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Aspects of linear and nonlinear Semigroup Theory applied to Cauchy problems for evolution equations

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Preface

This thesis is meant as an overview of the Theory of Semigroups and its applications to the abstract Cauchy Problem. Chapter 1 explains some motivation behind the definition of semigroups and how these could be used to solve partial differential equations. In Chapter 2 we will define what uniformly bounded C_0 semigroups are and describe how unbounded linear operators generate such. We will prove the famous Hille-Yosida theorem and show just how useful semigroups can be when it comes to a certain type of PDEs. In Chapter 3 we will generalize the results of Chapter 2 for nonlinear operators as far as possible - it will contain a variant of the celebrated Crandall-Liggett Theorem - and end with an illustration of the theory by some examples.

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Notations

$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$\{0, 1, 2, \dots\}$
$\mathbb{R}_+$	$\{t \in \mathbb{R} t \in (0, \infty)\}$
$\ \zeta\ _E$	Euclidean norm of $\zeta \in \mathbb{R}^n$
$(\zeta, \eta)_E$	Dot product of the two vectors $\zeta, \eta \in \mathbb{R}^n$
X	A real Banach space
X^*	Dual space of X ; $\{x^* : X \rightarrow \mathbb{R} x^* \text{ linear and bounded}\}$
(y, x^*)	Application of a function $x^* \in X^*$ to an element $y \in X$
$\ \cdot\ $	Norm of Banach space/Dual space
$\text{conv}(V)$	Convex hull of a subset $V \subseteq X$
$\overline{V}$	Closure with respect to the norm of the space X
B^k	Composition of the operator B k times with itself
$\mathcal{D}(B)$	Domain of the operator B
$B \subseteq A$	$\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and $Bx = Ax$ for all $x \in \mathcal{D}(B)$
$\mathbf{1}$	Identity map
$\mathcal{L}(X)$	Set of bounded linear operators from X to X
$\sigma(B)$	The spectrum of B ; $\{z \in \mathbb{C} (\mathbf{1} - zB)^{-1} \in \mathcal{L}(X)\}$
$\rho(B)$	The resolvent set; $\mathbb{C} - \sigma(B)$
$F(x)$	Duality Mapping; $\{x^* \in X^* (x, x^*) = \ x\ ^2 = \ x^*\ ^2\}$
$\frac{d}{dt}^+ f(t)$	Right derivative; $\lim_{h \searrow 0} \frac{f(t+h) - f(t)}{h}$
Ω	Open subset of $\mathbb{R}^n$
$\text{supp}(u)$	For a function $u : \Omega \rightarrow \mathbb{R}$ this is defined as the set $\overline{\{x \in \Omega u(x) \neq 0\}}$
$C_c^\infty(\Omega)$	$\{u : \Omega \rightarrow \mathbb{R} u \text{ is infinitely many times differentiable and } \text{supp}(u) \text{ is compact}\}$
$C_b^\infty(\Omega)$	$\{u : \Omega \rightarrow \mathbb{R} u \text{ is infinitely many times differentiable and has finite essential supremum}\}$
$L^2(\Omega)$	$\{u : \Omega \rightarrow \mathbb{R} u \text{ is measurable and } \int_\Omega u^2 < \infty\}$
$(u, v)_{L^2(\Omega)}$	$\int_\Omega uv$ for $u, v \in L^2(\Omega)$
$H_0^1(\Omega)$	Closure of $C_c^\infty(\Omega)$ under the norm $\ u\ _{H_0^1}^2 := \int_\Omega u^2 + \sum_i \int_\Omega \left(\frac{\partial}{\partial x_i} u\right)^2$

$C(I, X)$ Set of continuous functions from the intervall
 $I \subseteq \mathbb{R}$ into X
 $\mathbf{L}^1(I, X)$ $\{u : I \rightarrow X \text{ Bochner measurable} \mid \int_I \|f(t)\| < \infty\}$

CHAPTER 1

Some motivation

We will start this chapter by some heuristic considerations which will then motivate the definition of operator semigroups.

1.1 Viewing a PDE as an 'ODE' in a Banach space

At the heart of semigroup theory and its application to PDEs stands the observation that in certain circumstances one can view a PDE as an "ODE" in a suitable Banach space.

Suppose that we are trying to solve the heat equation

$$\frac{d}{dt}u(x, t) - \Delta u(x, t) = 0, \quad (x, t) \in \Omega \times (0, T], \quad (1.1)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T], \quad (1.2)$$

$$u(x, 0) = g(x), \quad x \in \Omega, \quad (1.3)$$

where Ω is an open and bounded subset of $\mathbb{R}^n$, with Lipschitz boundary $\partial\Omega$.

As a first attempt to determine solvability of this equation, we could see if the Lax-Milgram Lemma, where we just perceive the time variable as a "space" variable, applies. However this approach does not work, as the bilinear form will not be coercive. So a different method is necessary.

We could look at the equation from a different angle. One way is to view t as a "distinguished" variable. Instead of considering the function u mapping values from the space $\Omega \times [0, T]$ to $\mathbb{R}$, we consider a vector valued function v that maps every $t \in [0, T]$ to $v(t) = u(t, \cdot)$. Heuristically, the heat equation

rewrites as:

$$\frac{d}{dt}v(t) = Bv(t), \quad t \in (0, T], \quad (1.4)$$

$$v(0) = g, \quad (1.5)$$

where $B : \mathcal{D}(B) \rightarrow L^2(\Omega)$ is defined by

$$\mathcal{D}(B) := \{x \in H_0^1(\Omega) | \exists y \in L^2(\Omega) : (y, z)_{L^2(\Omega)} = -(\nabla x, \nabla z)_{L^2(\Omega)} \forall z \in H_0^1(\Omega)\}$$

$$Bx := y.$$

Remark. The "weak formulation" (1.4)-(1.5) is obtained by noticing that if $\varphi : \Omega \rightarrow \mathbb{R}$ is a test function - that is to say $\varphi \in C_c^\infty(\Omega)$ - then, multiplying (1.1) by $\varphi(x)$ and integrating, we get

$$\int_{\Omega} u_t \varphi = \int_{\Omega} \Delta u \varphi = - \int_{\Omega} \nabla u \nabla \varphi.$$

It is evident that many PDEs which include a time derivative can be rewritten in this form for an operator B .

All in all we are interested in the solvability of the so called Cauchy Problem associated to an operator $B : \mathcal{D}(X) \subseteq X \rightarrow X$:

$$\frac{d}{dt}v(t) = Bv(t), \quad t \in (0, T], \quad (1.6)$$

$$v(0) = x, \quad (1.7)$$

with initial value $x \in X$. From now on X will always denote a Banach space. So the question that immediately arises is, for which operator B are we able to construct a solution. This is the content of the upcoming chapters.

1.2 A first approach for bounded linear operators

To get an idea what a solution of the Cauchy Problem could look like, we look at the simplest case at first. This section follows loosely Chapter 1, Section 1 of Pazy [1].

From the theory of ODEs we know that the constant-coefficient linear initial value problem

$$u'(t) = bu(t), \quad t \in (0, T], \quad (1.8)$$

$$u(0) = u_0, \quad (1.9)$$

for some fixed $b \in \mathbb{R}$, has the unique solution $u(t) = e^{bt}u_0$. The important observation here is that $e^{bt}u_0$ is a solution of the problem above primarily because differentiating e^{bt} is the same as multiplying it by b . This follows from the fact that the exponential function has a series expansion, namely $e^{bt} = \sum_{k=0}^{\infty} \frac{(bt)^k}{k!}$. As this series is a power series, differentiation can be pulled in and, after rearranging the sum, one can take the constant b out, arriving at the expression be^{bt} .

It turns out that this approach works even if instead of a constant b we would have taken a linear bounded operator B , defined on the whole Banach space.

Motivated by the above, consider an operator $B \in \mathcal{L}(X)$.

Definition 1.1. We define for the bounded linear operator tB , where $t \in [0, \infty)$, the corresponding exponential function $e^{tB} : X \rightarrow X$, as the linear function

$$e^{tB}x := \sum_{k=0}^{\infty} \frac{1}{k!} (tB)^k x. \quad (1.10)$$

Remark. Actually we would have had to check before the definition if the right side of the definition makes sense. Indeed this is the case. On the one hand, tBx is in X . So iteratively $(tB)^k x$ is in X for any $k \in \mathbb{N}$. On the other hand, we have

$$\left\| \sum_{k=m}^n \frac{1}{k!} (tB)^k x \right\| \leq \|x\| \sum_{k=m}^n \frac{1}{k!} (t\|B\|)^k. \quad (1.11)$$

Therefore we get that $\sum_{k=0}^N \frac{1}{k!} (tB)^k x$ is a Cauchy sequence in a Banach space. So the limit exists and $e^{tB}x$ thus is well defined. The linearity follows directly from taking partial sums of e^{tB} and then passing to the limit.

Before turning to the question if $e^{tB}x$ actually solves the Cauchy Problem, we note some very important properties of this function.

Proposition 1. For any $x \in X$ and $s, t \in [0, \infty)$ we have

$$\begin{aligned} (1) \quad e^{(t+s)B} &= e^{sB} \circ e^{tB} & (2) \quad e^{0B} &\equiv \mathbf{1} \\ (3) \quad \lim_{t \searrow 0} e^{tB}x &= x & (4) \quad \|e^{tB}\| &\leq e^{t\|B\|}. \end{aligned}$$

Proof. (4) follows by a simple computation. We get, because $\|(tB)^k\| \leq \|tB\|^k$ holds, the following

$$\|e^{tB}x\| = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (tB)^k x \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} t^k \|B\|^k \|x\| = e^{t\|B\|} \|x\|.$$

(2) is trivial as when $t = 0$ only the identity operator is left in the sum. As for (3), we have the following estimate

$$\begin{aligned}\|e^{tB}x - x\| &= \left\| \sum_{k=1}^{\infty} \frac{1}{k!} (tB)^k x \right\| = \left\| \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (tB)^{k+1} x \right\| \\ &\leq \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \|(tB)^{k+1}\| \|x\| \\ &\leq t \|B\| e^{t\|B\|} \|x\|.\end{aligned}$$

So, for fixed x , we get (3) by letting $t \searrow 0$.

To prove (1) we rewrite $e^{sB} \circ e^{tB}$:

$$\sum_{k=0}^{\infty} \frac{1}{k!} (sB)^k \left(\sum_{l=0}^{\infty} \frac{1}{l!} (tB)^l x \right) = \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{1}{(k-l)!} (sB)^{k-l} \frac{1}{l!} (tB)^l.$$

This can be proven by using that e^{sB} is continuous - (4) - and arguing as it is done in a standard proof of the Cauchy Product Formula. We further deduce, by using the binomial theorem,

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{l=0}^k \frac{1}{(k-l)!} (sB)^{k-l} \frac{1}{l!} (tB)^l &= \sum_{k=0}^{\infty} \frac{1}{k!} B^k \sum_{l=0}^k \frac{k!}{(k-l)!} s^{k-l} t^l \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} B^k (t+s)^k.\end{aligned}$$

As the last sum is equal to $e^{(t+s)B}$, we are done. $\square$

We will now see how Proposition 1 can be used to solve the Cauchy Problem for an operator $B \in \mathcal{L}(X)$. For the difference quotient from above and below, in $t > 0$, of the function $t \mapsto e^{tB}x$, with $h > 0$, we get

$$\left\{ \begin{array}{c} \frac{e^{(t+h)B}x - e^{tB}x}{h} \\ \frac{e^{(t-h)B}x - e^{tB}x}{-h} \end{array} \right\} = \left\{ \begin{array}{c} \frac{e^{hB}e^{tB}x - e^{tB}x}{h} \\ \frac{e^{(t-h)B}x - e^{hB}e^{(t-h)B}x}{-h} \end{array} \right\} = \left\{ \begin{array}{c} \frac{e^{hB} - 1}{h} e^{tB}x \\ \frac{e^{hB} - 1}{h} e^{(t-h)B}x \end{array} \right\}.$$

So we see that the limit of the expression $\frac{1}{h}(e^{hB} - 1)$, when h goes to 0 from above, plays an important role in determining the derivative. As for any

$h > 0$, the estimate, in the operator norm,

$$\begin{aligned} \left\| \frac{e^{hB} - \mathbf{1}}{h} - B \right\| &= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} h^{k-1} B^k \right\| \leq \sum_{k=2}^{\infty} \frac{1}{k!} h^{k-1} \|B\|^k \\ &\leq \sum_{k=0}^{\infty} \frac{1}{(k+2)!} h^{k+1} \|B\|^{k+2} \\ &\leq h \|B\|^2 e^{h\|A\|} \end{aligned}$$

holds, we get for the right derivative

$$\frac{d^+}{dt} e^{tB} x = B e^{tB} x. \quad (1.12)$$

With the same estimate, and by using (3) and (4) of Proposition 1, we get that the left derivative is also equal to $B e^{tB} x$, as the following chain of inequalities shows

$$\begin{aligned} &\left\| \frac{e^{hB} - \mathbf{1}}{h} e^{(t-h)B} x - B e^{tB} x \right\| \\ &\leq \left\| \frac{e^{hB} - \mathbf{1}}{h} e^{(t-h)B} x - \frac{e^{hB} - \mathbf{1}}{h} e^{tB} x \right\| + \left\| \frac{e^{hB} - \mathbf{1}}{h} e^{tB} x - B e^{tB} x \right\| \\ &\leq \left\| \frac{e^{hB} - \mathbf{1}}{h} \right\| \|e^{(t-h)B}\| \|x - e^{hB} x\| + \left\| \frac{e^{hB} - \mathbf{1}}{h} - B \right\| \|e^{tB} x\| \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

All in all, we have proven the following theorem.

Theorem 1.1. *Given an operator $B \in \mathcal{L}(X)$ the function $t \mapsto e^{tB} x$ solves the Cauchy Problem for any initial value $x \in X$.*

Remark. One can also prove that $t \mapsto e^{tB} x$ is actually the unique solution. For this we refer to Chapter 3 or [1].

Example 1.1.

(1) Taking a matrix $B \in \mathbb{R}^{n \times n}$ and $X = \mathbb{R}^n$ we see, by what we have just said, that the solution to the corresponding Cauchy Problem is given by $u(t) = e^{tB} x$.

(2) Let $K : [0, 1]^2 \rightarrow \mathbb{R}$ be a continuous function. Then $B : \mathcal{C}([0, 1], \mathbb{R}) \rightarrow \mathcal{C}([0, 1], \mathbb{R})$, where $\mathcal{C}([0, 1], \mathbb{R})$ is endowed by the supremum norm $\|u\| := \sup_{x \in [0, 1]} |u(x)|$ for $u \in \mathcal{C}([0, 1], \mathbb{R})$, defined by

$$Bf(\cdot) := \int_0^1 K(\cdot, y) f(y) dy$$

is a bounded linear operator as

$$\|Bf\| \leq \int_0^1 \|K(\cdot, y)\| \|f(\cdot)\| dy \leq \max_{(x,y) \in [0,1]^2} \|K(x, y)\| \|f\|$$

holds. So we get that $u(t) = e^{tB}x$ solves the integro-differential equation

$$\frac{d}{dt}u(t, x) = \int_0^1 K(x, y)u(t, x)dy.$$

So now we are able to solve the Cauchy Problem for bounded linear operators. However, in a lot of applications the operator B may be linear but not bounded (for example, this is the case for the heat equation). It is therefore important to see what happens when we only assume linearity. This will be the content of the next chapter.

CHAPTER 2

Semigroups generated by unbounded linear operators

2.1 Introduction

As for unbounded linear operators the definition of $e^{tB}x$ by a series doesn't necessarily make sense - because $\sum_{k=0}^{\infty} \frac{1}{k!}(tB)^k x$ may not converge - another approach is needed. One way to do so is to define this "exponential" as the limit of a sequence of exponentials which stem from bounded linear operators. This also has the advantage that many proofs can be boiled down to the bounded case.

2.2 Uniformly bounded C_0 semigroups

We have seen in the previous chapter, more specifically in Proposition 1, that, for $B \in \mathcal{L}(X)$, the function $t \mapsto e^{tB}x$ has some interesting properties, which came in handy when dealing with the Cauchy Problem. To be able to solve the Cauchy Problem for unbounded linear operators, we therefore try to find a function with the same properties. On one hand this makes sense because it encapsulates the linear bounded case and on the other, when calculating its derivative, we want to be able to use the same "tricks" as in Chapter 1. We therefore define:

Definition 2.1. *Let X be a Banach space. A family $\{S(t) : X \rightarrow X | t \geq 0\}$ of linear operators is called a uniformly bounded C_0 semigroup if the following conditions*

- $$\begin{array}{ll} (1) & S(s+t)x = S(s)S(t)x \quad (2) \quad S(0) \equiv \mathbf{1} \\ (3) & \lim_{t \searrow 0} S(t)x = x \quad (4) \quad \|S(t)\| \leq 1 \end{array}$$

are fulfilled for any $s, t \in [0, \infty)$ and $x \in X$.

It may seem odd, as we are dealing with unbounded operators in this chapter, that we included in the definition that $S(t)$ must be bounded. However, as we already mentioned, we will construct the semigroup stemming from an unbounded linear operator as the pointwise limit of bounded operators. By the uniform boundedness principle this limit operator is therefore also bounded. Thus, this condition indeed does make sense.

Remark. The properties of this definition are the exact same properties than in Proposition 1. The only difference is that, for a given bounded linear operator, we look at the semigroup $e^{-t\|B\|}e^{tB}$ instead of just looking at e^{tB} , which also explains the upper bound in item (4) of Definition 2.1.

If the fourth condition is omitted one just speaks of a C_0 semigroup.

We have the following theorem.

Theorem 2.1. *If $S(t)$ is a C_0 semigroup then there exist constants $c \geq 0$ and $D \geq 1$ so that for all $t \in [0, \infty)$ we have*

$$\|S(t)\| \leq De^{ct}.$$

Proof. For a proof see [1]. □

Therefore, when dealing with a C_0 semigroup, one can use Theorem 2.1 and look at the uniformly bounded C_0 semigroup $(De^{ct})^{-1}S(t)$ instead. C_0 semigroups do not really generate any new results. We'll concentrate therefore on uniformly bounded C_0 semigroups and just mention the generalised results. From now on we only write semigroup when talking about uniformly bounded C_0 semigroups, unless stated otherwise.

Let's assume for a second that there is a solution of the Cauchy Problem, associated to an operator B , and in analogy to Chapter 1 this solution is a semigroup. Also in Chapter 1 we have seen that knowing $\lim_{t \searrow 0} \frac{S(t)x - x}{t}$ corresponded to the initial operator B evaluated in x . So the question now is, in what case do we have a semigroup $S(t)$, for a given linear unbounded operator B , such that the expression $\lim_{t \searrow 0} \frac{S(t)x - x}{t}$ can be written as Bx ? It therefore makes sense to introduce the following definition.

Definition 2.2. If $S(t)$ is a semigroup we define its infinitesimal generator by

$$Bx := \lim_{t \searrow 0} \frac{S(t)x - x}{t} \quad \text{whenever it exists}$$

$$\mathcal{D}(B) := \left\{ x \in X \left| \lim_{t \searrow 0} \frac{S(t)x - x}{t} \text{ exists} \right. \right\}.$$

Remark. It is easy to see that $\mathcal{D}(B)$ is a linear subspace.

First we check that, if given a semigroup and its infinitesimal generator, it actually fulfills the equation $\frac{d}{dt}S(t)x = BS(t)x$ for $x \in \mathcal{D}(B)$.

Theorem 2.2. Let $S(t)$ be a semigroup and B its infinitesimal generator. Then for any $x \in \mathcal{D}(B)$ we have $S(t)x \in \mathcal{D}(B)$ and

$$\frac{d}{dt}S(t)x = BS(t)x = S(t)Bx. \quad (2.1)$$

Furthermore $t \mapsto S(t)x$ is continuous for any $x \in X$.

Proof. This proof follows loosely the proofs of Corollary 2.3 and Theorem 2.4 of Chapter 1 of Pazy[1].

We will prove first that $t \mapsto S(t)x$ is continuous. For $h > 0$, small enough, we have

$$\|S(t+h)x - S(t)x\| = \|S(t)S(h)x - S(t)x\| \leq \underbrace{\|S(t)\|}_{\leq 1} \|S(h)x - x\|$$

and

$$\begin{aligned} \|S(t-h)x - S(t)x\| &= \|S(t-h)x - S(t-h)S(h)x\| \\ &\leq \underbrace{\|S(t-h)\|}_{\leq 1} \|S(h)x - x\|. \end{aligned}$$

And so, as the right hand side, in each case, goes to zero, we have that our function is indeed continuous.

Now as for the other part of the proof we look at the limit from above and below. So, take again $h > 0$ and a $x \in \mathcal{D}(B)$. Then we get

$$\frac{S(t+h)x - S(t)x}{h} = \frac{S(t)S(h)x - S(t)x}{h} = S(t) \frac{S(h)x - x}{h}.$$

Letting $h \searrow 0$ we therefore get $\left(\frac{d}{dt}\right)^+ S(t)x = S(t)Bx$. But, from the above it also follows that we can write

$$S(t)\frac{S(h)x - x}{h} = \frac{S(h) - \mathbf{1}}{h}S(t)x.$$

And so if $h \searrow 0$ then, as the limit on the left hand side exists, we get on the one hand $S(t)x \in \mathcal{D}(B)$ and on the other $BS(t)x = S(t)Bx$.

To see that the derivative from above is also equal to the derivative from below we write, for $h > 0$,

$$\frac{S(t-h)x - S(t)x}{-h} = \frac{S(t-h)x - S(t-h)S(h)x}{-h} = S(t-h)\frac{S(h)x - x}{h}.$$

Now as

$$\begin{aligned} & \left\| S(t-h)\frac{S(h)x - x}{h} - S(t)Bx \right\| \\ & \leq \left\| S(t-h)\frac{S(h)x - x}{h} - S(t-h)Bx \right\| + \|S(t-h)Bx - S(t)Bx\| \\ & \leq \underbrace{\|S(t-h)\|}_{\leq 1} \left\| \frac{S(h)x - x}{h} - Bx \right\| + \|S(t-h)Bx - S(t)Bx\| \end{aligned}$$

holds, and the right side of this estimate goes to zero, because $t \mapsto S(t)x$ is continuous and because $\left(\frac{d}{dt}\right)^+ S(t)x = S(t)Bx$ holds by above, we get

$$\lim_{h \searrow 0} \frac{S(t-h)x - S(t)x}{-h} = S(t)Bx.$$

This completes the proof. $\square$

Remark. So we see that a crucial ingredient so that the derivative of $S(t)x$ exists is, that $x \in \mathcal{D}(B)$ holds; this was also the case in Chapter 1 where $\mathcal{D}(B) = X$.

One implication of Theorem 2.2 is that the infinitesimal generator of a semigroup $S(t)$ must be closed. This means that, as we will see later on, for our construction of a semigroup stemming from an unbounded linear operator B to work, we must impose the condition of closedness upon B .

Corollary 2.1. Let $S(t)$ be a semigroup and B its infinitesimal generator. Then B is a closed operator.

Proof. The ideas of this proof are borrowed from Chapter 1, Section 1, Theorem 2.4 and Corollary 2.5, of Pazy [1].

Assume that we have a sequence $x_n \in \mathcal{D}(B)$ such that $x_n \rightarrow x$ and $Bx_n \rightarrow y$. Then integrating equation (2.1), which can be done for any x_n as $u \mapsto S(u)Bx_n$ is continuous, gives us

$$S(t)x_n - x_n = \int_0^t S(u)Bx_n du.$$

The left side now converges, as $S(t)$ is bounded, to $S(t)x - x$. Because

$$\|S(t)Bx_n - S(t)y\| \leq \underbrace{\|S(t)\|}_{\leq 1} \|Bx_n - y\| \leq \|Bx_n - y\|$$

holds, we see

$$\lim_{n \rightarrow \infty} \int_0^t S(u)Bx_n du = \int_0^t S(u)y du.$$

So, in total we have

$$S(t)x - x = \int_0^t S(u)y du. \quad (2.2)$$

Dividing this expression by $t > 0$ and letting t go to zero we get for the right side of (2.2)

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^t S(u)y du = y, \quad (2.3)$$

as

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t S(u)y du - y \right\| &= \left\| \frac{1}{t} \left[\int_0^t (S(u)y - y) du \right] \right\| \\ &\leq \sup_{u \in [0, t]} \|S(u)y - y\| \xrightarrow[t \searrow 0]{} 0 \end{aligned}$$

holds. And so finally we obtain

$$\lim_{t \searrow 0} \frac{S(t)x - x}{t} = y.$$

This means on the one hand $x \in \mathcal{D}(B)$ and on the other $Bx = y$. Therefore B is closed. $\square$

As Theorem 2.2 states, we get, for $x \in \mathcal{D}(B)$, that $S(t)x \in \mathcal{D}(B)$ holds. Therefore we might expect, as the Riemann Integral of the function $t \mapsto S(t)x$ is the limit of a sequence I_n of linear combinations which are all in the linear subspace $\mathcal{D}(B)$, that the integral $\int_0^T S(t)x dt$ is in $\overline{\mathcal{D}(B)}$. An even stronger statement is true, as the following Corollary shows.

Corollary 2.2. Let $S(t)$ be a semigroup and B its infinitesimal generator. The following hold for any $x \in X$ and $T \in [0, \infty)$:

- (1) $\lim_{h \rightarrow 0} \frac{1}{h} \int_T^{T+h} S(t)x dt = S(T)x.$
- (2) $\int_0^T S(t)x dt \in \mathcal{D}(B).$
- (3) $B \left(\int_0^T S(t)x dt \right) = S(T)x - x$
- (4) For any $x \in \mathcal{D}(B)$ and $t_0, t \geq 0$:

$$S(t)x - S(t_0)x = \int_{t_0}^t BS(s)x ds = \int_{t_0}^t S(s)Bx ds.$$

Proof. The ideas for the following proof and its implication are borrowed from the proof of Theorem 2.4 and Corollary 2.5 in Chapter 1 of Pazy [1].

(4) is a direct consequence of the previous theorem. The function $s \mapsto S(s)Bx$ is continuous in s on $[t, t_0]$ and so the Riemann Integral exists. By taking the integral of $\frac{d}{ds}S(s)x = S(s)Bx$ on both sides we are done.

Now to the proof of (1). Again, $t \mapsto S(t)x$ is continuous and so the integral exists. From the following

$$\begin{aligned} \left\| \frac{1}{h} \int_T^{T+h} S(t)x dt - S(T)x \right\| &= \left\| \frac{1}{h} \left[\int_T^{T+h} (S(t)x - S(T)x) dt \right] \right\| \\ &\leq \sup_{u \in [0, h]} \|S(T-u)x - S(T)x\| \\ &\leq \sup_{u \in [0, h]} \|S(u)x - x\|, \end{aligned}$$

where the last estimate follows the same way as in the proof of continuity of $t \mapsto S(t)x$, we deduce that, as the expression in the last line of the estimate goes to zero as $h \rightarrow 0$, that (1) holds.

(2) and (3) hold because for a $h > 0$ small enough we get, since $S(h)$ is a bounded linear operator,

$$\frac{S(h) - \mathbf{1}}{h} \int_0^T S(t)x dt = \frac{1}{h} \int_0^T (S(t+h)x - S(t)x) dt$$

and furthermore since

$$\frac{1}{h} \int_0^T S(t+h)x dt = \frac{1}{h} \int_T^{T+h} S(t)x dt + \frac{1}{h} \int_h^T S(t)x dt$$

holds, we have

$$\frac{S(h) - \mathbf{1}}{h} \int_0^T S(t)x dt = \frac{1}{h} \int_T^{T+h} S(t)x dt - \frac{1}{h} \int_0^h S(t)x dt.$$

Now letting $h \rightarrow 0$ and using (1), we have proven (2) and (3). $\square$

We see from Corollary 2.2, statement (1), that every $x \in X$ can be represented as the limit of $h^{-1} \int_0^h S(t)x dt$ where h goes to 0, since $S(0)x = x$. However this integral is by statement (2) of Corollary 2.2 always in $\mathcal{D}(B)$. Therefore $\mathcal{D}(B)$ is dense in X . So we have the following

Lemma 2.1. For the infinitesimal generator B of a semigroup $S(t)$ we have that the domain of B is dense in X , meaning $\overline{\mathcal{D}(B)} = X$.

We will close this section by a Lemma which shows that a semigroup is already determined by its infinitesimal generator B . For a proof we refer to [1].

Lemma 2.2. If two semigroups $S(t)$ and $T(t)$ have the same infinitesimal generator B then they are equal.

2.3 Yosida Approximation

We now turn our attention to the question of how we could actually construct a semigroup $S(t)$, given a linear unbounded operator $B : \mathcal{D}(X) \rightarrow X$, so that we might be able to solve the corresponding Cauchy Problem. As we have seen in Theorem 2.2, the operator B should correspond to its infinitesimal generator. So by Corollary 2.1 and Lemma 2.1 we can only consider operators B which are densely defined and closed.

As mentioned in the previous chapter, the idea is to boil down the construction of $S(t)$ to bounded linear operators. So how could we approximate pointwise an operator B by a family $\{B_\lambda\}_{\lambda>0}$ of bounded linear operators? One, and probably the most obvious way, is to try to find operators $C_\lambda : X \rightarrow X$, which converge pointwise to the identity map when λ goes to infinity and which have the property that $C_\lambda \circ B$ is bounded for any $\lambda > 0$.

Heuristically speaking, assume that we have a sequence $a_n \in \mathbb{R}_+$ with $a_n \nearrow +\infty$. One way to obtain a bounded sequence out of a_n by multiplying it

with another sequence c_n , is to choose c_n as a suitable function of a_n^{-1} . If for example we take $c_n = \left(1 + \frac{1}{\lambda}a_n\right)^{-1}$, we obtain $c_na_n \leq \lambda$ for all $n \in \mathbb{N}$. So with this choice of c_n we control the growth of c_na_n .

This works as well for operators. If we choose $C_\lambda = \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1}$, assuming $\mathbb{R}_+ \subseteq \rho(B)$, we are able to "control the growth" of $C_\lambda \circ B$ and obtain thus a bounded operator, for each $\lambda > 0$, since

$$B_\lambda x = \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} Bx = \lambda \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} \left(\frac{1}{\lambda}Bx - x + x\right) \quad (2.4)$$

$$= \lambda \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} x - \lambda \mathbf{1}x \quad (2.5)$$

holds.

Lemma 2.3. Assume that B is a unbounded linear closed operator that has a dense domain. Furthermore assume $\mathbb{R}_+ \subseteq \rho(B)$ and

$$\left\| \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} \right\| \leq 1 \quad \forall \lambda > 0. \quad (2.6)$$

Then, for any fixed $x \in X$, the following holds:

$$\lim_{\lambda \rightarrow \infty} \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} x = x. \quad (2.7)$$

Proof. The ideas of the proof are borrowed from Lemma 3.2. of Chapter 1 of Pazy [1].

For any $x \in \mathcal{D}(B)$ we get, by using the assumptions, that

$$\begin{aligned} \left\| \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} x - x \right\| &= \left\| \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} \left(\frac{1}{\lambda}Bx - \frac{1}{\lambda}Bx + x\right) - x \right\| \\ &= \left\| \frac{1}{\lambda} \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} Bx \right\| \\ &\leq \frac{1}{\lambda} \left\| \left(\mathbf{1} - \frac{1}{\lambda}B\right)^{-1} \right\| \|Bx\| \\ &\leq \frac{1}{\lambda} \|Bx\| \end{aligned}$$

holds. Now letting λ go to infinity the Lemma is proven for $x \in \mathcal{D}(B)$.

For a general $x \in X$ one just takes a sequence $x_n \in \mathcal{D}(B)$ with $x_n \rightarrow x$. This

can be done as the operator domain is dense. From the estimate

$$\begin{aligned}
\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} x - x \right\| &\leq \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} x - \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} x_n \right\| \\
&\quad + \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} x_n - x_n \right\| + \|x_n - x\| \\
&\leq \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} \right\| \|x - x_n\| + \frac{1}{\lambda} \|Bx_n\| + \|x_n - x\| \\
&\leq \|x - x_n\| + \frac{1}{\lambda} \|Bx_n\| + \|x_n - x\|,
\end{aligned}$$

where we used the estimate from before, we see that if n is chosen such that $\|x_n - x\| \leq \epsilon$, we get, by taking λ big enough,

$$\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} x - x \right\| \leq 3\epsilon.$$

As ϵ was arbitrarily small we are done. $\square$

Remark. One drawback of taking a family $(\mathbf{1} - \frac{1}{\lambda} B)^{-1}$ is that it is usually quite hard to determine if $\mathbb{R}_+ \subseteq \rho(B)$ holds and if this family is really uniformly bounded by 1.

Having this result at hand we now proceed as follows.

Definition 2.1. (Yosida Approximation) With the same assumptions as in Lemma 2.3, we define on $\mathcal{D}(B)$, for $\lambda > 0$, the Yosida Approximation of B as the operator

$$B_\lambda x := \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} Bx.$$

Now as for any $x \in \mathcal{D}(B)$ we have $Bx \in X$ we can use Lemma 2.3 to get

$$\lim_{\lambda \rightarrow \infty} \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} Bx = Bx. \tag{2.8}$$

The Yosida Approximation thus tells us that we can approximate pointwise, under certain conditions, an unbounded linear operator B , by bounded ones.

So as we want to construct a semigroup by pointwise approximation out of the family $\{e^{tB_\lambda}\}_{\lambda>0}$, we need to check that for any $x \in X$ the limit of $e^{tB_\lambda}x$, when $\lambda \rightarrow \infty$, actually exists. Also we need to check if the limit is a semigroup. The following two Lemmata address these issues.

Lemma 2.4. With the same assumptions as in Lemma 2.3 we get that the family $\{e^{tB_\lambda}\}_{t \geq 0}$, for fixed λ , is a semigroup and that

$$\|e^{tB_\lambda}x - e^{tB_\gamma}x\| \leq t \|B_\lambda x - B_\gamma x\|$$

holds for any $x \in X$ and $\lambda, \gamma > 0$.

Proof. We follow the proof of Lemma 3.4. in Chapter 1 of Pazy [1]. That e^{tB_λ} is a C_0 semigroup follows by Chapter 1 Proposition 1. For it to be a uniformly bounded C_0 semigroup we still need to check (4) of Definition 2.1. We get by using (2.4)-(2.5), the estimate

$$\|e^{tB_\lambda}\| = \|e^{t\lambda(\mathbf{1} - \frac{1}{\lambda}B)^{-1} - t\lambda\mathbf{1}}\| = \|e^{-t\lambda\mathbf{1}}e^{t\lambda(\mathbf{1} - \frac{1}{\lambda}B)^{-1}}\| \quad (2.9)$$

$$= \|e^{-t\lambda}\mathbf{1}e^{t\lambda(\mathbf{1} - \frac{1}{\lambda}B)^{-1}}\| \quad (2.10)$$

$$= \|e^{-t\lambda}e^{t\lambda(\mathbf{1} - \frac{1}{\lambda}B)^{-1}}\| \quad (2.11)$$

$$= e^{-t\lambda} \|e^{t\lambda(\mathbf{1} - \frac{1}{\lambda}B)^{-1}}\| \quad (2.12)$$

$$\leq e^{-t\lambda} e^{t\lambda\|(\mathbf{1} - \frac{1}{\lambda}B)^{-1}\|} \quad (2.13)$$

$$\leq 1. \quad (2.14)$$

That is to say, $\{e^{tB_\lambda}\}_{t \geq 0}$ is a semigroup.

For the second part of the proof we rewrite $e^{tB_\lambda}x - e^{tB_\gamma}x$ as

$$e^{tB_\lambda}x - e^{tB_\gamma}x = e^{tsB_\lambda}e^{t(1-s)B_\gamma}x \Big|_{s=0}^{s=1}.$$

Now by showing that $(\mathbf{1} - \frac{1}{\lambda}B)^{-1}$ and $(\mathbf{1} - \frac{1}{\gamma}B)^{-1}$ commute, we can show by using (2.4)-(2.5) that B_λ and B_γ commute. Therefore also $e^{tB_\lambda}, e^{tB_\gamma}, B_\lambda, B_\gamma$ commute. With this we get

$$\begin{aligned} \frac{d}{ds} e^{tsB_\lambda} e^{t(1-s)B_\gamma} x &= \frac{d}{ds} (e^{tsB_\lambda}) e^{t(1-s)B_\gamma} x + e^{tsB_\lambda} \frac{d}{ds} e^{t(1-s)B_\gamma} x \\ &= tB_\lambda e^{tsB_\lambda} e^{t(1-s)B_\gamma} x - t e^{tsB_\lambda} B_\gamma e^{t(1-s)B_\gamma} x \\ &= t e^{tsB_\lambda} e^{t(1-s)B_\gamma} (B_\lambda - B_\gamma) x. \end{aligned}$$

So, if we take the integral over this last term, which can be done as it is

continuous, we get

$$\begin{aligned}
\|e^{tB_\lambda}x - e^{tB_\gamma}x\| &= \left\| \int_0^1 \frac{d}{ds} e^{tsB_\lambda} e^{t(1-s)B_\gamma} x ds \right\| \\
&= \left\| \int_0^1 t e^{tsB_\lambda} e^{t(1-s)B_\gamma} (B_\lambda - B_\gamma) x ds \right\| \\
&\leq \int_0^1 \|t e^{tsB_\lambda} e^{t(1-s)B_\gamma} (B_\lambda - B_\gamma) x\| ds \\
&\leq t \|B_\lambda x - B_\gamma x\|,
\end{aligned}$$

where in the last line we used (2.14). $\square$

Lemma 2.5. Assume the same conditions as in Lemma 2.3. Then

$$S(t)x := \lim_{\lambda \rightarrow \infty} e^{tB_\lambda}x, \quad x \in X,$$

defines a semigroup.

Proof. This proof is borrowed from the proof of Theorem 3.1 of Chapter 1 of Pazy [1].

By Lemma 2.4 we directly get from

$$\|e^{tB_\lambda}x - e^{tB_\gamma}x\| \leq t \|B_\lambda x - B_\gamma x\| \leq t \|B_\lambda x - Bx\| + t \|Bx - B_\gamma x\|,$$

for fixed $t > 0$ and $x \in \mathcal{D}(B)$, that $e^{tB_\lambda}x$ is a Cauchy sequence in λ , since $\lim_{\lambda \rightarrow \infty} B_\lambda x = Bx$ holds. So $\lim_{\lambda \rightarrow \infty} e^{tB_\lambda}x$ exists.

We still need to prove that this actually constitutes a semigroup. A simple calculation shows

$$S(t+s)x = \lim_{\lambda \rightarrow \infty} e^{(t+s)B_\lambda}x = \lim_{\lambda \rightarrow \infty} e^{tB_\lambda} e^{sB_\lambda}x.$$

Now the right side is equal to $\lim_{\lambda \rightarrow \infty} e^{tB_\lambda} S(s)x$ as the following holds

$$\|e^{tB_\lambda} e^{sB_\lambda}x - e^{tB_\lambda} S(s)x\| \leq \underbrace{\|e^{tB_\lambda}\|}_{\leq 1} \underbrace{\|e^{sB_\lambda}x - S(s)x\|}_{\rightarrow 0}.$$

So we get

$$S(t+s)x = \lim_{\lambda \rightarrow \infty} e^{tB_\lambda} S(s)x = S(t)S(s)x.$$

$S(0)x = x$ for all x is obvious as $e^{0B_\lambda}x = x$.

The continuity in 0, that is to say $\lim_{t \searrow 0} S(t)x = x$, follows from the estimate

$$\begin{aligned}
\|S(t)x - x\| &\leq \|S(t)x - e^{tB_\lambda}x\| + \|e^{tB_\lambda}x - x\| \\
&\leq t \|B_\lambda x - Bx\| + \|e^{tB_\lambda}x - x\|,
\end{aligned}$$

where we used Lemma 2.4. By taking λ big enough such that $\|B_\lambda x - Bx\|$ is small and then taking $t > 0$ small enough such that $\|e^{tB_\lambda}x - x\|$ is also small, we see that the right side of the above estimate is also small. So continuity is proven.

Finally, since $\sup_{\|x\|=1} \|e^{tB_\lambda}x\| \leq 1$, as it was done in the proof of Lemma 2.4, holds, we get directly that $\|S(t)x\| \leq 1$ for $\|x\| = 1$, by letting λ go to infinity. Therefore $\|S(t)\| \leq 1$ is true. $\square$

We have seen in this section how one can generate a semigroup given a certain type of operator B . In the next section we will see exactly how a semigroup corresponds to an operator and the way we can use this to solve the linear Cauchy Problem.

2.4 The Hille-Yosida Theorem and the linear Cauchy Problem

In Section 2.2 we found out that if a semigroup $S(t)$ exists, its infinitesimal generator must be closed and densely defined. Furthermore, in Section 2.3, we've seen that given an unbounded linear operator B , fulfilling the assumptions of Lemma 2.3, we can construct a semigroup.

One question now is, if this also works the other way around. Meaning, that if given a semigroup $S(t)$, does this imply that $\mathbb{R}_+ \subseteq \rho(B)$ and $\left\| \left(1 - \frac{1}{\lambda}B\right)^{-1} \right\| \leq 1$ for all $\lambda > 0$, holds for the infinitesimal generator B ?

Another question is, if starting from a semigroup and going to its infinitesimal generator, is the semigroup obtained by applying the Yosida Approximation to the infinitesimal generator equal to the original one?

We will first answer the second question with the following two Lemmata.

Lemma 2.6. Assume that $S(t)$ is the semigroup of Lemma 2.5, that is to say, constructed by the Yosida Approximation B_λ . Then we have

$$\frac{d}{dt}S(t)x = S(t)Bx = BS(t)x, \quad x \in \mathcal{D}(B).$$

Proof. We follow Pazy [1].

By Theorem 2.2 we only need to prove that, for all $x \in \mathcal{D}(B)$, B is the infinitesimal generator of the semigroup. We have the following estimate

$$\left\| \frac{S(h)x - x}{h} - Bx \right\| \leq \left\| \frac{S(h)x - x}{h} - B_\lambda x \right\| + \|B_\lambda x - Bx\|.$$

The first term on the right side can be estimated by

$$\left\| \frac{S(h)x - x}{h} - \frac{e^{hB_\lambda}x - x}{h} \right\| + \left\| \frac{e^{hB_\lambda}x - x}{h} - B_\lambda x \right\|$$

and furthermore, using the estimate of Lemma 2.4, we get

$$\begin{aligned} \left\| \frac{S(h)x - x}{h} - \frac{e^{hB_\lambda}x - x}{h} \right\| &= \left\| \frac{S(h)x - e^{hB_\lambda}x}{h} \right\| \\ &\leq \|Bx - B_\lambda x\|. \end{aligned}$$

So, all in all, we have

$$\left\| \frac{S(h)x - x}{h} - Bx \right\| \leq 2\|Bx - B_\lambda x\| + \left\| \frac{e^{hB_\lambda}x - x}{h} - B_\lambda x \right\|.$$

By choosing λ big enough the first term will become arbitrarily small, as it is just the Yosida Approximation, and then by choosing $h > 0$ also small the second term also becomes arbitrarily small, because B_λ is the infinitesimal generator of e^{hB_λ} . We are therefore done with the proof. $\square$

Lemma 2.7. Assume the same conditions as in Lemma 2.3 for the operator B . Further let B be the infinitesimal generator of the semigroup $S(t)$. Then we have for any $x \in X$

$$S(t)x = \lim_{\lambda \rightarrow \infty} e^{tB_\lambda}x.$$

Proof. We follow Pazy [1].

By assumption we know, after Lemma 2.5, that $T(t)x := \lim_{\lambda \rightarrow \infty} e^{tB_\lambda}x$ is a semigroup. We will now show that $T(t) \equiv S(t)$ holds for all t . Looking at the derivative of the function $s \mapsto S(t-s)T(s)x$ at $0 \leq s \leq t$, for $x \in \mathcal{D}(B)$, we get

$$\begin{aligned} &\frac{d}{ds} S(t-s)T(s)x \\ &= \lim_{\hat{s} \rightarrow s} \frac{S(t-\hat{s})T(\hat{s})x - S(t-s)T(s)x}{\hat{s} - s} \\ &= \lim_{\hat{s} \rightarrow s} \frac{S(t-\hat{s})T(\hat{s})x - S(t-\hat{s})T(s)x + S(t-\hat{s})T(s)x - S(t-s)T(s)x}{\hat{s} - s} \\ &= \lim_{\hat{s} \rightarrow s} S(t-\hat{s}) \frac{T(\hat{s})x - T(s)x}{\hat{s} - s} + \lim_{\hat{s} \rightarrow s} \frac{S(t-\hat{s}) - S(t-s)}{\hat{s} - s} T(s)x \end{aligned}$$

Now for the first term we get

$$\lim_{\hat{s} \rightarrow s} S(t-\hat{s}) \frac{T(\hat{s})x - T(s)x}{\hat{s} - s} = S(t-s) \frac{d}{ds} T(s)x$$

as

$$\begin{aligned} & \left\| S(t - \hat{s}) \frac{T(\hat{s})x - T(s)x}{\hat{s} - s} - S(t - s) \frac{T(\hat{s})x - T(s)x}{\hat{s} - s} \right\| \\ & \leq \underbrace{\|S(t - \hat{s}) - S(t - s)\|}_{\rightarrow 0} \underbrace{\left\| \frac{T(\hat{s})x - T(s)x}{\hat{s} - s} \right\|}_{\rightarrow \frac{d}{ds}T(s)x} \end{aligned}$$

holds, for $x \in \mathcal{D}(B)$, by Lemma 2.6. And as $T(s)x \in \mathcal{D}(B)$, again by Lemma 2.6, we get in total

$$\begin{aligned} \frac{d}{ds}S(t - s)T(s)x &= S(t - s)\frac{d}{ds}T(s)x + \left(\frac{d}{ds}S(t - s)\right)T(s)x \\ &= S(t - s)BT(s)x - S(t - s)BT(s)x \\ &= 0. \end{aligned}$$

So, all in all, we have

$$T(t)x - S(t)x = S(0)T(t)x - S(t)T(0)x = \int_0^t \frac{d}{ds}S(t - s)T(s)x ds = 0.$$

As $T(t)x = S(t)x$ holds for any $x \in \mathcal{D}(B)$ and the domain is dense, we are done because $T(t)$ and $S(t)$ are bounded operators and as such it is easily deduced that they must be equal on the whole space X . $\square$

Finally we arrive at the Hille-Yosida Theorem which is of great importance. It shows how one can either start from a semigroup or an infinitesimal generator B to arrive at the other. It also provides an answer to our first question.

Theorem 2.3. (*Hille-Yosida Theorem*) *Let B be an unbounded linear operator. Then B is the infinitesimal generator of a semigroup $S(t)$ if and only if the following two conditions are fulfilled*

- (1) B is closed and $\mathcal{D}(B)$ is dense in X .
- (2) $\mathbb{R}_+ \subseteq \rho(B)$ and $\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} \right\| \leq 1$ for all $\lambda > 0$.

Proof. The proof is borrowed from Chapter 1, Theorem 3.1 of Pazy [1].

$\implies$ Let B be an unbounded operator which is the infinitesimal generator of the semigroup $S(t)$. We have by Lemma 2.1 and Corollary 2.1 that B must

be closed and densely defined. So only (2) needs to be proven. We make an Ansatz by defining

$$L(\lambda)x := \int_0^\infty \lambda e^{-\lambda t} S(t)x dt$$

for $x \in X$, and try to prove that this is the resolvent. $L(\lambda)$ is well defined because for $a, b \geq 0$ we get

$$\begin{aligned} \left\| \int_a^b \lambda e^{-\lambda t} S(t)x dt \right\| &\leq \int_a^b \left\| \lambda e^{-\lambda t} S(t)x \right\| dt \leq \int_a^b \lambda e^{-\lambda t} \|S(t)\| \|x\| dt \\ &\leq \|x\| \underbrace{\int_a^b \lambda e^{-\lambda t} dt}_{\leq 1}. \end{aligned}$$

So we see that $N \mapsto \int_0^N \lambda e^{-\lambda t} S(t)x dt$ is a Cauchy sequence and $L(\lambda)$ is well defined. From this we also see that $L(\lambda)$ is bounded for every $\lambda > 0$ and so we have to check, that $L(\lambda)(\mathbf{1} - \frac{1}{\lambda}B) = \mathbf{1}$ and $(\mathbf{1} - \frac{1}{\lambda}B)L(\lambda) = \mathbf{1}$ holds for every $\lambda > 0$, to be done with the proof.

We quickly motivate the definition of $L(\lambda)$. Assume that $B = b > 0$, then we have that $u(t) = e^{bt}u_0$ solves the Cauchy Problem. Now a straightforward computation shows $(1 - \frac{1}{\lambda}b)^{-1} = \int_0^\infty e^{(\frac{1}{\lambda}b-1)t} dt$, for λ big enough, and by a change of variable we get $(1 - \frac{1}{\lambda}b)^{-1} = \int_0^\infty \lambda e^{-\lambda t} e^{bt} dt = \int_0^\infty \lambda e^{-\lambda t} u(t) dt$. The right side is now exactly the definition from above but for an operator B .

Now let's continue with the proof. We get for $h > 0$, because $S(h)$ is continuous and because $\int_0^\infty \lambda e^{-\lambda t} S(t+h)x dt$ exists, that

$$\begin{aligned} \frac{S(h) - \mathbf{1}}{h} L(\lambda)x &= \frac{1}{h} \left[\int_0^\infty \lambda e^{-\lambda t} S(t+h)x dt - \int_0^\infty \lambda e^{-\lambda t} S(t)x dt \right] \\ &= \frac{1}{h} \left[\int_h^\infty \lambda e^{-\lambda(t-h)} S(t)x dt - \int_0^\infty \lambda e^{-\lambda t} S(t)x dt \right] \\ &= \frac{e^{\lambda h} - 1}{h} \int_0^\infty \lambda e^{-\lambda t} S(t)x dt - \frac{e^{\lambda h}}{h} \int_0^h \lambda e^{-\lambda t} S(t)x dt \end{aligned}$$

holds. Now letting $h \rightarrow 0$ we get $L(\lambda)x \in \mathcal{D}(B)$ and $BL(\lambda)x = \lambda L(\lambda)x - \lambda x$. So we have proven $(\mathbf{1} - \frac{1}{\lambda}B)L(\lambda)x = x$. Now let $x \in \mathcal{D}(B)$ then we get, by using Theorem 2.2,

$$\begin{aligned} L(\lambda)Bx &= \int_0^\infty \lambda e^{-\lambda t} S(t)Bx dt = \int_0^\infty \lambda e^{-\lambda t} BS(t)x dt = B \int_0^\infty \lambda e^{-\lambda t} S(t)x dt \\ &= BL(\lambda)x, \end{aligned}$$

where we used that B is closed in the forelast equation. Thus $L(\lambda)Bx = BL(\lambda)x = \lambda L(\lambda)x - \lambda x$ and so we are done.

$\Leftarrow$

Proving this direction is rather fast as we have already done most of the work. By Lemma 2.5 we have that for B there is a semigroup $S(t)$, obtained by the Yosida Approximation. We need to check now if B is also its infinitesimal generator. Let $\hat{B}$ denote its infinitesimal generator. Lemma 2.6 then tells us that for any $x \in \mathcal{D}(B)$ we have $x \in \mathcal{D}(\hat{B})$. So we have $B \subseteq \hat{B}$. By assumption of the Theorem we have $1 \in \rho(B)$ and so we get, by definition of the resolvent set, $(\mathbf{1} - B)\mathcal{D}(B) = X$. As $B \subseteq \hat{B}$ holds we deduce $X = (\mathbf{1} - B)\mathcal{D}(B) \subseteq (\mathbf{1} - \hat{B})\mathcal{D}(B)$. So also $(\mathbf{1} - \hat{B})\mathcal{D}(B) = X$ holds. But we know from the proof of the other direction that $1 \in \rho(\hat{B})$ holds. That means that we get $\mathcal{D}(\hat{B}) = (\mathbf{1} - \hat{B})^{-1}X$. All in all this means $\mathcal{D}(\hat{B}) = (\mathbf{1} - \hat{B})^{-1}X = \mathcal{D}(B)$. This finishes the proof because $B \equiv \hat{B}$ means that B is indeed also the infinitesimal generator of $S(t)$. $\square$

We remark that the following two extended results hold. For a proof see Pazy [1].

Theorem 2.4. *Let B be an unbounded linear operator. Then B is the infinitesimal generator of a C_0 semigroup $S(t)$, with $\|S(t)\| \leq e^{ct}$, if and only if the following two conditions are fulfilled*

- (1) B is closed and $\mathcal{D}(B)$ is dense in X .
- (2) $(c, \infty) \subseteq \rho(B)$

$$\text{and } \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} \right\| \leq \frac{\lambda}{\lambda - c} \quad \text{for all } \lambda > c.$$

Theorem 2.5. *Let B be an unbounded linear operator. Then B is the infinitesimal generator of a C_0 semigroup $S(t)$, with $\|S(t)\| \leq De^{ct}$, if and only if the following two conditions are fulfilled*

- (1) B is closed and $\mathcal{D}(B)$ is dense in X .
- (2) $(c, \infty) \subseteq \rho(B)$ and

$$\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-n} \right\| \leq D \left(\frac{\lambda}{\lambda - c} \right)^n \quad \text{for all } \lambda > c, n \in \mathbb{N}.$$

Before finishing this chapter we quickly want to talk about how we could use semigroups to solve the Cauchy Problem for an unbounded linear operator B . If the operator B satisfies condition (1) and (2) of the Hille-Yosida Theorem

we know that we can construct a semigroup $S(t)$. By Lemma 2.6 we know, that

$$\frac{d}{dt}S(t)x = BS(t)x$$

holds, for any $x \in \mathcal{D}(B)$ and $t > 0$. So if the initial value x is in $\mathcal{D}(B)$, we can at least find one solution. Motivated by this we define now what we actually mean by a solution.

Definition 2.2. A solution of the Cauchy Problem for a linear unbounded operator B , that satisfies conditions (1) and (2) of the Hille-Yosida Theorem, with initial value $x \in \mathcal{D}(B)$, is a function $t \mapsto u(t)$ that satisfies $u(0) = x$, solves $\frac{d}{dt}u(t) = Bu(t)$ for $t \in (0, \infty)$, is continuous on $[0, \infty)$ and is continuously differentiable in $(0, \infty)$.

So $S(t)x$ is a solution of the Cauchy Problem with initial value $x \in \mathcal{D}(X)$. Actually, this is also the unique solution. A proof of this will be provided in Chapter 3, Theorem 3.3.

2.5 Linear examples

2.5.1 The Heat equation

As a first example we look at the heat equation, which we have already seen in the first chapter,

$$\frac{d}{dt}u(x, t) = \Delta u(x, t), \quad (x, t) \in \Omega \times (0, \infty), \quad (2.15)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, \infty), \quad (2.16)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (2.17)$$

This equation describes the development of temperature u on an area with insulated boundary $\partial\Omega$, over time.

We will assume that $\Omega \subseteq \mathbb{R}^2$ is bounded, open and has a Lipschitz boundary. Defining the operator B by

$$\begin{aligned} \mathcal{D}(B) &:= \{u \in H_0^1(\Omega) \mid \exists w \in L^2(\Omega) : \\ &\quad (w, v)_{L^2(\Omega)} = -(\nabla u, \nabla v)_{L^2(\Omega)} \forall v \in H_0^1(\Omega)\} \\ Bu &:= w \end{aligned}$$

- where we remark that such a w is unique as $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ - we can "rewrite" the heat equation as a Cauchy Problem and use the Hille-Yosida Theorem to find the unique solution; given $u_0 \in \mathcal{D}(B)$. So we proceed

by checking the assumptions of the Hille-Yosida Theorem.

$\mathcal{D}(B)$ is a dense subspace:

It is trivial to see that $\mathcal{D}(B)$ is a linear subspace of $L^2(\Omega)$.

$\mathcal{D}(B)$ is dense in the latter as for any $u \in C_c^\infty(\Omega)$ we have $(\Delta u, v) = -(\nabla u, \nabla v)$, for all $v \in C_c^\infty(\Omega)$. By definition of $H_0^1(\Omega)$ we get that this also holds for all $v \in H_0^1(\Omega)$. So we get $u \in \mathcal{D}(B)$. As $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ we get that $\mathcal{D}(B)$ is dense in $L^2(\Omega)$.

B is closed:

Let $u_k \in \mathcal{D}(B)$ be a sequence such that $u_k \xrightarrow{L^2} u$ and $Bu_k \xrightarrow{L^2} w$, for some $u, w \in L^2(\Omega)$. As $u_k \in \mathcal{D}(B)$, we get

$$|(\nabla u_k, \nabla u_k)| = |(Bu_k, u_k)| \leq \|Bu_k\| \|u_k\| \rightarrow \|w\| \|u\|.$$

So we get $\sup_k \|\nabla u_k\| < \infty$. Because $L^2(\Omega)$ is reflexive we have that there exists a $\psi \in L^2(\Omega)$ such that ∇u_{k_l} converges weakly to ψ ; where u_{k_l} is a subsequence. All in all we have, for any $v \in C_c^\infty(\Omega)$,

$$(\psi, v) = \lim_{l \rightarrow \infty} (\nabla u_{k_l}, v) = \lim_{l \rightarrow \infty} -(u_{k_l}, \nabla v) = -(u, \nabla v).$$

This means that the weak derivative of u exists and that it is equal to ψ . For any $v \in H_0^1(\Omega)$ we get

$$-(\nabla u, \nabla v) = \lim_{l \rightarrow \infty} -(\nabla u_{k_l}, \nabla v) = \lim_{l \rightarrow \infty} (Bu_{k_l}, v) = (w, v).$$

So we have $u \in \mathcal{D}(B)$ and, as such a w is unique, we get $Bu = w$.

$\mathbb{R}_+ \subseteq \rho(B)$ and uniform boundedness :

We define a bilinear form on $H_0^1(\Omega)$ by $a(u, v) := (u, v) + \frac{1}{\lambda}(\nabla u, \nabla v)$. It is easy to see that this form is continuous and elliptic, with ellipticity constant $\min(1, \frac{1}{\lambda})$. By the Lax-Milgram theorem we get that for any $f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ that solves

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

This is equivalent to having $(\lambda(u - f), v) = -(\nabla u, \nabla v)$ for all $v \in H_0^1(\Omega)$. So by definition of B we get $Bu = \lambda(u - f)$, which is the same as $(1 - \frac{1}{\lambda}B)u = f$. So we get that the inverse of $(1 - \frac{1}{\lambda}B)$ exists on $L^2(\Omega)$.

We will now show the uniform boundedness of $(1 - \frac{1}{\lambda}B)^{-1}$ for any $\lambda \in \mathbb{R}_+$. By Poincaré's inequality there exists a constant $c > 0$, such that $\|u\| \leq c \|\nabla u\|$. This means that for any $u \in \mathcal{D}((1 - \frac{1}{\lambda}B))$ we have that $a(u, u) \geq (1 + \frac{1}{c\lambda}) \|u\|^2$ holds. As $a(u, u)$ can be estimated by $\|f\| \|u\|$, for f with $u = (1 - \frac{1}{\lambda}B)^{-1}f$,

we get in total the estimate

$$\|u\| \leq \frac{1}{1 + \frac{1}{c\lambda}} \|f\| \leq \|f\|.$$

So we have in total

$$\left\| \left(1 - \frac{1}{\lambda} B\right)^{-1} \right\| \leq 1.$$

By the Hille-Yosida Theorem we can thus find a semigroup which is the unique solution to the heat equation.

2.5.2 Advection–diffusion equation

As a second application of semigroup theory we will look at an advection-diffusion equation of the form

$$\begin{aligned} \frac{d}{dt}u(x, t) &= \nabla \cdot (A(x)\nabla u(x, t)) - \nabla \cdot (b(x)u(x, t)), & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary, $b(x) \in \mathbb{R}^n$ with $b_i \in C_b^\infty(\Omega)$ and $A(x) \in \mathbb{R}^{n \times n}$ is a symmetric matrix such that $A_{i,j} \in C_b^\infty(\Omega)$ holds. Furthermore we assume that we have $\|b\|_\infty > 0$ and that there exists a constant $c > 0$ such that $(A(x)\zeta, \zeta)_E \geq c(\zeta, \zeta)_E$ holds for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

Advection-diffusion equations appear in many different disciplines. For example u can describe the density of some quantity. The term $\nabla \cdot (A(x)\nabla u(x, t))$ then describes the diffusion, heuristically this means that if there is a maximum or minimum at a certain point, over time u will, if there is no source, level, meaning that u will become almost the same around where the maximum or minimum was before. The second term $\nabla \cdot (b(x)u(x, t))$ leads to a change of the density u .

To be able to treat this problem as a Cauchy Problem we introduce the operator B as

$$\begin{aligned} \mathcal{D}(B) &= \{u \in H_0^1(\Omega) | \exists w \in L^2(\Omega) : \\ &\quad (w, v)_{L^2(\Omega)} = -(A\nabla u, \nabla v)_{L^2(\Omega)} + (bu, \nabla v)_{L^2(\Omega)} \forall v \in H_0^1(\Omega)\} \\ Bu &:= w. \end{aligned}$$

We remark again that such a w is uniquely determined as $H_0^1(\Omega)$ is dense in $L^2(\Omega)$.

$\mathcal{D}(B)$ is a dense subspace:

$\overline{\mathcal{D}(B)}$ is a dense subset by the same arguments as in the example before, so we will go to the next part that has to be proven.

B is closed:

Let $u_k \in \mathcal{D}(B)$ be a sequence such that $u_k \xrightarrow{L^2} u$ and $Bu_k \xrightarrow{L^2} w$, for some $u, w \in L^2(\Omega)$.

First we prove $u \in H_0^1(\Omega)$. By using the assumptions on A we get

$$\begin{aligned} c(\nabla u_k, \nabla u_k) &\leq (A\nabla u_k, \nabla u_k) = -(Bu_k, u_k) + (bu_k, \nabla u_k) \\ &\leq -(Bu_k, u_k) + \frac{1}{4\epsilon} \|b\|_\infty^2 \|u_k\|^2 + \epsilon \|\nabla u_k\|^2, \end{aligned}$$

where we used the Cauchy inequality for some $0 < \epsilon < c$. Now, because $(Bu_k, u_k) \rightarrow (w, u)$ holds, we get that

$$\frac{1}{c - \epsilon} \left[-(Bu_k, u_k) + \frac{1}{4\epsilon} \|b\|_\infty^2 \|u_k\|^2 \right]$$

has a limit. This means that $\sup_k \|\nabla u_k\|$ is bounded and so, as $L^2(\Omega)$ is reflexive, that there exists a $\psi \in L^2(\Omega)$ such that ∇u_{k_l} converges weakly to it. As we have seen in the example before this means $u \in H_0^1(\Omega)$.

By assumption we get, for any $v \in C_c^\infty(\Omega)$, as A is symmetric and $\nabla \cdot (A\nabla v) \in C_c^\infty(\Omega)$,

$$-(A\nabla u_{k_l}, \nabla v) = (u_{k_l}, \nabla \cdot (A\nabla v)) \rightarrow (u, \nabla \cdot (A\nabla v)) = -(A\nabla u, \nabla v).$$

Furthermore we have $(bu_{k_l}, v) \rightarrow (bu, v)$, as

$$|(bu_{k_l} - bu, \nabla v)| \leq \|b\|_\infty \|\nabla v\| \|u - u_{k_l}\| \rightarrow 0$$

holds. So all in all we get

$$\begin{aligned} (w, v) &= \lim_l (Bu_{k_l}, v) = \lim_l (-(A\nabla u_{k_l}, \nabla v) + (bu_{k_l}, v)) \\ &= -(A\nabla u, \nabla v) + (bu, v). \end{aligned}$$

Since this holds for all $v \in C_c^\infty(\Omega)$, it will hold for all $v \in H_0^1(\Omega)$. This means that $u \in \mathcal{D}(B)$ and that $Bu = w$, by definition of B .

$(\|b\|^2 (4c)^{-1}, \infty) \subseteq \rho(B)$ and uniform boundedness :

First we show that for any $f \in L^2(\Omega)$, $(1 - \frac{1}{\lambda}B)u = f$ has a unique solution $u \in \mathcal{D}(B)$. We define a bilinear form on $H_0^1(\Omega)$ by $a(u, v) := (u, v) + \frac{1}{\lambda}(A\nabla u, \nabla v) - \frac{1}{\lambda}(bu, \nabla v)$, for a fixed $\lambda > \|b\|^2 (4c)^{-1}$. It is easy to

show that a is a continuous form, one just uses that A_{ij} and b_i are bounded, so we skip this part of the proof.

We show now that a is an elliptic form. By using the Cauchy-Schwarz inequality and the Cauchy inequality for ϵ , with $\|b\|^2 (4\lambda)^{-1} < \epsilon < c$, we get

$$\begin{aligned} (bu, \nabla u) &= \int_{\Omega} bu \nabla u \leq \int_{\Omega} \|bu\|_E \|\nabla u\|_E \leq \int_{\Omega} \frac{1}{4\epsilon} \|bu\|_E^2 + \epsilon \|\nabla u\|_E^2 \\ &= \frac{1}{4\epsilon} \|b\|_{\infty}^2 \|u\|^2 + \epsilon \|\nabla u\|^2. \end{aligned}$$

With this we deduce, also by using the assumption on A ,

$$\begin{aligned} a(u, u) &\geq \|u\|^2 + \frac{1}{\lambda} c \|\nabla u\|^2 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \|u\|^2 - \frac{\epsilon}{\lambda} \|\nabla u\|^2 \\ &= \|\nabla u\|^2 \left(\frac{1}{\lambda} c - \frac{\epsilon}{\lambda} \right) + \|u\|^2 \left(1 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \right) \\ &\geq \min \left(\frac{1}{\lambda} c - \frac{\epsilon}{\lambda}, 1 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \right) \|u\|_{H_0^1}^2. \end{aligned}$$

As we chose $\|b\|^2 (4\lambda)^{-1} < \epsilon < c$ we get that $\min \left(\frac{1}{\lambda} c - \frac{\epsilon}{\lambda}, 1 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \right)$ is strictly positive and therefore a is elliptic. By Lax-Milgram theorem we get that there exists a unique $u \in H_0^1(\Omega)$ such that $a(u, v) = (f, v)$ is fulfilled for all $v \in H_0^1(\Omega)$. This is equivalent to having $(\lambda(u - f), v) = -(A\nabla u, \nabla v) + (bu, \nabla v)$ for all $v \in H_0^1(\Omega)$. By definition of $\mathcal{D}(B)$, this means on one hand that $u \in \mathcal{D}(B)$ and on the other that $Bu = \lambda(u - f)$. The latter is equivalent to $(1 - \frac{1}{\lambda} B)u = f$. So all that is left to do is to show the boundedness.

As already shown before we get, for $\|b\|^2 (4\lambda)^{-1} < \epsilon < c$,

$$a(u, u) \geq \|\nabla u\|^2 \left(\frac{c}{\lambda} - \frac{\epsilon}{\lambda} \right) + \|u\|^2 \left(1 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \right). \quad (2.18)$$

Now by using that Ω is open, bounded and has a Lipschitz boundary, we get that there exists, by Poincaré's inequality, a constant $d > 0$ such that $\|\nabla u\|^2 \geq d \|u\|^2$ holds. So 2.18 becomes

$$a(u, u) \geq \|u\|^2 \left(\frac{1}{\lambda} dc - \frac{1}{\lambda} d\epsilon + 1 - \frac{1}{4\lambda\epsilon} \|b\|_{\infty}^2 \right).$$

As $a(u, u) = (f, u) \leq \|f\| \|u\|$ holds we arrive at

$$\frac{\|u\|}{\|f\|} \leq \frac{1}{1 - \frac{1}{\lambda} (d\epsilon - dc + \frac{1}{4\epsilon} \|b\|_{\infty}^2)}.$$

By letting $\epsilon \nearrow c$ we get

$$\frac{\|(\mathbf{1} - \frac{1}{\lambda}B)^{-1}f\|}{\|f\|} \leq \frac{1}{1 - \frac{1}{\lambda} \frac{1}{4c} \|b\|_{\infty}^2} = \frac{\lambda}{\lambda - \frac{1}{4c} \|b\|_{\infty}^2}.$$

By using the more general version of the Hille-Yosida Theorem, Theorem 2.4, we get that there exists a unique solution for the Advection-diffusion problem above.

CHAPTER 3

Semigroups generated by nonlinear operators

3.1 Introduction

In Chapter 2 we saw how we could construct a semigroup given an unbounded linear operator B with some properties. Now, in applications, a lot of time one deals with nonlinear operators. We will see in this chapter how a semigroup can be constructed, even if we drop the linearity condition. That this will not work without certain restrictions to the operator is clear. Crandall and Liggett proved in their paper [2], how and under what restrictions, this could be done. We describe their approach and, furthermore, we will see how such semigroups can be used to solve Cauchy problems for nonlinear operators.

3.2 The Crandall-Liggett Theorem

Analogously to Section 2.3 and 2.4, we would like to know under what kind of conditions we can find a semigroup for a nonlinear operator B .

Contrary to Section 2.2 we will not discuss what existence of a semigroup should imply for its infinitesimal generator. However, this theory should encapsulate the one for linear unbounded operators and by Section 2.2 we know that certain restrictions have to be imposed on the operator B . The following discussion will rather be heuristic and gives some motivation to comprehend the assumptions and the statement of the Crandall-Liggett Theorem.

As we cannot use the Yosida Approximation for a nonlinear operator in general, since it relies too much on the linearity of B , we need a different approach on how to construct a semigroup.

Because our main goal is to get a solution for the Cauchy Problem, we try the following. Let's assume for a moment that there actually exists a solution to

the Cauchy Problem for B . For any fixed $t > 0$ we can look at a equidistant partition $t_i = \frac{it}{N}$, for $i = 0, \dots, N$, of $[0, t]$, for a fixed $N \in \mathbb{N}$. For a solution $t \mapsto u(t)$ of the Cauchy Problem we get, as $\frac{d}{dt}u(t)$ can be approximated by a finite difference quotient - forward or backward - the two possibilities,

$$u(t_i) \approx \left(\mathbf{1} - \frac{t}{N}B \right) u(t_{i+1}) \quad (3.1)$$

$$u(t_{i+1}) \approx \left(\mathbf{1} + \frac{t}{N}B \right) u(t_i). \quad (3.2)$$

At the first look these two do not seem that useful to get an idea on what $u(t)$ could approximately look like, as in general we have no information on the behaviour of the operators $(\mathbf{1} + \frac{t}{N}B)$ and $(\mathbf{1} - \frac{t}{N}B)$. Nevertheless, if the operator B is linear, we know that we must assume conditions (1) and (2) of the Hille-Yosida Theorem. In that case we can rewrite the approximation (3.1) as

$$u(t_{i+1}) \approx \left(\mathbf{1} - \frac{t}{N}B \right)^{-1} u(t_i)$$

and so iteratively, as $(\mathbf{1} - \frac{t}{N}B)^{-1}$ is bounded,

$$u(t) \approx \left(\mathbf{1} - \frac{t}{N}B \right)^{-N} u(0).$$

So, for the general nonlinear case, we could try to see if $\lim_{N \rightarrow \infty} (\mathbf{1} - \frac{t}{N}B)^{-N}y$ defines a semigroup.

First of all we need the family of operators $(\mathbf{1} - \lambda B)$ to be injective, for all $\lambda > 0$, to be able to talk about their inverse on a suitable domain - surjectivity is not needed.

However, there has to be some restriction on what y can be. Looking at $(\mathbf{1} - \frac{t}{N}B)^{-N}y$ we need to have that y belongs to $\mathcal{D}((\mathbf{1} - \frac{t}{N}B)^{-1}) = (\mathbf{1} - \frac{t}{N}B)(\mathcal{D}(B))$. This means that for any $y \in (\mathbf{1} - \frac{t}{N}B)(\mathcal{D}(B))$ there exist an $x \in \mathcal{D}(B)$, such that $y = (\mathbf{1} - \frac{t}{N}B)x$.

So computing

$$(\mathbf{1} - \frac{t}{N}B)^{-N}y = (\mathbf{1} - \frac{t}{N}B)^{-N}(\mathbf{1} - \frac{t}{N}B)x = (\mathbf{1} - \frac{t}{N}B)^{-(N-1)}x$$

we see that we should also assume $\overline{\mathcal{D}(B)} \subseteq \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$ for all $\lambda > 0$; why we take the closure here will become apparent later.

Thirdly we need $N \mapsto (\mathbf{1} - \frac{t}{N}B)^{-N}y$ to be a Cauchy sequence so that the

limit exists. The question is now, if we have any information on $\|(\mathbf{1} - \frac{t}{N}B)^{-N}y - (\mathbf{1} - \frac{t}{N}B)^{-M}y\|$, for N, M .

Within the hypotheses of the Hille-Yosida Theorem, in the case of a linear unbounded operator, it turns out that we get by some reformulation,

$$\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} \right\| \leq 1 \quad (3.3)$$

$$\Leftrightarrow \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} y \right\| \leq \|y\| \quad \forall y \in X \quad (3.4)$$

$$\Leftrightarrow \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} y_1 - \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-1} y_2 \right\| \leq \|y_1 - y_2\| \quad \forall y_1, y_2 \in X \quad (3.5)$$

and so, for $N > M$, we deduce

$$\left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-N} y - \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-M} y \right\| \leq \left\| \left(\mathbf{1} - \frac{1}{\lambda} B \right)^{-(N-M)} y - y \right\| \quad (3.6)$$

for any $y \in X$.

As we will soon see in a more rigorous way even the term on the right side of (3.6) can be estimated in a "nice" way, so that one can show that we actually have a Cauchy sequence. The key to convergence will be to assume estimate (3.5), with the slight modification that we don't need it to hold for all $y_1, y_2 \in X$.

Before coming to the Crandall-Liggett Theorem we want to define what we mean by a semigroup on a subset V of X .

Definition 3.1. A semigroup on a subset V of X is a family of operators $\{S(t) : V \rightarrow X | t \in [0, \infty)\}$ that satisfies for all $t, s \in [0, \infty)$ and $x, y \in V$

$$\begin{aligned} (1) & S(t+s)x = S(t)S(s)x & (2) & S(0)x = x \\ (3) & \lim_{t \searrow 0} S(t)x = x & (4) & \|S(t)x - S(t)y\| \leq \|x - y\|. \end{aligned}$$

Remark. This definition has all the important properties which also the definition of semigroups of Chapter 2 had. To solve the Cauchy Problem, we do not need linearity, neither can we hope to achieve it when constructing a "semigroup" from a nonlinear operator B .

We will now get to the most important theorem of this chapter.

Theorem 3.1. (*Crandall-Liggett Theorem*) Let $B : \mathcal{D}(B) \rightarrow X$ be a nonlinear operator such that $\mathbf{1} - \lambda B$ is injective for all $\lambda \in (0, c]$, for some fixed $c > 0$. Furthermore assume that

$$\|(\mathbf{1} - \lambda B)^{-1} y_1 - (\mathbf{1} - \lambda B)^{-1} y_2\| \leq \|y_1 - y_2\| \quad \forall y_1, y_2 \in \mathcal{D}((\mathbf{1} - \lambda B)^{-1}) \quad (3.7)$$

and $\overline{\mathcal{D}(B)} \subseteq \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$ for all $\lambda \in (0, c]$, is satisfied. Then

$$\lim_{N \rightarrow \infty} \left(\mathbf{1} - \frac{t}{N} B \right)^{-N} x$$

exists for any $t \geq 0$ and $x \in \overline{\mathcal{D}(B)}$. Furthermore the function $t \mapsto S(t)$ defined by $S(t)x := \lim_{N \rightarrow \infty} \left(\mathbf{1} - \frac{t}{N} B \right)^{-N} x$ is a semigroup on the subset $\overline{\mathcal{D}(B)}$.

The proof of the Crandall-Liggett Theorem requires many estimates. We want to mention and prove these beforehand, so that the proof of the Crandall-Liggett Theorem will stay overseeable.

First off we mention some results that will allow us to estimate

$$\|(\mathbf{1} - \mu B)^{-n} x - (\mathbf{1} - \lambda B)^{-m} x\|$$

for $n, m \in \mathbb{N}$ and $\lambda, \mu \in (0, c]$.

Proposition 2. Suppose that the assumptions of the Crandall-Liggett Theorem are fulfilled. Then we have for $y \in \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$, $x \in \mathcal{D}(B)$, $\lambda, \mu \in (0, c]$, $m \in \mathbb{N}$:

- (1) $\|x - (\mathbf{1} - \lambda B)^{-1} x\| \leq \lambda \|Bx\|$
- (2) $\|x - (\mathbf{1} - \lambda B)^{-m} x\| \leq m \|x - (\mathbf{1} - \lambda B)^{-1} x\|$
- (3) $(\mathbf{1} - \lambda B)^{-1} y = (\mathbf{1} - \mu B)^{-1} \left(\frac{\mu}{\lambda} y + \frac{\lambda - \mu}{\lambda} (\mathbf{1} - \lambda B)^{-1} y \right).$

Proof. The following proof is an adaptation of the proof of Lemma 1.2 of Crandall and Liggett [2] and of the proof of Lemma 1.3, Chapter 3 of Barbu [3].

(1) By assumption we get for $x \in \mathcal{D}(B)$

$$\begin{aligned} \|x - (\mathbf{1} - \lambda B)^{-1} x\| &= \|(\mathbf{1} - \lambda B)^{-1} (\mathbf{1} - \lambda B) x - (\mathbf{1} - \lambda B)^{-1} x\| \\ &\leq \|(\mathbf{1} - \lambda B) x - x\| \\ &= \|\lambda Bx\| = \lambda \|Bx\|. \end{aligned}$$

(2) By the estimate (3.7) of the Crandall-Liggett Theorem we get

$$\begin{aligned}
\|x - (\mathbf{1} - \lambda B)^{-m} x\| &= \left\| \sum_{i=0}^{m-1} \left((\mathbf{1} - \lambda B)^{-(i+1)} x - (\mathbf{1} - \lambda B)^{-i} x \right) \right\| \\
&\leq \sum_{i=0}^{m-1} \left\| (\mathbf{1} - \lambda B)^{-(i+1)} x - (\mathbf{1} - \lambda B)^{-i} x \right\| \\
&\leq \sum_{i=0}^{m-1} \|x - (\mathbf{1} - \lambda B)^{-1} x\| \\
&= m \|x - (\mathbf{1} - \lambda B)^{-1} x\|.
\end{aligned}$$

(3) As we are trying to relate $(\mathbf{1} - \lambda B)^{-1}$ to $(\mathbf{1} - \mu B)^{-1}$ we take a $z \in \mathcal{D}(B)$ and then write the following

$$\begin{aligned}
(\mathbf{1} - \mu B) z &= z - \mu Bz + \lambda Bz - \lambda Bz \\
&= (\mathbf{1} - \lambda B) z + (\lambda - \mu) Bz.
\end{aligned}$$

We would like to factorize the last line, so we continue, to get to

$$\begin{aligned}
&(\mathbf{1} - \lambda B) z + (\lambda - \mu) \frac{\lambda}{\lambda} Bz + \frac{(\lambda - \mu)}{\lambda} z - \frac{(\lambda - \mu)}{\lambda} z \\
&= \frac{(\lambda - \mu)}{\lambda} \left[\frac{\lambda}{(\lambda - \mu)} (\mathbf{1} - \lambda B) z - (\mathbf{1} - \lambda B) z \right] + \frac{(\lambda - \mu)}{\lambda} z \\
&= \frac{(\lambda - \mu)}{\lambda} \left[\left(\frac{\lambda}{(\lambda - \mu)} - 1 \right) (\mathbf{1} - \lambda B) z \right] + \frac{(\lambda - \mu)}{\lambda} z \\
&= \left[\underbrace{\left(1 - \frac{(\lambda - \mu)}{\lambda} \right)}_{\frac{\mu}{\lambda}} (\mathbf{1} - \lambda B) z \right] + \frac{(\lambda - \mu)}{\lambda} z.
\end{aligned}$$

So in total we get

$$(\mathbf{1} - \mu B) z - \frac{(\lambda - \mu)}{\lambda} z = \frac{\mu}{\lambda} (\mathbf{1} - \lambda B) z.$$

Now as we did this for any $z \in \mathcal{D}(B)$, we can take, by assumption, z such that $y = (\mathbf{1} - \lambda B) z$. So we deduce

$$\begin{aligned}
&(\mathbf{1} - \mu B) (\mathbf{1} - \lambda B)^{-1} y - \frac{(\lambda - \mu)}{\lambda} (\mathbf{1} - \lambda B)^{-1} y = \frac{\mu}{\lambda} y \\
&\Leftrightarrow (\mathbf{1} - \mu B) (\mathbf{1} - \lambda B)^{-1} y = \frac{\mu}{\lambda} y + \frac{(\lambda - \mu)}{\lambda} (\mathbf{1} - \lambda B)^{-1} y.
\end{aligned}$$

Applying now $(\mathbf{1} - \mu B)^{-1}$ we are done. □

Proposition 3. Let $n \geq m > 0$ where $n, m \in \mathbb{N}$, $\alpha > 0$ and $\beta > 0$ be such that $\alpha + \beta = 1$, then we have the following

$$\begin{aligned}
(1) \quad & \sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} = (\alpha + \beta)^n = 1 \\
(2) \quad & \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j} = \alpha n (\alpha + \beta)^{n-1} = \alpha n \\
(3) \quad & \sum_{j=0}^n j^2 \binom{n}{j} \alpha^j \beta^{n-j} = \alpha^2 n(n-1) (\alpha + \beta)^{n-2} + \alpha n (\alpha + \beta)^{n-1} \\
& \quad \quad \quad = \alpha^2 n(n-1) + \alpha n \\
(4) \quad & \sum_{j=m}^{\infty} \binom{j-1}{m-1} \beta^{j-m} = \frac{1}{(1-\beta)^m} \\
(5) \quad & \sum_{j=m}^{\infty} (j-m) \binom{j-1}{m-1} \beta^{j-m} = \frac{m\beta}{(1-\beta)^{m+1}} \\
(6) \quad & \sum_{j=m}^{\infty} (j-m)^2 \binom{j-1}{m-1} \beta^{j-m} = m\beta \frac{m\beta + 1}{(1-\beta)^{m+2}} \\
(7) \quad & \sum_{j=0}^m \binom{n}{j} \alpha^j \beta^{n-j} (m-j) \leq \sqrt{(n\alpha - m)^2 + n\alpha\beta} \\
(8) \quad & \sum_{j=m}^n \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n-j) \leq \sqrt{\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2}.
\end{aligned}$$

Proof. This proof is a combination of the proof of Lemma 1.5, Chapter 3 of Barbu [3] and the proof of Lemma 1.4 of Crandall and Liggett [2].

(1) This is just the binomial theorem.

(2) is obtained by differentiating (1) and the multiplying it with α .

(3) is obtained by differentiating (2) and the multiplying it by α .

Now to (4). We get, because

$$\begin{aligned}
\binom{-m}{j} &= \frac{(-m)(-m-1)\cdots(m+j-1)}{j!} = (-1)^k \frac{m(m+1)\cdots(m+j-1)}{j!} \\
&= (-1)^j \frac{(m+j-1)!}{j!(m-1+j-j)!} = (-1)^j \binom{m+j-1}{j} \\
&= (-1)^j \binom{m+j-1}{m+j-1-j} = (-1)^j \binom{m+j-1}{j} = (-1)^j \binom{m+j-1}{m-1}
\end{aligned}$$

holds, by using the generalised binomial theorem and an index change,

$$\frac{1}{(1-\beta)^m} = \sum_{j=0}^{\infty} \binom{-m}{j} (-1)^j \beta^j = \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \beta^j = \sum_{j=m}^{\infty} \binom{j-1}{m-1} \beta^{j-m}.$$

So (4) is proven.

(5) now follows by differentiating both sides of (4) with respect to β and the multiplying by β .

Equivalently (6) follows by differentiating both sides of (5) with respect to β and the multiplying by β .

Now to (7). As $m \leq n$, we get for $j \leq m$:

$$\begin{aligned} \sum_{j=0}^m \binom{n}{j} \alpha^j \beta^{n-j} (m-j) &= \sum_{j=0}^m \binom{n}{j} \alpha^j \beta^{n-j} |m-j| \\ &\leq \sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} |m-j| \\ &= \sum_{j=0}^m \binom{n}{j}^{\frac{1}{2}} \alpha^{\frac{j}{2}} \beta^{\frac{n-j}{2}} \binom{n}{j}^{\frac{1}{2}} \alpha^{\frac{j}{2}} \beta^{\frac{n-j}{2}} |m-j| \\ &\leq \underbrace{\left(\sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} \right)^{\frac{1}{2}}}_{(\alpha+\beta)^{\frac{1}{2}}=1} \left(\sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} (m-j)^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where the last estimate follows by Cauchy–Schwarz inequality. Furthermore, by using (1), (2) and (3) we get

$$\begin{aligned} &\sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} (m-j)^2 \\ &= m^2 \sum_{j=0}^n \binom{n}{j} \alpha^j \beta^{n-j} - 2m \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j} + \sum_{j=0}^n j^2 \binom{n}{j} \alpha^j \beta^{n-j} \\ &= m^2 - 2m\alpha n + \alpha^2 n(n-1) + \alpha n = (m - \alpha n)^2 + \alpha n \underbrace{(1 - \alpha)}_{\beta}. \end{aligned}$$

So we are done with (7).

(8) is quite similar, using the Cauchy–Schwarz inequality and $j \leq n$, we have

$$\begin{aligned}
& \sum_{j=m}^n \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n-j) \\
&= \sum_{j=m}^n \binom{j-1}{m-1} \alpha^m \beta^{j-m} |n-j| \leq \sum_{j=m}^{\infty} \binom{j-1}{m-1} \alpha^m \beta^{j-m} |n-j| \\
&\leq \left(\sum_{j=m}^{\infty} \binom{j-1}{m-1} \alpha^m \beta^{j-m} \right)^{\frac{1}{2}} \left(\sum_{j=m}^{\infty} \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n-j)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The first term of the product is, by (4), equal to $(\alpha^m \frac{1}{(1-\beta)^m})^{\frac{1}{2}} = (\alpha^m \frac{1}{\alpha^m})^{\frac{1}{2}} = 1$. For the second term we get by using $(n-j)^2 = (n-m)^2 + (m-j)^2 - 2(n-m)(j-m)$, (4), (5), (6) and remembering $1-\beta = \alpha$,

$$\begin{aligned}
& \alpha^m \sum_{j=m}^{\infty} \binom{j-1}{m-1} \beta^{j-m} (n-j)^2 \\
&= \alpha^m (n-m)^2 \sum_{j=m}^{\infty} \binom{j-1}{m-1} \beta^{j-m} - \alpha^m 2(n-m) \sum_{j=m}^{\infty} (j-m) \binom{j-1}{m-1} \beta^{j-m} \\
&+ \alpha^m \sum_{j=m}^{\infty} (j-m)^2 \binom{j-1}{m-1} \beta^{j-m} \\
&= \alpha^m \left((n-m)^2 \frac{1}{(1-\beta)^m} - 2(n-m) \frac{m\beta}{(1-\beta)^{m+1}} + m\beta \frac{m\beta+1}{(1-\beta)^{m+2}} \right) \\
&= (n-m)^2 - 2(n-m) \frac{m\beta}{\alpha} + m\beta \frac{m\beta+1}{\alpha^2} = ((n-m) - \frac{\beta m}{\alpha})^2 + \frac{m\beta}{\alpha^2}.
\end{aligned}$$

By taking the square root of the last term we are done. $\square$

At last we also will need the following estimate.

Proposition 4. Assume that for a sequence $a_{m,n} : \mathbb{N}_0^2 \rightarrow \mathbb{R}_+$ we have

$$a_{m+1,n+1} \leq \alpha a_{m,n} + \beta a_{m+1,n} \quad (3.8)$$

for all $(m,n) \in \mathbb{N}_0^2$ and fixed $\alpha, \beta > 0$. Then we get for $1 \leq m \leq n$ the following estimate

$$a_{m,n} \leq \sum_{j=0}^m \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j,0} + \sum_{j=m}^n \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,n-j}. \quad (3.9)$$

For $m \geq n \geq 1$ we get

$$a_{m,n} \leq \sum_{j=0}^n \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j,0}. \quad (3.10)$$

Before the proof of this Proposition let's motivate quickly the two terms on the right side of (3.9). For $n \leq 3, m \leq n$, we can estimate $a_{m,n}$ by a sum of $a_{1,0}, a_{2,0}, a_{3,0}, a_{0,0}, a_{0,1}, a_{0,2}, a_{0,3}$ with coefficients shown in the next two tables.

n=2	$a_{0,0}$	$a_{1,0}$	$a_{2,0}$	n=3	$a_{0,0}$	$a_{1,0}$	$a_{2,0}$	$a_{3,0}$
m=1	$\beta\alpha$	β^2	0	m=1	$\alpha\beta^2$	β^3	0	0
m=2	α^2	$2\alpha\beta$	β^2	m=2	$2\alpha^2\beta$	$3\alpha\beta^2$	β^3	0
				m=3	α^3	$3\alpha^2\beta$	$3\alpha\beta^2$	β^3

We know by (3.8) that for fixed m only $a_{l,0}$, for $l = 0, \dots, m$, appear. Furthermore we see for each fixed m that the exponents of the product of α and β always adds up to n . For $j = 0, \dots, m$ we see that in the first and in the second table in the last line the coefficients of $\alpha^j \beta^{n-j} a_{m-j,0}$ are the binomial coefficients $\binom{n}{j}$ and that for m smaller than n the coefficients of $\alpha^j \beta^{n-j} a_{m-j,0}$ can be bounded by $\binom{n}{j}$. This motivates the expression

$$\sum_{j=0}^m \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j,0}.$$

The other term of (3.9) is motivated by the following. Let's look again at the table for the first couple of $a_{m,n}$.

n=2	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	n=3	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$
m=1	$\alpha\beta$	α	0	m=1	$\alpha\beta^2$	$\alpha\beta$	α	0
m=2	α^2	0	0	m=2	$2\alpha^2\beta$	α^2	0	0
				m=3	α^3	0	0	0

First of all we see that only coefficients $a_{0,l}$, with $l = 0, \dots, n - m$, seem to appear. Furthermore we see from both tables that the exponent of α seems to be m and that the exponents of the product of α and β for $a_{0,n-j}$, with $j = m, \dots, n$, seem to add to j . To get an idea on what the coefficient of $\alpha^m \beta^{j-m} a_{0,n-j}$ could look like and that they seem to be bounded by $\binom{j-1}{m-1}$ one would have calculate further $a_{m,n}$. This motivates the second sum of (3.9).

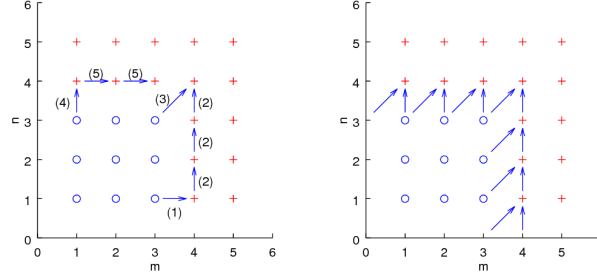


Figure 3.1: The blue circles represent the induction hypothesis. The red crosses represent the (m, n) for which the statement needs to be proven. The first figure represents the way the substeps will be carried out to get to $N + 1$, indicated by (1), (2), (3), (4) and (5). The second figure shows the terms involved in the estimate 3.8 of $a_{m,n}$.

The estimate (3.9) can be motivated in a similar way by looking at the coefficients of the first couple of (n, m) .

Proof. We will prove (3.9) and (3.10) at the same time by "induction". We will assume (3.9) and (3.10) to hold true for the set $Q_N := \{(m, n) | 1 \leq m \leq N, 1 \leq n \leq N\}$ and then prove it to be also true for Q_{N+1} . One "induction step" will be done by five "substeps", see figure 3.1.

Induction basis $N=1$: We only need to check $m = 1, n = 1$. So as

$$\begin{aligned} a_{1,1} &\leq \alpha a_{0,0} + \beta a_{1,0} = \alpha^0 \beta^1 \binom{1}{0} a_{1,0} + \alpha^1 \beta^0 \binom{1}{1} a_{0,0} \\ &\leq \alpha^0 \beta^1 \binom{1}{0} a_{1,0} + \alpha^1 \beta^0 \binom{1}{1} a_{0,0} + \alpha^1 \beta^0 \binom{0}{0} a_{0,0} \end{aligned}$$

holds, we get that (3.9) and (3.10) are fulfilled.

Inductive step $N \mapsto N + 1$:

First substep: $(N, 1) \mapsto (N + 1, 1)$

We want to prove (3.10) for $(N+1, 1)$. By (3.8) we get

$$a_{N+1,1} \leq \alpha a_{N,0} + \beta a_{N+1,0} = \sum_{j=0}^1 \alpha^j \beta^{1-j} \binom{1}{j} a_{N+1-j,0}$$

and so (3.10) is true.

Second substep: $(N+1, 1) \mapsto (N+1, 2) \mapsto \dots (N+1, N+1)$

We want to do an induction over $n = 1, \dots, N+1$ for $(N+1, n)$. That is to say, we want to prove (3.10) for $a_{N+1,n}$ with $n = 1, \dots, N+1$. As induction basis we use the first substep. So assume now that (3.10) holds for $a_{N+1,n-1}$ with $2 \leq n \leq N+1$ then we get, by using (3.8), $a_{N+1,n} \leq \alpha a_{N,n-1} + \beta a_{N+1,n-1}$. For the first term we get by using the induction hypothesis (3.10) of Q_N

$$\alpha a_{N,n-1} \leq \alpha \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a_{N-j,0}.$$

For the second term we have by induction hypothesis over n for fixed $N+1$

$$\begin{aligned} \beta a_{N+1,n-1} &\leq \beta \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a_{N+1-j,0} \\ &= \beta \alpha^0 \beta^{n-1} \binom{n-1}{0} a_{N+1,0} + \beta \sum_{j=1}^{n-1} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a_{N+1-j,0} \\ &= \underbrace{\beta \alpha^0 \beta^{n-1} \binom{n-1}{0} a_{N+1,0}}_{\beta^n} + \beta \sum_{j=0}^{n-2} \alpha^{j+1} \beta^{n-2-j} \binom{n-1}{j+1} a_{N-j,0}. \end{aligned}$$

So in total we get

$$\begin{aligned} &\alpha a_{N,n-1} + \beta a_{N+1,n-1} \\ &\leq \alpha \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} \binom{n-1}{j} a_{N-j,0} + \beta^n a_{N+1,0} \\ &\quad + \alpha \beta \sum_{j=0}^{n-2} \alpha^j \beta^{n-2-j} \binom{n-1}{j+1} a_{N-j,0} \\ &= \alpha \sum_{j=0}^{n-2} \alpha^j \beta^{n-1-j} \left(\binom{n-1}{j} + \binom{n-1}{j+1} \right) a_{N-j,0} \\ &\quad + \alpha^n \binom{n-1}{0} a_{N-(n-1),0} + \beta^n a_{N+1,0}. \end{aligned}$$

We can further write this as

$$\begin{aligned}
& \alpha \sum_{j=0}^{n-2} \alpha^j \beta^{n-1-j} \binom{n}{j+1} a_{N-j,0} + \alpha^n \binom{n-1}{0} a_{N-(n-1),0} + \beta^n a_{N+1,0} \\
&= \alpha \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} \binom{n}{j+1} a_{N-j,0} + \beta^n a_{N+1,0} \\
&= \sum_{j=1}^n \alpha^j \beta^{n-j} \binom{n}{j} a_{N-j+1,0} + \beta^n a_{N+1,0} = \sum_{j=0}^n \alpha^j \beta^{n-j} \binom{n}{j} a_{N-j+1,0},
\end{aligned}$$

where we used $\binom{n-1}{j} + \binom{n-1}{j+1} = \binom{n}{j+1}$ and in the forelast equality a change of index.

Third substep: $(N+1, N+1)$ for (3.9)

From the last substep it is easy to see that (3.9) holds for $(N+1, N+1)$ as

$$\begin{aligned}
a_{N+1,N+1} &\leq \sum_{j=0}^{N+1} \alpha^j \beta^{N+1-j} \binom{N+1}{j} a_{N-j+1,0} \\
&\leq \sum_{j=0}^{N+1} \alpha^j \beta^{N+1-j} \binom{N+1}{j} a_{N-j+1,0} + \alpha^{N+1} \beta^{j-(N+1)} \binom{j-1}{N+1-1} a_{0,N+1-j}.
\end{aligned}$$

Fourth substep: $(1, N) \mapsto (1, N+1)$

We want to prove (3.9) for $(1, N+1)$. By using (3.8) iteratively, we get

$$\begin{aligned}
a_{1,N+1} &\leq \alpha a_{0,N} + \beta a_{1,N} \leq \alpha a_{0,N} + \alpha \beta a_{0,N-1} + \beta^2 a_{1,N-1} \\
&\leq \alpha a_{0,N} + \alpha \beta a_{0,N-1} + \alpha \beta^2 a_{0,N-2} + \beta^3 a_{1,N-2} \\
&\leq \dots \\
&\leq \alpha \sum_{j=0}^N \beta^j a_{0,N-j} \\
&= \alpha \sum_{j=1}^{N+1} \beta^{j-1} a_{0,N+1-j} \\
&\leq \sum_{j=0}^1 \alpha^j \beta^{N+1-j} \binom{N+1}{j} a_{1-j,0} + \sum_{j=1}^{N+1} \alpha^1 \beta^{j-1} \binom{j-1}{1-1} a_{0,N+1-j}.
\end{aligned}$$

So we are done with this substep.

Fifth substep: $(1, N) \mapsto (2, N+1) \mapsto \dots (N, N+1)$

We want to do an induction over $m = 1, \dots, N$ for $(m, N+1)$. That is to say,

we want to prove (3.9) for $a_{m,N+1}$ with $m = 1, \dots, N$. As induction basis we use the fourth substep. So assume now that (3.9) holds for $a_{m-1,N+1}$, with $2 \leq m \leq N$. With the induction hypothesis (3.9) of Q_N we get

$$\begin{aligned}
& a_{m,N+1} \\
& \leq \alpha a_{m-1,N} + \beta a_{m,N} \\
& \leq \underbrace{\alpha \sum_{j=0}^{m-1} \alpha^j \beta^{N-j} \binom{N}{j} a_{m-1-j,0}}_{=:(1)} + \underbrace{\alpha \sum_{j=m-1}^N \alpha^{m-1} \beta^{j-m+1} \binom{j-1}{m-1-1} a_{0,N-j}}_{=:(2)} \\
& \quad + \underbrace{\beta \sum_{j=0}^m \alpha^j \beta^{N-j} \binom{N}{j} a_{m-j,0}}_{=:(3)} + \underbrace{\beta \sum_{j=m}^N \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,N-j}}_{=:(4)}.
\end{aligned}$$

On the one hand we have

$$\begin{aligned}
(1) + (3) &= \sum_{j=0}^{m-1} \alpha^{j+1} \beta^{N-j} \binom{N}{j} a_{m-1-j,0} + \beta \sum_{j=0}^m \alpha^j \beta^{N-j} \binom{N}{j} a_{m-j,0} \\
&= \sum_{j=1}^m \alpha^j \beta^{N-j+1} \binom{N}{j-1} a_{m-j,0} + \beta \sum_{j=0}^m \alpha^j \beta^{N-j} \binom{N}{j} a_{m-j,0} \\
&= \beta \sum_{j=1}^m \alpha^j \beta^{N-j} \binom{N}{j-1} a_{m-j,0} + \beta \sum_{j=0}^m \alpha^j \beta^{N-j} \binom{N}{j} a_{m-j,0} \\
&= \beta \sum_{j=1}^m \alpha^j \beta^{N-j} \left(\binom{N}{j-1} + \binom{N}{j} \right) a_{m-j,0} + \beta \alpha^0 \beta^N \binom{N}{0} a_{m,0} \\
&= \beta \sum_{j=1}^m \alpha^j \beta^{N-j} \binom{N+1}{j} a_{m-j,0} + \beta \alpha^0 \beta^N \binom{N}{0} a_{m,0} \\
&= \beta \sum_{j=0}^m \alpha^j \beta^{N-j} \binom{N+1}{j} a_{m-j,0}
\end{aligned}$$

where we used $\binom{N}{j-1} + \binom{N}{j} = \binom{N+1}{j}$. On the other hand we get, by using

$$\binom{j-1}{m-2} + \binom{j-1}{m-1} = \binom{j}{m-1},$$

$$(2) + (4)$$

$$\begin{aligned} &= \sum_{j=m-1}^N \alpha^m \beta^{j-m+1} \binom{j-1}{m-2} a_{0,N-j} + \sum_{j=m}^N \alpha^m \beta^{j-m+1} \binom{j-1}{m-1} a_{0,N-j} \\ &= \sum_{j=m}^N \alpha^m \beta^{j-m+1} \left(\binom{j-1}{m-2} + \binom{j-1}{m-1} \right) a_{0,N-j} + \alpha^m \beta^0 \binom{m-2}{m-2} a_{0,N-m+1} \\ &= \sum_{j=m}^N \alpha^m \beta^{j-m+1} \binom{j}{m-1} a_{0,N-j} + \alpha^m a_{0,N+1-m} \\ &= \sum_{j=m+1}^{N+1} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,N+1-j} + \alpha^m a_{0,N+1-m} \\ &= \sum_{j=m}^{N+1} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,N+1-j} \end{aligned}$$

Adding (1) + (3) and (2) + (4), we see that we are done with this substep. All in all this means that we have proven the Proposition. $\square$

We have now all the tools to prove the Crandall-Liggett Theorem.

Proof. (Crandall-Liggett Theorem) Our presentation relies on the proof of Theorem 1 of Crandall and Liggett [2].

For $\mu, \lambda > 0$, with $\lambda > \mu$, $x \in \mathcal{D}(B)$ and $m, n \in \mathbb{N}$, with $m \leq n$, we get, by (3) of Proposition 2, with $y = (\mathbf{1} - \lambda B)^{-(m-1)}x$

$$\begin{aligned} &(\mathbf{1} - \lambda B)^{-m} x \\ &= (\mathbf{1} - \lambda B)^{-1} (\mathbf{1} - \lambda B)^{-(m-1)} x \\ &= (\mathbf{1} - \mu B)^{-1} \left[\frac{\mu}{\lambda} (\mathbf{1} - \lambda B)^{-(m-1)} x + \frac{\lambda - \mu}{\lambda} (\mathbf{1} - \lambda B)^{-m} x \right]. \end{aligned}$$

We therefore obtain

$$\begin{aligned} a_{m,n} &:= \|(\mathbf{1} - \mu B)^{-n} x - (\mathbf{1} - \lambda B)^{-m} x\| \\ &= \left\| (\mathbf{1} - \mu B)^{-n} x \right. \\ &\quad \left. - (\mathbf{1} - \mu B)^{-1} \left[\frac{\mu}{\lambda} (\mathbf{1} - \lambda B)^{-(m-1)} x + \frac{\lambda - \mu}{\lambda} (\mathbf{1} - \lambda B)^{-m} x \right] \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \left(\mathbf{1} - \mu B \right)^{-(n-1)} x - \left[\frac{\mu}{\lambda} \left(\mathbf{1} - \lambda B \right)^{-(m-1)} x + \frac{\lambda - \mu}{\lambda} \left(\mathbf{1} - \lambda B \right)^{-m} x \right] \right\| \\
&\leq \left\| \left(\mathbf{1} - \mu B \right)^{-(n-1)} x \pm \frac{\mu}{\lambda} \left(\mathbf{1} - \mu B \right)^{-(n-1)} x - \left[\frac{\mu}{\lambda} \left(\mathbf{1} - \lambda B \right)^{-(m-1)} x + \frac{\lambda - \mu}{\lambda} \left(\mathbf{1} - \lambda B \right)^{-m} x \right] \right\| \\
&\leq \frac{\mu}{\lambda} \left\| \left(\mathbf{1} - \mu B \right)^{-(n-1)} x - \left(\mathbf{1} - \lambda B \right)^{-(m-1)} x \right\| \\
&\quad + \frac{|\lambda - \mu|}{\lambda} \left\| \left(\mathbf{1} - \mu B \right)^{-(n-1)} x - \left(\mathbf{1} - \lambda B \right)^{-m} x \right\| \\
&= \frac{\mu}{\lambda} a_{m-1, n-1} + \frac{\lambda - \mu}{\lambda} a_{m, n-1}.
\end{aligned}$$

Set $\alpha := \frac{\mu}{\lambda}$ and $\beta := \frac{\lambda - \mu}{\lambda}$. By (2) and (1) of Proposition 2 we have $a_{m-j, 0} \leq (m-j)\lambda \|Bx\|$ and $a_{0, n-j} \leq (n-j)\mu \|Bx\|$. We can therefore use (3.9) of Proposition 4 to obtain

$$\begin{aligned}
a_{m, n} &\leq \sum_{j=0}^m \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j, 0} + \sum_{j=m}^n \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0, n-j} \\
&\leq \|Bx\| \left(\lambda \sum_{j=0}^m \alpha^j \beta^{n-j} \binom{n}{j} (m-j) + \mu \sum_{j=m}^n \alpha^m \beta^{j-m} \binom{j-1}{m-1} (n-j) \right).
\end{aligned}$$

Now by (7) and (8) of Proposition 3 and remembering the definition of α, β we get

$$a_{m, n} \tag{3.11}$$

$$\leq \|Bx\| \left(\lambda \sqrt{(n\alpha - m)^2 + n\alpha\beta} + \mu \sqrt{\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2} \right) \tag{3.12}$$

$$= \|Bx\| \left(\sqrt{\lambda^2 \left(n\frac{\mu}{\lambda} - m\right)^2 + \lambda^2 n \frac{\mu}{\lambda} \frac{\lambda - \mu}{\lambda}} \right) \tag{3.13}$$

$$+ \|Bx\| \left(\sqrt{\mu^2 \frac{m \frac{\lambda - \mu}{\lambda}}{\left(\frac{\mu}{\lambda}\right)^2} + \mu^2 \left(\frac{m \frac{\lambda - \mu}{\lambda}}{\frac{\mu}{\lambda}} + m - n\right)^2} \right) \tag{3.14}$$

$$= \|Bx\| \left(\sqrt{(n\mu - m\lambda)^2 + n\mu(\lambda - \mu)} + \sqrt{m\lambda(\lambda - \mu) + (m\lambda - n\mu)^2} \right). \tag{3.15}$$

Finally if we take $\mu := \frac{t}{n}$ and $\lambda := \frac{t}{m}$, with $m \neq n$, then $\lambda > \mu$, so everything of the proof still holds and we get

$$\begin{aligned}
& \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x - \left(\mathbf{1} - \frac{t}{m} B \right)^{-m} x \right\| \\
& \leq \|Bx\| \sqrt{\underbrace{\left(\frac{n}{n} - m \frac{t}{m} \right)^2}_{=0} + n \frac{t}{n} \left(\frac{t}{m} - \frac{t}{n} \right)} \\
& \quad + \|Bx\| \sqrt{m \frac{t}{m} \left(\frac{t}{m} - \frac{t}{n} \right) + \underbrace{\left(m \frac{t}{m} - n \frac{t}{n} \right)^2}_{=0}} \\
& = \|Bx\| 2t \sqrt{\left(\frac{1}{m} - \frac{1}{n} \right)}.
\end{aligned}$$

So we deduce that $n \mapsto \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x$ is a Cauchy sequence.

We want to show now that the family $S(t)$, defined by

$S(t)x := \lim_{n \rightarrow \infty} \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x$ for $x \in \mathcal{D}(B)$, actually can be extended to $\overline{\mathcal{D}(B)}$ and that on $\overline{\mathcal{D}(B)}$ it is a semigroup. To do so we use the assumption (3.7) of the Crandall-Liggett Theorem to get

$$\left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x_1 - \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x_2 \right\| \leq \|x_1 - x_2\|.$$

So by letting $n \rightarrow \infty$ we get

$$\|S(t)x_1 - S(t)x_2\| \leq \|x_1 - x_2\|.$$

This proves that $S(t)$ satisfies (4) of Definition 3.1. Furthermore by the last inequality we see that, if we take $x_n \in \mathcal{D}(B)$ such that $x_n \rightarrow x$, for a fixed $x \in \overline{\mathcal{D}(B)}$,

$$\|S(t)x_n - S(t)x_m\| \leq \|x_m - x_n\|$$

holds. So $S(t)x_n$ is a Cauchy sequence and therefore we can define $S(t)x$ as its limit. (4) of Definition 3.1 therefore also holds for $x \in \overline{\mathcal{D}(B)}$.

Let's show next that $t \mapsto S(t)x$ is continuous for a fixed x , that is to say (3) of Definition 3.1 is fulfilled. First take $x \in \mathcal{D}(B)$. Now going back to

estimate (3.15) we get by taking $\mu := \frac{t}{n}$ and $\lambda := \frac{\tau}{n}$, with $\tau > t$, the estimate

$$\begin{aligned} & \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} x - \left(\mathbf{1} - \frac{\tau}{n} B \right)^{-n} x \right\| \\ & \leq \|Bx\| \left(\sqrt{(t-\tau)^2 + \frac{t}{n}(\tau-t)} + \sqrt{(t-\tau)^2 + \frac{\tau}{n}(\tau-t)} \right). \end{aligned}$$

So if we let $n \rightarrow \infty$, then $\|S(t)x - S(\tau)x\| \leq 2\|Bx\||t-\tau|$ holds. Therefore $t \mapsto S(t)x$ is continuous for $x \in \mathcal{D}(B)$.

Now for a general $x \in \overline{\mathcal{D}(B)}$, take a sequence $x_n \in \mathcal{D}(B)$ such that $x_n \rightarrow x$. Taking n big enough so that $\|x_n - x\|$ is small and then taking τ close enough to t so that also $\|Bx_n\||t-\tau|$ is small, we get, by

$$\begin{aligned} & \|S(t)x - S(\tau)x\| \\ & \leq \|S(t)x - S(t)x_n\| + \|S(t)x_n - S(\tau)x_n\| + \|S(\tau)x_n - S(\tau)x\| \\ & \leq \|x - x_n\| + 2\|Bx_n\||t-\tau| + \|x - x_n\| \end{aligned}$$

that $\|S(t)x - S(\tau)x\|$ is also small. This implies (3) of Definition 3.1 for $x \in \overline{\mathcal{D}(B)}$.

At last we will verify (1) of Definition 3.1. To do so we will first show by induction that for any $m \in \mathbb{N}$ and $x \in \mathcal{D}(B)$

$$\lim_{n \rightarrow \infty} \left(\left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \right)^m x = S(t)^m x \quad (3.16)$$

holds. We have already proven this for $m = 1$ so the induction basis holds true. Now assume that (3.16) holds for m . Then we get, with the definitions $y := S(t)^m x$ and $y_n := \left(\left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \right)^m x$,

$$\begin{aligned} & \left\| \left(\left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \right)^{m+1} x - S(t)^{m+1} x \right\| \\ & = \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \left(\left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \right)^m x - S(t) S(t)^m x \right\| \\ & = \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y_n - S(t) y \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y_n - \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y \right\| + \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y - S(t)y \right\| \\
&\leq \|y - y_n\| + \left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y - S(t)y \right\|.
\end{aligned}$$

Now we have by the induction hypothesis that $\|y - y_n\|$ goes to zero for $n \rightarrow \infty$. Also $\left\| \left(\mathbf{1} - \frac{t}{n} B \right)^{-n} y - S(t)y \right\|$ goes to zero by definition of $S(t)$. So we have proven (3.16).

With this result at hand we can deduce the following

$$\begin{aligned}
S(t)^m x &= \lim_{n \rightarrow \infty} \left(\left(\mathbf{1} - \frac{t}{n} B \right)^{-n} \right)^m x = \lim_{n \rightarrow \infty} \left(\mathbf{1} - \frac{t}{n} B \right)^{-nm} x \\
&= \lim_{n \rightarrow \infty} \left(\mathbf{1} - \frac{tm}{nm} B \right)^{-nm} x \\
&= \lim_{k \rightarrow \infty} \left(\mathbf{1} - \frac{tm}{k} B \right)^{-k} x.
\end{aligned}$$

So all in all we get $S(t)^m x = S(tm)x$. Furthermore if we take $m, n, r, s \in \mathbb{N}$ then we get, as $ms + rn \in \mathbb{N}$ holds, by what we have just proven,

$$\begin{aligned}
S\left(\frac{m}{n} + \frac{r}{s}\right) x &= S\left(\frac{1}{ns}(ms + rn)\right) x \\
&= S\left(\frac{1}{ns}\right)^{(ms+rn)} x = S\left(\frac{1}{ns}\right)^{ms} S\left(\frac{1}{ns}\right)^{rn} x \\
&= S\left(\frac{ms}{ns}\right) S\left(\frac{rn}{ns}\right) x = S\left(\frac{m}{n}\right) S\left(\frac{r}{s}\right) x.
\end{aligned}$$

So $S(t + \tau)x = S(t)S(\tau)x$ holds for $t, \tau \in \mathbb{Q} \cap \{s > 0\}$. As $t \rightarrow S(t)x$ is Lipschitz continuous we get that this also holds for $t, \tau \in \mathbb{R}_+$. Furthermore this also holds for $x \in \overline{\mathcal{D}(B)}$, because if we take $x_n \in \mathcal{D}(B)$ with $x_n \rightarrow x$, then we have

$$\begin{aligned}
&\|S(t + \tau)x - S(t)S(\tau)x\| \\
&\leq \|S(t + \tau)x - S(t + \tau)x_n\| + \|S(t + \tau)x_n - S(t)S(\tau)x\| \\
&\leq \|S(t + \tau)\| \|x_n - x\| + \|S(t)S(\tau)\| \|x_n - x\|.
\end{aligned}$$

The two terms of the last line now go to zero for $n \rightarrow \infty$. This means that we have proven (1) of Definition 3.1 for any $x \in \overline{\mathcal{D}(B)}$. So we are done with the proof. $\square$

Remark. We remark that we successfully constructed a semigroup without assuming B to be closed or even densely defined. However, this will come in play when discussing if this constructed semigroup actually solves the Cauchy Problem.

The following was proven on the way.

Corollary 3.1. Let the assumptions of the Crandall-Liggett Theorem be satisfied. Then we have for a fixed $x \in \mathcal{D}(B)$ that $t \rightarrow S(t)x$ is Lipschitz continuous.

We have seen how and under what conditions we can construct a semigroup on a subset of X . In the next chapter we will see if this semigroup satisfies the Cauchy Problem associated to the operator B .

3.3 Nonlinear Cauchy Problem

In this section we will discuss under what conditions the semigroup constructed in the Crandall-Liggett Theorem is the unique solution of the Cauchy Problem. First we will talk about differentiability, then uniqueness of $S(t)x$ and finally if $S(t)x$ is indeed a solution of the Cauchy Problem.

Before continuing we mention two Propositions which are motivated by the following two observations.

Assume we have given a nonlinear operator B with domain in a real Hilbert space H . Then we get, for all $t > 0$ and $y_i = (\mathbf{1} - tB)x_i$,

$$\begin{aligned} & \|(\mathbf{1} - tB)^{-1}y_1 - (\mathbf{1} - tB)^{-1}y_2\|^2 \leq \|y_1 - y_2\|^2 \\ \Leftrightarrow & \|x_1 - x_2\|^2 \leq \|x_1 - x_2 - tBx_1 + tBx_2\|^2 \\ \Leftrightarrow & \|x_1 - x_2\|^2 \leq \|x_1 - x_2\|^2 + t^2 \|Bx_1 - Bx_2\|^2 + 2t\langle x_1 - x_2, Bx_1 - Bx_2 \rangle \\ \Leftrightarrow & 0 \leq t \|Bx_1 - Bx_2\|^2 - 2\langle x_1 - x_2, Bx_1 - Bx_2 \rangle \\ \Leftrightarrow & \langle x_1 - x_2, Bx_1 - Bx_2 \rangle \leq 0. \end{aligned}$$

Furthermore we get for a differentiable function $t \mapsto u(t)$ with values in H ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} (\langle u(t+h), u(t+h) \rangle - \langle u(t), u(t) \rangle) \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} (\langle u(t+h) - u(t), u(t) \rangle + \langle u(t+h), u(t+h) - u(t) \rangle) \\ &= \frac{1}{2} \left\langle \frac{d}{dt} u(t), u(t) \right\rangle + \left\langle u(t), \frac{d}{dt} u(t) \right\rangle = \left\langle \frac{d}{dt} u(t), u(t) \right\rangle. \end{aligned}$$

We have the following Propositions.

Proposition 5. For an operator $B : \mathcal{D}(B) \rightarrow X$ we have the equivalence

$$\begin{aligned} \|x_1 - x_2\| &\leq \left\| x_1 - x_2 - \frac{1}{\lambda} Bx_1 + \frac{1}{\lambda} Bx_2 \right\| \quad \forall \lambda > 0, x_1, x_2 \in \mathcal{D}(B) \\ \Leftrightarrow \exists f \in F(x_1 - x_2) : (Bx_1 - Bx_2, f) &\leq 0 \quad \forall x_1, x_2 \in \mathcal{D}(B) \end{aligned}$$

Proof. For a proof see [3]; here F stands for the duality mapping (see the Notation section). $\square$

Remark. An operator B that fulfills one of the equivalent conditions of Proposition 5 is called a dissipative operator.

Proposition 6. Let $t \mapsto u(t)$ be a function on $[0, \infty)$ with values in a Banach space. Assume that $t \mapsto \|u(t)\|$ is differentiable on $(0, \infty)$, except on null set N , and that there exists a function $t \mapsto g(t)$ on $[0, \infty)$ such that $\frac{d}{dt}(u(t), x^*) = (g(t), x^*)$ holds for all $x^* \in X^*$, except on N , then we have

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 = (g(t), f) \quad \text{for all } f \in F(u(t)), t \notin N.$$

Proof. The proof is borrowed from the proof of Lemma 1.3 of Kato [6]. For $f \in F(u(t))$ we get, by Cauchy Schwarz Inequality and because $(u(t), f) = \|u(t)\|^2$ holds,

$$\begin{aligned} (u(t+h), f) &\leq \|u(t+h)\| \overbrace{\|f\|}^{=\|u(t)\|} \\ \Leftrightarrow (u(t+h) - u(t), f) &\leq \|u(t)\| (\|u(t+h)\| - \|u(t)\|). \end{aligned}$$

Dividing both sides by $h > 0$ and letting it go to zero we get $(g(t), f) \leq \frac{1}{2} \frac{d}{dt} \|u(t)\|^2$. The other direction follows if we divide by $h < 0$ and let it go to zero. $\square$

Remark. The duality mapping is motivated by the fact that for a Banach space one would like to be able to connect for an element $x \in X$ its norm $\|x\|$ with a linear map f , while still preserving the useful property that $(x, f) = \|x\|^2 = \|f\|^2$, as it's the case for a Hilbert space. In a Hilbert space there's only exactly one such a map, namely $y \mapsto \langle y, x \rangle$, by the Riesz representation theorem.

Now we want to discuss the differentiability of $t \mapsto S(t)x$. In Section 2.4 we've seen in Lemma 2.6 that if $S(t)$ is constructed by the Yosida Approximation then $t \mapsto S(t)x$ is differentiable. However for a nonlinear operator B we can not say much about the differentiability of $t \mapsto \left(1 - \frac{t}{n}B\right)^{-n}$. Even if this

function was differentiable it is hard to say what happens in the limit case. Nevertheless, we have seen, in Corollary 3.1, that the function $t \mapsto S(t)x$ is, for a fixed $x \in \mathcal{D}(B)$, Lipschitz continuous. This means that $t \mapsto S(t)x$ is a absolutely continuous function. In the case of $X = \mathbb{R}$ we know for a function $t \rightarrow u(t)$ that is absolutely continuous, that the derivative exists almost everywhere. This holds also for a general Banach space X under some restrictions.

Definition 3.2. A function $t \mapsto u(t)$ defined on an interval I with values in a Banach space X is called absolutely continuous if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that for any finite number of disjoint subset (t_i, t_{i+1}) of I we have

$$\sum_i |t_{i+1} - t_i| < \delta \Rightarrow \sum_i \|u(t_{i+1}) - u(t_i)\| < \epsilon.$$

How come one would expect an absolutely continuous function to be almost everywhere differentiable? Well heuristically speaking finding a derivative almost everywhere would follow, by the weaker statement, from being able to write $u(t) = u(t_0) + \int_{t_0}^t g(s)ds$, for a function $g \in \mathbf{L}^1(I, X)$, as, by the Lebesgue differentiation theorem which also holds for functions with values in a Banach space, we'd get

$$\lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} g(s)ds = g(t) \text{ a.e in } t.$$

If we could now define a "measure", constructed by the differences $u(t) - u(s)$, with values in a Banach space, we see by looking at the question, if a $g \in \mathbf{L}^1(I, X)$ exists such that $u(t) - u(t_0) = \int_{t_0}^t g(s)ds$ holds, that this is very similar to asking the question if a "Radon-Nikodým Theorem" holds for Banach spaces as well.

Furthermore we know that for $X = \mathbb{R}$ a construction of this measure is possible if u has finite total variation. Therefore we might expect that for a absolutely continuous function with values in a Banach space, as it has finite total variation, we are able to construct such a "measure" and so that we can then use the "Radon-Nikodým Theorem". However it turns out that a "Radon-Nikodým Theorem" doesn't hold, in general, in a Banach space X , but if we assume X to be reflexive, it does.

We will however not prove all theses things and directly prove the following.

Theorem 3.2. *Let X be a reflexive Banach space. Then any function $t \mapsto u(t)$ on $[0, \infty)$ that is absolutely continuous on every compact subset of $[0, \infty)$ is a.e differentiable on $(0, \infty)$ and*

$$u(t) = u(0) + \int_0^t \frac{d}{ds} u(s) ds \quad (3.17)$$

holds.

Proof. The following proof is borrowed from the proof of Theorem 2.1, Chapter 1 of Barbu [3].

We will do the proof only for the case where u is Lipschitz continuous, with constant L , as we will use this Theorem only for $t \mapsto S(t)x$. For the general case we refer to [3].

So let's prove that for any interval $[0, T]$ equation (3.17) holds. The first idea is to only look at the space $Y := \overline{\text{span}\{u(t) | 0 \leq t \leq T\}}$ - as we will only be interested to consider the difference quotient $\frac{u(t+h)-u(t)}{h}$, which always stays in Y . The second idea is to use the Fundamental theorem of calculus, which implies that a absolutely continuous function in $\mathbb{R}$ is a.e differentiable. So we'll be looking at $t \mapsto (u(t), y^*)$.

It is easy to show that Y is separable. Basically this follows from u being continuous and $[0, T]$ being separable. So as Y is a closed subspace of a reflexive space we know that Y is also reflexive. Furthermore it follows that Y^* is separable as Y is separable. We will now show that there exists a function $t \mapsto g(t)$ such that $\frac{d}{dt}(u(t), y^*) = (g(t), y^*)$ holds; that is to say g is the weak derivative of u .

As u is Lipschitz continuous we get that $t \mapsto (u(t), y^*)$ is Lipschitz continuous. So we have that $\frac{d}{dt}(u(t), y^*)$ exists a.e and is in $\mathbf{L}^1([0, T], \mathbb{R})$. Now assume that $\{y_n^*\}$ is a dense countable subset of Y^* . So we get that $\frac{d}{dt}(u(t), y_n^*)$ exists a.e. If N_n are the null sets such that $\frac{d}{dt}(u(t), y_n^*)$ doesn't exist, then we get for $t \in (0, T) - N$, with $N := \cup_n N_n$, that $\frac{d}{dt}(u(t), y_n^*)$ exists for any n .

Now if for any fixed $t \in (0, T)$ we choose a sequence $h_m \rightarrow 0$ we see that, as u is Lipschitz continuous, $\frac{u(t+h_m)-u(t)}{h_m} \in Y$ is bounded. So as Y is reflexive we have that there exists a subsequence h_{m_k} and a $g(t)$ such that we have, for $t \in (0, T) - N$ and for any n ,

$$\frac{d}{dt}(u(t), y_n^*) = \lim_{k \rightarrow \infty} \left(\frac{u(t + h_{m_k}) - u(t)}{h_{m_k}}, y_n^* \right) = (g(t), y_n^*).$$

We also note that g doesn't actually depend on the choice of the sequence, because if h would have been another such function, we would have gotten $\|g(t) - h(t)\| = \sup\{(g(t) - h(t), y^*) | \|y^*\| \leq 1\} = 0$. Now as $\frac{d}{dt}(u(t), y_n^*)$ is in $\mathbf{L}^1([0, T], \mathbb{R})$ we get especially that $t \mapsto (g(t), y_n^*)$ is measurable. For any y^* we have therefore, as we can find a sequence $y_n^* \rightarrow y^*$ and as $\lim_{n \rightarrow \infty} (g(t), y_n^*) = (g(t), y^*)$, that $t \mapsto (g(t), y^*)$ is also measurable. So as g is separably valued we get by Pettis' Theorem that g is measurable. Now as $g(t)$ is the weak

limit of $\lim_{k \rightarrow \infty} \frac{u(t+h_{n_k})-u(t)}{h_{n_k}}$ we have

$$\|g(t)\| \leq \liminf_{k \rightarrow \infty} \left\| \frac{u(t+h_{n_k})-u(t)}{h_{n_k}} \right\| \leq L \quad \text{for } t \in (0, T) - N.$$

So by Bochner's Theorem we get that g is Bochner integrable. So all in all we get

$$\begin{aligned} (u(t) - u(0), y^*) &= \lim_{n \rightarrow \infty} (u(t) - u(0), y_n^*) = \lim_{n \rightarrow \infty} \int_0^t \frac{d}{ds} (u(s), y_n^*) ds \\ &= \lim_{n \rightarrow \infty} \int_0^t (g(s), y_n^*) ds = \lim_{n \rightarrow \infty} \left(\int_0^t g(s) ds, y_n^* \right) = \left(\int_0^t g(s) ds, y^* \right). \end{aligned}$$

And as this holds for any y^* this means that $u(t) - u(0) = \int_0^t g(s) ds$ and we are done. $\square$

So we see that $t \mapsto S(t)x$ is a.e differentiable, with derivative in $\mathbf{L}^1(I, X)$. Even though we can not expect a function that is a solution of the Cauchy Problem to have a continuous derivative - as it was the case for linear unbounded operators B that satisfy the assumptions of the Hille-Yosida Theorem, see Lemma 2.6 - we would at least like to have that the derivative is in $\mathbf{L}^1(I, X)$.

Definition 3.3. A function $t \mapsto u(t)$ on $[0, \infty)$ is a solution to the nonlinear Cauchy Problem, with initial value $x \in \mathcal{D}(B)$, for an operator B satisfying the assumptions of the Crandall-Liggett Theorem if and only if u is Lipschitz continuous on compact subsets of $[0, \infty)$, it is almost everywhere differentiable, solves $\frac{d}{dt}u(t) = Bu(t)$ a.e on $(0, \infty)$ and fulfills $u(0) = x$.

Remark. We have already defined for linear unbounded operators that satisfy the conditions of the Hille-Yosida Theorem what we mean by a solution of the Cauchy Problem associated with it. However such operators satisfy also the Crandall-Liggett Theorem and as we had included in the Definition 2.2 that u should be continuously differentiable, it follows easily, that u is Lipschitz continuous on compact subsets of $[0, \infty)$. Therefore the Definition above is just an extension of the Definition 2.2. The uniqueness proof below therefore also covers this case.

Now we want to turn to the question of uniqueness.

Theorem 3.3. *If a solution of the nonlinear Cauchy Problem for an operator B satisfying the assumptions of the Crandall-Liggett Theorem exists, then it is unique.*

Proof. We follow Barbu [3]. Assume that $t \mapsto u(t)$ and $t \mapsto v(t)$ are solutions. Define $w(t) := u(t) - v(t)$. We have $\frac{d}{dt}w(t) = Bu(t) - Bv(t)$ a.e. Furthermore we get as the derivative of w exists a.e that $(\frac{d}{dt}w(t), x^*) = \frac{d}{dt}(w(t), x^*)$ holds for all $x^* \in X^*$. Now as w is also Lipschitz continuous we get that $t \mapsto \|w(t)\|$ is also Lipschitz continuous. So the derivative of $t \mapsto \|w(t)\|$ exists a.e, as $\mathbb{R}$ is reflexive. Therefore we get by Proposition 6 for $f \in F(w(t)) = F(u(t) - v(t))$ and for almost all t ,

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|^2 = (\frac{d}{dt}w(t), f) = (Bu(t) - Bv(t), f) \leq 0$$

where the last estimate follows from Proposition 5. So Integrating from 0 to t we get $\|w(t)\| \leq \|w(0)\|$. And as $w(0) = 0$ we have $w \equiv 0$. $\square$

Finally we will turn to a result that tells us under what conditions we get that $t \mapsto S(t)x$ solves the nonlinear Cauchy Problem. We'll make some heuristic observations first (with ideas borrowed from [5]). One question is, how could we get to $\frac{d}{dt}S(t)x = BS(t)x$ when we only know that $t \mapsto S(t)x$ is a.e differentiable and that $S(t)x = \lim_{n \rightarrow \infty} (\mathbf{1} - \lambda_n B)^{-n} x$ holds? Well one goal should be to make B "appear" outside of the $(\mathbf{1} - \lambda_n B)^{-n} x$. We can do the following

$$\begin{aligned} (\mathbf{1} - \lambda_n B)^{-(n-1)} x &= (\mathbf{1} - \lambda_n B)^{-n+1} x \\ &= (\mathbf{1} - \lambda_n B) (\mathbf{1} - \lambda_n B)^{-n} x \\ &= (\mathbf{1} - \lambda_n B)^{-n} x - \lambda_n B (\mathbf{1} - \lambda_n B)^{-n} x. \end{aligned}$$

So we see that we were able to factor out B . Furthermore we see that this is equivalent to

$$(\mathbf{1} - \lambda_n B)^{-n} x - (\mathbf{1} - \lambda_n B)^{-(n-1)} x = \lambda_n B (\mathbf{1} - \lambda_n B)^{-n} x. \quad (3.18)$$

At a first glance this looks as if we could get to $\frac{d}{dt}S(t)x = BS(t)x$, by choosing $\lambda_n = \frac{t}{n-1}$, dividing it by λ_n and then letting $n \rightarrow \infty$ - as we can estimate, by (3.15), $(\mathbf{1} - \lambda_n B)^{-n} x \approx S(t + \frac{t}{n-1})x$. However there are two problems. First B is usually not continuous and secondly the order of the approximation is $o(\sqrt{\frac{n}{(n-1)^2}})$ and so dividing by $\frac{t}{n-1}$ shows that an approximation of the difference quotient doesn't work in this way. Nevertheless we can do the following by using that B is dissipative and using equation (3.18),

$$\left((\mathbf{1} - \lambda_n B)^{-n} x - (\mathbf{1} - \lambda_n B)^{-(n-1)} x, f(t) \right) = (\lambda_n B (\mathbf{1} - \lambda_n B)^{-n} x, f(t)) \quad (3.19)$$

$$\leq \lambda_n (B\hat{x}, f(t)) \quad (3.20)$$

for $f(t) \in F((\mathbf{1} - \lambda_n B)^{-n} x - \hat{x})$ and any $\hat{x} \in \mathcal{D}(B)$. Using

$$((\mathbf{1} - \lambda_n B)^{-n} x - \hat{x}, f(t)) = \|(\mathbf{1} - \lambda_n B)^{-n} x - \hat{x}\|^2$$

and

$$\begin{aligned} & - \left((\mathbf{1} - \lambda_n B)^{-(n-1)} x - \hat{x}, f(t) \right) \\ & \geq - \left\| (\mathbf{1} - \lambda_n B)^{-(n-1)} x - \hat{x} \right\| \left\| (\mathbf{1} - \lambda_n B)^{-n} x - \hat{x} \right\| \end{aligned}$$

we get that (3.19), by using $a^2 - ab \geq \frac{1}{2}(a^2 - b^2)$ which follows from $(a+b)^2 \geq 0$ for $a, b \in \mathbb{R}$, is equivalent to

$$\begin{aligned} & \frac{1}{2} \left(\left\| (\mathbf{1} - \lambda_n B)^{-n} x - \hat{x} \right\|^2 - \left\| (\mathbf{1} - \lambda_n B)^{-(n-1)} x - \hat{x} \right\|^2 \right) \\ & \leq \left\| (\mathbf{1} - \lambda_n B)^{-n} x - \hat{x} \right\|^2 - \left\| (\mathbf{1} - \lambda_n B)^{-n} x - \hat{x} \right\| \left\| (\mathbf{1} - \lambda_n B)^{-(n-1)} x - \hat{x} \right\| \\ & = \left((\mathbf{1} - \lambda_n B)^{-n} x - (\mathbf{1} - \lambda_n B)^{-(n-1)} x, f(t) \right) \\ & \leq \lambda_n (B\hat{x}, f(t)). \end{aligned}$$

Under certain assumptions, which we will see in the next Theorem that is basically due to Brezis [4], one can then integrate with respect to t and let $n \rightarrow \infty$ to get to

Theorem 3.4. *Assume that the requirements of the Crandall-Liggett Theorem hold for a closed nonlinear operator B . Furthermore assume $\text{conv}(\mathcal{D}(B)) \subseteq \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$ for all $\lambda \in (0, c]$, for a fixed c , then we get for any $\hat{x} \in \mathcal{D}(B)$ and $x \in \mathcal{D}(B)$*

$$\begin{aligned} (1) \quad & \|S(t)x - \hat{x}\|^2 - \|S(r)x - \hat{x}\|^2 \leq 2 \int_r^t \sup\{(B\hat{x}, v) | v \in F(S(s)x - \hat{x})\} ds \\ (2) \quad & \limsup_{t \rightarrow 0} \sup \left\{ \left(\frac{S(t)\hat{x} - \hat{x}}{t}, v \right) \mid v \in F(\hat{x} - x) \right\} \\ & \leq \sup\{(Bx, v) | v \in F(\hat{x} - x)\}. \end{aligned}$$

Proof. For a detailed proof see [2], [3] or [4]. □

With the previous theorem we can prove the following theorem, which tells us under what conditions the semigroup constructed by the Crandall-Liggett Theorem actually is a solution, moreover its unique solution, of the Cauchy Problem associated with the operator B .

Theorem 3.5. *Assume that X is a reflexive Banach space. Furthermore let B be a closed nonlinear operator that satisfies the requirements of the Crandall-Liggett Theorem as well as $\overline{\text{conv}(\mathcal{D}(B))} \subseteq \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$ for all $\lambda \in (0, c]$, for a fixed c . Then we get that the semigroup $S(t)$ constructed in the Crandall-Liggett Theorem is the unique solution of the corresponding Cauchy Problem for any initial value $x \in \mathcal{D}(B)$.*

Proof. The proof of this Theorem is borrowed from the proof of Theorem 1.5, Chapter 3 of Barbu [3].

As we know from Theorem 3.2, $t \rightarrow S(t)x$ is differentiable. Therefore we have for $h > 0$ small enough $S(t-h)x = S(t)x - h \frac{d}{dt}S(t)x + o(\|h\|)$. So, as $S(t-h)x \in \overline{\mathcal{D}(B)}$ holds by construction and as $\overline{\text{conv}(\mathcal{D}(B))} \subseteq \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$ holds by assumption, we get that for each h there exists x_h such that $S(t-h)x = x_h - hBx_h$. So we have

$$S(t)x - h \frac{d}{dt}S(t)x + o(\|h\|) = x_h - hBx_h. \quad (3.21)$$

This is equivalent to $S(t)x - x_h = h \left(\frac{d}{dt}S(t)x - Bx_h \right) + o(\|h\|)$. By Theorem 3.4 we get by (2)

$$\begin{aligned} & \limsup_{s \rightarrow 0} \sup \left\{ \left(\frac{S(s)S(t)x - S(t)x}{s}, v \right) \middle| v \in F(S(t)x - x_h) \right\} \\ & \leq \sup \{ (Bx_h, v) | v \in F(S(t)x - x_h) \}. \end{aligned}$$

From this follows

$$\sup \left\{ \left(\frac{d}{dt}S(t)x, v \right) \middle| v \in F(S(t)x - x_h) \right\} \leq \sup \{ (Bx_h, v) | v \in F(S(t)x - x_h) \}. \quad (3.22)$$

Since $F(S(t)x - x_h)$ is weak* compact - this basically follows from the Banach-Alaoglu Theorem, see [5] - we have that there exists a $f_h \in F(S(t)x - x_h)$ such that $\sup \{ (Bx_h, v) | v \in F(S(t)x - x_h) \} = (Bx_h, f_h)$. With this we get from (3.22)

$$\begin{aligned} \left(\frac{d}{dt}S(t)x, f_h \right) & \leq (Bx_h, f_h) \Rightarrow h \left(\frac{d}{dt}S(t)x - Bx_h, f_h \right) \leq 0 \\ & \Rightarrow (S(t)x - x_h + o(\|h\|), f_h) \leq 0 \\ & \Rightarrow \|S(t)x - x_h\|^2 \leq (o(\|h\|), f_h) \\ & \Rightarrow \|S(t)x - x_h\| \leq o(\|h\|), \end{aligned}$$

where the forelast implication follows from the fact that $(S(t)x - x_h, f_h) = \|S(t)x - x_h\|^2$ holds. So we get $\lim_{h \rightarrow 0} x_h = S(t)x$. With this we get by using (3.21) also $\lim_{h \rightarrow 0} Bx_h = \frac{d}{dt}S(t)x$. Finally we have, as B is closed, that $BS(t)x = \frac{d}{dt}S(t)x$. The uniqueness now follows from Theorem 3.3. $\square$

3.4 Nonlinear examples

We will now see two nonlinear examples. In the following let $n > 3$.

3.4.1 Diffusion with nonlinear additive term

As a first example we look at the following semilinear equation

$$\begin{aligned} \frac{d}{dt}u(x, t) &= \Delta u(x, t) + g(u(x, t)), & (x, t) &\in \Omega \times (0, \infty), \\ u(x, t) &= 0, & (x, t) &\in \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x &\in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function that is monotonically decreasing, for example, $g(u) = -u^3 - u$.

One could think of this PDE as a diffusion process of particles where g would represent some potential. We will make use of the following result.

Theorem 3.6. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing locally Lipschitz continuous function and $f \in L^2(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega)$ to the problem*

$$\int_{\Omega} \nabla u \nabla v + \int_{\Omega} h(u)v = \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Proof. This theorem is basically proven by using Stampacchias Theorem, which is an extension of the Lax-Milgram Theorem. For a proof we refer to the proof of Theorem 4.7, Chapter 4, [9]. $\square$

We will now check all the assumptions of the Crandall-Liggett Theorem and Theorem 3.5 of Chapter 3 for the operator B defined by

$$\begin{aligned} \mathcal{D}(B) &= \{u \in H_0^1(\Omega) | \exists w \in L^2(\Omega) : \\ &\quad (w, v)_{L^2(\Omega)} = -(\nabla u, \nabla v)_{L^2(\Omega)} + (g(u), v)_{L^2(\Omega)} \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega)\} \\ Bu &:= w. \end{aligned}$$

Yet again B is well defined, because as $C_c^\infty(\Omega) \subseteq H_0^1(\Omega) \cap L^\infty(\Omega)$ holds we have, as $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, that such a w is unique. From this point on we will assume $u_0 \in \mathcal{D}(B)$.

First of all we show that $\mathbf{1} - \lambda B$ is injective for all $\lambda > 0$ small enough. $\mathbf{1} - \lambda B$ is injective:

Assume $(\mathbf{1} - \lambda B)u_1 = (\mathbf{1} - \lambda B)u_2$. So we get by definition of B ,

$$\begin{aligned} 0 &= \int_{\Omega} (u_1 - u_2)(\mathbf{1} - \lambda B)u_1 - \int_{\Omega} (u_1 - u_2)(\mathbf{1} - \lambda B)u_2 \\ &= \int_{\Omega} (u_1 - u_2)(u_1 - u_2) + \lambda \int_{\Omega} \nabla(u_1 - u_2) \nabla(u_1 - u_2) \\ &\quad - \lambda \int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2). \end{aligned}$$

As g is a decreasing function we get that $\mathbf{1} - \lambda g(\cdot)$ is an increasing function. This means that

$$\int_{\Omega} [(u_1 - u_2) - \lambda(g(u_1) - g(u_2))](u_1 - u_2) \geq 0$$

holds, as for any $x_1, x_2 \in \mathbb{R}$ we have $[(x_1 - \lambda g(x_1) - (x_2 - \lambda g(x_2)))](x_1 - x_2) \geq 0$. So in total we get

$$\begin{aligned} &\int_{\Omega} (u_1 - u_2)(u_1 - u_2) + \lambda \int_{\Omega} \nabla(u_1 - u_2) \nabla(u_1 - u_2) \\ &\quad - \lambda \int_{\Omega} (g(u_1) - g(u_2))(u_1 - u_2) \\ &\geq \lambda \int_{\Omega} \nabla(u_1 - u_2) \nabla(u_1 - u_2). \end{aligned}$$

As Ω is open, bounded and has a Lipschitz boundary, we get, by Poincaré's inequality, $\|\nabla u_1 - \nabla u_2\|^2 \geq d \|u_1 - u_2\|^2$, for a constant $d > 0$. As $0 \geq \lambda \|\nabla u_1 - \nabla u_2\|^2 \geq d\lambda \|u_1 - u_2\|^2$ this means $u_1 = u_2$. So $\mathbf{1} - \lambda B$ is injective. Now we will prove $\overline{\mathcal{D}(B)} \subseteq \cap_{\lambda > 0} \mathcal{D}((\mathbf{1} - \lambda B)^{-1})$. We will do this by showing $\mathcal{D}((\mathbf{1} - \lambda B)^{-1}) = L^2(\Omega)$.

$\mathcal{D}((\mathbf{1} - \lambda B)^{-1}) = L^2(\Omega)$:

As $\frac{1}{\lambda} \mathbf{1} - g(\cdot)$ is an increasing function on $\mathbb{R}$ we get, by Theorem 3.6, that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \nabla v + \int_{\Omega} \left(\frac{1}{\lambda} u - g(u)\right) v = \int_{\Omega} \frac{1}{\lambda} f v \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

holds. But this is equivalent to

$$-(\nabla u, \nabla v) + (g(u), v) = \left(\frac{1}{\lambda}(u - f), v\right) \quad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

meaning that $u \in \mathcal{D}(B)$ on the one hand and on the other $Bu = \frac{1}{\lambda}(u - f)$. So we have $(\mathbf{1} - \lambda B)u = f$. Therefore $\mathcal{D}((\mathbf{1} - \lambda B)^{-1}) = L^2(\Omega)$.

At last we will prove that B is a dissipative operator.

B is dissipative:

By Proposition 5 we need to prove $(Bu_1 - Bu_2, u_1 - u_2) \leq 0$ for any $u_1, u_2 \in \mathcal{D}(B)$. Using the definition of B we get,

$$\begin{aligned} (Bu_1 - Bu_2, u_1 - u_2) &= (Bu_1, u_1 - u_2) - (Bu_2, u_1 - u_2) \\ &= -(\nabla u_1 - \nabla u_2, \nabla u_1 - \nabla u_2) \\ &\quad + (g(u_1) - g(u_2), u_1 - u_2) \\ &\leq 0, \end{aligned}$$

where the last estimate follows from the fact that g is a decreasing function.

B is closed:

Let $u_k \in \mathcal{D}(B)$ be a sequence such that $u_k \xrightarrow{L^2} u$ and $Bu_k \xrightarrow{L^2} w$ for some $u, w \in L^2(\Omega)$. We get, as

$$(Bu_k, u_k) = -(\nabla u_k, \nabla u_k) + (g(u_k), u_k)$$

and $(Bu_k, u_k) \rightarrow (w, u)$ as well as $(g(u_k), u_k) \rightarrow (g(u), u)$ holds, because g is continuous, that, by the same reasoning as in the linear examples, we have $u \in H_0^1(\Omega)$ and furthermore $u \in \mathcal{D}(B)$ as well as $Bu = w$.

So, as $L^2(\Omega)$ is a reflexive space, by the Crandall-Liggett Theorem and Theorem 3.5 of Chapter 3, we know that the function $t \mapsto S(t)u_0$ is the unique solution of the Cauchy Problem associated with the operator B .

3.4.2 Nonlinear diffusion

We now look at quasi-linear equations of the form

$$\begin{aligned} \frac{d}{dt}u(x, t) &= \nabla \cdot \varphi(x, \nabla u(x, t)), & (x, t) &\in \Omega \times (0, \infty), \\ u(x, t) &= 0, & (x, t) &\in \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x &\in \Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open and bounded set with a Lipschitz boundary. We will assume that $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function that satisfies:

- (1) there is a $\alpha > 0$ such that for all $x \in \Omega : \|\varphi(x, b)\|_E \leq \alpha \|b\|_E^{p-1}$
- (2) there is a $\gamma > 0$ such that for all $x \in \Omega : \varphi(x, b) \cdot b \geq \gamma \|b\|_E^p$
- (3) for any $b_1, b_2 \in \mathbb{R}^2, b_1 \neq b_2, x \in \Omega : (\varphi(x, b_1) - \varphi(x, b_2)) \cdot (b_1 - b_2) > 0$,

for some fixed $p \in (1, \infty)$, for example, $\varphi(x, b) = ((e^{-b_1^2} + 1)b_1, b_2, \dots, b_n)$ and $p = 2$. Here $\|\cdot\|_E$ denotes the Euclidean norm.

Before proving the assumption of the Crandall-Liggett Theorem we mention the following result which we'll need.

Theorem 3.7. *Assume $\Omega \subseteq \mathbb{R}^n$ is an open and bounded set and that for a continuous function $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ the conditions (1), (2), (3) from above are satisfied. Then for any $f \in L^2(\Omega)$ we have that there exists a solution $u \in H_0^1(\Omega)$ to the following problem*

$$\int_{\Omega} \varphi(\cdot, \nabla u) \nabla v + uv = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega).$$

Proof. For a proof see [9], Theorem 5.11, Chapter 5. □

We define the operator B by

$$\begin{aligned} \mathcal{D}(B) &:= \{u \in H_0^1(\Omega) | \exists w \in L^2(\Omega) : \\ &\quad (w, v)_{L^2(\Omega)} = -(\varphi(\cdot, \nabla u), \nabla v)_{L^2(\Omega)} \forall v \in H_0^1(\Omega)\} \\ Bu &:= w. \end{aligned}$$

We remark again that such a w is unique as $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. In the following we have $\lambda > 0$ and we will now check the assumptions of the Crandall-Liggett Theorem.

$\mathbf{1} - \lambda B$ is injective:

Assume $(\mathbf{1} - \lambda B)u_1 = (\mathbf{1} - \lambda B)u_2$ for $u_1, u_2 \in \mathcal{D}(B)$. By using assumption (3) for φ we get

$$\begin{aligned} 0 &= ((\mathbf{1} - \lambda B)u_1 - (\mathbf{1} - \lambda B)u_2, u_1 - u_2) \\ &= (u_1 - u_2, u_1 - u_2) - \lambda(Bu_1, u_1 - u_2) + \lambda(Bu_2, u_1 - u_2) \\ &= (u_1 - u_2, u_1 - u_2) + \lambda(\varphi(\cdot, \nabla u_1), \nabla u_1 - \nabla u_2) - \lambda(\varphi(\cdot, \nabla u_2), \nabla u_1 - \nabla u_2) \\ &= (u_1 - u_2, u_1 - u_2) + \lambda(\varphi(\cdot, \nabla u_1) - \varphi(\cdot, \nabla u_2), \nabla u_1 - \nabla u_2) \\ &\geq \|u_1 - u_2\|^2. \end{aligned}$$

So it follows that $u_1 = u_2$ holds.

$\mathcal{D}((\mathbf{1} - \lambda B)^{-1}) = L^2(\Omega)$:

By the theorem above we get that there exists a $u \in H_0^1(\Omega)$ such that

$$(\lambda\varphi(\cdot, \nabla u), \nabla v) + (u, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

is fulfilled. As this is equivalent to

$$-(\varphi(\cdot, \nabla u), \nabla v) = \left(\frac{1}{\lambda}(u - f), v\right) \quad \forall v \in H_0^1(\Omega),$$

we get on the one hand $u \in \mathcal{D}(B)$ and on the other $Bu = \frac{1}{\lambda}(u - f)$. The last expression is the same as $(\mathbf{1} - \lambda B)u = f$.

B is dissipative:

We quickly prove $(Bu_1 - Bu_2, u_1 - u_2) \leq 0$ for any $u_1, u_2 \in \mathcal{D}(B)$. With assumption (3) on φ we get

$$\begin{aligned} & (Bu_1 - Bu_2, u_1 - u_2) \\ &= (Bu_1, u_1 - u_2) - (Bu_2, u_1 - u_2) \\ &= -(\varphi(\cdot, \nabla u_1), \nabla u_1 - \nabla u_2) + (\varphi(\cdot, \nabla u_2), \nabla u_1 - \nabla u_2) \\ &= -(\varphi(\cdot, \nabla u_1) - \varphi(\cdot, \nabla u_2), \nabla u_1 - \nabla u_2) \leq 0. \end{aligned}$$

So by the Crandall-Liggett Theorem we know that for $u_0 \in \mathcal{D}(B)$ there exists a semigroup $t \mapsto S(t)u_0$.

Outlook

Many extensions of the theory mentioned are possible. First of all, one could also look at semigroups where its "time" variable is in the complex numbers, as it is done in [1]. Another extension would be that the Crandall-Liggett Theorem also works for multivalued operators; this case was actually treated in their original paper [2], but wasn't done here for the sake of simplicity. We also didn't look at perturbations of operators which offers us an even richer toolbox when it comes to dealing with PDEs, see [1], [3] and [5]. Another important extension would be to look at time dependent operators. This is done for example in [2] or [1].

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Deutsche Zusammenfassung

Diese Masterarbeit handelt von Operatorhalbgruppen und ihren Anwendungen auf das abstrakte Cauchy-Problem. Den größten Teil der Arbeit nimmt die Konstruktion von Operatorhalbgruppen ein. Diese werden zuerst für beschränkte lineare Operatoren, dann für unbeschränkte lineare und schließlich für nichtlineare Operatoren konstruiert. Des Weiteren wird im Laufe der Kapitel immer eine Verbindung zu dem zugehörigen abstrakten Cauchy-Problem hergestellt. Die Nützlichkeit von Operatorhalbgruppen bei der Lösungsfindung von partiellen Differentialgleichungen wird anhand von Beispielen illustriert.

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