = \inf_{y>0}[v(y) + xy].$$

The functions u and v are continuously differentiable and strictly concave (respectively, convex) and satisfy

$$u'(0) = -v'(0) = \infty, \qquad u'(\infty) = v'(\infty) = 0.$$

(ii) For all x, $y > 0$, the solutions $\widehat{g}(x) \in \mathcal{C}(x)$ and $\widehat{h}(y) \in \mathcal{D}(y)$ exist, are unique and take their values a.s. in $(0,\infty)$. There are $\big(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)\big) \in \mathcal{A}(x)$ and $\big(\widehat{Y}^0(y), \widehat{Y}^1(y)\big) \in$

$\mathcal{B}(y)$ *such that*

$$V_T^{liq}\big(\widehat{\varphi}(x)\big) = \widehat{\varphi}_T^0 = \widehat{g}(x) \quad and \quad \widehat{Y}_T^0(y) = \widehat{h}(y). \tag{3.1.5}$$

(iii) If $x > 0$ *and* $y > 0$ *are related by* $u'(x) = y$*, or equivalently* $x = -v'(y)$*, then* $\widehat{g}(x)$ *and* $\widehat{h}(y)$ *are related by the first order conditions*

$$\widehat{h}(y) = U'\big(\widehat{g}(x)\big) \quad and \quad \widehat{g}(x) = -V'\big(\widehat{h}(y)\big),$$

and we have

$$\mathbb{E}\big[\widehat{g}(x)\widehat{h}(y)\big] = xy. \tag{3.1.6}$$

In particular, $\widehat{\varphi}^0(x)\widehat{Y}^0(y) + \widehat{\varphi}^1(x)\widehat{Y}^1(y)$ *is a* $\mathbf{P}$*-martingale for all* $\big(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)\big) \in \mathcal{A}(x)$ *and* $\big(\widehat{Y}^0(y), \widehat{Y}^1(y)\big) \in \mathcal{B}(y)$ *satisfying* (3.1.5).

(iv) Finally, we have

$$v(y) = \inf_{(Z^0, Z^1) \in \mathcal{Z}^{loc,\lambda}} \mathbb{E}\left[V(yZ_T^0)\right].$$

Remark 3.1.15. Similar to [63, Proposition 3.2] (see also [17, Appendix A]), the infimum of $v(y)$ defined in (3.1.4) could be approximated by the element from $\{Z_T^0 : Z \in \mathcal{Z}^{loc,\lambda}\}$, i.e.,

$$v(y) := \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)] = \inf_{Z \in \mathcal{Z}^{loc,\lambda}} \mathbb{E}[V(yZ_T^0)]. \tag{3.1.7}$$

We clarify here notation for our argument of stability in the following sections. For the utility maximization problem in the market under the physical probability $\mathbf{P}$, we employ

$$\mathcal{B}(y) = \mathcal{B}(y, \mathbf{P}),\ \mathcal{D}(y) = \mathcal{D}(y, \mathbf{P}),\ \mathcal{Z}^\lambda = \mathcal{Z}^\lambda(\mathbf{P}),\ \mathcal{Z}^{loc,\lambda} = \mathcal{Z}^{loc,\lambda}(\mathbf{P}).$$

The market parameterized by $(x, U; \mathbf{P})$ is treated as the original market.

3.2 Static stability

Given the inputs: a physical probability measure $\mathbf{P}$, a utility function U, an initial endowment x, then under the assumptions of Theorem 3.1.14, we obtained the optimal value $u(x)$ as well as the existence and uniqueness of the primal optimizer $\widehat{\varphi}_T^0(x)$. In this section, we shall consider the stability of the outcomes from the previous result, i.e., how does the model perturbation impact the solution of the utility maximization problem. Precisely, this gives rise to the following mathematical formulation of the problem: suppose that the utility maximization problems are associated with a sequence of physical probability measures $\{\mathbf{P}_n\}_{n=1}^\infty$, of initial endowments $\{x_n\}_{n=1}^\infty$ and of utility functions $\{U_n\}_{n=1}^\infty$, when $x_n \to x$, $U_n \to U$, $\mathbf{P}_n \sim \mathbf{P}$ and $\mathbf{P}_n \to \mathbf{P}$, shall we have

- the convergence of value functions, i.e., $u(x_n; U_n, \mathbf{P}_n) \longrightarrow u(x; U, \mathbf{P})$;

- the convergence of primal optimizers, i.e., $\widehat{\varphi}^0_T(x_n, U_n, \mathbf{P}_n) \longrightarrow \widehat{\varphi}^0_T(x; U, \mathbf{P})$?

Now let make the notation clear, set

$$u_n = u(x_n; U_n, \mathbf{P}_n),\ u = u(x; U, \mathbf{P}),\ v_n = v(y_n; V_n, \mathbf{P}_n),\ v = v(y; V, \mathbf{P}),$$

for each $n \in \mathbb{N}$, u_n and v_n have the same property as u and v described in Theorem 3.1.14. And let $\frac{\partial v_n}{\partial y}$, $\frac{\partial v}{\partial y}$ be the corresponding derivatives with respect to the first variable. Similarly, note

$$(\widehat{\varphi}^{0,n}, \widehat{\varphi}^{1,n}) = (\widehat{\varphi}^0(x_n; U_n, \mathbf{P}_n), \widehat{\varphi}^1(x_n; U_n, \mathbf{P}_n)),\ (\widehat{\varphi}^0, \widehat{\varphi}^1) = (\widehat{\varphi}^0(x; U, \mathbf{P}), \widehat{\varphi}^1(x; U, \mathbf{P})),$$
$$(\widehat{Y}^{0,n}, \widehat{Y}^{1,n}) = (\widehat{Y}^0(y_n; V_n, \mathbf{P}_n), \widehat{Y}^1(y_n; V_n, \mathbf{P}_n)),\ (\widehat{Y}^0, \widehat{Y}^1) = (\widehat{Y}^0(y; V, \mathbf{P}), \widehat{Y}^1(y; V, \mathbf{P})).$$

Before presenting our results, we should first precise the assumptions, which are in accordance with the ones in [62] for considering the frictionless case with random endowments.

Assumption 3.2.1. *For each $n \in \mathbb{N}$, $\mathbf{P}_n \sim \mathbf{P}$ and $\lim_{n\to\infty} \mathbf{P}_n = \mathbf{P}$ in total variation, $\lim_{n\to\infty} U_n = U$ pointwise and $x_n \to x > 0,\ y_n \to y > 0$.*

Remark 3.2.2. (1) Since $\mathbf{P}_n \sim \mathbf{P}$, the Radon-Nikodým derivatives exist, denoted by $\widetilde{Z}_n := \frac{d\mathbf{P}_n}{d\mathbf{P}}$. It is obvious that $\mathcal{Z}^{loc,\lambda}(\mathbf{P}) \neq \emptyset$ implies $\mathcal{Z}^{loc,\lambda}(\mathbf{P}_n) \neq \emptyset$, for each $n \in \mathbb{N}$.

(2) That $\lim_{n\to\infty} U_n = U$ pointwise implies the pointwise convergence of the sequence of their Legendre-Fenchel transforms, i.e., $\lim_{n\to\infty} V_n = V$ pointwise (see e.g. [79] for the general analysis or in the proof of Theorem 2.1 (p. 246) in [64]).

(3) The sequences $\{U_n\}_{n\in\mathbb{N}}, \{V_n\}_{n\in\mathbb{N}}$ as well as their derivatives $\{U'_n\}_{n\in\mathbb{N}}, \{V'_n\}_{n\in\mathbb{N}}$ converge *uniformly* on compact subsets of $(0, \infty)$ to their respective limits $U,\ V,\ U'$ and V' (see e.g. [78] for a general statement or [65, Proposition 3.9]).

In the frictionless case, Larsen mentioned in [64, Section 2.6] that the pointwise convergence of $\{U_n\}_{n\in\mathbb{N}}$ is not sufficient to prove the upper semicontinuity of the dual value function v, and more structure conditions on the converging sequence of $\{U_n\}_{n\in\mathbb{N}}$ should be imposed. In the present section, we introduce the following assumption, whose analogue in the frictionless case could be found in [62, condition (UI) and §2.2.3].

Assumption 3.2.3. *Define*

$$\mathcal{Z}^{loc}_T := \{Z^0_T : (Z^0, Z^1) \in \mathcal{Z}^{loc,\lambda}\}.$$

There exists $Z^0_T \in \mathcal{Z}^{loc}_T$, for all $y > 0$, $\left(\widetilde{Z}_n V^+_n\left(y\frac{Z^0_T}{\widetilde{Z}_n}\right)\right)_{n\in\mathbb{N}}$ is $\mathbf{P}$-uniformly integrable.

Theorem 3.2.4. *Under the assumptions of Duality Theorem (Theorem 3.1.14) and Assumptions 3.2.1, 3.2.3, we have the following limiting relationships for value functions and optimal solutions:*

$$\begin{aligned}
\lim_{n\to\infty} u_n = u, &\quad \lim_{n\to\infty} v_n = v;\\
\lim_{n\to\infty} \frac{\partial u_n}{\partial x} = \frac{\partial u}{\partial x}, &\quad \lim_{n\to\infty} \frac{\partial v_n}{\partial y} = \frac{\partial v}{\partial y};\\
\mathbf{L}^0(\mathbf{P}) - \lim_{n\to\infty} \widehat{\varphi}_T^{0,n} = \widehat{\varphi}_T^0, &\quad \mathbf{L}^0(\mathbf{P}) - \lim_{n\to\infty} \widehat{Y}_T^{0,n} = \widehat{Y}_T^0.
\end{aligned}$$

To prove this theorem, we first need to reformulate the dual problem v_n with the help of the following proposition.

Proposition 3.2.5. *Denote $c_n := \frac{y}{y_n}$. If $\widetilde{h} \in \mathcal{D}(y, \mathbf{P}) := \mathcal{D}(y)$, then for each $n \in \mathbb{N}$, $\frac{\widetilde{h}}{c_n \widetilde{Z}_n} \in \mathcal{D}(y_n, \mathbf{P}_n)$. Conversely, if $h \in \mathcal{D}(y_n, \mathbf{P}_n)$, then $c_n h \widetilde{Z}_n \in \mathcal{D}(y, \mathbf{P})$.*

Proof. Without loss of generality, let $h \in \mathcal{D}(1, \mathbf{P}_n)$. By definition of $\mathcal{D}(1, \mathbf{P}_n)$, there is a nonnegative $\mathbf{P}_n$-optional strong supermartingale (Y^0, Y^1) with $Y_0^0 = 1$, $Y_T^0 = h$, satisfying that $\frac{Y^1}{Y^0} \in [(1-\lambda)S, S]$, and $\varphi^0 Y^0 + \varphi^1 Y^1$ is a nonnegative $\mathbf{P}_n$-optional strong supermartingale for all $\varphi \in \mathcal{A}(1)$. Define

$$\widetilde{Z}_t^n := \mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{P}_n}{d\mathbf{P}}\middle| \mathcal{F}_t\right],$$

which is a $\mathbf{P}$-martingale. Indeed, $\widetilde{Z}_n = \widetilde{Z}_T^n = \frac{d\mathbf{P}_n}{d\mathbf{P}}$.

It suffices to show the supermartingale property of the process $\varphi^0 Y^0 \widetilde{Z}^n + \varphi^1 Y^1 \widetilde{Z}^n$ under $\mathbf{P}$, for all $\varphi \in \mathcal{A}(1)$. For $0 \le s < t \le T$ and $\varphi = (\varphi_t^0, \varphi_t^1,)_{0\le t\le T} \in \mathcal{A}(1)$, by Bayes' formula, we obtain

$$(\varphi_s^0 Y_s^0 + \varphi_s^1 Y_s^1)\widetilde{Z}_s^n \ge \mathbb{E}^{\mathbf{P}}\left[(\varphi_t^0 Y_t^0 + \varphi_t^1 Y_t^1)\widetilde{Z}_t^n \middle| \mathcal{F}_s\right].$$

∎

Therefore, the dual problem $v(y_n; V_n, \mathbf{P}_n)$ corresponding to the utility maximization problem parameterized by $(x_n; U_n, \mathbf{P}_n)$ can be reformulated with the parameters $(x; U_n, \mathbf{P})$. Precisely,

$$\begin{aligned}
v(y_n; V_n, \mathbf{P}_n) &= \inf_{h\in\mathcal{D}(y_n,\mathbf{P}_n)} \mathbb{E}^{\mathbf{P}_n}\left[V_n(h)\right] = \mathbb{E}^{\mathbf{P}_n}\left[V_n(\widehat{h}(y_n, \mathbf{P}_n))\right] &(3.2.1)\\
&= \inf_{\overline{h}\in\mathcal{D}(y,\mathbf{P})} \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n\left(\frac{\overline{h}}{\overline{Z}_n}\right)\right] = \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n\left(\frac{\widehat{h}_n(y, \mathbf{P})}{\overline{Z}_n}\right)\right], &(3.2.2)
\end{aligned}$$

where the ratio $c_n = \frac{y}{y_n} > 0$ converges to 1, and $\overline{Z}_n := \widetilde{Z}_n c_n$ converges to 1 in $\mathbf{L}^1(\mathbf{P})$ and also in $L^0(\mathbf{P})$, the dual optimizer $\widehat{h}_n(y; \mathbf{P}) \in \mathcal{D}(y, \mathbf{P})$ and $\widehat{h}(y_n; \mathbf{P}_n) \in \mathcal{D}(y_n, \mathbf{P}_n)$

attain the infimum of $v(y_n; V_n, \mathbf{P}_n)$ corresponding to the utility maximization problem parameterized by $(x_n; U_n, \mathbf{P}_n)$.

3.3 Proof of static stability

The proof of limiting relationships of static stability in our case with transaction costs, which is very similar as in [62] for the case in incomplete markets by setting the random endowments $q = 0$, will be separated into the following steps:

(1) The dual value function v is l.s.c., which does not need Assumption 3.2.3. Indeed, after we have the observation (3.2.2) that

$$v(y_n; V_n, \mathbf{P}_n) = \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n\left(\frac{\widehat{h}_n(y, \mathbf{P})}{\overline{Z}_n}\right)\right],$$

we can carry over verbatim the whole proof from [62, Section 3.2], because the proof only depends on the sequence of $\{V_n\}_{n\in\mathbb{N}}$.

(2) The dual value function v is also u.s.c. For any $Z_T^0 \in \mathcal{Z}_T^{loc}, y > 0$, we define

$$\widetilde{\mathcal{D}}(y, Z_T^0) := \left\{h \in \mathcal{D}(y, \mathbf{P}) : \frac{Z_T^0}{h} \in \mathbf{L}^\infty\right\}.$$

In [62, Section 3.3], two important lemmas are applied to show the upper semicontinuity property of v in frictionless markets. Passing to the transaction costs case, we have the analogous lemmas as [62, Lemma 3.3 and 3.4] which can be proved in an analogous way. Then by the same proof in [62, Section 3.3.2], we finish it.

(3) The continuity of the primal value function u comes from the continuity of the dual value function v.

(4) Finally, we prove $\mathbf{L}^0$ convergence of the dual optimizer $\widehat{Y}_T^0$. We can also follow the proof in [62, Section 3.5], because only the convexity of the function V, V_n and a more elaborate version of the method used in the proof of [22, Lemma A1.1] plays the key role.

In the following, we will provide the proof in details for the case of transaction costs for the convenience of the reader (compare with [62, §3]). The preparatory work described in the following subsection §3.3.1 is no difference from [62, §3.2.1].

3.3.1 Preparatory work

Without loss of generality, we assume that each of the utility functions has $U_n(1) = 0$ and $U_n'(1) = 1$. This means $V_n(1) = -1 = V_n'(1)$. To proceed with the proof, we define some helpful functions.

(I) For each $n \in \mathbb{N}$ and $\epsilon \in (0,1)$, define the function as follows: set $V_n^\epsilon(x) = V_n(x)$ for $x \geq \epsilon$, and extend V_n^ϵ to $[0,\epsilon)$ in affine and continuously differentiable way, that is match the 0th and first derivatives at $x = \epsilon$. In the similar way, we define V^ϵ by using the function V for all $x \in (0,1)$. Then we have

1. the function V_n^ϵ is unique defined, convex, decreasing.
2. For all $x > 0$, $V_n^\epsilon(x) \leq V_n(x)$. At the point $x = \epsilon$, $V_n^\epsilon(x) = V_n(x)$ and $(V_n^\epsilon)'(x) = (V_n)'(x)$.
3. $\lim_{\epsilon \downarrow 0} \downarrow V_n^\epsilon(x) = V_n(x), \ \forall \ x > 0$.
4. Remark 3.2.2 (3) shows

$$\lim_{n\to\infty} V_n(x) = V(x) \text{ and } \lim_{n\to\infty} V_n'(x) = V'(x) \text{ uniformly for } x \in [\epsilon, 1],$$

so we have

$$\lim_{n\to\infty} V_n^\epsilon(x) = V^\epsilon(x) \text{ uniformly for } x \in [0,1].$$

Notice the uniform convergence fails in general for $\epsilon = 0$, unless $(U_n)_{n\in\mathbb{N}}$ is uniformly bounded from above. It follows that

$$\forall \ \epsilon \in (0,1), \ \exists \ n_1(\epsilon) \in \mathbb{N}, \ \forall \ x \in [0,1], \ \forall \ n \geq n_1(\epsilon), \ V_n^\epsilon(x) \geq V^\epsilon(x) - \epsilon. \quad (3.3.1)$$

(II) Define $\widetilde{V}_n$ the *convex minor* of the family $\{V_n, V_{n+1}, \cdots\}$, in other words, $\widetilde{V}_n$ is the largest convex function that is dominated by all V_k for $k \geq n$, so $\widetilde{V}_n \leq \inf_{k\geq n} V_k$.

(1) Since V is the (pointwise) limit of the sequence V_n, then $\widetilde{V}_n(x) \leq V(x), \ \forall \ x > 0$.

(2) Each $\widetilde{V}_n$ is convex and decreasing. But the sequence is increasing, $\widetilde{V}_n(x) \leq \widetilde{V}_{n+1}(x)$, for a fixed x.

(3) Recall $V_n(1) = V_n'(1) = -1$, thus $V_n(x) \geq -x, \forall \ n \in \mathbb{N}$. Then $-x \leq \widetilde{V}_n(x) \leq V(x)$.

(4) $\widetilde{V}_n$ is the largest convex function that is dominated by all V_k for $k \geq n$, while $-1 + \int_1^x (\inf_{k\geq n} V_k'(\mu))\, d\mu$ is convex and dominated by all $V_k, \ \forall \ k \geq n$. Thus

$$-1 + \int_1^x \left(\inf_{k\geq n} V_k'(\mu) \right) d\mu \leq \widetilde{V}_n(x). \quad (3.3.2)$$

(5) We can present the following lemma.

Lemma 3.3.1. $\lim_{n\to\infty} \uparrow \widetilde{V}_n = V$ *uniformly on compact subsets of* $(0,\infty)$.

Proof. As $\lim_{n\to\infty} V_n' = V'$ uniformly on compact subsets of $(0,\infty)$, we have $\forall \ \varepsilon > 0, \ \exists N(\varepsilon), \ \forall \ n \geq N(\varepsilon), \ \forall \ \mu \in [1,x], |V_n'(\mu) - V'(\mu)| \leq \varepsilon$, i.e.,

$$V'(\mu) - \varepsilon \leq V_n'(\mu) \leq \varepsilon + V'(\mu).$$

Fix an n, for all $k \geq n$, we have $V'(\mu) - \varepsilon \leq V_k'(\mu)$, which shows

$$V'(\mu) - \varepsilon \leq \inf_{k \geq n} V_k'(\mu).$$

Therefore, $\forall\, \varepsilon > 0,\ \exists N(\varepsilon),\ \forall\, n \geq N(\varepsilon)$,

$$-1 + \int_1^x \left(\inf_{k \geq n} V_k'(\mu) \right) d\mu \geq -1 + \int_1^x \left(V'(\mu) - \varepsilon \right) d\mu = V(x) - \varepsilon(x-1).$$

This means $V(x) - \varepsilon(x-1) \leq \widetilde{V}_n(x) \leq V(x)$. Letting $n \longrightarrow \infty, \widetilde{V}_n(x) \nearrow V(x)$ pointwisely. By Dini's theorem, $\widetilde{V}_n(x) \nearrow V(x)$ uniformly on compact subsets of $(0, \infty)$. □

(6) $\widetilde{V}_n(1) = \widetilde{V}_n'(1) = -1$, because of the normalization $V_n(1) = V_n'(1) = -1$.

(III) Define the "average" functions $\overline{V}_n(x) := \frac{\widetilde{V}_n(x)}{x},\ x > 0$ for all $n \in \mathbb{N}$, as well as $\overline{V}(x) := \frac{V(x)}{x},\ x > 0$.

(a) For $x > 0$, we have $\overline{V}_n(x) \leq \overline{V}_{n+1}(x)$, since $\widetilde{V}_n(x) \leq \widetilde{V}_{n+1}(x)$.

(b) Following (II)(6) that $\widetilde{V}_n(1) = \widetilde{V}_n'(1) = -1$, and $V(1) = V'(1) = -1$, we see that $\overline{V},\ \overline{V}_n$ are increasing for $x \in [1, \infty)$.

(c) $\lim_{n\to\infty} \uparrow \overline{V}_n = \overline{V}$.

(d) The following stronger statement holds true.

Lemma 3.3.2 ([62, Lemma 3.1])**.** $\lim_{n\to\infty} \overline{V}_n = \overline{V}$ *uniformly on* $[1, \infty)$.

Proof. Define $\overline{V}_n(\infty) := \lim_{x\to\infty} \overline{V}_n(x)$, and $\overline{V}(\infty) := \lim_{x\to\infty} \overline{V}(x)$. Since

$$\lim_{x\to\infty} \overline{V}(x) = \lim_{x\to\infty} V'(x) = 0,$$

we have $\overline{V}(\infty) = 0$. Then we can pick $M > 1$ such that $\overline{V}(M) > -\frac{\varepsilon}{2}$ for arbitrary $\varepsilon > 0$. As $\overline{V_n} \nearrow \overline{V}$, there is an $n_2(\varepsilon, M) \in \mathbb{N}$ such that

$$\overline{V}_n(M) > \overline{V}(M) - \frac{\varepsilon}{2} > -\varepsilon,\ \forall\, n \geq n_2(\varepsilon, M).$$

Since $\overline{V}_n(x)$ is increasing for all $x \in [1, \infty)$, $\overline{V}_n(\infty) > \overline{V}_n(M) > -\varepsilon,\ \forall\, n \geq n_2(\varepsilon, M)$. Moreover, we have $\overline{V}_n(\infty) \leq 0$, as $\overline{V}_n(x) \leq 0, \forall x > 0$. Hence,

$$-\varepsilon < \overline{V}_n(\infty) \leq 0,$$

which implies

$$\lim_{n\to\infty} \overline{V}_n(\infty) = \overline{V}(\infty) = 0.$$

By Dini's theorem,

$$\lim_{n\to\infty} \overline{V}_n = \overline{V} \text{ uniformly on } [1, \infty).$$

$\square$

(IV) With the same method, we can also define $\overline{V}^\epsilon_n(x) := \frac{\widetilde{V}^\epsilon_n(x)}{x}$, $x > 0$ for all $n \in \mathbb{N}$ as well as $\overline{V}^\epsilon(x) := \frac{V^\epsilon(x)}{x}$, $x > 0$, where $\widetilde{V}^\epsilon_n(x)$ is the convex minor of the family $\{V^\epsilon_n, V^\epsilon_{n+1}, \cdots\}$.

a. Following Lemma 3.3.2, $\lim_{n\to\infty} \overline{V}^\epsilon_n = \overline{V}^\epsilon$ uniformly on $[1, \infty)$, which implies

$$\exists\ n_3(\epsilon) \in \mathbb{N}, \quad \text{s.t.} \quad \forall\ x \geq 1,\ n \geq n_3(\epsilon),\ \overline{V}^\epsilon_n(x) \geq \overline{V}^\epsilon(x) - \epsilon. \tag{3.3.3}$$

b. Now we can present the following lemma, which is the same as [62, Lemma 3.2].

Lemma 3.3.3 ([62, Lemma 3.2])**.** *The mapping $(z, y) \longmapsto zV^\epsilon(y/z)$ is convex in $(z, y) \in (0, \infty)^2$. Furthermore, for each $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that for all $n \geq n_0(\epsilon)$, we have*

$$zV^\epsilon_n(y/z) \geq zV^\epsilon(y/z) - \epsilon(y + z) \tag{3.3.4}$$

for all pair $(z, y) \in (0, \infty)^2$.

Proof. Since the function V^ϵ is convex, by [47, p. 90], we have the convexity of the mapping $(z, y) \longmapsto zV^\epsilon(y/z)$.

For the second statement, choose $n_0(\epsilon) := n_1(\epsilon) \vee n_3(\epsilon)$, where $n_1(\epsilon)$ is from (3.3.1) and $n_3(\epsilon)$ is from (3.3.3). For all $n \geq n_0(\epsilon)$, under the help of (3.3.1) that $V^\epsilon_n(x) \geq V^\epsilon(x) - \epsilon, \forall x \in [0, 1]$ and (3.3.3) that $\frac{\widetilde{V}^\epsilon_n(x)}{x} \geq \frac{V^\epsilon(x)}{x} - \epsilon, \forall x \geq 1$ as well as the fact that $V^\epsilon_n(x) \geq \widetilde{V}^\epsilon_n(x)$, we have

$$\begin{aligned}
zV^\epsilon_n(y/z) &= zV^\epsilon_n(y/z)\mathbf{1}_{\{y/z\in[0,1]\}} + zV^\epsilon_n(y/z)\mathbf{1}_{\{y/z\in(1,\infty)\}} \\
&\geq z(V^\epsilon(y/z) - \epsilon)\mathbf{1}_{\{y/z\in[0,1]\}} + y(\overline{V}^\epsilon(y/z) - \epsilon)\mathbf{1}_{\{y/z\in(1,\infty)\}} \\
&\geq zV^\epsilon(y/z)\mathbf{1}_{\{y/z\in[0,1]\}} + zV^\epsilon(y/z)\mathbf{1}_{\{y/z\in(1,\infty)\}} - \epsilon(z + y) \\
&= zV^\epsilon(y/z) - \epsilon(z + y).
\end{aligned}$$

$\square$

c. $\overline{V}^\epsilon$ is increasing on $[1, \infty)$ and satisfies $\overline{V}^\epsilon(1) = -1$ and $\overline{V}^\epsilon(\infty) = 0$.

(V) Choosing $M > 1$, such that $\overline{V}^\epsilon(M) = -\epsilon$, then construct a function by defining

$$\overline{V}^{\epsilon,M}(x) := \begin{cases} \overline{V}^\epsilon(x) & \text{for } x \in (0, M]; \\ -\epsilon & \text{for } x = M; \\ 0 & \text{for } x \geq M + 1; \end{cases}$$

and interpolating in a continuous way between M and $M+1$ such that $\overline{V}^\epsilon \leq \overline{V}^{\epsilon,M}$. It satisfies

$$-1 - \epsilon \leq \overline{V}^{\epsilon,M} - \epsilon \leq \overline{V}^\epsilon \leq \overline{V}^{\epsilon,M}.$$

3.3.2 Lower semicontinuity of the dual value function

We contribute to show $v(y) \leq \liminf_{n\to\infty} v(y_n; V_n, \mathbf{P}_n)$. If necessary, we will pass to an attaining subsequence so that the liminf will be regarded as an actual limit. Recall the observation (3.2.2) that

$$v(y_n; V_n, \mathbf{P}_n) = \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n\left(\frac{\widehat{h}_n(y, \mathbf{P})}{\overline{Z}_n}\right)\right],$$

where $c_n = \frac{y}{y_n} > 0$ converges to 1, and $\overline{Z}_n := \widetilde{Z}_n c_n$ converges to 1 in $\mathbf{L}^1(\mathbf{P})$ and in $\mathbf{L}^0(\mathbf{P})$, the dual optimizer $\widehat{h}_n := \widehat{h}_n(y; \mathbf{P}) \in \mathcal{D}(y, \mathbf{P})$ attains the infimum of $v(y_n; V_n, \mathbf{P}_n)$ corresponding to the utility maximization problem parameterized by $(x_n; U_n, \mathbf{P}_n)$. Then,

$$v(y_n; V_n, \mathbf{P}_n) = \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n(\frac{\widehat{h}_n}{\overline{Z}_n})\right] \geq \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right].$$

Fix $\epsilon \in (0, 1)$, according to (3.3.4) in Lemma 3.3.3, for each $\epsilon > 0$, $\exists\, n_0(\epsilon) \in \mathbb{N}$, $\forall\, n \geq n_0(\epsilon)$, we have

$$\overline{Z}_n V_n^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n}) \geq \overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n}) - \epsilon(\widehat{h}_n + \overline{Z}_n).$$

Taking expectation w.r.t. $\mathbf{P}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] &\geq \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - \epsilon\mathbb{E}^{\mathbf{P}}\left[\widehat{h}_n + \overline{Z}_n\right] \\
&\geq \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - c_n\epsilon(y+1).
\end{aligned}$$

Taking limits, passing to a subsequence so that the $\lim_{n\to\infty} v_n(y_n; V_n, \mathbf{P}_n)$ exists, then

$$\begin{aligned}
\lim_{n\to\infty} v_n(y_n; V_n, \mathbf{P}_n) &\geq \limsup_{n\to\infty} \frac{1}{c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] \\
&\geq \limsup_{n\to\infty} \frac{1}{c_n}\left(\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - c_n\epsilon(y+1)\right) \\
&= \limsup_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - \epsilon(y+1). \qquad (3.3.5)
\end{aligned}$$

Now we need to find an intermediate term such that the connection to compare $\lim_{n\to\infty} v_n(y_n; V_n, \mathbf{P}_n)$ and $\mathbb{E}\left[V(h)\right]$ with $h \in \mathcal{D}(y, \mathbf{P})$ can be built.

Define

$$\zeta_n := \sum_{k=n}^{m_n} \alpha_k^n \overline{Z}_k, \text{ and } h_n := \sum_{k=n}^{m_n} \alpha_k^n \widehat{h}_k,$$

where $m_n \geq n,\ 0 \leq \alpha_k^n \leq 1$ s.t. $\sum_{k=n}^{m_n} \alpha_k^n = 1$. It is easy to see ζ_n converges to 1 almost surely, and

$$h_n \in \operatorname{conv}\{\widehat{h}_n, \widehat{h}_{n+1}, \dots\} = \operatorname{conv}\{\widehat{h}_n(y, V_n, \mathbf{P}), \widehat{h}_{n+1}(y, V_{n+1}, \mathbf{P}), \dots\}.$$

Moreover, the set $\mathcal{D}(1, \mathbf{P})$ is convex and closed w.r.t. the topology of convergence in measure, so

$$\forall\, n \in \mathbb{N},\ \lim_{n\to\infty} h_n = h, \text{ a.s., and } h \in \mathcal{D}(y, \mathbf{P}).$$

The convexity of the mapping $(\overline{Z}, y) \longmapsto \overline{Z} V^\epsilon(y/\overline{Z})$ implies

$$\zeta_n V^\epsilon(h_n/\zeta_n) \leq \sum_{k=n}^{m_n} \alpha_k^n \overline{Z}_k V^\epsilon(\widehat{h}_k/\overline{Z}_k).$$

Taking expectation and the limsup, we have

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\zeta_n V^\epsilon(h_n/\zeta_n)\right] &\leq \limsup_{n\to\infty} \sum_{k=n}^{m_n} \alpha_k^n \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_k V^\epsilon(\widehat{h}_k/\overline{Z}_k)\right] \\
&\leq \limsup_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^\epsilon(\widehat{h}_n/\overline{Z}_n)\right]. \qquad (3.3.6)
\end{aligned}$$

Combining the inequalities (3.3.5) and (3.3.6), we have

$$\begin{aligned}
\lim_{n\to\infty} v_n(y_n; V_n, \mathbf{P}_n) &\geq \limsup_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^\epsilon(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - \epsilon(y+1) \\
&\geq \limsup_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[\zeta_n V^\epsilon(h_n/\zeta_n)\right] - \epsilon(y+1).
\end{aligned}$$

Since $\overline{V}^{\epsilon,M} - \epsilon \leq \overline{V}^\epsilon$, by the definition of V^ϵ,

$$\begin{aligned}
\zeta_n V^\epsilon(h_n/\zeta_n) &= \zeta_n \overline{V}^\epsilon(h_n/\zeta_n) h_n/\zeta_n = \overline{V}^\epsilon(h_n/\zeta_n) h_n \\
&\geq h_n \left(\overline{V}^{\epsilon,M}(h_n/\zeta_n) - \epsilon\right),
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}^{\mathbf{P}}\left[\zeta_n V^\epsilon(h_n/\zeta_n)\right] &\geq \mathbb{E}^{\mathbf{P}}\left[h_n \overline{V}^{\epsilon,M}(h_n/\zeta_n)\right] - \mathbb{E}^{\mathbf{P}}\left[h_n \epsilon\right] \\
&\geq \mathbb{E}^{\mathbf{P}}\left[h_n \overline{V}^{\epsilon,M}(h_n/\zeta_n)\right] - y\epsilon.
\end{aligned}$$

Now we verify the term $\left| h_n \overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n}) \right| \leq \zeta_n \max\{V^\epsilon(0), M+1\}$, so that the sequence

$\left\{h_n\overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n})\right\}_{n\in\mathbb{N}}$ is uniformly integrable. Indeed,

$$\frac{h_n}{\zeta_n}\overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n}) \leq V^{\epsilon}(0).$$

By the definition of $\overline{V}^{\epsilon,M}$, $\overline{V}^{\epsilon,M}(x) \geq -1,\ \forall\ x > 0$ and $\overline{V}^{\epsilon,M}(x) = 0,\ \forall\ x > M+1$, we have

$$\begin{aligned}
h_n\overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n}) &= h_n\overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n})\mathbf{1}_{\{\frac{h_n}{\zeta_n}\leq M+1\}} + h_n\overline{V}^{\epsilon,M}(\frac{h_n}{\zeta_n})\mathbf{1}_{\{\frac{h_n}{\zeta_n}> M+1\}} \\
&\geq -h_n\mathbf{1}_{\{\frac{h_n}{\zeta_n}\leq M+1\}} \\
&\geq -(M+1)\zeta_n.
\end{aligned}$$

As $\zeta_n = \sum_{k=n}^{m_n}\alpha_k^n c_n\widetilde{Z}_k$ converges a.s. to 1, where $\widetilde{Z}_k$ is uniformly integrable, and h_n converges a.s. to h, then

$$\mathbb{E}^{\mathbf{P}}\left[h_n\overline{V}^{\epsilon,M}(h_n/\zeta_n)\right] - y\epsilon \longrightarrow \mathbb{E}^{\mathbf{P}}\left[h\overline{V}^{\epsilon,M}(h)\right] - y\epsilon,$$

Therefore,

$$\limsup_{n\to\infty}\mathbb{E}^{\mathbf{P}}\left[\zeta_n V^{\epsilon}(h_n/\zeta_n)\right] \geq \mathbb{E}^{\mathbf{P}}\left[h\overline{V}^{\epsilon,M}(h)\right] - y\epsilon. \tag{3.3.7}$$

Combining the inequalities (3.3.5), (3.3.6) and (3.3.7), we have

$$\begin{aligned}
\lim_{n\to\infty} v_n(y_n;V_n,\mathbf{P}_n) &\geq \limsup_{n\to\infty}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V^{\epsilon}(\frac{\widehat{h}_n}{\overline{Z}_n})\right] - \epsilon(y+1) \\
&\geq \limsup_{n\to\infty}\mathbb{E}^{\mathbf{P}}\left[\zeta_n V^{\epsilon}(h_n/\zeta_n)\right] - \epsilon(y+1) \\
&\geq \mathbb{E}^{\mathbf{P}}\left[h\overline{V}^{\epsilon,M}(h)\right] - (2y+1)\epsilon \\
&\geq \mathbb{E}^{\mathbf{P}}\left[h\overline{V}^{\epsilon}(h)\right] - (2y+1)\epsilon \\
&= \mathbb{E}^{\mathbf{P}}\left[V^{\epsilon}(h)\right] - (2y+1)\epsilon.
\end{aligned}$$

Since $-h \leq V^{\epsilon}(h) \leq V(h)$ and $h \in \mathcal{D}(y,\mathbf{P}) \subset \mathbf{L}^1(\mathbf{P})$, using monotone convergence theorem, let $\epsilon \downarrow 0$,

$$\lim_{n\to\infty} v_n(y_n;V_n,\mathbf{P}_n) \geq \mathbb{E}^{\mathbf{P}}\left[V(h)\right] \geq \inf_{h\in\mathcal{D}(y,\mathbf{P})}\mathbb{E}^{\mathbf{P}}\left[V(h)\right] = v(y,V,\mathbf{P}).$$

3.3.3 Upper semicontinuity of the dual value function

Just as done in [62, Section 3.3.1], for any $Z_T^0 \in \mathcal{Z}_T^{loc},\ y>0$, we define

$$\widetilde{\mathcal{D}}(y, Z_T^0) := \left\{ h \in \mathcal{D}(y, \mathbf{P}) : \frac{Z_T^0}{h} \in \mathbf{L}^\infty \right\}.$$

Then, $\widetilde{\mathcal{D}}(y, Z_T^0) \neq \emptyset$, because $\frac{1}{k} y Z_T^0 + (1-\frac{1}{k})h \in \widetilde{\mathcal{D}}(y, Z_T^0)$ for all $h \in \mathcal{D}(y, \mathbf{P})$ and $k \in \mathbb{N}$.

Lemma 3.3.4 (compare [62, Lemma 3.3])**.** *Fix $y>0$, let $Z_T^0 \in \mathcal{Z}_T^{loc}$ be such that $V^+(yZ_T^0) \in \mathbf{L}^1(\mathbf{P})$, then*

$$v(y, \mathbf{P}) = \inf_{\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)} \mathbb{E}^{\mathbf{P}} \left[V(\widetilde{h}) \right].$$

Proof. Let $\widehat{h} \in \mathcal{D}(y, \mathbf{P})$ satisfy $v(y; \mathbf{P}) = \mathbb{E}\left[V(\widehat{h})\right]$. Define $h_k := \frac{1}{k} y Z_T^0 + (1-\frac{1}{k})\widehat{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)$ for all $k \in \mathbb{N}$. Moreover, $V^+(yZ_T^0) \in \mathbf{L}^1(\mathbf{P})$ and $V(yZ_T^0) \geq -yZ_T^0 \in \mathbf{L}^1(\mathbf{P})$ imply $V(yZ_T^0) \in \mathbf{L}^1(\mathbf{P})$. Then it follows from the convexity of V that

$$\mathbb{E}\left[V(h_k)\right] \leq \frac{1}{k}\mathbb{E}\left[V(yZ_T^0)\right] + (1-\frac{1}{k})\mathbb{E}\left[V(\widehat{h})\right].$$

Letting $k \to \infty$, we get $v(y, \mathbf{P}) = \lim_{k\to\infty} \mathbb{E}\left[V(h_k)\right].$ □

The following Lemma applied in the case of transaction costs here is as same as [62, Lemma 3.4].

Lemma 3.3.5 ([62, Lemma 3.4])**.** *Suppose the collection $\{\overline{Z}_n V_n^+(f/\overline{Z}_n)\}_{n\in\mathbb{N}}$ is $\mathbf{P}$-uniformly integrable for some $f \in \mathbf{L}^0_+$. Let $h \in \mathbf{L}^1(\mathbf{P})$ be such that $h \geq f$ a.s. Then*

$$\lim_{n\to\infty} \overline{Z}_n V_n(h/\overline{Z}_n) = V(h) \text{ in } \mathbf{L}^1(\mathbf{P}).$$

Proof. $\overline{Z}_n = c_n \widetilde{Z}_n$ converges to 1 in $\mathbf{L}^0(\mathbf{P})$, since $c_n\widetilde{Z}_n$ converges to 1 in $\mathbf{L}^1(\mathbf{P})$. Moreover $V_n \to V$ pointwisely, so we have

$$\lim_{n\to\infty} \overline{Z}_n V_n(h/\overline{Z}_n) = V(h) \text{ in } \mathbf{L}^0(\mathbf{P}).$$

We only need to show $\{\overline{Z}_n V_n(h/\overline{Z}_n)\}_{n\in\mathbb{N}}$ is $\mathbf{P}$-uniformly integrable.

Each V_n is decreasing, thus $\overline{Z}_n V_n^+(h/\overline{Z}_n) \leq \overline{Z}_n V_n^+(f/\overline{Z}_n)$, and $\{\overline{Z}_n V_n^+(f/\overline{Z}_n)\}_{n\in\mathbb{N}}$ is $\mathbf{P}$-uniformly integrable, then we obtain the $\mathbf{P}$-uniform integrability of $\{\overline{Z}_n V_n^+(h/\overline{Z}_n)\}_{n\in\mathbb{N}}$.

For the negative part of the V_n, because $V_n(x) \geq -x,\ \forall\, n \in \mathbb{N}$, we get $\overline{Z}_n V_n^-(h/\overline{Z}_n) \leq h \in \mathbf{L}^1(\mathbf{P})$, which means the uniform integrability of the $\{\overline{Z}_n V_n^-(h/\overline{Z}_n)\}_{n\in\mathbb{N}}$. □

To show the upper semicontinuity of the function v, we pick some $Z_T^0 \in \mathcal{Z}_T^{loc}$ such that Assumption 3.2.3 is satisfied, i.e., $\{\overline{Z}_n V_n^+(y\frac{Z_T^0}{\overline{Z}_n})\}_{n\in\mathbb{N}}$ is $\mathbf{P}$-uniformly integrable for all $y>0$. Since $\left\{\overline{Z}_n V_n^+(y\frac{Z_T^0}{\overline{Z}_n})\right\}_{n\in\mathbb{N}}$ converges to $V^+(yZ_T^0)$ in $L^0(\mathbf{P})$, then it converges to $V^+(yZ_T^0)$ in $\mathbf{L}^1(\mathbf{P})$. Moreover, $V^+(yZ_T^0) \in \mathbf{L}^1(\mathbf{P})$ and by Lemma 3.3.4, we have $v(y, \mathbf{P}) = \inf_{\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)} \mathbb{E}^{\mathbf{P}} \left[V(\widetilde{h}) \right]$.

For any $\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0) \subseteq \mathcal{D}(y, \mathbf{P})$, we have $\frac{Z_T^0}{\widetilde{h}} \in \mathbf{L}^\infty$. By Lemma 3.3.5,

$$\lim_{n \to \infty} \mathbb{E} \left[\overline{Z}_n V_n(\widetilde{h}/\overline{Z}_n) \right] = \mathbb{E} \left[V(\widetilde{h}) \right].$$

Let us again recall the observation (3.2.2),

$$\begin{aligned} v(y_n; V_n, \mathbf{P}_n) &= \inf_{h \in \mathcal{D}(y, \mathbf{P})} \frac{1}{c_n} \mathbb{E}^{\mathbf{P}} \left[\overline{Z}_n V_n(\frac{h}{\overline{Z}_n}) \right] \leq \inf_{\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)} \frac{1}{c_n} \mathbb{E}^{\mathbf{P}} \left[\overline{Z}_n V_n(\frac{\widetilde{h}}{\overline{Z}_n}) \right], \\ &\leq \frac{1}{c_n} \mathbb{E}^{\mathbf{P}} \left[\overline{Z}_n V_n(\frac{\widetilde{h}}{\overline{Z}_n}) \right], \end{aligned}$$

where $\widetilde{h}$ in the last inequality belongs to the set $\widetilde{\mathcal{D}}(y, Z_T^0)$. Therefore, for any $\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0) \subseteq \mathcal{D}(y, \mathbf{P})$,

$$\begin{aligned} \limsup_{n \to \infty} v(y_n; V_n, \mathbf{P}_n) &\leq \limsup_{n \to \infty} \frac{1}{c_n} \mathbb{E}^{\mathbf{P}} \left[\overline{Z}_n V_n(\frac{\widetilde{h}}{\overline{Z}_n}) \right] \\ &= \mathbb{E}^{\mathbf{P}} \left[V(\widetilde{h}) \right]. \end{aligned}$$

Since $v(y, \mathbf{P}) = \inf_{\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)} \mathbb{E}^{\mathbf{P}} \left[V(\widetilde{h}) \right]$, we have

$$\limsup_{n \to \infty} v(y_n; V_n, \mathbf{P}_n) \leq \inf_{\widetilde{h} \in \widetilde{\mathcal{D}}(y, Z_T^0)} \mathbb{E}^{\mathbf{P}} \left[V(\widetilde{h}) \right] = v(y; V, \mathbf{P}).$$

We can conclude that the dual value function is continuous, so that

$$\lim_{n \to \infty} v(y_n; V_n, \mathbf{P}_n) = v(y; V, \mathbf{P}) \quad \text{pointwisely.}$$

Thanks to the conjugate properties of the family of the convex functions $\{v_n\}_{n \in \mathbb{N}}$, v and the concave functions $\{u_n\}_{n \in \mathbb{N}}$, u, this fact yields the convergence of the concave primal value functions, i.e., $\lim_{n \to \infty} u(x_n; U_n, \mathbf{P}_n) = u(x; U, \mathbf{P})$ pointwisely.

3.3.4 Convergence of dual and primal optimizers

Though we consider the convergence of the dual and primal optimizers in markets with transaction costs, we can prove it by carrying over the proof in [62, Section 3.5]. We still write down the proof explicitly for the convenience of the reader.

Duality theorem (Theorem 3.1.14) points out that

$$V_T^{liq}(\widehat{\varphi}(x_n); U_n, \mathbf{P}_n) = -V_n'(\widehat{Y}_T^0(y_n; V_n, \mathbf{P}_n)), \text{ where } y_n = \frac{\partial u(x_n)}{\partial x},$$

so we only need to show the continuity of the dual optimizer, then by this equation, it is obvious that the primal optimizer is continuous.

To ease notation, we denote

$$h_n := \overline{Z}_n \widehat{Y}_T^0(y_n; V_n, \mathbf{P}_n) \quad \text{and} \quad h := \widehat{Y}_T^0(y; V, \mathbf{P}).$$

Then $h_n \in \mathcal{D}(y; \mathbf{P})$ satisfies $v(y_n; V_n, \mathbf{P}_n) = \mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{h_n}{\overline{Z}_n})\right]$ for each $n \in \mathbb{N}$. And $h \in \mathcal{D}(y; \mathbf{P})$ satisfies $v(y; V, \mathbf{P}) = \mathbb{E}^{\mathbf{P}}[V(h)]$. Obviously, h_n is bounded in $\mathbf{L}^0(\mathbf{P})$. To prove $\lim_{n\to\infty} \widehat{Y}_T^0(y_n; V_n, \mathbf{P}_n) = \widehat{Y}_T^0(y; V, \mathbf{P})$ in $\mathbf{L}^0(\mathbf{P})$, we need to prove $\lim_{n\to\infty} \frac{h_n}{\overline{Z}_n} = h$ in $\mathbf{L}^0(\mathbf{P})$. Assume that an arbitrary subsequence has already been extracted from $\{\frac{h_n}{\overline{Z}_n}\}_{n\in\mathbb{N}}$. It suffices to show that $\mathbf{L}^0(\mathbf{P})-\lim_{k\to\infty} \frac{h_{n_k}}{\overline{Z}_{n_k}} = h$ along some further subsequence $\{\frac{h_{n_k}}{\overline{Z}_{n_k}}\}_{k\in\mathbb{N}}$.

Pick some $Z_T^0 \in \mathcal{Z}_T^{loc}$ such that $f := yZ_T^0$ satisfies $V^+(f) \in \mathbf{L}^1(\mathbf{P})$. For each $n \in \mathbb{N}$, define $f_n := \frac{1}{n} f + (1 - \frac{1}{n})h \in \widetilde{\mathcal{D}}(y, Z_T^0)$, so the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^0$ and $\lim_{n\to\infty} f_n = h$ in $\mathbf{L}^0$. By the following lemma, we can conclude the sequence $\{\frac{h_n}{\overline{Z}_n}\}_{n\in\mathbb{N}}$ converges in $\mathbf{L}^0$.

Lemma 3.3.6 ([62, Claim 3.8])**.**

For any $m \in \mathbb{N}$, define

$$\mathcal{R}_m := \left\{(a, b) \in \mathbb{R}^2 : a \in [\frac{1}{m}, m],\ b \in [\frac{1}{m}, m], |a - b| > \frac{1}{m}\right\}.$$

Then there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ so that the subsequences $(h_{n_m})_{m\in\mathbb{N}}$ and $(f_{n_m})_{m\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$, respectively, satisfy

$$\lim_{m\to\infty} \mathbf{P}_{n_m}\left[(\frac{h_{n_m}}{\overline{Z}_{n_m}}, \frac{f_{n_m}}{\overline{Z}_{n_m}}) \in \mathcal{R}_m\right] = 0. \tag{3.3.8}$$

Remark 3.3.7 ([62, Remark 3.9]). In (3.3.8), $\mathbf{P}_{n_m}$ is instead of $\mathbf{P}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable sets.

$$\lim_{n\to\infty} \mathbf{P}_n(A_n) = 0 \text{ is equivalent to } \lim_{n\to\infty} \widetilde{Z}_n \mathbf{1}_{\{A_n\}} = 0 \text{ in } \mathbf{L}^0(\mathbf{P}),$$

which is equivalent to $\lim_{n\to\infty} \mathbf{1}_{\{A_n\}} = 0$ in $\mathbf{L}^0(\mathbf{P})$, or $\lim_{n\to\infty} \mathbf{P}(A_n) = 0$, since $\lim_{n\to\infty} \widetilde{Z}_n = 1$ in $\mathbf{L}^1(\mathbf{P})$. In our case, $\overline{Z}_n = c_n \widetilde{Z}_n$ converges to 1 in $\mathbf{L}^1(\mathbf{P})$.

Proof. [62, Proof of Claim 3.8] The conjugate function V is strictly convex, so there is $\beta_m > 0$ for any $m \in \mathbb{N}$ such that for all $(a, b) \in (0, \infty)^2$,

$$V(\frac{a+b}{2}) \le \frac{V(a) + V(b)}{2} - \beta_m \mathbf{1}_{\{\mathcal{R}_m\}}(a, b).$$

As the sequence $\{V_n\}_{n\in\mathbb{N}}$ converges uniformly to V on compact subsets of $(0,\infty)$, we can choose a possible lower but still strictly positive β_m, then we have

$$V_n(\frac{a+b}{2}) \leq \frac{V_n(a)+V_n(b)}{2} - \beta_m \mathbf{1}_{\{\mathcal{R}_m\}}(a,b),\ \forall\ n\in\mathbb{N},\ (a,b)\in(0,\infty)^2.$$

Multiplying both sides of $\widetilde{Z}_n$, then take expectation with respect to $\mathbf{P}$,

$$\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{a+b}{2})\right] \leq \frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(a)\right] + \frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(b)\right] - \mathbb{E}^{\mathbf{P}}\left[\beta_m \widetilde{Z}_n \mathbf{1}_{\{\mathcal{R}_m\}}(a,b)\right].$$

Set $a=\frac{h_n}{\overline{Z}_n}$ and $b=\frac{f_k}{\overline{Z}_n}$, we obtain

$$\beta_m \mathbf{P}_n\left[\left(\frac{h_n}{\overline{Z}_n},\frac{f_k}{\overline{Z}_n}\right)\in\mathcal{R}_m\right] \leq \frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{h_n}{\overline{Z}_n})\right] + \frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{f_k}{\overline{Z}_n})\right] - \mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{h_n}{2\overline{Z}_n}+\frac{f_k}{2\overline{Z}_n})\right].$$

It is known that

$$\frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{h_n}{\overline{Z}_n})\right] = \frac{1}{2}v(y_n;V_n,\mathbf{P}_n),$$

$$\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{h_n}{2\overline{Z}_n}+\frac{f_k}{2\overline{Z}_n})\right] \geq v(\frac{y_n+y}{2};V_n,\mathbf{P}_n),\ \text{since}\ \frac{h_n}{2\overline{Z}_n}+\frac{f_k}{2\overline{Z}_n}\in\mathcal{D}(\frac{y_n+y}{2},\mathbf{P}),$$

and by Lemma 3.3.5,

$$\lim_{n\to\infty}\frac{1}{2}\mathbb{E}^{\mathbf{P}}\left[\widetilde{Z}_n V_n(\frac{f_k}{\overline{Z}_n})\right] = \lim_{n\to\infty}\frac{1}{2c_n}\mathbb{E}^{\mathbf{P}}\left[\overline{Z}_n V_n(\frac{f_k}{\overline{Z}_n})\right] = \frac{1}{2}\mathbb{E}\left[V(f_k)\right],\ \text{for all fixed}\ k\in\mathbb{N}.$$

Therefore, we have

$$\beta_m \mathbf{P}_n\left[\left(\frac{h_n}{\overline{Z}_n},\frac{f_k}{\overline{Z}_n}\right)\in\mathcal{R}_m\right] \leq \frac{1}{2}v(y_n;V_n,\mathbf{P}_n) + \frac{1}{2}\mathbb{E}\left[V(f_k)\right] - v(\frac{y_n+y}{2};V_n,\mathbf{P}_n).$$

Furthermore, the proof of Lemma 3.3.4 shows $\lim_{n\to\infty}\mathbb{E}\left[V(f_k)\right]=\mathbb{E}\left[V(\widehat{h}(y,\mathbf{P}))\right]=v(y;V,\mathbf{P})$. As the sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to v uniformly on compact subsets of $(0,\infty)$, we can choose k_m and n_m large enough so that

$$\mathbf{P}_{n_m}\left[\left(\frac{h_{n_m}}{\overline{Z}_{n_m}},\frac{f_{k_m}}{\overline{Z}_{n_m}}\right)\in\mathcal{R}_m\right] \leq \frac{1}{m}.$$

Then one can choose a universal strictly increasing sequence $k_m=n_m, m\in\mathbb{N}$ with all the desired properties. □

3.4 Dynamic stability

In the previous section, our stability result is only on the terminal value of the wealth process as well as the terminal value of the optional strong supermartingale deflator attaining the dual optimality, that is why we call it static stability result. In the present section, we are concerned not only with the terminal value but with the whole process which produces the optimizer. In particular, we will discuss the so-called dynamic stability for the optimal dual processes, and construct the shadow price for the limiting problem by the sequence of shadow prices corresponding to the problems parameterized by $(x_n; U_n, \mathbf{P}_n)$.

Throughout this section, we need the following assumption, which is necessary for the existence of shadow price processes (see [19]).

Assumption 3.4.1. *Additionally, the stock price process S is continuous.*

3.4.1 Properties of optional strong supermartingale deflators

In what follows, we shall first study the properties of optional strong supermartingale deflators. For the simplicity of notation, we consider only $Y := (Y^0, Y^1) \in \mathcal{B}(1)$. Since the processes Y^0 and Y^1 are optional strong supermartingales, according to [70], they admit unique Mertens decomposition, which is an analogue of the Doob-Meyer type decomposition for càdlàg supermartingales, i.e.,

$$Y^i = M^i - A^i, \quad i = 0, 1, \tag{3.4.1}$$

where M^i is a càdlàg local martingale and A^i is a nondecreasing làdlàg predictable process. We now introduce the following proposition, which shows the relation between the nondecreasing parts A^0 and A^1, that is, symbolically,

$$(1-\lambda)S_t dA_t^0 \leq dA_t^1 \leq S_t dA_t^0.$$

Lemma 3.4.2. *Let Assumption 3.4.1 hold and $Y := (Y^0, Y^1) \in \mathcal{B}(1)$. The processes Y^0 and Y^1 admit the unique Mertens decomposition as (3.4.1). Let $\varepsilon + \lambda \leq 1$ and let σ be a stopping time taking values in $[0, T]$. Define*

$$\tau_\varepsilon := \inf\left\{t \geq \sigma \,\middle|\, \frac{S_t}{S_\sigma} = (1+\varepsilon) \text{ or } (1-\varepsilon)\right\} \wedge T,$$

and

$$\tau_{\varepsilon/2} := \inf\left\{t \geq \sigma \,\middle|\, \frac{S_t}{S_\sigma} = (1+\varepsilon/2) \text{ or } (1-\varepsilon/2)\right\} \wedge T.$$

Then, for all stopping time τ satisfying $\sigma \leq \tau \leq \tau_{\varepsilon/2}$,

$$\begin{aligned}(1-\varepsilon)(1-\lambda)S_\sigma \mathbb{E}\left[A_\tau^0 - A_\sigma^0 \middle| \mathcal{F}_\sigma\right] &\leq \mathbb{E}\left[A_\tau^1 - A_\sigma^1 \middle| \mathcal{F}_\sigma\right] \\ &\leq (1+\varepsilon)S_\sigma \mathbb{E}\left[A_\tau^0 - A_\sigma^0 \middle| \mathcal{F}_\sigma\right].\end{aligned}$$

Hence,

$$(1-\lambda)S_t dA_t^0 \leq dA_t^1 \leq S_t dA_t^0,$$

which is the symbolic differential notation for the integral inequality

$$\int_0^T (1-\lambda)S_t \mathbf{1}_D dA_t^0 \leq \int_0^T (1-\lambda)\mathbf{1}_D dA_t^1 \leq \int_0^T S_t \mathbf{1}_D dA_t^0,$$

which we require to hold true for every optional subset $D \subseteq \Omega \times [0,T]$.

Remark 3.4.3. A similar lemma can be found in [19, Lemma 3.5], which shows the same property between A^0 and A^1 for a particular supermartingale deflator constructed as the Fatou limit process of a sequence of local consistent price systems. We remark that our proof of the above proposition will not depend on the construction of Y, but only on the supermartingale property of the process $\varphi^0 Y^0 + \varphi^1 Y^1$, for all $\varphi \in \mathcal{A}(1)$.

Proof. Consider the following trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T} \in \mathcal{A}(1)$ defined by

$$(\varphi_t^0, \varphi_t^1) := \begin{cases} (1-(1-\lambda)(1-\varepsilon), 0), & 0 \leq t < \sigma; \\ \left(-(1-\lambda)(1-\varepsilon), \frac{1}{S_\sigma}\right), & \sigma \leq t < \tau_\varepsilon; \\ \left(V_{\tau_\varepsilon}^{liq}(\varphi), 0\right), & \tau_\varepsilon \leq t \leq T. \end{cases}$$

Obviously, this trading strategy defined as above is λ-self-financing and 0-admissible. Indeed, we only need to consider the worst case, i.e., the stock price attains $(1-\varepsilon)S_\sigma$, then we have for $\sigma < t \leq T$, the liquidation value

$$V_t^{liq}(\varphi) \geq -(1-\lambda)(1-\varepsilon) + (1-\lambda)\frac{1}{S_\sigma}(1-\varepsilon)S_\sigma = 0.$$

By the definition of the optional strong supermartingale deflator, we obtain that for $\tau \in [\![\sigma, \tau_{\varepsilon/2}]\!]$,

$$\begin{aligned} &-(1-\lambda)(1-\varepsilon)(M_\sigma^0 - A_\sigma^0) + \frac{1}{S_\sigma}(M_\sigma^1 - A_\sigma^1) \\ &\geq \mathbb{E}\left[-(1-\lambda)(1-\varepsilon)\left(M_\tau^0 - A_\tau^0\right) + \frac{1}{S_\sigma}\left(M_\tau^1 - A_\tau^1\right) \middle| \mathcal{F}_\sigma\right]. \end{aligned}$$

We may assume without loss of generality that M^0 and M^1 are true martingales, otherwise, stopping technique applies. Therefore, we obtain

$$\mathbb{E}\left[A_\tau^1 - A_\sigma^1 | \mathcal{F}_\sigma\right] \geq (1-\varepsilon)(1-\lambda)S_\sigma \mathbb{E}\left[A_\tau^0 - A_\sigma^0 | \mathcal{F}_\sigma\right].$$

For the other inequality, let us define another trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T} \in \mathcal{A}(1)$ by

$$(\varphi_t^0, \varphi_t^1) := \begin{cases} (\lambda+\varepsilon, 0), & 0 \leq t < \sigma; \\ \left((1-\lambda)+\lambda+\varepsilon, -\frac{1}{S_\sigma}\right), & \sigma \leq t < \tau_\varepsilon; \\ \left(V_{\tau_\varepsilon}^{liq}(\varphi), 0\right), & \tau_\varepsilon \leq t \leq T. \end{cases}$$

With the same argument, we have

$$(1+\varepsilon)S_\sigma \mathbb{E}\left[A_{\tau_\varepsilon}^0 - A_\sigma^0 | \mathcal{F}_\sigma\right] \geq \mathbb{E}\left[A_{\tau_\varepsilon}^1 - A_\sigma^1 | \mathcal{F}_\sigma\right],$$

which ends the proof. $\square$

The following corollary is straightforward.

Corollary 3.4.4. *For each optional strong supermartingale deflator* $\big(Y^0(y), Y^1(y)\big) \in \mathcal{B}(y)$*, if the first component* $Y^0(y)$ *is a local martingale, then the second component* $Y^1(y)$ *is also a local martingale.*

Proof. From Lemma 3.4.2, for each OSSD $\big(Y^0(y), Y^1(y)\big) \in \mathcal{B}(y)$ with the Mertens decomposition, we have

$$(1-\lambda)S_t dA_t^0 \leq dA_t^1 \leq S_t dA_t^0.$$

If $Y^0(y)$ is a local martingale, which implies $dA_t^0 \equiv 0$, then $dA_t^1 = 0$. This means $Y^1(y)$ is a local martingale. $\square$

Remark 3.4.5. Actually, we can still have Corollary 3.4.4 without Lemma 3.4.2. The proof is also provided here.

Another proof method of Corollary 3.4.4. Suppose the first component $Y^0(y)$ of an OSSD $\big(Y^0(y), Y^1(y)\big) \in \mathcal{B}(y)$ is a local martingale, then there is a sequence of almost surely increasing and diverging stopping times σ_n, such that the stopped process $(Y^0_{\sigma_n \wedge t}(y))_{t\in[0,T]}$ is a true martingale.

The continuity of S implies the locally boundedness of S, so define a sequence of stopping times

$$\rho_n := \inf\{t : |S_t| \geq n+1\}, \text{ for } n \in \mathbb{N}.$$

Take $\tau_n := \rho_n \wedge \sigma_n$ and a trading strategy $(\varphi_t^0, \varphi_t^1)_{0<t\leq T} = (1+\frac{1}{n}, -\frac{1}{n}), n \in \mathbb{N}$, which is 0-admissible with

$$1 + \frac{1}{n} - \frac{1}{n} S_{\tau_n} \geq 0.$$

Then for $0 \leq s < t \leq T$, by the definition of supermartingale deflator and the positivity of the optimal dual process, we have

$$\begin{aligned}(1+\frac{1}{n})Y^0_{s\wedge\tau_n}(y) - \frac{1}{n}Y^1_{s\wedge\tau_n}(y) &\geq \mathbb{E}\left[(1+\frac{1}{n})Y^0_{t\wedge\tau_n}(y) - \frac{1}{n}Y^1_{t\wedge\tau_n}(y)\Big|\mathcal{F}_s\right] \\ &= (1+\frac{1}{n})Y^0_{s\wedge\tau_n}(y) - \mathbb{E}\left[\frac{1}{n}Y^1_{t\wedge\tau_n}(y)\Big|\mathcal{F}_s\right]\end{aligned}$$

which shows $Y^1_{s\wedge\tau_n}(y) \leq \mathbb{E}\left[Y^1_{t\wedge\tau_n}(y)\big|\mathcal{F}_s\right]$.

For the other direction, set $(\varphi_t^0, \varphi_t^1)_{0<t\leq T} = (1-\frac{1}{n}, \frac{1}{n})$, we can obtain $Y^1_{s\wedge\tau_n}(y) \geq \mathbb{E}\left[Y^1_{t\wedge\tau_n}(y)\big|\mathcal{F}_s\right]$. Thus, $Y^1_{s\wedge\tau_n}(y) = \mathbb{E}\left[Y^1_{t\wedge\tau_n}(y)\big|\mathcal{F}_s\right]$, i.e., the stopped process $Y^1_{\cdot\wedge\tau_n}(y)$ is a martingale, which has a càdlàg version. Thus, $(Y_t^1(y))_{0\leq t\leq T}$ is a local martingale. $\square$

It follows from Duality Theorem that under appropriate assumptions, both the primal and dual problems are solvable, in particular, there exists a unique dual optimizer $\widehat{h}(y) \in \mathcal{D}(y)$ solving the dual problem, and moreover there exists an optional strong supermartingale deflator $(\widehat{Y}^0(y), \widehat{Y}^1(y)) \in \mathcal{B}(y)$ such that $\widehat{Y}^0_T(y) = \widehat{h}(y)$, where $y = u'(x)$.

Similarly as Definition 2.1.14 for the case in incomplete markets in §2.1.1, we define optimal dual and primal processes for the case with transaction costs by the same spirit as follows.

Definition 3.4.6. *The optional strong supermartingale deflator* $\big(\widehat{Y}^0(y), \widehat{Y}^1(y)\big) \in \mathcal{B}(y)$ *satisfying* $\widehat{Y}^0_T(y) = \widehat{h}(y)$*, where* $y = u'(x)$ *is called* ***optimal dual process (ODP)****. Similarly, we call the trading strategy* $\big(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)\big) \in \mathcal{A}(x)$ *satisfying* $V^{liq}_T\big(\widehat{\varphi}(x)\big) = \widehat{g}(x)$ ***optimal trading strategy****.*

In what follows, we shall study the existence of shadow price processes. Generally speaking, it states in [17, Proposition 3.7] that if the optimizer $\widehat{h}(y)$ in the Duality Theorem is induced by a local martingale $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{B}(y)$, then there exists a shadow price which could be defined as the quotient $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$. (Sometimes, we use the calibrated processes $\widehat{Z} := (\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{B}(1)$, and by abuse of definition, we still call it ODP.) In particular, Czichowsky, Schachermayer and Yang found a sufficient condition which guarantees this local martingale property of ODPs when S is continuous, that is, the liquidation value process of the optimal trading strategy is a.s. strictly positive, i.e., $\inf_{0\le t\le T} \widehat{V}^{liq}_t(\widehat{\varphi}) > 0$ a.s. We summarize the result of [19] in the following proposition.

Proposition 3.4.7 ([19, Proposition 3.3])**.** *Fix the level* $0 < \lambda < 1$ *of transaction costs. We assume that Assumption 3.4.1 and all assumptions for Duality Theorem (Theorem 3.1.14) are satisfied. In addition, suppose that the liquidation value process of the optimal trading strategy* $\widehat{\varphi} = (\widehat{\varphi}^0_t, \widehat{\varphi}^1_t)_{0\le t\le T} \in \mathcal{A}(x)$ *is strictly positive, i.e.,*

$$\inf_{0\le t\le T} \widehat{V}^{liq}_t(\widehat{\varphi}) := \inf_{0\le t\le T} \big\{\widehat{\varphi}^0_t + (1-\lambda)(\widehat{\varphi}^1_t)^+ S_t - (\widehat{\varphi}^1_t)^- S_t\big\} > 0, \ a.s. \tag{3.4.2}$$

Let $y = u'(x)$*, then there exists a local* λ*-consistent price system* $\widehat{Z} \in \mathcal{Z}^{loc,\lambda}$*, such that* $y\widehat{Z}^0_T = \widehat{h}$*.*

Remark 3.4.8. The continuity of the price process S is essential in the above proposition. Otherwise, counterexamples could be found, e.g., in [15].

Remark 3.4.9. It is shown in [19] that under the condition of "no unbounded profit with bounded risk" (NUPBR), the condition (3.4.2) is satisfied. Very recently, it is proved in the working paper [16] that even the condition (NUPBR) is replaced by a weaker condition of "two way crossing" (TWC), (3.4.2) still holds. We remark that the condition (TWC) is introduced by Bender in [3], which means that, for each stopping time σ, we have $\sigma_+ = \sigma_-$ a.s., where

$$\sigma_+ := \inf\{t > \sigma : S_t > S_\sigma\} \ \text{ and } \ \sigma_- := \inf\{t > \sigma : S_t < S_\sigma\}.$$

Besides, by [84, Theorem 5.11 (ii) and p.116], (TWC) implies that S satisfies (CPS^μ) locally for all $0 < \mu < 1$.

The proof of the proposition above in [19] is based on the strict positivity of the liquidation value of the optimal trading strategy and [19, Lemma 3.5] (see also Lemma

3.4.2). In the present paper, we shall give an alternative proof of this proposition, in which the martingale property of the product of the optimal trading strategy and ODP ((3.1.6) in Theorem 3.1.14) plays the central role. The same methodology applies in the next subsection.

Proof of Proposition 3.4.7. Fix $y > 0$ and assume without loss of generality that $y = 1$. From (3.1.7), we can choose a sequence $\{Z^n\}_{n=1}^\infty = \{(Z_t^{0,n}, Z_t^{1,n})_{0\le t\le T}\}_{n=1}^\infty \subseteq \mathcal{Z}^{loc,\lambda}$, such that

$$\mathbb{E}[V(Z_T^{0,n})] \searrow \mathbb{E}[V(\widehat{h})] = v(y).$$

Then, by [18, Theorem 2.7], there exists a subsequence of convex combinations of $\{Z^n\}_{n=1}^\infty$, still denoted by $\{Z^n\}_{n=1}^\infty$, and a positive optional strong supermartingale $(\widehat{Z}^0, \widehat{Z}^1)$, such that for each stopping time ρ taking values in $[0, T]$,

$$(Z_\rho^{0,n}, Z_\rho^{1,n}) \xrightarrow{\mathbf{P}} (\widehat{Z}_\rho^0, \widehat{Z}_\rho^1). \tag{3.4.3}$$

In particular, we have $Z_T^{0,n} \to \widehat{Z}_T^0$, a.s. Applying Fatou's lemma and [63, Lemma 3.2], we have

$$\mathbb{E}[V(\widehat{Z}_T^0)] \le \liminf_{n\to\infty} \mathbb{E}[V(Z_T^{0,n})] \le \lim_{n\to\infty} \mathbb{E}[V(Z_T^{0,n})] = \mathbb{E}[V(\widehat{h})].$$

Then, the infimum of $v(y)$ and the uniqueness of the dual optimizer show that $\widehat{Z}_T^0 = \widehat{h}$ a.s., which is the dual minimizer. It is easy to verify that $\widehat{Z} \in \mathcal{B}(1)$.

Since the stock price process S is continuous, S is locally bounded. We may assume without loss of generality that the liquidation value of the optimal trading strategy $(\widehat{V}_t^{liq})_{0\le t\le T}$ is predictable (otherwise, consider the càglàd version of the trading strategy), then from $\inf_{0\le t\le T} \widehat{V}_t^{liq} > 0$ a.s., we can find a sequence of a.s. increasing and diverging stopping times $\{\varrho_m\}_{m=1}^\infty$, such that, for each $m \in \mathbb{N}$ and $t \in [0, T]$, we have $\widehat{V}_{t\wedge\varrho_m}^{liq} \ge \frac{1}{m}$,

$$\tfrac{1}{m} \le (1-\lambda) S_{t\wedge\varrho_m} \le S_{t\wedge\varrho_m} \le m,$$

and $\{(Z_{\cdot\wedge\varrho_m}^{0,n}, Z_{\cdot\wedge\varrho_m}^{1,n})\}_{n=1}^\infty$ is a sequence of martingales.

For the optimal trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in \mathcal{A}(x)$, we have

$$\widehat{\varphi}_{\varrho_m}^0 + \widehat{\varphi}_{\varrho_m}^1 \widetilde{S}_{\varrho_m}^n := \widehat{\varphi}_{\varrho_m}^0 + \widehat{\varphi}_{\varrho_m}^1 \frac{Z_{\varrho_m}^{1,n}}{Z_{\varrho_m}^{0,n}} \ge \widehat{V}_{\varrho_m}^{liq} \ge \frac{1}{m};$$
$$\widehat{\varphi}_{\varrho_m}^0 + \widehat{\varphi}_{\varrho_m}^1 \widehat{S}_{\varrho_m} := \widehat{\varphi}_{\varrho_m}^0 + \widehat{\varphi}_{\varrho_m}^1 \frac{\widehat{Z}_{\varrho_m}^{1}}{\widehat{Z}_{\varrho_m}^{0}} \ge \widehat{V}_{\varrho_m}^{liq} \ge \frac{1}{m}.$$

From (3.4.3), we have moreover (up to a subsequence), for all $m \in \mathbb{N}$

$$\left(Z_{\varrho_m}^{0,n}, Z_{\varrho_m}^{1,n}\right) \longrightarrow \left(\widehat{Z}_{\varrho_m}^0, \widehat{Z}_{\varrho_m}^1\right), \quad a.s.$$

Since $\{Z^n\}_{n=1}^\infty \subseteq \mathcal{Z}^{loc,\lambda} \subseteq \mathcal{B}(1)$ and $\widehat{Z} \in \mathcal{B}(1)$, we can deduce

$$\mathbb{E}\left[Z^{0,n}_{\varrho_m}\left(\widehat{\varphi}^0_{\varrho_m} + \widehat{\varphi}^1_{\varrho_m}\frac{Z^{1,n}_{\varrho_m}}{Z^{0,n}_{\varrho_m}}\right)\right] \leq x;$$
$$\mathbb{E}\left[\widehat{Z}^0_{\varrho_m}\left(\widehat{\varphi}^0_{\varrho_m} + \widehat{\varphi}^1_{\varrho_m}\frac{\widehat{Z}^1_{\varrho_m}}{\widehat{Z}^0_{\varrho_m}}\right)\right] = x,$$

where the second equality is deduced from the martingale property of $\widehat{\varphi}^0\widehat{Z}^0 + \widehat{\varphi}^1\widehat{Z}^1$ (see Duality Theorem). Now we are ready to apply Lemma 5.1.1 and conclude that for each $m \in \mathbb{N}$, the sequence $\{Z^{0,n}_{\varrho_m}\}_{n=1}^\infty$ is uniformly integrable. Thus,

$$Z^{0,n}_{\varrho_m} \xrightarrow{\mathbf{L}^1} \widehat{Z}^0_{\varrho_m}. \tag{3.4.4}$$

In addition, for each $m \in \mathbb{N}$, $\frac{Z^{1,n}_{\varrho_m}}{Z^{0,n}_{\varrho_m}} \in \left[(1-\lambda)S_{\varrho_m}, S_{\varrho_m}\right]$ is uniformly bounded. Thus, $\{Z^{1,n}_{\varrho_m}\}_{n=1}^\infty$ is also uniformly integrable, so that

$$Z^{1,n}_{\varrho_m} \xrightarrow{\mathbf{L}^1} \widehat{Z}^1_{\varrho_m}. \tag{3.4.5}$$

It is obvious that (3.4.4) and (3.4.5) as well as the local martingale property of $Z^{0,n}_{\cdot\wedge\varrho_m}$ and $Z^{1,n}_{\cdot\wedge\varrho_m}$ imply that the stopped processes $(\widehat{Z}^0_{\cdot\wedge\varrho_m}, \widehat{Z}^1_{\cdot\wedge\varrho_m})$ are true martingales, for each $m \in \mathbb{N}$. Therefore, the process $(\widehat{Z}^0, \widehat{Z}^1)$ is a local martingale. Moreover, under the usual conditions of the filtration, they admit càdlàg modifications. □

Indeed, the result above could be sharpened and thus we provide the following proposition. We are grateful for the fruitful discussion with Walter Schachermayer.

Proposition 3.4.10. *Fix the level $0 < \lambda < 1$ of transaction costs. Under the assumptions of Proposition 3.4.7, let $y = u'(x)$ and $\widehat{Z} \in \mathcal{B}(1)$ be the optional strong supermartingale deflator associated with the dual optimizer in the sense that $y\widehat{Z}^0_T = \widehat{h}$ a.s. Then, $\widehat{Z}$ is a local λ-consistent price system. In other words, all λ-optional strong supermartingale deflator associated with the dual optimizer is a local λ-consistent price system.*

Remark 3.4.11. The result of Proposition 3.4.7 only provides the existence of such ODP, which is a local martingale, however, it is sufficient to construct a shadow price. The result of Proposition 3.4.10 above is a little stronger, since it states that all ODPs are local martingales. The proof of Proposition 3.4.10 will be more straightforward and general, nevertheless, we keep the proof of Proposition 3.4.7, because of its own interest for the next subsection. Besides, both proofs show explicitly the mechanism that the first coordinate optimal dual process will not "lose" its mass whenever the liquidation value is strictly positive, while in the frictionless case, the unique ODP has the same property when S is continuous (see e.g. [65, 38]). We remark that these arguments here are also applied in the frictionless case by simply replacing the liquidation value process by the wealth process.

Proof of Proposition 3.4.10. We pick up the sequence of stopping times as in the proof of Proposition 3.4.7. Fix the optimal trading strategy $\big(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)\big) \in \mathcal{A}(x)$. It is easy to verify that $\big(\widehat{\varphi}^0_{\cdot\wedge\varrho_m} - \frac{1}{m}, \widehat{\varphi}^1_{\cdot\wedge\varrho_m}\big) \in \mathcal{A}(x)$. Thus,

$$\left(\widehat{\varphi}^0_{\cdot\wedge\varrho_m} - \frac{1}{m}\right)\widehat{Z}^0 + \widehat{\varphi}^1_{\cdot\wedge\varrho_m}\widehat{Z}^1$$

is an optional strong supermartingale. Recall that $\widehat{\varphi}^0\widehat{Z}^0 + \widehat{\varphi}^1\widehat{Z}^1$ is a martingale, so that $\widehat{Z}^0$ is an optional strong submartingale up to ϱ_m. Furthermore, we deduce that $\widehat{Z}^0_{\cdot\wedge\varrho_m}$ is a martingale because $\widehat{Z} \in \mathcal{B}(1)$. The remainder of the proof goes in a similar way as in the previous one. $\square$

Remark 3.4.12. The optimal dual process may not be unique, even the first component of the optimal dual process may not be unique. This is quite different from the frictionless case. We refer the reader to [84, Remark 6.9] for the counterexample.

3.4.2 Convergence of optimal dual processes

In this subsection, we move to discuss the dynamic stability, which means the stability of the optimal dual processes. For the sake of simplicity, we first consider the utility maximization problem under perturbations of the initial endowments and the utility functions, while we keep the physical probability measure unchanged, i.e., $\mathbf{P}_n \equiv \mathbf{P}$. We assume all conditions of Proposition 3.4.7 for each utility maximization problem $u(x_n; U_n)$ as well as for the limiting problem $u(x;U)$ hold and moreover, let assumptions of Theorem 3.2.4 hold true. Then, by Theorem 3.2.4, we have

$$\begin{gathered}\lim_{n\to\infty} u_n = u, \ \lim_{n\to\infty} v_n = v;\\ \lim_{n\to\infty}\frac{\partial u_n}{\partial x} = \frac{\partial u}{\partial x}, \ \lim_{n\to\infty}\frac{\partial v_n}{\partial y} = \frac{\partial v}{\partial y}.\\ \mathbf{L}^0(\mathbf{P}) - \lim_{n\to\infty} V_T^{liq}(\widehat{\varphi}^n) = V_T^{liq}(\widehat{\varphi}), \quad \mathbf{L}^0(\mathbf{P}) - \lim_{n\to\infty}\widehat{h}^n = \widehat{h}.\end{gathered} \tag{3.4.6}$$

From the result in the previous subsection the optimal dual process $\widehat{Z}^n := (\widehat{Z}^{0,n}, \widehat{Z}^{1,n}) := (\widehat{Z}^0(y_n; V_n), \widehat{Z}^1(y_n; V_n))$ is $\mathbf{P}$-local martingale, i.e., $\widehat{Z}^n \in \mathcal{Z}^{loc,\lambda}(\mathbf{P})$.

Proposition 3.4.13. *Let $\mathbf{P}_n \equiv \mathbf{P}$. We assume all conditions of Proposition 3.4.7 for each utility maximization problem $u(x_n; U_n)$ as well as for the limiting problem $u(x;U)$ are satisfied. Moreover, let assumptions of Theorem 3.2.4 and Theorem 3.1.14 hold true. Then, for a sequence of ODPs $\widehat{Z}^n := (\widehat{Z}^{0,n}, \widehat{Z}^{1,n})$ that associated with the dual optimizers $\widehat{h}^n$ (i.e., $y_n\widehat{Z}^{n,0}_T = \widehat{h}^n$) corresponding to the dual problems $v(y_n; V_n)$, there exist a local martingale $\widehat{Z} := (\widehat{Z}^0, \widehat{Z}^1)$ and a subsequence of convex combinations of $\{\widehat{Z}^n\}_{n=0}^\infty$, denoted still by $\{\widehat{Z}^n\}_{n=0}^\infty$, such that for every $[0,T]$-valued stopping time τ,*

$$\big(\widehat{Z}^{0,n}_\tau, \widehat{Z}^{1,n}_\tau\big) \xrightarrow{\mathbf{P}} \big(\widehat{Z}^0_\tau, \widehat{Z}^1_\tau\big).$$

Moreover, the couple $\widehat{Z} \in \mathcal{Z}^{loc,\lambda}$ is an ODP corresponding to the limiting dual problem $v(y;V)$ with $y\widehat{Z}^0_T = \widehat{h}$.

Proof. Since for each n, $\widehat{Z}^n \in \mathcal{Z}^{loc,\lambda} \subseteq \mathcal{B}(1)$, then by [18, Theorem 2.7], there exist a subsequence of convex combinations of $\left\{\widehat{Z}^n\right\}_{n=1}^{\infty}$, still denoted by $\left\{\widehat{Z}^n\right\}_{n=1}^{\infty}$ and an optional strong supermartingale $\widehat{Z} = (\widehat{Z}^0, \widehat{Z}^1)$, for every stopping time τ taking values in $[0,T]$,

$$(\widehat{Z}^{0,n}_\tau, \widehat{Z}^{1,n}_\tau) \xrightarrow{\mathbf{P}} (\widehat{Z}^0_\tau, \widehat{Z}^1_\tau). \tag{3.4.7}$$

From the result of Theorem 3.2.4, $y_n \longrightarrow y$ and $y_n\widehat{Z}^{0,n}_T = \widehat{h}^n \xrightarrow{\mathbf{P}} \widehat{h} = y\widehat{Z}^0_T$. It is obvious that $\widehat{Z} \in \mathcal{B}(1)$ and thus an ODP corresponding to the utility maximization problem $u(x;U)$. The constructed process $\widehat{Z} = (\widehat{Z}^0, \widehat{Z}^1)$ is a local martingale, which thus admits càdlàg versions. Indeed, this could be done by first showing that for optimal trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ corresponding to the utility maximization problem $u(x;U)$, the process $\widehat{\varphi}^0\widehat{Z}^0 + \widehat{\varphi}^1\widehat{Z}^1$ is a true martingale, and then applying the methodology of Proposition 3.4.7. Mathematically,

$$\mathbb{E}\left[\widehat{\varphi}^0_T(x)\widehat{Z}^0_T + \widehat{\varphi}^1_T(x)\widehat{Z}^1_T\right] = \mathbb{E}\left[\frac{1}{y}\widehat{g}(x)\widehat{h}(y)\right] = x = \mathbb{E}\left[\widehat{\varphi}^0_0(x)\widehat{Z}^0_0 + \widehat{\varphi}^1_0(x)\widehat{Z}^1_0\right],$$

which ends the proof. □

Secondly, we consider the more general case with perturbation on the physical probability measures, i.e., for each n, we solve a utility maximization problem under $\mathbf{P}_n$, satisfying Assumption 3.2.1 and Assumption 3.2.3. Without loss of generality, let $y_n = y = 1$. For each n, the ODP $\left(\widehat{Z}^0(1;V_n,\mathbf{P}_n), \widehat{Z}^1(1;V_n,\mathbf{P}_n)\right) \in \mathcal{Z}^{loc,\lambda}(\mathbf{P}_n)$ are $\mathbf{P}_n$-local martingales. By change of measure, we obtain

$$(\widetilde{Z}^{0,n}, \widetilde{Z}^{1,n}) := \left(\widehat{Z}^0(1;V_n,\mathbf{P}_n)\mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{P}_n}{d\mathbf{P}}\middle|\mathcal{F}_\cdot\right], \widehat{Z}^1(1;V_n,\mathbf{P}_n)\mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{P}_n}{d\mathbf{P}}\middle|\mathcal{F}_\cdot\right]\right) \in \mathcal{Z}^{loc,\lambda}(\mathbf{P}),$$

which are $\mathbf{P}$-local martingales and belong to $\mathcal{B}(1,\mathbf{P})$. Moreover, by the fact $\frac{d\mathbf{P}_n}{d\mathbf{P}} \to 1$ in $\mathbf{L}^1(\mathbf{P})$ and Theorem 3.2.4, we have

$$\widehat{Z}^0_T(1;V_n,\mathbf{P}_n)\frac{d\mathbf{P}_n}{d\mathbf{P}} \xrightarrow{\mathbf{P}} \widehat{h}(1,V,\mathbf{P}),$$

which is the dual optimizer for $v(1,V,\mathbf{P})$. Applying Proposition 3.4.13, we deduce that there exists a subsequence of convex combinations, denoted also by $(\widetilde{Z}^{0,n}, \widetilde{Z}^{1,n})$, such that for every stopping time τ taking values in $[0,T]$,

$$\left(\widetilde{Z}^{0,n}_\tau, \widetilde{Z}^{1,n}_\tau\right) \xrightarrow{\mathbf{P}} \left(\widehat{Z}^0_\tau, \widehat{Z}^1_\tau\right),$$

where the limit process $\left(\widehat{Z}^0_\tau, \widehat{Z}^1_\tau\right) \in \mathcal{Z}^{loc,\lambda}(\mathbf{P})$ is the ODP for $v(y;V,\mathbf{P})$, which are $\mathbf{P}$-local martingales.

Remark 3.4.14. We emphasis here that the càdlàg property of the path of ODPs is important for the construction of a shadow price process in the next subsection.

3.4.3 Construction of shadow price processes

Under certain conditions, the original markets with transaction costs can be replaced by a frictionless market that yields the same optimal strategy and utility. Such frictionless market is called **shadow market** for the utility maximization problem. This subsection is devoted to the construction of shadow price processes for the limiting utility maximization problem.

Definition 3.4.15. *A semimartingale $\widetilde{S} = (\widetilde{S}_t)_{0\le t\le T}$ is called* ***shadow price process*** *for the optimization problem* (3.1.3) *if*

(i) *$\widetilde{S}$ takes its values in the bid-ask spread $[(1-\lambda)S, S]$.*

(ii) *The optimizer to the corresponding frictionless utility maximization problem* (2.1.8), *i.e.,*

$$\mathbb{E}[U(\widetilde{g})] \to \max!, \qquad \widetilde{g} \in \widetilde{\mathcal{C}}(x), \tag{3.4.8}$$

exists and coincides with the solution $\widehat{g}(x) \in \mathcal{C}(x)$ for the optimization problem (3.1.3) *under transaction costs, where $\widetilde{\mathcal{C}}(x)$ is defined in* (2.1.7).

(iii) *The optimal trading strategy $\widehat{H}$, which is predictable, $\widetilde{S}$-integrable process for the frictionless market $\widetilde{S}$ as described in §2.1.1 (or in* [63]*), is equal to the left-continuous version of the finite variation process $\widehat{\varphi}^1(x)$ of the unique optimizer $(\widehat{\varphi}^0_t(x), \widehat{\varphi}^1_t(x))_{0\le t\le T}$ of the optimization problem* (3.1.3).

In general, the shadow price process can be constructed by ODPs, which are local martingales. On the other hand, each shadow price process can be represented by the quotient of some ODPs. Here below are the results obtained in [17].

Proposition 3.4.16 ([17, Proposition 3.7])**.** *Under conditions for the Duality Theorem, fix $x > 0$. Assume that the dual optimizer $\widehat{h}(y)$ equals $\widehat{Y}^0_T(y)$, where $u'(x) = y$ and $\widehat{Y}(y) \in \mathcal{B}(y;\mathbf{P}) := (\widehat{Y}^0(y), \widehat{Y}^1(y))$ are* local **P**-martingales. *Then, the strictly positive semimartingale $\widehat{S} := \frac{\widehat{Y}^1(y)}{\widehat{Y}^0(y)}$ is a shadow price process for the optimization problem* (3.1.3).

Proposition 3.4.17 ([17, Proposition 3.8])**.** *If a shadow price $\widehat{S}$ exists, it is given by $\widehat{S} = \frac{\widehat{Y}^1(y)}{\widehat{Y}^0(y)}$ for an ODP $(\widehat{Y}^0(y), \widehat{Y}^1(y)) \in \mathcal{B}(y, \mathbf{P})$ of the dual problem* (3.1.4).

In what follows, we shall show how to construct a shadow price process for the limiting utility maximization with respect to the sequence of shadow price processes corresponding to the perturbed problems. First, we consider the case when the perturbation is not applied to the physical probability measure, i.e., $\mathbf{P}_n \equiv \mathbf{P}$.

Proposition 3.4.18. *Let* $\mathbf{P}_n \equiv \mathbf{P}$. *We assume all conditions of Proposition 3.4.7 for each utility maximization problem* $u(x_n; U_n)$ *as well as for the limiting problem* $u(x; U)$ *hold true and moreover, let assumptions for Theorem 3.2.4 and Theorem 3.1.14 hold. For each* n, *suppose that* $\widehat{S}^n = \widehat{S}(y_n; V_n)$ *is a shadow price process corresponding to the* n*th problem, which can be represented by* $\widehat{S}^n = \frac{\widehat{Y}^{1,n}}{\widehat{Y}^{0,n}}$, *where* $\widehat{Y}^n := (\widehat{Y}^{0,n}, \widehat{Y}^{1,n}) \in \mathcal{B}(y_n)$ *are* $\mathbf{P}$*-local martingales. Then, there exists a subsequence of convex combinations of* $\{\widehat{Y}^n\}_{n=0}^{\infty}$ *and a limit process* $\widehat{Y} \in \mathcal{B}(y)$ *such that* $\widehat{Y}^n$ *converges to* $\widehat{Y}$ *in the sense of* (3.4.7). *Moreover,* $\widehat{S} := \frac{\widehat{Y}^1}{\widehat{Y}^0}$ *defines a shadow price process for the limiting problem* $u(x; U)$.

Remark 3.4.19. This proposition can be proved simply by applying Proposition 3.4.13 and Proposition 3.4.17.

Remark 3.4.20. For the more complicated situation with the perturbation of the physical probability measures, we also assume all conditions of Proposition 3.4.7 for each utility maximization problem $u(x_n; U_n, \mathbf{P}_n)$ as well as for the limiting problem $u(x; U, \mathbf{P})$. Moreover, let assumptions for Theorem 3.2.4 hold, then the definition of the shadow price and the observation observation (3.2.2) of $v(y_n; V_n, \mathbf{P}_n)$ expressed by $\mathbf{P}$ show that for each n, a shadow price

$$\widehat{S}^n = \widehat{S}(y_n; V_n, \mathbf{P}_n) := \frac{\frac{\widehat{Y}^1(y; V_n, \mathbf{P})}{\frac{y}{y_n}\mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{P}_n}{d\mathbf{P}}\middle|\mathcal{F}_\cdot\right]}}{\frac{\widehat{Y}^0(y; V_n, \mathbf{P})}{\frac{y}{y_n}\mathbb{E}^{\mathbf{P}}\left[\frac{d\mathbf{P}_n}{d\mathbf{P}}\middle|\mathcal{F}_\cdot\right]}} = \frac{\widehat{Y}^1(y; V_n, \mathbf{P})}{\widehat{Y}^0(y; V_n, \mathbf{P})} \tag{3.4.9}$$

and $\widehat{S}(x; U, \mathbf{P}) := \frac{\widehat{Y}^1(y; V, \mathbf{P})}{\widehat{Y}^0(y; V, \mathbf{P})}$, where $\big(\widehat{Y}^0(y; V, \mathbf{P}), \widehat{Y}^1(y; V, \mathbf{P})\big) \in \mathcal{B}(y, \mathbf{P})$ is the limit process of the sequence of $\big(\widehat{Y}^0(y; V_n, \mathbf{P}), \widehat{Y}^1(y; V_n, \mathbf{P})\big) \in \mathcal{B}(y, \mathbf{P})$ in the sense of (3.4.7) ([18, Theorem 2.7]). The limit process is an ODP in the market parameterized by $(x; U, \mathbf{P})$.

Chapter 4

Utility maximization problems with transaction costs and random endowment

In this chapter, we discuss the utility maximization problem in markets with transaction costs and random endowments. Under the no-short-selling constraint, we prove that such utility maximization problem admits a unique solution, then shadow price processes are constructed directly by the derivatives of dynamic primal value functions. However, we provide an example to show that there may exist a shadow price if the random endowment becomes negative.

4.1 On the existence of shadow prices for optimal investment with random endowment

We shall briefly introduce the basic setting of the utility maximization problem in markets with nonnegative random endowment and transaction costs, then we construct a shadow price process. The reader, who has more interests in the details of these topics, is referred to [54, 55, 5, 15, 19].

4.1.1 Formulation of the problem

We consider a numéraire-based model in financial markets: the financial market consists of two assets, one bond and one stock, where the price of the bond B is constant and normalized to $B \equiv 1$. The market is equipped with proportional transaction costs $\lambda \in (0, 1)$ as in §3.1. The stock price process satisfies the Assumption 3.1.1.

We also assume that the agent is endowed with an exogenous random endowment $\overline{e}_T$ at the terminal time T, which is represented by an $\mathcal{F}_T$-measurable random variable.

Assumption 4.1.1. *The endowment $\overline{e}_T \in \mathcal{F}_T$ is a strictly positive and finite-valued random variable, which can be decomposed into a deterministic part and a random part,* *i.e., $\overline{e}_T = x + e_T$, where $x > 0$ and $e_T \geq 0$, a.s. We assume that x is the initial wealth of the agent and e_T is endowed at time T.*

Remark 4.1.2. In the financial market, e_T can be explained as a positive contingent claim, e.g., an option contract.

As the counterpart of martingale measures in the frictionless case, consistent price systems (CPSs) play a very important role in the framework with transaction costs (compare, e.g., [53, 83]). To establish the utility maximization problem, we adopt an extended notion – λ-supermartingale-CPSs, similarly defined as in [5].

Definition 4.1.3. *Fix $\lambda > 0$ and the price process $S = (S_t)_{0 \leq t \leq T}$ satisfying Assumption 3.1.1. A **λ-supermartingale-CPS** is a couple of two positive processes $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ consisting of two supermartingales Z^0 and Z^1, such that*

$$S_t^Z := \frac{Z_t^1}{Z_t^0} \in [(1 - \lambda) S_t, S_t], \quad a.s., \tag{4.1.1}$$

for all $0 \leq t \leq T$.

The set of all λ-supermartingale-CPSs is denoted by $\mathcal{Z}_{sup}^{\lambda}$.

We introduce now the following assumption on the existence of a supermartingale-CPS, which is an analogue to the existence of an equivalent supermartingale density in the frictionless setting.

Assumption 4.1.4. *For some $0 < \lambda' < \lambda$, we have that $\mathcal{Z}_{sup}^{\lambda'} \neq \emptyset$.*

Remark 4.1.5. We remark that this condition corresponds to the "no unbounded profit with bounded risk" with no-short-selling constraint on portfolios in the frictionless theory, see [56].

Remark 4.1.6. Instead of all $0 < \lambda' < \lambda$, we only need that the condition above holds for some $0 < \lambda' < \lambda$, similar to Lemma 3.1 in [83]. This condition is sufficient to show the convex compactness and the $\mathbf{L}^0$-boundedness of the set of admissible terminal wealth.

In markets under transaction costs, the following definition of self-financing trading strategies is commonly adopted, e.g., in [80, 82].

Definition 4.1.7. *Fix $\lambda \in (0, 1)$. A **self-financing** trading strategy under transaction costs λ is a predictable $\mathbb{R}^2$-valued finite variation process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that*

$$\int_s^t d\varphi_u^{0,\uparrow} = \int_s^t (1 - \lambda) S_u d\varphi_u^{1,\downarrow}, \quad \int_s^t d\varphi_u^{0,\downarrow} = \int_s^t S_u d\varphi_u^{1,\uparrow},$$

for all $0 \leq s \leq t \leq T$, where $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ denote the Jordan-Hahn decomposition of φ.

The processes φ_t^0 and φ_t^1 describe the amount of bond and the number of stock shares held at time $t \in [0, T]$, respectively.

For any process ψ of finite variation, we denote by $\psi = \psi_0 + \psi^{\uparrow} - \psi^{\downarrow}$ its Jordan-Hahn decomposition into two non-decreasing $\psi^{\uparrow}$, $\psi^{\downarrow}$ both null at zero. Any process ψ of finite variation is in particular làdlàg (with right and left limits). For any làdlàg process X, we have

$$X_t = X_t^c + \sum_{s<t} \Delta_+ X_s + \sum_{s \le t} \Delta X_s,$$

where $\Delta_+ X_t = X_{t+} - X_t$, and $\Delta X_t = X_t - X_{t-}$ are its right and left jumps respectively and X^c is the continuous part of X. Now we define the integrals for a finite variation process ψ and a làdlàg process X

$$\begin{aligned}
\int_0^t X_u(\omega) d\psi_u(\omega) &:= \int_0^t X_u(\omega) d\psi_u^c + \sum_{0<u\le t} X_{u-}(\omega)\Delta\psi_u(\omega) + \sum_{0\le u<t} X_u(\omega)\Delta_+\psi_u(\omega) \\
\psi \bullet X_t := \int_0^t \psi_u(\omega) dX_u(\omega) &= \int_0^t \psi_u^c dX_u(\omega) + \sum_{0<u\le t} \Delta\psi_u(\omega)(X_t(\omega) - X_{u-}(\omega)) \\
&\quad + \sum_{0\le u<t} \Delta_+\psi_u(\omega)(X_t(\omega) - X_u(\omega)) \qquad (4.1.2)
\end{aligned}$$

pathwise by using Riemann-Stieltjes integrals such that the integration by parts formula holds true.

$$\psi_t(\omega) X_t(\omega) = \psi_0(\omega) X_0(\omega) + \int_0^t X_u(\omega) d\psi_u(\omega) + \int_0^t \psi_u dX_u(\omega).$$

If X is a semimartingale and ψ is in addition predictable, (4.1.2) coincides with the classical stochastic integral. For more details we refer to, e.g., [17, p. 1894] or [18, Section 7].

We shall consider a utility maximization problem similar to the one in Benedetti et al. [5], where the agent is facing the no-short-selling constraint, which forces him to keep both the amount of bond and the number of stock shares positive. In other words, with the initial wealth x, the agent is only allowed to trade with the admissible (under no-short-selling constraint) strategies defined as follows:

Definition 4.1.8. *Under transaction costs $\lambda \in (0,1)$, a self-financing strategy $\varphi = (\varphi^0, \varphi^1)$ with $(\varphi_0^0, \varphi_0^1) = (x, 0)$ and $(\varphi_T^0, \varphi_T^1) = (\varphi_T^0, 0)$ is called **admissible under no-short-selling constraint** (for short **NSS-admissible**), if for each $t \in [0,T]$ we have that*

$$\varphi_t^0 \ge 0 \ \textit{ and } \ \varphi_t^1 \ge 0, \quad a.s.$$

We denote by $\mathcal{A}^\lambda(x)$ the collection of all such strategies. Moreover, we define

$$\mathcal{C}^\lambda(x) := \left\{ g \in \mathbf{L}_+^0 \,\middle|\, g \le \varphi_T^0, \ \textit{for some } (\varphi^0, \varphi^1) \in \mathcal{A}^\lambda(x) \right\}.$$

We suppose that the agent's preferences over terminal wealth are modeled by a utility function $U : (0, \infty) \to \mathbb{R}$, which satisfies Assumption 2.1.6. Without loss of generality, we may assume $U(\infty) > 0$ to simplify the analysis and we define $U(x) = -\infty$ whenever $x \leq 0$. Then, the primal problem for the agent is to maximize expected utility at terminal time T from his bond account derived from trading and the random endowment, i.e.,

$$u(x; e_T) := \sup_{g \in \mathcal{C}^\lambda(x)} \mathbb{E}[U(g + e_T)]. \tag{4.1.3}$$

For $Z \in \mathcal{Z}^\lambda_{sup}$, define $S^Z := \frac{Z^1}{Z^0}$. By the definition, S^Z is a positive semimartingale taking values in $[(1-\lambda)S, S]$. Then, we can construct a frictionless market consisting of one bond with zero interest rate and an underlying asset, whose price process is S^Z. In order to adapt the previous setting under transaction costs, we adopt the following notion of self-financing trading strategies.

Definition 4.1.9. *In the frictionless market associated with S^Z, an $\mathbb{R}^2$-valued predictable process $\widetilde{\varphi} := (\widetilde{\varphi}^0_t, \widetilde{\varphi}^1_t)$ starting from $(x, 0)$ is a **self-financing** trading strategy, if $\widetilde{\varphi}^1$ is S^Z-integrable and*

$$\widetilde{\varphi}^0_t + \widetilde{\varphi}^1_t S^Z_t = x + \int_0^t \widetilde{\varphi}^1_u dS^Z_u, \quad 0 \leq t \leq T.$$

Here, $\widetilde{\varphi}^0_t$ and $\widetilde{\varphi}^1_t$ describe the amount of bond and the number of stock shares held at time $t \in [0, T]$.

We shall formulate a utility maximization problem for the frictionless model with S^Z. In accordance with (4.1.3), we always assume that neither asset can be shorted, so that the maximization problem is established over all NSS-admissible strategies defined as follows.

Definition 4.1.10. *Let $S^Z := \frac{Z^1}{Z^0}$, for some $Z \in \mathcal{Z}^\lambda_{sup}$. A self-financing strategy $\widetilde{\varphi}$ is **admissible under no-short-selling constraint (NSS-admissible)** in the market driven by S^Z, if we have*

$$\widetilde{\varphi}^0_t \geq 0 \ \text{ and } \ \widetilde{\varphi}^1_t \geq 0, \quad a.s.,$$

for all $0 \leq t \leq T$.

We denote by $\mathcal{A}^Z(x)$ the collection of all such NSS-admissible trading strategies starting from $(x, 0)$. Moreover, we define

$$\mathcal{C}^Z(x) := \left\{ \widetilde{g} \in \mathbf{L}^0_+ \,\middle|\, \widetilde{g} \leq \widetilde{\varphi}^0_T + \widetilde{\varphi}^1_T S^Z_T, \ \text{for some } (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \in \mathcal{A}^Z(x) \right\}.$$

Lemma 4.1.11. *Any payoff in the original market S with transaction costs can be dominated by that in the potentially more favorable frictionless markets S^Z, where $Z \in \mathcal{Z}^\lambda_{sup}$. Namely, fix $Z \in \mathcal{Z}^\lambda_{sup}$ and let $(\varphi^0, \varphi^1) \in \mathcal{A}^\lambda(x)$ be arbitrary, then there exists a $(\widetilde{\varphi}^0, \widetilde{\varphi}^1) \in \mathcal{A}^Z(x)$ such that*

$$\widetilde{\varphi}^0_t \geq \varphi^0_t \ \text{ and } \ \widetilde{\varphi}^1_t \geq \varphi^1_t, \quad a.s., \tag{4.1.4}$$

for all $0 \leq t \leq T$.

Proof. For any $(\varphi^0, \varphi^1) \in \mathcal{A}^\lambda(x)$, since (φ^0, φ^1) is a λ-self-financing trading strategy, applying the integration by parts formula introduced in [18, §7], we have that

$$\begin{aligned}\varphi_t^0 + \varphi_t^1 S_t^Z &= x + \int_0^t d\varphi_u^0 + \int_0^t \varphi_u^1 dS_u^Z + \int_0^t S_u^Z d\varphi_u^1 \\ &= x + \int_0^t (d\varphi_u^0 + S_u^Z d\varphi_u^1) + \int_0^t \varphi_u^1 dS_u^Z \leq x + \int_0^t \varphi_u^1 dS_u^Z.\end{aligned}$$

Then, define a self-financing trading strategy in the frictionless market associated with S^Z by

$$\begin{cases} \widetilde{\varphi}_t^0 := x + \int_0^t \varphi_u^1 dS_u^Z - \varphi_t^1 S_t^Z \geq \varphi_t^0, \\ \widetilde{\varphi}_t^1 := \varphi_t^1, \end{cases}$$

which satisfies (4.1.4). □

Obviously, $\mathcal{C}^\lambda(x) \subseteq \mathcal{C}^Z(x)$, for any $Z \in \mathcal{Z}_{sup}^\lambda$. Therefore, letting

$$u^Z(x; e_T) := \sup_{\widetilde{g} \in \mathcal{C}^Z(x)} \mathbb{E}[U(\widetilde{g} + e_T)],$$

it follows that

$$u(x; e_T) \leq \inf_{Z \in \mathcal{Z}_{sup}^\lambda} u^Z(x; e_T),$$

which means each frictionless market with S^Z affords better, at least not worse, investment opportunity than the original market with transaction costs. An interesting question is whether there exists a least favorable $\widehat{Z} \in \mathcal{Z}_{sup}^\lambda$, such that the gap is closed, i.e., the inequality becomes equality. If so, the corresponding price process $S^{\widehat{Z}} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ is called shadow price. Below is the definition of the shadow price similar to [5, Definition 3.9].

Definition 4.1.12. *Fix the initial value x and the terminal random endowment e_T. We assume that short selling of either asset is not allowed. Then, the process $S^{\widehat{Z}}$ associated with some $\widehat{Z} := \widehat{Z}(x, e_T) \in \mathcal{Z}_{sup}^\lambda$ is called a* ***shadow price process****, if*

$$\sup_{g \in \mathcal{C}^\lambda(x)} \mathbb{E}[U(g + e_T)] = \sup_{\widetilde{g} \in \mathcal{C}^{\widehat{Z}}(x)} \mathbb{E}[U(\widetilde{g} + e_T)].$$

4.1.2 Solvability of the problem and existence of shadow prices

We shall present and prove our result here, that is, the solvability of (4.1.3) and the existence of shadow prices.

Solvability of the problem

In the previous chapters, the solution to the primal problem is obtained by duality methods. By contrast, we shall solve (4.1.3) directly by following the line of [5].

First of all, we display a superreplication theorem as an analogue of [5, Lemma 4.1]:

Lemma 4.1.13. *For any $Z \in \mathcal{Z}^{\lambda}_{sup}$, the process $Z^0\varphi^0 + Z^1\varphi^1$ is a positive supermartingale, for any $(\varphi^0, \varphi^1) \in \mathcal{A}^{\lambda}(x)$.*

Proof. As (φ^0, φ^1) is of finite variation and (Z^0, Z^1) is a supermartingale, again applying the integration by parts formula introduced in [18, §7], we obtain that

$$\begin{aligned} Z^0_t\varphi^0_t + Z^1_t\varphi^1_t &= (Z^0_0\varphi^0_0 + Z^1_0\varphi^1_0) + \int_0^t (\varphi^0_u dZ^0_u + \varphi^1_u dZ^1_u) + \int_0^t (Z^0_u d\varphi^0_u + Z^1_u d\varphi^1_u) \\ &= x + \int_0^t (\varphi^0_u dZ^0_u + \varphi^1_u dZ^1_u) + \int_0^t (Z^0_u d\varphi^0_u + Z^1_u d\varphi^1_u). \end{aligned}$$

The first integral defines a supermartingale due to the positivity of φ^0 and φ^1. The second integral defines a decreasing process by the fact that (φ^0, φ^1) is λ-self-financing and that $\frac{Z^1_u}{Z^0_u}$ takes values in $[(1-\lambda)S_u, S_u]$. Therefore, the process $Z^0\varphi^0 + Z^1\varphi^1$ is a positive supermartingale. $\square$

Remark 4.1.14. Comparing with [83, Theorem 1.4], we require less on the underlying asset price S for the superreplication theorem, since we are working with a smaller set of trading strategies.

Furthermore, we have some properties of the convex sets $\mathcal{A}^{\lambda}(x)$ and $\mathcal{C}^{\lambda}(x)$ as follows.

Lemma 4.1.15. *Under Assumptions 3.1.1 and 4.1.4, the total variation* $\mathrm{Var}(\varphi^0)$ *and* $\mathrm{Var}(\varphi^1)$ *remain bounded in $\mathbf{L}^0$, when φ runs through $\mathcal{A}^{\lambda}(x)$.*

Proof. Fix $0 < \lambda' < \lambda$, by Assumption 4.1.4 there is a couple of supermartingales $Z = (Z^0_t, Z^1_t)_{0\le t\le T}$ such that

$$S^Z_t := \frac{Z^1_t}{Z^0_t} \in [(1-\lambda')S_t, S_t] \quad \text{a.s.,} \quad 0 \le t \le T.$$

And the position in stock is assumed to be liquidated at the terminal time, i.e., $\varphi^1_T = 0$.

Fix a λ-self-financing, NSS-admissible trading strategy $\varphi \in \mathcal{A}^{\lambda}(x)$ with $(\varphi^0_0, \varphi^1_0) = (x, 0)$. Write $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ as the canonical differences of increasing processes. Then, we could define a strategy $\widetilde{\varphi}$ by

$$\widetilde{\varphi}_t := \left(\varphi^0_t + \frac{\lambda - \lambda'}{1-\lambda}\varphi^{0,\uparrow}_t, \varphi^1_t\right), \quad 0 \le t \le T.$$

It is clear that $\widetilde{\varphi}$ is λ'-self-financing, NSS-admissible. It follows from Lemma 4.1.13

$$\mathbb{E}[Z^0_T\widetilde{\varphi}^0_T + Z^1_T\widetilde{\varphi}^1_T] \le x, \ \forall \ Z \in \mathcal{Z}^{\lambda'}_{sup}$$

that

$$\frac{\lambda-\lambda'}{1-\lambda}\mathbb{E}\left[Z_T^0\varphi_T^{0,\uparrow}\right] \le \mathbb{E}[Z_T^0\varphi_T^0 + Z_T^1\varphi_T^1] + \frac{\lambda-\lambda'}{1-\lambda}\mathbb{E}\left[Z_T^0\varphi_T^{0,\uparrow}\right] \le x.$$

Then

$$\mathbb{E}\left[Z_T^0\varphi_T^{0,\uparrow}\right] \le x\frac{(1-\lambda)}{\lambda-\lambda'},\ Z \in \mathcal{Z}_{sup}^{\lambda'}.$$

As $0 \le Z_T^0\varphi_T^0 = Z_T^0\varphi_T^{0,\uparrow} - Z_T^0\varphi_T^{0,\downarrow}$ a.s., we have

$$\mathbb{E}\left[Z_T^0\varphi_T^{0,\downarrow}\right] \le \mathbb{E}\left[Z_T^0\varphi_T^{0,\uparrow}\right] \le x\frac{(1-\lambda)}{\lambda-\lambda'},$$

therefore

$$\mathbb{E}[Z_T^0(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})] \le 2x\frac{(1-\lambda)}{\lambda-\lambda'}.$$

Inspired by the proof of [7, Lemma 3.2], we construct a probability measure $\mathbf{Q}$ by defining

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \frac{g}{\mathbf{E}[g]}, \quad \text{where} \quad g := \frac{\inf_{t\in[0,T]} Z_t^0}{\mathbb{E}[Z_T^0]} > 0, \tag{4.1.5}$$

we have $\frac{Z_T^0}{\inf_{t\in[0,T]} Z_t^0} \ge 1$, a.s. Then,

$$\begin{aligned}
2x\frac{(1-\lambda)}{\lambda-\lambda'} &\ge \mathbb{E}^{\mathbf{Q}}\left[\frac{d\mathbf{P}}{d\mathbf{Q}}Z_T^0(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})\right] \\
&= \mathbb{E}^{\mathbf{Q}}\left[\frac{\mathbb{E}\left[g\right]}{\frac{\inf_{t\in[0,T]} Z_t^0}{\mathbb{E}[Z_T^0]}}Z_T^0(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})\right] \\
&= \mathbb{E}\left[g\right]\mathbb{E}[Z_T^0]\mathbb{E}^{\mathbf{Q}}\left[\frac{Z_T^0}{\inf_{t\in[0,T]} Z_t^0}(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})\right] \\
&\ge \mathbb{E}\left[\inf_{t\in[0,T]} Z_t^0\right]\mathbb{E}^{\mathbf{Q}}\left[(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})\right].
\end{aligned}$$

Thus, we have

$$\mathbb{E}^{\mathbf{Q}}\left[(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow})\right] \le 2x\frac{(1-\lambda)}{(\lambda-\lambda')\mathbb{E}\left[\inf_{t\in[0,T]} Z_t^0\right]}, \tag{4.1.6}$$

which shows $|\varphi_T^0|$ is $\mathbf{L}^1(\mathbf{Q})$-bounded.

The reminder of the proof, which is identical with the one of [83, Lemma 3.1], is still provided here.

For $\epsilon > 0$, there is a $\delta > 0$ such that for subsets $A \in \mathcal{F}$ with $\mathbf{Q}(A) < \delta$, we have $\mathbf{P}(A) < \epsilon$. Setting $K = 2x\frac{(1-\lambda)}{(\lambda-\lambda')\mathbb{E}\left[\inf_{t\in[0,T]} Z_t^0\right]}$, and applying Tschebyscheff to (4.1.6), we get

$$\mathbf{P}\left(\varphi_T^{0,\uparrow}+\varphi_T^{0,\downarrow}\geq K\right)<\epsilon, \tag{4.1.7}$$

which shows the total variation of the first component of $(\varphi_t^0,\varphi_t^1)_{0\leq t\leq T}$ is bounded in $\mathbf{L}^0(\mathbf{P})$.

As regards the boundedness of the total variation of φ_T^1 in $\mathbf{L}^0$, it follows from the self-financing definition that

$$d\varphi_t^{1,\uparrow}=\frac{d\varphi_t^{0,\downarrow}}{S_t}, \tag{4.1.8}$$

$$\text{precisely,}\quad d\varphi_t^{1,\uparrow,c}=\frac{d\varphi_t^{0,\downarrow,c}}{S_t},\ \Delta\varphi_t^{1,\uparrow}=\frac{\Delta\varphi_t^{0,\downarrow}}{S_{t-}},\ \Delta_+\varphi_t^{1,\uparrow}=\frac{\Delta_+\varphi_t^{0,\downarrow}}{S_t}.$$

Since the trajectories of S are strictly positive, we even have $\inf_{0\leq t\leq T}S_t(\omega)>0$, $\mathbf{P}$-a.s. for almost all trajectories $(S_t(\omega))_{0\leq t\leq T}$. Moreover, by the same argument as (2.2.13), we obtain $\inf_{0\leq t\leq T}S_t^Z(\omega)>0,\mathbf{P}$-a.s. Thus, for $\varepsilon>0$, there is $\delta>0$ such that

$$\mathbf{P}\left(\inf_{0\leq t\leq T}S_t<\delta\right)<\varepsilon,$$

hence we can control $\varphi_T^{1,\uparrow}$ by applying (4.1.8) and estimating $\varphi^{0,\downarrow}$ by (4.1.7). Finally, we can control $\varphi_T^{1,\downarrow}$ by the fact that $\varphi_T^{1,\uparrow}-\varphi_T^{1,\downarrow}=\varphi_T^1-\varphi_0^1=0$. □

Remark 4.1.16 ([83, Remark 3.2]). In the proof, we have shown $\varphi_T^{0,\uparrow},\varphi_T^{0,\downarrow},\varphi_T^{1,\uparrow},\varphi_T^{1,\downarrow}$ remain bounded in $\mathbf{L}^0(\mathbf{P})$, when φ runs through the NSS-admissible self-financing processes in the set $\mathcal{A}^\lambda(x)$. The proof also shows the convex combinations of the functions $\varphi_T^{0,\uparrow}$ etc. remain bounded in $\mathbf{L}^0(\mathbf{P})$. Indeed, the convex hull of the functions $\varphi_T^{0,\uparrow},\varphi_T^{0,\downarrow}$ are bounded in $\mathbf{L}^1(\mathbf{Q})$, where $\mathbf{Q}$ is defined in (4.1.5). For $\varphi_T^{1,\uparrow},\varphi_T^{1,\downarrow}$ the argument is similar.

Then, we state the following lemma. In spite of showing the closeness of $\mathcal{C}^\lambda(x)$, the proof is as same as the one of [83, Theorem 3.4]. We still present the proof for the convenience of the reader.

Lemma 4.1.17. *Under Assumptions 3.1.1 and 4.1.4, the set $\mathcal{C}^\lambda(x)$ is convex closed and bounded in $\mathbf{L}^0_+$.*

Proof. Lemma 4.1.15 and its subsequent remark show that $\varphi_T^{0,\uparrow},\varphi_T^{0,\downarrow},\varphi_T^{1,\uparrow},\varphi_T^{1,\downarrow}$ are bounded in $\mathbf{L}^0(\mathbf{P})$, so φ_T^0, φ_T^1 are bounded in $\mathbf{L}^0(\mathbf{P})$. Moreover, because of the no-short-selling constraint that $\varphi_T^0\geq 0$, $\varphi_T^1\geq 0$ a.s., they are bounded in $\mathbf{L}^0_+$.

Let $\left\{\varphi_T^{0,n}\right\}_{n\in\mathbb{N}}$ be a nonnegative sequence in $\mathcal{C}^\lambda(x)$ converging a.s. to some nonnegative $\varphi_T^0\in\mathbf{L}^0(\mathbb{R}_+)$. We have to show $\varphi_T^0\in\mathcal{C}^\lambda(x)$. We may find λ-self-financing, NSS-admissible trading strategies $\varphi^n=(\varphi_t^{0,n},\varphi_t^{1,n})_{0\leq t\leq T}\in\mathcal{A}^\lambda(x)$ starting at $\varphi_0^n=(x,0)$, ending with terminal values $(\varphi_T^{0,n},0)$. Decompose canonically these nonnegative processes into two nonnegative increasing processes as

$$\varphi_T^{0,n} = \varphi_T^{0,n,\uparrow} - \varphi_T^{0,n,\downarrow}.$$

We know from above $(\varphi_T^{0,n,\uparrow})_{n=1}^\infty, (\varphi_T^{0,n,\downarrow})_{n=1}^\infty$ as well as their convex combinations are bounded in $\mathbf{L}_+^0$, then by [22, Lemma A1.1a], we can find convex combinations converging a.s. to elements $\varphi_T^{0,\uparrow}, \varphi_T^{0,\downarrow}$. We claim that

$$\varphi_T^0 = \varphi_T^{0,\uparrow} - \varphi_T^{0,\downarrow} \in \mathcal{C}^\lambda(x).$$

By passing to convex combinations inductively, still denoted by the original sequences, we assume that $(\varphi_r^{0,n,\uparrow})_{n=1}^\infty, (\varphi_r^{0,n,\downarrow})_{n=1}^\infty, (\varphi_r^{1,n,\uparrow})_{n=1}^\infty, (\varphi_r^{1,n,\downarrow})_{n=1}^\infty$ converge to some elements $\widetilde{\varphi}_r^{0,\uparrow}, \widetilde{\varphi}_r^{0,\downarrow}, \widetilde{\varphi}_r^{1,\uparrow}, \widetilde{\varphi}_r^{1,\downarrow}$ in $\mathbf{L}_+^0$, for each rational number $r \in [0, T[$. Then passing to diagonal subsequence, we suppose that this convergence holds true for all rationals $r \in [0, T[$. The four limit processes $(\varphi_r^{0,\uparrow})_{r \in \mathbb{Q} \cap [0,T[}$ etc. are a.s. increasing, nonnegative.

We have to consider these processes at all time $t \in [0, T]$. To do this, we let

$$\bar{\varphi}_t^{0,\uparrow} = \lim_{r \searrow t, r \in \mathbb{Q}} \widetilde{\varphi}_r^{0,\uparrow}, \quad 0 \le t < T.$$

It is càdlàg with $\bar{\varphi}_0^{0,\uparrow} = 0$ and $\bar{\varphi}_T^{0,\uparrow} = \varphi_T^{0,\uparrow}$. Then we exhaust the jumps of the process $(\bar{\varphi}_t^{0,\uparrow})_{0 \le t \le T}$ by a sequence $(\tau_k)_{k=1}^\infty$ of stopping times. By passing once more to a sequence of convex combinations, still denoted by $(\bar{\varphi}^{0,n,\uparrow})_{n=1}^\infty$, we assume that $(\varphi_{\tau_k}^{0,n,\uparrow})_{n=1}^\infty$ converges almost surely for each $k \in \mathbb{N}$.

Define

$$\varphi_t^{0,\uparrow} = \begin{cases} \lim_{n\to\infty} \varphi_{\tau_k}^{0,n,\uparrow}, & \text{if } t = \tau_k, \text{ for some } k \in \mathbb{N}, \\ \bar{\varphi}_t^{0,\uparrow}, & \text{otherwise.} \end{cases}$$

The process is predictable. Indeed, there is a subset $A \subseteq \Omega$ of full measure $\mathbf{P}(A) = 1$, such that $(\varphi^{0,n,\uparrow})_{n=1}^\infty$ converges pointwisely to $\varphi^{0,\uparrow}$ everywhere on $A \times [0, T]$. The processes $\varphi^{0,\uparrow}$ is also a.s. increasing in $t \in [0, T]$. Therefore, we have found a predictable process $\varphi^{0,\uparrow} = (\varphi_t^{0,\uparrow})_{0 \le t \le T}$ such that the sequence $(\varphi_t^{0,n,\uparrow})_{0 \le t \le T}$ converges to $\varphi^{0,\uparrow} = (\varphi_t^{0,\uparrow})_{0 \le t \le T}$ a.s. for all $t \in [0, T]$.

In a similar way, we can find the other three limit processes $\varphi^{1,\uparrow}, \varphi^{1,\downarrow}, \varphi^{0,\downarrow}$. They are predictable, increasing and satisfy the self-financing conditions.

Finally, we define the process $(\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ as $(\varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow})_{0 \le t \le T}$. It is predictable, nonnegative and satisfies the self-financing conditions because of the convergence of the processes $(\varphi^n)_{n=1}^\infty$, for all $t \in [0, T]$. $\square$

In what follows, we shall establish the existence and uniqueness for the primal solution of (4.1.3). In the frictionless case, a similar result without random endowment has been proved in [80] (compare also [41] and [5]). The author of [80] applied a technical lemma ([80, Lemma 3.16]) to conduct a proof by contradiction and show that the limit of a maximizing sequence indeed solves the utility maximization problem. We find that the assumption $\{f_n\}_{n\in\mathbb{N}} \geq 0$ in [80, Lemma 3.16] is not essentially needed for proceeding the argument. Thus, we reorganize a lemma in the Appendix (Lemma 5.1.2) and for the sake of completeness, we also give the proof of the following theorem.

Theorem 4.1.18. *Let Assumptions 2.1.6, 3.1.1, 4.1.1 and 4.1.4 hold. Assume moreover that*

$$u(x; e_T) < \infty.$$

Then, the utility maximization problem (4.1.3) *admits a unique solution* $\widehat{g} \in \mathcal{C}^\lambda(x)$.

Proof. (i) Since $u(x; e_T) < \infty$, we could pick a maximizing sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\lambda(x)$ for (4.1.3), i.e.,

$$u(x; e_T) = \lim_{n\to\infty} \mathbb{E}[U(g_n + e_T)].$$

By passing to a sequence of convex combinations $\operatorname{conv}(g_n, g_{n+1}, \ldots)$, still denoted by g_n, and applying the Komlós' theorem (e.g. [22, Lemma A1.1a]) as well as Lemma 4.1.17, we may suppose that g_n converges a.s. to $\widehat{g} \in \mathcal{C}^\lambda(x)$.

(ii) We claim that $(U(\widehat{g} + e_T))^+$ is integrable and thus $\mathbb{E}[U(\widehat{g} + e_T)]$ exists. Without loss of generality, assume that $U(1) = 0$. For any $g \in \mathcal{C}^\lambda(x)$, $g + 1 \in \mathcal{C}^\lambda(x + 1)$. It is easy to verify that u is still a concave function in x and thus $u(x; e_T) < \infty$ implies $u(x + 1; e_T) < \infty$. We suppose $(U(\widehat{g} + e_T))^+$ is not integrable, then

$$\infty = \mathbb{E}[(U(\widehat{g} + e_T))^+] < \mathbb{E}[(U(\widehat{g} + 1 + e_T))^+] = \mathbb{E}[U(\widehat{g} + 1 + e_T)] \leq u(x + 1) < \infty,$$

which is contradiction. Therefore, $\mathbb{E}[U(\widehat{g} + e_T)]$ exists.

(iii) We now prove that $\widehat{g}$ is the primal optimizer by applying Lemma 5.1.2. If not, there exists an $\alpha \in (0, \infty]$ such that

$$\alpha = u(x; e_T) - \mathbb{E}[U(\widehat{g} + e_T)].$$

For each n, denote $f_n = U(g_n + e_T)$ and $f_0 = U(\widehat{g} + e_T)$. Fixing $\varepsilon > 0$, there exists an $m' \in \mathbb{N}$, such that for each $n \geq m'$,

$$u(x; e_T) - \mathbb{E}[f_n] \leq \varepsilon. \tag{4.1.9}$$

Since $AE(U) < 1$, by [63, Lemma 6.3], there exists some $\gamma > 1$, such that $U(\frac{x}{2}) > \frac{\gamma}{2}U(x)$, for all $x \geq x_0 > 0$. Note that for each $n \in \mathbb{N}$ and any $M > 0$,

$$\mathbf{P}\big[f_n \geq M\big] \leq \mathbf{P}\left[|f_n - f_0| \geq \frac{M}{2}\right] + \mathbf{P}\left[f_0^+ \geq \frac{M}{2}\right].$$

Thus, for any $\delta > 0$, we can choose sufficiently large $M > 0$ with $U^{-1}(M) \geq 2x_0$ and find a $m_0 \geq m'$ such that for any $n \geq m_0$, $\mathbf{P}[f_n \geq M] \leq \delta$. Due to the integrability of $f_{m'}$, for properly chosed δ, $\mathbb{E}\big[|f_{m'}|\mathbf{1}_{\{f_n\geq M\}}\big] \leq \varepsilon$ holds for any $n \geq m_0$. From Lemma 5.1.2, we fix $m > m_0$ such that

$$\mathbb{E}\big[f_m\mathbf{1}_{\{f_m\geq M\}}\big] \geq \alpha - \varepsilon \quad \text{and} \quad \mathbb{E}\big[|f_{m'}|\mathbf{1}_{\{f_m\geq M\}}\big] \leq \varepsilon. \tag{4.1.10}$$

Then,

$$\mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\right] = \mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\mathbf{1}_{\{f_m\geq M\}}\right] + \mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\mathbf{1}_{\{f_m< M\}}\right].$$

Furthermore, due to the positivity of $g_{m'}$, g_m and e_T, we have

$$\mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\mathbf{1}_{\{f_m\geq M\}}\right] \geq \frac{\gamma}{2}\mathbb{E}\left[U\left(g_{m'}+g_m+2e_T\right)\mathbf{1}_{\{f_m\geq M\}}\right] \geq \frac{\gamma}{2}\mathbb{E}\left[f_m\mathbf{1}_{\{f_m\geq M\}}\right]$$

and

$$\mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\mathbf{1}_{\{f_m< M\}}\right] \geq \frac{1}{2}\mathbb{E}\left[f_{m'}\mathbf{1}_{\{f_m< M\}}\right] + \frac{1}{2}\mathbb{E}\left[f_m\mathbf{1}_{\{f_m< M\}}\right].$$

Therefore, we can deduce from (4.1.9) and (4.1.10),

$$\begin{aligned}\mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\right] &\geq \frac{1}{2}\mathbb{E}\left[f_{m'}\mathbf{1}_{\{f_m< M\}}\right] + \frac{1}{2}\mathbb{E}\left[f_m\right] + \frac{\gamma-1}{2}\mathbb{E}\left[f_m\mathbf{1}_{\{f_m\geq M\}}\right] \\ &\geq u(x;e_T) + \frac{(\gamma-1)\alpha}{2} - \frac{\gamma+2}{2}\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\mathbb{E}\left[U\left(\frac{g_{m'}+g_m}{2}+e_T\right)\right] > u(x;e_T),$$

which is a contradiction to the maximality of $u(x;e_T)$.

The uniqueness is due to the strict concavity of U. Let $\widehat{f},\ \widehat{g}\in\mathcal{C}^\lambda(x)$ be two solutions to the primal problem (4.1.3). Fix $\alpha\in[0,1]$, then $\alpha\widehat{f}+(1-\alpha)\widehat{g}\in\mathcal{C}^\lambda(x)$, so we have

$$\mathbb{E}[U(\alpha\widehat{f}+(1-\alpha)\widehat{g})] > \alpha\mathbb{E}[U(\widehat{f})] + (1-\alpha)\mathbb{E}[U(\widehat{g})] = u(x),$$

which is contradiction to the optimality of $u(x)$. □

Existence of shadow prices

The existence of shadow prices for the utility maximization problem with neither the no-short-selling constraint nor random endowment has been studied in [55, 15, 17, 19] by duality methods. Here we shall construct shadow price directly from the derivatives of dynamic primal value functions.

Now we turn to consider the frictionless market associated with S^Z, for $Z\in\mathcal{Z}^\lambda_{sup}$. Similarly to Lemma 4.1.13, we have the supermartingale property of $Z^0\widetilde{\varphi}^0+Z^1\widetilde{\varphi}^1$ for $(\widetilde{\varphi}^0,\widetilde{\varphi}^1)\in\mathcal{A}^Z(x)$. The subsequent lemma has been reviewed in [5, Lemma 4.1]. However, for the convenience of the reader, we prove it in the numéraire-based case.

Lemma 4.1.19. *Fix $Z \in \mathcal{Z}^{\lambda}_{sup}$. The process $Z^0\widetilde{\varphi}^0 + Z^1\widetilde{\varphi}^1$ is a positive supermartingale, for any $(\widetilde{\varphi}^0, \widetilde{\varphi}^1) \in \mathcal{A}^Z(x)$.*

Proof. Note that $Z^0\widetilde{\varphi}^0 + Z^1\widetilde{\varphi}^1 = Z^0\big(\widetilde{\varphi}^0 + \widetilde{\varphi}^1 S^Z\big) = Z^0\big(x + \widetilde{\varphi}^1 \bullet S^Z\big)$ by the frictionless self-financing condition. Using Itô's formula and [36, Proposition A.1], we obtain that

$$\begin{aligned} Z^0_t\widetilde{\varphi}^0_t + Z^1_t\widetilde{\varphi}^1_t &= xZ^0_0 + \big((\widetilde{\varphi}^0_- + \widetilde{\varphi}^1_- S^Z_-) \bullet Z^0\big)_t + \big(Z^0_- \bullet (\widetilde{\varphi}^1 \bullet S^Z)\big)_t + \big[\widetilde{\varphi}^1 \bullet S^Z, Z^0\big]_t \\ &= xZ^0_0 + \big((\widetilde{\varphi}^0_- + \widetilde{\varphi}^1_- S^Z_-) \bullet Z^0\big)_t + \big(\widetilde{\varphi}^1 \bullet (Z^0_- \bullet S^Z)\big)_t + \big(\widetilde{\varphi}^1 \bullet \big[S^Z, Z^0\big]\big)_t. \end{aligned}$$

It follows from the frictionless self-financing condition again and [49, I.4.36] that

$$\Delta(\widetilde{\varphi}^0 + \widetilde{\varphi}^1 S^Z) = \Delta(\widetilde{\varphi}^1 \bullet S^Z) = \widetilde{\varphi}^1 \Delta S^Z,$$

therefore, $\widetilde{\varphi}^0_- + \widetilde{\varphi}^1_- S^Z_- = \widetilde{\varphi}^0 + \widetilde{\varphi}^1 S^Z_-$. By [49, I.4.37, Definition I.4.45], we obtain

$$\begin{aligned} Z^0_t\widetilde{\varphi}^0_t + Z^1_t\widetilde{\varphi}^1_t &= xZ^0_0 + \big(\widetilde{\varphi}^0 \bullet Z^0\big)_t + \big(\widetilde{\varphi}^1 \bullet \big(S^Z_- \bullet Z^0 + Z^0_- \bullet S^Z + [S^Z, Z^0]\big)\big)_t \\ &= xZ^0_0 + \big(\widetilde{\varphi}^0 \bullet Z^0\big)_t + \big(\widetilde{\varphi}^1 \bullet (S^Z Z^0)\big)_t \\ &= xZ^0_0 + \big(\widetilde{\varphi}^0 \bullet Z^0\big)_t + \big(\widetilde{\varphi}^1 \bullet Z^1\big)_t, \end{aligned}$$

which is a positive local supermartingale and hence a supermartingale. □

Then, it is easy to deduce that for each $\widetilde{g} \in \mathcal{C}^Z(x)$ and $Z \in \mathcal{Z}^{\lambda}_{sup}$,

$$\mathbb{E}[U(\widetilde{g} + e_T)] \leq \mathbb{E}[V(Z^0_T) + Z^0_T(\widetilde{g} + e_T)] \leq \mathbb{E}[V(Z^0_T)] + \mathbb{E}[Z^0_T e_T] + Z^0_0 x, \tag{4.1.11}$$

where V is the conjugate of U. Therefore, to prove the existence of shadow prices, it suffices to have the following lemma.

Lemma 4.1.20. *Let Assumptions 2.1.6, 3.1.1, 4.1.1 and 4.1.4 hold. There exists a $\widehat{Z} \in \mathcal{Z}^{\lambda}_{sup}$, such that*

(i) $\widehat{Z}^0_T = U'(\widehat{g} + e_T)$;

(ii) $\mathbb{E}[\widehat{Z}^0_T \widehat{g}] = \widehat{Z}^0_0 x$.

Theorem 4.1.21. *The λ-supermartingale-CPS $\widehat{Z} \in \mathcal{Z}^{\lambda}_{sup}$ satisfying Lemma 4.1.20 (i)-(ii) defines a shadow price $S^{\widehat{Z}} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$.*

Proof. Consider the frictionless market associated with $S^{\widehat{Z}}$. By Lemma 4.1.20, we have

$$\begin{aligned} u^{\widehat{Z}}(x; e_T) \geq u(x; e_T) = \mathbb{E}[U(\widehat{g} + e_T)] &= \mathbb{E}[V(\widehat{Z}^0_T) + \widehat{Z}^0_T(\widehat{g} + e_T)] \\ &= \mathbb{E}[V(\widehat{Z}^0_T)] + \mathbb{E}[\widehat{Z}^0_T e_T] + \widehat{Z}^0_0 x \geq u^{\widehat{Z}}(x; e_T), \end{aligned}$$

where the last inequality follows from (4.1.11). The inequality above implies $u^{\widehat{Z}}(x; e_T) = u(x; e_T)$, which proves that $S^{\widehat{Z}}$ is a shadow price for the problem (4.1.3). □

Remark 4.1.22. As has been indicated in the proof of Theorem 4.1.18, by the strict concavity of U, $\widehat{g}$ is the unique solution in $\mathcal{C}^{\widehat{Z}}(x)$ for the frictionless problem $u^{\widehat{Z}}(x;e_T)$. Moreover, the trading strategy $\widehat{\varphi}$, that attains the maximality in the market with transaction costs, does the same in the frictionless one associated with the shadow price $S^{\widehat{Z}}$. Therefore, the optimal trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ for S under transaction costs λ satisfies

$$\begin{aligned}\{d\widehat{\varphi}_t^1 > 0\} &\subseteq \{S_t^{\widehat{Z}} = S_t\},\\ \{d\widehat{\varphi}_t^1 < 0\} &\subseteq \{S_t^{\widehat{Z}} = (1-\lambda)S_t\},\end{aligned}$$

for all $0 \le t \le T$.

Remark 4.1.23. In our case, shadow prices are determined not only by the random endowment but also by its decomposition (see Assumption 4.1.1). The decomposition of the random endowment together with the no-short-selling constraints can be explained as the agent's trading rule created by his/her controller. Precisely, if the random endowment $\widetilde{e}_T$ that the agent will eventually receive is decomposed into $x + e_T$ by his/her controller, then it means that the agent is allowed to spend at most x in the bond market for trading the stock. Thus, the different ways of decomposition mean the different limits of short selling in bond, which lead to different maximal utilities and also shadow prices.

In what follows, we shall prove Lemma 4.1.20. Generally speaking, we shall follow the line of construction of a shadow price in [5, page 814-816]. However, we still give the details in order to show how it develops in the numéraire-based context and how a positive random endowment works. The proof is divided into several stages.

Firstly, similar to [5, Lemma 4.4], we have the following dynamic programming principle (see also [30, Theorem 1.17]), which could be proved in a direct way with the numéraire-based model.

Proposition 4.1.24. *Define*

$$\mathcal{U}_s(\varphi_s^0, \varphi_s^1) := \operatorname*{ess\,sup}_{(\psi^0,\psi^1)\in\mathcal{A}^\lambda_{s,T}(\varphi_s^0,\varphi_s^1)} \mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\,\mathcal{F}_s\right],$$

where $\mathcal{A}^\lambda_{s,T}(\varphi_s^0, \varphi_s^1)$ is the set of all NSS-admissible λ-self-financing trading strategies, which agree with $\varphi \in \mathcal{A}^\lambda(x)$ in $[0, s]$.

Then, the process $\big(\mathcal{U}_s(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)\big)_{0\le s\le T}$ is a martingale, i.e.,

$$\mathcal{U}_s(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1) = \mathbb{E}\left[\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)\middle|\,\mathcal{F}_s\right] \text{ a.s., } 0 \le s < t \le T,$$

for all optimal trading strategies $\widehat{\varphi}$ attaining $\widehat{g}$.

Proof. Without loss of generality, it suffices to verify the following claim

$$\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\,\mathcal{F}_s\right] \le \mathbb{E}\left[U(\widehat{\varphi}_T^0 + e_T)\middle|\,\mathcal{F}_s\right], \tag{4.1.12}$$

for all $(\psi^0, \psi^1) \in \mathcal{A}^\lambda_{s,T}(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)$.

To obtain a contradiction, we suppose that (4.1.12) is not true, i.e., there exists a $(\psi^0,\psi^1)\in\mathcal{A}^\lambda_{s,T}(\widehat{\varphi}^0_s,\widehat{\varphi}^1_s)$ and a set $A\subseteq\Omega$ with $\mathbf{P}(A)>0$ defined as

$$A:=\left\{\mathbb{E}[U(\psi^0_T+e_T)|\mathcal{F}_s]>\mathbb{E}[U(\widehat{\varphi}^0_T+e_T)|\mathcal{F}_s]\right\}\in\mathcal{F}_s. \tag{4.1.13}$$

Then define

$$(\psi^0,\psi^1)\mathbf{1}_A+(\widehat{\varphi}^0,\widehat{\varphi}^1)\mathbf{1}_{A^c}=:(\eta^0,\eta^1)\in\mathcal{A}^\lambda_{s,T}(\widehat{\varphi}^0_s,\widehat{\varphi}^1_s).$$

We have

$$\mathbb{E}\left[U(\eta^0_T+e_T)\right]>\mathbb{E}\left[U(\widehat{\varphi}^0_T+e_T)\right]=u(x;e_T),$$

which is in contradiction to the maximality of $\widehat{\varphi}$. Thus, by the definition of $\mathcal{U}_s(\varphi^0_s,\varphi^1_s)$ and (4.1.12), we obtain

$$\begin{aligned}\mathcal{U}_s(\widehat{\varphi}^0_s,\widehat{\varphi}^1_s)&=\operatorname*{ess\,sup}_{(\psi^0,\psi^1)\in\mathcal{A}^\lambda_{s,T}(\widehat{\varphi}^0_s,\widehat{\varphi}^1_s)}\mathbb{E}\left[U(\psi^0_T+e_T)\middle|\mathcal{F}_s\right]=\mathbb{E}\left[U(\widehat{\varphi}^0_T+e_T)\middle|\mathcal{F}_s\right]\\&=\mathbb{E}\left[\mathbb{E}\left[U(\widehat{\varphi}^0_T+e_T)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\mathcal{U}_t(\widehat{\varphi}^0_t,\widehat{\varphi}^1_t)\middle|\mathcal{F}_s\right].\end{aligned}$$

□

The next step goes in an exactly same way as in [5], i.e., we should first construct a pair $\widehat{Z}=(\widehat{Z}^0_t,\widehat{Z}^1_t)_{0\le t\le T}$, then verify the shadow price can be defined by $\frac{\widehat{Z}^1}{\widehat{Z}^0}$. The additional positive e_T in the dynamic will not alter the following results.

Proposition 4.1.25. *The following processes are well defined:*

$$\begin{cases}\widetilde{Z}^0_t:=\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(\mathcal{U}_t(\widehat{\varphi}^0_t+\varepsilon,\widehat{\varphi}^1_t)-\mathcal{U}_t(\widehat{\varphi}^0_t,\widehat{\varphi}^1_t)\big),\\ \widetilde{Z}^1_t:=\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(\mathcal{U}_t(\widehat{\varphi}^0_t,\widehat{\varphi}^1_t+\varepsilon)-\mathcal{U}_t(\widehat{\varphi}^0_t,\widehat{\varphi}^1_t)\big),\end{cases} \tag{4.1.14}$$

for $0\le t<T$, *and*

$$\begin{cases}\widetilde{Z}^0_T:=U'(\widehat{\varphi}_T+e_T),\\ \widetilde{Z}^1_T:=U'(\widehat{\varphi}_T+e_T)(1-\lambda)S_T.\end{cases} \tag{4.1.15}$$

Furthermore, define

$$\begin{cases}\widehat{Z}^i_t:=\lim_{\substack{s\searrow t\\ s\in\mathbb{Q}}}\widetilde{Z}^i_s, & 0\le t<T;\\ \widehat{Z}^i_t:=\widetilde{Z}^i_T, & t=T.\end{cases} \tag{4.1.16}$$

Then, the process $\widehat{Z}$ is a càdlàg supermartingale and moreover, for all $0\le t\le T$, we have

$$(1-\lambda)S_t\le\frac{\widehat{Z}^1_t}{\widehat{Z}^0_t}\le S_t,\quad a.s. \tag{4.1.17}$$

Consequently, $\widehat{Z}$ is a λ-supermartingale-CPS.

The proposition will be showed by the following lemmas.

Lemma 4.1.26. *For each $t \in [0,T]$, the random variables $\widetilde{Z}_t^i,\ i = 0,1$ are well defined as the limit of an increasing sequence.*

Proof. Consider $\varepsilon_1 > \varepsilon_2 > 0$, it follows from the the concavity of U and the definition of $\mathcal{U}_t$ that

$$\begin{aligned}\mathcal{U}_t(\widehat{\varphi}_t^0 + \varepsilon_2, \widehat{\varphi}_t^1) &= \mathcal{U}_t\left(\frac{\varepsilon_2}{\varepsilon_1}(\widehat{\varphi}_t^0 + \varepsilon_1) + (1 - \frac{\varepsilon_2}{\varepsilon_1})\widehat{\varphi}_t^0, \widehat{\varphi}_t^1\right) \\ &\geq \frac{\varepsilon_2}{\varepsilon_1}\mathcal{U}_t((\widehat{\varphi}_t^0 + \varepsilon_1), \widehat{\varphi}_t^1) + (1 - \frac{\varepsilon_2}{\varepsilon_1})\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1),\end{aligned}$$

which implies

$$\frac{\mathcal{U}_t(\widehat{\varphi}_t^0 + \varepsilon_2, \widehat{\varphi}_t^1) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_2} \geq \frac{\mathcal{U}_t(\widehat{\varphi}_t^0 + \varepsilon_1, \widehat{\varphi}_t^1) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_1}.$$

Similarly, we have

$$\frac{\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \varepsilon_2) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_2} \geq \frac{\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \varepsilon_1) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)}{\varepsilon_1}.$$

Thus, $\widetilde{Z}_t^0,\ \widetilde{Z}_t^1$ are well-defined as the limit of an increasing sequence. □

For simplicity, we rewrite $\widetilde{Z}_t^i := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)),\ i = 0,1,$ and $\ t \in [0,T]$.

Lemma 4.1.27. *The family*

$$\left\{\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] \,\middle|\, (\psi^0, \psi^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)\right\}$$

is directed upwards (see e.g., [32, Theorem A. 32]), i.e., for all (ψ^0, ψ^1) and (φ^0, φ^1) in $\mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)$, there exists $(\eta^0, \eta^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)$ such that

$$\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] \vee \mathbb{E}\left[U(\varphi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] \leq \mathbb{E}\left[U(\eta_T^0 + e_T)\middle|\, \mathcal{F}_t\right],$$

then there exists an increasing sequence $(\psi^n)_{n\in\mathbb{N}} = (\psi^{n,0}, \psi^{n,1})_{n\in\mathbb{N}} \subseteq \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)$ such that

$$\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) = \nearrow - \lim_{n\to\infty} \mathbb{E}\left[U\big(\psi_T^{n,0} + e_T\big)\middle|\, \mathcal{F}_t\right].$$

Proof. Define a set

$$A := \left\{\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] > \mathbb{E}\left[U(\varphi_T^0 + e_T)\middle|\, \mathcal{F}_t\right]\right\} \in \mathcal{F}_t$$

and the process

$$(\psi^0, \psi^1)\mathbf{1}_A + (\varphi^0, \varphi^1)\mathbf{1}_{A^c} =: (\eta^0, \eta^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i).$$

Easily, we have

$$\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] \vee \mathbb{E}\left[U(\varphi_T^0 + e_T)\middle|\, \mathcal{F}_t\right] \leq \mathbb{E}\left[U(\eta_T^0 + e_T)\middle|\, \mathcal{F}_t\right],$$

so the proof is completed. □

Proposition 4.1.28. *The process $\widetilde{Z}$ defined in* (4.1.14) *and* (4.1.15) *is a (not necessarily càdlàg) supermartingale.*

Proof. Clearly, $\mathcal{A}^{\lambda}_{t,T}(\widehat{\varphi}_t + \varepsilon e_i) \subseteq \mathcal{A}^{\lambda}_{s,T}(\widehat{\varphi}_t + \varepsilon e_i)$, for $0 \leq s \leq t < T$. Then we have

$$\begin{aligned}
\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) &= \operatorname*{ess\,sup}_{(\psi^0,\psi^1)\in\mathcal{A}^{\lambda}_{s,T}(\widehat{\varphi}_t+\varepsilon e_i)} \mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_s\right] \\
&\geq \operatorname*{ess\,sup}_{(\psi^0,\psi^1)\in\mathcal{A}^{\lambda}_{t,T}(\widehat{\varphi}_t+\varepsilon e_i)} \mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_s\right] \\
&= \operatorname*{ess\,sup}_{(\psi^0,\psi^1)\in\mathcal{A}^{\lambda}_{t,T}(\widehat{\varphi}_t+\varepsilon e_i)} \mathbb{E}\left[\mathbb{E}\left[U(\psi_T^0 + e_T)\middle|\, \mathcal{F}_t\right]\middle|\, \mathcal{F}_s\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[U(\psi_T^{n,0} + e_T)\middle|\, \mathcal{F}_t\right]\middle|\, \mathcal{F}_s\right].
\end{aligned}$$

By the monotone convergence theorem,

$$\begin{aligned}
\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) &\geq \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[U(\psi_T^{n,0} + e_T)\middle|\, \mathcal{F}_t\right]\middle|\, \mathcal{F}_s\right]. \\
&= \mathbb{E}\left[\lim_{n\to\infty} \mathbb{E}\left[U(\psi_T^{n,0} + e_T)\middle|\, \mathcal{F}_t\right]\middle|\, \mathcal{F}_s\right]. \\
&= \mathbb{E}\left[\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i)\middle|\, \mathcal{F}_s\right],
\end{aligned}$$

which shows

$$\frac{\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) - \mathcal{U}_s(\widehat{\varphi}_s)}{\varepsilon} \geq \mathbb{E}\left[\frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon}\middle|\, \mathcal{F}_s\right].$$

Therefore, for $0 \leq t < T$, by definition of $\widetilde{Z}^i_t, i = 0, 1$,

$$\begin{aligned}
\widetilde{Z}^i_s :&= \lim_{\varepsilon\searrow 0} \frac{\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) - \mathcal{U}_s(\widehat{\varphi}_s)}{\varepsilon} \\
&\geq \lim_{\varepsilon\searrow 0} \mathbb{E}\left[\frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon}\middle|\, \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\lim_{\varepsilon\searrow 0} \frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon}\middle|\, \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\widetilde{Z}^i_t\middle|\, \mathcal{F}_s\right].
\end{aligned}$$

Because of the different definition of $\widetilde{Z}$ at time T, we have to verify it at $t = T$. For $i = 0$, by the definition of $\mathcal{U}$ that

$$\mathcal{U}_s(\widehat{\varphi}^0_s + \varepsilon, \widehat{\varphi}^1_s) \geq \mathbb{E}\left[U(\widehat{\varphi}^0_T + \varepsilon + e_T)\middle|\, \mathcal{F}_s\right] \text{ and } \mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s) = \mathbb{E}\left[U(\widehat{\varphi}^0_T + e_T)\middle|\, \mathcal{F}_s\right]$$

and the monotone convergence theorem that

$$
\begin{aligned}
\widetilde{Z}^0_s &:= \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_s(\widehat{\varphi}^0_s + \varepsilon, \widehat{\varphi}^1_s) - \mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s)}{\varepsilon} \\
&\geq \lim_{\varepsilon \searrow 0} \mathbb{E}\left[\left. \frac{U(\widehat{\varphi}^0_T + \varepsilon + e_T) - U(\widehat{\varphi}^0_T + e_T)}{\varepsilon} \right| \mathcal{F}_s \right] \\
&= \mathbb{E}\left[U'(\widehat{\varphi}^0_T + e_T) \middle| \mathcal{F}_s \right] \\
&= \mathbb{E}\left[\widetilde{Z}^0_T \middle| \mathcal{F}_s \right].
\end{aligned}
$$

For the second component $\widetilde{Z}^1$, by the similar way, we have

$$
\mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s + \varepsilon) \geq \mathbb{E}\left[U(\widehat{\varphi}^0_T + \varepsilon(1-\lambda)S_T + e_T) \middle| \mathcal{F}_s \right] \text{ and } \mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s) = \mathbb{E}\left[U(\widehat{\varphi}^0_T + e_T) \middle| \mathcal{F}_s \right],
$$

so

$$
\begin{aligned}
\widetilde{Z}^1_s &:= \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s + \varepsilon) - \mathcal{U}_s(\widehat{\varphi}^0_s, \widehat{\varphi}^1_s)}{\varepsilon} \\
&\geq \lim_{\varepsilon \searrow 0} \mathbb{E}\left[\left. \frac{U(\widehat{\varphi}^0_T + \varepsilon(1-\lambda)S_T + e_T) - U(\widehat{\varphi}^0_T + e_T)}{\varepsilon} \right| \mathcal{F}_s \right] \\
&= \mathbb{E}\left[U'(\widehat{\varphi}^0_T + e_T)(1-\lambda)S_T \middle| \mathcal{F}_s \right] \\
&= \mathbb{E}\left[\widetilde{Z}^1_T \middle| \mathcal{F}_s \right].
\end{aligned}
$$

Now we should check $\widetilde{Z}^0_0$:

$$
\widetilde{Z}^0_0 := \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_0(\widehat{\varphi}^0_0 + \varepsilon, \widehat{\varphi}^1_0) - \mathcal{U}_0(\widehat{\varphi}^0_0, \widehat{\varphi}^1_0)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{u(x+\varepsilon; e_T) - u(x; e_T)}{\varepsilon} < \infty,
$$

for $x > 0$, as $u(x; e_T) < \infty$ and concave on $\mathbb{R}_+$. □

Proposition 4.1.29. *The process $\widetilde{Z}$ satisfies*

$$
\frac{\widetilde{Z}^1_t}{\widetilde{Z}^0_t} \in [(1-\lambda)S_t, S_t], \text{ a.s., for all } 0 \leq t \leq T.
$$

Proof. By (4.1.15), it is obvious that $\frac{\widetilde{Z}^1_T}{\widetilde{Z}^0_T} \in [(1-\lambda)S_T, S_T]$ at the terminal time T. Fix $0 \leq t < T$, let $(k^n_l)_{l \geq 0}$ be a partition of $[0, \infty)$, with mesh size decreasing to 0 as n goes to infinity. Note that for all $\epsilon > 0$, on the set $A_l := \{k^n_l < S_t \leq k^n_{l+1}\}$, we observe that

$$
\mathcal{U}_t(\widehat{\varphi}^0_t + \varepsilon, \widehat{\varphi}^1_t) \geq \mathcal{U}_t\left(\widehat{\varphi}^0_t, \widehat{\varphi}^1_t + \frac{\varepsilon}{S_t}\right) \geq \mathcal{U}_t\left(\widehat{\varphi}^0_t, \widehat{\varphi}^1_t + \frac{\varepsilon}{k^n_{l+1}}\right).
$$

Therefore,

$$
\frac{\mathcal{U}_t(\widehat{\varphi}^0_t + \varepsilon, \widehat{\varphi}^1_t) - \mathcal{U}_t(\widehat{\varphi}^0_t, \widehat{\varphi}^1_t)}{\varepsilon} \geq \frac{\mathcal{U}_t\left(\widehat{\varphi}^0_t, \widehat{\varphi}^1_t + \frac{\varepsilon}{k^n_{l+1}}\right) - \mathcal{U}_t(\widehat{\varphi}^0_t, \widehat{\varphi}^1_t)}{\varepsilon}.
$$

Again using monotone convergence, this implies

$$\widetilde{Z}_t^0 \mathbf{1}_{A_l} \geq \widetilde{Z}_t^1 \frac{1}{k_{l+1}^n} \mathbf{1}_{A_l},$$

and thus

$$\sum_{l \geq 0} \mathbf{1}_{A_l} \widetilde{Z}_t^0 \geq \sum_{l \geq 0} \mathbf{1}_{A_l} \frac{1}{k_{l+1}^n} \widetilde{Z}_t^1.$$

Then letting $n \to \infty$, we obtain $S_t \geq \frac{\widetilde{Z}_t^1}{\widetilde{Z}_t^0},\ 0 \leq t < T$.

For the other inequality, by the observation that

$$\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \varepsilon) \geq \mathcal{U}_t(\widehat{\varphi}_t^0 + \varepsilon(1-\lambda)S_t, \widehat{\varphi}_t^1).$$

With the similar method, we have $(1-\lambda)S_t \leq \frac{\widetilde{Z}_t^1}{\widetilde{Z}_t^0},\ 0 \leq t < T$. □

Proposition 4.1.30. *The process $\widehat{Z}$ belongs to $\mathcal{Z}_{sup}^\lambda$, i.e., it is a càdlàg supermartingale satisfying*

$$\frac{\widehat{Z}_t^1}{\widehat{Z}_t^0} \in [(1-\lambda)S_t, S_t],\ \textit{a.s., for all } 0 \leq t \leq T.$$

Proof. By [59, Proposition 1.3.14(iii)], (4.1.16) is a well-defined càdlàg supermartingale. In particular, $\widehat{Z}_0^0 \leq \widetilde{Z}_0^0$. As S is càdlàg, (4.1.17) holds true. □

Proof of Lemma 4.1.20. It remains to proof (2) in Lemma 4.1.20. Since $\left(\widehat{Z}^0, \widehat{Z}^1\right) \in \mathcal{Z}_{sup}^\lambda$, we have

$$\mathbb{E}\left[\widehat{Z}_T^0 \widehat{g}\right] \leq \widehat{Z}_0^0 x. \tag{4.1.18}$$

It remains to show the reversed inequality. For $0 < \alpha < 1$, we note that $u(\alpha x; e_T) \geq \mathbb{E}[U(\alpha \widehat{g} + e_T)]$. By the concavity of u, we obtain

$$\widetilde{Z}_0^0(x - \alpha x) \leq u(x; e_T) - u(\alpha x; e_T) \leq \mathbb{E}[U(\widehat{g} + e_T)] - \mathbb{E}[U(\alpha \widehat{g} + e_T)].$$

Therefore, it follows from the strict concavity and the continuous differentiability of U that

$$\widetilde{Z}_0^0 x \leq \mathbb{E}\left[\frac{U(\widehat{g} + e_T) - U(\alpha \widehat{g} + e_T)}{1-\alpha}\right] \leq \mathbb{E}\left[U'(\alpha \widehat{g} + e_T)\widehat{g}\right]. \tag{4.1.19}$$

Letting $\alpha \nearrow 1$, monotone convergence yields

$$\widehat{Z}_0^0 x \leq \widetilde{Z}_0^0 x \leq \mathbb{E}\left[U'(\widehat{g} + e_T)\widehat{g}\right] \leq \mathbb{E}\left[\widehat{Z}_T^0 \widehat{g}\right]. \tag{4.1.20}$$

We complete the proof by comparing (4.1.18) and (4.1.20). □

Remark 4.1.31. We have seen here that the nonnegativity of the random endowment is important. This ensures the strict positivity of $\alpha \widehat{g} + e_T$, for $0 < \alpha < 1$, such that $\mathbb{E}\left[U'(\alpha \widehat{g} + e_T)\widehat{g}\right]$ in (4.1.19) is well defined.

Remark 4.1.32. We study in this paper the numéraire-based two-asset model just for the sake of simplicity, however, the argument could adapt to the multi-currency setting as in [5] with no essential changes. Notice that both the present paper and [5] concern a univariate utility function.

4.2 Example: when the random endowment becomes negative

We have discussed the existence of shadow price processes for the utility maximization problem with positive random endowment under the no-short-selling constraint of each asset. Actually, the positivity constraint on the random endowment is a sufficient condition. However, when the random endowment becomes negative, the existence of shadow prices is not clear if we keep our setting. This will be an interesting problem for our future research. We need to mention here that with a different definition of admissibility for portfolios, the existence of sandwiched shadow price is proved in [2], in which the random endowment could be negative.

Here, we provide a nontrivial and heuristic example showing that even when the random endowment becomes negative, there may exist a shadow price for the utility maximization problem in the market under transaction costs. In particular, we discuss maximal trading strategies in the context with transaction costs. Notice that the random endowment constructed in our example is associated with such a maximal trading strategy.

4.2.1 Financial model

We consider the Black-Scholes model with finite time horizon. Assume that the market consists of a saving account with zero interest rate and a stock with price dynamics

$$S_t = \begin{cases} \exp\left(B_t + \frac{t}{2}\right), & 0 \le t \le T/2; \\ S_{T/2}, & T/2 < t \le T, \end{cases}$$

where $(B_t)_{t\ge 0}$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbf{P})$.

Now fix a level of transaction costs $\lambda \in (0,1)$. We assume that an agent is endowed with the initial capital $x = 1$ at time 0 and the random endowment $e_T := -\zeta(1-\lambda)S_{T/2}$ at terminal time T, where ζ is an $\mathcal{F}_T$-measurable $(0,1)$-valued uniformly distributed random variable which is independent of $\mathcal{F}_{T/2}$. Then, we have the following proposition.

Proposition 4.2.1. *The buy-hold-sell strategy*

$$\left(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1\right) := \begin{cases} (1,0), & t = 0; \\ (0,1), & 0 < t < T/2; \\ \left((1-\lambda)S_{T/2}, 0\right), & T/2 \le t \le T, \end{cases} \tag{4.2.1}$$

solves the utility maximization problem (4.1.3).

Before proceeding the proof of this proposition, we first introduce the notion of maximal trading strategy in this market with transaction costs. Notice that a similar notion in frictionless markets is created by Delbaen and Schachermayer in [22] for considering a superreplication problem.

Definition 4.2.2. *An element $\varphi^0_T \in \mathcal{C}^\lambda(1)$ is called* **maximal**, *if for $\phi^0_T \in \mathcal{C}^\lambda(1)$ satisfying $\phi^0_T \geq \varphi^0_T$, a.s., we have $\phi^0_T = \varphi^0_T$, a.s. A trading strategy (φ^0, φ^1) is called* ***maximal*** *in $\mathcal{A}^\lambda(1)$, if it is associated with $(\varphi^0_T, 0)$, where φ^0_T is a maximal element in $\mathcal{C}^\lambda(1)$.*

Remark 4.2.3. The existence of a maximal element in $\mathcal{C}^\lambda(1)$ could be deduced by transfinite induction, provided that $\mathcal{C}^\lambda(1)$ is bounded in $\mathbf{L}^0$ and closed w.r.t. convergence in probability (cf. Schachermayer [83]).

Indeed, the trading strategy defined by (4.2.1) is maximal in $\mathcal{A}^\lambda(1)$. The proof is postponed to the next subsection.

Proposition 4.2.4. *The random variable $(1-\lambda)S_{T/2}$ is a maximal element in $\mathcal{C}^\lambda(1)$. Therefore, the trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ defined by (4.2.1) is maximal in $\mathcal{A}^\lambda(1)$.*

Before proving the proposition above, we first give the proof of Proposition 4.2.1 with the help of the maximality of the trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ defined in (4.2.1).

Proof of Proposition 4.2.1. Consider an element $\phi^0_T \in \mathcal{C}^\lambda(1)$ such that $\phi^0_T \neq \widehat{\varphi}^0_T = (1-\lambda)S_{T/2}$. From the maximality of $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, we have $\mathbf{P}\big[\phi^0_T < \widehat{\varphi}^0_T\big] > 0$. Therefore, there exists $k \in \mathbb{N}$, such that

$$\mathbf{P}\left[\phi^0_T < \left(1 - \tfrac{1}{k}\right)(1-\lambda)S_{T/2}\right] > 0.$$

Note that $\left\{\phi^0_T < \left(1 - \frac{1}{k}\right)(1-\lambda)S_{T/2}\right\} \in \mathcal{F}_{T/2}$ and ζ is independent of $\mathcal{F}_{T/2}$. Then,

$$\mathbf{P}\left[\phi^0_T < \left(1 - \tfrac{1}{k}\right)(1-\lambda)S_{T/2} \text{ and } 1 - \zeta < \tfrac{1}{k}\right] > 0,$$

which implies $\mathbf{P}[\phi^0_T + e_T < 0] > 0$. Consequently, $\mathbb{E}[U(\phi^0_T + e_T)] = -\infty$.

On the other hand, by the definition of ζ, we have

$$\widehat{\varphi}^0_T + e_T = (1-\zeta)(1-\lambda)S_{T/2} > 0, \text{ a.s.}$$

Thus, $u(1; e_T) = \mathbb{E}[U(\widehat{\varphi}^0_T + e_T)] > -\infty$. The proof is completed. □

Remark 4.2.5. Even without the no-short-selling constraint, $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ is still the unique strategy on $[0, T/2]$ solving the problem (4.1.3). Indeed, we shall prove in the next subsection that $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ is a maximal trading strategy in a larger space introduced in [17, 19] (still denoted by $\mathcal{A}^\lambda(1)$).

Next, we shall construct a shadow price in the following corollary.

Corollary 4.2.6. *Define*

$$\widetilde{S}_t := \begin{cases} S_0, & t = 0; \\ \exp\left(B_t + \left(\frac{1}{2} + \frac{2\log(1-\lambda)}{T}\right)t\right), & 0 < t < T/2; \\ (1-\lambda)S_{T/2}, & T/2 \le t \le T. \end{cases}$$

Then, the process $\widetilde{S}$ is a shadow price of the utility maximization problem (4.1.3).

Proof. It follows from the definition of $\widetilde{S}$ that $\widetilde{S} \in [(1-\lambda)S, S]$.

Obviously, the wealth process associated with the buy-hold-sell strategy under $\widetilde{S}$ is $\widetilde{S}$ itself. Thus, we can find an equivalent martingale measure $\mathbf{Q} \in \mathcal{M}^e(\widetilde{S})$, under which this wealth process is a uniformly integrable martingale. By [24, Corollary 4.6], $\widehat{\varphi}_T^0 = \widetilde{S}_T$ is a maximal element in $\mathcal{C}^{\widetilde{S}}(1)$, where

$$\mathcal{C}^{\widetilde{S}}(1) := \left\{ \widetilde{X}_T = 1 + \left(H \bullet \widetilde{S}\right)_T \,\middle|\, H \text{ is admissible} \right\}.$$

Similarly to the proof of Proposition 4.2.1, one can see that $\widehat{\varphi}_T^0 = \widetilde{S}_T$ solves the utility maximization problem in the frictionless market $u^{\widetilde{S}}(1; e_T)$, and

$$\begin{aligned} u^{\widetilde{S}}(1; e_T) &= \sup_{\widetilde{X}_T \in \mathcal{C}^{\widetilde{S}}(1)} \mathbb{E}\left[U\left(\widetilde{X}_T + e_T\right)\right] = \mathbb{E}\left[U\left(\widetilde{S}_T - \zeta(1-\lambda)\widetilde{S}_T\right)\right] \\ &= \mathbb{E}\left[U\left((1-\zeta)(1-\lambda)S_T\right)\right] = u(1; e_T), \end{aligned}$$

which implies that $\widetilde{S}$ is the desired shadow price. $\square$

4.2.2 Proof of Proposition 4.2.4

In this subsection, we shall prove that (4.2.1) defines a maximal trading strategy in $\mathcal{A}^\lambda(1)$, i.e, Proposition 4.2.4. Here, $\mathcal{A}^\lambda(1)$ could be the set of admissible trading strategies defined in [17] without no-short-selling constraint.

Proof of Proposition 4.2.4. Let $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ be another trading strategy in $\mathcal{A}^\lambda(1)$. Since the stock-price process S is continuous and the trading strategies are pathwisely of finite variation, we could only consider those (φ^0, φ^1) with right-continuous paths (cf. [19]). In addition, we assume that the agent buys no more stocks after time $T/2$.

In what follows, we shall discuss trading strategies φ in two cases and furthermore, we shall prove that the following statement cannot hold

$$\varphi_T^0 \ge \widehat{\varphi}_T^0 = (1-\lambda)S_{T/2}, \text{ a.s. with } \mathbf{P}[\varphi_T^0 > \widehat{\varphi}_T^0] > 0, \tag{4.2.2}$$

and thus $\widehat{\varphi}$ is a maximal trading strategy in $\mathcal{A}^\lambda(1)$.

Case I. Suppose that there exists a $\gamma > 0$ and a set $B \in \mathcal{F}_{T/2}$ with $\mathbf{P}(B) > 0$ such that for $\omega \in B$, $\varphi_t^1(\omega) \le 1 - \gamma$, $0 \le t \le T/2$, $\mathbf{P}$-a.s. (i.e., the holding in stock never excesses

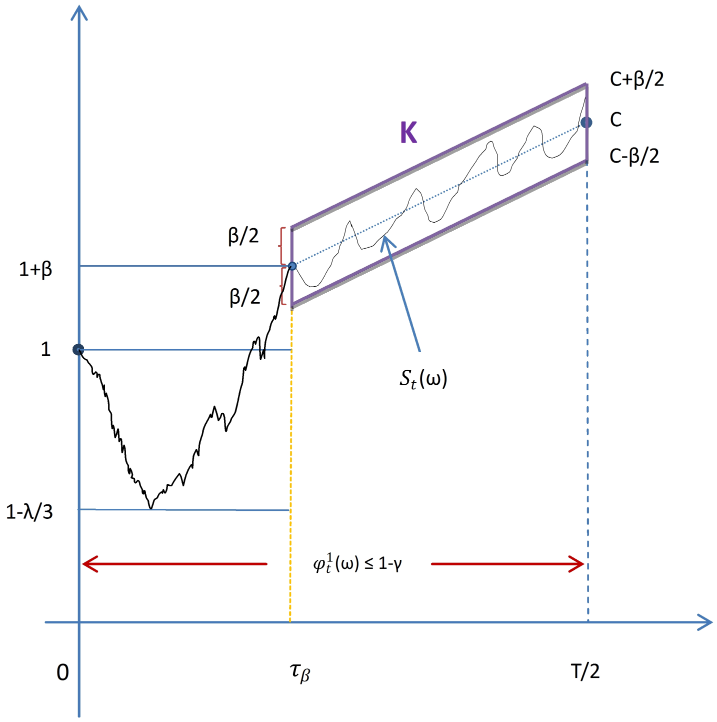

Figure 4.1: Case 1

$1-\gamma$). Our aim is to construct a set $\overline{B} \in \mathcal{F}_{T/2}$ with $\mathbf{P}(\overline{B}) > 0$, such that for $\omega \in \overline{B}$, $\varphi^0_T(\omega) < \widehat{\varphi}^0_T(\omega)$.

For $\beta > 0$, define

$$\tau_\beta := \inf\{t > 0 : S_t \geq 1+\beta\} \wedge T/2.$$

Then, by choosing $\beta \ll \lambda$ sufficiently small, we can find an $\mathcal{F}_{\tau_\beta}$-measurable set defined by

$$B' := \{\omega : \tau_\beta(\omega) < T/2\} \cap \{\omega : 1-\lambda/3 \leq S_t(\omega) \leq 1+\beta,\ 0 \leq t \leq \tau_\beta(\omega)\},$$

such that $\mathbf{P}(B') > 1 - \mathbf{P}(B)$. It is obvious that $\mathbf{P}(B' \cap B) > 0$ and for $\omega \in B'$, $S_{\tau_\beta(\omega)}(\omega) = 1+\beta$. Furthermore, we define

$$\widetilde{B}' := B' \cap \{\omega : \varphi^1_t(\omega) \leq 1-\gamma,\ 0 \leq t \leq \tau_\beta(\omega)\} \supseteq B' \cap B.$$

From $\mathbf{P}(B' \cap B) > 0$, we have $\mathbf{P}(\widetilde{B}') > 0$. We now construct a corridor K of stock price on the time interval $[\![\tau_\beta, T/2]\!]$ with width β (see Figure 4.1). The center point of the corridor at τ_β is $1+\beta$ while at $T/2$ is a sufficiently large number C. Obviously, the stock-price process S has conditional full support (see, e.g., [44]), which implies the set

$$\overline{B}' := \widetilde{B}' \cap \{S \text{ stays in the corridor } K \text{ on } [\![\tau_\beta, T/2]\!]\},$$

has a strictly positive measure, i.e., $\mathbf{P}(\overline{B}') > 0$. For $\omega \in \overline{B}'$, the most advisable trading strategy is to buy $1-\gamma$ stock shares at the lowest price, which should be at least $1-\lambda/3$, and hold them until $T/2$. Therefore,

$$\varphi_T^0(\omega) = V_{T/2}^{liq}(\varphi)(\omega) < (1-\lambda)\big(C + \tfrac{\beta}{2}\big)(1-\gamma) + 1.$$

On the other hand,

$$\widehat{\varphi}_T^0(\omega) = \widehat{\varphi}_{T/2}^0(\omega) > (1-\lambda)\big(C - \tfrac{\beta}{2}\big).$$

By properly choosing

$$C > \frac{1}{\gamma(1-\lambda)} + \frac{\beta(2-\gamma)}{2\gamma},$$

we could ensure that $\varphi_T^0(\omega) < \widehat{\varphi}_T^0(\omega)$.

Case II. Suppose that Case I cannot happen, then for almost every $\omega \in \Omega$, the following stopping time takes values in $[0, T/2]$:

$$\sigma := \inf\{t > 0 : \varphi_t^1 \geq 1 \text{ or } \varphi_{t-}^1 \geq 1\}.$$

For any stopping times τ and ρ taking values in $[0, T/2]$, $\rho \geq \tau$, we have:

$$\begin{aligned}
V_\rho^{liq}(\varphi) &= \varphi_\tau^0 + \varphi_\tau^1 S_\tau + \int_\tau^\rho \varphi_t^1 dS_t - \lambda \int_\tau^\rho S_t d\varphi_t^{1,\downarrow} - \lambda S_\rho (\varphi_\rho^1)^+ \\
&= \varphi_\tau^0 + \varphi_\tau^1 S_\tau + \int_\tau^\rho \varphi_t^1 dS_t - \lambda \int_\tau^\rho S_t d\varphi_t^{1,\downarrow} - \lambda S_\rho (\varphi_\rho^1)^+ + \lambda \int_\tau^\rho \varphi_t^1 dS_t - \lambda \int_\tau^\rho \varphi_t^1 dS_t \\
&= \varphi_\tau^0 + (1-\lambda)\varphi_\tau^1 S_\tau + (1-\lambda) \int_\tau^\rho \varphi_t^1 dS_t - \lambda S_\rho (\varphi_\rho^1)^- - \lambda \int_\tau^\rho S_t d\varphi_t^{1,\uparrow}. \qquad (4.2.3)
\end{aligned}$$

In the sequel, we shall consider two situations, i.e.,

(i) For almost every $\omega \in \Omega$,

$$\varphi_\sigma^0(\omega) + (1-\lambda)\varphi_\sigma^1(\omega) S_\sigma(\omega) \geq \widehat{\varphi}_\sigma^0(\omega) + (1-\lambda)\widehat{\varphi}_\sigma^1(\omega) S_\sigma(\omega);$$

(ii) There exists an $\mathcal{F}_\sigma$-measurable set D, with $\mathbf{P}(D) > 0$, for $\omega \in D$,

$$\varphi_\sigma^0(\omega) + (1-\lambda)\varphi_\sigma^1(\omega) S_\sigma(\omega) < \widehat{\varphi}_\sigma^0(\omega) + (1-\lambda)\widehat{\varphi}_\sigma^1(\omega) S_\sigma(\omega).$$

Assume that (i) holds, then we observe that for almost every $\omega \in \Omega$,

$$\varphi_\sigma^0(\omega) + (1-\lambda)\varphi_\sigma^1(\omega) S_\sigma(\omega) = \widehat{\varphi}_\sigma^0(\omega) + (1-\lambda)\widehat{\varphi}_\sigma^1(\omega) S_\sigma(\omega). \qquad (4.2.4)$$

Indeed, if $\varphi_{\sigma(\omega)}^1(\omega) \geq 1$, then

$$\varphi_\sigma^0(\omega) + (1-\lambda)\varphi_\sigma^1(\omega) S_\sigma(\omega) = V_{\sigma(\omega)}^{liq}(\varphi)(\omega) \leq V_{\sigma(\omega)-}^{liq}(\varphi)(\omega);$$

if $\varphi_{\sigma(\omega)}^1(\omega) < 1$, then $\varphi_{\sigma(\omega)-}^1(\omega) \geq 1$, and thus from the self-financing constraint,

$$\varphi_\sigma^0(\omega) + (1-\lambda)\varphi_\sigma^1(\omega) S_\sigma(\omega) \leq \varphi_{\sigma-}^0(\omega) + (1-\lambda)\varphi_{\sigma-}^1(\omega) S_\sigma(\omega) = V_{\sigma(\omega)-}^{liq}(\varphi)(\omega).$$

Consider a frictionless market in which the agent trades for $(1-\lambda)S$, where the agent gains better than trading in the market with transaction costs. Then, we can deduce

$$\begin{aligned}\widehat{\varphi}_0^0 + \widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^\sigma \widehat{\varphi}_t^1 dS_t &= \widehat{\varphi}_\sigma^0 + (1-\lambda)\widehat{\varphi}_\sigma^1 S_\sigma \\ &\le \varphi_\sigma^0 + (1-\lambda)\varphi_\sigma^1 S_\sigma \\ &\le V_{\sigma-}^{liq}(\varphi) \\ &\le \varphi_0^0 + \varphi_0^1 S_0 + (1-\lambda)\int_0^\sigma \varphi_t^1 dS_t.\end{aligned}$$

Since $\varphi_0^0 + \varphi_0^1(1-\lambda)S_0 = \widehat{\varphi}_0^0 + \widehat{\varphi}_0^1(1-\lambda)S_0 = 1$ and $\mathbf{1}$ is a maximal trading strategy when trading for $(1-\lambda)S$, we can prove (4.2.4).

Now we calculate $V_T^{liq}(\varphi)$. From (4.2.3), we obtain

$$\begin{aligned}V_T^{liq}(\varphi) &\le \varphi_\sigma^0 + (1-\lambda)\varphi_\sigma^1 S_\sigma + (1-\lambda)\int_\sigma^T \varphi_t^1 dS_t \\ &= \widehat{\varphi}_0^0 + (1-\lambda)\widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^\sigma \widehat{\varphi}_t^1 dS_t + (1-\lambda)\int_\sigma^T \varphi_t^1 dS_t \\ &= \widehat{\varphi}_0^0 + (1-\lambda)\widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^T \left(\widehat{\varphi}_t^1 \mathbf{1}_{[\![0,\sigma]\!]}(t) + \varphi_t^1 \mathbf{1}_{[\![\sigma,T]\!]}(t)\right) dS_t.\end{aligned} \tag{4.2.5}$$

It is obvious that trading for $(1-\lambda)S$, $((\widehat{\varphi}_\cdot^1 \mathbf{1}_{[\![0,\sigma]\!]}(\cdot) + \varphi_\cdot^1 \mathbf{1}_{[\![\sigma,T]\!]}(\cdot)) \bullet (1-\lambda)S)$ is uniformly bounded from below, and thus admissible. Comparing (4.2.5) with

$$V_T^{liq}(\widehat{\varphi}) = \widehat{\varphi}_0^0 + (1-\lambda)\widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^T \widehat{\varphi}_t^1 dS_t,$$

we can conclude from the maximality of $(1-\lambda)\int_0^T \mathbf{1} dS_t$ that (4.2.2) cannot happen.

On the other hand, if (ii) holds, then by similar reasoning, we have for $\omega \in D$,

$$\begin{aligned}V_T^{liq}(\varphi) &< \widehat{\varphi}_0^0 + (1-\lambda)\widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^\sigma \widehat{\varphi}_t^1 dS_t + (1-\lambda)\int_\sigma^T \varphi_t^1 dS_t \\ &= \widehat{\varphi}_0^0 + (1-\lambda)\widehat{\varphi}_0^1 S_0 + (1-\lambda)\int_0^T \left(\widehat{\varphi}_t^1 \mathbf{1}_{[\![0,\sigma]\!]}(t) + \varphi_t^1 \mathbf{1}_D \mathbf{1}_{[\![\sigma,T]\!]}(t)\right) dS_t.\end{aligned} \tag{4.2.6}$$

Again, $((\widehat{\varphi}^1 \mathbf{1}_{[\![0,\sigma]\!]} + \varphi^1 \mathbf{1}_D \mathbf{1}_{[\![\sigma,T]\!]}) \bullet (1-\lambda)S)$ is uniformly bounded from below when trading for $(1-\lambda)S$, and thus admissible. Comparing $\left((\widehat{\varphi}^1 \mathbf{1}_{[\![0,\sigma]\!]} + \varphi^1 \mathbf{1}_D \mathbf{1}_{[\![\sigma,T]\!]}) \bullet (1-\lambda)S\right)_T$ with $(\mathbf{1} \bullet (1-\lambda)S)_T$, we can conclude that either for almost every $\omega \in D$,

$$\left((\widehat{\varphi}^1 \mathbf{1}_{[\![0,\sigma]\!]} + \varphi^1 \mathbf{1}_D \mathbf{1}_{[\![\sigma,T]\!]}) \bullet (1-\lambda)S\right)_T = (\mathbf{1} \bullet (1-\lambda)S)_T,$$

or there exists a subset $D' \subset D$, $D' \in \mathcal{F}_{T/2}$, such that for $\omega \in D'$,

$$\left((\widehat{\varphi}^1 \mathbf{1}_{[\![0,\sigma]\!]} + \varphi^1 \mathbf{1}_D \mathbf{1}_{[\![\sigma,T]\!]}) \bullet (1-\lambda)S\right)_T < (\mathbf{1} \bullet (1-\lambda)S)_T.$$

In particular, we exclude (4.2.2) by recalling (4.2.6). $\square$

Chapter 5

Appendix

5.1 Referenced lemmas

Lemma 5.1.1. *Let* $\{Y^n\}_{n=1}^{\infty} \subset \mathbf{L}^1_+(\Omega, \mathcal{F}, \mathbf{P})$ *and* $\{X^n\}_{n=1}^{\infty} \subset \mathbf{L}^0(\Omega, \mathcal{F}, \mathbf{P})$*, where for each* n*,* $X^n \geq a > 0$*,* $\mathbf{P}$*-a.s.. If there exists another couple of random variables* $(Y, X) \in \mathbf{L}^1(\Omega, \mathcal{F}, \mathbf{P}) \times \mathbf{L}^0(\Omega, \mathcal{F}, \mathbf{P})$*, such that*

$$Y^n \to Y \text{ and } X \leq \liminf_n X^n, \ \mathbf{P} - a.s.,$$

$$\mathbb{E}^{\mathbf{P}}[YX] \geq \liminf_n \mathbb{E}^{\mathbf{P}}[Y^n X^n].$$

Then, $\{Y^n\}_{n=1}^{\infty}$ *is uniformly integrable.*

Proof. If the sequence $\{Y^n\}_{n=1}^{\infty}$ is not uniformly integrable, by passing to a subsequence if necessary, there exists $\varepsilon > 0$, for each $n \in \mathbb{N}$, there exists A_n such that

$$\mathbf{P}(A_n) < \frac{1}{2^n} \quad \text{and} \quad \mathbb{E}^{\mathbf{P}}\left[Y^n \mathbf{1}_{A_n}\right] \geq \varepsilon.$$

Define

$$\eta^n := Y^n \mathbf{1}_{A_n}, \ \xi^n := Y^n \mathbf{1}_{A_n^c}.$$

It is obvious that $\xi^n \to Y$, a.s., while by Fatou's Lemma, we have

$$\mathbb{E}^{\mathbf{P}}\left[YX\right] \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbf{P}}[\xi^n X^n] = \liminf_{n\to\infty} \mathbb{E}^{\mathbf{P}}\left[Y^n X^n - \eta^n X^n\right] \leq \mathbb{E}^{\mathbf{P}}\left[YX\right] - a\varepsilon,$$

which is a contradiction. $\square$

Lemma 5.1.2. *Suppose* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence in* $\mathbf{L}^1$*,* $f_n \to f_0 \in L^0$*, almost surely. Moreover,* $\lim_{n\to\infty} \mathbb{E}[f_n]$ *is finite and* f_0^+ *is integrable. We denote* $\alpha := \lim_{n\to\infty} \mathbb{E}[f_n] - \mathbb{E}[f_0]$*. In particular, if* $\mathbb{E}[f_0] = -\infty$*, we note* $\alpha := \infty$*. Then, we have*

(i) For any $M > 0$*,*

$$\limsup_{n\to\infty} \mathbb{E}\big[f_n \mathbf{1}_{\{f_n \geq M\}}\big] \geq \alpha. \tag{5.1.1}$$

(ii) For any $\alpha' < \alpha$ and $M > 0$, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a sequence of disjoint sets $(A_k)_{k\in\mathbb{N}}$, such that for each $k \in \mathbb{N}$, $f_{n_k} \geq M$ on A_k, and

$$\mathbb{E}\big[f_{n_k}\mathbf{1}_{A_k}\big] \geq \alpha'. \tag{5.1.2}$$

Proof. (i) We suppose, contrary to our claim (5.1.1), there exists $M > 0$, such that

$$\limsup_{n\to\infty} \mathbb{E}\big[f_n\mathbf{1}_{\{f_n\geq M\}}\big] =: \beta < \alpha, \tag{5.1.3}$$

which implies the boundedness of $\{\mathbb{E}[f_n\mathbf{1}_{\{f_n\geq M\}}]\}_{n\in\mathbb{N}}$. Suppose $\mathbb{E}[f_0] = -\infty$, then we have

$$\mathbb{E}\big[f_0\mathbf{1}_{\{f_0<M\}}\big] \leq \mathbb{E}[f_0] = -\infty.$$

Thus,

$$\limsup_{n\to\infty} \mathbb{E}\big[f_n\mathbf{1}_{\{f_n<M\}}\big] \leq \mathbb{E}\big[f_0\mathbf{1}_{\{f_0<M\}}\big] = -\infty. \tag{5.1.4}$$

From (5.1.4) and (5.1.3), we can conclude that $\lim_{n\to\infty}\{\mathbb{E}[f_n]\}_{n\in\mathbb{N}} = -\infty$, which contradicts to the assumption. Therefore, $\mathbb{E}[f_0] \in \mathbb{R}$. In this case, we have

$$\begin{aligned}
\mathbb{E}\big[f_0\mathbf{1}_{\{f_0<M\}}\big] &\geq \limsup_{n\to\infty} \mathbb{E}\big[f_n\mathbf{1}_{\{f_n<M\}}\big] \\
&= \lim_{n\to\infty} \mathbb{E}[f_n] - \liminf_{n\to\infty} \mathbb{E}\big[f_n\mathbf{1}_{\{f_n\geq M\}}\big] \\
&\geq \mathbb{E}[f_0] + \alpha - \beta,
\end{aligned} \tag{5.1.5}$$

where the equality is deduced from the convergence of $\{\mathbb{E}[f_n]\}_{n\in\mathbb{N}}$ and the boundedness of $\{\mathbb{E}[f_n\mathbf{1}_{\{f_n\geq M\}}]\}_{n\in\mathbb{N}}$. Obviously, (5.1.5) is a contradiction.

(ii) We first consider the case that $\alpha < \infty$. Fix α', M and denote by $\varepsilon := \alpha - \alpha'$. We now construct inductively a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ as well as a sequence $\{\mathcal{A}^m\}_{m\in\mathbb{N}}$, where $\mathcal{A}^m$ consists of m subsets of Ω, i.e.,

$$\mathcal{A}^m := \{A^m_1, A^m_2, \ldots, A^m_m\},\ m \in \mathbb{N}.$$

Let $M_1 = M$. From (i), we could choose n_1 such that $\mathbb{E}\big[f_{n_1}\mathbf{1}_{\{f_{n_1}\geq M_1\}}\big] \geq \alpha - \frac{\varepsilon}{2}$ and define $A^1_1 := \{f_{n_1} \geq M_1\}$. Suppose that $\{f_{n_k}\}_{k=1}^N$ and $\{\mathcal{A}^m\}_{m=1}^N$ are well-defined and we now define $f_{n_{N+1}}$ and $\mathcal{A}^{N+1}$. Note that for each $n \in \mathbb{N}$ and any $\widetilde{M} > 0$,

$$\mathbf{P}\big[f_n \geq \widetilde{M}\big] \leq \mathbf{P}\left[|f_n - f_0| \geq \frac{\widetilde{M}}{2}\right] + \mathbf{P}\left[f_0^+ \geq \frac{\widetilde{M}}{2}\right].$$

Thus, for any $\delta > 0$, we can choose sufficiently large $M_{N+1} > M_N$ and find a $n_0 \geq n_N$ such that for any $n \geq n_0$, $\mathbf{P}[f_n \geq M_{N+1}] \leq \delta$. Due to the integrability of f_{n_1}, f_{n_2}, $\ldots$,

f_{n_N}, for properly chosed δ,

$$\mathbb{E}\left[|f_{n_k}|\mathbf{1}_{\{f_n\geq M_{N+1}\}}\right] \leq \frac{\varepsilon}{2^{N+1}}, \quad k=1,2,\ldots,N, \tag{5.1.6}$$

holds for any $n \geq n_0$.

Then, we fix $n_{N+1} \geq n_N$ satisfying $\mathbb{E}[f_{n_{N+1}}\mathbf{1}_{\{f_{n_{N+1}}\geq M_{N+1}\}}] \geq \alpha - \frac{\varepsilon}{2}$ and define $A_{N+1}^{N+1} := \{f_{n_{N+1}} \geq M_{N+1}\}$ and $A_k^{N+1} := A_k^N \backslash A_{N+1}^{N+1}$. Note that the sequence $\{\mathcal{A}^m\}_{m\in\mathbb{N}}$ we defined above has the following properties:

(a) For each k, $\{A_k^m\}_{m\geq k}$ is decreasing in m;

(b) For each m, $A_1^m, A_2^m, \ldots, A_m^m$ are disjoint;

(c) For each k and $m \geq k$,

$$\mathbb{E}[f_{n_k}\mathbf{1}_{A_k^m}] \geq \alpha - \frac{\varepsilon}{2} - \sum_{i=k}^{m-1}\frac{\varepsilon}{2^{i+1}} > \alpha'.$$

Letting $A_k := \bigcap_{m\geq k} A_k^m$. It is easy to verify that the sequences $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $\{A_k\}_{k\in\mathbb{N}}$ are well defined and satisfy (5.1.2).

If $\alpha = \infty$, we could complete the proof by a similar argument. $\square$

5.2 Space of bounded and finitely additive measures

We state some known results in the space $(\mathbf{L}^\infty)^*$ for the convenience of the reader. A detailed discussion can be found in [29, 77, 89].

5.2.1 Finitely additive measure

A function Q from a field $\mathcal{F}$ of subsets of a set Ω to a Banach space $\mathscr{E}$ is called a **finitely additive measure** if

$$Q(A_1 \cup A_2) = Q(A_1) + Q(A_2),$$

whenever A_1 and A_2 are disjoint members of $\mathcal{F}$. We are concerned with the special case that $\mathscr{E} = \mathbb{R}$. The **total variation of a (finitely additive) measure** $Q : \mathcal{F} \to \mathbb{R}$ is the function $|Q| : \mathcal{F} \to [0,\infty]$ defined by

$$|Q|(A) := \sup \sum_{j=1}^{n} Q(A_j),$$

where the supremum is taken over all finite sequences $(A_j)_{j=1}^n$ of disjoint sets in $\mathcal{F}$ with $A_j \subseteq A$. A measure Q is said to have **bounded total variation** if $|Q|(\Omega) < \infty$. A measure Q is said to be bounded if $\sup\{|Q(A)| : A \in \mathcal{F}\} < \infty$. Indeed, we can show

$$\sup\left\{|Q(A)| : A \in \mathcal{F}\right\} \leq |Q|(\Omega) \leq 2\sup\left\{|Q(A)| : A \in \mathcal{F}\right\},$$

hence a measure is **bounded** if and only if it has bounded total variation. A measure Q is said to be **purely finitely additive**, if $0 \leq \mu \leq |Q|$ and μ is countably additive, implying $\mu = 0$. A measure Q is said to be **weakly absolutely continuous** with respect to $\mathbf{P}$, if $Q(A) = 0$ whenever $A \in \mathcal{F}$ and $\mathbf{P}(A) = 0$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the underlying probability space. Denote by $\mathrm{ba}(\mathbb{R}) = \mathrm{ba}(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{R})$ the space of bounded (and finitely additive) measures $Q : \mathcal{F} \to \mathbb{R}$ which are weakly absolutely continuous with respect to $\mathbf{P}$. Denote by $\mathrm{ca}(\mathbb{R})$ the subspace of countably additive members of $\mathrm{ba}(\mathbb{R})$. Equipped with the norm

$$\|Q\|_{\mathrm{ba}(\mathbb{R})} := |Q|(\Omega),$$

the spaces $\mathrm{ba}(\mathbb{R})$ and $\mathrm{ca}(\mathbb{R})$ are Banach spaces.

We shall see that the elements of $\mathrm{ba}(\mathbb{R})$ play a natural role as linear functionals on spaces of (essentially) bounded $\mathbb{R}$-valued random variables. We need some notations:

- Let $\mathbf{L}^0(\mathbb{R}) = \mathbf{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R})$ the space of $\mathbb{R}$-valued random variables (identified under the equivalence relation of a.s. equality).

- Let $\mathbf{L}^1(\mathbb{R})$ denote the subspace of $\mathbf{L}^0(\mathbb{R})$ consisting of those random variables X for which $\|X\|_{\mathbf{L}^1(\mathbb{R})} := \mathbb{E}\left[|X|\right] < \infty$.

- Let $\mathbf{L}^\infty(\mathbb{R})$ denote the subspace of $\mathbf{L}^0(\mathbb{R})$ consisting of those random variables X for which $\|X\|_{\mathbf{L}^\infty(\mathbb{R})} := \operatorname{ess\,sup}\{|X|\} < \infty$.

- Finally, denote by $\mathbf{L}^\infty(\mathbb{R})^*$ the dual space of $(\mathbf{L}^\infty(\mathbb{R}), \|\cdot\|_{\mathbf{L}^\infty(\mathbb{R})})$.

- Let $\mathbf{L}^0(\mathbb{R}_+)$ and $\mathbf{L}^\infty(\mathbb{R}_+)$ denote respectively the convex cones of random variables $\mathbf{L}^0(\mathbb{R})$ and $\mathbf{L}^\infty(\mathbb{R})$ which are $\mathbb{R}_+$-valued a.s.

- let $\mathrm{ba}(\mathbb{R}_+)$ denote the convex cone of $\mathbb{R}_+$-valued measures within $\mathrm{ba}(\mathbb{R})$.

We define the map $\Psi : \mathrm{ba}(\mathbb{R}) \longrightarrow (\mathbf{L}^\infty)^*$ by

$$(\Psi(Q)(X)) := \int_\Omega \langle X, dQ \rangle := \int_\Omega X dQ. \tag{5.2.1}$$

For details we refer to, for example, [77, Chapter 4]. We also define the map $\Xi : \mathrm{ca}(\mathbb{R}) \longrightarrow \mathbf{L}^1(\mathbb{R})$ by

$$\Xi(Q) := \frac{dQ}{d\mathbf{P}},$$

where $\frac{dQ}{d\mathbf{P}}$ the Radon–Nikodým derivative of the measure. Finally, we define the isometric embedding $l : \mathbf{L}^1 \longrightarrow (\mathbf{L}^\infty)^*$ by

$$(l(Y))(X) := \mathbb{E}\left[\langle X, Y \rangle\right].$$

Proposition 5.2.1. *The map Ψ and Ξ are isometric isomorphisms. Furthermore,*

$$l \circ \Xi = \Psi|_{\mathrm{ca}(\mathbb{R})}.$$

Proof. See, for example, [77, Theorem 4.7.10]. $\square$

In our case, given $Q \in \mathrm{ba}(\mathbb{R})$ and $X \in \mathbf{L}^\infty$, we write $Q(X)$ as an abbreviation of $(\Psi(Q))(X)$, and we define $\frac{dQ}{d\mathbf{P}} := \Xi(Q)$. Given $x \in \mathbb{R},\ A \in \mathcal{F}$, it follows from (5.2.1) that $Q(x\mathbf{1}_A) = \langle x, Q(A)\rangle$.

Note if $Q \in \mathrm{ba}(\mathbb{R}_+)$ and $X \in \mathbf{L}^\infty(\mathbb{R}_+)$, then $Q(X) \geq 0$ (see [77, Theorem 4.4.13]). This observation allows us to extend the definition of $Q(X)$ to cover the case where $Q \in \mathrm{ba}(\mathbb{R}_+)$ and $X \in \mathbf{L}^0(\mathbb{R}_+)$ by setting

$$Q(X) := \sup_{n\in\mathbb{N}} Q(X \wedge n). \tag{5.2.2}$$

It is trivial that the definition (5.2.2) is consistent with the definition of $Q(X)$ for $X \in \mathbf{L}^\infty(\mathbb{R})$. Furthermore, the supremum in (5.2.2) can be replaced by a limit because the sequence of numbers is increasing. It follows that given $Q_1, Q_2 \in \mathrm{ba}(\mathbb{R}_+),\ X_1, X_2 \in \mathbf{L}^0(\mathbb{R}_+)$ and $\alpha_1, \alpha_2, \mu_1, \mu_2 \geq 0$, we have

$$(\alpha_1 Q_1 + \alpha_2 Q_2)(\mu_1 X_1 + \mu_2 X_2) = \alpha_1\mu_1 Q_1 X_1 + \alpha_1\mu_2 Q_1 X_2 + \alpha_2\mu_1 Q_2 X_1 + \alpha_2\mu_2 Q_2 X_2.$$

Note that the final statement of Proposition 5.2.1 means that given $Q \in \mathrm{ca}(\mathbb{R})$ and $X \in \mathbf{L}^\infty(\mathbb{R})$, we have $Q(X) = \mathbb{E}\left[\langle X, \frac{dQ}{d\mathbf{P}}\rangle\right]$. It is easy to show that this property is also true under the extended definition (5.2.2).

The following proposition collects some properties of the space ba_+; more information can be found in Appendix of [13] and references there.

Proposition 5.2.2.

1. *The set ba_+ can be identified as the set of all nonnegative finitely additive bounded set functions on $\mathcal{F}$, which vanish on the $\mathbf{P}$-null sets.*

   (See, for example, [29, Theorem IV.8.16].)

2. *Every $Q \in ba_+$ admits a unique decomposition in the form of $Q = Q^r + Q^s$, where the regular part Q^r is the maximal countably additive measure on $\mathcal{F}$, that is dominated by Q, and the singular part Q^s is purely finitely additive and does not dominate any nontrivial countably additive measure.*

   (See, for example, [77, Theorem 10.2.1].)

3. *$Q \in ba_+$ is purely finitely additive, i.e., $Q^r = 0$, if and only if for every $\varepsilon > 0$, there exists a set $A_\varepsilon \in \mathcal{F}$ such that $\mathbf{P}(A_\varepsilon) > 1 - \varepsilon$ and $Q(A_\varepsilon) = 0$.*

   (See e.g. [13, Lemma A.1].)

4. *Suppose $(Q_n)_{n\in\mathbb{N}} \subseteq ba_+$ is a sequence such that $\frac{dQ_n^r}{d\mathbf{P}} \to f$ almost surely for some $f \geq 0$. Then any weak-star cluster point Q of $(Q_n)_{n\in\mathbb{N}}$ satisfies $\frac{dQ^r}{d\mathbf{P}} = f$ almost surely.*

   (See e.g. [13, Proposition A.1].)

Proposition 5.2.3. *Suppose $\widetilde{\mathcal{D}}$ is a convex subset of $\mathbf{L}^1_+$, which is also a subset of $(\mathbf{L}^\infty)^*_+$ via the canonical embedding. Denote by $\overline{\mathcal{D}}$ the weak-star closure of $\widetilde{\mathcal{D}}$ in $(\mathbf{L}^\infty)^*_+$. Then, for each $Q \in \overline{\mathcal{D}}$, there exists a sequence $\{Q^n\}_{n=1}^\infty \subset \widetilde{\mathcal{D}}$, such that $Q^n \to Q^r$, $\mathbf{P}$-a.s., as $n \to \infty$.*

Proof. Fixing $Q \in \overline{\mathcal{D}}$, for each $n \in \mathbb{N}$, there exists a set $A_n \in \mathcal{F}$, such that $\mathbf{P}(A_n) > 1 - \frac{1}{2^n}$ and Q^s is null on A_n. By the definition of $\overline{\mathcal{D}}$, we see that Q is a weak-star limit point of $\widetilde{\mathcal{D}}$ and thus, $Q|_{A_n} \in (\mathbf{L}^\infty)^*$ is also a weak-star limit point of $\widetilde{\mathcal{D}}|_{A_n}$, where $\widetilde{\mathcal{D}}|_{A_n} := \{\widetilde{Q}|_{A_n} : \widetilde{Q} \in \widetilde{\mathcal{D}}\}$. From $Q|_{A_n} = Q^r|_{A_n} \in \mathbf{L}^1$, we know that $Q^r|_{A_n}$ is a weak limit point of $\widetilde{\mathcal{D}}|_{A_n}$. Moreover, due to the fact that $\widetilde{\mathcal{D}}|_{A_n}$ is convex, $Q^r|_{A_n}$ belongs to the $\mathbf{L}^1$-closure of $\widetilde{\mathcal{D}}|_{A_n}$. Therefore, there exists an element $Q^n \in \widetilde{\mathcal{D}}$, such that

$$\|Q^n|_{A_n} - Q^r|_{A_n}\|_{\mathbf{L}^1} < \tfrac{1}{2^n}.$$

Finally,

$$\|Q^n|_{A_n} - Q^r\|_{\mathbf{L}^1} \leq \|Q^n|_{A_n} - Q^r|_{A_n}\|_{\mathbf{L}^1} + \|Q^r|_{A_n} - Q^r\|_{\mathbf{L}^1} \longrightarrow 0,$$

which yields $Q^n \to Q^r$, $\mathbf{P}$-a.s. up to a subsequence. $\square$

Corollary 5.2.4. *Suppose $\widetilde{\mathcal{D}}$ is a convex subset of $\mathbf{L}^1_+$, which is also a subset of $(\mathbf{L}^\infty)^*_+$ via the canonical embedding. Denote by $\overline{\mathcal{D}}$ the weak-star closure of $\widetilde{\mathcal{D}}$ in $(\mathbf{L}^\infty)^*_+$. Then, for each $Q \in \overline{\mathcal{D}}$, there exists a sequence $\{Q^n\}_{n=1}^\infty \subset \widetilde{\mathcal{D}}$, such that $Q^n \to Q^r$, $\mathbf{P}$-a.s., as $n \to \infty$, and for countably many $f_i \in \mathbf{L}^\infty$, $i \in \mathbb{N}$,*

$$\langle Q^n, f_i\rangle \longrightarrow \langle Q, f_i\rangle, \text{ as } n \to \infty.$$

Proof. For each n, define

$$\widetilde{\mathcal{D}}^n := \bigcap_{i=1}^n \left\{\widetilde{Q} \in \widetilde{\mathcal{D}} : |\langle \widetilde{Q}, f_i\rangle - \langle Q, f_i\rangle| < \tfrac{1}{n}\right\}.$$

It is evident that Q belongs to the weak-star closure of $\widetilde{\mathcal{D}}^n$. Applying the proposition above, one can find a sequence $\{Q^{n,m}\}_{m=1}^\infty \subset \widetilde{\mathcal{D}}^n$, such that $Q^{n,m} \to Q^r$, $\mathbf{P}$-a.s. By the diagonal argument, we complete the proof. $\square$

Remark 5.2.5. We remark that the assumption that $\widetilde{\mathcal{D}} \subset \mathbf{L}^1$ is crucial in the above proposition. For a general subset $\widetilde{D} \subset (\mathbf{L}^\infty)^*$ and an element Q in its weak-star closure $\overline{D}$, one may not find a sequence $\{Q^n\}_{n=1}^\infty$ from $\widetilde{D}$, such that $(Q^n)^r \to Q^r$. For instance, $\Omega = [0,1]$, $\mathcal{F}$ is the Lebesgue sigma-algebra and $\mathbf{P}$ is the Lebesgue measure. Define $\widetilde{D} := \{Q \in (\mathbf{L}^\infty)^*_+ : \|Q\|_{(\mathbf{L}^\infty)^*} = 1, Q = Q^s\}$, then we find that statement of Proposition 5.2.3 does not hold, since $\{Q \in \mathbf{L}^1_+ : \|Q\|_{(\mathbf{L}^\infty)^*} = 1\} \subset \{Q \in (\mathbf{L}^\infty)^*_+ : \|Q\|_{(\mathbf{L}^\infty)^*} = 1\} = \overline{D}$.

5.2.2 Properties of the dual domain

We present and prove the properties of the dual domain $\widetilde{\mathscr{D}}$ (2.2.2) for the case in incomplete markets with random endowment. Let repeat some notation here. Denote by $\widetilde{\mathscr{C}}_0$ the convex cone of random variables dominated by admissible stochastic integrals, and define

$$\begin{aligned}
\widetilde{\mathscr{C}}_0 &:= \{g \in \mathcal{F}_T : g \leq (H \bullet S)_T, \text{ for some admissible strategy } H\},\\
\widetilde{\mathscr{C}} &:= \widetilde{\mathscr{C}}_0 \cap \mathbf{L}^\infty,\\
\widetilde{\mathscr{D}} &:= \left\{Q \in (\mathbf{L}^\infty)^*_+ : \|Q\|_{(\mathbf{L}^\infty)^*} = 1, \langle Q, f\rangle \leq 0, \text{ for all } f \in \widetilde{\mathscr{C}}\right\},\\
\mathcal{M} \subseteq &\widetilde{\mathscr{D}}^r := \widetilde{\mathscr{D}} \cap \mathbf{L}^1.
\end{aligned}$$

Some properties of $\widetilde{\mathscr{D}}$:

1. $\widetilde{\mathscr{D}}$ is convex, $\sigma(ba, \mathbf{L}^\infty)$-compact.
2. $-\mathbf{L}^\infty_+ \subseteq \widetilde{\mathscr{C}}$, then $\widetilde{\mathscr{D}} \subseteq ba_+$, hence $\widetilde{\mathscr{D}}^r \subseteq \mathbf{L}^1_+$.
3. The set $\widetilde{\mathscr{D}}$ is the $\sigma(ba, \mathbf{L}^\infty)$-closure of $\mathcal{M}$.
4. $\widetilde{\mathscr{D}}^r$ is the set of absolutely continuous local martingale measures in the case of locally bounded S. (See [22, Theorem 5.6], see also [26] for the general case and the notation of equivalent sigma martingale measures.)
5. $\widetilde{\mathscr{D}}$ is the polar set of $\widetilde{\mathscr{C}}$ intersected with the norm-1 elements.

Remark 5.2.6. Let consider the properties of the dual domain with the same setting introduced in [13] but in markets with transaction costs. Denote by $\mathscr{A}(x)$ the set of all admissible self-financing trading strategies under transaction costs $\lambda \in (0, 1)$ with $(\varphi^0_0, \varphi^1_0) = (x, 0)$ and $\varphi^1_T = 0$. Then, define

$$\begin{aligned}
\mathscr{C}(x) &:= \left\{\varphi^0_T \,\middle|\, \varphi = (\varphi^0, \varphi^1) \in \mathscr{A}(x)\right\},\\
\mathscr{D} &:= \left\{Q \in ba_+ \,\middle|\, \|Q\| = 1 \text{ and } \langle Q, g\rangle \leq 0 \text{ for all } g \in \mathscr{C} \cap \mathbf{L}^\infty\right\},\\
\mathcal{Z}^\lambda_T &:= \left\{Z^0_T \in \mathbf{L}^1(\mathbf{P}) \,\middle|\, (Z^0, Z^1) \in \mathcal{Z}^\lambda(S)\right\},\\
\mathcal{Z}^\lambda_T \subseteq &\mathscr{D}^r := \mathscr{D} \cap \mathbf{L}^1.
\end{aligned}$$

We can still have the similar properties of the dual domain $\mathscr{D}$ as $\widetilde{\mathscr{D}}$. For more details, we refer to [37].

Proposition 5.2.7. *$\widetilde{\mathscr{D}}$ is convex.*

Proof. Let $Q_1, Q_2 \in \widetilde{\mathscr{D}}$, define $Q := \gamma Q_1 + (1-\gamma)Q_2,\ \gamma \in [0, 1]$

$$\begin{aligned}
\|Q\|_{(\mathbf{L}^\infty)^*} &= \langle Q, \mathbf{1_\Omega}\rangle = \gamma \langle Q_1, \mathbf{1_\Omega}\rangle + (1-\gamma)\langle Q_2, \mathbf{1_\Omega}\rangle\\
&= \gamma\|Q_1\|_{(\mathbf{L}^\infty)^*} + (1-\gamma)\|Q_2\|_{(\mathbf{L}^\infty)^*} = 1
\end{aligned}$$

It is obviously that $\langle Q, f\rangle \le 0$, for all $f \in \widetilde{\mathscr{C}}$. Thus, $Q \in \widetilde{\mathscr{D}}$. □

Proposition 5.2.8. *$\widetilde{\mathscr{D}}$ is weak star closed.*

Proof. Let $\{Q_\alpha\}_{\alpha\in I}$ be net in $\widetilde{\mathscr{D}}$, which converges to Q^* in weak star topology, i.e., for all $f \in \mathbf{L}^\infty$, $\langle Q_\alpha, f\rangle \longrightarrow \langle Q^*, f\rangle$. Since $g \in \widetilde{\mathscr{C}} = (\mathbf{L}^\infty \cap \widetilde{\mathscr{C}_0}) \subset \mathbf{L}^\infty$, so $0 \ge \langle Q_\alpha, g\rangle \longrightarrow \langle Q^*, g\rangle$. Thus, $\langle Q^*, g\rangle \le 0$.

Moreover,

$$\|Q^*\|_{(\mathbf{L}^\infty)^*} = \langle Q^*, \mathbf{1_\Omega}\rangle = \lim_{\alpha\in I} \langle Q_\alpha, \mathbf{1_\Omega}\rangle = \lim_{\alpha\in I} \|Q_\alpha\|_{(\mathbf{L}^\infty)^*} = 1$$

□

Proposition 5.2.9. *$\widetilde{\mathscr{D}}$ is weak star compact.*

Proof. Define

$$\widetilde{\mathscr{D}}^b := \Big\{Q \in (\mathbf{L}^\infty)^*_+ : \|Q\|_{(\mathbf{L}^\infty)^*} \le 1, \langle Q, f\rangle \le 0, \text{ for all } f \in \widetilde{\mathscr{C}}\Big\},$$

then $\widetilde{\mathscr{D}}^b$ is a closed unit ball in $(\mathbf{L}^\infty)^*$, so it is weak star compact by Alaoglu's Theorem. $\widetilde{\mathscr{D}}$ is a weak star closed subset of $\widetilde{\mathscr{D}}^b$, every closed subset of a compact space is compact, so $\widetilde{\mathscr{D}}$ is compact. □

Proposition 5.2.10. *The set $\mathcal{M}$ is a subset in $\widetilde{\mathscr{D}}$, i.e.,*

$$\mathcal{M} \subset \widetilde{\mathscr{D}}.$$

Proof. By definition, $\mathcal{M}$ is a subset in $\mathbf{L}^1(\mathbf{P})$, and we know $(\mathbf{L}^1(\mathbf{P}), \|\cdot\|_{\mathbf{L}^1(\mathbf{P})})$ is the norm-topology, so $\mathcal{M} \subseteq (\mathbf{L}^1(\mathbf{P}), \|\cdot\|_{\mathbf{L}^1(\mathbf{P})})$. Let $\mathbf{Q} \in \mathcal{M}$ be arbitrary, after embedding $\mathcal{M}$ into the space $((\mathbf{L}^\infty)^*, \|\cdot\|_{(\mathbf{L}^\infty)^*})$, we have

$$1 = \left\|\frac{d\mathbf{Q}}{d\mathbf{P}}\right\|_{\mathbf{L}^1} = \left\|\frac{d\mathbf{Q}}{d\mathbf{P}}\right\|_{(\mathbf{L}^\infty)^*} = 1.$$

□

Proposition 5.2.11. *$\widetilde{\mathscr{D}}$ is the weak star closure of $\mathcal{M}$, i.e.,*

$$\widetilde{\mathscr{D}} = \overline{\mathcal{M}}^{\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)} \tag{5.2.3}$$

Proof. Since $\mathcal{M} \subset \widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{D}}$ is weak star closed, we have

$$\overline{\mathcal{M}}^{\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)} \subseteq \widetilde{\mathscr{D}}.$$

To show the equality, we suppose, contrary to the claim, there exists a point $\widetilde{Q} \in \widetilde{\mathscr{D}}$ satisfying $\widetilde{Q} \notin \overline{\mathcal{M}}^{\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)}$. As $\widetilde{\mathscr{D}}$ is compact, applying Hahn-Banach separation theorem, one can find a linear functional $f \in \mathbf{L}^\infty$ and a constant $\alpha \in \mathbb{R}$ s.t.

$$\langle Q, f\rangle \le \alpha, \ \forall\, Q \in \overline{\mathcal{M}}^{\sigma((\mathbf{L}^\infty)^*, \mathbf{L}^\infty)},$$

but

$$\left\langle \widetilde{Q}, f \right\rangle > \alpha. \tag{5.2.4}$$

By superreplication theorem, for all $\mathbf{Q} \in \mathcal{M}$, $\langle \mathbf{Q}, f \rangle \leq \alpha$, we have $f - \alpha \in \widetilde{\mathscr{C}}$. Then by definition of $\widetilde{\mathscr{D}}$, we have $\left\langle \widetilde{Q}, f - \alpha \right\rangle \leq 0$, i.e., $\left\langle \widetilde{Q}, f \right\rangle \leq \alpha$ which is in contradiction to (5.2.4). Therefore, $\widetilde{\mathscr{D}} \subseteq \overline{\mathcal{M}}^{\sigma((\mathbf{L}^\infty)^*,\ \mathbf{L}^\infty)}$. □

5.3 FTAP

5.3.1 Incomplete markets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ be a stochastic basis satisfying the usual conditions.

Definition 5.3.1 (NFLVR)**.** *Given the (locally) bounded real-valued semimartingale S, we denote*

$$\mathcal{K}_0 = \left\{ (H \bullet S)_\infty : H \text{ admissible and } (H \bullet S)_\infty = \lim_{t \to \infty} (H \bullet S)_t \text{ exists a.s.} \right\} \subset \mathbf{L}^0,$$

and

$$C = (\mathcal{K}_0 - \mathbf{L}^0_+) \cap \mathbf{L}^\infty.$$

We say that S satisfies the condition ***no free lunch vanishing risk (NFLVR)****, if*

$$\overline{C} \cap \mathbf{L}^\infty_+ = \{0\},$$

where $\overline{C}$ denote the closure of C with respect to the norm topology of $\mathbf{L}^\infty$.

More details we refer to [22] or [27, Chapter 9].

Remark 5.3.2. Actually, the condition of (NFLVR) means: there is no sequence of final payoffs of admissible integrands, $f_n = (H^n \cdot S)_\infty$ such that the negative parts f_n^- tend to 0 uniformly and such that f_n tends a.s. to a $[0, \infty]$-valued function f_0 satisfying

$$\mathbf{P}[f_0 > 0] > 0.$$

In the property (NFLVR), one only requires that this risk is uniformly bounded below and that the variables f_n^- tend to 0 in probability.

Theorem 5.3.3. *Let S be a (locally) bounded real-valued semimartingale. There is an (ELMM) EMM for S if and only if S satisfies (NFLVR).*

Proof. See [22, Theorem 1.1 and Corollary 1.2]. □

Definition 5.3.4 (sigma-martingale)**.** *An $\mathbb{R}^d$- valued semimartingale S is called a **sigma-martingale**, if there is an $\mathbb{R}^d$-valued martingale M and an M-integrable predictable $\mathbb{R}_+$-valued process H such that $S = H \bullet M$.*

Theorem 5.3.5. *Let S be a real-valued semimartingale. Then S satisfies (NFLVR) if and only if there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that S is a sigma-martingale with respect to $\mathbf{Q}$.*

Proof. See [26, Theorem 1.1]. □

Definition 5.3.6. *An $\mathbb{R}$-valued predictable process H is called **1-admissible** trading strategy for $\mathbb{R}$-valued semimartingale S, if H is S integrable and $(H \bullet S)_t \geq -1$, a.s., $t \in [0, T]$.*

Definition 5.3.7 (NUPBR)**.** *The semimartingale S is said to satisfy the **no unbounded profit with bounded risk (NUPBR)** condition if the set*

$$\mathcal{K}_1 := \{(H \bullet S)_T : H \textit{ is a 1-admissible strategy for } S\}$$

is bounded in $\mathbf{L}^0$, i.e.,

$$\lim_{c \to \infty} \sup_{H \textit{ is 1-adm}} \mathbf{P}\left[|(H \bullet S)_T| > c\right] = 0.$$

Definition 5.3.8 (NA)**.** *The semimartingale S is said to satisfy the **no arbitrage (NA)** condition if 0 is the maximal element in the set $\mathcal{K}_1$, i.e., if there does not exist any 1-admissible strategy H such that*

$$(H \bullet S)_T \geq 0, \textit{ a.s., and } \mathbf{P}\left[(H \bullet S)_T > 0\right] > 0.$$

Remark 5.3.9. We may adopt the definition of (NA) from [27, Chapter 9.2.8] that a locally bounded semimartingale S satisfies (NA) if $C \cap \mathbf{L}^{\infty}_{+} = \{0\}$.

Definition 5.3.10. *An $\mathbb{R}$-valued càdlàg local martingale Z is called a **strict sigma-martingale density** for S if the following holds:*

1. *Z is strictly positive;*
2. *$\mathbb{E}\left[Z_0\right] < \infty$;*
3. *ZS is an $\mathbb{R}^d$-valued sigma-martingale.*

Theorem 5.3.11. *The semimartingale S satisfies (NUPBR) if and only if there exists a strict sigma-martingale density for S.*

Proof. See [86, Theorem 2.6]. □

Definition 5.3.12. *An $\mathbb{R}$-valued process Z is called an **equivalent local martingale deflator (ELMD)** or **strict martingale density** for semimartingale S if Z is a strictly positive localmartingale with $Z_0 = 1$ $\mathbf{P}$-a.s. such that ZS is a local martingale.*

Theorem 5.3.13. *The condition (NUPBR) is equivalent to the existence of at least one ELMD.*

Proof. See [61, Theorem 2.1]. $\square$

Remark 5.3.14 ([86, Remark 2.5]). Being the reciprocal of a strict sigma-martingale density is equivalent to being an ELMD.

Remark 5.3.15. NFLVR is equivalent to NA plus NUPBR (see e.g. [22, Corollary 3.8], [24, Theorem 2.9]), and it is also equivalent to NA plus the existence of a strict sigma-martingale density.

Remark 5.3.16. A result of [56, Theorem 4.12] proves that the condition of (NUPBR) is equivalent to the existence of an equivalent supermartingale deflator for the price process.

5.3.2 Markets with transaction costs

We recall the Fundamental Theorem of Asset Pricing for the continuous process in markets with transaction costs $\lambda \in (0,1)$. We adopt the Definitions 3.1.8 and 3.1.7 of CPS^λ in Chapter 3.

Assumption 5.3.17. *Assume the stock price process $S = (S_t)_{0\leq t\leq T}$ is adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$, with continuous and strictly positive paths.*

Definition 5.3.18 (Arbitrage)**.** *The process S admits arbitrage with λ-transaction costs, if there is an admissible self-financing trading strategy φ starting at $(\varphi_0^0, \varphi_0^1) = (0,0)$, such that the liquidation value*

$$V_T^{liq}(\varphi) \geq 0, \text{ a.s., and } \mathbf{P}[V_T^{liq}(\varphi) > 0] > 0.$$

Theorem 5.3.19. *Under Assumption 5.3.17, the following are equivalent:*

1. *For each $\mu > 0$, there exists an μ-CPS.*
2. *For each $\mu > 0$, there exists NA for μ-transaction costs.*

Proof. See [45, Theorem 4]. $\square$

We now give a similar local version of the Fundamental Theorem of Asset Pricing, which is described in e.g. [84, Theorem 5.11]. We shall use the subsequent variants of the concept of no arbitrage: no obvious arbitrage (NOA) from [45] with an additional boundedness condition in the case (b) below. We need this additional condition here as we now use a different notion of admissibility than in [45].

Definition 5.3.20. *Under Assumption 5.3.17, we say that S allows for an **obvious arbitrage** if there are $\alpha > 0$ and $[0,T] \cup \{\infty\}$-valued stopping times $\sigma \leq \tau$ with $\mathbf{P}[\sigma < \infty] = \mathbf{P}[\tau < \infty] > 0$, such that either*

(a) $S_\tau \geq (1+\alpha)S_\sigma$, *a.s. on* $\{\sigma < \infty\}$, *or*

(b) $S_\tau \le \frac{1}{(1+\alpha)} S_\sigma$, *a.s. on* $\{\sigma < \infty\}$.

In the case of (b) we also impose that $(S_t)_{\sigma \le t \le \tau}$ *is uniformly bounded.*

We say that S *allows for an* ***obvious immediate arbitrage*** *if, in addition, we have*

(a) $S_t \ge S_\sigma$, *for* $t \in [\![\sigma, \tau]\!]$, *a.s. on* $\{\sigma < \infty\}$, *or*

(b) $S_t \le S_\sigma$, *for* $t \in [\![\sigma, \tau]\!]$, *a.s. on* $\{\sigma < \infty\}$.

We say that S *satisfies the condition (NOA)(resp. (NOIA)) of no obvious arbitrage (resp. no obvious immediate arbitrage) if no such opportunity exists.*

In markets with sufficiently small transaction costs $0 < \lambda < 1$ (compare [45]), we will see how to make an arbitrage if (NOA) fails. Assuming e.g. condition (a), one goes long in the asset S at time σ and closes the position at time τ. In case of an obvious immediate arbitrage one is in addition assured that during such an operation the stock price will never fall under the initial value S_σ. In the case of condition (b), one does a similar operation by going short in the asset S. The boundedness condition in the case (b) of (NOA) makes sure that such a strategy is admissible.

Theorem 5.3.21. *Let* S *satisfy Assumption 5.3.17. The following assertions are equivalent.*

1. *Locally, there is no obvious immediate arbitrage (NOIA).*
2. *Locally, there is no obvious arbitrage (NOA).*
3. *Locally, for each* $0 < \mu < 1$, *there exists a* μ*-consistent price system.*
4. *For each* $0 < \mu < 1$, *there exists a local* μ*-consistent price system.*

Proof. See [16, Theorem 3.3]. □

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