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Bergman Kernel in several complex Variables

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I dedicate this work to the meomory of my parents Fateh died in 1992 and Baktha died in 2009. My best gratitude goes to my supervisor Mr Haslinger for his effort and his pertinent advices through this work and my study in this university I dedicate also this thesis to my brother-in law Riadh who unremittingly supported me and made this work possible, to the best sister in the world Zahra and her children Heithem, Mouhib and Schahd.

Abstract

This framework is devoted to the study of the Bergman kernel named after the American mathematician Stefan Bergman ($1895-1977$) whose work was focused on several complex variables, it represents namely the reproducing function of the Hilbert space of square integrable functions on a domain Ω in $\mathbb{C}^n$. Its definition is quite easy (using Riesz theorem on continuous linear functional in Hilbert spaces) but it is difficult to obtain a concrete computation for arbitrary domains except the case of an open ball or a polydsic.
To begin with,we recall the basic properties of holomorphic functions in several complex variables, so we start at an elementary level with standard results followed by a thorough discussion of the various fundamental concepts of holomorphy, Cauchy formula for the polydisc power series expansion and the Weierstrass theorem.
In the second part we recall some basic result from Hilbert spaces theory (orthonormal decomposition, convergence of series in Hilbert spaces, Riesz representation theorem).
In the third part, I give (without proof) an overview over the mean value theorem for harmonic functions, then we start with our main subject: the notion of Bergman spaces and its reproducing function called Bergman kernel, in the last part of this chapter I give and I analyse some special examples of how to compute the Bergman function of some special domains namely the open ball, the open polydisc,upper-half plane and the ellipsoid.Bergman metric and its invariance under biholomorphic maps will be briefly discussed. The Fock space in one and several variables will be discussed at the end of the third chapter .
Finally I give an overview of the so called d-bar operator and its relationship to our main subject the Bergman kernel, so we derive a solution to the d-bar problem in the unit disc of the complex plane: Canonical solution operator in terms of Bergman projection. We prove that this operator is compact and even a Hilbert Schmidt operator, then we investigate the solution operator S to $\bar{\partial}$ restricted to forms with coefficients in the Fock space (introduced in the last part of chapter 3), we will prove non-compactness of S restricted to

(0,1) forms with Fock-space coefficients.

Zusammenfassung

Thema dieser Arbeit ist eine Einführung in die Theorie des Bergman Kerns in mehreren Variablen. In den ersten zwei Kapiteln, sammeln wir alle benötigte Grundlagen aus der Komplexen Analysis in einer und mehrerer Vernderlichen sowie wichtige Resultate aus Funktionalanalysis.
Im dritten Kapitel, präsentieren wir die Eingenschaften des Hilbertraums $\mathbb{A}^2(\Omega)$, wobei Ω ein Gebiet in $\mathbb{C}^n$ ist. Wir beweisen die Existenz einer reproduzierenden Funktion des Raums $\mathbb{A}^2(\Omega)$, diese Funktion ist der Bergman Kern des Gebietes Ω.
Im nächsten Schritt, werden wir die Bergman Funktion einiger Gebiete explizit berechnen, wir betrachten den Fall der Einheitskugel in einer und mehreren Variablen, der Halbebene in $\mathbb{C}$ und schließlich den Fall des Ellipsoids in $\mathbb{C}^n$.
Am Ende des dritten Kapitels wird den Bergman Kern des Fockraums in einer und mehreren Veränderlichen berechnet.
Das letzte Kapitel ist dem Zusammenhang zwischen dem d-quer Operator und dem Bergman Kern gewidmet. Wir zeigen, dass der kanonische Lösungsoperator für die inhomogene Cauchy-Riemannsche Differentialgleichung eingeschränkt auf Elemente des Bergmn Raumes auf dem Einheitskreis ein Hilbert-Scmidt Operator ist.
Abschließend wird bewiesen, dass dieser Operator auf (0,1) Formen mit Koeffizienten aus dem Fock Raum sogar nicht kompakt ist.

Historical overview

The theory of integral operators and kernels has been known for a long time, however the characteristics of such kernels has been studied and applied since the beginning of the last century.
There have been and continue to be two trends in the consideration of those kernels.
The first is to consider a kernel as a function of two points in an abstract domain, say E , having the property, discovered by J.Mercer 1909, for each points $(y_j)_{1\leq j\leq n}$ and $(\zeta_j)_{1\leq j\leq n}$ complex numbers we have:

$$\sum_{i,j=1}^{n} k_{i,j}\zeta_i\overline{\zeta_j} \geq 0 \quad . \tag{0.0.1}$$

So K appears as a positive definite Hermitian matrix,

$$\mathscr{K} = (k_{i,j})_{i,j=1\ldots n} \quad , \quad k_{i,j} = K(y_i, y_j)\,.$$

E.H.Moore 1916 in his papers [28], considered functions having the property (0.0.1) and proved that to each such function (kernel)corresponds a well defined class of functions, say F , having the structure of a Hilbert space with a scalar product such that each element of this class can be reproduced by a scalar product i.e,

$$f(y) = \Big\langle f(x), K(x,y)\Big\rangle \quad . \tag{0.0.2}$$

The second trend was initiated by S.Zaremba in his papers [35] 1907. He was the first who introduced the notion of the kernel corresponding to a special class of functions and gave the reproducing formula (0.0.2). However he didn't develop general theory in this direction nor give any name to those kernel he introduced. It appears that nothing was done until the third decade of the 20^{th} century when S.Bergman 1920 introduced reproducing kernels of the spaces of analytic function, on a domain in one and several variables, with an integrability condition.Bergman studied spaces of the type,

$$\mathbb{A}^p(\Omega) = \Big\{ f\, holomorphic\, on\, \Omega\, and\, \int_\Omega |f|^p d\lambda < \infty \Big\} \quad .$$

He proved that in the special case when $p = 2$ this space is a Hilbert space, being a closed subspace of the Hilbert space $L^2(\Omega)$.
The continuity of the evaluation map (introduced in the beginning of chapter 3) leads to the existence of the Bergman Kernel and explicit computation of these kernels (of given domains) becomes a big challenge.
Several important results were achieved by the use of these kernels in the theory of the $\bar{\partial}$ Neumann operator, conformal mapping of simply and multiply connected domains and many other areas of mathematical research.

Chapter 1

Holomorphic functions

1.1 Notations

We introduce the basic notations and conventions used throughout this work.
(A) $\mathbb{C}^n = \{z : z = (z_1, z_2, ..., z_n), z_j \in \mathbb{C} for j = 1, 2, ..., n\}$ is the n-dimensional complex vector space. It is the product of n copies of $\mathbb{C}$, it carries the structure of an n-dimensional vector space. Its topology is identical with the one arising from the following identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.
Given an element $z = (z_1, z_2, ..., z_n)$in $\mathbb{C}^n$, each component z_j can be written in the form $z_j = x_j + iy_j$.
The mapping $z \to (x_1, x_2, \ldots, y_1, y_2, \ldots y_n)$ establishes an $\mathbb{R}$- isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$.
Via this identification we endow this space with the structure of a normed vector space and hence with a topology. All usual concepts of analysis in the Euclidean space $\mathbb{R}^{2n}$ carry immediately over $\mathbb{C}^n$.
All norms in $\mathbb{C}^n$ are equivalent, commonly we use the following two norms:
(i) Euclidean norm:$\|z\| := \{\sum_{j=1}^n |z_j|^2\}^{\frac{1}{2}}$.
(ii) The sup norm :$\|z\|_\infty := max_{j=1\ldots,n}|z_j|$.
(B) The open ball around $a \in \mathbb{C}^n$ with radius $r > 0$ is defined by
$B(a, r) = \{z \in \mathbb{C}^n / \|z - a\| < r\}$.
Recall that a set $D \subseteq \mathbb{C}^n$ is open if for every $a \in D$ there is an $r > 0$ such that the open ball $B(a; r) \subseteq D$.
(C) A domain is an open connected subset.
(D) For a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_j \in \mathbb{N}_0$. We define:
a) $\mid \alpha \mid = \alpha_1 + ... + \alpha_n$ is the length of α.
b) $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$.
c) $\alpha! = \alpha_1! \ldots \alpha_n!$.
d) $D^\alpha = \frac{\partial^{\alpha_1 + \ldots \alpha_n}}{\partial {z_1}^{\alpha_1} \ldots \partial {z_n}^{\alpha_n}}$.

E) For an open subset Ω in $\mathbb{C}^n$ we denote its boundary by bΩ rather than the usual symbol ∂ which has another meaning.
F) The Wirtinger notations:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}).$$

Where $\frac{\partial}{\partial x}$and $\frac{\partial}{\partial y}$ are the usual real partial derivatives.
We recall, in the one-dimensional theory, that for a continuous function $f : G \to \mathbb{C}$(G open subset in the complex plane)is holomorphic if it satisfies one the following equivalent conditions:
i) f is complex differentiable i.e $\frac{df}{dz}$ exists and coincides with $\frac{\partial f}{\partial z}$.
ii) Locally f is representable by a convergent power series.
iii) For each contractible (can be reduced to a point by a homotopy) piecewise smooth closed curve γ in G,$\int_\gamma f(z)dz = 0$ (Cauchy's theorem).
iv) f has a continuous partial derivative with respect to x and y at each point in G,and they satisfy the Cauchy -Riemann differential equations $\frac{\partial f}{\partial \bar{z}} = 0$ or

$$\begin{cases} U_x = V_y \\ U_y = -V_y \end{cases}$$

U and V are the real (respectively imaginary) part of f(regarded as a function of the real variables x and y).
Having that background in mind,we can generalize to the n-dimensional case.

1.2 Cauchy integral formula for polydiscs

In this section we investigate the notion of holomorphy in several complex variables by combining basic properties in one variable with calculus of several variables.

Definition 1.1. *Let Ω be an open subset in $\mathbb{C}^n$ and f a C^1 complex valued function.*
The function f is holomorphic on Ω, if for any $z = (z_1 \dots z_n) \in \Omega$,and any $j = 1 \dots n$, the function $f_j : \zeta \mapsto f(z_1 \dots z_{j-1}, \zeta, z_{j+1} \dots z_n)$ is holomorphic in the usual sense as a function of the one complex variable ζ.

We recall that in the one dimensional case a function f defined on a subset G of $\mathbb{C}$ is holomorphic, if and only if for every $z \in G$, f verifies the Cauchy -Riemann equation:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

For more details on holomorphic functions theory see [19].

Let us first recall the classical Cauchy integral formula in the one dimensional case.

Theorem 1.2. *Let G be a bounded set of $\mathbb{C}$ with C^1 boundary bG,and $f \in C^1(\bar{G})$ then for all $z \in G$ we have:*

$$f(z) = \frac{1}{2\pi i}\Big(\int_{bG} \frac{f(\zeta)}{\zeta - z} d\zeta + \iint_G \frac{\partial f/\partial\bar{\zeta}}{\zeta - z} d\zeta \wedge d\bar{\zeta}\Big). \tag{1.2.1}$$

In order to prove this theorem we need the Stokes'formula.
Let ω be a differential form defined on G then:

$$\int_{bG} \omega d\zeta = \iint_G d\omega \wedge d\zeta.$$

See [5]For a proof of this result.

Proof. Let $z \in G$ and $0 < r < dist(z, bG)$.Consider the 1-form defined on $G_r = G \setminus D(z,r)$ by,

$$\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - z} d\zeta \quad .$$

We apply now the $Stokes'$ formula for G_r and get,

$$\frac{1}{2\pi i}\int_{bGr} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\iint_{Gr} \frac{\partial}{\partial\bar{\zeta}}(\frac{f(\zeta)}{\zeta - z}) d\zeta \wedge d\bar{\zeta} \tag{1.2.2}$$

$$= \frac{1}{2\pi i}\iint_{Gr} \Big[\frac{\frac{\partial f}{\partial\bar{\zeta}}}{\zeta - z} + f(\zeta)\frac{\partial}{\partial\bar{\zeta}}(\frac{1}{\zeta - z})\Big] d\zeta \wedge d\bar{\zeta} \quad . \tag{1.2.3}$$

We know that the function $\zeta \to \frac{1}{\zeta - z}$ is holomorphic on G_r then the integral $\iint_{Gr} f(\zeta)\frac{\partial}{\partial\bar{\zeta}}(\frac{1}{\zeta-z}) d\zeta \wedge d\bar{\zeta} = 0$.
Let us compute the left hand side in (1.2.2) so we obtain,
$\frac{1}{2\pi i}\int_{bGr} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{bG} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\chi_r} \frac{f(\zeta)}{\zeta - z} d\zeta$,where $\chi_r$denotes the circle around z

with radius r.
For the second term we have,
$\int_{\chi_r} \frac{f(\zeta)}{\zeta-z}d\zeta = \int_{\chi_r} \frac{f(z)}{\zeta-z}d\zeta + \int_{\chi_r} \frac{f(\zeta)-f(z)}{\zeta-z}d\zeta$.the first integral in this equality is exactly $f(z)Ind(z,\chi_r) = -f(z)$ because the curve is negatively oriented.
For the second integral we have,
$|\int_{\chi_r} \frac{f(\zeta)-f(z)}{\zeta-z}d\zeta| \leqslant 2\pi r \max_{\zeta\in\chi_r} |f(\zeta)-f(z)| \to 0$ when $r \to 0$.
Now we have,
$\iint_{Gr} \frac{\frac{\partial f}{\partial\bar{\zeta}}}{\zeta-z}d\zeta \wedge d\bar{\zeta} = \iint_{G} \frac{\frac{\partial f}{\partial\bar{\zeta}}}{\zeta-z}d\zeta \wedge d\bar{\zeta} + \iint_{D_r(z)} \frac{\frac{\partial f}{\partial\bar{\zeta}}}{\zeta-z}d\zeta \wedge d\bar{\zeta}$.
For the second integral on the right hand side of this equation we use the polar coordinates and we obtain,
$\iint_{D_r(z)} \frac{1}{|\zeta-z|}d\zeta \wedge d\bar{\zeta} = -2i\int_0^{2\pi}\int_0^r \frac{1}{r'\exp{(i\theta)}}r'\exp{(i\theta)}dr'd\theta \to 0$, when $r \to 0$.
This gives the result. □

By this theorem it follows that every holomorphic function has nice properties in fact it can be represented locally by a convergent power series.
For several complex variables and for integral representation of a holomorphic function, the situation is different to the one dimensional case:integral formulas exist only for $\mathbb{C}^n$ domains which are products of C^1 domains it turns out that function theory of a ball in $\mathbb{C}^n$ is different from the one for polydisc. Let us now generalize this result to holomorphic function on special domains namely the polydsic.

Definition 1.3. *The open polydisc $P(a,r)$ of multi-radius $r = (r_1, \ldots\ldots, r_n)$ and center $a = (a_1 \ldots.., a_n), a \in \mathbb{C}^n$ is the product of n open discs.*
$P(a,r) = \{z \in \mathbb{C}^n, |z_j - a_j| < r_j, j = 1 \ldots\ldots, n\}$.
The distinguished boundary is the n dimensional torus defined by,

$$b_0P = \{z \in \mathbb{C}^n, |z_j - a_j| = r_j\} \quad .$$

We denote it some times by T_r, it is not the topological boundary.

Theorem 1.4. *Let U be an open set in $\mathbb{C}^n$ and let $P(a,r)$ be a polydisc, let f be a holomrphic function on $\bar{P} \subseteq U$.*
For all $z \in P$ we have:

$$f(z) = \frac{1}{(2\pi i)^n}\int_{b_0P} \frac{f(\zeta)d\zeta_1 \ldots\ldots d\zeta_n}{(\zeta_1 - z_1)\ldots\ldots(\zeta_n - z_n)} \quad . \tag{1.2.4}$$

Proof. By induction on the number of variables.
For $n = 1$ the result is already proved in theorem (1.2).
Suppose that the result holds for $n-1$ variables. We fix $z \in P$ and we

consider the function of the one complex variable $\omega \to f(\omega, z_2, \ldots, .., z_n)$. This function is holomorphic throughout a neighbourhood of the closed disc $\bar{D}_1(a_1, r_1)$. By Cauchy applied for z_1 we have:

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|z_1 - a_1| = r_1} \frac{f(\zeta_1, z_2 \ldots z_n)}{\zeta_1 - z_1} d\zeta_1 . \tag{1.2.5}$$

Fix now ζ_1 and apply the induction hypothesis to the function of $(n-1)$ variables $(z_2, \ldots, z_n) \to f(\zeta_1, z_2, ..., z_n)$ we obtain:

$$f(\zeta_1, z_2 \ldots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{b_0 P'} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_2 \ldots ... d\zeta_n}{(\zeta_2 - z_2) \ldots ... (\zeta_n - z_n)} \tag{1.2.6}$$

Now substitute (1.2.5) in (1.2.6) we obtain

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int_{|z_1 - a_1| = r_1} \frac{f(\zeta_1, z_2 \ldots z_n)}{\zeta_1 - z_1} d\zeta_1 \int_{b_0 P'} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_2 \ldots ... d\zeta_n}{(\zeta_2 - z_2) \ldots ... (\zeta_n - z_n)} \\ &= \frac{1}{(2\pi i)^n} \int_{b_0 P} \frac{f(\zeta) d\zeta_1 \ldots ... d\zeta_n}{(\zeta_1 - z_1) \ldots ... (\zeta_n - z_n)} \end{aligned}$$

In the last equality I used a parametrization of the boundary $b_0 P = \prod_{j=1}^n C_{r_j}(z_j)$, where $C_{r_j}(z_j)$ denotes the circle around z_j of radius r_j, with polar coordinates.
For every $1 \leq j \leq n$ we have the standard parametrization of C_r,

$$\zeta_j = z_j + r_j e^{i\theta_j} .$$

□

Remark 1.5. *All integrals involved in the last theorem were curve integrals. In the following we give a second alternative dealing with area integrals. Consider a holomorphic function on an open subset $G \subseteq \mathbb{C}$;let $z \in G$ and $r > 0$ such that $\overline{D(z, r)} \subseteq G$. Then,*

$$f(z) = \frac{1}{\pi r^2} \int_{D(r,z)} f(\omega) d\lambda(\omega) \quad . \tag{1.2.7}$$

where $d\lambda(\omega)$ denotes the Lebesgue measure on $\mathbb{C}$.
This formula follows directly from the Cauchy integral formula. Indeed,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + s\exp(i\theta)) d\theta \quad . \end{aligned}$$

C_s is the circle around z with radius s parametrised by $C_s(\theta) = z + s\exp(i\theta)$ A simple computation of the integral in (1.2.7) gives:

$$\int_{D(z,r)} f(\omega)d\lambda(\omega) = \int_0^r \int_0^{2\pi} f(z + s\exp(i\theta))sdsd\theta \tag{1.2.8}$$

$$= \pi r^2 \int_0^{2\pi} f(z + s\exp(i\theta))sdsd\theta \tag{1.2.9}$$

$$= \pi r^2 f(z). \tag{1.2.10}$$

Using again an iteration process we get a similar representation for holomorphic functions of several complex variables, applying the last formula for each f_j we obtain:

$$f(z) = \frac{1}{\pi^n r_1^2 \dots r_n^2} \int_{P(z,r)} f(\omega)d\lambda(\omega) \quad . \tag{1.2.11}$$

Corollary 1.6. *If f is holomorphic on $\Omega \subseteq \mathbb{C}^n$. Then f is C^∞.*

Proof. Like in the one dimensional case, take some radius r and apply Cauchy for the closed polydisc contained in Ω and differentiate under the integral sign to obtain the desired result. □

1.3 Multiple and power series in several complex variables

The main references I use in this section are [1], [3] and [29].
In the one dimensional case we declare that a series is convergent if the sequence of partial sums converges. Since $\mathbb{N}^n$ has no partial order, the convergence of a multiple series will be defined as follows:

Definition 1.7. *A power series at $a \in \mathbb{C}^n$ has the general form*

$$\sum_{\alpha_1=0}^{\infty} \dots \sum_{\alpha_n=0}^{\infty} \lambda_{\alpha_1 \dots \alpha_n}(z_1 - a_1)^{\alpha_1} \dots (z_n - a_n)^{\alpha_n} \quad , \tag{1.3.1}$$

where the $\lambda_{\alpha_1 \dots \alpha_n}$ are complex coefficients and the α_i are integers. By convention mentioned at the beginning it will be written in the form,

$$\sum_{\alpha} \lambda_\alpha (z - a)^\alpha \quad . \tag{1.3.2}$$

This series is said to be convergent at a point ω, if it converges with respect to every (hence arbitrary) order of summation, this is equivalent to the statement that it converges absolutely.

Remark 1.8. *In the one dimensional case the domain of convergence of a power series is always a disc however in several variables the domain of convergence of a multi-variable series can have a variety of shapes, consider for example the series: $\sum_{n=0}^{\infty}(z_1 z_2)^n$, the domain of convergence consists of an unbounded hyperbolic region in the absolute space.*

Definition 1.9. *A Reinhardt domain $\Omega \subseteq \mathbb{C}^n$ is one which, whenever it contains a point $(z_1 \dots z_n)$, also contains all points $(\lambda_1 z_1 \dots \lambda_n z_n)$ with $|\lambda_j| \leq 1, j = 1 \dots n$.*

In several complex variables, the domain of convergence of a power series is a Reinhardt domain. Furthermore ,every holomorphic function has a power series expansion valid throughout the domain, for more details see [23] or [25].

Definition 1.10. *Let $f_m : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}, m \in \mathbb{N}$ be a sequence of continuous functions.The series $\sum_{m=0}^{\infty} f_m$ converges normally on Ω if,*

$$\sum_{m=0}^{\infty}(\sup_{z\in K}|f_m|)$$

converges for every compact subset K such that $K \subseteq \Omega$.
The series in (1.3.2) converges normally if it converges in the sense of this definition with f_m replaced by $\lambda_\alpha(z-a)^\alpha$.

Remark 1.11. *Note that if the series (1.3.2) converges at some point ω such that $|\omega - a| = r$ with $r = (r_1 \dots r_n)$ a multi-radius,then the terms $\lambda_\alpha(\omega - a)^\alpha$ are bonded in the sense that there exists $M \geq 0$ such that, $|\lambda_\alpha| r_1^{\alpha_1} \dots r_n^{\alpha_n} \leq M$. We will prove even that the series, under the latter condition, converges absolutely throughout the polydisc $P(\omega, r)$.*

Proposition 1.12. *(Abel's lemma)*
Let $\omega \in \mathbb{C}^n$. If the set $\left\{\lambda_\alpha|\omega - a|^\alpha, |\alpha| \geq 0\right\}$ is bounded, then the series

$$\sum_{\alpha_1,\dots,\alpha_n=0}^{\infty} \lambda_{\alpha_1\dots\alpha_n}(z_1-a_1)^{\alpha_1}\dots(z_n-a_n)^{\alpha_n} \quad ,$$

converges normally in the polydisc:

$$P = \{z : |z-a| < r\}$$

where $r = (r_1 \dots r_n)$ and $r_j = |\omega_j|, j = 1, \dots n$.

Proof. Without lost of generality we can assume that $a = 0$. So there exists M such that $|\lambda_\alpha \omega^\alpha| \leq M$.
For $0 < \varepsilon < 1$ consider the family of polydiscs $P_\epsilon = \{z : |z_j| \leq \varepsilon |\omega_j|\}$.
For $z \in P_\epsilon$ we have:

$$|\lambda_\alpha z^\alpha| \leq M\varepsilon^{|\alpha|} \quad ,$$

and

$$\begin{aligned}\sum_\alpha^\infty M\varepsilon^{|\alpha|} &= M\Big(\sum_\alpha \varepsilon^{\alpha_1+\ldots+\alpha_n}\Big) \\ &= M\Big(\sum_{j=0}^\infty \varepsilon^j\Big)^n \\ &= M\Big(\frac{1}{1-\varepsilon}\Big)^n \quad .\end{aligned}$$

Hence the power series converges normally on the polycylinder P_ε.
Now let $K \subset P$ be a compact subset $(P_\varepsilon)_\varepsilon$ forms an open covering of P hence of K, but K is compact therefore there exists a finite sub-covering of K i.e, $K \subseteq P_{\varepsilon_1} \cup \ldots \cup P_{\varepsilon_k}$ let $q = max\{j = 1\ldots, k\}$,the series $\sum_\alpha z^\alpha$ converges normally on P_q hence on K.

□

Proposition 1.13. *Let r be a multi-radius and P the polydisc around zero with radius r,let T_r its distinguished boundary. Consider a continuous function $g : T_r \to \mathbb{R}$.Then the function:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{T_r} \frac{g(\zeta)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta \quad , \tag{1.3.3}$$

can be expanded as a convergent power series on the whole domain P.

Proof. Let $z \in P$ and $\zeta \in T$ fixed. Since $|\frac{z_j}{\zeta_j}| < 1$, we have

$$\frac{g(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)} = \frac{g(\zeta)}{\zeta_1\dots\zeta_n}\frac{1}{1-\frac{z_1}{\zeta_1}}\dots\frac{1}{1-\frac{z_1}{\zeta_1}} \tag{1.3.4}$$

$$= \frac{g(\zeta)}{\zeta_1\dots\zeta_n}\Big(\sum_{\alpha=0}^{\infty}\Big(\frac{z_1}{\zeta_1}\Big)^{\alpha_1}\Big)\dots\Big(\sum_{\alpha=0}^{\infty}\Big(\frac{z_n}{\zeta_n}\Big)^{\alpha_n}\Big) \tag{1.3.5}$$

$$= g(\zeta)\Big(\sum_{\alpha=0}^{\infty}\frac{z_1^{\alpha_1}}{\zeta_1^{\alpha_1}.\zeta_1}\Big)\dots\Big(\sum_{\alpha=0}^{\infty}\frac{z_n^{\alpha_n}}{\zeta_n^{\alpha_n}.\zeta_n}\Big) \tag{1.3.6}$$

$$= \sum_{\alpha=0}^{\infty}\frac{g(\zeta).z^{\alpha}}{\zeta_1^{\alpha_1+1}\dots\zeta_n^{\alpha_n+1}} \tag{1.3.7}$$

The function $\zeta \to \frac{g(\zeta)}{\zeta_1\dots\zeta_n}$ is continuous on T_r which is compact. Therefore it is bounded and there exists $M > 0$ such that $|\frac{g(\zeta)}{\zeta_1\dots\zeta_n}| \leq M$, for all $\zeta \in T_r$. The sum in(1.3.7) converges normally on P and we may interchange summation and integral to get,

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n}\int_{T_r}\frac{g(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}d\zeta \\ &= \frac{1}{(2\pi i)^n}\int_{T_r}\sum_{\alpha=0}^{\infty}\frac{g(\zeta).z^{\alpha}}{\zeta_1^{\alpha_1+1}\dots\zeta_n^{\alpha_n+1}}d\zeta \\ &= \sum_{\alpha=0}^{\infty}\lambda_\alpha z^\alpha \end{aligned}$$

where,

$$\lambda_\alpha = \frac{1}{(2\pi i)^n}\int_{T_r}\frac{g(\zeta)}{\zeta_1^{\alpha_1+1}\dots\zeta_n^{\alpha_n+1}}d\zeta \quad . \tag{1.3.8}$$

$\square$

Corollary 1.14. *Every holomorphic function on a subset Ω in $\mathbb{C}^n$ is locally representable by a convergent power series around every point.*
For any $\omega \in \Omega$ and every $r > 0$ such that $\overline{P(\omega, r)} \subseteq \Omega$ we have for $z \in P$,

$$f(z) = \sum_{\alpha}\lambda_\alpha(z-\omega)^\alpha. \tag{1.3.9}$$

with uniform convergence throughout P. The coefficients λ_α are given by the formula,

$$\lambda_\alpha = \frac{1}{(2\pi i)^n}\int_{T_r}\frac{f(\zeta)}{\zeta_1^{\alpha_1+1}\dots\zeta_n^{\alpha_n+1}}d\zeta \quad . \tag{1.3.10}$$

Proof. Without lost of generality we may assume that $\omega = 0$.Using the Cauchy integral formula for the polydisc $P(0,r)$ we have for $z \in P(0,r)$,

$$f(z) = \int_{T_r} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{1.3.11}$$

and the last proposition gives the desired result. ☐

Remark 1.15. *Using Taylor expansion for holomorphic functions of one complex variable and an iteration argument we obtain*
$D^\alpha f(0) = \alpha! \, \lambda_\alpha$.

1.4 The Weierstrass theorem

Theorem 1.16. *Let $\Omega \subseteq \mathbb{C}^n$ and $(f_n)_{n \geq 0}$ a sequence of holomorphic functions converging normally to a function f. Then f is holomorphic on Ω.*

Proof. Since all f_m are continuous then f is.In order to prove that f is holomorphic it suffices to show that f can be expanded to a convergent power series around each point $\omega \in \Omega$.
Without lost of generality we can assume that $\omega = 0$.
Choose an $r > 0$ such that $\overline{P(\omega, r)} \subseteq \Omega$, for $z \in P_r$. We have,

$$\begin{aligned} f(z) &= \lim_{m \to \infty} f_m(z) \\ &= \lim_{m \to \infty} \int_{T_r} \frac{f_m(\zeta)}{\zeta - z} d\zeta \quad . \end{aligned}$$

Our aim now is to interchange integration and limits.
The function $\zeta \to \zeta - z$ does not vanish on T_r thus, the function $\Phi : \zeta \to \frac{1}{\zeta - z}$ is continuous on T_r which is closed, hence it is bounded and there exists $M > 0$, such that $|\Phi(\zeta)| \leq M$.
We conclude that $\|\frac{f_m - f}{\Phi}\| \leq M \|f_m - f\| \to 0$ as $m \to \infty$ and $\frac{f_m}{\Phi}$ converges normally to $\frac{f}{\Phi}$ on $\mathbb{T}_r$.
Finally we get

$$f(z) = \int_{\mathbb{T}_r} \frac{f(\zeta)}{\zeta - z} d\zeta \quad . \tag{1.4.1}$$

By proposition (1.13) f can be expanded in a convergent power series on T_r hence it is holomorphic.

☐

1.5 Holomorphic Jacobian

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and

$$f : \Omega \to \mathbb{C}^n$$

$$z \to \big(f_1(z), \ldots, f_n(z)\big)$$

where $z = (z_1, \ldots, z_n)$,be a holomorphic function. In the sequel, we introduce the notion of the holomorphic Jacobian as presented in [3] and [29].

Definition 1.17. *The complex or holomorphic Jacobian of f at z is defined by the $n \times n$ matrix:*

$$\mathbb{J}_{\mathbb{C}} f(z) = \left(\frac{\partial f_k}{\partial z_j}\right)_{j=1,\ldots,n}^{k=1,\ldots n}$$

Set $f_j(z) = u_j(z) + iv_j(z)$and $z_j = x_j + iy_j$. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and considering f as a function of the 2n variables $x_1, y_1, \ldots x_n, y_n$, we can write the usual real Jacobian matrix.

$$\mathbb{J}_{\mathbb{R}} f = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} & \frac{\partial u_2}{\partial y_1} & \cdots & \frac{\partial u_2}{\partial y_n} \\ . & & & & & & \\ . & & & & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_1} & \cdots & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_n} \\ . & & & & & & \\ . & & & & & & \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial x_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_1} & \cdots & \frac{\partial v_n}{\partial y_n} \end{bmatrix}$$

Using the simple notations,

$$\frac{\partial U}{\partial X} = \left(\frac{\partial u_k}{\partial x_j}\right)_{j=1,\ldots,n}^{k=1,\ldots,n}$$

$$\frac{\partial U}{\partial Y} = \left(\frac{\partial u_k}{\partial x_j}\right)_{j=1,\ldots,n}^{k=1,\ldots,n}$$

$$\frac{\partial V}{\partial X} = \left(\frac{\partial v_k}{\partial x_j}\right)_{j=1,\ldots,n}^{k=1,\ldots,n}$$

$$\frac{\partial V}{\partial Y} = \left(\frac{\partial v_k}{\partial x_j}\right)_{j=1,\dots,n}^{k=1,\dots,n}$$

The matrix $\mathbb{J}_{\mathbb{R}}$ becomes:

$$\mathbb{J}_{\mathbb{R}}f = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{bmatrix}$$

Note that, each component $f_j = u_j + v_j$ of f is holomorphic, thus it satisfies the Cauchy-Riemann equations and we obtain,

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial Y} \quad (*)$$

and

$$\frac{\partial U}{\partial Y} = -\frac{\partial V}{\partial X} \quad (**)$$

Theorem 1.18. *With the same notations as in the above definition we have*

$$det\mathbb{J}_{\mathbb{R}}f = \left|det\mathbb{J}_{\mathbb{C}}f\right|^2 \quad .$$

Proof. Using the same notation as in the last definition, equations $(*)$ and $(**)$ and since the determinant does not change under linear combinations of lines(respectively rows)we can write:

$$\begin{aligned} det\mathbb{J}_{\mathbb{R}}f &= \begin{vmatrix} \frac{\partial U}{\partial X} & -\frac{\partial V}{\partial X} \\ \frac{\partial V}{\partial X} & \frac{\partial U}{\partial X} \end{vmatrix} \quad \Big(\mathbf{i} \times 2^{nd} \textbf{ line} + 1^{st} \textbf{ line}\Big) \\ &= \begin{vmatrix} \frac{\partial U}{\partial X} + i\frac{\partial V}{\partial X} & i\frac{\partial U}{\partial X} - \frac{\partial V}{\partial X} \\ i\frac{\partial V}{\partial X} & i\frac{\partial U}{\partial X} \end{vmatrix} \quad \Big(\textbf{-i} \times 1^{st} \textbf{ row} + 2^{nd} \textbf{ row}\Big) \\ &= \begin{vmatrix} -i\frac{\partial U}{\partial X} + \frac{\partial V}{\partial X} & 0 \\ \frac{\partial V}{\partial X} & i\frac{\partial U}{\partial X} + \frac{\partial V}{\partial X} \end{vmatrix} \\ &= \left|\frac{\partial U}{\partial X} + i\frac{\partial V}{\partial X}\right|\left|\frac{\partial U}{\partial X} - i\frac{\partial V}{\partial X}\right| \\ &= \left|det\mathbb{J}_{\mathbb{C}}f\right|^2 \end{aligned}$$

In the last equality, we used the fact that,

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \\ &= \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i(\underbrace{\frac{\partial u}{\partial y}}_{-\frac{\partial v}{\partial x}} + i\underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial u}{\partial x}})\Big) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad . \end{aligned}$$

□

1.6 Riemann mapping theorem

Now, we go back to the one dimensional complex analysis to state the fundamental Riemann theorem dealing with characterization of simply connected domains in $\mathbb{C}$. To begin with, let us recall some basic definitions of biholomorphic functions and simply connected domains.

Definition 1.19. *Let $\Omega \subseteq \mathbb{C}$ be an open subset in the complex plane.*
Ω is simply connected, if every closed curve can be continuously contracted to a point ***within*** *Ω.*
More details on the characterization of simply connected domains can be found on classical lecture courses on complex analysis, see for example[19](page 119 theorem 4.37).

Definition 1.20. *Let $\mathbb{U}, \mathbb{V} \subseteq \mathbb{C}$ be open.The holomorphic map $f : \mathbb{U} \to \mathbb{V}$ is biholomorphic or conformal if it is one-to-one, onto and if its inverse is holomorphic.*

Theorem 1.21. ***(Riemann mapping theorem)***
$\mathbb{D}$denotes as usual the unit disc around zero in $\mathbb{C}$.
Let $\Omega \subsetneqq \mathbb{C}$ be simply connected. There exists a biholomorphic map

$$f : \Omega \to \mathbb{D}$$

Furthermore, if $a \in \Omega$, then there exists a uniquely determined conformal map

$$\phi : \Omega \to \mathbb{D}$$

such that, $\phi(a) = 0$ and $\phi'(a) > 0$.

Proof. For all technical details,see [19]. □

Remark 1.22. *This result is of a great importance in the study of the Bergman Kernel of a simply connected domain in the complex plan. Indeed, one can gain the Riemann map of a simply connected region in terms of its Bergman function and vice versa. This will be studied in details in the third chapter.*

Chapter 2

Elementary Hilbert spaces theory

Hilbert space theory constitutes a part of a large branch in mathematics called functional analysis. Functional analysis is an important tool in the investigation of all kind of problems in pure mathematics, physics and even biology and economics. In fact it is hard to find a branch in science where functional analysis is not used. In this chapter we shall develop all Hilbert spaces properties related to the main subject of this work.
We collect all needed ingredients by a short overview of the main definitions and results namely the notion of orthogonality, projection on closed subspaces,the Cauchy-Schwarz inequality, the fundamental Riesz-Fisher representation theorem of continuous linear functional on Hilbert spaces and the fundamental theorem of vector decomposition in an orthonormal basis(called Fourier series decomposition in some text books over functional analysis). We introduce some powerful tools that will be used in the next chapter.
Throughout this chapter $\mathbb{H}$ denotes a complex vector space.
For the complex conjugate of a complex number z, I use either the standard notation $\bar{z}$,or z^*.
Main references of this chapter are [6], [16] and [31].

2.1 Basic definitions

We recall the basic definitions of normed and Hilbert vector spaces.

Definition 2.1. *Let $\mathbb{H}$ be a complex vector space. A norm on $\mathbb{H}$ is a map*

$$N : \mathbb{H} \to \mathbb{R}_+$$

with the following properties:
a)$N(x) = 0$ *implies* $x = 0$.
b) for all $\lambda \in \mathbb{C}$ *and* $x \in \mathbb{H}$ *we have* :$N(\lambda.x) = |\lambda|N(x)$.
c) for all $x; y \in \mathbb{H}$ *we have* :$N(x + y) \leq N(x) + N(y)$: *Triangle inequality.*

$\mathbb{H}$ becomes the structure of a normed vector space, commonly a norm is denoted by the symbol $||.||$.

Examples 2.2. *i)On the real line* $\mathbb{R}$ *we define the usual norm by,*

$$||x|| = |x|.$$

ii)In the Euclidean space $\mathbb{R}^n$ *we define the Euclidean norm by setting:*

$$||x|| = \big(\sum_{k=1}^{n} |x_i|^2\big)^{\frac{1}{2}}.$$

iii) Let $(\mathbb{X}, d\mu)$ *be a measure space,*
Let $\mathbb{L}^p, \quad p \geq 1$,*be the set of equivalence classes for the relation:*

$$\Big[f \quad R \quad g \Leftrightarrow f = g \qquad \text{almost everywhere}\Big].$$

On this space we define a norm by setting:

$$\|f\|_p = \Big(\int_{\mathbb{X}} |f|^p d\mu\Big)^{\frac{1}{p}}.$$

See [34] for further discussions on the theory of L^p*-spaces.*

Definition 2.3. *A metric on* $\mathbb{H}$ *is a map* $d : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_+$ *satisfying:*
a) $d(x, y) = 0$ *implies* $x = y$.
b) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathbb{H}$, *[Symmetry].*
c) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathbb{H}$, *[Triangle inequality].*

Remark 2.4. *Due to the triangle inequality, it is trivial to show that every norm on* $\mathbb{H}$ *induces a metric by setting* $d(x, y) = N(x - y)$.

Having a metric we can now introduce the notion of convergence of sequence.

Definition 2.5. *A sequence* $(x_n)_{n\geq 0}$ *in* $\mathbb{H}$ *is a Cauchy sequence, if for all* $\varepsilon > 0$ *there exists* $n_0(\varepsilon)$*, such that for all* $n, m \geq n_0$ *we have* $d(x_n, x_m) \leq \varepsilon$ *.*

Definition 2.6. *A sequence* $(x_n)_{n\geq 0}$*converges to* $x \in \mathbb{H}$ *if, for all* $\varepsilon > 0$*, it exists* $n_0(\varepsilon)$*, such that if* $n \geq n_0$ *we have:*$d(x_n, x) \leq \varepsilon$*.*
$\mathbb{H}$ *is complete, if every Cauchy sequence converges.*
A Banach space is a normed vector space complete with respect to the metric induced by its norm.

2.2 Hilbert spaces

Definition 2.7. *A scalar product on a vector space* $\mathbb{H}$ *is a map:*

$$(,) : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$$

subject to the following rules:
a) $(x, x) \geq 0$ *for all* $x \in \mathbb{H}$ *and* $(x, x) = 0 \iff x = 0$*.*
b) For all $\alpha, \beta \in \mathbb{C}$ *,* $(\alpha x + \beta y, z) = \alpha(x, y) + \beta(y, z)$*.*
c) $(x, y) = (y, x)^*$*,for every* $x, y \in \mathbb{H}$*.*

Remark 2.8. *Due to the item c, it is easy to verify that, for* $\alpha, \beta \in \mathbb{C}$ *and* $x, y,\ z \in \mathbb{H}$ *we have:* $(x, \alpha y + \beta z) = \alpha^*(x, y) + \beta^*(y, z)$*, thus an inner product is linear with respect to the first variable and conjugate linear for the second.*

A pre-Hilbert space is a vector space equipped with a scalar product.

Remark 2.9. *Every inner product induces in a natural way a norm defined by:* $\|x\|^2 = (x, x)$*.*

Definition 2.10. *A Hilbert space is a complete pre-Hilbert space for the norm induced by the inner product.*

Example 2.11. *a)The basic example of a Hilbert space is the space of the square integrable function on a measure space.*
Let $(\mathbb{X}, \mu)$ *be a measure space and consider the space,*

$$(\mathbb{L}^2, \mu) = \Big\{ f : \mathbb{X} \to \mathbb{C} \quad \textit{measurable}, \int_{\mathbb{X}} |f|^2 d\mu \leq \infty \Big\}.$$

Together with the inner product defined by,

$$(f, g) = \int_{\mathbb{X}} f\overline{g}d\mu$$

is a Hilbert space (see[31]).
b) $l^2(\mathbb{C}) = \{(a_n)_n : \sum_n |a_n|^2 < \infty\}$.
This space is a Hilbert space with respect to the inner product defined by:

$$< a, b >= \sum_n a_n b_n^* \quad ,$$

where $a := (a_n)_n$ *and* $b := (b_n)_n$.
c) Let Ω *be a domain in* $\mathbb{C}^n$ *and consider the Bergman space defined as follows:*

$$\mathbb{A}^2(\Omega) = \left\{ f : \mathbb{X} \to \mathbb{C} \quad holomorphic, \int_{\Omega} |f|^2 d\lambda \leq \infty \right\} \quad ,$$

equipped with the scalar product

$$(f, g) = \int_{\Omega} f\overline{g}d\lambda \quad ,$$

becomes the structure of a Hilbert space, more details will be developed in the next chapter when we deal with the Bergman kernel.
d) Hardy space $\mathbb{H}^2$ *is defined as follows.*
Let $\mathbb{D}$ *be the unit disc in* $\mathbb{C}$,

$$\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\},$$

We define the Hardy space by:

$$\mathbb{H}^2 = \{f : f\, holomorphic, f(z) = \sum_n a_n z^n \, and \, (a_n)_n \in l^2(\mathbb{C}\}$$

Let $f, g : \mathbb{D} \to \mathbb{C}$ *be holomorphic functions on* $\mathbb{D}$ *with Taylor expansion at* $z \in \mathbb{D}$*:*

$$f(z) = \sum_{n \geq 0} a_n z^n$$

and

$$g(z) = \sum_{n \geq 0} b_n z^n$$

The inner product is defined by the relation

$$(f,g) = \sum_{n\geq 0} a_n b_n^* \quad .$$

e) The Fock space consists of all analytic functions on $\mathbb{C}^n$ with the property:

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\lambda < \infty \quad .$$

It becomes a Hilbert space by defining the inner product

$$(f,g) = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} d\lambda \quad .$$

2.2.1 The Cauchy - Schwarz inequality

Proposition 2.12. *For every $x, y \in \mathbb{H}$ we have:*

$$|(x,y)| \leq \|x\|\|y\| \quad .$$

Proof. Let $(x,y) = |(x,y)|e^{i\theta}$, $0 \leq \theta \leq 2\pi$.
For $t \in \mathbb{R}$ we compute the following inner product

$$\begin{aligned}(x + te^{i\theta}y, x + te^{i\theta}y) &= \|x\|^2 + te^{-i\theta}(x,y) + te^{i\theta}(y,x) + t^2 e^{i\theta}e^{-i\theta}\|y\|^2 \\ &= \|x\|^2 + t^2\|y\|^2 + 2t|(x,y)| \quad .\end{aligned}$$

The last expression is a non-negative polynomial of degree 2 in t, it has therefore a negative discriminant,

$$\Delta = 4|(x,y)|^2 - 4\|x\|^2\|y\|^2 \leq 0 \quad .$$

This gives the desired result.

□

Remark 2.13. *This result leads to the triangle inequality:*

$$\|x + y\| \leq \|x\| + \|y\| \quad .$$

In fact

$$\|x+y\|^2 = \|x\|^2+\|y\|^2+(x,y)+(y,x) \leq \|x\|^2+\|y\|^2+2\,\|x\|\,\|y\| = \Big(\,\|x\|+\|y\|\Big)^2 \quad .$$

Lemma 2.14. *$\mathbb{H}$ denotes a complex Hilbert space, let $y \in \mathbb{H}$ then the following maps:*

$$\mathbb{H} \to \mathbb{C}$$
$$x \to (x,y)$$

and

$$\mathbb{H} \to \mathbb{R}_+$$
$$x \to \|x\|$$

are continuous.

Proof. By Cauchy-Schwarz we have, if $\zeta_1, \zeta_2 \in \mathbb{H}$.Then,
$|(\zeta_1, y) - (\zeta_2, y)| = |(\zeta_1 - \zeta_2, y)| \leq \|\zeta_1 - \zeta_2\|\,\|y\|$.
By the triangle inequality we have:
$|\,\|\zeta_1\| - \|\zeta_2\|\,| \leq \|\zeta_1 - \zeta_2\|$.
This shows that these maps are even uniformly continuous.

□

2.2.2 Orthogonality and the projection theorem

Definition 2.15. *Let $\mathbb{H}$ be a Hilbert space, $x, y \in \mathbb{H}$.*
i) x,y are orthogonal if $(x,y) = 0$.
ii)The orthogonal of a subset $\mathbb{M} \subseteq \mathbb{H}$ is defined by:$\mathbb{M}^\perp = \big\{y \in \mathbb{H}$, for all $x \in \mathbb{M}, (x,y) = 0\big\}$.

Remark 2.16. *Due to the continuity of the scalar product, we conclude that the orthogonal of a subset is closed in fact, if $(x_n)_n$ is a sequence in $M^\perp$ which converges to x we have,*

$$(x,y) = (\lim_{n\to\infty} x_n, y) = \lim_{n\to\infty} (x_n, y) = 0 \quad .$$

In the sequel we set the following problem:
Let $\mathbb{M}$ be a proper closed subset in $\mathbb{H}$ and let $x \in \mathbb{H}$ an arbitrary point. Does

there exist a point $x_0 \in \mathbb{M}$ closest to x ? If it exists, is it unique ?
The next theorem gives a positive answer to this question.

Theorem 2.17. *Let $\mathbb{M}$ be a convex closed subset in $\mathbb{H}$ and let $x \in \mathbb{H}$ an arbitrary point.*
Define $\delta = \inf\{\|x-y\|, y \in \mathbb{M}\}$. There exists a uniquely determined point $x_0 \in \mathbb{M}$ such that,

$$\delta = \|x - x_0\| \quad (*)$$

Proof. Throughout the proof we take advantage of the so called parallelogram law which reads as follows:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad .$$

a)<u>Uniqueness</u>:
Suppose that there are x_0 and x_0' satisfying $(*)$. Then by the parallelogram rule we have:

$$\begin{aligned} \|x_0 - x_0'\|^2 &= \|x_0 - x + x - x_0'\|^2 \\ &= 2\|x_0 - x\|^2 + 2\|x - x_0'\|^2 - 4\left\| x - \underbrace{\frac{x_0 + x_0'}{2}}_{\in \mathbb{M}} \right\|^2 \\ &\leq 2\delta^2 + 2\delta^2 - 4\delta^2 \end{aligned}$$

Thus $x_0 = x_0'$.
b)<u>Existence</u>:
By definition of δ there exists a sequence $(x_n)_n$ in $\mathbb{M}$ such that,

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad .$$

We will prove that it is Cauchy sequence and we identify its limits as being the sought after $x_0 \in \mathbb{M}$.
By the last step we see easily that for $n, m \in \mathbb{N}$

$$\|x_n - x_m\| \leq \|x_n - x\|^2 + 2\|x_m - x\|^2 - 4\delta^2.$$

Thus $(x_n)_n$ is a Cauchy sequence, it converges in $\mathbb{H}$ and setting its limits to be x_0 this achieves our proof of existence. $\square$

Remark 2.18. *x_0 is the unique element satisfying, for all $y \in \mathbb{M}$:*

$$(x - x_0) \perp y \quad .$$

Indeed, let $y \in \mathbb{M}$ and set:

$$f : \mathbb{R} \to \mathbb{R}$$
$$\lambda \to \|x - (x_0 + \lambda y)\|^2 \quad .$$

We have:

$$f(\lambda) = \|x - y\|^2 - 2\lambda \Re(x - x_0, y) + \lambda^2 \|y\|^2$$

which attains its minimum δ^2at $\lambda = 0$, furthermore we have,

$$f'(0) = \Re(x - x_0, y) = 0 \quad .$$

Similar considerations for the function:

$$g(\lambda) = \|x - (x_0 + i\lambda y)\|^2$$

yield to

$$\Im(x - x_0, y) = 0 \quad .$$

Hence $(x - x_0, y) = 0$ as claimed.
The existence of an other x_0' with that property yields to $(x_0 - x_0', y) = (x - x_0, y) + (x - x_0', y) = 0$.The choice of $y = x_0 - x_0'$ leads to $x_0 = x_0'$.

Remark 2.19. *If $\mathbb{M}$ is only closed then the set*

$$x + \mathbb{M} = \Big\{ x + y, y \in \mathbb{M} \Big\} \quad ,$$

is convex and closed thus theorem (2.17) ensures existence of the so called orthogonal projection of x on $\mathbb{M}$ denoted by x_0.

Corollary 2.20. *Let $\mathbb{M}$ be a closed subset in a Hilbert space $\mathbb{H}$. Then there exist two linear operators $P : \mathbb{H} \to \mathbb{M}$ and $Q : \mathbb{H} \to \mathbb{M}^\perp$ with the following properties*
(1)For all $x \in \mathbb{H}$ we have $x = Px + Qx$, i.e $\mathbb{H} = \mathbb{M} \bigoplus \mathbb{M}^\perp$.
(2)If $x \in \mathbb{M}$. Then $Px = x$ and $Qx = 0$ hence $P^2 = P$.
If $x \in \mathbb{M}^\perp$. Then $Px = 0$ and $Qx = x$ hence $Q^2 = Q$
(P and Q are idempotent).
(3)For any $x \in \mathbb{H}$ we have $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.
(4)P and Q are continuous and self- adjoint operators.

Proof. (1)Existence is ensured by the last theorem, the only statement to prove is that $Qx \in \mathbb{M}^{\perp}$. Indeed,
(i)If $y = 0$ there is nothing to show.
(ii)If $y \neq 0$. Then without lost of generality, we can assume that $\|y\| = 1$. Let $y \in \mathbb{M}$, and α a complex number, since Qx is of minimal norm in $x + \mathbb{M}$ and $Qx - \alpha y \in \mathbb{M}$ we have:

$$\begin{aligned} 0 \leq (Qx, Qx) &\leq (Qx - \alpha y, Qx - \alpha y) \\ &= \|Qx\|^2 - 2\overline{\alpha}(Qx, y) + |\alpha|^2. \end{aligned}$$

If we set $\alpha = (Qx, y)$ the last inequality becomes,
$0 \leq -|\alpha|^2$ thus $(Qx, y) = 0$.
For the uniqueness of this decomposition we use the fact that,
$\mathbb{M} \cap \mathbb{M}^{\perp} = \{0\}$.Indeed if $x = x_0 + x_1$ with $x_0 \in \mathbb{M}$ and $x_1 \in \mathbb{M}^{\perp}$ then

$$\underbrace{Px - x_0}_{\in \mathbb{M}\cap\mathbb{M}^{\perp}} = \underbrace{Qx - x_1}_{\in \mathbb{M}\cap\mathbb{M}^{\perp}} \quad .$$

Hence $Px = x_0$ and $Qx = x_1$.
(2)If $x \in \mathbb{M}$. Then $\underbrace{x - Px}_{\in \mathbb{M}} = \underbrace{Qx}_{\in \mathbb{M}^{\perp}} \Rightarrow x = Px$.
Using the same argument for Q,we obtain the remaining statement.
(3)Straightforward from the computation of the inner-product,
$(Px + Qx, Px + Qx) = \|Px\|^2 + \underbrace{(Px, Qx)}_{=0} + \underbrace{(Qx, Px)}_{=0} + \|Qx\|^2$.
(4)Let $x, y \in \mathbb{H}$ by definition of Q we have:

$$\|Q(x - y)\| = \min\{x - y + m, m \in \mathbb{M}\} \leq \|x - y\|.$$

Since $P = I - Q$,where I is the identity. Then P is continuous.
Self-adjointness follows from the fact that:

$$(Px, y) = (Px, Py + Qy) = (Px, Py) \quad ,$$

and

$$(x, Py) = (Px + Qx, Py) = (Px, Py) \quad .$$

Thus

$$(Px, y) = (x, Py) \quad ,$$

and $P^* = P$.
The same argument gives $Q^* = Q$. □

2.2.3 Orthogonal sets and orthonormal basis

In this paragraph I give a short overview of the notion of orthogonal basis and we state the fundamental theorem of Fourier expansion which gives an elegant way to represent any vector in a Hilbert space as the sum of a convergent series.

Definition 2.21. *A collection(finite or infinite, possibly uncountable)of vectors $\{x_i\}_{i\in\mathbb{I}}$ in $\mathbb{H}$ is said to be orthogonal if $(x_i, x_j) = 0$ whenever $i \neq j$.*
If in addition $\|x_i\| = 1$ for all $i \in \mathbb{I}$, then the set is called orthonormal.
A set of non-zero vectors $\{x_i\}$ can be transformed into an ON-set by replacing x_i with $\frac{x_i}{\|x_i\|}$.
Note that orthonormal sets are linearly independent.
An orthonormal basis is a maximal orthonormal set(maximality with respect to the inclusion order).
Zorn's lemma shows that every separable Hilbert space has an orthonormal basis.
The fundamental property of orthonormal bases is that they have a natural interpretation as orthogonal coordinates systems.

Lemma 2.22. *(**Bessel's inequality**) Let $\{e_j\}_{j\geq 1}$ be a countable or finite orthonormal set in Hilbert space $\mathbb{H}$. Then,*

$$\sum_{j\geq 1} |(x, e_j)|^2 \leq \|x\|^2$$

for all $x \in \mathbb{H}$.

Proof. For a finite subset $\{e_1, \dots e_n\}$ we have:

$$\begin{aligned}
0 \leq \left\| x - \sum_{j=1}^{n} (x, e_j) e_j \right\|^2 &= \left(x - \sum_{j=1}^{n} (x, e_j) e_j , x - \sum_{j=1}^{j=n} (x, e_j) e_j \right) \\
&= \|x\|^2 - \sum_{j=1}^{n} (x, e_j)(e_j, x) - \sum_{j=1}^{n} \overline{(x, e_j)}(x, e_j) \\
&+ \sum_{p=1}^{n} \sum_{j=1}^{n} (x, e_p)\overline{(x, e_j)}(e_p, e_j) \\
&= \|x\|^2 - \sum_{k=1}^{n} |(x, e_k)|^2 \quad .
\end{aligned}$$

The last line is due to the fact that,$(e_p, e_j) = \delta_{pj}$,where the δ_{pj} is the Kronecker's symbol. The inequality holds therefore for finite sums.
We have now,

$$\sum_{k=1}^{n} |(x, e_k)|^2 \leq \|x\|^2 \quad .$$

Since the right side is fixed,a limiting process ensures its validity even for infinite sums. □

Lemma 2.23. *As in the previous lemma. Let $\{e_j\}_{j\geq 1}$ be an orthonormal set in a Hilbert space $\mathbb{H}$.Then the following holds:*
*(**a**) The series: $\sum_{k=1}^{\infty} \alpha_k e_k$, converges in $\mathbb{H}$ if and only if,*

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty \quad .$$

*(**b**)If, $\sum_{k=1}^{\infty} \alpha_k e_k = \sum_{k=1}^{\infty} \beta e_k$, in the sense that both series converge to the same x in $\mathbb{H}$.*
Then $\alpha_k = \beta_k$ for all $k \geq 1$,and:

$$\|x\|^2 = \sum_{k=1}^{\infty} |(x, e_k)|^2.$$

Proof. (**a**) Suppose that $\lim_{N\to\infty} \sum_{k=1}^{k=N} \alpha_k = x$.
Then by the continuity of the scalar product we have

$$(x, e_k) = \left(\sum_{k=1}^{\infty} \alpha_k e_k, e_n \right) = \sum_{k=1}^{\infty} \alpha_k (e_k, e_n) = \alpha_n \quad .$$

Bessel's inequality ensures the convergence of the numerical series $\sum_{k=1}^{\infty} |\alpha_k|^2$.
Conversely if, $\sum_{k=1}^{\infty} |\alpha_k|^2 \leq \infty$.Let $S_n = \sum_{k=1}^{n} \alpha_k e_k$.
For $n > m \geq 1$ we have:

$$\begin{aligned}
\|S_n - S_m\|^2 &= (S_n - S_m, S_n - S_m) \\
&= \left(\sum_{k=m+1}^{n} \alpha_k e_k, \sum_{j=m+1}^{n} \alpha_j e_j \right) \\
&= \sum_{k=m+1}^{n} \sum_{j=m+1}^{n} \alpha_k \overline{\alpha_j} (e_k, e_j) = \sum_{k=m+1}^{n} |\alpha_k|^2
\end{aligned}$$

The last line is due to the orthonormality of the set $\{e_j\}_{j\geq 1}$.The sequence $(S_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{H}$ which is complete hence it converges.
(**b**) If $x = \sum_{k=1}^{\infty} \alpha_k e_k = \sum_{k=1}^{\infty} \beta e_k$ then,

$$0 = \lim_{N\to\infty} \{ \sum_{k=1}^{N} \alpha_k e_k - \sum_{k=1}^{N} \beta e_k \} = \lim_{N\to\infty} \sum_{k=1}^{N} (\alpha_k - \beta_k) e_k \quad .$$

By Bessel's inequality we obtain $\alpha_k = \beta_k$.
Now if $x = \sum_{k=1}^{\infty} \alpha_k e_k$ in the sense that the series converges in $\mathbb{H}$ to x.
By Cauchy-Schwarz we have,

$$\begin{aligned} \|x\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 &= (x, x - \sum_{k=1}^{n} \alpha_k e_k) + (x - \sum_{k=1}^{n} \alpha_k e_k, \sum_{k=1}^{n} \alpha_k e_k) \\ &\leq \left\| x - \sum_{k=1}^{n} \alpha_k e_k \right\| \left(\|x\| + \left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \right) \quad . \end{aligned}$$

Since $\sum_{k=1}^{n} \alpha_k e_k \to x$ as $n \to \infty$, the right hand side converges to zero. Therefore $\|x\|^2 \leq \sum_{k=1}^{\infty} |\alpha_k|^2$.
Combining this with Bessel's inequality we obtain the desired result. □

Theorem 2.24. ***(The best approximation)*** *Let $\{e_1 \ldots e_n\}$ be an orthonormal set in a Hilbert space $\mathbb{H}$ and $x \in \mathbb{H}$.Then for every choice of complex numbers: $\alpha_1, \ldots, \alpha_n$. We have,*

$$\left\| x - \sum_{k=1}^{n} (x, e_k) e_k \right\| \leq \left\| x - \sum_{k=1}^{n} \alpha_k e_k \right\|$$

with equality if and only if:$\alpha_k = (x, e_k)$.

Proof. In view of theorem(2.17), $\sum_{k=1}^{k=n} (x, e_k) e_k$ is identified as the orthogonal projection of x on the n-dimensional linear space spanned by $\{e_1 \ldots e_n\}$.
Indeed, by orthonormality we have:

$$\begin{aligned} (x - \sum_{k=1}^{n} (x, e_k) e_k, \sum_{k=1}^{n} \alpha_j e - j) &= \sum_{k=1}^{n} \overline{\alpha_k} (x, e_k) - \sum_{j=1}^{n} \sum_{k=1}^{n} (x, e_k) \overline{\alpha_j} (e_k, e_j) \\ &= \sum_{k=1}^{n} \overline{\alpha_k} (x, e_k) - \sum_{k=1}^{n} \overline{\alpha_k} (x, e_k) \\ &= 0 \quad . \end{aligned}$$

The statement follows at once from theorem (2.17). □

We are now ready to prove the fundamental theorem dealing with orthonormal bases in Hilbert spaces.

Theorem 2.25. *Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis in a Hilbert space $\mathbb{H}$. Then:*
*(**a**) For any $x \in \mathbb{H}$ we have:*

$$x = \sum_{n\geq 1}(x, e_n)e_n \quad .$$

*(**b**)For all$x, y \in \mathbb{H}$,*

$$(x, y) = \sum_{n\geq 1}(x, e_n)\overline{(y, e_n)} \quad .$$

*(**c**) (**Parseval identity**) For any $x \in \mathbb{H}$ we have:*

$$\|x\|^2 = \sum_{n\geq 1}|(x, e_n)|^2 \quad .$$

Proof. (a)**Claim** If $\mathbb{M}$ is a linear subspace. Then $(\mathbb{M}^{\perp})^{\perp} = \overline{\mathbb{M}}$.
If $\{e_n\}$ is a maximal orthonormal set, the only vector orthogonal to all of the e_n 's is zero. Set $\mathbb{M} = \textbf{span}\{e_n, n \geq 1\}$.We have $\mathbb{M}^{\perp} = \{0\}$.But then $(\mathbb{M}^{\perp})^{\perp} = \mathbb{H}$ and $\overline{\mathbb{M}} = \mathbb{H}$. Consequently, for every $x \in \mathbb{H}$, we can find a sequence $\{y_n\}$ of finite linear combinations of vectors from the orthonormal basis,

$$y_n = \sum_{j=1}^{N(n)} \alpha_{j,n}e_j$$

with $y_n \to x, n \to \infty$.
We may assume that $N(n) \geq n$. The best approximation yields:

$$\left\| x - \sum_{j=1}^{N(n)}(x, e_j)e_j \right\| \leq \left\| x - \sum_{j=1}^{N(n)} \alpha_{j,n}e_j \right\| = \|x - y_n\| \to 0, n \to \infty$$

Consequently,

$$x = \sum_{n\geq 1}(x, e_n)e_n \quad .$$

(b) For $x = \sum_{n\geq 1}(x, e_n)e_n$ and $y = \sum_{n\geq 1}(y, e_n)e_n$.We observe that,

$$\begin{aligned}(x, y) = (\sum_{n\geq 1}(x, e_n)e_n, \sum_{n\geq 1}(y, e_n)e_n &= \sum_{k\geq 1}\sum_{j\geq 1}(x, e_k)\overline{(y, e_j)}(e_k, e_j) \\ &= \sum_{n\geq 1}(x, e_n)\overline{(y, e_n)} \quad .\end{aligned}$$

(**c**)Straightforward by setting $x = y$ in (**b**). □

2.2.4 Riesz-Fischer theorem

Definition 2.26. *The dual space of a Hilbert space is the set of all linear-continuous functional defined on $\mathbb{H}$ i.e:*

$$\mathbb{H}^* = \{\mathbb{L} : \mathbb{H} \to \mathbb{C}, \mathbb{L} \text{ linear and continuous } \}$$

On this space we define a norm by:

$$\begin{aligned} \|\mathbb{L}\| &= \sup\Big\{|\mathbb{L}(x)|, x \in \mathbb{H}\Big\} \\ &= \sup\Big\{|\mathbb{L}(x)|, \|x\| = 1\Big\} \end{aligned}$$

$\mathbb{L}$ is bounded if: $\|\mathbb{L}\| < \infty$.
In Hilbert space continuity of a linear functional is equivalent to boundedness.

Remark 2.27. *By the Cauchy-Schwarz inequality we know that for every $y \in \mathbb{H}$. The map:*

$$\mathbb{L}_y : x \to (x, y)$$

is a bounded linear functional (with norm $\|y\|$).
The Riesz-Fischer theorem shows indeed, that every linear functional on a Hilbert space can be written in this way.

Theorem 2.28. *Let $\mathbb{L} : \mathbb{H} \to \mathbb{C}$ be a bounded linear functional. Then there is a uniquely determined vector $y \in \mathbb{H}$ such that:*

$$\mathbb{L}(x) = (x, y) \quad ,$$

for all $x \in \mathbb{H}$.

Proof. a) **Existence**
i) If $\mathbb{L} \equiv 0$.Then just take $y = 0$.
ii) If $\mathbb{L} \not\equiv 0$.Then,

$$Ker(\mathbb{L}) = \Big\{x \in \mathbb{H}, \mathbb{L}(x) = 0\Big\}$$

is a proper subspace in $\mathbb{H}$, therefore $Ker(\mathbb{L})^{\perp}$ contains a unit vector, say $\tilde{y}$.
Now remark that for every $x \in \mathbb{H}$,

$$\mathbb{L}(x)\tilde{y} - \mathbb{L}(\tilde{y})x \in \; Ker(\mathbb{L}) \quad .$$

Hence

$$\begin{aligned}0 &= \Big(\tilde{y}, \mathbb{L}(x)\tilde{y} - \mathbb{L}(\tilde{y})x\Big)\\ &= \mathbb{L}(x)^* \underbrace{(\tilde{y}, \tilde{y})}_{=1} - \mathbb{L}(\tilde{y})^*(\tilde{y}, f) \quad ,\end{aligned}$$

and

$$\mathbb{L}(x)^* = \mathbb{L}(\tilde{y})(\tilde{y}, x) \quad .$$

Now take $y = \mathbb{L}(\tilde{y})^*\tilde{y}$, the desired result follows at once.
b)**Uniqueness**
Suppose that there are $y_1, y_2 \in \mathbb{H}$ such that for every $x \in \mathbb{H}$ we have,

$$\mathbb{L}(x) = (x, y_1) = (x, y_2) \quad .$$

It follows that $0 = (x, y_1 - y_2)$, therefore $y_1 - y_2 \in \mathbb{H}^\perp = \{0\}$. □

Remark 2.29. *The last theorem shows that a Hilbert space is equivalent to its own dual space i.e* $\mathbb{H} \cong \mathbb{H}^*$ *via the map* $x \to (x, .)$*which is a conjugate linear isometric bijection between* $\mathbb{H}$ *and* $\mathbb{H}^*$*.*

Chapter 3

Bergman spaces and reproducing formula

3.1 Preliminaries

We recall in this section a standard result in harmonic analysis, the mean value property. For a proof of this result see for example [14].

Definition 3.1. *Let $U \subseteq \mathbb{R}^n$ an open subset, $f : U \to \mathbb{C}$ a C^2 function f is harmonic on U if it is a solution of the Laplace equation $\triangle f = 0$ where the differential operator Laplace is defined as follows,*

$$\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad .$$

Remark 3.2. *Every holomorphic function on a subset of the complex plane is harmonic considered as a function of the two real variables x and y, indeed if $f = u + iv$ where u and v are the real and imaginary parts,these function are C^2 because f is holomorphic in particular C^2, using the Cauchy -Riemann equalities and the Schawrz identity we obtain $\triangle u = u_{xx} + u_{yy} = (u_x)_x + (v_y)_y = \underbrace{v_{yx} - v_{xy}}_{\textit{Cauchy-Riemann}} = \underbrace{v_{xy} - v_{xy}}_{\textit{Schwarz}} = 0 \quad .$*

Next we derive the fundamental mean value theorem for harmonic functions which explains that, for every $x \in U$ and every ball $\overline{B(x,r)} \subseteq U$, $u(x)$ equals to the average over the sphere $S(x,r)$ and the average over the entire $B(x,r)$, these implicit representations have important consequences as we will see when we introduce the evaluation map on the Bergman space.

Theorem 3.3. *(Mean value theorem)*
Let $U \subseteq \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{C}$ a harmonic function,let $x \in U$ and $r > 0$ such that $\overline{B(x,r)} \subseteq U$ then

$$\begin{aligned} u(x) &= \frac{1}{S} \int_{S(x,r)} U(\zeta) d\sigma(\zeta) \\ &= \frac{1}{V} \int_{B(x,r)} u(\omega) d\lambda(\omega) \end{aligned}$$

λ is the usual Lebesgue measure and σ is the area on the surface of the sphere. S and V are respectively the area of the sphere $S(x,r)$ and the volume of the ball $B(x,r)$.

Proof. See [14]for a detailed proof. □

3.2 Bergman spaces

To begin with, let $\Omega \subset \mathbb{C}^n$ be a domain. As mentioned in the last chapter the Bergman space is defined as follows:

$$\begin{aligned} \mathbb{A}^2 &= \left\{ f : \Omega \to \mathbb{C}; f\, holomorphic\ and\ \int_\Omega |f(z)|^2 d\lambda(z) < \infty \right\} \\ &= \mathbb{L}^2(\Omega) \cap \mathbb{H}(\Omega) \end{aligned}$$

where $d\lambda$ denotes the Lebesgue measure and $\mathbb{H}(\Omega)$ is the space of all holomorphic functions on Ω. For the discussion below I use [13] and [36].

Proposition 3.4. *Let $K \subseteq \Omega$ be a compact subset and $f \in \mathbb{A}^2$.Then there exists a constant $C_{K,n}$ such that,*

$$\sup_{z \in K} |f(z)| \leq C_{K,n} \|f\|_{\mathbb{L}^2} \tag{3.2.1}$$

Proof. Since K is compact then for every $z \in K$ there is r_K such that $\overline{B(z, r_K)} \subseteq K$. By remark (3.2) f is harmonic on Ω hence it verifies the mean value theorem hence;

$$
\begin{aligned}
|f(z)| &= \frac{1}{|V(B(z,r_K))|}\left|\int_{B(z,r_K)} f(\omega)d\lambda(\omega)\right| \\
&\leq \frac{1}{|V(B(z,r_K))|}\int_{B(z,r_K)} |f(\omega)|\,d\lambda(\omega) \\
&= \frac{1}{|V(B(z,r_K))|}\int_{\Omega} \chi_{B(z,r_K)}(\omega)|f(\omega)|d\lambda(\omega) \\
&\leq \frac{1}{|V(B(z,r_K))|}\underbrace{\Big(\int_{\Omega} |\chi_{B(z,r_K)}(\omega)|^2 d\lambda(\omega)\Big)^{\frac{1}{2}}}_{=\sqrt{V(B(z,r_K))}}\underbrace{\Big(\int_{\Omega} |f(\omega)|^2 d\lambda(\omega)\Big)^{\frac{1}{2}}}_{=\|f\|_{\mathbb{L}^2}}
\end{aligned}
$$

In the last inequality I used the Cauchy -Schwartz inequality. Now if we set $C_{n,K} = \frac{1}{\sqrt{V(B(z,r_K)}}$ we obtain the desired estimate. □

Estimate(3.2.1) means in other words that convergence in the $\mathbb{L}^2$ norm implies uniform convergence on compact subsets.

Corollary 3.5. $\mathbb{A}^2(\Omega)$ *is a closed subspace of* $\mathbb{L}^2(\Omega)$ *and hence a Hilbert space.*

Proof. Let$\{f_n\}_{n\geq 0}$ be a sequence in $\mathbb{A}^2$ such that

$$f_n \to f \quad in\ \mathbb{L}^2$$

From functional analysis we know that there is a subsequence $f_{n,j}$ converging almost everywhere to f(see [31]).
On the other hand the last proposition ensures that this sequence is a Cauchy sequence with respect to the uniform convergence on compact subsets,indeed for $K \subseteq \Omega$ and $p \geq q$ we have

$$sup_{z\in K}|f_p(z) - f_q(z)| \leq C_{n,K}\,\|f_p - f_q\|_{\mathbb{L}^2} \to 0$$

This means that $\{f_n\}$ has a limit,say g,which is by the Weierstrass theorem analytic.
Combining both arguments we can identify f and g.
Since $\mathbb{L}^2$ is complete and $\mathbb{A}^2$ is closed,it follows that the Bergman space is a Hilbert space. □

Now we introduce the so-called point evaluation map defined by, for $z \in \Omega$ fixed:

$$
\begin{aligned}
Ev_z : \mathbb{A}^2 &\to \mathbb{C} \\
f &\to f(z)
\end{aligned}
$$

Lemma 3.6. *The point evaluation map is a linear bounded functional on* $\mathbb{A}^2$.

Proof. **a)**<u>**Linearity** :</u> Let $f, g \in \mathbb{A}^2$and $\alpha \in \mathbb{C}$ we have

$$Ev_z(f + \alpha g) = (f + \alpha g)(z) = f(z) + \alpha g(z) = Ev_z(f) + \alpha Ev_z(g)$$

b) <u>**Boundedness**</u> The set $\{z\}$ is a compact subset in Ω we have by proposition (3.4)

$$|E_z(f)| \leq C_{n,\{z\}} \|f\|_{\mathbb{L}^2}$$

Thus the point evaluation is bounded hence continuous. □

For fixed $z \in \Omega$ the point evaluation is a linear continuous functional thus by the Riesz representation theorem (2.28) there exists a uniquely determined element, denoted by K_z, satisfying: For all $f \in \mathbb{A}^2$,

$$Ev_z(f) = (f, K_z).$$

This means that for every $z \in \Omega$ and $f \in \mathbb{A}^2$ we have:

$$f(z) = \int_\Omega f(\omega)\overline{K_z(\omega)}d\lambda(\omega). \tag{3.2.2}$$

Definition 3.7. *The* ***Bergman Kernel*** *of* Ω *is defined by,*

$$K : \Omega \times \Omega \to \mathbb{C}$$
$$(z, \omega) \to K(z, \omega) := \overline{K_z(\omega)}$$

The formula 4.4 becomes

$$f(z) = \int_\Omega f(\omega)K(z, \omega)d\lambda(\omega)$$

This is the so-called: **reproducing formula** .
Now we collect fundamental properties of the Bergmana kernel.

Proposition 3.8. *The Bergman kernel is holomorphic in the first variable and anti-holomorphic in the second.*

Proof. For each $z \in \Omega$ we know that the function $\omega \to K_z(\omega)$ is holomorphic. Since $K(z, \omega) = \overline{K_z(\omega)}$, it follows that K is antiholomorphic in the second variables.
Claim We claim that for each $z, \omega \in \Omega$ we have:

$$K(z, \omega) = \overline{K(\omega, z)}$$

For $\omega \in \Omega$ fixed we apply the reproducing formula (4.4) to the function $K_\omega \in \mathbb{A}^2(\Omega)$:

$$\begin{aligned}
\overline{K(\omega, z)} = K_\omega(z) &= \int_\Omega K_\omega(\zeta) K(z, \zeta) d\lambda(\zeta) \\
&= \int_\Omega \overline{K_z(\zeta) K(\omega, \zeta)} d\lambda(\zeta) \\
&= \overline{\int_\Omega K_z(\zeta) K(\omega, \zeta) d\lambda(\zeta)} \\
&= \overline{K_z(\omega)} = K(z, \omega)
\end{aligned}$$

Hence $K(z, \omega) = \overline{K(\omega, z)}$ and the reproducing kernel is holomorphic with respect to the first component. □

Proposition 3.9. *The Bergman kernel is uniquely determined by the properties that it is an element of* $\mathbb{A}^2(\Omega)$ *holomorphic in z,conjugate symmetric and reproduces* $\mathbb{A}^2(\Omega)$*.*

Proof. Suppose that,there is another K' satisfying(4.4). For every $z \in \Omega$ and by its very definition, $K'_z \in \mathbb{A}^2$.Thus by the reproducing formula we have,

$$\begin{aligned}
\overline{K'(z, \omega)} &= K'_z(\omega) \\
&= \int_\Omega K'(\zeta) K(\omega, \zeta) d\lambda(\zeta) \\
&= \int_\Omega \overline{K'(z, \zeta)} K(\omega, \zeta) d\lambda(\zeta) \\
&= \overline{\int_\Omega K'(z, \zeta) \overline{K(\omega, \zeta)} d\lambda(\zeta)} \\
&= \overline{K(z, \omega)}
\end{aligned}$$

□

The next result gives an expansion of the Bergman kernel in an orthonormal basis of $\mathbb{A}^2(\Omega)$, it leads to a powerful method how to represent the Bergman Kernel in terms of a convergent series.

Proposition 3.10. *Let* $\mathbb{K} \subseteq \Omega$ *be a compact subset and* $\{\Phi_n\}_{n \geq 1}$ *be an orthonormal basis of the Hilbert space* $\mathbb{A}^2(\Omega)$*.*
The sum

$$\sum_{n \geq 1} \Phi_n(z) \overline{\Phi_n}(\Omega)$$

sums uniformly on $\mathbb{K}\times\mathbb{K}$ to the Bergman Kernel K i.e

$$K(z,\omega)=\sum_{n\geq 1}\Phi_n(z)\overline{\Phi_n}(\Omega)$$

Proof. Let $f\in\mathbb{A}^2$. Then

$$f=\sum_{j=1}^{\infty}a_j\Phi_j$$

Where

$$a_j=(f,\Phi_j)$$

By estimate(3.2.1) we know that there exists $C_{\mathbb{K}}$ such that

$$\sup_{z\in\mathbb{K}}|f(z)|\leq C_{\mathbb{K}}.$$

On the other hand, by the Riesz-Fischer representation theorem and proposition(3.4)we have:

$$\begin{aligned} sup\Big\{\Big(\sum_{j=1}^{\infty}\Big|\Phi_j(z)\Big|^2\Big)^{\frac{1}{2}},z\in\mathbb{K}\Big\} &= sup\Big\{\Big|\sum_{j=1}^{\infty}a_j\Phi_j(z)\Big|,z\in\mathbb{K},\Big(\sum_{j=1}^{\infty}|a_j|^2\Big)^{\frac{1}{2}}=1\Big\}\\ &= sup\Big\{|f(z)|,z\in\mathbb{K},\|f\|_{\mathbb{L}^2}=1\Big\}\leq C_{\mathbb{K}}\end{aligned}$$

This inequality shows that, for each $z\in\mathbb{K}$ the sequence of complex numbers $\{\Phi_n(z)\}_n\subset l^2$.
Now take $z,\zeta\in\mathbb{K}$, we may use the Cauchy-Schwarz inequality(proposition(2.12))to obtain:

$$\Big|\sum_{n=1}^{\infty}\Phi(z)\overline{\Phi_n(\omega)}\Big|\leq\underbrace{\Big(\sum_{n=1}^{\infty}\Big|\Phi_n(z)\Big|^2\Big)^{\frac{1}{2}}}_{\leq\mathbf{C}_{\mathbb{K}}}\underbrace{\Big(\sum_{n=1}^{\infty}\Big|\Phi_n(\omega)\Big|^2\Big)^{\frac{1}{2}}}_{\leq\mathbf{C}'_{\mathbb{K}}}$$

This proves the uniform convergence of the series$\sum_{n=1}^{\infty}\Phi(z)\overline{\Phi_n(\omega)}$ on $\mathbb{K}\times\mathbb{K}$ and that $\sum_{n=1}^{\infty}\Phi(z)\overline{\Phi_n(\omega)}\in\mathbb{A}^2$ as a function of ω.Now set

$$K'(z,\omega)=\sum_{n=1}^{\infty}\Phi(z)\overline{\Phi_n(\omega)}$$

Let $f\in\mathbb{A}^2(\Omega)$. By theorem(2.25) (a)we have for each $z\in\Omega$

$$f(z)=\sum_{n=1}^{\infty}(f,\Phi_n)\Phi_n(z)$$

which converges not only in $\mathbb{L}^2(\Omega)$, but also uniformly on compact subsets of Ω ($\mathbb{L}^2$ convergence dominates uniform convergence by proposition (3.4).

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}(f,\Phi_n)\Phi_n(z) \\
&= \sum_{n=1}^{\infty}\int_{\Omega} f(\zeta)\overline{\Phi_n(\zeta)}d\lambda(\zeta)\Phi_n(z) \\
&= \int_{\Omega} f(\zeta)\underbrace{\sum_{n=1}^{\infty}\overline{\Phi_n(\zeta)}\Phi_n(z)}_{\mathbf{K}'(z,\zeta)}\,d\lambda(\zeta) \\
&= \int_{\Omega} f(\zeta)K'(z,\zeta)d\lambda(\zeta) \quad (*)
\end{aligned}$$

By the reproducing formula we have:

$$f(z) = \int_{\Omega} f(\zeta)K(z,\zeta)d\lambda(\zeta) \quad (**)$$

Combining(*)and(**) and by the Riesz representation theorem we obtain $K(z,\zeta) = K'(z,\zeta)$.

□

Corollary 3.11. *Let $\mathbb{K} \subseteq \Omega$ be compact.Then for each $z \in \mathbb{K}$ we have*

$$K(z,z) > 0$$

Proof. We know that $K(z,z) = \sum_{j=0}^{\infty}|\Phi_j(z)|^2$. If in fact $K(z,z) = 0$, we must have $\Phi_j(z) = 0, \forall j \in \mathbb{N}$ which means that $f(z) = 0$ for each $f \in \mathbb{A}^2(\Omega)$ which is absurd. □

3.3 Bergman projection

In corollary(3.5) we have shown that $\mathbb{A}^2(\Omega)$ is closed in $L^2(\Omega)$.By the projection theorem there is a unique operator

$$P : L^2(\Omega) \to \mathbb{A}^2(\Omega)$$

satisfying:
a)$\forall f \in \mathbb{A}^2(\Omega)$, $Pf = f$.

b) $Pf - f \perp g, \forall g \in \mathbb{A}^2(\Omega)$.
This operator is called the Bergman projection. Our next aim is to express the Bergman projection of a domain in terms of its reproducing kernel.
Since $\forall f \in L^2(\Omega), Pf \in \mathbb{A}^2(\Omega)$ we can apply the reproducing formula to this function, so for each $z \in \Omega$ we have:

$$\begin{aligned}
Pf(z) &= \int_\Omega Pf(\omega)K(z,\omega)d\lambda(\omega) \\
&= \int_\Omega Pf(\omega)\overline{K_z(\omega)}d\lambda(\omega) \\
&= \Big\langle Pf, K_z \Big\rangle \\
&= \Big\langle f, P^* K_z \Big\rangle \\
&= \Big\langle f, K_z \Big\rangle \\
&= \int_\Omega f(\omega)K(z,\omega)d\lambda(\omega)
\end{aligned}$$

3.4 Bergman metric

In this section we describe the behaviour of the Bergman Kernel under biholomorphic maps. In what follows, K_Ω denotes the Bergman Kernel of the bounded domain Ω.

Theorem 3.12. *Let Ω_1,Ω_2 two bounded domains in $\mathbb{C}^n$ and let $f : \Omega_1 \to \Omega_2$ be a biholomorphic map. We have*

$$det\mathbb{J}_{\mathbb{C}}f(z)K_{\Omega_2}\Big(f(z), f(\omega)\Big)\overline{det\mathbb{J}_{\mathbb{C}}f(\omega)} = K_{\Omega_1}(z,\omega) \tag{3.4.1}$$

Proof. For the proof we use a combination between operator theory on Hilbert spaces and complex analysis.
Define

$$\begin{aligned}
T_f : L^2(\Omega_2) &\to L^2(\Omega_1) \\
\phi \to T_f\phi &= \Big(\phi \circ f\Big)det\mathbb{J}_{\mathbb{C}}f
\end{aligned}$$

We claim that, this map is an isometric isomorphism. The linearity of this map is obvious and for fixed $\phi \in L^2(\Omega_2)$ using the change of variable

$\zeta = f(z)$,we obtain:

$$\begin{aligned}\|T_f\phi\|^2_{L^2(\Omega_1)} &= \int_{\Omega_1} |\phi \circ f(z)|^2 |det\mathbb{J}_{\mathbb{C}} f(z)|^2 d\lambda(z)\\ &= \int_{\Omega_2} |\phi(\zeta)|^2 d\lambda(\zeta)\\ &= \|\phi\|^2_{L^2(\Omega_2)}\,.\end{aligned}$$

Thus T_f is an isometry with inverse $T_{f^{-1}}$.
Note that, if we differentiate

$$z = f^{-1}(f(z))$$

we obtain

$$1 = det\mathbb{J}_{\mathbb{C}} f^{-1}\Big(f(z)\Big) det\mathbb{J}_{\mathbb{C}} f(z)$$

.
Now, take $\psi \in A^2(\Omega_1) \subseteq L^2(\Omega_2)$ and apply the reproducing formula for $A^2(\Omega_2)$ to the function $T_{f^{-1}}\psi$,for $\zeta \in \Omega_2$ and by setting $\zeta = f(z)$ we obtain

$$\begin{aligned}T_{f^{-1}}\psi(\zeta) &= \int_{\Omega_2} T_{f^{-1}}\psi(\eta) K_{\Omega_2}(\zeta,\eta) d\lambda(\eta) && (3.4.2)\\ &= \psi(z)\Big[det\mathbb{J}_{\mathbb{C}} f(z)\Big]^{-1} && (3.4.3)\end{aligned}$$

Now since T_f is a unitary operator,then we have

$$T_{f^{-1}} = (T_f)^{-1} = T_f^*$$

hence

$$(T_{f^{-1}})^* = T_f$$

(3.4.2)becomes:

$$\begin{aligned}\Big\langle T_{f^{-1}}\psi, K_{\Omega_2}(.,\zeta)\Big\rangle_{L^2(\Omega_2)} &= \Big\langle \psi, T_f K_{\Omega_2}(.,\zeta)\Big\rangle_{L^2(\Omega_1)} && (3.4.4)\\ &= \int_{\Omega_1} \psi(\omega)\overline{K_{\Omega_2}(f(\omega),\zeta) det\mathbb{J}_{\mathbb{C}} f(\omega)} d\lambda(\omega) && (3.4.5)\\ &= \int_{\Omega_1} \psi(\omega) K_{\Omega_2}(f(z), f(\omega))\overline{det\mathbb{J}_{\mathbb{C}} f(\omega)} d\lambda(\omega) && (3.4.6)\end{aligned}$$

Identifying (3.4.3)and(3.4.6)we obtain

$$\psi(z) = \int_{\Omega_1} \psi(\omega) det\mathbb{J}_{\mathbb{C}}f(z) K_{\Omega_2}(f(z), f(\omega)) \overline{det\mathbb{J}_{\mathbb{C}}f(\omega)} d\lambda(\omega)$$

The right hand-side of the above equality has the same properties of the reproducing kernel of the domain Ω_1, so the equality of the two functions follows.This finishes the proof.

□

Remark 3.13. *The last theorem has a nice application in computation of the Bergman Kernel of some simply connected domain in $\mathbb{C}$ namely the upper-half plane (see next section). It allows us furthermore to express the Bergman projection of a domain in terms of bi-holomorphic maps.*

Corollary 3.14. *Let Ω_1,Ω_2 be two domains in $\mathbb{C}^n$. Denote P_1 respectively P_2 the Bergman projections operators. Let $f : \Omega_1 \to \Omega_2$ be a bi-holomorphic map. Then for all $g \in L^2(\Omega_2)$ we have*

$$P_1\Big((det\mathbb{J}_{\mathbb{C}}f)(g \circ f)\Big) = (det\mathbb{J}_{\mathbb{C}}f)\Big(P_2(g \circ f)\Big)$$

Proof. This identity follows directly from the last theorem and the formula for the Bergman projection. Indeed,

$$\begin{aligned}
P_1\Big((det\mathbb{J}_{\mathbb{C}}f)(g \circ f)\Big)(z) &= \int_{\Omega_1} (det\mathbb{J}_{\mathbb{C}}f(\omega))(g \circ f(\omega)) K_{\Omega_1}(z, \omega) d\lambda(\omega) \\
= \int_{\Omega_1} (det\mathbb{J}_{\mathbb{C}}f(\omega))(g \circ f(\omega)) & det\mathbb{J}_{\mathbb{C}}f(z) K_{\Omega_2}\Big(f(z), f(\omega)\Big) \overline{det\mathbb{J}_{\mathbb{C}}f(\omega)} d\lambda(\omega) \\
&= det\mathbb{J}_{\mathbb{C}}f(z) \int_{\Omega_1} \Big|(det\mathbb{J}_{\mathbb{C}}f(\omega))\Big|^2 (g \circ f)(\omega) K_{\Omega_2}\Big(f(z), f(\omega)\Big) d\lambda(\omega) \\
&\underset{\zeta = f(\omega)}{=} det\mathbb{J}_{\mathbb{C}}f(z) \int_{\Omega_2} K_{\Omega_2}\Big(f(z), \zeta\Big) g(\zeta) d\lambda(\zeta) \\
&= (det\mathbb{J}_{\mathbb{C}}f)\Big(P_2(g \circ f)\Big)
\end{aligned}$$

□

In what follows,we give a short introduction to Hermitian metrics and Bergman metric, for more details on this topic see[12].

Definition 3.15. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain.A Riemanian metric on Ω is a matrix $\mathscr{A}(x) = \Big(a_{i,j}(x)\Big)_{i,j\geq 1}^{n}$ satisfying the following condition*
For every $\zeta \in \mathbb{R}^n$ we have:

$$\zeta \mathscr{A} \zeta^\top \geq 0$$

$\zeta^\top$ stands for the transpose of the vector ζ.
In this way we assign to each vector $x \in \mathbb{R}^n$ a positive definite quadratic form. For each vector ζ we can define its length (for each x) by:

$$\|\zeta\|_x = \sqrt{\zeta \Big(a_{i,j}(x)\Big)_{i,j\geq 1}^{n} \zeta^\top}$$

A curve in Ω is a C^1 map $\gamma : [0,1] \to \Omega$.We usually define its length by:

$$l(\gamma) = \int_0^1 |\gamma'(t)| dt$$

In our case, the length of the curve with respect to the Riemanian metric $\mathscr{A}$ is defined by

$$l_{\mathscr{A}}(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt$$

In the same way we may define the distance, with respect to this metric,between $p, q \in \Omega$ by:

$$d_{\mathscr{A}}(p,q) = \inf \Big\{ l_{\mathscr{A}}(\gamma)\,, \gamma\, piecewise\, \mathcal{C}^1\, curve\, such\, that\, \gamma(0) = p\, and\, \gamma(1) = q \Big\}$$

Example 3.16. *a) Let $\Omega = \mathbb{D}$ be the unit disc in the plane and take*

$$\mathscr{A}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider the segment $\gamma(t) = (\lambda t, 0)\,, t \in [0,1]$.
We have$\gamma'(t) = (\lambda, 0)$ and

$$\begin{aligned} \|\gamma'(t)\|_{\mathscr{A}} &= \sqrt{(\lambda,0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \end{pmatrix}} \\ &= \lambda \end{aligned}$$

Therefore

$$l_{\mathscr{A}}(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt = \int_0^1 \lambda dt$$
$$= \lambda$$

b)We consider now the following metric defined on the unit disc of the complex plane by

$$\mathscr{A}(z) = \begin{pmatrix} \frac{1}{(1-|z|^2)^2} & 0 \\ 0 & \frac{1}{(1-|z|^2)^2} \end{pmatrix}$$

For the same curve as above we have

$$\begin{aligned} l_{\mathscr{A}}(\gamma) &= \int_0^1 \sqrt{(\lambda, 0)\begin{pmatrix} \frac{1}{(1-|\lambda t|^2)^2} & 0 \\ 0 & \frac{1}{(1-|\lambda t|^2)^2} \end{pmatrix}\begin{pmatrix} \lambda \\ 0 \end{pmatrix}} dt \\ &= \lambda \int_0^1 \frac{1}{1-(\lambda t)^2} dt \\ &= \frac{1}{2} log\Big(\frac{1+\lambda}{1-\lambda}\Big) \end{aligned}$$

This metric is a version of the Poincare metric on the disc, and we see that the length of the segment joining 0 and λ is unbounded when $\lambda \to 0$

Definition 3.17. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $K = K_\Omega$ its corresponding Bergman Kernel. The Bergman metric is defined via*

$$\mathscr{A}(z) = \Big(a_{i,j}(z)\Big)_{i,j=1}^n = \Big(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} logK(z,z)\Big)_{i,j=1}^n$$

This means that the length of a tangent vector $\zeta = (\zeta_1, \ldots, \zeta_n)$ at a point $z \in \Omega$ is given by

$$l_{\mathscr{A}}(\zeta) = \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_{i,j}(z) \zeta_i \bar{\zeta_j}}$$

This definition is meaningful since $K(z,z) \geq 0$.(see corollary (3.11)). The next result shows that the Bergman metric is invariant under biholomorphic maps.
Let Ω_1 and Ω_2 two domains in $\mathbb{C}^n$ and $\Phi : \Omega_1 \to \Omega_2$ a biholomorphic map.Let

also $\gamma : [0,1] \to \Omega_1$ be a C^1 curve, define its image γ^* by $\gamma^*(t) = \Phi \circ \gamma$.
γ^* is in fact a curve in Ω_2.
Denote $\mathscr{A}_{\Omega_1}(z) = \Big(a_{i,j}^{\Omega_1}(z)\Big)_{i,j=1}^{n}$ and $\mathscr{A}_{\Omega_2}(z) = \Big(a_{i,j}^{\Omega_2}(z)\Big)_{i,j=1}^{n}$ the Bergman metric of Ω_1 resp Ω_2

Proposition 3.18. *Under the same notations as above we have*

$$l_{\mathscr{A}_{\Omega_1}}(\gamma) = l_{\mathscr{A}_{\Omega_2}}(\gamma^*)$$

Proof. Our aim is to show that

$$\int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt = \int_0^1 \|(\gamma^*)'(t)\|_{\gamma^*(t)}\, dt$$

It suffices to show that the integrands are equal.
By the transformation formula (4.3.1) we have

$$det\mathbb{J}_{\mathbb{C}}\Phi(z) K_{\Omega_2}\Big(\Phi(z), \Phi(z)\Big)\overline{det\mathbb{J}_{\mathbb{C}}\Phi(z)} = K_{\Omega_1}(z,z)$$

Applying Logarithms on both sides we obtain

$$Log(det\mathbb{J}_{\mathbb{C}}\Phi(z)) + Log(K_{\Omega_2}\Big(\Phi(z), \Phi(z)\Big) + Log(\overline{det\mathbb{J}_{\mathbb{C}}\Phi(z)}) = Log(K_{\Omega_1}(z,z))$$

Apply now the mixed derivative, the right hand side is nothing but the Bergman metric of Ω_1. The first and the last terms of the left hand side vanish, and we are left only with the term

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} Log(K_{\Omega_2}\Big(\Phi(z), \Phi(z)\Big)$$

which is exactly

$$\sum_{l=1}^{n} \sum_{m=1}^{n} \underbrace{\frac{\partial^2}{\partial z_l \partial \bar{z_m}} Log(K_{\Omega_2}\Big(\Phi(z), \Phi(z)\Big)}_{\left(a_{l,m}^{\Omega_2}(\Phi(z)\right)} \frac{\partial \Phi_l(z)}{\partial z_i} \frac{\partial \Phi_m(z)}{\partial \bar{z}_j}$$

Therefore we have:

$$a_{i,j}^{\Omega_1}(z) = \sum_{l,m} a_{l,m}^{\Omega_2}(\Phi(z)) \frac{\partial \Phi_l(z)}{\partial z_i} \frac{\partial \Phi_m(z)}{\partial \bar{z}_j}$$

$\square$

3.5 Computation of the Bergman Kernel

We start now with the computation of the Bergman kernel of some special domains.

3.5.1 Bergman kernel of the unit disc

As usual we denote

$$\mathbb{D}(0,1) := \left\{z \in \mathbb{C}, |z| < 1\right\}$$

Lemma 3.19. *The normalized monomials* $\Phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$ *constitute an orthonormal basis of the space* $\mathbb{A}^2(\mathbb{D})$

Proof. **Orthonormality**
For $n \neq m$ using polar coordinates we obtain

$$\begin{aligned}
(\Phi_n, \Phi_m) &= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_{\mathbb{D}(0,1)} z^n \overline{z^m} d\lambda(z) \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m} e^{i(n-m)\theta} r dr d\theta \\
&= \frac{\sqrt{(n+1)(m+1)}}{\pi} \frac{2\pi}{n+m+2} \delta_{n,m} \quad ,
\end{aligned}$$

where,

$$\delta_{m,n} = \begin{cases} 1 & if \quad n \neq m \\ 0 & else \end{cases}$$

is the Kronecker symbol.
This shows that $(\Phi_n, \Phi_m) = 0$ if $n \neq m$.

Completeness
In order to show that the family $\{\Phi_n\}$ is completed, we have to show that:

$$\mathbb{M} = \overline{span\{\Phi_n\}} = \mathbb{A}^2(\mathbb{D})$$

which is equivalent to

$$\mathbb{M}^\perp = \{0\}$$

Let $f \in \mathbb{M}^\perp . f \perp z^n, \forall n$ i.e $0 = \int_{\mathbb{D}} f(z)\overline{z^n} d\lambda(z)$
Since f is entire, it admits a Taylor expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges uniformly on each compact subset of the unit disc.
By Lebesgue's dominated convergence theorem we may write:

$$\begin{aligned}
0 &= \int_{\mathbb{D}} f(z)\overline{z^n} d\lambda(z) \\
&= \lim_{\rho\to 1} \int_{\mathbb{D}(0,\rho)} \left(\sum_{k=0}^{\infty} a_k z^k\right) \overline{z^n} d\lambda(z) \\
&= \lim_{\rho\to 1} \sum_{k=0}^{\infty} a_k \underbrace{\int_0^{\rho}\int_0^{2\pi} r^{k+n+1} e^{i(k-n)\theta} d\theta dr\, 2\pi\delta_{n,k}}_{\frac{r^{k+n+2}}{k+n+2}} \\
&= a_n \lim_{\rho\to\infty} \frac{r^{2n+2}}{2n+2} 2\pi \\
&= a_n \frac{\pi}{n+1}
\end{aligned}$$

This implies that $a_n = 0, \forall n \in \mathbb{N}$ and thus $f \equiv 0$ □

Now applying(3.10) for this basis we obtain, for each $z, \omega \in \mathbb{D}$

$$\begin{aligned}
K_{\mathbb{D}}(z,\omega) &= \sum_{n=0}^{\infty} \Phi_n(z)\overline{\Phi_n(\omega)} && (3.5.1)\\
&= \sum_{n=0}^{\infty} \frac{(n+1)}{\pi} (z\bar{\omega})^n && (3.5.2)\\
&= \frac{1}{\pi}\frac{1}{(1-z\bar{\omega})^2} && (3.5.3)
\end{aligned}$$

Proposition 3.20. *The Bergman kernel of the polydisc $\mathbb{P}(0,1) \subseteq \mathbb{C}^n$ is given by*

$$K_{\mathbb{P}}(z,\omega) = \frac{1}{\pi^n}\prod_{k=1}^{n} \frac{1}{(1-z_j\bar{\omega_j})^2}$$

Proof. By the last lemma we know that, the Bergman kernel of the unit disc $\mathbb{B}(0,1) \subseteq \mathbb{C}$ has the following expression

$$K_{\mathbb{B}}(z_1,\omega_1) = \frac{1}{\pi}\frac{1}{(1-z_1\bar{\omega_1})^2}$$

Claim Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ and $\Omega = \Omega_1 \times \Omega_2$.Then

$$K_{\Omega} = K_{\Omega_1}.K_{\Omega_2}$$

Indeed,let $f \in \mathbb{A}^2(\Omega)$.By the reproducing formula we have

$$f(z) = \int_{\Omega} f(\omega) K_{\Omega}(z, \omega) d\lambda(\omega) \quad (*)$$

Where $z = (z_1, z_2)$ and $\omega = (\omega_1, \omega_2)$. Now fix z_2, and consider the map

$$\phi : \Omega_1 \to \mathbb{C}$$

$$z_1 \to f(z_1, z_2)$$

We see easily that $f \in \mathbb{A}(\Omega_1)$ Therefore,

$$\begin{aligned} f(z_1, z_2) &= \int_{\Omega_1} f(\omega_1, z_2) K_{\Omega_1}(z_1, \omega_1) d\lambda(\omega_1) \\ &= \int_{\Omega_1} \underbrace{\left(\int_{\Omega_2} f(\omega_1, \omega_2) K_{\Omega_2}(z_2, \omega_2) d\lambda(\omega_2) \right)}_{\textit{reproducing formula for } \Omega_2} K_{\Omega_1}(z_1, \omega_1) d\lambda(\omega_1) \\ &\overset{Fubini}{=} \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) K_{\Omega_2}(z_2, \omega_2) K_{\Omega_1}(z_1, \omega_1) d\lambda(\omega_1) d\lambda(\omega_2) \quad (**) \end{aligned}$$

Identifying $(*)$ and $(**)$ we obtain the result, and by an induction process we can prove even prove that, if $\Omega_1 \ldots \Omega_n \subseteq \mathbb{C}$.Then

$$K_{\prod_{i=1}^{n} \Omega_i} = \prod_{i=1}^{n} K_{\Omega_i}$$

Now since

$$\mathbb{P} = \prod_{i=1}^{n} \mathbb{B}(0, 1)$$

we obtain that, for $z = (z_1, \ldots z_n)$ and $\omega = (\omega_1, \ldots \omega_n)$

$$\begin{aligned} K_{\mathbb{P}}(z, \omega) &= \prod_{i=1}^{n} K_{\mathbb{B}}(z_i, \omega_i) \\ &= \frac{1}{\pi^n} \prod_{k=1}^{n} \frac{1}{(1 - z_j \bar{\omega_j})^2} \end{aligned}$$

□

3.5.2 Bergman kernel of the upper-half plane

In theorem (3.12), we saw how the Bergman Kernel behaves under biholomorphic maps so we will take advantage of this tool to derive an explicit formula of the kernel function of the upper half plane in $\mathbb{C}$.
We define the upper-half plane of the one dimensional complex plane by

$$\mathbb{U} = \Big\{ z \in \mathbb{C} : Imz > 0 \Big\}$$

This domain is simply connected by the Riemann mapping theorem, indeed if we set

$$\phi : \mathbb{U} \to \mathbb{D}$$

defined by

$$\phi(z) = \frac{z-i}{z+i}$$

we see immediately that ϕ is biholomorphic and even conformal.

Proposition 3.21. *For all $z, \omega \in \mathbb{U}$ we have*

$$K_{\mathbb{U}}(z,\omega) = \frac{-1}{\pi(z-\bar{\omega})}$$

Proof. By theorem (3.12) we have for each $z, \omega \in \mathbb{U}$:

$$K_{\mathbb{U}}(z,\omega) = K_{\mathbb{D}}\Big(\phi(z), \phi(\omega)\Big)\phi'(z)\overline{\phi'(\omega)}$$

$$\phi'(z) = \frac{2i}{(z+i)^2}$$

By(3.5.3)we have:

$$K_{\mathbb{D}}(\zeta,\eta) = \frac{1}{\pi(1-\zeta\bar{\eta})}$$

with some intermediary computation we get,

$$\begin{aligned} K_{\mathbb{U}}(z,\omega) &= \frac{1}{\pi\Big(1 - \frac{z-i}{z+i}\frac{(\bar{\omega}+i)}{(\bar{\omega}-i)}\Big)^2} \frac{2i}{(z+i)^2} \overline{\left\{\frac{2i}{(\omega+i)^2}\right\}} \\ &= \frac{4}{\pi(-2iz+2i\bar{\omega})^2} \\ &= \frac{-1}{\pi(z-\bar{\omega})} \end{aligned}$$

$\square$

Remark 3.22. *Knowing explicitly the Riemann function of a simply connected domain, one can compute the corresponding Bergman kernel and vice versa.The next result gives the relationship between the Riemann function and the reproducing kernel of a simply connected domain.*

Proposition 3.23. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, $a \in \Omega$ and let ϕ be the corresponding Riemann map with the property that $\phi(a) = 0$ and $\phi'(a) > 0$.*
Let K_Ω be the corresponding Bergman kernel. Then

$$\phi'(z) = \sqrt{\frac{\pi}{K_\Omega(a,a)}} K_\Omega(z,a)$$

Proof. By theorem(3.12)we have

$$K_\Omega(z,a) = K_{\mathbb{D}}\Big(\phi(z), \phi(a)\Big)\phi'(z)\overline{\phi'(a)} \tag{3.5.4}$$

$$= \frac{1}{\pi\Big(1 - \underbrace{\phi(z)\overline{\phi(a)}}_{=0}\Big)^2}\phi'(z)\phi'(a) \tag{3.5.5}$$

We have furthermore

$$K_\Omega(a,a) = \Big|\phi'(a)\Big|^2$$

and by corollary(3.11)we have

$$K_{\Omega)}(a,a) > 0$$

this leads to

$$\phi'(a) = \sqrt{K_\Omega(a,a)}$$

The desired result is obtained by inserting the last equality in(3.5.5).

Remark 3.24. *Practical application of this result is the computation of the Riemann function of a simply connected domain. Indeed, having an ON-basis of the space $\mathbb{A}^2(\Omega)$ leads to the desired map. At least, if the polynomials are dense in the Bergman space $\mathbb{A}^2(\Omega)$ one can get an ON-basis using the procedure of Gram-Schmidt applied to monomials.*

□

3.5.3 Bergman kernel of the ball in $\mathbb{C}^n$

Now, we restrict our attention to the unit ball $\mathbb{B} \subseteq \mathbb{C}^n$.

Lemma 3.25. *The set of monomials*

$$\{z^\alpha, \alpha = (\alpha_1 \dots \alpha_n), \alpha_j \in \mathbb{N}_0\} \quad ,$$

is a an orthogonal complete system of the space $\mathbb{A}^2(\mathbb{B})$.

Proof. We begin with completeness.
Let f $\in \mathbb{A}^2(\mathbb{B})$ such that for each α,

$$\left\langle f, z^\alpha \right\rangle = 0$$

we have to show that

$$f \equiv 0$$

Since f is analytic on $\mathbb{B}$, it can be locally represented by a power series (Taylor expansion) converging uniformly on compact subsets.
Let $z \in \mathbb{B}$ and $0 < \epsilon < 1$ consider the sets

$$I_\epsilon = \left\{z : \|z\| \leq 1 - \epsilon\right\}$$

$$j_\epsilon = \left\{z : 1 - \epsilon < \|z\| < 1\right\}$$

we have

$$\begin{aligned} 0 = \left\langle f, z^\alpha \right\rangle &= \int_{\mathbb{B}} f(z)\overline{z}^\alpha d\lambda(z) \\ &= \int_{I_\epsilon} f(z)\overline{z}^\alpha d\lambda(z) + \int_{J_\epsilon} f(z)\overline{z}^\alpha d\lambda(z) \end{aligned}$$

The two integrals on the right hand side have the same absolute value, by the triangular inequality we obtain

$$\left|\left\langle f, z^\alpha \right\rangle\right| \leq 2\left|\int_{J_\epsilon} f(z)\overline{z}^\alpha d\lambda(z)\right|$$

Now by the Cauchy -Schwarz inequality and using the fact that $|z^\alpha| < 1$ we obtain

$$\left|\int_{J_\epsilon} f(z)\overline{z}^\alpha d\lambda(z)\right| = \left|\int_{\mathbb{B}} f(z)\overline{z}^\alpha \chi_{J_\alpha}(z) d\lambda(z)\right| \tag{3.5.6}$$

$$\leq \underbrace{\Big(\int_{\mathbb{B}} |f(z)^2| d\lambda(z)\Big)^{\frac{1}{2}}}_{\|f\|_{\mathbb{A}^2}} \underbrace{\Big(\int_{\mathbb{B}} \chi_{J_\alpha}(z) d\lambda(z)\Big)^{\frac{1}{2}}}_{\sqrt{V(J_\epsilon)}} \to 0 \, as \, \epsilon \to 0 \tag{3.5.7}$$

$V(J_\epsilon)$denotes the volume of the region J_ϵ which tends to zero as ϵ goes to zero.

Now by Taylor expansion we have

$$f(z) = \sum_\mu C_\mu z^\mu$$

with uniform convergence on I_ϵ therefore we may interchange summation and integration to obtain:

$$\int_{I_\epsilon} f(z)\overline{z}^\alpha d\lambda(z) = \int_{I_\epsilon} \sum_\mu C_\mu z^\mu \overline{z}^\alpha d\lambda(z) \tag{3.5.8}$$

$$= C_\alpha \int_{I_\epsilon} |z^\alpha|^2 d\lambda(z) \tag{3.5.9}$$

The second line is due to the orthogonality of the monomials (the only contribution arises when $\mu = \alpha$).
Combining(3.5.7)and (3.5.9)we get

$$|C_\alpha| \, \|z^\alpha\|^2_{\mathbb{A}^2} \leq 0$$

Thus $C_\alpha = 0$ and since α was arbitrary, we conclude that $f \equiv 0$ □

Set

$$\gamma_\alpha = \int_{\mathbb{B}} |z^\alpha|^2 d\lambda(z)$$

we see that $\{\frac{z^\alpha}{\sqrt{\gamma_\alpha}}\}$ is an orthonormal basis of $\mathbb{A}^2(\mathbb{B})$. Thus

$$K_{\mathbb{B}}(z, \omega) = \sum_\alpha \frac{z^\alpha \bar{\omega}^\alpha}{\gamma_\alpha}$$

In order to compute the Bergman function of the ball explicitly we have to determine the coefficients γ_α.For this purpose we need some lemmas related to real analysis, these lemmas are formulated in $\mathbb{R}^N$. For more details see[34] (Topics in Real and Functional analysis pages 134-140).

Lemma 3.26. *We have that*

$$\int_{\mathbb{R}^N} e^{-\pi|x|^2} dx = 1$$

Proof. For $N = 1$ we have

$$I = \int_{\mathbb{R}} e^{-\pi|x|^2} dx = 2 \underbrace{\int_0^{\infty} e^{-\pi|x|^2} dx}_{\mathbf{J}}$$

By Fubini's theorem we have

$$\begin{aligned} J^2 &= \left(\int_0^{\infty} e^{-\pi|x|^2} dx \right)^2 \\ &= \int_0^{\infty} \int_0^{\infty} e^{-\pi(x^2+y^2)} dx dy \end{aligned}$$

Let now:

$$\begin{aligned} \varphi : [0, \frac{\pi}{2}] \times [0, \infty[&\to [0, \infty[\times [0, \infty[\\ (\theta, r) &\to (r\cos(\theta), r\sin(\theta)) \end{aligned}$$

φ is a C^{∞}- diffeomorphism and its Jacobian is

$$\mathbb{J}_{\varphi}(r, \theta) = \begin{vmatrix} \partial_{\theta}\varphi_1 & \partial_{\theta}\varphi_2 \\ \partial_r\varphi_1 & \partial_r\varphi_2 \end{vmatrix} = -r$$

By the measure transformation's theorem (or change of variables theorem)we obtain:

$$\begin{aligned} J^2 &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\pi(r^2\cos^2\theta + r^2\sin^2\theta)} |\mathbf{J}_{\varphi}| dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\pi r^2} r dr d\theta = \frac{1}{4} \end{aligned}$$

Thus $J = \frac{1}{2}$ and therefore $I = 1$.
For $N \geq 1$ applying Fubini once more we obtain

$$\int_{\mathbb{R}^N} e^{-\pi|x|^2} dx = \int_{\mathbb{R}} e^{-\pi x^2} dx \ldots \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$$

$\square$

Let σ the unique rotationally invariant area measure on $S_{N-1} = \partial\mathbb{B}_N$ i.e the measure of the unit sphere in $\mathbb{R}^N$ and let $\omega_{N-1} = \sigma(S_{N-1})$

Lemma 3.27. *We have*

$$\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Proof. For this proof I follow [25]. Using polar coordinates we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}dx \\
&= \underbrace{\int_{S^{N-1}} d\sigma}_{\omega_{N-1}} \int_0^\infty e^{-\pi r^2} r^{N-1}dr \\
&= \frac{\omega_{N-1}}{2\pi^{\frac{N}{2}}} \underbrace{\int_0^\infty e^{-s}s^{\frac{N}{2}-1}ds}_{\Gamma(\frac{N}{2})}
\end{aligned}$$

The last equality is obtained by setting $s = \pi r^2$. $\square$

Now return to $\mathbb{B} \subseteq \mathbb{C}^n$.We set

$$\eta(k) = \int_{\partial\mathbb{B}} |z_1|^{2k}d\sigma \quad , N(k) = \int_{\mathbb{B}} |z_1|^{2k}d\lambda(z)$$

Lemma 3.28. *We have*

$$\eta(k) = \pi^n \frac{2k!}{(k+n-1)!}$$

and

$$N(k) = \pi^n \frac{k!}{(k+n)!}$$

Proof. Polar coordinates again show that $\eta(k) = 2(k+n)N(k)$.So it suffices to compute the $N(k)$.

Let $z = (z_1 \dots z_n) = (z, z')$.

$$\begin{aligned}
N(k) &= \int_{|z|<1} |z_1| d\lambda(z) \\
&= \int_{|z'|<1} \left(\int_{|z_1| \le \sqrt{1-|z'|^2}} |z_1|^{2k} d\lambda(z_1) \right) d\lambda(z') \\
&= 2\pi \int_{|z'|<1} \int_0^{\sqrt{1-|z'|^2}} r^{2k} r dr d\lambda(z') \\
&= 2\pi \int_{|z'|<1} \frac{(1-|z'|^2)^{k+1}}{2k+2} d\lambda(z') \\
&= \frac{\pi}{k+1} \omega_{2n-3} \int_0^1 (1-r^2)^{k+1} r^{2n-3} dr \\
&= \frac{\pi}{2(k+1)} \omega_{2n-3} \int_0^1 (1-s)^{k+1} s^{n-1} \frac{ds}{2s} \\
&= \frac{\pi}{2(k+1)} \omega_{2n-3} \beta(n-1, k+2)
\end{aligned}$$

Where β is the classical Euler function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

and characterized by the fundamental identity

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Using this formula and the fact that $\Gamma(n+1) = n!$ we obtain

$$\begin{aligned}
N(k) &= \frac{\pi}{2(k+1)} \omega_{2n-3} \frac{\Gamma(n-1)\Gamma(k+2)}{\Gamma(n+k+1)} \\
&= \frac{\pi}{2(k+1)} \frac{2\pi^{n-1}}{\Gamma(n-1)} \frac{\Gamma(n-1)\Gamma(k+2)}{\Gamma(n+k+1)} \\
&= \frac{\pi^n k!}{(k+n)!}
\end{aligned}$$

This leads to the desired result.

□

Lemma 3.29. *Let $z \in \mathbb{B} \subseteq \mathbb{C}^n$ and $0 < r < 1$. The symbol $\mathbb{I}$ denotes the point $(1, 0, \dots 0)$. We have*

$$K_{\mathbb{B}}(z, r\mathbb{I}) = \frac{n!}{\pi^n} \frac{1}{(1-rz_1)^{n+1}}$$

Proof.

$$\begin{aligned}
K_{\mathbb{B}}(z, r\mathbb{I}) &= \sum_{\alpha} \frac{z^\alpha \overline{(r\mathbb{I})}^\alpha}{\gamma_\alpha} = \sum_{k=0}^{\infty} \frac{z_1^k r^k}{N(k)} \\
&= \frac{1}{\pi^n} \sum_{k=0}^{\infty} (rz_1)^k \frac{(k+n)!}{k!} \\
&= \frac{n!}{\pi^n} \sum_{K=0}^{\infty} (rz_1)^k \begin{pmatrix} k+n \\ n \end{pmatrix} \\
&= \frac{n!}{\pi^n} \frac{1}{(1-rz_1)^{n+1}}
\end{aligned}$$

In the last step we used the fact that $|rz_1| < 1$,hence

$$\frac{1}{1-rz_1} = \sum_{k=0}^{\infty} (rz_1)^k$$

Set $y = rz_1$ and differentiate n times with respect to this variable. □

Theorem 3.30. *For each $z, \omega \in \mathbb{B}$ we have*

$$K_{\mathbb{B}}(z, \omega) = \frac{n!}{\pi^n} \frac{1}{(1 - z\bar{\omega})^{n+1}}$$

where

$$z\bar{\omega} = z_1\bar{\omega}_1 \dots z_n\bar{\omega}_n$$

Proof. Let $z = r\tilde{z} \in \mathbb{B}$ where $r = |z|$ and $|\tilde{z}| = 1$.Fix also,$\omega \in \mathbb{B}$. Choose an unitary rotation ρ such that $\rho(\tilde{z}) = \mathbb{I}$,

$$\begin{aligned}
K_{\mathbb{B}}(z, \omega) = K_{\mathbb{B}}(r\tilde{z}, \omega) = K_{\mathbb{B}}(r\rho^{-1}\mathbb{I}, \omega) = K_{\mathbb{B}}(r\mathbb{I}, \rho\omega) &= \overline{K_{\mathbb{B}}(\rho\omega, r\mathbb{I})} \\
= \frac{n!}{\pi^n} \frac{1}{(1 - r(\overline{\rho\omega})_1)^{n+1}} &= \frac{n!}{\pi^n} \frac{1}{(1 - r\mathbb{I}(\overline{\rho\omega}))^{n+1}} \\
= \frac{n!}{\pi^n} \frac{1}{\Big(1 - (r\rho^{-1}\mathbb{I})(\overline{\omega})\Big)^{n+1}} &= \frac{n!}{\pi^n} \frac{1}{\Big(1 - z\overline{\omega}\Big)^{n+1}}
\end{aligned}$$

This gives the desired result. □

Remark 3.31. *An other method to compute the $\mathbb{A}^2$-norm of the monomials was presented in [11], we recall it in the next section with more details.*

Remark 3.32. *An other approach of the Bergman kernel of the ball in $\mathbb{C}^n$ can be found in [32].*

3.6 Bergman kernel of the ellipsoid

The main reference of this paragraph is[11]:
In the sequel we prove some results giving explicitly the Bergman kernel function of special domains namely ellipsoid and annulus.
Let p positive number. We consider the domain in Euclidean space $\mathbb{C}^{n+m}$ where m,n are integers,

$$\Omega_p = \Big\{(z,\omega) \in \mathbb{C}^n \times \mathbb{C}^m \,/\, \|z\|^2 + \|\omega\|^{2p} < 1\Big\}$$

For p=1, Ω_p is nothing than the unit ball in $\mathbb{C}^{n+m}$
In proposition (3.22) we have shown that whenever $\{\Phi_n\}$ is an ON basis of the the Hilbert space $\mathbb{A}^2(\Omega)$ then the orthonormal series,

$$\sum_n \Phi_n(z)\overline{\Phi_n}(\omega)$$

sums uniformly to the Bergman kernel function. We will use this method to compute the desired function.
For this purpose we need some preparation of calculus.
Notations
i) An element $\zeta \in \mathbb{C}^{n+m}$ will be denoted as follows $\zeta = (z,\omega)$ where $z \in \mathbb{C}^n$ and $\omega \in \mathbb{C}^m$.
ii)In the same way we denote a multi-index $\mu \in \mathbb{N}^{n+m}$ by $\mu = (\alpha,\gamma)$,$\alpha \in \mathbb{N}^n$ and $\gamma \in \mathbb{N}^m$.
According to these notations we write,$\zeta^\mu = (z^\alpha, \omega^\gamma)$.
iii) To avoid any confusion, for a multi-index $\alpha = (\alpha_1 \dots \alpha_n)$ we denote $\alpha + 1 = (\alpha_1 + 1 \dots \alpha_n + 1)$.

Definition 3.33. *The Euler Gamma function is defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

this function is well defined, continuous on its domain of definition $]0,\infty[$ *and even of class* $\mathcal{C}^\infty$.
This function has the basic property that, for $x \in]0,\infty[$,

$$\Gamma(x+1) = x\Gamma(x).$$

Thus, for every integer n,

$$\Gamma(n+1) = n!\,.$$

and for every $s, k \in \mathbb{R}^+$.

$$s(s+1)\dots(s+k-1)\Gamma(s) = \Gamma(s+k). \tag{3.6.1}$$

The Gamma function can be seen as an extension of the factorial function to non-negative real numbers.
For multi-index $\alpha = (\alpha_1 \dots \alpha_n)$, $\alpha_j \in \mathbb{R}_+$, *we set:*

$$\Gamma(\alpha) = \prod_{j=1}^{n} \Gamma(\alpha_j).$$

Lemma 3.34. *For every positive real numbers a,b. We have:*

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proof. By making the change of variable $t = s^2$

$$\Gamma(a) = 2\int_0^{\infty} s^{2a-1}e^{-s^2}$$

Now Fubini's theorem and polar coordinates give:

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= 4\int_0^{\infty}\int_0^{\infty} s^{2a-1}t^{2b-1}e^{-(s^2+t^2)}dsdt \\
&= 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} r^{2(a+b)-1}e^{-r^2}cos^{2a-1}(\theta)sin^{2b-1}(\theta)drd\theta \\
&= \underbrace{\left(2\int_0^{\infty} r^{2(a+b)-1}e^{-r^2}dr\right)}_{\Gamma(a+b)}\underbrace{\left(2\int_0^{\frac{\pi}{2}} cos^{2a-1}(\theta)sin^{2b-1}(\theta)drd\theta\right)}_{I}
\end{aligned}$$

Setting $y = sin\theta$, I becomes:

$$\begin{aligned}
I &= \int_0^1 (1-y^2)^{a-1}y^{2b-2}2ydy \\
&\overset{u=y^2}{=} \int_0^1 (1-u)^{a-1}u^{b-1}du = \beta(a,b).
\end{aligned}$$

□

Definition 3.35. *In* $\mathbb{R}^n$ *we consider the unit ball,*

$$B^n = \left\{X = (x_1 \dots, x_n) \,/\, \sum_{j=1}^{n} x_j^2 < 1\right\}$$

The unit sphere will be denoted by S^{n-1}. When we wish to consider the subset of these sets for which all the variables are positive, we use $+$ as a subscript. For $\alpha = (\alpha_1 \ldots \alpha_n)$, $\alpha_j \in \mathbb{R}_+$,we define:

$$\mathscr{B}(\alpha) = 2^n|\alpha| \int_{B^n} r^{2\alpha-1} dr.$$

For $n = 2$ and $\alpha = (\alpha_1, \alpha_2)$ we retrieve the usual β function. Indeed,

$$\begin{aligned}
\mathscr{B}(\alpha_1, \alpha_2) &= 4(\alpha_1 + \alpha_2) \int_{\mathbb{B}(0,1)} x^{2\alpha_1-1} y^{2\alpha_2-1} dxdy \\
&= 4(\alpha_1 + \alpha_2) \int_0^1 \int_0^{\sqrt{1-x^2}} x^{2\alpha_1-1} y^{2\alpha_2-1} dxdy \\
&= \frac{\alpha_1 + \alpha_2}{\alpha_2} \int_0^1 x^{2\alpha_1-2}(1 - x^2)^{\alpha_2} dx \\
&\overset{u=x^2}{=} \frac{\alpha_1 + \alpha_2}{\alpha_2} \underbrace{\int_0^1 u^{\alpha_1-1}(1 - u)^{\alpha_2} du}_{J(\alpha_1,\alpha_2)}.
\end{aligned}$$

$$\begin{aligned}
J(\alpha_1, \alpha_2) &= \beta(\alpha_1, \alpha_2 + 1) \\
&= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_1 + \alpha_2 + 1)} \\
&= \frac{\alpha_2\Gamma(\alpha_1)\Gamma(\alpha_2)}{(\alpha_1 + \alpha_2)\Gamma(\alpha_1 + \alpha_2)}
\end{aligned}$$

Hence

$$\mathscr{B}(\alpha_1, \alpha_2) = \beta(\alpha_1, \alpha_2).$$

Lemma 3.36. *For $\alpha = (\alpha_1 \ldots \alpha_n)$, $\alpha_j \in \mathbb{R}_+$ n-tuples of real positive numbers. We have,*

$$\mathscr{B}(\alpha) = \frac{\prod_{j=1}^n \Gamma(\alpha_j)}{\Gamma(|\alpha|)}.$$

Remark 3.37. *For the proof of this lemma we need the following identity*

$$\int_{B^n} f d\lambda = 2n \int_0^1 r^{2n-1} dr \int_{S^{n-1}} f(r\zeta) d\sigma(\zeta).$$

A proof of this statement can be found in the book of Rudin [32].

Proof. By definition we have:

$$\Gamma(\alpha) = \prod_{j=1}^{n} \Gamma(\alpha_j) \tag{3.6.2}$$

$$= 2^n \int_0^\infty x_1^{2\alpha_1} e^{-x_1^2} dx_1 \dots \int_0^\infty x_n^{2\alpha_n} e^{-x_n^2} dx_n \tag{3.6.3}$$

$$= 2^n \int_{]0,\infty[^n} x^{2\alpha-1} e^{-\|x\|^2} dx \tag{3.6.4}$$

$$= 2^n \int_0^\infty \int_{S_+^{n-1}} e^{-\rho^2} \rho^{2|\alpha|-1} \omega^{2\alpha-1} d\rho d\sigma(\omega) \tag{3.6.5}$$

$$= 2^{n-1} \Gamma(|\alpha|) \int_{S_+^{n-1}} \omega^{2\alpha-1} d\sigma(\omega). \tag{3.6.6}$$

Using now spherical coordinates: $X = \rho\omega$, where $\rho \in]0,1[$ and $\omega \in S^{n-1}$(the unit sphere in $\mathbb{R}^n$)we obtain,

$$\frac{\mathscr{B}(\alpha)}{2^n |\alpha|} = \int_{B_+^n} X^{2\alpha-1} dX \tag{3.6.7}$$

$$= \int_0^1 \int_{S^{n-1}} \rho^{2|\alpha|-1} \omega^{2\alpha-1} d\rho d\sigma(\omega) \tag{3.6.8}$$

$$= \frac{1}{2|\alpha|} \int_{S_+^{n-1}} \omega^{2\alpha-1} d\sigma(\omega). \tag{3.6.9}$$

Thus:

$$\int_{S_+^{n-1}} \omega^{2\alpha-1} d\sigma(\omega) = \frac{\mathscr{B}(\alpha)}{2^{n-1}}. \tag{3.6.10}$$

Combining (3.6.6)and (3.6.10) we obtain our equality. □

Next we derive some results related to Taylor expansion of some function helping us to compute the norm of the monomials in $L^2(\Omega_p)$.

Lemma 3.38. *For* $\|x\| < 1$ *and* $s > 0$ *we have*

$$\frac{1}{(1 - \|x\|^2)^s} = \sum_\alpha \frac{\Gamma(|\alpha| + s)}{\Gamma(s) \prod_{i=1}^n \Gamma(\alpha_i + 1)} x^{2\alpha}.$$

Proof. For $|t| < 1$ and $s > 0$.Consider the function $f(t) = \frac{1}{(1-t)^s}$.In a neighbourhood of zero, f has the Taylor expansion:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k \tag{3.6.11}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)\Gamma(k+1)} t^k. \tag{3.6.12}$$

This is since $f^{(k)}(0) = s(s+1)\dots(s+k) = \frac{\Gamma(s+k)}{\Gamma(s)}$.
Now using the fact that

$$\|x\|^{2k} = \sum_{|\alpha|=k} \frac{\Gamma(|\alpha|+1)}{\prod \Gamma(\alpha_i+1)} x^{2k},$$

and substitute t by $\|x\|^2$ in (3.6.12) we obtain:

$$\begin{aligned} f(\|x\|^2) &= \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)\Gamma(k+1)} \sum_{|\alpha|=k} \frac{\Gamma(|\alpha|+1)}{\prod_{i=1}^n \Gamma(\alpha_i+1)} x^{2k} \\ &= \sum_{\alpha} \frac{\Gamma(|\alpha|+1)\cancel{\Gamma(k+1)}}{\Gamma(s)\cancel{\Gamma(k+1)}\prod_{i=1}^n \Gamma(\alpha_i+1)} x^{2k}. \end{aligned}$$

∎

Lemma 3.39. *Let q_d be a polynomial of degree d in one variable. Then there are constants c_k such that*

$$\sum_{\gamma} \frac{\Gamma(|\gamma|+s)}{\prod_{i=1}^n \Gamma(\alpha_i+1)} q_d(|\gamma|) x^{2\gamma} = \sum_{k=0}^{d} c_k \Big(\frac{1}{1-\|x\|^2}\Big)^{s+k}. \tag{3.6.13}$$

Proof. If q_d is constant the previous lemma shows that:

$$\frac{\Gamma(s)}{(1-\|x\|^2)^s} = \sum_{\gamma} \frac{\Gamma(|\gamma|+s)}{\prod_{i=1}^n \Gamma(\gamma+1)} x^{2\alpha}.$$

For the vector space of polynomials of degree d in $|\gamma|$,the set

$$\{1, |\gamma|+s, \dots \prod_{l=0}^{d-1} (|\gamma|+s+l)\}$$

is a basis, since all entries are of different degree thus they are linearly independent.
There exist b_k such that

$$q_d(|\gamma|) = \sum_{k=-1}^{d-1} b_k \prod_{l=0}^{k} (|\gamma| + s + l)$$

with the convention that the product equals 1 for $k = -1$.
Now plug this equality in the left of (3.6.13)and write,

$$\prod_{l=0}^{k} (|\gamma| + s + l) = \frac{\Gamma(|\gamma| + s + k + 1)}{\Gamma(|\gamma| + s)}$$

we obtain:

$$\begin{aligned}
\sum_{\gamma} \frac{\Gamma(|\gamma| + s)}{\prod_{i=1}^{n} \Gamma(\alpha_i + 1)} q_d(|\gamma|) x^{2\gamma} &= \sum_{\gamma} \frac{\Gamma(|\gamma| + s)}{\prod_{i=1}^{n} \Gamma(\alpha_i + 1)} \sum_{k=-1}^{d-1} b_k \frac{\Gamma(|\gamma| + s + k + 1)}{\Gamma(|\gamma| + s)} x^{2\gamma} \\
&= \sum_{k=0}^{d} b_k \underbrace{\sum_{\gamma} \frac{\Gamma(|\gamma| + s + k)}{\prod_{i=1}^{n} \Gamma(\alpha_i + 1)} x^{2\gamma}}_{\frac{\Gamma(s+k)}{(1-\|x\|^2)^{s+k}}}
\end{aligned}$$

Our equality follows by setting $c_k = b_k \Gamma(s + k)$. □

Lemma 3.40. *Let $\zeta = (z, \omega) \in \mathbb{C}^{n+m}$ and consider the set of monomials $\{\zeta^\mu, \mu = (\gamma, \alpha), \gamma = (\gamma_1 \dots \gamma_n)\, and\, \alpha = (\alpha_1 \dots \alpha_m)\}$.
This set is an orthogonal complete set in $A^2(\Omega_p)$.*

Proof. Let $\mu = (\gamma, \alpha)$ and $\mu' = (\gamma', \alpha')$ two multi-indices. Then,

$$\begin{aligned}
\left\langle \zeta^\mu, \zeta^{\mu'} \right\rangle &= \int_{\mathbb{C}^{n+m}} \zeta^\mu \overline{\zeta^{\mu'}} d\lambda(\zeta) \\
&= \int_{\mathbb{C}^n} z^\alpha \overline{z}^{\alpha'} d\lambda_n(z) \int_{\mathbb{C}^m} \omega^\gamma \overline{\omega}^{\gamma'} d\lambda_m(z) \\
&= \int_{\mathbb{C}} z_1^{\alpha_1} \overline{z_1}^{\alpha'_1} dz_1 \dots \int_{\mathbb{C}} z_n^{\alpha_n} \overline{z_n}^{\alpha'_n} dz_n \int_{\mathbb{C}} \omega_1^{\gamma_1} \overline{\omega_1}^{\gamma'_1} d\omega_1 \dots \int_{\mathbb{C}} \omega_m^{\gamma_m} \overline{\omega_m}^{\gamma'_m} d\omega_m.
\end{aligned}$$

Each integral above can be computed using polar coordinates and it follows that, unless $\alpha_j = \alpha'_j$:

$$\int_{\mathbb{C}} z_j^{\alpha_j} \overline{z_j}^{\alpha'_j} dz_j = 0,$$

and unless $\gamma_k = \gamma_k'$:

$$\int_{\mathbb{C}} \omega_k^{\gamma_k} \overline{\omega_k}^{\gamma_k'} d\omega_k = 0.$$

This shows that the set is orthogonal.
It is straightforward to see that Ω_p is a Reinhardt domain, thus each holomorphic function has a power series expansion uniformly convergent throughout Ω_p.Hence

$$\begin{aligned}\left\langle f, \zeta^{\mu} \right\rangle &= \int_{\mathbb{C}^{n+m}} \sum_{\theta} C_{\theta} \zeta^{\theta} \overline{\zeta}^{\mu} d\lambda(\zeta) \\ &= C_{\mu} \int_{\mathbb{C}^{n+m}} \zeta^{\mu} \overline{\zeta}^{\mu} d\lambda(\zeta).\end{aligned}$$

Thus $\left\langle f, \zeta^{\mu} \right\rangle = 0$ implies $C_{\mu} = 0$. Hence $f \equiv 0$ and the set is complete. $\square$

Now our task is to compute the norm of the monomials in order to get an ON basis.

Lemma 3.41. *Let $\{z^{\alpha}\omega^{\gamma}\}$ be as above.Then*

$$\|z^{\alpha}\omega^{\gamma}\|^2_{L^2(\Omega_p)} = \frac{\pi^{n+m}}{p} \frac{\prod_{i=1}^n \Gamma(\alpha_i + 1) \prod_{i=1}^n \Gamma(\gamma_i + 1) \Gamma(\frac{|\gamma|+m}{p})}{\Gamma(|\gamma| + m)\Gamma(|\alpha| + n + 1 + \frac{|\gamma|+m}{p})}.$$

Proof. For the proof we will use the same techniques as before.

$$\|z^{\alpha}\omega^{\gamma}\|^2_{L^2} = \int_{\Omega_p} |z|^{2\alpha} |\omega|^{2\gamma} d\lambda(z) d\lambda(\omega). \tag{3.6.14}$$

Writing again,

$$z = r\exp(i\theta) \quad where \quad \begin{cases} r = (r_1, \ldots r_n) \\ \theta = (\theta_1 \ldots \theta_n) \end{cases}$$

and

$$\omega = s\exp(i\phi) \quad \begin{cases} s = (s_1, \ldots s_m) \\ \theta = (\phi_1 \ldots \phi_m) \end{cases}$$

After integrating out the angular variables,(3.6.14) becomes:

$$(2\pi)^{n+m} \int_{\|r\|^2 + \|s\|^{2p} < 1_+} r^{2\alpha+1} s^{2\gamma+1} d\lambda(r) d\lambda(s). \tag{3.6.15}$$

Introducing the spherical coordinates in the s variable

$$s = \phi(\rho, \omega) = \rho\omega,$$

$0 < \rho < 1$ and $\rho \in S_{+}^{m-1}$.
The Jacobian of this diffeomorphism is $|J\phi| = \rho^{m-1}$.
(3.6.15)becomes,

$$(2\pi)^{n+m} \int_{\|r\|^2+|\rho^{2p}<1_+} \int_{S_{+}^{m-1}} r^{2\alpha+1} \rho^{2|\gamma|+2m-1} \omega^{2\gamma+1} d\lambda(r) d\rho d\sigma(\omega). \tag{3.6.16}$$

Now we make the change the change of variable

$$x = \rho^p,$$

(3.6.16) becomes:

$$\frac{(2\pi)^{n+m}}{p} \int_{\|r\|^2+|x|^2<1_+} \int_{S_{+}^{m-1}} r^{2\alpha+1} x^{\frac{2|\gamma|+2m}{p}-1} \omega^{2\gamma+1} d\lambda(r) dx d\sigma(\omega). \tag{3.6.17}$$

For the integral over the sphere we use the equality (3.6.10)to obtain

$$\int_{S^{m-1}} \omega^{2(\gamma+1)-1} d\sigma(\omega) = \frac{\mathscr{B}(\gamma+1)}{2^{m-1}} \tag{3.6.18}$$

$$= \frac{1}{2^{m-1}} \frac{\prod_{i=1}^{m} \Gamma(\gamma_i+1)}{\Gamma(|\gamma|+m)}. \tag{3.6.19}$$

For the rest of the integral define,

$$\zeta = (\alpha+1, \frac{|\gamma|+m}{p}) \quad and \quad y = (r_1 \dots r_n, x),$$

and use(3.6.9)for the ball in $\mathbb{R}^{n+1}$ we obtain,

$$\int_{B_{+}^{n+1}} y^{2\zeta-1} d\lambda(y) = \frac{\mathscr{B}(\zeta)}{2^{n+1}|\zeta|}, \tag{3.6.20}$$

where

$$|\zeta| = |\alpha| + n + 1 + \frac{|\gamma|+m}{p}.$$

Thus(3.6.20)becomes:

$$\frac{\prod_{i=1}^{n} \Gamma(\alpha_i+1).\Gamma(\frac{|\gamma|+m}{p})}{\Gamma(|\alpha|+n+1+\frac{|\gamma|+m}{p})}. \tag{3.6.21}$$

(3.6.19)and(3.6.21)lead to the desired result. □

We are now ready to compute explicitly the Bergman kernel function of our domain.
In view of lemma(3.40), the set

$$\{\zeta^\mu,\, \zeta \in \mathbb{C}^{n+m},\, \mu \in \mathbb{N}^{n+m}\}$$

is an ON basis of $\mathbb{A}^2(\Omega_p)$. Thus

$$K_{\Omega_p}(\zeta,\bar{\zeta}) = \sum_\mu \frac{|\zeta^\mu|^2}{\|\zeta^\mu\|^2_{\mathbb{L}^2}}. \tag{3.6.22}$$

Theorem 3.42. *The Bergman kernel of Ω_p is given by,*

$$K_{\Omega_p}\Big((z,\omega),\overline{(z,\omega)}\Big) = \sum_{k=0}^{n+1} c_k \frac{\big(1-\|z\|^2\big)^{-n-1+\frac{1}{p}}}{\big((1-\|z^2\|)^{\frac{1}{p}}-\|\omega\|^2\big)^{m+k}}. \tag{3.6.23}$$

Proof. It remains to sum (3.6.22).For this purpose we will use our results. First we sum out the z:

$$\sum_{\alpha,\gamma} \frac{|z^\alpha\omega^\gamma|^2}{\|z^\alpha\omega^\gamma\|^2_{\mathbb{L}^2}} = \frac{p}{\pi^{n+m}} \sum_{\alpha,\gamma} \frac{\Gamma(|\gamma|+m)\Gamma\Big(|\alpha|+n+1+\frac{|\gamma|+m}{p}\Big)}{\prod_{i=1}^n \Gamma(\alpha_i+1)\prod_{i=1}^n\Gamma(\gamma_i+1)\Gamma(\frac{|\gamma|+m}{p})} |z|^{2\alpha}|\omega|^{2\gamma} \tag{3.6.24}$$

$$= \frac{p}{\pi^{n+m}} \sum_{\gamma} \frac{\Gamma(|\gamma|+m)\Gamma\Big(n+1+\frac{|\gamma|+m}{p}\Big)}{\prod_{i=1}^n\Gamma(\gamma_i+1)\Gamma(\frac{|\gamma|+m}{p})} |\omega|^{2\gamma} \frac{1}{(1-\|z\|)^{n+1+\frac{m+|\gamma|}{p}}}. \tag{3.6.25}$$

Now using(3.6.1)we obtain:

$$\Gamma\Big(n+1+\frac{|\gamma|+m}{p}\Big) = \Gamma\Big(\frac{|\gamma|+m}{p}\Big)\prod_{k=0}^{n}\Big(\frac{|\gamma|+m}{p}+k\Big) \tag{3.6.26}$$

$$= \Gamma\Big(\frac{|\gamma|+m}{p}\Big)\frac{1}{p^{n+1}}\prod_{k=0}^{n}\Big(|\gamma|+m+kp\Big) \tag{3.6.27}$$

The product in(3.6.27) is a ploynomial of degree n+1 in $|\gamma|$, say $q_{n+1}(|\gamma|)$.
By inserting(3.6.27)in(3.6.25)we have

$$\frac{p}{\pi^{n+m}} \sum_{\gamma} \frac{\Gamma(|\gamma|+m)}{\prod_{i=1}^n\Gamma(\gamma_i+1)} q_{n+1}(|\gamma|)|\omega|^{2\gamma} \frac{1}{(1-\|z\|^2)^{n+1+\frac{m+|\gamma|}{p}}} \tag{3.6.28}$$

$$= \frac{p}{\pi^{n+m}} \sum_{\gamma} \frac{\Gamma(|\gamma|+m)}{\prod_{i=1}^n\Gamma(\gamma_i+1)} q_{n+1}(|\gamma|) \left(\frac{|\omega|^2}{(1-\|z\|^2)^{\frac{1}{p}}}\right)^{\gamma} (1-\|z\|^2)^{-n-1-\frac{m}{p}} \tag{3.6.29}$$

The expression above conforms to the hypothesis of lemma(3.39). We can thus invoke the result to obtain the following expression

$$\frac{1}{(1-\|z\|^2)^{n+1+\frac{m}{p}}}\sum_{k=0}^{n+1} c_k\left(\frac{1}{1-\frac{\|\omega\|^2}{(1-\|z\|^2)^{\frac{1}{p}}}}\right)^{m+k}$$

Multiplying numerator and denominator of each term in the sum by

$$(1-\|z\|^2)^{\frac{m+k}{p}}$$

yields the desired result.
The information on the constants follows from the explicit nature of the polynomial $q_{n+1}(|\gamma|)$. For more details see[11] page 31 to 33. $\square$

3.7 The Fock space

In this paragraph, we study some examples of Fock spaces in one and several variables. This subject was studied in numerous books and papers, in this work we follow [8] and [8].

Definition 3.43. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain.*
(a)A weight function is a positive measurable function $f : \Omega \to \mathbb{R}^+$
(b)A weight is a positive measure μ on the Borel σ-algebra $\mathcal{B}(\Omega)$ such that, for each $\mathcal{A} \in \mathcal{B}(\Omega)$ we have

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} f(z)dA(z)$$

where dA is the usual Lebesgue area measure in the complex plane.
(c)The weighted space $L^2(\mathbb{C}^n, f)$ is the space of all square integrable functions with respect to the weighted measure $d\mu = f dA$
In order to get a non-trivial $L^2(\mathbb{C}^n)$ of entire functions, we need a reasonable decay of the weight function. If for example the weight has only a polynomial decay the space will be finite dimensional.
(d)Let $f(z) = e^{-\alpha|z|^2}$ (Gaussian weight)
The Fock space $\mathscr{F}^2_\alpha(\mathbb{C}^n, e^{-\alpha|z|^2})$is the space of all holonorphic square integrable functions with respect to the measure $d\lambda_\alpha = f dA$ in other words

$$\mathscr{F}^2_\alpha(\mathbb{C}^n, e^{-\alpha|z|^2}) = \left\{\varphi \text{ holomorphic on } \mathbb{C} \text{ and} \int_{\mathbb{C}^n} |\varphi(z)|^2 e^{-\alpha|z|^2} d\lambda_\alpha < \infty\right\}$$

On this space we define an inner product via

$$\left\langle f,g\right\rangle_{\mathscr{F}_\alpha}=\int_{\mathbb{C}^n}f(z)\overline{g(z)}d\lambda_\alpha$$

and

$$\|f\|^2_{\mathscr{F}_\alpha}=\left\langle f,f\right\rangle_{\mathscr{F}_\alpha}$$

We start with the one dimensional case.

Theorem 3.44. *With respect to the this inner product, $\mathscr{F}_\alpha(\mathbb{C},e^{-\alpha|z|^2})$ becomes a Hilbert space.*

In order to prove this theorem we need the following two results.

Proposition 3.45. *For each $f\in\mathscr{F}^2_\alpha(\mathbb{C},e^{-\alpha|z|^2})$ we have*

$$|f(z)|\le e^{\frac{\alpha}{2}|z|^2}\,\|f\|_{\mathscr{F}^2_\alpha}$$

For the proof we follow [37].

Proof. For $z=0$,the mean value theorem gives, for $r>0$

$$f(0)=\frac{1}{2\pi}\int_0^{2\pi}f(re^{it})ie^{it}dt$$

Thus

$$\begin{aligned}|f(0)|&\le\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|dt\\&\overset{Cauchy-Schwarz}{\le}\frac{1}{2\pi}\Big(\int_0^{2\pi}|f(re^{it})|^2dt\Big)^{\frac{1}{2}}\Big(\int_0^{2\pi}1dt\Big)^{\frac{1}{2}}\end{aligned}$$

Thus

$$|f(0)|^2\le\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^2dt$$

Now multiply by $re^{-\alpha r^2}$ and integrating over $(0,\infty)$ we obtain

$$|f(0)|^2\underbrace{\int_0^\infty re^{-\alpha r^2}dr}_{\frac{1}{2\alpha}}\le\frac{1}{2\pi}\underbrace{\int_0^\infty\int_0^{2\pi}|f(re^{it})|^2re^{-\alpha r^2}dtdr}_{\int_{\mathbb{C}}|f(z)|^2e^{-\alpha|z|^2}dz=\|f\|^2_{\mathscr{F}^2_\alpha}}$$

For an arbitrary z, define $G(\omega) = f(z-\omega)e^{\alpha\omega\overline{z}-\frac{\alpha}{2}|z|^2}$
By the preceding we have $|G(0)| \leq \|G\|_{\mathscr{F}^2_\alpha}$

$$\begin{aligned}
\|G\|^2_{\mathscr{F}^2_\alpha} &= \frac{\alpha}{\pi}\int_{\mathbb{C}}\left|f(z-\omega)\right|^2\left|e^{2\alpha\omega\overline{z}-\alpha|z|^2-\alpha|\omega|^2}\right|dA(\omega)\\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}\left|f(z-\omega)\right|^2 e^{-\alpha|\omega|^2+|z|^2-2Re(\omega\overline{z})}dA(\omega)\\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}\left|f(z-\omega)\right|^2 e^{-\alpha|z-\omega|^2}dA(\omega)\\
&\overset{\zeta=z-\omega}{=} \frac{\alpha}{\pi}\int_{\mathbb{C}}\left|f(\zeta)\right|^2 e^{-\alpha|\zeta|^2}dA(\omega)\\
&= \|f\|^2_{\mathscr{F}^2_\alpha}
\end{aligned}$$

Now replacing G(0) by $f(z)e^{-\frac{\alpha}{2}|z|^2}$, we obtain the desired estimate. □

Lemma 3.46. *The multiplication operator*

$$M : L^2(\mathbb{C}) \to L^2_\alpha(\mathbb{C})$$

defined via

$$M(f)(z) := \sqrt{\frac{\pi}{\alpha}}f(z)e^{\frac{\alpha}{2}|z|^2}$$

is an isometry between $L^2(\mathbb{C})$ and $L^2_\alpha(\mathbb{C})$

Proof. Just remark that

$$\begin{aligned}
\|M(f)\|^2_{L^2_\alpha} &= \frac{\pi}{\alpha}\frac{\alpha}{\pi}\int_{\mathbb{C}}|f(z)|^2 e^{\alpha|z|^2}e^{-\alpha|z|^2}dA(z)\\
&= \|f\|^2_{L^2}
\end{aligned}$$

and that the inverse operator M^{-1} is simply multiplication by $\sqrt{\frac{\alpha}{\pi}}e^{-\frac{\alpha}{2}|z|^2}$ □

Theorem 3.47. *The Fock space $\mathscr{F}^2_\alpha$ is complete hence a Hilbert space.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\mathscr{F}^2_\alpha$, proposition(3.45)shows that $(f_n)_n$ is uniformly Cauchy on each compact subset of $\mathbb{C}$. Indeed, let $K \subseteq \mathbb{C}$ be compact and set

$$M = sup\{e^{\alpha|z|^2}, z \in K\}$$

For $n > m \in \mathbb{N}$ we have

$$sup_{z\in K}|f_n(z) - f_m(z)| \leq M\,\|f_n - f_m\|_{\mathscr{F}^2_\alpha}$$

It follows that $(f_n)_n$ converges uniformly to some g which is analytic by the Weierstrass theorem. Our task now is to show that g is square integrable.
$(f_n)_n$ is Cauchy in L^2_α since $\mathscr{F}^2_\alpha \subseteq L^2_\alpha$.We will show that this sequence is Cauchy in $L^2(\mathbb{C})$.
Since the multiplication operator defined in lemma (3.46) is invertible, it follows that $(M^{-1}f_n)_n$ is Cauchy in $L^2(\mathbb{C})$ which is complete hence it converges in norm of L^2 to some function say $M^{-1}f$.This leads to the existence of a subsequence converging almost everywhere to $M^{-1}f$(see Theorem 3.12 in[30]). Thus $(f_n)_n$ converges to f almost everywhere, and $(f_n)_n$ converges to g on compact subsets, so we can identify f with an analytic function g. This achieves the proof. □

Remark 3.48. *Proposition (3.45) shows that the evaluation maps are bounded on $\mathscr{F}^2_\alpha$ which leads to the existence of the Bergman kernel of the space $\mathscr{F}^2_\alpha$ say K_α.*
Our next aim is to give an explicit formula for this kernel function by means of ON basis.

Proposition 3.49. *For any non-negative integer n.*
Let, $e_n(z) = \sqrt{\frac{\alpha^n}{n!}}z^n$.
Then the set $\{e_n\}$ *is an ON basis for* $\mathscr{F}^2_\alpha$

Proof. Polar coordinates show that this set is an orthonormal set.
Given $f \in \mathscr{F}^2_\alpha$ and $n \geq 0$, dominated convergence gives

$$\begin{aligned}\Big\langle f, e_n \Big\rangle_{\mathscr{F}^2_\alpha} &= \int_{\mathbb{C}} f(z)\overline{e_n(z)}d\lambda_\alpha \\ &= \lim_{R\to\infty} \int_{|z|<R} f(z)\overline{e_n(z)}d\lambda_\alpha\end{aligned}$$

Since the Taylor series

$$\sum_{k=0}^{\infty} a_k z^k$$

converges uniformly on $|z| < R$ we obtain,

$$\lim_{R\to\infty} \int_{|z|<R} f(z)\overline{e_n(z)}d\lambda_\alpha = \sum_{k=0}^{\infty} a_k \lim_{R\to\infty} \int_{|z|<R} z^k\overline{e_n(z)}d\lambda_\alpha$$

Using polar coordinates again, we obtain

$$\begin{aligned}\left\langle f, e_n \right\rangle_{\mathscr{F}^2_\alpha} &= \int_{\mathbb{C}} f(z)\overline{e_n(z)} d\lambda_\alpha \\ &= \lim_{R\to\infty} \int_{|z|<R} f(z)\overline{e_n(z)} d\lambda_\alpha \\ &= a_n \int_{\mathbb{C}} z^n \overline{e_n(z)} d\lambda_\alpha(z)\end{aligned}$$

Therefore, the condition that $\left\langle f, e_n \right\rangle_{\mathscr{F}^2_\alpha}$ for all $n \geq 0$ implies that $a_n = 0$ for all $n \geq 0$ which in turn implies that $f = 0$. This shows that the system $\{e_n\}$ is complete in $\mathscr{F}^2_\alpha$. □

Now, since the series

$$\sum_{n=0}^{\infty} e_n(z)\overline{e_n(\omega)}$$

sums uniformly to K_α on compact subsets of $\Omega \times \Omega$, we obtain,

$$\begin{aligned}K_\alpha(z, \omega) &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} z^n \overline{\omega}^n \\ &= e^{\alpha z \overline{\omega}}\end{aligned}$$

Now, we study an example of Fock space in several variables.
Consider the space

$$\mathscr{F}^2(\mathbb{C}^n, e^{-|z|^2}) = \left\{\varphi \, holomorphic \ on \ \mathbb{C}^n and \int_{\mathbb{C}} |\varphi(z)|^2 e^{-|z|^2} d\lambda_\alpha < \infty\right\}$$

Define the scalar product

$$\left\langle f, g \right\rangle_{\mathscr{F}} = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} d\lambda(z)$$

and

$$\|f\|^2_{\mathscr{F}} = \left\langle f, f \right\rangle_{\mathscr{F}}$$

Proposition 3.50. *With respect to the scalar product defined above, $\mathscr{F}^2(\mathbb{C}^n, e^{-|z|^2})$ becomes a Hilbert space.*

Proof. We show that the Fock space is a closed subspace of $L^2(\mathbb{C}^n, f)$ using the same arguments used in the proof of corollary(3.5), just replace the Lebesgue measure by the weighted measure $d\mu$. □

Similar discussion as in the second paragraph leads to the existence of the Bergman Kernel of the Fock space. Our next aim is to give an explicit formula of that kernel by means of an ON-basis of the Hilbert space $\mathscr{F}(\mathbb{C}^n, e^{-|z|^2})$.

Proposition 3.51. *The normalized monomials* $\left\{\phi_\alpha = \frac{z^\alpha}{\|z^\alpha\|}, \alpha = (\alpha_1 \ldots, \alpha_n)\right\}$ *form an ON-basis of the Fock space.*

Proof. The proof is analogous to the one done in(3.25). It is easy to show that each ϕ_α belongs to $\mathscr{F}$ and that the elements of this set are orthogonal to each other. □

We can even show that the series

$$\sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(\omega)}$$

sums uniformly on compact subsets of $\mathbb{C}^n$ to the Bergman kernel of the Fock space. Now,our task is to compute the norm of the monomials.

Lemma 3.52. *For every multi-index* $\alpha = (\alpha_1 \ldots, \alpha_n)$ *we have*

$$\|z^\alpha\|^2_{\mathscr{F}} = \alpha!$$

Proof. An easy computation shows that

$$\begin{aligned}
\|z^\alpha\|^2_{\mathscr{F}} &= \frac{1}{\pi^n}\int_{\mathbb{C}^n} |z|^{2\alpha} e^{-|z|^2} d\lambda(z) \\
&\overset{\text{Fubini}}{=} \frac{1}{\pi^n}\prod_{k=1}^{k=n}\int_{\mathbb{C}} |z_k|^{2\alpha_k} e^{-|z_k|^2} d\lambda(z_k) \\
&= \frac{1}{\pi^n}\pi^n \alpha_1! \ldots \alpha_n! \\
&= \alpha!
\end{aligned}$$

Polar coordinates gives that for every $0 \leq m \leq n$

$$\begin{aligned}
\int_{\mathbb{C}} |z|^{2\alpha_m} e^{-|z|^2} d\lambda(z) &= \int_0^{2\pi}\int_0^\infty r^{2\alpha_m} e^{-r^2} r dr d\theta \\
&= \pi \int_0^\infty r^{2\alpha_m} e^{-r^2} 2r dr \\
&\overset{s=r^2}{=} \pi \int_0^\infty s^{\alpha_m} e^{-s} ds \\
&= \pi\Gamma(\alpha_m + 1) = \pi\alpha_m!
\end{aligned}$$

□

Lemma 3.53. *The Bergman kernel of the Fock space is given by,* $K_{\mathscr{F}}(z,\omega) = e^{z\bar{\omega}}$, *for every* $z, \omega \in \mathbb{C}^n$.

Proof. The set $\{\frac{z^\alpha}{\|z^\alpha\|}, \alpha\, multi-index\}$ is an ON basis of the Fock space thus the Bergman kernel has the form

$$\begin{aligned} K_{\mathscr{F}}(z,\omega) &= \sum_\alpha \frac{z^\alpha}{\|z^\alpha\|} \frac{\overline{\omega}^\alpha}{\|\omega^\alpha\|} \\ &= \sum_\alpha \frac{(z\overline{\omega})^\alpha}{\alpha!} \\ &= e^{z\overline{\omega}} \end{aligned}$$

□

Chapter 4

Bergman Kernel and the d-bar problem

In this chapter we are going to give an overview of the d-bar Neumann problem in one and several complex variables, this theme still be a huge area of mathematical research and numerous books and papers were written in this subject. I give just a warm up to the subject with introducing its relationship with our main topic of the work namely the Bergman kernel. We will show how to take profit of the Bergman projection to solve the one dimensional inhomogeneous $\bar{\partial}$ problem on the unit disc of the plane. We will show that in this case the canonical solution operator is compact and even Hilbert-Schmidt. However, if the dimension is grater than one, this operator fails to be compact. We prove furthermore that the canonical solution operator on (0,1)-forms with Fock space coefficients is also non compact. Several investigations on compactness of the $\bar{\partial}$ were made, see for example [17], [**?**] , [38] and [33].

4.1 Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and consider $f : \Omega \to \mathbb{C}$ a C^1 map.For each component we write $z_j = x_j + iy_j$, let $p \in \Omega$, the differential of f at p is:

$$df(p) = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial x_j}(p) dx_j + \frac{\partial f}{\partial y_j}(p) dy_j \Big)$$

Write now

$$dz_j = dx_j + idy_j \, d\bar{z}_j = dx_j - idy_j$$

and using the Wirtinger symbols,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big)$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

We can rewrite the differential of f in the form

$$df(p) = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial z_j}(p)dz_j + \frac{\partial f}{\partial \bar{z}_j}(p)d\bar{z}_j\Big) = \partial f(p) + \bar{\partial} f(p)$$

Definition 4.1. *A general(p,q)-differential form is given by*

$$\omega = \sum_{|J|=p\,|K|=q}{}' a_{J,K} dz_J \wedge d\bar{z}_K$$

where the $a_{J,K}$ are C^1 functions on Ω. The prime symbol means that we sum over all increasing multi-indices

$$J = (j_1 \dots j_p) \quad \text{and} \quad K = (k_1 \dots k_q)$$

The derivative of this form is defined by

$$d\omega = \sum_{|J|=p\,|K|=q}{}' da_{J,K} \wedge dz_J \wedge d\bar{z}_K = \sum_{|J|=p\,|K|=q}{}' (\partial a_{J,K} + \bar{\partial} a_{J,K}) \wedge dz_J \wedge d\bar{z}_K$$

We denote

$$\partial\omega = \sum_{|J|=p\,|K|=q}{}' \partial a_{J,K} \wedge dz_J \wedge d\bar{z}_K$$

and

$$\bar{\partial}\omega = \sum_{|J|=p\,|K|=q}{}' \bar{\partial} a_{J,K} \wedge dz_J \wedge d\bar{z}_K$$

Definition 4.2. $C_0^\infty(\Omega)$*denotes the set of all infinitely differentiable maps with compact support in Ω. A function $f \in L^2(\Omega)$ has a derivative of order α in the distributional sense. If for each $\phi \in C_0^\infty(\Omega)$, there exists a function $g \in L^2(\Omega)$ such that*

$$\int_\Omega f \partial^\alpha \phi d\lambda = (-1)^{|\alpha|} \int_\Omega \phi g d\lambda$$

We denote

$$g := \partial^\alpha f$$

in the distributional sense.

Definition 4.3.

$$L^2_{(0,q)}(\Omega) = \Big\{ u = \sum_{|K|=q} {}' u_K d\bar{z}_K \quad , \quad u_K \in L^2(\Omega) \Big\}$$

The topology of this space is defined via the scalar product

$$< u, v >= \sum_{|K|=q} {}' \int_{\mathbb{C}^n} u_K \overline{v_K} d\lambda$$

Where

$$u = \sum_K{}' u_K d\bar{z_K} \quad and \quad v = \sum_K{}' v_K d\bar{z_K}$$

The $\bar{\partial}$ acts on this space in the following way

$$\bar{\partial} u = \sum_{j=0}^{n} \sum_{|K|=q} {}' \frac{\partial u_K}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_K \quad (*)$$

The derivative is again in the distributional sense.
The domains of this operator consists of all (0,q) forms with L^2 coefficients such that the right hand-side of $(*)$ belongs to $L^2_{(0,q+1)}(\Omega)$.
On its domain $\bar{\partial}$ acts as an unbounded, densely defined and closed operator therefore it has an adjoint operator defined on $L^2_{(0,q+1)}(\Omega)$,denoted by $\bar{\partial}^*$.
The complex Laplacian is defined by

$$\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

A self contained and an elegant discussion of the properties of $\Box$ can be found in[21].
In the sequel we focus on (0,1) forms.
For $f \in L^2(\Omega)$,the $\bar{\partial}$ acts as follows:

$$\bar{\partial} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial \bar{z}_i} d\bar{z}_i$$

4.2 Canonical solution operator with Bergman space coefficients

Let Ω be a bounded domain in $\mathbb{C}^n$.
As usual we consider:

$$\bar{\partial} : Dom(\bar{\partial}) \subseteq L^2(\Omega) \to L^2_{(0,1)}(\Omega)$$

We consider the problem:

$$\begin{cases} \bar{\partial} u = g \quad g = \sum_{j=1}^{n} g_j d\bar{z}_j, \quad g_j \in A^2(\Omega) \\ u \perp (Ker\bar{\partial}) \end{cases} \tag{4.2.1}$$

The second condition makes the solution unique.
A bounded operator

$$S_1 : L^2_{(0,1)}(\Omega) \to L^2(\Omega)$$

is a solution operator to $\bar{\partial}$ if

$$\bar{\partial} S_1 g = g, \; for\, every\, g \in L^2_{(0,1)}(\Omega)$$

If

$$S_1 g \perp Ker(\bar{\partial})$$

is valid as well, then we call S_1 the canonical solution operator.

Proposition 4.4. *Let $g = \sum_{j=1}^{n} g_j d\bar{z}_j$ be a (0,1)form such that $g_j \in A^2(\Omega)$. The canonical solution operator has the following form:*

$$S_1 g(z) = \int_\Omega K(z, \omega) < g(\omega), z - \omega > d\lambda(\omega). \tag{4.2.2}$$

K is the Bergman kernel of the domain and

$$< g(\omega), z - \omega >= \sum_{j=1}^{n} g_j(\omega)\overline{(z_j - \omega_j)}$$

.

Proof. Since $g_j \in A^2(\Omega)$ then

$$g_j(\omega) = \int_\Omega K(\omega, \zeta) g_j(\zeta) \lambda(\zeta) \quad (*)$$

Now set

$$v(z) = \sum_{j=1}^{n} z_j g_j(z)$$

we get:

$$\begin{aligned} \bar{\partial} v &= \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} (\bar{z}_j g_j) d\bar{z}_j \\ &= \sum_{j=1}^{n} g_j(z) d\bar{z}_j = g \end{aligned}$$

Hence the canonical solution operator has the form $S_1 g = (I - P)(\bar{z}g)$ where P is the Bergman projection defined in (3.3).
The identity ($*$) gives:

$$\begin{aligned} S_1 g(z) &= \bar{z}g(z) - P(\bar{z}g)(z) \\ &= \int_\Omega \sum_{j=1}^n \bar{z}_j K(z,\omega) g_j(\omega) d\lambda(\omega) - \int_\Omega K(z,\omega) \sum_{j=1}^n \bar{\omega_j} g_j(\omega) d\lambda(\omega) \\ &= \int_\Omega K(z,w) \sum_{j=1}^n g_j(\omega)(\bar{z_j} - \bar{\omega}) d\lambda(\omega) \\ &= \int_\Omega K(z,\omega) < g(\omega), z - \omega > d\lambda(\omega) \end{aligned}$$

$\square$

Definition 4.5. *Recall that the Bergman projection is defined as follows:*

$$P : L^2(\Omega) \to A^2(\Omega)$$

$$f \mapsto Pf$$

$$Pf(z) = \int_\Omega K(z,\omega) f(\omega) d\lambda(\omega)$$

K is the corresponding Bergman kernel of the domain.
We define the Hankel operator with symbol ψ (satisfying suitable conditions)by

$$H_\psi : A^2(\Omega) \to L^2(\Omega) \backslash A^2(\Omega)$$

$$g \mapsto H_\psi g := (I - P)(\psi g)$$

A simple calculation shows that

$$H_\psi g(z) = \int_\Omega \Big(\psi(z) - \psi(\omega)\Big) K(z,\omega) d\lambda(\omega)$$

Remark 4.6. *The canonical solution operator to $\bar{\partial}$ restricted to the Bergman space A^2 can be interpreted as a Hankel operator with symbol $\bar{z}$*

$$H_{\bar{z}} g = \Big(I - P\Big)(\bar{z}g)$$

Remark 4.7. *Investigations of the solution operator on (0,1)-forms with entire coefficients are made in[18].*

In the next section we will show that, for the unit disc in $\mathbb{C}$ the canonical solution operator restricted to $A^2(\mathbb{D})$ is a Hilbert Schmidt operator whereas for the unit ball in $\mathbb{C}^n$, $n \geq 2$, the canonical solution operator fails to be Hilbert Schmidt. A self contained and elegant survey on this subject can be found in[17].

4.3 Inhomogeneous Cauchy-Riemann equation in one dimension

The following contains a discussion of the inhomogeneous Cauchy-Riemann equation restricted to holomorphic functions in one complex variable.
Let $g \in \mathbb{A}^2(\mathbb{D})$ and consider the problem

$$\frac{\partial f}{\partial \overline{z}} = g \tag{4.3.1}$$

Let P be the Bergman projection defined in (3.3. We define an operator

$$\mathscr{S} : \mathbb{A}^2(\mathbb{D}) \to \mathscr{L}^2(\mathbb{D})$$

by $\mathscr{S}(g)(z) := \overline{z}g(z) - P(\tilde{g})(z)$ where $\tilde{g}(\omega) := \overline{\omega}g(\omega)$
We claim that,$\mathscr{S}(g)$is a solution to (4.3.1). Indeed

$$\begin{aligned}\frac{\partial \mathscr{S}(g)}{\partial \overline{z}}(z) &= \frac{\partial \overline{z}g}{\partial \overline{z}}(z) - \frac{\partial \tilde{g}}{\partial \overline{z}}(z)\\ &\overset{chain\,rule}{=} \underbrace{\frac{\partial \overline{z}}{\partial \overline{z}}}_{=1} g(z) + \Big(\overline{z} - 1\Big) \underbrace{\frac{\partial g}{\partial \overline{z}}(z)}_{=0}\\ &= g(z)\end{aligned}$$

$\mathscr{S}$ is called the canonical solution operator. In the following we will show that $\mathscr{S}$ is compact and even a Hilbert-Schmidt operator.
Let $\{\phi_n = \frac{z^n}{c_n}, c_n = \|z^n\|_{\mathbb{A}^2}, n \in \mathbb{N}\}$be an ON-basis of $\mathbb{A}^2(\mathbb{D})$ and define:

$$\tilde{\phi}(z) = \bar{z}\phi(z).$$

Theorem 4.8. *The canonical solution operator of (4.2.1)is compact.*

Proof. In order to prove the result, we will show that $\mathscr{S}^*\mathscr{S}$ is compact which implies that $\mathscr{S}$ is also compact(detailed discussion on compact operators and its properties can be found in[4]chapter 6).
First step
Recall that if $\psi \in \mathscr{L}^2(\mathbb{D})$ then

$$P(\psi)(z) = \int_{\mathbb{D}} K(z,\omega)\psi(\omega)d\lambda(\omega)$$

Where P is the Bergman projection.
Now let us compute $P(\tilde{\phi_n})$,

$$
\begin{aligned}
P(\tilde{\phi_n})(z) &= \int_{\mathbb{D}} K(z,\omega)\overline{\omega}\phi_n(\omega)d\lambda(\omega) \\
&= \int_{\mathbb{D}} \sum_{k\geq 0}^{\infty} \frac{z^k\overline{\omega}^k}{c_k^2}\overline{\omega}\frac{\omega^n}{c_n}d\lambda(\omega) \\
&= \sum_{k\geq 0}^{\infty} \frac{z^k}{c_k^2 c_n} \underbrace{\int_{\mathbb{D}} \overline{\omega}^{k+1}\omega^n d\lambda(\omega)}_{=0,\text{ unless } k=n-1} \\
&= \frac{z^{n-1}}{c_{n-1}^2 c_n} \underbrace{\int_{\mathbb{D}} \overline{\omega}^n\omega^n d\lambda(\omega)}_{c_n^2} \\
&= \frac{z^{n-1}c_n}{c_{n-1}^2}
\end{aligned}
$$

<u>**second step**</u>
Let us compute the adjoint operator of $\mathscr{S}$
Let $\phi \in \mathscr{L}^2(\mathbb{D})$, since $g\,,P(\tilde{g}) \in \mathbb{A}^2(\mathbb{D})$ we have:

$$
\begin{aligned}
\Big\langle \mathscr{S}g, \phi \Big\rangle_{\mathscr{L}^2(\mathbb{D})} &= \int_{\mathbb{D}} \mathscr{S}(g)(\omega)\overline{\phi}(\omega)d\lambda(\omega) \\
&= \int_{\mathbb{D}} \Big(\overline{\omega}g(\omega) - (P\tilde{g})(\omega)\Big)\overline{\phi}(\omega)d\lambda(\omega) \\
&\overset{Reproducing formula}{=} \int_{\mathbb{D}} \Bigg(\int_{\mathbb{D}} K(\omega,z)g(z)(\overline{\omega}-\overline{z})d\lambda(z)\Bigg)\overline{\phi}(\omega)d\lambda(\omega) \\
&\overset{Fubini}{=} \int_{\mathbb{D}} g(z)\overline{\Bigg(\int_{\mathbb{D}} K(z,\omega)(\omega - z)\phi(\omega)d\lambda(\omega)\Bigg)}d\lambda(z) \\
&= \Big\langle g, \mathscr{S}^*\phi \Big\rangle_{\mathscr{L}^2(\mathbb{D})}
\end{aligned}
$$

Where

$$
\mathscr{S}(\phi)(z) = \int_{\mathbb{D}} K(z,\omega)(\omega - z)\phi(\omega)d\lambda(\omega).
$$

For $n \geq 1$ we have

$$\begin{aligned}
\mathscr{S}^*\mathscr{S}(\phi_n)(\omega) &= \mathscr{S}^*\Big(\overline{z}\phi_n(z) - P(\tilde{\phi_n}(z))\Big)(\omega) \\
&= \mathscr{S}^*\Big(\overline{z}\frac{z^n}{c_n} - c_n\frac{z^{n-1}}{c_{n-1}^2}\Big) \\
&= \int_{\mathbb{D}} K(z,\omega)(z-\omega)\Big(\overline{z}\frac{z^n}{c_n} - c_n\frac{z^{n-1}}{c_{n-1}^2}\Big)d\lambda(z) \\
&= \int_{\mathbb{D}} \sum_{k\geq 0}\frac{\overline{z}^k\omega^k}{c_k^2}(z-\omega)\Big(\overline{z}\frac{z^n}{c_n} - c_n\frac{z^{n-1}}{c_{n-1}^2}\Big)d\lambda(z)
\end{aligned}$$

This integral will be computed in two steps, first we multiply by z, we may interchange integral and sum because of uniform convergence to get:

$$\begin{aligned}
\int_{\mathbb{D}} \sum_{k\geq 0}\frac{\overline{z}^k\omega^k}{c_k^2}\Big(\overline{z}\frac{z^{n+1}}{c_n} - c_n\frac{z^n}{c_{n-1}^2}\Big)d\lambda(z) = \sum_{k\geq 0}\frac{\omega^k}{c_k^2}\Big(&\underbrace{\int_{\mathbb{D}}\frac{z^{k+1}z^{n+1}}{c_n}d\lambda(z)}_{=0 \text{ unless } k=n} - \\
&\frac{c_n}{c_{n-1}^2}\underbrace{\int_{\mathbb{D}}\overline{z}^k z^n d\lambda(z)}_{=0 \text{ unless } k=n}\Big) \\
&= \frac{\omega^n c_{n+1}^2}{c_n^3} - \frac{\omega^n}{c_{n-1}^2}
\end{aligned} \tag{4.3.2}$$

Next we multiply by ω and we get,

$$\begin{aligned}
\int_{\mathbb{D}} \sum_{k\geq 0}\frac{\overline{z}^k\omega^{k+1}}{c_k^2}\Big(\overline{z}\frac{z^n}{c_n} - c_n\frac{z^{n-1}}{c_{n-1}^2}\Big)d\lambda(z) = \sum_{k\geq 0}\frac{\omega^{k+1}}{c_k^2}\Big(&\underbrace{\int_{\mathbb{D}}\frac{z^{k+1}z^n}{c_n}d\lambda(z)}_{=0 \text{ unless } k=n-1} - \\
&\frac{c_n}{c_{n-1}^2}\underbrace{\int_{\mathbb{D}}\overline{z}^k z^{n-1} d\lambda(z)}_{=0 \text{ unless } k=n-1}\Big) \\
&= \frac{\omega^n c_n}{c_{n-1}} - \frac{\omega^n c_n}{c_{n-1}^2} \\
&= 0
\end{aligned} \tag{4.3.3}$$

Finally we get:

$$\begin{aligned}\mathscr{S}^*\mathscr{S}(\phi_n)(\omega) &= \omega^n\Big[\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2}\Big] \\ &= \Big[\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\Big]\phi_n(\omega)\end{aligned}$$

Similar computations leads to

$$\mathscr{S}^*\mathscr{S}(\phi_0)(\omega) = \frac{c_1^2}{c_0^2}\phi_0(\omega)$$

Now let $f \in \mathbb{A}^2(\mathbb{D})$. With respect to the ON-basis $\{\phi_n\}$ f has the following representation:

$$f = \sum_{n\geq 0}\Big\langle f, \phi_n\Big\rangle\phi_n$$

Therefore,

$$\begin{aligned}\mathscr{S}^*\mathscr{S}(f) &= \sum_{n\geq 0}\Big\langle \mathscr{S}^*\mathscr{S}f, \phi_n\Big\rangle\phi_n \\ &= \mathscr{S}^*\mathscr{S}(\phi_0) + \sum_{n\geq 1}\Big\langle f, \mathscr{S}^*\mathscr{S}\phi_n\Big\rangle\phi_n \\ &= \frac{c_1^2}{c_0^2}\phi_0(\omega) + \underbrace{\Big[\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}\Big]}_{s_n^2}\sum_{n\geq 0}^{\infty}\Big\langle f, \phi_n\Big\rangle\phi_n\end{aligned}$$

Recall that

$$c_n^2 = \|z^n\|^2_{\mathbb{A}^2(\mathbb{D})} = \frac{\pi}{n+1}$$

Therefore

$$\begin{aligned}s_n^2 &= \frac{n+1}{n+2} - \frac{n}{n+1} \\ &= \frac{1}{(n+1)(n+2)} \to 0 \quad as \quad n \to \infty\end{aligned}$$

<u>**Conclusion**</u>

$\mathscr{S}^*\mathscr{S}$ is compact hence $\mathscr{S}$ is also compact. $\square$

Next we introduce the notion of Hilbert-Schmidt operators and we show that the canonical solution operator is Hilbert-Schmidt. In my discussion we follow[20].

Definition 4.9. *a)Let $\mathbb{H}$ be a Hilbert space and $T \in \mathscr{L}(H)$ a linear operator. The s-numbers of T are the eigenvalues of the linear operator T^*T.*
b)T is a Hilbert-Schmidt operator if its sequence of s-numbers belongs to l^2

Proposition 4.10. *The canonical solution operator of (4.2.1)is a Hilbert-Schmidt operator.*

Proof. We have already shown that the s-numbers of $\mathscr{S}$ are

$$s_n = \sqrt{\frac{1}{(n+1)(n+2)}}$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} < \infty$$

we obtain the desired result. $\square$

4.4 Canonical solution operator on (0,1) forms with Fock space coefficients

Our main purpose of this section is to show that the solution operator of the $\bar{\partial}$ restricted to (0,1) forms with coefficients in the Fock space $\mathscr{F}$ is non compact. For this purpose I use the papers [18] and the dissertation [33].
Recall that we understand under the Fock space $\mathscr{F}$ the space of holomorphic functions that are square integrable with respect to weight function $e^{-|z|^2}$.
Let consider the following notation :

$$c_k^2 = \int_{\mathbb{C}^n} |z|^{2k} e^{-|z|^2} d\lambda(z)$$

Here $k = (k_1 \dots k_n)$ is a multi-index.
By proposition(3.51)the set,

$$\left\{ \frac{z^\alpha}{\|z^\alpha\|}, \alpha = (\alpha_1 \dots \alpha_n) \right\}$$

is an ON-basis of the Fock space furthermore

$$\|z^\alpha\|^2 = \alpha! \quad .$$

The Bergman kernel of this space has the explicit form:

$$K(z,\omega) = e^{\bar{\omega}z}$$

Here $\bar{\omega}z = z_1\bar{\omega_1}\dots z_n\bar{\omega_n}$.

Definition 4.11. *The space $L^2_{(0,1)}(\mathbb{C}^n, e^{-|z|^2})$ is the set of all(0,1)-forms with coefficients in the weighted space $L^2(\mathbb{C}^n, e^{-|z|^2})$. Let $d\mu = e^{-|z|^2}d\lambda$.*
The topology of this space is defined via,

$$\|u\|^2 = \sum_{j=1}^{n}\int_{\mathbb{C}^n}|u_j|^2 d\mu \quad .$$

Here

$$u = \sum_{j=1}^{n} u_j d\bar{z}_j \quad .$$

In the same way we define $\mathscr{F}_{(0,1)}(\mathbb{C}^n)$ the space of(0,1)forms with coefficients in the Fock space $\mathscr{F}$.

In the sequel,we will show that the canonical solution operator of(4.2.1) with right hand side in $\mathscr{F}_{(0,1)}(\mathbb{C}^n)$ can be written as a Hankel operator with symbol $\bar{z}$.

Definition 4.12. *We define*

$$\sqrt{\mathscr{F}} = \left\{ f : f \in \mathscr{F} \, and \, \exists \, g \in \mathscr{F} \, such \, that \, : f^2 = g \right\}$$

Lemma 4.13. *Let $M_{\bar{z}}$ be the multiplication operator.*
Then

$$M_{\bar{z}}(\sqrt{\mathscr{F}}) \subseteq L^2(\mathbb{C}^n, e^{-|z|^2})$$

Proof. Let $f \in \sqrt{\mathscr{F}}$, then $\exists \, g \in \mathscr{F}$ such that $f^2 = g$
By Cauchy-Schwarz inequality in the Hilbert space $\mathscr{F}$ we obtain

$$\begin{aligned}\int_{\mathbb{C}^n}|f|^2|\bar{z}|^2 e^{-|z|^2}d\lambda(z) &\leq \Big(\int_{\mathbb{C}^n}|f|^4 e^{-|z|^2}d\lambda(z)\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{C}^n}|\bar{z}|^4 e^{-|z|^2}d\lambda(z)\Big)^{\frac{1}{2}} \\ &\leq \|g\|_{\mathscr{F}}\left\|z^2\right\|_{\mathscr{F}}\end{aligned}$$

$\square$

Remark 4.14. *Every monomial of the form z^k belongs to $\sqrt{\mathscr{F}}$ since z^k and z^{2k} belong to $\mathscr{F}$.Furthermore the set $\sqrt{F}$ is clearly dense in $\mathscr{F}$.*

Let us again define the Hankel operator with symbol f on the Fock space $\mathscr{F}$ by:

$$H_f : \mathscr{F} \to \mathscr{F}^{\perp}$$

$$g \mapsto (I - P)(fg)$$

P is now the Bergman projection from the weighted space $L^2(\mathbb{C}^n, |z|^2)$ onto the closed subspace $\mathscr{F}$.

Theorem 4.15. *The following holds*

$$H_{\bar{z}} \mid_{\sqrt{\mathscr{F}}} = S_1 \mid_{\sqrt{\mathscr{F}}}$$

Here S_1 denotes the canonical solution operator to $\bar{\partial}$ on (0,1)-forms with holomorphic coefficients. Moreover,S_1 can be interpreted as L^2 limit of the images of suitable elements of $\sqrt{\mathscr{F}}$.

Proof. Let as usual $g = \sum_{i=1}^{n} = g_i d\bar{z}_i$ be a (0,1)-form with holomorphic coefficients and define

$$v(z) = \sum_{i=1}^{n} \bar{z}_i g_i(z)$$

It follows that

$$\begin{aligned} \bar{\partial} v &= \sum_{i=1}^{n} \frac{\partial v}{\partial \bar{z}_i} d\bar{z}_i \\ &= \sum_{i=1}^{n} = g_i d\bar{z}_i = g \end{aligned}$$

The solution operator is orthogonal to the kernel of $\bar{\partial}$.The kernel is exactly the Fock space, so the Bergman projection and the projection onto the kernel of $\bar{\partial}$ coincide. The solution operator is well defined. It follows by Hörmander(see[23]) that the solution operator is bounded. It has to coincide with the Hankel operator on the dense set $\mathscr{F}$. ☐

Remark 4.16. *Proposition(4.4)gives an integral representation of the solution operator with A^2-coefficients. It turns out that this property remains valid if the right-hand side of (4.2.1) has coefficients in $\sqrt{\mathscr{F}}$.*
The proof is the same as in proposition (4.4).
Now we want to compute the projection of certain functions explicitly.For this

purpose, we take profit of the Taylor expansion of the reproducing kernel. Let us, before doing this, make the following conventions.
Let $k = (k_1 \dots k_n)$ be a multi-index.Then

$$k + 1_j = (k_1 \dots, k_j + 1, \dots k_n)$$

Lemma 4.17. *Let f the(0,1)-form defined by $F = z^j d\bar{z}_1$ then*

$$P(f\bar{z})(\omega) = \frac{c_j^2}{c_{j-1}^2}\omega^{j-1_1}$$

Proof. By orthogonality of the set $\{z^\alpha , \alpha \in \mathbb{N}^n\}$ we have:

$$\begin{aligned}
P(f\bar{z})(\omega) &= \int_{\mathbb{C}^n} K(\omega, z) < f(z), z > d\mu(z) \\
&= \int_{\mathbb{C}^n} \sum_m \frac{\bar{z}^m}{c_m^2}\omega^m(\bar{z}_1 z^j)d\mu(z) \\
&= \int_{\mathbb{C}^n} \sum_m \frac{\bar{z}^{m+1_1} z^j}{c_m^2}\omega^m d\mu(z) \\
&= \sum_m \frac{1}{c_m^2}\omega^m \int_{\mathbb{C}^n} \bar{z}^{m+1_1} z^j d\mu(z) \\
&= \sum_m \frac{1}{c_m^2}\omega^m < z^j, z^{m+1_1} >_{\mathscr{F}} \\
&= \sum_m \frac{c_{m+1_1}}{c_m^2}\delta_{m+1_1,j}\omega^m \\
&= \frac{c_j^2}{c_{j-1_1}^2}\omega^{j-1_1}
\end{aligned}$$

□

Lemma 4.18. *Let $S_1 : \mathscr{F}_{(0,1)} \to \mathscr{F}(\mathbb{C}^n)$ with adjoint S_1^*.*
For the special (0,1)-form defined by

$$g = \tilde{g}d\bar{z}_1 \,\textit{where}\, \tilde{g} \in \mathscr{F}(\mathbb{C}^n) \quad .$$

We have:

$$S_1^*(g)(\omega) = \Big(\int_{\mathbb{C}^n} K(\omega, z)h(z)(z_1 - \omega_1)d\mu(z)\Big)d\bar{\omega}_1$$

Proof. Let $h \in \mathscr{F}(\mathbb{C}^n)$. By(4.2.2)we have:

$$\begin{aligned}
< S_1(g), h >_{\mathscr{F}} &= \int_{\mathbb{C}^n} S_1(g)(z)\bar{h}(z)d\mu(z) \\
&= \int_{\mathbb{C}^n} \bar{h}(z)\Big(\int_{\mathbb{C}^n} K(z,\omega) < g, z-\omega > d\mu(\omega)\Big)d\mu(z) \\
&= \int_{\mathbb{C}^n} \bar{h}(\omega)\Big(\int_{\mathbb{C}^n} K(z,\omega)\tilde{g}(\omega)\overline{(z_1-\omega_1)}d\mu(\omega)\Big)d\mu(z) \\
&= \int_{\mathbb{C}^n} \tilde{g}(\omega)\overline{\Big(\int_{\mathbb{C}^n} K(\omega,z)h(z)(z_1-\omega_1)d\mu(z)\Big)}d\mu(\omega) \\
&=< g, S_1^* h >_{\mathscr{F}_{(0,1)}}
\end{aligned}$$

□

Remark 4.19. *Let$A : H_1 \to H_2$ be an operator between Hilbert spaces, it is well known that A^*A is compact if and only if A is,see for example [16], we are going to use this method in order to prove non-compactness of S_1.*

Theorem 4.20. *Let*

$$U_m(z) = \frac{z^m}{c_m} d\bar{z}_1$$

where $m = (m_1 \dots m_n)$ is a multi-index.
If $m_1 > 1$.Then

$$S_1^* S_1(U_m)(\omega) = \Big(\frac{c_{m+1_1}^2}{c_m} - \frac{c_m^2}{c_{m-1_1}^2}\Big)U_m(\omega)$$

Proof. We still know from lemma(4.17)that, if $f = z^m d\bar{z_1}$.Then

$$P(f\bar{z})(\omega) = \frac{c_m^2}{c_{m-1_1}^2}\omega^{m-1_1}$$

Hence

$$S_1(U_m)(z) = \bar{z_1}U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}$$

By the last lemma, it follows that

$$\begin{aligned}
S_1^* S_1(U_m)(\omega) &= \int_{\mathbb{C}^n} K(\omega, z)\Big(\bar{z}_1 U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}\Big)(z_1 - \omega_1) d\mu(z) \\
&= \int_{\mathbb{C}^n} \Big(\sum_k \frac{\bar{z}^k}{\sqrt{k!}} \frac{\omega^k}{\sqrt{k!}}\Big)\Big(\bar{z}_1 U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}\Big)(z_1 - \omega_1) d\mu(z) \\
&= \int_{\mathbb{C}^n} \Big(\sum_k \frac{\bar{z}^k}{\sqrt{k!}} \frac{\omega^k}{\sqrt{k!}}\Big)\Big(\bar{z}_1 U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}\Big)(z_1) d\mu(z) \\
&= \int_{\mathbb{C}^n} \frac{z^{m+1_1}}{c_m} \Big(\sum_k \frac{\omega^k}{\sqrt{k!}} \frac{\bar{z}^{k+1_1}}{\sqrt{k!}}\Big) d\mu(z) \\
&- \frac{c_m}{c_{m-1_1}^2} \int_{\mathbb{C}^n} z^m \Big(\sum_k \frac{\omega^k}{\sqrt{k!}} \frac{\bar{z}^{k+1_1}}{\sqrt{k!}}\Big) d\mu(z) \\
&= \frac{\omega^m}{c_m^3} \int_{\mathbb{C}^n} |z|^{2(m+1_1)} d\mu(z) - \frac{\omega^m}{c_{m-1_1}^2 c_m} \int_{\mathbb{C}^n} |z|^{2m} d\mu(z) \\
&= \Big(\frac{c_{m+1_1}^2}{c_m^2} - \frac{c_m^2}{c_{m-1_1}^2}\Big) U_m(\omega)
\end{aligned} \tag{4.4.1}$$

For the integral of the second line in the equality above we have of course,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \Big(\sum_k \frac{\bar{z}^k}{\sqrt{k!}} \frac{\omega^k}{\sqrt{k!}}\Big)\Big(\bar{z}_1 U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}\Big)(\omega_1) d\mu(z) \\
&= \omega_1 \int_{\mathbb{C}^n} \Big(\sum_k \frac{\bar{z}^k}{\sqrt{k!}} \frac{\omega^k}{\sqrt{k!}}\Big)\Big(\bar{z}_1 U_m(z) - \frac{c_m^2}{c_{m-1_1}^2} z^{m-1_1}\Big) d\mu(z) \\
&= \omega_1 P(I - P)(z_1 U_m(z)) = 0
\end{aligned}$$

□

Now we are ready to prove the main result of this section.

Corollary 4.21. *The canonical solution operator S_1 on (0,1)-forms with Fock-space coefficients is not compact.*

Proof. The set

$$\Big\{U_m(z) = \frac{z^m}{c_m} d\bar{z}_1 \, , m = (m_1 \dots m_n)\Big\}$$

is orthonormal.
Now remember the definition of the c_k,

$$\begin{aligned} c_k^2 &= \int_{\mathbb{C}^n} |z|^{2k} e^{-|z|^2} d\lambda(z) \\ &= \int_{\mathbb{C}^n} |z_1|^{2k_1} \dots |z_n|^{2k_n} e^{-(|z_1|^2 + \dots |z_n|^2)} d\lambda(z_1, \dots, z_n) \end{aligned}$$

By Fubini and using the Euler Γ function we obtain

$$c_k = k!$$

Now,

$$\begin{aligned} \frac{c_{m+1_1}^2}{c_m^2} - \frac{c_m^2}{c_{m-1_1}^2} &= \frac{(m+1_1)!}{m!} - \frac{m!}{(m-1_1)!} \\ &= m_1 + 1 - m_1 = 1 \end{aligned}$$

So we have,

$$S_1^* S_1 U_m(z) = U_m(z)$$

which means that the operator$S_1^* S_1$ is not compact (it has the special eigenvectors to the eigenvalue 1). For an excellent survey on characterisation of compact operators see [4] or [9].

□

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