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Abstract

The aim of this thesis is to introduce a new approach on Poisson transforms between vector bundle valued differential forms on a homogeneous parabolic geometry G/P and vector bundle valued differential forms on the corresponding Riemannian symmetric space G/K. Explicitly, these transforms are given by G-equivariant integral operators associated to a G-invariant differential form on a homogeneous space of G as its kernel, hence they are fully determined by invariant elements in a finite dimensional representation of a reductive Lie group. In addition, since this approach is tailored to the calculus of differential forms, we can express compositions of Poisson transforms with several G-equivariant differential operators again as Poisson transforms. In this way, we can explicitly design such operators with prescribed compatibility conditions via computations on the level of the corresponding representations. As a demonstration, we will construct a universal family of Poisson transforms Φ_k between real valued differential forms, whose images are coclosed and which reduces to the classical transform for smooth functions.

Afterwards, we apply our method to $G = \mathrm{SO}(n+1,1)_0$ and prove that the operators Φ_k are chain maps between the de Rham sequences on the conformal sphere and on the real hyperbolic space, respectively. Furthermore, we give an explicit construction of a family of Poisson transforms between standard tractor bundle valued differential forms, which factor to G-equivariant chain maps between the corresponding BGG-sequence on G/P and the twisted de Rham complex on G/K. Furthermore, the images of weighted differential forms under these operators are eigenforms for a covariant Laplace operator.

Finally, we turn to the case $G = \mathrm{SU}(n+1,1)$, where we obtain a finer decomposition of the space of Poisson kernels by exploiting the Kähler structure on the complex hyperbolic space G/K as well as the CR-structure on the corresponding boundary sphere G/P. This enables us to construct Poisson transforms between complex valued differential forms which descend to G-equivariant operators on sections of the complexified homology bundles on the CR-sphere. Subsequently, these will be used to design equivariant chain maps between the real valued BGG-sequence on G/P and the de Rham sequence on G/K.

Introduction

An important question in the theory of geometry on homogeneous spaces G/H is the existence of local invariants, i.e. G-invariant sections of homogeneous bundles, as well as differential operators which are equivariant with respect to the induced G-actions and therefore intrinsic to the given geometric structure. However, the existence and properties of such objects strongly depend on the Lie groups G and H. Indeed, since any Riemannian symmetric space G/K is reductive, every homogeneous vector bundle over G/K carries a canonical G-invariant connection, hence taking iterative derivatives and apply them to its curvature produce a vast amount of differential operators and local invariants, respectively.

On the other hand, the lack of G-invariant data on homogeneous vector bundles on homogeneous parabolic geometries G/P made the construction of local invariants and G-equivariant differential operators an intractable problem for a long time. This was ultimately solved by the development of a unified theory for parabolic geometries in terms of Cartan connections, which led to a tractor calculus on homogeneous vector bundles V over G/P associated to G-representations. Furthermore, via the canonical Lie algebra structure on the cotangent spaces of G/P and the induced homology differential ∂^* on $\Omega^*(G/P, V)$, called the Kostant codifferential, the concept of BGG-sequences were introduced in [**ČSS01**] and [**CD01**]. These are complexes of G-invariant differential operators D between sections of the corresponding homology bundles $\mathcal{H}_k(G/P, V) = \ker(\partial^*)/\operatorname{im}(\partial^*)$. Here, the crucial part is the existence of a splitting operator $L\colon \mathcal{H}_k(G/P, V) \to \ker(\partial^*)$ for the canonical projection, which is characterized by $\partial^* \circ d^\nabla \circ L = 0$ in terms of the covariant exterior derivative d^∇ induced by the tractor connection on V.

Naturally, the question arises if these two very different geometric structures can be related by a natural operator, which is answered by the Poisson transform. Initially, this was defined in the 1850's as an integral operator mapping continuous functions ϕ on the boundary ∂B of the unit ball B in $\mathbb{R}^n$ to smooth harmonic functions u on B, which satisfy $u|_{\partial B} = \phi$, c.f. [**GT01**, Theorem 2.6]. Otherwise said, the Poisson transform provided the solution of the corresponding Dirichlet problem on the unit ball. Afterwards, it was reformulated in terms of group theory as an operator from distributions on K/M to smooth functions on G/K, where the compact symmetric space K/M is interpreted as the Fürstenberg boundary of G/K, c.f. [**BJ06**, ch. 1.6]. In this setting, the main purpose of the Poisson transform was to produce joint eigenfunctions of the algebra of G-invariant differential operators on G/K, see [**Hel94**, Proposition II.3.14]. Afterwards, it was reinterpreted in [**KZ11**, section 4.1] as a G-equivariant differential operator mapping sections of principal series representations on G/P_0, where P_0 is a minimal parabolic subgroup of G, to smooth functions on G/K.

Due to its various important properties the Poisson transform was generalized to a map between sections of homogeneous vector bundles over $K/M \cong G/P_0$ and sections of homogeneous vector bundles over G/K, first for the case of real rank 1 in [**vdV94**] and afterwards in full generality in [**Olb95**] and [**Yan94**]. Here, the existence and the properties of such operators depend directly on an M-equivariant map between the representations underlying the natural bundles. Another approach in this direction was taken by P.Y-Gaillard in [**Gai86**], in which he defined Poisson transforms between density valued differential forms on the conformal sphere and real valued differential forms on the real hyperbolic space of the same degree. This was done by constructing an $\mathrm{SO}(n+1,1)$-invariant differential form on the product manifold using "visual maps" as in [**Thu**, chapter 11], see also section 2.1.3.

The aim of this thesis is to generalize Gaillard's approach to Poisson transforms. Explicitly, let G be a semisimple Lie group with finite centre, K its maximal compact subgroup, P a parabolic subgroup of G and let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional representations of P and K, respectively. Then we construct in section 2.2.1 G-equivariant Poisson transforms

$$\Phi\colon \Omega^k(G/P, G\times_P \mathbb{V}) \to \Omega^k(G/K, G\times_K \mathbb{W})$$

by considering a $(G\times_M L(\mathbb{V},\mathbb{W}))$-valued differential form ϕ on the product space $G/K\times G/P$ and define

$$\Phi(\alpha) := \fint_{G/P} \phi\wedge \pi_P^*\alpha$$

for all $\alpha\in\Omega^*(G/P, G\times_P\mathbb{V})$, where π_P is the canonical projection from the product manifold $G/K\times G/P$ onto its second factor and where $\fint_{G/P}$ denotes the fibre integral over G/P. Now the crucial observation that $G/K\times G/P$ is itself a homogeneous space isomorphic to G/M with $M = K\cap P$ implies that the Poisson kernel ϕ is determined entirely by an invariant element in a finite dimensional M-representation, c.f. Theorem 2.2.1.

Despite being a special case of the general vector valued version of Poisson transforms (c.f. Theorem 2.2.2), the advantage of our approach is that it is tailored specifically to the calculus of differential forms. In this way, the compositions of Φ with the covariant exterior derivative and the Kostant codifferential on G/P as well as with the standard differential operators in Hodge theory on G/K can be realized again as Poisson transforms in our sense, c.f. Proposition 2.2.3. In particular, we can determine the properties of these compositions on the level of the corresponding M-representations, i.e. via computations in finite dimensional vector spaces.

A special focus in this thesis is put on the construction of G-equivariant operators $\tilde{\Phi}$ defined on sections of the homology bundles $\mathcal{H}_k(G/P, G\times_P\mathbb{V})$ for irreducible G-representations $\mathbb{V}$. One way to obtain such operators is to first identify the sections of these bundles with the subcomplex of the twisted de Rham-complex determined by the Casimir eigenvalue of $\mathbb{V}$ via the splitting operator using [**ČS07**, Corollary 1], and afterwards apply a nontrivial Poisson transform. However, the explicit computation of the image of the splitting operator is a difficult task in general, meaning that we cannot relate these to properties of the Poisson kernel corresponding to Φ. Therefore, we choose to construct Poisson transforms which are trivial on the image of the Kostant codifferential, as they factor canonically to such G-equivariant operators which can be computed independently of the representative of a section of $\mathcal{H}_k(G/P, G\times_P\mathbb{V})$. Furthermore, in this picture the composition of $\tilde{\Phi}$ with the BGG-operators can be described in terms of the corresponding transform Φ as the composition with the covariant exterior derivative. The study of these BGG-compatible Poisson transforms culminates in Theorem 2.2.3 which provides a relation to the Laplace operator on G/K induced by the tractor connection.

THEOREM. *Let $\mathbb{V}$ be a G-representation. Then for any Poisson transform $\Phi\colon \Omega^k(G/P, G\times_P\mathbb{V})\to\Omega^k(G/K, G\times_K\mathbb{V})$ the following are equivalent:*

(i) The image of the transform Φ is contained in the harmonic differential forms.
(ii) The transform Φ is BGG-compatible.

Afterwards, we will show the efficiency of our approach by considering several explicit cases. First, using an easy analysis of the geometry of G/M we uniformly construct in section 2.3.2 a family of Poisson kernel whose associated Poisson transforms

$$\Phi\colon \Omega^k(G/P,\mathcal{E}[\lambda]) \to \Omega^k(G/K)$$

have the property that their images consist of coclosed differential forms. Next, we turn in section 3 to the case of $G = \mathrm{SO}(n+1,1)$, where we discuss Gaillard's transforms in a more conceptual way, recovering many of its properties. In addition, we present a new construction of a BGG-compatible vector valued Poisson kernel whose associated Poisson transform maps between standard tractor valued differential forms on the conformal sphere and standard tractor valued differential forms on the real hyperbolic space of the same degree. Then a careful analysis show that the induced G-equivariant operators $\tilde{\Phi}_k^{\mathbb{V}}$ on the corresponding homology bundles satisfy the following Theorem 3.4.4.

THEOREM. *Let $\mathbb{V}$ be the standard representation of $G = \mathrm{SO}(n+1,1)_0$. Then the G-equivariant operators*

$$\tilde{\Phi}_k^{\mathbb{V}}\colon \mathcal{H}_k(G/P, G\times_P \mathbb{V}[\lambda]) \to \Omega^k(G/K, G\times_K \mathbb{V})$$

satisfy the following properties:

(i) The image of $\tilde{\Phi}_k^{\mathbb{V}}$ consists of coclosed differential forms and eigenvectors of the twisted Laplace Beltrami operator with eigenvalue $-\lambda(n-2k+\lambda)$.

(ii) For $\lambda = 0$ let D be the BGG-operator on $\Gamma(\mathcal{H}_k(G/P, G\times_P \mathbb{V}))$ and d the covariant exterior derivative on $\Omega^k(G/K, G\times_K \mathbb{V})$ induced by the tractor connection on $G\times_K \mathbb{V}$. Then

$$(k+2)d\circ \tilde{\Phi}_k^{\mathbb{V}} = (-1)^{n-k} k(n-2k)\tilde{\Phi}_{k+1}^{\mathbb{V}}\circ D.$$

Finally, we consider the case $G = \mathrm{SU}(n+1,1)$ and construct in section 4.3.2 a family of BGG-compatible Poisson transforms between complex valued differential forms on the CR-sphere and differential forms on G/K with a specific (p,q)-type. Due to the different nature of the homology bundles on G/P these operators divide into two classes depending on the degree k of the differential forms in their domain, where for $k \leq n$ we will recover the transforms constructed by Gaillard in [**Gai78**]. For the other degrees, the author recently learned through a private conversation that these operators where obtained independently by M. Olbrich.

The thesis is divided as follows: After recalling some basic properties of complex and real semisimple Lie algebras in section 1.1 we will discuss in section 1.2 the correspondence between G-invariant data on homogeneous spaces G/H and H-invariant objects in finite dimensional vector spaces, which will be essential for the study of our version of Poisson transforms. Subsequently, we will apply this to the special case of Riemannian symmetric spaces G/K in section 1.3. Here, we will focus on the calculus for differential forms with values in homogeneous bundles associated to G-representations, which is in contrast to irreducible vector bundles usually considered in harmonic analysis. In particular, we will prove in Theorem 1.3.3 a Weitzenböck formula between the resulting Laplace operator and the differential operator induced by the Casimir element of the Lie algebra of G similar to [**ČS07**, Corollary 1].

Next, we turn to homogeneous parabolic geometries G/P and introduce the Kostant codifferential ∂^* in 1.4.2 on tractor-bundle valued differential forms, which will lead to the definition of BGG-sequences in 1.4.3. Furthermore, we establish a formula for the adjoint of ∂^* with respect to the wedge product of differential forms in Theorem 1.4.4, which will be needed to relate the composition of this operator and a Poisson transform to an associated Poisson kernel.

Section 2 is dedicated to the introduction and general discussion of our version of Poisson transforms. Explicitly, after their construction in section 2.2.1 we will prove alternative descriptions in terms of equivariant maps in section 2.2.2 and show in Proposition 2.2.3 that compositions of Poisson transforms with several G-equivariant differential operators are again Poisson transforms. This general discussion is completed by the explicit construction of a family of Poisson kernels whose associated transforms Φ_k map between real valued differential forms and whose images are contained in the coclosed forms on G/K, c.f. Theorem 2.3.2. Furthermore, we will see in Proposition 2.3.3 that this example reduces in the special case of minimal parabolic subgroups and real functions to the classical Poisson transform as in [**Hel94**, ch. II.4].

In section 3 we consider Poisson transforms between the conformal sphere and the real hyperbolic space corresponding to $G = \mathrm{SO}(n+1,1)_0$. After recalling the structure of the involved Lie groups and homogeneous spaces in sections 3.1 and 3.2 we focus in 3.3 on the analysis of the Poisson transforms Φ_k in this case, which coincide with Gaillard's transforms in [**Gai86**]. In particular, we recover many of their properties in a more efficient and conceptual way. Finally, in section 3.4 we construct vector valued BGG-compatible Poisson kernels whose associated transforms map between standard tractor bundle valued differential forms of the same degree. Subsequently, we will show that these operators are chain maps between the corresponding twisted de Rham sequences and that the induced G-equivariant operators defined on sections of the weighted homology bundles produce eigenforms for the twisted Laplace Beltrami operator on G/K.

Finally, in section 4 we turn to the case $G = \mathrm{SU}(n+1,1)$. After a discussion of the geometry of the corresponding homogeneous spaces in 4.1, the geometric properties of G/M will be used to define basic invariant forms in 4.2.2 as wedge products of invariant forms of low degree. Subsequently, these will serve in a sophisticated ansatz tailored to the structure of the homology bundles on G/P to construct BGG-compatible Poisson kernels whose associated Poisson transforms $\Phi_{p,q}$ have values in the space of differential forms of a specific (p,q)-type, c.f. section 4.3.2. Finally, in section 4.4 we will derive formulae for the M-equivariant maps corresponding to differential operators on G/K and use them to determine the properties of the images of $\Phi_{p,q}$ as well as to construct Poisson transforms between real valued differential forms, which have properties similar to Gaillard's operators.

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Contents

CHAPTER 1

Preliminaries

In this preliminary section we develop the tools needed to construct and study our approach to Poisson transforms. After recalling some of the theory of Lie algebras in 1.1 we turn to the discussion of the geometry of homogeneous spaces. Here, we will first consider the general case G/H in section 1.2, where G is a Lie group and H is any closed subgroup, and analyse natural geometric objects, such as homogeneous bundles and invariant connections, in terms of invariant data on finite dimensional H-representations.

Subsequently, we will apply these tools to two specific types of homogeneous spaces, starting with symmetric spaces G/K in section 1.3, where we focus on a calculus of differential forms with values in a tractor bundle. In particular, the existence of two flat connections on such natural bundles will induce a Laplace operator on such forms, which can be related to the differential operator induced by the Casimir element of the Lie algebra of $\mathfrak{g}$ by a Weitzenböck formula, c.f. Theorem 1.3.3.

Afterwards, we consider the case of homogeneous parabolic geometries G/P in section 1.4. After a brief discussion of the structure of $|k|$-graded Lie algebras will introduce the Kostant codifferential, i.e. a homology differential between differential forms with values in natural bundles which is tensorial and G-equivariant, and the BGG-operators, which is a series of natural differential operators between sections of the induced homology bundles.

Finally, we define in section 1.5 the integral of vector bundle valued differential forms over the fibres of a fibred manifold $E \to M$ and provide a description in a local coordinates on E. In addition, we will show compatibility of this operation with covariant exterior derivatives and derive estimates of the pointwise norms of the integrals in the case of Riemannian submersions.

1.1. Semisimple Lie algebras and their representations

We start our discussion by recalling some properties of complex and real semisimple Lie algebras, which are common knowledge and can be found in any book about Lie theory, e.g. [**Kna02**].

1.1.1. Complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let B denote its Killing form, which is an invariant, non-degenerate bilinear form on $\mathfrak{g}$. Choosing a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we denote by Δ the roots of $\mathfrak{g}$ and for any $\alpha \in \Delta$ by $\mathfrak{g}_\alpha$ the corresponding root space. It is well known that these spaces are 1-dimensional and that the bracket $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ equals $\mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta$ is a root and is trivial if $\alpha + \beta$ is not a root. In particular, this implies that the Killing form on $\mathfrak{g}$ induces a non-degenerate pairing of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

Let $\mathfrak{h}_0$ denote the real subspace of $\mathfrak{h}$ on which all roots are real valued. The Killing form induces a positive definite inner product $\langle\ ,\ \rangle$ on the real vector space $\mathfrak{h}_0^*$. If we fix a total ordering on $\mathfrak{h}_0^*$ we denote by Δ^+ and Δ^0 the positive and simple roots in Δ, respectively, which correspond to the choice of ordering. Then every positive root can be written as a linear combination of simple roots with positive integer coefficients.

For any finite dimensional representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$ of $\mathfrak{g}$ the Cartan algebra $\mathfrak{h}$ acts diagonalisable on $\mathbb{V}$. For each weight $\lambda \in \mathfrak{h}^*$ we denote by $\mathbb{V}_\lambda$ the corresponding weight space and by $n_\lambda(\mathbb{V}) = \dim(\mathbb{V}_\lambda)$ the multiplicity of the weight λ in $\mathbb{V}$. For all $\alpha \in \Delta$ the action of the root space $\mathfrak{g}_\alpha$ satisfies $\mathfrak{g}_\alpha \cdot \mathbb{V}_\lambda \subset \mathbb{V}_{\alpha+\lambda}$, where the latter space is trivial if $\alpha + \lambda$ is not a weight of $\mathbb{V}$. It is well known that every representation of $\mathfrak{g}$ is decomposable and splits into a direct sum of irreducible representations. Moreover, every irreducible representation is determined by its highest weight with respect to the total ordering on $\mathfrak{h}_0^*$.

Recall that the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$ is the unique unital associative algebra generated by the Lie algebra $\mathfrak{g}$. By construction, every $\mathfrak{g}$-representation is canonically a module over $\mathfrak{u}(\mathfrak{g})$. Moreover, the centre of $\mathfrak{u}(\mathfrak{g})$ commutes with every element in $\mathfrak{g}$, so by Schur's lemma acts by a scalar on irreducible representations. The Casimir C of $\mathfrak{g}$ is the unique element in $\mathfrak{u}(\mathfrak{g})$ which is the image of the identity on $\mathfrak{g}$ under the series of isomorphisms

$$\operatorname{End}(\mathfrak{g}) \xrightarrow{\cong} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\mathrm{id} \otimes B} \mathfrak{g} \otimes \mathfrak{g},$$

where we view the latter space as a subspace of the universal enveloping algebra of $\mathfrak{g}$. Explicitly, if $\{X_i\}$ is a basis of $\mathfrak{g}$ and $\{Y_i\}$ is the dual basis with respect to the Killing form, then the Casimir is given by $C = \sum_i Y_i \otimes X_i$.

PROPOSITION 1.1.1. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\langle\ ,\ \rangle$ and $\|\ \|$ denote the inner product and the norm on the real subspace $\mathfrak{h}_0^*$, respectively, which are both induced by the Killing form. Let $\mathbb{V}$ be an irreducible representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$. Then the Casimir $C \in \mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$ acts on $\mathbb{V}$ by the scalar*

$$\|\lambda\|^2 + 2\langle\lambda, \delta\rangle = \|\lambda + \delta\|^2 - \|\delta\|^2,$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$ is the lowest weight of $\mathfrak{g}$.

PROOF. [**Kna02**, Proposition 5.28]. □

1.1.2. Parabolic subalgebras of complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable Lie subalgebra, which is unique up to inner automorphisms of $\mathfrak{g}$. In order to construct it explicitly, first choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a positive subsystem Δ^+ of the roots of $\mathfrak{g}$. Afterwards, define the nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ as the direct sum of all roots spaces corresponding to positive roots. Then the space $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ is a maximal solvable subalgebra of $\mathfrak{g}$, called the *standard Borel subalgebra* with respect to the choice of $\mathfrak{h}$ and Δ^+.

A *(standard) parabolic subalgebra* is a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ which contains a (standard) Borel subalgebra. It can be shown that every parabolic subalgebra is a standard parabolic subalgebra for appropriate choices of a Cartan subalgebra and positive roots. Explicitly, there is a correspondence between parabolic subalgebras and subsets of simple roots of $\mathfrak{g}$.

PROPOSITION 1.1.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, Δ the corresponding set of roots and Δ^0 the set of simple roots for a given ordering on $\mathfrak{h}_0^*$. Then there is a bijective correspondence between subsets of Δ^0 and standard parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$.*

Explicitly, it is given by the map

$$\mathfrak{p} \mapsto \Sigma_{\mathfrak{p}} = \{\alpha \in \Delta^0 \mid \mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}\},$$

whose inverse assigns to every set Σ of simple roots the direct sum of the standard Borel subalgebra $\mathfrak{b}$ and all negative root spaces which can be written as linear combinations of elements in $\Delta^0 \setminus \Sigma$.

PROOF. [**ČS09**, Proposition 3.2.1] □

This correspondence can be used to associate a direct sum decomposition of the Lie algebra $\mathfrak{g}$, which is nicely compatible with the Lie bracket.

DEFINITION 1.1.2. Let $\mathfrak{g}$ be a (complex or real) semisimple Lie algebra and $k \geq 0$ an integer. A $|k|$-*grading* on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ into a direct sum of vector spaces so that

(a) $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_i = \{0\}$ if $|i| > k$,
(b) the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$ as a Lie algebra,
(c) $\mathfrak{g}_{-k} \neq \{0\}$ and $\mathfrak{g}_k \neq \{0\}$.

Now we explain how parabolic subalgebras are related to $|k|$-gradings on $\mathfrak{g}$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ corresponding to the subset Σ of the simple roots $\Delta^0 = \{\alpha_1, \ldots, \alpha_n\}$. Then we define the Σ-height $\mathrm{ht}_\Sigma(\alpha)$ of a root $\alpha \in \Delta$ via

$$\mathrm{ht}_\Sigma\left(\sum_i a_i\alpha_i\right) := \sum_{i:\alpha_i\in\Sigma} a_i.$$

Moreover, for all $i \in \mathbb{Z}$, $i \neq 0$ we define $\mathfrak{g}_i := \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$, and for $i = 0$ we put $\mathfrak{g}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha:\mathrm{ht}_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$.

THEOREM 1.1.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ with corresponding roots Δ. Choose a positive subsystem Δ^+ and let Δ^0 be the simple roots in Δ.*

(i) For any standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ corresponding to $\Sigma \subset \Delta^0$ the decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ according to Σ-height endows $\mathfrak{g}$ with a structure of a $|k|$-graded Lie algebra such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$.
(ii) Conversely, for any $|k|$-grading we can choose a Cartan subalgebra and positive roots so that $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is a standard parabolic subalgebra corresponding to Σ, and the grading is given by Σ-height.

PROOF. [**ČS09**, Theorem 3.2.1] □

Since $|k|$-gradings of semisimple Lie algebras will play a significant role in the study of parabolic geometries, we will analyse them further in section 1.4.1.

1.1.3. Real semisimple Lie algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let B be the Killing form of $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the corresponding Cartan decomposition, where $\mathfrak{k}$ and $\mathfrak{q}$ denote the eigenspaces of θ with eigenvalues $+1$ and -1, respectively. Determining directly the bracket relations with respect to the Cartan decomposition we obtain that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, the vector space $\mathfrak{q}$ is a $\mathfrak{k}$-module via the adjoint action and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$. Moreover, the restriction of the Killing form to $\mathfrak{k}$ is negative definite whereas its restriction to $\mathfrak{q}$ is positive definite.

From now on we assume that the subspace $\mathfrak{q}$ is nontrivial. Choosing a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{q}$ the maps $\mathrm{ad}(X)$ for $X \in \mathfrak{a}$ are simultaneously diagonalisable. We call the dimension of $\mathfrak{a}$ the *real rank* of the Lie algebra $\mathfrak{g}$. Moreover, we denote by $\Delta_r \subset \mathfrak{a}^*$ the set of restricted roots and for any such restricted root λ we define $\mathfrak{g}_\lambda$ to be the corresponding restricted root space.

If λ is a restricted root, then the Cartan involution induces a linear isomorphism between $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{-\lambda}$, which implies that also $-\lambda$ is a restricted root. Moreover, we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ for all λ, $\mu \in \Delta_r$. In particular, the space $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ which is invariant under the action of θ. Thus, we can decompose $\mathfrak{g}_0$ into the direct sum of $\mathfrak{m} := \mathfrak{g}_0 \cap \mathfrak{k}$, which is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $\mathfrak{g}_0 \cap \mathfrak{q}$, which equals $\mathfrak{a}$ by maximality.

Choosing a notion of positivity on $\mathfrak{a}^*$ we denote by Δ_r^+ and Δ_r^0 the positive and simple restricted roots, respectively. Let $\mathfrak{n}$ be the direct sum of all restricted root spaces corresponding to positive roots. Then by definition of the Cartan involution the space $\theta(\mathfrak{n})$ consists of all negative root spaces. Hence, we can decompose $\mathfrak{g}$ into the direct sum

$$\mathfrak{g} = \theta(\mathfrak{n}) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

of $\mathfrak{g}_0$-modules. In particular, every summand in the above decomposition is an $\mathfrak{m}$-module as well as an $\mathfrak{a}$-module. Using the relation $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ we deduce that both $\mathfrak{n}$ and $\theta(\mathfrak{n})$ are nilpotent Lie algebras. Moreover, the space $\mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{g}$ with bracket relation $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] \subset \mathfrak{n}$, meaning that $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie algebra.

Next, we want to see how the Cartan decomposition is related to the above eigenspace decomposition of $\mathfrak{g}$. The maximal compact subalgebra $\mathfrak{k}$ was defined as the $(+1)$-eigenspace of the Cartan involution. By definition, the Lie algebra $\mathfrak{m}$ is contained in $\mathfrak{k}$, whereas $\mathfrak{a}$ is a subalgebra of $\mathfrak{q}$. Moreover, we can write every element $X \in \mathfrak{n}$ as

$$X = \frac{1}{2}(X + \theta(X)) + \frac{1}{2}(X - \theta(X)),$$

where the first summand is contained in $\mathfrak{k}$, whereas the second summand lies in $\mathfrak{q}$. In particular, if we define

$$\mathrm{S}_\pm(\mathfrak{n}) := \{X \pm \theta(X) | X \in \mathfrak{n}\}$$

we see that $\mathfrak{k} = \mathrm{S}_+(\mathfrak{n}) \oplus \mathfrak{m}$ and $\mathfrak{q} = \mathrm{S}_-(\mathfrak{n}) \oplus \mathfrak{a}$. From this we immediately obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra $\mathfrak{g}$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\Theta\colon G \to G$ be the global Cartan involution, i.e. the unique automorphism of G whose derivative is θ. The group K of fixed points of Θ is a connected closed subgroup of G which contains the centre $Z(G)$ and has Lie algebra $\mathfrak{k}$. In particular, the group K is compact if and only if $Z(G)$ is finite. In this case, K is a maximal compact subgroup of G.

It is also possible to prove a global version of the Iwasawa decomposition. For that, let A and N be the analytic subgroups of G corresponding to the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ and let K be the fix point set of the global Cartan involution as above.

PROPOSITION 1.1.3 (Global Iwasawa decomposition). *Using the notation from above, the multiplication $K \times A \times N \to G$, $(k, a, n) \mapsto kan$, is a diffeomorphism.*

PROOF. [**Kna02**, Theorem 6.46] □

1.1.4. Parabolic subalgebras of real semisimple Lie algebras. In the realm of real semisimple Lie algebras $\mathfrak{g}$ we will define parabolic subalgebras in terms of their complexification in $\mathfrak{g}_\mathbb{C}$. For that, we first have to analyse the connection between restricted root systems of $\mathfrak{g}$ and roots in $\mathfrak{g}_\mathbb{C}$.

As before let $\mathfrak{g}$ be a real semisimple Lie algebra, θ a Cartan involution with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{a} \subset \mathfrak{q}$ a maximal abelian subalgebra and $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, i.e. a maximal abelian subalgebra so that its complexification $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. In order to get a connection between the roots of $\mathfrak{g}_\mathbb{C}$ and the restricted roots of $\mathfrak{g}$, we assume that $\mathfrak{h}$ is θ-stable. In this way, we can decompose $\mathfrak{h}$ into the direct sum $(\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{q})$, and we say that $\mathfrak{h}$ is *maximally noncompact* if the dimension of $\mathfrak{h} \cap \mathfrak{q}$ is maximal amongst all choices of Cartan subalgebras. We can realize a maximally noncompact θ-stable Cartan algebra by choosing a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{m}$ and defining $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$.

Let Δ denote the roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Then the restricted roots Δ_r are exactly the nontrivial restrictions of roots of $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{a}$. On the other hand, let $\Delta_c = \{\alpha \in \Delta : \alpha|_{\mathfrak{a}} = 0\}$ be the set of compact roots of $\mathfrak{g}_{\mathbb{C}}$. If we denote by σ the conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$, then a root is compact if and only if $\sigma^*\alpha = -\alpha$. We say that a positive subsystem Δ^+ of Δ is admissible if for $\alpha \in \Delta^+$ we either have $\sigma^*\alpha = -\alpha$ or $\sigma^*\alpha \in \Delta^+$.

Definition 1.1.4 (Parabolic subalgebra). Let $\mathfrak{g}$ be a real semisimple Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$, θ a Cartan involution, $\mathfrak{h}$ a θ-stable maximally noncompact Cartan subalgebra in $\mathfrak{g}$, Δ the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and Δ^+ and admissible positive subsystem.

A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a *standard parabolic subalgebra* with respect to $\mathfrak{h}$ and Δ^+ if and only if the complexification $\mathfrak{p}_{\mathbb{C}}$ is a standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ and Δ^+.

Analogously to the case of complex parabolic subalgebras, there is a connection between subsets of the restricted roots and standard parabolic subalgebras.

Proposition 1.1.4. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, θ a Cartan involution with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ a maximally noncompact θ-stable Cartan subalgebra. Let Δ be the roots of $\mathfrak{g}_{\mathbb{C}}$ and Δ^+ be an admissible positive subsystem.*

(i) Let $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{n} \subset \mathfrak{g}$ be the direct sum of all positive restricted root spaces. Then $\mathfrak{p}_0 := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{g}$, and the standard parabolic subalgebras of $\mathfrak{g}$ are exactly those containing $\mathfrak{p}_0$.

(ii) Let Δ_r^0 be the set of simple restricted roots. Then subsets of Δ_r^0 are in bijective correspondence with the set of all standard parabolic subalgebras of $\mathfrak{g}$. Explicitly, the correspondence is given by mapping $\Sigma \subset \Delta_r^0$ to the sum of $\mathfrak{p}_0$ and the restricted root space for those negative restricted roots which can be written as a linear combination of elements in $\Delta_r^0 \setminus \Sigma$.

Proof. [**ČS09**, Theorem 3.2.9] □

Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra of $\mathfrak{g}$ corresponding to $\Sigma \subset \Delta_r^0$. As before, we can introduce the Σ-height of restricted roots in Δ_r^0, which induces a grading of $\mathfrak{g}$ via the restricted root space decomposition. Thus, we are able to associate a $|k|$-grading on $\mathfrak{g}$ as in the case of complex Lie algebras.

Theorem 1.1.4. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and θ a Cartan involution on $\mathfrak{g}$. Let $\mathfrak{h}$ be a θ-stable maximally noncompact Cartan subalgebra of $\mathfrak{g}$ and choose an admissible positive subsystem Δ^+. Then we have:*

(i) For a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ corresponding to a subset $\Sigma \subset \Delta_r^0$, the Σ-height determines a $|k|$-grading of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$.

(ii) Given a $|k|$-grading of $\mathfrak{g}$, there is an automorphism $\phi \in \mathrm{Int}(\mathfrak{g})$ such that the image of $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ under ϕ is a standard parabolic subalgebra of $\mathfrak{g}$. Denoting by Σ the corresponding subset of Δ_r^0, the given grading on $\mathfrak{g}$ is induced by the grading by Σ-height via ϕ.

Proof. [**ČS09**, Proposition 3.2.9] □

1.2. Geometry of homogeneous spaces

In the sequel we will introduce a new method of constructing Poisson transforms between sections of homogeneous bundles on homogeneous parabolic geometries and Riemannian symmetric spaces, respectively, which will depend on invariant differential forms on a homogeneous space. Thus, for the analysis of these operators several tools developed in the study of the geometry on homogeneous spaces G/H can be used, to which this section is dedicated. First, we recall the notion of (sections of) homogeneous bundles and link these to invariant objects in finite dimensional H-representations in sections 1.2.1 and 1.2.2. Next, we turn our focus to invariant differential operators by discussing G-invariant connections on natural vector bundles and their local invariants in 1.2.3 as well as the corresponding covariant exterior derivative in section 1.2.4. Afterwards, we will introduce the adjoint tractor bundle in section 1.2.5 and show how it induces a notion of a fundamental derivative D for all natural bundles, which will be related in section 1.2.6 to the differential operator induced by the Casimir element of the Lie algebra of G.

1.2.1. Homogeneous bundles. Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. We will denote by $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint actions of G and $\mathfrak{g}$, respectively. In addition, the Maurer-Cartan form will be denoted by $\omega_G \in \Omega^1(G, \mathfrak{g})$.

For a closed subgroup H of G consider the homogeneous space $M := G/H$. Left multiplication on G induces a left G-action ℓ_g on the homogeneous space by diffeomorphisms. Moreover, the canonical projection $\pi_G\colon G \to M$ endows G with the structure of a principal H-bundle over M.

Let $\pi_E\colon E \to M$ be a homogeneous bundle with G-action $\tilde{\ell}$ on E. Then we obtain an induced H-action on the fibre $E_o = \pi_E^{-1}(o)$ over $o = eH$ by restriction. The associated bundle construction delivers an isomorphism $E \cong G \times_H E_o$, so the H-manifold E_o determines the bundle E completely.

Proposition 1.2.1. *Let $o = eH \in G/H$ and let E_o denote the fibre over o. The mappings $E \mapsto E_o$ and $f \mapsto f|_{E_o}$ induce an equivalence between the category of homogeneous fibre bundles over G/H and the category of manifolds endowed with left H-actions and H-equivariant smooth maps. In this equivalence, vector bundles correspond to H-representations.*

Proof. [**ČS09**, Proposition 1.4.3.] □

Since the associated bundle construction is natural, every natural bundle over a homogeneous space is again homogeneous. Furthermore, dual bundles, Whitney sums, tensor products and exterior powers of homogeneous bundles are homogeneous, and the corresponding representations are given by their counterpart on the level of vector spaces. For example, if V and W are homogeneous vector bundles with corresponding H-representations $\mathbb{V}$ and $\mathbb{W}$, then the bundle $V^* \oplus W$ is also homogeneous with underlying H-representation $\mathbb{V}^* \oplus \mathbb{W}$.

Finally, if we choose $\mathbb{V}$ to be a representation of the full group G, then we can view $\mathbb{V}$ also as a representation of the Lie group H by restriction. Therefore, we can form the associated vector bundle $G \times_H \mathbb{V}$ over G/H, which we will call a *tractor bundle* over G/H. We obtain a natural diffeomorphism $G \times_H \mathbb{V} \cong G/H \times \mathbb{V}$ by mapping $[\![g, v]\!]$ to the element $(gH, g \cdot v)$. This shows that on a homogeneous space every tractor bundle is globally trivial.

As an important example, we consider the tangent bundle TM of the homogeneous space M. Via the tangent maps of the G-action on M this is a homogeneous bundle. If we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively, Proposition 1.2.1 tells us that $TM \cong G \times_H \mathfrak{g}/\mathfrak{h}$, where we identified the tangent space T_oM at $o = eH$ with the H-representation $\mathfrak{g}/\mathfrak{h}$. Explicitly, if $\pi_{\mathfrak{h}}$ denotes the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, this diffeomorphism assigns to any tangent vector $\xi \in T_{gH}M$ the element $[\![g, \pi_{\mathfrak{h}}(\omega_G(\tilde{\xi}))]\!]$ in the associated bundle $G \times_H \mathfrak{g}/\mathfrak{h}$, where $\tilde{\xi} \in T_gG$ is a lift of ξ to G. Since two such lifts differ by an element of the vertical bundle VM, which is mapped to $\mathfrak{h}$ by ω_G, this map is indeed well defined. A direct computation shows that the induced action of $h \in H$ on the fibre over $o = eH$ is given by

$$h \cdot \pi_{\mathfrak{h}}(X) = \pi_{\mathfrak{h}}(\mathrm{Ad}(h)(X)) =: \underline{\mathrm{Ad}}(h)(X + \mathfrak{h})$$

for all $X \in \mathfrak{g}$.

By naturality of associated bundles we deduce that the cotangent bundle T^*M of TM is also a homogeneous bundle diffeomorphic to $G \times_H (\mathfrak{g}/\mathfrak{h})^*$. In addition, functoriality implies that the action of $h \in H$ on $\varphi \in (\mathfrak{g}/\mathfrak{h})^*$ is the dual of the action of H on $\mathfrak{g}/\mathfrak{h}$, i.e.

$$(h \cdot \varphi)(X + \mathfrak{h}) = \varphi(\underline{\mathrm{Ad}}(h^{-1})(X + \mathfrak{h}))$$

for all $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. In a similar way, we can determine the induced H-actions on $\Lambda^k(\mathfrak{g}/\mathfrak{h})^*$, which correspond to the homogeneous vector bundles $\Lambda^k T^*M$ for all $1 \leq k \leq \dim(M)$.

For the construction of Poisson transforms we have to analyse the pullback of a homogeneous vector bundle along diffeomorphisms between homogeneous spaces.

Corollary 1.2.1. *Let G be a Lie group, H and K closed subgroups of G with $K \subset H$ and let $\pi\colon G/K \to G/H$ be the canonical projection. For any H-representation $\mathbb{V}$ let V denote the associated vector bundle over G/H. Then the pullback bundle $\pi^*V \to G/K$ is a homogeneous vector bundle isomorphic to $G \times_K \mathbb{V}$, where $\mathbb{V}$ is viewed as a K-representation by restriction.*

Proof. By definition, the fibre of π^*V over a point $gK \in G/K$ is given by the set of all pairs $\xi = (gK, \eta)$ with $\eta \in V_{gH}$. Since V is a homogeneous bundle, we deduce that $g' \cdot \xi := (g'gK, g'\eta)$ is a well defined element in $(\pi^*V)_{gg'K}$ and thus defines a G-action on the pullback bundle compatible with the projection $\pi^*V \to G/K$. Therefore, we see that the pullback bundle is a homogeneous bundle over G/K. The corresponding K-action on the fibre of π^*V over $o = eK$ reads as $k \cdot (o, v) = (o, kv)$ for $v \in \mathbb{V}$, where we consider $\mathbb{V}$ with the induced K-action on the right hand side. Thus, we obtain a diffeomorphism $\pi^*V \cong G \times_K \mathbb{V}$. $\square$

1.2.2. Sections of homogeneous bundles. As in the last section, let G be a Lie group with Lie algebra $\mathfrak{g}$ and H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. Let $V \to M$ be a homogeneous vector bundle over $M := G/H$ with underlying H-representation $\mathbb{V}$. Then the space of smooth sections $\Gamma(V)$ of V inherits a G-action via $(g \cdot s)(g'H) := g \cdot s(g^{-1}g'H)$ for all $s \in \Gamma(V)$ and g, $g' \in G$. We call $\Gamma(V)$ the *induced representation* of G corresponding to the H-representation $\mathbb{V}$, and denote it by $\mathrm{Ind}_H^G(\mathbb{V})$.

Equivalently, we can describe the space of sections of V as the set of smooth maps $f\colon G \to \mathbb{V}$ which satisfy the equivariance condition $f(gh) = h^{-1} \cdot f(g)$ for all $g \in G$ and $h \in H$. Given such a map, we can define a section of V via $s(gH) = [\![g, f(g)]\!]$ for all $g \in G$, and this assignment is an isomorphism. In this picture, the G-action on $\Gamma(V)$ translates to $(g' \cdot f)(g) = f((g')^{-1}g)$ for all g, $g' \in G$.

Since every homogeneous vector bundle is determined by its underlying representation, we are able to relate G-invariant sections of the bundle with invariant elements in the vector space.

Theorem 1.2.2. *Let $V \to G/H$ be a homogeneous vector bundle with underlying H-representation $\mathbb{V}$, and let X be a smooth manifold endowed with a smooth left G-action. Then the evaluation at $o = eH$ induces a natural bijection between the set of G-equivariant maps $\Phi\colon X \to \Gamma(V)$ which are smooth as maps $\Phi_o\colon X \to E_o$, and the set of smooth, H-equivariant maps $X \to \mathbb{V}$. In particular, there is a natural linear bijection between the set $\Gamma(V)^G$ of G-invariant sections of V and the set $\mathbb{V}^H$*

of H-invariant elements in the standard fibre. Denoting by $\mathrm{Res}_H^G(\mathbb{V})$ the space $\mathbb{V}$ viewed as an H-representation, this implies that we get a linear isomorphism

$$\mathrm{Hom}_G(\mathbb{V}, \mathrm{Ind}_H^G(\mathbb{W})) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(\mathbb{V}), \mathbb{W}),$$

i.e., Res_H^G and Ind_H^G are adjoint functors.

PROOF. [**ČS09**, Theorem 1.4.4.]. Note that the original assumptions in the book do not guarantee that $f\colon X \to \mathbb{V}$ induces a map from X into the smooth sections of E. □

In section 2.2.1 we will define the Poisson transform by using an G-invariant differential form on a homogeneous space with values in a homogeneous vector bundle. Therefore, it will be important to determine the G-action on this space explicitly. As before, let V be a homogeneous vector bundle over M with corresponding H-representation $\mathbb{V}$ and denote the G-action on V by $\tilde{\ell}$. If α is a V-valued differential form of degree k on M, then for all $g \in G$, $g'H \in M$ and $\xi_1, \dots, \xi_k \in T_{g'H}M$ we get

$$\begin{aligned}(g \cdot \alpha)(g'H)(\xi_1, \dots, \xi_k) &= (g \cdot \alpha(g^{-1}g'H))(\xi_1, \dots, \xi_k)\\ &= \tilde{\ell}_g \alpha(g^{-1}g'H)(T_{g'H}\ell_{g^{-1}} \cdot \xi_1, \dots, T_{g'H}\ell_{g^{-1}} \cdot \xi_k)\\ &= \tilde{\ell}_g(\ell_{g^{-1}}^* \alpha)(g'H)(\xi_1, \dots, \xi_k),\end{aligned}$$

so we see that the G-action on α is given by the pullback along the inverse of ℓ_g composed with the action of g on the bundle V.

If α is G-invariant, then Theorem 1.2.2 tells us that $\alpha(o) \in \Lambda^k(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}$ is H-invariant. Here, the H-action on an element $\varphi \in \Lambda^k(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}$ reads as

$$(h \cdot \varphi)(X_1 + \mathfrak{h}, \dots, X_k + \mathfrak{h}) = h \cdot \varphi(\underline{\mathrm{Ad}}(h^{-1})(X_1 + \mathfrak{h}), \dots, \underline{\mathrm{Ad}}(h^{-1})(X_k + \mathfrak{h}))$$

for all $X_1, \dots, X_k \in \mathfrak{g}$. Conversely, if $\varphi \in \Lambda^k(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}$ is H-invariant the corresponding G-invariant k-form α_φ is given by

$$\alpha_\varphi(gH)(\xi_1, \dots, \xi_k) := \tilde{\ell}_g \varphi(T_{gH}\ell_{g^{-1}} \cdot \xi_1, \dots, T_{gH}\ell_{g^{-1}} \cdot \xi_k)$$

for all $gH \in M$ and $\xi_1, \dots, \xi_k \in T_{gH}M$.

1.2.3. Invariant connections. In order to relate differential operators to Poisson transforms it will be necessary to construct a calculus on vector valued differential forms on homogeneous spaces G/H. For that, we have to answer the question of existence of connections on homogeneous bundles which are compatible with the corresponding actions of the Lie group G. It turns out that such connections are again determined by data of the underlying H-representation.

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. Let $\rho\colon H \to \mathrm{GL}(\mathbb{V})$ be an H-representation and V be the homogeneous vector bundle associated to $\mathbb{V}$. Consider a linear connection ∇ on V, which is a linear map $\Gamma(V) \to \Omega^1(G/H, V)$ satisfying a Leibniz rule. Since we have seen before that the space of sections of a homogeneous bundle naturally inherits a G-action, we require this linear map to be G-equivariant. Explicitly, this means that

$$g(\nabla_\xi s) = \nabla_{g\xi}(gs)$$

for all $s \in \Gamma(V)$, $\xi \in \mathfrak{X}(G/H)$ and $g \in G$, and in this case we say that the linear connection ∇ is G-invariant.

For a G-invariant linear connection ∇ on V consider the curvature tensor $R \in \Omega^2(G/H, \mathrm{End}(V))$ given by the formula

$$R(\xi, \eta)(s) = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s$$

for all $s \in \Gamma(V)$ and vector fields ξ, η on G/H. By invariance of the connection the curvature is a G-invariant differential form and thus corresponds to an H-invariant element in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \operatorname{End}(\mathbb{V})$ due to Theorem 1.2.2.

In the special case of invariant connections on the tangent bundle of G/H there is another important tensorial quantity. Let ∇ be a connection on $T(G/H)$ and consider the torsion $T \in \Omega^2(G/H, T(G/H))$ defined by

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$$

for all vector fields ξ, $\eta \in \mathfrak{X}(G/H)$. If ∇ is G-invariant, then the torsion is an invariant section, and therefore by Theorem 1.2.2 corresponds to an H-invariant element in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}/\mathfrak{h}$.

Similarly as before, the G-invariance of a linear connection on a homogeneous vector bundle $G \times_H \mathbb{V}$ suggests that it is entirely determined by data given in terms of the representation $\mathbb{V}$.

THEOREM 1.2.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$. Let $\rho\colon H \to \mathrm{GL}(\mathbb{V})$ be an H-representation and let $V = G \times_H \mathbb{V}$ be the corresponding associated vector bundle over G/H.*

(i) G-invariant linear connections on V are in bijective correspondence with linear maps $\alpha\colon \mathfrak{g} \to \operatorname{End}(\mathbb{V})$ satisfying
(a) $\alpha|_\mathfrak{h} = \rho'$
(b) $\alpha(\operatorname{Ad}(h)(X)) = \rho(h) \circ \alpha(X) \circ \rho(h^{-1})$ for all $X \in \mathfrak{g}$ and $h \in H$.
Explicitly, if $s \in \Gamma(V)$ corresponds to the equivariant map $f\colon G \to \mathbb{V}$, then the H-equivariant map $\nabla f\colon G \to \mathrm{L}(\mathfrak{g}/\mathfrak{h}, \mathbb{V})$ corresponding to ∇s is given by

$$\nabla f(g)(X + \mathfrak{h}) = (L_X \cdot f)(g) + \alpha(X)(f(g))$$

for all $g \in G$ and $X \in \mathfrak{g}$. Here, L_X denotes the left-invariant vector field generated by $X \in \mathfrak{g}$.

(ii) The curvature of a G-invariant linear connection corresponding to the map α is the invariant section in $\Omega^2(G/H, \operatorname{End}(V))$ which is induced by the H-equivariant map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to \operatorname{End}(\mathbb{V})$ defined by

$$(X + \mathfrak{h}, Y + \mathfrak{h}) \mapsto \alpha(X) \circ \alpha(Y) - \alpha(Y) \circ \alpha(X) - \alpha([X, Y]).$$

(iii) In the case of a G-invariant linear connection ∇ on $T(G/H)$ corresponding to $\alpha\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g}/\mathfrak{h})$, consider the linear map $\Lambda^2\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ given by

$$(X, Y) \mapsto \alpha(X)(Y + \mathfrak{h}) - \alpha(Y)(X + \mathfrak{h}) - [X, Y] + \mathfrak{h}.$$

This map factors to an H-invariant map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{g}/\mathfrak{h}$, and the corresponding G-invariant section in $\Omega^2(G/H, T(G/H))$ is the torsion of ∇.

PROOF. [**ČS09**, Thm 1.4.7, Prop. 1.4.7, Prop 1.4.8] □

The formula in Theorem 1.2.3(i) shows that every linear connection is given by a first order operator plus an algebraic part, which is the action of $\mathfrak{g}$ on $\mathbb{V}$ via α. In particular, if the section $s \in \Gamma(V)$ is G-invariant, then the corresponding map f is constant and therefore $L_X \cdot f = 0$. This reflects the fact that G-invariant sections can be understood pointwise.

One way to obtain such invariant connections on V is to induce them from principal connections on G. Recall that $\pi\colon G \to G/H$ is a principal bundle with structure group H, and the principal action is simply the restriction of the right multiplication $\rho^g\colon h \mapsto hg$ on G. Assume that G is endowed with a *principal connection,* i.e. a horizontal distribution $\mathcal{H}$ on G such that $T_g\rho^h(\mathcal{H}_g) = \mathcal{H}_{gh}$ for all $g \in G$ and $h \in H$. Equivalently, this can be described by the *principal connection form* $\gamma \in \Omega^1(G, \mathfrak{h})$ which satisfies $(\rho^h)^*\gamma = \operatorname{Ad}(h^{-1}) \circ \gamma$ for all $h \in H$.

PROPOSITION 1.2.3. *Let G be a Lie group and H a closed subgroup of G. Then principal connections on $G \to G/H$ are in bijective correspondence with linear maps $\beta\colon \mathfrak{g} \to \mathfrak{h}$ such that*

(i) $\beta|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$

(ii) $\mathrm{Ad}(h^{-1}) \circ \beta \circ \mathrm{Ad}(h) = \beta$ *for all* $h \in H$.

In particular, if $\rho\colon H \to \mathrm{GL}(\mathbb{V})$ is any finite dimensional H-representation, then the linear map $\alpha := \rho' \circ \beta\colon \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ satisfies the properties of Theorem 1.2.3(i) and thus induces a linear connection on $G \times_H \mathbb{V}$.

PROOF. [**ČS09**, Theorem 1.4.5] applied in the case $H = K$ and $i = \mathrm{id}_H$. □

Note that the naturality of the induced connections on associated vector bundles implies that H-equivariant maps between H-representations induce G-equivariant bundles maps which are parallel. We will use this Proposition in section 1.3.1 to construct canonical connections on homogeneous vector bundles over symmetric spaces.

For the rest of this section let $\rho\colon G \to \mathrm{GL}(\mathbb{V})$ be a representation of the full group G. Then the derivative $\alpha := \rho'$ satisfies the properties of part (i) in Theorem 1.2.3 and therefore induces a canonical G-invariant linear connection ∇^V on the corresponding tractor bundle V, which we will call the *tractor connection*. Since the map α is a Lie algebra homomorphism, we deduce from part (ii) of the same Theorem that the curvature of ∇^V is trivial. Moreover, if $s \in \Gamma(V)^G$ is a G-invariant section corresponding to the H-invariant element $v \in \mathbb{V}^H$ we obtain that $\nabla^V s$ corresponds to the equivariant map $(X + \mathfrak{h}) \mapsto \rho'(X)(v)$.

Note that we can also relate the tractor connection to the standard exterior derivative. More explicitly, we have seen above that any tractor bundle is globally trivialisable, implying that smooth sections of V correspond to smooth maps $G/H \to \mathbb{V}$. Applying the exterior derivative to such functions defines a flat connection $d\colon C^\infty(G/H, \mathbb{V}) \to \Omega^1(G/H, \mathbb{V})$, and this corresponds precisely to the tractor connection in this trivialisation.

1.2.4. The covariant exterior derivative. Following the previous section, we use the existence of a G-invariant connection on a homogeneous vector bundle V over a homogeneous space G/H to construct differential operators between V-valued differential forms on G/H. In the case of a flat connection we obtain a complex of differential operators, and its cohomology is of major importance in the study of homogeneous spaces.

Using a G-invariant connection ∇ on V, we can mimic the formula for the standard exterior derivative of real valued differential forms by replacing the Lie derivative with the connection. In this way, we obtain the *covariant exterior derivative* d^∇, which is a G-equivariant differential operator on $\Omega^*(G/H, V)$ determined by $d^\nabla s = \nabla s$ for $s \in \Gamma(V)$ as well as

$$d^\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d^\nabla \beta$$

for all real valued forms α and V-valued forms β on G/H. Explicitly, on V-valued k-forms α the covariant exterior derivative is given by

$$\begin{aligned}(d^\nabla \alpha)(\xi_0, \dots, \xi_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{\xi_i}(\alpha(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k)\end{aligned}$$

for all $\xi_0, \dots, \xi_k \in \mathfrak{X}(G/H)$, where the hat denotes omission.

The covariant exterior derivative is compatible with natural constructions of vector bundles and induced connections. However, although the operator d^∇ is defined analogously to the exterior derivative, it is not a differential in general. We will summarize in the following Proposition some properties of the covariant exterior derivative, which will be important in the sequel.

PROPOSITION 1.2.4. *Let V and W be homogeneous vector bundles over a homogeneous space G/H and let ∇^V and ∇^W be connections on V and W, respectively. Then we have*

(i) For all V-valued forms α on G/H we have $d^{\nabla^V} d^{\nabla^V} \alpha = R^{\nabla^V} \wedge \alpha$, where $R^{\nabla^V} \in \Omega^2(G/H, \mathrm{End}(V))$ is the curvature of ∇^V.

(ii) For all V-valued forms α and W-valued forms β on G/H we have

$$d^{\nabla^{V\otimes W}}(\alpha \wedge \beta) = d^{\nabla^V}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d^{\nabla^W}\beta,$$

where we view $\alpha \wedge \beta$ as an $V \otimes W$-valued differential form on G/H. Here, $\nabla^{V\otimes W}$ is the induced connection on the homogeneous vector bundle $V \otimes W$.

PROOF. (i) [**KMS93**, Lemma 11.13(4)]

(ii) Follows immediately from the definition of the connection on $V \otimes W$ and the explicit formula for the covariant exterior derivative. □

For the rest of the section assume that $\mathbb{V}$ is a G-representation. Then the corresponding tractor bundle V carries a canonical flat linear connection, and we will denote the corresponding covariant exterior derivative by d^V. Note that Proposition 1.2.4(i) shows that d^V is always a differential and thus induces a complex

$$\Gamma(V) \xrightarrow{\nabla} \Omega^1(G/H, V) \xrightarrow{d^V} \cdots \xrightarrow{d^V} \Omega^{\dim(G/H)}(G/H, V),$$

called the *twisted de Rham complex.*

Finally, let α be a G-invariant V-valued k-form on G/H and ϕ be the corresponding invariant element in the H-representation $\Lambda^k(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}$ due to Theorem 1.2.2. Using G-equivariance, the image of α under the covariant exterior derivative is again G-invariant, and we can express the corresponding H-invariant element $d^V\phi$ in terms of ϕ. In order to do so, recall from section 1.2.3 that for $s \in \Gamma(V)^G$ corresponding to $v \in \mathbb{V}^H$ the invariant element $\nabla s \in \Omega^1(G/H, V)$ is obtained via the $\mathfrak{g}$-action on v. Using this, a direct computation shows that for all $X_i \in \mathfrak{g}$ we have

(1.2.4)
$$\begin{aligned}(d^V\phi)(X_0 + \mathfrak{h}, \dots, X_k + \mathfrak{h}) &= \sum_{i=0}^{k}(-1)^i X_i \cdot \phi(X_0 + \mathfrak{h}, \dots, \hat{\imath}, \dots, X_k + \mathfrak{h}) \\ &+ \sum_{i<j}(-1)^{i+j}\phi([X_i, X_j] + \mathfrak{h}, X_0 + \mathfrak{h}, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, X_k + \mathfrak{h}),\end{aligned}$$

where the dot denotes the action of X_i on $\mathbb{V}$. Note that this formula is well defined due to H-invariance of ϕ. Moreover, if ∂ denotes the standard codifferential in Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathbb{V}$, we can write this formula simply as

$$\pi_\mathfrak{h}^*(d^V\phi) = \partial(\pi_\mathfrak{h}^*\phi)$$

for all $\phi \in \Lambda^k(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}$, where $\pi_\mathfrak{h}$ is the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

1.2.5. The adjoint tractor bundle and the fundamental derivative. Having developed the general theory of homogeneous bundles and invariant connections we will introduce a canonical tractor bundle, called the adjoint tractor bundle, which will induce a canonical derivative on all homogeneous bundles.

Every Lie group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action, implying that we can always form the *adjoint tractor bundle* $\mathcal{A} := G \times_H \mathfrak{g}$ over G/H. Due to the definition as the bundle associated to the Lie algebra $\mathfrak{g}$, the adjoint tractor bundle carries additional structure. Explicitly, the Lie bracket on $\mathfrak{g}$ is equivariant with respect to the restriction of the adjoint action to H and thus gives rise to a G-equivariant bundle map $\{\ ,\ \}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. For all $x \in G/H$ this map turns the fibre $\mathcal{A}_x$ into a Lie algebra isomorphic to $\mathfrak{g}$. Moreover, the short exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of H-representations gives rise to an exact sequence

$$0 \longrightarrow G \times_H \mathfrak{h} \longrightarrow \mathcal{A} \xrightarrow{\ \Pi\ } T(G/H) \longrightarrow 0$$

of homogeneous bundles. Finally, we have seen that every tractor bundle is endowed with a canonical G-invariant connection, which we call the *adjoint tractor connection* in the case of the adjoint tractor bundle.

PROPOSITION 1.2.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$. Let V be a tractor bundle on G/H corresponding to the G-representation $\mathbb{V}$ with tractor connection ∇^V.*

(i) There is an isomorphism between the space $\Gamma(\mathcal{A})$ of sections of the adjoint tractor bundle and the space $\mathfrak{X}(G)^H$ of H-right invariant vector fields on G, which is induced by the map $\mathfrak{X}(G)^H \to C^\infty(G,\mathfrak{g})^H$ defined by $\xi \mapsto \omega_G(\xi)$.

(ii) The $\mathfrak{g}$-action on $\mathbb{V}$ induces a bundle map $\bullet\colon \mathcal{A} \times V \to V$ which endows every fibre V_x with the structure of an $\mathcal{A}_x$-module. This action satisfies for all s_1, $s_2 \in \Gamma(\mathcal{A})$ and $t \in \Gamma(V)$ that

$$\{s_1, s_2\} \bullet t = s_1 \bullet (s_2 \bullet t) - s_2 \bullet (s_1 \bullet t).$$

(iii) The operations $\{\ ,\ \}$ and $\bullet$ are parallel with respect to the connections induced by ∇^V and $\nabla^{\mathcal{A}}$. This means that

$$\nabla^{\mathcal{A}}_\xi \{s_1, s_2\} = \{\nabla^{\mathcal{A}}_\xi s_1, s_2\} + \{s_1, \nabla^{\mathcal{A}}_\xi s_2\},$$
$$\nabla^V_\xi (s \bullet t) = (\nabla^{\mathcal{A}}_\xi s) \bullet t + s \bullet (\nabla^V_\xi t)$$

for all sections s, s_1, $s_2 \in \Gamma(\mathcal{A})$ and $t \in \Gamma(V)$ and all vector fields $\xi \in \mathfrak{X}(G/H)$.

PROOF. [**ČS09**, Proposition 1.5.7(2),(3),(5); Proposition 1.5.8] □

Part (i) of Proposition 1.2.5 leads to a notion of derivative of any sections of natural vector bundles. Indeed, let $\mathbb{V}$ be an H-representation and V be the corresponding homogeneous vector bundle. We have seen in section 1.2.2 that sections of V correspond to H-equivariant maps $f\colon G \to \mathbb{V}$. If we derive this map with respect to an H-right invariant vector field $\xi \in \mathfrak{X}(G)^H$, then the resulting map will again be H-equivariant and thus corresponds to a section of V. On the other hand, part (i) of Proposition 1.2.5 shows that ξ corresponds to a section of the adjoint tractor bundle. In this way, we can associate to all $s \in \Gamma(\mathcal{A})$ and $t \in \Gamma(V)$ a section of V, which we will denote by $D_s t$. The resulting map

$$D\colon \Gamma(\mathcal{A}) \times \Gamma(V) \to \Gamma(V)$$

is called the *fundamental derivative*.

By definition, the map D is linear in the $\Gamma(\mathcal{A})$-slot, whereas is satisfies a Leibniz rule with respect to the section of V. Thus, we can indeed view D as a generalization of a derivative on any homogeneous vector bundle.

THEOREM 1.2.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$.*

(i) For all smooth maps $f\colon G/H \to \mathbb{R}$ and $s \in \Gamma(\mathcal{A})$ we have $D_s f = \Pi(s) \cdot f$, where $\Pi\colon \Gamma(\mathcal{A}) \to \mathfrak{X}(G/H)$ is the projection induced by $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

(ii) If s is a section of $G \times_H \mathfrak{h} \subset \mathcal{A}$, then the fundamental derivative satisfies $D_s t = -s \bullet t$ for all $t \in \Gamma(G \times_H \mathbb{V})$, where $\mathbb{V}$ is any H-representation and $\bullet$ is the G-equivariant bundle map induced by the $\mathfrak{h}$-action on $\mathbb{V}$.

(iii) The fundamental derivative is compatible with all G-invariant bundle maps induced by H-equivariant maps between H-representations. In particular, it is compatible with tensor products and contractions of sections of homogeneous vector bundles.

(iv) If V is a tractor bundle, then we can express the tractor connection ∇^V as

$$\nabla^V_{\Pi(s)} t = D_s t + s \bullet t$$

for all sections $s \in \Gamma(\mathcal{A})$ and $t \in \Gamma(V)$.

PROOF. [**ČS09**, Proposition 1.5.8; Theorem 1.5.8] □

1.2.6. The action of the Casimir. For any semisimple Lie group $\mathfrak{g}$ we have seen in section 1.1.1 that the Casimir C is a distinguished element in the centre of the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$. In this section we show how the Casimir induces a differential operator between sections of homogeneous vector bundles on any homogeneous space, which will be used to relate two Laplace operators on Riemannian symmetric spaces and homogeneous parabolic geometries via Poisson transforms, c.f. Theorem 2.2.3.

Let G be a semisimple Lie group, $H \subset G$ be a closed subgroup and $\mathbb{V}$ be an H-representation with corresponding homogeneous vector bundle V. Let $t \in \Gamma(V)$ be a smooth section of V and denote by $f\colon G \to \mathbb{V}$ the corresponding H-equivariant map. We have seen that G acts on f via $(g' \cdot f)(g) = f((g')^{-1}g)$ for all g, $g' \in G$. Therefore, the infinitesimal action of $X \in \mathfrak{g}$ on f is given by

$$X \cdot f := -T_g f \circ T_e \rho^g \cdot X = -R_X \cdot f,$$

where $R_X \in \mathfrak{X}(G)$ denotes the right invariant vector field induced by $X \in \mathfrak{g}$. The resulting map $X \cdot f$ is again H-equivariant and thus corresponds to a section of V. Therefore, every $X \in \mathfrak{g}$ defines a differential operator D_X on $\Gamma(V)$ of order 1, which is G-equivariant if and only if $\mathrm{Ad}(g)(X) = X$ for all $g \in G$. However, this is equivalent to X being in the centre of $\mathfrak{g}$, which is trivial by semisimplicity. Therefore, we cannot get G-equivariant differential operators in this way.

However, recall that every representation of a Lie algebra $\mathfrak{g}$ is canonically a representation of the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$. Moreover, the Lie algebra homomorphism $X \mapsto R_X$ lifts to a homomorphism on $\mathfrak{u}(\mathfrak{g})$, so we can analogously define differential operators D_X on $\Gamma(V)$ for all $X \in \mathfrak{u}(\mathfrak{g})$. In particular, the Casimir operator C is in the centre of $\mathfrak{u}(\mathfrak{g})$ and thus defines a G-equivariant differential operator D_C on sections of V. Explicitly, if X_i is a basis of $\mathfrak{g}$ and Y_i is its dual basis with respect to the Killing form B, then this operator is induced by the map

$$C \cdot f = \sum_i R_{Y_i} \cdot R_{X_i} \cdot f \in C^\infty(G, \mathbb{V})^H$$

for all H-equivariant maps $f\colon G \to \mathbb{V}$.

Now Proposition 1.2.5(i) implies that for all $X \in \mathfrak{g}$ there is a section s_X of the dual adjoint tractor bundle $\mathcal{A}$ on G/H corresponding to R_X. By definition, the section of V induced by the H-equivariant map $R_X \cdot f$ coincides with $D_{s_X} t$, where $t \in \Gamma(V)$ corresponds to $f \in C^\infty(G, \mathbb{V})^H$. In particular, we can express the differential operator induced by the Casimir in terms of the fundamental derivative.

PROPOSITION 1.2.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ be a closed subgroup of G. Let B be the G-equivariant bundle map on the adjoint tractor bundle $\mathcal{A}^*$ induced by the Killing form on $\mathfrak{g}$. Let $\mathbb{V}$ be an H-representation and V be the associated homogeneous vector bundle on G/H. Then we can write the G-equivariant differential operator $D_C \colon \Gamma(V) \to \Gamma(V)$ induced by the action of the Casimir as the composition*

$$\Gamma(V) \xrightarrow{D^2} \Gamma(\mathcal{A}^* \otimes \mathcal{A}^* \otimes V) \xrightarrow{B \otimes \mathrm{id}_V} \Gamma(V)$$

PROOF. [**ČS07**, Proposition 1]. Note that the proof in this paper is valid for any homogeneous space. □

1.3. Symmetric spaces

The Poisson transforms we construct in this thesis map between vector valued differential forms, whose underlying manifolds are given by two specific classes of homogeneous spaces whose geometric properties are disparate. The first of these is the class of Riemannian symmetric spaces, which has been intensively studied in the field of harmonic analysis. In section 1.3.1 we recall the basic properties of these structures and in particular show that homogeneous vector bundles over such spaces always carry invariant connections. Afterwards, the main focus in section 1.3.2 will lie in the analysis of invariant geometric objects on tractor bundles, which ultimately leads to a Laplace operator which satisfies a Weitzenböck formula with respect to the differential operator induced by the Casimir element of $\mathfrak{g}$, c.f. Theorem 1.3.3. This result will be essential for the proof of Theorem 2.2.3, where we relate harmonic eigenforms of this differential operator to Poisson transforms which are compatible with the BGG-sequence on the corresponding homogeneous parabolic geometry.

1.3.1. Definition and basic properties. Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and let θ be a Cartan involution of $\mathfrak{g}$. We have seen that the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ is orthogonal with respect to the Killing form B and the restriction of B to $\mathfrak{k}$ and $\mathfrak{q}$ is negative and positive definite, respectively. Moreover, since θ is a Lie algebra homomorphism the bracket relations read as $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$. Therefore, $\mathfrak{k}$ is a compact Lie subalgebra of $\mathfrak{g}$, whereas $\mathfrak{q}$ is a $\mathfrak{k}$-module via the adjoint action. In addition, by invariance of the Killing form, the restriction of B to $\mathfrak{q}$ is $\mathfrak{k}$-invariant.

Consider the Lie subgroup K of G whose Lie algebra is $\mathfrak{k}$. Since the centre of G is finite the group K is a maximal compact subgroup, and both Lie algebras $\mathfrak{k}$ and $\mathfrak{q}$ are K-modules via the adjoint action. By definition, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ is K-equivariant, so we can identify the tangent bundle of G/K with $G \times_K \mathfrak{q}$. The restriction of the Killing form to $\mathfrak{q}$ is a K-invariant inner product, which in turn endows G/K with a G-invariant Riemannian metric g. Moreover, the adjoint tractor bundle $\mathcal{A}(G/K)$ of G/K decomposes into the Whitney sum $(G \times_K \mathfrak{k}) \oplus T(G/K)$ of G-invariant bundles.

On the other hand, the K-invariant subspace $\mathfrak{q}$ of $\mathfrak{g}$ induces a distribution Q on the tangent bundle $TG \cong G \times \mathfrak{g}$, which is a principal connection for $G \to G/K$. By Proposition 1.2.3 we obtain induced linear connections ∇ on any homogeneous vector bundle over G/K, which are naturally compatible with G-invariant operations. In particular, for any K-equivariant map between finite dimensional K-representations the induced G-equivariant bundle map will be parallel.

DEFINITION 1.3.1. Let G be a semisimple Lie group with finite centre, $K \subset G$ the maximal compact subgroup and $\mathbb{V}$ a finite dimensional K-representation with associated homogeneous vector bundle $V \to G/K$. Then we call the connection $\nabla\colon \Gamma(V) \to \Omega^1(G/K, V)$ the *canonical connection* on V.

In the next Proposition we relate the canonical connection to the fundamental derivative defined in section 1.2.5 as well as to the tractor connection on tractor bundles over G/K.

PROPOSITION 1.3.1. *Let G be a semisimple Lie group with finite centre and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be a Cartan decomposition of its Lie algebra. Let $K \subset G$ be the maximal compact subgroup with Lie algebra $\mathfrak{k}$ and $\rho\colon K \to \mathrm{End}(\mathbb{V})$ be a finite dimensional K-representation with corresponding homogeneous vector bundle V over G/K.*

(i) The linear map $\alpha\colon \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ from Theorem 1.2.3 corresponding to the canonical connection on V is determined by $\alpha|_{\mathfrak{q}} = 0$. In particular, the curvature of ∇ is induced by the map $R\colon \Lambda^2\mathfrak{q} \to \mathrm{End}(\mathbb{V})$ given by $(X, Y) \mapsto -\rho'([X, Y])$.

(ii) The canonical connection on V is the restriction of the fundamental derivative to the subbundle $T(G/K)$ of the adjoint tractor bundle.

(iii) Let V be a tractor bundle and ∇^V its tractor connection. Then for all $\xi \in \mathfrak{X}(G/K)$ and $t \in \Gamma(V)$ we have

$$\nabla^V_\xi t = \nabla_\xi t + \xi \bullet t.$$

PROOF. (i) Let $Q \subset TG$ be the principal connection defined by the K-module $\mathfrak{q}$. Then the corresponding connection form $\gamma \in \Omega^1(G, \mathfrak{k})$ is given by the vertical projection of tangent vectors in TG followed by left translation to $\mathfrak{k}$. In view of Proposition 1.2.3 this shows that the map $\beta\colon \mathfrak{g} \to \mathfrak{k}$ corresponding to γ is simply given by projection onto the first factor of the Cartan decomposition. Thus, it follows that the K-equivariant linear map $\alpha\colon \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ corresponding to the canonical connection satisfies $\alpha|_{\mathfrak{q}} = 0$ and $\alpha|_{\mathfrak{k}} = \rho'$. The second part follows immediately from part (ii) of Theorem 1.2.3.

(ii) Since the tangent bundle $T(G/K)$ is a subbundle of the adjoint tractor bundle we can restrict the fundamental derivative D to vector fields on G/K. Moreover, the fundamental derivative satisfies a Leibniz rule and is linear in the $\Gamma(\mathcal{A})$-slot, so this restriction defines a connection $\tilde{\nabla}$ on $\Gamma(V)$. Using the correspondence between sections of $\mathcal{A}$ and K-right invariant vector fields, part (i) immediately implies that $\tilde{\nabla}$ coincides with the canonical connection.

(iii) The formula follows from part (iv) of Theorem 1.2.5. □

In particular, we can consider the canonical connection ∇ on the tangent bundle $T(G/K)$, which is exactly the Levi Civita connection on the Riemannian symmetric space G/K.

COROLLARY 1.3.1. *Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and let $K \subset G$ be a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the corresponding Cartan decomposition and let g be the G-invariant Riemannian metric on G/K induced by the Killing form. Then the Levi-Civita connection induced by g coincides with the canonical connection on the tangent bundle of G/K.*

In particular, the Riemann curvature tensor $R \in \Omega^2(G/K, \mathrm{End}(T(G/K)))$ is induced by the map $\Lambda^2\mathfrak{q} \to \mathrm{End}(\mathfrak{q})$ defined by

$$R(X, Y)(Z) = -[[X, Y], Z]$$

for all X, Y, $Z \in \mathfrak{q}$.

PROOF. By uniqueness it suffices to show that the canonical connection on the tangent bundle is torsion free and metric. However, the Riemannian metric g on G/K is induced by the restriction of the Killing form to $\mathfrak{q}$, so by naturality we obtain $\nabla g = 0$. Moreover, the Lie bracket maps elements of $\mathfrak{q}$ into $\mathfrak{k}$, so combining Proposition 1.3.1(i) and Theorem 1.2.3(iii) shows that the torsion of ∇ is trivial. $\square$

In the proof of the last Corollary we used that the Riemannian metric is parallel since it is induced by the restriction of the Killing form to $\mathfrak{q}$. However, the same argument holds for any K-invariant inner product on $\mathfrak{q}$, so we have actually proven that the canonical connection is the Levi Civita of any G-invariant Riemannian structure on G/K. For a classical proof of this statement see [**Hel01**, Corollary 4.3].

1.3.2. Tractor calculus on symmetric spaces. After the general discussion our aim is to establish a calculus for tractor bundles over symmetric spaces. First, we will show the existence of a second G-invariant flat connection on tractor bundles over G/K, which in turn give rise to a formal adjoint of the covariant exterior derivative with respect to a special L^2-norm on tractor valued differential forms. The resulting vector valued Laplace Beltrami operator will satisfy a Weitzenböck formula with respect to the differential operator induced by the Casimir element of G, which will be the main result in section 1.3.3.

Throughout this section let $\rho\colon G \to \mathrm{GL}(\mathbb{V})$ be a finite dimensional representation of G with associated tractor bundle V over G/K. Since the group K is compact, we can find (many) K-invariant inner products on $\mathbb{V}$, and each choice induces a G-invariant Euclidean bundle metric h on V. Combining this with the Riemannian metric on the tangent bundle of G/K induces a G-invariant inner product $\langle\ ,\ \rangle$ on $\Omega^*(G/K, V)$, which allows us to define a Hodge star operator on tractor valued differential forms.

Since we are in the category of G-representations it would be natural to use an inner product on $\mathbb{V}$ which is invariant with respect to the G-action. However, this would mean that the image of ρ is in $\mathrm{SO}(\mathbb{V})$ which is only possible if G is a compact Lie group. Therefore, we have to slightly modify the inner product by bringing the Cartan involution θ of $\mathfrak{g}$ into the game.

LEMMA 1.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan involution θ and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Then there is a positive definite inner product $\langle\ ,\ \rangle_{\mathbb{V}}$ on every finite dimensional $\mathfrak{g}$-representation $\mathbb{V}$ which satisfies*

$$\langle X \cdot v, w\rangle_{\mathbb{V}} = -\langle v, \theta(X) \cdot w\rangle_{\mathbb{V}}$$

for all $X \in \mathfrak{g}$ and $v,\ w \in \mathbb{V}$. Moreover, if $\mathbb{V}$ is irreducible, then the inner product is unique up to a positive factor.

PROOF. The existence of the inner product is shown in the proof of [**ČS09**, Prop. 3.3.1]. To obtain uniqueness, let $\Phi\colon \mathbb{V} \to \mathbb{V}^*$ be the linear isomorphism induced by the inner product, which satisfies $\Phi(\theta(X)v) = X \cdot \Phi(v)$ for all $v \in \mathbb{V}$ and $X \in \mathfrak{g}$ by definition. If there is another inner product on $\mathbb{V}$ with induced map $\tilde{\Phi}$, then the composition $\Phi^{-1} \circ \tilde{\Phi}$ is a $\mathfrak{g}$-equivariant automorphism of $\mathbb{V}$. Therefore, if $\mathbb{V}$ is irreducible, Schur's Lemma implies that this map is an isomorphism. Moreover, if we complexify $\mathfrak{g}$ and $\mathbb{V}$, then the corresponding complexified map $\Phi_{\mathbb{C}}^{-1} \circ \tilde{\Phi}_{\mathbb{C}}$ is a complex multiple of the identity. By construction, this means that the two complex bilinear products corresponding to $\Phi_{\mathbb{C}}$ and $\tilde{\Phi}_{\mathbb{C}}$ differ by a complex number μ. However, their restrictions to the subspace $\mathbb{V} \otimes 1 \subset \mathbb{V}_{\mathbb{C}}$ have to be positive definite, implying that μ is a positive real number. $\square$

Recall from section 1.2.4 that the tractor bundle V carries a canonical G-invariant flat connection ∇^V. Moreover, the inner product on $\mathbb{V}$ from Lemma 1.3.2 is in particular K-equivariant, so it induces a G-invariant Euclidean bundle metric h on V. Therefore, we can define a connection on V which is adjoint to the tractor connection with respect to the bundle metric h.

DEFINITION 1.3.2 (θ-twisted tractor connection). Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and $K \subset G$ a maximal compact subgroup with respect to a Cartan involution θ on $\mathfrak{g}$. Let $\mathbb{V}$ be a finite dimensional G-representation and V the associated tractor bundle with tractor connection ∇^V. Let $\langle\ ,\ \rangle_{\mathbb{V}}$ be the inner product on $\mathbb{V}$ from Lemma 1.3.2 and denote by h the induced G-invariant Euclidean bundle metric on V.

The *θ-twisted tractor connection* $\nabla^\theta\colon \Gamma(V) \to \Omega^1(G/K, V)$ is defined as the adjoint of the tractor connection with respect to h, i.e. the uniquely determined connection so that

$$h(s, \nabla^\theta_\xi t) = \xi \cdot h(s,t) - h(\nabla^V_\xi s, t)$$

for all vector fields $\xi \in \mathfrak{X}(G/K)$ and sections $s,\ t \in \Gamma(V)$.

In the next Proposition we will clarify the relation between the θ-twisted tractor connection and the canonical connection introduced in section 1.3.1.

PROPOSITION 1.3.2. *Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$, let $K \subset G$ be the maximal compact subgroup corresponding to the Cartan involution θ on $\mathfrak{g}$. Let $\rho\colon G \to \mathrm{GL}(\mathbb{V})$ be a finite dimensional G-representation with associated tractor bundle V on G/K be its associated tractor bundle and h be the bundle map $V \times V \to \mathbb{R}$ induced by the inner product $\langle\ ,\ \rangle_{\mathbb{V}}$ from Lemma 1.3.2. Moreover, let ∇^V and ∇^θ be the tractor connection and the θ-twisted tractor connection on V, respectively.*

(i) For all vector fields $\xi \in \mathfrak{X}(G/K)$ and sections $s \in \Gamma(V)$ we have

$$\nabla^\theta_\xi s = \nabla_\xi s - \xi \bullet s$$

via the identification of $T(G/K)$ with $G \times_K \mathfrak{q}$.

(ii) The connection ∇^θ is a G-invariant connection on V. In view of Theorem 1.2.3, the corresponding equivariant map $\alpha\colon \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ is given by the composition $\alpha := \rho' \circ \theta$. In particular, the connection ∇^θ is flat.

PROOF. (i) Since the bundle metric h is induced by a K-invariant inner product on $\mathbb{V}$ it follows by naturality of the canonical connection that $\nabla h = 0$. Moreover, since we identified the tangent bundle of G/K with $G \times_K \mathfrak{q}$ the fundamental property of the inner product on $\mathbb{V}$ implies that $h(\xi \bullet s, t) = h(s, \xi \bullet t)$ for all $s,\ t \in \Gamma(V)$ and $\xi \in \mathfrak{X}(G/K)$. Thus, from Proposition 1.3.1(iii) we obtain

$$\begin{aligned} h(s, \nabla^\theta_\xi t) &= \xi \cdot h(s,t) - h(\nabla^V_\xi s, t) \\ &= h(\nabla_\xi s, t) + h(s, \nabla_\xi t) - h(\nabla^V_\xi s, t) \\ &= h(s, \nabla_\xi t - \xi \bullet t) \end{aligned}$$

for all section $s, t \in \Gamma(V)$ and vector fields $\xi \in \mathfrak{X}(G/K)$.

(ii) The bundle map $\bullet$ is G-equivariant and ∇ is G-invariant, so it follows that ∇^θ is a G-invariant connection. Moreover, the corresponding map $\alpha\colon \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ is determined by its restriction to $\mathfrak{q}$, on which it equals $-\rho'$. Thus, we obtain that $\alpha = \rho' \circ \theta$, and flatness of ∇^θ follows directly from Theorem 1.2.3(ii). $\square$

Now we are ready to construct a calculus on differential forms on G/K with values in a tractor bundle. For that, let $\langle\ ,\ \rangle$ denote the G-invariant inner product on $\Omega^k(G/K, V)$ induced by the Riemannian metric on G/K and the bundle metric h on V and let vol be the G-invariant volume form on G/K. Then we can define for α, $\beta \in \Omega^k(G/K, V)$ with α compactly supported their L^2-*inner product* by

$$\langle\!\langle \alpha, \beta \rangle\!\rangle_V := \int_{G/K} \langle \alpha, \beta \rangle \operatorname{vol}.$$

Note that this definition sometimes also makes sense for differential forms without compact support, and we extend the definition of the L^2-inner product to such forms. By abuse of notation, we will write $\langle\!\langle \alpha, \beta \rangle\!\rangle$ for α, $\beta \in \Omega^k(G/K, V)$, implicitly assuming that the L^2-inner product is well defined for these differential forms.

Recall from section 1.2.4 that the tractor connection ∇^V on V induces the covariant exterior derivative d^V on V-valued differential forms on G/K, which is a local operator. Our aim is to construct an adjoint of the covariant exterior derivative on compactly supported differential forms with respect to the L^2-inner product. For that, we define for V-valued differential forms α and β on G/K a real valued differential form $\alpha \wedge_h \beta$, which is the usual wedge product on differential forms combined with the contraction $V \times V \to C^\infty(G/K)$ induced by the metric h.

Subsequently, we obtain an Hodge star operator

$$*\colon \Omega^k(G/K, V) \to \Omega^{\dim(G/K)-k}(G/K, V)$$

by the formula

$$\alpha \wedge_h *\beta = \langle \alpha, \beta \rangle \operatorname{vol}$$

for all α, $\beta \in \Omega^k(G/K, V)$. Note that this operator reduces on decomposable elements to $*(\omega \otimes v) = (*\omega) \otimes v$ for all $\omega \in \Omega^k(G/K)$ and $v \in \Gamma(V)$, where on the right hand side the operator $*$ is the usual Hodge star on k-forms on G/K.

Let d^θ be the covariant exterior derivative on V-valued differential forms on G/K induced by the θ-twisted tractor connection. Then we define the *covariant codifferential* $\delta^V : \Omega^k(G/K, V) \to \Omega^{k-1}(G/K, V)$ by the formula

$$\delta^V \alpha = (-1)^k *^{-1} d^\theta * \alpha$$

and subsequently define the *covariant Laplace* operator on $\Omega^k(G/K, V)$ by $\Delta^V := d^V \delta^V + \delta^V d^V$.

In the next Theorem we will show that the covariant codifferential is an adjoint map for the covariant exterior derivative with respect to the L^2-inner product on compactly supported differential forms. Moreover, we will establish a formula for δ^V in terms of the connection on $\Lambda^k T^*(G/K) \otimes V$ induced by the canonical connection on $\Lambda^k T^*(G/K)$ and the tractor connection on V, which will be needed in the next section to clarify the relation between the covariant Laplace and the differential operator induced by the Casimir element of $\mathfrak{g}$.

THEOREM 1.3.2. *Let G be a semisimple Lie group with finite centre, $K \subset G$ a maximal compact subgroup and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathbb{V}$ be a finite dimensional G-representation with associated tractor bundle V over G/K and denote by h the G-invariant bundle metric on V induced by the K-invariant inner product from Lemma 1.3.2. Moreover, let ∇^V be the tractor connection and ∇^θ the θ-twisted tractor connection on V.*

(i) Let $\alpha \in \Omega^{k-1}(G/K, V)$ be compactly supported and let $\beta \in \Omega^k(G/K, V)$. Then we obtain that

$$\langle\!\langle d^V \alpha, \beta \rangle\!\rangle = \langle\!\langle \alpha, \delta^V \beta \rangle\!\rangle,$$

so the covariant codifferential is L^2-adjoint to the covariant exterior derivative.

(ii) *Let ∇^{θ_k} denote the connection on $\Lambda^k T^*(G/K) \otimes V$ induced by the canonical connection on $T(G/K)$ and the connection ∇^θ on V. Then we have*

$$\delta^V = -\sum_{i=1}^n \iota_{e_i} \nabla^{\theta_k}_{e_i},$$

where $e_1, \dots, e_n$ is a local orthonormal frame of G/K.

Proof. Since the covariant codifferential is linear it suffices to prove the formulae for decomposable elements $\beta = \theta \otimes t$, where $\theta \in \Omega^k(G/K)$ and $t \in \Gamma(V)$.

(i) Let $\alpha = \omega \otimes s$ with $\omega \in \Omega^{k-1}(G/K)$ and $s \in \Gamma(V)$ and assume that α has compact support. Since the covariant exterior derivative is a local operator, the support of $d^V\alpha$ is also compact, so the left hand side of the claimed equation is well defined.

By definition of the θ-twisted tractor connection we have

$$dh(s,t) = h(\nabla^V s, t) + h(s, \nabla^\theta t).$$

Therefore, applying the exterior derivative to the wedge product $\alpha \wedge_h *\beta$ yields

$$\begin{aligned} d(\alpha \wedge_h *\beta) &= d(h(s,t)\omega \wedge (*\theta)) \\ &= (dh(s,t)) \wedge \omega \wedge (*\theta) + h(s,t) d(\omega \wedge (*\theta)) \\ &= h(\nabla^V s, t) \wedge \omega \wedge (*\theta) + h(s,t)(d\omega) \wedge (*\theta) \\ &\quad + h(s, \nabla^\theta t) \wedge \omega \wedge (*\theta) + (-1)^{k-1} h(s,t)\omega \wedge d(*\theta) \\ &= (d^V\alpha) \wedge_h (*\beta) + (-1)^{k-1}\alpha \wedge d^\theta(*\beta). \end{aligned}$$

Integrating both sides and using Stokes' Theorem we see that

$$\langle\langle d^V\alpha, \beta \rangle\rangle = \langle\langle \alpha, \delta^V\beta \rangle\rangle .$$

(ii) From [**Sch95**, Formula 2.19, p. 25] we know that the Riemannian codifferential on G/K can be expressed as

$$\delta\theta = -\sum_{i=1}^n \iota_{e_i} \nabla_{e_i}\theta.$$

By definition, we get

$$\delta^V\beta = (-1)^k *^{-1} d^\theta * (\theta \otimes t) = (\delta\theta) \otimes t + (-1)^n *^{-1} \left((*\theta) \wedge \nabla^\theta t\right).$$

Since the local frame $\{e_i\}$ is orthonormal, we can rewrite $\nabla^\theta t$ in the second summand as $\sum_{i=1}^n e_i^\flat \wedge \nabla^\theta_{e_i} t$, where ${}^\flat \colon \mathfrak{X}(G/K) \to \Omega^1(G/K)$ is the isomorphism induced by the Riemannian metric. Next, we move the 1-forms $e_i^\flat$ to the left, which adds the sign $(-1)^{n-k}$, and finally apply the formula $e_i^\flat \wedge *\theta = (-1)^{k-1} * (\iota_{e_i}\theta)$, which holds on a general Riemannian manifold. All in all, we deduce that

$$\delta^V\beta = -\sum_{i=1}^n \iota_{e_i} \left(\nabla_{e_i}\theta \otimes t + (-1)^k \theta \wedge \nabla^\theta_{e_i} t\right) = -\sum_{i=1}^n \iota_{e_i} \nabla^{\theta_k}_{e_i}\beta,$$

where we used the definition of ∇^{θ_k} for the last equality. □

1.3.3. Laplacian type operators and the Casimir. Continuing the discussion of symmetric spaces we now analyse the relation between two Laplacian type operators as well as the differential operator on vector valued differential forms induced by Casimir element. This will enable us in section 2.2.3 to relate eigenvalues of the Laplacian type operator $\Delta^V = d^V\delta^V + \delta^V d^V$ on G/K with eigenvalues of the curved Box operator on homogeneous parabolic geometries, which we will define in section 1.4.2.

Let $\mathbb{V}$ be a K-representation with associated vector bundle V on G/K. In Proposition 1.2.6 we have seen that the differential operator on $\Gamma(V)$ induced by the Casimir operator can be expressed in terms of the fundamental derivative. In order to compute it explicitly on symmetric spaces, we first show that we can always find a local frame which is adapted to the specific structure of the adjoint tractor bundle.

LEMMA 1.3.3. *Let G be a semisimple Lie group with finite centre, $K \subset G$ a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ the associated Cartan decomposition. Let B be the G-invariant bundle map on the adjoint tractor bundle $\mathcal{A}$ on G/K induced by the Killing form. Let U be an open subset of G/K on which the bundle $G \to G/K$ is trivial. Then we can find a local frame $\{X_i\}$ of $G \times_K \mathfrak{k}$, which is orthonormal with respect to $-B$, as well as an orthonormal frame $\{Y_i\}$ of $T(G/K)$ with respect to B, which satisfies $B(X_i, Y_j) = 0$ for all $i = 1, \dots, \dim(\mathfrak{k})$ and $j = 1, \dots, \dim(\mathfrak{q})$.*

PROOF. We have seen before that the Cartan decomposition is K-invariant and orthogonal with respect to the Killing form. Therefore, the induced decomposition of $\mathcal{A}$ into the Whitney sum of the G-invariant subbundles $(G \times_K \mathfrak{k})$ and $T(G/K)$ is orthogonal with respect to the bundle map B. Moreover, in every point this map induces a Euclidean inner product on the tangent spaces of G/K and a negative definite bilinear product on the fibres of $G \times_K \mathfrak{k}$.

Since G is trivial over U, so are all bundles associated to G. Choose any local frame $\{X_i\}$ of $G \times_K \mathfrak{k}$ and $\{Y_i\}$ of $T(G/K)$ over U. Since the values of $B(X_i, X_j)$ depend smoothly on the base point, we can apply in every point the Gram-Schmidt procedure to obtain a local orthonormal frame of $T(G/K)$. Similarly, $-B$ is a Riemannian metric on $G \times_K \mathfrak{k}$, so we can proceed analogously. $\square$

We will refer to a local frame as in Lemma 1.3.3 as an *adapted frame for $\mathcal{A}$*. Using these we are able to determine the differential operator induced by the Casimir operator on sections of homogeneous bundles. Explicitly, we can express this operator in terms of the trace of the canonical connection on V and the action of the Casimir element of the Lie group K.

PROPOSITION 1.3.3. *Let G be a semisimple Lie group with finite centre, K a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ the corresponding Cartan decomposition. Let $\mathbb{V}$ be a K-representation and V be the corresponding associated vector bundle on G/K. Let $D_C \colon \Gamma(V) \to \Gamma(V)$ be the differential operator induced by the Casimir element C on $\mathfrak{g}$ as in section 1.2.6 and denote by ϕ_{C_K} the tensorial, G-equivariant map $V \to V$ induced by the action of the Casimir element C_K of $\mathfrak{k}$. Then for all $t \in \Gamma(V)$ we obtain that*

$$D_C t = \operatorname{tr}(\nabla^2 t) + \phi_{C_K}(t),$$

where ∇^2 is the composition of the canonical connections on the vector bundles V and $T^*(G/K) \otimes V$.

In particular, the Casimir operator on $\Gamma(V)$ splits into the sum of a differential operator of order 2 and the algebraic action of the K-Casimir on $\mathbb{V}$.

PROOF. Let $\{X_i\}$ and $\{Y_j\}$ be a local adapted frame for $\mathcal{A}$ and let t be a local section of V. Using Proposition 1.2.6 we can compute

$$D_C(t) = -\sum_i (D^2 t)(X_i, X_i) + \sum_j (D^2 t)(Y_j, Y_j) = (*).$$

Moreover, by naturality of the fundamental derivative we obtain that

$$(D^2 t)(s_1, s_2) = D_{s_1}(D_{s_2} t) - D_{D_{s_1} s_2} t$$

for all local sections $s_1,\, s_2 \in \Gamma(\mathcal{A})$.

The fundamental derivative in direction of a section of $G\times_K \mathfrak{k}$ is the negative of the algebraic action due to Theorem 1.2.5(ii). Since the action of $G\times_K \mathfrak{k}$ on itself is given by the algebraic bracket we deduce that $D_{X_i}X_i = \{X_i, X_i\} = 0$. Therefore, we can write the first summand of $(*)$ as

$$-\sum_i (D^2 t)(X_i, X_i) = -\sum_i D_{X_i}(D_{X_i} t) = -\sum_i X_i \bullet (X_i \bullet t) = \phi_{C_K}(t).$$

For the other summand, we use that the restriction of the fundamental derivative to the subbundle $T(G/K)$ coincides with the canonical connection, see part (ii) of Proposition 1.3.1. It follows that

$$\begin{aligned}\sum_j (D^2 t)(Y_j, Y_j) &= \sum_j \Big(D_{Y_j} D_{Y_j} - D_{D_{Y_j} Y_j}\Big)\, t \\ &= \sum_j \Big(D_{Y_j} \nabla_{Y_j} - D_{\nabla_{Y_j} Y_j}\Big)\, t \quad = \sum_j \Big(\nabla_{Y_j} \nabla_{Y_j} - \nabla_{\nabla_{Y_j} Y_j}\Big)\, t,\end{aligned}$$

which is $\operatorname{tr}(\nabla^2 t)$ due to naturality of the canonical connection. $\square$

In the case of a tractor bundle we have seen in section 1.3.2 that there is another differential operator of order 2 on V-valued differential forms on G/K given by the covariant Laplacian Δ^V. Similarly to 1.3.3, we can prove a Weitzenböck formula between this operator and the differential operator D_C induced by the Casimir element.

Theorem 1.3.3. *Let G be a semisimple Lie group with finite centre and $K \subset G$ a maximal compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the Cartan decomposition of $\mathfrak{g}$ so that $\mathfrak{k}$ is the Lie algebra of K. Let $\mathbb{V}$ be a G-representation with corresponding tractor bundle V on G/K. Let C be the Casimir operator on $\mathfrak{g}$, let $D_C\colon \Omega^*(G/K, V) \to \Omega^*(G/K, V)$ be the differential operator induced by C as in section 1.2.6 and ϕ_C^V be the tensorial map on $\Omega^*(G/K, V)$ which is induced by the action of the Casimir C on $\mathbb{V}$ tensorized with the identity on $\Lambda^k\mathfrak{q}^*$. Then on V-valued differential forms α on G/K we obtain*

$$D_C \alpha = -\Delta^V \alpha + \phi_C^V(\alpha).$$

Proof. Let $\{X_1, \dots, X_N\}$ a local frame of $G\times_K \mathfrak{k}$ and $\{Y_1, \dots, Y_n\}$ be a local frame of $T(G/K)$ so that the union is a local adapted frame for $\mathcal{A}$. Moreover, let B be the Euclidean bundle metric on $\mathcal{A}$ induced by the Killing form. If we denote by $\{Y^i\}$ the dual frame of the local frame $\{Y_i\}$, then we can write

$$d^V \alpha = \sum_{i=1}^n Y^i \wedge \nabla^{V_k}_{Y_i} \alpha, \qquad\qquad \delta^V \alpha = -\sum_{i=1}^n \iota_{Y_i} \nabla^{\theta_k}_{Y_i} \alpha$$

for all $\alpha \in \Omega^k(G/K, V)$, where ∇^{V_k} and ∇^{θ_k} are the connections on the bundle $\Lambda^k T^*(G/K) \otimes V$ induced by the pairings of the canonical connection on $T(G/K)$ with the connections ∇^V and ∇^θ on V, respectively. From part (iii) of Proposition 1.3.1 we know that the tractor connection is given by the sum of the canonical connection and the algebraic action on the vector bundle V. Therefore, naturality of the canonical connection implies

$$\nabla^{V_k}_Y \alpha = \nabla_Y \alpha + \rho^V_Y(\alpha)$$

for all vector fields $Y \in \mathfrak{X}(G/K)$, where ρ^V is induced by the linear map $\mathfrak{g} \to \operatorname{End}\left(\Lambda^k \mathfrak{q}^* \otimes \mathbb{V}\right)$ given by the tensor product of the identity on the exterior power and the action of $\mathfrak{g}$ on $\mathbb{V}$, and similarly for ∇^{θ_k}.

Therefore, the covariant exterior derivative and the covariant codifferential can be written for all $1 \leq i_0, \ldots, i_k \leq n$ as

$$(d^V\alpha)(Y_{i_0}, \ldots, Y_{i_k}) = \sum_{j=0}^{k} (-1)^j (\nabla_{Y_{i_j}}\alpha + \rho^V_{Y_{i_j}}(\alpha))(Y_{i_0}, \ldots, \widehat{Y_{i_j}}, \ldots, Y_{i_k})$$

as well as

$$(\delta^V\alpha)(Y_{i_1}, \ldots, Y_{i_{k-1}}) = -\sum_{\ell=1}^{n} (\nabla_{Y_\ell}\alpha - \rho^V_{Y_\ell}(\alpha))(Y_\ell, Y_{i_1}, \ldots, Y_{i_{k-1}}),$$

c.f. Theorem 1.3.2(ii).

Combining these two formulae a direct computation shows that

$$\begin{aligned}
(\Delta^V\alpha)(Y_{i_1}, \ldots, Y_{i_k}) &= (\phi^V_{C_\mathfrak{q}}(\alpha) - \mathrm{tr}(\nabla^2\alpha))(Y_{i_1}, \ldots, Y_{i_k}) \\
&+ \sum_{\ell=1}^{n}\sum_{j=1}^{k} (-1)^{j+1} \Bigg(\left(\rho^V_{\left(\{Y_{i_j}, Y_\ell\} + \nabla_{Y_{i_j}} Y_\ell\right)}(\alpha)\right)(Y_\ell, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k}) \\
&+ (\rho^V_{Y_\ell}(\alpha) - \nabla_{Y_\ell}\alpha)(\nabla_{Y_{i_j}} Y_\ell, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k}) \\
&+ (D_{\ell,j}\alpha)(Y_\ell, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k}) \Bigg) \\
&= (*),
\end{aligned}$$

where $\phi_{C_\mathfrak{q}}$ is the tensorial, G-equivariant map induced by the action of the $\mathfrak{q}$-part $C_\mathfrak{q}$ of the Casimir of $\mathfrak{g}$ on $\mathbb{V}$ tensorized with the identity on $\Lambda^k\mathfrak{q}^*$, and $D_{\ell,j}$ denotes the differential operator

$$D_{\ell,j}(\alpha) = \nabla_{Y_\ell}\nabla_{Y_{i_j}}\alpha - \nabla_{Y_{i_j}}\nabla_{Y_\ell}\alpha - \nabla_{\nabla_{Y_\ell}Y_{i_j}}\alpha.$$

For the following computations, we will use Roman numbers to refer to the corresponding lines in $(*)$.

First, we want to rewrite the third line (III) of $(*)$. In order to do so, recall that the Riemannian metric B on G/K is induced by the Killing form, so for the orthonormal frame $\{Y_m\}$ this implies that

$$B(\nabla_{Y_k}Y_i, Y_j) + B(Y_i, \nabla_{Y_k}Y_j) = Y_k \cdot B(Y_i, Y_j) = 0.$$

Thus, if we expand the vector field $\nabla_{Y_{i_j}}Y_\ell$ in the frame $\{Y_m\}$, we obtain that

$$\nabla_{Y_{i_j}}Y_\ell = \sum_{m=1}^{n} B(\nabla_{Y_{i_j}}Y_\ell, Y_m)Y_m = -\sum_{m=1}^{n} B(\nabla_{Y_{i_j}}Y_m, Y_\ell)Y_m.$$

We insert this into the third line of $(*)$ and use multilinearity of α as well as linearity of ∇ in the lower entry to move the coefficients of $\nabla_{Y_{i_j}}Y_\ell$ in the above expansion to the vector field Y_ℓ, which produces $\nabla_{Y_{i_j}}Y_m$. Therefore, after renaming indices we obtain that

$$(III) = \sum_{\ell=1}^{n}\sum_{j=1}^{k} (-1)^j \left(\rho^V_{\nabla_{Y_{i_j}}Y_\ell}(\alpha) - \nabla_{\nabla_{Y_{i_j}}Y_\ell}\alpha\right)(Y_\ell, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k})$$

The first sum cancels with the corresponding expression in the second line of $(*)$. Moreover, the second summand of (III) together with $D_{\ell,j}$ gives

$$\nabla_{Y_\ell}\nabla_{Y_{i_j}}\alpha - \nabla_{Y_{i_j}}\nabla_{Y_\ell}\alpha - \nabla_{\nabla_{Y_\ell}Y_{i_j}}\alpha + \nabla_{\nabla_{Y_{i_j}}Y_\ell}\alpha.$$

By Corollary 1.3.1 the canonical connection on $T(G/K)$ is torsionfree, so the above expression reduces to

$$\nabla_{Y_\ell}\nabla_{Y_{i_j}}\alpha - \nabla_{Y_{i_j}}\nabla_{Y_\ell}\alpha - \nabla_{[Y_\ell, Y_{i_j}]}\alpha = R(Y_\ell, Y_{i_j})(\alpha),$$

where R denotes the curvature of the canonical connection on $\Lambda^k T^*(G/K) \otimes V$. Now from part (i) of Proposition 1.3.1 it follows that the curvature is given by the algebraic action of $-\{Y_\ell, Y_{i_j}\}$ on α. All in all, we obtain that

$$\begin{aligned}(*) &= (\phi^V_{C_\mathfrak{q}}(\alpha) - \operatorname{tr}(\nabla^2\alpha))(Y_{i_1}, \ldots, Y_{i_k})\\ &\quad + \sum_{\ell=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\left(\rho^V_{\{Y_{i_j}, Y_\ell\}}(\alpha) + \rho_{\{Y_{i_j}, Y_\ell\}}(\alpha)\right)(Y_\ell, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k}),\end{aligned}$$

where the map ρ denotes the tensorial map induced by the $\mathfrak{k}$-action on $\Lambda^k\mathfrak{q}^* \otimes \mathbb{V}$.

Since the algebraic bracket $\{Y_{i_j}, Y_\ell\}$ is a section of $\Gamma(G \times_K \mathfrak{k})$ we can expand it in the pseudo-orthonormal local frame $\{X_m\}$. The invariance property of the Killing form implies that

$$\{Y_{i_j}, Y_\ell\} = -\sum_{m=1}^{N} B(\{Y_{i_j}, Y_\ell\}, X_m)X_m = \sum_{m=1}^{N} B(Y_\ell, \{Y_{i_j}, X_m\})X_m,$$

and inserting this into $(*)$ gives

$$\begin{aligned}(*) &= (\phi^V_{C_\mathfrak{q}}(\alpha) - \operatorname{tr}(\nabla^2\alpha))(Y_{i_1}, \ldots, Y_{i_k})\\ &\quad + \sum_{m=1}^{N}\sum_{j=1}^{k}(-1)^{j+1}\left(\rho^V_{X_m}(\alpha) + \rho_{X_m}(\alpha)\right)(\{Y_{i_j}, X_m\}, Y_{i_1}, \ldots, \hat{\jmath}, \ldots, Y_{i_k}).\end{aligned}$$

Let ϕ_{C_K} be the tensorial, G-equivariant map induced by the action of the Casimir element C_K of $\mathfrak{k}$ on $\Lambda^k\mathfrak{q}^* \otimes \mathbb{V}$ and $\phi^V_{C_K}$ the tensorial, G-equivariant map induced by the action of C_K on $\mathbb{V}$ tensorized with the identity on $\Lambda^k\mathfrak{q}^*$. Then by definition of the algebraic action of $G \times_K \mathfrak{k}$ on α we deduce that

$$\Delta^V\alpha = \phi^V_{C_\mathfrak{q}}(\alpha) + \phi^V_{C_K}(\alpha) - \operatorname{tr}(\nabla^2\alpha) - \phi_{C_K}(\alpha).$$

The first two summands add up to $\phi^V_C(\alpha)$, whereas the last two expressions sum up to $-D_C\alpha$ by Proposition 1.3.3. Thus, we have

$$D_C\alpha = -\Delta^V\alpha + \rho^V_C(\alpha).$$

□

1.4. Parabolic geometries

In this section we turn to the analysis of homogeneous parabolic geometries, which are the second class of homogeneous space appearing in our construction of Poisson transforms. In contrast to Riemannian symmetric spaces, these geometric structures are quite exotic and were usually studied case by case until the development of curved BGG-operators in [**ČSS01**].

After a brief discussion of the structure of $|k|$-graded Lie algebras in section 1.4.1 we use the canonical Lie algebra structure on the cotangent spaces of G/P to define in 1.4.2 the Kostant codifferential, which is a tensorial G-equivariant homology differential. Subsequently, we will introduce the BGG-operators in 1.4.3 as a distinguished family of differential operators between the corresponding homology groups. Finally, we will analyse in section 1.4.5 density bundles over such manifolds which are of crucial importance in this field.

1.4.1. Structure of $|k|$-graded Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra. We have seen in the Theorems 1.1.2 and 1.1.4 that parabolic subalgebras of $\mathfrak{g}$ are in bijective correspondence with $|k|$-gradings on $\mathfrak{g}$. Since this structure will be of major importance in the analysis of parabolic geometries, we will dedicate this chapter to give some basic properties about graded Lie algebras. As a reference for this chapter, see [**ČS09**, Chapter 3.1].

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_k$ be a $|k|$-graded Lie algebra. The compatibility with the Lie bracket shows that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$, which is reductive, and each $\mathfrak{g}_i$ is canonically a $\mathfrak{g}_0$-module via the adjoint action. Moreover, the Jacobi identity implies that the Lie bracket $[\ ,\ ]\colon \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ is a $\mathfrak{g}_0$-homomorphism. Moreover, the spaces $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ are both nilpotent subalgebras of $\mathfrak{g}$, the former being an ideal in the parabolic subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{p}_+$. We will refer to $\mathfrak{p}$ as the parabolic Lie algebra associated to the $|k|$-grading on $\mathfrak{g}$.

There is a unique element E, called the grading element, which lies in the centre $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ and satisfies $\mathrm{ad}(E)|_{g_j} = j \cdot \mathrm{id}_{\mathfrak{g}_j}$ for all $j = -k, \ldots, k$. Therefore, we can recover the index of the grading components via the action of the grading element.

Next, the $|k|$-grading determines a filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k \supset 0$ of $\mathfrak{g}$ via

$$\mathfrak{g}^l := \bigoplus_{i=l}^{k} \mathfrak{g}_i.$$

In particular, we see that $\mathfrak{g}^0 = \mathfrak{p}$, $\mathfrak{g}^1 = \mathfrak{p}_+$ and the grading property implies that each $\mathfrak{g}^i$ is a $\mathfrak{p}$-module. Since we are interested in $\mathfrak{p}$-invariant data the filtration of $\mathfrak{g}$ will be of major importance.

Let B denote the Killing form on $\mathfrak{g}$, which is non-degenerate by semisimplicity of $\mathfrak{g}$. Then B is compatible with the grading and the filtration of $\mathfrak{g}$ and thus induces bilinear pairings between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ as $\mathfrak{g}_0$-modules for all $i = 0, \ldots, k$. In addition, it induces a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{g}^{-i+1}$ and $\mathfrak{g}^i$, so in particular we obtain an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of $\mathfrak{p}$-modules.

We now turn to the group level. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define a subgroup P of G via

$$P = \{g \in G \mid \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for all } i = 1, \ldots, k\}.$$

Then P is a closed subgroup of G with Lie algebra $\mathfrak{p}$. We call any subgroup between P and its connected component a *parabolic subgroup*. For a fixed parabolic subgroup P we define the *Levi subgroup* G_0 via

$$G_0 = \{g \in P \mid \mathrm{Ad}(g)\mathfrak{g}_i \subset \mathfrak{g}_i \text{ for all } i = -k, \ldots, k\}.$$

Then G_0 is a reductive Lie group with Lie algebra $\mathfrak{g}_0$. In addition, it can be shown that the restriction of the exponential map to $\mathfrak{p}_+$ is a diffeomorphism into its image, which is a closed, nilpotent and contractible subgroup $\exp(\mathfrak{p}_+) =: P_+$ of G. Moreover, P_+ is a normal subgroup of P and and the groups G_0 and P/P_+ are canonically isomorphic. In particular, we see that P is the semidirect product $G_0 \ltimes P_+ \cong G_0 \ltimes \mathfrak{p}_+$, the latter isomorphism being induced by the exponential map.

DEFINITION 1.4.1. Let G be a semisimple Lie group with parabolic subgroup $P \subset G$. Then we call the homogeneous space G/P the *homogeneous model of parabolic geometries of type* (G, P) or shortly a *homogeneous parabolic geometry.*

The general notion of a parabolic geometry consists of a principal P-bundle $\mathcal{G}$ over a manifold M, together with a distinguished $\mathfrak{g}$-valued 1-form on $\mathcal{G}$ which induces a global parallelism and is compatible with the P-structure on $\mathcal{G}$ and $\mathfrak{g}$. In our case the bundle over $M = G/P$ is simple given by the Lie group G and the distinguished 1-form is the Maurer Cartan form.

For the rest of this section, we want to analyse finite dimensional representations of the parabolic Lie algebra $\mathfrak{p}$.

PROPOSITION 1.4.1. *Let* $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ *be a* $|k|$*-graded semisimple Lie algebra,* $\mathfrak{p} = \mathfrak{g}^0$ *the corresponding parabolic subalgebra and* $E \in \mathfrak{z}(\mathfrak{g}_0)$ *the grading element.*

(i) *Any finite dimensional completely reducible representation* $\mathbb{V}$ *of* $\mathfrak{p}$ *is obtained by trivially extending a completely reducible representation of* $\mathfrak{g}_0$ *to* $\mathfrak{p}$. *Moreover, the grading element acts by a scalar on every irreducible component of* $\mathbb{V}$.

(ii) *Let* $\mathbb{V}$ *be a finite dimensional representation of* $\mathfrak{p}$ *such that* $\mathfrak{z}(\mathfrak{g}_0)$ *acts diagonalisably. Then* $\mathbb{V}$ *admits a* $\mathfrak{p}$*-invariant filtration* $\mathbb{V} = \mathbb{V}^0 \supset \cdots \supset \mathbb{V}^N \supset \mathbb{V}^{N+1} = \{0\}$ *such that each of the quotients* $\mathbb{V}^i/\mathbb{V}^{i+1}$ *is completely reducible.*

PROOF. [**ČS09**, Proposition 3.2.12] $\square$

Now consider a finite dimensional irreducible representation $\mathbb{V}$ of the full Lie algebra $\mathfrak{g}$. Then the grading element E acts diagonalisably on $\mathbb{V}$. Moreover, for any eigenvector $v \in \mathbb{V}$ with eigenvalue i and any $X \in \mathfrak{g}_j$ we obtain that

$$E \cdot X \cdot v = X \cdot E \cdot v + [E, X] \cdot v = (i + j)X \cdot v,$$

so $X \cdot v$ is an eigenvector of the action of E with eigenvalue $i + j$. Hence by irreducibility of $\mathbb{V}$ these eigenvalues form an unbroken string of integers $i_0, \ldots, i_0+r$. If we denote by $\mathbb{V}_j$ the eigenspace to the eigenvalue j then the compatibility with the grading reads as

$$\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j},$$

where we agree that $\mathbb{V}_k = \{0\}$ if k is not an eigenvalue of the action of E. Moreover, the filtration induced by the grading coincides with the filtration obtained from Proposition 1.4.1(ii) by viewing $\mathbb{V}$ as a $\mathfrak{p}$-representation.

1.4.2. The Kostant codifferential. The main feature of parabolic geometries is that any cotangent space carries a canonical structure of a Lie algebra so that the group P acts by Lie algebra homomorphisms. Therefore, we can form the homology differential on the fibres, which induces a tensorial operator on k-forms on G/P. Together with the covariant exterior derivative, this enables us to generate a sequence of strongly invariant operators between sections of the homology bundles, which in the homogeneous case computes the twisted de Rham cohomology.

As before, let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$ be a $|k|$-graded semisimple Lie algebra with parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. Since the Killing form on $\mathfrak{g}$ is nondegenerate it induces a P-equivariant duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$, so we can identify the cotangent space $T^*(G/P)$ with the bundle of Lie algebras associated to the P-representation $\mathfrak{p}_+$. Consider the standard complex for Lie algebra homology $H_*(\mathfrak{p}_+, \mathbb{V})$ of $\mathfrak{p}_+$ with values in a G-representation $\mathbb{V}$. The k-chain space in this complex is given by the P-module $C_k(\mathfrak{p}_+, \mathbb{V}) := \Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$, and the differential is the map $\partial^* \colon C_k(\mathfrak{p}_+, \mathbb{V}) \to C_{k-1}(\mathfrak{p}_+, \mathbb{V})$ defined by

$$\begin{aligned}\partial^*(Z_1 \wedge \ldots \wedge Z_k \otimes v) := &\sum_{i=1}^{k} (-1)^i Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \otimes (Z_i \cdot v) \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge \hat{\jmath} \wedge \ldots \wedge Z_k \otimes v.\end{aligned}$$

It is easily verified that ∂^* is P-equivariant and satisfies $\partial^* \circ \partial^* = 0$. Although the map ∂^* is a Lie algebra homology differential it is usually called the *Kostant codifferential* for historical reasons. Furthermore, by P-equivariance the subspaces $\ker(\partial^*)$ and $\operatorname{im}(\partial^*)$ of $C_k(\mathfrak{p}_+, \mathbb{V})$ are P-invariant, implying that the associated homology groups $H_k(\mathfrak{p}_+, \mathbb{V}) = \ker(\partial^*)/\operatorname{im}(\partial^*)$ are P-modules. Since $\mathfrak{p}_+$ acts trivial on these modules, we see that $H_*(\mathfrak{g}_-, \mathbb{V})$ is completely reducible due to Proposition 1.4.1(i).

There is another construction which can be done using the graded structure on $\mathfrak{g}$. Consider the standard complex for the Lie algebra cohomology $H^*(\mathfrak{g}_-, \mathbb{V})$

of $\mathfrak{g}_-$ with values in a $\mathfrak{g}$-representation $\mathbb{V}$. Here, the k-cochain spaces are given by the P-modules $C^k(\mathfrak{g}_-, \mathbb{V}) := \Lambda^k \mathfrak{g}_-^* \otimes \mathbb{V}$, which we identify with multilinear alternating maps $(\mathfrak{g}_-)^k \to \mathbb{V}$. Moreover, the Lie algebra cohomology differential $\partial\colon C^k(\mathfrak{g}_-, \mathbb{V}) \to C^{k+1}(\mathfrak{g}_-, \mathbb{V})$ is defined by

$$\begin{aligned}\partial(f)(X_0, \ldots, X_k) = & \sum_{i=0}^{k}(-1)^i X_i \cdot f(X_0, \ldots, \hat{\imath}, \ldots, X_k) \\ & + \sum_{i<j}(-1)^{i+j} f([X_i, X_j], X_0, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, X_k)\end{aligned}$$

for all $X_0, \ldots, X_k \in \mathfrak{g}_-$, where the hat denotes omission. It is well known that $\partial \circ \partial = 0$, which means that $(C^k(\mathfrak{g}_-, \mathbb{V}), \partial)$ is indeed a complex. However, the cohomology differential is only a homomorphism of G_0-modules.

In [**ČS09**, Proposition 3.3.1] it is shown that ∂ and ∂^* are adjoint maps with respect to an inner product on $C_k(\mathfrak{p}_+, \mathbb{V})$, which immediately induces a Hodge-decomposition of these cochain spaces. Explicitly, we define the *Kostant Laplacian* $\square$ as the self-adjoint linear map $\square := \partial \circ \partial^* + \partial^* \circ \partial$. Then we have the following result.

THEOREM 1.4.2. *Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$ be a $|k|$-graded Lie algebra and $\mathbb{V}$ a representation of $\mathfrak{g}$. Then for each $k = 0, \ldots, \dim(\mathfrak{g}_-)$ we obtain a decomposition*

$$C_k(\mathfrak{p}_+, \mathbb{V}) = \operatorname{im}(\partial^*) \oplus \ker(\square) \oplus \operatorname{im}(\partial),$$

where the first two spaces add up to $\ker(\partial)$*, whereas the last two summands add up to* $\ker(\partial^*)$.

PROOF. The proof of [**ČS09**, Theorem 3.3.1] applies without changes. $\square$

1.4.3. BGG-sequences. Now we turn to the global picture and show how the operators from the previous section induce differential operators between homology bundles.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let P be the subgroup of G with Lie algebra $\mathfrak{p}$. Let $\mathbb{V}$ be a G-representation and V be the corresponding tractor bundle. Since V-valued k-forms on G/P are sections of the bundle associated to the P-representation $\Lambda^k\mathfrak{p}_+ \otimes \mathbb{V}$, the P-equivariant map ∂^* induces a G-invariant tensorial operator on $\Omega^k(G/P, V)$ for all k. Moreover, the P-invariant subspaces $\ker(\partial^*)$ and $\operatorname{im}(\partial^*)$ give rise to two subbundles of $\Lambda^*T(G/P) \otimes V$, which we will denote by the same symbols. By naturality, their quotient bundle $\mathcal{H}_k(G/P, V)$ is given by the bundle associated to the P-representation $H_k(\mathfrak{p}_+, \mathbb{V})$, so the canonical projection induces a natural operator π from the space of sections of the bundle $\ker(\partial^*)$ to sections of the bundle $\mathcal{H}_k(G/P, V)$.

On the other hand, the operators ∂ and $\square$ are only $\mathfrak{g}_0$-equivariant, so they do not have immediate geometric counterparts. However, via a careful analysis of the interaction between the Kostant Laplacian $\square$ and the curved Laplace operator $\square^R := d^V\partial^* + \partial^* d^V$, where d^V denotes the covariant exterior derivative, we can construct an operator which splits the canonical projection π onto the sections of the homology bundle.

THEOREM 1.4.3. *Let G be a semisimple Lie group, $P \subset G$ a parabolic subgroup and let V be the associated bundle corresponding to a G-representation $\mathbb{V}$. Let ∂^* denote the Kostant codifferential on $C_*(\mathfrak{g}_-, \mathbb{V})$ and let us denote the induced operators on the bundles $\Lambda^*T^*(G/P) \otimes V$ and their sections by the same symbol. Let π denote the natural operator $ker(\partial^*) \to \Gamma(\mathcal{H}_k(G/P, V))$. Then there is a natural differential operator*

$$L\colon \Gamma(\mathcal{H}_k(G/P, V)) \to \Gamma(\ker(\partial^*)) \subset \Omega^k(G/P, V)$$

which splits the projection π. Moreover, the operator L is uniquely characterized by the following conditions: For all $\alpha \in \Gamma(\mathcal{H}_k(G/P, V))$ we have

(i) $\pi(L(\alpha)) = \alpha$,
(ii) $d^V(L(\alpha)) \in \Gamma(\ker(\partial^*))$.

PROOF. [**ČSS01**, Section 4]; [**CD01**] □

The last Theorem allows us to construct natural invariant differential operators. Indeed, we have the following diagram

$$\begin{tikzcd}
& \Gamma(\ker(\partial^*)) \arrow[d, shift right, "\pi"'] & & \Gamma(\ker(\partial^*)) \arrow[d, shift right, "\pi"'] & & {} \\
\cdots \arrow[ur, "d^V \circ L"] \arrow[r, "D^V"] & \Gamma(\mathcal{H}_k(G/P, V)) \arrow[u, shift right, "L"'] \arrow[urr, "d^V \circ L"] \arrow[rr, "D^V"] & & \Gamma(\mathcal{H}_{k+1}(G/P, V)) \arrow[u, shift right, "L"'] \arrow[urr, "d^V \circ L"] \arrow[r, "D^V"] & \cdots &
\end{tikzcd},$$

where the induced operators $D^V : \Gamma(\mathcal{H}_k(G/P, V)) \to \Gamma(\mathcal{H}_{k+1}(G/P, V))$ defined by $D^V = \pi \circ d^V \circ L$ are natural differential operators. Moreover, since the tractor connection is flat we deduce from Proposition 1.2.4(i) that $D^V \circ D^V = 0$.

DEFINITION 1.4.3. Let G/P be a homogeneous parabolic geometry and let $\mathbb{V}$ be a G-module with associated tractor bundle V on G/P. We call the sequence of strongly invariant operators

$$0 \longrightarrow \Gamma(\mathcal{H}_0(G/P, V)) \xrightarrow{D^V} \cdots \xrightarrow{D^V} \Gamma(\mathcal{H}_{\dim(G/P)}(G/P, V)) \longrightarrow 0$$

the *Bernstein-Gelfand-Gelfand sequence* or *BGG-sequence.*

In the definition of BGG-sequences we used the covariant exterior derivative d^V as defined in section 1.2.4. However, a precise analysis shows that this construction can be applied to a wider range of differential operators between V-valued differential forms on G/P, so called *compressible* operators. This idea was recently developed in the setting of relative BGG-sequences in [**ČS15**], which also includes the classical case above. However, for our purpose it will be sufficient to stick to the above definition of BGG-sequences.

Since the BGG-sequence is a complex on the homogeneous space G/P, we can compute its cohomology, which turns out to be precisely the twisted de Rham cohomology as defined in section 1.2.4.

COROLLARY 1.4.3. *For any representation $\mathbb{V}$ of G the BGG-sequence*

$$0 \longrightarrow \Gamma(\mathcal{H}_0(G/P, V)) \xrightarrow{D^V} \cdots \xrightarrow{D^V} \Gamma(\mathcal{H}_{\dim(G/P)}(G/P, V)) \longrightarrow 0$$

is a complex, which computes the twisted de Rham cohomology of G/P with values in the bundle V.

PROOF. [**ČSS01**, Corollary 4.15] □

In general, the bundles appearing in the definition of the BGG-sequence are much smaller than those in the twisted de Rham complex. Therefore, the above Corollary gives an easier way to compute the de Rham cohomology of a homogeneous parabolic geometry.

Finally, each section of $\mathcal{H}_k(G/P, V)$ possesses a distinguished representative in $\Gamma(\ker(\partial^*))$, which can be shown using Theorem 1.4.2

PROPOSITION 1.4.3. *Let G/P be a homogeneous parabolic geometry and let V be a tractor bundle on G/P. Then every homology class in $\mathcal{H}_k(G/P,V)$ has a unique representative in the kernel of $\Box^R$. Therefore, we obtain an isomorphism*

$$\Gamma(\mathcal{H}_k(G/P,V)) \cong \Gamma(\ker(\Box^R)),$$

which is given by the splitting operator from Theorem 1.4.3.

PROOF. Using that $\Box^R$ is invertible on the image of ∂^* (see [**CD01**, Theorem 5.2]) this is a simple consequence of Theorem 1.4.3. □

1.4.4. An adjoint for the Kostant codifferential. In section 2.2.1 we will define a Poisson transform on differential forms on G/P with values in a tractor bundle as an integral operator whose kernel is a vector valued differential form. In this context, it will be necessary to determine an operator on such kernels which is adjoint to the Kostant codifferential with respect to the wedge product. More explicitly, let $\mathbb{V}$ be a G-representation and consider two chains $\alpha \in C_k(\mathfrak{p}_+,\mathbb{V}^*)$ and $\beta \in C_{n+1-k}(\mathfrak{p}_+,\mathbb{V})$, where $\dim(\mathfrak{g}/\mathfrak{p}) = n$. Our aim is to find an operator ∂^*_A which satisfies

$$\partial^*_A\alpha \wedge \beta = (-1)^{|\alpha|}\alpha \wedge \partial^*\beta,$$

where the wedge product is formed by applying the contraction $\mathbb{V}^* \otimes \mathbb{V} \to \mathbb{R}$. In this section we will derive this operator by finding an expression for the Kostant codifferential in terms of operators which are compatible with the wedge product.

Recall that the Killing form induces a duality between $\mathfrak{p}_+$ and $\mathfrak{g}/\mathfrak{p}$ as P-modules, so we can identify $\Lambda^k\mathfrak{p}_+ \otimes \mathbb{V}$ with the space of k-linear alternating maps $\mathfrak{g}^k \to \mathbb{V}$ which vanish after insertion of any element in $\mathfrak{p}$. Explicitly, for all $W_1,\dots,W_k \in \mathfrak{g}$, $Z_1,\dots,Z_k \in \mathfrak{p}_+$ and $v \in \mathbb{V}$ we have

$$(Z_1 \wedge \ldots \wedge Z_k \otimes v)(W_1,\dots,W_k) = \det(B(Z_i,W_j)_{i,j}) \cdot v,$$

and since the Killing form is trivial on $\mathfrak{p} \times \mathfrak{p}_+$ this expression vanishes if $W_i \in \mathfrak{p}$ for any $i = 1,\dots,k$.

In this way, we can define two operations on chains. First, for every $\eta \in \mathfrak{g}$ we can form the insertion operator ι_η, which is a map from k-chains to $(k-1)$-chains. On the other hand, we define for every $\xi \in \mathfrak{p}_+$ the operator $\epsilon_\xi \colon \Lambda^k\mathfrak{p}_+\otimes\mathbb{V} \to \Lambda^{k+1}\mathfrak{p}_+\otimes\mathbb{V}$, which is given by $\epsilon_\xi(\alpha) = \xi \wedge \alpha$. By definition, we obtain for all $\eta \in \mathfrak{g}$ and $\xi \in \mathfrak{p}_+$ that

$$\iota_\eta\epsilon_\xi\alpha = B(\xi,\eta)\alpha - \epsilon_\xi\iota_\eta\alpha.$$

In particular, if ξ and η are orthogonal with respect to the Killing form, then the operators ι_η and ϵ_ξ anticommute.

Every element $Z \in \mathfrak{p}_+$ acts canonically on both $\mathbb{V}$ and $\Lambda^*\mathfrak{p}_+$, the latter action is given by the exterior power of the adjoint action. We denote by $\rho^{\mathfrak{p}_+}$ the action on the chain space $\Lambda^*\mathfrak{p}_+ \otimes \mathbb{V}$ given by the adjoint action on $\Lambda^*\mathfrak{p}_+$ tensorised with the identity on $\mathbb{V}$, and similarly by $\rho^{\mathbb{V}}$ the action of $\mathfrak{p}_+$ on $\mathbb{V}$ tensorised with the identity on $\Lambda^*\mathfrak{p}_+$.

PROPOSITION 1.4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}_-$ denote the negative part of the grading on $\mathfrak{g}$ induced by the choice of parabolic. Let $\mathbb{V}$ be a $\mathfrak{p}$-representation and let $\phi \in \Lambda^k\mathfrak{p}_+ \otimes \mathbb{V}$, which we view as a multilinear, alternating map $\mathfrak{g}^k \to \mathbb{V}$ which vanishes after insertion of any vector in $\mathfrak{p}$. Let $\xi_1,\dots,\xi_n$ be a basis of $\mathfrak{p}_+$ and let $\eta_1,\dots,\eta_n \in \mathfrak{g}_-$ be the dual basis with respect to the Killing form. Then*

$$\partial^*\phi = -\sum_{\ell=1}^{n}\left(\rho^{\mathbb{V}}_{\xi_\ell} + \frac{1}{2}\rho^{\mathfrak{p}_+}_{\xi_\ell}\right)\iota_{\eta_\ell}\phi.$$

PROOF. By linearity of the Kostant codifferential it suffices to prove the formula for decomposable elements $\phi = Z_1 \wedge \ldots \wedge Z_k \otimes v$ with $Z_1, \ldots, Z_k \in \mathfrak{p}_+$ and $v \in \mathbb{V}$. Moreover the defining formula for the Kostant codifferential splits into the sum $\partial^* = \partial^*_{(1)} + \partial^*_{(2)}$, where the first term corresponds to the summand which contains the $\mathfrak{p}_+$-action on $\mathbb{V}$, and $\partial^*_{(2)} = \partial^* - \partial^*_{(1)}$. Since the bases $\{\xi_\ell\}$ and $\{\eta_\ell\}$ are dual with respect to the Killing form, every element $Z \in \mathfrak{p}_+$ can be written as $Z = \sum_\ell B(\eta_\ell, Z)\xi_\ell$.

The first part of the claimed formula can be computed as

$$\begin{aligned} -\sum_{\ell=1}^{n} \rho^{\mathbb{V}}_{\xi_\ell} \iota_{\eta_\ell} \phi &= \sum_{\ell=1}^{n}\sum_{i=1}^{k} (-1)^i B(\eta_\ell, Z_i) Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \otimes (\xi_\ell \cdot v) \\ &= \sum_{i=1}^{k} (-1)^i Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \otimes (Z_i \cdot v), \end{aligned}$$

which is exactly $\partial^*_{(1)}(\phi)$. Similarly, for the second part we deduce

$$\begin{aligned} -\frac{1}{2}\sum_{\ell=1}^{n} \rho^{\mathfrak{p}_+}_{\xi_\ell} \iota_{\eta_\ell}(\phi) &= \frac{1}{2}\sum_{\ell=1}^{n}\sum_{i=1}^{k} (-1)^i B(\eta_\ell, Z_i) \rho^{\mathfrak{p}_+}_{\xi_\ell} (Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \otimes v) \\ = \frac{1}{2}\sum_{\ell=1}^{n}\sum_{i<j}^{k} &(-1)^{i+j} B(\eta_\ell, Z_i)[\xi_\ell, Z_j] \wedge Z_1 \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge Z_k \otimes v \\ &+ \frac{1}{2}\sum_{\ell=1}^{n}\sum_{i>j}^{k} (-1)^{i+j+1} B(\eta_\ell, Z_i)[\xi_\ell, Z_j] \wedge Z_1 \wedge \ldots \hat{\jmath} \ldots \hat{\imath} \ldots \wedge Z_k \otimes v. \end{aligned}$$

If we move the Killing form into the Lie bracket and sum over all ℓ, we obtain the expression $[Z_i, Z_j]$. Next, if we exchange the indices i and j in the second sum and use skew-symmetry of the Lie bracket we see that the two sums coincide. All in all, we get

$$-\frac{1}{2}\sum_{\ell=1}^{n} \rho^{\mathfrak{p}_+}_{\xi_\ell} \iota_{\eta_\ell}(\phi) = \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge Z_k \otimes v,$$

which equals $\partial^*_{(2)}(\phi)$. □

As mentioned at the beginning of this section, our aim is to construct an operator which is adjoint to the Kostant codifferential with respect to the wedge product in a special case. For that, it will be necessary to rewrite the action $\rho^{\mathfrak{p}_+}$ in an appropriate way.

LEMMA 1.4.4. *Let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathfrak{p}_+$ and let $\{\eta_1, \ldots, \eta_n\}$ be the dual basis for $\mathfrak{g}_-$ with respect to the Killing form. Then for all $\alpha \in C_k(\mathfrak{p}_+, \mathbb{V})$ and $X \in \mathfrak{p}_+$ we have*

$$\rho^{\mathfrak{p}_+}_X \alpha = -\sum_{t=1}^{n} \epsilon_{\xi_t} \iota_{[X, \eta_t]} \alpha,$$

where we view α as a k-linear alternating map $\mathfrak{g}^k \to \mathbb{V}$ which vanishes after insertion of an element in $\mathfrak{p}$.

PROOF. By linearity of the operators it suffices to consider $\alpha = Z_1 \wedge \ldots \wedge Z_k \otimes v$ with $Z_1, \ldots, Z_k \in \mathfrak{p}_+$ and $v \in \mathbb{V}$. The Killing form induces a nondegenerate pairing between $\mathfrak{g}_-$ and $\mathfrak{p}_+$, so the right hand side of the claimed formula yields

$$
\begin{aligned}
-\sum_{t=1}^{n} \epsilon_{\xi_t} \iota_{[X,\eta_t]}\alpha &= \sum_{i=1}^{k}\sum_{t=1}^{n}(-1)^i B([X,\eta_t],Z_i)\xi_t \wedge Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \\
&= \sum_{i=1}^{k}\sum_{t=1}^{n}(-1)^{i+1} B(\eta_t,[X,Z_i])\xi_t \wedge Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k \\
&= \sum_{i=1}^{k}(-1)^{i+1}[X,Z_i] \wedge Z_1 \wedge \ldots \wedge \hat{\imath} \wedge \ldots \wedge Z_k,
\end{aligned}
$$

which is exactly the $\mathfrak{p}_+$-action of X on α. □

We can insert the formula from the last Lemma into Proposition 1.4.4 to obtain an expression for the Kostant codifferential which is compatible with the wedge product. In this way, we can prove

THEOREM 1.4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}_-$ denote the negative part of the grading on $\mathfrak{g}$ induced by the choice of parabolic and $\mathbb{V}$ be a $\mathfrak{p}$-representation. Let $\alpha \in C_k(\mathfrak{p}_+,\mathbb{V}^*)$ and $\beta \in C_{n+1-k}(\mathfrak{p}_+,\mathbb{V})$ be two chains, where $n = \dim(\mathfrak{g}/\mathfrak{p})$. Then we obtain that*

$$\partial^*\alpha \wedge \beta = (-1)^{|\alpha|}\alpha \wedge \partial^*\beta,$$

where the wedge product is formed by applying the contraction $\mathbb{V}^* \otimes \mathbb{V} \to \mathbb{R}$.

PROOF. Let $\{\xi_1,\ldots,\xi_n\}$ be a basis for $\mathfrak{p}_+$ and $\{\eta_1,\ldots,\eta_n\}$ the basis of $\mathfrak{g}_-$ which is dual with respect to the Killing form. Combining Proposition 1.4.4 and Lemma 1.4.4 we deduce that

$$\alpha \wedge \partial^*\beta = -\sum_{s=1}^{n} \alpha \wedge \rho^{\mathbb{V}}_{\xi_s}\iota_{\eta_s}\beta + \frac{1}{2}\sum_{s,t=1}^{n}\alpha \wedge \epsilon_{\xi_t}\iota_{[\xi_s,\eta_t]}\iota_{\eta_s}\beta,$$

where we view the chains α and β as multilinear, alternating, $\mathbb{V}$-valued maps on $\mathfrak{g}^k$ which vanish after insertion of an element of $\mathfrak{p}$.

For the first sum, note that the insertion operator commutes with the action $\rho^{\mathbb{V}}$ by definition. Therefore, we first move ι_{η_s} to the left, which produces the sign $(-1)^{|\alpha|+1}$. Now we apply the definition of the dual representation to obtain

$$\alpha \wedge \rho^{\mathbb{V}}_{\xi_s}\iota_{\eta_s}\beta = (-1)^{|\alpha|}\left(\rho^{\mathbb{V}^*}_{\xi_s}\iota_{\eta_s}\alpha\right) \wedge \beta = (*).$$

For the second sum, we can first move the operator ϵ_{ξ_t} to the left, where we pick up the sign $(-1)^{|\alpha|}$. Next, we use the fact that the wedge product of ξ_t, α and β is a chain of degree $n+2$, so we can move $\iota_{[\xi_s,\eta_t]}$ to the left, obtaining the sign $(-1)^{|\alpha|}$. Finally, the last insertion operator produces $(-1)^{|\alpha|+1}$ when moved to the left. Using that two interior products anticommute we deduce that

$$\alpha \wedge \epsilon_{\xi_t}\iota_{[\xi_s,\eta_t]}\iota_{\eta_s}\beta = (-1)^{|\alpha|}(\iota_{[\xi_s,\eta_t]}\iota_{\eta_s}\epsilon_{\xi_t}\alpha) \wedge \beta.$$

We want to simplify the last expression by commuting ϵ_{ξ_t} with the interior products. For that, we will make a specific choice of basis for the subalgebra $\mathfrak{p}_+$.

Assume that the basis $\{\xi_1,\ldots,\xi_n\}$ is the union of bases of the positive grading components. First, by duality of the bases we see that $B(\xi_s,\eta_t) \neq 0$ if and only if $s = t$, so we can easily anticommute ϵ_{ξ_t} and ι_{η_s} for all $s \neq t$. However, if the indices are the same, the Lie bracket $[\eta_s,\xi_s]$ is contained in $\mathfrak{g}_0 \subset \mathfrak{p}$ by the compatibility condition of the Lie bracket and thus inserts trivially into α. For the other interior product, note that

$$B([\xi_s,\eta_t],\xi_t) = B(\xi_s,[\eta_t,\xi_t]) = 0,$$

since $\xi_s \in \mathfrak{p}_+$ and $[\eta_t, \xi_t] \in \mathfrak{g}_0$. Thus, we can anticommute both interior products with ϵ_{ξ_t}, which does not change the sign of $(*)$. Putting all together we obtain that

$$\alpha \wedge \partial^* \beta = (-1)^{|\alpha|} \partial^* \alpha \wedge \beta,$$

so we see that the Kostant codifferential is self-adjoint with respect to the wedge product. □

1.4.5. Densities on G/P. In the last chapter of this section we will introduce and study density bundles over G/P, which are a tool of major importance in the realms of parabolic geometries.

Let G be a semisimple Lie group and H a closed subgroup of G. Consider an orientable, homogeneous line bundle $\mathcal{E}$ over G/H with associated 1-dimensional H-representation $\mathbb{E}$ and assume that G acts by orientation preserving bundle maps. Then the action of H on $\mathbb{E}$ is determined by a smooth character $\chi \colon H \to \mathbb{R}_+$. In this case, we denote the line bundle by $\mathcal{E}[\chi]$ and the associated 1-dimensional H-representation by $\mathbb{R}[\chi]$. We call any section of $\mathcal{E}[\chi]$ a *χ-density* and we refer to the bundle as the *bundle of χ-densities* on G/H. Moreover, if $V \to G/H$ is any vector bundle, then we define $V[\chi] := V \otimes \mathcal{E}[\chi]$. Clearly, if V is a homogeneous vector bundle associated to the H-representation $\mathbb{V}$, then $V[\chi]$ is again homogeneous with corresponding H-representation $\mathbb{V} \otimes \mathbb{R}[\chi] := \mathbb{V}[\chi]$.

PROPOSITION 1.4.5. *Let G be a Lie group and H a closed subgroup.*

(i) Let χ and χ' be two characters $H \to \mathbb{R}_+$ and V and W be two vector bundles over G/H. Then there is a canonical isomorphism

$$V[\chi] \otimes W[\chi'] \cong (V \otimes W)[\chi + \chi'].$$

(ii) If χ is the trivial character, then $V[\chi] \cong V$ for all vector bundles V over G/H

(iii) For every character $\chi \colon H \to \mathbb{R}_+$ and every vector bundle $V \to G/H$ the dual bundle of $V[\chi]$ is isomorphic to $V^*[-\chi]$.

PROOF. This follows directly from the naturality of the associated bundle construction. □

The existence of nontrivial density bundles depends strongly on the Lie group H. Indeed, if $\chi \colon H \to \mathbb{R}_+$ is a character on a compact subgroup H, then $\chi(H)$ is a compact subgroup of $\mathbb{R}_+$ and thus has to be trivial. In particular, there are no interesting density bundles over Riemannian symmetric spaces.

On the other hand, density bundles are crucial in the theory of invariant differential operators on homogeneous parabolic geometries. In this setting they can be fully described by a character on the Levi subgroup.

THEOREM 1.4.5. *Let G be a semisimple Lie group, $P \subset G$ a parabolic subgroup and G_0 the Levi subgroup of P. Then any density bundle $\mathcal{E}$ over G/P is determined by a character $G_0 \to \mathbb{R}_+$. Furthermore, if P is a minimal parabolic subgroup, then $\mathcal{E}$ is already determined by a linear functional $\lambda \in \mathfrak{a}_0^*$.*

PROOF. Let $\chi \colon P \to \mathbb{R}_+$ be the character associated to the density bundle $\mathcal{E}$. Since the corresponding P-representation $\mathbb{R}[\chi]$ is 1-dimensional we obtain from Proposition 1.4.1(i) that the nilpotent subgroup P_+ of P acts trivially on $\mathbb{R}[\chi]$, implying that χ factors to a character $\chi' \colon P/P_+ \to \mathbb{R}_+$. Furthermore, the quotient group P/P_+ is canonically isomorphic to G_0, so we can view χ' as a character on G_0, which by construction is just the restriction of χ to the Levi subgroup.

Now assume that P is a minimal parabolic subgroup of G with Lie algebra $\mathfrak{p}_0$. Then we can find a Cartan involution on the Lie algebra of G so that $P = MAN$, where $G = KAN$ is the Iwasawa decomposition of G and M is the centralizer of the Lie algebra $\mathfrak{a}_0$ of A in K. In this case, we have $G_0 = MA$ and $P_+ = N$, so the density bundle is determined by a character on MA. However, since M is compact this character is trivial on M, and thus its restriction to $A = \exp(\mathfrak{a}_0)$ is determined by a linear functional on $\mathfrak{a}_0$. $\square$

1.5. Fibre integration

In this section we will introduce the integral over the fibre of a vector valued differential form on a fibred manifold, which will be essential for our construction of Poisson transforms. Here we will give a broader and more detailed discussion of this tool since results in this context are usually only formulated for real valued differential forms, c.f. **[GHV72]**.

Explicitly, we define fibre integrals in section 1.5.1 and provide a description in terms of local coordinates. Moreover, we will prove that this operation is compatible with pullbacks as well as covariant exterior derivatives. Afterwards, we focus in section 1.5.2 on fibre integrals in the category of Riemannian manifolds, which will be used in section 3.3.3 to derive estimates on the pointwise norms of the images of Poisson transform in the special case of the real hyperbolic space.

1.5.1. Definition and first properties of fibre integration. Let M and F be smooth manifolds of dimensions m and n, respectively, and let $\pi\colon E \to M$ be a smooth (locally trivial) fibre bundle with standard fibre F. For all $x \in M$ we denote by $E_x = \pi^{-1}(x)$ the corresponding fibre, which is a submanifold of E via the inclusion $j_x\colon E_x \hookrightarrow E$. Moreover, the union of all tangent spaces of the fibres forms the vertical bundle $\mathcal{V}E := \ker(T\pi) \subset TE$.

We say that the fibre bundle E is *orientable* if there exists a differential form $o \in \Omega^n(E)$ such that the pullback $j_x^* o$ orients the fibre E_x for all $x \in M$. Two such differential forms are said to be *equivalent* if they induce the same orientation on the fibres and any equivalence class is called an *orientation* on the bundle E. For trivial bundles $E \cong M \times F$, orientability of E is equivalent to orientability of the standard fibre F.

Let α be a differential form on E and denote by $\operatorname{supp}(\alpha)$ the support of α. Then α is said to have *fibre-compact support* if for every compact subset $K \subset M$ the intersection $\pi^{-1}(K) \cap \operatorname{supp}(\alpha)$ is compact. We denote by $\Omega_F(E)$ the space of fibre-compact supported differential forms on E. If we denote by $\Omega_c(E)$ the space of differential forms with compact support, we have the following chain of inclusions

$$\Omega_c(E) \subseteq \Omega_F(E) \subseteq \Omega(E),$$

where the first two spaces coincide iff M is compact, whereas the last two coincide iff F is compact.

Let $p\colon V \to M$ be a vector bundle over M and denote by $\pi^*V \to E$ the *pullback bundle* of V along π, i.e. the bundle

$$\pi^*V = \{(z, v) \in E \times V \mid \pi(z) = p(v)\}.$$

Then any section $s \in \Gamma(V)$ induces a section π^*s of the pullback bundle π^*V, called the *pullback section* of s, which is defined via $(\pi^*s)(z) = s(\pi(z))$ for all $z \in E$. In particular, if $\{s_i\}$ is a local frame of V over $U \subset M$, then we can write every local section $\sigma \in \Gamma(\pi^*V|_U)$ as $\sigma = \sum_i f_i \pi^* s_i$ for smooth functions $f_i \in C^\infty(E|_U, \mathbb{R})$.

Now we have collected all the ingredients for the definition of the fibre integral. Let α be a π^*V-valued differential form on E of degree $n+p$ for any $0 \le p \le m$, which has fibre-compact support. For any $x \in M$ we fix an element $z \in E_x$ as well as tangent vectors $\xi_1, \dots, \xi_p \in T_xM$ and $\eta_1, \dots, \eta_n$ in the vertical bundle $\mathcal{V}_zE$. We choose local lifts $\tilde{\xi}_i$ of the tangent vectors ξ_i to T_zE for all $i = 1, \dots, p$. Then by dimensional reasons the element $\alpha(z)(\tilde{\xi}_1, \dots, \tilde{\xi}_p, \eta_1, \dots, \eta_n) \in (\pi^*V)_z$ is independent of the choice of lifts of the vector fields ξ_i. This allows us to define an n-form $\alpha_x \in \Omega^n(E_x, \Lambda^p T_x^* M \otimes V_x)$ via

$$\alpha_x(z)(\eta_1, \dots, \eta_n)(\xi_1, \dots, \xi_p) := \alpha(z)(\tilde{\xi}_1, \dots, \tilde{\xi}_p, \eta_1, \dots, \eta_n)$$

for all $\eta_1, \dots, \eta_n \in V_z E$ and $\xi_1, \dots, \xi_p \in T_x M$.

Definition 1.5.1. Let $\pi\colon E \to M$ be an oriented fibre bundle over M with smooth fibre F of dimension n and $V \to M$ a vector bundle. Then for all fibre compact differential forms $\alpha \in \Omega_F^{n+p}(E, \pi^* V)$ we define a V-valued differential form $⨍_F \alpha \in \Omega^p(M, V)$ via

$$\left(⨍_F \alpha \right)(x) := \int_{E_x} \alpha_x.$$

The resulting linear map

$$⨍_F \colon \Omega_F(E, \pi^* V) \to \Omega(M, V)$$

is called the *integration over the fibre* F or shortly the *fibre integral over* F.

Note that if α is a differential form on E which vanishes after insertion of n vertical vector fields, then its fibre integral is trivial by definition. Moreover, we extend the fibre integral to differential forms on E of degree $< n$ trivially. We have already mentioned that if the fibre F is compact, then every differential form has fibre compact support and thus $⨍_F$ is defined for all differential forms.

In order to give a local description of the fibre integral, we will start our discussion with the special case of the product bundle $E = M \times F$. The tangent maps of the canonical projections $\pi_M\colon E \to M$ and $\pi_F\colon E \to F$ induce a linear isomorphism

$$T_{(x,y)}(M \times F) \cong T_x M \oplus T_y F$$

for all $x \in M$ and $y \in F$. We call a tangent vector $\xi \in T_z E$ *horizontal* (respectively *vertical*) if its image under $T_z \pi_F$ (respectively $T_z \pi_M$) is trivial. By naturality of the exterior product the decomposition of TE induces an isomorphism

$$\Lambda^k T^* E \cong \bigoplus_{p+q=k} \Lambda^p T^* M \otimes \Lambda^q T^* F,$$

and we define the bundle $\Lambda^{p,q} T^* E$ to be the preimage of $\Lambda^p T^* M \otimes \Lambda^q T^* F$ under this isomorphism. The space of smooth sections of $\Lambda^{p,q} T^* E$ is called the space $\Omega^{p,q}(E)$ of *differential forms of degree* (p,q) and consists of those differential forms on E which vanish after insertion of $p+1$ horizontal or $q+1$ vertical vectors. Moreover, the space of differential forms on E of degree k decomposes into the direct sum

$$\Omega^k(E) = \bigoplus_{p+q=k} \Omega^{p,q}(E).$$

for all $k = 0, \dots, m+n$.

Assume now that the product bundle $\pi\colon E = M \times F \to M$ is oriented, which is equivalent to the standard fibre F being oriented, and let $V \to M$ be a vector bundle. For any differential form $\alpha \in \Omega^{n+p}(E, \pi^* V)$ the definition of the fibre integral implies that $⨍_F \alpha$ is trivial unless $\alpha \in \Omega^{p,n}(E, \pi^* V)$ for any $0 \le p \le m$.

We can easily get a local description of the fibre integral in this case. Let U be an open subset of M and choose a local coframe $e^1, \dots, e^m$ on U. Moreover, let $dy \in \Omega^n(F)$ be an orientation on F and let $\{s_j\}$ be a local frame of $V|_U$. Then we can write the restriction of $\alpha \in \Omega^{p,n}(E)$ to $U \times F$ as

$$\alpha|_{U \times F} = \sum_{I,j} f_{I,j}(x,y) e^{i_1} \wedge \ldots \wedge e^{i_p} \wedge dy \otimes \pi^* s_j,$$

where $f_{I,j} \in C^\infty(U \times F)$ and the sum is indexed over all ordered tuples $I = (i_1, \ldots, i_p)$. If we abbreviate $e^I = e^{i_1} \wedge \ldots \wedge e^{i_p}$, then the fibre integral at any point $x \in U$ is simply given by

$$\left(\fint_F \alpha\right)(x) = \sum_I \left(\int_F f_I(x,y)dy\right) e^I(x) \otimes s_I(x).$$

In the next Proposition we will show that the fibre integral is compatible with isomorphisms of fibre bundles. Together with partitions of unity, this will be used to obtain a local description of the fibre integral on any bundle E. In particular, many properties which hold for integration of smooth functions can be translated to fibre integrals in this way.

PROPOSITION 1.5.1. *Let $\pi\colon E \to M$ and $\tilde{\pi}\colon \tilde{E} \to \tilde{M}$ be two oriented fibre bundle with standard fibres F respectively $\tilde{F}$ and let $\tilde{V} \to \tilde{M}$ be a vector bundle over $\tilde{M}$. Let $\Psi\colon E \to \tilde{E}$ be a fibre bundle isomorphism covering a diffeomorphism $\psi\colon M \to \tilde{M}$ and assume that for every $x \in M$ the map $\Psi_x\colon E_x \to \tilde{E}_{\psi(x)}$ is orientation preserving.*

Then for all $\alpha \in \Omega^{n+p}_{\tilde{F}}(\tilde{E}, \tilde{\pi}^*\tilde{V})$ the form $\Psi^*\alpha \in \Omega^{n+p}(E, \Psi^*\tilde{\pi}^*\tilde{V})$ has again fibre compact support. Moreover,

$$\fint_F \Psi^*\alpha = \psi^*\left(\fint_{\tilde{F}} \alpha\right).$$

PROOF. We will generalize the proof in [**GHV72**, ch. VII, Prop. VIII] to vector valued differential forms. First, we show that $\Psi^*\alpha$ has fibre compact support. Since Ψ is an isomorphism of fibre bundles, the base map ψ is a diffeomorphism. Therefore, we can write every compact subset K of M as $\psi^{-1}(\tilde{K})$ for a compact subset $\tilde{K}$ in $\tilde{M}$. Since Ψ is a fibre bundle map we deduce that $\pi^{-1}(K) = \Psi^{-1}\tilde{\pi}^{-1}(\tilde{K})$. On the other hand, we have $\operatorname{supp}(\Psi^*\alpha) = \Psi^{-1}(\operatorname{supp}(\alpha))$ by definition of the pullback of differential forms. Thus, we obtain

$$\pi^{-1}(K) \cap \operatorname{supp}(\Psi^*\alpha) = \Psi^{-1}\left(\tilde{\pi}^{-1}(\tilde{K}) \cap \operatorname{supp}(\alpha)\right),$$

and since Ψ^{-1} is a diffeomorphism it follows that $\Psi^*\alpha$ has fibre compact support.

Therefore, the left hand side of the claimed equation is well defined. For $x \in M$ let $(\Psi^*\alpha)_x \in \Omega^n(E_x, \Lambda^p T^*_x M \otimes (\psi^*V)_x)$ be the induced differential form on the fibre E_x used in the definition of the fibre integral. The diffeomorphism ψ induces a linear isomorphism $\Lambda^p T^*_{\psi(x)}\tilde{M} \to \Lambda^p T^*_x M$ and thus a linear map

$$\psi^*_x\colon \Omega^n(E_x, \Lambda^p T^*_{\psi(x)}\tilde{M} \otimes V_{\psi(x)}) \to \Omega^n(E_x, \Lambda^p T^*_x M \otimes (\psi^*V)_x).$$

From the definitions it is immediate that

$$(\Psi^*\alpha)_x = (\psi^*_x \circ \Psi^*_x)\left(\alpha_{\psi(x)}\right)$$

for all $x \in M$. Now the map $\Psi_x\colon E_x \to \tilde{E}_{\psi(x)}$ is orientation preserving, so we can apply the transformation formula for the integral to obtain that

$$\int_{E_x} (\Psi^*\alpha)_x = \int_{E_x} (\psi^*_x \circ \Psi^*_x)\alpha_{\psi(x)} = \psi^*_x\left(\int_{\tilde{E}_{\psi(x)}} \alpha_{\psi(x)}\right),$$

where we can move ψ^*_x outside of the integral since it only acts on the values of the differential form $\Psi^*_x\alpha_{\psi(x)}$ on E_x. Thus, by definition of the fibre integral the above equation implies

$$\fint_F \Psi^*\alpha = \psi^*\left(\fint_{\tilde{F}} \alpha\right)$$

as claimed. □

For the remainder of this section we want to prove that the fibre integral commutes with the covariant exterior derivative. For that, let $\pi\colon E \to M$ be a fibre bundle with standard fibre F and let $V \to M$ be a vector bundle over M. If V is endowed with a linear connection $\nabla\colon \Gamma(V) \to \Omega^1(M, V)$, then we can construct an induced connection $\pi^*\nabla\colon \Gamma(\pi^*V) \to \Omega^1(E, \pi^*V)$ on π^*V, called the *pullback connection*. This connection is determined by the property that the following diagram commutes:

$$\begin{tikzcd} \Gamma(V) \arrow[r, "\nabla"] \arrow[d, "\pi^*"] & \Omega^1(M,V) \arrow[d, "\pi^*"] \\ \Gamma(\pi^*V) \arrow[r, "\pi^*\nabla"] & \Omega^1(E,\pi^*V) \end{tikzcd}$$

Otherwise said, we have that $\pi^*(\nabla_{T\pi\cdot\xi}s) = (\pi^*\nabla)_\xi(\pi^*s)$ for all sections $s \in \Gamma(V)$ and vector fields $\xi \in \mathfrak{X}(E)$. In particular, we can form the covariant exterior derivatives d^∇ on $\Omega^*(M, V)$ and $d^{\pi^*\nabla}$ on $\Omega^*(E, \pi^*V)$ as in section 1.2.4, which by naturality are again compatible with pullbacks of differential forms along the projection π, i.e. these operators satisfy

$$d^\nabla \circ \pi^* = \pi^* \circ d^{\pi^*\nabla}.$$

Moreover, if $R^\nabla \in \Omega^2(M, \mathrm{End}(V))$ denotes the curvature of ∇ as defined in section 1.2.3 we have $d^\nabla d^\nabla \alpha = R^\nabla \wedge \alpha$ for all $\alpha \in \Omega^k(M, V)$, c.f. Proposition 1.2.4. Since we can locally find dual frames of π^*V consisting of pullbacks it follows that $R^{\pi^*\nabla} = \pi^*R^\nabla$.

In the next Theorem we show that the fibre integral commutes with the covariant exterior derivatives. In the case of real valued differential forms or a flat connection on V, this will imply that $⨍_F$ descends to a map in de Rham cohomology, which was the initial motivation to study fibre integrals.

THEOREM 1.5.1. *Let $\pi\colon E \to M$ be an oriented fibre bundle with standard fibre F and let $V \to M$ be a vector bundle over M. Let ∇ be a connection on V and denote by $\pi^*\nabla$ the pullback connection on the pullback bundle $\pi^*V \to E$. Then the fibre integral commutes with the covariant exterior derivative, i.e.*

$$d^\nabla ⨍_F \alpha = ⨍_F d^{\pi^*\nabla}\alpha.$$

for all $\alpha \in \Omega_F(E, \pi^*V)$.

PROOF. For real valued forms, the commutation relation between the fibre integral and the exterior derivative is proven in [**GHV72**, ch. VII, Prop. X]. However, we will a give a full proof for the general case of fibre integrals of (π^*V)-valued differential forms on E.

For all $\alpha \in \Omega^{n+p}_F(E, \pi^*V)$ the value of $⨍_F \alpha$ at $x \in M$ only depends on the restriction of α to the fibre E_x, so if U is an open neighbourhood of x in M and if α_U denotes the restriction of α to $E|_U := \pi^{-1}(U)$, then the values of the fibre integrals of α and α_U coincide at $x \in M$. Thus, after possibly shrinking the open subset we can assume that U is a coordinate neighbourhood of x in M and that $\Phi\colon U \times F \to E|_U$ is a local parametrization of E. Then the smooth map Φ is a bundle isomorphism covering the identity, so Proposition 1.5.1 shows that

$$⨍_F \alpha_U = ⨍_F \Phi^*\alpha_U.$$

Therefore, we can assume that $E = M \times F$ is a trivial bundle and that there are a global coordinates $x^1, \ldots, x^m$ on M. We have seen that the space of differential forms on $M \times F$ decompose into forms of degree (p, q), and the fibre integral of such a form is trivial unless $q = n$. Moreover, we can split the exterior derivative on E into the direct sum $d^{\pi^*\nabla} = d_M + d_F$, where a (p, q)-form is mapped to a $(p+1, q)$-form by d_M and to a $(p, q+1)$-form by d_F. Then it follows that the fibre integral of the exterior derivative of $\alpha \in \Omega^{p,q}(E)$ is trivial unless $q = n-1$ or $q = n$.

In the first case consider for $x \in M$ the $(\Lambda^p T_x^* M \otimes V_x)$-valued differential form α_x on E_x of degree $n-1$ induced by α. By construction we deduce that the form α_x has compact support and satisfies the relation $d(\alpha_x) = (-1)^p (d_F \alpha)_x$. But then Stokes' Theorem implies

$$\left(\fint_F d_F \alpha \right)(x) = (-1)^p \int_{E_x} d\alpha_x = 0$$

for all $x \in M$.

Thus, it suffices to show that

$$\fint_F d_M \alpha = d^\nabla \left(\fint_F \alpha \right)$$

for all $\alpha \in \Omega^{p,n}(E)$, which we will prove in local coordinates. If $dy \in \Omega^n(F)$ is an orientation on F and $\{s_j\}$ is a local frame of V, then we can write

$$\alpha = \sum_{I,j} f_{I,j} dx^I \wedge dy \otimes \pi^* s_j,$$

where the $f_{I,j}$ are smooth functions on E, $s_I \in \Gamma(V)$ and I runs over all ordered tuples indices $(i_1, \ldots, i_p)$. We want to apply the derivative d_M to this differential form. Using the relation $(\pi^*\nabla)(\pi^* s_j) = \pi^*(\nabla s_j)$ and the antiderivation property of the covariant exterior derivative we obtain that

$$d_M \alpha = \sum_{I,j} \sum_{i=1}^{m} \frac{\partial f_{I,j}}{\partial x^i} dx^i \wedge dx^I \wedge dy \otimes \pi^* s_j + (-1)^{p+n} f_{I,j} dx^I \wedge dy \wedge \pi^*(\nabla s_j).$$

Since the partial derivatives in direction of ∂_{x^i} commute with the integral over F we can write the integral of this form as

$$\left(\fint_F d_M \alpha \right)(x) = d^\nabla \left(\sum_{I,j} \left(\int_F f_{I,j}(-, y) dy \right) dx^I \otimes s_j \right)(x) = \left(d^\nabla \fint_F \alpha \right)(x),$$

proving the claim. □

The last Theorem has an immediate consequence for the twisted de Rham cohomology on M and E. Explicitly, let the vector bundle $V \to M$ be endowed with a flat connection $\nabla \colon \Gamma(V) \to \Omega^1(M, V)$. Then the twisted de Rham sequence

$$\Gamma(V) \xrightarrow{\nabla} \Omega^1(M, V) \xrightarrow{d^\nabla} \ldots \xrightarrow{d^\nabla} \Omega^m(M, V)$$

is a complex (c.f. Proposition 1.2.4(i)) and thus we can form its corresponding cohomology groups $H^i(M, V)$. Moreover, the pullback connection on the pullback bundle is also flat, so we also obtain twisted de Rham cohomology groups $H^j(E, \pi^* V)$.

Corollary 1.5.1. *Let $E \to M$ be an oriented fibre bundle and let $V \to M$ be a vector bundle endowed with a flat connection. Then the fibre integral descends to a linear map $\fint_F^\sharp \colon H_F^{n+p}(E, \pi^* V) \to H^p(M, V)$.*

Proof. Follows immediately from Theorem 1.5.1. □

1.5.2. Fibre integration on Riemannian products. Following the general discussion we will now study fibre integrals on bundles $\pi\colon E \to M$ in the category of Riemannian manifolds. In particular, our aim is to deduce an estimate of the norm on M of fibre integrals in terms of the integral of the norm of the corresponding differential form on E.

As before, let M be a smooth manifold of dimension m, let $\pi\colon E \to M$ be a fibre bundle with n-dimensional fibre F and assume that (M, g_M) and (E, g_E) are Riemannian manifolds. For all $z \in E$ we obtain a distribution H on TE by defining $H_z \subset T_zE$ as the orthogonal complement of $\ker(T_z\pi) = \mathcal{V}_zE$ with respect to g_E. Otherwise said, H defines an *Ehresmann connection* on E, and we call a tangent vector $\xi \in T_zE$ *horizontal* if $\xi \in H_z$. Moreover, if we define the bundle $\Lambda^{p,q}T^*E := \Lambda^p H^* \otimes \Lambda^q \mathcal{V}E$, then by naturality of the exterior power we obtain a Whitney sum decomposition

$$\Lambda^k T^*E = \bigoplus_{p+q=k} \Lambda^{p,q}T^*E.$$

We denote the space of sections of $\Lambda^{p,q}T^*E$ by $\Omega^{p,q}(E)$ and call any elements in this space a *differential form of degree* (p,q). By definition, $\Omega^{p,q}(E)$ consists of those differential forms on E of degree $p+q$ which vanish after insertion of $p+1$ horizontal or $q+1$ vertical vectors. From above we also obtain a decomposition

$$\Omega^k(E) = \bigoplus_{p+q=k} \Omega^{p,q}(E)$$

on the level of sections, and every differential form α of degree (p,q) can be locally written as a sum of elements of the form $\phi \wedge \psi$, where ϕ is a local section of $\Lambda^p H^*$ and ψ is a local section of $\Lambda^q \mathcal{V}E$.

The Riemannian metric on E induces a bundle metric $\langle\ ,\ \rangle_E$ on $\Lambda^k T^*E$ for all $1 \leq k \leq m$. Since the subbundles H and VE are orthogonal in TE with respect to g_E, so are $\Lambda^p H^*$ and $\Lambda^p V^*E$ in $\Lambda^p T^*E$ for all p. Therefore, the bundle $\Lambda^{p,q}T^*E$ is orthogonal to $\Lambda^{p',q'}T^*E$ for all $(p,q) \neq (p',q')$ and $\langle\ ,\ \rangle_E$ restricts to a bundle metric $\Lambda^{p,q}E$ for all $0 \leq p \leq m$ and $0 \leq q \leq n$. Moreover, the bundle metrics on $\Lambda^k T^*E$ and $\Lambda^{p,q}T^*E$ induce pointwise inner products on the corresponding spaces of sections, which we will denote by the same symbol, as well as a pointwise norm $|\ |_E$. By construction it follows that $\Omega^{p,q}(E)$ is orthogonal to $\Omega^{p',q'}(E)$ for all $(p,q) \neq (p',q')$ as before.

Since we want the Riemannian structures on M and E to be compatible we assume further that π is a *Riemannian submersion,* i.e. that the tangent maps $T_z\pi\colon H_z \to T_{\pi(z)}M$ are orthogonal for all $z \in E$. Then the restriction of g_E to the horizontal subbundle is exactly the pullback metric of g_M along the projection.

Assume that both M and E are orientable and let vol_M and vol_E denote the corresponding volume forms. Then vol_E is a nowhere vanishing element in $\Omega^{m,n}(E)$, where $m = \dim(M)$ and $n = \dim(F)$ as before. For $x \in M$ let $\xi_1, \ldots, \xi_m$ be an orthonormal basis of T_xM. Since π is a Riemannian submersion the pullback of these vectors to any $z \in E_x$ form an orthonormal basis of H_z. Thus, insertion of these vectors into vol_E defines a vertical differential form ω_x of degree n on the fibre E_x via

$$\omega_x(z)(\eta_1, \ldots, \eta_n) := \mathrm{vol}_E(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$$

for all $\eta_1, \ldots, \eta_n \in V_zE$. In this way, we obtain a smooth differential form $\omega \in \Omega^{0,n}(E)$ of norm 1 which satisfies

$$\mathrm{vol}_E = \pi^* \mathrm{vol}_M \wedge \omega$$

by construction.

THEOREM 1.5.2. *Let (M, g_M) be an oriented Riemannian manifold and F be an n-dimensional smooth manifold. Let $\pi\colon E \to M$ be a fibre bundle with standard fibre F and assume that (E, g_E) is a Riemannian manifold so that $E \to M$ is a Riemannian submersion. Let $V \to M$ be an Euclidean bundle with bundle metric h and denote the pullback metric on π^*V by π^*h. Let $|\ |_M$ and $|\ |_E$ denote the pointwise norms on $\Omega^*(M, V)$ and $\Omega^*(E, \pi^*V)$ induced by g_M and h respectively g_E and π^*h.*

Then for all $\alpha \in \Omega^{p,n}(E, \pi^*V)$ we obtain that

$$\left| ⨍_F \alpha \right|_M \leq ⨍_F |\alpha|_E \, \omega.$$

PROOF. Since both sides of the inequality only depend on a point in M it suffices to show the inequality locally around any $x \in M$. Let U be an open subset of M with $x \in U$ so that there is a fibre bundle parametrization $\Phi_U \colon U \times F \to E|_U$ as well as a parametrization for the vector bundle V. On $U \times F$ we define the Riemannian metric g_U as the pullback of the restriction of g_E to $E|_U$ along Φ_U. Since the diffeomorphism Φ_U preserves the fibres it maps the kernel of the tangent map of the first projection $\mathrm{pr}_1 \colon U \times F \to U$ to the kernel of $T\pi$. Therefore, by definition of the metric g the same holds for its orthogonal complement, and the equation $\mathrm{pr}_1 = \pi \circ \Phi_U$ shows that the projection onto U is a Riemannian submersion. Together with Proposition 1.5.1 this implies that it suffices to derive the claimed inequality for product manifolds.

Therefore, let M be a Riemannian manifold and let $e^1, \ldots, e^m$ be a global orthonormal coframe for M. Let $E = M \times F$ be the product bundle endowed with a Riemannian metric so that the projection π onto the first factor is a Riemannian submersion. Then we can write every (π^*V)-valued (p, n)-form α on E as

$$\alpha = \sum_I \sum_j f_{I,j} e^I \wedge \omega \otimes \pi^* s_j,$$

where the $f_{I,j}$ are smooth functions on E, $s_1, \ldots, s_r$ is a global orthonormal frame of V and I runs over all ordered tuples indices $(i_1, \ldots, i_p)$.

Then the norm of the fibre integral of α at x simply reads as

$$\begin{aligned}
\left| ⨍_F \alpha \right|_M (x) &= \left| \sum_{I,j} \left(\int_{E_x} f_{I,j}(x, y) \omega_x \right) e^I(x) \otimes s_j(x) \right|_M \\
&= \left(\sum_I \sum_j \left(\int_{E_x} f_{I,j}(x, y) \omega_x \right)^2 \right)^{\frac{1}{2}} \\
&= (*)
\end{aligned}$$

Now we can interpret the collection of the maps $f_{I,j}(x, -)$ as a single smooth map $f \colon F \to \Lambda^k T_x^* M \otimes V \cong \mathbb{R}^{\mathrm{rank}(\Lambda^k TM) \cdot \mathrm{rank}(V)}$. Therefore, using the well known fact that $\| \int_W f(y) dy \| \leq \int_W \|f(y)\| dy$, where $\|\ \|$ is the standard Euclidean norm on the target of f, we further deduce

$$\begin{aligned}
(*) = \;&\leq \int_{E_x} \left(\sum_I \sum_j f_{I,j}(x,y)^2 \right)^{\frac{1}{2}} \omega_x \\
&= \int_{E_x} \left| \sum_I \sum_j f_{I,j}(x,y) e^I(x) \wedge \omega_x \otimes s_j(x) \right|_E \omega_x \\
&= \int_{E_x} |\alpha|_E \,(x,y)\ \omega_x \\
&= \left(\fint_F |\alpha|_E \,\omega \right)(x).
\end{aligned}$$

□

CHAPTER 2

Poisson transforms for differential forms

The aim of this thesis is to introduce a new version of Poisson transforms on vector-bundle valued differential forms, which is an immediate modification of Gaillards approach in [**Gai86**] in terms of geometry of homogeneous spaces. Explicitly, let G be a semisimple Lie group with finite center, K its maximal compact subgroup and P a parabolic subgroup of G. Then we construct Poisson transforms via an integral transform, whose kernel is a G-invariant differential forms on $G/K \times G/P$. However, since the latter is diffeomorphic to G/M with $M = K \cap P$, these correspond by Theorem 1.2.2 to invariant elements in a finite dimensional M-representation, c.f. Theorem 2.2.1.

Although by Theorem 2.2.2 our transform is a special case of the vector valued Poisson transforms in the sense of Definition 2.1.2, our approach is taylored to the calculus of differential forms, which allows us to control the behaviour of the composition of Poisson transforms with differential operators on G/P and G/K again at the level of the corresponding M-representations, c.f. Proposition 2.2.3. In particular, we will be able to relate the BGG-sequences of homology bundles on G/P with the de Rham complex on the Riemannian symmetric space in a G-equivariant way, c.f. Theorem 2.2.3.

Finally, as a demonstration of the strength of our approach, we present a uniform family of Poisson transforms between real valued differential forms in section 2.3.2 by explicitly constructing a Poisson kernel in terms of geometric data on G/M, and show that the images of these operators are contained in the space of coclosed differential forms on G/K. Finally, we will prove that in the special case of functions these reduce to the classical version of Poisson transforms.

2.1. Previous results on Poisson transforms

This section is dedicated to give an overview on previous results on Poisson transforms. First we introduce the classical transforms between real valued functions and afterwards consider its generalisation to sections of homogeneous vector bundles. Finally, we will present a more detailed discussion of the work of Gaillard, whose construction of Poisson transforms between differential forms was the inspiration for the topic of this thesis.

Throughout this section let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ the Cartan decomposition of $\mathfrak{g}$ with respect to a Cartan involution θ, $\mathfrak{a}_0$ a maximal abelian subspace of $\mathfrak{q}$ and $\mathfrak{m}_0$ the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}$. Let $\mathfrak{a}_0^+ \subset \mathfrak{a}_0$ be a choice of a positive Weyl chamber, which determines a positive subsystem Δ_r^+ of the restricted roots Δ_r, and let $\mathfrak{n}_0$ be the direct sum of all positive restricted root spaces. Let W be the Weyl group of $\mathfrak{g}$ determined by the choice of $\mathfrak{a}_0$ and define $\rho := \frac{1}{2}\sum_{\alpha\in\Delta_r^+} \dim(\mathfrak{g}_\alpha)\alpha \in \mathfrak{a}_0^*$. Let G be a connected semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and let K, A_0, M_0 and N_0 be the Lie subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}_0$, $\mathfrak{m}_0$ and $\mathfrak{n}_0$, respectively. Let Θ be the global Cartan involution corresponding to θ. We denote by $g = \kappa(g)\exp(H(g))\eta(g)$ the Iwasawa decomposition of $g \in G$, where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}_0$ and $\eta(g) \in \mathfrak{n}_0$. Moreover, the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G with Lie algebra $\mathfrak{p}_0 := \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$.

2.1.1. Real valued Poisson transforms. In the field of harmonic analysis the objects of interest are differential operators $D\colon C^\infty(G/K) \to C^\infty(G/K)$ on symmetric spaces G/K which are equivariant with respect to the induced G-actions. In particular, some of the main points in the program proposed by Helgason (see [**Hel77**]) were to find joint eigenfunctions for the space $\mathbf{D}(G/K)$ of all G-invariant differential operators as well as to describe the joint eigenspaces E_χ for a given eigenfunction $\chi\colon \mathbf{D}(G/K) \to \mathbb{C}$ as G-representations. Via the Harish-Chandra isomorphism Ξ ([**Hel94**, Theorem II.2.1]) we can identify $\mathbf{D}(G/K)$ with the space $\mathfrak{u}(\mathfrak{a}_0)^W$ of elements in the universal enveloping algebra of $\mathfrak{a}_0$ which are invariant under the Weyl group. In particular, every eigenfunction is determined by a linear functional $\lambda \in (\mathfrak{a}_0)^*_\mathbb{C}$ via $\chi_\lambda(D) := \Xi(D)(\lambda)$.

We can interpret the homogeneous space $G/P_0 \cong K/M$ as the Fürstenberg boundary of the symmetric space G/K, see [**BJ06**, ch. I.6]. Now the idea of the Poisson transform is to extend functions on this boundary to G/K via an integral transform.

Definition 2.1.1. [**Hel94**, p.119] For every $\lambda \in (\mathfrak{a}_0)^*_\mathbb{C}$ we define the *Poisson transform* as the linear map $P_\lambda\colon C(K/M) \to C^\infty(G/K)$ defined by

$$P_\lambda(f)(gK) = \int_{K/M} e^{-(i\lambda+\rho)(H(g^{-1}k))} f(kM) dkM,$$

where dkM is the normalized volume on the symmetric space K/M.

Note that this definition can be naturally extended to hyperfunctions on the Fürstenberg boundary. Moreover, we will see in Proposition 2.3.3 that this transform can be interpreted as a G-equivariant map defined on sections of density bundles on G/P_0. In particular, the parameter λ origins from a principal series representation on the parabolic boundary geometry, see also [**KZ11**, section 4.1].

The key property of the Poisson transform is that it produces joint eigenfunctions for the space of G-invariant differential operators on G/K. More explicitly, we have the following result.

Proposition 2.1.1. *For every $\lambda \in (\mathfrak{a}_0)^*_\mathbb{C}$ the Poisson transform P_λ maps $C(K/M)$ into the space*

$$E_\lambda := \{f \in C^\infty(G/K) \mid D(f) = \Xi(D)(i\lambda)(f) \text{ for all } D \in \boldsymbol{D}(G/K)\}.$$

In particular, the images of the Poisson transform are eigenvectors of the Laplace Beltrami operator on G/K with eigenvalue $\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle$.

Proof. Follows from [**Hel94**, Proposition II.3.14] as well as the explicit formula for Ξ. □

A natural question concerns the bijectivity of the Poisson transform depending on the linear functional λ. This was first answered for the case of real rank 1 by Helgason in [**Hel74**] and in the general case by Kashiwara et al in [**KKM+78**]. Here, the bijectivity of the Poisson transform is related to the value of the Harish-Chandra $\mathbf{c}$-function defined by

$$\mathbf{c}(\lambda) := \int_{\Theta(N)} e^{-(i\lambda+\rho)(H(\overline{n}))} d\overline{n}.$$

We will not need the explicit formulation of the Theorem on the bijectivity in the sequel; However, as a small result in this direction we will state the result showing the existence of boundary values of Poisson transforms.

THEOREM 2.1.1. *Suppose $\lambda \in (\mathfrak{a}_0)^*_{\mathbb{C}}$ satisfies* $\mathrm{re}(\langle i\lambda, \alpha\rangle) > 0$ *for all* $\alpha \in \Delta_r^+$. *Then if* $H \in \mathfrak{a}_0^+$ *and* $a_t = \exp(tH)$ *for* $t \in \mathbb{R}_+$, *then*

$$\lim_{t\to+\infty} e^{(-i\lambda+\rho)(tH)} P_\lambda(f)(a_t K) = \boldsymbol{c}(\lambda) f(eM)$$

for all $f \in C(K/M)$.

PROOF. [**Hel94**, Theorem II.3.16]. □

2.1.2. Vector valued Poisson transforms. Due to its unique properties the Poisson transform was generalized to a map between sections of homogeneous vector bundles. Explicitly, let $\mathbb{W}_\tau$ be a finite dimensional K-representation with induced homogeneous vector bundle $W_\tau \to G/K$. Then due to section 1.2.2 its space of sections $\Gamma(W_\tau)$ can be identified with the space $C^\infty(G, \mathbb{W}_\tau)^K$ of smooth K-equivariant maps $G \to \mathbb{W}_\tau$. Moreover, let $\mathbb{V}_\sigma$ be a finite dimensional P-representation with associated vector bundle $V_\sigma \to G/P$ and consider the induced representation

$$C^\infty(G, \mathbb{V}_\sigma)^{P,\rho} = \{f \colon G \to \mathbb{V}_\sigma \mid f(gp) = e^{-\rho(H(p))}\sigma(p)^{-1} f(g)\},$$

where we included a ρ-shift following standard conventions. By Theorem 1.4.5, this representation is isomorphic to the space of sections of the bundle $V_\sigma \otimes \mathcal{E}[\rho] \to G/P$, where $\mathcal{E}[\rho]$ is the density bundle over G/P determined by $\rho \in \mathfrak{a}_0^*$.

The following definition of Poisson transforms was used by van der Ven in [**vdV94**] for the case of $\mathrm{rank}(G/K) = 1$ and in general by Yang in [**Yan94**]. Moreover, via the arguments used in the proof of Proposition 2.3.3 it can be shown to be equivalent to the definition given by Olbrich in [**Olb95**].

DEFINITION 2.1.2. Let $\phi \in \mathrm{Hom}_M(\mathbb{V}_\sigma, \mathbb{W}_\tau)$ be an M-equivariant linear map. Then we define the *Poisson transform* $P_\phi \colon C^\infty(G, \mathbb{V}_\sigma)^{P,\rho} \to C^\infty(G, \mathbb{W}_\tau)^K$ associated to ϕ via

$$\Phi_\phi(f)(gK) = \int_K (\tau(k) \circ \phi)(f(gk))\, dk,$$

where dk is the normalized Haar measure on the compact Lie group K.

Note that the M-invariance of the integrand allows us to change the integration to the symmetric space K/M, endowed with the K-left invariant measure induced by the Haar measures on K and M. We will see in Theorem 2.2.2 that our version of Poisson transform between vector valued differential forms is a special case of Definition 2.1.2.

For vector valued Poisson transforms the same questions regarding eigendecompositions and bijectivity were asked. Again, in our context it will be sufficient to give the result on the existence of boundary values. For that, consider the generalized Harish-Chandra **c**-function defined by

$$\mathbf{c}(\lambda) = \int_{\Theta(N)} \tau(k(\overline{n})) e^{-(\lambda+\rho)(H(\overline{n}))} d\overline{n}$$

for all $\lambda \in (\mathfrak{a}_0^*)_{\mathbb{C}}$.

THEOREM 2.1.2. *Let* $\lambda \in (\mathfrak{a}_0^*)_{\mathbb{C}}$ *be such that* $\mathrm{re}(\langle \lambda, \alpha\rangle) > 0$ *for all* $\alpha \in \Delta_r^+$. *Then for all* $\phi \in \mathrm{Hom}_M(\mathbb{V}_\sigma, \mathbb{W}_\tau)$ *and* $H \in \mathfrak{a}_0^+$ *we have*

$$\lim_{t\to+\infty} e^{(-\lambda+\rho)(tH)} P_\phi(f)(g\exp(tH)) = \boldsymbol{c}(\lambda)\phi(f(g)).$$

PROOF. [**Yan94**, Prop. 2.4]. □

Note that in the last section we used the parameter $i\lambda \in (\mathfrak{a}_0^*)_{\mathbb{C}}$, whereas in the vector valued case the multiplication with the imaginary unit is usually dropped.

In particular, we can apply this construction to the representations $\mathbb{W}_\tau = \Lambda^\ell(\mathfrak{g}/\mathfrak{k})^*$ and $\mathbb{V}_\sigma = \Lambda^k(\mathfrak{g}/\mathfrak{p})^*[-\rho]$, obtaining for every $\phi \in \operatorname{Hom}_M(\mathbb{V}_\sigma, \mathbb{W}_\tau)$ a Poisson transform $P_\phi \colon \Omega^k(G/P) \to \Omega^\ell(G/K)$. However, from this point of view it is not clear if such a M-invariant linear map ϕ exists. Furthermore, for a given ϕ the relation between the Poisson transform P_ϕ and various differential operators used in the exterior calculus on G/P and G/K is not obvious.

2.1.3. Gaillard's transform. A different version of a Poisson transform on differential forms in the case of even dimensional real hyperbolic space was developed by Pierre-Yves Gaillard in this dissertation, see [**Gai86**]. Specifically built in the language of exterior calculus, his approach enabled him to relate his Poisson transform with several differential operators quite easily. Since his construction was the main inspiration for the point of view we pursue in this thesis, we will give a more detailed description of his work.

The key ingredient for the construction of Gaillard's transform was an idea by Thurston in [**Thu**, chapter 11], where he studied the interaction between the geometries of the hyperbolic space H^n and its associated boundary ∂H^n via so called visual maps. In order to define them, consider the Poincaré ball model of the hyperbolic space with its boundary given by the unit sphere in $\mathbb{R}^n$, c.f. section 3.1.2. To every $x \in H^n$ and every ξ in the unit sphere S_x of the tangent space at x, the visual map V_x assigns the boundary point $y \in \partial H^n$ given by the (Euclidean) limit of the unique geodesic ray $\gamma_\xi \colon \mathbb{R}_+ \to H^n$ defined by ξ. Due to the negative curvature of the hyperbolic space, the maps V_x are G-isomorphisms for the natural action of the group $G = \mathrm{SO}(n,1)$, c.f. section 3.2.3. Thus, they can be used to relate the natural conformal structure on ∂H^n with the Riemannian structure on H^n.

Definition 2.1.3. Let α be a k-form on ∂H^n and for $x \in H^n$ let $\xi_1, \dots, \xi_k$ be vectors in $T_x H^n$. Choose a point $u \in S_x$, translate the vectors ξ_i to u and project them orthogonally to obtain vectors $\zeta_1, \dots, \zeta_k$ in $T_u S_x$. Let $|S^{n-1}|$ be the volume of the $(n-1)$-dimensional unit sphere, vol be the hyperbolic volume form on H^n and ν be the normal unit vector field on S_x. Then we define the *Poisson transform* $P_k \colon \Omega^k(\partial H^n) \to \Omega^k(H^n)$ via

$$P_k(\alpha)(x)(\xi_1, \dots, \xi_k) := \frac{1}{|S^{n-1}|} \int_{S_x} (V_x^* \alpha)(\zeta_1, \dots, \zeta_k) \ (\iota_\nu \operatorname{vol})(x).$$

Gaillard extended this definition to hyperforms on ∂H^n with values in density bundles. Subsequently, he showed that it can be expressed as an integral transform, whose kernel ϕ_k is an $\mathrm{O}(n)$-invariant differential form of degree $(k, n-1-k)$ on $H^n \times \partial H^n$, i.e.,

$$P_k(\alpha) = \int_{\partial H^n} \phi_k \wedge \alpha,$$

see [**Gai86**, Lemme 1]. Moreover, he explicitly computed the kernel in terms of coordinates on the ball model and used it to obtain information on the behaviour of his transform with respect to exterior derivatives and the Riemannian codifferential. In particular, he showed that the image of his transform is contained in the space of harmonic differential forms on H^n.

The main tool for his studies was the "lemme fondamentale" [**Gai86**, chapter 4], which expresses the Poisson transform in terms of the polar decomposition of H^n on the Poincaré ball model. Using this, he was able to show that his operator has boundary values for all degrees lower than $n/2$. Moreover, using the explicit formula for the Poisson kernel J. Lott showed by a direct computation the following condition on L^2-integrability of Poisson transforms.

THEOREM 2.1.3. *If n is even then up to a constant, $P_{\frac{n}{2}}$ is an isometric isomorphism between exact $\frac{n}{2}$-forms on S^{n-1} which are Sobolev $H^{-\frac{1}{2}}$-regular, and L^2-harmonic $\frac{n}{2}$-forms on H^n.*

PROOF. [**Lot00**, Theorem 1]. □

In order to get a clearer connection to the classical definition of Poisson transforms on smooth functions A. Juhl reformulated in [**Juh87**, Definition 1] Gaillard's operator in terms of equivariant maps. In this way, he suggested that this construction should be generalizable to other hyperbolic spaces as well as differential forms with values in a natural vector bundle. In particular, this reformulation showed that Gaillard's transform is a special case of the vector valued version of the Poisson transform as in Definition 2.1.2.

It should also be noted that Pierre Julg proved in his thèse d'État [**Jul91**] that there is a boundary value operator on L^2-harmonic forms in middle degree on real and complex hyperbolic space, which is an isometry for a Sobolev norm on the boundary. In [**Pan93**], Pansu discussed these operators and posed the question whether there is a Poisson transform on differential forms on the complex hyperbolic space, which is also G-equivariant, commutes with the exterior derivative and has boundary values, and whose image consists of all harmonic forms on G/K, generalizing Gaillard's operator. This was partly answered by Gaillard himself in [**Gai78**] for differential forms with degree smaller than half of the dimension of complex hyperbolic space, and recently by M. Olbrich, which was communicated through a private conversation. We will address this question in section 4.

2.2. Construction and properties of Poisson transforms

This section is dedicated to the introduction and general discussion of a new approach to Poisson transforms between vector valued differential forms on homogeneous parabolic geometries G/P and vector valued differential forms on Riemannian symmetric spaces G/K, which is the main aim of this thesis. Explicitly, these will be defined in 2.2.1 as G-equivariant integral operators, whose kernels are G-invariant vector valued differential forms on the product space $G/K \times G/P$. Since the latter space is itself a homogeneous space, Poisson transforms are in bijective correspondence with invariant elements in finite dimensional representations of semisimple Lie groups, c.f. Theorem 2.2.1. Afterwards, we will provide in section 2.2.2 two alternative descriptions of these operators, which will imply that our Poisson transforms are special case of the vector valued transforms as in 2.1.2. However, since our approach is tailored to the calculus of differential forms, we can express compositions of Poisson transforms with several differential operators again as Poisson transforms, c.f. Proposition 2.2.3, and thus compute their properties on the level of the corresponding representations. In particular, this enables us to explicitly construct transforms which factor to G-equivariant operators defined on sections of the corresponding homology bundles over G/P, whose image is automatically contained in the space of harmonic differential forms on G/K, c.f. Theorem 2.2.3.

2.2.1. General construction. Let G be a connected semisimple Lie group with finite centre and let θ be a Cartan involution on its Lie algebra $\mathfrak{g}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ with respect to a θ-stable Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be the corresponding $|k|$-grading of $\mathfrak{g}$. Let $K \subset G$ be the maximal compact subgroup with Lie algebra $\mathfrak{k}$ and P be the parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Then G/K is a Riemannian symmetric space and G/P is a homogeneous parabolic geometry. Furthermore, using the Iwasawa decomposition of G we see that the compact group K acts transitively on G/P. (see e.g. [**ČS09**, Proposition 3.2.6]). This action induces a K-isomorphism $G/P \cong K/M$, where M denotes the Lie group $K \cap P$ with Lie algebra $\mathfrak{m}$. Explicitly, this isomorphism is given by $gP \mapsto \kappa(g)M$, where $g = \kappa(g)a(g)\eta(g)$ is the Iwasawa decomposition of the element $g \in G$. In particular, the homogeneous space G/P is compact.

The G-action on the product space $G/K \times G/P$ is therefore seen to be transitive. Indeed, for all $g_1,\ g_2 \in G$ the element $h := g_1\kappa(g_1^{-1}g_2)$ satisfies $hK = g_1K$ as well as $hP = g_1\kappa(g_1^{-1}g_2)P = g_2P$. Moreover, the kernel of the point (eK, eP) is exactly M, so we get a diffeomorphism $\Psi\colon G/M \xrightarrow{\sim} G/K \times G/P$ of G-manifolds and a double fibration

$$\begin{tikzcd} & G/M \arrow[dl, "\pi_K"'] \arrow[dr, "\pi_P"] & \\ G/K & & G/P \end{tikzcd}$$

The derivative of Ψ induces a splitting of $\mathfrak{g}/\mathfrak{m}$ into the direct sum of the M-representations $(\mathfrak{g}/\mathfrak{m})^{1,0}$ and $(\mathfrak{g}/\mathfrak{m})^{0,1}$ corresponding to $\mathfrak{g}/\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{p}$, respectively. Therefore, we obtain a decomposition of the tangent bundle $T(G/M)$ into the Whitney sum of the two G-invariant subbundles $T^{1,0} := G \times_M (\mathfrak{g}/\mathfrak{m})^{1,0}$ and $T^{0,1} := G \times_M (\mathfrak{g}/\mathfrak{m})^{0,1}$. By construction, the tangent maps of the canonical projections π_K and π_P are pointwise isomorphisms between $T^{1,0}$ and $T(G/K)$ as well as $T^{0,1}$ and $T(G/P)$, respectively. Otherwise put, the distribution $T^{0,1}$ is the vertical bundle of the fibre bundle $G/M \to G/K$, whereas $T^{1,0}$ defines an Ehresmann connection on $T(G/M)$.

Via the projections of $T(G/M)$ onto its factors we can view the vector bundles $\Lambda^{p,q}T^*(G/M) := \Lambda^pT^{1,0*} \otimes \Lambda^qT^{0,1*}$ as subbundles in $\Lambda^{p+q}T(G/M)$. We say that any element in $\Lambda^{p,q}T^*(G/M)$ is of *degree* (p,q). By compatibility of the exterior product with the Whitney sum we obtain a decomposition

$$\Lambda^kT^*(G/M) = \bigoplus_{p+q=k} \Lambda^{p,q}T^*(G/M)$$

for all $0 \le k \le \dim(G/M)$. Moreover, we denote by $\Omega^{p,q}(G/M)$ the space of sections of the bundle $\Lambda^{p,q}T^*(G/M)$ and say that a differential form $\omega \in \Omega^{p,q}(G/M)$ has *degree* (p,q), as well as K-*degree* $|\omega|_K := p$ and P-*degree* $|\omega|_P := q$. Note that the differential forms of degree (p,q) are exactly those $(p+q)$-forms on G/M which vanish at every point upon insertion of either $p+1$ horizontal or $q+1$ vertical vectors.

Let $\mathbb{V}$ be a P-representation with associated bundle V over G/P and $\mathbb{W}$ be a K-representation with associated bundle W over G/K. Moreover, denote the bundles associated to $\mathbb{V}$ and $\mathbb{W}$ over G/M by V_M and W_M, respectively. Then the vector space $\mathrm{L}(\mathbb{V},\mathbb{W})$ of linear maps from $\mathbb{V}$ to $\mathbb{W}$ is canonically an M-representation, and we denote by $\mathrm{L}(V_M, W_M)$ its associated bundle over G/M. If we consider a k-form α on G/P with values in V, then we can form the pullback of α along the projection π_P to get a $(0,k)$-form on G/M with values in the pullback bundle π_P^*V, which by Corollary 1.2.1 coincides with the associated bundle V_M.

Let $\chi\colon P \to \mathbb{R}_+$ be a character on P and let $\mathcal{E}[\chi]$ be the corresponding density bundle, c.f. section 1.4.5. Every χ-density $\sigma \in \Gamma(\mathcal{E}[\chi])$ corresponds to a smooth, P-equivariant map $f\colon G \to \mathbb{R}[\chi]$. From Theorem 1.4.5 we obtain that f can be viewed as an M-invariant map, and we call the induced smooth function $\pi_P^*\sigma \in C^\infty(G/M)$ given by $\pi_P^*\sigma(gM) := f(g)$ the *pullback of the* χ-*density* σ. In particular, if we define $V[\chi] := V \otimes \mathcal{E}[\chi]$ for all homogeneous vector bundles $V \to G/P$, we can form the pullback of any section $t \in \Gamma(V[\chi])$ along the projection π_P, which is a section of the vector bundle V_M over G/M and compatible with the G-actions.

Let $\alpha \in \Omega^k(G/P, V[\chi])$ and $\phi_{k,\ell}$ be a $(\ell, n-k)$-form on G/M with values in $\mathrm{Hom}(V_M, W_M)$, where n is the dimension of the homogeneous space G/P. Then we can interpret the wedge product $\phi_{k,\ell} \wedge \pi_P^*\alpha$ as a W_M-valued differential form on G/M of degree (ℓ, n) simply by applying $\phi_{k,\ell}$ to $\pi_P^*\alpha$ after insertion of vector fields on G/M. Moreover, the manifold G/P is compact, so if we assume that it is oriented we can integrate this form along the fibre to get a k-form on G/K with values in the vector bundle W.

In this way, we obtain a linear transform

$$\Phi_{k,\ell} \colon \Omega^k(G/P, V[\chi]) \to \Omega^\ell(G/K, W)$$

via the formula

$$\Phi_{k,\ell}(\alpha) := \fint_{G/P} \phi_{k,\ell} \wedge \pi_P^*\alpha.$$

We call $\phi_{k,\ell}$ the *kernel* of the transform $\Phi_{k,\ell}$.

As we have seen in section 1.2.2 we obtain an induced G-action on the space of $\mathrm{Hom}(V_M, W_M)$-valued differential forms on G/M. The next lemma derives a correspondence between G-equivariant integral transformations of the above kind and invariance properties of the kernel of the operator.

LEMMA 2.2.1. *Using the notation from above, the integral transform*

$$\Phi_{k,\ell} \colon \Omega^k(G/P, V[\chi]) \to \Omega^\ell(G/K, W)$$

is G-equivariant if and only if its kernel $\phi_{k,\ell}$ is G-invariant.

PROOF. The projection $\pi_P \colon G/M \to G/P$ is G-equivariant, so the pullback along this map commutes with the G-actions on the sections of the bundles $V[\chi]$ and $\pi_P^*(V[\chi]) = V_M$ as well as with the pullback of differential forms along the G-actions on G/M and G/P, respectively. Next, for sections $s \in \Gamma(V_M)$ and $\varphi \in \Gamma(\mathrm{Hom}(V_M, W_M))$ we obtain that $g \cdot \varphi(s) = (g \cdot \varphi)(g \cdot s)$ for all $g \in G$, where the dots denote the actions on the appropriate associated bundles. Finally, Proposition 1.5.1 shows that the fibre integral commutes with pullbacks along smooth functions.

Therefore, if $\phi_{k,\ell}$ is G-invariant, we obtain for all $g \in G$ and $V[\chi]$-valued k-forms α on G/P that

$$g \cdot \Phi_{k,\ell}(\alpha) = \fint_{G/P} (g \cdot \phi_{k,\ell}) \wedge (g \cdot \pi_P^*\alpha) = \fint_{G/P} \phi_{k,\ell} \wedge \pi_P^*(g \cdot \alpha) = \Phi_{k,\ell}(g \cdot \alpha),$$

so $\Phi_{k,\ell}$ is G-equivariant.

On the other hand, if the transformation $\Phi_{k,\ell}$ is G-equivariant, we see that

$$\fint_{G/P} (g \cdot \phi_{k,\ell}) \wedge (g \cdot \pi_P^*\alpha) = g \cdot \Phi_{k,\ell}(\alpha) = \Phi_{k,\ell}(g \cdot \alpha) = \fint_{G/P} \phi_{k,\ell} \wedge (g \cdot \pi_P^*\alpha)$$

for all $V[\chi]$-valued k-forms α on G/P. Since the wedge product is non-degenerate, this implies that the kernel $\phi_{k,\ell}$ is G-invariant. $\square$

DEFINITION 2.2.1. With the notation from above, we call a G-invariant differential form $\phi_{k,\ell} \in \Omega^{\ell, n-k}(G/M, \mathrm{Hom}(V_M, W_M))$ a *Poisson kernel* and the corresponding G-equivariant integral transformation

$$\Phi_{k,\ell} \colon \Omega^k(G/P, V[\chi]) \to \Omega^\ell(G/K, W), \qquad \alpha \mapsto \fint_{G/P} \phi_{k,\ell} \wedge \pi_P^*\alpha$$

a *Poisson transform.*

As a G-invariant section of a homogeneous vector bundle over G/M every Poisson kernel is determined by its value over the base point $eM \in G/M$ due to Theorem 1.2.2. In particular, we immediately obtain the following Theorem.

THEOREM 2.2.1. *There is a bijective correspondence between Poisson transforms* $\Phi_{k,\ell}\colon \Omega^k(G/P, V[\chi]) \to \Omega^\ell(G/K, W)$ *and the vector space of M-invariant elements in the finite dimensional representation* $\Lambda^{\ell,n-k}(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{L}(\mathbb{V}, \mathbb{W})$.

Using the last Theorem we obtain Poisson transforms simply by constructing invariant vectors in a finite dimensional M-representation. Moreover, we will show in Proposition 2.2.3 that the composition of a Poisson transform with several differential operators is again a Poisson transform, whose kernel can be determined at the level of the corresponding M-module.

2.2.2. Alternative descriptions of Poisson transforms. We continue our discussion by giving two alternative descriptions of our Poisson transforms. One of them provides a geometric interpretation and thereby creates a link to Gaillard's initial construction in terms of visual maps as in Definition 2.1.3. From this we derive an equivalent description in terms of equivariant functions, proving that our definition can be seen as a special case of the vector valued Poisson transform in section 2.1.2.

First we have to show that there is a unique vertical form on G/M which serves as a volume form for every fibre of $G/M \to G/K$.

LEMMA 2.2.2. *There exists a Poisson kernel* $\mathrm{vol} \in \Omega^{0,n}(G/M)$ *which satisfies* $\int_{K/M} \mathrm{vol} = 1$.

PROOF. By Theorem 1.2.2 it suffices to construct M-invariant elements in the 1-dimensional representation $\Lambda^n(\mathfrak{g}/\mathfrak{m})^{0,1*}$. However, compactness of M and orientability of G/P implies that every element in this representation is M-invariant, so we obtain a family of G-invariant differential forms of degree $(0, n)$ on G/M which differ by a constant factor. In particular, there is a unique form vol whose integral over $\pi_K^{-1}(eK) = K/M$ equals 1. $\square$

Let $\mathbb{V}$ be a finite dimensional P-representation with associated vector bundle $V \to G/P$ and let $\mathbb{W}$ be a finite dimensional K-representation with associated vector bundle $W \to G/K$. If $p_{k,\ell}\colon \Lambda^k T^{0,1*} \otimes V_M^* \to \Lambda^\ell T^{1,0*} \otimes W_M$ is a G-equivariant bundle map we define for all $\alpha \in \Omega^k(G/P, V[\chi])$ the W-valued differential form $P_{k,\ell}(\alpha)$ on G/K of degree ℓ via

$$P_{k,\ell}(\alpha) := \int_{G/P} p_{k,\ell}(\pi_P^* \alpha) \otimes \mathrm{vol}\,.$$

Then invariance of the volume form vol on G/P implies that the induced linear map $P_{k,\ell}\colon \Omega^k(G/P, V[\chi]) \to \Omega^\ell(G/K, W)$ is G-equivariant.

PROPOSITION 2.2.2. *The set of* $\mathrm{Hom}(V_M, W_M)$*-valued Poisson kernels on G/M of degree $(\ell, n-k)$ is in bijective correspondence with the space of G-invariant elements in*

$$\mathrm{Hom}_{k,\ell} := \mathrm{Hom}(\Lambda^k T^{0,1*} \otimes V_M, \Lambda^\ell T^{1,0*} \otimes W_M).$$

Moreover, if ϕ is such a Poisson kernel with associated Poisson transform Φ and corresponding G-equivariant bundle map $p \in \mathrm{Hom}_{k,\ell}^G$, then

$$\Phi(\alpha) = \int_{G/P} p(\pi_P^* \alpha) \otimes \mathrm{vol}$$

for all $\alpha \in \Omega^k(G/P, V[\chi])$.

PROOF. For two M-representations X and Y with $\dim(X) = n$ consider the two linear maps

$$X \otimes Y^* \to \mathrm{L}(X^*, Y^*), \qquad v \otimes \phi \mapsto (\psi \mapsto \psi(v)\phi)$$
$$\Lambda^k X \otimes \Lambda^n X^* \to \Lambda^{n-k} X^*, \qquad v \otimes \alpha \mapsto \iota_v \alpha,$$

which both are M-equivariant and bijective. Composing these linear maps we obtain an isomorphism

$$\mathrm{L}(\Lambda^k X^*, \Lambda^\ell Y^*) \otimes \Lambda^n X^* \to \Lambda^{n-k} X^* \otimes \Lambda^\ell Y^*.$$

of M-modules.

In particular, for $X = (\mathfrak{g}/\mathfrak{m})^{0,1}$ and $Y = (\mathfrak{g}/\mathfrak{m})^{1,0}$ this yields a G-equivariant isomorphism

$$\Gamma(\mathrm{Hom}_{k,\ell}) \otimes \Omega^{0,n}(G/M) \to \Omega^{\ell,n-k}(G/M, \mathrm{L}(V_M, W_M)).$$

For $p \in \mathrm{Hom}^G_{k,\ell}$ and vol the G-invariant $(0,n)$-form from Lemma 2.2.2 the tensor product $p \otimes \mathrm{vol}$ is G-invariant and therefore also its image under the above isomorphism, which defines a Poisson kernel ϕ of degree $(\ell, n-k)$. Moreover, by definition we have

$$p(\pi_P^* \alpha) \otimes \mathrm{vol} = \phi \wedge \pi_P^* \alpha$$

for all $\alpha \in \Omega^k(G/P, V[\chi])$, and integrating both sides yields the claimed formula. □

We can use Proposition 2.2.2 to construct Poisson transforms between real valued differential forms as follows: For all $z \in G/M$ we can identify the vector spaces $T_z^{1,0}$ with $T_{\pi_K(z)}(G/K)$ as well as $T_z^{0,1}$ with $T_{\pi_P(z)}(G/P)$ in a G-equivariant way. Then Proposition 2.2.2 implies that every Poisson transform between real valued differential forms is obtained by specifying a G-equivariant family of linear maps $p_{x,y} \colon \Lambda^k T_y^*(G/P) \to \Lambda^\ell T_x^*(G/K)$, which depends smoothly on $x \in G/K$ and $y \in G/P$. In order to construct this set of maps we can use geometric objects and methods. As an example, Gaillard used the visual maps on the real hyperbolic space to construct such a family, c.f. section 2.1.3. Note that since G acts transitively on the product manifold $G/K \times G/P$, the family $\{p_{x,y}\}$ is determined by a single M-equivariant map $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \to \Lambda^\ell(\mathfrak{g}/\mathfrak{k})^*$ due to Theorem 1.2.2.

Furthermore, we can use Proposition 2.2.2 to obtain an expression of our Poisson transform in terms of equivariant maps. For that, let $\alpha \in \Omega^k(G/P, V[\chi])$ correspond to the P-equivariant map $f \colon G \to \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{V}[\chi]$. Then the M-equivariant map $\pi_P^* f$ corresponding to the pullback of α along the canonical projection is given by the composition of f with the isomorphism $\mathfrak{g}/\mathfrak{p} \cong (\mathfrak{g}/\mathfrak{m})^{0,1}$. Moreover, for any $p \in \mathrm{Hom}^G_{k,\ell}$ the composition $p \circ \pi_P^* f$ defines an M-equivariant map $G \to \Lambda^\ell(\mathfrak{g}/\mathfrak{m})^{1,0*} \otimes \mathbb{W}$, whose target space is a K-representation σ via the identification of $(\mathfrak{g}/\mathfrak{m})^{1,0}$ with $\mathfrak{g}/\mathfrak{k}$. Since K is compact, we can average the K-action on $p \circ \pi_P^* f$ and obtain a smooth, K-equivariant map

$$\Phi(f) \colon G \to \Lambda^k(\mathfrak{g}/\mathfrak{k}) \otimes \mathbb{W},$$
$$g \mapsto \int_K (\sigma(k) \circ p) \pi_P^* f(gk)\, dk,$$

where dk is the normalized Haar measure on the Lie group K.

THEOREM 2.2.2. *Let ϕ be a $\mathrm{Hom}(V_M, W_M)$-valued Poisson kernel of degree $(\ell, n-k)$ with corresponding Poisson transform Φ and corresponding G-equivariant map $p \in \mathrm{Hom}^G_{k,\ell}$ as in Proposition 2.2.2. Let α be a $V[\chi]$-valued differential form on G/P with associated P-equivariant map f. Let σ be the induced representation of K on $\Lambda^k(\mathfrak{g}/\mathfrak{m})^{1,0*} \otimes \mathbb{W}$ and dk the normalized Haar measure on K.*

Then the K-equivariant map $G \to \Lambda^\ell(\mathfrak{g}/\mathfrak{k})^* \otimes \mathbb{W}$ corresponding to $\Phi_{k,\ell}(\alpha)$ is given by

$$g \mapsto \int_K (\sigma(k) \circ p)\pi_P^* f(gk)\ dk.$$

Proof. Let $\tilde{f}\colon G \to \Lambda^k(\mathfrak{g}/\mathfrak{k})^* \otimes \mathbb{W}$ be the smooth, K-equivariant map corresponding to $\Phi(\alpha)$, which can be computed via $\tilde{f}(g) = \big(g^{-1} \cdot \Phi(\alpha)\big)(eK)$ for all $g \in G$, where the dot denotes the induced G-action on sections of the homogeneous vector bundle $\Lambda^k T^*(G/K) \otimes W$, c.f. section 1.2.2. Applying Proposition 2.2.2 we obtain that

$$\tilde{f}(g) = \left(g^{-1} \cdot \left(\fint_{G/P} p(\pi_P^*\alpha) \otimes \mathrm{vol}\right)\right)(eK) = (*),$$

where vol is the unique real-valued Poisson kernel of degree $(0,n)$ constructed in Lemma 2.2.2. Now we have seen in section 1.2.2 that the G-action on $\Phi(\alpha)$ is given by the composition of the pullback along left translations and the induced action on the homogeneous bundle W, so by Proposition 1.5.1 we can commute this action with the fibre integral. Thus, we can write

$$(*) = \int_{K/M} \big(g^{-1} \cdot (p(\pi_P^*\alpha) \otimes \mathrm{vol})\big)(kM),$$

where we used that the fibre of $G/M \to G/K$ over the base point eK equals K/M. By G-invariance of the volume form the above expression becomes

$$(*) = \int_{K/M} \big(g^{-1} \cdot p(\pi_P^*\alpha)\big)(kM) \otimes \mathrm{vol}_{K/M},$$

and the integrand is equal to $(\sigma(k) \circ p)(\pi_P^* f)(gk)$ in terms of the corresponding M-equivariant maps. Finally, the volume form $\mathrm{vol}_{K/M}$ induces a K-left invariant measure on K/M so that its volume equals 1. However, it is well known (see e.g. [**Hel84**, Prop. I.1.8 and Theorem I.1.9]) that such a measure on K/M is constructed from normalized Haar measures of the compact Lie groups K and M, respectively, and that integrals over K can be expressed in terms of integrals over K/M and M. Since the function $p \circ f$ is M-equivariant we obtain that

$$\tilde{f}(g) = \int_K (\sigma(k) \circ p)\pi_P^* f(gk)\ dk.$$

□

In particular, Theorem 2.2.2 shows that our construction is a special case of the classical vector valued Poisson transform in the sense of Definition 2.1.2.

As a Corollary, we want to apply the above formula to the Poisson kernel vol of degree $(0,n)$ defined in Lemma 2.2.2 in the case of a minimal parabolic subgroup P, which will be used later to prove Proposition 2.3.3. Recall from Theorem 1.4.5 that in the case of a minimal parabolic subgroup P of a semisimple Lie group G any density bundle on G/P is determined by a linear functional $\lambda \in \mathfrak{a}_0^*$. In this case we denote the corresponding 1-dimensional P-representation by $\mathbb{R}[\lambda]$, its associated bundle by $\mathcal{E}[\lambda]$ and call any section of $\mathcal{E}[\lambda]$ a λ-*density*. Explicitly, if $p = m\exp(H(p))n$ is the Iwasawa decomposition of $p \in P$ with $m \in K$, $H(p) \in \mathfrak{a}_0$ and $n \in N$, then $p \cdot t = e^{\lambda(H(p))}t$ for all $t \in \mathbb{R}[\lambda]$.

Corollary 2.2.2. *Let G be a semisimple Lie group with finite centre, K its maximal compact subgroup, P be its minimal parabolic subgroup and let $M := K \cap P$. Let* vol *be the real valued Poisson kernel of degree $(0,n)$ defined in Lemma 2.2.2 and Φ be the Poisson transform associated to* vol*. Denote by dkM the normalized,*

K-left invariant measure on K/M. Let $g = \kappa(g)\exp(H(g))\eta(g)$ the Iwasawa decomposition of $g \in G$ with $\kappa(g) \in K$, $H(g) \in \mathfrak{a}_0$ and $\eta(g) \in N$ and denote by $\rho \in \mathfrak{a}_0^*$ the lowest form of $\mathfrak{g}$.

Then for all λ-densities σ on G/P the smooth function $\Phi(\sigma) \in C^\infty(G/K)$ is given by

$$\Phi(\sigma)(gK) = \int_{K/M} e^{(\lambda-2\rho)(H(g^{-1}k))}\pi_P^*\sigma(kM)\ dkM.$$

PROOF. From Proposition 2.2.2 we deduce that the Poisson kernel vol corresponds to the identity in $\mathrm{Hom}(G/M \times \mathbb{R}, G/M \times \mathbb{R})$. Therefore, we obtain from Theorem 2.2.2 that the smooth function $\Phi(\sigma)$ is given by

$$\Phi(\sigma)(gK) = \int_K (\pi_P^* f)(gk)dk,$$

where dk is the normalized Haar measure on K and $f\colon G \to \mathbb{R}[\lambda]$ is the smooth, P-equivariant map corresponding to σ.

Next, we apply the Iwasawa decomposition to gk and use P-equivariance of the smooth map $\pi_P^* f$ to obtain

$$\pi_P^* f(gk) = \pi_P^* f(\kappa(gk)\exp(H(gk))\eta(gk)) = e^{-\lambda(H(gk))}\pi_P^* f(\kappa(gk)).$$

But the map $\kappa_g\colon k \mapsto \kappa(gk)$ is a diffeomorphism of K with inverse $\kappa_g^{-1} = \kappa_{g^{-1}}$ (see [**Hel84**, Lemma I.5.19]), and the change of the Haar measure on K is given by $\kappa_g^* dk = e^{-2\rho(H(gk))}dk$. Inserting this into the above formula and using the relation $H(g\kappa(g^{-1}k)) = -H(g^{-1}k)$ we obtain that

$$\Phi(\sigma)(gK) = \int_K e^{(\lambda-2\rho)(H(g^{-1}k))}\pi_P^* f(k)dk.$$

Finally, since the integrand is invariant under the map $k \mapsto km$ for all $m \in M$ we can change the integral to the homogeneous space K/M, obtaining the claimed formula. □

2.2.3. Poisson transforms and differential operators. In section 2.2.1 we have seen how to construct Poisson transforms by reducing the problem to computations in finite dimensional M-representations. In the next step we focus on their compatibility with several differential operators on G/P and G/K.

Let $\mathbb{V}$ and $\mathbb{W}$ be two G-representations with associated vector bundles V_M and W_M over G/M. Then the vector space $\mathrm{L}(\mathbb{V}, \mathbb{W})$ inherits a natural G-action, so its associated bundle $Z := \mathrm{L}(V_M, W_M)$ is a tractor bundle over G/M. Recall from section 1.2.3 that Z is endowed with the G-invariant tractor connection, which in turn allows us to form the covariant exterior derivative d^Z on the space of Z-valued differential forms on G/M due to section 1.2.4.

Since the space of differential forms on G/M is naturally bigraded we obtain a splitting of the covariant exterior derivative into $d^Z = d_K + d_P$, where the first operator increases the K-degree, whereas the second summand increases the P-degree. We call d_K and d_P the *partial derivatives* in direction G/K and G/P, or simply the *K-derivative* and *P-derivative*, respectively. Since $d^Z \circ d^Z = 0$ we also obtain that $d_K^2 = 0$, $d_P^2 = 0$ and $d_K d_P = -d_P d_K$. Note that we can also twist the G-action on $\mathbb{W}$ with the Cartan involution θ on $\mathfrak{g}$ as in section 1.3.2. In this way, we obtain another covariant derivative $d^{Z,\theta}$, which splits into two operators $d_K^\theta + d_P^\theta$ in the same way as above. However, since the subbundle $T^{0,1}(G/M)$ is associated to the M-representation $(\mathfrak{g}/\mathfrak{m})^{0,1*} = \mathfrak{k}/\mathfrak{m}$ it follows from formula (1.2.4) that $d_P^\theta = d_P$ due to θ-invariance of $\mathfrak{k}$.

Furthermore, we have constructed in section 1.4.2 the Kostant codifferential ∂^* on homogeneous parabolic geometries and have shown in Theorem 1.4.4 that it is selfadjoint with respect to the wedge product. Now the pointwise identification of $T(G/M)$ with the Whitney sum $T(G/K) \oplus T(G/P)$ allows us to define a tensorial operator

$$\partial_P \colon \Lambda^{p,q}T^*(G/M) \otimes Z \to \Lambda^{p,q-1}T^*(G/M) \otimes Z,$$

called the *P-codifferential*. Explicitly, for $x \in G/M$ we define this operator on decomposable elements $\alpha \wedge \beta$ with $\alpha \in \Lambda^{p,0}T^*_x(G/M) \otimes (V_M)_x$ and $\beta \in \Lambda^{0,q}T^*_x(G/M) \otimes (W_M)^*_x$ via

$$\partial_P(\alpha \wedge \beta) := (-1)^p \alpha \wedge \partial^* \beta$$

and extend this operator linearly, where we used the pointwise isomorphism between $T^{0,1}$ and $T(G/P)$ induced by the projection π_P. Subsequently, we can define the *P-Laplace* $\Box_P := d_P \partial_P + \partial_P d_P$ on $\Omega^{k,\ell}(G/M, Z)$.

On the other hand, we have defined in section 1.3.1 the Hodge star operator $*^W$ on W-valued differential forms on G/K. Similarly as for the P-codifferential we obtain a tensorial operator

$$*_K \colon \Lambda^{p,q}T^*(G/M) \otimes Z \to \Omega^{\dim(G/K)-p,q}T^*(G/M) \otimes Z,$$

which we will call the *K-Hodge star operator*. Explicitly, for $x \in G/M$ this operator is defined on decomposable elements $\alpha \wedge \beta$ with $\alpha \in \Lambda^{p,0}T^*_x(G/M) \otimes (V_M)_x$ and $\beta \in \Lambda^{0,q}T^*_x(G/M) \otimes (W_M)^*_x$ by

$$*_K(\alpha \wedge \beta) := (*\alpha) \wedge \beta,$$

where we used the identification of $T^{1,0}$ with $T(G/K)$ induced by the projection π_K. Moreover, in analogy to Theorem 1.3.2 we define

$$\delta_K := (-1)^{\dim(G/K)(p-1)+1} *_K d^\theta_K *_K,$$

which is a G-equivariant operator lowering the K-degree by 1. Finally, we define the *K-Laplace* $\Delta_K := d_K \delta_K + \delta_K d_K$.

By naturality, all the above maps are G-equivariant and thus have M-equivariant counterparts on the level of the corresponding M-representations, which we will denote by the same symbols. In particular, for any Poisson kernel we can determine its image under the above operators via computations at the level of the corresponding M-representations.

PROPOSITION 2.2.3. *Let G be a semisimple Lie group with finite centre, K its maximal compact subgroup, P a parabolic subgroup of G and define $M := K \cap P$. Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional G-representations with corresponding tractor bundles V over G/P and W over G/K and covariant exterior derivatives d^V on V and d^W on W, respectively, which are induced by the respective tractor connections. Let Z be the homogeneous vector bundle over G/M associated to the M-representation $L(\mathbb{V}, \mathbb{W})$. Let ϕ be a Z-valued Poisson kernel of degree $(\ell, n-k)$ and let*

$$\Phi \colon \Omega^k(G/P, V) \to \Omega^\ell(G/K, W)$$

be the corresponding Poisson transform.

- *(i) The compositions $d^W \circ \Phi$, $* \circ \Phi$ and $\delta^W \circ \Phi$ are again Poisson transforms with kernels $d_K\phi$, $*_K\phi$ and $\delta_K\phi$, respectively.*
- *(ii) The compositions $\Phi \circ d^V$ and $\Phi \circ \partial^*$ are again Poisson transforms with kernels $(-1)^{n-k+\ell+1} d_P \phi$ and $(-1)^{n-k+\ell} \partial_P \phi$, respectively.*

In particular, we have $*_K \circ d_P = (-1)^{\dim(G/K)} d_P \circ *_K$, and operators d_P and ∂_P anticommute with both operators d_K and δ_K.

PROOF. (i) From Theorem 1.5.1 we know that the covariant exterior derivative commutes with the fibre integral. Moreover, using Proposition 1.2.4(ii) and the isomorphism $\mathrm{L}(\mathbb{V},\mathbb{W}) \cong \mathbb{V}^* \otimes \mathbb{W}$ we can express d^{W_M} in terms of d^Z and d^{V_M}, which are all induced by the respective tractor connections. Therefore, for $\alpha \in \Omega^k(G/P,V)$ we deduce that

$$\begin{aligned} d^W\Phi(\alpha) &= d^W\left(\fint_{G/P} \phi \wedge \pi_P^*\alpha\right) \\ &= \fint_{G/P} d^{W_M}\left(\phi \wedge \pi_P^*\alpha\right) \\ &= \fint_{G/P} \left(\left(d^Z\phi\right) \wedge \pi_P^*\alpha + (-1)^{k+n-\ell}\phi \wedge \left(d^{V_M}\pi_P^*\alpha\right)\right) \\ &= \fint_{G/P} (d_K\phi) \wedge \pi_P^*\alpha, \end{aligned}$$

where we used in the last line that any differential form on G/M with P-degree $n+1$ is trivial.

For the second operator note that we have

$$*^W\Phi(\alpha) = \fint_{G/P} (*_K\phi) \wedge \pi_P^*\alpha,$$

which follows immediately from the local description of the fibre integral in section 1.5.1 and the tensoriality of the vector valued Hodge star on G/K. Therefore, we deduce that

$$\delta^W\Phi(\alpha) = \fint_{G/P} (\delta_K\phi) \wedge \pi_P^*\alpha.$$

(ii) By naturality of the covariant exterior derivative we deduce for all $\alpha \in \Omega^{k-1}(G/P,V)$ that

$$\begin{aligned} \Phi(d^V\alpha) &= \fint_{G/P} \phi \wedge \pi_P^*(d^V\alpha) \\ &= \fint_{G/P} \phi \wedge (d^{V_M}\pi_P^*\alpha) \\ &= (-1)^{n-k+\ell}\left(d^W \fint_{G/P} \phi \wedge \pi_P^*\alpha - \fint_{G/P} (d^Z\phi) \wedge \pi_P^*\alpha\right) \\ &= (-1)^{n-k+\ell+1} \fint_{G/P} (d_P\phi) \wedge \pi_P^*\alpha, \end{aligned}$$

where we used again Proposition 1.2.4 in the third line as well as the fact that integration over a differential form on G/M with P-degree $(n-1)$ is trivial.

For the other operator, the selfadjointness of the Kostant codifferential from Theorem 1.4.4 implies for all $\alpha \in \Omega^{k+1}(G/P,V)$ that

$$\Phi(\partial^*\alpha) = \int_{G/P} \phi \wedge \partial_P\pi_P^*\alpha = (-1)^{n-k+\ell}\int_{G/P} (\partial_P\phi) \wedge \pi_P^*\alpha.$$

The last part of the Proposition follows by composing the Poisson transform with pairs of operators and determining the induced Poisson kernels. As an example, we consider the Poisson transform $* \circ \Phi \circ d^V$. The corresponding Poisson kernel is given by $(-1)^r *_K d_P\phi$ with $r := n-k+\ell+1$, where we first moved the covariant derivative to the Poisson kernel and afterwards applied the K-Hodge star. However, if we apply the operators to the Poisson kernel ϕ in the other way round, we obtain the equivalent expression $(-1)^{\dim(G/K)+r} d_P *_K \phi$ for the kernel of this transform. Therefore, it follows that $*_K d_P = (-1)^{\dim(G/K)} d_P *_K$. $\square$

For the remainder of this section we want to relate the covariant Laplace operator on G/K from section 1.3.2 and the operator $\square^R$ on G/P from section 1.4.3 via Poisson transforms. For that, recall from section 1.2.6 that the Casimir element of $\mathfrak{g}$ induces a differential operator D_C on differential forms on every homogeneous space with values in a homogeneous bundle, which is constructed via the infinitesimal action of $\mathfrak{g}$. In particular, if Φ is a Poisson transform, then by G-equivariance we obtain that

$$D_C\Phi(\alpha) = \Phi(D_C\alpha)$$

for all $\alpha \in \Omega^k(G/P, V)$.

LEMMA 2.2.3. *Let G be a semisimple Lie group with finite centre, K its maximal compact subgroup and P a parabolic subgroup of G. Let $\mathbb{V}$ and $\mathbb{W}$ be irreducible G-representations, let $c_{\mathbb{V}}$ and $c_{\mathbb{W}}$ be their Casimir eigenvalues and V and W be the corresponding tractor bundles over G/P and G/K, respectively. Then every Poisson transform Φ between $\Omega^k(G/P, V)$ and $\Omega^\ell(G/K, W)$ satisfies*

$$2\Phi(\square^R\alpha) + \Delta^W\Phi(\alpha) = (c_{\mathbb{W}} - c_{\mathbb{V}})\Phi(\alpha)$$

for all $\alpha \in \Omega^k(G/P, V)$.

PROOF. In [**ČS07**, Corollary 1] it was proved that the differential operator D_C induced by the Casimir element on $\mathfrak{g}$ acts on V-valued differential forms on G/P via $D_C = 2\square^R + c_{\mathbb{V}}$. On the other hand, we have seen in Theorem 1.3.3 that its action on W-valued differential forms on G/K is given by $D_C = -\Delta^W + c_{\mathbb{W}}$, so the claimed formula follows immediately from $D_C(\Phi(\alpha)) = \Phi(D_C\alpha)$. $\square$

One of the major achievements in recent years in the theory of parabolic geometries was the development of BGG-sequences, which are of crucial importance for the geometric study of such manifolds. In order to relate these to geometric objects on their corresponding symmetric spaces, we aim to construct equivariant operators between sections of homology bundles on homogeneous parabolic geometries and vector valued differential forms on Riemannian symmetric spaces which are compatible with the BGG-operators.

First, we recall from section 1.4.3 the definition of BGG-sequences. Let G be a semisimple Lie group, $P \subset G$ a parabolic subgroup and let $\mathbb{V}$ be a G-representation with associated tractor bundle V over G/P. Let ∂^* be the Kostant codifferential and denote the induced tensorial map on the space of V-valued k-forms on G/P by the same symbol, c.f. section 1.4.2. Then we defined the homology bundles $\mathcal{H}_k(G/P, V)$ over G/P as the homogeneous vector bundles associated to the P-representations $H_k(G/P, \mathbb{V}) = \ker(\partial^*)/\operatorname{im}(\partial^*)$. Moreover, in Theorem 1.4.3 we have seen that there exists a splitting operator $L\colon \mathcal{H}_k(G/P, V) \to \ker(\partial^*)$, which is a section of the canonical projection $\pi\colon \ker(\partial^*) \to \mathcal{H}_k(G/P, V)$, and which assigns to each section of a homology bundle its unique representative in the kernel of $\square^R$, c.f. Proposition 1.4.3. Finally, we defined the k-th BGG-operator $D\colon \mathcal{H}_k(G/P, V) \to \mathcal{H}_{k+1}(G/P, V)$ by $D := \pi \circ d^V \circ L$, where d^V is the restriction of the covariant exterior derivative with respect to the tractor connection to sections of the kernel of the Kostant codifferential.

Our aim is construct G-equivariant operators between sections of the homology bundles $\mathcal{H}_k(G/P, V)$ and tractor bundle valued differential forms on the symmetric space G/K, which are compatible with BGG-operators. In order to do so, we could first identify $\Gamma(\mathcal{H}_k(G/P, V))$ with sections of the kernel of $\square^R$ via the splitting operator L, which coincides with the subcomplex of the twisted de Rham sequence on G/P determined by the Casimir eigenvalue of $\mathbb{V}$ due to [**ČS07**, Corollary 1], and afterwards apply a nontrivial Poisson transform. However, the explicit computation of the image of the splitting operator is a difficult task in general, so in particular we cannot relate these to properties of the corresponding Poisson kernel.

Therefore, we will follow another approach by constructing a Poisson transform Φ which is trivial on the image of the Kostant codifferential. In this way, this operator factors canonically to a G-equivariant map $\tilde{\Phi}$ on sections of the corresponding homology bundles via $\tilde{\Phi} \circ \pi := \Phi$. Furthermore, we also want to consider the composition of the equivariant operator $\tilde{\Phi}$ with the BGG-operators $D = \pi \circ d^V \circ L$, which should again be computable independently of the choice of a representative of a section of the homology bundle. By definition, this is satisfied if and only if the transform Φ is trivial on the image of $d^V \circ \partial^*$. This leads to the following definition.

DEFINITION 2.2.3. Let G be a semisimple Lie group with finite centre, $K \subset G$ a maximal compact subgroup and $P \subset G$ a parabolic subgroup. Let $\mathbb{V}$ and $\mathbb{W}$ be G-representations, denote by $V = G \times_P \mathbb{V}$ and $W = G \times_K \mathbb{W}$ the associated bundles over G/P and G/K, respectively, and let V_M and W_M be the vector bundles on G/M associated to $\mathbb{V}$ and $\mathbb{W}$, respectively. Let d^V be the covariant exterior derivative on V-valued differential forms on G/P induced by the tractor connection.

(i) We say that a Poisson transform $\Phi\colon \Omega^k(G/P, V) \to \Omega^\ell(G/K, W)$ is *BGG-compatible* iff it is trivial on the images of the operators ∂^* and $d^V \circ \partial^*$.
(ii) We say that an $\mathrm{End}(V_M, W_M)$-valued Poisson kernel ϕ on G/M is *BGG-compatible* iff its associated Poisson transform has this property. By Theorem 2.2.3(ii) this is equivalent to $\partial_P \phi = 0$ and $\partial_P d_P \phi = 0$.

The following Theorem proves that BGG-compatibility of a Poisson transform has strong consequences on the geometric behaviour of its images, which is mainly due to the relations between the Casimir operator and the Laplace $\Box^R$ on tractor valued forms on G/P as well as the twisted Laplace Beltrami operator Δ^W.

THEOREM 2.2.3. *Let G be a semisimple Lie group with finite centre, $K \subset G$ a maximal compact subgroup and $P \subset G$ a parabolic subgroup. Let $\mathbb{V}$ be an irreducible G-representation and let V_K and V_P be the corresponding associated bundles over G/K and G/P, respectively. Then for a Poisson transform $\Phi\colon \Omega^k(G/P, V_P) \to \Omega^k(G/K, V_K)$ the following are equivalent:*

(i) The image of the transform Φ is contained in the harmonic differential forms.
(ii) The transform Φ is trivial on the image of the Laplace operator $\Box^R$.
(iii) The transform Φ is BGG-compatible.

PROOF. The equivalence of (i) and (ii) follows immediately from Lemma 2.2.3, whereas the implication (iii) $\Rightarrow$ (ii) is clear by definition of the operator $\Box^R = \partial^* d^{V_P} + d^{V_P} \partial^*$. Thus, we have to show that (ii) $\Rightarrow$ (iii).

In [**CD01**, Theorem 5.2] it is proven that $\Box^R$ is invertible on the space of sections of $\mathrm{im}(\partial^*)$. Therefore, we obtain that

$$\Phi \circ \partial^* = \Phi \circ \Box^R \circ (\Box^R)^{-1} \circ \partial^* = 0,$$

i.e. Φ is trivial on the image of the Kostant codifferential. Finally, by the defining equation of $\Box^R$ it follows that

$$\Phi \circ d^{V_P} \circ \partial^* = \Phi \circ \Box^R - \Phi \circ \partial^* \circ d^{V_P} = 0,$$

so our Poisson transform is BGG-compatible. $\Box$

As a Corollary we show that in the case of $|1|$-graded Lie algebras the image of every Poisson transform between real valued differential forms is contained in the space of harmonic forms.

COROLLARY 2.2.3. *Let G be a semisimple Lie group with finite centre and assume that the Lie algebra $\mathfrak{g}$ of G is $|1|$-graded. Let $K \subset G$ be a maximal compact subgroup and $P \subset G$ a parabolic subgroup compatible with the given grading of $\mathfrak{g}$. Then the image of any Poisson transform $\Phi\colon \Omega^k(G/P) \to \Omega^k(G/K)$ is contained in the space of harmonic differential forms on G/K.*

PROOF. Since the Lie algebra $\mathfrak{g}$ is $|1|$-graded it follows that Kostant codifferential is trivial on $\Omega^*(G/P)$ by definition. In particular, the Poisson transform Φ is BGG-compatible and the claim follows from Theorem 2.2.3. □

2.3. A uniform construction of a Poisson kernel

We will conclude this section with a demonstration of the efficiency of our method by constructing a uniform family of Poisson kernels ϕ_k whose associated Poisson transforms map between real valued differential forms. In order to do so, we first analyse in 2.3.1 the decomposition of the tangent bundle of G/M induced by the $|k|$-grading of $\mathfrak{g}$ and identify the horizontal and vertical subbundles therein. As a result, we construct in section 2.3.2 a Poisson kernel ϕ_k of degree $(k, n-k)$ in terms of the grading element of $\mathfrak{g}$ and show that its induced Poisson transform $\Phi_k\colon \Omega^k(G/P, \mathcal{E}[\chi]) \to \Omega^k(G/K)$ has values in the space of coclosed differential forms on G/K. Finally, in the last section 2.3.3 we prove that in the special case of smooth functions the operator Φ_k coincides with the classical Poisson transform in the sense of Definition 2.1.1.

2.3.1. The geometry of the space G/M. As a first step towards the construction of a uniform family of Poisson kernels we will identify the geometric properties of the tangent bundle of G/M by analysing the structure of $\mathfrak{g}/\mathfrak{m}$ as an M-module.

Consider the $|k|$-grading $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ and let θ be the Cartan involution on $\mathfrak{g}$, which is M-equivariant and maps $\mathfrak{g}_i$ to $\mathfrak{g}_{-i}$ for all $i = -k, \dots, k$. In particular, the subalgebra $\mathfrak{g}_0$ is θ-stable and thus decomposes into the direct sum of $\mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{g}_0 \cap \mathfrak{q}$. Since $\mathfrak{p}_+$ consists of positive restricted root spaces its intersection with $\mathfrak{k}$ is trivial, implying that $\mathfrak{g}_0 \cap \mathfrak{k}$ coincides with the subalgebra $\mathfrak{m}$. In particular, this shows that every grading component is an M-module via the restriction of the adjoint action of the Levi subgroup G_0. Moreover, the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{m}$ is an M-equivariant map, so the image of its restriction to every grading component $\mathfrak{g}_i$ is again an M-subrepresentation $(\mathfrak{g}/\mathfrak{m})_i$ of $\mathfrak{g}/\mathfrak{m}$. Thus, passing to the quotient $\mathfrak{g}/\mathfrak{m}$ we obtain an induced decomposition

$$\mathfrak{g}/\mathfrak{m} = (\mathfrak{g}/\mathfrak{m})_{-k} \oplus \ldots \oplus (\mathfrak{g}/\mathfrak{m})_k$$

into M-submodules. By construction, each factor $(\mathfrak{g}/\mathfrak{m})_i$ is isomorphic to $\mathfrak{g}_i$ for $i \neq 0$, whereas $(\mathfrak{g}/\mathfrak{m})_0$ is isomorphic to $\mathfrak{g}_0 \cap \mathfrak{q}$. In addition, the Cartan involution factorizes to $\mathfrak{g}/\mathfrak{m}$ which maps $(\mathfrak{g}/\mathfrak{m})_{-i}$ to $(\mathfrak{g}/\mathfrak{m})_i$ for all i.

PROPOSITION 2.3.1. *Let G be a real semisimple Lie group with finite centre and $K \subset G$ be its maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let θ be the Cartan involution on $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ with respect to a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be the $|k|$-grading determined by $\mathfrak{p}$ and let P be a parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Let M be the intersection of K and P and let $\mathfrak{m}$ be its Lie algebra.*

Then the Cartan involution induces a G-equivariant involutive bundle map Θ_M on $T(G/M)$. Moreover, we obtain a Whitney sum decomposition of the tangent bundle of G/M into G-invariant Euclidean subbundles

$$T(G/M) \cong T^{-k}(G/M) \oplus \cdots \oplus T^k(G/M).$$

The rank of the bundle $T^i(G/M)$ equals the dimension of the i-th grading component for all $i \neq 0$, whereas for $i = 0$ it coincides with the dimension of $\mathfrak{g}_0 \cap \mathfrak{q}$. Moreover, the G-invariant bundles $T^i(G/M)$ and $T^{-i}(G/M)$ are canonically isomorphic via Θ_M for all $i = 0, \dots, k$.

PROOF. Recall that the tangent bundle $T(G/M)$ is the homogeneous bundle associated to $\mathfrak{g}/\mathfrak{m}$. Since each subspace $(\mathfrak{g}/\mathfrak{m})_i$ is M-invariant it gives rise to a G-invariant subbundle $T^i(G/M) \subset T(G/M)$ via the associated bundle construction for all $i = -k, \dots, k$. Therefore, we obtain a decomposition of the tangent bundle into the Whitney sum

$$T(G/M) \cong T^{-k}(G/M) \oplus \ldots \oplus T^k(G/M),$$

where the rank of $T^i(G/M)$ equals the dimension of $\mathfrak{g}_i$ for all $i \neq 0$ as well as $\dim(\mathfrak{g}_0 \cap \mathfrak{q})$ for $i = 0$. Moreover, the Cartan involution gives rise to a canonical G-equivariant bundle map $\Theta_M : T(G/M) \to T(G/M)$, which restricts to an isomorphism $T^i(G/M) \cong T^{-i}(G/M)$ of homogeneous bundles for all $i = -k, \dots, k$. □

Recall from section 2.3 that the product structure of G/M induces a decomposition of $\Omega^k(G/M)$ into forms of degree (p, q). In order to keep track of the bidegree of G-invariant forms on G/M we have to determine how the subbundles $T^{1,0}$ and $T^{0,1}$ are related to the splitting of $T(G/M)$ from Proposition 2.3.1. For that, it suffices to determine how the M-subrepresentations $(\mathfrak{g}/\mathfrak{m})^{1,0}$ and $(\mathfrak{g}/\mathfrak{m})^{0,1}$ sit inside $\mathfrak{g}/\mathfrak{m}$. The derivative of the G-isomorphism $\Psi : G/M \cong G/K \times G/P$ is an M-equivariant map which is given by the direct sum of the canonical projections $\pi_{\mathfrak{k}} : \mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{k}$ and $\pi_{\mathfrak{p}} : \mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{p}$, and the horizontal and vertical subspaces correspond to the kernels of these maps. Therefore, we deduce that $(\mathfrak{g}/\mathfrak{m})^{1,0}$ equals $\mathfrak{p}/\mathfrak{m}$, whereas the second representation $(\mathfrak{g}/\mathfrak{m})^{0,1}$ coincides with $\mathfrak{k}/\mathfrak{m}$.

Thus, we have to determine how the subalgebras $\mathfrak{k}$ and $\mathfrak{p}$ sit inside the $|k|$-grading of $\mathfrak{g}$. By definition, the subalgebra $\mathfrak{p}$ is given by the sum of all nonnegative grading components, which immediately shows that

$$(\mathfrak{g}/\mathfrak{m})^{1,0} = \bigoplus_{i=0}^{k} (\mathfrak{g}/\mathfrak{m})_i.$$

On the other hand, we can determine $\mathfrak{k}$ as the $+1$-eigenspace of the Cartan involution of $\mathfrak{g}$, which maps $\mathfrak{g}_i$ to $\mathfrak{g}_{-i}$ for all $i \geq 0$. Therefore, the algebra $\mathfrak{k}$ is given by the sum of $\mathfrak{m}$ together with the symmetrisations

$$S_\theta(\mathfrak{g}_i) : \{X + \theta(X) \mid X \in \mathfrak{g}_i\}$$

for all $i < 0$. Since $\mathfrak{m}$ itself is θ-stable we deduce that the projection of the M-representation $S_\theta(\mathfrak{g}_i)$ to $\mathfrak{g}/\mathfrak{m}$ coincides with the symmetrisation $S_{\theta_\mathfrak{m}}((\mathfrak{g}/\mathfrak{m})_i)$, which is defined analogously for the M-equivariant map $\theta_\mathfrak{m} : \mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{m}$ induced by the Cartan involution.

THEOREM 2.3.1. *Let G be a real semisimple Lie group with finite centre and K be a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let θ be the Cartan involution on $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ with respect to a θ-stable Cartan algebra and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ the associated $|k|$-grading on $\mathfrak{g}$. Let P be a parabolic subgroup of G with Lie algebra $\mathfrak{p}$ and M be the intersection of K and P.*

Then the horizontal and vertical distributions of the tangent bundle of $G/M \to G/K$ are given by

$$T^{1,0} = T^0(G/M) \oplus \ldots \oplus T^k(G/M), \qquad T^{0,1} = \bigoplus_{i=1}^{k} S_{\Theta_M}(T^{-i}(G/M)),$$

where $S_{\Theta_M}(T^{-i}(G/M)) = \{\xi + \Theta_M(\xi) \mid \xi \in T^{-i}(G/M)\}$ denotes the fibrewise symmetrisation with respect to the G-equivariant map $\Theta_M \colon T(G/M) \to T(G/M)$ induced by the Cartan involution.

PROOF. Follows immediately from the discussion above by applying the associated bundle construction. □

Since the group M is compact, the homogeneous space G/M can be endowed with the structure of a Riemannian symmetric space. However, since $\mathfrak{g}/\mathfrak{m}$ is not irreducible as an M-representation, there are numerous choices of possible G-invariant Riemannian metrics, which are induced by M-invariant inner products on $\mathfrak{g}/\mathfrak{m}$. We will now specify a class of metrics which is associated to the given product structure of G/M.

We have already seen that the projection $\pi_K \colon G/M \to G/K$ induces a pointwise isomorphism of $T^{1,0}_{gM}$ and $T_{gK}(G/K)$ for all $g \in G$. Since the symmetric space G/K naturally carries a Riemannian metric g_K we can form the pullback metric $g_M = \pi_K^* g_K$, which is a Riemannian metric on the horizontal subbundle $T^{1,0}$. Moreover, we define the metric to be trivial on pairings between horizontal and vertical vectors so that the Whitney sum decomposition $T(G/M) = T^{1,0} \oplus T^{0,1}$ is orthogonal. In this way, the canonical projection $\pi_K \colon G/M \to G/K$ becomes a Riemannian submersion. However, we will not specify a metric on $T^{0,1}$ for now.

2.3.2. A Poisson transform on differential forms. We will finish the discussion of the general theory of Poisson transform by explicitly constructing a Poisson kernel of degree $(k, n-k)$. This will give a first demonstration of the nature of the computations and serves as a foretaste for the explicit examples of real and complex hyperbolic spaces in sections 3.3 and 4. For the construction we will heavily use the geometric structure of the tangent bundle of G/M developed in section 2.3.1.

We start this discussion by constructing an $\mathfrak{m}$-invariant element in $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ of degree 1 and determine its derivative. In order to do keep track of the bidegree in future computations, we will split $\mathfrak{g}/\mathfrak{m}$ into the direct sum of the horizontal and vertical subspace according to Theorem 2.3.1. For all $X \in \mathfrak{p}_+$ we denote by F_X the horizontal vector $X + \mathfrak{m} \in (\mathfrak{g}/\mathfrak{m})^{1,0}$. Similarly, for all $Y \in \mathfrak{g}_-$ we define the vertical vector $G_Y \in (\mathfrak{g}/\mathfrak{m})^{0,1}$ via $G_Y := S_{\theta_\mathfrak{m}}(Y) + \mathfrak{m}$.

PROPOSITION 2.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and θ a Cartan involution on $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic subalgebra with respect to a θ-stable Cartan subalgebra of $\mathfrak{g}$ and denote by $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ the $|k|$-grading associated to $\mathfrak{p}$. Let $\mathfrak{m}$ be the intersection of $\mathfrak{k}$ and $\mathfrak{p}$ and let B be the Killing form on $\mathfrak{g}$.*

Then the grading element of $\mathfrak{g}$ induces a nontrivial $\mathfrak{m}$-invariant element $E^* \in (\mathfrak{g}/\mathfrak{m})^*$ of degree $(1,0)$. Moreover, the K-derivative of E^* is trivial, whereas its P-derivative is given by

$$d_P E^*(F_X, G_Y) = i \left(\sum_{j=-k}^{k} j^2 \dim(\mathfrak{g}_j) \right)^{-1} B(X, Y)$$

for all $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_{-i}$. In particular, it induces non-degenerate $\mathfrak{m}$-invariant pairings $(\mathfrak{g}/\mathfrak{m})_{-i} \times (\mathfrak{g}/\mathfrak{m})_i \to \mathbb{R}$ for all $i = 1, \dots, k$.

PROOF. The grading element of $\mathfrak{g}$ is contained in the centre of $\mathfrak{g}_0$, so in particular it commutes with the subalgebra $\mathfrak{m}$. Moreover, for all $X \in \mathfrak{g}_i$ we have

$$[\theta(E), X] = \theta([E, \theta(X)]) = -iX,$$

which implies that $\theta(E) = -E$ and therefore $E \in \mathfrak{q}$. Since $\mathfrak{g}_0$ is θ-stable, this shows that E is contained in the subalgebra $\mathfrak{g}_0 \cap \mathfrak{q}$, which has trivial intersection with $\mathfrak{m}$. Therefore, we can project the grading element to a distinguished nonzero element in $(\mathfrak{g}/\mathfrak{m})_0$, which we will denote by the same symbol. Since the grading element is in the centralizer of $\mathfrak{m}$, its dual $E^* \in (\mathfrak{g}/\mathfrak{m})^*$ is $\mathfrak{m}$-invariant and by construction of degree $(1, 0)$.

Using formula (1.2.4) we can compute the derivative of E^* via

$$dE^*(X + \mathfrak{m}, Y + \mathfrak{m}) = -E^*([X, Y] + \mathfrak{m})$$

for all X, $Y \in \mathfrak{g}$, which is well defined due to $\mathfrak{m}$-invariance of E^*. Moreover, the compatibility of the Lie bracket on $\mathfrak{g}$ with the $|k|$-grading shows that the derivative is trivial unless $X \in \mathfrak{g}_{-i}$ and $Y \in \mathfrak{g}_i$ for all $i = -k, \dots, k$. Therefore, we have to determine the projections of the Lie brackets on $\mathfrak{g}$ onto the subalgebra generated by E.

For the subalgebra $\mathfrak{g}_0$ it suffices to compute the bracket between two vectors X and Y in $\mathfrak{g}_0 \cap \mathfrak{q}$. But we have seen in section 1.1.3 that $[X, Y]$ is an element in $\mathfrak{k}$, which implies that dE^* is trivial on $\mathfrak{g}_0$.

For the other components, recall that the Killing form B of $\mathfrak{g}$ induces a non-degenerate pairing $\mathfrak{g}_{-i} \times \mathfrak{g}_i \to \mathbb{R}$. Moreover, the decomposition of $\mathfrak{g}_0$ into the direct sum $(\mathfrak{g}_0 \cap \mathfrak{q}) \oplus \mathfrak{m}$ is orthogonal with respect to B and the restriction of the Killing form to $\mathfrak{g}_0 \cap \mathfrak{q}$ and $\mathfrak{m}$ is positive and negative definite, respectively. For $X \in \mathfrak{g}_{-i}$ we choose $Y \in \mathfrak{g}_i$ with $B(X, Y) \neq 0$. By the invariance property of the Killing form we obtain that

$$B(E, [X, Y]) = B([E, X], Y) = -iB(X, Y) \neq 0.$$

Extend the grading element $E = E_1$ to an orthogonal basis $\{E_1, E_2, \dots, E_\ell\}$ of $\mathfrak{g}_0 \cap \mathfrak{q}$. Then the dual basis with respect to the Killing form is given by $\{\lambda_1 E_1, \dots, \lambda_\ell E_\ell\}$, where $\lambda_j = B(E_j, E_j)^{-1} \neq 0$, and every vector $Z \in \mathfrak{g}_0 \cap \mathfrak{q}$ can be expressed as $Z = \sum_{j=1}^{\ell} \lambda_j B(E_j, Z) E_j$. Since the grading element acts on $\mathfrak{g}_i$ by multiplication with i we deduce that

$$\lambda_1 = B(E, E) = \sum_{j=-k}^{k} j^2 \dim(\mathfrak{g}_j).$$

For $X \in \mathfrak{g}_{-i}$ and $Y \in \mathfrak{g}_i$ we obtain from above that

$$dE^*(X + \mathfrak{m}, Y + \mathfrak{m}) = -E^*([X, Y] + \mathfrak{m}) = i\lambda_1 B(X, Y) \neq 0.$$

This shows that the restriction of dE^* to $(\mathfrak{g}/\mathfrak{m})_{-i} \times (\mathfrak{g}/\mathfrak{m})_i$ is nondegenerate for all $1 \leq i \leq k$. In particular, by formula (1.2.4) the derivative is trivial on the horizontal subspace $(\mathfrak{g}/\mathfrak{m})^{1,0}$, hence the K-derivative of E^* is trivial. □

If we assume that the invariant form constructed in Proposition 2.3.2 is also M-invariant, it can be used to produce a wide number of Poisson kernels on $\mathfrak{g}/\mathfrak{m}$ by applying operators from 2.2.3 and forming wedge products. In particular, the group M has this property if it is connected or if P is a minimal parabolic subgroup. However, in order to get a Poisson transform which preserves the degree of the differential forms we need to make an additional assumption on the M-representation $(\mathfrak{g}/\mathfrak{m})_0$.

Let $d = \dim((\mathfrak{g}/\mathfrak{m})_0)$ and assume that the group M acts trivially on the 1-dimensional representation $\Lambda^d(\mathfrak{g}/\mathfrak{m})_0^*$, e.g. M is connected or P is a minimal parabolic subgroup. Let τ be any element in this space and denote its trivial extension to all of $\mathfrak{g}/\mathfrak{m}$ by the same symbol. Then the K-derivative of τ is trivial. Indeed, from formula (1.2.4) we see that in order to compute the K-derivative of τ we have to insert vectors of the form $[X,Y]+\mathfrak{m}$ with X, $Y \in \mathfrak{p}$. However, the compatibility of the $|k|$-grading with the Lie bracket shows that $[X,Y] \in \mathfrak{g}_0$ if and only if X and $Y \in \mathfrak{g}_0$. But this means that we have to insert $d+1$ vectors of $(\mathfrak{g}/\mathfrak{m})_0$ into the form $d_K\tau$, which is trivial by dimensional reasons.

We will from now on assume that M acts trivial on the 1-form E^* constructed from the grading element in Proposition 2.3.2 as well as on the 1-dimensional representation $\Lambda^d(\mathfrak{g}/\mathfrak{m})_0^*$. Note that this is satisfied if M is connected or if P is a minimal parabolic subgroup, in which case M acts trivially on $(\mathfrak{g}/\mathfrak{m})_0 \cong \mathfrak{a}_0$.

Before we explicitly construct the Poisson transform we will prove a Lemma in which we compute the K-derivative of the pullback of densities on G/P along the projection π_P.

LEMMA 2.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and θ a Cartan involution on $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic with respect to a θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be the $|k|$-grading of $\mathfrak{g}$ corresponding to the choice of parabolic subalgebra. Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and let K and P be the subgroups of G with Lie algebras $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Let $M = K \cap P$ with Lie algebra $\mathfrak{m}$ and let G_0 be the Levi subgroup of P.*

Any character $\chi \colon G_0 \to \mathbb{R}_+$ induces an M-invariant element $\tilde{\chi}$ in $(\mathfrak{g}/\mathfrak{m})_0^*$. Moreover, if we denote the induced G-invariant 1-form on G/M by the same symbol, then for every χ-density $\sigma \in \Gamma(\mathcal{E}[\chi])$ on G/P we have

$$d_K \pi_P^* \sigma = -\pi_P^* \sigma \cdot \tilde{\chi}.$$

PROOF. The subgroup M is compact and contained in G_0, so the restriction of the character χ to M is trivial. Therefore, its derivative $\chi' \colon \mathfrak{g}_0 \to \mathbb{R}$ is trivial on the subalgebra $\mathfrak{m}$ and thus factors to an element $\tilde{\chi} \in (\mathfrak{g}/\mathfrak{m})_0^* \to \mathbb{R}$, which is M-invariant by definition.

Via Theorem 2.3.1 we can determine the K-derivative of the pullback $\pi_P^*\sigma \in C^\infty(G/M)$ of any χ-density σ by deriving it in direction of (local) sections ξ of the subbundle $T^i(G/M)$ for all $i \geq 0$. Equivalently, we can determine the derivative of the P-equivariant function $f \colon G \to \mathbb{R}$ corresponding to the pullback of σ in direction of local lifts of ξ. Since the derivative only depends on the value of ξ in the point gM, we can choose a local lift of the form $L_X(g)$ with $X \in \mathfrak{g}_i$, where $L_X \in \mathfrak{X}(G)$ denotes the left invariant vector field induced by X. In this way, we can use the infinitesimal version of the P-equivariance of f which is given by $(L_X \cdot f)(g) = -X \cdot f(g)$ for all $g \in G$ and $X \in \mathfrak{p}$.

If $X \in \mathfrak{p}_+$, then X acts trivial on $\mathbb{R}$, meaning that the K-derivative in direction of sections of $T^i(G/M)$ for $i > 0$ vanishes. In the remaining case $X \in \mathfrak{g}_0$ we have

$$-X \cdot f(g) = \left.\frac{d}{dt}\right|_{t=0} e^{-tX} \cdot f(g) = \left.\frac{d}{dt}\right|_{t=0} \chi(e^{-tX}) f(g) = -\chi'(X) f(g).$$

On the level of smooth functions on G/M this means that

$$d_K \pi_P^* \sigma = -\pi_P^* \sigma \cdot \tilde{\chi}$$

for all $\sigma \in \Gamma(\mathcal{E}[\chi])$. □

Let E^* be the M-invariant form constructed in Proposition 2.3.2 and let $\tau \in \Lambda^d((\mathfrak{g}/\mathfrak{m})_0^*)$ be any M-invariant element, which is unique up to a constant, and extend it trivially to an invariant d-form on $\mathfrak{g}/\mathfrak{m}$. Moreover, denote the induced G-invariant differential forms on G/M by the same symbol. Then we consider the Poisson kernel

$$\phi_k := *_K(\tau \wedge (d_P E^*)^{n-k})$$

which is of degree $(k, n-k)$ and thus induces a Poisson transform

$$\Phi_k \colon \Omega^k(G/P, \mathcal{E}[\chi]) \to \Omega^k(G/K)$$

for all $k = 0, \dots, n$ and any density bundle $\mathcal{E}[\chi]$ over G/P.

THEOREM 2.3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and θ a Cartan involution on $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let $\mathfrak{p}$ be a standard parabolic with respect to a θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ the $|k|$-grading of $\mathfrak{g}$ induced by $\mathfrak{p}$. Let G be a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$ and let K and P be the subgroups of G with Lie algebras $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Let $M = K \cap P$ with Lie algebra $\mathfrak{m}$ and denote by $(\mathfrak{g}/\mathfrak{m})_0$ the image of $\mathfrak{g}_0$ under the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{m}$. Assume that M acts trivial on the element $E^* \in (\mathfrak{g}/\mathfrak{m})^*$ from Proposition 2.3.2 as well as on the representation $\Lambda^d(\mathfrak{g}/\mathfrak{m})_0^*$ with $d = \dim((\mathfrak{g}/\mathfrak{m})_0)$. Let $\chi \colon P \to \mathbb{R}_+$ be a character on P and denote by $\mathcal{E}[\chi]$ the corresponding bundle of χ-densities.*

Then for all $k = 0, \dots, \dim(G/P)$ the images of the Poisson transform Φ_k are coclosed, i.e.

$$\delta\Phi_k(\sigma) = 0$$

for all $\sigma \in \Omega^k(G/P, \mathcal{E}[\chi])$.

PROOF. Let σ be a χ-density on G/P. By Proposition 2.2.3 we have to determine the image of the differential form $\phi_{k,\chi} := \phi_k \wedge \pi_P^* \sigma$ under the K-codifferential, which is computed by applying $(-1)^{\dim(G/K)(k-1)+1}$-times the operator $*_K d_K *_K$. By definition, the first K-Hodge star just affects the Poisson kernel ϕ_k. Next, using the antiderivation property of the K-derivative and Lemma 2.3.2 the differential form $d_K *_K \phi_{k,\chi}$ splits into the sum

$$d_K *_K \phi_{k,\chi} = (d_K *_K \phi_k - \tilde{\chi} \wedge (*_K \phi_k)) \wedge \pi_P^* \sigma = (*).$$

In order to simplify $(*)$, recall that the K-derivatives of the differential forms E^* and τ are trivial. Furthermore, since $d^2 = 0$ we also obtain that $d_K d_P E^* = -d_P d_K E^* = 0$, which implies that the Poisson kernel ϕ_k is K-coclosed and thus the first summand of $(*)$ is trivial. For the second summand, note that $\tilde{\chi}$ is only nontrivial on $(\mathfrak{g}/\mathfrak{m})_0$, and since τ is a form of maximal degree on this space, their wedge product is trivial. Thus, the K-codifferential of $\phi_{k,\chi}$ is zero. □

We will see in section 3.3 that the Poisson kernel constructed in Theorem 2.3.2 will be the unique Poisson transform preserving the degree in the case of the special orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1,1)$. In particular, this shows that we cannot expect any other Poisson kernel in general apart from those built from the invariant 1-form E^*.

However, Proposition 2.3.2 implies that the ring of Poisson kernels is more complicated in general, implying that a fine-tuned analysis of the ring of Poisson kernels for specific examples could lead to transforms with different properties. Indeed, first we could restrict the derivative $d_P E^*$ to the subspaces $(\mathfrak{g}/\mathfrak{m})_i \times (\mathfrak{g}/\mathfrak{m})_{-i}$ and extend them trivially, obtaining a family of M-invariant $(1,1)$-forms on $\mathfrak{g}/\mathfrak{m}$, whose number depends on the length of the grading on $\mathfrak{g}$. Moreover, there might be other central elements in $\mathfrak{g}_0$ which in the same way as the grading element induce invariant 1-forms on $\mathfrak{g}/\mathfrak{m}$. Finally, depending on the choice of G there could be additional geometric structures on G/K or G/P compatible with the G-actions, which are reflected by the existence of additional M-invariant forms on some grading components.

If P is a minimal parabolic subgroup we have seen that the subalgebra $\mathfrak{g}_0 \cap \mathfrak{q}$ equals $\mathfrak{a}_0$, which is also the centre of $\mathfrak{g}_0$, whereas the algebra $\mathfrak{m}$ is exactly the centralizer $\mathfrak{m}_0$ of $\mathfrak{a}_0$ in $\mathfrak{k}$ and coincides with the semisimple part of $\mathfrak{g}_0$. In particular, if the real rank of $\mathfrak{g}$ is 1, then the centre of $\mathfrak{g}_0$ is generated by E, so there are no other invariant 1-forms on $\mathfrak{g}/\mathfrak{m}$ induced by central elements in $\mathfrak{g}_0$. Subsequently, the ring of Poisson kernels will be "small", and therefore we will focus on these cases in the last chapters of this thesis.

2.3.3. The classical Poisson transform. In the case of a minimal parabolic subgroup we will show in this section that the Poisson transform constructed in Theorem 2.3.2 reduces to the classical Poisson transform between smooth functions.

Let P be a minimal parabolic subgroup and consider for $\lambda \in \mathfrak{a}_0^*$ the Poisson transform $\Phi_0 \colon \Gamma(\mathcal{E}[\lambda]) \to C^\infty(G/K)$ from Theorem 2.3.2. Since the corresponding Poisson kernel ϕ_0 is a G-invariant differential form on G/M of degree $(0,n)$, it can be normalized so that its integral over every fibre of $G/M \to G/K$ equals 1. Note that in this way the form coincides with the Poisson kernel vol constructed in Lemma 2.2.2.

PROPOSITION 2.3.3. *Let $\rho \in \mathfrak{a}_0^*$ be the lowest form on $\mathfrak{g}$. Then for all $\lambda \in \mathfrak{a}_0^*$ the map $\Phi_0 \colon \Gamma(\mathcal{E}[\rho - \lambda]) \to C^\infty(G/K)$ is the classical Poisson transform.*

PROOF. Since we normalized the Poisson kernel ϕ_0 so that it coincides with the invariant form vol from Lemma 2.2.2, the claim follows from Corollary 2.2.2 $\square$

We have seen in Definition 2.1.1 that the classical Poisson transform is usually defined for smooth functions on K/M and depends on a parameter $\lambda \in \mathfrak{a}_0^*$. However, for $f \in C^\infty(K/M)$ let $\tilde{f} \colon K \to \mathbb{R}$ be the corresponding M-equivariant map given by $\tilde{f}(k) = f(kM)$. For every $\lambda \in \mathfrak{a}_0^*$ we can define a smooth map $\chi_\lambda \colon G \to \mathbb{R}[\lambda]$ via $\chi_\lambda(g) = e^{-\lambda(H(g))}\tilde{f}(\kappa(g))$, where $\kappa(g) \in K$ and $H(g) \in \mathfrak{a}_0$ are the corresponding elements in the Iwasawa decomposition of $g \in G$. By definition, this map is P-equivariant and therefore determines a λ-density on G/P. Thus, we obtain an induced isomorphism

$$\chi_\lambda \colon C^\infty(K/M) \to \Gamma(\mathcal{E}[\lambda])$$

for all $\lambda \in \mathfrak{a}_0^*$. Using this identification we can write the classical Poisson transform in Definition 2.1.1 as $P_{-i\lambda} = \Phi_0 \circ \chi_{\rho-\lambda}$.

As an illustration of the efficiency of our methods we compute the eigenvalues of the Poisson transform Φ_0, reproving Proposition 2.1.1 in this way.

THEOREM 2.3.3. *Let $\Phi_0 \colon \Gamma(\mathcal{E}[\lambda]) \to C^\infty(G/K)$ be the Poisson transform from Proposition 2.3.3. Then for all λ-densities σ the image $\Phi_0(\sigma)$ is an eigenvector of the K-Laplacian. Explicitly, we have*

$$\Delta_K \Phi_0(\sigma) = -\langle \lambda, \lambda - 2\rho \rangle \Phi_0(\sigma).$$

PROOF. Using Proposition 1.3.3 and G-equivariance of the Poisson transform we can compute $\Delta_K \Phi_0(\sigma)$ as $-\Phi_0(D_C \sigma)$, where D_C is the differential operator induced by the Casimir element of $\mathfrak{g}$. Moreover, from [**ČS07**, Theorem 1] we see that D_C acts by a scalar on σ. Explicitly, if $\{Z_i\}$ and $\{X_i\}$ are bases of $\mathfrak{n}_0$ and $\theta(\mathfrak{n}_0)$, respectively, which are dual with respect to the Killing form B, and if $\{A_\ell\}$ is a basis of $\mathfrak{g}_0$ with dual basis $\{A^\ell\}$ with respect to B, then this scalar is computed via the formula

$$-\sum_i [Z_i, X_i] \cdot t + \sum_\ell A_\ell \cdot A^\ell \cdot t, \tag{2.3.3}$$

where t is an element in the P-representation $\mathbb{R}[\lambda]$. In this paper, this expression was evaluated by complexifying the representation as well as the Lie algebra. However, for our purpose it will be convenient to stay in the real picture.

The space $\mathfrak{n}_0$ consists of all restricted root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_r^+$. Choose a basis $\{E_\alpha^1, \dots, E_\alpha^{m_\alpha}\}$ of $\mathfrak{g}_\alpha$, where m_α is the dimension of $\mathfrak{g}_\alpha$, and a basis $\{F_\alpha^1, \dots, F_\alpha^{m_\alpha}\}$ of $\mathfrak{g}_{-\alpha}$ so that $B(E_\alpha^i, F_\alpha^j) = \delta_{ij}$. Then for all $i = 1, \dots, m_\alpha$ the Lie bracket $[E_\alpha^i, F_\alpha^i]$ equals the unique vector $H_\alpha \in \mathfrak{a}_0^*$, which is dual to the restricted root α via the Killing form. Moreover, the subspace $\mathfrak{g}_0$ splits into the orthogonal direct sum $\mathfrak{m}_0 \oplus \mathfrak{a}_0$, and we choose an orthonormal basis A_ℓ of $\mathfrak{a}_0$. Since the subalgebra $\mathfrak{m}_0$ acts trivially on $\mathbb{R}[\lambda]$ we can compute (2.3.3) as

$$-\sum_{\alpha \in \Delta_r^+} \sum_{i=1}^{m_\alpha} [E_\alpha^i, F_\alpha^i] \cdot t + \sum_\ell A_\ell \cdot A_\ell \cdot t.$$

Since λ is the only weight of $\mathbb{R}[\lambda]$, the first sum contributes by the factor $-2\langle \lambda, \rho \rangle$. For the other sum, we use that A_ℓ is an orthonormal basis to obtain the factor $\langle \lambda, \lambda \rangle$. □

Note that in the classical theory one usually considers complex line bundles and chooses the weight $\rho - i\lambda$ for $\lambda \in (\mathfrak{a}_0)^*_{\mathbb{C}}$. In this way, the eigenvalue of Φ_0 on $\Gamma(\mathcal{E}[\rho - i\lambda])$ equals $\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle$, compare with Proposition 2.1.1. Note that we used the geometric version of the Laplacian, which differs by a sign from the analytic version usually regarded in the theory of harmonic analysis.

CHAPTER 3

Poisson transforms for real hyperbolic space

In section 2.3.2 we have constructed a family of Poisson transforms between real valued differential forms on homogeneous parabolic geometries G/P and Riemannian symmetric spaces G/K, which exists for a wide range of cases. However, we also mentioned that additional properties of these operators as well as the existence of other transforms depend strongly on the specific structure of the Lie algebra $\mathfrak{g}$, implying that a fine-tuned analysis for special choices of G is inevitable. For the rest of this thesis we will illustrate in several examples how such a case by case analysis leads to additional information about Poisson transforms.

In this section we consider Poisson transforms between the conformal sphere of dimension n and the $(n+1)$-dimensional real hyperbolic space, whose common automorphism group $G = \mathrm{SO}(n+1,1)_0$ is of real rank 1 and its Lie algebra $\mathfrak{g}$ naturally carries a $|1|$-grading, c.f. section 3.1.1. In particular, Corollary 2.2.3 shows that the images of the operators Φ_k from Theorem 2.3.2 consist of harmonic differential forms on G/K. Furthermore, due to the simple structure of the M-module $\mathfrak{g}/\mathfrak{m}$ these transforms are also chain maps between the de Rham sequences on G/P and G/K. Finally, we are able to determine global norm estimates of the images of Φ_k on the Poincaré Ball model of hyperbolic space in section 3.3.3.

In addition, we will compute in Theorem 3.3.2 the dimension of the space of Poisson kernels in this case, implying that the operators Φ_k are a multiple of Gaillard's Poisson transform from section 2.1.3. Thus, we reproved some of the properties of Gaillard's Poisson transform shown in [**Gai86**] in a more conceptual and simple way. In addition, via the interpretation of the homogeneous space G/M as a sphere bundle in section 3.2.3 we can recover Gaillard's initial definition in terms of visual maps, c.f. Definition 2.1.3.

In the second part of this section we will illustrate the construction of a Poisson transform between differential forms with values in the standard tractor bundle. In order to construct Poisson kernels in this case, we proceed as in 2.3 and first analyse the geometries of the vector bundles $T(G/M)$ and $G \times_M \mathbb{V}$ over G/M, where $\mathbb{V} = \mathbb{R}^{n+1,1}$ is the standard representation of G, by identifying the M-module structure of $\mathfrak{g}/\mathfrak{m}$ and $\mathbb{V}$, respectively, in section 3.2. Afterwards, we will present an explicit construction of a BGG-compatible Poisson kernel $\phi_k^{\mathbb{V}}$ in section 3.4.1 and show that its induced Poisson transform $\Phi_k^{\mathbb{V}}$ has properties similar to the operator Φ_k. Finally, we will prove that the distinguished Poisson transforms induce chain maps between the BGG-sequence on G/P and the twisted de Rham sequence on G/K.

3.1. Preliminaries about real hyperbolic space

3.1.1. The Lie groups and Lie algebras. Let $\mathbb{V}$ be the real vector space $\mathbb{R}^{n+2}$ endowed with the nondegenerate inner product corresponding to the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{id}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can decompose this Lorentzian vector space into the disjoint union $\mathbb{V} = \mathbb{V}_- \cup \mathbb{V}_n \cup \mathbb{V}_+$, where $\mathbb{V}_-$, $\mathbb{V}_n$ and $\mathbb{V}_+$ denote the set of all negative, null and positive vectors of $\mathbb{V}$, respectively. By bilinearity of the inner product each of these sets is a cone in $\mathbb{V}$, so we obtain an action of the Lie group $\mathbb{R}_+$ on the set of negative vectors as well as on the light cone $\mathcal{C} := \mathbb{V}_n \setminus \{0\}$. We define the *real hyperbolic space* $\mathcal{H}$ of dimension n as the space of negative lines in $\mathbb{V}_-$, which is diffeomorphic to the set of all vectors $v = (v^0, \dots, v^{n+1}) \in \mathbb{V}_-$ with $\langle v, v\rangle = -1$ and $v^0 - v^{n+1} > 0$ simply by mapping any generator of a line to its normalization. Similarly, we denote by $P(\mathcal{C}) := \mathcal{C}/\mathbb{R}_+$ the projectivization of the light cone and call it the *boundary at infinity* $\partial\mathcal{H}$ of the hyperbolic space. Note that we can identify these manifolds as homogeneous spaces by considering the special orthogonal group $\mathrm{SO}(n+1,1)$, which we realize as the set of all $g \in \mathrm{GL}(\mathbb{V})$ satisfying $g^tSg = S$ and $\det(g) = 1$. Then its connected component G acts transitively on both the hyperbolic space and its boundary.

If ℓ_- denotes the negative line in $\mathbb{V}_-$ defined by the difference of the first and last basis vector, then by Silvester's law of inertia the Lorentzian inner product on $\mathbb{V}$ induces a positive definite inner product g_K on $(\ell_-)^\perp \cong T_{\ell_-}\mathcal{H}$. Furthermore, the isotropy group K of the negative line also stabilizes this inner product and thus has to coincide with the maximal compact subgroup $\mathrm{SO}(n+1)$ in G. In particular, we deduce that the real hyperbolic space is isomorphic to the symmetric space G/K, and the K-invariant inner product g_K on $T_{\ell_-}\mathcal{H} \cong \mathfrak{q}$ induces a Riemannian metric on the real hyperbolic space, called the *hyperbolic metric*. Note that under the identifications $\mathfrak{q} \cong \mathbb{R}^{n+1}$ and $K \cong \mathrm{SO}(n+1)$ the inner product g_K corresponds to the standard Euclidean product on $\mathbb{R}^{n+1}$, which differs from the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{q}$ by a constant factor, c.f. Proposition 3.1.1.

Similarly, let ℓ_0 be the null line in $P(\mathcal{C})$ generated by the first standard basis vector of $\mathbb{V}$ and let P be its stabilizer in G. Then in the representation determined by the matrix S this group is realized as

$$P = \left\{ \begin{pmatrix} a & -aY^tB & -\frac{1}{2}a\,\|Y\|^2 \\ 0 & B & Y \\ 0 & 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R}_+, Y \in \mathbb{R}^n, B \in \mathrm{SO}(n) \right\}.$$

In particular, $P \cong \mathrm{CSO}(n) \times \mathbb{R}^n$ is a minimal parabolic subgroup of G and ∂H is isomorphic to the homogeneous parabolic geometry G/P. Furthermore, the Langlands decomposition $P = MAN$ of the parabolic subgroup is also nicely visible in this matrix representation. Indeed, the group $M \cong \mathrm{SO}(n)$ corresponds to the matrices $a = 1$ and $Y = 0$, the abelian subgroup $A \cong \mathbb{R}_+$ corresponds to $Y = 0$ and $B = \mathrm{id}_n$ and the nilpotent subgroup N are those matrices with $a = 1$ and $B = \mathrm{id}_n$.

Turning to the infinitesimal picture, the Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1,1)$ is given by all $g \in \mathrm{End}(\mathbb{V})$ which satisfy $g^tS = -Sg$. If we write elements of $\mathfrak{g}$ as real block matrices whose blocks have the same sizes than those of S, a direct computation shows that

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & -Y^t & 0 \\ X & B & Y \\ 0 & -X^t & -a \end{pmatrix} \middle| a \in \mathbb{R},\ X, Y \in \mathbb{R}^n,\ B \in \mathfrak{so}(n) \right\}.$$

PROPOSITION 3.1.1. *Let $G = \mathrm{SO}(n+1,1)_0$ with Lie algebra $\mathfrak{g}$ and denote by $\mathfrak{k}$ and $\mathfrak{p}$ the Lie algebras of the maximal compact subgroup $K \cong \mathrm{SO}(n+1)$ and the minimal parabolic subgroup $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ of G, respectively. Let $P = MAN$ be the Langlands decomposition of the minimal parabolic subgroup and denote by $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ the Lie algebras of M, A and N, respectively.*

(i) *The Lie algebra $\mathfrak{g}$ carries a natural $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ so that the parabolic subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ coincides with the Lie algebra of P.*

(ii) *The Lie algebra* $\mathfrak{m}$ *is isomorphic to* $\mathfrak{so}(n)$*, whereas the Lie algebra* $\mathfrak{a}$ *is* 1*-dimensional. In particular, we obtain an isomorphism* $\mathfrak{g}_0 \cong \mathfrak{so}(n) \oplus \mathbb{R}$ *as Lie algebras.*

(iii) *The Lie algebra* $\mathfrak{k}$ *consists of all matrices in* $\mathfrak{g}$ *whose* $\mathfrak{a}$*-part is trivial and which are symmetric in* $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$*.*

(iv) *The Killing form* B *on* $\mathfrak{g}$ *is determined by the following nondegenerate pairings:*

$$\begin{aligned} \mathfrak{g}_{-1} \otimes \mathfrak{g}_1 &\to \mathbb{R}, & B(X,Y) &= -2n\langle X,Y\rangle, \\ \mathfrak{a} \otimes \mathfrak{a} &\to \mathbb{R}, & B(E,E) &= 2n, \\ \mathfrak{m} \otimes \mathfrak{m} &\to \mathbb{R}, & B(A_1,A_2) &= n\,\mathrm{tr}(A_1A_2). \end{aligned}$$

PROOF. (i) If we define the subspaces $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ to be generated by the entries $X \in \mathbb{R}^n$, $(a,B) \in \mathfrak{cso}(n)$ and $Y \in \mathbb{R}^n$, then the block form of $\mathfrak{g}$ in our matrix representation immediately implies the claim.

(ii) From the matrix representation of M we immediately deduce that its Lie algebra $\mathfrak{m}$ is given in terms of the $|1|$-grading by all elements of the form $(0,(B,0),0)$ with $B \in \mathrm{SO}(n)$, whereas the Lie algebra $\mathfrak{a}$ of A consists of all matrices $(0,(0,a),0)$ with $a \in \mathbb{R}$. In particular, we obtain that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ as Lie algebras.

(iii) In the matrix representation of G the global Cartan involution is given by $g \mapsto (g^t)^{-1}$. Therefore, applying its derivative $g \mapsto -g^t$ immediately implies that $\mathfrak{k}$ is of the claimed form.

(iv) The Killing form on $\mathfrak{so}(n+1,1)$ is a multiple of the trace form of the matrix group $\mathfrak{g}$, and the factor is determined by the value of $B(E,E)$, which can be computed directly. □

For later computations, we derive explicit formulae for the Lie bracket on $\mathfrak{g}$ in terms of our matrix representation. Via the identifications of $\mathfrak{g}_{\pm 1}$ with $\mathbb{R}^n$ and $\mathfrak{g}_0$ with $\mathfrak{so}(n) \oplus \mathbb{R}$ we obtain the following nontrivial pairings:

$$\begin{aligned} \mathfrak{g}_0 \times \mathfrak{g}_{-1} &\to \mathfrak{g}_{-1}, & [(B,a),X] &= BX - aX, \\ \mathfrak{g}_0 \times \mathfrak{g}_0 &\to \mathfrak{g}_0, & [(B_1,a_1),(B_2,a_2)] &= ([B_1,B_2],0), \\ \mathfrak{g}_0 \times \mathfrak{g}_1 &\to \mathfrak{g}_1, & [(B,a),Y] &= BY + aY, \\ \mathfrak{g}_{-1} \times \mathfrak{g}_1 &\to \mathfrak{g}_0, & [X,Y] &= (YX^t - XY^t, \langle X,Y\rangle). \end{aligned}$$

For the rest of this section we want to analyse the geometric structure of the boundary at infinity of the real hyperbolic space. The tangent space of $\mathcal{C}$ at $v \in \mathcal{C}$ is given by $v^\perp$; however, since the vector v is null it is contained in its orthogonal complement. Thus, the tangent map of the canonical projection $\pi_n \colon \mathcal{C} \to \partial\mathcal{H}$ at v factors to a linear isomorphism $v^\perp/\mathbb{R}v \to T_{\pi_n(v)}P(\mathcal{C})$. Moreover, the Lorentzian inner product on $\mathbb{V}$ induces a positive definite inner product on $v^\perp/\mathbb{R}v$, which can be carried over to $T_{\pi_n(v)}P(\mathcal{C})$. If we replace v by λv for any $\lambda \in \mathbb{R}_+$ this inner product is multiplied by the factor λ^2, obtaining a conformal metric on $\partial\mathcal{H}$.

We will rephrase the conformal metric as a section of a weighted natural bundle, where we use the identification of $\partial\mathcal{H}$ as a homogeneous parabolic geometry G/P with P a minimal parabolic subgroup. Recall from Theorem 1.4.5 that density bundles on G/P are determined by linear functionals on $\mathfrak{a}^*$. Since the real rank of $\mathfrak{g}$ is 1, we can express each such functional as $\lambda \cdot E^*$ for $\lambda \in \mathbb{R}$, where $E^* \in \mathfrak{a}^*$ is the dual of the grading element E of $\mathfrak{g}$. Following standing conventions, we will write $\mathcal{E}[\lambda]$ for the line bundle $\mathcal{E}[-\lambda E^*]$ over G/P and call any section of this bundle a λ-*density*. Moreover, if W is a vector bundle over G/P, then we define $W[\lambda] := W \otimes \mathcal{E}[\lambda]$ and say that $W[\lambda]$ has *weight* λ. Note that if $\mathbb{R}[\lambda]$ is the 1-dimensional P-representation underlying $\mathcal{E}[\lambda]$, then for all $p \in P$ with $p = m \exp(tE)n$ for $m \in M$, $t \in \mathbb{R}$ and $n \in N$ we obtain by construction that $p \cdot \sigma = e^{-\lambda t}\sigma$ for all $\sigma \in \mathbb{R}[\lambda]$.

THEOREM 3.1.1. *Let $G = \mathrm{SO}(n+1,1)_0$ and $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ the minimal parabolic subgroup of G. Then the homogeneous space G/P carries a canonical G-invariant conformal metric $\boldsymbol{g}_{ab} \in \Gamma(S^2T^*(G/P)[2])$.*

PROOF. In the matrix representation of G we immediately see that the P-action on the vector space $\mathfrak{g}/\mathfrak{p}$ is determined by the action of the Levi subgroup $G_0 = MA$. Explicitly, under the identifications $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$, $M \cong \mathrm{SO}(n)$ and $A \cong \mathbb{R}_+$ this action is given by

$$(Be^t) \cdot Y = e^{-t}BY$$

for all $Y \in \mathbb{R}^n$, $B \in \mathrm{SO}(n)$ and $t \in \mathbb{R}$. In particular, the standard inner product on $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$ defines a P-invariant element in $S^2(\mathfrak{g}/\mathfrak{p})^*[2]$, which induces a G-invariant section $\mathbf{g}_{ab} \in \Gamma(S^2T^*(G/P)[2])$ due to Theorem 1.2.2. $\square$

3.1.2. The Poincaré ball model. An important feature of the real hyperbolic space is the existence of the Poincaré ball model, which provides a strong connection between Euclidean and hyperbolic geometry. In particular, it is a convenient setting to derive norm estimates for Poisson transforms in section 3.3.3, so we will recall its basic features in this section.

Let B^{n+1} be the open unit ball in $\mathbb{R}^{n+1}$ endowed with the Riemannian metric $g_{\mathrm{hyp}} = \rho^{-2}g_{\mathrm{euc}}$, where g_{euc} is the Euclidean metric on B^{n+1} and $\rho \in C^\infty(B^{n+1}, \mathbb{R})$ is the smooth map $\rho(x) = \frac{1}{2}(1-|x|^2)$, which is a defining function for the boundary sphere. If we write a point $x \in B^{n+1}$ as $x = (x^0, \tilde{x})$ with $x^0 \in \mathbb{R}$ and $\tilde{x} \in \mathbb{R}^n$, and if we identify the hyperbolic space $\mathcal{H}$ with the set of all negative vectors $v = (v^0, \dots, v^{n+1}) \in \mathbb{V}_-$ with $\langle v, v\rangle = -1$ and $v^0 - v^{n+1} > 0$, a direct computation shows that

$$f\colon (B^{n+1}, g_{\mathrm{hyp}}) \to (\mathcal{H}, g_K)$$
$$(x^0, \tilde{x}) \mapsto \frac{1}{2(1-|x|^2)} \begin{pmatrix} 1 + 2x^0 + |x|^2 \\ 2\sqrt{2}\tilde{x} \\ -(1 - 2x^0 + |x|^2) \end{pmatrix}$$

is an isomorphism of Riemannian manifolds. We call $(B^{n+1}, g_{\mathrm{hyp}})$ the *Poincaré ball model* of the $(n+1)$-dimensional real hyperbolic space.

Furthermore, in this picture the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is identified with the boundary at infinity $\partial\mathcal{H}$ as follows: If $y = (y^0, \tilde{y})$ is a unit vector in $\mathbb{R}^{n+1}$ with $y^0 \in \mathbb{R}$ and $\tilde{y} \in \mathbb{R}^n$, then the vector $v(y) = \frac{1}{\sqrt{2}}(y^0 + 1, \tilde{y}^t, y^0 - 1) \in \mathbb{V}$ is null with respect to the Lorentzian inner product on $\mathbb{V}$. Now if $[v]$ denotes the line generated by $v \in \mathcal{C}$, then a direct computation shows that the map $y \mapsto [v(y)]$ defines a diffeomorphism $S^n \to \partial\mathcal{H}$ so that the conformal structure on the boundary at infinity of the hyperbolic space corresponds to the equivalence class of all metrics which are a positive multiple of the round metric on S^n.

The advantage of the Poincaré ball model is that the hyperbolic metric is given by a conformal change of the Euclidean metric and therefore easy to understand. However, it is not quite obvious how the group G acts in this picture. In order to obtain estimates on the pointwise norm of the Poisson transform on the Poincaré ball model in terms of a defining function of the boundary in section 3.3.3 we need to express the A-part in the Iwasawa decomposition of elements in G in terms of coordinates on B^{n+1} via the action on its origin.

LEMMA 3.1.2. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup and denote by E the grading element in the Lie algebra of G. For all*

$g \in G$ let $g = \kappa(g)\exp(\tau(g)E)\exp(Y(g))$ be its Iwasawa decomposition, where $\kappa(g) \in K$, $\tau(g) \in \mathbb{R}$ and $Y(g) \in \mathfrak{n} \cong \mathbb{R}^n$. Let B^{n+1} be the Poincaré ball model of the real hyperbolic space G/K and let $\mathbf{0}$ be the origin in B^{n+1}. If $x \in B^{n+1}$ satisfies $g \cdot x = \mathbf{0}$, then

$$e^{\tau(g)} = \frac{1 - 2x^0 + |x|^2}{1 - |x|^2}.$$

Proof. Under the isomorphism $f\colon B^{n+1} \to \mathcal{H}$ the origin $\mathbf{0}$ is mapped to $\frac{1}{2}$ times the difference of the first and last basis vector in $\mathbb{R}^{n+1,1}$. Since this vector is invariant under the action of the maximal compact subgroup we can assume without loss of generality that $\kappa(g) = 0$, hence in terms of the matrix representation of G from section 3.1.1 the element g is given by

$$g = \begin{pmatrix} e^{\tau(g)} & -e^{\tau(g)}Y(g)^t & -\frac{e^{\tau(g)}}{2}\|Y(g)\|^2 \\ 0 & \mathrm{id}_n & Y(g) \\ 0 & 0 & e^{-\tau(g)} \end{pmatrix}.$$

In particular, the last component of the equation $g \cdot f(x) = f(\mathbf{0})$ reads as

$$-e^{-\tau(g)}\frac{1 - 2x^0 + |x|^2}{2(1 - |x|^2)} = -\frac{1}{2},$$

showing the claim. □

The previous result is used to derive norm estimates for the Poisson transforms Φ_k constructed in section 2.3.2 in the special case of the real hyperbolic space as follows: We will show in section 3.3.3 that for all $g \in G$ the pointwise hyperbolic norm of $\Phi_k(gK)$ is bounded from above by the integral of a power of $e^{-\tau(g^{-1}h)}$ over $hM \in K/M$. Now Lemma 3.1.2 allows us to rewrite this smooth function as the classical Poisson kernel on the unit ball in the following Proposition, for which several integral identities are available in the literature.

Proposition 3.1.2. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup and denote by E the grading element in the Lie algebra of G. For all $g \in G$ let $g = \kappa(g)\exp(\tau(g)E)\eta(g)$ be its Iwasawa decomposition, where $\kappa(g) \in K$, $\tau(g) \in \mathbb{R}$ and $\eta(g) \in N$. Let B^{n+1} be the Poincaré ball model of the real hyperbolic space G/K and let $\mathbf{0}$ be the origin in B^{n+1}. For $g \in G$ and $h \in K$ let $x = g \cdot \mathbf{0} \in B^{n+1}$ and $u = h \cdot \mathbf{1}$ be in the unit sphere S^n, where $\mathbf{1} \in S^n$ denotes the first unit vector in $\mathbb{R}^{n+1}$. Then we obtain that*

$$e^{-\tau(g^{-1}h)} = P(x,u) := \frac{1 - |x|^2}{|x - u|^2}$$

equals the classical Poisson kernel on the unit ball in $\mathbb{R}^{n+1}$.

Proof. We can apply Lemma 3.1.2 to the element $h^{-1}x$, obtaining that

$$e^{-\tau(g^{-1}h)} = \frac{1 - |h^{-1}x|^2}{1 - 2(h^{-1}x)^0 + |h^{-1}x|^2}.$$

A direct computation shows that the restriction of the G-action on B^{n+1} to the maximal compact subgroup K coincides with the standard action of $\mathrm{SO}(n+1)$, implying that that $|h^{-1}x|^2 = |x|^2$. Furthermore, if the matrix $B \in \mathrm{SO}(n+1)$ corresponds to the element $h \in K$ we deduce that the first coordinate of $h^{-1}x$ coincides with the standard inner product of x with the first column vector of B and thus by construction with $\langle x, u\rangle$. Since the element u is in the unit circle the above expression simplifies to

$$e^{-\tau(g^{-1}h)} = \frac{1 - |x|^2}{|u|^2 - 2\langle x, u\rangle + |x|^2} = P(x,u). \qquad \square$$

3.2. The geometry of G/M and the standard tractor bundle

For the construction of Poisson kernels we will proceed as in section 2.3 and analyse in 3.2.1 the geometry of the tangent bundle of G/M by identifying the structure of $\mathfrak{g}/\mathfrak{m}$ as an M-representation. In particular, we will identify in 3.2.3 the bundle $G/M \to G/K$ as a sphere bundle in terms of the Poincaré ball model of hyperbolic space. Moreover, in section 3.2.2 we also determine the geometric properties of the homogeneous vector bundles over G/M, G/K and G/P, which are induced by the standard representation $\mathbb{V}$ of G. This analysis will be essential in section 3.4 where we demonstrate the efficiency of our approach by constructing a BGG-compatible Poisson kernel whose associated Poisson transform map between vector valued differential forms.

3.2.1. The geometry of $T(G/M)$. First, we want to identify the decomposition of the tangent bundle of G/M into G-invariant subbundles as in Proposition 2.3.1. For this, we have to express the decomposition of the M-module $\mathfrak{g}/\mathfrak{m}$ into the direct sum $(\mathfrak{g}/\mathfrak{m})_{-1} \oplus (\mathfrak{g}/\mathfrak{m})_0 \oplus (\mathfrak{g}/\mathfrak{m})_1$ induced by the $|1|$-grading of $\mathfrak{g}$ in terms of its matrix representation. Recall from Proposition 3.1.1(ii) that the subalgebra $\mathfrak{m}$ consists of those matrices in $\mathfrak{g}$ which contain an element of $\mathfrak{so}(n)$ in their middle block as the only nontrivial entry. Therefore, we can represent $\mathfrak{g}/\mathfrak{m}$ as the set

$$\mathfrak{g}/\mathfrak{m} = \left\{ \begin{pmatrix} a & -Y^t & 0 \\ X & 0 & Y \\ 0 & -X^t & -a \end{pmatrix} + \mathfrak{m} \,\middle|\, a \in \mathbb{R}, X, Y \in \mathbb{R}^n \right\},$$

and for $\xi \in \mathfrak{g}/\mathfrak{m}$ the induced map $\xi \mapsto (X, a, Y)$ corresponds to the direct sum decomposition of $\mathfrak{g}/\mathfrak{m}$ as above.

Next, we determine the action of M on $\mathfrak{g}/\mathfrak{m}$ with respect to this decomposition. Let $m \in M$ correspond to the matrix $B \in \mathrm{SO}(n)$ and $\xi \in \mathfrak{g}/\mathfrak{m}$ to the triple (X, a, Y) with X, $Y \in \mathbb{R}^n$ and $a \in \mathbb{R}$. Then a direct computation shows that the action of m on ξ is given by $\underline{\mathrm{Ad}}(m)(\xi) = (BX, a, BY)$. Otherwise said, after identifying $(\mathfrak{g}/\mathfrak{m})_{\pm 1}$ with $\mathbb{R}^n$ the group $M \cong \mathrm{SO}(n)$ acts by the standard action on $(\mathfrak{g}/\mathfrak{m})_{-1}$ and $(\mathfrak{g}/\mathfrak{m})_1$ and trivial on the M-invariant subspace $(\mathfrak{g}/\mathfrak{m})_0$. Applying Proposition 1.2.1 this immediately implies

LEMMA 3.2.1. *The tangent bundle of G/M decomposes into a direct sum of G-invariant subbundles*

$$T(G/M) = T^{-1}(G/M) \oplus T^0(G/M) \oplus T^1(G/M),$$

where $T^{\pm 1}(G/M)$ are subbundles of rank n, whereas $T^0(G/M)$ is a line bundle.

For the computations needed in the construction of Poisson kernels in sections 3.3 and 3.4 we define generators for the M-module $\mathfrak{g}/\mathfrak{m}$ which are compatible with the decomposition of $T(G/M)$ into horizontal and vertical distributions from Theorem 2.3.1. In order to do so, recall from section 2.3.1 that these bundles are associated to the M-submodules $(\mathfrak{g}/\mathfrak{m})^{1,0} = \mathfrak{p}/\mathfrak{m}$ and $(\mathfrak{g}/\mathfrak{m})^{0,1} = \mathfrak{k}/\mathfrak{m}$ of $\mathfrak{g}/\mathfrak{m}$, respectively. Thus, with respect to the decomposition of $\mathfrak{g}/\mathfrak{m}$ into irreducible M-modules we define for all $X \in \mathbb{R}^n$ the vectors

$$E := (0,1,0) \in \mathfrak{p}/\mathfrak{m}, \qquad F_X := (0,0,X) \in \mathfrak{p}/\mathfrak{m}, \qquad G_X := (X,0,X) \in \mathfrak{k}/\mathfrak{m},$$

and these elements generate $\mathfrak{g}/\mathfrak{m}$ as a vector space.

Moreover, in order to determine the properties of the Poisson kernels in the sequel we will heavily use the calculus on $\Lambda^*(\mathfrak{g}/\mathfrak{m})$ induced by the structure of G/K as a Riemannian manifold. Thus, it is necessary to compute the pullback of the hyperbolic metric on G/K to the horizontal subbundle $T^{1,0} = G \times_M (\mathfrak{p}/\mathfrak{m})$.

PROPOSITION 3.2.1. *Let $G = \mathrm{SO}(n+1,1)_0$ be represented as in section 3.1.1, $\mathfrak{g}$ its Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the Cartan decomposition with respect to the involution $g \mapsto -g^t$. Let $K \cong \mathrm{SO}(n+1)$ be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, let $M \cong \mathrm{SO}(n)$ be the intersection of K with the minimal parabolic P in G and $\mathfrak{m}$ the Lie algebra of M. Then the pullback of the hyperbolic metric along the canonical projection $\pi_K \colon G/M \to G/K$ is induced by the M-invariant inner product g_M on the M-subrepresentation $\mathfrak{p}/\mathfrak{m}$ in $\mathfrak{g}/\mathfrak{m}$ which is determined by*

$$g_M(E,E) = 1, \qquad\qquad g_M(F_X, F_Y) = \frac{1}{2}\langle X, Y\rangle_{\mathbb{R}^n}$$

for all X, $Y \in \mathbb{R}^n$.

PROOF. Recall from section 3.1.1 that the hyperbolic metric on G/K was induced by the standard inner product of $K \cong \mathrm{SO}(n+1)$ on $\mathfrak{q} \cong \mathbb{R}^{n+1}$. Thus, in order to determine the M-invariant inner product on $\mathfrak{p}/\mathfrak{m}$ we first have to apply the projection $\mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{k}$ and identify the result with $\mathfrak{q}$. Doing this for the base elements, a direct computation shows that E is mapped to the first basis vector, whereas F_X corresponds to $-\sqrt{2}^{-1}(0, X^t)^t$ in $\mathfrak{q}$. $\square$

Note that we could extend g_M to $\mathfrak{k}/\mathfrak{m}$ via

$$g_M(G_Z, G_W) = \tau\langle Z, W\rangle$$

for an arbitrary positive real number τ, obtaining an M-invariant inner product on $\mathfrak{g}/\mathfrak{m}$ with the property that the horizontal and vertical subspaces are orthogonal. Here, the fact that the factor τ can be chosen arbitrarily reflects that G/P carries the structure of a conformal manifold, c.f. Theorem 3.1.1. We will use such an extension of g_M in section 3.3.3 to derive norm estimates on the images of the Poisson transform Φ_k from Theorem 2.3.2 on the Poincaré ball model of real hyperbolic space.

As a last step, we deduce formulae for calculus on forms on $\mathfrak{g}/\mathfrak{m}$ which will be needed to analyse the ring of Poisson kernels. The invariant inner product g_M on $\mathfrak{p}/\mathfrak{m}$ induces an isomorphism $^\flat \colon \mathfrak{p}/\mathfrak{m} \to (\mathfrak{p}/\mathfrak{m})^*$ via $\xi^\flat(\eta) = g_M(\xi, \eta)$ for all ξ, $\eta \in \mathfrak{p}/\mathfrak{m}$. Moreover, we denote by $*_K$ the operator on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ induced by the Hodge star operator on G/K as defined in section 2.2.3. Note that if $\langle\ ,\ \rangle_M$ denotes the M-invariant inner product on $\Lambda^k(\mathfrak{p}/\mathfrak{m})^*$ induced by g_M, then this operator is characterized by the equation

$$\langle \alpha, \beta\rangle_M = *_K\left(\alpha \wedge *_K \beta\right)$$

for all α, $\beta \in \Lambda^k(\mathfrak{p}/\mathfrak{m})^*$.

COROLLARY 3.2.1. *Let ω be a form on $\mathfrak{g}/\mathfrak{m}$ of degree (k, ℓ).*

(i) For all $\xi \in \mathfrak{p}/\mathfrak{m}$ we have $\iota_\xi(*_K\omega) = (-1)^k *_K (\xi^\flat \wedge \omega)$ as well as $\xi^\flat \wedge *_K\omega = (-1)^{k+1} *_K (\iota_\xi\omega)$.

(ii) For all $\eta \in \mathfrak{k}/\mathfrak{m}$ we get $\iota_\eta(*_K\omega) = (-1)^{n+1} *_K (\iota_\eta\omega)$.

PROOF. It suffices to consider decomposable elements $\omega = \alpha \wedge \beta$, where α and β are forms on $\mathfrak{g}/\mathfrak{m}$ of degrees $(k, 0)$ and $(0, \ell)$, respectively.

(i) Let $\langle\ ,\ \rangle_M$ denote the inner product on $\Lambda^*(\mathfrak{p}/\mathfrak{m})^*$ induced by g_M. By construction, we have $\langle \gamma, \xi^\flat \wedge \alpha\rangle_M = \langle \iota_\xi\gamma, \alpha\rangle_M$ for all $\xi \in \mathfrak{p}/\mathfrak{m}$ and $\gamma \in \Lambda^{k+1}(\mathfrak{p}/\mathfrak{m})^*$. Therefore, the defining equation for the K-Hodge star yields

$$*_K\left(\gamma \wedge *_K\left(\alpha \wedge \xi^\flat\right)\right) = \langle \gamma, \alpha \wedge \xi^\flat\rangle_M = (-1)^k\langle \iota_\xi\gamma, \alpha\rangle_M = *_K\left(\gamma \wedge \iota_\xi *_K \alpha\right),$$

showing that $\iota_\xi *_K \alpha = *_K(\alpha \wedge \xi^\flat)$. For the form ω, this means that

$$\iota_\xi *_K \omega = \iota_\xi(*_K\alpha) \wedge \beta = *_K(\alpha \wedge \xi^\flat) \wedge \beta = (-1)^k *_K (\xi^\flat \wedge \omega).$$

The second equation is obtained by applying the first relation to the form $*_K\omega$ and afterwards the K-Hodge star to both sides of the resulting equation.

(ii) The antiderivation property of the interior product implies

$$\iota_\eta *_K \omega = (-1)^{n+1-k}(*_K\alpha) \wedge \iota_\eta\beta = (-1)^{n+1} *_K \iota_\eta\omega,$$

where we used that ι_η is trivial on both α and its image under the K-Hodge star. □

3.2.2. The standard tractor bundle. Although we have defined Poisson transforms for differential forms with values in general homogeneous vector bundles in section 2.2.1, we have so far only constructed explicit examples in the real valued case. Therefore we want to demonstrate the efficiency of our method by constructing in section 3.4 a Poisson kernel whose associated transform maps between vector valued differential forms. For that, we need to choose appropriate homogeneous vector bundles over G/P and G/K, which by 1.2.1 are determined by representations of the Lie groups P and K, respectively.

Here, the simplest examples are given by representations of the full group G, which induce tractor bundles over all homogeneous spaces G/H for any closed subgroup H of G, c.f. section 1.2.1. Hence we consider the homogeneous bundle over G/H associated to the standard representation $\mathbb{V}$ of G, to which we will refer as the *standard tractor bundle (over G/H)*. In this section, we will identify the geometric structure of the standard tractor bundles over the homogeneous spaces G/M, G/K and G/P.

Let $\mathbb{V} := \mathbb{R}^{n+1,1}$ be the standard representation of G. Then by Proposition 1.4.1(i) the grading element E of $\mathfrak{g}$ acts diagonalisable on $\mathbb{V}$, inducing an eigendecomposition $\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1$, where $\mathbb{V}_j$ is the eigenspace with eigenvalue j. Explicitly, the spaces $\mathbb{V}_{-1}$ and $\mathbb{V}_1$ are generated by the first and last standard basis vector in $\mathbb{V}$, respectively, whereas $\mathbb{V}_0$ is isomorphic to $\mathbb{R}^n$. With respect to this decomposition we define elements

$$v_1 := \begin{pmatrix}1\\0\\0\end{pmatrix} \in \mathbb{V}_1, \qquad v_Z := \begin{pmatrix}0\\Z\\0\end{pmatrix} \in \mathbb{V}_0, \qquad v_{-1} := \begin{pmatrix}0\\0\\1\end{pmatrix} \in \mathbb{V}_{-1}$$

for all $Z \in \mathbb{R}^n$.

THEOREM 3.2.2. *Let $G = \mathrm{SO}(n+1,1)_0$ be realized as a matrix group as in 3.1.1 and let $M \cong \mathrm{SO}(n)$ the diagonal subgroup. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of G and V be the standard tractor bundle over G/M. Then V splits into a Whitney sum*

$$V = V_{-1} \oplus V_0 \oplus V_1$$

of G-invariant subbundles. Here, V_{-1} and V_1 are trivial line bundles, whereas V_0 is a vector bundle of rank n.

PROOF. We have to analyse the structure of $\mathbb{V}$ as an M-module. By definition of the matrix representation of G we immediately see that the decomposition of $\mathbb{V}$ into eigenspaces of the action of the grading element is M-invariant. Explicitly, the group M acts trivially on $\mathbb{V}_1$ and $\mathbb{V}_{-1}$, whereas $\mathbb{V}_0 \cong \mathbb{R}^n$ is the standard representation of $M \cong \mathrm{SO}(n)$. Applying the associated bundle construction implies the claimed decomposition of V. □

In the construction of the Poisson transform we also consider the standard tractor bundles over the Riemannian symmetric space G/K as well as the conformal sphere G/P, so we have to analyse the structure of the standard representation as a K-module and a P-module as well.

PROPOSITION 3.2.2. *Let* $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ *its maximal compact subgroup,* $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ *its minimal parabolic subgroup and* $\mathbb{V} = \mathbb{R}^{n+1,1}$ *be the standard representation of* G.

(i) The standard tractor bundle $V_P = G \times_P \mathbb{V}$ *over* G/P *is endowed with a filtration*

$$V_P = V^{-1} \supset V^0 \supset V^1 \supset \{0\}$$

of G*-invariant subbundles. The induced quotient bundles are irreducible and isomorphic to*

$$V^{-1}/V^0 \cong \mathcal{E}[1], \qquad V^0/V^1 \cong T(G/P)[-1], \qquad V^1 \cong \mathcal{E}[-1]$$

as homogeneous bundles over G/P.

(ii) The standard tractor bundle $V_K = G \times_K \mathbb{V}$ *over* G/K *splits into a Whitney sum of* G*-invariant subbundles*

$$V_K \cong T(G/K) \oplus L,$$

where L *is a trivial line bundle.*

PROOF. (i) Via the associated bundle construction it suffices to prove the corresponding properties of the P-representation $\mathbb{V}$. In part (ii) of Proposition 1.4.1 we have seen that $\mathbb{V}$ carries a P-invariant filtration

$$\mathbb{V} = \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1 \supset \{0\},$$

where the i-th filtration component is defined by $\mathbb{V}^i := \oplus_{j \geq i} \mathbb{V}_i$. Moreover, the nilpotent subgroup P_+ of P maps $\mathbb{V}^i$ to $\mathbb{V}^{i+1}$ for $i \in \{-1, 0, 1\}$, implying that it acts trivially on the induced representations $\mathbb{V}^i/\mathbb{V}^{i+1}$. By Proposition 1.4.1(i), it follows that these quotients are irreducible P-representations, so the P-action is already determined by its Levi subgroup $G_0 \cong \mathrm{CSO}(n)$. Explicitly, if $(b, B) \in \mathrm{CSO}(n)$ with $b \in \mathbb{R}^*$ and $B \in \mathrm{SO}(n)$ corresponds to $g \in G_0$, then a direct computation shows that

$$g \cdot \begin{pmatrix} \lambda \\ X \\ \mu \end{pmatrix} = \begin{pmatrix} b\lambda \\ BX \\ b^{-1}\mu \end{pmatrix}.$$

Comparing this with the formula for the G_0-action on $\mathfrak{g}/\mathfrak{p}$ in the proof of Theorem 3.1.1 we deduce that $\mathbb{V}^{-1}/\mathbb{V}^0 \cong \mathbb{R}[1]$, $\mathbb{V}^0/\mathbb{V}^1 \cong (\mathfrak{g}/\mathfrak{p})[-1]$ and $\mathbb{V}^1 \cong \mathbb{R}[-1]$ as P-modules, and from Proposition 1.2.1 we obtain the corresponding isomorphisms on the level of vector bundles.

(ii) By definition, the maximal compact subgroup K is the stabilizer of a negative line in $\mathbb{V}$ as well as its orthogonal complement with respect to the standard Lorentzian product. In this way, we obtain a decomposition $\mathbb{V} = \mathbb{R}^{n+1} \oplus \mathbb{R}$ as a K-module, where the group K acts on $\mathbb{R}^{n+1}$ by the standard representation and trivial on the 1-dimensional part. Explicitly, the trivial representation is generated by the vector $v_1 - v_{-1}$, whereas $\mathbb{R}^{n+1}$ is the direct sum of $\mathbb{V}_0$ and the line generated by $v_1 + v_{-1}$. Since $\mathfrak{g}/\mathfrak{k}$ is also isomorphic to the standard representation of K the claim follows by applying the associated bundle construction. □

Finally, for computations in section 3.4 we need to determine formulae for the induced $\mathfrak{g}$-action on $\mathbb{V}$ in terms of its decomposition into irreducible M-modules. By definition, the $|1|$-grading of $\mathfrak{g}$ is compatible with the eigendecomposition of $\mathbb{V}$ in the sense that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$, where we agree that $\mathbb{V}_{i+j}$ is trivial if $|i + j| > 1$.

Now if we identify $\mathfrak{g}_{\pm 1}$ with $\mathbb{R}^n$ and $\mathfrak{g}_0$ with $\mathbb{R} \oplus \mathfrak{so}(n)$ as in section 3.1.1, a direct computation shows that the only nontrivial relations are given by

$$\begin{aligned}
\mathfrak{g}_{-1} \otimes \mathbb{V}_0 &\to \mathbb{V}_{-1}, & X \otimes v_Z &\mapsto -\langle X, Z\rangle v_{-1},\\
\mathfrak{g}_{-1} \otimes \mathbb{V}_1 &\to \mathbb{V}_0, & X \otimes v_1 &\mapsto v_X,\\
\mathfrak{g}_0 \otimes \mathbb{V}_1 &\to \mathbb{V}_1, & (B,a) \otimes v_1 &\mapsto a v_1,\\
\mathfrak{g}_0 \otimes \mathbb{V}_0 &\to \mathbb{V}_0, & (B,a) \otimes v_Z &\mapsto v_{B\cdot Z},\\
\mathfrak{g}_0 \otimes \mathbb{V}_{-1} &\to \mathbb{V}_{-1}, & (B,a) \otimes v_{-1} &\mapsto -a v_{-1},\\
\mathfrak{g}_1 \otimes \mathbb{V}_0 &\to \mathbb{V}_1, & Y \otimes v_Z &\mapsto -\langle Y, Z\rangle v_1,\\
\mathfrak{g}_1 \otimes \mathbb{V}_{-1} &\to \mathbb{V}_0, & Y \otimes v_{-1} &\mapsto v_Y
\end{aligned}$$

for all X, Y, $Z \in \mathbb{R}^n$, $a \in \mathbb{R}$ and $B \in \mathrm{SO}(n)$.

3.2.3. Geometric interpretation of the bundle $G/M \to G/K$. This section is dedicated to give a geometric interpretation of the homogeneous space G/M as a sphere bundle over G/K in terms of the Poincaré ball model (c.f. section 3.1.2). This will lead us in section 3.3.2 back to Gaillard's initial definition of Poisson transforms. In order to do so, we will use some additional background on the geometric properties of the Poincaré ball model in the sequel, which can be found in [**CFKW97**].

Let B be the open unit ball in $\mathbb{R}^{n+1}$ endowed with the hyperbolic metric and denote its boundary sphere by ∂B. For all $x \in B$ we denote by S^r_x the sphere of radius $r > 0$ in the tangent space T_xB. The union $S^r := \bigcup_{x\in B} S^r_x$ of all tangent spheres inherits the structure of a smooth manifold so that the canonical projection $\pi\colon S^r \to B$ is a surjective submersion. We call the fibre bundle $S^r \to B$ the *r-sphere bundle* of B. Note that the action of G of B also induces an action on the sphere bundle S^r, which is locally given by $g \cdot (x, u) = (gx, T_x\ell_g \cdot u)$ for $x \in B$, $u \in S^r_x$ and $g \in G$.

The Levi-Civia connection ∇ on B induces a splitting of the tangent bundle $TS^r_x \subset TTB$ into a Whitney sum $H \oplus V$, where V is the kernel of $T\pi$ and H is the horizontal subbundle determined by the connection. Using this, we can endow the sphere bundle S^r with the structure of a Riemannian manifold as follows: First, we assume that the subbundles H and V are orthogonal. Moreover, in every point $x \in B$ the tangent map $T_x\pi$ is a linear isomorphism $H_x \cong T_xB$, so we can pullback the bundle metric of the tangent bundle of B to a metric on the distribution H. By construction, this means that the canonical projection is a Riemannian submersion. Finally, on the vertical bundle we define the metric to be induced by the inclusion of the r-sphere into T_xB, where we used the identification of the tangent spaces of T_xB with T_xB itself. Therefore, the sphere bundle S^r is a Riemannian manifold, and we denote by g^{S^r} its Riemannian metric.

LEMMA 3.2.3. *Let (B, g_{hyp}) be the Poincaré ball model of $(n+1)$-dimensional hyperbolic space and for $r > 0$ let $\pi\colon S^r \to B$ be the r-sphere bundle of B. Then the Riemannian metric g^{S^r} on S^r is invariant with respect to the induced action of $G = \mathrm{SO}(n+1,1)_0$.*

PROOF. For $x \in B$ fix a point $u \in S^r_x$ and let ξ, $\eta \in T_uS^r$ be tangent vectors. Since the subbundles H and V are orthogonal it suffices to assume that ξ and η are both either in H or in V. First, if both vectors are in the horizontal distribution, then

$$\begin{aligned} g^{S^r}_{gu}(g \cdot \xi, g \cdot \eta) &= g_{\pi(gu)}(T_{gu}\pi \circ T_u(T_{\pi(u)}\ell_g) \cdot \xi, T_{gu}\pi \circ T_u(T_{\pi(u)}\ell_g) \cdot \eta) \\ &= g_{g\pi(u)}(T_{\pi(u)}\ell_g \circ T_u\pi \cdot \xi, T_{\pi(u)}\ell_g \circ T_u\pi \cdot \eta) \\ &= g^{S^r}_u(\xi, \eta), \end{aligned}$$

where we used G-equivariance of the projection π as well as G-invariance of the Riemannian metric on B.

On the other hand, if both vectors ξ and η are in the vertical distribution, we have to identify those vectors with elements in T_xB. Explicitly, we can find vectors v, $w \in T_xB$ with $\xi = \partial_t|_{t=0}(u + tv) =: \chi_u(v)$ and $\eta = \chi_u(w)$. By construction, the map χ_u satisfies

$$\chi_{gu}(gv) = \partial_t|_{t=0}T_x\ell_g \cdot (u + tv) = T_u(T_x\ell_g) \cdot \partial_t|_{t=0}(u + tv) = g \cdot \chi_u(v)$$

for all $v \in T_xB$. In particular, for $\xi = \chi_u(v)$ and $\eta = \chi_u(w)$ this means that

$$g^{S^r}_{gu}(g \cdot \xi, g \cdot \eta) = g^{S^r}_{gu}(g \cdot \chi_u(v), g \cdot \chi_u(w)) = g_{gx}(T_x\ell_g \cdot v, T_x\ell_g \cdot w) = g^{S^r}_u(\xi, \eta).$$

All in all, we see that the metric g^{S^r} is G-invariant. □

In the next Proposition we will show that the r-sphere bundle S^r over B is diffeomorphic to the product bundle $B \times \partial B$ as a G-manifold, and that the isomorphism is induced by the so called visual maps. In order to construct them, recall from [**Thu**, ch. 2.1] that for all $x \in B$ and $y \in \partial B$ there is a unique (hyperbolic) geodesic ray $\Gamma_{x,y} \colon \mathbb{R}_{\geq 0} \to B$ which starts at $\Gamma_{x,y}(0) = x$ and approaches the point y in the limit. In the Euclidean picture, the ray $\Gamma_{x,y}$ is a circle segment starting at x, which is orthogonal to the boundary in the point y. This means that for all $y \in \partial B$ there is a unique vector $\gamma^r_x(y) \in S^r_x$ so that the geodesic ray determined by this vector equals $\Gamma_{x,y}$. Moreover, it can be shown that the induced map $\gamma^r_x \colon \partial B \to S^r_x$ given by $y \mapsto \gamma^r_x(y)$ is a diffeomorphism. Thus, for every $r > 0$ the induced map

$$\gamma^r \colon B \times \partial B \to S^r, \qquad (x, y) \mapsto \gamma^r_x(y)$$

is a bundle isomorphism covering the identity on B. The inverse of this map is given by $(x, \xi) \mapsto V^r_x(\xi)$, where $V^r_x \colon S^r_x \to \partial B$ is the *visual map* defined by $\xi \mapsto \lim_{t\to\infty} \gamma_\xi(t)$. Note that this is exactly the definition of the visual map suggested by Thurston and used by Gaillard to construct his Poisson transform, c.f. section 2.1.3.

PROPOSITION 3.2.3. *Let (B, g_{hyp}) be the Poincaré ball model of hyperbolic space and let ∂B denote its boundary sphere. For $r > 0$ let $S^r \to B$ be the r-sphere bundle and let $\gamma^r \colon B \times \partial B \to S^r$ be the bundle isomorphism defined above. Then γ^r is an equivariant map for the induced actions of the group $G = \mathrm{SO}(n + 1, 1)$.*

PROOF. Let $\Gamma_{x,y} \colon \mathbb{R}_{\geq 0} \to B$ be the unique geodesic ray which start at $x \in B$ and tends to $y \in \partial B$. We have already seen that the group G acts transitively on the product $B \times \partial B \cong G/M$. Moreover, for all $g \in G$ the geodesic ray $\Gamma_{gx,gy}$ coincides with the translation $g \circ \Gamma_{x,y}$. If we derive this equation we obtain that

$$\gamma^r_{gx}(gy) = \Gamma'_{gx,gy}(0) = T_x\ell_g \cdot \Gamma'_{x,y}(0) = T_x\ell_g \cdot \gamma^r_x(y),$$

which shows that γ^r is G-equivariant. □

Thus, we see that G-invariant Riemannian metrics on the product manifold $G/K \times G/P$ are obtained via the identification with the sphere bundles over $B \cong G/K$. Moreover, via the G-isomorphism $G/M \cong G/K \times G/P$ we see that the visual map is just the identification of the fibre $(G/M)_x$ of the bundle $G/M \to B$ over $x \in B$ with the fibre of the r-sphere bundle S^r_x. Here, the ambiguity given by the choice of the radius $r > 0$ of the spheres is reflected in the conformal choice of the metric on the vertical subbundle $T^{0,1}$ of $T(G/M)$.

3.3. Poisson transforms between real valued forms

In this section we further analyse the Poisson transforms Φ_k constructed in Theorem 2.3.2 as operators between the conformal sphere and the real hyperbolic space. First, due to the simple structure of the Lie algebra $\mathfrak{so}(n+1,1)$ the image of these operators consists of eigenforms for the Laplace Beltrami operator induced by the hyperbolic metric, and these Poisson transforms are chain maps between the de Rham complexes of G/P and G/K, c.f. Theorem 3.3.1. Afterwards, the explicit computation of the dimension of the space of Poisson kernels in Theorem 3.3.2 implies that the operators Φ_k are essentially unique and thus coincide with Gaillard's Poisson transforms, c.f. section 2.1.3. Finally, the existence of a geometric model describing both the hyperbolic space and its boundary in a unified picture allows us to gain norm estimates on the images of Φ_k in section 3.3.3. In particular, we will reprove many of Gaillard's results in his dissertation [**Gai86**] in a simpler and more conceptual way.

3.3.1. The Poisson transform Φ_k. Recall from Theorem 2.3.2 that we have constructed a Poisson kernel ϕ_k of degree $(k, n-k)$, whose associated transform Φ_k maps k-forms on the conformal sphere to coclosed differential forms of degree k on the real hyperbolic space. Since in the special case $G = \mathrm{SO}(n+1,1)_0$ the Lie algebra $\mathfrak{g}$ has a simple structure, these operators possess even more interesting properties, which are summarized in the following Theorem.

THEOREM 3.3.1. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup and $P = \mathrm{CSO}(n) \ltimes \mathbb{R}^n$ the minimal parabolic subgroup of G. Then for all $k = 0, \dots, n$ and $\lambda \in \mathbb{R}$ the operator*

$$\Phi_k \colon \Omega^k(G/P, \mathcal{E}[\lambda]) \to \Omega^k(G/K)$$

constructed in Theorem 2.3.2 is the unique Poisson transform up to a multiple which preserves the degree. Moreover, these operators satisfy the following relations:

(i) The image of Φ_k consists of coclosed forms for all $k = 0, \dots, n$ and all $\lambda \in \mathbb{R}$.

(ii) Let $\mathrm{vol}_K \in \Omega^{n+1,0}(G/M)$ denotes the K-volume form on G/M, i.e. the pullback of the hyperbolic volume form along the canonical projection. Then for all $\alpha \in \Omega^n(G/P, \mathcal{E}[\lambda])$ we have

$$d\Phi_n(\alpha) = (\lambda - n) \fint_{G/P} \pi_P^* \alpha \wedge \mathrm{vol}_K \,.$$

(iii) For all $\alpha \in \Omega^k(G/P, \mathcal{E}[\lambda])$ the differential form $\Phi_k(\alpha)$ is an eigenform of the Laplace-Beltrami operator on G/K with eigenvalue $-\lambda(n - 2k + \lambda)$.

Moreover, if $\lambda = 0$, then we have

$$d \circ \Phi_k = (-1)^{n-k}(n-2k)\Phi_{k+1} \circ d.$$

for all $k = 0, \dots, n-1$.

PROOF. We have already shown (i) in a more general setting in Theorem 2.3.2. Furthermore, we have seen in section 3.1.1 that the Lie algebra $\mathfrak{g}$ is $|1|$-graded, so Corollary 2.2.3 implies part (iii) in the case $\lambda = 0$. Finally, the claim on uniqueness of these transforms will follow from Theorem 3.3.2.

We translate the remaining claims into properties of the corresponding Poisson transforms via Proposition 2.2.3, which we will prove in the remainder of this section. Explicitly, the relation between the exterior derivatives follow from Lemma 3.3.1 whereas (ii) and (iii) is equivalent to Proposition 3.3.1. ∎

Before we continue we briefly recall the definition of the Poisson kernels ϕ_k from section 2.3.2. Let E be the M-invariant vector in $\mathfrak{g}/\mathfrak{m}$ induced by the grading element and let $E^* \in (\mathfrak{g}/\mathfrak{m})^{1,0*}$ be its dual element. Then the Poisson kernels ϕ_k are defined by

$$\phi_k = *_K(E^* \wedge (d_P E^*)^{n-k})$$

for all $k = 0, \dots, n$ and satisfy the relations $\delta_K \phi_k = 0$ as well as $\iota_E \phi_k = 0$. Moreover, recall that we showed in Proposition 2.3.2 that $d_K E^* = 0$ and computed a formula for the P-derivative of E^* in a more general setting. Due to the explicit description of the Killing form in in Proposition 3.1.1(iv) this simplifies to $d_P E^*(F_X, G_Y) = \langle X, Y\rangle$ for all X, $Y \in \mathbb{R}^n$ in this case.

First, we will focus on the case of unweighted differential forms on G/P and compute the partial derivatives of the Poisson kernels ϕ_k. For that, let $\mathrm{vol}_K := *_K(1)$ be the M-invariant $(n+1, 0)$-form of unit length, which we will call the *K-volume form*. By construction, this is the pullback of the hyperbolic volume form on G/K, so by naturality we obtain that

$$\alpha \wedge *_K \beta = \langle \alpha, \beta\rangle_M \, \mathrm{vol}_K$$

for all $(\ell, 0)$-forms α, β on $\mathfrak{g}/\mathfrak{m}$, where $\langle\ ,\ \rangle_M$ denotes the M-invariant inner product on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ induced by g_M as in section 3.2.1.

LEMMA 3.3.1. *For all $k = 0, \dots, n$ let ϕ_k be the Poisson kernel defined above.*

(i) The K-derivative of the form ϕ_k is given by

$$d_K \phi_k = (n - 2k) E^* \wedge \phi_k.$$

In particular, if $n = 2p$ is even then the form ϕ_p is K-closed, whereas for $k = n$ we have $d_K \phi_n = -n \, \mathrm{vol}_K$.

(ii) The P-derivative of the form ϕ_k equals

$$d_P \phi_k = (-1)^k E^* \wedge \phi_{k-1}.$$

In particular, for all $0 \le k \le n-1$ the partial derivatives are related via

$$d_K \phi_k = (-1)^{k+1} (n - 2k) d_P \phi_{k+1}.$$

PROOF. (i) In order to compute the K-derivative of ϕ_k we use formula (1.2.4) together with the formulae for the Lie bracket on $\mathfrak{g}$ from section 3.1.1. If we would only insert vectors of the form F_X and G_Y into $d_K\phi$, then the Lie bracket produces the element $E \in \mathfrak{g}/\mathfrak{m}$, which contributes trivially. Therefore, we see that $d_K \phi_k = E^* \wedge \iota_E d_K \phi_k$ which by a direct computation equals $d_K \phi_k = (n - 2k) E^* \wedge \phi_k$.

Moreover, the vector E is normalized due to Proposition 3.2.1 and hence its dual coincides with $E^\flat$. Therefore, in the case $k = n$ the formula for the K-derivative reduces to

$$d_K \phi_n = -n E^* \wedge *_K(E^*) = -n \, \mathrm{vol}_K,$$

where we used the defining equation of the K-Hodge star.

(ii) Since $\dim(G/K) = n + 1$ the P-derivative satisfies the commutator relation $d_P *_K = (-1)^{n+1} *_K d_P$ due to Proposition 2.2.3. Thus, by definition of the K-Hodge star we deduce that

$$d_P \phi_k = (-1)^{n+1} *_K \left((d_P E^*)^{n-k+1}\right) = (-1)^{n+1} E^* \wedge \iota_E *_K \left((d_P E^*)^{n-k+1}\right),$$

and application of Corollary 3.2.1(i) yields the claimed formula. □

Now recall from section 1.4.5 that density bundles over G/P are associated to one dimensional P-representations, so they do not carry canonical G-invariant connections. In particular, the partial derivative d_P on G/M has no geometric counterpart on weighted differential forms on the conformal sphere. However, we can still determine the image of Laplace-Beltrami operator of the Poisson transform of such differential forms, which by Proposition 2.2.3 boils down to determining the K-Laplace of the pullback of λ-densities to G/M.

PROPOSITION 3.3.1. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup, $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ its minimal parabolic and $M := K \cap P$. For $\lambda \in \mathbb{R}$ let $\mathcal{E}[\lambda]$ denote the bundle of λ-densities over G/P and let $\phi_k \in \Omega^{k,n-k}(G/M)$ be the G-invariant Poisson kernel defined above. Then for all $\alpha \in \Omega^k(G/P, \mathcal{E}[\lambda])$ we have*

(i) $d_K\left(\phi_k \wedge \pi_P^*\alpha\right) = (d_K\phi_k + \lambda E^* \wedge \phi_k) \wedge \pi_P^*\alpha$,

(ii) $\delta_K(\phi_k \wedge \pi_P^*\alpha) = (\delta_K\phi_k - \lambda\iota_E\phi_k) \wedge \pi_P^*\alpha$.

In particular, the differential form $\phi_k \wedge \pi_P^*\alpha$ on G/M is an eigenvector of the K-Laplace with eigenvalue $-\lambda(n-2k+\lambda)$.

PROOF. (i) Follows immediately from the antiderivation property of d_K and Lemma 2.3.2.

(ii) It suffices to consider differential forms $\alpha = \omega \otimes \sigma$, where ω is a k-form and σ is a λ-density on G/P. Then by naturality, the pullback of α along the projection π_P can be expressed as $(\pi_P^*\sigma)\pi_P^*\omega$.

First, we apply the K-Hodge star to the form $\phi_\lambda := \phi \wedge \pi_P^*\alpha$, which by definition only acts on the form ϕ. Next, using the antiderivation property of d_K, naturality of the pullback of ω and the formula in part (i) we deduce that

$$d_K *_K \phi_\lambda = (d_K *_K \phi + \lambda E^* \wedge (*_K\phi)) \wedge \pi_P^*\alpha.$$

Finally, we have to apply $(-1)^{(n+1)(\ell-1)+1}$ times the K-Hodge star to this form. The first summand in the bracket equals the K-codifferential of ϕ. For the second summand, we can use part (i) of Corollary 3.2.1 to move $E^* = E^\flat$ out of the K-Hodge star, where we pick up the sign $(-1)^{n+1-\ell}$. The resulting operator $*_K^2$ equals $(-1)^{\ell(n+1-\ell)}$, so all in all we get

$$\delta_K\phi_\lambda = (\delta_K\phi - \lambda\iota_E\phi) \wedge \pi_P^*\alpha$$

as claimed.

For the last part of the Proposition we will apply parts (i) and (ii) to the case of the Poisson kernels ϕ_k. First, recall from section 2.3.2 that $\delta_K\phi_k = 0$ and $\iota_E\phi_k = 0$ in general, which implies that $\phi_{k,\lambda} := \phi_k \wedge \pi_P^*\alpha$ is trivial. Note that this just a special case of the computation in the proof of Theorem 2.3.2.

On the other hand, combining part (i) with Lemma 3.3.1(i) we get

$$d_K\phi_{k,\lambda} = (n-2k+\lambda)E^* \wedge \phi_{k,\lambda}.$$

Since the P-differential commutes with the K-codifferential up to a sign, it follows from Lemma 3.3.1 that $\delta_K(E^* \wedge \phi_k) = 0$. Therefore, applying again the formula for the K-codifferential from part (ii) we deduce that

$$\delta_K d_K \phi_{k,\lambda} = -\lambda(n-2k+\lambda)\phi_{k,\lambda},$$

so $\phi_{k,\lambda}$ is an eigenvector of Δ_K. □

In the special case $k = 0$ we have already analysed the Poisson transform $\Phi_0 \colon \Gamma(\mathcal{E}[\lambda]) \to C^\infty(G/K)$ induced by ϕ_0 in section 2.3.3. In particular, we have seen in Theorem 2.3.3 that this transform produces eigenvectors of the Laplace operator Δ^B on G/K induced by the Killing form with eigenvalues $-B(-\lambda E^*, -\lambda E^* - 2\rho)$. On the real hyperbolic space, the lowest restricted root ρ is given by $\frac{1}{2}n \cdot E^*$, and the hyperbolic Laplacian is given by $B(E,E)^{-1}$ times the Killing Laplace Δ^B.

Inserting this we obtain from Theorem 2.3.3 that Φ_0 produces eigenvectors of Δ with eigenvalue $-\lambda(n+\lambda)$, which is consistent with the formula in Theorem 3.3.1(iii).

Note that Theorem 3.3.1 also shows that for fixed degree k of the differential form on G/P the largest eigenvalue of the Laplace-Beltrami operator is obtained for the weight $\lambda = -\frac{n-2k}{2}$, in which case the eigenvalue equals λ^2. This is exactly the lower bound of the continuous L^2-spectrum of the Laplace-Beltrami operator on the real hyperbolic space, see [**CP04**, Theorem 2.4].

3.3.2. Uniqueness and geometric interpretation. Next, we want to prove uniqueness as well as give a geometric interpretation of the Poisson kernels ϕ_k, which will be done by identifying their corresponding G-equivariant bundle maps as in Proposition 2.2.2. Furthermore, we will show how these can be obtained by several geometric constructions on the Poincaré ball model, which are equivalent in the case of the real hyperbolic space but differ in more general settings. In this way, we also recover the initial definition of Gaillard's transforms in terms of visual maps, c.f. Definition 2.1.3.

First, we determine the dimension of the space of M-invariant vectors in $\Lambda^*(\mathfrak{g}/\mathfrak{m})$ through a simple exercise in representation theory.

THEOREM 3.3.2. *Let $G = \mathrm{SO}(n+1,1)_0$, $M \cong \mathrm{SO}(n)$ the intersection of the maximal compact and minimal parabolic subgroup of G and denote by $\mathfrak{g}$ and $\mathfrak{m}$ the Lie algebras of G and M, respectively. Then the dimension of the M-invariant (ℓ,k)-forms is given by*

$$\dim\left(\left(\Lambda^{\ell,k}(\mathfrak{g}/\mathfrak{m})^*\right)^M\right) = \begin{cases} 2 & n = 2k, \ell \in \{k, k+1\} \\ 1 & n \neq 2k, \ell \in \{k, k+1, n-k, n-k+1\} \\ 0 & \textit{otherwise} \end{cases}.$$

PROOF. We have seen in section 3.2.1 that under the identification $M \cong \mathrm{SO}(n)$ the vertical subspace $(\mathfrak{g}/\mathfrak{m})^{0,1} = \mathfrak{k}/\mathfrak{m}$ is isomorphic to the standard representation, whereas the horizontal subspace $(\mathfrak{g}/\mathfrak{m})^{1,0} = \mathfrak{p}/\mathfrak{m}$ is the direct sum of the standard and the trivial representation. In particular, after dualizing $\mathbb{R}^n$ via the standard inner product we obtain that the space of Poisson kernels of bidegree (ℓ,k) is isomorphic to

$$\left(\Lambda^{\ell,k}(\mathfrak{g}/\mathfrak{m})^*\right)^M \cong \mathrm{Hom}_{\mathrm{SO}(n)}(\Lambda^k\mathbb{R}^n, \Lambda^\ell\mathbb{R}^n) \oplus \mathrm{Hom}_{\mathrm{SO}(n)}(\Lambda^k\mathbb{R}^n, \Lambda^{\ell-1}\mathbb{R}^n).$$

Therefore, it suffices to determine the dimension of the space of $\mathrm{SO}(n)$-equivariant maps between exterior powers of the standard representation.

First, it is well known (see e.g. [**Pro09**, ch. 11.6]) that $\Lambda^{n-k}\mathbb{R}^n$ is isomorphic to $\Lambda^k\mathbb{R}^n$ as an $\mathrm{SO}(n)$-representation for all $0 \le k \le n$, it suffices to consider the case $0 \le k, \ell \le \frac{n}{2}$. Furthermore, by [**Pro09**, Remark ch. 11.6] we know that the $\mathrm{SO}(n)$-representations $\Lambda^k\mathbb{R}^n$ are pairwise non-isomorphic for all $0 \le k \le \frac{n}{2}$ and irreducible for $k \neq \frac{n}{2}$, whereas for $k = \frac{n}{2}$ the exterior power splits into the direct sum of two irreducible representations. Therefore, Schur's Lemma implies that $\mathrm{Hom}_{\mathrm{SO}(n)}(\Lambda^k\mathbb{R}^n, \Lambda^\ell\mathbb{R}^n)$ is trivial unless $k = \ell$, in which case this space is of dimension 1 for $k \neq \frac{n}{2}$ and of dimension 2 if $k = \frac{n}{2}$. □

Note that Theorem 3.3.2 immediately implies that any Poisson transform which map between differential forms of the same degree has to be a multiple of Φ_k. In turn, this uniqueness result can be used to determine their corresponding G-invariant bundle maps $p_k \colon \Lambda^k T^{0,1*} \to \Lambda^k T^{1,0*}$ as in Proposition 2.2.2. Recall that if $\mathrm{vol} \in \Omega^{0,n}(G/M)$ denotes the G-invariant differential form from Lemma 2.2.2, then these maps are characterized by $\phi_k \wedge \pi_P^*\alpha = p_k(\pi_P^*\alpha) \wedge \mathrm{vol}$ for all $\alpha \in \Omega^k(G/P)$. Now by Proposition 1.2.1, these maps p_k are induced by M-equivariant maps $\Lambda^k(\mathfrak{k}/\mathfrak{m})^* \to \Lambda^k(\mathfrak{p}/\mathfrak{m})^*$, which by uniqueness have to be a multiple of the M-equivariant map induced by

$$\chi\colon \mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m},$$

which is defined by $E \mapsto 0$ and $F_X \mapsto G_X$ for all $X \in \mathbb{R}^n$.

Corollary 3.3.2. *Let $\chi^*\colon \Lambda^*(\mathfrak{p}/\mathfrak{m})^* \to \Lambda^*(\mathfrak{k}/\mathfrak{m})^*$ be the M-equivariant map induced by χ and denote the corresponding G-equivariant bundle map by the same symbol. Then for all $\alpha \in \Omega^k(G/P)$ we obtain that*

$$\phi_k \wedge \pi_P^*\alpha = c_k\phi_0 \wedge (\chi^*\pi_P^*\alpha)$$

with $c_k \in \mathbb{R}$.

Proof. Since the G-invariant $(0,n)$-forms vol and ϕ_0 are proportional, the claim follows immediately from Theorem 3.3.2. □

For later computations we will derive an explicit formula for the relation between the Poisson kernels ϕ_k and ϕ_{k+1} induced by the map χ.

Lemma 3.3.2. *For all $k = 0,\dots,n$ and $\xi \in \mathfrak{p}/\mathfrak{m}$ we have*

$$\iota_{\chi(\xi)}\phi_k = (-1)^k 2(n-k)\iota_\xi\phi_{k+1}.$$

Proof. We have already seen that $\iota_E\phi_k$ is trivial for all k, so it suffices to prove the equation for $\xi = F_X$ with $X \in \mathbb{R}^n$. From the formulae for the M-invariant inner product on $\mathfrak{p}/\mathfrak{m}$ in Proposition 3.2.1 we obtain for all vectors X, $Y \in \mathbb{R}^n$ that

$$(\iota_{G_X} d_P E^*)(F_Y) = -\langle X, Y\rangle = -2F_X^\flat(F_Y).$$

Therefore, the formulae for the calculus on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ in Corollary 3.2.1 imply that

$$\begin{aligned}\iota_{G_X}\phi_k &= (-1)^{n+1} *_K \iota_{G_X}(E^* \wedge (d_P E^*)^{n-k})\\ &= (-1)^n(n-k) *_K (E^* \wedge (d_P E^*)^{n-k-1} \wedge \iota_{G_X} d_P E^*)\\ &= (-1)^{n+1}2(n-k) *_K (E^* \wedge (d_P E^*)^{n-k-1} \wedge F_X^\flat)\\ &= (-1)^k 2(n-k)\iota_{F_X}\phi_{k+1}\end{aligned}$$

as claimed. □

For the rest of this section we want to illustrate how geometric methods can be used to construct the M-equivariant map χ on the real hyperbolic space. Note that these approaches also make sense in more general settings, where they usually differ and thus serve as possible sources for Poisson kernels.

First, recall from section 3.2.3 that we can identify the homogeneous space G/M with the r-sphere bundle S^r over G/K in terms of the Poincaré ball model B. This was done via the G-equivariant visual maps V_x^r, which assigned to every $x \in G/K$ and $\xi \in S_x^r$ the limit of the unique geodesic ray determined by ξ. For the base point $\mathbf{0} = eK$ the geodesic ray determined by the first unit vector e_0 in $\mathfrak{g}/\mathfrak{k} \cong T_{\mathbf{0}}B$ is the line $\gamma_{\mathbf{0}}\colon t \mapsto (t,0,\dots,0)$, which approaches the point $\mathbf{1} = eP \in G/P$ in the limit. Thus, if we identify the tangent space $T_{e_0}S^r$ with its natural embedding into $T_{\mathbf{0}}(G/K) \cong \mathfrak{g}/\mathfrak{k}$, the visual map $V_{\mathbf{0}}$ induces a linear map $\mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m}$, which is M-equivariant by construction and thus has to be a multiple of χ. This is exactly the point of view of Gaillard in his initial definition of Poisson transforms on the real hyperbolic space, c.f. Definition 2.1.3.

On the other hand, we have seen in section 1.4.1 that the Killing form on $\mathfrak{g}$ induces a P-equivariant isomorphism between $(\mathfrak{g}/\mathfrak{p})^*$ and $\mathfrak{p}_+$. Next, every $X \in \mathfrak{p}_+$ defines a fundamental vector field $\zeta_X \in \mathfrak{X}(G/K)$, which is by definition a Killing field for the Riemannian metric on G/K associated to the Killing form. Moreover, the map $\zeta\colon \mathfrak{p}_+ \to \mathfrak{X}(G/K)$ is G-equivariant with respect to the adjoint action on $\mathfrak{g}$ and the natural G-action on vector fields on G/K, so its composition with the musical operator $\flat\colon \mathfrak{X}(G/K) \to \Omega^1(G/K)$ is a P-equivariant map $(\mathfrak{g}/\mathfrak{p})^* \to \Omega^1(G/K)$. Hence evaluation of these one forms at the base point $eK \in G/K$ induces an M-module homomorphism $(\mathfrak{k}/\mathfrak{m})^* \to (\mathfrak{p}/\mathfrak{m})^*$, which by uniqueness has to differ from the dual of χ by a multiple.

3.3.3. Growth estimates of Poisson transforms. In this final part of the study of the Poisson transform Φ_k we want to derive growth estimates for the pointwise hyperbolic norm of its images in terms of a defining function for the boundary sphere in the Poincaré ball model of real hyperbolic space. For this, we will use the results on fibre integrals from section 1.5.2.

We have already seen in section 2.2.1 that the maximal compact subgroup K acts transitively on the conformal sphere G/P, which induces a diffeomorphism $G/P \cong K/M$. Explicitly, if $\kappa\colon G \to K$ denotes the K-part of the Iwasawa decomposition of G, then this isomorphism is given by $gP \mapsto \kappa(g)M$. Otherwise said, this corresponds to a reduction of the holonomy group of the conformal sphere to the subgroup $M \cong \mathrm{SO}(n)$, i.e. to a choice of a Riemannian metric in the conformal class of metrics on G/P. For our purpose, we choose on the symmetric space $K/M \cong S^n$ the Riemannian metric induced by the restriction of the standard Euclidean metric on $\mathbb{R}^{n+1}$, which is K-invariant by definition.

Subsequently, we obtain an isomorphism between G/M and $G/K \times K/M$ as G-manifolds, where the action ℓ of G on the right hand side is given by $\ell_g(g'K, hM) = (gg'K, \kappa(gh)M)$ for all g, $g' \in G$ and $h \in K$. Moreover, the metric on K/M together with the hyperbolic metric on G/K induce a G-invariant Riemannian metric on the product manifold $G/K \times K/M$, which we will again denote by g_M, so that the decomposition of its tangent bundle into the Whitney sum of $T(G/K)$ and $T(K/M)$ is orthogonal. In this way, if vol is the normed G-invariant $(0, n)$-form on $G/K \times K/M$ constructed in Lemma 2.2.2 and if we denote its restriction to the fibre over $gK \in G/K$ by vol_{gK}, then by construction vol_{eK} coincides with the Euclidean volume form on K/M.

In the following Lemma we compare the norm of the pullback of differential forms on G/P along the projection $\pi_P\colon G/K \times K/M \to G/P$ induced by the isomorphism $K/M \cong G/P$ as well as determine the change in the measure induced by vol on different fibres over G/K.

Lemma 3.3.3. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ the maximal compact subgroup, P the minimal parabolic subgroup of G and M the intersection of K and P. For all $g \in G$ let $g = \kappa(g)\exp(\tau(g)E)\eta(g)$ be its Iwasawa decomposition with $\kappa(g) \in K$, $\tau(g) \in \mathbb{R}$ and $\eta(g) \in N$, where E is the grading element of the Lie algebra of $\mathfrak{g}$. For $\lambda \in \mathbb{R}$ denote by $\mathcal{E}[\lambda]$ the bundle of λ-densities over G/P with underlying 1-dimensional P-representation $\mathbb{R}[\lambda]$.*

(i) For all $\alpha \in \Lambda^k T^*_{hP}(G/P)[\lambda]$ with $h \in K$ we obtain that

$$g_M(\pi_P^*\alpha, \pi_P^*\alpha)(gK, hM) = e^{2(k-\lambda)\tau(g^{-1}h)} g_M(\pi_P^*\alpha, \pi_P^*\alpha)(eK, hM).$$

(ii) Let $|\ |_M$ denote the pointwise norm on $\Omega^*(G/M)$ induced by the metric g_M. Then for all $\mathcal{E}[\lambda]$-valued k-forms α on G/P we have

$$\int_{K/M} |\pi_P^*\alpha|_M(gK, hM)\,\mathrm{vol}_{gK}$$
$$= \int_{K/M} e^{(k-\lambda-n)\tau(g^{-1}h)} |\pi_P^*\alpha|_M(eK, hM)\ \mathrm{vol}_{eK}.$$

Proof. (i) The action of the element $h_g := g\kappa(g^{-1}h)h^{-1} \in G$ maps the point $(eK, hM) \in G/K \times K/M$ in the fibre over eK to the point (gK, hM).

Furthermore, it can be rewritten as

$$h_g = h\eta(g^{-1}h)\exp(-\tau(g^{-1}h)E)h^{-1}$$

using the Iwasawa decomposition of G. Then G-invariance of the metric g_M implies that

$$g_M(\pi_P^*\alpha, \pi_P^*\alpha)(gK, hM) = g_M(h_g^{-1}\cdot\pi_P^*\alpha, h_g^{-1}\cdot\pi_P^*\alpha)(eK, hM) = (*),$$

and since the canonical projection π_P is G-equivariant we can commute the action of h_g^{-1} with the pullback.

In order to rewrite $(*)$ we need to simplify the element

$$h_g^{-1}\cdot\alpha = h\exp(\tau(g^{-1}h)E)\eta(g^{-1}h)^{-1}h^{-1}\cdot\alpha.$$

For this, note that the covector $h^{-1}\cdot\alpha$ is an element in $\Lambda^k T_{eP}^*(G/P)[\lambda] \cong \Lambda^k(\mathfrak{g}/\mathfrak{p})^*[\lambda]$, on which $\exp(\tau(g^{-1}h)) \in A$ acts by the scalar $e^{(k-\lambda)\tau(g^{-1}h)}$, whereas the group N acts trivially. Thus, we obtain that

$$h_g^{-1}\cdot\alpha = e^{(k-\lambda)\tau(g^{-1}h)}\alpha,$$

and inserting this back into $(*)$ yields

$$(*) = e^{2(k-\lambda)\tau(g^{-1}h)}g_M(\pi_P^*\alpha, \pi_P^*\alpha)(eK, hM).$$

(ii) In order to prove this formula we need to compare the volume forms vol_{gK} and vol_{eK} on the homogeneous space K/M. Consider for all $g \in G$ the map $\kappa_g\colon K/M \to K/M$ defined by $hM \mapsto \kappa(gh)M$, which is an isomorphism with inverse $\kappa_{g^{-1}}$, c.f. [**Hel94**, Lemma I.5.19]). Then by definition of the G-action the map $\ell_g \circ \kappa_{g^{-1}}$ is the identity on K/M, so applying the transformation formula for integrals we deduce that

$$\begin{aligned}\int_{K/M} &|\pi_P^*\alpha|_M(gK, hM)\ \mathrm{vol}_{gK}\\ &= \int_{K/M} |\pi_P^*\alpha|_M(gK, hM)\ \kappa_{g^{-1}}^*\ell_g^*\,\mathrm{vol}_{gK} = (**).\end{aligned}$$

By G-invariance of vol, the pullback of vol_{gK} along ℓ_g coincides with vol_{eK}. Moreover, the latter differential form is a multiple of the K-invariant volume form on K/M induced by the Haar measure on K, so we can apply [**Hel84**, Lemma I.5.19] and obtain $\kappa_{g^{-1}}^*\,\mathrm{vol}_{eK} = e^{-n\tau(g^{-1}h)}\,\mathrm{vol}_{eK}$. Inserting this into the expression $(**)$ and using (i) we obtain the claimed formula for the integral. □

The previous Lemma allows us to derive growth estimates on the hyperbolic norm of $\Phi_k(\alpha)$ for all $\alpha \in \Omega^k(G/P, \mathcal{E}[\lambda])$. Indeed, recall that we chose the G-invariant Riemannian metric on the tangent bundle of $G/K \times K/M$ so that the projection onto the first factor is a Riemannian submersion, so Theorem 1.5.2 is applicable. Thus, we obtain for all $g \in G$ that

$$|\Phi_k(\alpha)|\,(gK) \le \int_{K/M} |\phi_k \wedge \pi_P^*\alpha|_M\,(gK, hM)\,\mathrm{vol}_{gK} = (*),$$

and we can find differential forms on G/P so that this inequality is sharp. Moreover, applying Corollary 3.3.2 to the above equation and using orthogonality of the horizontal and vertical subbundle of the tangent bundle of $G/K \times K/M$ we can rewrite this as

$$(*) = c_k \int_{K/M} |\phi_0|_M\,(gK, hM)\cdot|\chi^*\pi_P^*\alpha|_M\,(gK, hM)\ \ \mathrm{vol}_{gK}$$

for some constant $c_k \neq 0$. Since both the differential form ϕ_0 and the metric g_M are G-invariant, the expression $|\phi_0|_M$ is constant and thus can be moved outside of the integral. Moreover, we have chosen the Riemannian metric on the vertical bundle so that the map χ^* satisfies $|\chi^*\pi_P^*\alpha|_M = 2^k|\pi_P^*\alpha|_M$. Finally, we can apply Lemma 3.3.3(ii) to transform the remaining expression into an integral over the fibre $\pi_K^{-1}(eK)$. All in all, we have

$$(*) = c_k 2^k |\phi_0|_M \int_{K/M} e^{(k-\lambda-n)\tau(g^{-1}h)}\, |\pi_P^*\alpha|_M\, (eK, hM) dhM,$$

where dhM is the measure on K/M induced by the round metric.

Our aim is to obtain growth estimates in terms of a defining function for the boundary sphere on the Poincaré ball model of real hyperbolic space. So let $x = g\cdot\mathbf{0} \in B^{n+1}$ and let $u = h\cdot\mathbf{1}$ in the boundary sphere. Then we have computed in Proposition 3.1.2 that the expression $e^{-\tau(g^{-1}h)}$ translates into the classical Poisson kernel $P(x,u)$ on the unit ball in $\mathbb{R}^{n+1}$. Since the Riemannian metric and the vertical volume form vol_{eK} on $\{eK\}\times K/M$ coincide with the Euclidean metric and the Euclidean volume form on S^n, respectively, we obtain for all $\lambda\in\mathbb{R}$, $\alpha \in \Omega^k(S^n)[\lambda]$ and $x \in B$ that

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C\cdot \int_{S^n} P(x,u)^{n-k+\lambda}|\alpha|_{\mathrm{euc}}(u)du \leq C|\alpha|_\infty \int_{S^n} P(x,u)^{n-k+\lambda}du,$$

and these inequalities are sharp. Here, C is a constant and $|\alpha|_\infty$ is the maximum of the pointwise norm of α on the compact space S^n.

In summary, we can estimate the norm of the Poisson transform of $\Phi_k(\alpha)$ via the average of appropriate powers of the Poisson kernel on the unit sphere S^n. Moreover, it is well known that

$$\text{(3.3.3)} \qquad \frac{1}{|S^n|}\int_{S^n} \frac{1-|x|^2}{|x-u|^{n+1}}\, du = 1, \qquad \frac{1}{|S^n|}\int_{S^n}\frac{1}{|x-u|^{n-1}}\, du = 1,$$

see [**GT01**, chapters 2.4 and 2.5]. Using this, we can prove

PROPOSITION 3.3.3. *Let B be the Poincaré ball model of the real hyperbolic space of dimension $n+1$ endowed with the hyperbolic metric g_{hyp}. Let g_{euc} be the Euclidean metric on B and denote by $\rho\colon B\to\mathbb{R}$ the defining function $\rho(x) = \frac{1}{2}(1-|x|^2)$. Let $\Phi_k\colon S^n \to B$ the Poisson transform from Theorem 3.3.1 and let $\alpha\in\Omega^k(S^n)[\lambda]$ for any $\lambda\in\mathbb{R}$.*

(i) For $k-\lambda = \frac{n+1}{2}$ we obtain that

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C\rho(x)^{k-\lambda-1}$$

for all $x\in B$, and this inequality is sharp. In particular, we see that the Euclidean norm of $\Phi_k(\alpha)(x)$ is bounded by a constant multiple of $\rho(x)^{-\lambda-1}$.

(ii) If $k-\lambda \leq \frac{n-1}{2}$, then the hyperbolic norm of $\Phi_k(\alpha)$ satisfies

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C\rho(x)^{k-\lambda}$$

for all $x\in B$. In particular, the Euclidean norm of $\Phi_k(\alpha)(x)$ is bounded by $\rho(x)^{-\lambda}$.

PROOF. From above we obtain that the hyperbolic norm of $\Phi_k(\alpha)$ for all $\alpha \in \Omega^k(S^n)[\lambda]$ satisfies

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C\int_{S^n} P(x,u)^{n-k+\lambda}du$$

for a constant $C\in\mathbb{R}$, and this inequality is sharp.

(i) Since $n+1-2(k-\lambda)=0$ we immediately obtain that

$$P(x,u)^{n-k+\lambda} = \frac{(1-|x|^2)^{n-k+\lambda}}{|x-u|^{n-1}},$$

and inserting this into the above estimate and use the second integral formula from (3.3.3) yields

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C \int_{S^n} \frac{(1-|x|^2)^{n-k+\lambda}}{|x-u|^{n-1}} du = C2^{n-k}|S^n|\rho(x)^{n-k+\lambda}.$$

Moreover, the Euclidean norm of $\Phi_k(\alpha)$ is obtained by multiplying the hyperbolic norm with $\rho(x)^{-k}$. Thus, it follows that

$$|\Phi_k(\alpha)|_{\mathrm{euc}}(x) \leq \tilde{C}\rho(x)^{-\lambda-1},$$

where $\tilde{C}$ is an appropriate constant, and this inequality is sharp.

(ii) Fix a point $x \in B$ and write it as $x = rv$ with $r \in [0,1)$ and $v \in S^n$. For all $u \in S^n$ we obtain that

$$|x-u|^2 = |x|^2 - 2\langle x,u\rangle + 1 \geq r^2 - 2r + 1 = (1-r)^2.$$

Since we want to compare the norm of $x-u$ with the value of the defining function at x we further estimate

$$(1-r)^2 > 0 \Leftrightarrow 1-r > \frac{1}{2}(1-r^2),$$

and since $|x| = r < 1$ this implies that

$$\rho(x) < |x-u| < 2 \Leftrightarrow \frac{1}{2} < \frac{1}{|x-u|} < \rho(x)^{-1}.$$

Applying this estimate to the Poisson kernel we obtain for $k-\lambda \leq \frac{n-1}{2}$ that

$$P(x,u)^{n-k+\lambda} \leq 2^{n-k+\lambda-1}\rho(x)^{k-\lambda}\frac{1-|x|^2}{|x-u|^{n+1}},$$

so integrating both sides over S^n and using the first formula from (3.3.3) yields

$$|\Phi_k(\alpha)|_{\mathrm{hyp}}(x) \leq C\rho(x)^{k-\lambda}$$

for an appropriate constant $C \in \mathbb{R}$. In particular, this implies that the Euclidean norm of $\Phi_k(\alpha)$ is bounded by $\rho^{-\lambda}$ on B from above. $\square$

3.4. Poisson transforms between tractor valued forms

Having concluded the study of real valued Poisson kernels, our aim in this section is to demonstrate the efficiency of our method by constructing in section 3.4.1 a family of Poisson transforms between differential forms on G/P and G/K with values in the respective standard tractor bundles, which are BGG-compatible in the sense of Definition 2.2.3. Remarkably, it turns out that these operators are again a chain maps between the corresponding twisted de Rham sequences on G/K and G/P, respectively, c.f. Theorem 3.4.3. Moreover, their extensions to weighted differential forms factor to G-equivariant operators between sections of the corresponding homology bundles on G/P and standard tractor valued differential forms on G/K which are eigenforms for the twisted Laplace Beltrami operator. Furthermore, in the unweighted case these operators are chained maps between the BGG-sequences on G/P and the twisted de Rham sequence on G/K, c.f. Theorem 3.4.4.

3.4.1. Construction of a BGG-compatible Poisson kernel. In this section we present an explicit construction of a BGG-compatible Poisson transform which maps differential forms on the conformal sphere with values in $V_P := G \times_P \mathbb{V}$ to differential forms on the real hyperbolic space with values in the standard tractor bundle $V_K := G \times_K \mathbb{V}$. In order to do so, we will first recall the specific geometric properties of these vector bundles established in section 3.2.2 and subsequently use them to find an appropriate ansatz for the corresponding Poisson kernel.

First, recall from Proposition 3.2.2 that V_K decomposes as a homogeneous vector bundle into the direct sum of the tangent bundle of G/K and a line bundle. Since the projection to each of these summands is tensorial we obtain an induced decomposition

$$\Omega^\ell(G/K, V_K) = \Omega^\ell(G/K, T(G/K)) \oplus \Omega^\ell(G/K)$$

for all $0 \leq \ell \leq n+1$, implying that it suffices to construct Poisson transforms with values in each of these summands separately. Note that by naturality of the pullback this decomposition is reflected on the space of $\mathrm{End}(G \times_M \mathbb{V})$-valued differential forms on G/M. In addition, we have seen in the same Proposition that V_P carries a G-invariant filtration $V_P = V^{-1} \supset V^0 \supset V^1$ whose associated graded vector bundle $\mathrm{gr}(V_P)$ decomposes into

$$\mathrm{gr}(V_P) \cong \mathcal{E}[-1] \oplus T(G/P)[-1] \oplus \mathcal{E}[1],$$

where $\mathcal{E}[\lambda]$ denotes the bundle of λ-densities as before. In particular, we can identify the space of real valued functions on G/P with the space of sections of the subbundle $V^1[-1]$ of the weighted tractor bundle $V_P[-1]$. Therefore, to avoid recovering the operators Φ_k between real valued differential forms as in Theorem 3.3.1, we will focus on constructing Poisson transforms between V_P-valued k-forms on G/P and tangent bundle valued ℓ-forms on the real hyperbolic space G/K in the sequel. By Theorem 2.2.1, these correspond to M-invariant elements in the finite dimensional representation $\Lambda^{\ell,n-k}(\mathfrak{g}/\mathfrak{m})^* \otimes L(\mathbb{V}, \mathbb{V}_T)$, where $\mathbb{V}_T$ denotes the K-invariant subspace of the standard representation $\mathbb{V}$ corresponding to the tangent bundle $T(G/K)$.

Furthermore, we want the constructed Poisson kernel to be BGG-compatible, i.e. to be contained in the kernels of ∂_P and $\partial_P d_P$, c.f. Definition 2.2.3, for which we have to develop a formula for the P-codifferential on M-invariant elements in $\Lambda^*(\mathfrak{g}/\mathfrak{m}) \otimes \mathrm{End}(\mathbb{V})$. In order to do so, recall from section 3.2.2 that the standard representation decomposes as an M-module into $\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1$, where $\mathbb{V}_{\pm 1}$ are copies of the trivial representation and $\mathbb{V}_0 \cong \mathbb{R}^n$ is the standard representation of $M \cong \mathrm{SO}(n)$. Furthermore, this decomposition is compatible with the $|1|$-grading of $\mathfrak{g}$ in the sense that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$, where we agree that $\mathbb{V}_k = \{0\}$ if $|k| > 1$.

Proposition 3.4.1. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup, $P \cong \mathrm{CSO}(n) \ltimes \mathbb{R}^n$ its minimal parabolic subgroup and $M \cong \mathrm{SO}(n)$ be the intersection of K and P. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{m}$ be the Lie algebras of G, K, P and M, respectively. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of G, let ρ be the tensor product of the $\mathfrak{p}$ action on $\mathbb{V}$ with the identity on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ and let $\rho^{\mathbb{V}^*}$ be the tensor product of the $\mathfrak{p}$-action on $\mathbb{V}^*$ with the identity on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathbb{V}$. Then the P-codifferential of $\phi \in \Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ is given by*

$$\partial_P \phi = -\frac{1}{2n} \sum_{j=1}^{n} \iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \phi, \tag{3.4.1}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with respect to the standard inner product and $\xi_{e_j} \in \mathfrak{g}_1 \cong \mathbb{R}^n$ and $G_{e_j} \in \mathfrak{k}/\mathfrak{m} \cong \mathbb{R}^n$ are the vectors corresponding to $e_j \in \mathbb{R}^n$. Moreover, *(i) If $E \in \mathfrak{g}/\mathfrak{m}$ denotes the M-invariant vector induced by the grading element of $\mathfrak{g}$, then for all $\phi \in \Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ we have*

$$\partial_P \iota_E \phi = -\iota_E \partial_P \phi, \qquad \partial_P(E^* \wedge \phi) = -E^* \wedge \partial_P \phi.$$

(ii) Let $\mathbb{W}$ be any subspace of $\mathbb{V}$. Then the image of $\phi \in \Lambda^*(\mathfrak{g}/\mathfrak{p})^* \otimes L(\mathbb{V}_i, \mathbb{W})$ under the P-codifferential has values in the subspace $L(\mathbb{V}_{i-1}, \mathbb{W})$. In particular, if ϕ has values in $L(\mathbb{V}_{-1}, \mathbb{W})$, then it is contained in the kernel of ∂_P.

PROOF. In section 2.2.3 we defined the P-codifferential by the formula $\partial_P(\alpha \wedge \beta) = (-1)^\ell \alpha \wedge \partial^* \beta$ for all $\alpha \in \Lambda^{\ell,0}(\mathfrak{g}/\mathfrak{m})^* \otimes \mathbb{V}$ and $\beta \in \Lambda^{0,n-k}(\mathfrak{g}/\mathfrak{m})^* \otimes \mathbb{V}^*$, where we used the identification $\mathfrak{k}/\mathfrak{m} \cong \mathfrak{g}/\mathfrak{p}$ induced by the restriction of the canonical projection $\mathfrak{g}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{p}$.

Moreover, by Proposition 3.1.1(i) the Lie algebra $\mathfrak{g}$ is $|1|$-graded, so the formula for the Kostant codifferential proved in Proposition 1.4.4 reduces to the alternation of the $\mathfrak{p}_+$-action on $\mathbb{V}$. Explicitly, if $\{\xi_j\}$ is a basis for $\mathfrak{p}_+ = \mathfrak{g}_1$ with dual basis $\{\eta_j\}$ for $\mathfrak{g}_{-1}$ with respect to the Killing form, then for all $\beta \in \Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}$ we have

$$\partial^* \beta = -\sum_{j=1}^n \rho^{\mathbb{V}}_{\xi_j} \iota_{\eta_j} \beta,$$

where $\rho^{\mathbb{V}}$ is the tensor product of the identity on $\Lambda^k \mathfrak{p}_+$ with the $\mathfrak{p}_+$-action on $\mathbb{V}$ and where we interpret β as a multilinear, alternating map $\mathfrak{g}^k \to \mathbb{V}$ which vanishes after insertion of any vector in $\mathfrak{p}$.

Now if we interpret the above formula for the Kostant codifferential in terms of the M-module $\mathfrak{g}/\mathfrak{m}$, the subspace $\mathfrak{g}_{-1} \cong \mathfrak{g}/\mathfrak{p}$ corresponds to $\mathfrak{k}/\mathfrak{m}$, which is generated by vectors G_{e_j} for $j = 1, \dots, n$. Moreover, we have seen in Proposition 3.1.1(iv) that the restriction of the Killing form to $\mathfrak{g}_{-1} \times \mathfrak{g}_1$ equals $(2n)^{-1}$ times the standard inner product on $\mathbb{R}^n$ under the identifications $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^n$, implying that we can replace in the above formula for ∂^* the vector ξ_j by ξ_{e_j} and η_j by $(2n)^{-1} G_{e_j}$, respectively.

Thus, on decomposable elements $\phi = \alpha \wedge \beta$ we deduce that

$$\partial_P \phi = (-1)^{\ell+1} \frac{1}{2n} \sum_{j=1}^n \alpha \wedge \rho_{\xi_{e_j}} \iota_{G_{e_j}} \beta = -\frac{1}{2n} \sum_{j=1}^n \iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \phi,$$

where we used that α is of bidegree $(\ell, 0)$ and thus trivial upon insertion of the vectors G_{e_j} for all $j = 1, \dots, n$. Since ∂_P is linear this formula holds for all elements in $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$.

(i) Follows immediately from (3.4.1) by using the antiderivation property of the interior product.

(ii) If ϕ is of degree k, then the $\mathfrak{p}$-action $\rho^{\mathbb{V}^*}$ in the formula (3.4.1) is given by

$$\left(\rho^{\mathbb{V}^*}_\xi \phi\right)(\eta_1, \dots, \eta_k)(v) := -\phi(\eta_1, \dots, \eta_k)(\xi \cdot v)$$

for all $\xi \in \mathfrak{p}$, $\eta_1, \dots, \eta_k \in \mathfrak{g}/\mathfrak{m}$ and $v \in \mathbb{V}$. In particular, for $\xi \in \mathfrak{p}_+ = \mathfrak{g}_1$ the values of the Poisson kernels $\rho^{\mathbb{V}^*}_\xi \phi$ lie in the space $L(\mathbb{V}_{i-1}, \mathbb{W})$. Since we only act with elements in $\mathfrak{g}_1$ in the formula for the P-codifferential the claim follows. $\square$

In the next step we will choose a natural ansatz for the BGG-compatible Poisson kernel. For that, recall from section 3.3.1 that we defined a distinguished real-valued Poisson kernel ϕ_k of degree $(k, n-k)$ by

$$\phi_k = *_K(E^* \wedge (d_P E^*)^{n-k}),$$

where E^* is the M-invariant element in $(\mathfrak{g}/\mathfrak{m})^*$ of bidegree $(1,0)$ induced by the grading element of $\mathfrak{g}$ as in Proposition 2.3.2. Therefore, as an ansatz for a vector-valued BGG-compatible Poisson kernel of degree $(k, n-k)$ we choose

$$*_K(E^* \wedge (d_P E^*)^{n-k-1} \wedge \tau),$$

where τ is a Poisson kernel on $\mathfrak{g}/\mathfrak{m}$ of degree $(1,1)$. Furthermore, we already mentioned that we want this form to have values in $L(\mathbb{V}, \mathbb{V}_T)$, so by definition of the K-Hodge star in section 2.2.3 this is satisfied if

$$\tau \in \Lambda^{1,1}(\mathfrak{g}/\mathfrak{m})^* \otimes L(\mathbb{V}, \mathbb{V}_T). \tag{R1}$$

Next, for BGG-compatibility we need our ensure that our ansatz is contained in the kernel of the P-codifferential. Recall that ∂_P commutes with $*_K$ up to a sign due to their definitions in section 2.2.3, and by Proposition 3.4.1(i) the same holds for ∂_P and the wedge product with E^*. Hence our ansatz form is contained in $\ker(\partial_P)$ if

$$\partial_P\left((d_P E^*)^{n-k-1} \wedge \tau\right) = 0. \tag{R2}$$

Furthermore, we want the G-equivariant operator on sections of the twisted homology bundles on G/P induced by the Poisson transform not to be trivial, which means that our initial form should not be contained in the image of ∂_P. Regarding Proposition 3.4.1(ii) we therefore assume that it is trivial on the M-submodule $\mathbb{V}_1$ of $\mathbb{V}$, hence also

$$\tau(\xi_1, \xi_2)|_{\mathbb{V}_1} = 0 \tag{R3}$$

for all ξ_1, $\xi_2 \in \mathfrak{g}/\mathfrak{m}$.

In order to ensure BGG-compatibility we will also need to determine the P-derivative of our ansatz and apply the P-codifferential to the result. Since the P-differential satisfies $d_P *_K = (-1)^{n+1} *_K d_P$ due to Proposition 2.2.3 and is an antiderivation we obtain that its image under d_P is given by

$$(-1)^{n+1}\left(*_K((d_P E^*)^{n-k} \wedge \tau) - *_K(E^* \wedge (d_P E^*)^{n-k-1} \wedge d_P\tau)\right).$$

Hence, if we assume that

$$d_P\tau = 0, \tag{R4}$$

then we can compute the image of our ansatz under the operators ∂_P and $\partial_P d_P$ in a unified way using the following Lemma.

LEMMA 3.4.1. *Let $E^* \in \mathfrak{g}/\mathfrak{m}$ be the M-invariant $(1,0)$-form induced by the grading element of $\mathfrak{g}$ and τ be a $L(\mathbb{V}, \mathbb{V}_T)$-valued Poisson kernel of degree $(1,1)$ which is trivial on $\mathbb{V}_1$. Write τ in the general ansatz*

$$\tau(F_X, G_Y)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} b w_{-1}\langle X, Y\rangle \\ a_1\langle X, Y\rangle W + a_2\langle X, W\rangle Y + a_3\langle Y, W\rangle X \\ b w_{-1}\langle X, Y\rangle \end{pmatrix}$$

$$\tau(E, G_Y)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} d\langle Y, W\rangle \\ c w_{-1} Y \\ d\langle Y, W\rangle \end{pmatrix}.$$

for all X, Y, $W \in \mathbb{R}^n$ and w_1, $w_{-1} \in \mathbb{R}$, where a_i, b, c and d are arbitrary real numbers.

(i) The invariant form $(d_P E^*)^r \wedge \tau$ is contained in the kernel of ∂_P if and only if

$$(r+1)a_1 + a_2 + (n-r)a_3 = 0, \qquad (n-r)d = 0.$$

Moreover, ∂_P is surjective onto the Poisson kernels $(d_P E^*)^r \wedge \sigma$ for any Poisson kernel σ of degree $(1,0)$ with values in $L(\mathbb{V}_{-1}, \mathbb{V}_T)$.

(ii) The invariant form $(d_P E^*)^r \wedge \tau$ *is in the image of* ∂_P *if in addition* $a_3 = 0$.

PROOF. Let ϕ be an $\mathrm{End}(\mathbb{V})$-valued Poisson kernel of degree (ℓ, k) and let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ with corresponding elements $\xi_{e_j} \in \mathfrak{g}_1$. If we apply formula (3.4.1) for the P-codifferential and distribute the interior product over the wedge product via the antiderivation property we obtain that

$$\partial_P \left((d_P E^*)^r \wedge \phi\right) = -\frac{1}{2n}(d_P E^*)^{r-1} \wedge \tilde{\phi},$$

where the $(\ell+1, k)$-form $\tilde{\phi}$ is defined by

$$\tilde{\phi} := \sum_{j=1}^{n} r\left(\iota_{G_{e_j}} d_P E^*\right) \wedge \left(\rho^{\mathbb{V}^*}_{\xi_{e_j}} \phi\right) + (d_P E^*) \wedge \left(\iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \phi\right).$$

(i) Since the Poisson kernel $d_P E^*$ is nondegenerate due to Proposition 2.3.2 it follows that $(d_P E^*)^r \wedge \tau$ lies in the kernel of the P-codifferential if and only the corresponding $(2,1)$-form $\tilde{\tau}$ is trivial. Our aim is to translate this condition into equations for the parameters of the Poisson kernel τ.

First, we know from Proposition 3.4.1(ii) that the P-codifferential is trivial on M-invariant elements with values in $L(\mathbb{V}_{-1}, \mathbb{V})$, implying that the parameters b and c of τ can be chosen arbitrarily. Furthermore, $\{e_j\}$ is an orthonormal basis for $\mathbb{R}^n$, so we can write every $X \in \mathbb{R}^n$ as $X = \sum_{j=1}^n \langle X, e_j\rangle e_j$. Therefore, from the formulae for the $\mathfrak{g}$-action on the standard representation in section 3.2.2 we deduce for all $X \in \mathbb{R}^n$ that

$$\begin{aligned}\sum_{j=1}^{n} \left(\rho^{\mathbb{V}^*}_{\xi_{e_j}} \tau\right)(F_X, G_{e_j})(v_{-1}) &= -(a_1 + a_2 + n a_3) v_X,\\ \sum_{j=1}^{n} \left(\rho^{\mathbb{V}^*}_{\xi_{e_j}} \tau\right)(E, G_{e_j})(v_{-1}) &= dn(v_1 + v_{-1}),\end{aligned}$$

where v_{-1} and v_1 are the base element of $\mathbb{V}_{-1}$ and $\mathbb{V}_1$, respectively, corresponding to $1 \in \mathbb{R}$, and where $v_X \in \mathbb{V}_0$ corresponds to $X \in \mathbb{R}$, c.f. section 3.2.2. On the other hand, recall from section 3.3.1 that the real valued $(1,1)$-form $d_P E^*$ was defined by $d_P E^*(F_X, G_Y) = \langle X, Y\rangle$ for all X, $Y \in \mathbb{R}^n$. Therefore, we obtain for all $X \in \mathbb{R}^n$ that

$$\sum_{j=1}^{n} (\iota_{F_X} \iota_{G_{e_j}} d_P E^*) \wedge \left(\rho^{\mathbb{V}^*}_{\xi_{e_j}} \tau\right) = -\rho^{\mathbb{V}^*}_{\xi_X} \tau.$$

Inserting this into the $(2,1)$-form $\tilde{\tau}$ a direct computation shows that we can write $\tilde{\tau} = (d_P E^*) \wedge \sigma$, where σ is the M-invariant $(1,0)$-form defined by

$$\begin{aligned}\sigma(F_X)(v_{-1}) &= \left((r+1)a_1 + a_2 + (n-r)a_3\right) v_X,\\ \sigma(E)(v_{-1}) &= (n-r)d(v_1 + v_{-1}).\end{aligned}$$

for all $X \in \mathbb{R}^n$. Now by M-invariance it is easy to see that any $(1,0)$-form with values in $L(\mathbb{V}_{-1}, \mathbb{V}_T)$ has this form, so surjectivity of ∂_P follows. Furthermore, since $d_P E^*$ is nondegenerate we deduce that $\tilde{\tau}$ is trivial if and only if the equations

$$(r+1)a_1 + a_2 + (n-r)a_3 = 0, \qquad\qquad (n-r)d = 0$$

on the parameter of τ are satisfied.

(ii) Now we assume that $(d_P E^*)^r \wedge \tau = \partial_P((d_P E^*)^r \wedge \theta)$ is in the image of the P-codifferential, where we can choose the $(1,2)$-form θ to have values in $L(\mathbb{V}, \mathbb{V}_T)$ and to be trivial on $\mathbb{V}_{-1}$ due to Corollary 3.4.1(ii). Since the real valued $(1,1)$-form $d_P E^*$ is nondegenerate due to Proposition 2.3.2 the corresponding $(2,2)$-form $\tilde{\theta}$ appearing in the formula for $\partial_P((d_P E^*)^r \wedge \theta)$ has to satisfy $\tilde{\theta} = d_P E^* \wedge \tau$, and we want to translate this equality into linear equations for the parameters of τ.

First, note that by M-invariance of θ it can be written as

$$\theta = d_P E^* \wedge \theta_1 + E^* \wedge \theta_2,$$

where θ_1 and θ_2 are M-invariant forms of degree $(0,1)$ and $(0,2)$, respectively, determined by

$$\theta_1(G_X)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} \beta\langle X, W\rangle \\ \alpha w_1 X \\ \beta\langle X, W\rangle \end{pmatrix},$$

$$\theta_2(G_X, G_Y)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \gamma\left(\langle Y, W\rangle X - \langle X, W\rangle Y\right) \\ 0 \end{pmatrix}$$

for all X, Y, $W \in \mathbb{R}^n$ and w_1, $w_{-1} \in \mathbb{R}$, where α, β and γ are arbitrary real numbers. In this decomposition, the corresponding $(2,2)$-form $\tilde{\theta}$ can be expressed as

$$\begin{aligned} \tilde{\theta} = \sum_{j=1}^{n} &(r+1)(d_P E^*) \wedge (\iota_{G_{e_j}} d_P E^*) \wedge \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_1 + (d_P E^*)^2 \wedge \iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_1 \\ &- E^* \wedge \left(r\ (\iota_{G_{e_j}} d_P E^*) \wedge \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_2 + (d_P E^*) \wedge \iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_2 \right). \end{aligned}$$

In order to write $\tilde{\theta}$ as a wedge product of $d_P E^*$ and an $\mathrm{End}(\mathbb{V})$-valued $(1,1)$-form τ we need to rewrite the first summand in the second line. Using the formulae for the $\mathfrak{g}$-action on $\mathbb{V}$ from section 3.2.2 an explicit computation shows that

$$-rE^* \wedge (\iota_{G_{e_j}} d_P E^*) \wedge \rho^{\mathbb{V}^*}_{\xi_j} \theta_2 = d_P E^* \wedge \tau_3,$$

where τ_3 is the $(1,1)$-form whose only nontrivial parameter is $c = -r\gamma$.

The other summands in the expansion of $\tilde{\theta}$ are already of the desired form, so we need to determine the parameters of the occurring Poisson kernels

$$\begin{aligned} \tau_1 &:= (r+1) \sum_j (\iota_{G_{e_j}} d_P E^*) \wedge \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_1 \\ \tau_2 &:= \sum_{j=1}^{n} (d_P E^*) \wedge \left(\iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_1 \right) \\ \tau_4 &:= -\sum_{j=1}^{n} E^* \wedge \left(\iota_{G_{e_j}} \rho^{\mathbb{V}^*}_{\xi_{e_j}} \theta_2 \right) \end{aligned}$$

of degree $(1,1)$. Since the basis $\{e_j\}$ is orthonormal a direct computation using the formulae for the $\mathfrak{g}$-action on $\mathbb{V}$ from section 3.2.2 shows that τ_1 corresponds to the nontrivial parameters $a_2 = -(r+1)\alpha$ and $b = (r+1)\beta$, the kernel τ_2 corresponds to $a_1 = \alpha$ and $b = -n\beta$ and finally τ_4 corresponds to the nontrivial parameter $c = (n-1)\gamma$.

Inserting this into the above expansion for $\tilde{\theta}$ shows that $\tilde{\theta} = d_P E^* \wedge \tau$, where the parameter of the Poisson kernel $\tau = \sum_{i=1}^{4} \tau_i$ are given by

$$\begin{aligned} a_1 &= \alpha, & a_2 &= -(r+1)\alpha, & a_3 &= 0, \\ b &= (r-(n-1))\beta, & c &= -(r-(n-1))\gamma, & d &= 0. \end{aligned}$$

In particular, we see that in comparison to (i) the only additional restriction is given by $a_3 = 0$. $\square$

Therefore, we require the $L(\mathbb{V}, \mathbb{V}_T)$-valued Poisson kernel τ of degree $(1,1)$ to satisfy the linear relations from part (i) of Theorem 3.4.1, but not those of part (ii).

Finally, in analogy to the real valued case in section 3.3 it is convenient to choose our ansatz to be contained in the kernel of the K-codifferential. Recall from section 2.2.3 that this operator is defined on $\mathrm{End}(\mathbb{V})$-valued Poisson kernels of degree $(k, n-k)$ via

$$\delta_K = (-1)^{(n+1)(k+1)+1} *_K d_K^\theta *_K .$$

Here d_K^θ denotes the covariant exterior derivative on $\Omega^*(G/M, \mathrm{End}(\mathbb{V}))$ with respect to the θ-twisted tractor connection as defined in 1.3.2, which is explicitly given on invariant maps $f \in \mathrm{End}(\mathbb{V})^M$ by

$$(d_K^\theta f)(\xi + \mathfrak{m})(v) = \theta(\xi) \cdot f(v) - f(\xi \cdot v) \tag{3.4.1}$$

for all $v \in \mathbb{V}$ and $\xi \in \mathfrak{p}$, where θ denotes the Cartan involution on $\mathfrak{g}$. In particular, for our ansatz we obtain that

$$\delta_K *_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge \tau\right) = \pm *_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge d_K^\theta \tau\right),$$

and thus we want the invariant form τ to satisfy

$$d_K^\theta \tau = 0. \tag{R5}$$

Regarding (R4) and (R5), we now construct τ as a linear combination of M-invariant $(1,1)$-forms which are contained in the kernels of d_P and d_K^θ by construction:

- For all $f \in \mathrm{End}(\mathbb{V})^M$ the M-invariant form $d_P d_K^\theta f$ is of degree $(1,1)$ and clearly contained in the kernels of d_P and d_K^θ. In order to obtain such M-equivariant maps, we consider the decomposition of the standard representation into irreducible M-modules in section 3.2.2. Then it follows that for all $i, j \in \{-1, 1\}$ the maps $f_j^i \colon \mathbb{V}_i \to \mathbb{V}_j$ induced by the identity on $\mathbb{R}$ are in $\mathrm{End}(\mathbb{V})^M$.
- Let $S \in \mathrm{End}(\mathbb{V})$ be the M-equivariant map which induced the matrix representation of $\mathfrak{g}$ in section 3.1.1. Otherwise said, this map satisfies $\theta(X) \cdot S(v) = S(X \cdot v)$ for all $v \in \mathbb{V}$ and $X \in \mathfrak{g}$ by construction, which by the formulae (3.4.1) and (1.2.4) for the (twisted) covariant exterior derivatives imply that the M-invariant $(1,1)$-form $(d_P E^*) \otimes S$ is contained in the kernels of d_K^θ and the P-codifferential.

Note that if we define τ as a linear combination of these M-invariant $(1,1)$-forms, then it will be automatically d_P-exact. Via a direct computation we deduce that the P-derivative of the M-invariant $(1,0)$-form

$$\sigma_k := (k+1) E^* \otimes S - (k+1) d_K^\theta f_1^1 + (n+2) d_K^\theta f_1^{-1}$$

satisfies (R1) - (R5). Explicitly, in terms of the general ansatz in Theorem 3.4.1 the M-invariant $(1,1)$-form $d_P \sigma_k$ corresponds to the parameter

$$a_1 = a_2 = k+1, \quad a_3 = -(n-k+1), \quad b = n+2, \quad c = -2(n+2), \quad d = 0,$$

and thus satisfies the linear relations in part (i) of the same Theorem in the case $r = n-k-1$.

Therefore, with the choice of $\tau = d_P \sigma_k$ our initial M-invariant $(k, n-k)$-form given by

$$*_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge d_P \sigma_k\right)$$

has values in $L(\mathbb{V}, \mathbb{V}_T)$, is trivial on $\mathbb{V}_1$ and contained in the kernels of the P-codifferential and the K-codifferential.

Next, for BGG-compatibility we also have to ensure that the image of our ansatz lies in the kernel of $\partial_P d_P$. Since we have seen in the proof of Proposition 2.2.3 that $d_P *_K = (-1)^{n+1} *_K d_P$ we can easily compute that

$$d_P *_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge d_P \sigma_k\right) = (-1)^{n+1} *_K \left((d_P E^*)^{n-k} \wedge d_P \sigma_k\right).$$

Now comparing the parameter of $d_P\sigma_k$ to the linear equations of Theorem 3.4.1(i) we deduce that this form is not contained in the kernel of the P-codifferential. However, note that if we exchanged σ_k by σ_{k-1} on the right hand side, then the resulting form would be in $\ker(\partial_P)$.

Therefore, we have to add an appropriate correction term to our general ansatz so that the resulting Poisson kernel is contained in the kernel of $\partial_P d_P$. Furthermore, we have to choose this correction term to be contained in the kernel of the P-codifferential, which by 3.4.1(ii) can be achieved if we consider invariant elements with values in $L(\mathbb{V}_{-1}, \mathbb{V}_T)$.

Regarding the definition of σ_k the M-invariant form σ_{-1} of degree $(1,0)$ has values in $L(\mathbb{V}_{-1}, \mathbb{V}_T)$ and is trivial under d_K^θ by construction. Moreover, a direct computation shows that

$$k\sigma_k + \sigma_{-1} = (k+1)\sigma_{k-1},$$

and by linearity the same equation holds for their P-derivative.

Thus, we define for all $k = 0, \ldots, n-1$ the Poisson kernel $\phi_k^{\mathbb{V}}$ of degree $(k, n-k)$ by

$$\phi_k^{\mathbb{V}} := k *_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge d_P \sigma_k\right) + *_K \left((d_P E^*)^{n-k} \wedge \sigma_{-1}\right).$$

Then this invariant form has again values in $L(\mathbb{V}, \mathbb{V}_T)$, is trivial on $\mathbb{V}_1$ and contained in the kernels of ∂_P and δ_K by construction. Furthermore, a direct computation shows that

$$d_P \phi_k^{\mathbb{V}} = (-1)^{n+1}(k+1) *_K \left((d_P E^*)^{n-k} \wedge d_P \sigma_{k-1}\right),$$

so since the P-codifferential commutes with the K-Hodge star up to a sign the $(k, n-k+1)$-form $d_P\phi_k^{\mathbb{V}}$ is contained in the kernel of ∂_P due to Theorem 3.4.1(i).

DEFINITION 3.4.1. For all $0 \le k \le n-1$ we call the $L(\mathbb{V}, \mathbb{V}_T)$-valued Poisson kernel $\phi_k^{\mathbb{V}}$ on $\mathfrak{g}/\mathfrak{m}$ of degree $(k, n-k)$ the *distinguished Poisson kernel* and the corresponding integral transformation

$$\Phi_k^{\mathbb{V}} \colon \Omega^k(G/P, G \times_P \mathbb{V}) \to \Omega^k(G/K, T(G/K))$$

the *distinguished Poisson transform.*

By construction of the distinguished Poisson kernels we immediately obtain the following Theorem.

THEOREM 3.4.1. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup and $P \cong \mathrm{CSO}(n) \ltimes \mathbb{R}^n$ its minimal parabolic subgroup. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of G and denote by V_K and V_P the standard tractor bundles over G/K and G/P, respectively.*

Then for all $k = 0, \ldots, n-1$ the distinguished Poisson transforms

$$\Phi_k^{\mathbb{V}} \colon \Omega^k(G/P, V_P) \to \Omega^k(G/K, T(G/K))$$

are BGG-compatible. Furthermore, their values lie in the space of harmonic and coclosed $T(G/K)$-valued differential forms on G/K.

PROOF. We have constructed the Poisson kernels $\phi_k^{\mathbb{V}}$ to be BGG-compatible and K-coclosed, so the claim follows immediately from Proposition 2.2.3 and Theorem 2.2.3. □

Another strategy we could have followed is to classify the full space of M-invariant elements in $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ via a similar analysis as in section 3.3.2, which was done by the author beforehand, and afterwards determine those elements which are BGG-compatible. Doing this, it turns out that $\Phi_k^{\mathbb{V}}$ is the unique BGG-compatible Poisson transform up to a multiple, which preserves the degree. However, the space of M-invariant elements is significantly larger than in the real valued case, so such a classification is attended by lengthy and mindless computations which go beyond the scope of this thesis.

For the remainder of this section we will prove a Corollary of Lemma 3.4.1 which will be needed to show that the G-equivariant operator on sections of weighted homology bundles on G/P induced by the distinguished Poisson transform produces eigenforms of the twisted Laplace operator on G/K, c.f. Theorem 3.4.4.

Corollary 3.4.1. *Let $\phi_k^{\mathbb{V}}$ be the distinguished Poisson transform as above.*

(i) $\phi_k^{\mathbb{V}}|_{\mathbb{V}_{-1}} \in \mathrm{im}(\partial_P)$,

(ii) $\iota_E \phi_k^{\mathbb{V}} \in \mathrm{im}(\partial_P)$.

Proof. (i) The restriction of $\phi_k^{\mathbb{V}}$ to $\mathbb{V}_{-1}$ can be written as a linear combination of the Poisson kernels

$$*_K \left(E^* \wedge (d_P E^*)^{n-k-1} \wedge \left(d_P \sigma_k|_{\mathbb{V}_{-1}}\right)\right), \qquad *_K \left((d_P E^*)^{n-k} \wedge \left(\sigma_{-1}|_{\mathbb{V}_{-1}}\right)\right)$$

by definition. Furthermore, by Proposition 2.2.3 and Corollary 3.4.1(i) the P-codifferential commutes with the K-Hodge star as well as with the wedge product with E^* up to a sign. Therefore, it suffices to show that both $(d_P E^*)^{n-k-1} \wedge \left(d_P \sigma_k|_{\mathbb{V}_{-1}}\right)$ and $(d_P E^*)^{n-k} \wedge \left(\sigma_{-1}|_{\mathbb{V}_{-1}}\right)$ are in the image of the P-codifferential, which we have shown in Theorem 3.4.1.

(ii) Using the calculus on $\mathfrak{g}/\mathfrak{m}$ developed in Corollary 3.2.1(i) we can move the interior product with the M-invariant vector $E \in \mathfrak{g}/\mathfrak{m}$ inside of the K-Hodge star, where it transforms into the wedge product with $E^\flat = E^*$. Thus, by definition of the distinguished Poisson kernel we obtain that

$$\iota_E \phi_k^{\mathbb{V}} = \iota_E *_K \left((d_P E^*)^{n-k} \wedge \sigma_{-1}\right).$$

But this is the interior product of E with the second Poisson kernel in the proof of part (i), so since $\iota_E \partial_P = -\partial_P \iota_E$ we immediately obtain that $\iota_E \phi_k^{\mathbb{V}}$ is in the image of the P-codifferential. □

3.4.2. The $\mathfrak{p}$-action on the distinguished Poisson kernel. In this rather technical section we will compute a formula for the $\mathfrak{p}$-action on the values of $\phi_k^{\mathbb{V}}$, which will be needed in Proposition 3.4.3 to compute the K-derivative of the distinguished Poisson kernels.

Let ρ be the $\mathfrak{g}$-representation on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ given by the tensor product of the identity on the exterior power and the induced action on $\mathrm{End}(\mathbb{V})$. Our aim in this section is to determine $\rho_\xi \phi_k^{\mathbb{V}}$ for all $\xi \in \mathfrak{p}$, which is part of formula (1.2.4) for the K-derivative. As a first step, note that by definition of the K-Hodge star in section 2.2.3 it follows that $\rho \circ *_K = *_K \circ \rho$. Now if define for all $0 \leq k \leq n$ the M-invariant form φ_k of degree $(2,1)$ via

$$\varphi_k := kE^* \wedge d_P \sigma_k + (d_P E^*) \wedge \sigma_{-1},$$

then we have

$$\phi_k^{\mathbb{V}} = *_K \left((d_P E^*)^{n-k-1} \wedge \varphi_k\right)$$

and therefore

$$\rho_\xi \phi_k^{\mathbb{V}} = *_K \left((d_P E^*)^{n-k-1} \wedge \rho_\xi \varphi_k\right)$$

for all $\xi \in \mathfrak{g}$. Hence, we will start this section by computing the $\mathfrak{p}$-action on the M-invariant $(2,1)$-form φ_k explicitly.

In order to state this result we need two more ingredients. First, recall from section 3.3.2 that we defined an M-invariant map $\chi\colon \mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m}$ via $\chi(F_X) = G_X$ for all $X \in \mathbb{R}^n$ and $\chi(E) = 0$, which was subsequently used to establish a geometric interpretation of the real valued Poisson kernels.

Furthermore, we define for all $\ell \in \mathbb{R}$ the M-invariant forms $\nu_\ell \in \Lambda^{0,1}(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ by

$$\nu_\ell(G_Y)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} -(\ell+1)\langle Y, W\rangle \\ (n-\ell+1)w_{-1}Y \\ 0 \end{pmatrix}$$

for all $Y \in \mathbb{R}^n$. Note that unless $\ell = -1$, the image of $\nu_\ell(G_Y)$ is not symmetric in the first and last entry and therefore not contained in the K-invariant subspace $L(\mathbb{V}, \mathbb{V}_T)$ of $\mathrm{End}(\mathbb{V})$. Furthermore, the element $\chi^*\nu_{-1}$ coincides with the M-invariant $(1,0)$-form σ_{-1} used in the construction of the distinguished Poisson kernel.

LEMMA 3.4.2. *Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $\mathfrak{m} \cong \mathfrak{so}(n)$ the intersection of its maximal compact subalgebra $\mathfrak{k}$ and minimal parabolic subalgebra $\mathfrak{p}$. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of $\mathfrak{g}$ and denote by ρ the induced $\mathfrak{g}$-action on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$. For $U \in \mathbb{R}^n$ let ξ_U be the corresponding element in $\mathfrak{p}_+$ and F_U its projection to $\mathfrak{p}_+/\mathfrak{m}$. Then the M-invariant $L(\mathbb{V}, \mathbb{V}_T)$-valued $(2,1)$-form φ_k defined above satisfies*

(i) For all $U \in \mathbb{R}^n$ we have

$$k\iota_E\left(E^* \wedge \rho_{\xi_U}\varphi_k\right) = \iota_E\left(F_U^\flat \wedge \rho_E\varphi_k\right).$$

(ii) For all $U \in \mathbb{R}^n$ we obtain that

$$\begin{aligned} E^* \wedge \iota_E\rho_{\xi_U}\varphi_k = {} & kE^* \wedge \left(-\left(\iota_{F_U} d_P E^*\right) \wedge \chi^*\nu_k + (d_P E^*) \wedge \iota_{F_U}\chi^*\nu_k\right) \\ & - 2kE^* \wedge F_U^\flat \wedge \nu_{n-k}, \end{aligned}$$

where $\chi^*\nu_k \in (\mathfrak{p}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ denotes the pullback of ν_k along the M-equivariant map $\chi\colon \mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m}$.

PROOF. We will prove these formulae by explicit computations in terms of the decomposition of the standard representation into irreducible $\mathfrak{m}$-modules as in section 3.2.2. Recall that for all $U \in \mathbb{R}^n$ the actions of the vector $\xi_U \in \mathfrak{g}_1$ and the grading element E on the standard representation are given by

$$\xi_U \cdot \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} = \begin{pmatrix} -\langle U, W\rangle \\ w_{-1}U \\ 0 \end{pmatrix}, \qquad E \cdot \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} = \begin{pmatrix} w_1 \\ 0 \\ -w_{-1} \end{pmatrix}.$$

Furthermore, in this picture the M-invariant forms σ_{-1} and $d_P\sigma_k$ satisfy

$$\sigma_{-1}(F_X)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ (n+2)w_{-1}X \\ 0 \end{pmatrix}$$

$$d_P\sigma_k(F_X, G_Y)\colon \begin{pmatrix} w_1 \\ W \\ w_{-1} \end{pmatrix} \mapsto \begin{pmatrix} k(n+2)w_{-1}\langle X, Y\rangle \\ k(k+1)\left(\langle X, Y\rangle W + \langle Y, W\rangle Y\right) - k(n-k+1)\langle Y, W\rangle X \\ k(n+2)w_{-1}\langle X, Y\rangle \end{pmatrix}$$

for all X, Y, $W \in \mathbb{R}^n$ and w_1, $w_{-1} \in \mathbb{R}$, compare with section 3.4.1.

(i) By definition of the invariant $(2,1)$-form φ_k we conclude that the left hand side of the claimed equation can be rewritten as

$$k\iota_E\left(E^* \wedge \rho_{\xi_U}\varphi_k\right) = k(d_P E^*) \wedge \rho_{\xi_U}\sigma_{-1},$$

where σ_{-1} is the $L(\mathbb{V}_{-1}, \mathbb{V}_0)$-valued M-invariant form of degree $(0,1)$ used in the correction term for the construction of the distinguished Poisson kernel. In particular, since the action of $\mathfrak{g}_1 = \mathfrak{p}_+$ on $\mathbb{V}$ raises homogeneity by 1 it follows from the explicit description that the action of ξ_U on this Poisson kernel is trivial unless it is applied to a vector in $\mathbb{V}_{-1}$. Explicitly, if we denote by $v_{\pm 1}$ the base elements of $\mathbb{V}_{\pm 1}$ corresponding to $1 \in \mathbb{R}$ (c.f. section 3.2.2), then we deduce for all $X \in \mathbb{R}^n$ that

$$\begin{aligned}\left(\rho_{\xi_U}\sigma_{-1}\right)(F_X)(v_{-1}) &= \xi_U \cdot \left(\sigma_{-1}(F_X)(v_{-1})\right)\\ &= -(n+2)\langle U, X\rangle v_1\\ &= -2(n+2)\left(F_U^\flat \otimes f_1^{-1}\right)(F_X)(v_{-1}),\end{aligned}$$

where used that $F_U^\flat(F_X) = \frac{1}{2}\langle U, X\rangle$ for all U, $X \in \mathbb{R}^n$ due to Proposition 3.2.1. All in all, we see that

$$k\iota_E\left(E^* \wedge \rho_{\xi_U}\varphi_k\right) = -2k(n+2)F_U^\flat \wedge (d_P E^*) \otimes f_1^{-1}.$$

On the other hand, the action of the grading element on $\mathbb{V}_j$ is given by multiplication with j. Therefore, it follows that the expression $\rho_E\iota_E\varphi_k$ is only nontrivial upon insertion of $v_{-1} \in \mathbb{V}_{-1}$, which equals

$$\begin{aligned}\rho_E\iota_E\varphi_k(F_X, G_Z)(v_{-1}) &= \rho_E d_P\sigma_k(F_X, G_Y)(v_{-1})\\ &= 2k(n+2)\langle X, Z\rangle v_1\\ &= 2k(n+2)\left(d_P E^* \otimes f_1^{-1}\right)(F_X, G_Z)(v_{-1}).\end{aligned}$$

Thus, we obtain

$$\iota_E\left(F_U^\flat \wedge \rho_E\varphi_k\right) = -2k(n+2)F_U^\flat \wedge (d_P E^*) \otimes f_1^{-1},$$

which coincides with the formula for $k\iota_E\left(E^* \wedge \rho_{\xi_U}\varphi_k\right)$.

(ii) First, by definition of φ_k we obtain that $\varphi_k(E, F_X, G_Y) = d_P\sigma_k(F_X, G_Y)$ for all X, $Y \in \mathbb{R}^n$. Furthermore, using the formulae at the beginning of the proof shows that $\rho_{\xi_U} d_P\sigma_k(F_X, G_Y) \in \operatorname{End}(\mathbb{V})$ maps the vector $\begin{pmatrix} w_1 \\ W \\ w_{-1}\end{pmatrix}$ in $\mathbb{V}$ to

$$k\begin{pmatrix}(n-k+1)\langle X, U\rangle\langle Y, W\rangle - (k+1)\left(\langle X, Y\rangle\langle W, U\rangle + \langle X, W\rangle\langle Y, U\rangle\right)\\ -(k+1)w_{-1}\langle X, U\rangle Y + (n-k+1)w_{-1}\left(\langle X, Y\rangle U + \langle Y, U\rangle X\right)\\ 0\end{pmatrix}$$

for all X, Y, W, $U \in \mathbb{R}^n$ and w_{-1}, $w_1 \in \mathbb{R}$. Comparing this with the definition of the $(0,1)$-form ν_k we see that the first summands in each component of $\iota_E\rho_{\xi_U}\varphi_k$ can be rewritten as $-2F_U^\flat \wedge \nu_{n-k}$, whereas the second and last summands equal $d_P E^* \wedge \iota_{F_U}\chi^*\nu_k$ and $-\left(\iota_{F_U} d_P E^*\right) \wedge \chi^*\nu_k$, respectively, all evaluated at F_X and G_Y. $\square$

For the computation of the K-differential of the distinguished Poisson kernel in Proposition 3.4.3 we will apply formula (1.2.4) for the infinitesimal version of the covariant exterior derivative, in which we have to alternate over the action of vectors in $\mathfrak{p}$. In order to bring the resulting expression into an appropriate form, we need to rewrite this action in terms of interior products with the corresponding vectors in $\mathfrak{p}/\mathfrak{m}$.

PROPOSITION 3.4.2. *Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $\mathfrak{m} \cong \mathfrak{so}(n)$ the intersection of its maximal compact subalgebra $\mathfrak{k}$ and minimal parabolic subalgebra $\mathfrak{p}$. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of $\mathfrak{g}$ and denote by ρ the $\mathfrak{g}$-action on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ given by the tensor product of the identity on the exterior power and the induced action on $\mathrm{End}(\mathbb{V})$. Let $\phi_k^{\mathbb{V}}$ be the distinguished Poisson kernel as in Definition 3.4.1 and ϕ_k the real valued Poisson kernels of degree $(k, n-k)$ defined in section 3.3.1. Moreover, for all $X \in \mathbb{R}^n$ let ξ_X be the corresponding element in $\mathfrak{p}_+ = \mathfrak{g}_1$ and F_X its projection onto $\mathfrak{p}_+/\mathfrak{m}$.*

(i) For all $X \in \mathbb{R}^n$ we have

$$k\iota_E \rho_{\xi_X} \phi_k^{\mathbb{V}} = \iota_E \left(E^* \wedge \iota_{F_X} \rho_E \phi_k^{\mathbb{V}}\right)$$

(ii) For all $X \in \mathbb{R}^n$ we have

$$\begin{aligned}\iota_E \left(E^* \wedge \rho_{\xi_X} \phi_k^{\mathbb{V}}\right) = &-\frac{k}{n-k} *_K \iota_{F_X} \left(E^* \wedge (d_P E^*)^{n-k} \wedge \chi^* \nu_k\right)\\ &+ k\left(\frac{n-k+1}{n-k}\right) \phi_k \otimes (\iota_{F_X} \chi^* \nu_k)\\ &+ 2k(-1)^{n-k} \left(\iota_{F_X} \phi_{k+1}\right) \wedge \nu_{n-k},\end{aligned}$$

where ϕ_k denotes the real valued Poisson kernel of degree $(k, n-k)$ defined in section 3.3.1 and $\chi \colon \mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m}$ is the M-equivariant map defined in section 3.3.2.

PROOF. (i) As mentioned at the beginning of this section, the operator $*_K$ commutes with the $\mathfrak{g}$-action ρ, so after insertion the definition of the distinguished Poisson kernel we obtain that

$$\begin{aligned}k\iota_E \rho_{\xi_X} \phi_k^{\mathbb{V}} &= k\iota_E *_K \left((d_P E^*)^{n-k-1} \wedge \rho_{\xi_X} \varphi_k\right)\\ &= k\iota_E *_K \left((d_P E^*)^{n-k-1} \wedge \iota_E \left(E^* \wedge \rho_{\xi_X} \varphi_k\right)\right)\\ &= (*),\end{aligned}$$

where we used for the second equality that $\iota_E *_K (E^* \wedge \alpha) = 0$ for all M-invariant forms α by Corollary 3.2.1. Next, we apply Lemma 3.4.2(i) and afterwards use the calculus on $\mathfrak{g}/\mathfrak{m}$ from Corollary 3.2.1 to move ι_E as well as $F_U^\flat$ out of the K-Hodge star. All in all, we get

$$\begin{aligned}(*) &= \iota_E *_K \left((d_P E^*)^{n-k-1} \wedge \iota_E \left(F_X^\flat \wedge \rho_E \varphi_k\right)\right)\\ &= (-1)^{n-k+1} \iota_E \left(E^* \wedge *_K \left(F_X^\flat \wedge (d_P E^*)^{n-k-1} \wedge \rho_E \varphi_k\right)\right)\\ &= \iota_E \left(E^* \wedge \iota_{F_X} \rho_E \phi_k^{\mathbb{V}}\right).\end{aligned}$$

(ii) Similar to (i) we use the calculus on $\mathfrak{g}/\mathfrak{m}$ from Corollary 3.2.1(i) to rewrite

$$\begin{aligned}\iota_E \left(E^* \wedge \rho_{\xi_X} \phi_k^{\mathbb{V}}\right) &= \iota_E \left(E^* \wedge *_K \left((d_P E^*)^{n-k-1} \wedge \rho_{\xi_X} \varphi_k\right)\right)\\ &= *_K \left((d_P E^*)^{n-k-1} \wedge \left(E^* \wedge \iota_E \rho_{\xi_X} \varphi_k\right)\right)\\ &= (**).\end{aligned}$$

Next, we insert the formula from Lemma 3.4.2(ii) and obtain that

$$\begin{aligned}(**) = &-k *_K \left((d_P E^*)^{n-k-1} \wedge E^* \wedge (\iota_{F_X} d_P E^*) \wedge \chi^* \nu_k\right)\\ &+ k *_K \left((d_P E^*)^{n-k} \wedge E^* \otimes (\iota_{F_X} \chi^* \nu_k)\right)\\ &- 2k *_K \left((d_P E^*)^{n-k-1} \wedge E^* \wedge F_X^\flat \wedge \nu_{n-k}\right).\end{aligned}$$

In order to simply this expression, we first use the antiderivation property of the interior product to write the first line as a sum of two expressions, where

ι_{F_X} is either applied to the whole argument of the K-Hodge star or only to the $(1,0)$-form $\chi^*\nu_k$, respectively, and afterwards move the M-invariant map $\iota_{F_X}\chi^*\nu_k \in \mathrm{End}(\mathbb{V})$ out of the K-Hodge star in the first and second line. Moreover, in the third line we can move the $(0,1)$-form ν_{n-k} out of the Hodge star as well as apply Corollary 3.2.1(i) to $F_X^\flat$. All in all, we get

$$\begin{aligned}(**) = & -\frac{k}{n-k} *_K \iota_{F_X}\left(E^* \wedge (d_P E^*)^{n-k} \wedge \chi^*\nu_k\right)\\ &+\left(\frac{k}{n-k}+k\right)\phi_k \otimes (\iota_{F_X}\chi^*\nu_k) + 2k(-1)^{n-k}\left(\iota_{F_X}\phi_{k+1}\right)\wedge \nu_{n-k},\end{aligned}$$

which coincides with the claimed formula. □

3.4.3. The distinguished Poisson transforms and exterior derivatives. One of the main features of the Poisson transforms between real valued differential forms in section 3.3 is that it is a chain map between the de Rham sequences on G/P and G/K, respectively. Using the technical results from the previous section we will now prove that the distinguished Poisson transform $\Phi_k^{\mathbb{V}}$ has the same property with respect to the covariant exterior derivatives induced by the tractor connections on the standard tractor bundles on G/P and G/K, respectively. By Proposition 2.2.3, it suffices to show that the invariant form $d_K\phi_k^{\mathbb{V}}$ is a multiple of the P-derivative of $\phi_{k+1}^{\mathbb{V}}$. Note that the latter has already been computed in section 3.4.1.

In order to determine the K-derivative of the distinguished Poisson kernel we will apply formula (1.2.4) for infinitesimal version of the covariant exterior derivative. In particular, we will need to alternate over the action of vectors $\xi_X \in \mathfrak{g}_1$ on $\phi_k^{\mathbb{V}}$, which we explicitly determined in Proposition 3.4.2. To simplify the formula in part (ii) of this Proposition, we will prove a relation between the wedge products of the real valued Poisson kernels ϕ_k and the M-invariant elements ν_ℓ and $\chi^*\nu_\ell$ of degree $(0,1)$ and $(1,0)$, respectively, which is the vector-valued generalization of Lemma 3.3.2.

For that, we need to introduce some notation. By a *k-vector* $\mathbf{X}$ on $\mathbb{R}^n$ we mean a k-fold wedge product $X_1 \wedge \ldots \wedge X_k$ of vectors in $\mathbb{R}^n$. Moreover, we denote by $F_{\mathbf{X}}$ the wedge product $F_{X_1} \wedge \ldots \wedge F_{X_k}$, and similarly for $G_{\mathbf{X}}$. Finally, for $1 \le i \le k$ we write $F_{\mathbf{X}}^i$ for the wedge product obtained from $F_{\mathbf{X}}$ by omitting the i-th factor.

LEMMA 3.4.3. *Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$, $\mathfrak{m} \cong \mathfrak{so}(n)$ the intersection of its maximal compact subalgebra $\mathfrak{k}$ and minimal parabolic subalgebra $\mathfrak{p}$ and $\mathbb{V} = \mathbb{R}^{n+1,1}$ the standard representation of $\mathfrak{g}$. Let $\phi_k \in \Lambda^{k,n-k}(\mathfrak{g}/\mathfrak{m})^*$ be the real valued Poisson kernel on $\mathfrak{g}/\mathfrak{m}$ defined in section 3.3.1. Let ν be an M-invariant $\mathrm{End}(\mathbb{V})$-valued $(0,1)$-form on $\mathfrak{g}/\mathfrak{m}$ and let $\chi^*\nu$ be the corresponding invariant form of degree $(1,0)$ defined by $(\chi^*\nu)(\xi) = \nu(\chi(\xi))$ for all $\xi \in \mathfrak{p}/\mathfrak{m}$. Then we have*

$$\phi_k \wedge \chi^*\nu = (-1)^{k+1}2(n-k)\phi_{k+1}\wedge\nu,$$

where $\chi\colon \mathfrak{p}/\mathfrak{m} \to \mathfrak{k}/\mathfrak{m}$ is the M-equivariant map defined in section 3.3.2.

PROOF. Recall from Lemma 3.3.2 that the real valued Poisson kernels ϕ_k are related in different degrees by

$$\iota_{G_X}\phi_k = (-1)^k 2(n-k)\iota_{F_X}\phi_{k+1}$$

for all $X \in \mathbb{R}^n$. Furthermore, we have seen in section 3.3.1 that $\phi_n = \iota_E \operatorname{vol}_K$, where $\operatorname{vol}_K = *_K(1) \in \Lambda^{n+1,0}(\mathfrak{g}/\mathfrak{m})^*$ is the K-volume form on $\mathfrak{g}/\mathfrak{m}$. Therefore, applying the above formula inductively we obtain that

$$\phi_k(F_{\mathbf{X}}, G_{\mathbf{Y}}) = \lambda_k \iota_E \operatorname{vol}_K(F_{\mathbf{X}}, F_{\mathbf{Y}})$$

for all k-vectors $\mathbf{X}$ and $(n-k)$-vectors $\mathbf{Y}$ on $\mathbb{R}^n$, where the coefficient λ_k satisfies $\lambda_k\lambda_{k+1}^{-1} = (-1)^k 2(n-k)$.

We have seen in section 3.2.1 that the M-modules $(\mathfrak{g}/\mathfrak{m})_1$ and $(\mathfrak{g}/\mathfrak{m})_{-1}$ are both isomorphic to $\mathbb{R}^n$, and under this identification the K-volume form is a multiple of the determinant. Now by a simple application of the Laplace expansion formula for matrices we deduce that if $v_1, \dots, v_{n+1}, w$ are arbitrary vectors in $\mathbb{R}^n$ and if $\langle\ ,\ \rangle$ is an inner product on $\mathbb{R}^n$, then

$$\sum_{i=1}^{n+1}(-1)^{i+1}\langle v_i, w\rangle \det(v_1, \dots, \hat{\imath}, \dots, v_{n+1}) = 0. \tag{3.4.3}$$

Due to M-invariance, for all $X \in \mathbb{R}^n$ the linear map $\nu(G_X) \in \mathrm{End}(\mathbb{V})$ is built up by the $\mathrm{SO}(n)$-invariant parings of X with $W \in \mathbb{V}_0 \cong \mathbb{R}^n$ as well as multiplication on X with elements in $\mathbb{V}_{\pm 1} \cong \mathbb{R}$. Therefore, we can apply equation (3.4.3) to each of the summands of the decomposition of $\mathbb{V}$ into irreducible $\mathfrak{m}$-modules. Otherwise said, we obtain that

$$\sum_{i=1}^{n+1}(-1)^i \iota_E \operatorname{vol}_K(F_{X_1}, \dots, \widehat{F_{X_i}}, \dots, F_{X_{n+1}})\ \chi^*\nu(F_{X_i}) = 0$$

for all vectors $X_1, \dots, X_{n+1} \in \mathbb{R}^n$, where the hat denotes omission.

Thus, for all $k+1$-vectors $\mathbf{X}$ and $(n-k)$-vectors $\mathbf{Y}$ on $\mathbb{R}^n$ we deduce

$$\begin{aligned}
(\phi_k \wedge (\chi^*\nu))(F_{\mathbf{X}}, G_{\mathbf{Y}}) &= \sum_{i=1}^{k+1}(-1)^{n+1+i}\phi_k(F^i_{\mathbf{X}}, G_{\mathbf{Y}}) \otimes \iota_{F_{X_i}}(\chi^*\nu) \\
&= (-1)^n \lambda_k \sum_{i=1}^{k+1}(-1)^{i+1}\iota_E \operatorname{vol}_K(F^i_{\mathbf{X}}, F_{\mathbf{Y}}) \otimes \iota_{F_{X_i}}(\chi^*\nu) \\
&= (-1)^{n-k+1}\lambda_k \sum_{j=1}^{n-k}(-1)^j \iota_E \operatorname{vol}_K(F_{\mathbf{X}}, F^j_{\mathbf{Y}}) \otimes \iota_{F_{Y_j}}(\chi^*\nu) \\
&= (-1)^{n+1}2(n-k)\sum_{j=1}^{n-k}(-1)^j \phi_{k+1}(F_{\mathbf{X}}, G^j_{\mathbf{Y}}) \otimes \iota_{G_{Y_j}}\nu \\
&= (-1)^{k+1}2(n-k)(\phi_{k+1}\wedge\nu)(F_{\mathbf{X}}, G_{\mathbf{Y}}),
\end{aligned}$$

which proves the claimed equation. □

Using the formulae for the $\mathfrak{p}$-action on $\phi_k^{\mathbb{V}}$ in Proposition 3.4.2 as well as Lemma 3.4.3 we are finally able to compute the K-derivative of the distinguished Poisson kernel.

Proposition 3.4.3. *Let $\phi_k^{\mathbb{V}}$ be the distinguished Poisson kernel as in Definition 3.4.1. Then its image under the K-derivative satisfies*

(i) $E^* \wedge \iota_E d_K \phi_k^{\mathbb{V}} = (n-2k)E^* \wedge \phi_k^{\mathbb{V}}$.

(ii) $\iota_E\left(E^* \wedge d_K\phi_k^{\mathbb{V}}\right) = (-1)^{n-k}k(n-2k) *_K \left(E^* \wedge (d_P E^*)^{n-k} \wedge \iota_E d_P \sigma_k\right)$,

where σ_k is the M-invariant element $(1,0)$-form used in the construction of ϕ_k in section 3.4.1. In particular, we have

$$d_K\phi_k^{\mathbb{V}} = (-1)^{n-k}k(n-2k) *_K \left((d_P E^*)^{n-k} \wedge d_P\sigma_k\right).$$

Proof. In order to prove the Proposition we will use formula (1.2.4) for the infinitesimal version of the covariant exterior derivative as well as the vector notation introduced at the beginning of this section. Moreover, for all $X,\ Y \in \mathbb{R}^n$ we denote by ξ_X and η_Y the corresponding elements in $\mathfrak{p}_+$ and $\mathfrak{k}$ and by $F_X \in \mathfrak{p}_+/\mathfrak{m}$ and $G_Y \in \mathfrak{k}/\mathfrak{m}$ their projection to $\mathfrak{g}/\mathfrak{m}$, respectively.

(i) Let $\mathbf{X}$ be a k-vector and $\mathbf{Y}$ be an $(n-k)$-vector on $\mathbb{R}^n$. First, we want to determine

$$E^* \wedge \iota_E d_K \phi_k^{\mathbb{V}}(E, F_{\mathbf{X}}, G_{\mathbf{Y}}) = d_K \phi_k^{\mathbb{V}}(E, F_{\mathbf{X}}, G_{\mathbf{Y}}) = (*)$$

by applying formula (1.2.4), which splits into $(*) = (I) + (II)$, where (I) is the sum of all expressions containing Lie brackets of representatives of the vectors in $\mathfrak{g}/\mathfrak{m}$ and (II) is the sum of all expressions containing the action of $\mathfrak{g}$ on $\mathrm{End}(\mathbb{V})$.

For the first summand we obtain from the bracket relations on $\mathfrak{g}$ in section 3.1.1 that the only pairings which contribute nontrivially to the K-derivative are those between the grading element and the representatives ξ_{X_i} and η_{Y_j}, which equal ξ_{X_i} and $2\xi_{Y_j} - \eta_{Y_j}$, respectively. Thus, the bracket part of $(*)$ equals

$$(I) = (n-2k)\left(E^* \wedge \phi_k^{\mathbb{V}}\right)(E, F_{\mathbf{X}}, G_{\mathbf{Y}}).$$

For the other summand of $(*)$ we have to alternate over the actions of the representatives of the vectors in $\mathfrak{p}/\mathfrak{m}$, i.e. we have to determine

$$(II) = \rho_E \phi_k^{\mathbb{V}}(F_{\mathbf{X}}, G_{\mathbf{Y}}) + \sum_{i=1}^{k} (-1)^{i+1} \left(\rho_{\xi_{X_i}} \phi_k^{\mathbb{V}}\right)(E, F_{\mathbf{X}}^i, G_{\mathbf{Y}}).$$

However, we can apply Proposition 3.4.2(i) to the second summand of (II), obtaining

$$\begin{aligned}
(II) &= \rho_E \phi_k^{\mathbb{V}}(F_{\mathbf{X}}, G_{\mathbf{Y}}) + \frac{1}{k}\sum_{i=1}^{k} (-1)^i \left(E^* \wedge \iota_{F_{X_i}} \rho_E \phi_k^{\mathbb{V}}\right)(E, F_{\mathbf{X}}^i, G_{\mathbf{Y}}) \\
&= \rho_E \phi_k^{\mathbb{V}}(F_{\mathbf{X}}, G_{\mathbf{Y}}) - \frac{1}{k}\sum_{i=1}^{k} \rho_E \phi_k^{\mathbb{V}}(F_{\mathbf{X}}, G_{\mathbf{Y}}) \\
&= 0
\end{aligned}$$

and therefore

$$E^* \wedge \iota_E d_K \phi_k^{\mathbb{V}} = (n-2k) E^* \wedge \phi_k^{\mathbb{V}}.$$

(ii) By definition, the expression $\iota_E\left(E^* \wedge d_K \phi_k^{\mathbb{V}}\right)$ is only nontrivial upon insertion of $F_{\mathbf{X}}$ and $G_{\mathbf{Y}}$ for any $(k+1)$-vector $\mathbf{X}$ and $(n-k)$-vector $\mathbf{Y}$. Therefore, we have to compute the expression

$$\iota_E\left(E^* \wedge d_K \phi_k^{\mathbb{V}}\right)(F_{\mathbf{X}}, G_{\mathbf{Y}}) = d_K \phi_k^{\mathbb{V}}(F_{\mathbf{X}}, G_{\mathbf{Y}}) = (**),$$

for which we will again use formula (1.2.4). Note that the Lie bracket between two elements in $\mathfrak{p}_+$ is trivial, and that $[\xi_X, \eta_Y] = -\langle X, Y\rangle E$ is not contained in $\mathfrak{k}$. Therefore, the bracket part of $(**)$ is trivial, obtaining

$$\begin{aligned}
(**) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{\xi_{X_i}} \phi_k^{\mathbb{V}}\left(F_{\mathbf{X}}^i, G_{\mathbf{Y}}\right) \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \iota_E\left(E^* \wedge \rho_{\xi_{X_i}} \phi_k^{\mathbb{V}}\right)\left(F_{\mathbf{X}}^i, G_{\mathbf{Y}}\right).
\end{aligned}$$

Applying Proposition 3.4.2(ii) to $(**)$ and moving the vectors F_{X_i} back to the right position we obtain that

$$\begin{aligned}(**) = &-\frac{k}{n-k}\sum_{i=1}^{k+1}(-1)^{i+1} *_K \iota_{F_{X_i}}\left(E^*\wedge(d_PE^*)^{n-k}\wedge\chi^*\nu_k\right)(F^i_{\mathbf{X}},G_{\mathbf{Y}})\\ &+(-1)^n k\left(\frac{n-k+1}{n-k}\right)(\phi_k\wedge\chi^*\nu_k)(F_{\mathbf{X}},G_{\mathbf{Y}})\\ &+(-1)^{n-k}2k(k+1)(\phi_{k+1}\wedge\nu_{n-k})(F_{\mathbf{X}},G_{\mathbf{Y}}).\end{aligned}$$

For the first line, note that by Corollary 3.2.1(i) we can move the interior product with F_X outside of the Hodge star, which transforms into the wedge product with $F^\flat_{X_i}$ up to a sign. Now if we insert F_{X_j} into this $(1,0)$-form, the result is symmetric in i and j, hence the corresponding summands cancel. Therefore, the first line of $(*)$ is trivial. Moreover, applying Lemma 3.4.3 to the second line we can write

$$\iota_E\left(E^*\wedge d_K\phi^{\mathbb{V}}_k\right) = (-1)^{n-k}2k\phi_{k+1}\wedge((k+1)\nu_{n-k}-(n-k+1)\nu_k).$$

Finally, in order to simplify the expression in the bracket, recall from section 3.4.1 that the invariant $(0,1)$-forms ν_ℓ were defined by

$$\nu_\ell(G_Y)\colon \begin{pmatrix} w_1\\ W\\ w_{-1}\end{pmatrix}\mapsto\begin{pmatrix}-(\ell+1)\langle Y,W\rangle\\ (n-\ell+1)w_{-1}Y\\ 0\end{pmatrix}.$$

for all Y, $W\in\mathbb{R}^n$ and w_{-1}, $w_1\in\mathbb{R}$. Using this, a direct computation shows that

$$2\left((k+1)\nu_{n-k}-(n-k+1)\nu_k\right)=(n-2k)\iota_E d_P\sigma_k.$$

Therefore, if we move the $(0,1)$-form $\iota_E d_P\sigma_k$ back into the K-Hodge star we obtain

$$\iota_E\left(E^*\wedge d_K\phi^{\mathbb{V}}_k\right)=(-1)^{n-k}k(n-2k) *_K\left(E^*\wedge(d_PE^*)^{n-k}\wedge\iota_E d_P\sigma_k\right).$$

To prove the final statement we first use Corollary 3.2.1(i) and afterwards the identity $\iota_E\varphi_k = kd_P\sigma_k - kE^*\wedge\iota_E d_P\sigma_k$ to write

$$\begin{aligned}d_K\phi^{\mathbb{V}}_k &= E^*\wedge\iota_E d_K\phi^{\mathbb{V}}_k+\iota_E\left(E^*\wedge d_K\phi^{\mathbb{V}}_k\right)\\ &=(n-2k)\Big(E^*\wedge\phi^{\mathbb{V}}_k+(-1)^{n-k}k *_K\left(E^*\wedge(d_PE^*)^{n-k}\wedge\iota_E d_P\sigma_k\right)\Big)\\ &=(-1)^{n-k}k(n-2k) *_K\left((d_PE^*)^{n-k-1}\wedge\iota_E d_P\sigma_k\right).\end{aligned}$$

□

Using the relation between the image of Poisson kernels under the partial derivatives and compositions of the corresponding Poisson transforms with covariant exterior derivatives provided in Proposition 2.2.3 we immediately obtain the following Theorem.

THEOREM 3.4.3. *Let $G=\mathrm{SO}(n+1,1)_0$, $K\cong\mathrm{SO}(n+1)$ its maximal compact subgroup, $P\cong\mathrm{CSO}(n)\rtimes\mathbb{R}^n$ its minimal parabolic subgroup and $M=K\cap P$. Let $\mathbb{V}=\mathbb{R}^{n+1,1}$ be the standard representation of G and denote by V_K and V_P the corresponding standard tractor bundles over G/K and G/P with covariant exterior derivatives d^{V_K} and d^{V_P} induced by the tractor connections, respectively. Then for all $k=1,\dots,n-1$ the distinguished Poisson transform $\Phi^{\mathbb{V}}_k$ satisfies*

$$(k+2)d^{V_K}\circ\Phi^{\mathbb{V}}_k=(-1)^{n-k}k(n-2k)\Phi_{k+1}\circ d^{V_P}.$$

PROOF. In section 3.4.1 we have computed that

$$d_P \phi^{\mathbb{V}}_{k+1} = (-1)^{n+1}(k+2) *_K \left((d_P E^*)^{n-k} \wedge d_P \sigma_k\right).$$

Comparing this with the formula for the K-derivative of the distinguished Poisson kernel in Proposition 3.4.3 we deduce that

$$(k+2) d_K \phi^{\mathbb{V}}_k = (-1)^{k+1} k(n-2k) d_P \phi^{\mathbb{V}}_{k+1},$$

which by Proposition 2.2.3 translates into the claimed global formula. □

3.4.4. The Poisson transform on weighted differential forms. In the next step we want to extend the domain of the distinguished Poisson transform to differential forms on G/P with values in the weighted standard tractor bundles $V_P[\lambda]$ for $\lambda \in \mathbb{R}$ and consider their induced G-equivariant operators $\tilde{\Phi}^{\mathbb{V}}_k$ on sections of the homology bundles $\mathcal{H}_k(G/P, V_P[\lambda])$. We will show in Theorem 3.4.4 that the image of $\tilde{\Phi}^{\mathbb{V}}_k$ consists of eigenforms of the covariant Laplace operator on G/K as defined in section 1.3.2. Furthermore, in the unweighted case $\lambda = 0$ these transforms will be chain maps between the BGG-sequence on G/P with values in V_P and the twisted de Rham sequence on G/K with values in $T(G/K)$. Note that in contrast to the case of Poisson transform between real valued differential forms the images of weighted differential forms under $\Phi^{\mathbb{V}}_k$ will not be eigenforms of the Laplace operator in general.

We start this section by computing the image of the distinguished Poisson kernel under two M-equivariant operators which will be needed for the proof of Proposition 3.4.4. In order to do so, we define the $\mathfrak{g}$-actions ρ and ρ^θ on $f \in \mathrm{End}(\mathbb{V})$ by

$$(\rho_X f)(v) = X \cdot f(v) - f(X \cdot v), \qquad \left(\rho^\theta_X f\right)(v) = \theta(X) \cdot f(v) - f(X \cdot v)$$

for all $v \in \mathbb{V}$ and $X \in \mathfrak{g}$. Furthermore, we denote their trivial extensions to $\Lambda^k(\mathfrak{g}/\mathfrak{m})^* \otimes \mathrm{End}(\mathbb{V})$ by the same symbols.

LEMMA 3.4.4. *Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $\mathfrak{m} \cong \mathfrak{so}(n)$ the intersection of the maximal compact and the minimal parabolic subalgebra with respect to the Cartan involution θ. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of $\mathfrak{g}$ and denote by ρ and ρ^θ the $\mathfrak{g}$-action on $\Lambda^*(\mathfrak{g}/\mathfrak{m}) \otimes \mathrm{End}(\mathbb{V})$ defined above. Then the distinguished Poisson kernel $\phi^{\mathbb{V}}_k$ satisfies*

(i) $d_K \iota_E \phi^{\mathbb{V}}_k + \iota_E d_K \phi^{\mathbb{V}}_k = (n-2k)\phi^{\mathbb{V}}_k + \rho_E(\phi^{\mathbb{V}}_k)$,
(ii) $\delta_K(E^* \wedge \phi^{\mathbb{V}}_k) + E^* \wedge \delta_K \phi^{\mathbb{V}}_k = -\rho^\theta_E(\phi^{\mathbb{V}}_k)$,
(iii) $\rho_E(\phi^{\mathbb{V}}_k) + \rho^\theta_E(\phi^{\mathbb{V}}_k) = 2\left(\phi^{\mathbb{V}}_k|_{\mathbb{V}_{-1}}\right)$.

PROOF. (i) If we apply the formula (1.2.4) for the infinitesimal version of the covariant exterior derivative we obtain that $\iota_E d_K \phi^{\mathbb{V}}_k$ splits into two summands. The first one is given by all summands containing the Lie bracket of the grading element and representatives of vectors in $\mathfrak{g}/\mathfrak{m}$, which equals $(n-2k)\phi^{\mathbb{V}}_k$ using the formulae from section 3.1.1. The other contains the actions on the values of $\phi^{\mathbb{V}}_k$, which cancels with the corresponding expression in $d_K \iota_E \phi^{\mathbb{V}}_k$ except for $\rho_E(\phi^{\mathbb{V}}_k)$.

(ii) The distinguished Poisson kernels is K-coclosed by construction, so the second summand of the left hand side the claimed formula is trivial. For the other summand, we first insert the definition of the K-codifferential, then apply Corollary 3.2.1(i) to move the 1-form E^* inside of the K-Hodge star and afterwards use that $*_K^2$ is simply given by multiplication with $(-1)^{(n-k)(k+1)}$

on all M-invariant forms of K-degree $n-k$. All in all, we get

$$\begin{aligned}\delta_K(E^* \wedge \phi_k^{\mathbb{V}}) &= (-1)^{(n+1)(k+2)+1} *_K d_K^\theta *_K \left(E^* \wedge \phi_k^{\mathbb{V}}\right) \\ &= (-1)^{nk+n+1} *_K d_K^\theta *_K^2 \left((d_P E^*)^{n-k-1} \wedge \iota_E \varphi_k\right) \\ &= - *_K \left((d_P E^*)^{n-k-1} \wedge d_K^\theta \iota_E \varphi_k\right),\end{aligned}$$

where we used in the last line that $d_P E^*$ is K-closed. Finally, since the invariant $(2,1)$-form φ_k is d_K^θ-closed by construction, formula (1.2.4) for the covariant exterior derivative implies that $d_K^\theta \iota_E \varphi_k = \rho_E^\theta(\varphi_k)$, proving the claim.

(iii) Recall that the actions ρ_E and ρ_E^θ are defined by

$$\begin{aligned}&\left(\rho_E \phi_k^{\mathbb{V}}\right)(\xi_1, \dots, \xi_n)(v) = E \cdot \left(\phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_n)(v)\right) - \phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_n)(E \cdot v), \\ &\left(\rho_E^\theta \phi_k^{\mathbb{V}}\right)(\xi_1, \dots, \xi_\ell)(v) = \theta(E) \cdot \left(\phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_\ell)(v)\right) - \phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_n)(E \cdot v)\end{aligned}$$

for all $v \in \mathbb{V}$ and $\xi_1, \dots, \xi_n \in \mathfrak{g}/\mathfrak{m}$. Now the grading element is mapped to $-E$ by the Cartan involution, so the first summand of $\rho_E \phi_k^{\mathbb{V}}$ is the negative of the first summand of $\rho_E^\theta \phi_k^{\mathbb{V}}$. Furthermore, from the formulae for the $\mathfrak{g}$-action on the standard representation in section 3.2.2 we know that the grading element acts by multiplication with j on the M-submodule $\mathbb{V}_j$ of the standard representation. Since we constructed the distinguished Poisson kernel to be trivial on $\mathbb{V}_1$ we obtain that

$$\begin{aligned}\left(\rho_E \phi_k^{\mathbb{V}} + \rho_E^\theta \phi_k^{\mathbb{V}}\right)(\xi_1, \dots, \xi_n)(v) &= -2\phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_n)(E \cdot v) \\ &= 2\phi_k^{\mathbb{V}}(\xi_1, \dots, \xi_n)|_{\mathbb{V}_{-1}}(v)\end{aligned}$$

for all $\xi_1, \dots, \xi_n \in \mathfrak{g}/\mathfrak{m}$ and $v \in \mathbb{V}$. □

Since the distinguished Poisson transform has similar properties as the Poisson transform $\Phi_k \colon \Omega^k(G/P) \to \Omega^k(G/K)$ from Theorem 3.3.1 we would also expect the images of $V_P[\lambda]$-valued forms on G/P under $\Phi_k^{\mathbb{V}}$ to be eigenforms of the Laplace Beltrami operator on G/K. By Proposition 2.2.3, this is equivalent to the differential forms $\phi_k^{\mathbb{V}} \wedge \pi_P^* \alpha \in \Omega^{k,n}(G/M, G \times_M \mathbb{V})$ being eigenforms of the K-Laplace for all $V_P[\lambda]$-valued k-forms α on G/P. However, the following Proposition shows that this is only satisfied in the case $\lambda = 0$.

PROPOSITION 3.4.4. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its the maximal compact subgroup, $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ its minimal parabolic subgroup and $M \cong \mathrm{SO}(n)$ the intersection of K and P. Let $\mathbb{V}$ be the standard representation of G and denote by V and V_P its associated vector bundles over G/M and G/P, respectively. Moreover, for all $\lambda \in \mathbb{R}$ let $\mathcal{E}[\lambda]$ be the bundle of λ-densities over G/P and put $V_P[\lambda] := V_P \otimes \mathcal{E}[\lambda]$. Let $\phi_k^{\mathbb{V}}$ be the distinguished Poisson kernel and denote the corresponding G-invariant differential form by the same symbol. Then for all $\alpha \in \Omega^k(G/P, V_P[\lambda])$ we obtain that*

$$\Delta_K(\phi_k^{\mathbb{V}} \wedge \pi_P^* \alpha) = -\lambda(n - 2k + \lambda)\phi_k^{\mathbb{V}} \wedge \pi_P^* \alpha - 2\lambda \left(\phi_k^{\mathbb{V}}\big|_{\mathbb{V}_{-1}}\right) \wedge \pi_P^* \alpha,$$

where $\phi_k^{\mathbb{V}}\big|_{\mathbb{V}_{-1}}$ denotes the restriction of the distinguished Poisson kernel to $\mathbb{V}_{-1}$.

PROOF. For all $\alpha \in \Omega^k(G/P, V_P[\lambda])$ we define the (k,n)-form $\phi_{k,\lambda} := \phi_k^{\mathbb{V}} \wedge \pi_P^* \alpha$ on G/M. Then the same computations as in the proof of part (ii) of Proposition 3.3.1 show that

$$\begin{aligned}d_K \phi_{k,\lambda} &= \left(d_K \phi_k^{\mathbb{V}} + \lambda E^* \wedge \phi_k^{\mathbb{V}}\right) \wedge \pi_P^* \alpha, \\ \delta_K \phi_{k,\lambda} &= \left(\delta_K \phi_k^{\mathbb{V}} - \lambda \iota_E \phi_k^{\mathbb{V}}\right) \wedge \pi_P^* \alpha,\end{aligned}$$

where E is the G-invariant vector field on G/M induced by the grading element of the Lie algebra of G. Combining these two formulae we obtain

$$\begin{aligned}\Delta_K \phi_{k,\lambda} = &\left(\Delta_K \phi_k^{\mathbb{V}}\right) \wedge \pi_P^* \alpha - \lambda^2 \phi_{k,\lambda} - \lambda \left(\left(\iota_E d_K + d_K \iota_E \right) \phi_k^{\mathbb{V}} \right) \wedge \pi_P^* \alpha \\ &+ \lambda \left(\delta_K \left(E^* \wedge \phi_k^{\mathbb{V}} \right) + E^* \wedge \delta_K \phi_k^{\mathbb{V}} \right) \wedge \pi_P^* \alpha,\end{aligned}$$

which by K-harmonicity of the distinguished Poisson kernel in Theorem 3.4.1 as well as Lemma 3.4.4 reduces to the claimed formula. □

In particular, combining Proposition 3.4.4 with Proposition 2.2.3 we see that for $\alpha \in \Omega^k(G/P, V_P[\lambda])$ the image $\Phi_k^{\mathbb{V}}(\alpha)$ is in general not an eigenform of the twisted Laplace operator Δ^{V_K} on G/K. However, the induced G-equivariant operator on sections of the corresponding homology bundles will have this property, and the proof of this fact is our aim for the rest of this section.

Recall from section 1.4.3 that the the homology bundles $\mathcal{H}_k(G/P, V_P[\lambda])$ on the conformal sphere G/P are defined as the homogeneous bundles associated to the P-representations $H_k(G/P, V_P[\lambda]) = \ker(\partial^*)/\operatorname{im}(\partial^*)$, where ∂^* denotes the Kostant codifferential. Since we constructed the distinguished Poisson transforms to be BGG-compatible, their restriction to the kernel of the Kostant codifferential factor to linear, G-equivariant operators

$$\tilde{\Phi}_k^{\mathbb{V}} \colon \Gamma(\mathcal{H}_k(G/P, V_P[\lambda])) \to \Omega^k(G/K, T(G/K)), \qquad \tilde{\Phi}_k^{\mathbb{V}}(\alpha + \operatorname{im}(\partial^*)) := \Phi_k^{\mathbb{V}}(\alpha),$$

for all $\alpha \in \Gamma(\ker(\partial^*))$. Moreover, by Theorem 1.4.3 there are natural splitting operators

$$L \colon \Gamma(\mathcal{H}_k(G/P, V_P)) \to \Gamma(\ker(\partial^*)),$$

which are sections of the canonical projection $\pi \colon \Gamma(\ker(\partial^*)) \to \Gamma(\mathcal{H}_k(G/P, V_P))$. Subsequently, these led to the definition of the BGG-operators

$$\begin{aligned} D^V \colon \Gamma(\mathcal{H}_k(G/P, V_P)) &\to \Gamma(\mathcal{H}_{k+1}(G/P, V_P)), \\ \alpha &\mapsto (\pi \circ d^V \circ L)(\alpha)\end{aligned}$$

for all $k = 0, \ldots, \dim(G/P) - 1$.

THEOREM 3.4.4. *Let $G = \mathrm{SO}(n+1,1)_0$, $K \cong \mathrm{SO}(n+1)$ its maximal compact subgroup and $P \cong \mathrm{CSO}(n) \rtimes \mathbb{R}^n$ its minimal parabolic subgroup. Let $\mathbb{V} = \mathbb{R}^{n+1,1}$ be the standard representation of G and $V_K := G \times_K \mathbb{V}$ the standard tractor bundle over G/K. For $\lambda \in \mathbb{R}$ let $V_P[\lambda]$ be the tensor product of the standard tractor bundle on G/P with the bundle of λ-densities and $\mathcal{H}_k(G/P, V_P[\lambda])$ the corresponding homology bundle in degree k. Let*

$$\tilde{\Phi}_k^{\mathbb{V}} \colon \Gamma(\mathcal{H}_k(G/P, V_P[\lambda])) \to \Omega^k(G/K, T(G/K))$$

be the G-equivariant linear operators induced by the distinguished Poisson transform. Then we have:

(i) The image of $\tilde{\Phi}_k^{\mathbb{V}}$ consists of coclosed differential forms and eigenvectors of the twisted Laplace Beltrami operator with eigenvalue $-\lambda(n - 2k + \lambda)$.

(ii) For $\lambda = 0$ let D^{V_P} be the BGG-operator on $\Gamma(\mathcal{H}_k(G/P, V_P))$ and d^{V_K} the covariant exterior derivative on $\Omega^k(G/K, V_K)$ induced by the tractor connection on V_K. Then the transform $\tilde{\Phi}_k^{\mathbb{V}}$ satisfies the relation

$$(k+2) d^{V_K} \circ \tilde{\Phi}_k^{\mathbb{V}} = (-1)^{n-k} k(n-2k) \tilde{\Phi}_{k+1}^{\mathbb{V}} \circ D^{V_P}.$$

PROOF. (i) Let σ be a section of $\mathcal{H}_k(G/P, V_P[\lambda])$ and choose any representative $\alpha \in \Omega^k(G/P, V_P[\lambda])$ in the kernel of the Kostant codifferential. Moreover, denote by $\phi_{k,\lambda}$ the contraction of the distinguished Poisson kernel with $\pi_P^* \alpha \in \Omega^{0,k}(G/M, G \times_M \mathbb{V})$.

By Proposition 2.2.3, the image of the V_K-valued differential form $\tilde{\Phi}_k^{\mathbb{V}}(\sigma)$ on G/K under the covariant codifferential δ^{V_K} and the covariant Laplace operator Δ^{V_K} is given by the fibre integral of the differential forms $\delta_K\phi_{k,\lambda}$ and $\Delta_K\phi_{k,\lambda}$, respectively. Explicitly, we have computed in Proposition 3.4.4 that

$$\begin{aligned}\delta_K\phi_{k,\lambda} &= -\lambda\left(\iota_E\phi_k^{\mathbb{V}}\right)\wedge\pi_P^*\alpha,\\ \Delta_K\phi_{k,\lambda} &= -\lambda(n-2k+\lambda)\phi_{k,\lambda}-2\lambda\left(\phi_k^{\mathbb{V}}\big|_{\mathbb{V}_{-1}}\right)\wedge\pi_P^*\alpha,\end{aligned}$$

where the Poisson kernel $\phi_k^{\mathbb{V}}\big|_{\mathbb{V}_{-1}}$ denotes the restriction of the distinguished Poisson kernel to the M-invariant subspace $\mathbb{V}_{-1}$ of $\mathbb{V}$. Therefore, we have to show that the Poisson transforms associated to the kernels $\iota_E\phi_k^{\mathbb{V}}$ and $\phi_k^{\mathbb{V}}\big|_{\mathbb{V}_{-1}}$ are trivial on the kernel of the Kostant codifferential. Furthermore, by Proposition 2.2.3(ii) it suffices to prove that these two Poisson kernels are in the image of the P-codifferential, which has already been done in Corollary 3.4.1.

(ii) By Theorem 3.4.3 we obtain that

$$\begin{aligned}(-1)^{n-k}k(n-2k)(\tilde{\Phi}_{k+1}^{\mathbb{V}}\circ D^{V_P}) &= (-1)^{n-k}k(n-2k)\left(\Phi_{k+1}^{\mathbb{V}}\circ d^{V_P}\circ L\right)\\ &= (k+2)\left(d^{V_K}\circ\Phi_k^{\mathbb{V}}\circ L\right),\end{aligned}$$

and since the distinguished Poisson transform is BGG-compatible we have $\Phi_k^{\mathbb{V}}\circ L=\tilde{\Phi}_k^{\mathbb{V}}$. □

CHAPTER 4

Poisson transform for complex hyperbolic space

The last part of this thesis is dedicated to the construction of Poisson transforms between the CR-sphere and the complex hyperbolic space, which are both homogeneous spaces of the complex Lie group $G = \mathrm{SU}(n+1,1)$. In this case, the homogeneous spaces G/K and G/P have a richer geometric structure (c.f. sections 4.1.2 and 4.1.3), which induces a finer decomposition of the space of invariant forms on the complexified M-representation $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$, c.f. section 4.1.4. Using this, we define a family of basic invariant forms in 4.2, which serve afterwards as an ansatz in the construction of BGG-compatible Poisson kernels in 4.3.2. Finally, we determine the properties of the corresponding BGG-compatible Poisson transforms in section 4.4.3, which will lead in 4.4.4 to the construction of a family of Poisson transforms mapping between real valued differential forms whose properties are similar to Gaillard's operators on the real hyperbolic space.

4.1. Preliminaries about complex hyperbolic space

4.1.1. The Lie groups and Lie algebras. Let $\mathbb{V}$ be the complex vector space $\mathbb{C}^{n+2}$ endowed with the Hermitian product of signature $(n+1,1)$ corresponding to the matrix

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{id}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then the complex hyperbolic space $\mathcal{H}$ of dimension n is the complex manifold given by the set of negative complex lines in $\mathbb{V}$, whereas its boundary $\partial\mathcal{H}$ is defined to be the set of all complex null lines in $\mathbb{V}$. In order to identify these manifolds as homogeneous spaces, consider the special unitary group $G = \mathrm{SU}(n+1,1)$, which we realize as the group of all matrices $g \in \mathrm{GL}(\mathbb{V})$ so that $g^*Sg = S$ and $\det(g) = 1$. Then it follows by construction that G acts transitively on the complex hyperbolic space as well as its boundary.

Let ℓ_- be the negative line in $\mathbb{V}$ generated by the difference of the first and last standard basis vectors. Since the Hermitian product on the vector space $\mathbb{V}$ has signature $(n+1,1)$, Silvester's law of inertia implies that it induces a positive definite Hermitian product $\tilde{b}$ on $(\ell_-)^\perp \cong T_{\ell_-}\mathcal{H}$ by restriction. Furthermore, the isotropy group K of the negative line ℓ_- also stabilizes this Hermitian product and therefore coincides with the maximal compact subgroup $\mathrm{U}(\tilde{b}) \cong \mathrm{U}(n+1)$ in G. In particular, we have identified the $(n+1)$-dimensional complex hyperbolic space as the Riemannian symmetric space G/K, and the K-invariant inner product induces a Riemannian metric on $T_{\ell_-}\mathcal{H} \cong \mathfrak{q}$.

Similarly, let ℓ_0 be the complex null line in $\mathbb{V}$ generated by the first standard basis vector and let P be its stabilizer in G. If we write elements of G as block matrices with the same sizes as the matrix S we obtain that

$$P = \left\{ \begin{pmatrix} a & -aY^*B & \frac{a}{2}\left(b - \|Y\|^2\right) \\ 0 & B & Y \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{C}^*, b \in \mathbb{C}, Y \in \mathbb{C}^n, B \in \mathrm{U}(n) \\ \mathrm{re}(b) = 1, \det(B) = \overline{a}a^{-1} \end{array} \right\},$$

showing that $P \cong \mathrm{CSU}(n) \rtimes (\mathbb{C}^n \oplus \mathbb{C})$ is a minimal parabolic subgroup of G. All in all, we see that the boundary ∂H of the complex hyperbolic space is isomorphic to the homogeneous parabolic geometry G/P.

In the above matrix representation, the subgroup $M = K \cap P$ consists of all block matrices with $Y = 0$, $b = 0$ and $a \in \mathrm{U}(1)$. In particular, we see that M is isomorphic to the $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. Moreover, the abelian subgroup A is given by all matrices whose only nontrivial entries are given by $a \in \mathbb{R}_+$ and $B = \mathrm{id}_n$, whereas the nilpotent subgroup N is the subgroup of matrices satisfying $a = 1$ and $B = \mathrm{id}_n$.

Turning to the infinitesimal picture, a direct computation shows that the real Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1,1)$ of G has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} b & -Y^* & y \\ X & B & Y \\ x & -X^* & -\overline{b} \end{pmatrix} \middle| \begin{array}{c} b \in \mathbb{C},\ x, y \in i\mathbb{R},\ X, Y \in \mathbb{C}^n, \\ B \in \mathfrak{u}(n),\ \mathrm{tr}(B) = 2\,\mathrm{im}(b) \end{array} \right\},$$

and we refer to such an element by $(x, X, (B, b), Y, y)$.

Proposition 4.1.1. *Let $G = \mathrm{SU}(n+1,1)$, $K \cong \mathrm{U}(n+1)$ be its maximal compact subgroup and $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively.*

(i) The Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1,1)$ naturally carries a $|2|$-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{su}(n)$, $\mathfrak{g}_{\pm 1} \cong \mathbb{C}^n$ and $\mathfrak{g}_{\pm 2} \cong i\mathbb{R}$.

(ii) The Lie algebra $\mathfrak{m}$ of M is isomorphic to $i\mathbb{R} \oplus \mathfrak{su}(n)$, whereas the Lie algebra $\mathfrak{a}$ of A is generated by the grading element $E := (0, 0, (0, 1), 0, 0)$.

(iii) The Lie algebra $\mathfrak{k}$ is given by all matrices of the form $(x, X, (B, b), X, x)$ with $x, b \in i\mathbb{R}$, $X \in \mathbb{C}^n$ and $B \in \mathfrak{u}(n)$.

(iv) The Killing form on $\mathfrak{g}$ is determined by the following nondegenerate pairings:

$$\begin{aligned} &\mathfrak{a} \times \mathfrak{a} \to \mathbb{R}, & B(E, E) &= 4(n+2), \\ &\mathfrak{m} \times \mathfrak{m} \to \mathbb{R}, & B((b_1, B_1), (b_2, B_2)) &= 2(n+2)(b_1 b_2 + \mathrm{tr}(B_1 B_2)), \\ &\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathbb{R}, & B(X, Y) &= -4(n+2)\langle X, Y \rangle, \\ &\mathfrak{g}_{-2} \times \mathfrak{g}_2 \to \mathbb{R}, & B(x, y) &= -2(n+2)xy, \end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the standard Kähler metric on $\mathbb{C}^n$.

Proof. (i) The block form of $\mathfrak{g}$ immediately induces a $|2|$-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\mathfrak{g}_{\pm 1} \cong \mathbb{C}^n$ and $\mathfrak{g}_{\pm 2} \cong i\mathbb{R}$. Furthermore, every matrix $A \in \mathfrak{u}(n)$ can be written as

$$A = \left(A - \frac{\mathrm{tr}(A)}{n} \mathrm{id}_n \right) + \frac{\mathrm{tr}(A)}{n} \mathrm{id}_n,$$

and computing the Lie bracket in this decomposition immediately implies that $\mathfrak{g}_0$ is isomorphic to $\mathbb{C} \oplus \mathfrak{su}(n)$ as a Lie algebra.

(ii) This follows immediately from the matrix representations of M and A.

(iii) In the given matrix representation of G we can express the Lie group K as the stabilizer of the global Cartan involution $g \mapsto (g^*)^{-1}$ on G. Thus, applying its derivative $g \mapsto -g^*$ to the matrix representation of $\mathfrak{g}$ immediately shows that $\mathfrak{k}$ is of the claimed form.

(iv) The Killing form is given by a multiple of the real part of the trace form on $\mathfrak{g}$ and therefore determined by $B(E,E)$, which can be computed explicitly. □

Finally, for later computations we want to determine explicit formulae for the Lie bracket on $\mathfrak{g}$ in terms of the $|2|$-grading. A direct computation shows that the only nontrivial relations are given by

$$\begin{aligned}
\mathfrak{g}_0 \times \mathfrak{g}_{-2} &\to \mathfrak{g}_{-2}, & [(A,a),x] &= -2\operatorname{re}(a)x,\\
\mathfrak{g}_0 \times \mathfrak{g}_{-1} &\to \mathfrak{g}_{-1}, & [(A,a),X] &= AX - aX,\\
\mathfrak{g}_0 \times \mathfrak{g}_{0} &\to \mathfrak{g}_{0}, & [(A,a),(B,b)] &= ([A,B],0),\\
\mathfrak{g}_0 \times \mathfrak{g}_{1} &\to \mathfrak{g}_{1}, & [(A,a),Y] &= AY + \overline{a}Y,\\
\mathfrak{g}_0 \times \mathfrak{g}_{2} &\to \mathfrak{g}_{2}, & [(A,a),y] &= 2\operatorname{re}(a)y,\\
\mathfrak{g}_{-1} \times \mathfrak{g}_{1} &\to \mathfrak{g}_{0}, & [X,Y] &= (YX^* - XY^*, \langle X,Y\rangle),\\
\mathfrak{g}_{-2} \times \mathfrak{g}_{2} &\to \mathfrak{g}_{0}, & [x,y] &= (0,-xy),\\
\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} &\to \mathfrak{g}_{-2}, & [X_1,X_2] &= 2i\operatorname{im}(\langle X_1,X_2\rangle),\\
\mathfrak{g}_{1} \times \mathfrak{g}_{1} &\to \mathfrak{g}_{2}, & [Y_1,Y_2] &= 2i\operatorname{im}(\langle Y_1,Y_2\rangle),\\
\mathfrak{g}_{-1} \times \mathfrak{g}_{2} &\to \mathfrak{g}_{1}, & [X,y] &= yX,\\
\mathfrak{g}_{-2} \times \mathfrak{g}_{1} &\to \mathfrak{g}_{-1}, & [x,Y] &= -xY,
\end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian product on $\mathbb{C}^n$.

4.1.2. The geometry of G/K. We have seen in section 4.1.1 that the symmetric space G/K naturally carries the structure of a complex manifold, which induces a Hermitian metric on its complexified tangent bundle and as well as a decomposition of the exterior power of the complexified cotangent bundle into (p,q)-types. Since these objects will be used extensively in the study of Poisson kernels on G/M in sections 4.2 and 4.3, we will now recall their construction.

First, we have seen in section 1.3.1 that the tangent bundle of G/K is associated to $\mathfrak{q}$, so we have to explicitly determine the action of K on this vector space. We could do this in terms of the representation of $\mathfrak{g}$ as in the last section; however, it will be more suitable to consider a different matrix representation in this case. Consider the complex vector space $\mathbb{C}^{n+2}$ endowed with the standard inner product of signature $(n+1,1)$, which corresponds to the symmetric matrix

$$\tilde{S} = \begin{pmatrix} \mathrm{id}_{n+1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Writing G again as block matrices with the same sizes as the matrix $\tilde{S}$, the maximal compact subgroup K is given by

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \middle| A \in \mathrm{U}(n+1) \right\} \cong \mathrm{U}(n+1),$$

which can also be expressed as the stabilizer of the Cartan involution $g \mapsto (g^*)^{-1}$.

Differentiating the defining equations for G we deduce that the Lie algebra $\mathfrak{g}$ is represented as

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & X \\ X^* & a \end{pmatrix} \middle| A \in \mathfrak{u}(n+1),\ a = -\operatorname{tr}(A) \in i\mathbb{R},\ X \in \mathbb{C}^{n+1} \right\}.$$

The Lie algebra $\mathfrak{k}$ of K correspond to those matrices with trivial $\mathbb{C}^n$-part, which is exactly the stabilizer of the Cartan involution $\theta\colon g \mapsto -g^*$ on $\mathfrak{g}$. On the other hand, the (-1)-eigenspace $\mathfrak{q}$ of θ consists of those matrices with $A = 0$ and $a = 0$ and thus is isomorphic to $\mathbb{C}^{n+1}$, viewed as a real vector space.

In this picture we immediately see that the adjoint action of K on $\mathfrak{q}$ corresponds to the action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ given by $B \cdot v := \det(B)^{-1}Bv$. Otherwise said, this is the standard action of $\mathrm{SU}(n+1)$ combined with the surjective group homomorphism $\mathrm{U}(n+1) \to \mathrm{SU}(n+1)$ mapping B to $\det(B)^{-1}B$. In particular, the inner product on $\mathfrak{q}$ corresponding to the standard Kähler metric on $\mathbb{C}^{n+1}$ is K-invariant.

Definition 4.1.2. We define the hyperbolic metric g_{hyp} on G/K as the G-invariant Riemannian metric corresponding to the standard Kähler metric on $\mathbb{C}^{n+1}$ under the identifications $K \cong \mathrm{U}(n+1)$ and $\mathfrak{q} \cong \mathbb{C}^{n+1}$.

In contrast to the real hyperbolic space we obtain additional G-invariant geometric structure on the tangent bundle of G/K. Indeed, the K-invariant complex structure J on $\mathfrak{q}$ obtained by multiplication with the imaginary unit induces a G-invariant almost complex structure $T(G/K) \to T(G/K)$, which we denote by the same symbol. Moreover, this bundle map is orthogonal with respect to the hyperbolic metric, showing that G/K is a *Hermitian manifold*. In particular, we obtain an induced decomposition of its complexified tangent bundle into a Whitney sum of G-invariant subbundles, which we will prove in the following Lemma by constructing the corresponding K-invariant decomposition of the representation $\mathfrak{q}$. For a general discussion of Hermitian manifolds we refer to [**Wel73**].

Lemma 4.1.2. *Let $G = \mathrm{SU}(n+1,1)$ and $K \cong \mathrm{U}(n+1)$ its maximal compact subgroup. Then the complex linear extension of the almost complex structure J on $T(G/K)$ to the complexified tangent bundle induces a Whitney sum decomposition of $T(G/K) \otimes \mathbb{C}$ into subbundles $T^{1,0}(G/K) \oplus T^{0,1}(G/K)$. In particular, we obtain for all $0 \leq k \leq 2n+2$ a decomposition*

$$\Lambda^k T^*(G/K) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} T^*(G/K),$$

where

$$\Lambda^{p,q} T^*(G/K) := \Lambda^p T^{1,0*}(G/K) \otimes \Lambda^q T^{0,1*}(G/K)$$

is the bundle of forms of (p,q)-type.

Proof. By Proposition 1.2.1 it suffices to show the corresponding decompositions of the K-module $\mathfrak{q}$. First, the complex linear extension of J to $\mathfrak{q} \otimes_{\mathbb{R}} \mathbb{C}$ is diagonalizable with eigenvalues i and $-i$, and we denote by $\mathfrak{q}^{1,0}$ and $\mathfrak{q}^{0,1}$ the corresponding eigenspaces. By construction, these are both K-invariant vector spaces and thus induce the corresponding decomposition of $T(G/K) \otimes \mathbb{C}$.

Furthermore, if we define the K-representations $\Lambda^{p,q}\mathfrak{q}^* := \Lambda^p \mathfrak{q}^{1,0*} \otimes \Lambda^q \mathfrak{q}^{0,1*}$, then by naturality of the complexification we obtain a direct sum decomposition

$$\Lambda^k \mathfrak{q}^* \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \mathfrak{q}^*$$

as K-representations, and applying Proposition 1.2.1 yields the corresponding decomposition of $\Lambda^k T^*(G/K) \otimes \mathbb{C}$ into (p,q)-types. ☐

Following the previous Lemma we will introduce some notation. First, we denote the decomposition of a vector $\xi \in T(G/K) \otimes \mathbb{C}$ as in Lemma 4.1.2 by $\xi = \xi^{1,0} + \xi^{0,1}$ and call $\xi^{1,0}$ and $\xi^{0,1}$ the *holomorphic* and *antiholomorphic* part of ξ, respectively. In particular, if we identify $T(G/K)$ with its image $T(G/K) \otimes 1$ in the complexified tangent bundle, then the holomorphic part of a tangent vector $\xi \in T(G/K)$ is given by $\xi^{1,0} = \frac{1}{2}(\xi - iJ\xi)$, whereas its antiholomorphic part reads as $\xi^{0,1} = \frac{1}{2}(\xi + iJ\xi)$. Next, the splitting of $\Lambda^k T^*(G/K) \otimes \mathbb{C}$ from Lemma 4.1.2 induces a decomposition

$$\Omega^k(G/K, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(G/K)$$

on the level of sections. We call any element $\omega \in \Omega^{p,q}(G/K)$ a *differential form of type* (p,q). By definition, a complex valued differential form on G/K is of type (p,q) if and only if it vanishes after insertion of $p+1$ holomorphic or $q+1$ antiholomorphic vectors. Moreover, the complex conjugate of a form of type (p,q) is of type (q,p), showing that real forms of type (p,q) can only exist if $q = p$.

Next, we want to bring the hyperbolic metric into the game. We have seen above that the almost complex structure on the tangent bundle of G/K is orthogonal with respect to the hyperbolic metric, so we can define the *fundamental form* $\omega \in \Omega^2(G/K)$ as $\omega(\xi, \eta) := g(J\xi, \eta)$ for all ξ, $\eta \in \mathfrak{X}(G/K)$. Furthermore, we extend the hyperbolic metric complex bilinearly to $T(G/K) \otimes \mathbb{C}$ and denote this extension by the same symbol.

Proposition 4.1.2. *Let $G = \mathrm{SU}(n+1,1)$ and $K \cong \mathrm{U}(n+1)$ its maximal compact subgroup. Let J be the complex structure on $T(G/K)$ and g be the hyperbolic metric on G/K. Let $\omega \in \Omega^2(G/K)$ be the associated fundamental form and denote the complex bilinear extension of the hyperbolic metric to the complexified tangent bundle of G/K by the same symbol.*

(i) The restriction of g to the pairing $T^{1,0}(G/K) \times T^{0,1}(G/K) \to \mathbb{C}$ is nondegenerate, whereas both $T^{1,0}(G/K)$ and $T^{0,1}(G/K)$ are Lagrangian subbundles of the complexified tangent bundle.

(ii) The fundamental form $\omega \in \Omega^2(G/K)$ is a G-invariant, nondegenerate, closed differential form of type $(1,1)$. In particular, the almost complex structure on $T(G/K)$ is integrable and G/K is a Kähler manifold.

Proof. (i) Let g be the hyperbolic metric on G/K and denote the induced K-invariant inner product on $\mathfrak{q}$ as well as its complex linear extension to $\mathfrak{q} \otimes \mathbb{C}$ by the same symbols. Let $\mathfrak{q} \otimes \mathbb{C} = \mathfrak{q}^{1,0} \oplus \mathfrak{q}^{0,1}$ be the decomposition corresponding to Lemma 4.1.2. Then by Proposition 1.2.1 and Theorem 1.2.2 it suffices to show that the restrictions of g_K to $\mathfrak{q}^{1,0}$ and $\mathfrak{q}^{0,1}$ are trivial, whereas on $\mathfrak{q}^{1,0} \times \mathfrak{q}^{0,1}$ it is nondegenerate. But if J denotes the complex structure on $\mathfrak{q} \otimes \mathbb{C}$, we can write every vector in $\mathfrak{q}^{1,0}$ as $\frac{1}{2}(X - iJX)$ for $X \in \mathfrak{q}$ and similarly for $\mathfrak{q}^{0,1}$, hence the claim follows from orthogonality of J.

(ii) Since the hyperbolic metric is nondegenerate and G-invariant and J is a G-equivariant isomorphism of vector bundles, the fundamental form is a G-invariant, nondegenerate differential form, which is of type $(1,1)$ by definition. In order to compute its derivative, we apply formula (1.2.4) to the K-invariant element in $\Lambda^2(\mathfrak{g}/\mathfrak{k})^*$ corresponding to ω due to Theorem 1.2.2, which we will denote by the same symbol. However, the Lie bracket on $\mathfrak{g}$ satisfies $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$, implying that $d\omega = 0$. Therefore, by Theorem 1.2.2 the fundamental form is closed, so J is integrable and G/K is a Kähler manifold. $\square$

The (complexified) Riemannian metric on the complexified tangent bundle of G/K induces an isomorphism $^\flat \colon T(G/K) \otimes \mathbb{C} \to T^*(G/K) \otimes \mathbb{C}$ defined by $\xi^\flat(\eta) := g(\eta, \xi)$ for all ξ, $\eta \in T(G/K) \otimes \mathbb{C}$, which by naturality coincides with the complex linear extension of the flat operator of the underlying Riemannian manifold. Following Proposition 4.1.2(i), if ξ is an element in $T^{1,0}(G/K)$, then its image under the flat operator is an element in $T^{0,1*}(G/K)$, and similarly for their complex conjugates.

Second, we proved in Proposition 4.1.2(ii) that the Kähler form ω on G/K is nondegenerate, which implies that ω^{n+1} is a nontrivial differential form, and computing its norm shows that $\mathrm{vol} := \frac{1}{(n+1)!}\omega^{n+1}$ defines a volume form on G/K. Next, let $*$ be the Hodge star operator on G/K and denote its complex linear extension to complex valued differential forms by the same symbol. Then this operator is determined by the equation

$$\alpha \wedge *\beta = g(\alpha, \beta)\,\mathrm{vol}$$

for all α, $\beta \in \Omega^k(G/K, \mathbb{C})$, where we consider vol as a differential form on G/K of type $(n+1, n+1)$. Comparing the types of the forms involved in the characteristic equation for $*$ we see that the Hodge star restricts to a G-equivariant map

$$*\colon \Omega^{p,q}(G/K, \mathbb{C}) \to \Omega^{n+1-q,n+1-p}(G/K, \mathbb{C}).$$

Furthermore, we define the *Lefschetz map* $\mathcal{L}\colon \Omega^{p,q}(G/K) \to \Omega^{p+1,q+1}(G/K)$ as the wedge product $\alpha \mapsto \omega \wedge \alpha$ with the Kähler form on G/K. Note that since ω is real, the Lefschetz map preserves real differential forms. Subsequently, using the Hermitian metric on G/K we can define its adjoint

$$\mathcal{L}^*\colon \Omega^{p,q}(G/K) \to \Omega^{p-1,q-1}(G/K)$$

with respect to the hyperbolic metric via $g(\mathcal{L}(\alpha), \beta) = g(\alpha, \mathcal{L}^*(\beta))$ for all forms α and β of types $(p-1, q-1)$ and (p, q), respectively. By definition of the Hodge star operator this map is explicitly given by $\mathcal{L}^* = *^{-1} \circ \mathcal{L} \circ *$. Moreover, we say that a differential form α on G/K is *primitive* if $\mathcal{L}^*(\alpha) = 0$. It is well known ([**Wel73**, Theorem 3.11(b)]) that primitive forms can only exist for degrees up to the complex dimension of G/K.

Finally, we want to show how the decomposition of the k-th exterior power of the complexified cotangent bundle induces a refinement of the de Rham sequence of differential forms. Consider the complex linear extension of the exterior derivative on G/K, which we will denote by the same symbol. Since G/K is a complex manifold, we know from [**Wel73**, Theorem 3.7] that this operator splits into the sum $d = \partial + \overline{\partial}$, where the first and second operator map a differential form of type (p, q) to a form of type $(p+1, q)$ and $(p, q+1)$, respectively. By definition, these operators are related via $\overline{\partial\alpha} = \overline{\partial}\overline{\alpha}$ for all $\alpha \in \Omega^k(G/K, \mathbb{C})$. Furthermore, both operators are differentials and satisfy the commutator relation $\partial\overline{\partial} = -\overline{\partial}\partial$. We say that a differential form $\alpha \in \Omega^{p+q}(G/K, \mathbb{C})$ is *holomorphic* (respectively, *antiholomorphic*) if $\overline{\partial}\alpha = 0$ (respectively, $\partial\alpha = 0$). Similarly, we can decompose the Riemannian codifferential $\delta = \partial^* + \overline{\partial^*}$, where the first operator is defined on $\Omega^{p,q}(G/K)$ by $\partial^* := (-1)^{p+q} * \overline{\partial} *$ and similar for $\overline{\partial^*}$. Again, we have $(\partial^*)^2 = \left(\overline{\partial^*}\right)^2 = 0$ and these two operators anticommute. Finally, we can define the associated Laplace operator $\Box := \partial\partial^* + \partial^*\partial$.

THEOREM 4.1.2. *Let $G = \mathrm{SU}(n+1, 1)$ and $K \cong \mathrm{U}(n+1)$ be its maximal compact subgroup. Then the Laplace operators Δ and $\Box$ are related via*

$$\Delta = 2\Box.$$

In particular, the restriction of Δ to $\Omega^{p,q}(G/K, \mathbb{C})$ maps into the space of forms of type (p, q).

PROOF. We have seen in Proposition 4.1.2 that G/K is a Kähler manifold, so the claim follows from [**Wel73**, Theorem 4.7]. $\Box$

4.1.3. The geometry of G/P. We continue the discussion of the geometric framework with the analysis of the tangent bundle of the homogeneous parabolic geometry G/P. Explicitly, our aim is to show that G/P is naturally endowed with the structure of a CR-manifold.

DEFINITION 4.1.3. Let M be a smooth manifold of dimension $2n+1$ and let (L, J) be a complex subbundle of TM of rank $2n$. Let $L \otimes_{\mathbb{R}} \mathbb{C} = L^{1,0} \oplus L^{0,1}$ be the decomposition of the complex vector bundle $L \otimes_{\mathbb{R}} \mathbb{C}$ into eigenspaces for $J \otimes \mathbb{C}$ with eigenvalues i and $-i$, respectively. Then (M, L) is called a *(hypersurface type) CR-manifold* if the subbundle $L^{1,0}$ is involutive.

We will show in Proposition 4.1.3 that G/P is a CR-manifold and afterwards in Theorem 4.1.3 explicitly describe its BGG-complex with values in the trivial representation, which is also known in this case as the *Rumin complex*. For a general discussion of CR-manifolds we refer to [**DT06**].

First, we will show how the $|2|$-grading of the Lie algebra $\mathfrak{g}$ from Proposition 4.1.1(i) defines a complex distribution of the tangent bundle of G/P of complex rank n. Recall from section 1.4.1 that the grading on $\mathfrak{g}$ induces a P-invariant filtration

$$\mathfrak{g} = \mathfrak{g}^{-2} \supset \cdots \supset \mathfrak{g}^{2},$$

where $\mathfrak{g}^{\ell}$ is defined as the direct sum of all grading components $\mathfrak{g}_j$ for $j \geq \ell$. In particular, the vector space $\mathbb{L} := \mathfrak{g}^{-1}/\mathfrak{p}$ is P-invariant and thus defines a subrepresentation of $\mathfrak{g}/\mathfrak{p}$ isomorphic to $\mathbb{C}^n$. Moreover, since the nilpotent subgroup P_+ of P maps $\mathfrak{g}^{\ell}$ to $\mathfrak{g}^{\ell'}$ for $\ell' > \ell$ we deduce that the action of P on $\mathbb{L}$ is determined by the Levi subgroup $G_0 \cong \mathrm{CSU}(n)$. In particular, by Proposition 1.4.1(i) the P-representation $\mathbb{L}$ is irreducible.

Using the matrix representations of P and $\mathfrak{g}$ from section 4.1.1 a direct computation shows that if $p \in G_0$ corresponds to $(a, B) \in \mathrm{CSU}(n)$ and if $\xi \in \mathbb{L}$ corresponds to $X \in \mathbb{C}^n$, then the action of p on ξ corresponds to the vector $a^{-1}BX \in \mathbb{C}^n$. In particular, the induced complex structure on $\mathbb{L}$ is P-invariant.

PROPOSITION 4.1.3. *Let $G = \mathrm{SU}(n+1,1)$, P its minimal parabolic subgroup and let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P, respectively. Let $(\mathbb{L}, J)$ be the P-invariant complex subrepresentation of $\mathfrak{g}/\mathfrak{p}$ defined above and let $L \to G/P$ be its associated bundle. Then $(G/P, L)$ is a CR-manifold.*

PROOF. Since $\mathbb{L} \subset \mathfrak{g}/\mathfrak{p}$ is a P-invariant subspace of dimension $2n$ endowed with a P-invariant complex structure, we obtain by Proposition 1.2.1 and Theorem 1.2.2 that L is a complex subbundle of $T(G/P)$ of dimension $2n$. Thus, if $L^{1,0} \oplus L^{0,1}$ is the decomposition of its complexification $L \otimes \mathbb{C}$ into eigenspaces of the almost complex structure induced by J, we have to show that $L^{1,0}$ is involutive.

The complex structure on $\mathbb{L}$ is P-invariant, which implies that $L^{1,0}$ and $L^{0,1}$ are the bundles associated to the eigenspaces $\mathbb{L}^{1,0}$ and $\mathbb{L}^{0,1}$ of the complexification of J on $\mathbb{L} \otimes \mathbb{C}$ to the eigenvalues i and $-i$, respectively. In particular, for two section ξ_1, ξ_2 of $L^{1,0}$ we can compute their Lie bracket $[\xi_1, \xi_2]$ via $\tilde{\xi_1} \cdot f_2 - \tilde{\xi_2} \cdot f_1$, where $\tilde{\xi_i} \in \Gamma(TG \otimes \mathbb{C})$ are local lifts of ξ_i and $f_i \colon G \to \mathbb{L}^{1,0}$ is the P-equivariant map corresponding to ξ_i for $i = 1, 2$. However, the derivative of f_i is again a smooth map on G with values in $\mathbb{L}^{1,0}$, so the Lie bracket $[\xi_1, \xi_2]$ is again a section of $L^{1,0}$. $\square$

One feature of CR-manifolds is the existence of a distinguished complex of differential forms, called the *Rumin complex*. First defined in [**Rum90**], it was later shown in [**BEGN**] to be the BGG-sequence on G/P with values in the trivial representation. Since we want to construct Poisson transforms which are compatible with this complex in section 4.3, we will determine the homology groups on G/P explicitly.

Consider the quotient representation $(\mathfrak{g}/\mathfrak{p})/\mathbb{L}$, which is by construction a P-representation isomorphic to $\mathfrak{g}/\mathfrak{g}^{-1}$. Since the Killing form on $\mathfrak{g}$ induces isomorphisms $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ and $(\mathfrak{g}/\mathfrak{g}^{-1})^* \cong \mathfrak{g}_2$ of P-modules (c.f. section 1.4.1), the exact sequence defining the quotient $\mathfrak{g}^{-1}/\mathfrak{p}$ dualises to the short exact sequence

$$0 \longrightarrow \mathfrak{g}_2 \longrightarrow \mathfrak{p}_+ \longrightarrow \mathbb{L}^* \longrightarrow 0$$

of P-modules. Moreover, we have seen in Proposition 4.1.1(i) that the grading component $\mathfrak{g}_2$ is 1-dimensional, so by naturality of the exterior power we obtain the short exact sequence of P-representations

$$0 \longrightarrow \Lambda^{k-1}\mathbb{L}^* \otimes \mathfrak{g}_2 \longrightarrow \Lambda^k \mathfrak{p}_+ \longrightarrow \Lambda^k \mathbb{L}^* \longrightarrow 0$$

for all $k = 1, \dots, 2n+1$.

THEOREM 4.1.3. *Let $G = \mathrm{SU}(n+1,1)$ and P its minimal parabolic subgroup. Let $L \subset T(G/P)$ be the CR-distribution on G/P and define $Q := T(G/P)/L$. Then the Kostant codifferential ∂^* on $\Lambda^k T^*(G/P)$ factors to a G-equivariant bundle map $\underline{\partial}^* \colon \Lambda^k L^* \to \Lambda^{k-2} L^* \otimes Q^*$. Furthermore, if $\mathcal{H}_k(G/P)$ denotes the k-th homology bundle as defined in section 1.4.3, then we obtain that $\mathcal{H}_k(G/P) = \ker(\underline{\partial}^*)$ for all $k \leq n$, whereas $\mathcal{H}_k(G/P) = \left(\Lambda^{k-1} L^* \otimes Q^*\right)/\operatorname{im}(\underline{\partial}^*)$ for all $k \geq n+1$.*

PROOF. Recall from section 1.4.2 that the Kostant codifferential on $\Lambda^k T^*(G/P)$ was induced by the P-equivariant map $\partial^* \colon \Lambda^k \mathfrak{p}_+ \to \Lambda^{k-1}\mathfrak{p}_+$ defined by

$$\partial^*(Z_1 \wedge \ldots \wedge Z_k) = \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \ldots \wedge \widehat{Z_i} \wedge \ldots \wedge \widehat{Z_j} \wedge \ldots \wedge Z_k$$

for all $Z_1, \cdots, Z_k \in \mathfrak{p}_+$, where the hat denotes omission. Now the compatibility of the Lie bracket with the $|2|$-grading on $\mathfrak{g}$ implies that the Kostant codifferential factors to a P-equivariant map $\underline{\partial}^* \colon \Lambda^k \mathbb{L}^* \to \Lambda^{k-2}\mathbb{L}^* \otimes \mathfrak{g}_2$, which integrates to a G-equivariant bundle map $\underline{\partial}^* \colon \Lambda^k L^* \to \Lambda^{k-2} L^* \otimes Q^*$ due to Proposition 1.2.1. Moreover, due to dimensional reasons this map is surjective for $k \leq n+1$ and injective for $k \geq n+1$.

Next, recall from section 1.4.3 that the k-th homology bundle $\mathcal{H}_k(G/P)$ is associated to the P-representation $H_k(G/P) := \ker(\partial^*)/\operatorname{im}(\partial^*)$. For $k \leq n$ we obtain that $H_k(G/P)$ is the projection of $\ker(\partial^*)$ to $\Lambda^k \mathbb{L}^*$, which coincides with $\ker(\underline{\partial}^*)$ by construction. On the other hand, for $k \geq n+1$ the kernel of ∂^* equals $\Lambda^{k-1}\mathbb{L}^* \otimes Q^*$, so since the images of ∂^* and $\underline{\partial}^*$ coincide we obtain that $H_k(G/P) = \left(\Lambda^{k-1}\mathbb{L}^* \otimes Q^*\right)/\operatorname{im}(\underline{\partial}^*)$. Applying Proposition 1.2.1 in both cases yields the claimed expressions for $\mathcal{H}_k(G/P)$. □

4.1.4. The structure of G/M. Finally, we have to analyse the geometric structure of the homogeneous space G/M in order to construct Poisson kernels in the sequel. After decomposing $T(G/M)$ into G-invariant subbundles as in Proposition 2.3.1 and identifying the Euclidean bundle metric on the horizontal distribution $T^{1,0}$ induced by the hyperbolic metric on G/K, we will consider the complexified tangent bundle of G/M and its decomposition induced by the Kähler structure on G/K and the CR-structure on G/P. In this way, we obtain a finer grading of differential forms, which will be beneficial for the construction of BGG-compatible Poisson kernels in section 4.3.

We have seen in Proposition 4.1.1 that the Lie algebra $\mathfrak{g}$ is $|2|$-graded, so denoting by $(\mathfrak{g}/\mathfrak{m})_i$ the image of the i-th grading component under the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{m}$ we obtain a decomposition

$$\mathfrak{g}/\mathfrak{m} = (\mathfrak{g}/\mathfrak{m})_{-2} \oplus (\mathfrak{g}/\mathfrak{m})_{-1} \oplus (\mathfrak{g}/\mathfrak{m})_0 \oplus (\mathfrak{g}/\mathfrak{m})_1 \oplus (\mathfrak{g}/\mathfrak{m})_2$$

as M-modules. To write this in terms of the matrix representation of $\mathfrak{g}$, recall from Proposition 4.1.1(ii) that the subalgebra $\mathfrak{m}$ is isomorphic to $i\mathbb{R} \oplus \mathfrak{so}(n)$ embedded diagonally into $\mathfrak{g}$. Therefore, we can represent the M-module $\mathfrak{g}/\mathfrak{m}$ as

$$\mathfrak{g}/\mathfrak{m} = \left\{ \begin{pmatrix} a & -Y^* & y \\ X & 0 & Y \\ x & -X^* & -a \end{pmatrix} + \mathfrak{m} \,\middle|\, a \in \mathbb{R}, x, y \in i\mathbb{R}, X, Y \in \mathbb{C}^n \right\},$$

and mapping such an element $\xi \in \mathfrak{g}/\mathfrak{m}$ to the tuple (x, X, a, Y, y) corresponds exactly to the decomposition of $\mathfrak{g}/\mathfrak{m}$ as above.

Next, we want to determine the action of M on $\mathfrak{g}/\mathfrak{m}$ in this picture. For that, recall from section 4.1.1 that the Lie group M is given by all block diagonal matrices in G with $B \in \mathrm{U}(n)$ in the middle and an element $b \in \mathrm{U}(1)$ in the upper left and lower right corner satisfying the relation $b^2 \det(B) = 1$. Thus, if $m \in M$ corresponds to the matrix (B, b) and if $\xi \in \mathfrak{g}/\mathfrak{m}$ is given by $\xi = (x, X, a, Y, y)$, then the action of m on ξ equals $(x, b^{-1}BX, a, b^{-1}BY, y)$, i.e. M acts trivial on $(\mathfrak{g}/\mathfrak{m})_{-2}$, $(\mathfrak{g}/\mathfrak{m})_2$ and $(\mathfrak{g}/\mathfrak{m})_0$ and by a twisted action on $(\mathfrak{g}/\mathfrak{m})_1$ and $(\mathfrak{g}/\mathfrak{m})_{-1}$. In particular, the complex structures on the latter spaces are M-equivariant.

Applying the associated bundle construction yields

LEMMA 4.1.4. *The tangent bundle of G/M decomposes into a Whitney sum of irreducible subbundles*

$$T(G/M) = T^{-2}(G/M) \oplus T^{-1}(G/M) \oplus T^0(G/M) \oplus T^1(G/M) \oplus T^2(G/M),$$

where $T^i(G/M)$ are line bundles for $i \in \{-2, 0, 2\}$, whereas $T^{\pm 1}(G/M)$ are vector bundles of rank $2n$ which are endowed with a G-equivariant complex structure.

In order to obtain the bigrading on the space of Poisson kernels induced by the isomorphism $G/M \cong G/K \times G/P$ we will define generators of the tangent bundle of G/M with are compatible with this product. For that, recall from section 2.3.1 that the horizontal and vertical distributions of $G/M \to G/K$ are those associated to the M-modules $\mathfrak{p}/\mathfrak{m}$ and $\mathfrak{k}/\mathfrak{m}$, respectively. Thus, in terms of the above decomposition of $\mathfrak{g}/\mathfrak{m}$ we define the M-invariant vectors

$$E := (0, 0, 1, 0, 0) \in \mathfrak{p}/\mathfrak{m}, \quad H := (0, 0, 0, 0, i) \in \mathfrak{p}/\mathfrak{m}, \quad I := (i, 0, 0, 0, i) \in \mathfrak{k}/\mathfrak{m}$$

as well as for all X, $Y \in \mathbb{C}^n$ the elements

$$F_X := (0, 0, 0, X, 0) \in \mathfrak{p}/\mathfrak{m}, \qquad G_Y := (0, Y, 0, Y, 0) \in \mathfrak{k}/\mathfrak{m}.$$

Furthermore, for the definition of the K-Hodge star we need to determine the G-invariant bundle metric on the horizontal distribution $G \times_M (\mathfrak{p}/\mathfrak{m})$ obtained by the pullback of the hyperbolic metric on G/K.

PROPOSITION 4.1.4. *Let $G = \mathrm{SU}(n+1, 1)$, K its maximal compact subgroup, $P \subset G$ the minimal parabolic subgroup and let $M = K \cap P$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{m}$ be the Lie algebras of G, K, P and M, respectively, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the Cartan decomposition corresponding to the choice of maximal compact subgroup.*

Then the pullback g_M of the hyperbolic metric on G/K along the canonical projection $\pi_K \colon G/M \to G/K$ is induced by the M-invariant inner product on $\mathfrak{p}/\mathfrak{m}$ determined by

$$g_M(E, E) = 1, \qquad g_M(F_X, F_Y) = \frac{1}{2} \operatorname{re}(\langle X, Y \rangle), \qquad g_M(H, H) = \frac{1}{4}$$

for all X, $Y \in \mathbb{C}^n$, where $\langle\ ,\ \rangle$ is the standard Hermitian product on $\mathbb{C}^n$.

PROOF. In Definition 4.1.2 we obtained the hyperbolic metric as the G-invariant Riemannian metric associated to the standard Hermitian product on $\mathfrak{q} \cong \mathbb{C}^{n+1}$, which is K-invariant. In order to determine the corresponding M-invariant inner product on $\mathfrak{p}/\mathfrak{m}$ we first apply the projection $\mathfrak{p}/\mathfrak{m} \to \mathfrak{g}/\mathfrak{k}$ then identify this space with the subalgebra $\mathfrak{q}$. An explicit computation shows that in this way E is mapped to the first unit vector in $\mathbb{C}^{n+1}$, F_X is mapped to $-\sqrt{2}^{-1}(0, X^t)^t$ and H is mapped to $-\frac{1}{2}Je_0$. □

We could start now to construct BGG-compatible Poisson kernels as we did in the case of real hyperbolic space, c.f. section 3.4.1. However, we have seen in Proposition 4.1.2(ii) that G/K is a Kähler manifold, so the corresponding decomposition of the space of differential forms into (p,q)-types is reflected on the horizontal distribution of $T(G/M)$. Similarly, Proposition 4.1.3 showed that G/P is a CR-manifold, so we obtain a corresponding decomposition of partial differential forms defined on sections of the CR-distribution. In order to include these geometric structures into our computations we consider complex valued Poisson kernels defined on the M-module $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$. In this way, we obtain that the horizontal subspace $(\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ is a Hermitian vector space with respect to the complex bilinear extension of g_M, which we denote by the same symbol, whereas $\mathfrak{k}/\mathfrak{m}$ contains a subrepresentation $\mathbb{L}$ endowed with an M-invariant complex structure, which corresponds to the CR-distribution on G/P. In particular, the induced decompositions of $\Lambda^k(\mathfrak{p}/\mathfrak{m})^*_{\mathbb{C}}$ and $\Lambda^k\mathbb{L}^*_{\mathbb{C}}$ into (p,q)-types are M-invariant.

Definition 4.1.4. Let ω be an M-invariant form of bidegree $(p+q, r+s)$ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$.

(i) We say that ω is of *K-type* (p,q) if ω vanishes after insertion of $p+1$ holomorphic or $q+1$ antiholomorphic vectors in $(\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$.
(ii) We say that ω is of *P-type* (r,s) if ω vanishes after insertion of $r+1$ holomorphic or $s+1$ antiholomorphic vectors in $\mathbb{L}_{\mathbb{C}}$.

For future computations we need to adapt our choice of generators of $\mathfrak{p}/\mathfrak{m}$ and $\mathbb{L} \subset \mathfrak{k}/\mathfrak{m}$ to such vectors which are compatible with the complex structures on these spaces. For the first space, the complex structure was obtained by the identification of $\mathfrak{p}/\mathfrak{m}$ with the vector space $\mathfrak{q} \cong \mathbb{C}^{n+1}$. We have seen in the proof of Proposition 4.1.4 that the first unit vector e_0 in $\mathbb{C}^{n+1}$ is mapped to $E \in \mathfrak{p}/\mathfrak{m}$, whereas Je_0 corresponds to $-2H$. Therefore, we define the elements

$$Z := \frac{1}{2}(E + 2iH), \qquad \overline{Z} := \frac{1}{2}(E - 2iH),$$

which are both M-invariant vectors in the complexified horizontal subspace $(\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$, the first being holomorphic and the second being antiholomorphic with respect to the complex structure induced by $\mathfrak{q} \cong \mathbb{C}^{n+1}$. Here, the factors are chosen so that the pairing $g_M(Z, \overline{Z})$ equals $\frac{1}{2}$, c.f. Proposition 4.1.4. Moreover, for all $X \in \mathbb{C}^n$ we split the vector $F_X \in (\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ into its holomorphic part $F_X^{1,0}$ and antiholomorphic part $F_X^{0,1}$, and similarly for the vector $G_X \in \mathbb{L}_{\mathbb{C}}$. However, since a complex structure only exists on the CR-subspace $\mathbb{L} \subset \mathfrak{k}/\mathfrak{m}$, the complex vector $I \hookrightarrow I \otimes 1 \in (\mathfrak{k}/\mathfrak{m})_{\mathbb{C}}$ has to be treated in a special way.

Finally, we need to derive formulae for the differential calculus of elements in $\Lambda^*(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$, which we will use heavily in the construction of Poisson kernels in the sequel. In order to do so, we define for any element $\xi \in (\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ the covector $\xi^\flat \in (\mathfrak{p}/\mathfrak{m})^*_{\mathbb{C}}$ via $\xi^\flat(\eta) = g_M(\eta, \xi)$, which coincides with the complex linear extension of the flat operator induced by g_M. In addition, the K-Hodge star $*_K$ has the property that it maps M-invariant forms of K-type (p,q) to those of K-type $(n+1-q, n+1-p)$ (c.f. section 4.1.2), leaving the P-type unchanged.

Corollary 4.1.4. *Let ω be a form of degree (k,ℓ) on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$.*

(i) For all $\xi \in (\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ we have $\iota_\xi(*_K\omega) = (-1)^k *_K(\xi^\flat \wedge \omega)$ as well as $\xi^\flat \wedge *_K\omega = (-1)^{k+1} *_K (\iota_\xi\omega)$.

(ii) For all $\eta \in (\mathfrak{k}/\mathfrak{m})_{\mathbb{C}}$ *we have* $\iota_\eta(*_K\omega) = *_K(\iota_\eta\omega)$.

PROOF. It suffices to prove the formulae for elements of the form $\omega = \alpha \wedge \beta$, where α and β are forms on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ of degree $(k,0)$ and $(0,\ell)$, respectively.

(i) From the definition of the flat operator we obtain for all $\xi \in (\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$ that $\iota_\xi *_K \alpha = *_K(\alpha \wedge \xi^\flat)$. Thus, using that the $(0,\ell)$-form β is not affected by the K-Hodge star we deduce

$$\iota_\xi *_K \omega = (\iota_\xi *_K \alpha) \wedge \beta = \left(*_K(\alpha \wedge \xi^\flat)\right) \wedge \beta = (-1)^k *_K (\xi^\flat \wedge \omega).$$

The second equation is obtained by applying the first relation to $*_K\omega$ and afterwards the K-Hodge star to both sides of the resulting equation.

(ii) Since the dimension of $\mathfrak{g}/\mathfrak{k}$ equals $2n+2$ we obtain for all $\eta \in (\mathfrak{k}/\mathfrak{m})_{\mathbb{C}}$ that

$$\iota_\eta *_K \omega = (-1)^{2n+2-k}(*_K\alpha) \wedge \iota_\eta\beta = (-1)^k *_K (\alpha \wedge \iota_\eta\beta) = *_K\iota_\eta\omega,$$

and by linearity of ι_η and the K-Hodge star the claim follows. □

4.2. Poisson kernels on G/M

Following the previous discussion of the geometric properties of G/M we now turn to the construction of Poisson kernels in this case. In order to do so, recall from section 4.1.4 that the geometric structures on the Kähler manifold G/K and the CR-manifold G/P induce a finer decomposition of the space of M-invariant forms on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ into K-types and P-types, respectively. As by definition two such forms with different K or P-type have to be linear independent, these additional bigradings will be advantageous for the construction of the BGG-compatible Poisson kernels. Therefore, we will focus on M-invariant forms on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ in the sequel, where we start with forms of degree 1 and 2 in 4.2.1 and afterwards use them as building blocks for a family of Poisson kernels of higher degree in 4.2.2.

4.2.1. Poisson kernels of low degree. We start our analysis of the space of Poisson kernels by defining M-invariant elements in $(\mathfrak{g}/\mathfrak{m})^*_{\mathbb{C}}$. In order to do so, recall from section 4.1.4 that there are three M-invariant vectors in $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$, so their dual elements Z^*, $\overline{Z}^*$ and I^* will be M-invariant 1-forms:

TABLE 1. Invariant forms of degree 1

form	bidegree	K-type	P-type
Z^*	$(1,0)$	$(1,0)$	$(0,0)$
$\overline{Z}^*$	$(1,0)$	$(0,1)$	$(0,0)$
I^*	$(0,1)$	$(0,0)$	$(0,0)$

In order to obtain invariant forms of degree 2 we can compute the images of these invariant 1-forms under the (complex linear extension of the) partial derivatives. However, from Proposition 4.1.2(ii) we know that G/K is a Kähler manifold, which induces an M-equivariant splitting of d_K into $\partial_K + \overline{\partial}_K$, where the first and second operator map an invariant form of K-type (p,q) to an element of K-type $(p+1,q)$ and $(p,q+1)$, respectively. Note that these two operators are related via $\overline{\partial}_K\overline{\phi} = \overline{(\partial_K\phi)}$ for all $\phi \in \Lambda^*(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$.

Now we define

$$\omega_K := -i\left(\partial_K\overline{Z}^* + Z^* \wedge \overline{Z}^*\right), \quad \omega_P := \frac{1}{2}d_P I^*, \quad \omega_{\mathrm{tw}} := \frac{1}{2}\left(d_P Z^* - 2iZ^* \wedge I^*\right),$$

which are M-invariant forms of degree 2 on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ which are trivial upon insertion of any M-invariant vector in $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$. Explicitly, if re$\langle\ ,\ \rangle$ and im$\langle\ ,\ \rangle$ denote the complex bilinear extensions of the real and imaginary part of the standard Hermitian product on $\mathbb{C}^n$, respectively, then the formulae for the Lie bracket on $\mathfrak{g}$ at the end of section 4.1.1 show that these elements satisfy for all X, $Y \in \mathbb{C}^n$:

TABLE 2. Invariant forms of degree 2

form	bidegree	K-type	P-type	formula
ω_K	$(2,0)$	$(1,1)$	$(0,0)$	$\omega_K(F_X^{1,0}, F_Y^{0,1}) = -\operatorname{im}\langle X^{1,0}, Y^{0,1}\rangle$
ω_P	$(0,2)$	$(0,0)$	$(1,1)$	$\omega_P(G_X^{1,0}, G_Y^{0,1}) = -\operatorname{im}\langle X^{1,0}, Y^{0,1}\rangle$
ω_{tw}	$(1,1)$	$(1,0)$	$(0,1)$	$\omega_{\mathrm{tw}}(F_X^{1,0}, G_Y^{0,1}) = \operatorname{re}\langle X^{1,0}, Y^{0,1}\rangle$
$\overline{\omega_{\mathrm{tw}}}$	$(1,1)$	$(0,1)$	$(1,0)$	$\overline{\omega_{\mathrm{tw}}}(F_X^{0,1}, G_Y^{1,0}) = \operatorname{re}\langle Y^{1,0}, X^{0,1}\rangle$

In particular, these invariant 2-forms can be related via the interior product, which in turn will lead to a nontrivial linear relation between M-invariant forms of higher degree in Theorem 4.2.2.

LEMMA 4.2.1. *The Poisson kernels ω_K, ω_{tw}, $\overline{\omega_{\mathrm{tw}}}$ and ω_P are related via*

$$\iota_{F_X^{1,0}}\omega_K = -i\iota_{G_X^{1,0}}\overline{\omega_{\mathrm{tw}}}, \qquad \iota_{F_X^{0,1}}\omega_K = i\iota_{G_X^{0,1}}\omega_{\mathrm{tw}},$$
$$\iota_{F_X^{1,0}}\omega_{\mathrm{tw}} = -i\iota_{G_X^{1,0}}\omega_P, \qquad \iota_{F_X^{0,1}}\overline{\omega_{\mathrm{tw}}} = i\iota_{G_X^{0,1}}\omega_P$$

for all X, $Y \in \mathbb{C}^n$.

PROOF. Follows immediately from $\operatorname{re}\langle X^{1,0}, Y^{0,1}\rangle = i\operatorname{im}\langle X^{1,0}, Y^{0,1}\rangle$. □

For the sake of completeness we will compute the partial exterior derivatives of all the invariant 1 and 2-forms defined above. Using the equations $\partial_K^2 = \overline{\partial}_K^2 = d_P^2 = 0$ as well as the anticommutativity of these three operators we obtain that

$$\partial_K Z^* = 0, \qquad d_P Z^* = 2\omega_{\mathrm{tw}} + 2iZ^* \wedge I^*,$$
$$\partial_K \overline{Z}^* = \overline{Z}^* \wedge Z^* + i\omega_K, \qquad d_P \overline{Z}^* = 2\overline{\omega_{\mathrm{tw}}} - 2i\overline{Z}^* \wedge I^*,$$
$$\partial_K I^* = Z^* \wedge I^*, \qquad d_P I^* = 2\omega_P.$$

as well as

$$\partial_K \omega_K = -Z^* \wedge \omega_K, \qquad d_P \omega_K = 2iZ^* \wedge \overline{\omega_{\mathrm{tw}}} - 2i\overline{Z}^* \wedge \omega_{\mathrm{tw}},$$
$$\partial_K \omega_{\mathrm{tw}} = 0, \qquad d_P \omega_{\mathrm{tw}} = -2iI^* \wedge \omega_{\mathrm{tw}} + 2iZ^* \wedge \omega_P,$$
$$\partial_K \overline{\omega_{\mathrm{tw}}} = -I^* \wedge \omega_K, \qquad d_P \overline{\omega_{\mathrm{tw}}} = 2iI^* \wedge \overline{\omega_{\mathrm{tw}}} - 2i\overline{Z}^* \wedge \omega_P,$$
$$\partial_K \omega_P = Z^* \wedge \omega_P - I^* \wedge \omega_{\mathrm{tw}}, \qquad d_P \omega_P = 0.$$

In particular, we do not get new M-invariant forms in this way.

Finally, for later computations we need to relate the invariant elements defined in this section with the the flat operator $^\flat : \xi \mapsto g_M(\ ,\xi)$ induced by the M-invariant inner product on $(\mathfrak{p}/\mathfrak{m})_{\mathbb{C}}$.

PROPOSITION 4.2.1. *The invariant forms of degree 1 are related to the flat operator by*

$$Z^* = 2\overline{Z}^\flat, \qquad \overline{Z}^* = 2Z^\flat.$$

Moreover, the invariant forms of degree 2 satisfy

$$\iota_{G_X^{0,1}}\omega_{\mathrm{tw}} = -2F_X^{0,1\flat}, \qquad \iota_{G_X^{1,0}}\overline{\omega_{\mathrm{tw}}} = -2F_X^{1,0\flat},$$
$$\iota_{F_X^{0,1}}\omega_K = -2iF_X^{0,1\flat}, \qquad \iota_{F_X^{1,0}}\omega_K = 2iF_X^{1,0\flat}$$

for all $X \in \mathbb{C}^n$.

PROOF. Recall from section 4.1.4 that the M-invariant vectors Z and $\overline{Z}$ satisfy $g_M(Z,\overline{Z}) = \frac{1}{2}$ by construction, implying the first two identities. Moreover, using Proposition 4.1.4 we obtain that

$$\iota_{G_X^{0,1}}\omega_{\mathrm{tw}}(F_Y^{1,0}) = -\operatorname{re}\langle Y^{1,0}, X^{0,1}\rangle = -2g_M(F_Y^{1,0}, F_X^{0,1}) = -2F_X^{0,1\flat}(F_Y^{1,0}).$$

Furthermore, we can apply complex conjugation to this equation to obtain the similar statement for the Poisson kernel $\overline{\omega_{\mathrm{tw}}}$. Finally, using the relations in Lemma 4.2.1 the other two identities follow immediately. □

4.2.2. Basic invariant forms. In the next step we construct Poisson kernels of higher degree by forming wedge products between the invariant forms of degree 2 defined in section 4.2.1. In order to do so, recall from section 4.1.4 that the space of Poisson kernels can be decomposed into K-types and P-types, which are induced by the complex structures on the Kähler manifold G/K and the CR-distribution in $T(G/P)$, respectively. Therefore, our aim is to construct Poisson transforms with given K-type (p,q) and P-type (r,s) for $0 \le p,q,r,s \le n$.

Consider for non-negative integers a,b,c and d the wedge product

$$\omega := \omega_K^a \wedge \omega_P^b \wedge \omega_{\mathrm{tw}}^c \wedge \overline{\omega_{\mathrm{tw}}}^d,$$

which is an M-invariant form on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ of degree $(2a+c+d, 2b+c+d)$. Moreover, regarding the complex structures on $\mathfrak{p}/\mathfrak{m}$ and $\mathbb{L} \subset \mathfrak{k}/\mathfrak{m}$ its K-type is $(a+c, a+d)$, its P-type is $(b+d, b+c)$ and it is trivial after insertion of any M-invariant vector in $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$. For the parameters of the form ω this means that

$$a+c = p, \qquad a+d = q, \qquad b+d = r, \qquad b+c = s,$$

and adding the first and the third as well as the second and the last equation implies that

$$k := a+b+c+d = p+r = q+s,$$

so the P-type of ω is $(k-p, k-q)$. Moreover, the restrictions on the K-type and the P-type of ω yield the condition $\max\{p,q\} \le k \le \min\{n+p, n+q\}$ on the sum of the parameters of ω. In particular, we see that the choices of a K-type (p,q) and the parameter k already determine the P-type. Explicitly, if we put $j := a$, the above equations show that every Poisson kernel of K-type (p,q) and P-type $(k-p, k-q)$, which is a wedge product of the invariant 2-forms from section 4.2.1 is given by

$$\omega_j^{p,q;k} := \omega_K^j \wedge \omega_P^{k-(p+q)+j} \wedge \omega_{\mathrm{tw}}^{p-j} \wedge \overline{\omega_{\mathrm{tw}}}^{q-j}$$

for all indices $\max\{0, (p+q)-k\} \le j \le \min\{p,q\}$.

DEFINITION 4.2.2. Let p, q and k be integers so that $0 \le p, q, k-p, k-q \le n$. Then the Poisson kernel $\omega_j^{p,q;k}$ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ is called a *basic invariant form.* Moreover, we call the range of the index j so that $\omega_j^{p,q;k}$ is well defined the *admissible range.*

The notation we chose for the basic invariant forms is nicely compatible with the wedge product. Indeed, we have

LEMMA 4.2.2. *The basic invariant forms satisfy the following relations:*

(i) $\overline{\omega_j^{p,q;k}} = \omega_j^{q,p;k}$

(ii) $\omega_j^{p,q;k} \wedge \omega_{j'}^{p',q';k'} = \omega_{j+j'}^{p+p',q+q';k+k'}$.

(iii) The powers of the Poisson kernels of degree 2 *are retrieved via the formulae*

$$\omega_K^j = \omega_j^{j,j;j}, \qquad \omega_P^j = \omega_0^{0,0;j}, \qquad \omega_{\mathrm{tw}}^j = \omega_0^{j,0;j}, \qquad \overline{\omega_{\mathrm{tw}}}^j = \omega_0^{0,j;j}.$$

for all $0 \le j \le n$.

PROOF. Follows immediately from the definition. □

In section 4.3.2 we will explicitly construct Poisson kernels on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ which are BGG-compatible in the sense of Definition 2.2.3. There, we will proceed by considering a general linear combination of basic invariant forms and afterwards apply the operators ∂_P and $\partial_P d_P$, which will induce linear relations on the coefficients of our ansatz. For this, it will be necessary to derive a nontrivial relation between such basic invariant forms, which we will do in Theorem 4.2.2.

For the proof of this result we need to introduce some notation. Using the identification of $\mathbb{C}^n$ with $\mathbb{C}^n \otimes 1 \subset \mathbb{C}^n \otimes \mathbb{C}$ we can decompose any vector $X \in \mathbb{C}^n$ into its holomorphic part $X^{1,0}$ and antiholomorphic part $X^{0,1}$. Then we define a *holomorphic k-vector on $\mathbb{C}^n$* to be a k-fold wedge product $\mathbf{X}^{1,0} := X_1^{1,0} \wedge \ldots \wedge X_k^{1,0}$ of vectors $X_1, \ldots, X_k$ in $\mathbb{C}^n$. Moreover, we denote by $F_{\mathbf{X}}^{1,0}$ the wedge product $F_{X_1}^{1,0} \wedge \ldots \wedge F_{X_k}^{1,0}$, which is well defined since $F_X^{1,0} = F_{X^{1,0}}$ for all $X \in \mathbb{C}^n$. Similarly, we define an *antiholomorphic k-vector* on $\mathbb{C}^n$ and denote these elements by $\mathbf{X}^{0,1}$, as well as define the wedge products $G_{\mathbf{X}}^{1,0}$ and $G_{\mathbf{X}}^{0,1}$.

THEOREM 4.2.2. *For all $0 \leq p, q \leq n$ we define the rational numbers*

$$\kappa_j^{p,q;k} := \binom{k}{p}\binom{p}{j}\binom{k-p}{q-j},$$

where we agree that $\kappa_j^{p,q;k} = 0$ if one of the binomial coefficients is not defined. Then the sum

$$R^{p,q;k} := \sum_{j\in\mathbb{Z}} \kappa_j^{p,q;k} \omega_j^{p,q;k}$$

is trivial for all $k > n$.

PROOF. Let $\mathbf{X}^{1,0}$ be a holomorphic p-vector and $\mathbf{Y}^{0,1}$ be an antiholomorphic q-vector on $\mathbb{C}^n$. Applying the relation $\iota_{G_X^{1,0}}\omega_P = i\iota_{F_X^{1,0}}\omega_{\mathrm{tw}}$ from Lemma 4.2.1 iteratively we compute that

$$\iota_{G_{\mathbf{Y}^{0,1}}}\iota_{G_{\mathbf{X}^{1,0}}}\omega_P^k = \binom{k}{p} i^p \iota_{G_{\mathbf{Y}^{0,1}}}\iota_{F_{\mathbf{Y}^{1,0}}}\left(\omega_P^{k-p}\wedge\omega_{\mathrm{tw}}^p\right) = (*).$$

Next, we want to transform the interior products with the antiholomorphic vectors $G_{Y^{0,1}}$ into expressions involving the interior product with the corresponding vectors $F_{Y^{0,1}}$ using again Lemma 4.2.1. For that, note that if we move j of the vectors of $G_{\mathbf{Y}}^{0,1}$ into ω_{tw}^p, we have to use $q-j$ for the wedge product ω_P^{k-p}. Now a direct computation shows that

$$(*) = \sum_{j=\max\{0,p+q-k\}}^{\min\{p,q\}} \kappa_j^{p,q;k} i^{p-q} \iota_{F_{\mathbf{Y}^{0,1}}}\iota_{F_{\mathbf{X}^{1,0}}}\omega_j^{p,q;k}.$$

In particular, for $k > n$ we have $\omega_P^k = 0$, implying that $R^{p,q;k} = 0$. □

For an effective application of Theorem 4.2.2 we will now prove a characterization of the numbers $\kappa_j^{p,q;k}$ in terms of linear recursions.

PROPOSITION 4.2.2. *Let p, q and k be integers so that $0 \leq p, q, k-p, k-q \leq n$ and define $j_0 := \min\{0, (p+q)-k\}$. Let $\{\kappa_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers which satisfies the recursion*

$$(\kappa) \qquad \kappa_j(p-j)(q-j) = \kappa_{j+1}(j+1)(k-(p+q)+j+1)$$

for all $j \in \mathbb{Z}$. Then $\kappa_j = C\kappa_j^{p,q;k}$ for all $j \in \mathbb{Z}$, where $C \in \mathbb{C}$ is a constant independent of j.

PROOF. A direct computation shows that the complex numbers $\kappa_j^{p,q;k}$ satisfy the relations (κ). On the other hand, for $j = j_0 - 1$ we obtain that the right hand side of relation (κ) is trivial, whereas left hand side equals $\kappa_{j_0-1}(p+1-j_0)(q+1-j_0)$. Since the numbers in the brackets are always positive, this implies that $\kappa_{j_0-1} = 0$ and inductively $\kappa_j = 0$ for all $j < j_0$. Similarly, for $j = \min\{p, q\}$ the left hand side of relation (κ) is trivial, whereas the coefficients on the right hand side are nontrivial. Inductively, this implies that $\kappa_j = 0$ for all $j > \min\{p, q\}$. Therefore, the sequence $\{\kappa_j\}$ is possibly nontrivial only for $j_0 \leq j \leq \min\{p, q\}$, where we can compute κ_j from κ_{j-1} recursively. Since the relations (κ) are linear, this implies that $\{\kappa_j\}$ is a constant multiple of $\{\kappa_j^{p,q;k}\}$. □

4.3. BGG-compatible Poisson transforms

In this section we present an explicit construction of BGG-compatible Poisson transforms which map between complex valued differential forms on the CR-sphere of degree $p + q$ and complex valued differential forms on G/K of type (p, q) for all $0 \leq p, q \leq n$. This will be done in section 4.3.2 by choosing an ansatz for the corresponding Poisson kernels in terms of linear combinations of basic invariant forms, which is adapted to the structure of the homology bundles on G/P. Afterwards, using the explicit formula for the P-codifferential in Theorem 4.3.1 we translate the condition of being BGG-compatible into linear relations on the coefficients of our ansatz, which can be solved easily. This family of operators will be used in section 4.4.4 to construct a family of real valued Poisson transforms, which resemble Gaillard's operators on the real hyperbolic space.

4.3.1. Preliminary results. Recall from Definition 2.2.3 that a Poisson transform $\Psi\colon \Omega^k(G/P) \to \Omega^\ell(G/K)$ is called BGG-compatible iff it is trivial on sections of the subbundles $\operatorname{im}(\partial^*)$ and $\operatorname{im}(\partial^* d)$ of $\Omega^k(G/P)$. By Proposition 2.2.3, this can be equivalently formulated in terms of its associated kernel ψ by $\partial_P\psi = 0$ and $\partial_P d_P \psi = 0$. Moreover, note that in this case the Poisson transform factors to a G-equivariant linear operator defined on sections of the homology bundles $\mathcal{H}_k(G/P)$ on G/P so that its composition with the k-th BGG-operator is again induced by a Poisson transform.

In order to construct such operators explicitly we first want to bring the geometric structure of G/K into the game. For that, recall from Lemma 4.1.2 that the space of complex valued differential forms on G/K splits into (p, q)-types so that the canonical projection $\pi_{p,q}\colon \Omega^{p+q}(G/K, \mathbb{C}) \to \Omega^{p,q}(G/K)$ is tensorial and equivariant with respect to the induced G-actions. In particular, this implies that the composition $\Psi_{p,q} := \pi_{p,q} \circ \Psi$ is again a Poisson transform for all $0 \leq p, q \leq n$ with $p + q = \ell$.

LEMMA 4.3.1. *Let $G = \mathrm{SU}(n+1, 1)$, $K \cong \mathrm{U}(n+1)$ its maximal compact subgroup and $P \subset G$ its minimal parabolic subgroup. Let $\Psi\colon \Omega^k(G/P, \mathbb{C}) \to \Omega^\ell(G/K, \mathbb{C})$ be a Poisson transform and $\Psi_{p,q}$ be its composition with the canonical projection onto $\Omega^{p,q}(G/K)$ for all $p + q = \ell$. Then Ψ is BGG-compatible if and only if $\Psi_{p,q}$ is BGG-compatible for all $p + q = \ell$.*

PROOF. By construction we can write $\Psi = \sum_{p+q=\ell} \Psi_{p,q}$, hence composing this equation with any linear operator D from the right implies that $\Psi \circ D = 0$ if and only if $\Psi_{p,q} \circ D = 0$ due to linear independence of differential forms on G/K with different (p, q)-type. In particular, for $D = \partial^*$ and $D = \partial^* \circ d$ this shows that Ψ is BGG-compatible if and only if $\Psi_{p,q}$ is BGG-compatible for all $p + q = \ell$. □

By Lemma 4.3.1 it suffices to construct BGG-compatible Poisson transforms whose values have a specific (p, q)-type, which in turn correspond to Poisson kernels

ψ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ of K-type (p,q) which satisfy $\partial_P\psi = 0$ and $\partial_P d_P \psi = 0$. Thus, in the next step we need to determine an explicit formula for the complex linear extension of the P-codifferential, which we denote by the same symbol. For that, we will interpret the formula for the Kostant codifferential from section 1.4.2 in terms of the homogeneous space G/M in the case $G = \mathrm{SU}(n+1,1)$.

PROPOSITION 4.3.1. *Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitian product. Then for any Poisson kernel ψ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ we have*

$$\partial_P\psi = \frac{i}{4(n+2)}\sum_{\ell=1}^n I^* \wedge (\iota_{G_{e_\ell}^{0,1}}\iota_{G_{e_\ell}^{1,0}} - \iota_{G_{e_\ell}^{1,0}}\iota_{G_{e_\ell}^{0,1}})\psi.$$

In particular, the P-codifferential is trivial on the ideal generated by I^*.

PROOF. In section 2.2.3 we defined the P-codifferential on decomposable elements by $\partial_P(\alpha\wedge\beta) = (-1)^\ell\alpha\wedge\partial^*\beta$ for $\alpha \in \Lambda^{\ell,0}(\mathfrak{g}/\mathfrak{m})^*$ and $\beta\in\Lambda^{0,k}(\mathfrak{g}/\mathfrak{m})^*$, where we used the identification of $\mathfrak{k}/\mathfrak{m}$ with $\mathfrak{g}/\mathfrak{p}$ induced by the restriction of the canonical projection $\mathfrak{g}/\mathfrak{m}\to\mathfrak{g}/\mathfrak{p}$. Furthermore, we have seen in section 1.4.1 that the Killing form on $\mathfrak{g}$ induces an isomorphism $(\mathfrak{g}/\mathfrak{p})^*\cong\mathfrak{p}_+$ of P-modules, so in particular the Lie algebra $\mathfrak{p}_+$ acts naturally on β.

First, we consider β as an element in $\Lambda^k(\mathfrak{g}/\mathfrak{p})^*$, which we interpret as a multilinear, alternating map $\mathfrak{g}^k\to\mathbb{R}$ which vanishes upon insertion of any element in $\mathfrak{p}$. Then we have proven in Proposition 1.4.4 that the Kostant codifferential can be expressed by a sum over the $\mathfrak{p}_+$-action on β. Explicitly, if $\{\xi_1,\dots,\xi_{2n+1}\}$ is a basis of $\mathfrak{p}_+$ with dual basis $\{\eta_1,\dots,\eta_{2n+1}\}$ of $\mathfrak{g}_-$ with respect to the Killing form, then combining Proposition 1.4.4 and Lemma 1.4.4 yields

$$\partial^*\beta = \frac{1}{2}\sum_{s=1}^{2n+1}\sum_{t=1}^{2n+1}\epsilon_{\xi_t}\iota_{[\xi_s,\eta_t]}\iota_{\eta_s}\beta.$$

Note that by naturality of the complexification the same formula holds for the complex linear extension of the Kostant codifferential.

Next, by choosing particular bases of $\mathfrak{p}_+$ we can use the $|2|$-grading on $\mathfrak{g}$ from Proposition 4.1.1(i) to simplify this formula. For $X\in\mathbb{C}^n$ denote by ξ_X and η_X the corresponding elements in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, respectively, and let $\nu_\pm$ be the elements in $\mathfrak{g}_{\pm2}$ corresponding to the imaginary unit. Then $\{\xi_{e_\ell},\xi_{Je_\ell},\nu_+\}$ is a real basis of $\mathfrak{p}_+$, and from Proposition 4.1.1(iv) we deduce that $\{-(2\mu)^{-1}\eta_{e_\ell}, -(2\mu)^{-1}\eta_{Je_\ell}, -\mu^{-1}\nu_-\}$ is the dual basis of $\mathfrak{g}_-$ with respect to the Killing form, where $\mu := 2(n+2)$.

Using this basis to compute $\partial^*\beta$ we deduce that the Lie bracket between base elements of $\mathfrak{g}_-$ and $\mathfrak{p}_+$ is contained in $\mathfrak{p}$ except for $[-\mu^{-1}\nu_-,\xi_{e_\ell}]$ and $[-\mu^{-1}\nu_-,\xi_{Je_\ell}]$, which equals $-\mu^{-1}\eta_{Je_\ell}$ and $\mu^{-1}\eta_{e_\ell}$, respectively. Since β is trivial upon insertion of vectors in $\mathfrak{p}$ it follows that

$$\partial^*\beta = \frac{1}{2\mu^2}\sum_{\ell=1}^n \epsilon_{\nu_+}\iota_{\eta_{Je_\ell}}\iota_{\eta_{e_\ell}}\beta.$$

In order to obtain the corresponding expression for the P-codifferential we have to interpret the formula for ∂^* in terms of the M-module $\mathfrak{g}/\mathfrak{m}$, meaning that we identify $\mathfrak{g}_-$ with $\mathfrak{k}/\mathfrak{m}$ as well as $\mathfrak{p}_+$ with $(\mathfrak{k}/\mathfrak{m})^*$ via the Killing form. In terms of the generators of $\mathfrak{g}/\mathfrak{m}$ defined in section 4.1.4 we deduce that η_{e_ℓ} corresponds to the element $G_{\eta_j}\in\mathfrak{k}/\mathfrak{m}$, whereas ν_+ corresponds to the invariant form μI^*. Therefore, we obtain that

$$\partial_P\psi = \frac{1}{4(n+2)}\sum_{\ell=1}^n I^*\wedge\iota_{G_{Je_\ell}}\iota_{G_{e_\ell}}\beta,$$

and decomposing the base vectors G_{e_ℓ} into holomorphic and antiholomorphic part yields the claimed formula. $\square$

From the formula in Proposition 4.3.1 it follows immediately that ∂_P anticommutes with the interior products of the vectors Z and $\overline{Z}$ as well as with the wedge products of Z^* and $\overline{Z^*}$. Therefore, for the construction of BGG-compatible Poisson kernels it suffices to compute the P-codifferential of the basic invariant forms explicitly.

THEOREM 4.3.1. *Let p, q and k be integers with $0 \le p, q, k-p, k-q \le n$ and let j be in the admissible range. Then*

$$\partial_P \omega_j^{p,q;k} = \frac{1}{4(n+2)}\big((k-(p+q)+j)(k-j-(n+1))I^* \wedge \omega_j^{p,q;k-1} \\ + (p-j)(q-j)I^* \wedge \omega_{j+1}^{p,q;k-1}\big).$$

PROOF. By Proposition 4.3.1 the image of $\omega_j^{p,q;k}$ under ∂_P is computed by taking pairwise interior products with the vectors $\iota_{G^{1,0}_{e_\ell}}$ and $\iota_{G^{0,1}_{e_\ell}}$ and their complex conjugates, where $\{e_\ell\}$ is an orthonormal basis for $\mathbb{C}^n$. Using the antiderivation property of the interior product we deduce that $\partial_P \omega_j^{p,q;k}$ can be written as a sum with each summand having the form

$$I^* \wedge \omega_j^{p',q';k'} \wedge \left(\sum_{\ell=1}^{n} \left(\iota_{G^*_{e_\ell}} \omega_1\right) \wedge \left(\iota_{G^*_{e_\ell}} \omega_2\right) \right),$$

where ω_1 and ω_2 are contained in the set $\{\omega_P,\ \omega_{\mathrm{tw}},\ \overline{\omega_{\mathrm{tw}}}\}$. Therefore, it suffices to determine the sum in the parentheses, where we benefit from the fact that the vectors e_ℓ are orthonormal with respect to the standard Hermitian metric on $\mathbb{C}^n$.

As an example, we compute

$$\sum_{\ell=1}^{n} \left((\iota_{G_{e_\ell^{0,1}}} \omega_P) \wedge (\iota_{G_{e_\ell^{1,0}}} \omega_P) - (\iota_{G_{e_\ell^{1,0}}} \omega_P) \wedge (\iota_{G_{e_\ell^{0,1}}} \omega_P) \right) (G_X^{1,0}, G_Y^{0,1}) =$$
$$= -\sum_{s=1}^{n} i \left(\mathrm{re}\langle e_\ell^{1,0}, Y^{0,1}\rangle \,\mathrm{im}\langle e_\ell^{0,1}, X^{1,0}\rangle - \mathrm{re}\langle e_\ell^{0,1}, X^{1,0}\rangle \,\mathrm{im}\langle e_\ell^{1,0}, Y^{0,1}\rangle \right)$$
$$= -i\omega_P(G_X^{1,0}, G_Y^{0,1})$$

for all X, $Y \in \mathbb{C}^n$, and the other cases can be determined similarly, obtaining the claimed formula. $\square$

4.3.2. The construction of a BGG-compatible Poisson kernel. Using the preliminary results in section 4.3.1 we will now construct BGG-compatible Poisson kernels by choosing a special ansatz consisting of linear combinations of basic invariant forms defined in 4.2.2 and afterwards translate the condition on the BGG-compatibility into linear recursions for its coefficients.

In order to choose our ansatz for the Poisson kernel in an efficient way we will focus on those invariant elements whose associated Poisson transforms preserve the degrees of the differential forms and have values in $\Omega^{p,q}(G/K)$ for fixed $0 \le p, q \le n$. By construction, this means that we are interested in Poisson kernels on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ of bidegree $(p+q, 2n+1-(p+q))$ and K-type (p,q). Since we have seen in section 4.2.2 that the basic invariant forms $\omega_j^{p,q;k}$ are of even total degree for all parameters p, q and k, we are forced to form wedge products of these elements with an odd number of M-invariant forms of degree 1 in order to meet our requirements.

Furthermore, since we want to construct our Poisson kernels to be BGG-compatible, we will adapt our ansatz to the structure of the homology bundles over G/P. Explicitly, recall from Theorem 4.1.3 that if L denotes the CR-distribution on G/P, then $\mathcal{H}_k(G/P)$ is isomorphic to a subbundle of $\Lambda^k L^*$ for all $k \leq n$, whereas for $k \geq n+1$ it is isomorphic to a quotient of $\Lambda^{k-1}L^* \otimes Q^*$, where Q denotes the quotient bundle $T(G/P)/L$. In terms of the homogeneous space G/M, the distribution L corresponds to the subbundle of the vertical bundle of $\pi_P : G/M \to G/P$ which is associated to the M-representation $\mathbb{L} \subset (\mathfrak{k}/\mathfrak{m})_{\mathbb{C}}$. Since the homology bundles split into two different families, we need to discuss the cases $k \leq n$ and $k \geq n+1$ in the construction of our Poisson kernels separately.

In the following construction we proceed in 4 steps:

(1) Choose an appropriate ansatz,
(2) Determine the coefficients of this ansatz so that it is contained in the kernel of ∂_P,
(3) Find an appropriate error term for the ansatz contained in the kernel of ∂_P and translate BGG-compatibility of the resulting form into linear relations on the coefficients,
(4) Solve these linear relations explicitly.

For step (4), recall from Theorem 4.2.2 that we defined for all p, q and k with $0 \leq p, q, k-p, k-q \leq n$ the complex numbers $\kappa_j^{p,q;k}$ via

$$\kappa_j^{p,q;k} = \binom{k}{p}\binom{p}{j}\binom{k-p}{q-j},$$

and showed that a certain linear combination of the basic invariant forms involving these coefficients is trivial. Furthermore, we have proven in Proposition 4.2.2 that these coefficients are determined by a linear recursion up to a constant multiple.

Case 1: Let $0 \leq p, q \leq n$ with $p + q \leq n$.

(1) The pullback of a section of $\mathcal{H}_{p+q}(G/P)$ along π_P is a section of the homogeneous bundle on G/M associated to the M-representation $\Lambda^k \mathbb{L}^*$. Therefore, it follows that the wedge product of ψ with such a pullback is nontrivial if and only if $I^* \wedge \psi = 0$ and $\iota_I \psi \neq 0$.

   Hence we consider the ansatz

$$\psi = \sum_{j \in \mathbb{Z}} \alpha_j I^* \wedge \omega_j^{p,q;n} + \sum_{j \in \mathbb{Z}} \beta_j I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;n-1},$$

   where α_j and β_j are complex numbers for all $j \in \mathbb{Z}$ which satisfy $\alpha_j = 0$ for all $j < 0$ and $j > \min\{p, q\}$, whereas $\beta_j = 0$ for $j < 0$ and $j > \min\{p-1, q-1\}$.
(2) Since ψ is contained in the ideal generated by I^* by definition, it follows immediately from Proposition 4.3.1 that $\partial_P \psi = 0$.
(3) Now we need to ensure that ψ is also contained in the kernel of the operator $\partial_P d_P$. Using that the P-differential is an antiderivation and the formulae in section 4.2.1 a direct computation shows

$$d_P \psi \equiv \sum_{j \in \mathbb{Z}} 2\alpha_j \omega_j^{p,q;n+1} + \sum_{j \in \mathbb{Z}} 2\beta_j Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;n} \mod \langle I^* \rangle.$$

   Since by Proposition 4.3.1 the ideal generated by I^* is contained in the kernel of the P-codifferential, it suffices to choose the coefficients α_j and β_j appropriately. Applying Theorem 4.3.1 we deduce that $\partial_P d_P \psi = 0$ if and only if the recursions

$$(j+1)(n+1-(p+q)-j)\alpha_{j+1} = (p-j)(q-j)\alpha_j$$
$$(j+1)(n+1-(p+q)-j)\beta_j = (p-j)(q-j)\beta_{j-1}$$

   are satisfied for all $j \in \mathbb{Z}$.
(4) In the last step we explicitly solve the above recursions. First, if we compare the relations of α_j and β_j with Proposition 4.2.2 we obtain that

$$\alpha_j = \alpha \kappa_j^{p,q;n+1}, \qquad \beta_j = \beta \kappa_{j+1}^{p,q;n+1}$$

for all $j \in \mathbb{Z}$, where α and β are complex numbers. Here, the left equation is well defined for all $j \in \mathbb{Z}$ and all $\alpha \in \mathbb{C}$. However, for $j = -1$ we have $\beta_{-1} = 0$ and $\kappa_0^{p,q;n+1} \neq 0$ by definition, so the second relation implies that $\beta = 0$ and therefore $\beta_j = 0$ for all $j \in \mathbb{Z}$.

In particular, the above congruence for the P-derivative of ψ reads as

$$d_P\psi \equiv \alpha \sum_{j\in\mathbb{Z}} \kappa_j^{p,q;n+1} I^* \wedge \omega_j^{p,q;n+1} \mod \langle I^* \rangle,$$

and by Theorem 4.2.2 the sum on the right hand side is trivial. Therefore, we deduce that $d_P\psi$ is already contained in the ideal generated by I^*.

Thus, we define for all $0 \leq p, q, \leq n$ with $p+q \leq n$ the BGG-compatible Poisson kernel

$$\psi_{p,q} := \sum_{j\in\mathbb{Z}} \kappa_j^{p,q;n+1} I^* \wedge \omega_j^{p,q;n}.$$

Furthermore, we denote its associated Poisson transform by $\Psi_{p,q}$, which factors to a G-equivariant operator $\tilde{\Psi}_{p,q}$ defined on sections of the homology bundles $\mathcal{H}_{p+q}(G/P) \otimes \mathbb{C}$.

Case 2: Let $0 \leq p, q \leq n$ with $p + q \geq n + 1$.

(1) The pullback of a section of $\mathcal{H}_{p+q}(G/P)$ along π_P is a section of the homogeneous bundle over G/M associated to the M-representation $\Lambda^{k-1}\mathbb{L}^* \otimes ((\mathfrak{k}/\mathfrak{m})/\mathbb{L})^*$ by construction. Moreover, by section 4.1.4 we can identify the M-module $((\mathfrak{k}/\mathfrak{m})/\mathbb{L})^*$ with the M-invariant subspace of $(\mathfrak{g}/\mathfrak{m})^*$ generated by I^*. Therefore, we deduce that the wedge product ψ with of the pullback of a section of the homology bundle is trivial unless $I^* \wedge \psi \neq 0$.

Furthermore, we choose our ansatz to consist of wedge products of basic invariant forms with an odd number of M-invariant 1-forms, so we need to ensure that either $Z^* \wedge \omega_j^{p-1,q;n}$ or $\overline{Z}^* \wedge \omega_j^{p,q-1;n}$ are summands contained in our ansatz. Since the latter can be written as the complex conjugate of the former with the parameter p and q exchanged, so we will consider the ansatz

$$\psi = \sum_{j\in\mathbb{Z}} \alpha_j Z^* \wedge \omega_j^{p-1,q;n}$$

where α_j are complex numbers which are trivial for all $j < p + q - (n+1)$ and $j > \min\{p-1, q\}$.

(2) In order to guarantee that $\partial_P\psi = 0$ we apply the explicit formula in Theorem 4.3.1 to our ansatz and shift the index j properly, which yields the condition

$$(j+2)((n+1) - (p+q) + j + 1)\alpha_{j+1} = (p-j-1)(q-j)\alpha_j$$

for all $j \in \mathbb{Z}$. Now comparing this with Proposition 4.2.2 shows that

$$\alpha_j = \alpha\kappa_{j+1}^{p,q+1;n+1}$$

for all $j \in \mathbb{Z}$, where α is an arbitrary complex number. In order to simplify the following computations, we will choose $\alpha = 1$.

(3) Next, we compute the image of our ansatz under the P-differential. Since d_P is an antiderivation we can use the formulae for the derivatives of the invariant forms of low degree in section 4.2.1 to deduce that

$$d_P\psi \equiv 2\sum_{j\in\mathbb{Z}} \alpha_j\omega_j^{p,q;n+1} + 2i\sum_{j\in\mathbb{Z}} \tilde{\alpha}_j Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;n} \mod \langle I^* \rangle,$$

where the coefficient $\tilde{\alpha}_j$ is given by

$$\tilde{\alpha}_j := (j+1)\alpha_{j+1} + (q-j)\alpha_j.$$

Since $\langle I^* \rangle$ is contained in the kernel of ∂_P due to Proposition 4.3.1 we can compute $\partial_P d_P \psi$ by applying the formulae for the P-codifferential of the basic invariant forms in Theorem 4.3.1 to the two sums on right hand side of the above congruence. In this way, a direct computation shows that $\partial_P d_P \psi$ is nontrivial.

Thus, we have to find an appropriate error term ϵ for ψ contained in the kernel of the P-codifferential which makes our ansatz BGG-compatible. By Proposition 4.3.1 we will choose ϵ to lie in the ideal generated by I^*, i.e.

$$\epsilon = \sum_{j \in \mathbb{Z}} \gamma_j I^* \wedge \omega_j^{p,q;n} + \sum_{j \in \mathbb{Z}} i\delta_j I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;n-1},$$

for appropriate coefficients γ_j and $\delta_j \in \mathbb{C}$. Explicitly, with $j_0 := p+q-(n+1)$ these satisfy $\gamma_j = 0$ for $j < j_0+1$ an $j > \min\{p,q\}$, whereas $\delta_j = 0$ for $j < j_0$ and $j > \min\{p-1, q-1\}$.

Using the formula for the P-derivative of such a form from case 1, we immediately obtain the following congruence modulo $\langle I^* \rangle$:

$$d_P(\psi + \epsilon) \equiv \sum_{j \in \mathbb{Z}} 2(\alpha_j + \gamma_j)\omega_j^{p,q;n+1} + 2i(\tilde{\alpha}_j + \delta_j) Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;n}.$$

Therefore, if we apply the formula for ∂_P from Theorem 4.3.1 we deduce that $\partial_P d_P(\psi+\epsilon) = 0$ if and only if the recursions

$$(\alpha_{j+1} + \gamma_{j+1})\,(j+1)(n+1-(p+q)+j+1) = (p-j)(q-j)\,(\alpha_j+\gamma_j)$$
$$(\tilde{\alpha}_j + \delta_j)\,(j+1)(n+1-(p+q)+j+1) = (p-j)(q-j)\,(\tilde{\alpha}_{j-1}+\delta_{j-1})$$

are satisfied for all $j \in \mathbb{Z}$.

(4) Thus, our final task is to solve the above recursions for the coefficients γ_j and δ_j. First, from Proposition 4.2.2 we deduce that

$$\gamma_j + \alpha_j = \gamma \kappa_j^{p,q;n+1}, \qquad \delta_j + \tilde{\alpha}_j = \delta \kappa_{j+1}^{p,q;n+1}$$

for complex numbers γ and δ, so if we insert $\alpha_j = \kappa_{j+1}^{p,q+1;n+1}$ as well as the definition of $\tilde{\alpha_j}$ we obtain

$$\gamma_j = \gamma\kappa_j^{p,q;n+1} - \kappa_{j+1}^{p,q+1;n+1},$$
$$\delta_j = \delta\kappa_{j+1}^{p,q;n+1} - (j+1)\kappa_{j+2}^{p,q+1;n+1} - (q-j)\kappa_{j+1}^{p,q+1;n+1}.$$

For the first equation, recall that we assumed that $\gamma_{j_0} = 0$, so for $j = j_0$ a direct computation shows that

$$\gamma = \left(\kappa_{j_0}^{p,q;n+1}\right)^{-1} \kappa_{j_0+1}^{p,q+1;n+1} = \frac{(n+1-q)}{p+q-n}.$$

By definition of the coefficients $\kappa_j^{p,q;k}$ this implies

$$\gamma_j = \frac{(n+1-q)}{p+q-n}\kappa_j^{p,q;n+1} - \kappa_{j+1}^{p,q+1;n+1} = \frac{p+1}{p+q-n}\kappa_{j+1}^{p+1,q+1;n+1}.$$

Similarly, if we insert $j = j_0 - 1$ into the expression for δ_j we use the assumption $\delta_{j_0-1} = 0$ as well as $\kappa_{j_0}^{p,q+1;n+1} = 0$ to obtain that

$$\delta = j_0 \left(\kappa_{j_0}^{p,q;n+1}\right)^{-1} \kappa_{j_0+1}^{p,q+1;n+1} = \frac{j_0(n+1-q)}{p+q-n}.$$

Using the definition of the complex numbers $\kappa_j^{p,q;k}$ a direct computation shows that

$$\begin{aligned}\delta_j &= \frac{j_0(n+1-q)}{p+q-n}\kappa_{j+1}^{p,q;n+1} - (j+1)\kappa_{j+2}^{p,q+1;n+1} - (q-j)\kappa_{j+1}^{p,q+1;n+1}\\ &= -\frac{p+1}{p+q-n}\kappa_{j+2}^{p+1,q+1;n+1}.\end{aligned}$$

In particular, we see that the error term ϵ is uniquely determined.

Thus, for all $0 \le p, q \le n$ with $p+q \ge n+1$ we define the BGG-compatible Poisson kernel

$$\begin{aligned}\psi_{p,q} := \sum_j \kappa_{j+1}^{p,q+1;n+1} Z^* \wedge \omega_j^{p-1,q;n} &+ \frac{p+1}{p+q-n}\kappa_{j+1}^{p+1,q+1;n+1} I^* \wedge \omega_j^{p,q;n}\\ &- i\frac{p+1}{p+q-n}\kappa_{j+2}^{p+1,q+1;n+1} I^* \wedge Z^* \wedge \overline{Z^*} \wedge \omega_j^{p-1,q-1;n-1}\end{aligned}$$

and denote by $\Psi_{p,q}^1$ its associated Poisson transform and by $\tilde{\Psi}_{p,q}^1$ its induced G-equivariant operator defined on sections of the $(p+q)$-th complexified homology bundle over G/P. Note that by construction the complex conjugate of $\psi_{p,q}$ is also BGG-compatible, and we denote its corresponding Poisson transform and G-equivariant factorization to the homology bundle by $\Psi_{p,q}^2$ and $\tilde{\Psi}_{p,q}^2$, respectively.

THEOREM 4.3.2. *Let $G = \mathrm{SU}(n+1,1)$, $K \cong U(n+1)$ its maximal compact subgroup and P its minimal parabolic subgroup.*

(i) If $p+q \le n$, then the image of the operator $\tilde{\Psi}_{p,q}$ is contained in the space of harmonic differential forms on G/K.
(ii) For $p+q \ge n+1$, then both operators $\tilde{\Psi}_{p,q}^1$ and $\tilde{\Psi}_{p,q}^2$ have values in the space of harmonic differential forms on G/K.

PROOF. Follows immediately from Theorem 2.2.3. □

4.4. Properties of the BGG-compatible Poisson transforms

Following the construction of BGG-compatible Poisson kernels our next aim is to analyse the image of the corresponding Poisson transforms with respect to differential operators on G/K. For that, we will first analyse the image of the basic invariant forms under the K-Hodge star in 4.4.1, which will afterwards lead to explicit formulae for the K-codifferential as well as for the M-equivariant map $\mathcal{L}_K^*$ corresponding to the adjoint of the Lefschetz map. Finally, we use these in 4.4.3 to determine the properties of the images of the BGG-compatible Poisson transforms constructed in 4.3.2. Resulting from this, we will also determine a family of BGG-compatible Poisson transforms between real valued differential forms which has properties similar to Gaillard's operators on the real hyperbolic space.

4.4.1. The K-Hodge dual of the basic invariant forms. In order to obtain formulae for the K-codifferential and the operator $\mathcal{L}_K^*$ we have to analyse the image of the basic invariant forms under the K-Hodge star. Explicitly, if p, q and k are positive integers satisfying $0 \le p, q, k-p, k-q \le n$, it will be necessary to derive a formula for $\omega \wedge *_K \omega_j^{p,q;k}$, where ω is any of the M-invariant forms of degree 2 from section 4.2.1.

We will start by identifying the pullback of the Kähler form on G/K along the canonical projection $\pi_K\colon G/M \to G/K$, which corresponds to an M-invariant form on $\mathfrak{g}/\mathfrak{m}$ of degree $(2,0)$ and K-type $(1,1)$. In this way, we also obtain an expression for the K-volume form $\mathrm{vol}_K := *_K(1)$ on the horizontal subspace of $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$.

LEMMA 4.4.1. *Let $G = \mathrm{SU}(n+1,1)$, $K \subset G$ its maximal compact subgroup and $M \subset G$ the intersection of K and the minimal parabolic subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{m}$ be the Lie algebras of G, K and M, respectively. Then the pullback of the Kähler form of G/K to G/M corresponds to the Poisson kernel*

$$\omega_M := \frac{1}{2}\left(iZ^* \wedge \overline{Z}^* + \omega_K\right)$$

on $\mathfrak{g}/\mathfrak{m}$. In particular, the (complexified) K-volume form vol_K is given by

$$\mathrm{vol}_K = \frac{1}{2^{n+1}n!} iZ^* \wedge \overline{Z^*} \wedge \omega_K^n.$$

PROOF. We have seen in the proof of Proposition 4.1.4 that the complex structure J on $\mathfrak{p}/\mathfrak{m}$ induced by the isomorphism $\mathfrak{p}/\mathfrak{m} \cong \mathfrak{q}$ is given by the complex structure on the M-module $\mathfrak{g}_1/\mathfrak{m} \cong \mathbb{C}^n$ as well as the relations $JE = -2H$ and $JH = \frac{1}{2}E$. Since the Euclidean metric on $\mathfrak{p}/\mathfrak{m}$ is the pullback of the K-invariant metric on $\mathfrak{q}$ we deduce that the M-invariant form ω_M can be determined via $\omega_M(\xi,\eta) = g_M(J\xi,\eta)$ for all ξ, $\eta \in \mathfrak{p}/\mathfrak{m}$. Using the formulae for g_M from Proposition 4.1.4 a direct computation shows

$$\omega_M = \frac{1}{2}\left(iZ^* \wedge \overline{Z}^* + \omega_K\right).$$

In particular, we can express the K-volume form via $\mathrm{vol}_K = \frac{1}{(n+1)!}\omega_M^{n+1}$ by construction, and expanding this wedge product we obtain the claimed formula. □

In the next step we determine the main tool in the proof of Theorem 4.4.1 which is a formula expressing the M-invariant form $*_K\omega_j^{p,q;k}$ in terms of powers of the invariant 2-form ω_K. For that, note that the M-invariant form ω_P is of degree $(0,2)$ and thus commutes with the K-Hodge star. In particular, it suffices to consider the K-Hodge star of those basic invariant forms which do not contain the $(0,2)$-form ω_P as a factor. By Definition 4.2.2 this means that the parameters of $*_K\omega_j^{p,q;k}$ satisfy the relation $j = p+q-k \geq 0$. In particular, for any choice of p, q and k there is exactly one such M-invariant form.

PROPOSITION 4.4.1. *Let p, q and k be integers so that $0 \leq p, q, k-p, k-q \leq n$. Let $\boldsymbol{X}^{1,0}$ be a holomorphic $(k-p)$-vector and $\boldsymbol{Y}^{0,1}$ an antiholomorphic $(k-q)$-vector on $\mathbb{C}^n$, respectively. Then we have*

$$\iota_{G_{\boldsymbol{X}}^{1,0}}\iota_{G_{\boldsymbol{Y}}^{0,1}} *_K \omega_{p+q-k}^{p,q;k} = \varepsilon^{p,q;k} iZ^* \wedge \overline{Z^*} \wedge \iota_{F_{\boldsymbol{X}}^{1,0}}\iota_{F_{\boldsymbol{Y}}^{0,1}}\omega_K^{n-(p+q)+k}, \tag{4.4.1}$$

where the factor $\varepsilon^{p,q;k}$ is given by

$$\varepsilon^{p,q;k} = (-1)^{\frac{1}{2}(p+q)(p+q+1)-(p+q-k)}2^{p+q-(n+1)}\frac{(p+q-k)!(k-p)!(k-q)!}{(n-(p+q)+k)!}.$$

PROOF. We will prove this formula by induction on the P-type of the M-invariant form $*_K\omega_{p+q-k}^{p,q;k}$. If the P-type equals $(0,0)$, then $k=p=q$ and thus the interior products in the claimed formula disappear. Therefore, the left hand side becomes $*_K\omega_k^{k,k;k}$, which by Lemma 4.2.2(iii) equals $*_K\omega_k^k$. Moreover, from the defining equation for the K-Hodge star and the expression for the K-volume form in Lemma 4.4.1 we deduce that

$$*_K\omega_K^k = \frac{k!}{(n-k)!}2^{2k-n-1}iZ^* \wedge \overline{Z}^* \wedge \omega_K^{n-k}.$$

Now assume that the formula is proven for the invariant form $*_K\omega_{p+q-k}^{p,q;k}$ of P-type $(k-p,k-q)$ and consider the form $*_K\omega_{p+q-k}^{p,q+1;k+1}$, which by definition is of P-type $(k-p+1,k-q)$. A direct computation using Lemma 4.2.1 shows that

$$\iota_{G_Y^{1,0}} *_K \omega_{p+q-k}^{p,q+1;k+1} = (-1)^{p+q+1}2(k-p+1)\iota_{F_Y^{1,0}} *_K \omega_{p+q-k}^{p,q;k}$$

for all $Y \in \mathbb{C}^n$, and we apply this equation to (4.4.1) for the interior product with $G^{1,0}_{Y_1}$. Next, we can move the resulting interior product with $F^{1,0}_{Y_1}$ to the left, apply the induction hypothesis and finally use the relation

$$(-1)^{p+q+1} 2(k-p+1)\varepsilon^{p,q;k} = \varepsilon^{p,q+1;k+1},$$

thereby obtaining (4.4.1). Since our formula is symmetric in p and q the same line of arguments can be applied to the Poisson kernel $*_K \omega^{p+1,q;k+1}_{p+q-k}$ of P-degree $(k-p, k-q+1)$. □

Using Proposition 4.4.1 we are now able to present the main result of this section which plays an essential role in the derivation of explicit formulae for several M-equivariant operators in the following section. However, note that the proof of Theorem 4.4.1 is very technical and involves long computations.

THEOREM 4.4.1. *Let p, q and k be integers so that $0 \leq p, q, k-p, k-q \leq n$. Then we have:*

(i) $\omega_{\mathrm{tw}} \wedge *_K \omega^{p,q;k}_j = (-1)^{p+q} 2i \left(j *_K \omega^{p,q-1;k}_{j-1} + (q-j) *_K \omega^{p,q-1;k}_j \right).$

(ii) $\overline{\omega_{\mathrm{tw}}} \wedge *_K \omega^{p,q;k}_j = (-1)^{p+q+1} 2i \left(j *_K \omega^{p-1,q;k}_{j-1} + (p-j) *_K \omega^{p-1,q;k}_j \right).$

(iii)
$$\begin{aligned} \omega_K \wedge *_K \omega^{p,q;k}_j &= 4j(n+1-(p+q)+j) *_K \omega^{p-1,q-1;k-1}_{j-1}. \\ &\quad - 4(p-j)(q-j) *_K \omega^{p-1,q-1;k-1}_j \end{aligned}$$

PROOF. In all cases we can move the Poisson kernels ω_P of degree $(0,2)$ outside of the K-Hodge star, so it suffices to prove the formulae in the case that those forms do not appear. Otherwise said, the parameters on the left hand sides of (i) - (iii) satisfy the relation $j = p+q-k$. Note that in this way, Proposition 4.4.1 is applicable.

(i) Let $\mathbf{X}^{1,0}$ and $\mathbf{Z}^{1,0}$ be holomorphic vectors on $\mathbb{C}^n$ of length $n-q+1$ and $k-p$, respectively, and let $\mathbf{Y}^{0,1}$ and $\mathbf{W}^{0,1}$ be antiholomorphic vectors of length $n-p$ and $k-q+1$, respectively. Our aim is to rewrite the expression

$$\left(\omega_{\mathrm{tw}} \wedge *_K \omega^{p,q;k}_{p+q-k}\right)(Z, \overline{Z}, F^{1,0}_{\mathbf{X}}, F^{0,1}_{\mathbf{Y}}, G^{1,0}_{\mathbf{Z}}, G^{0,1}_{\mathbf{W}}) = (*)$$

as the image of a basic invariant form under the K-Hodge star.

In order to do so, we first expand the wedge product by inserting the vectors $F^{1,0}_{X_s}$ and $G^{0,1}_{W_t}$ into ω_{tw}, where we pick up the signs $(-1)^{s+1}$ from the first vector and $(-1)^{t-k+q+1}$ from the second vector. Next, by Lemma 4.2.1 the expression $\omega_{\mathrm{tw}}(F^{1,0}_{X_s}, G^{0,1}_{W_t})$ coincides with $-i\omega_K(F^{1,0}_{X_s}, F^{0,1}_{W_t})$, and we apply Proposition 4.4.1 to $*_K \omega^{p,q;k}_{p+q-k}$, which adds the coefficient $i\varepsilon^{p,q;k}$. In this way we obtain that

$$(*) = \sum_{s,t} (-1)^{k-q+s+t} \varepsilon^{p,q;k} \omega_K(F^{1,0}_{X_s}, F^{0,1}_{W_t}) \omega_K^{n-(p+q)+k} \left(F^{1,0;s}_{\mathbf{X}}, F^{0,1}_{\mathbf{Y}}, F^{1,0}_{\mathbf{Z}}, F^{0,1;t}_{\mathbf{W}} \right),$$

where the wedge product $F^{1,0;s}_{\mathbf{X}}$ of length $(n-q)$ is obtained from $F^{1,0}_{\mathbf{X}}$ by omitting the s-th factor, and similarly for $F^{0,1;t}_{\mathbf{W}}$.

Using that the Poisson kernel ω_K is of K-type $(1,1)$, an easy computation shows that $(*)$ splits into the sum $A+B$, where

$$A := \frac{k-q+1}{n-(p+q)+k+1} \varepsilon^{p,q;k} \omega_K^{n-(p+q)+k+1} (F^{1,0}_{\mathbf{X}}, F^{0,1}_{\mathbf{Y}}, F^{1,0}_{\mathbf{Z}}, F^{0,1}_{\mathbf{W}})$$

$$B := \sum_{r=1}^{k-p} \sum_{t=1}^{k-q+1} (-1)^{k-p+r+t} \varepsilon^{p,q;k} \omega_K(F^{1,0}_{Z_r}, F^{0,1}_{W_t}) \omega_K^{n-(p+q)+k} (F^{1,0}_{\mathbf{X}}, F^{0,1}_{\mathbf{Y}}, F^{1,0;r}_{\mathbf{Z}}, F^{0,1;t}_{\mathbf{W}}).$$

Our aim is to manipulate the expressions A and B so that the occurring Poisson kernels have the same types and degrees as $\omega_{\mathrm{tw}} \wedge *_K\omega_{p+q-k}^{p,q;k}$.

Since A is given by a power of ω_K we can immediately apply Proposition 4.4.1, which allows us to write it as a multiple of the kernel $*_K\omega_{p+q-k-1}^{p,q-1;k}$ evaluated in the same arguments as $(*)$. Moreover, for the resulting coefficient of A we obtain by definition that

$$\frac{k-q+1}{n-(p+q)+k+1}\ \varepsilon^{p,q;k}\left(i\varepsilon^{p,q-1;k}\right)^{-1} = (-1)^{p+q}2i(p+q-k),$$

so we see that

$$A = (-1)^{p+q}2i(p+q-k)\left(*_K\omega_{p+q-k-1}^{p,q-1;k}\right)(Z,\overline{Z},F_{\mathbf{X}}^{1,0},F_{\mathbf{Y}}^{0,1},G_{\mathbf{Z}}^{1,0},G_{\mathbf{W}}^{0,1}).$$

In order to bring B into the right shape, we first apply Proposition 4.4.1 to change the power of ω_K to the Poisson kernel $*_K\omega_{p+q-k}^{p,q-1;k-1}$. The resulting coefficient can be simplified using

$$\varepsilon^{p,q;k}\left(i\varepsilon^{p,q-1;k-1}\right)^{-1} = (-1)^{p+q+1}2i(k-p).$$

Furthermore, we use Lemma 4.2.1 to replace $\omega_K(F_{Z_r}^{1,0},F_{W_t}^{0,1})$ with the expression $\omega_P(G_{Z_r}^{1,0},G_{W_t}^{0,1})$. Since the Poisson kernel ω_P is of degree $(0,2)$ it can be moved into the K-Hodge star, so by Lemma 4.2.2 we obtain

$$B = (-1)^{p+q}2i(k-p)\left(*_K\omega_{p+q-k}^{p,q-1;k}\right)(Z,\overline{Z},F_{\mathbf{X}}^{1,0},F_{\mathbf{Y}}^{0,1},G_{\mathbf{Z}}^{1,0},G_{\mathbf{W}}^{0,1}).$$

Since both expressions A and B have the same arguments as $(*)$, the corresponding M-invariant forms have to coincide, proving part (i).

(ii) Since the complex conjugate of $*_K\omega_j^{p,q;k}$ is $*_K\omega_j^{q,p;k}$ the formula follows immediately from (i).

(iii) Let $\mathbf{X}^{1,0}$ and $\mathbf{Z}^{1,0}$ be holomorphic vectors on $\mathbb{C}^n$ of length $n-q+1$ and $k-p$, respectively, and $\mathbf{Y}^{0,1}$ and $\mathbf{W}^{0,1}$ be antiholomorphic vectors on $\mathbb{C}^n$ of length $n-p+1$ and $k-q$, respectively. We proceed similar as in (i) and consider

$$\left(\omega_K \wedge *_K\omega_{p+q-k}^{p,q;k}\right)(Z,\overline{Z},F_{\mathbf{X}}^{1,0},F_{\mathbf{Y}}^{0,1},G_{\mathbf{Z}}^{1,0},G_{\mathbf{W}}^{0,1}) = (**),$$

which we want to rewrite as the image of the K-Hodge star using Proposition 4.4.1.

First, we expand the wedge product in $(**)$ by inserting the vectors $F_{X_r}^{1,0}$ and $F_{Y_s}^{0,1}$ into ω_K, which adds the sign $(-1)^{n-q+r+s}$. Next, we apply Proposition 4.4.1, thereby changing $*_K\omega_{p+q-k}^{p,q;k}$ to a multiple of $\omega_K^{n-(p+q)-k}$. All in all we obtain that

$$(**) = \sum_{r,s}(-1)^{n-q+r+s}i\varepsilon^{p,q;k}\omega_K(F_{X_r}^{1,0},F_{Y_s}^{0,1})\omega_K^{n-(p+q)-k}\left(F_{\mathbf{X}}^{1,0;r},F_{\mathbf{Y}}^{0,1;s},G_{\mathbf{Z}}^{1,0},G_{\mathbf{W}}^{0,1}\right).$$

By a direct computation we can write $(**)$ as the sum $C+D$, where

$$C := \frac{n-q+1}{n-(p+q)+k+1}i\varepsilon^{p,q;k}\omega_K^{n-(p+q)+k+1}(F_{\mathbf{X}}^{1,0},F_{\mathbf{Y}}^{0,1},F_{\mathbf{Z}}^{1,0},F_{\mathbf{W}}^{0,1}),$$
$$D := \sum_{r,t} i\varepsilon^{p,q;k}(-1)^{k-q+r+t+1}\omega_K(F_{X_r}^{1,0},F_{W_t}^{0,1})\omega_K^{n-(p+q)+k}(F_{\mathbf{X}}^{1,0;r},F_{\mathbf{Y}}^{0,1},F_{\mathbf{Z}}^{1,0},F_{\mathbf{W}}^{0,1;t}),$$

and we want to rewrite these two expression so that the occurring Poisson kernels have the same degrees and types as our initial form.

First, we can immediately apply Proposition 4.4.1 to expression C, and the resulting coefficients simplify via

$$\frac{n-q+1}{n-(p+q)+k+1} i\varepsilon^{p,q;k}\left(i\varepsilon^{p-1,q-1;k-1}\right)^{-1} = 4(n-q+1)(p+q-k).$$

On the other hand, the expression D has the same form as expression $(*)$ from part (i), and using

$$i\varepsilon^{p,q;k}\left(\varepsilon^{p-1,q;k-1}\right)^{-1} = (-1)^{p+q}2i(k-q)$$

we obtain via Proposition 4.4.1 that

$$D = (-1)^{p+q+1}2i(k-q)\left(\omega_{\mathrm{tw}} \wedge *_K\omega^{p-1,q;k-1}_{p+q-k}\right)(Z, \overline{Z}, F^{1,0}_{\mathbf{X}}, F^{0,1}_{\mathbf{Y}}, G^{1,0}_{\mathbf{Z}}, G^{0,1}_{\mathbf{W}}).$$

All in all we get

$$\begin{aligned}\omega_K \wedge *_K\omega^{p,q;k}_{p+q-k} &= 4(n-q+1)(p+q-k) *_K \omega^{p-1,q-1;k-1}_{p+q-k-1} \\ &\quad + (-1)^{p+q+1}2i(k-q)\omega_{\mathrm{tw}} \wedge *_K\omega^{p-1,q;k-1}_{p+q-k},\end{aligned}$$

so the claimed formula follows from (i). □

As an immediate Corollary we obtain that every M-invariant form $*_K\omega^{p,q;k}_j$ can be expressed as a linear combination of basic invariant forms. In particular, for the construction of BGG-compatible Poisson transforms in section 4.3 it sufficed to only consider those operators induced by wedge products of the basic invariant forms.

4.4.2. The invariant operators from G/K. In the next step we compute explicit formulae for the images of the basic invariant forms under M-equivariant maps which are induced by certain G-equivariant differential operators on G/K. These will be needed in section 4.3 to determine the geometric behaviour of the images of the BGG-compatible Poisson transforms constructed in section 4.3.2.

First, recall from section 4.2.1 that the decomposition of Poisson kernels into K-types induces a splitting of d_K into the sum of the two operators ∂_K and $\overline{\partial}_K$, which map Poisson kernels of K-type (p,q) to those of K-type $(p+1,q)$ and $(p,q+1)$, respectively. Geometrically, this corresponds to the decomposition $d = \partial + \overline{\partial}$ of the exterior derivative on G/K as in section 4.1.2. Indeed, the same arguments as in Proposition 2.2.3 show that if Ψ is any Poisson transform with kernel ψ, then $\partial \circ \Psi$ is again a Poisson transform with kernel $\partial_K\psi$, and similarly for its complex conjugate. Furthermore, the operators ∂_K and $\overline{\partial_K}$ are again antiderivations, square to zero, anticommute with each other and are related via $\overline{\partial}_K\overline{\psi} = \overline{\partial_K\psi}$ for all M-invariant forms ψ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$. In particular, we can compute the image of the basic invariant forms under these operators using the corresponding formulae for the Poisson kernels of low degree from section 4.2.1.

Next, we want to compute the K-codifferential of the basic invariant forms $\omega^{p,q;k}_j$. Since the dimension of the symmetric space G/K is even we obtain by definition that $\delta_K = - *_K d_K *_K$. Furthermore, the decomposition of the K-derivative induces a splitting of δ_K into $\partial^*_K + \overline{\partial}^*_K$, where $\partial^*_K = - *_K \overline{\partial}_K *_K$ maps M-invariant forms on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$ of K-type (p,q) to those of K-type $(p-1,q)$, and similarly for $\overline{\partial}^*_K$. In particular, since the K-Hodge star is a complex linear map these operators are again related via $\overline{\partial^*_K\psi} = \overline{\partial}^*_K\overline{\psi}$ for all M-invariant elements $\psi \in \Lambda^*(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$.

In section 4.3 we constructed BGG-compatible Poisson kernels as a linear combination of wedge products of an odd number of M-invariant forms of degree 1 and a basic invariant form $\omega^{p,q;k}_j$ for specific choices of p, q and k. We will determine formulae for the K-codifferential of these M-invariant forms in the following Theorem.

THEOREM 4.4.2. *Let p, q and k be integers so that $0 \leq p, q, k-p, k-q \leq n$ and let j be in the admissible rangle. Then we have*

$$\begin{aligned}\partial_K^*\big(Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}\big) = {} & 2(n-k)\overline{Z}^* \wedge \omega_j^{p,q;k} \\ & 2ij(k-(p+q)+j)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_{j-1}^{p-1,q;k-1} \\ & -2i(p-j)(n-k+p-j+1)I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1}.\end{aligned}$$

In particular, for the wedge products of a basic invariant form with an odd number of M-invariant 1-forms this implies that

$$\begin{aligned}\partial_K^*(Z^* \wedge \omega_j^{p,q;k}) = {} & 2(n+1-k)\omega_j^{p,q;k} \\ & + 2ij(n+1-(p+q)+j)Z^* \wedge \overline{Z}^* \wedge \omega_{j-1}^{p-1,q-1;k-1} \\ & - 2i(p-j)(q-j)Z^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;k-1} \\ & + 2ij(k-(p+q)+j)I^* \wedge Z^* \wedge \omega_{j-1}^{p-1,q;k-1} \\ & - 2i(p-j)(n-k+q-j+1)I^* \wedge Z^* \wedge \omega_j^{p-1,q;k-1}, \\ \partial_K^*(\overline{Z}^* \wedge \omega_j^{p,q;k}) = {} & 2ij(k-(p+q)+j)I^* \wedge \overline{Z}^* \wedge \omega_{j-1}^{p-1,q;k-1} \\ & - 2i(p-j)(n-k+q-j+1)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q;k-1},\end{aligned}$$

as well as

$$\begin{aligned}\partial_K^*(I^* \wedge \omega_j^{p,q;k}) = {} & 2ij(n+1-(p+q)+j)I^* \wedge \overline{Z}^* \wedge \omega_{j-1}^{p-1,q-1;k-1} \\ & - 2i(p-j)(q-j)I^* \wedge \overline{Z}^* \wedge \omega_j^{p-1,q-1;k-1}, \\ \partial_K^*(I^* \wedge Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}) = {} & 2(k-n+1)I^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}.\end{aligned}$$

PROOF. In order to compute the image of $\omega := Z^* \wedge \overline{Z}^* \wedge \omega_j^{p,q;k}$ under ∂_K^* we have to first apply the K-Hodge star and afterwards the operator $\overline{\partial_K}$, which we want to determine using formula (1.2.4) for the infinitesimal version of the covariant exterior derivative. For X, $Y \in \mathbb{C}^n$ let ξ_X and η_Y be the representatives of F_X and G_Y in $\mathfrak{p}$ and $\mathfrak{k}$, respectively, whose $\mathfrak{m}$-part is trivial. Then if we denote the representatives of Z, $\overline{Z}$ and I with trivial $\mathfrak{m}$-part by the same symbol, the bracket relations between these vectors are given by

$$[Z,\xi_X] = \frac{1}{2}\xi_X, \qquad [Z,\eta_Y^{1,0}] = 2\xi_Y^{1,0} - \frac{1}{2}\eta_Y^{1,0}, \qquad [I,\xi_X^{1,0}] = i(\xi_X^{1,0} - \eta_X^{1,0}).$$

for all X, $Y \in \mathbb{C}^n$ via the formulae at the end of section 4.1.1. Since the Poisson kernel $*_K\omega$ is trivial upon insertion of any invariant vector in $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$, formula (1.2.4) together with these bracket relations imply that

$$\overline{\partial}_K *_K \omega = \overline{Z}^* \wedge \iota_{\overline{Z}}\overline{\partial}_K *_K \omega + I^* \wedge \iota_I \overline{\partial}_K *_K \omega.$$

We will first compute the expression for the special case that the M-invariant form ω_P does not appear as a factor in $\omega_j^{p,q;k}$. By Definition 4.2.2, this means that $j = p+q-k$ and that Theorem 4.4.1 and Proposition 4.4.1 are applicable. Afterwards, we will prove the general case by using the antiderivation property of the K-derivative.

(i) Therefore, assume for now that $j = p+q-k \geq 0$. If we insert the bracket relations from above into the formula (1.2.4) for the exterior derivative we immediately obtain that

$$\iota_{\overline{Z}}\overline{\partial}_K *_K \omega = (k-n) *_K \omega.$$

Thus, it remains to determine $\iota_I\overline{\partial}_K *_K \omega$. Since the Lie bracket with the vector I preserves holomorphic and antiholomorphic vectors, it suffices to consider the case of inserting $k-q-1$ antiholomorphic vectors in $\mathbb{L}_{\mathbb{C}} \subset (\mathfrak{k}/\mathfrak{m})_{\mathbb{C}}$ into this form. So let $\mathbf{X}^{1,0}$ and $\mathbf{Z}^{1,0}$ be holomorphic vectors on $\mathbb{C}^n$ of length $n-p$ and $k-p$, respectively, and let $\mathbf{Y}^{0,1}$ and $\mathbf{W}^{0,1}$ be antiholomorphic vectors on $\mathbb{C}^n$ of length $n-q+1$ and $k-q-1$, respectively. Then the formula (1.2.4) of the infinitesimal version of the exterior derivative yields

$$\begin{aligned}(\iota_I\overline{\partial}_K *_K \omega)(F^{1,0}_{\mathbf{X}},F^{0,1}_{\mathbf{Y}},G^{1,0}_{\mathbf{Z}},G^{0,1}_{\mathbf{W}}) &= \sum_\ell (-1)^{n-p+\ell}\,(*_K\omega)\left(G^{0,1}_{iY_\ell},F^{1,0}_{\mathbf{X}},F^{0,1;\ell}_{\mathbf{Y}},G^{1,0}_{\mathbf{Z}},G^{0,1}_{\mathbf{W}}\right)\\ &= (*),\end{aligned}$$

where $F^{0,1;\ell}_{\mathbf{Y}}$ is obtained from the wedge product $F^{0,1}_{\mathbf{Y}}$ by omitting the ℓ-th factor. From Lemma 4.2.1 it follows that

$$\iota_{G^{0,1}_Y} *_K \omega = (-1)^{p+q}2(k-q)\iota_{F^{0,1}_Y}\left(Z^*\wedge\overline{Z}^*\wedge *_K\omega^{p-1,q;k-1}_{p+q-k}\right),$$

and inserting this into $(*)$ and moving the vector $F^{0,1}_{Y_\ell}$ to the right place shows

$$\iota_I\overline{\partial}_K *_K\omega = (-1)^{p+q+1}2i(k-q)(n-q+1)*_K\left(Z^*\wedge\overline{Z}^*\wedge\omega^{p-1,q;k-1}_{p+q-k}\right).$$

All in all we have determined the expression $\overline{\partial}_K *_K \omega$, and applying the negative of the K-Hodge star yields

$$\begin{aligned}\partial^*_K\omega &= 2(n-k)\overline{Z}^*\wedge\omega^{p,q;k}_j\\ &\quad - 2i(k-q)(n-q+1)I^*\wedge Z^*\wedge\overline{Z}^*\wedge\omega^{p-1,q;k-1}_j.\end{aligned}$$

(ii) In the general case we use Lemma 4.2.2 to decompose our Poisson kernel into the wedge product

$$\omega = \left(Z^*\wedge\overline{Z}^*\wedge\omega^{p,q;p+q-j}_j\right)\wedge\omega_P^{k-(p+q)+j},$$

where the first factor on the right hand side is of the same form as in (i). If we apply the definition of ∂^*_K to this decomposition we can first move the power of ω_P outside of the K-Hodge star since this kernel is of degree $(0,2)$. Next, being an antiderivation we can distribute the operator $\overline{\partial}_K$ over the wedge product, where the image of the first factor was computed in (i), whereas the image of the power of ω_P can be computed via the formulae in section 4.2.1. Afterwards, we use Theorem 4.4.1(ii) to write the resulting Poisson kernel as an image of $*_K$ and finally apply Corollary 4.1.4(i) to move the M-invariant 1-forms outside of the K-Hodge star.

The formulae for the images of the other Poisson kernels under ∂^*_K follow by combining the above with the calculus on $\mathfrak{g}/\mathfrak{m}$ from Corollary 4.1.4 and the K-derivative of the Poisson kernels of low degree from 4.2.1. As an example, we compute the K-codifferential of the Poisson kernel $\overline{Z}^*\wedge\omega^{p,q;k}_j$. In order to do so, first rewrite this form as $\iota_Z\omega$ and apply the K-Hodge star, which equals $\frac{1}{2}(-1)^{p+q}\overline{Z}^*\wedge *_K\omega$ using Corollary 4.1.4 and the relation $2Z^\flat = \overline{Z}^*$. Next, we apply the operator $\overline{\partial}_K$, where we make use of the fact that $\overline{\partial}_K\overline{Z}^* = 0$ (c.f. section 4.2.1). All in all we obtain

$$\partial^*_K\left(\overline{Z}^*\wedge\omega^{p,q;k}_j\right) = \frac{1}{2}(-1)^{p+q}*_K\left(\overline{Z}^*\wedge\overline{\partial}_K *_K\omega\right) = -\iota_Z\partial^*_K\omega,$$

where we applied Corollary 4.1.4(i) for the last equality. □

Finally, recall from section 4.1.2 that we introduced the Lefschetz map $\mathcal{L}$ on differential forms on G/K as the wedge product with the Kähler form ω. Analogously, we define the K-Lefschetz map $\mathcal{L}_K\colon \Omega^k(G/M)\to\Omega^{k+2}(G/M)$ as the wedge product with $\pi^*_K\omega$, where π_K is the canonical projection $G/M\to G/K$ as before.

By construction, this map is G-equivariant, and we denote the induced M-invariant operator on $\Lambda^*(\mathfrak{g}/\mathfrak{m})^*$ as well as on its complexification by the same symbols. Furthermore the image of an M-invariant form of K-type (p,q) under $\mathcal{L}_K$ is of K-type $(p+1,q+1)$. Geometrically, if Ψ is a Poisson transform with Poisson kernel ψ, then the transform associated to the kernel $\mathcal{L}_K\psi$ coincides with the composition $\mathcal{L}\circ\Psi$, which can be shown using the same arguments as in the proof of Proposition 2.2.3.

Subsequently, we defined in section 4.1.2 the G-equivariant operator $\mathcal{L}^*$ as the adjoint of the Lefschetz map with respect to the Hermitian product, which can also be computed using the Hodge star operator on G/K. By naturality, we obtain a corresponding G-equivariant operator $\mathcal{L}_K^*\colon \Omega^k(G/M)\to\Omega^{k-2}(G/M)$ by $\mathcal{L}_K^* = *_K^{-1}\mathcal{L}_K *_K^{-1}$, and we denote its induced M-equivariant map by the same symbol. By construction, if Ψ is a Poisson transform with Poisson kernel ψ, then the transform associated to the kernel $\mathcal{L}_K^*\psi$ coincides with the composition $\mathcal{L}^*\circ\Psi$,

Proposition 4.4.2. *Let p,q and k be integers so that $0\le p,q,k-p,k-q\le n$ and let j be in the admissible range. Then the image of the corresponding basic invariant under the operator $\mathcal{L}_K^*$ is given by*

$$\mathcal{L}_K^*\omega_j^{p,q;k} = 2j(n+1-(p+q)+j)\omega_{j-1}^{p-1,q-1;k-1} - 2(p-j)(q-j)\omega_j^{p-1,q-1;k-1}.$$

Furthermore, we have

$$\mathcal{L}_K^*(Z^*\wedge\omega_j^{p,q;k}) = Z^*\wedge\mathcal{L}_K^*\omega_j^{p,q;k} \qquad \mathcal{L}_K^*(\overline{Z}^*\wedge\omega_j^{p,q;k}) = \overline{Z}^*\wedge\mathcal{L}_K^*\omega_j^{p,q;k}$$
$$\mathcal{L}_K^*(I^*\wedge\omega_j^{p,q;k}) = I^*\wedge\mathcal{L}_K^*\omega_j^{p,q;k}$$

as well as

$$\mathcal{L}_K^*(I^*\wedge Z^*\wedge\overline{Z}^*\wedge\omega_j^{p,q;k}) = I^*\wedge Z^*\wedge\overline{Z}^*\wedge\mathcal{L}_K^*\omega_j^{p,q;k} - 2iI^*\wedge\omega_j^{p,q;k}.$$

Proof. Recall from Lemma 4.4.1 that the pullback of the Kähler form on G/K along the canonical projection π_K corresponds to the M-invariant form

$$\omega_M = \frac{1}{2}(iZ^*\wedge\overline{Z}^* + \omega_K),$$

which is by definition of degree $(2,0)$ and K-type $(1,1)$. Therefore, if we apply the operator $\mathcal{L}_K^* = *_K^{-1}\mathcal{L}_K *_K$ to any M-invariant form ψ on $(\mathfrak{g}/\mathfrak{m})_{\mathbb{C}}$, then from the calculus on $\mathfrak{g}/\mathfrak{m}$ in Corollary 4.1.4(i) and Proposition 4.2.1 we obtain that

$$*_K^{-1}\left(Z^*\wedge\overline{Z}^*\wedge *_K\psi\right) = 4\iota_Z\iota_{\overline{Z}}\psi.$$

On the other hand, we can express the wedge product $\omega_K\wedge *_K\psi$ again as an image under the K-Hodge star using Theorem 4.4.1(iii).

For $\psi=\omega_j^{p,q;k}$ the interior product with Z and $\overline{Z}$ is trivial, so the formula for $\mathcal{L}_K^*\psi$ reduces to

$$\begin{aligned}\mathcal{L}_K^*\omega_j^{p,q;k} &= \frac{1}{2}*_K^{-1}\left(\omega_K\wedge *_K\omega_j^{p,q;k}\right)\\ &= 2j(n+1-(p+q)+j)\omega_{j-1}^{p-1,q-1;k-1} - 2(p-j)(q-j)\omega_j^{p-1,q-1;k-1}.\end{aligned}$$

The expressions for the other Poisson kernels follow analogously. □

4.4.3. The image of the BGG-compatible Poisson kernels. Using the formulae from section 4.4.2 we are able to derive properties of the composition of the BGG-compatible Poisson transforms from section 4.3 by applying the corresponding M-equivariant operators to their Poisson kernels.

Proposition 4.4.3. *For all $0\le p,q\le n$ let $\psi_{p,q}$ be the BGG-compatible Poisson kernel defined in section 4.3.2.*

(i) If $p+q\leq n$, then the Poisson kernels $\psi_{p,q}$ satisfy

$$\delta_K\psi_{p,q}=0, \qquad \mathcal{L}_K^*\psi_{p,q}=0.$$

Furthermore, their partial derivatives are given by the formulae

$$\begin{aligned}\partial_K\psi_{p,q}&=(n+1-(p+q))Z^*\wedge\psi_{p,q},\\ d_P\psi_{p,q}&=2i\Big((n+2-p)Z^*\wedge\psi_{p-1,q}-(n+2-q)\overline{Z}^*\wedge\psi_{p,q-1}\Big).\end{aligned}$$

(ii) If $p+q\geq n+1$, then the Poisson kernel $\psi_{p,q}$ defined in section 4.3.2 satisfy

$$\partial_K\psi_{p,q}=0, \qquad \overline{\partial}_K^*\psi_{p,q},=0 \qquad \mathcal{L}_K\psi_{p,q}=0.$$

Furthermore, the K-derivative of $\psi_{p,q}$ is given by

$$d_K\psi_{p,q}=(n-(p+q))\overline{Z}^*\wedge\psi_{p,q}-(q+2)I^*\wedge\psi_{p,q+1},$$

whereas its P-derivative satisfies for $p+q\geq n+2$ that

$$d_P\psi_{p,q}=\frac{(n+1-q)}{p+q-n}2i\Big((n+1-(p+q))\overline{Z}^*\wedge\psi_{p,q-1}-(q+1)I^*\wedge\psi_{p,q}\Big),$$

and for $p+q=n+1$ that

$$d_P\psi_{p,q}=2ip\Big((q+1)Z^*\wedge\psi_{p-1,q}-(p+1)\overline{Z}^*\wedge\psi_{p,q-1}\Big).$$

Finally, its K-codifferential is given by

$$\begin{aligned}\delta_K\psi_{p,q}=&\ 2(p+q-n)\left(\iota_Z\psi_{p,q}+(q+2)p^{-1}\iota_{\overline{Z}}\psi_{p-1,q+1}\right)\\ &-2p^{-1}(p+q-(n+1))^2\iota_I\psi_{p-1,q}.\end{aligned}$$

Proof. (i) Recall from section 4.3.2 in the case $0\leq p+q\leq n$ the Poisson kernel $\psi_{p,q}$ was defined by

$$\psi_{p,q}=\sum_{j\in\mathbb{Z}}\kappa_j^{p,q;n+1}I^*\wedge\omega_j^{p,q;n}.$$

First, using the formulae for the holomorphic part of the K-derivative of the invariant forms of low degree we immediately obtain that

$$\partial_K\psi_{p,q}=(n+1-(p+q))Z^*\wedge\psi_{p,q}.$$

Similarly, from the formulae for the P-derivatives of low degree and the relation

$$(j+1)\kappa_{j+1}^{p,q;n+1}+(p-j)\kappa_j^{p,q;n+1}=(n+2-p)\kappa_j^{p-1,q;n+1}$$

we deduce that

$$\begin{aligned}d_P\psi_{p,q}=&\sum_{j\in\mathbb{Z}}\kappa_j^{p,q;n+1}\omega_j^{p,q;n+1}\\ &+2i\Big((n+2-p)Z^*\wedge\psi_{p-1,q}-(n+2-q)\overline{Z}^*\wedge\psi_{p,q-1}\Big),\end{aligned}$$

and the first sum is trivial due to Theorem 4.2.2.

Next, we apply the formulae for the operator ∂_K^* from Theorem 4.4.2 and shift the index j appropriately, obtaining that

$$\partial_K^*\psi_{p,q}=2i\sum_{j\in\mathbb{Z}}\tilde{\kappa}_jI^*\wedge\overline{Z}^*\wedge\omega_j^{p-1,q-1;n-1},$$

where the coefficients are given by

$$\tilde{\kappa}_j=(j+1)(n+1-(p+q)+j+1)\kappa_{j+1}^{p,q;n+1}-(p-j)(q-j)\kappa_j^{p,q;n+1}.$$

However, due to the linear recursion (κ) from Proposition 4.2.2 it follows that $\tilde{\kappa}_j = 0$ for all $j \in \mathbb{Z}$, hence $\partial_K^* \psi_{p,q} = 0$. Furthermore, the complex conjugate of $\psi_{p,q}$ equals $\psi_{q,p}$, which implies

$$\overline{\partial_K^*}\psi_{p,q} = \overline{\partial_K^* \psi_{q,p}} = 0$$

and therefore also $\delta_K \psi_{p,q} = 0$.

Finally, using the formulae for the adjoint of the K-Lefschetz map from Proposition 4.4.2 and shifting the index j appropriately we obtain in the same way as above that

$$\mathcal{L}_K^* \psi_{p,q} = 2\sum_{j\in\mathbb{Z}} \tilde{\kappa}_j I^* \wedge \omega_j^{p-1,q-1;n-1} = 0.$$

(ii) If $p+q \geq n+1$, then the Poisson kernel $\psi_{p,q}$ was defined in section 4.3.2 by

$$\begin{aligned}\psi_{p,q} = &\sum_j \kappa_{j+1}^{p,q+1;n+1} Z^* \wedge \omega_j^{p-1,q;n} + \frac{p+1}{p+q-n}\kappa_{j+1}^{p+1,q+1;n+1} I^* \wedge \omega_j^{p,q;n} \\ &- i\frac{p+1}{p+q-n}\kappa_{j+2}^{p+1,q+1;n+1} I^* \wedge Z^* \wedge \overline{Z^*} \wedge \omega_j^{p-1,q-1;n-1}.\end{aligned}$$

In order to determine its image under the operators ∂_K and ∂_K^* we will rewrite this form as

$$\psi_{p,q} = Z^* \wedge \vartheta_{p-1,q} + \frac{p+1}{p+q-n} I^* \wedge \vartheta_{p,q},$$

where

$$\vartheta_{p,q} := \sum_{j\in\mathbb{Z}} \kappa_{j+1}^{p+1,q+1;n+1}(\omega_K - iZ^* \wedge \overline{Z}^*) \wedge \omega_{j-1}^{p-1,q-1;n-1}.$$

Using the formulae for the partial derivatives of the invariant forms of low degree in section 4.2.1 as well as the definition of the coefficients $\kappa_j^{p,q;k}$ a direct computation shows that

$$\begin{aligned}\partial_K(Z^* \wedge \vartheta_{p-1,q}) &= -(p+1)Z^* \wedge \psi_{p,q}, \\ \partial_K\left(\frac{p+1}{p+q-n} I^* \wedge \vartheta_{p,q}\right) &= (p+1)Z^* \wedge \psi_{p,q},\end{aligned}$$

hence by linearity it follows that $\partial_K \psi_{p,q} = 0$.

For the antiholomorphic part of K-derivative of $\psi_{p,q}$ we apply the formulae in section 4.2.1 as well as the relation

$$(n+1-(p+q)+j)\kappa_{j+1}^{p,q+1;n+1} + (p-j)\kappa_j^{p,q+1;n+1} = (q+2)\kappa_{j+1}^{p,q+2;n+1}$$

to obtain

$$\begin{aligned}\overline{\partial}_K(Z^* \wedge \vartheta_{p-1,q}) = &(n-(p+q))\overline{Z}^* \wedge \psi_{p,q} - (q+2)I^* \wedge \psi_{p,q+1} \\ &- (p+1)\sum_{j\in\mathbb{Z}} \kappa_{j+1}^{p+1,q+1;n+1} I^* \wedge \overline{Z}^* \wedge \psi_{p,q}.\end{aligned}$$

On the other hand, we have

$$\overline{\partial}_K\left(\frac{p+1}{p+q-n} I^* \wedge \vartheta_{p,q}\right) = (p+1)\sum_{j\in\mathbb{Z}} \kappa_{j+1}^{p+1,q+1;n+1} I^* \wedge \overline{Z}^* \wedge \psi_{p,q},$$

so by linearity of $\overline{\partial}_K$ we obtain the claimed formula.

Similarly, using the formulae for the P-derivative of the invariant forms of low degree in section 4.2.1 as well as the definition of the coefficients $\kappa_j^{p,q;k}$ from Theorem 4.2.2 a direct computation shows that

$$\begin{aligned} d_P\psi_{p,q} = 2i\frac{n+1-q}{p+q-n}\sum_{j\in\mathbb{Z}}\Bigg(& (n+1-(p+q))\kappa_{j+1}^{p,q;n+1}\overline{Z}^* \wedge Z^* \wedge \omega_j^{p-1,q-1;n} \\ & - (p+1)\kappa_{j+1}^{p+1,q;n+1}\overline{Z}^* \wedge I^* \wedge \omega_j^{p,q-1;n} \\ & - (q+1)\kappa_{j+1}^{p,q+1;n+1} I^* \wedge Z^* \wedge \omega_j^{p-1,q;n}\Bigg). \end{aligned}$$

Now for $p+q \geq n+2$ the claim immediately follows from the definition of the Poisson kernels $\psi_{p,q}$ for $p+q \geq n+1$. Moreover, in the case $p+q = n+1$ we first apply the equation $\kappa_{j+1}^{p+1,q;n+1} = \kappa_j^{p,q-1;n+1}$, and comparing the result to the definition of $\psi_{p,q}$ for $p+q = n$ yields the claim.

Next, we determine the K-codifferential of $\psi_{p,q}$. Here we can directly apply the formulae developed in Theorem 4.4.2 and afterwards rewrite the coefficients of the resulting linear combination via the definition of the complex numbers $\kappa_j^{p,q;k}$. Similarly, we obtain that $\overline{\partial}_K^*\psi_{p,q}$ is a linear combination of the invariant forms $I^* \wedge Z^* \wedge \omega_j^{p-1,q-1;n}$. Using the identity

$$\kappa_{j+1}^{p,q+1;n+1} + \frac{p+1}{p+q-n}\kappa_{j+1}^{p+1,q+1;n+1} = \frac{n-q+1}{p+q-n}\kappa_j^{p,q;n+1}$$

the corresponding coefficients can be written as a multiple of

$$(j+1)(n+1-(p+q)+j+1)\kappa_{j+1}^{p,q;n+1} - (p-j)(q-j)\kappa_j^{p,q;n+1},$$

which are trivial due to the relations (κ) in Proposition 4.2.2.

Finally, the K-Lefschetz map is given by the wedge product with the pullback of the Kähler form of G/K along the canonical projection, which by Lemma 4.4.1 coincides with the invariant form $\frac{1}{2}(iZ^*\wedge\overline{Z}^* + \omega_K)$. Applying this to our Poisson kernel $\psi_{p,q}$ we deduce that the wedge product of first summand $\frac{i}{2}Z^* \wedge \overline{Z}^*$ is only nontrivial for the second sum of $\psi_{p,q}$. On the other hand, applying Theorem 4.2.2 to $\frac{1}{2}\omega_K \wedge \psi_{p,q}$ only the third summand is nontrivial, and after shifting the index j appropriately we obtain that $\mathcal{L}_K\psi_{p,q} = 0$. □

Using Proposition 2.2.3 we can translate the formulae from the last Proposition into properties of the corresponding G-equivariant operators

$$\Gamma(\mathcal{H}_{p+q}(G/P) \otimes \mathbb{C}) \to \Omega^{p,q}(G/K)$$

for all $0 \leq p, q \leq n$.

THEOREM 4.4.3. *Let $G = \mathrm{SU}(n+1,1)$, $K \cong U(n+1)$ its maximal compact subgroup and P its minimal parabolic subgroup. Let p and q be integers with $0 \leq p, q \leq n$.*

(i) For $p+q \leq n$ let $\tilde{\Psi}_{p,q}$ be the G-equivariant operator induced by the BGG-compatible Poisson kernel $\psi_{p,q}$. Then this operator satisfies

$$\delta \circ \tilde{\Psi}_{p,q} = 0, \qquad \mathcal{L}^* \circ \tilde{\Psi}_{p,q} = 0.$$

(ii) For $p+q \geq n+1$ let $\tilde{\Psi}_{p,q}^1$ and $\tilde{\Psi}_{p,q}^2$ be the G-equivariant operators on the homology bundles induce by the kernels $\psi_{p,q}$ and $\overline{\psi_{q,p}}$, respectively. Then the first operator satisfies

$$\mathcal{L} \circ \tilde{\Psi}_{p,q}^1 = 0, \qquad \partial \circ \tilde{\Psi}_{p,q}^1 = 0, \qquad \overline{\partial}^* \circ \tilde{\Psi}_{p,q}^1 = 0,$$

whereas the second operator has the properties

$$\mathcal{L} \circ \tilde{\Psi}^2_{p,q} = 0, \qquad \overline{\partial} \circ \tilde{\Psi}^2_{p,q} = 0, \qquad \partial^* \circ \tilde{\Psi}^2_{p,q} = 0.$$

PROOF. (i) By definition, for all sections α of $\ker(\partial^*)$ the differential forms $\tilde{\Psi}_{p,q}(\pi(\alpha))$ and $\Psi_{p,q}(\alpha)$ coincide, so it suffices to prove the claimed properties for the Poisson transform $\Psi_{p,q}$. Moreover, by Proposition 2.2.3 we can translate all the other properties to relations on the Poisson kernels $\psi_{p,q}$, which have been proven in Proposition 4.4.3(i).

(ii) Follows similarly to (i)

□

4.4.4. Real valued BGG-compatible Poisson transforms. So far we have considered BGG-compatible Poisson transforms which map between complex valued differential forms on the CR-sphere and the complex valued differential forms on the complex hyperbolic space. However, our initial setting was phrased in terms of real manifolds, hence our aim in this section is to construct BGG-compatible Poisson kernels ψ_k whose associated transforms Ψ_k map between real valued differential forms on G/P and real valued differential forms on G/K of the same degree.

First, recall from Lemma 4.3.1 that a BGG-compatible Poisson transform Ψ between real valued differential forms of degree k is BGG-compatible if and only if the Poisson transforms $\pi_{p,q} \circ \Psi$ have the same property for all $p + q = k$, where $\pi_{p,q} \colon \Omega^{p+q}(G/K, \mathbb{C}) \to \Omega^{p,q}(G/K)$ denotes the canonical projection onto the space of differential forms of type (p, q). Thus, we will define Ψ_k as a linear combinations of the operators constructed in section 4.3.2 so that it maps into the space of real differential forms. In terms of the corresponding Poisson kernels this can be ensured by requiring that

$$\overline{\psi_k} = \psi_k \tag{R1}$$

for all $0 \leq k \leq 2n + 1$.

Moreover, due to BGG-compatibility the Poisson transforms Ψ_k factor to G-equivariant operators $\tilde{\Psi}_k$ defined on sections of the homology bundles over G/P, and we want these operators to be chain maps between the BGG-sequence and the de Rham complex on G/K. Otherwise said, we will construct the Poisson kernels ψ_k so that they satisfy

$$d_K \psi_k = c_k d_P \psi_{k+1} \tag{R2}$$

for all $0 \leq k \leq 2n$, where $c_k \in \mathbb{R}$ is a suitable constant depending on k.

Finally, recall from Proposition 4.4.3 that for $p + q \leq n$ the Poisson kernels $\psi_{p,q}$ are K-coclosed, which implies that their associated Poisson transforms satisfy $\delta \circ \Psi_{p,q} = 0$ due to Proposition 2.2.3. In particular, for all $0 \leq k \leq n$ the operator Ψ_k will have the same property. In order to get a uniform picture we assume that

$$\delta_K \psi_k = 0 \tag{R3}$$

for all $0 \leq k \leq 2n + 1$.

In order to construct Poisson kernels ψ_k which satisfy the above requirements we have to distinguish two different cases.

(1) First, let $0 \leq k \leq n$ and recall from section 4.3.2 that the complex conjugate of the form $\psi_{p,q}$ with $p + q = k$ is given by $\psi_{q,p}$. Thus, it suffices to consider the ansatz

$$\psi_k := \sum_{p+q=k} \lambda_{p,q} \psi_{p,q},$$

where $\lambda_{p,q}$ are complex numbers satisfying $\lambda_{p,q} = \overline{\lambda_{q,p}}$. Then this invariant form is real by construction and trivial under the K-codifferential due to Proposition 4.4.3(i).

Thus, we have to choose the coefficients of ψ_k in such a way that (R2) is satisfied, which by the formulae for the partial derivatives of $\psi_{p,q}$ in Proposition 4.4.3(i) is the case if and only if

$$(n+1-k)\lambda_{p,q} = c_k 2i(n+1-p)\lambda_{p+1,q} = -c_k 2i(n+1-q)\lambda_{p,q+1}$$

for all $0 \leq p, q \leq n-1$. In order to obtain simple coefficients we choose $c_k := n+1-k$ and define

$$\lambda_{p,q} := 2^{-k} i^{p-q} \frac{(n+1-p)!(n+1-q)!}{(n+1)!^2}.$$

In this way, the Poisson kernel ψ_k satisfies our requirements (R6) - (R3) for all $0 \leq k < n$.

(2) Similarly, for all $n+1 \leq k \leq 2n+1$ we consider the ansatz

$$\psi_k := \sum_{p+q=k} \frac{1}{2}\left(\lambda_{p,q}\psi_{p,q} + \overline{\lambda_{q,p}\psi_{q,p}}\right),$$

which is a real Poisson kernel of degree $(k, 2n+1-k)$ by construction. First, applying the formula for the K-codifferential of the Poisson kernels $\psi_{p,q}$ in Proposition 4.4.3 and using linearity of δ_K it follows that ψ_k is K-coclosed if and only if its coefficients satisfy the relation

$$\lambda_{p+1,q} + \overline{\lambda_{q+1,p}} = 0$$

for all $p+q \geq n+1$.

Furthermore, we want to choose the coefficients so that

$$d_K\psi_k = c_k d_P \psi_{k+1}$$

for all $k \geq n+1$. In order to do so, note that Proposition 4.4.3 immediately implies

$$(p+q-n+1)d_P\psi_{p,q+1} = 2i(n-q)d_K\psi_{p,q}.$$

Thus, using linearity of the partial derivatives we deduce that the coefficients of ψ_k have to satisfy

$$(k-n+1)\lambda_{p,q} = c_k 2i(n-q)\lambda_{p,q+1}$$

for all p, q. In order to obtain an explicit solution we choose $c_k := k-n+1$ and define

$$\lambda_{p,q} := 2^{-k} i^{p-q} \frac{(n+1-p)!(n-q)!}{(n+1)!^2}.$$

In this way, the Poisson kernel ψ_k satisfies our requirements for all $n+1 \leq k \leq 2n+1$. Note that with these choices of coefficients the Poisson kernels also satisfy

$$d_K\psi_n = d_P\psi_{n+1}$$

due to Proposition 4.4.3.

Let $\Psi_k \colon \Omega^k(G/P) \to \Omega^k(G/K)$ denote the Poisson transform associated to the kernel ψ_k for all $0 \leq k \leq 2n+1$ and let $\tilde{\Psi}_k$ denote the induced G-equivariant operator on sections of the corresponding homology bundles. Then the connection between M-equivariant maps on the space of Poisson kernels and the composition of differential operators on G/K with Poisson transforms provided by Proposition 2.2.3 leads to the following Theorem.

THEOREM 4.4.4. *For $0 \leq k \leq 2n+1$ the Poisson transform Ψ_k factors to a G-equivariant operator*

$$\tilde{\Psi}_k \colon \Gamma(\mathcal{H}_k(G/P)) \to \Omega^k(G/K),$$

which has the following properties:

(i) The image of $\tilde{\Psi}_k$ consists of harmonic and coclosed differential forms.
(ii) The images of $\tilde{\Psi}_k$ are primitive for $0 \leq k \leq n$ and trivial under the Lefschetz map for $n+1 \leq k \leq 2n+1$.
(iii) For all $0 \leq k \leq 2n+1$ we have

$$d \circ \tilde{\Psi}_k = (|n-k|+1)\tilde{\Psi}_{k+1} \circ D_k,$$

where $D_k \colon \Gamma(\mathcal{H}_k(G/P)) \to \Gamma(\mathcal{H}_{k+1}(G/P))$ denotes the k-th BGG-operator.

PROOF. (i) Since the operator $\tilde{\Psi}_k$ is defined by $\tilde{\Psi}_k(\pi(\alpha)) = \Psi_k(\alpha)$ for all $\alpha \in \Gamma(\ker(\partial^*))$, it suffices to prove that the same properties hold for Ψ_k. First, by Theorem 4.3.2 the images of the Poisson transforms $\Psi_{p,q}$ consist of harmonic differential forms, so by linearity the same is true for the images of Ψ_k. Moreover, we constructed the Poisson kernels ψ_k so that they satisfy $\delta_K \psi_k = 0$ for all $0 \leq k \leq 2n+1$, hence Proposition 2.2.3 implies that the images of Ψ_k are coclosed.

(ii) Follows from Theorem 4.3.2.

(iii) Recall that we defined the Poisson kernels ψ_k so that

$$d_K \psi_k = (|n-k|+1) d_P \psi_{k+1}$$

for all $0 \leq k \leq 2n$. Since the dimension of G/K is even, we obtain from Proposition 2.2.3 that $d \circ \Psi_k = (|n-k|+1)\Psi_k \circ d$ and therefore

$$(|n-k|+1)(\tilde{\Psi}_k \circ D_k) = (|n-k|+1)\,(\Psi_k \circ d \circ L) = d \circ \tilde{\Psi}_k,$$

where we used that $\tilde{\Psi}_k = \Psi_k \circ L$ due to BGG-compatibility. $\square$

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Zusammenfassung

Das Ziel dieser Dissertation ist die Einführung eines neuen Zugangs zu Poissontransformationen zwischen vektorbündelwertigen Differentialformen auf homogenen parabolischen Geometrien G/P und vektorbündelwertigen Differentialformen auf dem entsprechenden Riemannschen symmetrischen Raum G/K. Explizit sind diese Abbildungen gegeben als Integraloperatoren, die zu G-invarianten Differentialformen auf einem homogenen Raum von G assoziiert sind, und damit vollständig durch invariante Elemente in einer endlich dimensionalen Darstellung einer reduktiven Lie Gruppe bestimmt. Weiters erlaubt der auf den Differentialformenkalkül zugeschnittene Ansatz dass Kompositionen von Poissontransformationen mit bestimmten G-äquivarianten Differentialoperatoren wieder als solche ausgedrückt werden können. Dadurch können wir Operatoren mit vorgegebenen Kompatibilitätsbedingungen durch Berechnungen auf dem Level der zugehörigen Darstellungen explizit konstruieren. Als Beispiel kreieren wir eine universelle Familie von Poissontransformationen Φ_k zwischen reellwertigen Differentialformen, deren Bilder kogeschlossen sind und die sich für glatte Funktionen auf die klassische Transformation reduzieren.

Danach wenden wir unseren Ansatz auf den Fall $G = \mathrm{SO}(n+1,1)_0$ an und zeigen, dass die Operatoren Φ_k Kettenabbildungen zwischen den de Rham Sequenzen auf der konformen Sphäre und dem reell hyperbolischen Raum sind. Zusätzlich präsentieren wir eine explizite Konstruktion einer Familie von Poissontransformation zwischen Differentialformen mit Werten in den jeweiligen Standardtraktorbündeln, die G-äquivariante Kettenabbildungen zwischen der zugehörigen BGG-Sequenz auf G/P und dem getwisteten de Rham Komplex auf G/K induzieren. Ferner sind die Bilder von gewichteten Differentialformen unter diesen Operatoren Eigenformen für einen kovarianten Laplace Operator.

Schlussendlich betrachten wir den Fall $G = \mathrm{SU}(n+1,1)$, wo wir eine feinere Zerlegung des Raumes der Poissonkerne erhalten, indem wir die Kähler Struktur auf dem komplex hyperbolischen Raum und die CR-Struktur auf der zugehörigen Randsphäre G/P ausnutzen. Dies ermöglicht es in Folge Poissontransformationen zwischen komplexwertigen Differentialformen zu konstruieren, die zu G-äquivarianten Operatoren auf Schnitten der komplexifizierten Homologiebündel auf der CR-Sphäre faktorisieren. Im weiteren Verlauf werden diese dazu verwendet äquivariante Kettenabbildungen zwischen dem reellwertigen BGG-Komplex auf G/P und dem de Rham Komplex auf G/K zu definieren.

Curriculum Vitae

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