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"Local behaviour of vacant set of random walk on large girth expander graphs "

verfasst von/submitted by

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# Abstract

In this thesis we construct a coupling of the trace left by a lazy random walk on a $d$-regular large girth expander graph with random interlacements on the infinite $d$-regular tree. This coupling can be achieved on balls of mesoscopic volume. Our main tool is the coupling of two Markov chains on a finite state space, based on the technique of *soft local times*. It yields an estimate on the error of the coupling, by controlling the mixing time and the transition probability's density. The two Markov chains on the boundary of two balls are characterized by the encoding of the random walk's trajectories into the ball, for the expander graph and the tree respectively.

# Zusammenfassung

In dieser Arbeit koppeln wir den Pfad einer Irrfahrt auf einem Large Girth Expander Graph mit Random Interlacements auf dem unendlichen Baum. Wir setzen voraus, dass beide Graphen $d$-regulär sind. Diese Kopplung funktioniert auf Bällen von mesoskopischer Größe. Wir verwenden eine spezielle Kopplung von zwei Markovketten auf dem endlichen Zustandsraum, welche auf der so genannten Soft-Local-Time-Technik basiert. Diese liefert eine Fehlerabschätzung, indem wir die Mischzeit und die Varianz der Übergangswahrscheinlichkeiten kontrollieren. Beide Markovketten - sowohl für den Expander Graph, als auch für den Baum - erhalten wir als Kodierung der Trajektorien der Irrfahrten, welche den Ball treffen.

# Contents

# Chapter 1

# Introduction

## 1.1 Main result

In this thesis we study the trace of a lazy random walk on a large girth expander graph. We are interested in the local behaviour of a set of vertices, that has not been visited up to a certain time, called the vacant set of the random walk. Locally the expander is isomorphic to a regular tree, and as we will show the corresponding local model is the random interlacements on such a tree. In order to understand the vacant set of the random walk we try to find a local coupling with the vacant set of random interlacements on a infinite tree. Coupling the vacant sets is only interesting for large expanders. Thus we consider asymptotics on a diverging sequence of graphs. This coupling can be achieved on a mesoscopic subset of the large girth expander graph.

Let us precise the setting. Consider the $d$-regular, connected simple graph $G_n := G_n(V_n, E_n)$ with $n$ vertices and $d \geq 3$. Let $X = (X_k)_{k\geq 0}$ be the lazy random walk on $G_n$, i.e. the discrete-time Markov chain on the state space $V_n$, which at each step stays put with probability $\frac{1}{2}$ and otherwise chooses its next state uniformly among all neighbors of the current state in the graph. For the graph $G_n$ and a parameter $u \geq 0$, the set $\mathcal{V}_n^u$ of vertices not visited by the random walk until time $un$, is called *vacant set of the random walk on level* $u$, i.e.

$$\mathcal{V}_n^u = V_n \backslash \{X_0, ..., X_{\lfloor un \rfloor}\}. \tag{1.1}$$

The density of the random walk trace is governed by the parameter $u$. For small $u$, the vacant set occupies a large proportion of the graph $G_n$ and vice versa.

Let $\mathbb{T} := \mathbb{T}(\mathbb{V}, \mathbb{E})$ be the infinite $d$-regular tree. The model of random interlacements can be described as a Poisson point process of doubly infinite random walk trajectories modulo time shift on the tree $\mathbb{T}$. The intensity of the Poisson point process is driven by a multiplicative parameter $u \geq 0$. The union of the random walk trajectories' range is called the *interlacement set on level* $u$, denoted by $\mathcal{I}^u$. The complement of $\mathcal{I}^u$ is called *vacant set of random interlacements on level* $u$, i.e

$$\mathcal{V}^u := \mathbb{V} \setminus \mathcal{I}^u. \tag{1.2}$$

Let $\mathbb{G} := (G_n(V_n, E_n))_{n \geq 1}$ be a sequence of $d$-regular, connected simple graphs, such that the number $n = |V_n|$ of vertices tends to infinity, as $n \to \infty$. We call the sequence $\mathbb{G}$ a *family of large girth expander graphs*, if

1. $G_n$ is an expander graph, i.e. for $n \geq 1$, the spectral gap $\lambda_n$ of $G_n$ (see definition below, (2.1)) is uniformly bounded from below by a constant $\lambda > 0$.

2. $G_n$ is a large girth graph, i.e. for some $0 < \alpha \leq 1$ and $n \geq 1$, the length of the shortest cycle in the graph $G_n$ is bounded from below by $2\alpha \log_{d-1} n$.

We now come to the precise statement of our result. Consider a family of large girth expander graphs $\mathbb{G}$, with $d \geq 3$, $\lambda > 0$ and $0 < \alpha \leq 1$. Let $\bar{y}_n \in V_n$ and $\beta < \frac{\alpha}{2}$, such that

$$G_n \setminus B_n^{\bar{y}_n} \quad \text{is connected,} \tag{1.3}$$

for the ball

$$B_n^{\bar{y}_n} := B(\bar{y}_n, \beta \log_{d-1} n) \subset V_n. \tag{1.4}$$

Furthermore, consider the infinite $d$-regular tree $\mathbb{T}$ and the ball $B_n^o$ with some fixed arbitrary root $o \in \mathbb{V}$,

$$B_n^o = B(o, \beta \log_{d-1} n) \subset \mathbb{V}. \tag{1.5}$$

For the balls

$$A_n^{\bar{y}_n} = B(\bar{y}_n, \alpha \log_{d-1} n) \subset V_n \quad \text{and} \quad A_n^o = B(o, \alpha \log_{d-1} n) \subset \mathbb{V}, \tag{1.6}$$

we fix $\phi_{y_n}$, an arbitrary graph isomorphism

$$\phi_{\bar{y}_n} : A_n^{\bar{y}_n} \to A_n^o, \quad \text{with } \phi_{\bar{y}_n}(\bar{y}_n) = o. \tag{1.7}$$

**Theorem 1.1.** *Assume* $u > 0$, *and let* $\epsilon_n$ *be a sequence, s.t. for some* $\delta' > 0$ *and some sufficiently small* $c \leq \frac{1}{2}$

$$n^{\frac{1}{2}(\delta' - \beta)} \leq \epsilon_n \leq c. \tag{1.8}$$

*Then there exists a coupling $\mathbb{Q}_n$ of the vacant set $\mathcal{V}_n^u$ with $\mathcal{V}^{u(1\pm\epsilon_n)}$, such that for every $n$ large enough,*

$$\mathbb{Q}_n[(\mathcal{V}^{u(1+\epsilon_n)} \cap B_n^o) \subseteq \phi_{\bar{y}_n}(\mathcal{V}_n^u \cap B_n^{\bar{y}_n}) \subseteq (\mathcal{V}^{u(1-\epsilon_n)} \cap B_n^o)] \geq 1 - c_1 e^{-c_2 n^\delta}, \quad (1.9)$$

*for some constants $\delta > 0$, and $c_2, c_1 \in (0, \infty)$ depending on $\alpha, \beta, d, \lambda$.*

**Remark 1.2.** *We will mostly identify the vertices of $G_n$ and of $\mathbb{T}$ linked by the isomorphism described in* (1.7) *and omit the $\phi_{\bar{y}_n}$ from the notation. For the balls $A_n^y, B_n^y$ we usually omit the $y$ and the $n$ in the definition.*

*Let $g(G_n)$ denote the girth of the graph $G_n$, i.e. the length of the shortest cycle in $G_n$. For d-regular graphs $G_n^d$ we can easily derive the asymptotic upper bound $g(G_n^d) \leq 2\log_{d-1} n$. Thus in* (2) *we need to choose $\alpha \leq 1$. Lubotzky-Phillips-Sarnak [LPS88] gave explicit examples for d-regular graphs $G_n^d$, with $g(G_n^d) \geq \frac{4}{3}\log_{d-1} n$. Thus we can apply our coupling to Lubotzky-Phillips-Sarnak graphs while taking $\alpha = \frac{2}{3}$.*

## 1.2 Previous results and applications

The properties of the vacant set of the random walk on finite graphs have been studied in several recent works. In [BS06] Benjamini and Sznitman showed that, for the torus $(\mathbb{Z}/N\mathbb{Z})^d$ with large dimension $d$ and small enough $u > 0$, the vacant set has a unique connected component with a non-negligible density.

In [Szn10] Sznitman introduced the random interlacements on $\mathbb{Z}^d$. It was motivated by the idea to have an infinite volume analogue for the problem of fragmentation of the random walk on $(\mathbb{Z}/N\mathbb{Z})^d$. He proved a phase transition similar to Bernoulli site percolation on $\mathbb{Z}^d$ (see [Szn10, SS09]).

In [T$^+$09] Teixeira extended the construction to the more general setting of transient weighted graphs. When the graph under consideration is a tree, the vacant set containing some fixed point can be characterized in terms of a Bernoulli site percolation. For the specific case of $d$-regular trees, $d \geq 3$, there exists an explicit formula for the critical value $u^*$ of the phase transition.

Teixeira and Windisch (see [TW11]) used a coupling between $\mathbb{Z}^d$ and the torus, to show that in all dimensions $d \geq 3$ the volume of the vacant set exhibits a phase transition, i.e. for some $0 < u_1 \leq u_2 < \infty$ with high probability as $n$ tends to infinity,

- for $u < u_1$, the largest connected component of $\mathcal{V}_n(u)$ is of size of order $|V_n|$,
- for $u > u_2$, the largest connected component of $\mathcal{V}_n(u)$ is of size of order $\log^\kappa |V_n|$, for some $\kappa > 0$.

Although the conjecture of a sharp phase transition, i.e. $u_1 = u_2$, is still an open problem, Černý and Teixeira proved a sharp phase transition for the diameter of the vacant set containing a given point. In [ČT14] they apply a variant of the soft-local time coupling technique, in order to construct a coupling on macroscopic subsets of the torus. The proof of our main result is strongly motivated by this paper and a description of the proof will follow in the next subsection. Before that we finish this section with a result on expanders and the infinite tree.

A local coupling between the vacant set of the random walk on expanders $G_n$ and the vacant set of random interlacements on the d-regular infinite tree $\mathbb{T}$ can be found in [ČTW11]. In this paper Černý, Teixeira, Windisch investigate the percolative properties of the vacant set on $d$-regular, large girth expanders and $d$-regular random graphs. They show that the vacant set of these graphs undergo a phase transition in $u^* > 0$, the critical value of random interlacements on the infinite $d$-regular tree $\mathbb{T}$. More precisely, it was shown that with high probability as $n$ tends to infinity,

- for $u < u^*$, the vacant set has a unique component with volume of order $|V_n|$,
- for $u > u^*$, the largest component of the vacant set only has a volume of order $\log |V_n|$

The coupling is used to construct a sufficient amount of mesoscopic clusters in the supercritical phase $u < u^*$.

## 1.3 Overview of the proof

Let us now describe the idea of the proof and the organization of the thesis.

In Chapter 2 we start with a detailed introduction of the lazy random walk $X$ on large girth expander graphs $G_n$ and the infinite tree $\mathbb{T}$(see Section 2.1- 2.2). In Section 2.3 we introduce random interlacements and we use the capacity of finite sets $B_n \subset \mathbb{V}$, in order to characterize the law of the vacant set of random interlacements. The main principal tool for the proof of the main result is a coupling of two Markov chains on a finite state space (see Section 2.4), based on the technique of the so-called *soft local times* [PT12]. The corresponding statement (see Theorem 2.1), [ČT14, Theorem 3.1.]) provides an estimate on the error of the coupling, by controlling the mixing times and the transition probability's density of both chains. Theorem 2.1 will be used later in Section 5.1.

In Chapter 3 we proceed with some general estimates for the random walk and random interlacements. These results will be used in Chapter 4 to bound the relevant coupling quantities. The results include estimates for the capacity on the ball $B_n$ and the escape probabilities from $B_n$ (see Section 3.1-3.2). In Section 3.3 we give an approximation for the hitting probability of the boundary points of $B_n$ for the random walk on $G_n$ started on $B_n^c$.

In Chapter 4 we continue with an explicit construction of two Markov chains $Y$ and $Z$ on the finite state space $\partial B_n \times \partial A_n^c$. These encode the trajectories of the random walk on the expander into the ball $B_n$, and the trajectories of random interlacements into $B_n$. (see Section 4.1). In Section 4.2 we show the required equality of the stationary measures of both Markov chains $Y$ and $Z$, and estimate all relevant coupling quantities. These include the variance of the arrival density (see Section 4.3), the mixing time (see Section 4.4) and the number of excursions (see Section 4.5). The number of excursions provide a useful length scaling for the Markov chains $Y$ and $Z$.

In Chapter 5 we use Theorem 2.1 and the estimates from Chapter 4 to get the desired coupling on $\partial B \times \partial A^c$, for $Y$ and $Z$ (see Section 5.1). In Section 5.2 we re-decorate $Y$ and $Z$ to obtain a coupling of the vacant sets restricted to $B_n$. We define a process as the concatenation of excursions on the large girth expander graph. Similarly we define a sequence of processes as the concatenation of excursions on the tree. Applying the coupling of $Y$ and $Z$ from Section 4.5 and results from Section 5.1, finishes the proof of Theorem 1.1.

## 1.4 Notation

By $c, c_i, c'$ we denote positive finite constants, whose values might change during the computations. For two sequences $a = (a_n)_{n\geq 0}, b = (b_n)_{n\geq 0}$, we write

$$a \asymp b \quad :\Longleftrightarrow \quad c^{-1}a_n \leq b_n \leq ca_n \quad \text{for some } c > 0. \tag{1.10}$$

Given an arbitrary measure space $(\Omega, \mathcal{A})$, we write $\delta_x(\cdot)$ for the Dirac measure on $x \in \Omega$.

# Chapter 2

# Definition and results

In this section we introduce all required definitions as some basic results for the considered models.

## 2.1 Expanders and the tree

The graph denoted by $G_n := G(V_n, E_n)$ is always a *simple graph*, i.e. it is undirected, without loops and without multiple edges. Furthermore $G_n$ is connected, $d$-regular and has $n = |V_n|$ vertices.

Let $M_n$ be the adjacency matrix of $G_n$. The eigenvalues of the matrix $\frac{1}{d}M_n - I$ are denoted by

$$0 = \lambda_n^1 < \lambda_n^2 \leq \lambda_n^3 \leq \cdots \leq \lambda_n^n. \tag{2.1}$$

Then $\lambda_n^2 =: \lambda_n$ is the *spectral gap* of $G_n$. The object of our investigations is the family of large girth expander graphs $\mathbb{G} = (G_n)_{n\geq 1}$ (or simply family of expanders), introduced in Section 1.1. Remember the uniform lower bound $\lambda > 0$ of the spectral gap $\lambda_n$ for $n \geq 1$. We call $\lambda$ the *spectral constant* of $\mathbb{G}$ and $G_n \in \mathbb{G}$ an *large girth expander graph* (or simply expander).

From now on we fix $d \geq 3$, $0 < \alpha \leq 1$ and the spectral constant $\lambda$ for the family of expanders $\mathbb{G} = (G_n)_{n\geq 1}$, and therefore omit $d$, $\lambda$ and $\alpha$ from the notation. For $G_n$ we sometimes also omit the $n$ in the definition and write simply $G$.

Due to the regularity, we know that $G_n$ has $\frac{dn}{2}$ edges, i.e. $G_n$ is sparse for $n \geq 1$. On the other hand, the spectral gap of $G_n$ is uniformly bounded away from 0. At least qualitatively, the spectral gap is tightly linked to high connectivity. These observations induce a more intuitive characterization of expanders. For a simple graph $G(V,E)$, let

$$h(G) = \min\left\{\frac{|\partial_e A|}{|A|} : A \subset V, |A| \leq \frac{|V|}{2}\right\} \tag{2.2}$$

be the *edge expansion* (or isoperimtric constant). The set $\partial_e A \subset E$ is called the edge boundary of the set $A$, i.e.

$$\partial_e A := \{(x,y) \in E : x \in A, y \in A^c\}. \tag{2.3}$$

The intuition that spectral gap is related to edge expansion is made precise by the following inequality.

**Theorem 2.1** (Cheeger's inequality)**.** *Let $G(V,E)$ be a $d$-regular, connected simple graph with spectral gap $\lambda$ and edge expansion $h(G)$, then*

$$\frac{\lambda}{2} \leq h(G) \leq \sqrt{2d\lambda}. \tag{2.4}$$

*Proof.* See [LW02, Theorem 4.9.]. □

The notations below are valid for an arbitrary graph $G(V,E)$. For two vertices $x,y \in V$, we write $x \sim y$, if $x$ is a *neighbor* of $y$. By a *path* we mean a sequence of vertices $x_1,\ldots,x_i$ such that $x_{k+1} \sim x_k$ for all $1 \leq k \leq i-1$ and we write $x_1 \leftrightarrow x_i$. The metric we use is the *graph distance* $\mathrm{dist}(\cdot,\cdot)$. It is characterized by the length of the shortest path between any two vertices. We write $\mathrm{diam}(G)$ for the diameter of the graph $G$, i.e. $\mathrm{diam}(G) = \max\{\mathrm{dist}(x,y) : x,y \in V\}$. We denote $B(x,r)$ the ball centered at $x \in V$ with radius $r$,

$$B(x,r) = \{y \in V : \mathrm{dist}(x,y) \leq r\}. \tag{2.5}$$

The boundary of $A \subset V$ is given by

$$\partial A = \{x \in A : x \sim y \text{ for some } y \in A^c\}, \tag{2.6}$$

where $A^c$ is the complement of $A$ in $V$. We denote $G \cap A$ and $G \setminus A$ the subgraphs induced by $A$ respectively $A^c$. By abuse of notation we sometimes write $A \subset G$, if $A \subset V$ for $G(V,E)$. For $A \subset V$, $A$ finite, we use the notation $A \subset\subset V$.

Recall the definition of the infinite, connected, $d$-regular tree $\mathbb{T} := \mathbb{T}(\mathbb{V},\mathbb{E})$. A distinct vertex $o \in \mathbb{V}$, is called the root of $\mathbb{T}$. For every $x \in \mathbb{V}$ we define the descendants in the tree by

$$V_x = \{y \in \mathbb{V} \setminus B(o, \mathrm{dist}(o,x)-1) : y \leftrightarrow x \text{ on } \mathbb{T} \setminus B(o,\mathrm{dist}(o,x)-1)\}. \tag{2.7}$$

and the subgraph $\mathbb{T}_x$ induced by $V_x$.

## 2.2 Lazy random walk (LRW)

The *lazy random walk (LRW)* on $G$, is the Markov process in discrete time with generator given by

$$(\Delta f)(x) = \sum_{y \in V} (f(y) - f(x)) p_{xy} \quad \text{for } f : V \to \mathbb{R}, \quad x \in V, \tag{2.8}$$

where $p_{xy} = \frac{1}{2d}$ if $x \sim y$ and $p_{xy} = 0$ otherwise. Note that $\Delta = \frac{1}{2d} M - \frac{1}{2} I$, where $M$ is the adjacency matrix of $G$. We use $P_x$ to denote the law of the lazy random walk on $G$ started at $x \in V$. The process $X = (X_k)_{k \geq 0}$ is the canonical process on $G$ and $(\mathcal{F}_k)_{k \geq 0}$ the canonical filtration. We write $E_x$ for the corresponding expectation. Keep in mind, that there exists a unique stationary distribution $\pi$ for the random walk $X$, which satisfies $\pi(x) p_{xy} = \pi(y) p_{yx}$, i.e. $\pi$ is reversible. Since the graphs $G$ are $d$-regular, the measure $\pi$ is actually uniform. Starting the random walk $X$ in $\pi$, we use the notation $P := P_\pi$.

By $\theta_i$, $i \geq 0$, we denote the canonical shift for the walk, defined on $V^{\mathbb{N}}$, i.e.,

$$\theta_i(x_0, x_1, \dots) = (x_i, x_{i+1}, \dots). \tag{2.9}$$

For the law $P_\pi$, with uniform measure $\pi$ on $V$, the canonical shifts $\theta_i$ are invariant transformations on $V^{\mathbb{N}}$, i.e. $P_\pi \circ \theta_i = P_\pi$ for all $i \geq 0$. Let $\lambda$ be the spectral gap of $G$, then from [SC97, p.328] it follows that

$$\sup_{x,y \in V} |P_x[X_k = y] - \pi(y)| \leq e^{-\lambda k} \quad k \geq 0. \tag{2.10}$$

Consider the random walk $X$ killed on hitting $B$ with generator $\Delta^B$ given by

$$(\Delta^B f)(x) = \sum_{y \in V \setminus B} (f(y) - f(x)) p_{xy} \quad \text{for } f : V \setminus B \to \mathbb{R}, \quad x \in V \setminus B, \tag{2.11}$$

where $p_{xy}$ are as above. We denote

$$0 < \lambda_B^1 < \lambda_B^2 \leq \cdots \leq \lambda_B^{|V \setminus B|} \tag{2.12}$$

the eigenvalues of $\Delta^B$. We further define the *quasi-stationary distribution* $\sigma_B$, for $B \subset V$, on the expander $G$. The distribution $\sigma_B$ is the normalized right-eigenvector $v_B^1$ of $\Delta^B$ corresponding to the eigenvalue $\lambda_B^1$.

Since $\mathbb{T}$ is locally finite, actually $d$-regular, we define the lazy random walk on $\mathbb{T}$ in the same manner. We write $P_x^{\mathbb{T}}$ for the canonical law of the lazy random walk on $\mathbb{T}$ started from $x$, and $(X_k)_{k \geq 0}$ for the canonical process as well. Writing $P_x^o$, we mean the law $P_x^{\mathbb{T}}$ for the walk on $\mathbb{T}$ either the law $P_x$ for the walk on $G$.

In order to construct random interlacements we need the definition of the normalized equilibrium measure and the capacity for finite subsets $B$ on $\mathbb{T}$.

**Definition 2.1.** *Let $B \subset\subset \mathbb{T}$, $d \geq 3$ and $x \in \mathbb{T}$. We set*

$$e_B(x) := P_x[\tilde{H}_B = \infty]\mathbf{1}_{\{x \in B\}}, \tag{2.13}$$

*and denote $e_B$ the equilibrium measure of $B$. Its total mass*

$$\text{cap}(B) := \sum_{x \in B} e_B(x) \tag{2.14}$$

*is called the capacity of $B$. The measure $\bar{e}_B$ denotes the normalized equilibrium measure on $B$, and is given by*

$$\bar{e}_B(x) := \frac{e_B(x)}{\text{cap}(B)}. \tag{2.15}$$

Note that $\bar{e}_B$ is supported on the boundary $\partial B \subset B$. The capacity for any finite subset is nontrivial, only if $d \geq 3$.

Let us define the normalized equilibrium measure on $B$, for finite and infinite graphs as well.

**Definition 2.2.** *Let $G(V, E)$ be a d-regular, connected, simple graph and $P_x$ the law for the lazy random walk started in $x \in V$. For $B \subset A \subset\subset V$ and $x \in B$ we set*

$$e_B^{A^c}(x) := P_x[\tilde{H}_B > H_{B^c}]\mathbf{1}_{\{x \in B\}}, \tag{2.16}$$

*and*

$$\text{cap}_{A^c}(K) := \sum_{x \in B} e_B^{A^c}(x). \tag{2.17}$$

*The measure $\bar{e}_B^{A^c}$ denotes the normalized equilibrium measure on $B$, for the walk killed on $A^c$, and is given by*

$$\bar{e}_B^{A^c}(x) := \frac{e_B^{A^c}(x)}{\text{cap}_{A^c}(B)}. \tag{2.18}$$

We finish the section with some well known results for a finite connected graph $G(V, E)$ and the corresponding lazy random walk with law $P_x$. We call a function $h : V \to \mathbb{R}$ harmonic on $A$, if $\Delta h(x) = 0$ for all $x \in A \subset V$. For functions $f, g : V \to \mathbb{R}$ we define the Dirichlet form

$$\mathcal{D}(f, g) = \frac{1}{2} \sum_{x,y \in V} \big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\pi(x)p_{xy}. \tag{2.19}$$

**Theorem 2.3.** *Let $A, C$ be two non-empty disjoint subsets of $V$. Then there exists a unique function $g^*_{A,C}$, s.t.*

$$\Delta g^*_{A,C}(x) = 0 \quad \forall x \in V, \tag{2.20}$$

$$g^*_{A,C}|_A = 1 \quad \text{and} \quad g^*_{A,C}|_C = 0. \tag{2.21}$$

*We call $g^*_{A,C}$ the equilibrium potential. It is given by*

$$g^*_{A,C}(x) = P_x[H_A \leq H_C] \quad \forall x \in V. \tag{2.22}$$

*Furthermore,*

$$\mathcal{D}(g^*_{A,C}, g^*_{A,C}) = \sum_{x \in A} P_x[\tilde{H}_A > H_C]\pi(x). \tag{2.23}$$

*Proof.* See [AF02, Lemma 2.27, Theorem 3.36, Corollary 3.37]. □

## 2.3 Random interlacements

We now introduce random interlacements on the infinite tree $\mathbb{T}$. We define the local vacant set $\mathcal{V}^u \cap K$, $K \subset\subset \mathbb{T}$, which possesses, as we will see at the end of the section, a particularly useful representation (see 2.42).

We begin with the introduction of the measurable space $(W^*, \mathcal{W}^*)$ of doubly infinite lazy random walk trajectories modulo time shifts on $\mathbb{T}$ and the $\sigma$-finite measure $\nu$ on it. Let $w = (\dots, w(k-1), w(k), w(k+1), \dots)$, then

$$W = \left\{ \begin{matrix} w : \mathbb{Z} \to \mathbb{T} : \text{dist}(w(k), w(k+1)) \leq 1 \text{ for all } k \in \mathbb{Z} \\ \text{and } \text{dist}(w(k), o)) \to \infty \text{ as } k \to \pm\infty \} \end{matrix} \right\}$$

is the space of doubly infinite nearest neighbor trajectories which visit every finite subset of $\mathbb{T}$ only finitely many times, and for $w = (w(0), w(1), \dots)$

$$W_+ = \left\{ \begin{matrix} w : \mathbb{N} \to \mathbb{T} : \text{dist}(w(k), w(k+1)) \leq 1 \text{ for all } k \in \mathbb{N}, \\ \text{and } \text{dist}(w(k), o)) \to \infty \text{ as } k \to \infty \end{matrix} \right\}$$

the space of forward trajectories which spend finite time in finite subsets of $\mathbb{T}$. We denote by $X_k$ the canonical coordinates on $W$ and $W_+$, i.e., $X_k(w) = w(k)$. We write $\mathcal{W}$ for the $\sigma$-algebra on $W$ generated by $(X_k)_{k \in \mathbb{Z}}$, and $\mathcal{W}_+$ for the $\sigma$-algebra on $W_+$ generated by $(X_k)_{k \in \mathbb{N}}$.

**Definition 2.1.** *Let $\sim$ be the equivalence relation on $W$ defined by*

$$w \sim w' \iff \exists i \in \mathbb{Z} : w' = \theta_i(w), \tag{2.24}$$

*i.e., $w$ and $w'$ are equivalent, if $w'$ can be obtained from $w$ by a time shift. The quotient space $W/\sim$ is denoted by $W^*$. We write*

$$\text{proj} : W \to W^* \tag{2.25}$$

*for the canonical projection which assigns to a trajectory $w \in W$ its $\sim$- equivalence class* $\text{proj}(w) \in W^*$. *The natural $\sigma$-algebra $\mathcal{W}^*$ is defined by*

$$A \in \mathcal{W}^* \iff (\text{proj})^{-1}(A) \in \mathcal{W}. \tag{2.26}$$

In other words, two trajectories are in the same equivalence class, if their paths coincide.

For any $K \subset\subset \mathbb{T}$, we define

$$W_K = \{w \in W : X_k(w) \in K \text{ for some } i \in \mathbb{Z}\} \in \mathcal{W} \tag{2.27}$$

to be the set of trajectories in $W$ that hit $K$, and let $W_K^* = \text{proj}(W_K) \in \mathcal{W}^*$. It will be helpful to partition $W_K$ according to the first entrance time of trajectories in $K$. For this purpose we define for $w \in W$, $k \in \mathbb{Z}$ and $K \subset\subset \mathbb{T}$,

$$W_K^k = \{w \in W : H_K(w) = k\} \in \mathcal{W}. \tag{2.28}$$

The sets $(W_K^k)_{k\in\mathbb{Z}}$ are disjoint and

$$W_K = \cup_{k\in\mathbb{Z}} W_K^k, \tag{2.29}$$

$$W_K^* = \text{proj}(W_K^k) \quad \forall k \in \mathbb{Z}. \tag{2.30}$$

Recall from Section 2.2, that $P_x^{\mathbb{T}}$ denotes the law of the lazy random walk starting in $x$. Consider $P_x^{\mathbb{T}}$ as a probability measure on $W_+$. We will proof later, that for $d \geq 3$ the random walk is transient, i.e., $P_x^{\mathbb{T}}[W_+] = 1$. Using the notions of the hitting time $\tilde{H}_K$ and the normalized equilibrium measure $\bar{e}_K$ of $K \subset\subset \mathbb{T}$ from Section 2.2, we define the measure $Q_K$ on $(W, \mathcal{W})$ by the formula

$$Q_K[(X_{-k})_{k\geq 0} \in A, X_0 = x, (X_k)_{k\geq 0} \in B] = P_x^{\mathbb{T}}[A|\tilde{H}_K = \infty]\bar{e}_K(x)P_x^{\mathbb{T}}[B] \tag{2.31}$$

for any $A, B \in \mathcal{W}_+$ and $x \in \mathbb{T}$. Note that we defined $Q_K$ only on sets of form

$$A \times \{X_0 = x\} \times B \in \mathcal{W}, \tag{2.32}$$

but the sigma-algebra $\mathcal{W}$ is generated by events of this form, so $Q_K$ can be uniquely extended to all $\mathcal{W}$ -measurable subsets of $W$ . For any $K \subset\subset \mathbb{T}$,

$$Q_K[W] = Q_K[W_K] = Q_K[W_K^0] = \sum_{x\in K} Q_K[X_0 = x] = \sum_{x\in K} \bar{e}_K(x) = \text{cap}(K). \tag{2.33}$$

In particular, the measure $Q_K$ is finite, and $\frac{1}{\mathrm{cap}(K)}Q_K$ is a probability measure on $(W, \mathcal{W})$ supported on $W_K^0$. The following theorem yields a $\sigma$-finite measure $\nu$ on the measurable space $(W^*, \mathcal{W}^*)$.

**Theorem 2.2.** *There exists a unique $\sigma$-finite measure $\nu$ on $(W^*, \mathcal{W}^*)$, such that for all $K \subset\subset \mathbb{T}$,*

$$\forall A \in \mathcal{W}^*, A \subset \mathcal{W}_K^* : \quad \nu(A) = Q_K[(\mathrm{proj})^{-1}(A)]. \tag{2.34}$$

*Proof.* See [DRS14, Theorem 6.2]. □

We further define the random interlacements point process on the space $W^* \times \mathbb{R}_+$ of labeled doubly-infinite trajectories modulo time shift. We endow this product space with the product $\sigma$-algebra $\mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+)$, and define the measure $\nu \otimes du$, where $\nu$ is the measure constructed in Theorem 2.2, and $du$ is the Lebesgue measure on $\mathbb{R}_+$. Note that for any $K \subset\subset \mathbb{T}$ and $u \geq 0$,

$$(\nu \otimes du)(W_K^* \times [0,u]) = \nu(W_K^*)u = \mathrm{cap}(K)u < \infty. \tag{2.35}$$

Thus, the measure $\nu \otimes du$ is $\sigma$-finite on $(W^* \times \mathbb{R}_+, \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+))$, and can be regarded as an intensity measure for a Poisson point process on $W^* \times \mathbb{R}_+$. It will be useful to consider this Poisson point process on the canonical probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where

$$\Omega := \left\{ \begin{array}{c} \omega = \sum_{k\geq 0} \delta_{(w_k^*, u_k)} : (w_k^*, u_n) \in W^* \times \mathbb{R}_+ \text{ for any } k \geq 0 \\ \text{and } \omega(W_K^* \times [0,u]) < \infty \text{ for any } K \subset\subset \mathbb{T}, u \geq 0 \end{array} \right\} \tag{2.36}$$

is the space of locally finite point measures on $W^* \times \mathbb{R}_+$, the $\sigma$-algebra $\mathcal{A}$ is generated by the evaluation maps

$$\omega \mapsto \omega(D) = \sum_{k \geq 0} \mathbf{1}_{\{(w_k^*, u_k) \in D\}}, \quad D \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+), \tag{2.37}$$

and $\mathbb{P}$ is the probability measure on $(\Omega, \mathcal{A})$, such that

$$\omega := \sum_{k \geq 0} \delta_{(w_k^*, u_k)} \tag{2.38}$$

is the *Poisson point process with intensity $\nu \otimes du$ on $(W^* \times \mathbb{R}_+, \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}))$ under $\mathbb{P}$.*

**Definition 2.3.** *We call the random set $\mathcal{I}^u \subset \mathbb{V}$* random interlacements *at level $u$, if*

$$\mathcal{I}^u(\omega) := \bigcup_{u_n \leq u} \mathrm{range}(w_n^*), \quad \textit{for } \omega = \sum_{k \geq 0} \delta_{(w_k^*, u_k)} \in \Omega, \tag{2.39}$$

*where*

$$\text{range}(w^*) = \{X_k(w) : w \in (\text{proj})^{-1}(w^*), k \in \mathbb{Z}\} \subset \mathbb{T} \tag{2.40}$$

*is the set of all vertices of* $\mathbb{T}$ *visited by* $w^*$*. The* vacant set of random interlacements at level $u$ *is defined as*

$$\mathcal{V}^u(\omega) := \mathbb{T} \setminus \mathcal{I}^u(\omega). \tag{2.41}$$

We finish the section with a simple representation of the set $\mathcal{V}^u \cap K$. Let $J_K^u$ be a Poisson random variable with parameter $u\text{cap}(K)$, and $(X^{(i)})_{i\geq 0}$ an i.i.d. sequence of simple random walks on $\mathbb{T}$ with law $P_{\bar{e}_K}$, independent from $J_K^u$. Then

$$\mathcal{V}^u \cap K \stackrel{\text{law}}{=} K \setminus \bigcup_{1\leq i\leq J_K^u} \bigcup_{k\geq 0} \{X_k^{(i)}\}. \tag{2.42}$$

As we will see in Section 4.1, this representation makes the encoding of excursions of the sequence $(X^{(i)})_{i\geq 0}$ useful.

## 2.4 Coupling the ranges of Markov chains

In this section we construct a coupling of two Markov chains on a finite state space such that their ranges almost coincide. For these Markov chains with equal stationary measure, the difference of their ranges can be controlled by the mixing time and the arrival density's variance. The theorem of the coupling is abstract and will be applied later for two processes on the set $\partial B \times \partial A^c$.

Let us now precise the setting of this section. For $i \in \{1, 2\}$ and the finite state space $\Sigma$, let $P_i = (p^i(x, y))_{x,y\in\Sigma}$ be a Markov transition matrix, and $\nu_i$ a distribution on $\Sigma$. We assume that $P_i$ is irreducible, and that there exists a unique $P_i$-invariant distribution $\pi$ for both $P_1$ and $P_2$ on $\Sigma$. The *mixing time* $T_i$ corresponding to $P_i$ is defined by

$$T_i = \min\{n \geq 0 : \max_{x\in\Sigma}\{\|P_i^n(x, \cdot\,) - \pi(\cdot)\|_{TV}\} \leq \frac{1}{4}. \tag{2.43}$$

where $\|\cdot\|_{TV}$ denotes the total variation distance, i.e.,

$$\|\nu^i - \nu'^i\|_{TV} := (1/2) \sum_{x\in\Sigma} |\nu^i(x) - \nu'^i(x)|. \tag{2.44}$$

Let $\mu$ be an *apriori* measure on $\Sigma$ with full support. This measure is introduced for convenience only. Let $g : \Sigma \to [0, \infty)$ be the density of $\pi$ with respect to $\mu$,

$$g(x) = \frac{\pi(x)}{\mu(x)}, \quad \forall x \in \Sigma, \tag{2.45}$$

and let further $\rho^i : \Sigma^2 \to [0,\infty)$ be the transition density with respect to $\mu$, i.e.,

$$\rho^i(x,y) = \frac{p^i(x,y)}{\mu(y)}, \quad \forall x,y \in \Sigma. \tag{2.46}$$

We use $\rho^i_y$ to denote the function $x \mapsto \rho^i(x,y)$ giving the arrival probability density at $y$ as we vary the starting point. For any function $f : \Sigma \to \mathbb{R}$, let $E_\pi[f] = \Sigma_{x\in\Sigma}\pi(x)f(x)$, and $\mathrm{Var}_\pi[f] = E_\pi[(f - E_\pi(f))^2]$.

The following theorem provides a coupling of two Markov chains so that their ranges almost coincide.

**Theorem 2.1.** *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where one can define Markov chains $Z^1$,$Z^2$ with respective transition matrices $P_1$, $P_2$ and starting distributions $\nu_1, \nu_2$ such that for every $\epsilon$ satisfying*

$$0 < \epsilon \le \frac{1}{2} \wedge \min_{i=1,2} \min_{z\in\Sigma} \frac{\mathrm{Var}_\pi \rho^i_z}{2||\rho^i_z||_\infty g(z)}, \tag{2.47}$$

*and $n \ge 2k(\epsilon)(T_1 \vee T_2)$ we have*

$$\begin{aligned} \mathbb{Q}[\mathcal{G}(n,\epsilon)^c] \le C \sum_{i=1,2}\sum_{z\in\Sigma} \Big[ &\exp(-cn\epsilon^2) \\ &+ \exp\Big(-\frac{cn\epsilon\pi(z)}{\nu_i(z)}\Big) + \exp\Big(-\frac{c\epsilon^2 g(z)^2}{\mathrm{Var}_\pi \rho^i_z}\frac{n}{k(\epsilon)T_i}\Big)\Big], \end{aligned} \tag{2.48}$$

*where $c, C \in (0,\infty)$ are absolute constants, $\mathcal{G}(n,\epsilon)$ is the event*

$$\mathcal{G}(n,\epsilon) = \Big\{ \bigcup_{1\le i\le n(1-\epsilon)} Z^1_i \subset \bigcup_{1\le i\le n} Z^2_i \subset \bigcup_{1\le i\le n(1+\epsilon)} Z^1_i \Big\}, \tag{2.49}$$

*and*

$$k(\epsilon) = -\min_{z\in\Sigma} \log_2 \frac{\epsilon^2 g(z)^2 \min_{x\in\Sigma}\pi(x)}{6\mathrm{Var}_\pi(\rho^i_z)}. \tag{2.50}$$

*Proof.* See [ČT14, Section 3]. □

# Chapter 3

# Local behaviour of LRW

In the following section we show some general properties for the random walk on $G$ and $\mathbb{T}$. These properties will be used to prove all relevant coupling quantities.

Let us recall some assumptions. For $n \geq 1$, the graph $G(V, E) = G_n(V_n, E_n)$ is a large girth expander graph with fixed $d \geq 3, \lambda > 0$ and $0 < \alpha \leq 1$. The subgraph $G \cap A_n^y$ is cycle-free for all $y = y_n \in V$, where $A_n^y = B(y, \alpha \log_{d-1} n)$. We choose $\bar{y}_n = \bar{y} \in V$ and $\beta < \frac{\alpha}{2}$, s.t. the induced graph $G \setminus B$ is connected, where $B = B_n^{\bar{y}} = B(\bar{y}, \beta \log_{d-1} n)$ and $A = A_n^{\bar{y}}$.

## 3.1 LRW killed outside the ball

Due to the isomorphism between $G \cap A$ and $\mathbb{T} \cap A$ the laws of the random walks started in $A$ killed on $A^c$ are equal for the expander $G$ and the tree $\mathbb{T}$. Recall that $P_x^o$ can stand for the laws $P_x^{\mathbb{T}}$ and $P_x$.

The following lemma gives us information about the hitting probability of the sets $\partial A^c$ and $\partial B$, starting in $x \in A \setminus B$.

**Lemma 3.1.** *Let $B(y, R) \cap G$ be cycle-free and $0 \leq r < R$. Then for all $x \in B(y, R)$ with $r(x) := \text{dist}(y, x) \geq r$*

$$P_x^o[H_{B(y,r)} < H_{B^c(y,R)}] = \frac{(d-1)^{R-r(x)} - 1}{(d-1)^{R-r} - 1}. \tag{3.1}$$

*Proof.* Since $G \cap B(y, R)$ is a tree, the probability doing one step to the direction of $\partial B(y, R)$, started from any $x \in \text{int}(B(y, R))$ is given by

$$P_x^o[X_1 \in \partial B(y, r(x) + 1)] = \frac{d-1}{2d}, \tag{3.2}$$

Now consider $X'$ the lazy random walk in $\mathbb{Z}$ with drift $\frac{d-1}{2d}$ on the space $(\Omega, \mathcal{A}, P')$. Then

$$P'_x[X'_1 = x+1] = \frac{d-1}{2d} \quad \forall x \in \mathbb{Z}, \tag{3.3}$$

and $f(x) = (d-1)^{-x}$ is harmonic for $P'$, i.e. $(P'f)(x) = f(x)$ for all $x \in \mathbb{Z}$. Then $f(X'_{T_{[r+1,R-1]\wedge n}})$ is a martingale and by the optional stopping theorem [LPW09, Theorem 17.6]

$$E_x[(f(X'_{T_{[r+1,R-1]}})] = f(x) \quad r \le r(x) \le R. \tag{3.4}$$

Since the event $\{X'_{T_{[r+1,R-1]}} = x\}$ is supported for $x \in \{r, R\}$,

$$P'_x[X'_{T_{[r+1,R-1]}} = r] = \frac{(d-1)^{R-x} - 1}{d-1)^{R-r} - 1}. \tag{3.5}$$

Since (3.2) holds for any $x \in B(y,R) \setminus \partial B(y,R)$,

$$P'_x[X'_{T_{[r+1,R-1]}} = r] = P^o_x[H_{B(y,r)} < H_{B(y,R)}], \tag{3.6}$$

and the lemma follows. □

We now compute the escape probability with respect to $B$, and the capacity of $B$. Note that

$$|\partial B| = \frac{d}{d-1} n^\beta \quad \text{and} \quad |\partial A| = \frac{d}{d-1} n^\alpha. \tag{3.7}$$

**Lemma 3.2.** *For the tree $\mathbb{T}$ and the expander $G$, $n \ge 1$,*

$$P^o_x[\tilde{H}_B > H_{A^c}] = \frac{d-2}{2d}(1 - (d-1)^{-1} n^{\beta-\alpha})^{-1} \quad \forall x \in \partial B \tag{3.8}$$

*and*

$$\text{cap}_{A^c}(B) = \frac{d-2}{2(d-1)} \frac{n^\beta}{1-(d-1)^{-1} n^{\beta-\alpha}}. \tag{3.9}$$

*Proof.* We apply Lemma 3.1 and get

$$\begin{aligned} P^o_{x\in\partial B}[\tilde{H}_B > H_{A^c}] &= \frac{d-1}{2d} P^o_{x\in\partial B^c}[H_{\partial B} > H_{\partial A^c}] \\ &= \frac{d-2}{2d}(1-(d-1)^{-1} n^{\beta-\alpha})^{-1}. \end{aligned} \tag{3.10}$$

Using (3.10), (3.7) and the definition of $\text{cap}_{A^c}(\,\cdot\,)$ proves (3.9). □

Consider equation (3.10) for the tree $\mathbb{T}$, assume $B = \{x\}$ and let $\alpha$ converge to infinity, then

$$P_x^{\mathbb{T}}[\tilde{H}_x = \infty] = \frac{d-2}{2d}. \tag{3.11}$$

This implies the transience of the random walk on $\mathbb{T}$ for $d > 2$.

The escape probabilities with respect to $B$ for the random walk killed on $A^c$ are equal for all initial points $x \in \partial B$. Therefore the normalized equilibrium measure $\bar{e}_B^{A^c}$ is the uniform measure on $\partial B$. Due to the transience of $X$ on $\mathbb{T}$ the normalized equilibrium measure $\bar{e}_B$ exists and equals $\bar{e}_B^{A^c}$. Using (3.7), we get the following lemma.

**Lemma 3.3.** *For the tree $\mathbb{T}$ and the expander $G$, $n \geq 1$,*

$$\bar{e}_B(x) = \frac{d-1}{d} n^{-\beta} \quad \forall x \in \partial B. \tag{3.12}$$

## 3.2 LRW on the tree

Recall the definition (2.7) of the induced subtree $\mathbb{T}_x$, for any $x \in \mathbb{V}$.

**Lemma 3.1.** *For the tree $\mathbb{T}$ and for all $z \in \partial B$ and $x \in \partial A$,*

$$P_x^{\mathbb{T}}[H_B = \infty] = 1 - n^{\beta-\alpha}, \tag{3.13}$$

$$P_x^{\mathbb{T}}[X_{H_B} = z, H_B < \infty] = \mathbf{1}_{\{x \in \mathbb{T}_z\}} n^{\beta-\alpha}, \tag{3.14}$$

$$\mathrm{cap}(B) = \frac{d-2}{2(d-1)} n^{\beta}. \tag{3.15}$$

*Proof.* Assume $x \in \partial A$ and $R > \alpha \log_{d-1} n$. Using Lemma 3.1, we get

$$\begin{aligned} P_x^{\mathbb{T}}[H_B = \infty] &= \lim_{R \to \infty} P_x^{\mathbb{T}}[H_B \geq H_{B(y,R)}] \\ &= \lim_{R \to \infty} 1 - \frac{(d-1)^{R - \alpha \log_{d-1} n} - 1}{(d-1)^{R - \beta \log_{d-1} n} - 1} = 1 - n^{\beta - \alpha}. \end{aligned} \tag{3.16}$$

Since (3.13) holds,

$$\begin{aligned} P_x^{\mathbb{T}}[X_{H_B} = z, H_B < \infty] &= P_x^{\mathbb{T}}[X_{H_B} = z | H_B < \infty] P_x^{\mathbb{T}}[H_B < \infty] \\ &= \mathbf{1}_{\{x \in \mathbb{T}_z\}} n^{\beta - \alpha}. \end{aligned} \tag{3.17}$$

Since (3.7) and (3.11), (3.15) follows. □

## 3.3 LRW on expanders

In the last two sections we investigated the random walk for the tree $\mathbb{T}$ and the random walk killed on $A^c$ for the graphs $\mathbb{T}$ and $G$. Although computing all relevant quantities for this cases was not too hard, it's more challenging to control the hitting probabilities for the boundary points of $B$, for the random walk on large girth expander graphs $G$. We know that $G$ is cycle-free on $A$. But outside of $A$ the informations are rare. Just the $d$-regularity and the uniform lower bound of the spectral gap are known. But these properties tempt to assert two important facts:

- Most of the trajectories of the random walk started on $A^c$ killed on $B$, are 'quiet long'.
- Stopping the random walk $X$ after a 'quiet long' time $t'$, the coordinate $X_{t'}$ is nearly uniformly distributed.

These observations motivate Lemma 3.4, and the key idea of its proof.

In the first part of this section we show some general statements, concerning the entrance time of $B$ and the quasi-stationary distribution on $B^c$.

Let us recall some definitions. For the random walk $X$ killed on hitting $B$, we write $\Delta^B$ for its generator and $\sigma_B$ for the quasi-stationary distribution on $B^c$ (see Section 2.2). Because of (3.8), the Dirichlet form $\mathcal{D}$ (see (2.23)) of the equilibrium potential $g^*_{B,A^c}$ (see (2.22)) is given by

$$\mathcal{D}(g^*_{B,A^c}, g^*_{B,A^c}) = \sum_{x\in B} P_x[\tilde{H}_B > H_{A^c}]\pi(x) \asymp n^{\beta-1}. \tag{3.18}$$

We start with an estimate of the expected entrance time for the random walk $X$ with the initial distribution $\pi$.

**Lemma 3.1.**

$$E[H_B] \asymp n^{1-\beta}. \tag{3.19}$$

*Proof.* The expected entrance time can be expressed by the following variational formula (see [AF02, Proposition 3.41].

$$E[H_B]^{-1} = \inf\{\mathcal{D}(f,f) : f : V \to \mathbb{R}, f|_B = 1, E[f] = 0\}, \tag{3.20}$$

with the minimizing function $f^*$ given by

$$f^*(x) = 1 - \frac{E_x[H_B]}{E[H_B]}. \tag{3.21}$$

Applying the variational formula, we obtain the following estimate (see [ČTW11, Proposition 3.2]).

$$\mathcal{D}(g^*_{B,A^c}, g^*_{B,A^c})\big(1 - 2\sup_{x\in A^c} |f^*(x)|\big) \le \frac{1}{E[H_B]} \le \mathcal{D}(g^*_{B,A^c}, g^*_{B,A^c})\pi(A^c)^{-2}. \tag{3.22}$$

In order to estimate the left-hand side of (3.22), we use

$$\sup_{x\in A^c} |f^*(x)| \le c|B|n^{\beta-\alpha}\log^4 n, \quad \text{for some } c > 0, \tag{3.23}$$

(see [ČTW11, Proposition 3.5]). Since $|B| \asymp n^\beta$, $\pi(A^c) \asymp c > 0$ and $2\beta < \alpha$, (3.22) reads

$$n^{\beta-1}(1 - cn^{2\beta-\alpha}\log^4 n) \le E[H_B]^{-1} \le c'n^{\beta-1}, \quad \text{for some } c, c' > 0, \tag{3.24}$$

and (3.19) follows. □

**Lemma 3.2.** *Let $\delta > 0$, then*

$$P[H_B > n^{1-\beta+\delta}] \le e^{-n^\delta}. \tag{3.25}$$

*Proof.* By [AB92, (1) and Theorem 3],

$$P[H_B > t] \ge \Big(1 - \frac{1}{\lambda E_{\sigma_B}[H_B]}\Big)\exp\Big(-\frac{t}{E_{\sigma_B}[H_B]}\Big) \tag{3.26}$$

and

$$P[H_B > t] \le \big(1 - \pi(B)\big)\exp\Big(-\frac{t}{E_{\sigma_B}[H_B]}\Big) \tag{3.27}$$

Integrating (3.26) over $t$ yields

$$E[H_B] \ge E_{\sigma_B}[H_B] - \lambda^{-1}. \tag{3.28}$$

Set $t = n^{1-\beta+\delta}$ in (3.27), using (3.28) and $E[H_B] \asymp n^\beta$, gives

$$P[H_B > n^{1-\beta+\delta}] \le \big(1 - \pi(B)\big)\exp\Big(-\frac{n^{1-\beta+\delta}}{E_{\sigma_B}[H_B]}\Big) \le \exp(-cn^\delta), \tag{3.29}$$

for some $c > 0$. □

Note that the quasi-stationary distribution $\sigma_B$ of the random walk $X$ killed on $B$ can be characterized by

$$P_{\sigma_B}[X_k = y | H_B > k] = \sigma_B(y), \quad \forall k > 0, \tag{3.30}$$

(see [AF02, see remarks in Section 3.6.5]). Now we show that the random walk at time $t^* = \lfloor \log^2 n \rfloor$ conditioned not to have visited $B$ and started in $x \in B^c$ is close to the quasi-stationary distribution.

**Lemma 3.3.** *For some* $c, c' > 0$ *and* $t^* = \log^2 n$,

$$\sup_{x,y\in B^c} |P_x[X_{t^*} = y | H_B > t^*] - \sigma_B(y)| \leq ce^{-c't^*}. \tag{3.31}$$

*Proof.* If we can show that

$$e^{-t^*(\lambda_B^2 - \lambda_B^1)} |B^c| \Big( \sup_{x\in B^c} \frac{\sigma_{B^c}(x)}{\pi(x)^{\frac{1}{2}}} \Big)^2 \Big( \inf_{x\in B^c} \frac{\sigma_{B^c}(x)}{\pi(x)^{\frac{1}{2}}} \Big)^{-1} \leq ce^{-c't^*}, \tag{3.32}$$

then (3.31) follows from [ČT13, Appendix: Lemma A.2.].

The generator $\Delta^B$ with corresponding eigenvalues

$$0 < \lambda_B^1 < \lambda_B^2 \leq \dots \leq \lambda_B^{|V\setminus B|} \tag{3.33}$$

can be viewed as a sub-matrix of the generator $\Delta$ (see (2.8)) with spectral gap $\lambda_n^2$. Thus by the eigenvalue interlacing inequality (see [Hae95, Corollary 2.2]) we have $\lambda_B^2 \geq \lambda_n^2$. On the other hand, by [AB92, Lemma 2 and the paragraph following equation (12)],

$$\lambda_B^1 = \frac{1}{E_{\sigma_B}[H_B]} \leq \frac{1}{E[H_B]}. \tag{3.34}$$

Combining these two inequalities we get

$$\lambda_B^2 - \lambda_B^1 \geq \lambda_n^2 - \frac{1}{E[H_B]}. \tag{3.35}$$

Since $E[H_B] \asymp n^{1-\beta}$ and $\lambda_n^2 \geq \lambda > 0$, for $n \geq 1$,

$$\lambda_B^2 - \lambda_B^1 \geq \lambda_B, \quad \text{for some constant} \lambda_B > 0. \tag{3.36}$$

We now show a lower bound for the quasi-stationary distribution on $B^c$. Let $x \in B^c$ and $k \geq 0$. By reversibility, for all $x' \in B^c$,

$$P_{x'}[X_k = x | H_B > k] = P_x[X_k = x' | H_B > k] \frac{P_x[H_B > k]}{P_{x'}[H_B > k]}. \tag{3.37}$$

In order to bound the above ratio, note that

$$P_x[H_B > k] \geq P_x[H_{x'} < H_B, H_B \circ \theta_{H_{x'}} > k] = P_x[H_{x'} < H_B] P_{x'}[H_B > k]. \tag{3.38}$$

By assumption the graph $G \setminus B$ is connected and

$$\max\{\deg(x) : x \in G \setminus B\} = d. \tag{3.39}$$

Applying [Kow16, Proposition 3.1.5] and (3.36), we get

$$\operatorname{diam}(G \setminus B) \leq \frac{c' \log |G \setminus B|}{\log(1 + \frac{\lambda_B}{d})} \leq c \log n, \quad \text{for some } c, c' > 0. \tag{3.40}$$

That is, we can find a path of length at most $c \log n$, connecting $x$ and $x'$ and not passing through $B$. That implies

$$P_x[H_{x'} < H_B] \geq 2d^{-c \log n} \geq cn^{-c'}, \quad \text{for some } c, c' > 0. \tag{3.41}$$

From [ČT13, Lemma A.2.], $P_x[X_k = x' | H_B > k] \overset{k\to\infty}{\longrightarrow} \sigma_B(x')$ uniformly for all $x, x' \in G \setminus B$. Therefore, taking the limit $k \to \infty$ in (3.37), together with (3.38) and (3.41),

$$\exists c, c' > 0 : \quad \sigma_{B^c}(x) \geq c\sigma_{B^c}(x') n^{-c'}, \quad \forall x, x' \in B^c \tag{3.42}$$

and since $\sigma_B$ is a probability measure,

$$\inf_{x \in B^c} \sigma_B(x) \geq cn^{-c'}. \tag{3.43}$$

The estimates (3.36) and (3.43) show (3.32), and the lemma follows. □

Finally, we control the hitting probabilities of boundary points of $B$.

**Lemma 3.4.** *Let $x \in \partial A^c$ and $y \in \partial B$, then*

$$P_x[X_{H_B} = y] \asymp n^{-\beta}. \tag{3.44}$$

*Proof.* By [ČTW11, Lemma 3.4.]), we can control the probability that the random walk started in $x \in A^c$ visits the ball $B$ before time $t$. More precisely, let $t > 0$, then for some $c, c' > 0$,

$$P_x[H_B < t] \leq ctn^{\beta - \alpha} + e^{-c't}, \quad \text{for all } x \in A^c. \tag{3.45}$$

Taking $t = t^* = \lfloor \log^2 n \rfloor$ in (3.45), for all $x \in A^c$,

$$P_x[H_B > t^*] \geq 1 - o(1). \tag{3.46}$$

Let $x \in \partial A^c$ and $y \in \partial B$. Using (3.46), Lemma 3.3 and the Markov property, gives

$$\begin{aligned} P_x[X_{H_B} = y] &\geq P_x[H_B > t^*] P_x[X_{H_B} = y | H_B > t^*] \\ &\geq c \sum_{z \in B^c} P_x[X_{H_B} = y, X_{t^*} = z | H_B > t^*] \\ &\geq c \sum_{z \in B^c} P_x[X_{t^*} = z | H_B > t^*] P_z[X_{H_B} = y] \\ &\geq c P_{\sigma_B}[X_{H_B} = y], \quad \text{for some } c > 0. \end{aligned} \tag{3.47}$$

We now estimate the distribution $P_{\sigma_B}[X_{H_B} = \cdot]$. Consider the probability that the random walk $X$ started in $x \in \partial B$ stays outside of $B$ for a time interval at least $t^* = \log^2 n$. Since $X$ is reversible with respect to the uniform distribution on $G$, the probability can be written as

$$\sum_{y \in \partial B \setminus \{x\}} P_x[\tilde{H}_B > t^*, X_{\tilde{H}_B} = y] = \sum_{y \in \partial B \setminus \{x\}} P_y[\tilde{H}_B > t^*, X_{\tilde{H}_B} = x]. \quad (3.48)$$

By the Markov property,

$$\begin{aligned} P_x[\tilde{H}_B > t^*, X_{\tilde{H}_B} = y] &= \sum_{z \in B^c} P_x[\tilde{H}_B > t^*, X_{t^*} = z, X_{\tilde{H}_B} = y] \\ &= \sum_{z \in B^c} P_x[X_{t^*} = z | \tilde{H}_B > t^*] P_x[\tilde{H}_B > t^*] P_z[X_{\tilde{H}_B} = y]. \end{aligned} \quad (3.49)$$

Moreover, the distribution $P_x[X_{t^*} = \cdot | \tilde{H}_B > t^*]$ can be approximated by the quasi-stationary distribution $\sigma_B$ (3.31), i.e., for some $c, c' > 0$,

$$\Big| P_x[\tilde{H}_B > t^*, X_{\tilde{H}_B} = y] - P_x[\tilde{H}_B > t^*] \sum_{z \in B^c} \sigma_B(z) P_z[X_{\tilde{H}_B} = y] \Big| \leq c e^{-c' t^*}. \quad (3.50)$$

Combining (3.50) and (3.48), we obtain

$$\Big| P_x[\tilde{H}_B > t^*] P_{\sigma_B}[X_{\tilde{H}_B} \neq x] - P_{\sigma_B}[X_{\tilde{H}_B} = x] \sum_{y \in \partial B \setminus \{x\}} P_y[\tilde{H}_B > t^*] \Big| \leq c e^{-c' t^*}, \quad (3.51)$$

or equivalently,

$$\frac{P_x[\tilde{H}_B > t^*] - c e^{-c' t^*}}{\sum_{y \in \partial B} P_y[\tilde{H}_B > t^*]} \leq P_{\sigma_B}[X_{\tilde{H}_B} = y] \leq \frac{P_x[\tilde{H}_B > t^*] + c e^{-c' t^*}}{\sum_{y \in \partial B} P_y[\tilde{H}_B > t^*]}. \quad (3.52)$$

Applying the escape probability (3.9) and the Markov property, we get

$$\begin{aligned} P_x[\tilde{H}_B > t^*] &\geq P_x[H_{A^c} < \tilde{H}_B] P_x[\tilde{H}_B > t^* | H_{A^c} < \tilde{H}_B] \\ &\geq P_x[H_{A^c} < \tilde{H}_B] \inf_{x \in \partial A^c} P_x[H_B > t^*] \\ &\geq \frac{d-2}{d} \frac{1}{1 - c n^{\beta - \alpha}} (1 - c' t^* n^{\beta - \alpha}) \geq c > 0, \end{aligned} \quad (3.53)$$

Combining (3.52) and (3.53), yields, for some $c > 0$,

$$P_{\sigma_B}[X_{\tilde{H}_B} = y] \geq c n^{-\beta}, \quad \forall y \in \partial B. \quad (3.54)$$

Due to (3.47),(3.54) and $|\partial B| \asymp n^{-\beta}$, Lemma 3.4 follows. □

# Chapter 4

# Coupling quantities

In order to use Theorem 2.1 for the proof of our main result, in Section 4.1 we construct two Markov chains $Y, Z$ on $\Sigma := \partial B \times \partial A^c$, which encode the behaviour of the random walk on $B \subset V$ and the random interlacements on $B \subset \mathbb{V}$, respectively. In Sections 4.2-4.5 we will estimate all relevant coupling quantities occurring in the theorem.

## 4.1 Encoding excursions

Consider the random walk $X$ on the expander $G$. For $B, A^c \subset V$, we define inductively two sequences of stopping times $R_i, D_i$, which describe the times of returns to the set $B$, and the times of departures of the set $A^c$, respectively. More precisely, $D_0 = H_{A^c}$ and for $i \geq 1$

$$R_i = H_B \circ \theta_{D_{i-1}} + D_{i-1}, \tag{4.1}$$

$$D_i = H_{A^c} \circ \theta_{R_i} + R_i. \tag{4.2}$$

For $i \geq 1$, the random walk $X$ between the return time $R_i$ and the successive departure time $D_i$ is called *excursion*. By the strong Markov property of $X$, $(Y_i)_{i\geq 1} = (X_{R_i}, X_{D_i})_{i\geq 1}$ is a Markov chain on $\Sigma := \partial B \times \partial A^c$ with transition probabilities

$$P[Y_{i+1} = \boldsymbol{y} | Y_i = \boldsymbol{x}] = P_{x_2}[X_{H_B} = y_1] P_{y_1}[X_{H_{A^c}} = y_2], \tag{4.3}$$

for every $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2) \in \Sigma$, and with initial distribution

$$\nu_Y(x) = P[X_{R_1} = x_1, X_{D_1} = x_2] = P[X_{R_1} = x_1] P_{x_1}[X_{H_{A^c}} = x_2]. \tag{4.4}$$

The Markov chain $Y$ encodes the excursions of the random walk $X$.

The second Markov chain, which encodes the behaviour of random interlacements in $B$, is defined similarly by considering separately the excursions of any random walk trajectories of random interlacements which enters $B$. More precisely, recall the representation of random interlacements on the finite set $B$, using an i.i.d. sequence of lazy random walks $(X^{(i)})_{i\geq 1}$ on $\mathbb{T}$ with law $P^{\mathbb{T}}_{\bar{e}_B}$ (2.42). For every $i \geq 1$, set $R^{(i)}_1 = 0$ and define $D^{(i)}_j$, $R^{(i)}_j$, $j \geq 1$ analogously to (4.1) and (4.2), to be the successive departure and return times between $B$ and $A$ of the random walk $X^{(i)}$. We set

$$T^{(i)} = \sup\{j : R^{(i)}_j < \infty\}, \tag{4.5}$$

to be the number of excursions of $X^{(i)}$ between $B$ and $A^c$. Finally, let $Z = (Z_k)_{k\geq 1}$ be the sequence of the starting and ending points of these excursions,

$$\begin{aligned} &Z_k = \big(X^{(i)}_{R^{(i)}_j}, X^{(i)}_{D^{(i)}_j}\big), \quad \text{for } i \geq 1, \\ &\text{and } 1 \leq j \leq T^{(i)}, \quad \text{given by } k = \sum_{l=1}^{i-1} T^{(l)} + j. \end{aligned} \tag{4.6}$$

The strong Markov property for $X^{(i)}$'s and their independence imply that $Z$ is a Markov chain on $\Sigma$ with transition distribution

$$\begin{aligned} P[Z_{k+1} = \boldsymbol{y} | Z_k = \boldsymbol{x}] &= P^{\mathbb{T}}_{y_1}[X_{H_{A^c}} = y_2] \\ &\cdot \big(P^{\mathbb{T}}_{x_2}[H_B < \infty, X_{H_B} = y_1] + P^{\mathbb{T}}_{x_2}[H_B = \infty]\bar{e}_B(y_1)\big), \end{aligned} \tag{4.7}$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \partial B \times \partial A^c$, and with initial distribution

$$\nu_Z(\boldsymbol{x}) = \bar{e}_B(x_1) P^{\mathbb{T}}_{x_1}[X_{H_{A^c}} = x_2]. \tag{4.8}$$

The construction above, together with (2.42), yields

$$\mathcal{V}^u \cap B \overset{\text{law}}{=} B \setminus \bigcup_{i=1}^{J^u_B} \bigcup_{k=1}^{T^{(i)}} \{X^{(i)}_{R^{(i)}_k}, \ldots, X^{(i)}_{D^{(i)}_k}\}, \tag{4.9}$$

where $J^u_B$ is a Poisson random variable with parameter $u\text{cap}(B)$. The Markov chain $Z$ encodes the excursions of the sequence of random walks $X^{(i)}$.

Due to the transience of the random walk on $\mathbb{T}$ (see (3.13)), we already know, that the number of excursions $T^{(i)}$ of the random walk $X^{(i)}$ is finite almost sure. The next lemma provides us the expected number of visits of any $x \in \partial B$. Later in Section 4.2, we apply this lemma to compute the stationary measure for the Markov chain $Z$ on $\partial B \times \partial A^c$.

**Lemma 4.1.** *For $x \in \partial B$ and $i \geq 1$*

$$E^{\mathbb{T}}_{\bar{e}_B}\Big[\sum_{j=1}^{T^{(i)}} \mathbf{1}_{\{X_{R^{(i)}_j}=x\}}\Big] = \frac{P^{\mathbb{T}}_x[\tilde{H}_B > H_{A^c}]}{\mathrm{cap}(B)}. \tag{4.10}$$

*Proof.* To simplify the notation we write $T, X, R_j$ for $T^{(i)}, X^{(i)}, R^{(i)}_j$. We extend $X$ to a two-sided random walk on $\mathbb{T}$ by requiring the law of $(X_{-i})_{i\geq 0}$ to be $P^{\mathbb{T}}_{X_0}[\cdot\,|\tilde{H}_B = \infty]$, conditionally independent of $(X_i)_{i\geq 0}$. We denote by $L = \sup\{n : X_n \in B\}$ the time of the last visit of $X$ to $B$. Then,

$$\begin{aligned} E^{\mathbb{T}}_{\bar{e}_B}\Big[\sum_{j=1}^{T} \mathbf{1}_{\{X_{R_j}=x\}}\Big] &= \sum_{y\in\partial B}\sum_{z\in\partial B} \bar{e}_B(y) E^{\mathbb{T}}_y\Big[\mathbf{1}_{\{X_L=z\}} \sum_{j=1}^{T} \mathbf{1}_{\{X_{R_j}=x\}}\Big] \\ &= \sum_{y\in\partial B}\sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(y) P^{\mathbb{T}}_y \begin{bmatrix} X_n = x, X_L = z, \exists m \in \mathbb{Z} : m < n, X_m \in A^c, \\ \{X_{m+1},\dots,X_{n-1}\} \subset A\setminus B \end{bmatrix}. \end{aligned} \tag{4.11}$$

According to [Szn12, Proposition 1.8.] under $P^{\mathbb{T}}_{\bar{e}_B}$, $X_L$ has also distribution $\bar{e}_B$. Hence, by reversibility, this equals

$$\begin{aligned} &= \sum_{y\in\partial B}\sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P^{\mathbb{T}}_z \begin{bmatrix} X_n = x, X_L = y, \exists m \in \mathbb{Z} : m < n, X_m \in A^c, \\ \{X_{m+1},\dots,X_{n-1}\} \subset A\setminus B \end{bmatrix} \\ &= \sum_{y\in\partial B}\sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P^{\mathbb{T}}_z \begin{bmatrix} X_n = x, \exists m \in \mathbb{Z} : m < n, X_m \in A^c, \\ \{X_{m+1},\dots,X_{n-1}\} \subset A\setminus B \end{bmatrix} \\ &= \sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P^{\mathbb{T}}_z[X_n = x] P_x[\tilde{H}_B > H_{A^c}]. \end{aligned} \tag{4.12}$$

Introducing the Green function $g(x,y) = \sum_{k=0}^{\infty} P^{\mathbb{T}}_x[X_k = y]$ and using the identity $\sum_{z\in\partial B} \bar{e}_B(z) g(z,x) = 1$ (see [Szn12, Proposition 1.8.]), this equals to

$$\sum_{z\in\partial B} \bar{e}_B(z) g(z,x) P_x[\tilde{H}_B > H_{A^c}] = \frac{P_x[\tilde{H}_B > H_{A^c}]}{\mathrm{cap}(B)}. \tag{4.13}$$

This completes the proof. □

Summing equation (4.10) over $x \in \partial B$, we obtain the expected number of excursions into $B$, i.e.

$$E^{\mathbb{T}}_{\bar{e}_B}[T^{(i)}] = \frac{\mathrm{cap}_{A^c}(B)}{\mathrm{cap}(B)}. \tag{4.14}$$

## 4.2 Equilibrium measure

In this section we show that the equilibrium measures of the Markov chains $Y$ and $Z$, defined in Section 4.1, coincide.

**Lemma 4.1.** *Let $\pi$ be the probability measure on $\Sigma$ given by*

$$\pi(\boldsymbol{x}) = \bar{e}_B(x_1)P_{x_1}[X_{H_{A^c}} = x_2], \quad \boldsymbol{x} = (x_1, x_2) \in \Sigma. \tag{4.15}$$

*Then $\pi$ is the invariant measure for both $Y$ and $Z$.*

*Proof.* To see that $\pi$ is invariant for $Y$, consider the stationary random walk $(X_i)_{i\in\mathbb{Z}}$ on $G$. Let $\mathcal{R}$ be the set of returns to $B$ for this walk,

$$\mathcal{R} = \{n \in \mathbb{Z} : X_n \in B, \exists m < n, X_m \in A^c, \{X_{m+1}, ..., X_{n-1}\} \subset A \setminus B\}, \tag{4.16}$$

$\mathcal{D}$ the set of departures

$$\mathcal{D} = \left\{ \begin{matrix} n \in \mathbb{Z} : X_n \in A^c, \exists m \in \mathcal{R}, m < n, \\ \{X_m, ..., X_{n-1}\} \subset A \setminus B \end{matrix} \right\}, \tag{4.17}$$

and write $\mathcal{R} = \{\bar{R}_i\}_{i\in\mathbb{Z}}$, $\mathcal{D} = \{\bar{D}_i\}_{i\in\mathbb{Z}}$ so that $\bar{R}_i < \bar{D}_i < \bar{R}_{i+1}$, $i \in \mathbb{Z}$, and

$$\bar{R}_0 < \inf\{i \geq 0 : X_i \in A^c\} < \bar{R}_1. \tag{4.18}$$

Observe that by this convention the sequence $(\bar{R}_i, \bar{D}_i)_{i\geq 1}$ agrees with $(R_i, D_i)_{i\geq 1}$, defined in Section (4.1). Remark also that $\bar{R}_0$ might be non-negative in general, but $\bar{R}_{-1} < 0$. Due to the stationarity and the reversibility of $X$, for every $\boldsymbol{x} = (x_1, x_2)$,

$$\begin{aligned} P[n \in \mathcal{R}, X_n = x_1] &= P\begin{bmatrix} X_n = x_1, \exists m < n, X_m \in A^c, \\ \{X_{m+1}, ..., X_{n-1}\} \subset A \setminus B \end{bmatrix} \\ &= P[X_0 = x_1]P_{x_1}[\tilde{H}_B > H_{A^c}] \\ &= n^{-1}P_{x_1}[\tilde{H}_B > H_{A^c}]. \end{aligned} \tag{4.19}$$

By the ergodic theorem (see [Szn12, Theorem 4.16]), the stationary measure $\pi_Y$ of $Y$ satisfies

$$\begin{aligned} \pi_Y(\{x_1\} \times \partial A) &= \lim_{k\to\infty} k^{-1} \sum_{i=1}^{k} \mathbf{1}_{\{X_{R_i} = x_1\}} \\ &= \lim_{m\to\infty} \frac{m^{-1}\sum_{n=1}^{m} \mathbf{1}_{\{n\in\mathcal{R}, X_n = x_1\}}}{m^{-1}\sum_{n=1}^{m} \mathbf{1}_{\{n\in\mathcal{R}\}}}, \end{aligned} \tag{4.20}$$

where we used the observation below (4.18) for the last equality. Applying the ergodic theorem for the numerator and denominator separately and using (4.19) yields

$$\pi_Y(\{x_1\} \times \partial A^c) = \frac{P_{x_1}[\tilde{H}_B > H_{A^c}]}{\sum_{y \in \partial B} P_y[\tilde{H}_B > H_{A^c}]} = \bar{e}_B(x_1). \tag{4.21}$$

By the strong Markov property, $\pi_Y(\boldsymbol{x}) = \pi_Y(\{x_1\} \times \partial A^c) P_{x_1}[H_{A^c} = x_2]$ and thus $\pi_Y = \pi$ as claimed.

We now consider the Markov chain $Z$. This chain is defined from the i.i.d. sequence of random walks $X^{(i)}$. Each of these random walks give rise to a random-length block of excursions distributed as

$$\big\{\big(X^{(1)}_{R^{(1)}_j}, X^{(1)}_{D^{(1)}_j}\big) : j = 1, ..., T^{(1)}\big\}. \tag{4.22}$$

The invariant measure $\pi_Z$ of $Z$ can thus be written as

$$\pi_Z(\boldsymbol{x}) = \frac{1}{E^{\mathbb{T}}_{\bar{e}_B}[T^{(1)}]} E^{\mathbb{T}}_{\bar{e}_B}\Big[\sum_{j=1}^{T^{(1)}} \mathbf{1}_{\{X^{(1)}_{R^{(1)}_j} = x_1\}}\Big] P_{x_1}[X_{H_{A^c}} = x_2], \quad \boldsymbol{x} = (x_1, x_2). \tag{4.23}$$

Due to Lemma 4.1 and (4.14), $\pi = \pi_Z$ follows. □

## 4.3 Variance estimates

We start with some definitions and then show Lemma 4.1, which yields the asymptotic behavior of the arrival densities' variance for both Markov chains $Y, Z$. Remember, $\beta < \frac{\alpha}{2}$.

Let us fix the base measure $\mu$ on $\Sigma$, such that $\pi(x_1, x_2) = \bar{e}_B(x_1)\mu(x_1, x_2)$, i.e.,

$$\mu(\boldsymbol{x}) = P_{x_1}[X_{H_{A^c}} = x_2] = P^{\mathbb{T}}_{x_1}[X_{H_{A^c}} = x_2], \quad \boldsymbol{x} := (x_1, x_2) \in \Sigma. \tag{4.24}$$

then we get the transition densities $\rho^Y$ of $P$ with respect to $\mu$, and $\rho^Z$ of $P^{\mathbb{T}}$ with respect to $\mu$, i.e.

$$\rho^Y(\boldsymbol{y}, \boldsymbol{x}) = P_{y_2}[X_{H_B} = x_1], \quad (\boldsymbol{y}, \boldsymbol{x}) \in \Sigma^2 \tag{4.25}$$

and

$$\rho^Z(\boldsymbol{y}, \boldsymbol{x}) = P^{\mathbb{T}}_{y_2}[X_{H_B} = x_1, H_B < \infty] + P^{\mathbb{T}}_{y_2}[X_{H_B} = \infty]\bar{e}_B(x_1), \quad (\boldsymbol{y}, \boldsymbol{x}) \in \Sigma^2. \tag{4.26}$$

Since the densities $\rho^o$ only depend on $y_2$ and $x_1$ for $o = Y, Z$, we write

$$\rho^o(\boldsymbol{y}, \boldsymbol{x}) =: \rho^o(y_2, x_1). \tag{4.27}$$

The arrival density $\rho^o_{\boldsymbol{x}}$ for $o = Y, Z$ is given by the function

$$\rho^o_{\boldsymbol{x}} : \Sigma \to [0,1] \quad \boldsymbol{y} \mapsto \rho^o(\boldsymbol{y}, \boldsymbol{x}). \tag{4.28}$$

Before we state the lemma, we capture the following properties. Since $\pi$ is invariant for the Markov chain $Y$,

$$\begin{aligned} E_\pi[\rho^Y_{\boldsymbol{x}}] &= \sum_{\boldsymbol{y}\in\Sigma} \pi(\boldsymbol{y})\rho^Y(\boldsymbol{y}, \boldsymbol{x}) \\ &= \sum_{\boldsymbol{y}\in\Sigma} \pi(\boldsymbol{y}) \frac{P_{\boldsymbol{x}}[Y_1 = \boldsymbol{y}]}{\mu(\boldsymbol{y})} \\ &= \frac{\pi(\boldsymbol{x})}{\mu(\boldsymbol{x})} = \bar{e}_B(x_1) = \frac{d-1}{d} n^{-\beta}. \end{aligned} \tag{4.29}$$

Since $\pi$ is invariant for the Markov chain $Z$ as well, as above

$$E_\pi[\rho^Z_{\boldsymbol{x}}] = \frac{d-1}{d} n^{-\beta}. \tag{4.30}$$

Because $\pi(\partial B \times \{\,\cdot\,\})$ is uniform on $\partial A^c$, for $o = Y, Z$

$$\begin{aligned} E_\pi[(\rho^o_{\boldsymbol{x}})^2] &= \sum_{\boldsymbol{y}\in\Sigma} \pi(\boldsymbol{y})\rho^o(\boldsymbol{y}, \boldsymbol{x})^2 \\ &= \sum_{y_2\in\partial A} \pi(\partial B \times \{y_2\})\rho^o(y_2, x_1)^2 \\ &= |\partial A^c|^{-1} \sum_{y_2\in\partial A^c} \rho^o(y_2, x_1)^2. \end{aligned} \tag{4.31}$$

**Lemma 4.1.** *Let $\beta < \frac{\alpha}{2}$. Then there exist constants $c_1, c_2 \in (0,\infty)$, such that for every $\boldsymbol{x} \in \Sigma$*

$$c_1 n^{-2\beta} \le \mathrm{Var}_\pi(\rho^Y_{\boldsymbol{x}}) \le c_2 n^{-2\beta}, \tag{4.32}$$

$$c_1 n^{-2\beta} \le \mathrm{Var}_\pi(\rho^Z_{\boldsymbol{x}}) \le c_2 n^{-2\beta}. \tag{4.33}$$

*Proof.* Because of Lemma 3.4, $\rho^Y(y_2, x_1) \asymp n^{-\beta}$. Combining this and the above equations (4.31), (4.29), the claim (4.32) follows.

We continue with the proof of (4.33). Since $\beta < \frac{\alpha}{2}$,

$$\begin{aligned} \rho^Z(y_2, x_1) &= P^{\mathbb{T}}_{y_2}(H_B = \infty)\bar{e}_B(x_1) + P^{\mathbb{T}}_{y_2}(X_{H_B} = x_1, H_B < \infty) \\ &= c(1 - n^{\beta-\alpha})n^{-\beta} + \mathbf{1}_{\{y_2\in\mathbb{T}_{x_1}\}} n^{\beta-\alpha} \\ &\asymp n^{-\beta}. \end{aligned} \tag{4.34}$$

Combining (4.34), (4.30) and (4.31), we obtain that

$$\mathrm{Var}_\pi(\rho^Z_x) \asymp n^{-\alpha} \sum_{y_2 \in \partial A^c} \rho^Z(y_2, x_1)^2 \asymp n^{-2\beta}. \tag{4.35}$$

□

## 4.4 Mixing times

Recall the definition of the mixing time $T$ of a Markov chain (2.43). We denote $T_Y, T_Z$ the mixing times of $Y$ respectively $Z$.

**Lemma 4.1.** *There exists a constant c, such that*

$$T_Z \leq c, \tag{4.36}$$

$$T_Y \leq c. \tag{4.37}$$

Let us start with some preliminary considerations for the proof. For the Markov chain $Y$ on $\Sigma = \partial B \times \partial A^c$, we know the transition probabilities

$$P[Y_{i+1} = \boldsymbol{y} | Y_i = \boldsymbol{x}] = P_{x_2}[X_{H_B} = y_1] P_{y_1}[X_{H_{A^c}} = y_2], \tag{4.38}$$

for $\boldsymbol{x} = (x_1, x_2) \in \Sigma$ and $\boldsymbol{y} = (y_1, y_2) \in \Sigma$. That is, we can achieve the mixing on the set $\partial A^c$, as on $\partial B$, separately. Since the induced graph $G \cap A$ is a tree, the distribution $P_{x_1}[H_{A^c} = \cdot\,]$ is nothing at all uniform on $\partial A^c$. On the other hand, Lemma 3.4 gives us a strong result for the distribution $P_{x_2}[H_B = \cdot\,]$ on $\partial B$, which is nearly uniform.

Similar considerations work for the Markov chain $Z$. For the random walk $X^{(i)}$ the number of excursions between $B$ and $A^c$ is finite a.s., and $X^{(i+1)}$ starts uniformly on $\partial B$.

It forces on, to use these facts in the proof of Lemma 4.1, i.e. we try to couple both chains on the set $\partial B$. In order to bound the mixing times we use the following lemma.

**Lemma 4.2.** *Let $(X_i)_{i \geq 0} = X$ be an arbitrary Markov chain on a finite state space $\Sigma$. Assume that for every $x, y \in \Sigma$ there exists a coupling $Q_{x,y}$ of two copies $X^1, X^2$ of $X$ starting respectively from $x$ and $y$, such that*

$$\max_{x,y \in \Sigma} Q_{x,y}[X^1_n \neq X^2_n] \leq 1/4. \tag{4.39}$$

*Then $T_X \leq n$.*

*Proof.* See [LPW09, Corollary 5.3] □

*Proof of Lemma 4.4.1.* To show (4.36), we consider two copies $Z^1, Z^2$ of the Markov chain $Z$ starting respectively in $\boldsymbol{x}, \boldsymbol{x}' \in \Sigma$ and define the coupling $Q_{\boldsymbol{x},\boldsymbol{x}'}$ between them as follows. Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d. Bernoulli random variables with parameter $P[\xi_i = 1] = 1 - n^{\beta-\alpha}$ (see (3.13)). Given $Z_i^1 = \boldsymbol{x}, Z_i^2 = \boldsymbol{x}'$ and $\xi_i = 1$, then $Z_{i+1}^1 = Z_{i+1}^2$ are distributed as $\pi(\boldsymbol{x}) = \bar{e}_B(x_1)P_{x_1}[X_{H_{A^c}} = x_2]$. For $\xi_i = 0$, we choose $Z_{i+1}^1$ and $Z_{i+1}^2$ independently with distribution $\mu_{\boldsymbol{x}}, \mu_{\boldsymbol{x}'}$ where (see Section 4.1)

$$\mu_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{P_{x_2}^{\mathbb{T}}[X_{H_B} = y_1, H_B < \infty]P_{y_1}^{\mathbb{T}}[X_{H_{A^c}} = y_2]}{1 - (1 - n^{\beta-\alpha})} \tag{4.40}$$

If $Z_i^1 = Z_i^2$ for some $i$, then we let them move together, i.e., $Z_j^1 = Z_j^2$ for all $j \geq i$. It follows that

$$\max_{\boldsymbol{x},\boldsymbol{x}'} Q_{\boldsymbol{x},\boldsymbol{x}'}[Z_i^1 \neq Z_i^2] \leq P[\xi_j = 0 \forall j < i] = (1 - (1 - n^{\beta-\alpha}))^i \tag{4.41}$$

Choosing now $i$ sufficiently large, but independent of $n$ and using Lemma 4.2, (4.36) follows.

Now we show (4.37). Assume $x_2 \in \partial A^c$ and $\mu$ is the sub-probability on $\partial B$, given by $\mu(y_1) = \inf_{x_2 \in \partial A^c} P_{x_2}[X_{H_B} = y_1]$. Since $\partial B \asymp n^\beta$ and because of Lemma 3.4, $\mu(\partial B) \geq c_1$ for some $c_1 \in (0, 1)$.
We can now construct the coupling required for the application of Lemma 4.2. Let $\boldsymbol{x}(0), \boldsymbol{x}'(0) \in \Sigma$ and define the coupling $Q_{\boldsymbol{x},\boldsymbol{x}'}$ of two copies $Y^1, Y^2$ of $Y$ as follows. Let $Y_0^1 = \boldsymbol{x}, Y_0^2 = \boldsymbol{x}'$, and let $(\xi_i)_{i\geq 0}$ be an independent sequence of i.i.d. Bernoulli random variables with $P[\xi = 1] = \mu(\partial B)$. Given $Y_i^1 = \boldsymbol{x}, Y_i^2 = \boldsymbol{x}'$ and $\xi_i = 1$, then $Y_{i+1}^1 = Y_{i+1}^2$ are distributed as $\pi(\boldsymbol{x}) = \bar{e}_B(x_1)P_{x_1}[X_{H_{A^c}} = x_2]$. For $\xi_i = 0$, we choose $Y_{i+1}^1$ and $Y_{i+1}^2$ independently with distribution $\nu_{\boldsymbol{x}}, \nu_{\boldsymbol{x}'}$ where (see Section 4.1)

$$\nu_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{\left(P_{x_2}^{\mathbb{T}}[X_{H_B} = y_1] - \mu(y_1)\right)P_{y_1}^{\mathbb{T}}[X_{H_{A^c}} = y_2]}{1 - \mu(\partial B)}. \tag{4.42}$$

If $Y_i^1 = Y_i^2$ for some $i$, then we let them move together, i.e., $Y_j^1 = Y_j^2$ for all $j \geq i$.

These steps construct two copies of $Y$, started from $\boldsymbol{x}$ and $\boldsymbol{x}'$ respectively. Since

$$Q_{\boldsymbol{x},\boldsymbol{x}'}[Y_i^1 \neq Y_i^2] \leq P[\xi_j = 0 \forall j < i] = (1 - \mu(\partial B)^{i-1}, \tag{4.43}$$

and $\mu(\partial B) \geq c_1$, we can choose $i$ independent of $n$, such that right part of (4.43) is $\leq \frac{1}{4}$. Applying Lemma 4.2, (4.37) follows. □

## 4.5 Number of excursions

Consider the random walk on the expander. Define

$$\mathcal{N}(t) = \sup\{i : R_i < t\}, \tag{4.44}$$

to be the number of excursions started before $t$. We show that $\mathcal{N}(t)$ concentrates around its expectation.

**Lemma 4.1.** *Let $u > 0$ be fixed. There exist constants $c_1, c_2$ depending on $\alpha, \beta$, such that for every $n \geq 1$*

$$P[|\mathcal{N}(un) - u\text{cap}_{A^c}(B)| > \eta\text{cap}_{A^c}(B)] \leq c_1 e^{-c_2\eta^2 n^{c_2}}. \tag{4.45}$$

*Proof.* Let's start with the computation of the expectation of $\mathcal{N}(t)$ and $E_{\bar{e}_B}(R_1)$. Recall (4.16)-(4.18), the returns and departures $(\bar{R}_i, \bar{D}_i)$ of the stationary random walk $(X_n)_{n\in\mathbb{Z}}$. Let $\bar{\mathcal{N}}(t) = \sup\{i : \bar{R}_i < t\}$. Recall equality (4.19). Summing it over $x_1 \in \partial B$, we obtain

$$P[k \in \bar{\mathcal{R}}] = n^{-1}\text{cap}_{A^c}(B), \quad k \geq 0. \tag{4.46}$$

Summing again over $0 \leq k < t$,

$$E[\bar{\mathcal{N}}(t)] = tn^{-1}\text{cap}_{A^c}(B) \tag{4.47}$$

follows. By the observation below (4.18), $|\bar{\mathcal{N}}(t) - \mathcal{N}(t)| \leq 1$. Combining this and equality (4.47) , we obtain

$$|E[\mathcal{N}(t)] - tn^{-1}\text{cap}_{A^c}(B)| \leq 1, \quad \forall t \in \mathbb{N}. \tag{4.48}$$

Since (4.48), the fact that every $X_{R_k}$ is $\bar{e}_B$-distributed at stationary, and the ergodic theorem,

$$E_{\bar{e}_B}(R_1) = \frac{n}{\text{cap}_{A^c}(B)}. \tag{4.49}$$

It is more convenient to show a concentration result for the return times $R_i$ instead of $\mathcal{N}(t)$. Observing that for any $t > 0$ and $b > 0$,

$$\{|\mathcal{N}(t) - E(\mathcal{N}(t))| > b\} \subset \{R_{\lceil E(\mathcal{N}(t)-b\rceil} > t\} \cup \{R_{\lfloor E(\mathcal{N}(t))+b\rfloor < t}\}, \tag{4.50}$$

and therefore

$$\begin{aligned} P[|\mathcal{N}(un) - u\text{cap}_{A^c}(B)| > \eta\text{cap}_{A^c}] \leq P[&R_{\lceil (u-\eta)\text{cap}_{A^c}(B)\rceil} > un] \\ &+ P[R_{\lfloor (u+\eta)\text{cap}_{A^c}(B)\rfloor} < un]. \end{aligned} \tag{4.51}$$

Let $\epsilon > 0$ be a constant that will be fixed later, and set $l = \lfloor n^\epsilon \rfloor$. In order to estimate the right-hand side of (4.51), we study the typical size of $R_{m_\pm l}$ where

$$m_- = \lceil l^{-1}(u-\eta)\mathrm{cap}_{A^c}(B)\rceil, \quad \text{and} \quad m_+ = \lceil l^{-1}(u+\eta)\mathrm{cap}_{A^c}(B)\rceil. \tag{4.52}$$

Since $\mathrm{cap}_{A^c}(B) \asymp n^\beta$, it follows that

$$m_\pm \asymp n^{\beta-\epsilon}. \tag{4.53}$$

Let $\mathcal{G}_i = \sigma(X_k : k \le R_{il})$. Using the standard properties of the mixing time (see [LPW09, Section 4.5.]) and the strong Markov property,

$$\|P[(X_{R_{il}}, X_{D_{il}}) \in \cdot\,|\mathcal{G}_{i-1}] - \pi(\cdot)\|_{TV} \le 2^{-n^\epsilon}. \tag{4.54}$$

Since $\pi(\{\cdot\} \times \partial A^c)$ is uniformly distributed on $\partial B$,

$$\Big|\frac{P[X_{R_{il}} = y|\mathcal{G}_{i-1}]}{\bar{e}_B^{A^c}(y)} - 1\Big| \le c2^{-n^{\epsilon/2}}, \quad \forall i \ge 1. \tag{4.55}$$

Let $m$ stand for $m_+$ or $m_-$, we write

$$R_{ml} = \sum_{j=1}^{m} Z_j, \text{ where } Z_j = R_{jl} - R_{(j-1)l} \text{ and } R_0 := 0. \tag{4.56}$$

For every $j \ge 2$, by (4.55),

$$P[Z_j > t|\mathcal{G}_{j-2}] \le (1 + c2^{-n^{\epsilon/2}})P_{\bar{e}_B^{A^c}}[R_l > t] \le 2lP_{\bar{e}_B^{A^c}}[R_1 > t/l]. \tag{4.57}$$

Using 3.25 for any $\delta > 0$, yields

$$P[R_1 > n^{1-\beta+\delta}] \le e^{-cn^\delta}, \tag{4.58}$$

and thus

$$P[Z_j > ln^{1-\beta+\delta}|\mathcal{G}_{j-2}] \le 2lP_{\bar{e}_B^{A^c}}[R_1 > n^{1-\beta+\delta}] \le ce^{-n^{c'\delta}}. \tag{4.59}$$

Analogous reasoning proves also that

$$P[Z_1 \ge ln^{1-\beta+\delta}] \le ce^{-n^{c'\delta}}. \tag{4.60}$$

By (4.55) again,

$$|E[Z_j] - E[Z_j|G_{j-1}]| \le c2^{-n^{\epsilon/2}}E(Z_j). \tag{4.61}$$

Hence,

$$P[|R_{ml} - E(R_{ml}) > \eta E(R_{ml})|] = P\Big[\Big|\sum_{j=1}^{m}(Z_j - E[Zj])\Big| > \eta E(R_{ml})\Big]$$

$$\leq P[Z_1 \geq \eta E(R_{ml}/4] + \tag{4.62}$$

$$\sum_{n\in\{0,1\}} P\Big[\Big|\sum_{\substack{j=n \bmod 2\\ 1\leq j\leq m}}^{m}(Z_j - E[Z_j|\mathcal{G}_{j-2}])\Big| > \eta E(R_{ml}/4)\Big].$$

Setting $\tilde{Z} = Z_j \wedge n^{1-\beta+\delta}l$, which by (4.59) satisfies

$$|E[\tilde{Z}_j|\mathcal{G}_{j-2}] - E[Z_j|G_{j-2}]| = \int_{n^{1-\beta+\delta}l}^{\infty} P[Z_j > t|\mathcal{G}_{j-2}]dt \leq ce^{-n^{c'\delta}}, \tag{4.63}$$

the right-hand side of (4.62) can be bounded by

$$\leq cm \exp -n^{c'\delta} +$$

$$\sum_{n\in\{0,1\}} P\Big[\Big|\sum_{\substack{j=n \bmod 2\\ 1\leq j\leq m}}^{m}(Z_j - E[Z_j|\mathcal{G}_{j-2}])\Big| > \eta E(R_{ml}/4)\Big]. \tag{4.64}$$

Applying Azuma's inequality, (4.64) can be bounded by

$$\leq cm \exp(-n^{c'\delta}) + 4\exp\Big(-\frac{2c_1(\eta E(R_{ml}/4))^2}{m(n^{1-\beta+\delta}l)^2}\Big), \tag{4.65}$$

and together with $E[R_{ml}] \asymp n$, $m_\pm \asymp n^{\beta-\epsilon}$, and $l = \lfloor n^\epsilon \rfloor$, (4.65) can be bounded by

$$\leq cm \exp(-n^{c'\delta}) + 4\exp\Big(-c_1\eta^2 n^{\beta-\epsilon-2\delta}\Big). \tag{4.66}$$

It is possible to fix $\delta$ and $\epsilon$ sufficiently small, so that the exponent of $n$ on the right-hand side of the last display is positive. Altogether the above decays at least as $c_1 \exp(-c_2\eta^2 n^{c_3})$ as $n$ tends to infinity, finishing the proof of the lemma. □

We now count the number of excursions of random interlacements at level $u$ into $B$. Let $(J_u^n)_{u\geq 0}$ be the Poisson process with intensity cap$(B)$ driving the excursions of random interlacements to $B$. Recall the definition of random variables $T^{(i)}$ (4.5), giving the number of excursions of $i$-th random walk between $B$ and $A^c$. Given those, denote by $\mathcal{N}'(u)$ the number of steps of Markov chain $Z$ corresponding to the level $u$ of random interlacements,

$$\mathcal{N}'(u) = \sum_{i=1}^{J_u^n} T^{(i)}. \tag{4.67}$$

**Lemma 4.2.** *There exist constants $c_1, c_2$ depending on $\alpha, \beta$ and $u$, such that for every $u > 0$*

$$P[|\mathcal{N}'(u) - u\mathrm{cap}_{A^c}(B)| \geq \eta u\mathrm{cap}_{A^c}(B)] \leq c_1 e^{-c_2\eta^2 n^{c_2}}. \tag{4.68}$$

*Proof.* By the definition of random interlacements, $J_u^n$ is a Poisson random variable with parameter $u\mathrm{cap}(B)$, and thus, by Chernov estimate,

$$P[|J_u^n - u\mathrm{cap}(B)| \geq \eta u\mathrm{cap}(B)] \leq e^{-c\eta^2 n^c}. \tag{4.69}$$

The random variables $T^{(i)}$ are i.i.d, geometrically distributed and due to (4.14),

$$E_{\bar{e}_B}^{\mathbb{T}}[T^{(i)}] = \frac{\mathrm{cap}_{A^c}(B)}{\mathrm{cap}(B)}. \tag{4.70}$$

Applying Chernov bound again for $v = (1 \pm \frac{\eta}{2})u\mathrm{cap}(B)$,

$$P\Big[\Big|\sum_{i=1}^{v} T^{(i)} - \frac{v\mathrm{cap}_A(B)}{\mathrm{cap}(B)}\Big| \geq \frac{\eta}{2}\frac{v\mathrm{cap}_A(B)}{\mathrm{cap}(B)}\Big] \leq c_1 e^{-c_2\eta^2 n^{c_2}} \tag{4.71}$$

for some constants $c_1$ and $c_2$ depending on $\alpha, \beta$. The proof is completed by combining (4.69) and (4.71). □

# Chapter 5

# Proof of the main result

In this chapter we show the local coupling of the vacant set of the random walk on expanders with the vacant set of random interlacements.

Before that we need to finish our investigations on the Markov chains $Y$ and $Z$. In the following section we show, that their ranges almost coincide.

## 5.1 Coupling encoded excursions

Remember the state space $\Sigma = \partial B \times \partial A^c$ of the Markov chains $Y$ and $Z$ is finite, and the stationary distributions are equal for both chains. Thus we can apply Theorem 2.1 to construct a coupling of the two chains on some probability space $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$.

We summarize the estimates from the last two chapters:

- $l := u\mathrm{cap}_{A^c}(B) \asymp n^{\beta}$,
- $g(x_1, x_2) = \bar{e}_B^{A^c}(x_1) \asymp n^{-\beta}$,
- $T_Y = T_Z \le c$ for some $c > 0$,
- $\mathrm{Var}_\pi(\rho_{\boldsymbol{x}}^Y) \asymp \mathrm{Var}_\pi(\rho_{\boldsymbol{x}}^Z) \asymp n^{-2\beta}$,
- $\|\rho_{\boldsymbol{x}}^Y\|_\infty \asymp n^{-\beta}$,
- $\|\rho_{\boldsymbol{x}}^Z\|_\infty \asymp n^{-\beta}$.

For the last two estimates we use Lemma 3.4 to get

$$\|\rho_{\boldsymbol{x}}^Y\|_\infty = \sup_{\boldsymbol{y} \in \Sigma} P_{y_2}[X_{H_B} = x_1] \asymp n^{-\beta}, \tag{5.1}$$

and due to $\beta < \frac{\alpha}{2}$,

$$\begin{aligned} \|\rho_{\boldsymbol{x}}^Z\|_\infty &= \sup_{y_2 \in \partial A^c} \{P_{y_2}^{\mathbb{T}}[H_B = \infty]\bar{e}_B(x_1) + P_{y_2}^{\mathbb{T}}[X_{H_B} = x_1, H_B < \infty]\} \\ &\asymp \sup_{y_2 \in \partial A^c} \{n^{-\beta} + \mathbf{1}_{\{y_1 \in \mathbb{T}_{x_1}\}} n^{\beta-\alpha}\} \\ &= n^{-\beta} + n^{\beta-\alpha} \asymp n^{-\beta}. \end{aligned} \tag{5.2}$$

We take the length $l = u\mathrm{cap}_{A^c}(B)$, since this is (with a negligible difference) the expected number of excursions of the random walk on $G$ with length $un$. In section 5.2 we need this fact to redecorate the Markov chains $Y$ and $Z$.

Let's now estimate all requirements for Theorem 2.1. Since

$$\min_{i=1,2} \min_{\boldsymbol{x} \in \Sigma} \frac{\mathrm{Var}_\pi \rho_x^i}{2\|\rho_z^i\|_\infty g(x)} \asymp c, \quad \text{for } \alpha > 2\beta, \tag{5.3}$$

and the condition (2.47), we need for some sufficiently small $c > 0$,

$$0 < \epsilon_n \leq c. \tag{5.4}$$

Due to 2.50, for some $c_1, c_2 > 0$

$$k(\epsilon_n) \asymp c_1 \log n - c_2 \log(\epsilon_n). \tag{5.5}$$

Now we can apply Theorem 2.1, which yields

$$\mathbb{Q}[\mathcal{G}(l, \epsilon_n)^c] \leq c_1 \exp\left(\frac{-c_2 \epsilon_n^2 n^\beta}{c_3 \log n - c_4 \log(\epsilon_n)}\right) \tag{5.6}$$

for

$$\mathcal{G}(l, \epsilon_n) = \Big\{ \bigcup_{1 \leq i \leq l(1-\epsilon_n)} Z_i \subset \bigcup_{1 \leq i \leq l} Y_i \subset \bigcup_{1 \leq i \leq l(1+\epsilon_n)} Z_i \Big\}. \tag{5.7}$$

With a sensible lower bound for the sequence $\epsilon_n$, we get the desired convergence. Let $\epsilon_n^2 \geq n^{\delta'-\beta}$ for some $\delta' > 0$, then

$$\frac{\epsilon_n^2}{c_3 \log n - c_4 \log(\epsilon_n)} \geq \frac{n^{\delta'-\beta}}{c \log n} > cn^{\delta-\beta}, \tag{5.8}$$

for some $\delta < \delta'$ and $c > 0$, and the following lemma is proved.

**Lemma 5.1.** *Let $Y$ and $Z$ be the Markov chains defined in Section 4.1. Suppose that $\beta < \frac{\alpha}{2}$, $u > 0$ and $\epsilon_n$ is a sequence satisfying $n^{\frac{1}{2}(\delta'-\beta)} \leq \epsilon_n < c$ for some $\delta' > 0$ and sufficiently small $c > 0$. Let*

$$\mathcal{G}(l, \epsilon_n) = \Big\{ \bigcup_{1 \leq i \leq l(1-\epsilon_n)} Z_i \subset \bigcup_{1 \leq i \leq l} Y_i \subset \bigcup_{1 \leq i \leq l(1+\epsilon_n)} Z_i \Big\}, \quad \textit{with } l = u\mathrm{cap}_{A^c}(B). \tag{5.9}$$

*Then there exist probability spaces* $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$*, such that for large enough* $n$

$$\mathbb{Q}_n[\mathcal{G}(l, \epsilon_n)^c] \leq c_1 e^{-c_2 n^{\delta}}, \tag{5.10}$$

*for some* $0 < \delta < \delta'$ *and* $c_1, c_2 > 0$*.*

## 5.2 Coupling the vacant sets

We now re-decorate $Y$ and $Z$ to obtain a coupling of the vacant sets restricted to $B_n^{\bar{y}_n} = B_n$.

Let $\Gamma$ be the space of all finite-length nearest-neighbor paths on $G_n$. For $\gamma \in \Gamma$ we use $s(\gamma)$ to denote its length and write $\gamma$ as $(\gamma_0, \ldots, \gamma_{s(\gamma)})$.
To construct the vacant set of the random walk, we define on the same probability space $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$ the sequence of excursions $(\mathcal{E}_i)_{i\geq 1}$ and bridges $(\tilde{\mathcal{E}}_i)_{i\geq 0}$, whose distribution is uniquely determined by the following properties.

- Given $Y = \big((Y_{i,1}, Y_{i,2})\big)$ and $Z = \big((Z_{i,1}, Z_{i,2})\big)$, $(\mathcal{E}_i)_{i\geq 1}$ and $(\tilde{\mathcal{E}}_i)_{i\geq 0}$ are conditionally independent sequences of conditionally independent random variables.

- For every $i \geq 1$, the random variable $\mathcal{E}_i$ is $\Gamma$-valued and for every $\gamma \in \Gamma$,

$$\mathbb{Q}_n[\mathcal{E}_i = \gamma | Y, Z] = P_{Y_{i,1}}[H_{A_n^c} = s(\gamma), X_i = \gamma_i \forall i \leq s(\gamma) | X_{H_{A_n^c}} = Y_{i,2}]. \tag{5.11}$$

- For every $i \geq 1$, the random variable $\tilde{\mathcal{E}}_i$ is $\Gamma$-valued and for every $\gamma \in \Gamma$,

$$\mathbb{Q}_n[\tilde{\mathcal{E}}_i = \gamma | Y, Z] = P_{Y_{i,2}}[H_B = s(\gamma), X_i = \gamma_i \forall i \leq s(\gamma) | X_{H_B} = Y_{i+1,1}]. \tag{5.12}$$

- The random variable $\tilde{\mathcal{E}}_0$ is $\Gamma$-valued and

$$\mathbb{Q}_n[\tilde{\mathcal{E}}_0 = \gamma | Y, Z] = P[R_1 = s(\gamma), X_i = \gamma_i \forall i \leq s(\gamma) | X_{R_1} = Y_{1,1}]. \tag{5.13}$$

By concatenation of $\tilde{\mathcal{E}}_0, \mathcal{E}_1, \tilde{\mathcal{E}}_1, \mathcal{E}_2, \ldots$ we define a process $X = (X_k)_{k\geq 0}$ on $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$. From the construction it follows, that $X$ is a lazy random walk on $G_n$ started from the uniform distribution. Finally we write $R_1 = s(\tilde{\mathcal{E}}_0), D_1 = s(\tilde{\mathcal{E}}_0) + s(\mathcal{E}_1), \ldots$, which is consistent with the previous notation, and set, as before, $\mathcal{N}(un) = \sup\{i : R_i < un\}$. Finally, we fix an arbitrary constant $\xi > 0$ and define the vacant set of random walk on $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$ by

$$\mathcal{V}_n^u = G_n \backslash \{X_{\xi n}, \ldots, X_{(\xi+u)n}\}, \tag{5.14}$$

which has the same distribution as the vacant set introduced in (1.1), since $X$ is a stationary Markov chain.

In order to construct the vacant set of random interlacements intersected with $B_n$, let $\mathcal{I}_0 = \emptyset$ and for $i \geq 1$ inductively

$$\begin{aligned} \iota_i &= \inf\{j \geq 1 : j \notin \mathcal{I}_{i-1}, Y_j = Z_i\}, \\ \mathcal{E}_i^{RI} &= \mathcal{E}_{\iota_i}, \\ \mathcal{I}_i &= \mathcal{I}_{i-1} \cup \{\iota_i\}. \end{aligned} \tag{5.15}$$

Let further $(U_i)_{i\geq 1}$ be a sequence of conditionally independent Bernoulli random variables with

$$P[U_i = 1] = \frac{P^{\mathbb{T}}_{Z_{i,2}}[H_{B_n} = \infty]\bar{e}_{B_n}(Z_{i+1,1})}{P^{\mathbb{T}}_{Z_{i,2}}[H_{B_n} < \infty, X_{H_{B_n}} = Z_{i+1,1}] + P^{\mathbb{T}}_{Z_{i,2}}[H_{B_n} = \infty]\bar{e}_{B_n}(Z_{i+1,1})}. \tag{5.16}$$

The event $\{U_i = 1\}$ heuristically corresponds to the event "after the excursion $Z_i$ the random walk leaves to infinity and the excursion of random interlacements corresponding to $Z_{i+1}$ is a part of another random walk trajectory". We set $V_0 = 0$ and inductively for $i \geq 1$, $V_i = \inf\{i > V_{i-1} : U_i = 1\}$. Then, by construction, for every $i \geq 1, (\mathcal{E}_j^{RI})_{V_{i-1}<j\leq V_i}$ has the same distribution as the sequence of excursions of random walk $X^{(i)}$ into $B_n$, (see Section 4.1). Finally, as in Section 2.3, we let $(J_u^n)_{u\geq 0}$ to stand for a Poisson process with intensity $\mathrm{cap}(B_n)$, defined on $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$, independent of all previous randomness, and set

$$\mathcal{N}'(u) = V_{J_u^n}. \tag{5.17}$$

This is again consistent with previous notation. Finally, for $\xi$ as above, we can construct the random variables having the law of the vacant set of random interlacements at levels $u + \epsilon_n$ and $u - \epsilon_n$ intersected with $B_n$,

$$B_n \cap \mathcal{V}^{u\pm\epsilon_n} = B_n \setminus \bigcup_{i=\mathcal{N}'(\xi\mp\epsilon_n/2)}^{\mathcal{N}'(\xi+u\pm\epsilon_n/2)} \mathrm{Range}(\mathcal{E}_i^{RI}). \tag{5.18}$$

*Proof of Theorem 1.1.* Let $\mathcal{K}_n = \mathrm{cap}_{A_n^c}(B_n)$ and $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$.

Consider the random walk $X$ on $G_n$. Then the number of excursions started before time $nu$ concentrates around its expectation $u\mathcal{K}_n$ (see Lemma 4.1). Since $\mathrm{Range}(\tilde{\mathcal{E}}_i) \cap B_n = \varnothing$, and (5.14),

$$\begin{aligned} &\mathbb{Q}_n\Big[B_n \setminus \bigcup_{i=(\xi-\epsilon_n/4)\mathcal{K}_n}^{(\xi+u+\epsilon_n/4)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i) \subset \mathcal{V}_n^u \cap B_n \subset B_n \setminus \bigcup_{i=(\xi+\epsilon_n/4)\mathcal{K}_n}^{(\xi+u-\epsilon_n/4)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i)\Big] \\ &\geq 1 - c_1 e^{-c_1\epsilon_n^2 n^{c_2}}. \end{aligned} \tag{5.19}$$

Recall the coupling of the encoded excursions (Lemma 5.1), and the construction of $\mathcal{E}_i^{RI}$ (5.15). Assume $n^{\frac{1}{2}(\delta'-\beta)} \le \epsilon_n \le c$ for sufficiently small $c > 0$ and $0 < \delta' < \beta$. Then for some $\delta < \delta'$,

$$\mathbb{Q}_n\Big[B_n \setminus \bigcup_{i=(\beta-\epsilon_n/4)\mathcal{K}_n}^{(\beta+u+\epsilon_n/4)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i) \supset B_n \setminus \bigcup_{i=(\beta-\epsilon_n/3)\mathcal{K}_n}^{(\beta+u+\epsilon_n/3)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i^{RI})\Big] \ge 1 - c_1 e^{-c_2 n^{\delta}}, \tag{5.20}$$

and

$$\mathbb{Q}_n\Big[B_n \setminus \bigcup_{i=(\beta+\epsilon_n/4)\mathcal{K}_n}^{(\beta+u-\epsilon_n/4)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i) \subset B_n \setminus \bigcup_{i=(\beta+\epsilon_n/3)\mathcal{K}_n}^{(\beta+u-\epsilon_n/3)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i^{RI})\Big] \ge 1 - c_1 e^{-c_2 n^{\delta}}. \tag{5.21}$$

Consider random interlacements at level $u$ on $\mathbb{T}$. Due to Lemma 4.2 we know that the number of excursions concentrates around its expectation $u\mathcal{K}_n$. Together with (5.18), we get

$$\mathbb{Q}_n\Big[\mathcal{V}^{u+\epsilon_n/2} \cap B_n \subset B_n \setminus \bigcup_{i=(\beta-\epsilon_n/3)\mathcal{K}_n}^{(\beta+u+\epsilon_n/3)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i^{RI})\Big] \ge 1 - c_1 e^{-c_2 \epsilon_n^2 n^{c_2}}, \tag{5.22}$$

and

$$\mathbb{Q}_n\Big[\mathcal{V}^{u-\epsilon_n/2} \cap B_n \supset B_n \setminus \bigcup_{i=(\beta+\epsilon_n/3)\mathcal{K}_n}^{(\beta+u-\epsilon_n/3)\mathcal{K}_n} \mathrm{Range}(\mathcal{E}_i^{RI})\Big] \ge 1 - c_1 e^{-c_2 \epsilon_n^2 n^{c_2}}. \tag{5.23}$$

Theorem 1.1 then follows by combining 5.19-5.23. □

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