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New residually amenable groups, permanence properties, and metric approximations

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Abstract

Residually amenable groups arise as a common generalisation of amenable and residually finite groups. These two classes of groups, that share little in common, are deeply rooted in modern group theory and connect it with many other branches of mathematics. Recently, residually amenable groups attracted considerable attention for their relation to soficity, a notion introduced by Gromov to tackle Gottschalk's surjunctivity conjecture in dynamical systems.

In this thesis we systematically study the class of residually amenable groups. Our approach focuses on one side on the structural properties of the class, and on the other on a quantitative description of residual amenability.

We prove that residual amenability is preserved by taking free products and, more generally, by graph products. We introduce the notion of proamenable topology on a discrete group, and we take advantage of its properties to provide necessary and sufficient conditions, for certain HNN extensions and amalgamated free products of residually amenable groups, to be again residually amenable. This is far from being true in general, and exploiting these obtained conditions we construct countably many pairwise non-isomorphic finitely presented groups that are not residually amenable.

We study an asymptotic invariant for finitely generated groups, the residually amenable profile, which quantifies how much a given group is residually amenable. This generalises the well-known notion of Følner functions for amenable groups to the residually amenable setting. We analyse the behavior of the residually amenable profile with respect to group-theoretic operations that preserve the class of residually amenable groups, as for direct products, free products, and extensions with amenable quotient.

Lastly, we investigate the class of sofic groups, producing a new, independent, proof of the soficity of any group extension with sofic kernel and amenable quotient. This proof, as well as the original one of Elek and Szabó, underlines how the amenability assumption might be the optimal one for results in this direction. We then propose a strengthened notion for soficity, that is, conjugacy soficity, and we study its properties. In this new context, and in contrast to soficity, the sofic approximations cannot anymore be homomorphisms onto finite symmetric groups, but are forced to be maps that are not homomorphisms.

Zusammenfassung

Residuell mittelbare Gruppen treten als gemeinsame Verallgemenierung mittelbarer und residuell endlicher Gruppen auf. Diese beiden Klassen von Gruppen, die wenig miteinander gemeinsam haben, sind tief in der modernen Gruppentheorie verwurzelt und verbinden sie mit vielen anderen Zweigen der Mathematik. Vor kurzem erlangten residuell mittelbare Gruppen große Beachtung aufgrund ihrer Beziehung zur Eigenschaft "Soficity" (ein Begriff, der von Gromov eingeführt wurde, um Gottschalks Surjunctivity-Vermutung dynamischer Systeme zu behandeln).

In dieser Arbeit untersuchen wir systematisch die Klasse der residuell mittelbaren Gruppen. Unser Ansatz konzentriert sich einerseits auf die strukturellen Eigenschaften dieser Klasse und andererseits auf eine quantitative Beschreibung der residuellen Mittelbarkeit.

Wir beweisen, dass die Eigenschaft der residuellen Mittelbarkeit durch das Bilden freier Produkte, und allgemeiner durch Graph-Produkte, erhalten bleibt. Wir führen den Begriff einer pro-mittelbaren Topologie auf diskreten Gruppen ein und nutzen die Eigenschaften solcher Gruppen, um notwendige und hinreichende Bedingungen zu finden, die sicherstellen, dass bestimmte HNN-Erweiterungen und amalgamiert freie Produkte residuell mittelbarer Gruppen wieder residuell mittelbar sind. Im Allgemeinen gilt dies bei weitem nicht; durch diese Bedingungen konstruieren wir abzählbar viele paarweise nichtisomorphe endlich erzeugte Gruppen, die nicht residuell mittelbar sind.

Wir untersuchen eine asymptotische Invariante für endlich erzeugte Gruppen, das residuell mittelbare Profil, das quantifiziert, wie sehr eine bestimmte Gruppe residuell mittelbar ist. Dies verallgemeinert den bekannten Begriff von Følner-Funktionen für mittelbare Gruppen auf die Klasse der residuell mittelbaren Gruppen. Wir analysieren das Verhalten des residuell mittelbaren Profiles im Bezug auf gruppentheoretische Operationen, die die Klasse der residuell mittelbaren Gruppe erhalten, wie direkte Produkte, freie Produkte und Erweiterungen mit mittelbarem Quotienten.

Schließlich untersuchen wir die Klasse von Sofic-Gruppen und bringen einen neuen, unabhängigen Beweis für die Tatsache, dass jede Gruppenerweiterung mit einem Sofic-Kern und mittelbarem Quotient wiederum eine Sofic-Gruppe ist. Dieser Beweis zeigt, wie auch der ursprüngliche von Elek und Szabó, wie die Voraussetzung der Mittelbarkeit die bestmöglichste für Resultate in diese Richtung sein könnte. Wir schlagen einen verschärften Begriff für Soficity vor, nämlich Konjugations-Soficity, und untersuchen die Eigenschaften dieses Begriffes. In diesem neuen Kontext können (im Gegensatz zur Soficity) die Soficity-Approximationen nicht mehr Homomorphismen auf endliche symmetrische Gruppen sein, sondern vielmehr Karten, die keine Homomorphismen sind.

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[1] Fontana di aga dal me país.
A no é aga pí fres-cia che ta'l me país.
Fontana di rustic amòur.
Fontana di aga di un país no me.
A no é aga pí vecia che ta chel país.
Fontana di amòur par nissun.

groups: limits, curvature, and randomness".

Contents

Chapter 1

Introduction

The natural idea of approaching more difficult situations through our understanding of easier cases is ubiquitous in mathematics and life in general. At the beginning of last century, this paradigm found at least two applications in the study of infinite discrete groups, giving birth to the notions of *residually finite* and *amenable* groups.

On one side, residually finite groups are groups whose elements can always be distinguished in suitable finite quotients, and hence groups where an understanding of the finite images - or, equivalently, the subgroups of finite index - yields back information about the original group. On the other side, an amenable group is approximated by a sequence of finite sets, called *Følner sets*, which exhaust the group and become more and more invariant under the natural group action on itself by left multiplication.

In both these situations, the philosophy is to approximate an infinite group that one wants to investigate through its pieces - being finite images or almost invariant finite subsets - over which one has more control and a much better understanding.

Amenable groups were defined in 1929 by von Neumann [94] in his work on the Banach-Tarski paradox and, since then, the study of amenability has been impressively fruitful in a variety of branches of mathematics. Amenable groups were originally defined as groups admitting an invariant finitely additive probability measure. Following this first definition, many more equivalent characterisations of them have emerged. Tarski [110, 111] proved that a group is amenable if and only if it admits no paradoxical decomposition. Følner [54] gave a characterisation of amenability in terms of almost invariant subsets, that later assumed the already mentioned name of Følner sets. This rich variety of equivalent descriptions just underlines the central role of amenable groups in modern group theory, and shows how deeply they are intertwined with other, seemingly distant, mathematical areas.

Von Neumann already noted that abelian groups are amenable, and that the class of amenable groups satisfies the following *permanence properties*: it is closed under the operations of taking subgroups, quotients, extensions and direct limits. Moreover, he recorded that a group that contains a non-abelian free subgroup cannot be amenable. Whether or not this was another characterisation of amenability became known as the *von Neumann conjecture*, even if its statement as a conjecture is due to Day [40] in 1957.

This question later proved to have a negative answer: in 1980 Ol'shanskii [95] proved that the so-called Tarski monsters are not amenable. Shortly after, Adian [1] showed that free Burnside groups with at least two generators and odd exponent bigger than 665 are also non-amenable. Both these examples consist of torsion groups, and hence cannot contain a free subgroup. Nevertheless, these groups are not finitely presented, and for some years the speculation was that the conjecture could hold for finitely presented groups. In 2002, Ol'shanskii and Sapir [96] gave the first finitely presented counterexamples to the von Neumann conjecture. Working on a result of Monod [91], Lodha and

Moore [81] recently isolated a counterexample with just three generators and nine relators.

In 1930 Levi [80] proved that every non-trivial element of a free group can be mapped non-trivially into some finite quotient, even if this had been already implicitly showed in 1927 by Schreier [107, page 169] in his proof of the solution of the word problem for free groups. Soon after, the notion of residual property was considered by Mal'cev [84] and independently by P. Hall [68]. Residually finite groups, in particular, received considerable attention.

Mal'cev, starting from the 1940s, proved that finitely generated linear groups are residually finite, and that finitely generated residually finite groups are hopfian [84]. This was the basis for the revolutionary result of Baumslag and Solitar [12], who provided the first example of a non-residually finite one-relator group, destroying several conjectures long and strongly held by many, and some published and unpublished "theorems"[1].

Since the early 1960s, the study of residual properties expanded from residual finiteness to include residually free, fully residually free (also known as Sela's limit groups or ω-residually free groups), and residually solvable groups.

In 1997 Vershik and Gordon [115], with ideas going back to Mal'cev, introduced a slight generalisation of residual finiteness: the concept of LEF groups, that is, groups that are *locally embeddable* into the class of *finite* groups. These are precisely the groups where every finite subset has a partial multiplication table that can be recovered in a finite group. Obviously, this definition is more general and other classes of approximating groups might be considered instead of just finite ones. Hence, a group is LE-$\mathcal{C}$ if each of its finite subsets has a partial multiplication table that can be witnessed in a group of the given class $\mathcal{C}$. This is the first appearance of *residual* approximations that are not constructed using surjective homomorphisms, but that are just maps having a *good* behavior on prescribed finite sets. This peculiarity will be exploited to its maximum a few years later, by Gromov.

The most recent application of the approximation paradigm in the realm of discrete groups indeed came to light in 1999, with the work of Gromov [62] towards Gottschalk's surjunctivity conjecture in dynamical systems [58]. Gromov defined the concept of *sofic* groups (although the name sofic is coined a year later by Weiss [116]): as residually finite groups are groups that can be studied through their finite quotients, sofic groups ought to be thought as groups that can be approximated by finite symmetric groups. In this case, the approximation - which is a *metric* approximation - is much weaker and hence covers a much wider family of examples.

The idea is to consider finite symmetric groups equipped with a metric, which in this case is the bi-invariant normalised Hamming distance. A group G is sofic if for any given positive ε - the *error* allowed in the approximation - and for any finite subset $K \subseteq G$, there exist a finite symmetric group $\mathrm{Sym}(F)$ on a finite set F and a map $\varphi \colon G \to \mathrm{Sym}(F)$ that behaves like a homomorphism on K, up to the error ε, and which injects K in a strong sense into the symmetric group. This means that different elements are sent to permutations whose distance is one, up to the error ε.

It is clear, and instructive, to prove that residually finite groups and amenable groups are sofic. For (residually) finite groups, one should apply Cayley's theorem, while for amenable groups a careful choice of a Følner set gives the desired approximation.

Gromov [62] considered and defined sofic groups because these metric approximations are sufficient for a proof of surjunctivity, that is, a proof that sofic groups satisfy the aforementioned Gottschalk's conjecture. Since then, the theory of sofic groups flourished, and now they are known to satisfy Connes' embedding conjecture [47], Kaplansky's direct and stable finiteness conjectures over any division ring [46], the determinant conjecture [47], and the algebraic eigenvalue conjecture [112].

Moreover, soficity is known to be preserved by several group constructions: direct and free products [48] and more generally graph products [34], free products and HNN extensions with amalgama-

[1]These are the words of B. H. Neumann that can be found in the MathSciNet review of the article [12] of Baumslag and Solitar.

tion over amenable subgroups [49, 98], restricted wreath products [70], extensions with sofic kernel and amenable quotient.

Whether or not all groups are sofic is still an open question, but some experts suggest that Higman's group [71], or Deligne's group [41], might be good candidates for finitely presented non-sofic groups.

We stress that both answers to this question, a positive or a negative one, would be very interesting. If all groups are sofic, this would immediately confirm many famous conjectures on soficity. On the other hand, if there are groups that are not sofic - as many believe - these could shed some light on these unresolved conjectures and, maybe, disprove them.

To overcome these difficulties and attack these open problems, weak notions of soficity have recently emerged: the classes of weakly sofic groups [56], weakly hyperlinear groups, linear sofic groups [9], are all generalisations of the original notion introduced by Gromov. This was done on one hand in an attempt to highlight possible ways and methods that would produce a non-sofic group, and on the other hand to give new alternative points of view on the class of sofic groups, admitting one can prove that these notions are equivalent to soficity.

Residually amenable groups are a natural, common generalisation of amenable and residually finite groups. By definition, the elements of such groups can always be distinguished in appropriate amenable quotients. Amenable groups and residually finite groups are particular examples of residually amenable groups, but this class is in fact much richer. They appeared in the literature in 1999, when Clair [35] extended to the residually amenable setting what was known to hold in the residually finite case [83] and in the amenable case [44]: the L^2-Betti numbers of the universal cover $\widetilde{X}$ of a finite simplicial complex X with residually amenable fundamental group can be approximated by the L^2-Betti numbers of the amenable coverings of X (which in turn are approximated by the ordinary Betti numbers of a sequence of Følner subsets exhausting the amenable cover).

These groups are now interesting because they represent a huge subfamily of the class of sofic groups (nowadays it is well known that there exist finitely presented sofic groups that are not residually amenable [37, 76]), but still their approximations are well understood. Indeed, sofic approximations of a residually amenable group can *always* be obtained by first projecting onto an appropriate amenable quotient, and then by exploiting the approximations of this amenable quotient. These are, by far, the most natural to consider, being governed by the Følner sets of the amenable group.

Outline of the thesis

In this thesis we present a systematic study of the class of residually amenable groups. We are particularly concerned with the *structural* characteristics of the class, and therefore we study its permanence properties. The approach we undertake is general, and deals with abstract classes of groups $\mathcal{C}$ satisfying some mild conditions that are met, in particular, by the class of amenable groups. This allows us to unify and recover the known facts concerning residual properties, while also proving new results in this direction.

In Chapter 2 we recall the definitions and basics of amenable groups and of residual properties, with a particular emphasis on residual finiteness. After this material, Chapter 3 is mainly devoted to free products. In Section 3.1 we explain why the classical machinery introduced by Gruenberg in his study of free products of residually $\mathcal{C}$ groups, that is, his notion of root classes, does not work in the context of residual amenability.

Before properly considering free products, we turn our attention to finitely generated, non-simple, groups without non-trivial finite quotients, and we inquire whether or not there exists such a group that, in addition, is residually amenable. Our result in this direction is the following:

Proposition 3.2.2 (Berlai)**.** *Let G be a finitely presented group without non-trivial finite quotients. Then the only LEF quotient of G is the trivial group.*

As an immediate application of this proposition, we conclude that the famous Higman's group does not admit any non-trivial LEF quotient. In particular, it cannot surject onto the simple, infinite, amenable groups recently constructed. For these groups Matui [90] proved simplicity, Grigorchuk and Medynets [61] that they are LEF (see also an alternative proof provided by Elek [50]), and Juschenko and Monod amenability [79].

This is further evidence that, despite being SQ-universal, Higman's group might have no amenable quotients, which is currently a well-known open problem.

With Example 3.2.9 we establish the existence of finitely generated LEF groups (and therefore, locally embeddable into amenable groups, and sofic) without any non-trivial finite quotient, which are also SQ-universal. In particular, this shows that the "finitely presented" assumption in Proposition 3.2.2 is optimal and cannot be relaxed to "finitely generated".

The last two sections of Chapter 3, that contribute to the published paper [19], focus on free products of residually $\mathcal{C}$, and LE-$\mathcal{C}$, groups.

Theorem 3.3.5 (Berlai [19]). *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of residually $\mathcal{C}$ groups is closed under taking free products.*

This theorem recovers the known facts about free products of residually finite, residually p-finite or residually solvable groups. Moreover, it extends the result to residually elementary amenable and residually amenable groups, and is followed by a similar result concerning local embeddings.

Theorem 3.4.1 (Berlai [19]). *Let $\mathcal{C}$ be a class of groups, and suppose either that $\mathcal{C}$ is closed under taking free products, or that $\mathcal{C}$ is closed under finite direct products and the free product of residually $\mathcal{C}$ groups is residually $\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under taking free products.*

Notably, this also applies to groups that are locally embeddable into free groups, whilst Theorem 3.3.5 does not apply to residually free groups, and indeed its conclusion is false in that case.

In Chapter 4 we introduce the notion of *pro-$\mathcal{C}$ topology* and take advantage of it to further produce permanence results for the class of residually amenable groups. The content of the first two sections of this chapter also appear in the published paper [19].

Given a group G, its pro-$\mathcal{C}$ topology is defined by letting all normal subgroups with quotient in $\mathcal{C}$ be the base at e_G for the topology. Under some natural conditions, namely if the class $\mathcal{C}$ is closed under taking subgroups and finite direct products, this is a *group topology*, meaning that the group operations are continuous maps.

Using this machinery, we precisely characterise those special HNN extensions and doubles of groups that are residually amenable. An HNN extension is called special if the amalgamating isomorphism is the identity map, while an amalgamated free product is a double if its free factors are isomorphic copies of the same group and the amalgamation isomorphism is nothing more than the map that sends a subgroup of one of the factors element-wise to its image in the other.

Theorem 4.2.2 (Berlai [19]). *Let A be a group, $H \leqslant A$ and consider the special HNN extension $G := A*_{\mathrm{id}}$ amalgamating the subgroup H. Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.*

Then G is residually $\mathcal{C}$ if and only if A is residually $\mathcal{C}$ and H is closed in the pro-$\mathcal{C}$ topology of A.

Theorem 4.2.6 (Berlai [19]). *Let A be a group, $H \leqslant A$ a subgroup and consider the double $G := A *_H A$. Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.*

Then G is residually $\mathcal{C}$ if and only if A is residually $\mathcal{C}$ and H is closed in the pro-$\mathcal{C}$ topology of A.

In the third section of Chapter 4 we obtain a direct application of the aforementioned results concerning the proamenable topology of a group. Generalising a construction of Kar and Nikolov [76], we produce countably many pairwise non-isomorphic finitely presented groups that are not residually amenable. Namely, let $n \geq 3$, $p > 2$ be a prime number, Γ and $\bar{\Gamma}$ be two isomorphic copies of the finitely presented linear group $\mathrm{SL}_n\big(\mathbb{Z}[1/p]\big)$, and consider the matrices

$$a = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}.$$

If F_m denotes the free subgroup of Γ of rank m generated by $\{a, bab^{-1}, \ldots, b^{m-1}ab^{-m+1}\}$, then the group $\Gamma *_{F_m} \bar{\Gamma}$ is not residually amenable, as proved in Corollary 4.3.2. For each fixed n, the countable family $\{\Gamma *_{F_m} \bar{\Gamma}\}_{m\geq 2}$ consists of pairwise non-isomorphic groups.

Exploiting Theorem 4.2.2 and Theorem 4.2.6, this result immediately translates into a statement for special HNN extensions: the finitely presented group $\Gamma *_{F_m}$ is not residually amenable, as proved in Corollary 4.3.4, and pairwise non-isomorphic as m varies.

It is not known whether or not these groups are sofic, but some believe that amalgamated products of sofic groups over free non-abelian subgroups could be the critical construction for producing non-sofic groups. Higman's group is an example, being the amalgamated product of two residually solvable groups over a free non-abelian subgroup.

Chapter 4 ends with partial results concerning HNN extensions and amalgamated products that are not special HNN extensions or doubles. Of these, Lemma 4.4.3 is in collaboration with Michal Ferov.

Graph products are a common generalisation of free and direct products, and Chapter 5 of this thesis is devoted to them. Its first three sections appear in the published paper [20], written jointly with Michal Ferov.

Very often, a property that is preserved under taking free products and direct products is preserved by graph products as well. This is indeed the case for residual finiteness and residual p-finiteness [59], soficity [34], (hereditary) conjugacy separability [53], Tits' alternatives [3], Haagerup's property, or finiteness of asymptotic dimension [2]. We confirm this intuition about properties inherited by graph products, proving that graph products also preserve residual elementary amenability, and residual amenability.

Theorem 5.2.3 (Berlai, Ferov [20]). *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of residually $\mathcal{C}$ groups is closed under taking graph products.*

The statement is proved also in the analogous case of local embeddings. The proof, in contrast with Theorem 3.3.5 and Theorem 3.4.1, does not plainly generalise the one of Theorem 5.2.3, and this time a careful treatment of *reduced words* in graph products is required.

Theorem 5.3.1 (Berlai, Ferov [20]). *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that graph products of residually $\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under taking graph products.*

In the fourth section of Chapter 5 we apply the *Reidemeister-Schreier method* to obtain presentations for specific subgroups of graph products. This is inspired by the analogy with the theory of raags and their subgroups, and by the known fact that there are finitely generated subgroups of raags that are not raags themselves, occurring as kernels of projections onto the infinite cyclic group.

Our strategy is to consider specific graph products, and suitable projections onto $\mathbb{Z}$. We prove that the presentations of the obtained kernels correspond to graph products associated with a trivial graph, that is a graph with one vertex and no edges. This naturally leads to the problem of *rigidity*, or uniqueness, of graph products decompositions: given two isomorphic graph products - or two different presentations of the same group - does it follow that the underlying graphs are isomorphic, and does this graph isomorphism induce an isomorphism of the vertex groups?

This is the content of the fifth section of Chapter 5. As one soon understands, the answer to this general question is no. Hence, we discuss some sufficient, but somehow very restrictive, conditions that guarantee such uniqueness.

After having dealt with structural properties of the class of residually amenable groups, in Chapter 6 we focus on *quantitative* aspects. In the realm of amenable groups, *Følner functions* [114] are an asymptotic invariant that describe how fast the cardinality of ε-Følner sets grows, as we consider smaller and smaller ε. Following Arzhantseva and Cherix [6], we export this machinery into the context of residually amenable groups and we consider the notion of *residually amenable profile* for a group.

Our approach is novel, because we do not necessarily equip a countable group with a word metric, *but* we also consider groups equipped with *proper* metrics, that is, metrics in which all metric balls of finite radius are finite subsets. This has the clear advantage of including groups that are not finitely generated, and this wider point of view is beneficial, and often necessary, for the description of the residually amenable profile of group extensions.

In an amenable group Q, let $A_Q(n)$ denote the smallest possible cardinality of a $\frac{1}{n}$-Følner set for the finite ball of radius n. Given a residually amenable group G, its residually amenable profile is defined by

$$RA_G(n) = \min\bigl\{A_Q(n) \mid Q \text{ amenable quotient of } G,\ B_G(n) \hookrightarrow Q\bigr\}.$$

This is reminiscent of the notion of *full residual finiteness growth* introduced by Bou-Rabee and McRaynolds [22] in the context of residually finite groups, where a natural number n is mapped to the smallest cardinality of any quotient of G in which the ball of radius n injects.

After proving that the residually amenable profile for a finitely generated group is invariant within word metrics associated to finite generating sets, we discuss its asymptotic behavior under group extensions. We give a bound for the residually amenable profile of an extension of a residually solvable group by an amenable one.

Theorem 6.2.7 (Berlai)**.** *Let G be a finitely generated residually amenable group, generated by the finite set X. Let $N \trianglelefteq G$ be residually solvable, with amenable quotient $Q = G/N$, and equip N with the proper metric induced by the word metric of G.*

There exists a sequence of natural numbers $\{s_n\}_{n\in\mathbb{N}}$, which depends only on G and N, such that

$$RA_G(n) \leqslant |A_n| \cdot |B_n|,$$

where $N^{(s_n)}$ denotes the s_n-th derived subgroup of N, $|B_n| = A_{N/N^{(s_n)}}\bigl(2n^2|A_n|^2(2|A_n|_{\bar{X}} + n)\bigr)$ and $|A_n| = A_Q(2n)$.

In Chapter 7 we turn our attention to sofic groups. One of the leading questions in the field is the extension problem: given two sofic groups, is their extension again sofic? This question is known to have a positive answer in the very particular case of an extension of a sofic group by an amenable one [48].

A classical result, known as the *Kaloujnine-Krasner Theorem*, guarantees that each extension of two groups G and H is isomorphic to a certain subgroup of the unrestricted wreath product $G \wr H$. Independently from the methods of [48] and in collaboration with Goulnara Arzhantseva, Martin Finn-Sell, and Lev Glebsky [5], we prove that the unrestricted wreath product $G \wr H$ is a sofic group whenever G is sofic and H is amenable. This provides a general, alternative proof of the soficity of any extension with sofic kernel and amenable quotient, because soficity is stable under taking subgroups.

This new approach also makes explicit how amenability is used in an essential way, and how the amenability assumption may be the optimal one.

In the second part of this last chapter, we propose a natural strengthening of the notions of soficity and weak soficity, which we call *conjugacy soficity* and *conjugacy weakly soficity*. These results are from ongoing collaboration with Michal Ferov and Martin Finn-Sell. Inspired by the analogy between residually finite groups and conjugacy separable ones, to define conjugacy soficity we simply impose an additional condition: elements that lie in distinct conjugacy classes of the group G should be mapped to permutations that, up to an error of ε, live in different conjugacy classes of the finite symmetric group.

As a consequence of this definition, a conjugacy sofic group G always admits an injective homomorphism into a metric ultraproduct of finite symmetric groups equipped with normalised Hamming distances, and this embedding satisfies the following property: if two elements of G are not conjugated in G, then their images are not conjugated in the metric ultraproduct.

However, we prove that this seemingly innocuous additional condition has dramatic consequences for the sofic approximations one is allowed to consider: these maps cannot no longer be homomorphisms.

Chapter 2

Amenable groups and residual properties

In this chapter we fix the main notation and we recall basic facts about amenable groups and residual properties, with a particular emphasis on the classical notion of residual finiteness.

Notation

A *class* of groups is a collection of groups closed under isomorphic images. For a *non-trivial* class of groups we intend a class which is not empty and does not consist just of the trivial group.

The identity of a group G is denoted by e_G, or simply by e if the group is clear from the context. If G is a group and $g, h \in G$, then by g^h we mean the conjugate hgh^{-1}, and the conjugacy class of g is denoted by g^G. If H is a subgroup of G and $g \in G$, by $H^g = gHg^{-1}$ we mean the conjugate of H by the element g, defined to be the subgroup consisting of all the elements of the form ghg^{-1}, where $h \in H$. The commutator of two elements is $[g,h] = ghg^{-1}h^{-1}$, and if H, K are two subgroups of G, their commutator subgroup $[H,K]$ is the subgroup generated by all the elements of the form $[h,k]$, where $h \in H$ and $k \in K$.

The *solvable length* of a solvable group is the length of the derived series of the group. It is the minimal length for any abelian subnormal series of the group.

We use the standard notation of an $\mathcal{A}$-by-$\mathcal{B}$ group to denote a group G with a normal subgroup $N \trianglelefteq G$ such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are given classes of groups (e.g. $\mathcal{A}$ being the free groups and $\mathcal{B}$ being the amenable groups, in the case of a free-by-amenable group).

The free group on n free generators is denoted by F_n, and if X is a set, the free group freely generated by X is denoted by $F(X)$.

If X is a set, $\mathrm{Sym}(X)$ denotes the group of permutations of X, $\mathrm{Sym}_0(X)$ the group of permutations of X with finite support, and $\mathrm{Alt}(X)$ denotes the subgroup of $\mathrm{Sym}_0(X)$ consisting of all even permutations.

If $\{G_i\}_{i \in I}$ is a family of groups, then the *direct product* is denoted by

$$\prod_{i \in I} G_i := \Big\{ (g_i)_{i \in I} \mid g_i \in G_i \Big\},$$

while the *direct sum* is denoted by

$$\bigoplus_{i \in I} G_i := \Big\{ (g_i)_{i \in I} \mid g_i \in G_i,\, g_i \neq e_{G_i} \text{ for only finitely many } i \in I \Big\} \leqslant \prod_{i \in I} G_i.$$

Given two groups G and H, the (*restricted*) *wreath product* $G \wr H$ is the semidirect product

$$G \wr H := \Big(\bigoplus_{h\in H} G\Big) \rtimes H,$$

where H acts on the direct sum shifting the coordinates:

$$\big((g_x)_{x\in H}, h\big) \cdot \big((g'_x)_{x\in H}, h'\big) = \big((g_x g'_{h^{-1}x})_{x\in H}, hh'\big).$$

The *unrestricted wreath product* $G \wr\wr H$ is the semidirect product

$$G \wr\wr H := \Big(\prod_{h\in H} G\Big) \rtimes H,$$

and H acts on $\prod_{h\in H} G$ by shifting the coordinates, analogously to the action in the above restricted wreath product $G \wr H$.

Let $K \subseteq G$ be a subset of a group G. The *Cayley graph* $\Gamma(G, K)$ of G with respect to K is the graph constructed as follows. The vertex set consists of the elements of the group G, and two vertices g and g' are connected if there exists an element $k \in K \cup K^{-1}$ such that $g' = gk$. It is a fact that $\Gamma(G, K)$ is connected if and only if K is a generating set for the group G. If K is a generating set, we define the *word metric* $d = d_{G,K}$ (which depends on K) on the group G: the distance $d(g, g')$ is defined to be the minimal number of edges necessary to reach the vertex g' from the vertex g in the graph $\Gamma(G, K)$. Moreover, with $|g|_d$ we intend the distance $d(g, e_G)$, and if d is a word metric associated to the generating set K, then the notation $|g|_d$ might be replaced by $|g|_K$.

By $B_{G,K}(n)$ we denote the set

$$\begin{aligned} B_{G,K}(n) :&= \{k_1 \cdots k_r \mid k_i \in K \cup K^{-1},\ \forall i = 1, \dots, r \text{ and } 0 \leqslant r \leqslant i\} \\ &= \{g \in G \mid d_{G,K}(g, e_G) \leqslant n\}. \end{aligned}$$

If K is a generating set for G, then $B_{G,K}(n)$ is the usual ball of radius n in the Cayley graph $\Gamma(G, K)$. If the group is equipped with a metric d then the ball of radius n with respect to this metric is denoted by $B_{G,d}(n)$. Let K and F be finite subsets of a group G. By $\partial_K F$ we denote the set

$$\partial_K F := \big\{f \in F \mid d_{G,K}(f, G \setminus F) = 1\big\}.$$

A metric on a group is called *proper* if all balls of finite radius are finite subsets of the group. If the group G is generated by the finite set K, then the associated word metric is proper.

All graphs considered, except for the just mentioned Cayley graphs, that is, all graphs considered in the chapter regarding graph products, are simplicial graphs, even if not explicitly stated. This means that there are no loops, nor multiple edges. Whilst Cayley graphs never have multiple edges, it might happen that they have a loop on every vertex: this is the case if the trivial element belongs to the considered generating set.

If Γ is a graph, then its vertex set is denoted by $V(\Gamma)$, and its edge set is denoted by $E(\Gamma)$. If the graph is clear from the context, then these two sets might be denoted also by V and E.

With $\mathcal{P}_f(G)$ we denote the collection of finite subsets of the group G. Given two functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we write $f \preccurlyeq g$ if there exists a constant $C > 0$ such that $f(n) \leqslant C \cdot g(Cn)$ for all $n \in \mathbb{N}$. If $f \preccurlyeq g$ and $g \preccurlyeq f$ then we write $f \sim g$.

2.1 Amenable groups

Amenable groups were defined in 1929 by von Neumann in his work on the Banach-Tarski paradox. Since then, the notion of amenability found applications and surprising connections in a variety of branches of mathematics.

We collect here a few equivalent characterisations of amenability that will be relevant in the thesis.

Definition 2.1.1. A group G is *amenable* if it satisfies one of the following equivalent definitions:

1. there exists a left-invariant finitely additive probability measure $\mu\colon \mathcal{P}(G) \to [0,1]$;

2. for every finite subset $K \subseteq G$ and every real number $\varepsilon > 0$ there exists a non-empty finite subset $F \subseteq G$ such that $|F \setminus kF| < \varepsilon|F|$, for all $k \in K$;

3. for every finite subset $K \subseteq G$ and every real number $\varepsilon > 0$ there exists a non-empty finite subset $F \subseteq G$ such that $|F \vartriangle kF| < \varepsilon|F|$, for all $k \in K$;

4. for every finite subset $K \subseteq G$ and every real number $\varepsilon > 0$ there exists a non-empty finite subset $F \subseteq G$ such that $|\partial_K F| < \varepsilon|F|$.

The finite sets F of the previous definition are called *Følner sets* for the group G, or more precisely ε-Følner sets for K. We stress that this notion of Følner set do depend on which of the equivalent conditions 2., 3. or 4. we are considering.

The list in Definition 2.1.1 is far from being complete. For example, these conditions are also equivalent to not admitting a paradoxical decomposition, or to the isomorphism of the reduced C^*-algebra of the group with the full C^*-algebra.

We collect basic, well known, permanence properties of the class of amenable groups in the following theorem. We refer to [31] for proofs.

Theorem 2.1.2. *The class of amenable groups is closed under taking subgroups, extensions, quotients, and direct limits.*

We stress that in this thesis we will only consider *discrete* amenable groups. Indeed, for the notion of amenability there is also a well developed theory in the context of locally compact topological groups.

In that case, the cardinalities appearing in Definition 2.1.1 are replaced using the *Haar measure* of the given locally compact group, and finite sets are replaced by *compact* sets. Theorem 2.1.2 has its equivalent reformulation in this setting of topological groups, but a notable difference is that topological amenability is preserved just passing to *closed* subgroups.

Finite groups are amenable, as well as abelian groups. Hence, it follows that every virtually solvable group is amenable. It is immediate to produce a paradoxical decomposition for free non-abelian groups [31, Example 4.8.2]. It thus follows that groups that contain free non-abelian subgroups are not amenable.

Definition 2.1.3. Let $\mathcal{E}$ be the smallest class that contains all finite groups and all abelian groups, and that is closed under taking subgroups, extensions, quotients and direct limits. A group in $\mathcal{E}$ is called *elementary amenable.*

Following this definition, virtually solvable groups are elementary amenable. By now, there are many examples of amenable groups which are not elementary amenable: the first was presented by Grigorchuk [60] in 1984.

Elementary amenable groups were defined in 1980 by Chou [33], when he proved that $\mathcal{E}$ equals the smallest class of groups containing finite and abelian groups, and that is closed under just taking extensions and direct unions. We record the following easy consequence of this.

Lemma 2.1.4. *Let G be a finitely generated elementary amenable group. Then G has non-trivial finite quotients.*

Følner functions on amenable groups

Let G be a countable group endowed with a proper metric d, so that every ball of finite radius is a finite subset of G. If G is amenable, Definition 2.1.1 implies that for every $n \in \mathbb{N}$ there exists a $\frac{1}{n}$-Følner set for the finite subset $B_{G,d}(n)$.

Define the function $A_{G,d} \colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ as

$$A_{G,d}(n) := \begin{cases} \min\bigl\{|A| \mid |A \setminus gA| < \frac{1}{n}|A|, \ \forall g \in B_{G,d}(n)\bigr\} & \text{if such sets exist,} \\ +\infty & \text{otherwise.} \end{cases}$$

Analogously, define the functions $A^{\triangle}_{G,d} \colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ as

$$A^{\triangle}_{G,d}(n) := \begin{cases} \min\bigl\{|A| \mid |A \triangle gA| < \frac{1}{n}|A|, \ \forall g \in B_{G,d}(n)\bigr\} & \text{if such sets exist,} \\ +\infty & \text{otherwise,} \end{cases}$$

and $\mathrm{Føl}_{G,d} \colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ as

$$\mathrm{Føl}_{G,d}(n) := \begin{cases} \min\bigl\{|A| \mid |\partial_n A| < \frac{1}{n}|A|\bigr\} & \text{if such sets exist,} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\partial_n A$ stands for $\partial_{B_{G,d}(n)} A$.

If the proper metric d is a word metric associated to the finite generating set X, then we denote these three functions also by $A_{G,X}$, $A^{\triangle}_{G,X}$ and $\mathrm{Føl}_{G,X}$. In this case, if Y is another finite generating set, it is an easy fact that there exists a constant $C > 0$ such that $A_{G,X}(n) \leqslant C \cdot A_{G,Y}(Cn)$ for all $n \in \mathbb{N}$.

Remark 2.1.5. Let G be a group generated by the finite set X. In contrast with our definition, Erschler [51] defines $\mathrm{Føl}_{G,X}(n)$ to be the minimal cardinality of a finite set A for which $|\partial_X A|$ is smaller than $1/n|A|$, that is, at each $n \in \mathbb{N}$ the same boundary ∂_X is considered, instead of taking ∂_n. Hence, at each level $n \in \mathbb{N}$ our function $\mathrm{Føl}_{G,d}(n)$ is as in Erschler's work, but with the new finite generating set $B_{G,d}(n)$.

We stress that defining the Følner function with respect to increasing balls is rather reasonable. Indeed, it allows us to extend this classical notion of Følner functions to groups that are not necessarily finitely generated. If the group is indeed not finitely generated and Y is an infinite generating set, then Equation (2.4) would not hold for the set $B_{G,Y}(1)$, because amenability produces Følner sets for given ε and given *finite* sets.

For a finite generating set X of the group G, we have that

$$A_{G,X}(|X \cup X^{-1}| \cdot n) \leqslant \mathrm{Føl}_{G,X}(n) \leqslant A_{G,X}(n) \tag{2.1}$$

and that

$$A^{\triangle}_{G,X}(n) \leqslant A_{G,X}(2n), \qquad A_{G,X}(n) \leqslant A^{\triangle}_{G,X}(n). \tag{2.2}$$

Hence, when the group G is finitely generated, the asymptotic behavior of these three functions is the same. If G is not finitely generated, we have that

$$\mathrm{Føl}_{G,d} \preccurlyeq A_{G,d} \sim A^{\triangle}_{G,d}. \tag{2.3}$$

The next theorem is a slight modification of a result of Cavaleri [30, Theorem 4], and refers to Følner sets A satisfying the condition

$$|A \setminus gA| < \frac{1}{n}|A| \qquad \forall g \in B_{G,d}(n), \tag{2.4}$$

where the natural number n will be specified. In the following statement, $|A|_d$ denotes the maximum of $|a|_d$ over all the finitely many $a \in A$. Moreover, given a group G generated by the finite set X and the canonical projection $\pi\colon G \twoheadrightarrow G/N$ onto a quotient, with $\bar{X}$ we denote the finite set $\pi(X) \subseteq G/N$, which is a finite generating set for the quotient group.

Theorem 2.1.6. *Let G be an amenable group generated by the finite set X, let N be a normal subgroup of G, d be the restriction on N of the word metric associated to X, and fix a natural number $n \in \mathbb{N}$. Then there exists a $\frac{1}{n}$-Følner set in G for $B_{G,X}(n)$ that is of the form $A \cdot B \subseteq G$, where*

- *$A \subseteq G$ is such that $|A| = |A'|$ and $A \subseteq \pi^{-1}(A')$;*
- *$A' \subseteq G/N$ is a $\frac{1}{2n}$-Følner set for $B_{G/N,\bar{X}}(2n)$;*
- *$B \subseteq N$ is a $\frac{1}{n^*}$-Følner set for $B_{N,d}(n^*)$, where $n^* = \big(2n^2|A'|^2 \cdot (2|A'|_{\bar{X}} + n)\big)$.*

The modification is due to the fact that Cavaleri's Følner functions, like Erschler's, are with respect to the same finite set (which is the finite generating set of the group G) at each level $n \in \mathbb{N}$, while ours are with respect to the increasing family of finite sets $\{B_{G,d}(n)\}_{n\in\mathbb{N}}$. Up to noticing this fact, the proof of Theorem 2.1.6 is the one that can be found in [30, Theorem 4].

2.2 Residual properties

In this section we first give the general definition of residual property, and then specify it for residually finite groups. Hence, we introduce the Gruenberg condition.

Definition 2.2.1. Let $\mathcal{C}$ be a class of groups. A group G is *residually* $\mathcal{C}$ if for every non-trivial element $g \in G$ there exist a group $C \in \mathcal{C}$ and a surjective homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $\varphi(g) \neq e_C$.

This definition has several equivalent reformulations, of which we collect some in the following lemma. We refer to [31] for the proofs.

Lemma 2.2.2. *Let $\mathcal{C}$ be a class of groups and let G be a group. The following properties are equivalent:*

1. *the group G is residually $\mathcal{C}$;*
2. *for every non-trivial element $g \in G$ there exists a normal subgroup $N \trianglelefteq G$ such that $g \notin N$ and $G/N \in \mathcal{C}$;*
3. *the intersection of all normal subgroups N of G with $G/N \in \mathcal{C}$ is equal to $\{e_G\}$.*

If the class $\mathcal{C}$ is closed under taking subgroups, then each of the above properties is also equivalent to:

4. *there exist groups $C_i \in \mathcal{C}$ and a family of indices I such that G is a subgroup of $\prod_{i\in I} C_i$.*

When the class $\mathcal{C}$ is closed under taking subgroups, then in Definition 2.2.1 the requirement that φ is a *surjective* homomorphism may be dropped, as whenever there exists a homomorphism $\varphi\colon G \to C$ such that $\varphi(g) \neq e_C$, then the restriction $\varphi'\colon G \to \varphi(G)$ is surjective, does not map g to the trivial element of $\varphi(G)$, and $\varphi(G) \in \mathcal{C}$ being a subgroup of $C \in \mathcal{C}$.

Definition 2.2.3. Let $\mathcal{C}$ be a class of groups. A group G is *fully residually* $\mathcal{C}$ if for each finite subset $F \subseteq G \setminus \{e_G\}$ there exist a group $C \in \mathcal{C}$ and a surjective homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $\varphi\restriction_F$ is injective.

By definition, a fully residually $\mathcal{C}$ group is always residually $\mathcal{C}$, as one may consider finite subsets in G consisting of only one element. If, in addition, the class $\mathcal{C}$ is closed under taking finite direct products and subgroups, then also the converse is true.

Lemma 2.2.4. *Let $\mathcal{C}$ be a class of groups closed under taking finite direct products and subgroups. Then a group G is residually $\mathcal{C}$ if and only if it is fully residually $\mathcal{C}$.*

Proof. Suppose that G is residually $\mathcal{C}$, and that the class $\mathcal{C}$ is closed under finite direct products. We prove that G is fully residually $\mathcal{C}$.

Let $F \subseteq G$ be a finite subset, and consider

$$K := \{fh^{-1} \mid f, h \in F \text{ and } f \neq h\}.$$

The group G is residually $\mathcal{C}$ and thus for every element $k \in K$ there exist a group $C_k \in \mathcal{C}$ and a homomorphism $\varphi_k \colon G \to C_k$ such that $\varphi_k(k) \neq e_{C_k}$. As $\mathcal{C}$ is closed under finite direct products and K is finite, it follows that $C = \prod_{k \in K} C_k$ is a group belonging to $\mathcal{C}$.

Consider the group homomorphism $\varphi \colon G \to C$, defined as $\varphi(g) := \big(\varphi_k(k)\big)_{k \in K}$ for all $g \in G$. Suppose that f and h are distinct elements of F, so that $k = fh^{-1} \in K$. By definition we have that

$$\varphi_k(k) = \varphi_k\big(fh^{-1}\big) \neq e_{C_k},$$

hence $\varphi(fh^{-1}) \neq e_C$. As φ is a homomorphism, it follows that $\varphi(f) \neq \varphi(h)$.

As $\mathcal{C}$ is closed under taking subgroups, this is enough to prove that G is fully residually $\mathcal{C}$. □

We now collect two interesting examples. In the first one, we consider a class of groups $\mathcal{C}$ which is not closed under finite direct products, and we provide an example of a residually $\mathcal{C}$ group that is not fully residually $\mathcal{C}$.

In the second example, we consider a class $\mathcal{D}$ for which every group that is fully residually $\mathcal{D}$ is already a group belonging $\mathcal{D}$. This is not known to hold for residually $\mathcal{D}$ groups.

Example 2.2.5. Let $\mathcal{C}$ be the class of free groups, so that $\mathcal{C}$ is not closed under finite direct products and Lemma 2.2.4 cannot be applied. To see that the classes of residually free groups and fully residually free groups are distinct, notice that fully residually free groups are *commutative transitive* (see, for instance, [32, Corollary 2.11]). This means that, given three non-trivial elements a, b, c in a fully residually free group, if $[a, b] = [b, c] = e$ then also $[a, c] = e$.

Consider now the group $F_2 \times \mathbb{Z}$, the direct product of a non-abelian free group with an infinite cyclic group. Let a and b be generators for F_2, and let z be a generator for $\mathbb{Z}$.

The group $F_2 \times \mathbb{Z}$ is residually free, being the direct product of two residually free groups. Nevertheless, it is not commutative transitive. Indeed, we have that

$$[a, z] = [z, b] = e,$$

but the element $[a, b]$ is not trivial in $F_2 \times \mathbb{Z}$.

We deduce that $F_2 \times \mathbb{Z}$ is not a fully residually free group.

For the next example, we need the following definition. Let D be a division ring and G be a group. The *group ring* $D[G]$ is the ring defined as follows. As a set,

$$D[G] := \Big\{\sum_{g \in G} r_g g \mid g \in G, r_g \in D,\ r_g = 0 \text{ for all but finitely many } g\Big\}.$$

The sum is defined component-wise:

$$\Big(\sum_{g\in G} r_g g\Big) + \Big(\sum_{g\in G} s_g g\Big) := \sum_{g\in G}(r_g + s_g)g,$$

while the multiplication is

$$\Big(\sum_{g\in G} r_g g\Big) \cdot \Big(\sum_{g\in G} s_g g\Big) := \sum_{g\in G} t_g g, \qquad \text{where} \quad t_g = \sum_{h\in G} r_h s_{h^{-1}g}.$$

A ring (in particular, a group ring $D[G]$) is called *directly finite* if for every element x and y of the ring, the equality $xy = 1$ implies that also $yx = 1$.

Example 2.2.6. Let D be a division ring and consider the class

$$\mathcal{D} := \{\text{groups } G \text{ such that the group ring } D[G] \text{ is directly finite}\}.$$

Whether or not, given $G, H \in \mathcal{D}$, also the group $G \times H$ is directly finite is a well known open problem. This is related to the famous Kaplansky's direct and stable finiteness conjectures [75, pages 122 – 123] (see also [42, 43] for recent developments over the topic).

Nevertheless (as mentioned in [93]) if a group G is fully residually $\mathcal{D}$ then $G \in \mathcal{D}$. To prove this, let us consider two elements $x, y \in D[G]$ such that $xy = 1$. If $yx \neq 1$, then we can write

$$yx - 1 = \sum_{g\in G} a_g g, \qquad a_g \in D, \text{ with only finitely many } a_g \neq 0. \tag{2.5}$$

Let $F = \{g_1, \ldots, g_n\}$ be the set of all elements of G for which $a_g \neq 0$. As G is fully residually $\mathcal{D}$, there exist a group $Q \in \mathcal{C}$ and a homomorphism $\varphi\colon G \to Q$ such that $\varphi\restriction_F$ is an injective map. Moreover, the homomorphism $\varphi\colon G \to Q$ canonically induces a homomorphism $\Phi\colon D[G] \to D[Q]$. Let $\Phi(x) = \bar{x}$, $\Phi(y) = \bar{y}$ and $\Phi(g) = \varphi(g) = \bar{g}$ for all $g \in G$.

As φ is injective on the set F and the ring elements $\{a_{g_1}, \ldots, a_{g_n}\}$ are not equal to 0, from Equation (2.5) we derive that

$$\bar{y}\bar{x} - 1 = \sum_{i=1}^{n} a_{g_i}\bar{g}_i \neq 0 \tag{2.6}$$

in $D[Q]$.

As $xy = 1$ in $D[G]$, we have that $\Phi(xy) = \bar{x}\bar{y} = 1$ in $D[Q]$. But $D[Q]$ is directly finite, and hence also the equality $\bar{y}\bar{x} = 1$ holds. This contradicts Equation (2.6).

Hence, a group which is fully residually $\mathcal{D}$ is in fact already a group in $\mathcal{D}$.

Residually finite groups

Here we collect some useful properties of residually finite groups.

It is well known that free groups are residually finite. This can be proved in the following manner. Using the Ping-Pong Lemma, one sees that the free group $F_2 = F(a, b)$ on two generators is isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the two matrices

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The linear group $\mathrm{SL}_2(\mathbb{Z})$ is residually finite (consider the projections $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{N}$), and therefore it follows that F_2 is residually finite as well.

Every finitely generated free group is a subgroup of $F_2 = F(a,b)$ (see, for instance, [69, II.A.20], or consider the subgroup generated by the elements $\{b^i ab^{-i}\}_{i=1}^n$), and hence they are residually finite as well. Free groups which are not finitely generated (countable or not) are again residually finite, because any element w has only finitely many generators appearing in its reduced form, and the problem of finding a suitable finite quotient can be reduced to the residual finiteness of the finitely generated free group generated by all the generators appearing in the element w.

In fact, one can prove more than this:

Lemma 2.2.7. *Free groups are*

1. *residually {finitely generated torsion-free nilpotent} [87];*
2. *residually p-finite for all prime numbers p [64].*

We present here the famous result of Mal'cev [86] on split extensions of residually finite groups. We also provide a proof in order to shed light on how to exploit root properties to prove the permanence of residual properties. Moreover, we give the proof of an analogous (but not quite the same) result for residually amenable groups in Proposition 3.1.3.

Proposition 2.2.8. *Let G, H be two residually finite groups, let $\alpha\colon G \to \mathrm{Aut}(H)$ be a homomorphism and suppose that G is finitely generated. Then the semidirect product $G \ltimes_\alpha H$ is residually finite.*

Proof. Let (g,h) be a non-trivial element of $G \ltimes_\alpha H$. If $h \neq e_H$ then $\pi_2\big((g,h)\big) = h \neq e_H$. As H is residually finite, there exists a normal subgroup $N_h \trianglelefteq H$ of finite index such that $h \notin N_h$. Hence, $G \ltimes_\alpha H \twoheadrightarrow H/N_h$ is a surjective homomorphism onto a finite group that maps h non-trivially.

If $h = e_H$ then it must be that $g \neq e_G$. As G is residually finite, there exists a normal subgroup $N_g \trianglelefteq G$ of finite index such that $g \notin N_g$. As G is finitely generated, it has only finitely many subgroups of a given finite index. Let

$$K = \bigcap \big\{ N \trianglelefteq G \mid [G:N] = [G:N_g] \big\}.$$

As K is the intersection of finitely many finite index subgroups, it also has finite index in the group G, as $[C : A \cap B] \leqslant [C:A] \cdot [C:B]$. By construction it is a characteristic subgroup of G, and hence it is normal in $G \ltimes_\alpha H$.

Hence the group $(G \ltimes_\alpha H)/K$, isomorphic to $G/K \ltimes_\alpha H$, has a finite-index subgroup that is residually finite, and $(g, e_H)K$ is not the trivial element. A virtually residually finite group is itself residually finite, and hence there exists a finite quotient $G/K \ltimes_\alpha H \twoheadrightarrow Q$ such that the element $(g, e_H)K$ is mapped non-trivially into Q by the canonical projection.

This concludes the proof that $G \ltimes_\alpha H$ is residually finite. $\square$

We stress that the finite generation of the normal subgroup in Proposition 2.2.8 cannot be dropped, see Remark 4.3.5. Moreover, Proposition 2.2.8 does not extend to extensions that are not split, as the next Example shows.

Example 2.2.9. A famous example considered by Deligne [41] is the following. Let $n \geq 2$ be an integer and $\mathrm{Sp}_{2n}(\mathbb{Z})$ be the symplectic group defined as

$$\mathrm{Sp}_{2n}(\mathbb{Z}) := \Big\{ S \in \mathrm{SL}_{2n}(\mathbb{Z}) \mid S^t J S = J \Big\}, \qquad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

Let $\widetilde{\mathrm{Sp}_{2n}(\mathbb{R})}$ be the universal cover of $\mathrm{Sp}_{2n}(\mathbb{R})$, and consider the preimage of $\mathrm{Sp}_{2n}(\mathbb{Z})$ in it, which we denote by $\widetilde{\mathrm{Sp}_{2n}(\mathbb{Z})}$. It is proved in [41] that $\widetilde{\mathrm{Sp}_{2n}(\mathbb{Z})}$ is given by the following central[1] extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}_{2n}(\mathbb{Z})} \longrightarrow \mathrm{Sp}_{2n}(\mathbb{Z}) \longrightarrow 1. \tag{2.7}$$

[1] An extension $1 \to N \to G \to Q \to 1$ is called *central* if the image of N is contained in the center of G.

The group $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{R})$ is thus an extension of two finitely generated residually finite groups. Nevertheless the extension is not split, and the group $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{R})$ is *not* itself residually finite, as proved in [41].

Moreover, $\widetilde{\mathrm{Sp}_{2n}}(\mathbb{R})$ satisfies Kazhdan's property (T) [17, Example 1.7.13 (iii)], and this immediately implies that it is not a residually amenable group.

Gruenberg condition and root classes

In 1957, Gruenberg [64] introduced the concept of *root property*, or *root class* of groups, to give a general condition ensuring that the free product of residually $\mathcal{C}$ groups is again residually $\mathcal{C}$.

Definition 2.2.10. A non-trivial class $\mathcal{R}$ is a root class if

1. $\mathcal{R}$ is closed under taking subgroups;
2. $\mathcal{R}$ is closed under taking finite direct products;
3. *Gruenberg Condition*: for any chain $K \trianglelefteq H \trianglelefteq G$ such that G/H and H/K are in $\mathcal{R}$, there exists $L \trianglelefteq G$ such that L is contained in K and $G/L \in \mathcal{R}$.

Finite groups, finite p-groups and solvable groups are known to satisfy the Gruenberg condition [64], while nilpotent groups do not satisfy it. Using the fact that finite groups form a root class, Gruenberg proved that the free product of residually finite groups is residually finite [64, Theorem 4.1].

In the following lemma we collect some easy properties implied by the Gruenberg condition. The second claim is false in general if the class $\mathcal{C}$ does not satisfy the Gruenberg condition. Indeed, Proposition 3.1.7 and Example 3.1.6 provide a counterexample, when $\mathcal{C} = \{\text{amenable groups}\}$. The second fact is proved in [64, Lemma 1.5].

Lemma 2.2.11. *If a non-trivial class $\mathcal{R}$ satisfies the Gruenberg condition then:*

1. *$\mathcal{R}$ is closed under extensions;*
2. *a group which is (residually $\mathcal{R}$)-by-$\mathcal{R}$ is residually $\mathcal{R}$;*
3. *free groups are residually $\mathcal{R}$.*

Proof. To prove that $\mathcal{C}$ is closed under taking extensions, consider a group G with a normal subgroup N such that N and G/N are in $\mathcal{R}$. Apply the Gruenberg condition to the subnormal chain $\{e\} \trianglelefteq N \trianglelefteq G$ to conclude that $G \in \mathcal{R}$.

For the second claim, let G be a (residually $\mathcal{R}$)-by-$\mathcal{R}$ group, with a normal residually $\mathcal{R}$ subgroup N such that $G/N \in \mathcal{R}$.

Consider an element $g \in G \setminus \{e_G\}$. If $g \notin N$ then the canonical projection $\pi\colon G \twoheadrightarrow G/N$ maps g to the non-trivial element $gN \in G/N \in \mathcal{R}$. If $g \in N$ then there exist a group $C \in \mathcal{R}$ and a surjective homomorphism $\varphi\colon N \twoheadrightarrow C$ such that $\varphi(g) \neq e_C$, that is, $g \notin K = \ker\varphi$. As φ is surjective, we have that $N/K \cong C \in \mathcal{R}$, so in the subnormal chain $K \trianglelefteq N \trianglelefteq G$ both quotients are groups in $\mathcal{R}$.

Thus, applying the Gruenberg condition, we obtain a subgroup $L \trianglelefteq G$ contained in K such that $G/L \in \mathcal{R}$. As $L \subseteq K$, we have that $g \notin L$, and the canonical projection $G \twoheadrightarrow G/L$ does not map g to the identity of G/L. This means that G is residually $\mathcal{R}$.

To conclude, let $G \in \mathcal{R}$ be a non-trivial group and g be a non-trivial element of G. Then either g has infinite or finite order. In the first case, $\mathcal{R}$ contains all finitely generated torsion-free nilpotent groups. In the second, $\mathcal{R}$ contains all finite p-groups, for p a prime dividing the order of g.

Free groups are both residually finitely generated torsion-free nilpotent and residually p-finite, as Lemma 2.2.7 recalls. Hence, it follows that they are residually $\mathcal{R}$. $\square$

In particular, if $\mathcal{R}$ satisfies the Gruenberg condition then $\mathcal{R}$ is closed under taking finite direct products. This shows that the second condition in the definition of root class is redundant.

We take advantage of the following characterisation of the Gruenberg condition, which is provided in [109, Theorem 1].

Theorem 2.2.12. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups. The following conditions are equivalent:*

1. *$\mathcal{C}$ satisfies the Gruenberg condition;*
2. *$\mathcal{C}$ is closed under taking unrestricted wreath products;*
3. *$\mathcal{C}$ is closed under taking extensions and for any two groups $G, H \in \mathcal{C}$ the group $\prod_{h\in H} G$ is again in $\mathcal{C}$.*

2.3 Local embeddings

A related notion is the one of local embeddability, which is a local form of a residual property. An established terminology for the following concepts is the one of [31], in particular of Sections 7.1 and 7.2. In this thesis we follow instead a terminology that is closer to the one of [4].

Definition 2.3.1. Let G and C be two groups and $K \subseteq G$ a finite subset. A map $\varphi\colon G \to C$ is called a K-*approximation* if it satisfies the following conditions:

($a1$) (*almost homomorphism*) $\varphi(k_1k_2) = \varphi(k_1)\varphi(k_2)$ for all k_1, $k_2 \in K$;

($a2$) $\varphi\restriction_K$ is injective.

With this definition in mind, we define the concept of local embeddability into a class of groups $\mathcal{C}$.

Definition 2.3.2. Let $\mathcal{C}$ be a class of groups. A group G is *locally embeddable into the class* $\mathcal{C}$ (or G is an LE-$\mathcal{C}$ group) if for each finite subset $K \subseteq G$ there exist a group $C \in \mathcal{C}$ and a K-approximation $\varphi\colon G \to C$.

Restricting this definition to $\mathcal{C} = \{\text{finite groups}\}$, we obtain the concept of *LEF groups*, locally embeddable into finite groups, which were defined by Vershik and Gordon in [115]. If $\mathcal{C} = \{\text{amenable groups}\}$, then we get the notion of groups which are locally embeddable into amenable groups, also known as *LEA groups*. These groups were considered in 1999 by Gromov [62, page 133] in his work towards the famous Gottschalk's surjunctivity conjecture, and called initially subamenable groups.

With very mild assumptions on the class of group $\mathcal{C}$, we have that

$$\text{residually } \mathcal{C} \quad \Rightarrow \quad \text{LE-}\mathcal{C}. \tag{2.8}$$

Indeed, this is the case if the class $\mathcal{C}$ is closed under finite direct products [31, Corollary 7.1.14].

We record here that, if the class $\mathcal{C}$ is closed under taking subgroups, then a finitely presented LE-$\mathcal{C}$ group is residually $\mathcal{C}$ [31, Corollary 7.1.22], and hence the implication of Equation (2.8) can be reversed.

Remark 2.3.3. Deligne's group considered in Example 2.2.9 is finitely presented and is not residually amenable. Hence, it is not an LEA group.

The following example, which can be found in [31, Proposition 7.3.9], provides a finitely generated LEF group that is not residually finite. In particular, this group is not finitely presented.

Example 2.3.4. The subgroup $G = \langle a, T \rangle \leqslant \mathrm{Sym}(\mathbb{Z})$, where a is the transposition $(0\ 1)$ and T is the translation $T\colon n \mapsto n+1$, is LEF. The translation T shifts a, so that $T^i a T^{-i} = (i\ i+1)$ is an element of G. The set of transpositions $\{(i\ i+1)\}_{i\in\mathbb{Z}}$ generates $\mathrm{Sym}_0(\mathbb{Z})$, which contains the infinite simple subgroup $\mathrm{Alt}(\mathbb{Z})$. Hence G, containing an infinite simple group, cannot be residually finite.

The following proposition can be found in [31, Proposition 7.1.6].

Proposition 2.3.5. *Let G be a group and $\mathcal{C}$ be a class that is closed under taking subgroups. The following conditions are equivalent:*

- *G is LE-$\mathcal{C}$;*
- *for every finite $K \subseteq G$ there exists $K \subseteq C \subseteq G$ and a binary operation $\diamond\colon C \times C \to C$ such that $(C, \diamond) \in \mathcal{C}$ and $k_1 \diamond k_2 = k_1 k_2$ for all $k_1, k_2 \in K$.*

As a consequence, it produces an interesting characterisation of LEA groups, which we record in the following proposition:

Proposition 2.3.6. *Let G be a group. The following conditions are equivalent:*

- *G is LEA;*
- *for every finite $K \subseteq G$ there exists $K \subseteq A \subseteq G$ and a binary operation $\diamond\colon A \times A \to A$ such that $(A, \diamond)$ is an amenable group and $k_1 \diamond k_2 = k_1 k_2$ for all $k_1, k_2 \in K$.*

Chapter 3

Free products

In this chapter we prove that the class of residually amenable groups is closed under taking free products. Our approach is more general, and we present sufficient conditions, for a class of groups $\mathcal{C}$, implying that the class of residually $\mathcal{C}$ groups is closed under taking free products. In the first section we collect basic properties for the class of residually amenable groups, and a nice counterexample of two nilpotent groups whose free product fails to be residually nilpotent.

The content of the last two sections appear in the published paper [19].

3.1 Basic permanence properties

Specialising Definition 2.2.1, we say that a group G is *residually amenable* if for every non-trivial element $g \in G$ there exist an amenable group A and a surjective homomorphism $\varphi\colon G \to A$ such that $\varphi(g) \neq e_A$. As noted in the previous chapter, being the class of amenable groups closed under taking subgroups, we may drop the "surjectivity"condition without any change in the notion of residual amenability.

The proof of the following lemma can be adapted from [31, §2.2].

Lemma 3.1.1. *The class of residually amenable groups is closed under taking subgroups, direct products, direct sums, inverse limits.*

Lemma 3.1.2. *Let G be a virtually residually amenable group. Then G is residually amenable.*

Proof. Let $H \leqslant G$ be a residually amenable subgroup of finite index of G. Then there exists $N \trianglelefteq G$ of finite index in G such that $N \leqslant H$. Hence N is residually amenable.

Let $g \in G \setminus \{e\}$ be a non-trivial element, we want to find a normal subgroup of G not containing g such that the quotient is amenable. If $g \in G \setminus N$ then it is sufficient to consider the finite quotient G/N.

Suppose that $g \in N \setminus \{e\}$. As N is residually amenable, there exists $K \trianglelefteq N$ such that $g \notin K$ and N/K is amenable. Let

$$G/N = \{g_1N, g_2N, \ldots, g_rN\}$$

and define $L := \bigcap_{i=1}^{r} g_iKg_i^{-1}$. Then $L \trianglelefteq G$, $g \notin L$ and N/L has finite index in G/L.

We now show that N/L is amenable, that is, G/L is virtually amenable. In fact, as $N \trianglelefteq G$, we have that

$$g_iKg_i^{-1} \leqslant g_iNg_i^{-1} = N, \qquad \forall i = 1, \ldots, r,$$

and so every conjugate $g_iKg_i^{-1}$ is again a normal subgroup of N. It follows that

$$N/L \leqslant \left(N/g_1Kg_1^{-1}\right) \times \cdots \times \left(N/g_rKg_r^{-1}\right) \cong \left(N/K\right)^r,$$

Thus N/L is amenable because it is a subgroup of an amenable group.

Hence G/L is indeed virtually amenable, that is, amenable. This concludes the proof, as we have shown that for every non-trivial element g there exists $L \trianglelefteq G$ which does not contain g and with amenable quotient G/L. □

Following the proof of Proposition 2.2.8, from Lemma 3.1.2 we obtain the following:

Proposition 3.1.3. *Let G be a finitely generated, residually finite group, and H be a residually amenable group. Let $\alpha\colon G \to \mathrm{Aut}(H)$ be a homomorphism. Then the semidirect product $G \ltimes_\alpha H$ is residually amenable.*

We collect here an instructive, negative result: the free product of nilpotent groups may fail to be residually nilpotent.

Proposition 3.1.4. *The free product of the nilpotent groups $\langle a \mid a^2\rangle$ and $\langle b \mid b^3\rangle$ is not residually nilpotent. In particular, the class of residually nilpotent groups is not closed under taking free products.*

Proof. Suppose that the free product $G = \langle a, b \mid a^2, b^3\rangle$ is residually nilpotent, and in particular residually {finitely generated nilpotent}. A finitely generated nilpotent group is residually finite [72]. Moreover, quotients of nilpotent groups are again nilpotent, hence a finitely generated nilpotent group is residually {finite nilpotent}. We will use the fact that a finite group is nilpotent if and only if it isomorphic to the direct product of its p-Sylow subgroups.

Suppose that the element $g = [a, b] \in G$ is mapped non-trivially, by the surjective homomorphism π, into a finite nilpotent quotient of G. As the element a has order two in G, and b has order three, it follows that either the order of $\pi(a)$ or the order of $\pi(b)$ will be coprime with p, for each p-Sylow subgroup appearing in the direct product decomposition of the finite nilpotent group. In any case, it must be that the element $[a, b]$ is trivial, and this contradicts the assumption that the free product is residually nilpotent. □

Proposition 3.1.4 shows that the free product of nilpotent groups is, in general, not residually nilpotent. Nevertheless, the result can be proved if we restrict our attention to torsion-free nilpotent groups [85], as recorded in Theorem 3.3.7.

We now prove that the class of amenable groups is *not* a root class. The following proposition, jointly with Theorem 2.2.12, already implies the claim. In Example 3.1.6 we present a group which is an extension of a residually amenable group by an amenable group, and prove that the group itself is not residually amenable, contradicting Gruenberg's condition (confront Definition 2.2.10).

Proposition 3.1.5. *The group $\prod_{\mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z})$ has free non-abelian subgroups. In particular it is not amenable.*

Proof. We produce an embedding of the free group F_2 on two generators into $\prod_{\mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z})$. Let a and b be two letters, and consider the group $F_2 = F(a, b)$. This group is residually finite, so for all $n \in \mathbb{N}$ there exists a finite quotient Q_n of F_2 in which the ball $B_{F_2,\{a,b\}}(n)$ is injected, that is π_n restricted to the finite set $B_{F_2,\{a,b\}}(n)$ is an injective map, where $\pi_n\colon F_2 \twoheadrightarrow Q_n$ is the canonical projection onto the quotient.

For all $n \in \mathbb{N}$ fix a bijection $b_n\colon Q_n \to Z_n$ of Q_n with a finite set $Z_n \subseteq \mathbb{Z}$. As the groups Q_n are finite, they are injected into $\mathrm{Sym}_0(\mathbb{Z})$ via the injective homomorphism $\iota_n \circ L_n$

$$Q_n \xrightarrow{L_n} \mathrm{Sym}(Q_n) \xrightarrow{\iota_n} \mathrm{Sym}_0(\mathbb{Z}),$$

where the permutation $L_n(q)\colon Q_n \to Q_n$ is the left-multiplication in Q_n induced by the element $q \in Q_n$, and $\iota_n\colon \mathrm{Sym}(Q_n) \to \mathrm{Sym}_0(\mathbb{Z})$ is the injective homomorphism that identifies the set Q_n with the set $Z_n \subseteq \mathbb{Z}$. It is defined as

$$\big(\iota_n(\sigma)\big)(z) = \begin{cases} b_n\big(\sigma(b_n^{-1}(z)\big) & \text{if } z \in Z_n, \\ z & \text{if } z \notin Z_n, \end{cases} \qquad \forall \sigma \in \mathrm{Sym}(Q_n),\ \forall z \in \mathbb{Z}.$$

The injective homomorphism $L_n\colon Q_n \to \mathrm{Sym}(Q_n)$ is classically called the *Cayley map*.

Proceding in this manner, we see that for all $n \in \mathbb{N}$ the ball $B_{F_2,\{a,b\}}(n)$ is mapped injectively by $\iota_n \circ L_n \circ \pi_n$ into the n-th copy of $\mathrm{Sym}_0(\mathbb{Z})$. Let $\varphi_n\colon F_2 \to \mathrm{Sym}_0(\mathbb{Z})$ denote the composition $\pi_n \circ \iota_n \circ L_n$, and let $\Phi\colon F_2 \to \prod_{z\in\mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z})$ be the homomorphism defined as

$$\Phi(x) := \big(\dots, 0, 0, \varphi_0(x), \varphi_1(x), \varphi_2(x), \dots, \varphi_n(x), \dots\big) \in \prod_{\mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z}), \qquad \forall x \in F_2,$$

where $\varphi_i(x)$ is in the i-th coordinate.

By construction Φ is an injective homomorphism, and hence $\Phi(F_2) \cong F_2$ is a free non-abelian subgroup of $\prod_{\mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z})$. □

Thus, there exist (elementary) amenable groups G such that the direct product $\prod_{\mathbb{Z}} G$ contains free non-abelian subgroups. Notice that this cannot occur if G is finite, as in this case the group $\prod_{\mathbb{Z}} G$ is locally finite, and hence elementary amenable.

With the following example, we show that the Gruenberg condition does not hold for the class of (elementary) amenable groups. This means that amenable groups, and elementary amenable groups, do not form a root class.

Example 3.1.6. Let $A = \mathrm{Sym}_0(\mathbb{Z})$, consider the unrestricted wreath product $G = A \wr \mathbb{Z}$, and define

$$N = \{(a_x)_{x\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid a_0 = e_A\} \trianglelefteqslant A^{\mathbb{Z}}. \tag{3.1}$$

Then the subnormal chain of subgroups $N \trianglelefteqslant A^{\mathbb{Z}} \trianglelefteqslant G$ does not satisfy the Gruenberg condition, and the group G is not residually amenable.

We write $\mathbb{Z}$ additively, so that if $z \in \mathbb{Z}$ then its inverse is $-z$, and $0 \in \mathbb{Z}$ is the identity element. Notice that the two quotients $G/A^{\mathbb{Z}} \cong \mathbb{Z}$ and $A^{\mathbb{Z}}/N \cong A$ are both elementary amenable. Suppose now that the Gruenberg condition holds, so that there exists a normal subgroup $L \trianglelefteqslant G$ such that $L \subseteq N$ and G/L is amenable.

As $L \trianglelefteqslant G$, we have that

$$L \subseteq \bigcap_{\substack{g=(e,z)\\ z\in\mathbb{Z}}} N^g.$$

Let $g = (e, z)$, then

$$\begin{aligned} N^g &= \{(e,z)\big((a_x)_{x\in\mathbb{Z}}, 0\big)(e,z)^{-1} \mid (a_x) \in N\} = \big\{\big((a_{x-z})_{x\in\mathbb{Z}}, z\big)(e,-z) \mid (a_x) \in N\big\} \\ &= \big\{\big((a_{x-z})_{x\in\mathbb{Z}}, 0\big) \mid (a_x) \in N\big\} = \{(a_x)_{x\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid a_z = e_A\}, \end{aligned}$$

thus L is contained in

$$\{e_{A^{\mathbb{Z}}}\} = \bigcap_{z\in\mathbb{Z}} \{(a_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}} \mid a_z = e_A\},$$

that is, L is the trivial subgroup. This implies that $G/L = G$ is amenable, and this contradicts Proposition 3.1.5.

We now prove that G is not residually amenable. Consider the alternating group $B = \mathrm{Alt}(\mathbb{Z}) \leqslant A$, which is an infinite simple group, and it has index two in A [31, C.4]. The quotient group $A^{\mathbb{Z}}/B^{\mathbb{Z}} \cong \mathbb{Z}_2^{\mathbb{Z}}$ is amenable because it is abelian. This implies that $B^{\mathbb{Z}}$ is not amenable, because amenability is closed under extensions and $A^{\mathbb{Z}}$ is not amenable by Proposition 3.1.5.

The group $B \wr \mathbb{Z}$ is a subgroup of $G = A \wr \mathbb{Z}$, so if $B \wr \mathbb{Z}$ is not residually amenable then neither is G. Let $N \trianglelefteqslant B \wr \mathbb{Z}$ be a normal subgroup with amenable quotient, then the normal subgroup $N \cap B^{\mathbb{Z}} \trianglelefteqslant B \wr \mathbb{Z}$ is not trivial. Let $e_G \neq \big((a_x), 0\big) \in N \cap B^{\mathbb{Z}}$, so that $a_{x'} \neq e_B$ for some $x' \in \mathbb{Z}$. The center of B is trivial because B is simple and non-abelian, so there exists $b \in B$ such that $a := [b, a_x]$ is not the trivial element of B.

Consider the new element $\beta = (\beta_x)_{x \in \mathbb{Z}} \in B^{\mathbb{Z}}$ defined as

$$\beta_x = \begin{cases} e_B & \text{if } x \neq x', \\ b & \text{if } x = x', \end{cases}$$

then $\alpha = (\alpha_x)_{x \in \mathbb{Z}} = [(a_x), \beta]$ is an element of $B^{\mathbb{Z}}$ with only one non-trivial component. Indeed, we have that

$$\alpha_x = \begin{cases} e_B & \text{if } x \neq x', \\ a & \text{if } x = x'. \end{cases}$$

An easy computation shows that

$$\big((g_x), z\big)\big((\alpha_x), 0\big)\big((g_x), z\big)^{-1} = \big((g_x \alpha_{x-z} g_x^{-1})_{x \in \mathbb{Z}}, 0\big), \qquad \forall \big((g_x), z\big) \in B \wr \mathbb{Z}.$$

These elements lie in $N \cap B^{\mathbb{Z}}$, because $N \cap B^{\mathbb{Z}} \trianglelefteqslant B \wr \mathbb{Z}$.

As $e_B \neq a \in B$ and B is simple, we have that

$$\{hah^{-1} \mid h \in B\} = B$$

and that

$$\begin{aligned} B^{\mathbb{Z}} &\supseteq N \cap B^{\mathbb{Z}} \supseteq \big\{(g, z)(\alpha, 0)(g, z)^{-1} \mid (g, z) \in B \wr \mathbb{Z}\big\} \\ &= \big\{(g \sigma_z(\alpha) g^{-1}, 0) \mid (g, z) \in B \wr \mathbb{Z}\big\} = \big\{\sigma_z(g \alpha g^{-1}) \mid (g, z) \in B \wr \mathbb{Z}\big\} = B^{\mathbb{Z}}, \end{aligned}$$

that is, $N \cap B^{\mathbb{Z}} = B^{\mathbb{Z}}$.

This means that $B \wr \mathbb{Z}$ is not residually amenable, because for the elements $(b, 0) \in B \wr \mathbb{Z}$ there is no normal subgroup N with amenable quotient, such that $(b, 0) \notin N$. Hence the group

$$A \wr \mathbb{Z} = \Big(\prod_{z \in \mathbb{Z}} \mathrm{Sym}_0(\mathbb{Z})\Big) \rtimes \mathbb{Z}$$

is not residually amenable.

The group in Example 3.1.6 is an extension of a residually amenable group by an amenable one, but it is not residually amenable. This highlights the difference between residual amenability and residual finiteness: as we saw in Lemma 2.2.11 any extension of a residually finite group by a finite one is again residually finite.

Therefore:

Lemma 3.1.7. *The class of residually amenable groups is not closed under extensions by amenable groups. The class of residually elementary amenable groups is not closed under of extensions by elementary amenable groups.*

3.2 Groups with no finite quotients

In this section we collect some examples of finitely generated groups without[1] finite quotients. As a finitely generated elementary amenable group does have finite quotients, it follows that these groups with no finite quotient are not residually elementary amenable in a strong sense.

Lemma 3.2.1. *Let G be a finitely generated group without finite quotients. Then the only elementary amenable quotient of G is the trivial group and, hence, G is not residually elementary amenable.*

Proof. The claim follows from Lemma 2.1.4. □

The following proposition is of particular interest for Higman's group, which will be described below.

Proposition 3.2.2. *Let G be a finitely presented group without finite quotients. Then the only LEF quotient of G is the trivial group.*

Proof. Suppose there exists a surjective homomorphism $\pi\colon G \twoheadrightarrow Q$, where Q is a non-trivial LEF group. As G is finitely generated, so is Q. Let $G = \langle X_G \mid R_G \rangle$ and $Q = \langle X_Q \mid R_Q \rangle$, where $R_Q = R_G \sqcup R'_Q$, X_G is finite and $X_Q = \pi(X_G)$. The group Q is non-trivial and generated by the set $\pi(X_G)$, so at least one of the elements of X_G is mapped non-trivially.

Let $F = F(X_G)$ be a free group freely generated by the finite set X_G, and consider the set of reduced words $R_G = \{r_1, \dots, r_s\} \subseteq F$. As G is finitely presented and hence R_G is a finite set, there exists

$$M := \max\{|r_i|_F \mid r_i \in R_G\}.$$

Let $B = B_{Q,X_Q}(M)$ be the ball in Q of radius M. This subset is finite as Q is finitely generated and so, by hypotheses, there exists a B-approximation $\varphi\colon Q \to A$ into a finite group A.

Consider the map $\Phi\colon G \to A$ defined as

$$\Phi(w) := \varphi(\pi(x_1)) \dots \varphi(\pi(x_r)) \qquad \forall\, w = x_1 \dots x_r \in G,\, x_i \in X_G \cup X_G^{-1}.$$

As $\pi(X_G) \subseteq X_Q$, we have that $\pi\big(B_{G,X_G}(n)\big) \subseteq B$. Thus

$$\Phi(xy) = \varphi(\pi(x))\varphi(\pi(y)) = \Phi(x)\Phi(y) \qquad \forall x, y \in X_G \cup X_G^{-1},$$

and $\Phi\restriction_{B_{G,X_G}(M)} = \varphi \circ \pi$.

By construction, we have that Φ maps all the relators of R_G into the trivial element. Hence, $\Phi\colon G \to A$ is a homomorphism from G to a finite group A. By hypotheses, it must be that $\Phi(G) = \{e_A\}$, contradicting the injectivity of $\varphi\restriction_B$.

Hence, the proposition is proved. □

Example 3.2.3. Higman's group is given by the presentation

$$H := \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle. \tag{3.2}$$

It was defined in [71], where it is proved that H is infinite and has no finite quotients. By Lemma 3.2.1, Higman's group has no elementary amenable quotients either. Nevertheless, it is *SQ-universal* [108], that is, every countable group C is a subgroup of some quotient of H.

A recent major advance in its understanding was obtained in [89], where Martin proved several fascinating results about this group: every non-trivial endomorphism of Higman's group is in fact an automorphism, and hence the group is both hopfian and cohopfian; the automorphisms of the group, up the cyclic permutations of the four generators, are just inner automorphisms.

[1] From now on, for a "group without finite quotients" we mean a group whose only finite quotient is the trivial group.

Applying Proposition 3.2.2 we exclude the possibility for H to surject onto the simple infinite amenable LEF groups constructed as commutator subgroups of topological full groups of Cantor systems. Matui [90] provided a proof for simplicity, Grigorchuk and Medynets [61] for LEF, whilst amenability was proved by Juschenko and Monod [79].

Corollary 3.2.4. *Higman's group does not project onto LEF groups, and in particular onto the simple amenable LEF groups constructed as commutator subgroups of full topological groups of Cantor systems.*

This leads to the following well-known open question:

Question 3.2.5. Does Higman's group admit amenable quotients?

To the author's knowledge, there is no known example in the literature of amenable groups without any LEF quotient, and we wonder if such groups can exist at all.

Question 3.2.6. Does there exist a finitely generated amenable group without any LEF quotient?

If the answer to Question 3.2.6 is no, that is, all finitely generated amenable groups have non-trivial LEF quotients, then it would follow that Higman group does not admit any amenable quotient.

A slight variation of this construction of Higman yields finitely presented groups which indeed do not have any amenable quotient group [92]. One of the key features of [92] is to consider infinite groups which have Kazhdan's property (T) (instead of the groups $\mathbb{Z}$), and starting from these apply the construction of Higman of Equation (3.2).

The following construction is due to Baumslag and Miller III [14, Theorem D]. It is a finitely presented group G that contains a finitely generated non-trivial subgroup K with the property that $K \cong K \times K$. Other finitely generated groups that are isomorphic to their own square are constructed in [78].

Example 3.2.7. To construct this group G, consider

$$A = \langle a, h, t \mid [a, h], t^{-1}a^2t = a^3, t^{-1}h^2t = h^3\rangle, \qquad B = \langle b, s \mid s^{-1}b^2s = b^3\rangle.$$

Notice that $B = \mathrm{BS}(2, 3)$ is a Baumslag-Solitar group, whilst A is a two-dimensional variation of it.

The subgroups $\langle t, [a, t^{-1}at]\rangle$ of A and $\langle s, [b, s^{-1}bs]\rangle$ of B are free groups on two generators. The finitely presented amalgamated free product

$$C = \langle A, B \mid t = [b, s^{-1}bs], [a, t^{-1}at] = s\rangle$$

has the peculiar property of admitting a surjective homomorphism onto $C \times C$, [14, Theorem C]. This is achieved considering the surjective (but not injective) homomorphism $\vartheta\colon C \twoheadrightarrow C_0 \times C_1$ defined on the generators of C to be

$$\vartheta(a) = a_0a_1, \quad \vartheta(b) = b_0b_1, \quad \vartheta(s) = s_0s_1, \quad \vartheta(t) = t_0t_1, \quad \vartheta(h) = a_1^{-1}h_0^2h_1^2,$$

where C_0 and C_1 are two isomorphic copies of C and, if $x \in C$, then x_0 and x_1 represent the image of x in C_0 and C_1 respectively. Let λ be the automorphism of $C_0 \times C_1$ that switches the element x_0y_1 into the element x_1y_0.

The Baumslag-Miller III group G is defined to be a double HNN extension of the group $C_0 \times C_1$:

$$G := \langle C_0 \times C_1, p, q \mid p^{-1}xp = \lambda(x) \quad \forall x \in C_0 \times C_1, \quad q^{-1}yq = \vartheta(y) \quad \forall y \in C_0\rangle.$$

Since ϑ is not injective, C_0 and C_1 are not injected in G. Let K_i be the image of C_i in G, and let $K = \langle K_0, K_1\rangle \cong K_0 \times K_1$. As the group C is finitely generated, also K_0, K_1, and K are.

As $C_0 \cong C_1$, we have that $K_0 \cong K_1$. Moreover, $K \cong K_0$ [14, Theorem D], and thus $K \cong K \times K$.

We cannot expect a result as strong as Proposition 3.2.2 for the Baumslag-Miller III group, because it maps onto the free group $\langle p, q\rangle$, which is residually finite. Nevertheless, the same argument gives the following:

Proposition 3.2.8. *Let G be the Baumslag-Miller III group described in Example 3.2.7. Suppose that $\pi\colon G \twoheadrightarrow Q$ is a surjective homomorphism with Q a LEF group. Then $K \subseteq \ker\pi$.*

Proof. Following the argument of the proof of Proposition 3.2.2, we can construct a homomorphism $\Phi\colon G \to L$ from G to a finite group L. To prove that the homomorphism $\Phi\restriction_K\colon K \to L$ has trivial image, we exploit the fact that $K \cong K \times K$ and that K is finitely generated. As recorded in [14, Proposition 2], suppose that there exists a subgroup of a given finite index. As K is finitely generated, it has only finitely many, say k, subgroups of this given index, but the isomorphism $K \cong K\times K$ implies that K should also have two times this finite number of subgroups. This can happen only if k is equal to zero, that is, there is no subgroup of finite index. □

The final example of this section comes from classical small cancellation theory [101, §2].

Example 3.2.9. Let a and b be two letters, for all $n \in \mathbb{N}$ let U_n, V_n be words in $\{a^n, b^n\}$, and consider the group

$$G = \langle a, b \mid aU_1, bV_1, aU_2, bV_2, \dots, aU_n, bV_n, \dots\rangle. \tag{3.3}$$

To ensure that the group G obtained in this way is not trivial, we need to rely on small cancellation theory: the set of words $\{U_n, V_n\}_{n\in\mathbb{N}}$ can be chosen is such a way that the resulting group G is an infinite small cancellation group. Now, suppose that N is a finite-index normal subgroup of G. As G/N is a finite group by assumption, there exists $m \in \mathbb{N}$ such that a^m and b^m are elements of N. This implies that the words U_m and V_m become trivial in the quotient group G/N, and hence the generators a and b are too, by means of Equation (3.3). This surely implies that the quotient G/N is trivial, and hence that the group G has no finite quotient. Nevertheless, notice that G is SQ-universal [63].

Every truncated presentation

$$G_n = \langle a, b \mid aU_1, bV_1, aU_2, bV_2, \dots, aU_n, bV_n\rangle$$

yields a finitely presented residually finite group (confront, for instance, [8, Example (i),§1]). Therefore, the group G is a limit, in the space of marked groups, of residually finite groups [32, §2.3, (a)]. A residually finite group is in turn limit of its finite quotients, and therefore G is a limit of finite groups in the space of marked groups.

This implies that G is a finitely generated LEF group, and in particular LEA. Nevertheless, it has no finite quotient.

We end this section with the following intriguing question:

Question 3.2.10. Does there exist a residually amenable non-simple group without finite quotients?

The requirement "non-simple" is necessary to discard the simple infinite amenable groups already discussed in this section, whilst the question should be read in the context of groups with abundantly many amenable quotients, none of which is finite.

3.3 Free products of residually $\mathcal{C}$ groups

We saw in the first section of this chapter that amenability is not a root property, thus [64, Theorem 4.1] cannot be applied to show that residual amenability is preserved under taking free products. We adopt here a general strategy that dodges this problem, and allows us to prove that residual amenability is indeed preserved by taking free products. In what follows, a root class is generally denoted by $\mathcal{R}$, to distinguish it from $\mathcal{C}$ (which, in the applications, will be the class of amenable groups).

Lemma 3.3.1. *Let $\mathcal{C}$ be a class of groups that contains a root class $\mathcal{R}$. Suppose that*

1. *$\mathcal{C}$ is closed under taking subgroups;*
2. *every $\mathcal{R}$-by-$\mathcal{C}$ group belongs to $\mathcal{C}$;*
3. *for every group in $\mathcal{C}$ there is a group in $\mathcal{R}$ of the same cardinality.*

Then a (residually $\mathcal{R}$)-by-$\mathcal{C}$ group is residually $\mathcal{C}$.

Proof. Let G be a (residually $\mathcal{R}$)-by-$\mathcal{C}$ group, with a residually $\mathcal{R}$ normal subgroup N such that $G/N \in \mathcal{C}$.

Consider a non-trivial element $g \in G$. If $g \notin N$ then g is not mapped to the identity of G/N by the canonical projection. If $g \in N \setminus \{e\}$, then there exists a normal subgroup $K \trianglelefteq N$ such that $g \notin K$ and $N/K \in \mathcal{R}$. Let S be a set of representatives of N in G, then the group $G/N \in \mathcal{C}$ has the same cardinality of S. The hypotheses imply that there exists a group $\Gamma \in \mathcal{R}$ of the same cardinality of G/N, so

$$|\Gamma| = |G/N| = |S|.$$

Consider the normal subgroup $L := \bigcap_{s\in S} K^s$ of G, then

$$N/L \hookrightarrow \prod_{s\in S} N/K^s \cong \prod_{s\in S} N/K.$$

The third condition of Theorem 2.2.12 implies that $\prod_{s\in S} N/K \in \mathcal{R}$, because $|S| = |\Gamma|$ and $\Gamma \in \mathcal{R}$. As $\mathcal{R}$ is closed under taking subgroups, this implies that $N/L \in \mathcal{R}$.

By the Third Isomorphism Theorem for groups

$$\frac{G/L}{N/L} \cong G/N,$$

so the group G/L is the extension of the group N/L, which is in $\mathcal{R}$, by the group G/N, which is in $\mathcal{C}$. Thus $G/L \in \mathcal{C}$.

As $L \leqslant N$ we have that $g \notin L$. Thus, the surjective homomorphism

$$\varphi\colon G \twoheadrightarrow G/L, \qquad \varphi(g) := gL \quad \forall g \in G$$

maps g to a non-trivial element of G/L, and G is residually $\mathcal{C}$. □

Corollary 3.3.2. *Let $\mathcal{C}$ be a class of groups that contains a root class $\mathcal{R}$. Suppose that*

1. *$\mathcal{C}$ is closed under taking subgroups*
2. *every $\mathcal{R}$-by-$\mathcal{C}$ group belongs to $\mathcal{C}$;*
3. *for every group in $\mathcal{C}$ there is a group in $\mathcal{R}$ of the same cardinality.*

If a group is free-by-$\mathcal{C}$ then it is residually $\mathcal{C}$.

Proof. Free groups are residually solvable, as recalled in Lemma 2.2.7. □

We apply Lemma 3.3.1 in the case when $\mathcal{R}$ is the class of solvable groups and $\mathcal{C}$ is the class of amenable groups.

Lemma 3.3.3. *If a group is (residually solvable)-by-amenable then it is residually amenable.*

If a group is (residually solvable)-by-(elementary amenable) then it is residually elementary amenable.

Proof. Solvable groups form a root class, as proved by Gruenberg. To apply Lemma 3.3.1, we note that both the classes of amenable groups and elementary amenable groups are closed under extensions, and they contain the class of solvable groups. In particular the first two conditions are satisfied. Moreover, for every cardinal number there exist solvable groups of that cardinality (for instance, given κ, the solvable group $\prod_\kappa \mathbb{Z}$), so the hypotheses are satisfied and we can conclude. □

As a particular case, we have that

Corollary 3.3.4. *If a group is free-by-amenable then it is residually amenable.*

We are now ready to prove the main theorem of the section, which is about free products. In view of Corollary 3.3.2, we can state the result without any reference to root classes of groups.

Theorem 3.3.5. *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of residually $\mathcal{C}$ groups is closed under taking free products.*

Proof. Let $\{G_i\}_{i\in I}$ be a family of residually $\mathcal{C}$ groups. We want to prove that the free product $*_{i\in I}G_i$ is residually $\mathcal{C}$.

Without loss of generality we can suppose that I is finite. Indeed, if I is not finite, let $w \in *_{i\in I}G_i$ be a non-trivial reduced word and let $J \subseteq I$ be the finite set of indices of I that appear in w. Consider then the surjective homomorphism

$$\psi\colon *_{i\in I} G_i \twoheadrightarrow *_{j\in J}G_j$$

such that $\psi(g_j) = g_j$ for $j \in J$ and $\psi(g_i) = e$ for $i \in I \setminus J$. Via ψ we see that if the theorem holds when the index set is finite, it will hold in full generality. We prove the theorem for $|I| = 2$, as the same argument applies to the other cases.

Consider a reduced non-trivial word $w \in G_1 * G_2$ containing letters from both groups. Let

$$A_i := \{g \in G_i \mid g \text{ appears in } w\}, \qquad \text{for } i = 1, 2.$$

The sets A_i are finite, non-empty subsets of G_i, and they do not contain the identity of G_i. The groups G_i are residually $\mathcal{C}$, and $\mathcal{C}$ is closed under taking finite direct products and subgroups. Using Lemma 2.2.4, we have that these groups are fully residually $\mathcal{C}$, that is, there exist groups $C_1, C_2 \in \mathcal{C}$ and surjective homomorphisms

$$\varphi_1 \colon G_1 \twoheadrightarrow C_1, \qquad \varphi_2 \colon G_2 \twoheadrightarrow C_2$$

such that

$$\varphi_1(A_1) \subseteq C_1 \setminus \{e_{G_1}\}, \qquad \varphi_2(A_2) \subseteq C_2 \setminus \{e_{G_2}\}$$

and that $\varphi_i \restriction_{A_i}$ is injective. Define the surjective homomorphism $\varphi\colon G_1 * G_2 \twoheadrightarrow C_1 * C_2$ such that

$$\varphi \restriction_{G_i} = \varphi_i \qquad \text{and} \qquad \varphi(g_1 \dots g_r) = \varphi(g_1) \dots \varphi(g_r) \quad \forall g_i \in G_{\iota_i}, \forall r \in \mathbb{N}.$$

This surjective homomorphism φ maps w to a non-trivial element of $C_1 * C_2$, which is a free product of groups belonging to $\mathcal{C}$.

Consider the canonical projection

$$\pi\colon C_1 * C_2 \twoheadrightarrow C_1 \times C_2.$$

The class $\mathcal{C}$ is closed under finite direct products, so $C_1 \times C_2 \in \mathcal{C}$. Let $K = \ker \pi$, then K is free because it is normal, has trivial intersection with each C_i, and therefore acts freely on the Bass-Serre tree associated to the free product $C_1 * C_2$. Hence $C_1 * C_2$ is an extension of a free group by an amenable one, and therefore it is a residually $\mathcal{C}$ group by hypotheses. This implies that $G_1 * G_2$ is residually $\mathcal{C}$. □

In particular, this is the case if the class $\mathcal{C}$ satisfies the hypotheses of Corollary 3.3.2.

Corollary 3.3.6. *The classes of residually amenable groups and of residually elementary amenable groups are closed under taking free products.*

Proof. Let $\mathcal{C}$ be the class of amenable groups, or the one of elementary amenable groups, $\mathcal{R}$ be the class of solvable groups and apply Theorem 3.3.5. □

Theorem 3.3.5 guarantees sufficient conditions for a free product of residually $\mathcal{C}$ groups to be again residually $\mathcal{C}$. In general, this is not always the case. A pathological counterexample is given by the class of abelian groups, and another one is given by the class of nilpotent groups, as seen in Proposition 3.1.4.

For a further example when the conclusion of Theorem 3.3.5 fails, [11, Theorem 6] guarantees that a free product of two non-trivial groups is residually free if and only if the two groups are fully residually free. So consider a residually free group that is not fully residually free, for instance $G = F_2 \times \mathbb{Z}$. Then $G * G$ is a free product of two residually free groups, and it is not residually free. Note that, in both cases - abelian groups or free groups - the two classes considered are not closed under extensions.

The approach used in this section cannot be applied to the family of torsion-free nilpotent groups, as they do not contain a root class with the properties of Lemma 3.3.1. Nevertheless, it is true that the free product of residually torsion-free nilpotent groups is again residually torsion-free nilpotent, as proved by Mal'cev [85] (compare also [16, Theorem 1.2]).

Theorem 3.3.7. *The free product of residually torsion-free nilpotent groups is residually torsion-free nilpotent.*

Remark 3.3.8. In [73] (see also [69, III. 13.]) the following class of groups is considered. Let $\mathcal{C}$ be the smallest class of groups satisfying the following conditions:

($c1$) $\mathcal{C}$ contains all amenable groups;

($c2$) if $G, H \in \mathcal{C}$ then $G * H \in \mathcal{C}$;

($c3$) if G is virtually $\mathcal{C}$ then $G \in \mathcal{C}$.

It is known that the fundamental group

$$G_k := \left\langle a_1, b_1, \dots, a_k, b_k \mid \prod_{i=1}^{k} [a_i, b_i] \right\rangle \tag{3.4}$$

of a closed orientable surface of genus $k \geqslant 2$ does not belong to $\mathcal{C}$ [73].

Corollary 3.3.6 and Lemma 3.1.2 imply that the class of residually amenable groups satisfy the previous conditions ($c2$) and ($c3$), and condition ($c1$) is trivially satisfied. Thus

$$\mathcal{C} \subsetneq \{\text{residually amenable groups}\},$$

where the inclusion is strict because the groups given by the presentation (3.4) are residually finite [13, §1.3], hence residually amenable.

Moreover

$$\mathcal{C} \nsubseteq \{\text{residually elementary amenable groups}\},$$

in fact this inclusion is not true because there exist finitely generated simple amenable groups [79] that are not elementary amenable (hence, they are not residually elementary amenable). These groups are not elementary amenable because they are infinite and [33, Corollary 2.4] guarantees that a finitely generated simple elementary amenable group is finite.

Also the reverse inclusion does not hold, that is

$$\{\text{residually elementary amenable groups}\} \nsubseteq \mathcal{C},$$

because the groups given by Equation (3.4) are residually finite, hence residually elementary amenable.

3.4 Free products of LE-$\mathcal{C}$ groups

In this section we consider local embeddings, and we prove the analogous of Theorem 3.3.5.

Although the class of residually free groups is not closed under taking free products, Theorem 3.4.1 can be applied also to the class of free groups. Hence, it follows that a free product of groups that are locally embeddable into free groups is again locally embeddable into free groups. In particular, the class of fully residually free groups is closed under taking free products.

Theorem 3.4.1. *Let $\mathcal{C}$ be a class of groups, and suppose that*

(*a*) *$\mathcal{C}$ is closed under taking free products*

or

(*b*) *$\mathcal{C}$ is closed under finite direct products and the free product of residually $\mathcal{C}$ groups is residually $\mathcal{C}$.*

Then the class of LE-$\mathcal{C}$ groups is closed under taking free products.

Proof. Let $\{G_i\}$ be a family of groups locally embeddable into $\mathcal{C}$. As in the proof of Theorem 3.3.5, we can suppose that the index set I is finite, and the argument for $|I| = 2$ applies to all other cases. Let $K = \{w_1, \dots, w_r\}$ be a finite set of reduced words of $G_1 * G_2$, and consider the finite subsets

$$K_i := \{e_{G_i}\} \cup \{g \in G_i \mid g \text{ appears in some } w_j \text{ for } j = 1, \dots, r\} \subseteq G_i, \qquad \text{for } i = 1, 2.$$

The groups G_i are locally embeddable into $\mathcal{C}$, hence there exist $C_i \in \mathcal{C}$ and K_i-approximations $\varphi_i \colon G_i \to C_i$. Moreover $e_{G_i} \in K_i$, thus $\varphi_i(e_{G_i}) = e_{C_i}$. This implies that

$$\varphi_i(g) \neq e_{C_i} \quad \forall g \in K_i \setminus \{e_{G_i}\}. \tag{3.5}$$

Then the map $\varphi \colon G_1 * G_2 \to C_1 * C_2$ defined by

$$\varphi \restriction_{G_i} = \varphi_i \qquad \text{and} \qquad \varphi(g_1 \dots g_s) = \varphi(g_1) \dots \varphi(g_s) \quad \forall g_i \in G_{\iota_i}, \forall s \in \mathbb{N}$$

is well defined in view of Equation (3.5). Let us see that φ is a K-approximation.

Consider the reduced words

$$w = g_1 \dots g_s \in K, \qquad g_j \in G_{\iota_j}\ \forall j \qquad \text{and} \qquad w' = h_1 \dots h_t \in K, \qquad h_j \in G_{\kappa_j}\ \forall j,$$

where ι_j, $\kappa_j \in \{1, 2\}$, and suppose that $\varphi(w) = \varphi(w')$. It follows that $s = t$. Moreover, as w and w' are reduced words, also $\varphi(w)$ and $\varphi(w')$ are reduced. This implies that $\iota_j = \kappa_j$ for all $j = 1, \dots, s$ and that

$$\varphi_{\iota_1}(g_1) = \varphi_{\iota_1}(h_1), \quad \dots, \quad \varphi_{\iota_s}(g_s) = \varphi_{\iota_s}(h_s).$$

As the maps φ_{ι_j} are K_j-approximations and $g_j, h_j \in K_{\iota_j}$, it follows that $g_j = h_j$ for all $j = 1, \dots, s = t$. This means that $w = w'$, that is, $\varphi \restriction_K$ is injective.

Moreover, condition ($a1$) of the definition of K-approximation is satisfied, because the maps φ_j satisfy the same conditions for K_j, respectively. Thus φ is a K-approximation.

If condition (a) holds, then $C_1 * C_2 \in \mathcal{C}$ and so $G_1 * G_2$ is locally embeddable into $\mathcal{C}$. On the other hand, if condition (b) holds then $C_1 * C_2$ is residually $\mathcal{C}$. As the class $\mathcal{C}$ is closed under finite direct products, the group $C_1 * C_2$ is fully residually $\mathcal{C}$, that is, there exists a surjective homomorphism $\psi \colon C_1 * C_2 \twoheadrightarrow D \in \mathcal{C}$ which is injective on the finite subset $\varphi(K)$ of $C_1 * C_2$. Thus, the composition $\psi \circ \varphi \colon G_1 * G_2 \to D$ is a K-approximation, and $G_1 * G_2$ is LE-$\mathcal{C}$.

□

Conditions (a) and (b) of the previous theorem are independent one from the other. In fact the class of free groups satisfies the first one but not the second, and the class of finite (or amenable, etc.) groups satisfies the second but not the first. As for the residual case, a class of groups $\mathcal{C}$ such that being locally embeddable into $\mathcal{C}$ is not preserved under free products is the class of abelian groups.

Applying Theorem 3.4.1 to specific classes of groups, we obtain the following corollary.

Corollary 3.4.2. *Let $\mathcal{C}$ be one of the following classes:*

1. *finite groups;*
2. *finite p-groups;*
3. *finite-solvable groups;*
4. *solvable groups;*
5. *free groups;*
6. *elementary amenable groups;*
7. *amenable groups.*

Then the class of LE-$\mathcal{C}$ groups is closed under taking free products.

Chapter 4

Proamenable topology

In this chapter we introduce a topology suited for the study of residually amenable groups, namely the *proamenable topology*. Our approach is in fact more general, as we work in the context of pro-$\mathcal{C}$ topologies, for a class of groups $\mathcal{C}$. The results of the first two sections are published in the paper [19].

4.1 The pro-$\mathcal{C}$ topology

Given a class $\mathcal{C}$, the idea of characterising residually $\mathcal{C}$ groups looking at a topology these groups carry goes back to M. Hall [67], who introduces the profinite topology as a tool to understand residually finite groups. Strictly related to this is the concept of profinite (or pro-$\mathcal{C}$) completion of a group.

In [7] the analogous *true*[1] *prosoluble completion* is studied, and in [53, 104] pro-$\mathcal{C}$ groups are considered, where $\mathcal{C}$ are various classes of finite groups.

To define the proamenable and the pro-$\mathcal{C}$ topologies, we begin with the following notion.

Definition 4.1.1. A *subdirect product* of the family of groups $\{G_i\}_{i\in I}$ is a subgroup H of their direct product $\prod_{i\in I} G_i$, such that the projection $H \to G_i$ on each coordinate $i \in I$ is surjective.

The direct product of groups is itself a subdirect product. Given a group G, the diagonal product $\big\{(g, g) \mid g \in G\big\} \leqslant G \times G$ is another example of subdirect product. More generally, if $\pi\colon G \twoheadrightarrow H$ is a surjective homomorphism, then the group $\big\{(g, \pi(g)) \mid g \in G\big\} \leqslant G \times H$ is a subdirect product of G and H.

Lemma 4.1.2. *Let G be a group and N_1, N_2 be two normal subgroups in G. The (isomorphic copy in $\frac{G}{N_1} \times \frac{G}{N_2}$ of the) group $G/(N_1 \cap N_2)$ is a subdirect product of the family $\{\frac{G}{N_1}, \frac{G}{N_2}\}$.*

Proof. The group $G/(N_1 \cap N_2)$ is isomorphic to a subgroup of $\frac{G}{N_1} \times \frac{G}{N_2}$, because of the injective homomorphism

$$G/(N_1 \cap N_2) \to \frac{G}{N_1} \times \frac{G}{N_2}, \qquad g(N_1 \cap N_2) \longmapsto (gN_1, gN_2), \qquad \forall g \in G.$$

Let $H \leqslant \frac{G}{N_1} \times \frac{G}{N_2}$ be the isomorphic image of $G/(N_1 \cap N_2)$ under the above mentioned injective map. By construction, the two projections $H \to G/N_1$ and $H \to G/N_2$ are surjective homomorphisms. Hence H is a subdirect product. □

[1] The term *true* appears here to stress that the completion is taken with respect to *all* solvable quotients of the given group, and not just the finite-and-solvable quotients. In prior works considering prosoluble completions, the authors usually took the completion with respect to all finite-and-solvable quotients only.

Given a class $\mathcal{C}$ and a group G, we define the family of subgroups

$$\mathcal{N}_{\mathcal{C}}(G) := \{N \trianglelefteqslant G \mid G/N \in \mathcal{C}\}. \tag{4.1}$$

Remark 4.1.3. Consider the following closure properties for a class of groups $\mathcal{C}$:

1. $\mathcal{C}$ is closed under taking finite subdirect products;
2. $\mathcal{C}$ is closed under taking subgroups;
3. $\mathcal{C}$ is closed under taking finite direct products.

Notice that, from Definition 4.1.1, we have

$$1 \Rightarrow 3 \qquad \text{and} \qquad 2 \;\&\; 3 \Rightarrow 1.$$

If the class $\mathcal{C}$ of Equation (4.1) satisfies the first condition, that is $\mathcal{C}$ is closed under taking finite subdirect produts, then the set $\mathcal{N}_{\mathcal{C}}(G)$ is closed under intersections for every group $G \in \mathcal{C}$, as proved in Lemma 4.1.2. This implies that $\mathcal{N}_{\mathcal{C}}(G)$ is a base at e_G for a topology on G.

Definition 4.1.4. Let $\mathcal{C}$ be a class of groups. The *pro-$\mathcal{C}$ topology* of a group G is the topology that has $\mathcal{N}_{\mathcal{C}}(G)$ as basis at e_G. A normal subgroup $N \in \mathcal{N}_{\mathcal{C}}(G)$ is also called a *co-$\mathcal{C}$ subgroup*.

By definition of basis for a topology, the open sets of the pro-$\mathcal{C}$ topology are

$$\mathcal{O} = \Big\{\bigcup_{i\in I} O_i \mid O_i \in \{gN \mid g \in G,\, N \in \mathcal{N}_{\mathcal{C}}(G)\}\Big\}.$$

The pro-$\mathcal{C}$ topology of a group is a group topology, namely, the group operations

$$\mu\colon G \times G \to G, \qquad \mu(g,h) = gh$$

and

$$\iota\colon G \to G, \qquad \iota(g) = g^{-1}$$

are continuous maps, where $G \times G$ is endowed with the product topology.

If the class $\mathcal{C}$ satisfies conditions two and three of Remark 4.1.3, or equivalently, conditions one and two, then equipping a group G with its pro-$\mathcal{C}$ topology is a faithful functor from the category of groups to the category of topological groups. That is to say, for groups G and H every homomorphism $\varphi\colon G \to H$ is a continuous map with respect to the corresponding pro-$\mathcal{C}$ topologies.

A set $X \subseteq G$ is *$\mathcal{C}$-closed* in G if X is closed in the pro-$\mathcal{C}$ topology of G: for every $g \notin X$ there exists $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that the open set gN does not intersect X, that is, $gN \cap X = \emptyset$. Accordingly, a set is *$\mathcal{C}$-open* in G if it is open in the pro-$\mathcal{C}$ topology of G.

Definition 4.1.5. Let $\mathcal{C}$ be a class of groups. The *pro-$\mathcal{C}$ completion* of a group G is the inverse limit

$$\widehat{G} := \varprojlim_{N \in \mathcal{N}_{\mathcal{C}}(G)} G/N$$

with respect to the canonical surjective homomorphisms $G/N \twoheadrightarrow G/M$, for $N \leqslant M$ in $\mathcal{N}_{\mathcal{C}}(G)$.

Let η be the natural homomorphism

$$\eta\colon G \to \widehat{G}, \qquad \eta(g) := \big(gN\big)_{N \in \mathcal{N}_{\mathcal{C}}(G)}.$$

We have the following characterisations.

Theorem 4.1.6. *Let G be a group and $\mathcal{C}$ be a class closed under taking subgroups and finite direct products, or, equivalently, closed under subgroups and subdirect products. The following conditions are equivalent:*

(c1) $G \in \mathcal{C}$;

(c2) $\{e_G\} \in \mathcal{N}_{\mathcal{C}}(G)$, *that is, the pro-$\mathcal{C}$ topology over G is the discrete topology;*

(c3) $\widehat{G} = G$ *and* $\eta \colon G \to \widehat{G}$ *is the identity map.*

Also the following conditions are equivalent:

(rc1) *G is residually $\mathcal{C}$;*

(rc2) *the pro-$\mathcal{C}$ topology over G is a Hausdorff topology;*

(rc3) *the homomorphism $\eta \colon G \to \widehat{G}$ is injective.*

For the rest of the section, also if not explicitly stated, $\mathcal{C}$ is assumed to be closed under subgroups and finite direct products. The proof of the following lemma is adapted from the proof of [67, Theorem 3.1], where M. Hall proved the same statement for the profinite topology.

Lemma 4.1.7. *Let G be a residually $\mathcal{C}$ group and consider its pro-$\mathcal{C}$ topology. A subgroup H contains an open set if and only if it contains a subgroup $N \in \mathcal{N}_{\mathcal{C}}(G)$.*

Moreover, in such a case, H is both open and closed.

Proof. If H is open in the pro-$\mathcal{C}$ topology, then there exist a family of indices I, elements $g_i \in G$ and subgroups $N_i \in \mathcal{N}_{\mathcal{C}}(G)$ such that

$$H = \bigcup_{i \in I} g_i N_i. \tag{4.2}$$

As $e_G \in H$, there exists $j \in I$ such that $e_G \in g_j N_j$, so that $g_j^{-1} \in N_j$. As N_j is a subgroup, it follows that $g_j N_j = N_j$, and thus $N_j \subseteq H$ by Equation (4.2).

Suppose now that H contains $N \in \mathcal{N}_{\mathcal{C}}(G)$. It follows that

$$H = \bigcup_{h \in H} hN. \tag{4.3}$$

Equation (4.3) implies that H is opes, because it is the union of the open subsets $\{hN\}_{h \in H}$. But, in the same manner, we see that the complement of H in G equals

$$G \setminus H = \bigcup_{g \notin H} gH, \tag{4.4}$$

and hence it is itself open. Equation (4.4) implies that the complement of H is open, so that H is closed. $\square$

We underline the following immediate corollary:

Corollary 4.1.8. *Let G be a residually $\mathcal{C}$ group, H a subgroup and $N \in \mathcal{N}_{\mathcal{C}}(G)$. Then the subgroup HN is both open and closed in the pro-$\mathcal{C}$ topology.*

In the following lemma we characterise subgroups that are closed in the pro-$\mathcal{C}$ topology.

Lemma 4.1.9. *Let H be a subgroup of the group G. The following conditions are equivalent:*

1. *H is closed in the pro-$\mathcal{C}$ topology of G;*

2. *for every $g \in G \setminus H$ there exists a surjective homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $C \in \mathcal{C}$ and $\varphi(g) \notin \varphi(H)$;*

3. *for every $g \in G \setminus H$ there exists $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $g \notin HN$;*

4. *$H = \bigcap\{HN \mid N \in \mathcal{N}_{\mathcal{C}}(G)\}$.*

Proof.

1.⇒2. Suppose that H is closed in the pro-$\mathcal{C}$ topology of G, so that there exists an open set gN, with $N \in \mathcal{N}_{\mathcal{C}}(G)$ satisfying

$$H \cap gN = \emptyset. \tag{4.5}$$

Consider the surjective homomorphism $\varphi\colon G \twoheadrightarrow G/N$. Equation (4.5) implies that the intersection $\varphi(g)\varphi(N) \cap \varphi(H)$ is empty. But N is the kernel of φ, so this means that $\varphi(g) \notin \varphi(H)$.

2.⇔3. The second and the third condition are clearly equivalent, setting $N = \ker\varphi$.

3.⇒4. The existance of an element

$$\tilde{g} \in \left(\bigcap\{HN \mid N \in \mathcal{N}_{\mathcal{C}}(G)\}\right) \setminus H$$

is a contradiction with condition three.

4.⇒1. Condition four means that H is the intersection of closed sets (confront Corollary 4.1.8). Hence, it is itself closed. □

When $\mathcal{C}$ is the class of finite groups, a subgroup satisfying one of the above conditions is sometimes called *separable* in the profinite topology.

Remark 4.1.10. If the class of groups $\mathcal{C}$ is such that a residually $\mathcal{C}$ group is fully residually $\mathcal{C}$ then, in Lemma 4.1.9, conditions two and three can be replaced respectively by:

2.$^+$ for every finite subset $F \subseteq G \setminus H$ there exists a surjective homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $C \in \mathcal{C}$ and $\varphi(F) \notin \varphi(H)$;

3.$^+$ for every finite subset $F \subseteq G \setminus H$ there exists $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $F \cap HN = \emptyset$.

We will exploit these conditions, when possible, even if referring to Lemma 4.1.9.

We end this section with the following proposition. In the particular case when the class $\mathcal{C}$ contains only finite groups, the statement is proved in [104, Lemma 3.1.5]. To state it, we need to define the notion of retract. A *retract* of a group G is a subgroup $R \leqslant G$ admitting a surjective homomorphism $\rho\colon G \twoheadrightarrow R$, called a *retraction*, such that $\rho(r) = r$ for all $r \in R$.

Proposition 4.1.11. *Let $\mathcal{C}$ be a class of groups closed under subgroups and finite direct products, and let G be a residually $\mathcal{C}$ group. A retract R of G is closed in the pro-$\mathcal{C}$ topology of G.*

Proof. Let $\rho\colon G \twoheadrightarrow R$ be the retraction corresponding to R and let $g \in G \setminus R$ be an element. As g is not an element of R, we have that $\rho(g) \neq g$, that is, $\rho(g)g^{-1}$ is not the trivial element of G. As the group G is residually $\mathcal{C}$, there exist a group $C \in \mathcal{C}$ and a homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $\varphi\big(\rho(g)g^{-1}\big) \neq e_C$, that is, $\varphi(\rho(g)) \neq \varphi(g)$.

Consider the homomorphism

$$\Phi\colon G \to C \times C, \qquad \Phi(g) = \big(\varphi(g), \varphi(\rho(g))\big), \quad \forall g \in G.$$

Let $D = \{(c, c) \mid c \in C\} \leqslant C \times C$ be the diagonal subgroup of $C \times C$. As R is a retract and $\rho(g) = g$ for all $g \in R$, we have that $\Phi(R) \leqslant D$. Moreover, $\Phi(g) \notin D$, thus $\Phi(g) \notin \Phi(R)$. As the class $\mathcal{C}$ is closed under taking subgroups and direct products, we have that $\Phi(G) \in \mathcal{C}$.

Hence, $\ker(\Phi) \in \mathcal{N}_{\mathcal{C}}(G)$ and $g \notin R\ker(\Phi)$. Lemma 4.1.9 implies that R is closed in the pro-$\mathcal{C}$ topology of G, and the proof is completed. $\square$

4.2 Special HNN extensions and amalgamated products

To fix the notation and recall some terminology, we start with the following classical definition.

Definition 4.2.1. Given a group A, two subgroups H and K and an isomorphism $\varphi\colon H \to K$, the HNN extension with base group A and amalgamated subgroups H and K is the group given by the presentation

$$A*_{\varphi} := \langle A, t \mid tht^{-1} = \varphi(h) \quad \forall h \in H\rangle.$$

We call the HNN extension *special* if $H = K$ and $\varphi = \mathrm{id}_H$.

In the next theorem we extend [113, Theorem 4.2] exploiting Lemma 3.3.1 to classes of groups $\mathcal{C}$ that are closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.

Remember that, in view of Corollary 3.3.2, the hypotheses of Theorem 4.2.2 are satisfied if the class $\mathcal{C}$ contains a root class $\mathcal{R}$, every $\mathcal{R}$-by-$\mathcal{C}$ group belongs to $\mathcal{C}$, and for every group in $\mathcal{C}$ there is one of $\mathcal{R}$ of the same cardinality.

Theorem 4.2.2. *Let A be a group, $H \leqslant A$ and consider the special HNN extension $G := A*_{\mathrm{id}}$ amalgamating the subgroup H. Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.*

Then G is residually $\mathcal{C}$ if and only if A is residually $\mathcal{C}$ and H is closed in the pro-$\mathcal{C}$ topology of A.

Proof.

$\boxed{\Leftarrow}$ We first assume that $A \in \mathcal{C}$ and we prove that the HNN extension G is residually $\mathcal{C}$. In this case the pro-$\mathcal{C}$ topology of A is discrete, as noticed in Theorem 4.1.6. Hence, all the subgroups are closed.

Consider the surjective homomorphism $\pi\colon G \twoheadrightarrow A$ defined by $\pi(a) = a$ for all $a \in A$ and $\pi(t) = e_A$. This is a homomorphism because the considered HNN extension is special. Let $K := \ker \pi$, then $A \cap K = \{e_A\}$, and for all $g \in G$ we have that

$$g^{-1}Ag \cap K = g^{-1}Ag \cap g^{-1}Kg = g^{-1}(A \cap K)g = \{e_A\}.$$

Hence, K is a free group because it acts freely on the Bass-Serre tree of the HNN extension $A*_{\mathrm{id}}$. This means that G is free-by-$\mathcal{C}$ and, by hypotheses, we conclude that G is residually $\mathcal{C}$.

Suppose that the base group A is residually $\mathcal{C}$ and that H is closed in A. We want to prove that G is residually $\mathcal{C}$. Let g be a non-trivial element of G. By Britton's lemma [82, §IV.2, page 181], g has the following reduced form

$$g = a_0 t^{\varepsilon_1} a_1 \dots t^{\varepsilon_n} a_n, \tag{4.6}$$

where the elements a_i are in A, $\varepsilon_i = \pm 1$ for all $i = 0, \dots, n$, and if $\varepsilon_i \varepsilon_{i+1} = -1$ for some $i = 1, \dots, n-1$ then $a_i \notin H$.

If $g = t^{\varepsilon}$ for some $\varepsilon \in \mathbb{Z} \setminus \{0\}$, then for all normal subgroups $N \trianglelefteq A$ the element g is mapped non-trivially by $\pi\colon G \twoheadrightarrow G_N$ to the special HNN extension

$$G_N := A/N*_{\bar{\mathrm{id}}} = \langle A/N, t \mid txt^{-1} = x \quad \forall x \in HN/N\rangle,$$

where $\bar{\mathrm{id}}\colon HN/N \to HN/N$ is the identity map. Let $N \in \mathcal{N}_{\mathcal{C}}(G)$, then G_N is residually $\mathcal{C}$ because the base group A/N is in $\mathcal{C}$. Hence there exists a surjective homomorphism $\varphi\colon G_N \twoheadrightarrow B$, where $B \in \mathcal{C}$ is such that $\varphi(t^{\varepsilon}) \neq e_B$. Thus, the composition $\varphi \circ \pi$ maps g to a non-trivial element of the group $B \in \mathcal{C}$.

If g is not of the form t^{ε}, then in Equation (4.6) there are some a_i which are not the trivial element of A. As H is closed in the pro-$\mathcal{C}$ topology of A, there exists a surjective homomorphism $\varphi\colon A \twoheadrightarrow B$, with $B \in \mathcal{C}$, such that for every non-trivial of these a_i we have that

$$\varphi(a_i) \notin \varphi(H) = HN/N, \qquad \text{where } N := \ker\varphi. \tag{4.7}$$

Extend $\varphi\colon A \twoheadrightarrow B$ to $\bar{\varphi}\colon A{*_{\mathrm{id}}} \twoheadrightarrow B{*_{\mathrm{id}}}$, where in $B{*_{\mathrm{id}}}$ the identity map is $\mathrm{id}\colon \varphi(H) \to \varphi(H)$, that is, the identity map of HN/N. The element $\bar{\varphi}(g)$ is non-trivial in $B{*_{\mathrm{id}}}$, because of the conditions in Equation (4.7), and $B{*_{\mathrm{id}}}$ is residually $\mathcal{C}$ by the first part of the theorem, because $B \in \mathcal{C}$.

This means that G is residually (residually $\mathcal{C}$), that is, G is residually $\mathcal{C}$.

$\boxed{\Rightarrow}$ Suppose that the special HNN extension $G = A{*_{\mathrm{id}}}$ is residually $\mathcal{C}$, we want to show that A is residually $\mathcal{C}$ and that the subgroup H is closed in the pro-$\mathcal{C}$ topology of A.

As $A \leqslant G$, it is residually $\mathcal{C}$. Suppose that H is not closed in A, so there exists an element $\tilde{a} \in A \setminus H$ such that $\tilde{a} \in HN$ for all the normal subgroups $N \in \mathcal{N}_{\mathcal{C}}(G)$. Consider the non-trivial element $g = t\tilde{a}t^{-1}\tilde{a}^{-1} \in G$. The group G is residually $\mathcal{C}$, so there exists a normal subgroup $K \trianglelefteq G$ such that $g \notin K$ and $G/K \in \mathcal{C}$.

Let $N := K \cap A \trianglelefteq A$, then $A/N \in \mathcal{C}$, because

$$A/N = \frac{A}{A \cap K} \cong \frac{AK}{K} \leqslant G/K.$$

Extend the canonical projection $\pi\colon A \twoheadrightarrow A/N$ to

$$\bar{\pi}\colon A{*_{\mathrm{id}}} \twoheadrightarrow \big(A/N\big){*_{\mathrm{id}}},$$

where in the second HNN extension the identity map is $\mathrm{id}\colon HN/N \to HN/N$. As $A/N \in \mathcal{C}$, it follows that $\tilde{a} \in HN$. So there exist $h \in H$ and $n \in N$ such that $\tilde{a} = hn$. Thus

$$\bar{\pi}(\tilde{a}) = \pi(\tilde{a}) = \pi(h)\pi(n) = \pi(h) \in \pi(H) = HN/N$$

and hence

$$\begin{aligned}\bar{\pi}(g) = \bar{\pi}(t\tilde{a}t^{-1}\tilde{a}^{-1}) &= t\pi(h)t^{-1}\pi(h^{-1})\\ &= \pi(h)\pi(h^{-1}) = \pi(hh^{-1}) = e.\end{aligned}$$

Thus $g \in \ker\pi_N \leqslant K$ and in particular $g \in K$. This is a contradiction with the initial choice of the normal subgroup K, and this contradiction arises from supposing that H is not closed in A.

□

As corollaries, we obtain:

Corollary 4.2.3. *Let $\mathcal{C}$ be a class of groups as in the previous theorem, A a residually $\mathcal{C}$ group and $H \leqslant A$ a finite subgroup of A. Then the special HNN extension $A{*_{\mathrm{id}}}$ with associated subgroup H is residually $\mathcal{C}$.*

Corollary 4.2.4. *A special HNN extension $G := A{*_{\mathrm{id}}}$ is residually amenable if and only if A is residually amenable and the amalgamated subgroup is closed in the proamenable topology of A.*

A similar statement is true for amalgamated free products.

Definition 4.2.5. Given the groups A and B, two subgroups $H \leqslant A$ and $K \leqslant B$ and an isomorphism $\varphi\colon H \to K$, the amalgamated free product of A and B along φ is the group given by the presentation

$$A *_{\varphi} B := \langle A, B \mid h = \varphi(h) \quad \forall h \in H \rangle. \tag{4.8}$$

We call the amalgamated free product $A *_{\varphi} B$ a *double* if $A = B$ and $\varphi = \mathrm{id}_H$. In this case, we write $A *_H A$ for $A *_{\varphi} A$, and we denote by $\bar{A}$ the right hand side copy of A in the double, so that Equation (4.8) reads as

$$A *_H A = \langle A, \bar{A} \mid h = \bar{h} \quad \forall h \in H \rangle.$$

Theorem 4.2.6. *Let A be a group, $H \leqslant A$ a subgroup and consider the double $G := A *_H A$. Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.*

Then G is residually $\mathcal{C}$ if and only if A is residually $\mathcal{C}$ and H is closed in the pro-$\mathcal{C}$ topology of A.

Proof.

$\boxed{\Leftarrow}$ We start with the case when $A \in \mathcal{C}$. Let $\bar{\mathrm{id}}\colon \bar{A} \to A$ be the isomorphism sending $\bar{a}$ to a. By the universal property of amalgamated free products, there exists a unique homomorphism $\varphi\colon G \to A$ that extends $\mathrm{id}\colon A \to A$ and $\bar{\mathrm{id}}\colon \bar{A} \to A$, so that $\varphi\restriction_A = \mathrm{id}$ and $\varphi\restriction_{\bar{A}} = \bar{\mathrm{id}}$.

Let $K = \ker\varphi$, then $K \cap A = K \cap \bar{A} = \{e_G\}$. This means that K is a free group. The homomorphism φ is surjective and $A \in \mathcal{C}$. Hence, it follows that the group G is free-by-$\mathcal{C}$ and thus residually $\mathcal{C}$ by Corollary 3.3.2.

Suppose that A is a residually $\mathcal{C}$ group and that $H \leqslant A$ is closed in the pro-$\mathcal{C}$ topology of A. Let $g \in G$ be a non-trivial element of the double $A *_H A$. If $g \in A$ then $\varphi(g) = g \in A \setminus \{e_A\}$. The group A is residually $\mathcal{C}$, so there exists a group $B \in \mathcal{C}$ and a surjective homomorphism $\psi\colon A \twoheadrightarrow B$ such that $\psi(g) \neq e_B$. The same argument works if $g \in \bar{A}$.

If $g \notin A \cup \bar{A}$ then it can be expressed [88, Corollary 4.4.1] as

$$g = a_1 b_1 \dots a_n b_n,$$

where a_i are elements of A, not in H for $i \geq 2$, and b_i are elements of $\bar{A}$, not in $\bar{H}$ for $i \leqslant n-1$.

The subgroup H is closed in the pro-$\mathcal{C}$ topology of A, thus there exist a group $C \in \mathcal{C}$ and a surjective homomorphism $\vartheta\colon A \twoheadrightarrow C$ such that

$$\vartheta(a_i) \notin \vartheta(H), \qquad \bar{\vartheta}(\bar{a}_i) \notin \bar{\vartheta}(\bar{H}), \qquad \forall i = 1, \dots, n.$$

Consider the surjective homomorphism

$$\Theta\colon G \twoheadrightarrow C *_{\vartheta(H)} C$$

such that

$$g \mapsto \Theta(g) = \vartheta(a_1)\bar{\vartheta}(b_1)\dots\vartheta(a_n)\bar{\vartheta}(b_n).$$

As $\vartheta(a_i) \notin \vartheta(H)$ and $\bar{\vartheta}(b_i) \notin \bar{\vartheta}(\bar{H})$, it follows that $\Theta(g)$ is not trivial in $C *_{\vartheta(H)} C$. By the first part of this proof $C *_{\vartheta(H)} C$ is residually $\mathcal{C}$, hence there exists a quotient $B \in \mathcal{C}$ of G where g is mapped non-trivially.

$\boxed{\Rightarrow}$ Suppose now that the amalgamated free product G is residually $\mathcal{C}$ and thus that A is residually $\mathcal{C}$. We need to prove that the amalgamated subgroup H is closed in the pro-$\mathcal{C}$ topology. Suppose it is not, then there exists an element $a \in A \setminus H$ such that $a \in HN$ for all normal subgroups $N \trianglelefteqslant A$ with $A/N \in \mathcal{C}$, as Lemma 4.1.9 shows. Consider the element $[\bar{a}, a] \in G \setminus (A \cup \bar{A})$.

As the group G is residually $\mathcal{C}$, there exists a normal subgroup $K \trianglelefteq G$ such that $[a, \bar{a}] \notin K$ and $G/K \in \mathcal{C}$. Let $N := K \cap A$, then $N \trianglelefteq A$ and the quotient

$$A/N \cong \frac{AK}{K} \leqslant \frac{G}{K} \in \mathcal{C}.$$

Thus $a \in HN$, so there exist $h \in H$ and $k \in N$ such that $a = hk$. Moreover $N \trianglelefteq A$, so there exist elements $n, m \in N$ such that $hk = nh$ and $\bar{k}^{-1}\bar{h}^{-1} = \bar{h}^{-1}m$. This means, remembering that $h = \bar{h}$ in G for all $h \in H$, that

$$\begin{aligned} [\bar{a}, a] = [\bar{h}\bar{k}, hk] &= \bar{h}\bar{k}(hk)(\bar{k}^{-1}\bar{h}^{-1})k^{-1}h^{-1} = \bar{h}\bar{k}(nh)(\bar{h}^{-1}m)k^{-1}h^{-1} \\ &= \bar{h}\bar{k}nmk^{-1}h^{-1} = h(\bar{k}nmk^{-1})h^{-1}. \end{aligned} \tag{4.9}$$

The element $\bar{k}nmk^{-1}$ is an element of N because each of the four elements appearing in the product lies in N. Hence, Equation (4.9) implies that

$$[\bar{a}, a] \in hNh^{-1} = N = K \cap A \leqslant K.$$

This is a contradiction, because $[\bar{a}, a]$ is mapped non-trivially in the quotient G/K. Hence the subgroup H is closed in the pro-$\mathcal{C}$ topology of A, and the proof is completed. □

Corollary 4.2.7. *Let $\mathcal{C}$ be a class of groups as in the previous theorem, A a residually $\mathcal{C}$ group and $H \leqslant A$ a finite subgroup of A. Then the double $A *_H A$ is residually $\mathcal{C}$.*

Corollary 4.2.8. *A double $A *_H A$ is residually amenable if and only if A is residually amenable and H is closed in the proamenable topology of A.*

Our results are optimal. Indeed, Theorem 4.2.6 is no longer valid if the amalgamated free product is not a double. In [26] infinite simple groups are constructed as amalgamated free products of free groups amalgamating subgroups of finite index (hence, closed in the profinite and proamenable topologies of the given free groups).

Thus an amalgamated free product of residually amenable groups, amalgamating two closed subgroups, may be not residually amenable in general.

Consider a solvable group A with two isomorphic subgroups $\varphi\colon H \to K$, and suppose that φ is the restriction of an automorphism of A. It was proved [103, Lemma 1.2] that, under these hypotheses, the HNN extension $A*_\varphi$ is residually solvable. Proposition 4.2.9 is the generalization of the just-mentioned fact.

Proposition 4.2.9. *Let A be a group, H, K two subgroups of A and $\varphi\colon H \to K$ an isomorphism. Suppose that there exists an automorphism $\alpha\colon A \to A$ such that $\alpha\restriction_H = \varphi$. Let $\mathcal{C}$ be a class of groups such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$, and such that every split extension $\mathcal{C}$-by-$\mathbb{Z}$ belongs to $\mathcal{C}$.*

If $A \in \mathcal{C}$ then $A*_\varphi$ is residually $\mathcal{C}$.

Proof. Consider the two HNN extensions

$$G = \langle A, t \mid tht^{-1} = \varphi(h) \quad \forall h \in H\rangle, \qquad G^\star = \langle A, t \mid tat^{-1} = \alpha(a) \quad \forall a \in A\rangle.$$

As $H \subseteq A$ and $\alpha\restriction_H = \varphi$, the map $\rho\colon G \to G^\star$ defined by $\rho(a) = a$ for all $a \in A$ and $\rho(t) = t$ is a well-defined surjective homomorphism. Moreover, $G^\star$ is isomorphic to the semidirect product $A \rtimes_\alpha \langle t\rangle$, hence it is a group in $\mathcal{C}$, as implied by the hypotheses.

Let $K = \ker\rho$, we have that $A \cap K = \{e_A\}$, thus K intersects trivially each conjugate of A in G because K is normal in G. Hence K is a free group, because it acts freely on the Bass-Serre tree of the HNN extension $A*_\varphi$. This means that G is free-by-$\mathcal{C}$. Hypotheses two and three mean that we can apply Corollary 3.3.2, so that G is residually $\mathcal{C}$. □

Corollary 4.2.10. *Let A be a group, H, K two subgroups of A and $\varphi\colon H \to K$ an isomorphism. Suppose that there exists an automorphism $\alpha\colon A \to A$ such that $\alpha\restriction_H = \varphi$. If A is amenable then the HNN extension $A*_\varphi$ is residually amenable.*

With the same hypotheses of Proposition 4.2.9, if the base group is residually amenable then the HNN extension need not be residually amenable: Example 3.1.6 gives the counterexample.

4.3 Finitely presented non-residually amenable groups

Here we apply the results of the previous section, Theorem 4.2.2 and Theorem 4.2.6, to provide examples of finitely presented groups which are not residually amenable.

From now on, if Γ is a group then $\bar{\Gamma}$ denotes an isomorphic copy of Γ. If $\Gamma = \langle X \mid R\rangle$ is a presentation of the group Γ, let $\bar{X}$ and $\bar{R}$ denote the same generators and relators in the isomorphic copy $\bar{\Gamma}$.

Let $p > 2$ be a prime number and $n \geq 3$. The group $\mathrm{SL}_n(\mathbb{Z}[1/p])$ is finitely presented [66, Theorem 4.3.21] and has Kazhdan's property (T) [17]. Moreover, it satisfies the *congruence subgroup property* [10]. This means that every finite index subgroup $H \leqslant \mathrm{SL}_n(\mathbb{Z}[1/p])$ contains the kernel of the natural projection

$$\pi_q\colon \mathrm{SL}_n(\mathbb{Z}[1/p]) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}), \tag{4.10}$$

for some q coprime with p. In particular, if the finite index subgroup H is normal in $\mathrm{SL}_n(\mathbb{Z}[1/p])$, it follows that

$$\frac{\mathrm{SL}_n(\mathbb{Z}[1/p])}{H} \cong \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}) \qquad \text{or} \qquad \frac{\mathrm{SL}_n(\mathbb{Z}[1/p])}{H} \cong \mathrm{PSL}_n(\mathbb{Z}/q\mathbb{Z}), \tag{4.11}$$

for exactly one q coprime with p.

In what follows, given an element $x \in \mathrm{SL}_n(\mathbb{Z}[1/p])$ and a projection π from $\mathrm{SL}_n(\mathbb{Z}[1/p])$ onto $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ or $\mathrm{PSL}_n(\mathbb{Z}/q\mathbb{Z})$, we denote the order of $\pi(x)$ by o_x.

The proof of the following theorem is adapted from [76, Theorem 1], where an analogous fact is proved, but in the case when the amalgamated subgroup is infinite cyclic.

Theorem 4.3.1. *Let $p > 2$ be a prime number, $n \geq 3$, let $\Gamma := \mathrm{SL}_n(\mathbb{Z}[1/p]) = \langle X \mid R\rangle$. Let $\langle a, b\rangle = F \leqslant \Gamma$ be the subgroup generated by the matrices*

$$a = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix},$$

where I_{n-2} is the identity matrix of dimension $n-2$. Then the group

$$G := \Gamma *_F \Gamma = \langle X, \bar{X} \mid R, \bar{R}, a = \bar{a}, b = \bar{b}\rangle$$

is not LEA.

Proof. The group G is finitely presented. Hence it is sufficient to prove that it is not residually amenable. Let

$$x = \begin{pmatrix} 1 & \frac{2}{p} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}$$

and consider the element $g = [x, \bar{x}] \in G$. Since $x \notin F$, using normal forms for the elements of the amalgamated free product [82, I.11], it follows that $g \neq e_G$.

Let $\pi\colon G \twoheadrightarrow A$ be a surjective homomorphism with A amenable. We claim that $\pi(g) = e_A$. Indeed, consider the restriction $\pi\restriction_\Gamma\colon \Gamma \to \pi(\Gamma)$. The group $\pi(\Gamma) \leqslant A$ is amenable and moreover it is a quotient of Γ, which is a group with Kazhdan's property (T). Hence $\pi(\Gamma)$ is finite and, in particular, $\pi(x)$ has finite order o_x.

The element x is *unipotent*, that is, there exists an integer m such that $(x - \mathrm{I}_n)^m = 0$, so $\pi(x)$ is unipotent too. As the group Γ satisfies the congruence subgroup property, it follows that $\pi(x)$ is an element of some $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ or $\mathrm{PSL}_n(\mathbb{Z}/q\mathbb{Z})$, for q coprime with p. As $\pi(x)$ is a unipotent element, the order o_x divides a power of q, [117]. Moreover $\gcd(p, q) = 1$, so $\gcd(p, o_x) = 1$.

As $x^p = a$, we have that $\langle\pi(a)\rangle \leqslant \langle\pi(x)\rangle$ and that

$$o_a = o_{x^p} = \frac{o_x}{\gcd(p, o_x)} = o_x.$$

This implies that the two finite groups $\langle\pi(a)\rangle$ and $\langle\pi(x)\rangle$ have the same cardinality, and so $\langle\pi(a)\rangle = \langle\pi(x)\rangle$.

The same argument applies to the elements $\bar{x}$ and $\bar{a}$, so $\langle\pi(\bar{a})\rangle = \langle\pi(\bar{x})\rangle$. As in the group G we have $a = \bar{a}$, it follows that $\langle\pi(x)\rangle = \langle\pi(\bar{x})\rangle$, and so $\pi(g) = [\pi(x), \pi(\bar{x})] = e_A$. That is, the element g is mapped to the trivial element in all amenable quotients of G. Thus, G is not residually amenable. $\square$

Corollary 4.3.2. *With the notation of the previous theorem, for $r \geq 2$ let $F_r \leqslant \Gamma$ be generated by $\{b^i a b^{-i} \mid i = 0, \dots, r-1\}$. Then the groups*

$$G_r := \Gamma *_{F_r} \Gamma = \langle X, \bar{X} \mid R, \bar{R}, a = \bar{a}, \dots, b^{r-1} a b^{-r+1} = \bar{b}^{r-1} \bar{a} \bar{b}^{-r+1}\rangle$$

are pairwise non-isomorphic and are not LEA.

Proof. The subgroup F_r is a free group of rank r. The argument of the proof of Theorem 4.3.1 shows that the element $g = [x, \bar{x}]$ is mapped to the trivial element in all amenable quotients of G_r. Hence G_r is not LEA.

To prove that the family $\{G_r\}_{r\geq 2}$ consists of pairwise non-isomorphic finitely presented groups, recall that the *deficiency* $\mathrm{def}(G)$ of a finitely presented group G is defined as the maximum of $|X| - |R|$, over all the finite presentations $G = \langle X \mid R\rangle$. It is invariant under isomorphism, and we have

$$\mathrm{def}(G_r) = 2 \cdot \mathrm{def}(\mathrm{SL}_m(\mathbb{Z}[1/p]) - r.$$

Therefore, the groups $\{G_r\}_{r\geq 2}$ are pairwise non-isomorphic. $\square$

Remark 4.3.3. The groups G of Theorem 4.3.1 and G_r of Corollary 4.3.2 are not residually amenable because they are constructed starting from groups that satisfy property (T). Nevertheless, they do not have themselves property (T) [17, Remark 2.3.5 and Theorem 2.3.6], because they split as amalgamated free products.

From Corollary 4.3.2 and the results of the previous section we obtain:

Corollary 4.3.4. *Let $p > 2$ be a prime number, $n \geq 3$ and $\mathrm{SL}_n(\mathbb{Z}[1/p]) = \langle X \mid R\rangle$. For $r \geq 2$ the groups*

$$\Gamma_r := \langle X, t \mid R,\, tat^{-1} = a,\, t(bab^{-1})t^{-1} = bab^{-1}, \dots,\, t(b^{r-1}ab^{-(r-1)})t^{-1} = b^{r-1}ab^{-(r-1)}\rangle$$

are pairwise non-isomorphic and are not LEA.

Proof. As proved in Corollary 4.3.2, the group $G_r := \Gamma *_{F_r} \Gamma$ is not residually amenable. As the group Γ is residually amenable, by Corollary 4.2.8 we have that F_r is not closed in the proamenable topology of Γ.

Hence, by Corollary 4.2.4, the group Γ_r is not residually amenable. As the groups Γ_r are finitely presented, it follows that they are not LEA. □

Remark 4.3.5. The groups Γ_r of Corollary 4.3.4 are isomorphic to a semidirect product, where the normal subgroup is a free group and the second subgroup is $\Gamma = \mathrm{SL}_n(\mathbb{Z}[1/p])$. As Γ_r are not residually finite, it follows that these normal free subgroups cannot be finitely generated, because this would be a contradiction to Proposition 2.2.8.

To see that Γ_r is a split extension and that the kernel is indeed a free group, let us consider the surjective homomorphism $\varphi\colon \Gamma_r \twoheadrightarrow \Gamma$ that extends $\mathrm{id}\colon \Gamma \to \Gamma$ and $\bar{\mathrm{id}}\colon \bar{\Gamma} \to \Gamma$. Let $K = \ker \varphi$, then $K \cap \Gamma = K \cap \bar{\Gamma} = \{e\}$. This means that K acts freely on the Bass-Serre tree associated to the amalgamated free product Γ_r, that is to say, K is a free group.

The extension is split because the homomorphism $s\colon \Gamma \to \Gamma_r$, which maps Γ to the left hand side copy of Γ inside $\Gamma_r = \Gamma *_{F_r} \Gamma$, is such that $\varphi \circ s = \mathrm{id}_\Gamma$.

4.4 Amalgamated products that are not doubles

In this final section, we collect permanence results that concern free amalgamated products and HNN extensions that are not doubles, or special HNN extensions.

Proposition 4.4.1. *Let $\mathcal{C}$ be a class closed under taking quotients, and such that free products of $\mathcal{C}$ groups are residually C. Let A, B be residually $\mathcal{C}$ groups, and $H \trianglelefteq A$, $K \trianglelefteq B$ be normal, finite subgroups of A and B respectively, with isomorphism $\varphi\colon H \to K$. The amalgamated free product $A *_\varphi B$ is residually $\mathcal{C}$.*

Proof. Let $g \in G = A *_\varphi B$ be a non-trivial element. Let T_H be a transversal of H in A and T_K be a transversal of K in B. The element g can be uniquely expressed as

$$g = ha_1b_1a_2b_2 \dots a_nb_n, \qquad h \in H,\ a_i \in T_H,\ b_i \in T_K. \tag{4.12}$$

The subgroups H and K are finite, hence we can find appropriate $\mathcal{C}$ quotients of A and B, where $H \cup \{a_1, \dots, a_n\}$ and $K \cup \{b_1, \dots, b_n\}$ are injected, respectively. So, assume without loss of generality that A and B are groups in $\mathcal{C}$.

If $g \in A$ (respectively $g \in B$), then the surjective homomorphism $G \twoheadrightarrow A$ (respectively $G \twoheadrightarrow B$) maps g to a non-trivial element of $A \in \mathcal{C}$ (respectvely of $B \in \mathcal{C}$). If $g \notin A \cup B$, then consider the projection $\pi\colon G \twoheadrightarrow A/H * B/K$. Equation (4.12) implies that

$$\pi(g) = (a_1H)(b_1K) \dots (a_nH)(b_nK) \neq e.$$

By hypotheses, the groups A/H and B/K are in $\mathcal{C}$, and the free product $A/H * B/K$ is residually $\mathcal{C}$. This concludes the proof. □

Proposition 4.4.1 goes towards an answer to [76, Question 2], where it is asked if the amalgamated free product of residually amenable groups, amalgamated over any finite subgroup (no normality assumption here) is again residually amenable.

Corollary 4.4.2. *Let A and B be residually amenable groups, with isomorphic normal, finite subgroups $H \trianglelefteq A$ and $K \trianglelefteq B$ respectively. The amalgamated free product $A *_\varphi B$ is residually amenable, where $\varphi\colon H \to K$ is an isomorphism.*

The previous results concern normal subgroups. In what follows, we focus on retracts.

Lemma 4.4.3. *Let $\mathcal{C}$ be a class of groups which is closed under taking subdirect products, and let A, B be two groups in $\mathcal{C}$, with a common retract R. The amalgamated free product $A *_R B$ is free-by-$\mathcal{C}$.*

Proof. The subgroup R is a retract of both A and B, and hence there exist retractions $\rho_A\colon A \twoheadrightarrow R$ and $\rho_B\colon B \twoheadrightarrow R$. Moreover, $A \cong K_A \rtimes R$ and $B \cong K_B \rtimes R$, where $K_A = \ker \rho_A$ and $K_B = \ker \rho_B$. These isomorphisms imply that $A *_R B \cong (K_A * K_B) \rtimes R$. Hence, an element g of $A *_R B$ can uniquely be expressed as

$$g = a_1 b_1 \dots a_n b_n r, \qquad a_i \in K_A,\ b_i \in K_B,\ r \in R. \tag{4.13}$$

Consider the surjective homomorphism

$$\pi\colon A *_R B \twoheadrightarrow \big\{(a, b) \in A \times B \mid \rho_A(a) = \rho_B(b)\big\}$$

defined, for an element g expressed as in Equation (4.13), as

$$\pi(g) = \pi(a_1 b_1 \dots a_n b_n r) = \big(a_1 a_2 \dots a_n r, b_1 b_2 \dots b_n r\big).$$

The group $\{(a, b) \in A \times B \mid \rho_A(a) = \rho_B(b)\} \leqslant A \times B$ is a subdirect product of A and B, which are groups in $\mathcal{C}$. Hence, the hypotheses imply that it is also a group belonging to $\mathcal{C}$.

Moreover, it is immediate to check that $A \cap \ker \pi = B \cap \ker \pi = \{e\}$. Hence, $K = \ker \pi$ acts freely on the Bass-Serre tree of the amalgamated free product $A *_R B$. This concludes the proof. □

Corollary 4.4.4. *Let A and B be amenable groups, with a common retract R. The amalgamated free product $A *_R B$ is residually amenable.*

Lemma 4.4.3 does not readily extend from groups in $\mathcal{C}$ to residually $\mathcal{C}$ groups. If A and B are residually $\mathcal{C}$ groups, in view of Proposition 4.1.11 we can find surjective homomorphisms π_1 and π_2 from A and B onto Q_1 and Q_2 respectively, such that these two groups belong to $\mathcal{C}$ and such that

$$\pi_1(a_i) \notin \pi_1(R), \qquad \pi_2(b_i) \notin \pi_2(R),$$

for all elements appearing in Equation 4.13. This can be exploited to reduce the problem to the setting of Lemma 4.4.3 *only if* $\pi_1(R) \cong \pi_2(R)$.

Chapter 5

Graph products

Graph products were introduced by Green [59] in her PhD thesis, with ideas going back to unpublished work of Bergman[1] [18], and are a common generalisation of direct and free products.

In this chapter we consider graph products of groups, and we extend the results of the previous chapters from the free products context to the graph product context. The first three sections of this chapter correspond to the published paper [20], which is joint work with Michal Ferov.

5.1 Graph products of groups

Definition 5.1.1. Let $\Gamma = (V, E)$ be a simplicial graph. Let $\mathcal{G} = \{G_v\}_{v \in V}$ be a family of groups indexed by the vertex set V. The *graph product* $\Gamma\mathcal{G}$ of the groups $\mathcal{G}$ with respect to the graph Γ is defined as the quotient of the free product $*_{v \in V} G_v$ obtained by adding all the relations of the form

$$g_u g_v = g_v g_u \quad \forall g_u \in G_u, g_v \in G_v, \{u, v\} \in E.$$

The groups $G_v \in \mathcal{G}$ are called the *vertex groups* of $\Gamma\mathcal{G}$.

We stress that we are not making any assumption on the cardinality of the vertex set V, which could be finite, countable or uncountable.

Let $G = \Gamma\mathcal{G}$ be a graph product. Every element $g \in G$ can be obtained as a product of a sequence $W \equiv (g_1, g_2, \ldots, g_n)$, where each g_i belongs to some $G_{v_i} \in \mathcal{G}$. We say that W is a *word* in G and that the elements g_i are its *syllables*. The *length* of a word is the number of its syllables, and it is denoted by $|W|$.

Transformations of the three following types can be defined on words in graph products:

(T1) remove the syllable g_i if $g_i = e_{G_v}$, where $v \in V$ and $g_i \in G_v$;

(T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group G_v and replace them by the single syllable $g_i g_{i+1} \in G_v$;

(T3) interchange two consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u, v\} \in E$.

The last transformation is also called *syllable shuffling*. Note that transformations (T1) and (T2) decrease the length of a word, whereas transformations (T3) preserve it. Thus, applying finitely many of these transformations to a word W, we obtain a word W' which is of minimal length and that represents the same element in G.

[1] We thank Warren Dicks for mentioning this unpublished work.

For $1 \leqslant i < j \leqslant n$, we say that syllables g_i, g_j can be *joined together* if they belong to the same vertex group and 'everything in between commutes with them'. More formally: $g_i, g_j \in G_v$ for some $v \in V$ and for all $i < k < j$ we have that $g_k \in G_{v_k}$ for some $v_k \in \operatorname{link}(v) := \{u \in V \mid \{u, v\} \in E\}$. In this case the words

$$W \equiv (g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_n)$$

and

$$W' \equiv (g_1, \dots, g_{i-1}, g_i g_j, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$$

represent the same group element in G, and the length of the word W' is strictly shorter than W.

We say that a word $W \equiv (g_1, g_2, \dots, g_n)$ is *reduced* if it is either the empty word, or if $g_i \neq e$ for all i and no two distinct syllables can be joined together. As it turns out, the notion of being reduced and the notion of being of minimal length coincide, as it was proved by Green [59, Theorem 3.9]:

Theorem 5.1.2 (Normal Form Theorem)**.** *Every element g of a graph product G can be represented by a reduced word. Moreover, if two reduced words W, W' represent the same element in the group G, then W can be obtained from W' by a finite sequence of syllable shufflings. In particular, the length of a reduced word is minimal among all words representing g, and a reduced word represents the trivial element if and only if it is the empty word.*

Thanks to Theorem 5.1.2 the following are well defined. Let $g \in G$ and let $W \equiv (g_1, \dots, g_n)$ be a reduced word representing g. We define the *length* of g in G to be $|g| = n$ and the *support* of g in G to be

$$\operatorname{supp}(g) = \{v \in V \mid \exists i \in \{1, \dots, n\} \text{ such that } g_i \in G_v \setminus \{e\}\}.$$

Let $x, y \in G$ and let $W_x \equiv (x_1, \dots, x_n), W_y \equiv (y_1 \dots, y_m)$ be reduced expressions for x and y, respectively. We say that the product xy is a reduced product if the word $(x_1, \dots, x_n, y_1, \dots, y_m)$ is reduced. The element xy is a reduced product if and only if $|xy| = |x| + |y|$. We can naturally extend this definition: for $g_1, \dots, g_n \in G$ we say that the product $g_1 \dots g_n$ is reduced if $|g_1 \dots g_n| = |g_1| + \cdots + |g_n|$.

A subset $X \subseteq V$ induces the *full subgraph* Γ_X of Γ, which is defined to be the graph Γ_X whose vertex set is X, and whose edge set consists of all edges in Γ connecting two vertices of $X \subseteq V$. Let G_X be the subgroup of $G = \Gamma\mathcal{G}$ generated by the vertex groups corresponding to X and, by convention, let $G_\emptyset$ be the trivial subgroup. It follows from Theorem 5.1.2 that G_X is isomorphic to the graph product of the family $\mathcal{G}_X = \{G_v \in \mathcal{G} \mid v \in X\}$ with respect to the full subgraph Γ_X. Subgroups of G that can be obtained in this way are called *full subgroups.* For such subgroups, there is a canonical retraction $\rho_X \colon G \twoheadrightarrow G_X$ defined on the vertex groups as

$$\rho_X(g) = \begin{cases} g & \text{if } g \in G_v \text{ and } v \in X, \\ e & \text{otherwise.} \end{cases}$$

Thus, G splits as the semidirect product $G \cong \ker(\rho_X) \rtimes G_X$, and full subgroups are retracts of G.

Let $B \leqslant A$ and C be groups, we define $A \star_B C$, the *special amalgam* of A and C over B, to be the free product with amalgamation

$$A \star_B C := A *_B (B \times C) = \langle A, C \mid [b, c] = e \quad \forall b \in B, \forall c \in C \rangle. \tag{5.1}$$

Special amalgams generalise the notion of special HNN extensions: every special HNN extension

$$A *_{\mathrm{id}_B} = \langle A, t \mid tbt^{-1} = b \quad \forall b \in B \rangle$$

is isomorphic to $A \star_B \langle t \rangle = A *_B (B \times \mathbb{Z})$.

Graph products can be seen in a natural way as special amalgams of their full subgroups:

Fact 5.1.3. *Let $G = \Gamma\mathcal{G}$ be a graph product. Then, for every $v \in V$, we have that $G \cong G_A \star_{G_B} G_C$, where $A = V \setminus \{v\}$, $B = \mathrm{link}(v)$ and $C = \{v\}$.*

Consider $G = A \star_B C$, then every element $g \in G$ can be represented as a product $a_0c_1a_1 \dots c_na_n$, where $a_i \in A$ for $i = 0, 1, \dots, n$ and $c_j \in C$ for $j = 1, \dots, n$. We say that $g = a_0c_1a_1 \dots c_na_n$ is in a *reduced form* if $a_i \notin B$ for $i = 1, \dots, n-1$ and $c_j \neq e$ for $j = 1, \dots, n$.

The following lemma is proved in [53, Lemma 5.3], and describes reduced forms for elements of special amalgams.

Lemma 5.1.4. *Let $B \leqslant A$ and C be groups, and consider $G = A \star_B C$. Suppose that $g = a_0c_1a_1 \dots c_na_n$ is in reduced form, where $a_i \in A$, $c_j \in C$ and $n \geqslant 1$. Then g is not the trivial element of G.*

Moreover, suppose that $f = x_0y_1x_1 \dots y_mx_m$ is in reduced form, where $x_i \in A$, $y_j \in C$. If $f = g$ then $m = n$ and $c_i = y_i$ for all $i = 1, \dots, n$.

Special amalgams satisfy a functorial property.

Fact 5.1.5. *Let $B \leqslant A, C, P, Q$ be groups and let $\psi_A \colon A \to P$, $\psi_C \colon C \to Q$ be homomorphisms. By the universal property of amalgamated free products, the homomorphisms ψ_A and ψ_C uniquely extend to the homomorphism $\psi \colon G \to H$, defined on the generators by*

$$\psi(g) = \begin{cases} \psi_A(a) & \text{if } g = a \text{ for some } a \in A, \\ \psi_C(c) & \text{if } g = c \text{ for some } c \in C, \end{cases}$$

where $G := A \star_B C$ and $H := P \star_{\psi_A(B)} Q$.

5.2 Graph products of residually $\mathcal{C}$ groups

The following lemma will be useful for later proofs. It is noted in [53, Lemma 6.6], and we report a short proof of it.

Lemma 5.2.1. *Let $\mathcal{C}$ be a class of groups closed under finite direct products, let $A, C \in \mathcal{C}$ and suppose that $B \leqslant A$. Then the special amalgam $G = A \star_B C$ is free-by-$\mathcal{C}$.*

Proof. The class $\mathcal{C}$ is closed under taking finite direct products, and hence $A \times C$ belongs to it. Consider the canonical surjective homomorphism $\pi \colon A \star_B C \twoheadrightarrow A \times C$, and let K be its kernel. We have that $K \cap A = \{e\} = K \cap C$. From this, being K a normal subgroup in G, we deduce that K acts freely on the Bass-Serre tree associated to the amalgamated free product $A \star_B C$. Hence, it is a free group, and it thus follows that G is free-by-$\mathcal{C}$. □

In the following proposition we characterise precisely which special amalgams are residually $\mathcal{C}$. This should be compared with Theorem 4.2.2 and Theorem 4.2.6.

Proposition 5.2.2. *Let $B \leqslant A, C$ be groups and suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, finite direct products and that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. The group $G = A \star_B C$ is residually $\mathcal{C}$ if and only if A, C are residually $\mathcal{C}$ and B is closed in the pro-$\mathcal{C}$ topology of A.*

Proof.

$\boxed{\Leftarrow}$ Suppose that the groups A and C are residually $\mathcal{C}$ and that the subgroup B is closed in A. We need to prove that the group G is residually $\mathcal{C}$.

Let $g \in G \setminus \{e\}$ be arbitrary and let $g = a_0c_1a_1 \dots c_na_n$, where $a_i \in A$ for $i = 0, \dots, n$, $c_j \in C$ for $j = 1, \dots, n$, be a reduced expression.

There are two cases to consider. If $n = 0$, then $g = a_0 \in A \setminus \{e\}$. Note that A is a retract of G and thus for the canonical retraction $\rho_A \colon G \twoheadrightarrow A$ we have $\rho_A(a_0) = a_0 \neq e$ in A. The group A is residually $\mathcal{C}$, so there is a group $H \in \mathcal{C}$ and a surjective homomorphism $\varphi \colon A \twoheadrightarrow H$ such that $\varphi(a_0) \neq e_H$. We see that $(\varphi \circ \rho_A)(g) \neq e_H$.

Suppose now that $n \geqslant 1$. As B is closed in the pro-$\mathcal{C}$ topology of A, there is a group $Q \in \mathcal{C}$ and a surjective homomorphism $\alpha \colon A \twoheadrightarrow Q$ such that $\alpha(a_i) \notin \alpha(B)$ for all $i = 1, \dots, n-1$. Moreover, C is residually $\mathcal{C}$, so there exists a group $S \in \mathcal{C}$ and a surjective homomorphism $\gamma \colon C \twoheadrightarrow S$ such that $\gamma(c_i) \neq e_S$ for all $i = 1, \dots, n$. Let $\psi \colon G \to P$, where $P = Q \star_{\alpha(B)} S$, be the canonical extension of α and γ given by Fact 5.1.5. It follows that

$$\psi(g) = \alpha(a_0)\gamma(c_1)\alpha(a_1)\dots\gamma(c_n)\alpha(a_n)$$

is a reduced expression for $\psi(g)$ in P. Hence $\psi(g) \neq e_P$ by Lemma 5.1.4.

The group P is free-by-$\mathcal{C}$ by Lemma 5.2.1, and thus residually $\mathcal{C}$ by assumption. Hence, G is residually $\mathcal{C}$.

$\boxed{\Rightarrow}$ To prove the other implication, suppose now that G is residually $\mathcal{C}$. As $A, C \leqslant G$ and $\mathcal{C}$ is closed under subgroups, it follows that the groups A and C are residually $\mathcal{C}$.

Looking for a contradiction, suppose that B is not closed in A. Then there exists an element $a \in A \setminus B$ such that $\varphi(a) \in \varphi(B)$ for all surjective homomorphisms $\varphi \colon A \twoheadrightarrow Q$, with $Q \in \mathcal{C}$, as proved in Lemma 4.1.9.

Let $c \in C$ be a non-trivial element, then the element $g := [a, c] \in G$ is not trivial, as $a \notin B$ and C only commutes with B.

The group G is residually $\mathcal{C}$, hence there exist a group $Q \in \mathcal{C}$ and a surjective homomorphism $\varphi \colon G \twoheadrightarrow Q$ such that $\varphi(g) \neq e_Q$.

By the choice of the element a, it follows that $\varphi(a) \in \varphi(B)$. Moreover, B and C commute elementwise in G, so $\varphi(B)$ and $\varphi(C)$ commute elementwise in Q. This implies that

$$\varphi(g) = [\varphi(a), \varphi(c)] \in [\varphi(B), \varphi(C)] = \{e_Q\},$$

contradicting the assumption $\varphi(g) \neq e_Q$. Hence, B is closed in the pro-$\mathcal{C}$ topology of A. □

Theorem 5.2.3. *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of residually $\mathcal{C}$ groups is closed under taking graph products.*

Proof. Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\}$ be a family of residually $\mathcal{C}$ groups. We want to prove that the graph product $G := \Gamma\mathcal{G}$ is residually $\mathcal{C}$.

Let $g \in G$ be a non-trivial element and set $S = \operatorname{supp}(g)$. Consider the retraction $\rho_S \colon G \twoheadrightarrow G_S$ onto the graph product associated to the finite graph Γ_S. As $\rho_S(g) = g \neq e$, without loss of generality we can assume that the graph Γ is itself finite.

We proceed by induction on $|V|$. If $|V| = 1$ then $G = G_v$ is residually $\mathcal{C}$ by assumption. Suppose now that $|V| = r > 1$ and that the statement holds for all graph products on graphs with at most $r-1$ vertices.

Fix a vertex $v \in V$ and let

$$A := V \setminus \{v\}, \qquad B := \operatorname{link}(v), \qquad C := \{v\}.$$

From Fact 5.1.3 it follows that $G = G_A \star_{G_B} G_C$. Moreover, G_A is a graph product of residually $\mathcal{C}$ groups with respect to a graph with $r-1$ vertices, hence G_A is residually $\mathcal{C}$ by the induction hypothesis. Note that G_C is a vertex group, thus it is residually $\mathcal{C}$ by assumption. Finally, G_B is a retract of G_A and thus G_B is closed in the pro-$\mathcal{C}$ topology of G_A by Proposition 4.1.11.

Hence, applying Proposition 5.2.2, we see that G is residually $\mathcal{C}$. □

Corollary 5.2.4. *Let $\mathcal{R}$ be a root class. Then the class of residually $\mathcal{R}$ groups is closed under taking graph products.*

Proof. Root classes are closed under taking subgroups, finite direct products and free-by-$\mathcal{R}$ groups are residually $\mathcal{R}$. $\square$

Using this, we recover Green's result that residually finite and residually p-finite groups are closed under taking graph products [59, Corollary 5.4, Theorem 5.6]. Moreover, Corollary 5.2.4 covers [53, Lemma 6.8] for the class of residually finite solvable groups, and yields the same statement for residually solvable groups:

Corollary 5.2.5. *Graph products of residually finite groups are residually finite. The same holds for the classes of residually p-finite groups, residually finite solvable groups and residually solvable groups.*

Proof. All the classes considered are root classes, so the claims follow from Corollary 5.2.4. $\square$

Moreover, we have the desired result for residual amenability, and residual elementary amenability:

Corollary 5.2.6. *Graph products of residually amenable groups are residually amenable. The same holds for residually elementary amenable groups.*

Proof. Free-by-amenable groups are residually amenable in view of Corollary 3.3.4, hence all the hypotheses of Theorem 5.2.3 are satisfied. $\square$

5.3 Graph products of LE-$\mathcal{C}$ groups

Theorem 5.2.3 has an analogue for graph products of LE-$\mathcal{C}$ groups.

Theorem 5.3.1. *Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups, finite direct products, and such that graph products of residually $\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under taking graph products.*

Proof. Let $\Gamma = (V, E)$ be a graph, $\mathcal{G} = \{G_v \mid v \in V\}$ be a family of LE-$\mathcal{C}$ groups and let $G := \Gamma\mathcal{G}$ be the graph product of $\mathcal{G}$ with respect to Γ. Let $K \subseteq G$ be a finite subset of G. The set $\cup_{k \in K}\mathrm{supp}(k)$ is a finite subset of V, thus without loss of generality we can suppose that V itself is finite. Set

$$K' = K^{-1}K = \{k^{-1}k' \mid k, k' \in K \cup \{e_G\}\}$$

and suppose that $K' = \{g_1, \dots, g_r\}$. Let $W_1, \dots, W_r$ be reduced words in G representing the elements $g_1, \dots, g_r$.

For every $v \in V$ consider the finite subset

$$K_v := \{e_{G_v}\} \cup \{g \in G_v \mid g \text{ is a syllable of some } W_i\} \subseteq G_v.$$

By assumption, the vertex group G_v is LE-$\mathcal{C}$ for every $v \in V$. Hence, there exist a family of groups $\mathcal{F} = \{F_v \in \mathcal{C} \mid v \in V\}$ and a family of K_v-approximations $\{\varphi_v \colon G_v \to F_v \mid v \in V\}$.

As $e_{G_v} \in K_v$ it follows $\varphi_v(e_{G_v}) = e_{F_v}$. This implies that

$$\varphi_v(g) \neq e_{F_v} \quad \forall g \in K_v \setminus \{e_{G_v}\}.$$

Let $F := \Gamma\mathcal{F}$ be the graph product of $\mathcal{F}$ with respect to Γ.

Let $\mathcal{W}_G$ and $\mathcal{W}_F$ denote the set of all the words in G and F respectively. We define the function $\tilde{\varphi}\colon \mathcal{W}_G \to \mathcal{W}_F$ in the following manner: for $W \equiv (g_1, \ldots, g_n)$, where $g_i \in G_{v_i}$ for some $v_i \in V$ for $i = 1, \ldots, n$, we set

$$\tilde{\varphi}(W) \equiv (\varphi_{v_1}(g_i), \ldots, \varphi_{v_n}(g_n)).$$

By definition, if W is the empty word in G then $\tilde{\varphi}(W)$ is the empty word in F. Let us note that the map $\tilde{\varphi}$ is compatible with concatenation: for all $U, V \in \mathcal{W}_G$ we have $\tilde{\varphi}(UV) = \tilde{\varphi}(U)\tilde{\varphi}(V)$.

Let $g \in G$ be an arbitrary element and let $W \equiv (g_1, \ldots g_n)$, $W' \equiv (g'_1, \ldots, g'_m)$ be two reduced words representing g in G. By Theorem 5.1.2 we see that $m = n$ and that the word W can be transformed to W' by a finite sequence of syllable shufflings. Since the groups G, F are graph products with respect to the same graph, it can be easily seen that the word $\tilde{\varphi}(W)$ can be transformed into $\tilde{\varphi}(W')$ (using the same sequence of syllable shufflings). Hence the words $\tilde{\varphi}(W)$ and $\tilde{\varphi}(W')$ represent the same element in F.

We see that the map $\tilde{\varphi}$ induces a well defined map $\varphi\colon G \to F$ given by

$$\varphi(g) = \varphi_{v_1}(g_1) \ldots \varphi_{v_n}(g_n).$$

Clearly, $\varphi \restriction_{G_v} = \varphi_v$ for every $v \in V$ and thus it makes sense to omit the subscripts and write

$$\varphi(g) = \varphi(g_1) \ldots \varphi(g_n).$$

We claim that φ is a K-approximation, that is, $\varphi \restriction_K$ is an injective map and $\varphi(kk') = \varphi(k)\varphi(k')$ for all $k, k' \in K$.

First of all, let us show that if the reduced word $W_k \equiv (f_1, \ldots, f_n)$ represents $k \in K'$ in the group G, then the word $\tilde{\varphi}(W_k) \equiv (\varphi(f_1), \ldots, \varphi(f_n))$, which represents $\varphi(k)$ in F, is a reduced word in F.

As the maps φ_v are K_v-approximations for every $v \in V$, it follows that $\varphi(f_i) \neq e$ in F for $i = 1, \ldots, n$, so no syllable of $\tilde{\varphi}(W_k)$ is trivial. Suppose that $\tilde{\varphi}(W_k)$ is not reduced in F. This means that there exist $i < j \in \{1, \ldots, n\}$ such that the syllables $\varphi(f_i)$ and $\varphi(f_j)$ can be joined together. However, this implies that the syllables f_i and f_j can be joined in the word W_k, which contradicts the fact that W_k is reduced. Hence $\tilde{\varphi}(W_k)$ is reduced.

Now, let us prove that $\varphi(kk') = \varphi(k)\varphi(k')$ for all $k, k' \in K'$. Let $k, k' \in K$ be arbitrary and let W, W' be reduced words representing k and k' respectively. We want to show that the word $\tilde{\varphi}(WW') \equiv \tilde{\varphi}(W)\tilde{\varphi}(W')$ represents the element $\varphi(kk')$.

Suppose that the product kk' is reduced, i.e. the concatenation WW', which is a word representing kk' in G, is reduced. Using a similar argument as above, we see that the word $\tilde{\varphi}(WW') \equiv \tilde{\varphi}(W)\tilde{\varphi}(W')$ is reduced. The word $\tilde{\varphi}(W)\tilde{\varphi}(W)$ represents $\varphi(k)\varphi(k')$ in F by definition, but at the same time we see that the word $\tilde{\varphi}(WW')$ represents $\varphi(kk')$ in F, and thus $\varphi(kk') = \varphi(k)\varphi(k')$.

Now, suppose that the product kk' is not reduced. Let $c, f, g \in G$ be such that k factorises as a reduced product $k = fc$, k' factorises as a reduced product $k' = c^{-1}g$ and $|c|$ is maximal. Clearly, $kk' = fg$. Without loss of generality we may assume that

$$W \equiv (f_1, \ldots, f_n, c_1, \ldots c_l) \qquad \text{and} \qquad W' \equiv (c_l^{-1}, \ldots, c_1^{-1}, g_1, \ldots, g_m),$$

where $c = c_1 \ldots c_l$, $f = f_1 \ldots f_n$ and $g = g_1 \ldots g_m$.

Consider the word $X \in \mathcal{W}_F$, where X is

$$(\varphi(f_1), \ldots, \varphi(f_n), \varphi(c_1), \ldots, \varphi(c_l), \varphi(c_l^{-1}), \ldots, \varphi(c_1^{-1}), \varphi(g_1), \ldots, \varphi(g_m)).$$

Note that $X = \tilde{\varphi}(WW') = \tilde{\varphi}(W)\tilde{\varphi}(W')$. The syllable $\varphi(c_l)$ can be joined with syllable $\varphi(c_l^{-1})$. Obviously, $c_l \in G_u$ for some $u \in V$. As $\varphi \restriction_{G_u}$ is a K_u-approximation and $c_l, c_l^{-1} \in K_u$ we see that

$\varphi(c_l)\varphi(c_l^{-1}) = \varphi(c_l c_l^{-1}) = \varphi(e_{G_u})$. As stated before, $\varphi(e_{G_v}) = e_{F_v}$ for every $v \in V$ and thus we can remove the trivial syllable. Note that this transformation is compatible with the function $\tilde{\varphi}$:

$$\varphi(k)\varphi(k') = \varphi(f_1 \dots f_n c_1 \dots c_{l-1})\varphi(c_{l-1}^{-1} \dots c_1^{-1} g_1 \dots g_m).$$

Repeating these two steps $l-1$ more times, the word X can be rewritten as

$$X' \equiv (\varphi(f_1), \dots, \varphi(f_n), \varphi(g_1), \dots, \varphi(g_m))$$

and thus we see that $\varphi(k)\varphi(k') = \varphi(f)\varphi(g)$.

Note that the word X' is reduced in F if and only if the word

$$(f_1, \dots, f_n, g_1, \dots, g_m)$$

is reduced in G. Suppose that the word X' is reduced. It then follows that

$$\varphi(k)\varphi(k') = \varphi(f)\varphi(g) = \varphi(fg) = \varphi(kk').$$

Suppose that the word X' is not reduced. As all the syllables of X' are nontrivial we see that two syllables of the word X' can be joined together. The word $(\varphi(f_1), \dots, \varphi(f_n))$ is a subword of $\tilde{\varphi}(W)$, which is a reduced word, and thus it is reduced, hence no two syllables of $(\varphi(f_1), \dots, \varphi(f_n))$ can be joined together. The same argument applies to $(\varphi(g_1), \dots, \varphi(g_m))$. Hence, we see that there exist $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that the syllables $\varphi(f_i)$ and $\varphi(g_j)$ can be joined together in X'. Again, $f_i, g_j \in G_u$ for some $u \in V$ and thus $\varphi(f_i)\varphi(g_j) = \varphi(f_i g_j)$ as $f_i, g_j \in K_u$. By the assumptions (as $\varphi \restriction_{K_u}$ is injective), $\varphi(f_i g_j) = e_{F_u}$ if and only if $f_i g_j = e_{G_u}$. However, $f_i = g_j^{-1}$ would contradict the maximality of $|c|$, hence $\varphi(f_i)\varphi(g_j) \neq e_{F_u}$. As $\varphi \restriction_{G_u}$ is a K_u-approximation we see that joining the syllable $\varphi(f_i)$ with the syllable $\varphi(g_j)$ is compatible with the map $\tilde{\varphi}$.

Suppose that the syllable $\varphi(f_i g_j)$ can be joined with some $\varphi(f_k)$. By definition, this means that $\varphi(f_k)$ and $\varphi(f_i)$ could have been joined in $\tilde{\varphi}(W)$. This contradicts the fact that $\tilde{\varphi}(W)$ is reduced. By an analogous argument, the syllable $\varphi(f_i g_j)$ cannot be joined with any syllable $\varphi(g_p)$.

By iterating the previous step at most $\min\{n, m\}$ times, we obtain a sequence of transformations compatible with the map $\tilde{\varphi}$. All together, we have shown that the word $\tilde{\varphi}(W)\tilde{\varphi}(W')$ can be rewritten as a reduced word X'', that represents the element $\varphi(k)\varphi(k')$ in F, and each rewriting step is compatible with the map $\tilde{\varphi}$: if we applied the analogous transformations to the word WW' we would obtain a reduced word U, that represents the element kk' in G, such that $\tilde{\varphi}(U) \equiv X''$. It follows that $\varphi(kk') = \varphi(k)\varphi(k')$.

To finish, we need to prove that $\varphi \restriction_K$ is an injective map. Let $k, k' \in K \subseteq K'$ be arbitrary such that $k \neq k'$, or equivalently $k'k^{-1} \neq e_G$. We have already shown that $\varphi(k'k^{-1}) = \varphi(k')\varphi(k^{-1})$. Consider a reduced word $W_{k'k^{-1}}$ representing the element $k'k^{-1} \in K'$. Note that by the construction of the function φ it follows that $\varphi(k) = \varphi(k)^{-1}$ for all $k \in K$. By the previous argumentation, the word $\tilde{\varphi}(W_{k'k^{-1}})$ is reduced in F and thus by Theorem 5.1.2 we see that $\varphi(k'k^{-1}) = \varphi(k')\varphi(k)^{-1} \neq e_F$. It follows that $\varphi(k) \neq \varphi(k')$.

Thus, we proved that φ is a K-approximation.

The graph product $F = \Gamma\mathcal{F}$ is residually $\mathcal{C}$ by assumption. Hence, there exists a surjective homomorphism $\psi\colon F \twoheadrightarrow D \in \mathcal{C}$ which is injective on the finite subset $\varphi(K) \subseteq F$. Thus, the composition $\psi \circ \varphi\colon G \to D$ is a K-approximation, and G is LE-$\mathcal{C}$. □

As an immediate corollary we get the following.

Corollary 5.3.2. *Let $\mathcal{R}$ be a root class. Then the class of LE-$\mathcal{R}$ groups is closed under graph products.*

Corollary 5.3.3. *Graph products of LEF groups are LEF. The same holds for the classes of LE-(finite p-groups), LE-(finite solvable) groups and LE-solvable groups.*

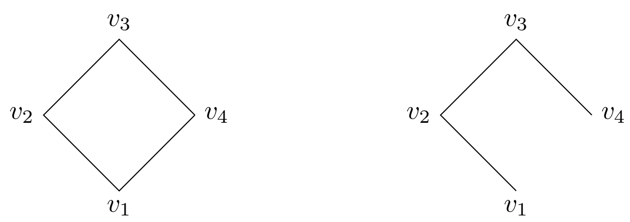
Proof. All the classes considered are root classes, so the claims follow from Corollary 5.3.2. □

Corollary 5.3.4. *Graph products of LEA groups are LEA. The same holds for LE-(elementary amenable) groups.*

Proof. Graph products of residually amenable groups are residually amenable in view of Corollary 5.2.6, hence all the hypotheses of Theorem 5.3.1 are satisfied. □

5.4 Subgroups of graph products

In this section we compute some explicit examples of residually amenable groups, using Theorem 5.2.3. The idea is to draw an analogy with subgroups of right-angled Artin groups (raags[2]). It is well-known that, given a finitely generated raag with associated graph Γ, all its finitely generated subgroups are again raags if and only if Γ does not contain, as an induced subgraph, a square on four vertices, or a line on four vertices:



A notable example of a finitely generated subgroup of a raag which is not itself a raag is the kernel of the surjective homomorphism

$$\varphi\colon G = \langle v_1, v_2, v_3, v_4 \mid [v_1, v_2] = [v_2, v_3] = [v_3, v_4] = [v_4, v_1] = e_G \rangle \twoheadrightarrow \mathbb{Z}$$

that sends each generator of G to $1 \in \mathbb{Z}$. Then, it is a fact that $\ker \varphi$ is finitely generated but *not* finitely presented, and hence it cannot be a raag.

In a similar fashion, in what follows we compute the presentation for such a kernel, when the vertex groups are Baumslag-Solitar groups. The presentation obtained in this manner corresponds to a finitely generated, not finitely presented, group. Moreover, we prove that it is associated to a *trivial* graph product, that is, a graph product associated to a graph with only one vertex.

We use the *Reidemeister-Schreier method* to produce a presentation for the subgroups we are interested in. We recall this powerful method here, and we give reference to [15] for more details.

Definition 5.4.1. Let F be a free group, freely generated by a set X. Let $H \leqslant F$ and T be a right transversal of H in F. The set T is called a *Schreier transversal* (confront [15, page 47]) if

$$x_1^{e_1} \dots x_{i-1}^{e_{i-1}} x_i^{e_i} \in T \qquad \text{implies} \qquad x_1^{e_1} \dots x_{i-1}^{e_{i-1}} \in T, \qquad e_j \in \{-1, +1\}, \quad x_i \in X,$$

that is, every initial segment of an element of T is again an element of T.

For an element $w \in F$, let $\bar{w}$ be the unique element in the Schreier transversal T so that $wT = \bar{w}T$. We now state the *Reidemeister-Schreier Method* [15, III. 6].

[2] We share an idea first heard from Warren Dicks, and we believe that the acronyms raag and raags should be written as normal text, and not as RAAG, or RaAg, or raAg, which will all eventually became old fashioned.

Theorem 5.4.2. *Let $G = \langle X \mid R\rangle$ be a group with a subgroup H, and let $\pi\colon F(X) \twoheadrightarrow G$ denote the canonical surjection. If T is a Schreier transversal for $\pi^{-1}(H)$ in $F(X)$, then the group H admits the presentation $\langle Y \mid S\rangle$, where*

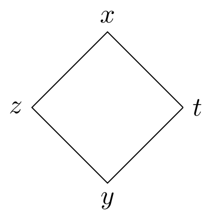
$$Y := \{tx(\overline{tx})^{-1} \mid t \in T, x \in X\} \setminus \{e\}$$

and

$$S := \{t^{-1}rt \mid t \in T, r \in R\} \qquad \text{(expressed as words in } Y\text{)}.$$

The graph is a square

Let $1 \leqslant p < q \in \mathbb{Z}$ be two natural numbers, and define the graph Γ to be



Consider the family of groups $\mathcal{G} = \{G_i\}_{i\in\{x,y,z,t\}}$, where each group is isomorphic[3] to the Baumslag-Solitar group $\mathrm{BS}(p, q)$. For convenience in the notation, let us write

$$G_x = \langle a, x \mid x^{-1}a^p x = a^q\rangle, \qquad G_y = \langle b, y \mid y^{-1}b^p y = b^q\rangle$$

and

$$G_z = \langle c, z \mid z^{-1}c^p z = c^q\rangle, \qquad G_t = \langle d, t \mid t^{-1}d^p t = d^q\rangle.$$

Notice that each $G_i = \langle g\rangle *_{g^p=g^q}$ is an HNN extension, and it admits a surjective homomorphism onto $\mathbb{Z}$ that sends the stable letter of this HNN extension to 1 and the other generator to 0. Hence $G = \Gamma\mathcal{G}$ admits the surjective homomorphism $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ defined on the generators as

$$\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = 0$$

and

$$\varphi(x) = \varphi(y) = \varphi(z) = \varphi(t) = 1.$$

The group G is residually solvable because it is a graph product of residually solvable groups. Moreover, it is not residually finite because the vertex groups are not residually finite.

Let $K = \ker\varphi$ be the kernel of this projection. Using the Reidemeister-Schreier Method we now produce a presentation for this group.

Let $G = \langle X \mid R\rangle$, where $X = \{a, b, c, d, x, y, z, t\}$ and R is given by the four relations from the groups $\mathcal{G}$, plus the sixteen commutation-relations induced on the generators of G from the graph Γ. Let K' be the preimage of the group K in $F(X)$, and consider the set $T = \{x^i\}_{i\in\mathbb{Z}} \subseteq F(X)$. Let us see that T is a Schreier transversal for K' in $F(X)$ (confront Definition 5.4.1). To do so, notice that

$$K' = \{w \in F(X) \mid \varepsilon(w) = 0\},$$

where $\varepsilon\colon F(X) \to \mathbb{Z}$ is the function that associates to an element $w \in F(X)$ the sum of all the exponents of the generators x, y, z and t appearing in the element w. To show that T is a transversal, it suffices to understand that $xK' = yK'$. This is surely the case because $y^{-1}x \in K'$, and hence

$$yK' = y(y^{-1}x)K' = xK'.$$

[3] The computations that follow do not require, in principle, the vertex groups to be isomorphic. Nevertheless, we consider this particular case to ease the notation, and because doing so simplifies the shape of the Schreier transversal.

By induction on $\varepsilon(w)$ one shows that, given an arbitrary element $w \in F(X)$, we have that

$$wK' = x^{\varepsilon(w)}K'.$$

Hence we proved that T is a right transversal for K' in $F(X)$. Given this, the fact that it is a *Schreier* transversal follows by the definition of T.

The Reidemeister-Schreier method guarantees that K satisfies the presentation $\langle Y \mid S \rangle$, where

$$Y = \{tx(\overline{tx})^{-1} \mid t \in T, x \in X\} \setminus \{e\} \subseteq F(X),$$

$$S = \{trt^{-1} \mid t \in T, r \in R\} \subseteq F(X)$$

and the elements of S are expressed as words in the alphabet Y.

For $i \in \mathbb{Z}$ and $w \in \{a, b, c, d, y, z, t\}$ let $w_i := x^i w x^{-i}$. Moreover, let

$$\alpha_i := y_i x^{-1}, \qquad \beta_i := z_i x^{-1}, \qquad \gamma_i := t_i x^{-1}. \tag{5.2}$$

In the list that follows we compute the elements of the generating set Y. The boxed letter l means that we are considering the case of $x^i \in T$ and $l \in X$.

$\boxed{a}$ In this case we have that

$$x^i a(\overline{x^i a})^{-1} = x^i a x^{-i} = a_i.$$

$\boxed{b}$ Analogously, for the element b we have that

$$x^i b(\overline{x^i b})^{-1} = x^i b x^{-i} = b_i.$$

$\boxed{c}$ Similarly

$$x^i c(\overline{x^i c})^{-1} = x^i c x^{-i} = c_i.$$

$\boxed{d}$ Moreover

$$x^i d(\overline{x^i d})^{-1} = x^i d x^{-i} = d_i.$$

$\boxed{x}$ This choice of $x^i \in T$ and $x \in X$ gives the trivial element of $F(X)$, which is discarted.

$\boxed{y}$ In this case we have that

$$x^i y(\overline{x^i y})^{-1} = x^i y x^{-i-1} = y_i x^{-1}.$$

$\boxed{z}$ Here we have that

$$x^i z(\overline{x^i z})^{-1} = x^i z x^{-i-1} = z_i x^{-1}.$$

$\boxed{t}$ Concluding,

$$x^i t(\overline{x^i t})^{-1} = x^i t x^{-i-1} = t_i x^{-1}.$$

Summarising, we have that the group K is generated by the set

$$Y = \{a_i, b_i, c_i, d_i, y_i x^{-1}, z_i x^{-1}, t_i x^{-1}\}_{i \in \mathbb{Z}},$$

which can be rewritten as

$$Y = \{a_i, b_i, c_i, d_i, \alpha_i, \beta_i, \gamma_i\}_{i \in \mathbb{Z}} \tag{5.3}$$

in view of Equation (5.2).

Now, let us compute the relations of the group, that is, we express the elements of the set S in terms of the alphabet Y. Again, the boxed word r in the following list means that we are considering the case of $x^i \in T$ and $r \in R$.

If r is one of the four words

$$\{[a,c],[a,d],[b,c],[b,d]\}$$

then

$$x^i r x^{-i} \in \{[a_i,c_i],[a_i,d_i],[b_i,c_i],[b_i,d_i]\}.$$

$\boxed{[x,c]}$ In this case

$$x^i[x,c]x^{-i} = x^{i-1}c^{-1}(x^{-(i-1)}x^{i-1})xcx^{-i} = c_{i-1}^{-1}c_i.$$

$\boxed{[x,d]}$ As in the previous case, we have that

$$x^i[x,d]x^{-i} = d_{i-1}^{-1}d_i.$$

$\boxed{[x,z]}$ In this case

$$\begin{aligned} x^i[x,z]x^{-i} &= x^{i-1}z^{-1}xzx^{-i} = xx^{i-2}z^{-1}(x^{-(i-2)}x^{i-2})xzx^{-i} \\ &= (xx^{i-2}z^{-1}x^{-(i-2)})(x^{i-1}zx^{-(i-1)}x^{-1}) = \beta_{i-2}^{-1}\beta_{i-1}. \end{aligned}$$

$\boxed{[x,t]}$ Analogously one has that

$$x^i[x,t]x^{-i} = \gamma_{i-2}^{-1}\gamma_{i-1}.$$

$\boxed{[a,z]}$ We have that

$$\begin{aligned} x^i[a,z]x^{-i} &= x^ia^{-1}z^{-1}azx^{-i} = x^ia^{-1}(x^{-i}x^i)z^{-1}(x^{-(i-1)}x^{i-1})a(x^{-(i-1)}x^{i-1})zx^{-i} \\ &= (x^ia^{-1}x^{-i})(xx^{i-1}z^{-1}x^{-(i-1)})(x^{i-1}ax^{-(i-1)})(x^{i-1}zx^{-(i-1)}x^{-1}) \\ &= a_i^{-1}(xz_{i-1}^{-1})a_{i-1}(z_{i-1}x^{-1}) = a_i^{-1}\beta_{i-1}^{-1}a_{i-1}\beta_{i-1}. \end{aligned}$$

$\boxed{[a,t]}$ Analogously as in the previous case, we have that

$$x^i[a,t]x^{-i} = a_i^{-1}\gamma_{i-1}^{-1}a_{i-1}\gamma_{i-1}.$$

$\boxed{[b,z]}$ Also here

$$x^i[b,z]x^{-i} = b_i^{-1}\beta_{i-1}^{-1}b_{i-1}\beta_{i-1}.$$

$\boxed{[b,t]}$ And

$$x^i[b,t]x^{-i} = b_i^{-1}\gamma_{i-1}^{-1}b_{i-1}\gamma_{i-1}.$$

$\boxed{[c,y]}$ As before, we have that

$$x^i[c,y]x^{-i} = c_i^{-1}\alpha_{i-1}^{-1}c_{i-1}\alpha_{i-1}.$$

$\boxed{[d,y]}$ Moreover

$$x^i[d,y]x^{-i} = d_i^{-1}\alpha_{i-1}^{-1}d_{i-1}\alpha_{i-1}.$$

$\boxed{[y,z]}$ For this choice of the element in R we have that

$$\begin{aligned}
x^i[y,z]x^{-i} &= x^iy^{-1}(x^{-(i-1)}x^{i-1})z^{-1}(x^{-(i-2)}x^{i-2})y(x^{-(i-1)}x^{i-1})zx^{-i}\\
&= (x^iy^{-1}x^{-(i-1)})(x^{i-1}z^{-1}x^{-(i-2)})(x^{i-2}yx^{-(i-2)}x^{-1})(x^{i-1}zx^{-i})\\
&= \alpha_{i-1}^{-1}\beta_{i-2}^{-1}\alpha_{i-2}\beta_{i-1}.
\end{aligned}$$

$\boxed{[y,t]}$ For the analogous choice of $[y,t]$, we have that

$$x^i[y,t]x^{-i} = \alpha_{i-1}^{-1}\gamma_{i-2}^{-1}\alpha_{i-2}\gamma_{i-1}.$$

$\boxed{yb^py^{-1}b^{-q}}$ In this case

$$\begin{aligned}
x^iyb^py^{-1}b^{-q}x^{-i} &= x^iy(x^{-(i+1)}x^{i+1})b^p(x^{-(i+1)}x^{i+1})y^{-1}(x^{-i}x^i)b^{-q}x^{-i}\\
&= (x^iyx^{-(i+1)})(x^{i+1}b^px^{-(i+1)})(x^{i+1}y^{-1}x^{-i})(x^ib^{-q}x^{-i})\\
&= \alpha_ib_{i+1}^p\alpha_i^{-1}b_i^{-q}.
\end{aligned}$$

$\boxed{zc^pz^{-1}c^{-q}}$ Analogously

$$x^izc^pz^{-1}c^{-q}x^{-i} = \beta_ic_{i+1}^p\beta_i^{-1}c_i^{-q}.$$

$\boxed{td^pt^{-1}d^{-q}}$ With the same computations we have that

$$x^itd^pt^{-1}d^{-q}x^{-i} = \gamma_id_{i+1}^p\gamma_i^{-1}d_i^{-q}.$$

$\boxed{xa^px^{-1}a^{-q}}$ Finally

$$x^ixa^px^{-1}a^{-q}x^{-i} = x^{i+1}a^p(x^{-(i+1)}x^{i+1})x^{-1}a^{-q}x^{-i} = a_{i+1}^pa_i^{-q}.$$

Collecting all the cases discussed, we have that

$$\begin{aligned}
S = \{&[a_i,c_i],[a_i,d_i],[b_i,c_i],[b_i,d_i],c_{i-1}^{-1}c_i,d_{i-1}^{-1}d_i,\beta_{i-2}^{-1}\beta_{i-1},\gamma_{i-2}^{-1}\gamma_{i-1},\\
&a_i^{-1}\beta_{i-1}^{-1}a_{i-1}\beta_{i-1},a_i^{-1}\gamma_{i-1}^{-1}a_{i-1}\gamma_{i-1},b_i^{-1}\beta_{i-1}^{-1}b_{i-1}\beta_{i-1},b_i^{-1}\gamma_{i-1}^{-1}b_{i-1}\gamma_{i-1},\\
&c_i^{-1}\alpha_{i-1}^{-1}c_{i-1}\alpha_{i-1},d_i^{-1}\alpha_{i-1}^{-1}d_i\alpha_{i-1},\alpha_{i-1}^{-1}\beta_{i-2}^{-1}\alpha_{i-2}\beta_{i-1},\alpha_{i-1}^{-1}\gamma_{i-2}^{-1}\alpha_{i-2}\gamma_{i-1},\\
&\alpha_ib_{i+1}^p\alpha_i^{-1}b_i^{-q},\beta_ic_{i+1}^p\beta_i^{-1}c_i^{-q},\gamma_id_{i+1}^p\gamma_i^{-1}d_i^{-q},a_{i+1}^pa_i^{-q}\}_{i\in\mathbb{Z}}.
\end{aligned}\tag{5.4}$$

Using Tietze transformations [15, III. 5, Proposition 2] we can significantly simplify the presentation of the group K. In fact, this group is finitely generated.

Let $a,b,c,d,\alpha,\beta,\gamma$ denote $a_0,b_0,c_0,d_0,\alpha_0,\beta_0,\gamma_0$ respectively. Using induction, we notice that relations five through eight in Equation (5.4) imply that

$$c_i = c, \quad d_i = d, \quad ,\beta_i = \beta, \quad \gamma_i = \gamma, \qquad \forall i\in\mathbb{Z}.$$

Hence, applying Tietze transformations of type 1 to Equation (5.3), we obtain that an equivalent presentation of K is given by $\langle Y_1 \mid S\rangle$, where

$$Y_1 = \{a_i,b_i,c,d,\alpha_i,\beta,\gamma\}_{i\in\mathbb{Z}}.\tag{5.5}$$

Applying now Tietze transformations of type 2 to the elements of Equation (5.5), we obtain that $K = \langle Y_1 \mid S_1 \rangle$, where

$$\begin{aligned} S_1 = \{&[a_i, c], [a_i, d], [b_i, c], [b_i, d], a_i^{-1}\beta^{-1}a_{i-1}\beta, a_i^{-1}\gamma^{-1}a_{i-1}\gamma, b_i^{-1}\beta^{-1}b_{i-1}\beta, \\ &b_i^{-1}\gamma^{-1}b_{i-1}\gamma, c^{-1}\alpha_i^{-1}c\alpha_i, d^{-1}\alpha_i^{-1}d\alpha_i, \alpha_i^{-1}\beta^{-1}\alpha_{i-1}\beta, \alpha_i^{-1}\gamma^{-1}\alpha_{i-1}\gamma, \\ &\alpha_i b^p \alpha_i^{-1} b^{-q}, \beta c^p \beta^{-1} c^{-q}, \gamma d^p \gamma^{-1} d^{-q}, a_{i+1}^p a_i^{-q}\}_{i\in\mathbb{Z}}. \end{aligned} \tag{5.6}$$

Rearranging the elements in S_1 and writing the commutators at the beginning, we have that

$$\begin{aligned} S_1 = \{&[a_i, c], [a_i, d], [b_i, c], [b_i, d], [\alpha_i, c], [\alpha_i, d], a_i^{-1}\beta^{-1}a_{i-1}\beta, a_i^{-1}\gamma^{-1}a_{i-1}\gamma, \\ &b_i^{-1}\beta^{-1}b_{i-1}\beta, b_i^{-1}\gamma^{-1}b_{i-1}\gamma, \alpha_i^{-1}\beta^{-1}\alpha_{i-1}\beta, \alpha_i^{-1}\gamma^{-1}\alpha_{i-1}\gamma, \\ &\alpha_i b^p \alpha_i^{-1} b^{-q}, \beta c^p \beta^{-1} c^{-q}, \gamma d^p \gamma^{-1} d^{-q}, a_{i+1}^p a_i^{-q}\}_{i\in\mathbb{Z}}. \end{aligned}$$

Let us split S_1 in a convenient way:

$$\begin{aligned} S_1 =&\{[a_i, c], [a_i, d], [b_i, c], [b_i, d], [\alpha_i, c], [\alpha_i, d]\}_{i\in\mathbb{Z}} \cup \{a_i^{-1}\beta^{-1}a_{i-1}\beta, a_i^{-1}\gamma^{-1}a_{i-1}\gamma\}_{i\in\mathbb{Z}} \cup \\ &\cup \{b_i^{-1}\beta^{-1}b_{i-1}\beta, b_i^{-1}\gamma^{-1}b_{i-1}\gamma\}_{i\in\mathbb{Z}} \cup \{\alpha_i^{-1}\beta^{-1}\alpha_{i-1}\beta, \alpha_i^{-1}\gamma^{-1}\alpha_{i-1}\gamma\}_{i\in\mathbb{Z}} \cup \\ &\cup \{\alpha_i b^p \alpha_i^{-1} b^{-q}, \beta c^p \beta^{-1} c^{-q}, \gamma d^p \gamma^{-1} d^{-q}, a_{i+1}^p a_i^{-q}\}_{i\in\mathbb{Z}}. \end{aligned} \tag{5.7}$$

The second set appearing in Equation (5.7) can be replaced, again using Tietze transformations of type 2, by the relations

$$\begin{cases} a_i = (\beta\gamma^{-1})^{-1} a_i (\beta\gamma^{-1}), & \forall i \in \mathbb{Z}, \\ a_i = \beta^{-1} a_{i-1} \beta, & \forall i \in \mathbb{Z}. \end{cases} \tag{5.8}$$

An induction argument then shows that

$$a_i = \beta^{-i} a \beta^i \in \langle a, \beta \rangle, \qquad \forall i \in \mathbb{Z}. \tag{5.9}$$

The same rewriting processes can be now applied to the third and fourth sets appearing in Equation (5.7). Analogously to what is shown in Equation (5.9), we have that

$$b_i = \beta^{-i} b \beta^i \in \langle b, \beta \rangle, \qquad \alpha_i = \beta^{-i} \alpha \beta^i \in \langle \alpha, \beta \rangle, \qquad \forall i \in \mathbb{Z}. \tag{5.10}$$

Hence, combining Equations from (5.7) to (5.10), we obtain that the presentation $\langle Y_1 \mid S_1 \rangle$ of K is equivalent to

$$\langle Y_2 \mid S_2 \rangle, \tag{5.11}$$

where

$$Y_2 = \{a, b, c, d, \alpha, \beta, \gamma\} \tag{5.12}$$

and

$$\begin{aligned} S_2 =&\{[\beta^{-i}a\beta^i, c], [\beta^{-i}a\beta^i, d], [\beta^{-i}b\beta^i, c], [\beta^{-i}b\beta^i, d], [\beta^{-i}\alpha\beta^i, c], [\beta^{-i}\alpha\beta^i, d]\}_{i\in\mathbb{Z}} \cup \\ &\cup \{\beta^{-i}a\beta^i = (\beta\gamma^{-1})^{-1}\beta^{-i}a\beta^i(\beta\gamma^{-1})\}_{i\in\mathbb{Z}} \cup \\ &\cup \{\beta^{-i}b\beta^i = (\beta\gamma^{-1})^{-1}\beta^{-i}b\beta^i(\beta\gamma^{-1})\}_{i\in\mathbb{Z}} \cup \\ &\cup \{\beta^{-i}\alpha\beta^i = (\beta\gamma^{-1})^{-1}\beta^{-i}\alpha\beta^i(\beta\gamma^{-1})\}_{i\in\mathbb{Z}} \cup \\ &\cup \{\alpha(\beta^i b \beta^{-i})^p \alpha^{-1} = (\beta^i b \beta^{-i})^q, \beta c^p \beta^{-1} = c^q, \gamma d^p \gamma^{-1} = d^q, \beta^{-1} a^p \beta = a^q\}_{i\in\mathbb{Z}}. \end{aligned} \tag{5.13}$$

Proposition 5.4.3. *Suppose that the presentation $\langle Y_2 \mid S_2 \rangle$ of the group $K \leqslant \Gamma\mathcal{G}$ given by Equation* (5.11) *corresponds to a graph product $\Theta\mathcal{H}$, for some graph Θ and some family of groups $\mathcal{H}$. Then the graph Θ has one vertex.*

Proof. To prove the claim, we inspect the presentation $K = \langle Y_2 \mid S_2 \rangle$.

First of all, notice that K is finitely generated. Hence, Θ is a finite graph. Inspecting the last set contributing to S_2 in Equation (5.13), we notice that the relation $\beta c^p \beta^{-1} = c^q$ implies that β and c belong to the same vertex group, because it is not a commutator, and hence it cannot possibly be induced by the graph Θ. Call this vertex group $H_1 \in \mathcal{H}$.

Analogously, $\beta^{-1} a^p \beta = a^q$ means that the generators β and a belong to the same vertex group, and in particular also $a \in G_1$.

As for all $i \in \mathbb{Z}$ we have that $\alpha(\beta^i b \beta^{-i})^p \alpha^{-1} = (\beta^i b \beta^{-i})^q$, we can also conclude that $\alpha, b \in H_1$. From the very last relation, $\gamma d^p \gamma^{-1} = d^q$, we obtain that the generators γ and d belong to the same vertex group, call it $H_2 \in \mathcal{H}$.

It remains to show that these two vertex groups H_1 and H_2 are, in fact, attached to the same vertex $v \in V(\Theta)$. It would hence follow that all elements of the generating set Y_2 lie in the same vertex group, and hence that $\Theta = \{v\}$. This claim is indeed implied by the relation $\beta^{-i} \alpha \beta^i = (\beta\gamma^{-1})^{-1} \beta^{-i} \alpha \beta^i (\beta \gamma^{-1})$ involving the letters α, β, γ. This, jointly with the fact that the element γ does not commute with α or with β in the group K, implies that this relation cannot be possibly induced by the graph Θ, but it is indeed induced by one of the vertex groups, $H_1 = H_2$.

Hence, all the generators of K lie in the same vertex group, that is, $\Theta = \{v\}$ is the trivial graph. $\square$

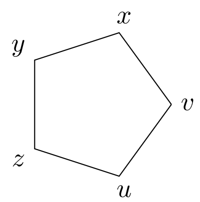
The graph is a pentagon

It may be noticed that the graph Γ of the previous example is bipartite, and therefore the group G is isomorphic to

$$G \cong \big(\mathrm{BS}(p,q) * \mathrm{BS}(p,q)\big) \times \big(\mathrm{BS}(p,q) * \mathrm{BS}(p,q)\big).$$

In particular, the fact that G is residually solvable follows from Theorem 3.3.6, and not for the more general result about graph products, Theorem 5.2.3.

Let Δ be the graph



and associate to each vertex a Baumslag-Solitar group $G_i \cong \mathrm{BS}(p,q)$, so that

$$G_x = \langle a, x \mid x^{-1} a^p x = a^q \rangle, \qquad G_y = \langle b, y \mid y^{-1} b^p y = b^q \rangle, \qquad G_z = \langle c, z \mid z^{-1} c^p z = c^q \rangle$$

and

$$G_u = \langle d, u \mid u^{-1} d^p u = d^q \rangle, \qquad G_v = \langle f, v \mid v^{-1} f^p v = f^q \rangle.$$

Let $\mathcal{G}$ denote the family of these five groups. Analogously to what is shown in the previous example, when the graph was a square, one can prove that the kernel K of the canonical projection $\Delta\mathcal{G} \twoheadrightarrow \mathbb{Z}$ satisfies the presentation

$$K = \langle X' \mid R' \rangle, \tag{5.14}$$

where

$$X' = \{a, b, c, d, f, \alpha, \beta, \gamma, \delta\} \tag{5.15}$$

and

$$
\begin{aligned}
R' = \Big\{ & [\delta^{-i}a\delta^{i}, b], [\delta^{-i}a\delta^{i}, f], [\delta^{-i}d\delta^{i}, f], [\delta^{-i}\gamma\delta^{i}, f], [\alpha^{-i}c\alpha^{i}, \delta^{-i}d\delta^{i}], [\alpha^{-i}c\alpha^{i}, b], [\alpha^{-i}\beta\alpha^{i}, b],\\
& \delta a^{p}\delta^{-1} = a^{q}, \alpha^{-1}b^{p}\alpha = b^{q}, (\alpha\beta^{-1})c^{p}(\alpha\beta^{-1})^{-1} = c^{q}, (\gamma\delta^{-1})^{-1}d^{p}(\gamma\delta^{-1}) = d^{q}, \delta^{-1}f^{p}\delta = f^{q},\\
& \delta^{-i}\gamma\delta^{i} = \alpha^{-(i-1)}\beta\alpha^{i-1}\delta^{-(i-1)}\gamma\delta^{i-1}\alpha^{-i}\beta\alpha^{i}, \delta^{-(i+1)}a\delta^{i+1} = \alpha^{-1}\delta^{-(i-1)}a\delta^{i-1}\alpha,\\
& \delta^{-i}d\delta^{i} = \alpha^{-(i-1)}\beta\alpha^{i-1}\delta^{-(i-1)}d\delta^{i-1}\alpha^{-(i-1)}\beta\alpha^{i-1},\\
& \alpha^{-i}c\alpha^{i} = \delta^{-(i-1)}\gamma^{-1}\delta^{-(i-1)}c\alpha^{i-1}\delta^{-(i-1)}\gamma\delta^{i-1}\Big\}_{i\in\mathbb{Z}}.
\end{aligned}
\tag{5.16}
$$

As in Proposition 5.4.3, a careful inspection of the presentation $\langle X' \mid R' \rangle$ reveals that it is associated to a graph product with only one vertex.

Proposition 5.4.4. *Suppose that the presentation $\langle X' \mid R' \rangle$ of the group $K \leqslant \Delta\mathcal{G}$ given by Equation* (5.14) *corresponds to a graph product $\Theta\mathcal{H}$, for some graph Θ and some family of groups $\mathcal{H}$. Then the graph Θ has just one vertex.*

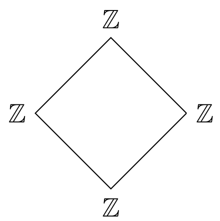
5.5 Rigidity of graph product decompositions

Our Proposition 5.4.3 and Proposition 5.4.4 lead to the following natural question:

Question 5.5.1. Suppose that the group K given by the presentation of Equation (5.14) is isomorphic to the graph product $\Theta\mathcal{H}$. Does it follows that Θ is the trivial graph with one vertex and no edges?

A more general problem is to determine sufficient conditions for *rigidity* of graph product decompositions: given a group G, is its graph product decomposition unique?

Fact 5.1.3 immediately suggests that one should impose restrictions on the vertex groups appearing in the graph product decompositions, as every graph product can be expressed as an amalgamated free product. For instance, the raag associated to a square is isomorphic, as a group, to the graph product with trivial graph and $(\mathbb{Z} \times F_2) *_{F_2} (F_2 \times \mathbb{Z})$ as vertex group:



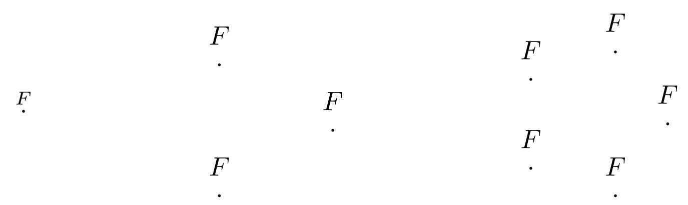
$\cdot \; (\mathbb{Z} \times F_2) *_{F_2} (F_2 \times \mathbb{Z})$

It is well known [45] that, for finitely generated right-angled Artin groups, the isomorphism of two such groups is equivalent to the graph isomorphism of the underlying finite graphs. Hence, in this case, the graph product decomposition is rigid. As we already stated, this is far from being true in general, even for right-angled Artin groups that are not finitely generated.

Example 5.5.2. Let F be a free group with countably many generators, so that

$$F \cong F * F \cong \ldots \cong *_{i=1}^{n} F, \qquad \forall n \in \mathbb{N}. \tag{5.17}$$

The isomorphisms of Equation (5.17) carry over to graph product decompositions, so that F, as a graph product, can be associated to any totally disconnected graph on n vertices, with each vertex group isomorphic to F.

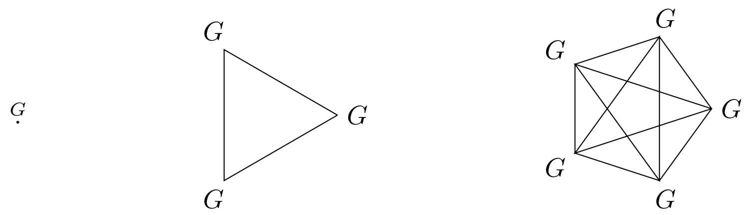


Example 5.5.2 suggests that one should restrict the attention to groups that are finitely generated. On the other hand, the next example shows that one should also require the vertex groups to be *directly indecomposable*, that is, the vertex groups should not be expressable as $A \times B$, with both the factors A and B non-trivial groups.

Example 5.5.3. Let G be a finitely generated group that is isomorphic to its square [14, 78] (see also Section 3.2). Analogously to Equation 5.17, we have that

$$G \cong G \times G \cong \ldots \cong \bigoplus_{i=1}^{n} G, \qquad \forall n \in \mathbb{N}. \tag{5.18}$$

Hence, in this case, G can be associated to any complete graph on n vertices, with each vertex group isomorphic to the group G.



It is proved in [102] that, whenever a group can be expressed as a graph product of finitely many finite, directly indecomposable vertex groups, then this decomposition is unique. The same result holds if the vertex groups are cyclic (finite or not) [65].

With the next example, which in [55] is attributed to Kuroš, we notice that direct indecomposability of the vertex groups is, in general, not enough to guarantee uniqueness of graph product decomposition, even when the vertex groups are finitely presented.

Example 5.5.4. Let

$$A = \langle a_1, a_2 \mid a_1^2 = a_2^2 \rangle, \qquad B = \langle b_1, b_2 \mid b_1^3 = b_2^3 \rangle$$

and

$$C = \langle c_1, c_2, c_3, c_4 \mid c_1^2 = c_2^2 = c_3^3 = c_4^3 \rangle, \qquad D = \langle d \rangle.$$

Then these four groups are directly indecomposable, pairwise non-isomorphic and $A \times B \cong C \times D$. Hence, the group $A \times B$, as a graph product, does not determine uniquely the vertex groups.

In [55] it is proved that, given two finite graphs Γ and Δ and two finite families of groups $\mathcal{G}$ and $\mathcal{H}$ that are directly indecomposable and satisfy Serre's property FA, the graph product $\Gamma\mathcal{G}$ is isomorphic to $\Delta\mathcal{H}$ if and only if the graphs Γ and Δ are isomorphic and the groups in $\mathcal{G}$ are isomorphic to the ones in $\mathcal{H}$.

Property FA seems to be too strong, because in [55] it is just used for this consequence: a group satisfying FA cannot split as an amalgamated free product. In the treatment of rigidity for graph product decompositions, we only need to exclude the case of groups splitting as in Fact 5.1.3, that is, we only deal with very peculiar amalgamated free products splittings.

Notice that the group of Equation (5.11) does not satisfy property FA, because its abelianisation $\mathbb{Z}^3 \times \left(\frac{\mathbb{Z}}{(q-p)\mathbb{Z}}\right)^4$ is infinite. For the same reason, the group of Equation (5.14) does not satisfy property FA either.

It is reasonable, to the author's view, to ask whether the rigidity results proved for infinite cyclic vertex groups [45], for finite groups [102], or for cyclic groups [65], could be extended to the setting of amenable groups, or even to groups without free subgroups. This setting, of course, excludes the possibility of shrinking the graph attached to the graph product by considering amalgamated free products as vertex groups, as showed just before of Example 5.5.2.

Conjecture 5.5.5. *Let Γ, Δ be two finite graphs and let $\mathcal{G} = \{G_i\}_{i\in V(\Gamma)}$, $\mathcal{H} = \{H_j\}_{j\in V(\Delta)}$ be family of directly indecomposable finitely generated groups without free subgroups. If the graph products $\Gamma\mathcal{G}$ and $\Delta\mathcal{H}$ are isomorphic, then there exists an isomorphism of graphs $\varphi\colon \Gamma \to \Delta$ that induces isomorphisms of groups $G_i \cong H_{\varphi(i)}$.*

Notice that, by [55], this can hold also if the vertex groups do contain free subgroups. For instance, the linear group $\mathrm{SL}_3(\mathbb{Z})$ satisfies Serre's property FA, and hence falls in the family of groups considered in [55].

Chapter 6

Quantification of residual amenability

In this chapter we discuss how to *quantify* residual amenability of groups. There are mainly two approaches to quantification of residual properties, and here we follow the definitions coming from the unpublished work of Arzhantseva and Cherix [6].

The first article on the subject that appeared in the literature in 2010 is due to Bou-Rabee [21], and it was devoted to residual finiteness. The idea is as follows: given a residually finite group G with a finite generating set X and a non-trivial element $g \in G$, let $\varphi_{G,X}(g)$ be the smallest cardinality of any finite quotient of G where the element g is mapped non-trivially, and let $F_{G,X}(n)$ be the maximum of $\varphi_{G,X}(g)$, where g ranges through all the elements of the ball $B_{G,X}(n)$. This viewpoint generated a quite active direction of research.

A closely related, but different, approach to the quantification problem for a residually finite group, is to study the function $\Phi_{G,X}(n)$ mapping n to the smallest cardinality of any finite quotient of G where the *whole* ball $B_{G,X}(n)$ of radius n is injected. This notion was introduced by Bou-Rabee and McRaynolds [22], and is called the *full residual finiteness growth* of a group by Bou-Rabee and Studenmund [24]. A slight variation is the one of *normal systolic growth*, studied by Bou-Rabee and de Cornulier [23, 39].

6.1 Residually amenable profile

Let G be a group equipped with a proper metric d. In this chapter all groups are assumed to be countable, because we want to exploit the fact that the finite metric balls $\{B_{G,d}(n)\}_{n\in\mathbb{N}}$ exhaust the group.

If $f \in \{\text{Føl}, A, A^{\Delta}\}$ is one of the three Følner functions introduced in Section 2.1, then define $Rf_{G,d}\colon \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ as the function

$$Rf_{G,d}(n) := \min\{f_{Q,\bar{d}}(n) \mid Q \text{ amenable quotient of } G, B_{G,d}(n) \hookrightarrow Q\},$$

where $\bar{d}$ is the metric on the quotient $Q = G/N$ induced by d:

$$\bar{d}(q,q') := \min\bigl\{d(g,g') \mid gN = q,\ g'N = q'\bigr\}, \qquad \forall q,q' \in Q.$$

The metric $\bar{d}$ is proper, because d is.

Thus, we have defined the three functions $R\text{Føl}_{G,d}$, $RA_{G,d}$ and $RA^{\Delta}_{G,d}$. If the proper metric d is a word metric, corresponding to a finite generating set X of G, then we may indicate these functions by

$R\mathrm{F\o l}_{G,X}$, $RA_{G,X}$ and $RA^{\triangle}_{G,X}$. Notice that, in this case, the metric $\bar{d}$ coincides with the word metric induced by (the embedded image of) X on the quotient Q.

Following Equation (2.3) we have that

$$R\mathrm{F\o l}_{G,d} \preccurlyeq RA_{G,d} \sim RA^{\triangle}_{G,d}. \tag{6.1}$$

Moreover, when d is a word metric induced by the finite generating set X, Equation (2.1) and Equation (2.2) imply that

$$R\mathrm{F\o l}_{G,X} \sim RA_{G,X} \sim RA^{\triangle}_{G,X} \tag{6.2}$$

Hence, the following definition is well posed:

Definition 6.1.1. The *residually amenable profile* of a finitely generated group G is the $\sim$-equivalence class of the functions $R\mathrm{F\o l}_{G,X} \sim RA_{G,X} \sim RA^{\triangle}_{G,X}$, where X is a finite generating set.

The notion of residually amenable profile is well defined because, asymptotically, it does not depend on the choice of the finite generating set.

Lemma 6.1.2. *Let G be a finitely generated group, X and Y be two finite generating sets. Then $RA_{G,X} \sim RA_{G,Y}$.*

Proof. Without loss of generality we can assume that G is residually amenable.

If we consider the constant $C_1 = \max_{x\in X}|x|_Y$, we have that $B_{G,X}(n) \subseteq B_{G,Y}(C_1 n)$ for all $n \in \mathbb{N}$. Let Q be an amenable quotient of the group G where the finite set $B_{G,Y}(C_1 n)$ injects, and such that there exists a finite subset $A \subseteq Q$ satisfying $|A| = RA_{G,Y}(C_1 n)$. By definition, the following inequalities hold:

$$|A \setminus gA| \leqslant \frac{1}{C_1 n}|A|, \quad \forall g \in B_{Q,\bar{Y}}(C_1 n). \tag{6.3}$$

As $B_{G,X}(n)$ is contained in $B_{G,Y}(C_1 n)$, also the finite set $B_{G,X}(n)$ is injected into the quotient Q. Moreover, being $1/C_1 n$ strictly smaller than $1/n$, from Equation (6.3) we can conclude that

$$|A \setminus gA| \leqslant \frac{1}{n}|A|, \quad \forall g \in B_{Q,\bar{X}}(n).$$

This surely implies that $RA_{G,X}(n) \leqslant |A|$, which was chosen to be equal to $RA_{G,Y}(C_1 n)$.

For the other inequality, one can reverse the roles of the two generating sets, and consider the new constant $C_2 = \max_{y\in Y}|y|_X$. ☐

Analogously, for $f \in \{\mathrm{F\o l}, A, A^{\triangle}\}$ we define the functions

$$LEf_{G,d} := \min\{f_{H,\bar{d}} \mid \exists\ B_{G,d}(n)\text{-approximation into the amenable group } H\}, \tag{6.4}$$

and the *LEA profile* of a finitely generated group G to be the $\sim$-equivalence class of (one of) these functions.

Remark 6.1.3. In principle, it is not required for the amenable group H appearing in Equation (6.4) to be finitely generated. However, this can always be achieved considering the finitely generated subgroup of H that is generated by the images of the elements of $B_{G,d}(n)$, because H is not required to be a quotient of G, and the $B_{G,d}(n)$-approximation from G to H restricts to a map with the same required properties from G to the finitely generated subgroup of H.

If the group G is amenable, then there is a natural relation between its residually amenable profile and its Følner function: the former is not bigger than the latter.

Lemma 6.1.4. *Let G be an amenable group and d be a proper metric on it. Then $RA_{G,d}(n) \leqslant A_{G,d}(n)$ for all $n \in \mathbb{N}$.*

Proof. The group G is an amenable quotient of itself. □

In analogy to Lemma 6.1.4, if the group G is residually finite, then the residually amenable profile is dominated by the full residual finiteness growth, which is defined to be the function

$$\Phi_{G,d}(n) := \min\big\{|Q| \mid Q \text{ finite quotient of } G,\ B_{G,d}(n) \hookrightarrow Q\big\}, \qquad \forall n \in \mathbb{N}.$$

Lemma 6.1.5. *Let G be a residually finite group and d be a proper metric on it. Then $RA_{G,d}(n) \leqslant \Phi_{G,d}(n)$ for all $n \in \mathbb{N}$.*

Proof. Let Q be a finite quotient of G such that the ball $B_{G,d}(n)$ injects, and of minimal cardinality among the quotients with such property. Then $A_{Q,\bar{d}}(n) \leqslant |Q|$, so that $RA_{G,d}(n) \leqslant |Q| = \Phi_{G,d}(n)$. □

In [51, Lemma 4] it is proved that, if $H \leqslant G$ are two finitely generated amenable groups and Y, X are two finite generating sets such that $Y \subseteq X$, then $\mathrm{Føl}_{H,Y}(n) \leqslant \mathrm{Føl}_{G,X}(n)$. This immedialy extends to a similar statement for residually amenable groups: if $H \leqslant G$ are two finitely generated residually amenable groups, Y, X two finite generating sets such that $Y \subseteq X$, then $R\mathrm{Føl}_{H,Y}(n) \leqslant R\mathrm{Føl}_{G,X}(n)$ for all $n \in \mathbb{N}$.

From this fact and Equation (6.2) it follows that $RA_{H,Y} \preccurlyeq RA_{G,X}$ and that $RA^{\vartriangle}_{H,Y} \preccurlyeq RA^{\vartriangle}_{G,X}$ when G and H are finitely generated groups.

In fact, this holds also in the case when the subgroup H is not finitely generated, but is equipped with a proper metric that agrees with the word metric of G:

Lemma 6.1.6. *Let $H \leqslant G$ be two groups, d' be a proper metric for H and d be a proper metric for G such that $d \restriction_H = d'$. Then $RFøl_{H,d'}(n) \leqslant RFøl_{G,d}(n)$.*

It is natural to ask how this invariant behaves under taking direct products, and in the next lemma we describe this behaviour. If d is a proper metric on the group G and d' is a proper metric on the group H, a natural proper metric d_M on the direct product $G \times H$ is obtained as the maximum over d and d':

$$d_M\big((g,h),(g',h')\big) := \max\{d(g,g'), d'(h,h')\}, \qquad \forall g, g' \in G,\ \forall h, h' \in H. \tag{6.5}$$

This metric is not artificial at all. Indeed, if the two groups are finitely generated, generated respectively by the finite sets X and Y, and d and d' are the respective word metrics, then the metric d_M is nothing but the word metric associated to the finite generating set

$$Z = \big\{(x,y) \mid x \in X \cup \{e_G\},\ y \in Y \cup \{e_H\},\ (x,y) \neq (e_G, e_H)\big\}$$

of the group $G \times H$.

Lemma 6.1.7. *Let G and H be two groups equipped with the proper metrics d and d' respectively. We have that*

$$RFøl_{G\times H,d_M}(n) \leqslant RFøl_{G,d}(n) \cdot RFøl_{H,d'}(n), \qquad \forall n \in \mathbb{N}. \tag{6.6}$$

Proof. By Equation (6.5) we have that

$$B_{G\times H,d_M}(n) = \big\{(x,y) \mid x \in B_{G,d}(n),\ y \in B_{H,d'}(n)\big\}.$$

If G or H is not residually amenable, then both sides of Equation (6.6) are equal to $+\infty$, and the claim holds.

If both the groups G and H are residually amenable, the same is true for $G \times H$. In this case, if Q_1 is a quotient of G where $B_{G,d}(n)$ is injected, and Q_2 is a quotient of H where $B_{H,d'}(n)$ is injected, it then follows that $B_{G\times H,d_M}(n)$ is injected into $Q_1 \times Q_2$, which is a quotient of $G \times H$. Hence

$$R\mathrm{F}\o\mathrm{l}_{G\times H,d_M}(n) \leqslant R\mathrm{F}\o\mathrm{l}_{G,d}(n) \cdot R\mathrm{F}\o\mathrm{l}_{H,d'}(n),$$

as desired. □

6.2 Extensions

In the previous section we defined the residually amenable profile, and we described its behavior under direct products. In this section we consider the case of group extensions.

It is not true, in general, that a group extension with residually amenable kernel and amenable quotient is again residually amenable, as recorded in Lemma 3.1.7. Nevertheless, if we restrict our attention to residually solvable kernels, then the resulting extension is residually amenable, Lemma 3.3.3.

We start our discussion focusing on solvable and residually solvable groups, which are remarkable for the following fact (see [51, 97]):

Fact 6.2.1. *Let G be a solvable group of solvable length l, equipped with a proper metric d. The Følner function of G is bounded from above by an iterated exponentiation:*

$$F\o l_{G,d}(n) \preccurlyeq \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{l\ times}, \qquad \forall n \in \mathbb{N}. \tag{6.7}$$

Let G be a group and d a proper metric on it. Let us consider the following function, called the *residually solvable profile* of the group G:

$$\ell_{G,d}(n) = \begin{cases} \min\left\{ l \in \mathbb{N} \,\middle|\, \begin{array}{l} B_{G,d}(n) \hookrightarrow Q,\ Q \text{ solvable} \\ \text{quotient of } G \text{ of length } l \end{array} \right\} & \text{if such quotient exists,} \\ +\infty & \text{otherwise.} \end{cases} \tag{6.8}$$

It is useful to define the above function more generally, and not just for balls in G of finite radius. Hence, consider the function $\tilde{\ell}_G \colon \mathcal{P}_f(G) \to \mathbb{N}$ defined by

$$\tilde{\ell}_G(F) = \min\{l \in \mathbb{N} \mid F \hookrightarrow Q,\ Q \text{ solvable quotient of } G \text{ of length } l\}. \tag{6.9}$$

An analogous proof to the one of Lemma 6.1.2 gives the following result:

Lemma 6.2.2. *Let G be a finitely generated residually solvable group, X and Y be two finite generating sets. Then $\ell_{G,X} \sim \ell_{G,Y}$.*

Theorem 6.2.3. *Let G be a group generated by the finite set X, and suppose $N \trianglelefteqslant G$ is a normal residually solvable subgroup, such that $Q = G/N$ is solvable of length ℓ_Q. Then*

$$\ell_{G,X}(n) \leqslant \tilde{\ell}_N\big(B_{G,X}(2n) \cap N\big) + \ell_Q. \tag{6.10}$$

Proof. Notice that G is residually solvable by Lemma 2.2.11, and hence its residually solvable profile always takes finite values. Let $B = B_{G,X}(n)$ be a finite ball in G, and consider the finite subset $B_{G,X}(2n) \cap N$ of the subgroup N. This set contains all elements of the form xy^{-1}, where $x, y \in B$ and $\pi(x) = \pi(y)$, where $\pi \colon G \twoheadrightarrow Q$ is the canonical surjective homomorphism.

As N is residually solvable, there exists a normal subgroup $L \trianglelefteq N$ such that N/L is solvable and the finite set $B_{G,X}(2n) \cap N$ injects in this solvable quotient. Let L be chosen so that the solvable length of N/L is minimal, and hence equal to $\tilde{\ell}_N\big(B_{G,X}(2n) \cap N\big)$ by definition.

From L, construct its characteristic core:

$$K = \bigcap\{L' \mid L' \subseteq N,\, L' \cong L\}, \tag{6.11}$$

which is a characteristic subgroup of N. The group N/K embeds canonically into $\prod_{L'} N/L'$, which is a solvable group with the same solvable length as N/L.

Hence, the solvable length of N/K is at most the one of N/L. If it was smaller, we would have found a solvable quotient of N/L where the finite set $B_{G,X}(2n) \cap N$ injects, and this would be a contradiction with the choice of the subgroup L.

As K is characteristic in N, it is normal in G. Moreover, G/K is an extension of the group N/K by the quotient

$$Q \cong (G/K)/(N/K). \tag{6.12}$$

Hence, it is a solvable group of solvable length at most $\tilde{\ell}_N\big(B_{G,X}(2n) \cap N\big) + \ell_Q$.

It remains to be shown that $B_{G,X}(n)$ injects into this solvable quotient G/K. Let $x, y \in B$ be two elements such that $xK = yK$: we need to prove that the equality $x = y$ holds already in the group G.

The quotient map $G \twoheadrightarrow Q = G/N$ factors through the group G/K because the group Q is a quotient of G/K, by Equation (6.12). Hence, the elements x and y are mapped to the same element of the quotient G/N. From this, it follows that $xy^{-1} \in N \cap B_{G,X}(2n)$. This set is injected into N/K by construction, and $e \in N \cap B_{G,X}(2n)$. Hence, from the equality $xK = yK$ we conclude that $x = y$ must hold, and the proof is complete. □

Remark 6.2.4. The function $\tilde{\ell}_G$ has been introduced to obtain a better bound in Theorem 6.2.3.

Even if the subgroup N is finitely generated, we are not equipping it with a word metric coming from a generating set. Considering a word metric on the finitely generated subgroup N would mean to introduce inside the formula for the bound the *distortion function* Δ_Y^X of the subgroup N in the group G. Recall that the distortion function $\Delta_Y^X \colon \mathbb{N} \to \mathbb{N}$ is defined to be

$$\Delta_Y^X(n) = \max\big\{|g|_Y \mid g \in B_{G,X}(n) \cap N\big\}.$$

This would significantly worsen the bound of Equation (6.10), as the new bound would be

$$\ell_{G,X}(n) \leqslant \ell_{N,Y}\big(\Delta_Y^X(2n)\big) + \ell_Q.$$

In a sense, in fact, we are implicitly using the trivial distortion function from the group G to its subgroup with the induced metric, so that $\Delta_d^X(n) = n$ for all $n \in \mathbb{N}$. This falls out of the classical setting for the study of distortion functions, where only word metrics coming from finite generating sets are considered.

As a corollary of Theorem 6.2.3 we obtain the following bound for the residually amenable profile of a residually solvable group:

Corollary 6.2.5. *Let G be a residually solvable group with a normal subgroup N and a solvable quotient $Q = G/N$ of solvable length ℓ_Q. Then*

$$RF\o l_{G,X}(n) \preccurlyeq \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{l(n)\ times},$$

where $l(n) = \tilde{\ell}_N\big(B_{G,X}(2n) \cap N\big) + \ell_Q$.

Remark 6.2.6. Another upper bound for the Følner function of a finitely generated residually solvable group can be obtained considering the values of the Følner function for free solvable groups, which are computed in [105, Theorem 7.4]. Let $S_{l,r}$ denote the free object in the cathegory of r-generated groups of solvable length l. The group $S_{l,r}$ is called the *free solvable group* of rank r and degree l.

Let $\log_{[1]}(n) := \log(1+n)$ and define inductively $\log_{[x+1]}(n) := \log\bigl(1+\log_{[x]}(n)\bigr)$ for all $x \in \mathbb{N}$. Then $\mathrm{Føl}_{S_{l,r}}$ is $\sim$-equivalent to the inverse function of

$$v \longmapsto \Bigl(\frac{\log_{[l-1]}(v)}{\log_{[l]}(v)}\Bigr)^{\frac{1}{r}}.$$

The function $\mathrm{Føl}_{G,X}$ decreases when passing to quotients [52, Lemma 2.2], hence it follows that

$$R\mathrm{Føl}_{G,X}(n) \leqslant \mathrm{Føl}_{S_{l(n),|X|}}(n),$$

where

$$l(n) = \tilde{\ell}_N\bigl(B_{G,X}(2n) \cap N\bigr) + \ell_Q.$$

The proof of Theorem 6.2.3 can be generalised to the case when the quotient G/N is amenable and not just solvable, at the cost of getting a slight worse bound. In this case, the strategy is to apply Theorem 2.1.6 to a particular solvable-by-amenable quotient to obtain the desired Følner set.

In general, if the normal subgroup N is supposed to be residually amenable, two problems can occurr. Firstly, when considering the characteristic subgroup of Equation (6.11), it may happen that the resulting quotient N/K is not amenable. Secondly, supposing that this quotient group N/K is in fact amenable, the knowledge of the Følner sets (and so, the Følner function) in N/L does not guarantee any knowledge on the Følner sets in N/K, because of the embedding of N/K into $\prod_{L'} N/L'$.

In the next theorem we use the function $RA_{G,X}$, and not $R\mathrm{Føl}_{G,X}$, because the result about extensions of Følner sets that we need to exploit uses the Følner function $A_{G,X}$. Nevertheless, these functions are $\sim$-equivalent. On the other hand, previously we used $R\mathrm{Føl}_{G,X}$ because the Følner function $\mathrm{Føl}_{G,X}$ is the one which is classically considered [51, 52].

Theorem 6.2.7. *Let G be a finitely generated residually amenable group, generated by the finite set X. Let $N \trianglelefteq G$ be residually solvable, with amenable quotient $Q = G/N$, and equip N with the proper metric d induced by the word metric of G.*

There exists a sequence of natural numbers $\{s_n\}_{n\in\mathbb{N}}$, which depends only on G and N, such that

$$RA_{G,X}(n) \leqslant |A_n| \cdot |B_n|, \tag{6.13}$$

where $N^{(s_n)}$ denotes the s_n-th derived subgroup of N, $|B_n| = A_{N/N^{(s_n)},\bar{d}}\bigl(2n^2|A_n|^2(2|A_n|_{\bar{X}}+n)\bigr)$ and $|A_n| = A_{Q,\bar{X}}(2n)$.

Proof. The number $s_n \in \mathbb{N}$ is the smallest natural number such that the finite set $N \cap B_{G,X}(2n)$ injects into the solvable quotient $N/N^{(s_n)}$. It always exists because N is residually solvable.

The proof follows the lines of Theorem 6.2.3, using Cavaleri's Theorem 2.1.6 to bound the Følner function of the solvable-by-amenable extension $G/N^{(s_n)}$. □

Remark 6.2.8. We stress that in Equation (6.13), the value for $|B_n|$ might be strictly larger than

$$RA_{N,d}\bigl(2n^2|A_n|^2(2|A_n|_{\bar{X}}+n\bigr),$$

and the former value cannot be replaced by the latter. Indeed, in order to be able to consider the quotient group G/K we need to work with a characteristic subgroup K of N.

For instance, if N is a free group of countable rank, then its residually amenable profile is bounded by its full residual finiteness growth in view of Lemma 6.1.5, which is obtained considering finite quotients, that is, normal subgroups of finite index. Nevertheless, characteristic subgroups of free groups of countable rank have infinite index, and hence the bound of Theorem 6.2.7 is obtained considering an extension of an infinite solvable group by an amenable quotient.

With this result, we can approach the free product case: let G and H be two finitely generated residually amenable groups, generated by the finite sets X and Y respectively, and consider the ball $B_{G*H,Z}(n)$, where $Z = X \sqcup Y$. Let Q_1 be an amenable quotient of G where $B_{G,X}(n)$ injects, and Q_2 be an amenable quotient of H where $B_{H,Y}(n)$ injects. Then, it follows that $B_{G*H,Z}(n)$ is injected into $Q_1 * Q_2$, which is the extension of the free group $[Q_1, Q_2]$ by the amenable quotient $Q_1 \times Q_2$.

Hence, we are in the situation of Theorem 6.2.7, and so we can find an amenable quotient of $Q_1 * Q_2$ where the (image in $Q_1 * Q_2$ of the) ball $B_{G*H,Z}(n)$ injects, and with a Følner set of cardinality bounded by Equation (6.13).

Example 6.2.9. Consider a non-amenable Baumslag-Solitar group

$$G := BS(p,q) = \langle a, t \mid t^{-1}a^p t = a^q \rangle,$$

where $2 \leqslant p \leqslant q$.

Consider the surjective homomorphism $\pi\colon G \twoheadrightarrow \mathbb{Z}$ defined on the generators by $\pi(a) = 0$ and $\pi(t) = 1$. The kernel $N = \ker \pi$ is the normal closure of a in G, and has the following presentation, neither finitely presented nor finitely generated (confront with [27], for instance):

$$\begin{aligned} N &= \langle \{a_i\}_{i\in\mathbb{Z}} \mid \{a_i^p = a_{i+1}^q\}_{i\in\mathbb{Z}} \rangle, \qquad \text{where } a_i := t^i a t^{-i} \text{ for all } i \in \mathbb{Z} \\ &\cong \cdots *_{a_{-3}^p = a_{-2}^q} \langle a_{-2} \rangle *_{a_{-2}^p = a_{-1}^q} \langle a_{-1} \rangle *_{a_{-1}^p = a_0^q} \langle a_0 \rangle *_{a_0^p = a_1^q} \langle a_1 \rangle *_{a_1^p = a_2^q} \langle a_2 \rangle *_{a_2^p = a_3^q} \cdots \qquad (6.14) \\ &= \cdots *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \cdots \end{aligned}$$

As $G/N = \mathbb{Z}$ is the abelianisation of G, it follows that N is the first derived subgroup $[G, G]$ of G. By a result of Kropholler [77] the second derived subgroup of G is a free group of countable rank. Let $D = [N, N]$ be the second derived subgroup of G. We have that

$$N/D \cong \frac{\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}}{\langle \{a_i^p = a_{i+1}^q\}_{i\in\mathbb{Z}} \rangle},$$

where the element a_i generates the group $\mathbb{Z}$ in the i-th coordinate.

Hence, the group $BS(p,q)$ is the extension of a free group of countable rank by the metabelian group Q given by the short exact sequence

$$\{e\} \longrightarrow N/D \hookrightarrow Q \twoheadrightarrow \mathbb{Z} \longrightarrow \{e\},$$

and its residually amenable profile is bounded as in Theorem 6.2.7.

In the next example we consider Houghton's groups, which are in fact (locally finite)-by-metabilian groups, and therefore elementary amenable.

Example 6.2.10. Let $n \geqslant 1$ and consider $S = \mathbb{N} \times \{1, \dots, n\}$. Let H_n be the subset of $\mathrm{Sym}(S)$ containing all the elements σ such that, on each $\mathbb{N} \times \{i\}$, the permutation σ is eventually a translation. It is a fact that the subset H_n forms a subgroup of $\mathrm{Sym}(S)$, called *Houghton's group* of rank n. They were defined by Houghton [74] (see also [25, 36]). It is known that, for $n \geqslant 3$ these groups are finitely presented [25], that their abelianisation is $\mathbb{Z}^n$, and that their second derived subgroup is $\mathrm{Alt}(S)$, the alternating group over the infinite set S, which is an infinite, locally finite, *not* finitely generated, simple group.

Therefore, H_n is an extension of the locally finite group $\mathrm{Alt}(S)$ by the metabelian group $H_n/\mathrm{Alt}(S)$.

The difficulty of obtaining *explicit* computations in these examples is at least twofold. In Example 6.2.10, one needs to deal with a locally finite, not finitely generated group, whose proper metric is induced by the Houghton's group H_n. A crude estimation of Følner sets can be obtained as follows: let B be the finite metric ball in $\mathrm{Alt}(S)$ of radius n. It contains only finitely many permutations of finite support, and hence the subset of elements of S which is permuted by any of the elements of B

$$\mathrm{supp}(B) = \big\{ s \in S \mid \exists \sigma \in B \text{ such that } \sigma(b) \neq b \big\}$$

is finite. Hence, a Følner set for B is surely given by $\mathrm{Sym}\big(\mathrm{supp}(B)\big)$, which is a finite symmetric group containing the subset B.

In Example 6.2.9, the obstacle is that the solvable group $N/N^{(s_n)}$, regardless of being finitely generated or not, is not equipped with a word metric, and that in principle its Følner sets are not the ones obtained with the residually amenable profile of N. Therefore, the knowledge of the residually amenable profile of N does not guarantee bounds for the Følner sets in $N/N^{(s_n)}$.

Chapter 7

Sofic groups

In this final chapter we consider sofic groups and some of their generalisations. The proof of Theorem 7.1.10 is fruit of collaboration with Goulnara Arzhantseva, Martin Finn-Sell and Lev Glebsky [5], while the second section is a joint work with Michal Ferov and Martin Finn-Sell.

7.1 Sofic groups

Sofic groups were introduced in 1999 by Gromov [62] (the name is coined by Weiss a year later [116]), in his work towards a solution of Gottschalk's surjunctivity conjecture in dynamical systems [58].

To define them, we need the concept of Hamming distance on a finite symmetric group. Let A be a finite set and consider, on the symmetric group $\mathrm{Sym}(A)$, the following distance, called the *normalised Hamming distance*:

$$d_A(\alpha, \beta) := \frac{1}{|A|}\Big|\Big\{a \in A \mid \alpha(a) \neq \beta(a)\Big\}\Big|. \tag{7.1}$$

The Hamming distance is bi-invariant, in the sense that, for any $\alpha, \beta, \gamma \in \mathrm{Sym}(A)$ we have that

$$d_A(\gamma\alpha, \gamma\beta) = d_A(\alpha, \beta) = d_A(\alpha\gamma, \beta\gamma).$$

Another way of expressing the Hamming distance between two permutations α and β is to count the elements in the set A which are *not* fixed by $\alpha^{-1}\beta$:

$$d_A(\alpha, \beta) = \frac{1}{|A|}\Big|\Big\{a \in A \mid (\alpha^{-1}\beta)(a) \neq a\Big\}\Big|.$$

Definition 7.1.1. A group G is *sofic* if for every finite subset $K \subseteq G$ and for every $\varepsilon > 0$ there exists a finite set A and a map $\varphi\colon G \to \mathrm{Sym}(A)$ satisfying:

(s1) (*almost homomorphism*) for all $k_1, k_2 \in K$ we have that $d_A\big(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)\big) \leqslant \varepsilon$;

(s2) (*uniform injectivity*) for all distinct $k_1, k_2 \in K$ we have that $d_A\big(\varphi(k_1), \varphi(k_2)\big) \geqslant 1 - \varepsilon$.

Historically, these conditions were proved to be equivalent to Gromov's original definition of soficity by Elek and Szabó in [46]. This equivalent definition of soficity has the advantage of *not* requiring the group G to be countable.

We call such a map $\varphi\colon G \to \mathrm{Sym}(A)$ a (K, ε)-*approximation* for the group G, and when K and ε are clear from the context, we refer to these maps as to approximations for the group.

From the definitions, we see that a $(K, 0)$-approximation is a K-approximation in the sense of Definition 2.3.1.

The following question drives the current interest in the topic:

Question 7.1.2. Does there exist a group which is not sofic?

As we will see, LEF groups (Proposition 7.1.5) and LEA groups (Proposition 7.1.7) are sofic, and thus a non-sofic group cannot be constructed starting from amenable *components*.

Remark 7.1.3. In Definition 7.1.1 one can replace the symmetric groups and their normalised Hamming distances with finite groups equipped with some bi-invariant distances (F, d). This leads to the notion of *weakly sofic groups*, defined in [56]. As the Hamming distance on $\mathrm{Sym}(A)$ is a bi-invariant distance, it follows that every sofic group is weakly sofic. The other implication is a known open problem.

We record the following immediate remark.

Remark 7.1.4. Let $\varepsilon \leqslant \delta$, G be a group, $K \subseteq K' \subseteq G$ be finite subsets and consider a (K, ε)-approximation $\varphi\colon G \to \mathrm{Sym}(A)$. Then φ is also a (K', δ)-approximation.

In view of this lemma, it suffices to consider constants ε such that $0 < \varepsilon < 1$. We collect here the two first classes of groups which satisfy the above definition.

Proposition 7.1.5. *Finite groups are sofic. Consequently, LEF groups are sofic.*

Proof. Let G be a finite group. By Cayley's Theorem, G embeds into the finite symmetric group $\mathrm{Sym}(G)$ via the map $L\colon G \to \mathrm{Sym}(G)$, where $L(g) = L_g$ is the permutation $L_g(h) = gh$ of G. As L is a homomorphism, we have that

$$d_G\big(L(gh), L(g)L(h)\big) = 0, \qquad \forall g, h \in G. \tag{7.2}$$

Moreover, if g and h are distinct elements of G, we have that

$$L_g(k) = gk \neq hk = L_h(k), \qquad \forall k \in G.$$

This implies that

$$d_G\big(L(g), L(h)\big) = 1, \qquad \forall g, h \in G,\, g \neq h. \tag{7.3}$$

Equations (7.2) and (7.3) imply that a finite group is sofic, and the approximations can always be chosen with $\varepsilon = 0$.

Suppose now that G is locally embeddable into finite groups, and let $K \subseteq G$ be a finite subset. Then, there exists a map $\varphi\colon G \to G_f$ into a finite group G_f that is injective on K and such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in K$. As G_f is a finite group, by the previous case we know that $L\colon G_f \to \mathrm{Sym}(G_f)$ is a $(G_f, 0)$-approximation. It is now immediate to check that

$$\Phi := L \circ \varphi\colon G \to \mathrm{Sym}(G_f)$$

is a $(K, 0)$-approximation. □

Proposition 7.1.5 shows that the approximations for LEF groups can always be chosen with $\varepsilon = 0$. Notice that the finitely generated group without finite quotients we considered in Example 3.2.9 is LEF, and hence sofic in view of Proposition 7.1.5. This group is not finitely presented, and the fact that it does not admit finite quotients exploits a limiting process for its infinitely related presentation. Strictly related to this, we pose the following question.

Question 7.1.6. Does there exists a finitely presented sofic group without finite quotients?

This question would have a positive answer if a finitely presented, infinite, simple sofic group existed. This is not known, and the Burger-Mozes simple groups constructed as amalgamated free products of free groups [26], are among candidates. Even more intriguing would be the existence of a finitely presented sofic group without finite quotients, which is in addition SQ-universal. Higman's group is a potential example to Question 7.1.6.

We now reproduce the proof that amenable groups are sofic, following [31, Proposition 7.5.6], because the understanding of the argument is required for the proof of Theorem 7.1.10.

Proposition 7.1.7. *Amenable groups are sofic. Consequently, LEA groups are sofic.*

Proof. Let $K \subseteq G$ be a finite subset and fix $\varepsilon > 0$. To symmetrise the set K, let us consider the finite subset

$$S = \big(\{e_G\} \cup K \cup K^{-1}\big)^2, \tag{7.4}$$

and set $\varepsilon' = \frac{\varepsilon}{|S|}$. As G is amenable, there exists a finite non-empty set $F \subseteq G$ such that

$$|F \setminus sF| \leqslant \varepsilon'|F|, \qquad \forall s \in S. \tag{7.5}$$

Consider the set $E = \bigcap_{s \in S} sF$. As the trivial element e_G is in S, we have that $E \subseteq F$. Moreover, as $S = S^{-1}$ is symmetric, we also have that

$$sE \subseteq F, \qquad \forall s \in S. \tag{7.6}$$

By construction, we obtain that

$$\begin{aligned} |F| - |E| = |F \setminus E| = \big|F \setminus \bigcap_{s\in S} sF\big| = \big|\bigcup_{s \in S}(F \setminus sF)\big| \\ \leqslant \sum_{s\in S}|F \setminus sF| \leqslant \sum_{s \in S}\varepsilon'|F| \leqslant \varepsilon|F|. \end{aligned} \tag{7.7}$$

Hence, it follows that

$$|E| = |F| - |F \setminus E| \geqslant (1-\varepsilon)|F|. \tag{7.8}$$

We now construct the map $\varphi\colon G \to \mathrm{Sym}(F)$, and prove that it is a (K, ε)-approximation.

As $|F| = |gF|$ for any element $g \in G$, we have that $|F \setminus gF| = |gF \setminus F|$. Hence, we can fix an arbitrary bijection $\alpha_g\colon gF \setminus F \to F \setminus gF$ for each $g \in G$. Consider then the map $\varphi\colon G \to \mathrm{Sym}(F)$ defined as

$$\varphi(g)(f) = \begin{cases} gf & \text{if } gf \in F, \\ \alpha_g(gf) & \text{if } gf \in gF \setminus F, \end{cases} \qquad \forall g \in G,\ \forall f \in F.$$

If $k_1, k_2 \in K$ and $f \in E$, we have that $k_2, k_1k_2 \in S$ by Equation (7.4), and that $k_2f, k_1k_2f \in F$ by Equation (7.6). This means that the permutations $\varphi(k_1k_2)$ and $\varphi(k_1)\varphi(k_2)$ coincide on $E \subseteq F$. Hence, it follows that

$$d_F\big(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)\big) \leqslant \frac{|F \setminus E|}{|F|} \leqslant \varepsilon, \qquad \forall k_1, k_2 \in K, \tag{7.9}$$

where the last inequality is due to Equation (7.7).

Suppose now that $k_1, k_2 \in K$ are distinct elements, and $f \in E$. By Equation (7.6) we have that

$$\varphi(k_1)(f) = k_1 f \neq k_2 f = \varphi(k_2)(f).$$

Hence, we have that

$$d_F\big(\varphi(k_1), \varphi(k_2)\big) \geqslant \frac{|E|}{|F|} \geqslant 1 - \varepsilon. \tag{7.10}$$

Equation (7.9) and Equation (7.10) imply that the amenable group G is sofic.

As in the proof of Proposition 7.1.5, from the amenable case we can immediately conclude the claim for LEA groups (that is, locally embeddable into amenable groups). □

It is clear that soficity is a local property, and hence a group that is locally embeddable into sofic groups is itself sofic. With the next theorem we collect the known stability results for the class of sofic groups. We refer to [28, 31, 34, 70] for the proofs.

Theorem 7.1.8. *The class of sofic groups is closed under taking subgroups, direct products, free products with amalgamation over amenable subgroups, HNN extensions over amenable subgroups, graph products, restricted wreath products, extensions with sofic kernel and amenable quotient.*

The finitely presented groups of Corollary 4.3.2 and of Corollary 4.3.4 are not known to be sofic, except for the case when the amalgamated subgroup is cyclic (case when $r = 1$). In all cases, they are not residually amenable nor LEA (as these notions coincide for finitely presented groups). If soficity will be proved for these groups, there will be further evidence that an amalgamated free product of sofic groups is again sofic, regardless of the amenability of the amalgamated subgroup, whilst now this permanence result is known only if the amalgamated subgroup is amenable, as remarked in Theorem 7.1.8.

The group G of Example 3.1.6 is sofic, because it is a split extension of a residually amenable group by an amenable group, but it is not itself residually amenable. Being an unrestricted wreath product of infinite groups, it is uncountable. A finitely generated split extension of a residually amenable group by an amenable group, which is itself not residually amenable, is given by Elek and Szabó in [48, §3]. As in Corollary 4.3.2 and in Corollary 4.3.4, Kazhdan's property (T) is crucial for the non residual amenability of this last example of Elek and Szabó.

Remark 7.1.9. While every solvable group is sofic by Proposition 7.1.7, it is an open problem to determine if an extension with solvable kernel and sofic quotient is again sofic. This extension problem is even open when the kernel is finite (solvable or not). Hence, the hypotheses of Theorem 3.3.5 and of Theorem 5.2.3 are not known to be satisfied for the class of sofic groups, and those results cannot be applied to yield an alternative proof for the soficity of free products and graph products.

We now present a new proof of the fact that any extension with sofic kernel and amenable quotient is again a sofic group. Our approach exploits unrestricted wreath products and the classical Kaloujnine-Krasner Theorem:

Kaloujnine-Krasner Theorem. *Let G and H be two groups, and let $\pi\colon G \wr H \twoheadrightarrow H$ denote the canonical projection onto the quotient H. There is a bijection of sets $E \mapsto E$ between*

$$\frac{\{E \mid e \to G \hookrightarrow E \twoheadrightarrow H \to e\}}{\textit{isomorphism of extensions}}$$

and

$$\frac{\{E \leqslant G \wr H \mid \pi(E) = H,\ E \cap \ker \pi \cong G \ \textit{via}\ \prod_H G \ni (g_x) \mapsto g_e \in G\}}{\textit{conjugacy of subgroups of}\ G \wr H}$$

that induces a group isomorphism between the extension E and the subgroup $E \leqslant G \wr H$.

The alternative proof of the soficity of any group extension with sofic kernel and amenable quotient combines the Kaloujnine-Krasner Theorem and Theorem 7.1.10, where we are going to prove that an unrestricted wreath product of a sofic group with an amenable group is again sofic.

For notational convenience in the next proof, given a set C, an element $c \in C$ and two permutations $\sigma, \tau \in \mathrm{Sym}(C)$, by $\big(\sigma\tau\big)(c)$ we mean $\tau\big(\sigma(c)\big)$, that is, the composition of permutations in $\mathrm{Sym}(C)$ is from the right to the left.

Theorem 7.1.10. *Let G be a sofic group and H be an amenable group. Then the unrestricted wreath product $G \wr H$ is sofic.*

Proof. Consider a finite subset $K \subseteq G \wr H$ and $\varepsilon > 0$. Without loss of generality, we may suppose that the identity element of $G \wr H$ belongs to K, and that $0 < \varepsilon < 1$.

Let us define the maps

$$\pi_1 \colon G \wr H \twoheadrightarrow \prod_H G, \qquad \pi_2 \colon G \wr H \twoheadrightarrow H$$

to be the projections onto the first and second component of the semidirect product $G \wr H$, respectively. The map π_1 is not a group homomorphism.

The set K is finite, hence there exists a finite separating set $I \subseteq H$ for $\pi_1(K) \subseteq \prod_H G$. This means that, given two distinct sequences (g_x) and (g'_x) in $\pi_1(K)$, there exists an index $i \in I$ that detects this difference, so that $g_i \neq g'_i$.

As the group H is amenable, its approximations can be constructed starting from Følner sets for the set we want to approximate (confront Proposition 7.1.7). Let

$$K_H = \pi_2(K) \cup I$$

and choose an ε_H such that

$$\varepsilon_H < \varepsilon. \tag{7.11}$$

By Proposition 7.1.7, there exist a finite subset $B \subseteq H$ and a (K_H, ε_H)-approximation $\varphi_H \colon H \to \mathrm{Sym}(B)$, such that

$$\varphi_H(h)b = \begin{cases} bh & \text{if } bh \in B \\ \alpha_h(bh) & \text{if } bh \in Bh \setminus B, \end{cases} \tag{7.12}$$

where $\alpha_h \colon Bh \setminus B \to B \setminus Bh$ is an arbitrarily chosen bijection. Moreover, there exists a set $E_B \subseteq B$ such that $|E_B| \geqslant (1 - \varepsilon_H)|B|$ and

$$\varphi_H(h)b = bh \qquad \text{and} \qquad \big(\varphi_H(h)\varphi_H(h')\big)b = bhh' = \varphi_H(hh')b \qquad \forall h, h' \in K_H, \forall b \in E_B. \tag{7.13}$$

By Equation (7.5), we have that

$$|B \setminus Bh| < \varepsilon_H |B|, \qquad \forall h \in K_H = I \cup \pi_2(K). \tag{7.14}$$

Concerning the approximation of G, consider

$$K_G = \{g_x \in G \mid (g_x) \in \pi_1(K),\, x \in B^{-1} \cdot B\}.$$

Choose an ε_G such that

$$1 - (1 - \varepsilon_H)(1 - \varepsilon_G)^{|B|} \leqslant \varepsilon. \tag{7.15}$$

This condition is equivalent to require that

$$(1 - \varepsilon_G)^{|B|} \geqslant \frac{1 - \varepsilon}{1 - \varepsilon_H}. \tag{7.16}$$

As $\varepsilon_H < \varepsilon$ by Equation (7.11), the right hand side of Equation (7.16) is strictly smaller than one. Hence, as the constants ε and ε_H - and consequently also $|B|$ - are fixed, there exists $\varepsilon_G > 0$ satisfying Equation (7.15).

As G is sofic, there exists a finite set A and a (K_G, ε_G)-approximation $\varphi_G\colon G \to \mathrm{Sym}(A)$. Moreover, there exists a set $E_A \subseteq A$ such that $|E_A| \geqslant (1-\varepsilon_G)|A|$ and

$$\big(\varphi_G(g)\varphi_G(g')\big)a = \varphi_G(gg')a, \qquad \forall g, g' \in K_G, \forall a \in E_A. \tag{7.17}$$

Given the approximations φ_G and φ_H, let us consider the finite set $C = B \times A^B$ and define the map $\varphi\colon G \wr H \to \mathrm{Sym}(C)$ as follows. For an element $(g,h) \in G \wr H$ and $(b,\tau) \in C$, let

$$\varphi\big((g,h)\big)(b,\tau) := \big(\varphi_H(h)b, \bar{\tau}\big), \tag{7.18}$$

where $\bar{\tau}\colon B \to A$ is the function defined by

$$\bar{\tau}(i) := \varphi_G(g_{b^{-1}i})\tau(i), \qquad \forall i \in B. \tag{7.19}$$

We now prove that φ satisfies the first condition for soficity, meaning that for (g,h) and (g',h') in the set F the inequality

$$d_C\big(\varphi(g,h)\varphi(g',h'), \varphi((g_x g'_{h^{-1}x}), hh')\big) \leqslant \varepsilon$$

holds. Hence, let $(b,\tau) \in C$. We have that

$$\varphi(\underbrace{(g_x g'_{h^{-1}x})}_{=(\gamma_x)}, hh')(b,\tau) = (\varphi_H(hh')b, \tau_1), \tag{7.20}$$

where

$$\tau_1(i) = \varphi_G(\gamma_{b^{-1}i})\tau(i) = \varphi_G(g_{b^{-1}i} g'_{h^{-1}b^{-1}i})\tau(i). \tag{7.21}$$

Moreover

$$\big(\varphi(g,h)\varphi(g',h')\big)(b,\tau) = \varphi(g',h')(\varphi_H(h)b, \tau_2) = \big((\varphi_H(h)\varphi_H(h'))b, \tau_3\big), \tag{7.22}$$

where

$$\tau_2(i) = \varphi_G(g_{b^{-1}i})\tau(i) \tag{7.23}$$

and

$$\tau_3(i) = \varphi_G(g'_{(\varphi_H(h)b)^{-1}i})\tau_2(i) = \big(\varphi_G(g_{b^{-1}i})\varphi_G(g'_{(\varphi_H(h)b)^{-1}i})\big)\tau(i). \tag{7.24}$$

Notice that if $b \in E_B$ (see Equation (7.13)), then $\varphi_H(h)b = bh \in B$. In this case, from Equation (7.24) we obtain that

$$\tau_3(i) = \big(\varphi_G(g_{b^{-1}i})\varphi_G(g'_{(bh)^{-1}i})\big)\tau(i) = \big(\varphi_G(g_{b^{-1}i})\varphi_G(g'_{h^{-1}b^{-1}i})\big)\tau(i). \tag{7.25}$$

Confronting Equations (7.21) and (7.25), we notice that the functions τ_1 and τ_3 will surely be equal if, in addition to $b \in E_B$, we have that $\tau(i) \in E_A$ for all $i \in B$ (see Equation (7.17)).

Hence, we conclude that

$$\begin{aligned} d_C\big(\varphi(g,h)\varphi(g',h'), \varphi((g_x g'_{h^{-1}x}), hh')\big) &\leqslant 1 - \frac{1}{|C|}\Big(|E_B| \cdot |E_A|^{|B|}\Big) \\ &= 1 - \frac{|E_B|}{|B|} \cdot \Big(\frac{|E_A|}{|A|}\Big)^{|B|} \\ &\leqslant 1 - (1-\varepsilon_H)(1-\varepsilon_G)^{|B|}. \end{aligned}$$

This is smaller or equal to ε in view of Equation (7.15).

It remains to be shown that φ satisfies also the second condition for soficity. Hence, let (g,h) and (g',h') be different elements of the finite set F. Two cases may occur: $h \neq h'$ or $h = h'$.

If $h \neq h'$ then

$$d_C\big(\varphi(g,h),\varphi(g',h')\big) \geqslant d_B\big(\varphi_H(h),\varphi_H(h')\big) \geqslant 1-\varepsilon_H \geqslant 1-\varepsilon, \tag{7.26}$$

where the last inequality is due to Equation (7.11).

If $h = h'$ then it must be that $(g_i) \neq (g'_i)$, and hence there exists an index i in the finite set I for which $g_i \neq g'_i$. To proceed, for every choice of $b \in B$ we want to be able to express this coordinate i as $b^{-1}b_i$, where $b_i \in B$ is an element which depends on i and on b. We have that

$$i = b^{-1}b_i \quad \Leftrightarrow \quad b^{-1} = ib_i^{-1} \quad \Leftrightarrow \quad b^{-1} \in B^{-1} \cap iB^{-1}.$$

As $|B^{-1} \cap iB^{-1}| = |B \cap Bi|$ and $|B \setminus Bi| \leqslant \varepsilon_H|B|$ for every $i \in I$ (by Equation (7.14)), we have that

$$|B^{-1} \cap iB^{-1}| = |B \cap Bi| = |B| - |B \setminus Bi| \geqslant |B| - \varepsilon_H|B| = (1-\varepsilon_H)|B|.$$

This means that, given $i \in I$, this coordinate can be expressed as $i = b^{-1}b_i$ with $b, b_i \in B$ for almost all choices of $b \in B$ (almost all meaning $\geqslant (1-\varepsilon_H)|B|$).

Suppose now that $(b,\tau) \in C$ and b is such that $i = b^{-1}b_i$, for $b_i \in B$. We have that

$$\varphi(g,h)(b,\tau) = (\varphi_H(h)b, \tau_4),$$

where $\tau_4(j) = \varphi_G(g_{b^{-1}j})\tau(j)$ for all $j \in B$, and

$$\varphi(g',h)(b,\tau) = (\varphi_H(h)b, \tau_5),$$

where $\tau_5(j) = \varphi_G(g'_{b^{-1}j})\tau(j)$ for all $j \in B$.

As G is sofic, $i = b^{-1}b_i$ and $g_{b^{-1}b_i}, g'_{b^{-1}b_i}$ are different elements, there exists $X \subseteq A$ such that $|X| \geqslant (1-\varepsilon_G)|A|$ and for which, for all $a \in X$, we have that $\varphi_G(g_{b^{-1}b_i})a \neq \varphi_G(g'_{b^{-1}b_i})a$.

Hence, the two functions τ_4 and τ_5 will surely be different if $\tau(b_i) \in X$. This implies that

$$\begin{aligned} d_C\big(\varphi(g,h),\varphi(g',h)\big) &\geqslant \frac{1}{|C|}\Big((1-\varepsilon_H)|B| \cdot (1-\varepsilon_G)|A| \cdot |A|^{|B|-1}\Big) \\ &= \frac{|B| \cdot |A|^{|B|}(1-\varepsilon_H)(1-\varepsilon_G)}{|B| \cdot |A|^{|B|}} = (1-\varepsilon_H)(1-\varepsilon_G) \geqslant 1-\varepsilon \end{aligned} \tag{7.27}$$

where the last inequality is due to Equation (7.15) and to $1-\varepsilon_G \geqslant (1-\varepsilon_G)^{|B|}$.

To conclude, combine Equation (7.26) and Equation (7.27) to see that φ satisfies condition $(s2)$ of Definition 7.1.1, as required. $\square$

Remark 7.1.11. The proof of Theorem 7.1.10 no longer works if we relax the condition on H from amenability to residual amenability (or to local embeddability into amenable groups, or to residual finiteness). Suppose that H is residually amenable. Using Proposition 2.3.6, we obtain a set $K_H \subseteq L \subseteq H$ and a binary operation $\diamond$ such that $(L,\diamond)$ is an amenable group, and such that $\diamond$ corresponds to the group operation of H when restricted to the finitely many elements of K_H.

This causes many problems, in particular when comparing Equations (7.21) and (7.24), because in this case one needs to restrict the attention to $b \in E_B \cap K_H$ (and not just $b \in E_B$, as in the amenable case). For everything to hold, we would need to impose, instead of Equation (7.15), the new condition

$$1 - \frac{|E_B \cap K_H|}{|B|}(1-\varepsilon_G)^{|B|} \leqslant \varepsilon.$$

on ε_G.

This, of course, cannot happen, because this new condition is equivalent to

$$(1-\varepsilon_G)^{|B|} \geqslant \frac{1-\varepsilon}{|E_B \cap K_H|/|B|}.$$

and we would therefore require the quantity $(1-\varepsilon_G)^{|B|}$ (which is smaller than one) to be bigger than $\frac{1-\varepsilon}{|E_B \cap K_H|/|B|}$. However, this last quantity is bigger than one, as $|K_H|/|B| \ll 1-\varepsilon$. This is a contradiction that exposes how amenability for the group H is the optimal condition in this approach (as well as in the proof of [48]), and it cannot be relaxed to residual amenability.

Remark 7.1.12. The strategy in the proof of Theorem 7.1.10 is not applicable to the (a priori) more general context of weak soficity either. Indeed, it is strictly related to soficity and symmetric groups equipped with the normalised Hamming distance.

The reason for this is that the sofic approximation for the group $G \wr H$ constructed in Equation (7.18) produces, starting from the maps $\varphi_G \colon G \to \mathrm{Sym}(A)$ and $\varphi_H \colon H \to \mathrm{Sym}(B)$, the new map $\varphi\colon G \wr H \to \mathrm{Sym}\big(B \times A^B\big)$. This new map is not diagonal, that is, its image is not contained in

$$\mathrm{Sym}(B) \times \mathrm{Sym}(A^B) \leqslant \mathrm{Sym}\big(B \times A^B\big).$$

This makes the whole procedure that follows incompatible with finite groups with normalised bi-invariant metrics, other than symmetric groups and their subgroups equipped with Hamming distances.

We record here that, similarly to the quantification of residual amenability of Chapter 6, there are several notions of quantification for soficity. One such is the *sofic dimension growth*, defined by Arzhantseva and Cherix [6], and recently considered by Cavaleri [29]. This notion studies the asymptotic behaviour of the function $K_{G,X} \colon \mathbb{N} \to \mathbb{N}$, defined as

$$K_{G,X}(n) := \min\Big\{|F| \mid \exists\ (B_{G,X}(n), \frac{1}{n})\text{-approximation } G \to \mathrm{Sym}(F)\Big\},$$

where G is a group generated by the finite set X.

Another quantitative notion for soficity is the *sofic profile*, introduced by de Cornulier [38]. This profile is much coarser than the sofic dimension growth, it is indeed bounded - that is trivial - on LEF groups. Nevertheless, this behavior seems to exactly grasp and convoy the fact that the approximations of LEF groups can *always* be taken with error ε equal to zero.

This is greatly beneficial for concrete examples. In particular, the group of Example 6.2.9 has a sofic profile that is $\sim$-equivalent to the Følner function of $\mathbb{Z}$, whilst the groups of Example 6.2.10 have a sofic profile that is $\sim$-equivalent to the Følner function of a metabelian group.

7.2 Conjugacy sofic groups

In this section we consider a strengthening of soficity and weak soficity.

Definition 7.2.1. A group G is *conjugacy sofic* if for every finite subset $K \subseteq G$ and for every $\varepsilon > 0$ there exist a finite set A and a map $\varphi\colon G \to \mathrm{Sym}(A)$ satisfying the following three conditions:

($s1$) for all $k_1, k_2 \in K$ we have that $d_A(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)) \leqslant \varepsilon$;

($s2$) for all distinct $k_1, k_2 \in K$ we have that $d_A(\varphi(k_1), \varphi(k_2)) \geqslant 1-\varepsilon$;

(cs) for all $\sigma \in \mathrm{Sym}(A)$, for all $k_1, k_2 \in K$ that are not conjugated in G, we have that

$$d_A(\varphi(k_1), \varphi(k_2)^\sigma) \geqslant 1-\varepsilon.$$

Notice that, as $\mathrm{id}_A \in \mathrm{Sym}(A)$, condition $(s2)$ is just a particular case of condition (cs) in the above definition. We call such a map φ a *conjugacy approximation.*

Remark 7.2.2. As in Remark 7.1.3, if we replace the symmetric group $\mathrm{Sym}(A)$ with a finite group F equipped with a bi-invariant metric d, we obtain the concept of *conjugacy weakly sofic groups.*

Classical conjugacy separable groups provide examples of conjugacy weakly sofic groups:

Definition 7.2.3. A group G is *conjugacy separable* if for every $g, h \in G$ that are not conjugated in G, there exists a finite quotient Q of G where the images of g and h are not conjugated.

Equivalently, a group is conjugacy separable if for every finite subset $K \subseteq G$ there exists a finite quotient Q such that K is injected into Q, and whenever $g, h \in K$ are not conjugated in G, then their images are not conjugated in Q. In particular, if $\pi\colon G \twoheadrightarrow Q$ denotes the projection, we have that $\pi\restriction_K$ is an injective map.

As finitely presented residually finite groups have a solvable word problem, finitely presented conjugacy separable groups were considered because they have a solvable conjugacy problem.

Lemma 7.2.4. *Let G be a conjugacy separable group. Then G is conjugacy weakly sofic.*

Proof. Let $\varepsilon > 0$ and $K \subseteq G$ be a finite subset. Let Q be the finite quotient of G that witnesses the conjugacy separability of G for the finite set K, and let $\pi\colon G \twoheadrightarrow Q$ be the projection onto the quotient Q.

Consider on Q the trivial bi-invariant metric

$$d(q, q') = \begin{cases} 0 & \text{if } q = q', \\ 1 & \text{if } q \neq q'. \end{cases}$$

Condition $(s1)$ of Definition 7.2.1 is immediately satisfied because π is a homomorphism, so

$$d(\pi(gh), \pi(g)\pi(h)) = 0 \qquad \forall g, h \in K.$$

The second and third conditions are also immediate, because all the elements involved are different, so their distance will always be equal to 1, and in particular greater than $1 - \varepsilon$. $\square$

In analogy with soficity, conjugacy soficity is a local property, and therefore groups that are locally embeddable into conjugacy (weakly) sofic groups are conjugacy (weakly) sofic. Moreover, the class of conjugacy (weakly) sofic groups is closed under taking subgroups and direct products.

One characterisation of soficity for countable groups is obtained in terms of metric ultraproducts of finite symmetric groups [31, Theorem 7.6.6]: a group G is sofic if and only if it can be embedded into a metric ultraproduct of finite symmetric groups equipped with Hamming distances, in such a way that different elements of G are sent to elements of maximal distance in the metric ultraproduct. This result was originally proved in [47].

A metric ultraproduct is defined in terms of an indexing set I, an ultrafilter ω on I, and a family $\{F_i\}_{i \in I}$ of nonempty finite sets, and it is the group

$$G_T = \frac{\prod_{i \in I} \mathrm{Sym}(F_i)}{N},$$

where T denotes the triple $(I, \omega, \{F_i\}_{i \in I})$ and N is the normal subgroup

$$\Big\{\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \mathrm{Sym}(F_i) \mid \lim_{i \to \omega} d_{F_i}(\mathrm{id}_{F_i}, \alpha_i) = 0\Big\} \trianglelefteq \prod_{i \in I} \mathrm{Sym}(F_i).$$

We define a partial order $\preccurlyeq$ on the index set I: $(K', \eta) \preccurlyeq (K, \varepsilon)$ if $K' \supseteq K$ and $\eta \leqslant \varepsilon$. Moreover, given the ultrafilter ω on I, the metric ultraproduct G_T inherits the bi-invariant metric

$$\delta_\omega(\alpha, \beta) := \lim_{i \to \omega} d_{F_i}(\alpha_i, \beta_i) \in [0, 1], \tag{7.28}$$

where $\alpha = (\alpha_i)_{i \in I}$ and $\beta = (\beta_i)_{i \in I}$. The limit of Equation (7.28) always exists, and it is unique, because $0 \leqslant d_{F_i}(\alpha_i, \beta_i) \leqslant 1$ is a bounded sequence in a Haussdorf space, and ω is an ultrafilter.

The embedding of a sofic group into a metric ultraproduct is obtained considering the index set I given by all pairs (K, ε), where K is a finite subset of G and $\varepsilon > 0$. For each $i = (K, \varepsilon)$ then there exist a finite set F_i and a (K, ε)-approximation $\varphi_i : G \to \mathrm{Sym}(F_i)$. Therefore, the injective homomorphism $\Phi : G \to G_T$ is given by the composition $\pi \circ \widetilde{\varphi}$, where

$$\widetilde{\varphi} = \prod_{i \in I} \varphi_i : G \to \prod_{i \in i} \mathrm{Sym}(F_i)$$

is the product map and $\pi : \prod_{i \in I} \mathrm{Sym}(F_i) \twoheadrightarrow G_T$ is the canonical projection. As recalled previously, different elements of G are sent by Φ to elements at distance one in G_T.

In Theorem 7.2.5 we prove that a conjugacy sofic group G can be embedded in the same fashion into a metric ultraproduct of finite symmetric groups, with the additional property that whenever two elements are not conjugated in G then their images are not conjugated in G_T.

Theorem 7.2.5. *Let G be a group. The following are equivalent:*

1. *G is conjugacy sofic;*

2. *there exist a triple $T = \big(I, \omega, \{F_i\}_{i \in I}\big)$ and an injective homomorphism $\Phi : G \to G_T$ that sends different elements of G to elements of G_T at distance one. Moreover, if $g, h \in G$ are not conjugated in G, then $\Phi(g)$ and $\Phi(h)$ are not conjugated in G_T, and $\Phi(g)$ and $\Phi(h)^\gamma$ have distance one in G_T for all $\gamma \in G_T$.*

Proof. If the second condition is satisfied, then G is necessarily conjugacy sofic.

Suppose hence that G is conjugacy sofic. We closely follow the construction of [31, Theorem 7.6.6], and we consider the set I consisting of all pairs (K, ε), where K is a finite subset of G and $\varepsilon > 0$. The group G is conjugacy sofic, therefore for each $i = (K, \varepsilon) \in I$ there exists a finite set F_i and a (K, ε)-approximation $\varphi_i : G \to \mathrm{Sym}(F_i)$ satisfying, in addition, condition (cs).

Then, the map $\Phi : G \to G_T$ obtained composing the product map $\prod_{i \in I} \varphi_i$ with the projection from $\prod_{i \in I} \mathrm{Sym}(F_i)$ onto G_T is an injective homomorphism, as proved in [31, Theorem 7.6.6], and different elements of G have images whose distance is one in G_T.

Suppose that g, h are two non-conjugated elements of G, and consider $\gamma = (\gamma_i)_{i \in I} N \in G_T$. We want to prove that $\Phi(g)$ and $\gamma \Phi(h) \gamma^{-1}$ have distance one in G_T.

Let $K = \{g, h\} \subseteq G$, $\varepsilon > 0$ and consider $i_0 = (K, \varepsilon) \in I$. Then, as G is conjugacy sofic, we have that

$$d_{F_i}\big(\varphi_i(g), \gamma_i \varphi_i(h) \gamma_i^{-1}\big) \geqslant 1 - \eta \geqslant 1 - \varepsilon, \qquad \forall i = (K', \eta) \preccurlyeq i_0.$$

This implies that

$$\delta_\omega\big(\Phi(g), \gamma \Phi(h) \gamma^{-1}\big) = \lim_{i \to \omega} d_{F_i}\big(\varphi_i(g), \gamma_i \varphi_i(h) \gamma_i^{-1}\big) = 1.$$

This concludes the proof. $\square$

We now see how the addition of condition (cs) in Definition 7.2.1 seriously changes the approximations that one is allowed to consider:

Proposition 7.2.6. *Let p be a prime number. If the cyclic group $\mathbb{Z}_p$ is conjugacy sofic, then its conjugacy approximations cannot be homomorphisms.*

Proof. Let G denote the cyclic group of order p. As G is cyclic, all its elements are pairwise not conjugated. Suppose that a homomorphism $\varphi\colon G \to \mathrm{Sym}(A)$ is a conjugacy approximation for the group G, and let $g, h \in G \setminus \{e_G\}$ be non-trivial different elements. As G is cyclic, there exists $n \in \{2, \dots, p-1\}$ such that $h = g^n$. As the order of g and h is the prime number p and φ is a homomorphism, it follows that this is the order of $\varphi(g)$ and $\varphi(h)$ too.

In $\mathrm{Sym}(A)$ the following holds: two permutations are conjugated if and only if they can be decomposed with the same number of disjoint cycles of the same lengths [31, Proposition C.3.2].

We apply this to $\varphi(g)$ and $\varphi(h)$: express $\varphi(g)$ as a product of commuting cycles

$$\varphi(g) = \sigma_1 \dots \sigma_k, \qquad \sigma_i \in \mathrm{Sym}(A).$$

As g has order p in G, we have that each σ_i has order, that is length, p. Moreover, as φ is a homomorphism, we have that

$$\varphi(h) = \varphi(g^n) = \varphi(g)^n = \sigma_1^n \dots \sigma_k^n$$

and these cycles σ_i^n are again of length p, because n is coprime with p. Hence the above fact implies that $\varphi(g)$ and $\varphi(h)$ are conjugated in $\mathrm{Sym}(A)$, which is a contradiction. □

Proposition 7.2.7. *If the group $\mathbb{Z}$ is conjugacy sofic, then its conjugacy approximations cannot be homomorphisms.*

Proof. Looking for a contradiction, suppose we have a finite set $K \subseteq \mathbb{Z}$ and an $\varepsilon > 0$, and that there exists a homomorphism $\varphi\colon \mathbb{Z} \to \mathrm{Sym}(A)$ into a finite symmetric group satisfying conditions $(s1)$, $(s2)$ and (cs).

As φ is a homomorphism, the image $\varphi(\mathbb{Z})$ is a finite cyclic group: let $\iota\colon \varphi(\mathbb{Z}) \to \mathrm{Sym}(A)$ be the subgroup inclusion. Properties $(s1)$, $(s2)$ and (cs) satisfied by φ (with respect to K and ε) imply that

1. for all different $h, k \in \varphi(\mathbb{Z})$ we have that $d_A(\iota(h), \iota(k)) \geqslant 1 - \varepsilon$;

2. for all different $h, k \in \varphi(\mathbb{Z})$, for all $\sigma \in \mathrm{Sym}(A)$ we have that $d_A(\iota(h), \iota(k)^\sigma) \geqslant 1 - \varepsilon$.

The second condition is a contradiction with Proposition 7.2.6. □

These results imply the following corollary.

Corollary 7.2.8. *Conjugacy approximations cannot be homomorphisms.*

Proof. Let $\varphi\colon G \to \mathrm{Sym}(A)$ be a homomorphism which is a $(K, 0)$-conjugacy approximation, for some finite subset $K \subseteq G$.

If G has elements of finite order (other than the trivial element), then it has an element of order a prime number. This element generates a cyclic subgroup of prime order, and Proposition 7.2.6 produces a contradiction for the existence of such a φ.

If G is torsion-free, then it has $\mathbb{Z}$ as a subgroup, and Proposition 7.2.7 again shows a contradiction for φ. □

We conclude with the following remark. We defined and considered conjugacy sofic groups because they are naturally related to the known concept of conjugacy separable groups. Nevertheless, Definition 7.2.1 can be generalised to the following setting:

Definition 7.2.9. Let $F = F(a, b)$ be the free group on two generators, and consider a reduced word $v \in F$. A group G is *v-sofic* (or *verbally sofic*) if for every finite subset $K \subseteq G$ and for every $\varepsilon > 0$ there exist a finite set A and a map $\varphi\colon G \to \mathrm{Sym}(A)$ satisfying the following conditions:

($s1$) for all $k_1, k_2 \in K$ we have that $d_A(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)) \leqslant \varepsilon$;

($s2$) for all $k_1, k_2 \in K$ we have that $d_A(\varphi(k_1), \varphi(k_2)) \geqslant 1 - \varepsilon$;

(vs) for all $k_1, k_2 \in K$ such that $k_1 \neq v(x, k_2)$ for all $x \in G$, we have that $d_A\big(\varphi(k_1), v(y, \varphi(k_2))\big) \geqslant 1-\varepsilon$ for all $y \in \mathrm{Sym}(A)$.

If the map φ satisfies these conditions, we call it a *v-approximation.*

Therefore, the conjugacy sofic groups discussed in this section are v-sofic groups for the choice of the word

$$v = aba^{-1} \in F_2.$$

As we saw in Corollary 7.2.8, for this choice of the word aba^{-1} the v-approximations cannot be homomorphisms, because the conjugacy classes in finite symmetric groups are determined by the cycle decompositions of the permutations. It is not, therefore, clear how to construct these conjugacy approximations in basic examples such as (finite or infinite) cyclic groups, or not even if the additional condition (cs) of Definition 7.2.1 forces the group G to be trivial.

On the other hand, could it happen that a group fails to be conjugacy sofic, but it is conjugacy weakly sofic? This is a well-known open problem in the context of soficity and weak soficity. Or, for a word v other than aba^{-1}, could it happen that a group is not v-sofic, but it is weakly v-sofic (we did not define explicitly weakly v-sofic groups. Analogously to weakly sofic groups, they are obtained from Definition 7.2.9 replacing symmetric groups by finite groups equipped with bi-invariant metrics)?

Question 7.2.10. Can a group be conjugacy weakly sofic, but not conjugacy sofic?

We did not pursue this direction of research, but we notice that natural choices for the word v in the context of v-soficity could be (iterated) commutators. Would these other choices for the word v imply a result along the lines of Corollary 7.2.8?

All this leads to the following problem:

Problem 7.2.11. *Let $v \in F_2$ be a reduced word. Give sufficient conditions so that the class of v-sofic groups is not trivial.*

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- residually amenable groups
- amenable groups, weak amenability notions
- sofic groups, generalisations
- space of marked groups
- rigidity of graph product decompositions
- one-relator groups
- Kaplansky's direct/stable finiteness conjecture

Totally disconnected locally compact groups
- algebraic/topological entropy of automorphisms
- Willis' scale function

Publications

Published, accepted papers

1. **Scale function vs topological entropy**. (joint with D. Dikranjan and A. Giordano Bruno) Topology Appl. 160 (2013), no. 18, 2314 – 2334;
2. **Residual properties of free products**. Comm. Algebra 44 (2016), no. 7, 2959 – 2980;
3. **Residual properties of graph products of groups**. (joint with M. Ferov) J. Group Theory 19 (2016), no. 2, 217 – 231;

Preprints, papers in preparation

4. **Groups satisfying Kaplansky's stable finiteness conjecture**. Preprint (2015), 9 pages (ArXiv: http://arxiv.org/abs/1501.02893);
5. **Unrestricted wreath products and sofic groups**. (joint with G. Arzhantseva, M. Finn-Sell and L. Glebsky) Preprint (2016);
6. **Residually amenable profile**. In preparation;
7. **Conjugacy sofic groups**. (joint with M. Ferov and M. Finn-Sell) in preparation.

Relevant, recent coursework

Advanced, semester courses by Goulnara Arzhantseva

- more infos at http://www.mat.univie.ac.at/~gagt/GAGTCourse.html

- **Introduction to RAAGs**;
  - Salvetti complex, subgroups, graph products, CAT(0) cubical complexes
- **Geometry and analysis on free groups**;
  - random subgroups, Stallings' foldings, (non-)amenability, C^*-simplicity, weak amenability properties
- **Coarse embeddings of infinite graphs and groups**;
  - coarse amenability, coarse embeddings, box spaces, property (T), expanders

Research seminars, conference talks

7 May '13 *Scale function and topological entropy in locally compact totally disconnected groups*, seminar, University of Vienna;

14 May '13 *Groupoids: an introduction and various examples*, seminar, University of Vienna;

17 Dec '13 *Residually amenable groups: stability properties*, seminar, University of Vienna;

28 Jan '14 *von Neumann algebras and II_1 factors*, seminar, University of Vienna;

1 July '14 *Scale function and topological entropy*, invited talk at the special session "Algebraic entropy and topological entropy" of the "First joint international meeting RSME-SCM-SEMA-SIMAI-UMI", Bilbao;

28 Oct '14 *Stably finite groups*, seminar, University of Vienna;

8 Apr '15 *Graph products of residually amenable groups*, invited seminar, University of Udine;

30 June '15 *Graph products of residually amenable groups*, conference talk at "Workshop on the Hanna Neumann conjecture", Bilbao;

6 Aug '15 *Residual properties of graph products of groups*, conference talk at "Summer school - Probability on graphs and groups", Lausanne.

Conferences, workshops, schools

Apr '13 *Locally compact groups beyond Lie theory*. Spa, Belgium;

Apr '13 *Word maps and stability of representations*. Erwin Schrödinger International Institute, Vienna, Austria;

Apr '13 *ESI anniversary*, Erwin Schrödinger International Institute, Vienna, Austria;

June '13 *Introductory workshop and summer school on the geometry of outer space: investigated through its analogy with Teichmüller space*. Marseille, France;

Sept '13 *Geometric and analytic group theory*. Ventotene, Italy;

Apr '14 *Geometry of computations in groups*. Erwin Schrödinger International Institute, Vienna, Austria;

June '14 *Growth in groups*. Neuchâtel, Switzerland;

July '14 *First joint international meeting RSME-SCM-SEMA-SIMAI-UMI*. Bilbao, Spain;

Dec '14 *Geometric and combinatorial group theory*. Jerusalem, Israel;

Jan '15 *Young geometric group theory IV*. Spa, Belgium;

July '15 *Workshop on the Hanna Neumann conjecture*. Bilbao, Spain;

Aug '15 *Summer school - Probability on graphs and groups*. Lausanne, Switzerland;

Jan - Mar '16 *Measured group theory* trimester, Erwin Schrödinger International Institute, Vienna, Austria;
ESI winter school *Measured group theory*, 1 – 12 Feb '16;
ESI conference *Measured group theory*, 15 – 19 Feb '16;

Mar '16 *Ischia group theory 2016*. Ischia, Italy.

Jul '16 Two-week visit at University of the Basque Country, Bilbao, Spain;

Sept '16 *XI encuentro en teoría de grupos*. Barcelona, Spain.

Languages

Italian: native
Friulian: native
English: fluent
Castillian: fluent
German: basic

Computer languages, programs

Familiar with C++, MATLAB, Maxima. Experienced with GAP and GeoGebra.

Other activities

- For one year (Oct '09 - Sept '10) I was a students representative at the University of Udine in the *Consiglio di Facoltà*, *Consiglio degli Studenti*, and *Comitato per lo sport universitario*.
- For two years (Sept '11 - Mar '13) I was coach and assistant coach for junior basketball teams.