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„On the symmetry of equatorial water waves with constant vorticity and stagnation points"

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Contents

Abstract

In recent years there have been many published results which prove symmetry of various types of water waves (for example: [4], [11], [18]). In [18] the author shows that gravity waves which are monotonic between crests and troughs with constant vorticity and allowing for stagnation points are symmetric. In this thesis we extend this proof and show that the same is true for equatorial water waves with constant vorticity and stagnation points.

CHAPTER 1

Introduction

In this thesis we consider periodic equatorial waves with one crest and one trough per period and prove that they are symmetric if certain assumptions are fulfilled. We work in the framework of nonlinear theory which allows for waves that are not sinusoidal but have the form that is observed in nature, namely the crest is higher and narrower and the trough is broader and less deep.

In Chapter 2 we present the derivation of the governing equations of gravity water waves. Such waves arise through the restoring action of gravity on water displaced from the equilibrium level (still water). Their equations of motion are the equation of mass conservation and Euler's equations supplemented by suitable boundary conditions. This thesis, however, concerns geophysical equatorial water waves. These are waves trapped near the equator and decay rapidly away from it.

The free surface of the flow is an unknown in the Euler equations (such problems are called free-boundary problems). We then reformulate the system which we derived in terms of the so-called stream function ([5]). This allows for a simplification of the problem and can be shown that the two systems are equivalent.

Another essential feature of most water flows is the vorticity. The vorticity describes the swirling motion of the water. In this thesis the vorticity is taken to be a constant different than zero. Physically this means from the water flow is shear-like.

In Chapter 3 we give a short introduction to maximum principles. Maximum principles are a very important tool in the study of partial differential equations and most of them are statements of the form that if a certain inequality is satisfied, then a function attains its maximum only on the boundary. We state four important theorems, the weak maximum principle, the strong maximum principle, Hopf's maximum principle and Serrin's edge-point lemma. The proof for the first three is also provided.

In Chapter 4 we move on to discuss the existence geophysical equatorial water flows allowing for stagnation points and a constant non-vanishing vorticity following the work laid out in [15]. Their existence is guaranteed by means of bifurcation theory, something which is not discussed in great detail in this thesis, since it is beyond its scope, and is merely quoted. Furthermore, we discuss the existence of stagnation points. Such points appear often in water flows. Their mathematical description is that for such points their horizontal velocity is equal to that with which the wave travels. The existence of such points is related to the appearance of critical layers and it may happen that the wave "breaks". Finally, we prove the claimed symmetry property.

Although symmetry in rotational water waves is quite counter-intuitive, we prove that under specific assumptions, they do exhibit this feature, something which is quite fascinating. Such results exist for various types of waves and different types of vorticity ([4]).

CHAPTER 2

The Governing Equations

2.1 Gravity water waves

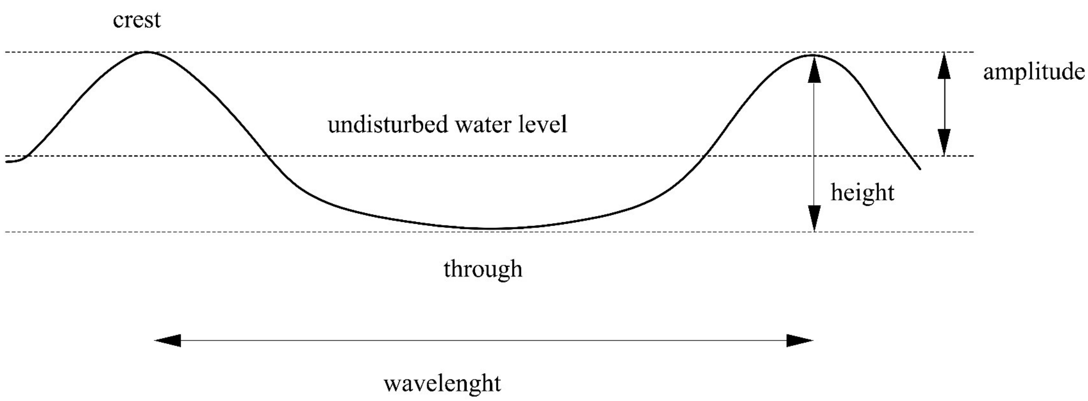

Figure 2.1: The setting (Figure found in [3])

Water waves are not confined in a closed domain. In fact, we can consider the ocean to be infinitely big for our purposes (the numerical results agree with the theoretical ones so we are in the right). We state some water wave terminology which will be used throughout this thesis.

The part of the wave which is most elevated from the undisturbed water level is called a *crest*, while the lowest part of the wave beneath the undisturbed water level is called a *trough*; the maximum deviation of the wave from the undisturbed water level, be it either crest or trough, is called the *wave amplitude*. The vertical distance from crest to trough is called the *wave height*, while the distance between two consecutive crests (or troughs) is the *wavelength*. Finally, the time required for a two consecutive crests or troughs to pass through a fixed point in space is the *wave period* and the ratio of wave length to wave period is called the *wave speed*.

2.1.1 Assumptions

As always when developing a model, we need to make some assumptions in order to simplify things.

We consider that water flows over a flat area called the *ocean bed* where no interaction between the water molecules and the Earth takes place. Such a flat ocean bed is said to be *rigid*.

We assume here that water has a constant density, which will be denoted by ρ.

Furthermore, we consider water to be an inviscid fluid and our aim will be to derive the Euler equations for inviscid fluids (in fact, these equations are the same as the Navier-Stokes equations without the viscosity term).

The water waves we will be considering at the final chapter of this thesis will be regular wave trains. This means that the motion is identical in any direction parallel to the crest line and which propagate at constant speed in a fixed direction. Therefore, such waves can be thought to be the free surface of a a two-dimensional flow. However, we derive the governing equations in the three dimensional general case and we will simply drop the third dimension when we concentrate ourselves at the problem at hand.

Finally, the $x-$ direction points due east, the $y-$ direction due north and the $z-$ direction points upward.

2.1.2 Mass conservation

We consider a volume of water V bounded by a C^1-surface S. The only possible changes that can occur inside V are either the increase or decrease of water inside it, or that water flows out of it. The total increase or decrease of water inside our volume is given by

$$\int_V \rho dx,$$

where ρ is the density of the water. The amount of water which flows out of V is

$$\int_S \rho \mathbf{u} \cdot \nu dS(x),$$

where $\mathbf{u}$ is the velocity vector of the water moving inside V. The rate of change of water mass in V is:

$$\frac{\partial}{\partial t} \int_V \rho dx = - \int_S \rho \mathbf{u} \cdot \nu dS(x).$$

We can now apply Gauss's theorem on the right hand side and interchange integration and derivation on the left hand side, to get:

$$\int_V \frac{\partial}{\partial t} \rho dx = - \int_V \nabla \cdot (\rho \boldsymbol{u}) dx.$$

We derived this equality for an arbitrary volume of water V and so the integrands must be equal, i.e:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{2.1}$$

This fundamental result is called the *continuity equation*. Summarizing the above considerations, this equation states that the rate at which mass enters the volume (the system, more generally) equals the rate at which mass leaves the volume (system).

As already stated, water is considered an incompressible fluid ([3]), which means that its density ρ is taken to be a constant; ergo, the time derivative of the density will equal to 0. Thus, equation (2.1) becomes

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.2}$$

This is the so-called *mass conservation* equation and it will be one of the equations of our system.

2.1.3 The Euler equations

The goal of this section is to apply Newton's second law of motion to our system. In high-school physics we learn that this law states that the force which acts on the system equals the total mass of the system times its acceleration, in symbols:

$$F = m \cdot a.$$

Originally, however, Newton formulated this law as follows: the rate of change of momentum of a volume element is equal to the net force acting on the element. This is the formulation we will use. Our task is to identify which types of forces act on our system and to translate the term momentum in our case.

There are two types of forces acting on a volume of water: *external* and *internal* ones. The first kind will be due to gravity and the second due to the pressure.

Let us have a closer look at gravity. It is given by

$$F = (0, 0, -g);$$

There is no gravity acting on the $x-$ and $y-$ directions but only in the $z-$ direction. The minus represents the attracting nature of this force toward the centre of the Earth and g is taken to be equal to $9,8m/sec^2$.

Consider again a volume of water V bounded by a C^1 surface S. We split the volume into infinitesimal volumina V_{ijk}; then the total contribution of this external force to our system is:

$$\lim_{n,m,p\to\infty} \sum_{i,j,k=1}^{m,n,p} \rho F V_{ijk} = \int_V \rho F dx.$$

(We abuse notation here and write dx when we should have written $dxdydz$.)

We now turn to the internal forces which are only contributed by the pressure of the system. As a physical quantity, pressure is a scalar; the total amount of it in our system is

$$\int_S -P\nu dS(x) = \int_V -\nabla P dx,$$

where ν is the outward pointing vector from the surface and we once again use dx instead of $dxdydz$ for simplification. The equality of the two integrals is granted by the gradient theorem.

So the total amount of force acting on our moving body of water is:

$$F_{\text{total}} = \int_V (\rho F - \nabla P)dx.$$

The right hand side of Newton's second law of motion is given by

$$\int_V \rho \frac{d^2\boldsymbol{x}(t)}{dt^2} dx;$$

the integral over the density of the volume is the mass (density equals mass over volume) and $\boldsymbol{x}(t) = (x(t), y(t), z(t))$ is the function which describes the position of a water particle at time t, so its second derivative is the acceleration of the particle. Since we are working with the vector $\boldsymbol{u}$ which gives the velocity of the particle, we rewrite the acceleration term as $\frac{d\boldsymbol{u}(\boldsymbol{x}(t),t)}{dt}$. So, Newton's second law of motion in our case reads:

$$\int_V (\rho F - \nabla P)dx = \int_V \rho \frac{d(\boldsymbol{u}(\boldsymbol{x}(t), t)}{dt} dx \tag{2.3}$$

Since the volume of water V was arbitrary, we can again simply equate the integrands. Before we do this, however, we need to do some calculations.

The velocity vector $\boldsymbol{u}(\boldsymbol{x}(t), t)$ can be written component-wise as

$$\boldsymbol{u}(\boldsymbol{x}(t), t) = \begin{pmatrix} u(\boldsymbol{x}(t), t) \\ v(\boldsymbol{x}(t), t) \\ w(\boldsymbol{x}(t), t) \end{pmatrix}.$$

We now consider only the first component $u(\boldsymbol{x}(t), t)$ and remember that the position vector is given by $\boldsymbol{x}(t) = (x(t), y(t), z(t))$. We compute the first time derivative and apply the chain-rule to do so:

$$\begin{aligned} \frac{d}{dt}u(\boldsymbol{x}(t), t) &= u_x(\boldsymbol{x}(t), t)\dot{x}(t) + u_y(\boldsymbol{x}(t), t)\dot{y}(t) + u_z(\boldsymbol{x}(t), t)\dot{z}(t) + u_t(\boldsymbol{x}(t), t) \\ &= u_x u + u_y v + u_z w + u_t, \end{aligned} \tag{2.4}$$

where the dot above the coordinate functions denotes the time-derivative and, by definition, $\dot{x}(t) = u(x(t), y(t), z(t))$, $\dot{y}(t) = v(x(t), y(t), z(t))$, $\dot{z}(t) =$

$w(x(t), y(t), z(t))$. The last line of the equation is just a simplification of notation.

We now perform the same calculation for the second and third arguments of the velocity vector:

$$\frac{d}{dt}v(\boldsymbol{x}(t), t) = v_x u + v_y v + v_z w + v_t \tag{2.5}$$

and

$$\frac{d}{dt}w(\boldsymbol{x}(t), t) = w_x u + w_y v + w_z w + w_t. \tag{2.6}$$

Recall that we can drop the integrals in equation (2.3) and write

$$\rho F - \nabla P = \rho \frac{d\boldsymbol{u}(\boldsymbol{x}(t), t)}{dt} \Leftrightarrow \rho(0, 0, -g) - (P_x, P_y, P_z) = \rho \frac{d\boldsymbol{u}(\boldsymbol{x}(t), t)}{dt} \Leftrightarrow$$
$$(0, 0 - g) - \left(\frac{P_x}{\rho}, \frac{P_y}{\rho}, \frac{P_z}{\rho}\right) = \frac{d\boldsymbol{u}(\boldsymbol{x}(t), t)}{dt}.$$

Using this and the three equations (2.4), (2.5), (2.6), we get the componentwise *Euler's equations*:

$$\begin{aligned} u_t + u_x u + u_y v + u_z w &= -\frac{P_x}{\rho} \\ v_t + v_x u + v_y v + v_z w &= -\frac{P_y}{\rho} \\ w_t + w_x u + w_y v + w_z w &= -\frac{P_z}{\rho} - g. \end{aligned}$$

By using the fact that

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \begin{pmatrix} (u, v, w) \cdot \nabla u \\ (u, v, w) \cdot \nabla v \\ (u, v, w) \cdot \nabla w \end{pmatrix},$$

we can write Euler's equations in a concise vector equation as:

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P + (0, 0, -g). \tag{2.7}$$

It is customary to write

$$\frac{D}{Dt} = \partial_t + (\boldsymbol{u} \cdot \nabla).$$

This operator is called the *total derivative* (also known as *material derivative*). Then the equation of motion becomes,

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla P + (0, 0, -g). \tag{2.8}$$

2.1.4 The boundary conditions

There are two boundaries of the fluid region: the rigid flat bed and the water's *free surface.*

The *kinematic boundary conditions* describe the fact that once a water particle is on a boundary (the free surface or the bed) at some time t_0, it will stay there at $t > t_0$. Assume that the C^1 free surface is given by $S(\boldsymbol{x}, t) = 0$, where $S : \mathbb{R}^4 \to \mathbb{R}$ is a C^k function for some integer $k \geq 1$. We want S to be differentiable and that it has non-vanishing gradient, i.e $\nabla S \neq 0$.

Choose a particle lying in position $\boldsymbol{x}_0$ at time t_0 on the free surface; then the subsequent location of the particle on the free surface will be the unique solution of

$$\boldsymbol{x}' = \boldsymbol{u}(\boldsymbol{x}(t), t),$$

with initial data $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ satisfying $S(\boldsymbol{x}(t), t) = 0$ for $t \geq t_0$. Differentiate the free surface with respect to t to get:

$$\frac{d}{dt} S(x(t), y(t), z(t), t) = 0 \Leftrightarrow \frac{DS}{Dt} = (S_x u + S_y v + S_z w + S_t) = 0.$$

This is a necessary and sufficient condition ([3]) for a particle to continue to lie on the surface at all times.

Now for our special case, the free surface is given by $z = h(x, y, t)$. We get $S(x, y, z, t) = h(x, y, t) - z = 0$. Then:

$$\frac{DS}{Dt} = 0 \Rightarrow S_x u + S_y v + S_z w + S_t = h_x u + h_y v - w + h_t = 0 \Rightarrow$$

$$w = h_t + u h_x + v h_y \text{ on } z = h(x, y, t). \tag{2.9}$$

This is the *kinematic boundary condition on the free surface.* On the ocean bed, we have

$$z = -d \Rightarrow S(x, y, z, t) := z + d = 0,$$

which implies

$$w = 0 \tag{2.10}$$

on the ocean bed $z = -d$. This is the the *kinematic boundary condition on the ocean bed.*

The *dynamic boundary condition* is given by the equation

$$P = P_{atm} \text{ on } z = h(x, y, t). \tag{2.11}$$

Since we already postulated that there is no interaction between the water and the bottom, there is no such condition there. Equation (2.11) separates the wave from the air above it.

In summary, the general description of the water wave problem for gravity waves is described by Euler's equation, the equation of mass conservation together with the three boundary conditions:

$$\begin{cases} \dfrac{D\boldsymbol{u}}{Dt} = -\dfrac{1}{\rho}\nabla P + (0,0,-g) & \\ \nabla \cdot \boldsymbol{u} = 0 & \\ w = h_t + uh_x + vh_y & \text{on } z = h(x,y,t) \\ w = 0 & \text{on } z = -d \\ P = P_{atm} & \text{on } z = h(x,y,t) \end{cases} \tag{2.12}$$

2.1.5 Vorticity

A fundamental property of a fluid flow is the *vorticity*[1]

$$\tilde{\omega} = \nabla \times \boldsymbol{u},$$

which measures the local spin or rotation of a fluid element (thus producing a swirling motion). This fact, that vorticity has a local nature, is not obvious. It is not very difficult to show that the flow in a neighbourhood of some fixed point $\boldsymbol{x}_0$ of the fluid domain can be viewed as a combination of infinitesimal translation, rotation and deformation, with the vorticity accounting for the rotational part. However, since this is a rather long calculation and of little importance to the results of this thesis, we refer to [3].

From the definition of vorticity it is immediate that,

$$\tilde{\omega} = (w_y - v_z, u_z - w_x, v_x - u_y).$$

We already mentioned in section 2.1.1 that we will deal with two dimensional flows; this would amount to setting $v = 0$ and the $y-$ derivatives to vanish. In this case, the vorticity is

$$\tilde{\omega} = (0, u_z - w_x, 0)$$

and we identify it with the scalar $\tilde{\omega}$.

Of greater interest to our cause is the fact that vorticity can be considered to be a function of the so-called stream function ψ (see next section).

A flow which has zero vorticity is called *irrotational.* Throughout this thesis, we will consider the vorticity of the flow to be **constant**, but non-zero.

2.2 Equatorial water waves

All the above was done in the setting of gravity water waves. This thesis, however, deals with equatorial water waves so we have add into the equations of motion the force acted by the rotation of the Earth. Consider the following:

[1] In the literature, the vorticity is denoted by ω; however, as will be evident later, we reserve the letter ω for the Earth's rotational speed.

a person stands on the Earth at position r and is observed by a person in space (who is on a fixed position). The observer sees

$$\boldsymbol{u}_F = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}.$$

This states that the moving person's velocity, as is measured from the fixed frame in space, equals to his velocity in the rotating frame plus the rotational velocity of the Earth. Differentiating the above expression with respect to time ([10]) we get:

$$\frac{d\boldsymbol{u}_F}{dt} = \frac{d}{dt}\left(\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}\right) + \boldsymbol{\Omega} \times \left(\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}\right)$$

which implies

$$\frac{d\boldsymbol{u}_F}{dt} = \frac{d\boldsymbol{u}_R}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \tag{2.13}$$

This means that the acceleration measured in the fixed frame has two additional contributions: the first one is given by the second term on the right-hand side of equation (2.13) and is called the *Coriolis acceleration* and the second one is given by the third term which accounts for the centrifugal acceleration. The latter acts perpendicular to the Earth's rotation axis and is constant in time. This is why it is possible to absorb this term into the gravity term and neglect it thereafter (see [13]). The Coriolis force can be seen to depend on the velocity in the rotating frame. Because it is a cross product, it acts perpendicular to the velocity causing a change in velocity direction but not on speed. This implies that the Coriolis force does no work.

We consider the Earth to be a perfect sphere of radius $6371km$ which rotates only on the $y-$ axis at a constant rotational speed $\omega = 73\cdot 10^{-6} rad/sec$. Thus, for us $\Omega = (0,\omega,0)$[2]. In analogy to gravity waves discussed in the previous section, the total rate of change of momentum is now given by

$$\frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u},$$

(we dropped the subscript R which is now implied), which is the left-hand side of Euler's equations. They can now be written in component-wise form as:

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + 2\omega w &= -\frac{P_x}{\rho} \\ v_t + uv_x + vv_y + wv_z &= -\frac{P_y}{\rho} \\ w_t + uw_x + vw_y + ww_z - 2\omega u &= -\frac{P_z}{\rho} - g. \end{aligned} \tag{2.14}$$

This is the system that we are going to be working with from now on with boundary conditions identical to the ones for gravity waves.

[2]This is the so-called $f-$ plane approximation, where the horizontal and vertical components of the Coriolis term vanish.

2.3 Reformulation of the problem

Until now, the convention was the the $x-$ axis points due east, the $y-$ axis due north and the $z-$ axis upward. We wish to consider two-dimensional flows, however, i.e waves for which the motion is identical in any direction parallel to the crest line and which propagate at constant speed c in a fixed direction. We achieve that by setting the second coordinate equal to zero. In all that follows, we should have written z instead of y (for example, the flat bed is actually given by $z = 0$). However, we the change the notation a bit a switch the letters y and z. From now on, the flat bed is $y = 0$ but this still means th third coordinate (i.e. the axis pointing upward). Our system now looks like this:

$$\begin{cases} u_t + uu_x + vu_y + 2\omega v = -\dfrac{P_x}{\rho} \\ v_t + uv_x + vv_y - 2\omega u = -\dfrac{P_y}{\rho} - g \\ u_x + v_y = 0 \\ v = \eta_t + u\eta_x \qquad \text{on } y = \eta(x,t) \\ v = 0 \qquad \text{on } y = 0 \\ P = P_{atm} \qquad \text{on } y = \eta(x,t) \end{cases} \tag{2.15}$$

The system (2.15) can be difficult to handle; this is due to its non-linearity and the fact that the free surface $\eta(x, t)$ is an unknown. This is why we wish to reformulate the problem to a simpler one.

Given $c > 0$, we consider wave trains travelling at speed c. This means that the (x, t) space-time dependence of the free surface, the pressure and the velocity field has the form $(x - ct)$ and that P, u, v and η all display a periodic dependence upon the $x-$ variable of minimal period $L > 0$ (L is some prescribed number). We can perform a change of frame $(x, y) \mapsto (x-ct, y)$ and thus eliminate the time-dependence. The new frame also moves with speed c. After this coordinate change we can rewrite the system (2.15) after the change of frame as

$$\begin{cases} (u-c)u_x + vu_y + 2\omega v = -\dfrac{P_x}{\rho} \\ (u-c)v_x + vv_y - 2\omega u = -\dfrac{P_y}{\rho} - g \\ u_x + v_y = 0 \\ v = (u-c)\eta_x \qquad \text{on } y = \eta(x) \\ v = 0 \qquad \text{on } y = 0 \\ P = P_{atm} \qquad \text{on } y = \eta(x), \end{cases} \tag{2.16}$$

cf. [3]. We will now introduce a new function $\psi : \mathbb{R}^2 \to \mathbb{R}$, called the *stream function*, in terms of which we will reformulate everything. We can measure the mass flux across a line $x = x_0$ relative to the flow at speed c by

$$\int_0^{\eta(x_0)} (u(x_0, \xi) - c) d\xi.$$

Using the kinematic boundary conditions (2.9) and (2.10), we see that the above expression is independent of x_0. We therefore define the *relative mass flux* by

$$m := -\int_0^{\eta(x)} (u(x, \xi) - c) d\xi.$$

Now consider the vector field $U = (-v, u - c, 0)$ and take its curl:

$$\nabla \times U = \nabla \times \begin{pmatrix} -v \\ u - c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u_x + v_y \end{pmatrix} = 0,$$

due to mass conservation. Furthermore, we know that if the domain of a curl-free vector field is simply connected, there exists a well-defined potential function ψ such that $\nabla \psi = U$. In this case, we can define ψ by its derivatives:

$$\psi_x = -v \quad \psi_y = u - c \tag{2.17}$$

and give an explicit formula:

$$\psi(x, y) = \psi_0 + \int_0^y (u(x, \xi) - c) d\xi.$$

Our goal is achieved: we found a function according to which we can reformulate our problem, namely the function ψ defined by the above expression. Another short calculation shows that

$$\Delta \psi = u_y - v_x := \gamma(\psi).$$

This last claim, that the vorticity can be written as a function of ψ, is shown in [5]. As already stated, the vorticity here will be a constant different from zero, so we set $\gamma(\psi) \equiv \gamma \neq 0$.

For the boundary conditions we impose on ψ, observe that the kinematic boundary conditions (2.9) and (2.10), state $v = 0$ on $y = 0$ and on the free surface $y = \eta(x)$. This forces ψ to be constant on the bed and on the free surface. If we normalize ψ to be zero on the the free surface (this is done by choosing $\psi_0 = m$), then the definition of the relative mass flux forces $\psi = m$ on $y = 0$. Using (2.17) and keeping in mind the change of frame we performed, the system (2.16) becomes:

$$\begin{cases} \psi_y\psi_{xy} - \psi_x\psi_{yy} - 2\omega\psi_x = -\dfrac{P_x}{\rho} & \text{in } 0 \le y \le \eta(x) \\ -\psi_y\psi_{xx} + \psi_x\psi_{xy} - 2\omega(\psi_y + c) = -\dfrac{P_y}{\rho} - g & \text{in } 0 \le y \le \eta(x) \\ \Delta\psi \equiv \gamma & \text{in } 0 \le y \le \eta(x) \\ \psi = 0 & \text{on } y = \eta(x) \\ \psi = m & \text{on } y = 0 \\ P = P_{atm} & \text{on } y = \eta(x) \end{cases} \tag{2.18}$$

Through this system, we can now find a function ψ from which we can calculate all the original unknowns u, v, P.

Furthermore, Bernoulli's law states that

$$E = \frac{1}{2}((u-c)^2 - v^2) + (g - 2\omega c)y + P - \Gamma(\psi) = \frac{|\nabla\psi|^2}{2} + (g - 2\omega c)y + P - \Gamma(\psi),$$

where $\Gamma(s) := \int_0^s \gamma(-\xi)d\xi$ for $s \in [0, m]$. This equation says that the total energy is constant within the fluid by taking the curl of the first two equations in (2.18) (cf. [11]). This equation implies that the dynamic boundary condition can be written as

$$\frac{|\nabla\psi|^2}{2} + (g - 2\omega c)y = Q, \text{ on } y = \eta(x),$$

where $Q = const.$

So we can further simplify (2.18) and write it as,

$$\begin{cases} \Delta\psi \equiv \gamma & \text{in } 0 \le y \le \eta(x) \\ \dfrac{|\nabla\psi|^2}{2} + (g - 2\omega c)y = Q & \text{on } y = \eta(x) \\ \psi = 0 & \text{on } y = \eta(x) \\ \psi = m & \text{on } y = 0 \end{cases} \tag{2.19}$$

So, we have to find $\psi(x, y) \in C^2(\mathbb{R}^2)$ and $\eta \in C^3(\mathbb{R}^2)$ which solve problem (2.19).

CHAPTER 3

Maximum Principles

In this chapter we put forth the basic mathematical background we will be needing: maximum principles. Maximum principles are one the most useful and best known tools employed in the study of partial differential equations. They are a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ achieves its maximum value at one of the endpoints of the interval. Solutions of the inequality $f'' \geq$ are said to satisfy a *maximum principle.* More generally, functions which satisfy a differential inequality in a domain D and, because of it, achieve their maxima on the boundary of D are said to possess a maximum principle. Let us begin by putting the above-mentioned simplest case into context.

Consider a function $f : (a, b) \to \mathbb{R}$ which is continuous on the closed interval $[a, b] \subseteq \mathbb{R}$. Such a function takes on its maximum at a point on this interval. Now, if $f(x)$ has a continuous second derivative, it is a well known fact that at a local extremum, say the point $c \in (a, b)$, we have

$$f'(c) = 0 \text{ and } f''(c) \leq 0. \tag{3.1}$$

Suppose now that in an open interval (a, b), our function f satisfies a differential inequality of the form

$$f'' + g(x)f' > 0, \tag{3.2}$$

where g is some bounded function. It is obvious that relations (3.1) and (3.2) cannot be both satisfied simultaneously at any given point c in (a, b). This implies that the maximum of the function f cannot be attained in the interior of the closed interval; thus, it is attained on the the boundary points a or b. This is a simple case for a so-called maximum principle: we found a condition which f has to satisfy in order to attain its maximum inside a closed interval and then found another which cannot hold if the previous condition applies; we then concluded that a maximum is only attained on the boundary of the domain of interest if both conditions should apply.

We are interested in four particular maximum principles: the weak maximum principle, the strong maximum principle, Hopf's maximum principle and the so-called Serrin's edge-point lemma. In the following we will present and proof all of the above (except the edge-point lemma which we give without proof).

All the figures in this chapter are copies found in [8].

3.1 Definitions

First of all, we want to define what exactly we mean by *maximum* of a function. From now on, we denote by $B_\rho(x_0)$ the open ball of radius ρ (for which we require $0 < \rho < \infty$) around the point x_0, i.e:

$$B_\rho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}.$$

Definition 3.1. *Consider a function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$.*

(i) f has a (global) maximum *if and only if there is a point $p \in A$ such that $f(x) \leq f(p)$ whenever $x \in A$, and* $\max_{x \in A} f(x) := f(p)$.

(ii) f has a local maximum *at a point $q \in A$ if and only if there is a number $\delta > 0$ such that $f(x) \leq f(q)$ whenever $x \in B_\delta(q) \cap A$.*

(iii) If f is bounded, the supremum $\sup_{x \in A} f(x)$ *(which always exists) is the unique element s of $\mathbb{R}$ such that*

(a) $f(x) \leq s$ whenever $x \in A$,

(b) if $t < s$, then there is a point $y \in A$ at which $f(y) > t$.

(iv) The minimum *of a function is defined as* $\min_{x \in A} f(x) := -max_{x \in A}\{-f(x)\}$, *while the* infimum *is defined as* $\inf_{x \in A} f(x) := -\sup_{x \in A}\{-f(x)\}$.

According to this definition, a maximum is a supremum that is attained at some point.

Before we continue, we are interested in giving an example.

Example 3.1. Consider a function $u \in C^2[a, b]$ which satisfies $u'' \geq 0$ on $[a, b]$. Then the following hold:

1. Either, $u(x) < \max\{u(a), u(b)\}$ for all $x \in (a, b)$, or u is constant on $[a, b]$.
2. If $u(b) \geq u(a)$, then either the outward derivative $u' > 0$, or u is constant on $[a, b]$.
3. If $u(a) > u(b)$, then the outward derivative $u'(a) < 0$.

Proof. Use that $u(x) = u(a) + \int_a^x u'(t)dt$ and integrate by parts to get

$$\begin{aligned} u(x) &= u(a) + \int_a^x u'(t)dt \\ &= u(a) + u'(a)(x - a) + \int_a^x u''(t)(x - t)dt, \end{aligned} \tag{3.3}$$

for $a \leq x \leq b$. We have some information about u'' and $u(a)$ and $u(b)$ are considered to be 'known', so the next step will be to remove the term $u'(a)$. To do that, set $x = b$ in (3.3):

$$u'(a) = \frac{1}{b-a}\left[u(b) - u(a) - \int_a^b u''(t)(b-t)dt\right]. \tag{3.4}$$

Substitution of this in (3.3) gives[1]

$$u(x) = v(x) - \int_a^b G(x,t)u''(t)dt, \tag{3.5}$$

where

$$v(x) := u(a) + \frac{u(b)-u(a)}{b-a}(x-a),$$

and

$$G(x,t) := \begin{cases} \frac{(b-x)(t-a)}{b-a} & \text{if } x \geq t \\ \frac{(x-a)(b-t)}{b-a} & \text{if } x \leq t \end{cases}.$$

Observe that G is defined on the square $[a,b] \times [a,b]$ and is strictly positive in $(a,b) \times (a,b)$, while it is zero on the boundary. We also note, that the function $v(x)$ defined as part of the formula (3.5), is the solution of $v'' = 0$.

Now, since $u'' \geq 0$ on $[a,b]$, it follows from (3.5), that $u(x) \leq v(x)$ for $a \leq x \leq b$ (the quantity in the integral is always greater or equal to zero). Strict inequality holds for $x \in (a,b)$ if $u''(x_0) > 0$ at some point $x_0 \in [a,b]$, because then u'' is an interval of positive length, by the cointinuity of u''.

We now move on to the actual proof of the claims of our example.

(i) To prove the first assertion, we have to consider three cases.

 (a) If $u(a) \neq u(b)$, then $u(x) \leq v(x) < max\{u(a), u(b)\}$ for $x \in (a,b)$. This is obvious.

 (b) If $u(a) = u(b)$ and $u''(x_0) > 0$ at some $x_0 \in [a,b]$, then, by the discussion above, $u(x) < v(x) = u(a)$ for $x \in (a,b)$.

 (c) If $u(a) = u(b)$ and $u'' = 0$ on $[a,b]$, then $u(x) = v(x) = u(a)$ for $x \in [a,b]$.

(ii) In order to prove the second assertion, we do an analogous procedure to that of (3.4) for $u'(b)$ and obtain

$$u'(b) = \frac{1}{b-a}\left[u(b) - u(a) - \int_a^b u''(t)(t-a)dt\right] \geq 0, \text{ when } u(b) \geq u(a),$$

with equality only for the case $u(a) = u(b)$ and $u'' = 0$ on $[a,b]$ as in the proof of (i).

[1] It is reasonable to question where this formula comes from; a straightforward calculation easily gives the definition of the term $v(x)$ while for $G(x,t)$, one must simply split the integral $\int_a^b u''(t)(b-t)dt$ to $\int_a^x u''(t)(b-t)dt + \int_x^b u''(t)(b-t)dt$ and perform the basic algebra that is needed to arrive at the result.

(iii) If $u(a) > u(b)$, then $u'(a) < 0$, by (3.4).

□

Another easy example we can give is the following, which is the implication of a strict differential inequality.

Example 3.2. If Ω is open, $u \in C^2(\Omega)$ and $\Delta u > 0$ in Ω, then u cannot have a local, and thus a global, maximum at a point of Ω.

Proof. Assume that u has a local maximum at some point $q \in \Omega$. Then for each $j \in \{1, \ldots, n\}$ we have $\partial_j u(q) = 0$ and $(\partial_j^2 u)(q) \leq 0$, which implies that $(\Delta u)(q) \leq 0$. But this is a contradiction. □

The next order of business is to define what we mean by elliptic operators (since our problem is an elliptic one). Let Ω be an open non-empty subset of $\mathbb{R}^n$.

Definition 3.2 (Elliptic operators). *Let $u \in C^2(\Omega)$.*

(i) The operator L, defined by

$$Lu(x) := \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x)$$

whenever $x \in \Omega$, is called a linear partial differential operator of order two. *Here*

$$a = (a_{ij}) : \Omega \to \mathbb{R}^{n^2}, \quad b = (b_j) : \Omega \to \mathbb{R}^n, \quad c : \Omega \to \mathbb{R}$$

are given bounded measurable functions (i.e. all belong to $L^\infty(\Omega)$). The $n \times n$ matrix a is symmetric, i.e. $a_{ij}(x) = a_{ji}(x)$ for all $i, j \in \{1, \ldots, n\}$ and all $x \in \Omega$.

(ii) The operator L is called elliptic at $x \in \Omega$ *if and only if there is a number $\lambda(x) > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2 \textit{ for all } \xi \in \mathbb{R}^n.$$

L is elliptic in Ω *if and only if it is elliptic at every $x \in \Omega$. L is* uniformly elliptic *in Ω if and only if there is a constant $\lambda_0 > 0$ such that $\lambda(x) \geq \lambda_0$ for all $x \in \Omega$.*

Ellipticity means that the symmetric $n \times n$ matrix $a = (a_{ij})$ is positive-definite with smallest eigenvalue greater than or equal to $\lambda(x)$.

Example 3.3. The most standard example of an elliptic operator is the Laplacian $\Delta = \sum_{i=1}^n \partial_i^2$. Another typical example is the $p-$ Laplacian; for some non-negative number p, it is defined by $L(u) = -\sum_{i=1}^n \partial_i(|\nabla u|^{p-2}\partial_i u)$.

Definition 3.3. *We say that u is a C^2-*subsolution (resp. *supersolution*) relative to L and Ω *if and only if $u \in C^2(\Omega)$ and $Lu \geq 0$ (resp. $(Lu \leq 0)$ in Ω.*

3.2 The weak maximum principle

We now present the first of the four maximum principles we will be needing.

We begin by stating the weak maximum principle for the operator

$$L_0 = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{j=1}^{n} b_j(x)\partial_j$$

(observe that $L = L_0 + c(x)$) in the form of a Lemma without proof in order to be able to give a more elegant proof of the weak maximum principle for L.

Lemma 3.1 (The weak maximum principle for the operator L_0). Suppose that Ω is bounded, $u \in C(\overline{\Omega})$ and that u is a C^2-subsolution relative to L_0 and Ω. Then the supremum of u is attained on the boundary:

$$\max_{x\in\overline{\Omega}} u = \max_{\partial\Omega} u.$$

We are now ready to present and prove our theorem.

Theorem 3.1 (The weak maximum principle)**.** *Suppose that Ω is bounded, $u \in C(\overline{\Omega})$ and that u is a C^2-subsolution relative to L and Ω. Then*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where $u^+ := \max_{x\in\Omega}\{0, u(x)\}$.

Proof. Let $\Omega^+ := \{x \in \Omega | u(x) > 0\}$. We claim that this set is open in $\mathbb{R}^n$. Indeed, if $y \in \Omega^+$, say $u(y) = \alpha > 0$, then there is a number $\delta > 0$ such that both $B_\delta(y) \subset \Omega$ (since Ω is open by assumption) and $u(x) > \alpha/2$ whenever $x \in B_\delta(y)$ (since u is continuous), so that $B_\delta(y) \subset \Omega^+$.

If Ω^+ is empty, then $\max_{\overline{\Omega}} u \leq 0$ and the theorem holds.

Suppose then that Ω^+ is non-empty. The hypotheses that u is a C^2-subsolution relative to L and Ω can be rewritten as

$$L_0 u \geq -c(x)u;$$

this combined with the fact that $c(x) \leq 0$ in Ω (by assumption for our differential operator), imply that $L_0 u \geq 0$ in Ω^+. Now, by Lemma 3.1, the maximum of u over $\overline{\Omega^+}$ equals that over $\partial\Omega^+$; hence, there is a point

$$x_0 \in \partial\Omega^+ \text{ such that } u(x_0) = max_{\overline{\Omega^+}} u > 0.$$

If $x_0 \in \Omega$ (Figure 4.3) we have a contradiction: indeed, by continuity, $u > 0$ in $B_\rho(x_0)$ for some $\rho > 0$; on the other hand, $B_\rho(x_0)$ contains points of $\Omega \setminus \Omega^+$, because $x_0 \in \partial\Omega^+$ and $u \leq 0$ at such points. Therefore $x_0 \in \partial\Omega$. □

In other words, if our assumptions are met, the weak maximum principle tells us that a subsolution of a uniformly elliptic operator attains a nonnegative maximum at the boundary.

Remark. The condition $c \leq 0$ *cannot* be omitted from Theorem 3.1 to allow $c > 0$. In such a case, the existence of positive eigenvalues κ to the problem $\Delta u + \kappa u = 0$ in Ω, $u = 0$ on $\partial\Omega$ (see [8] and [9]).

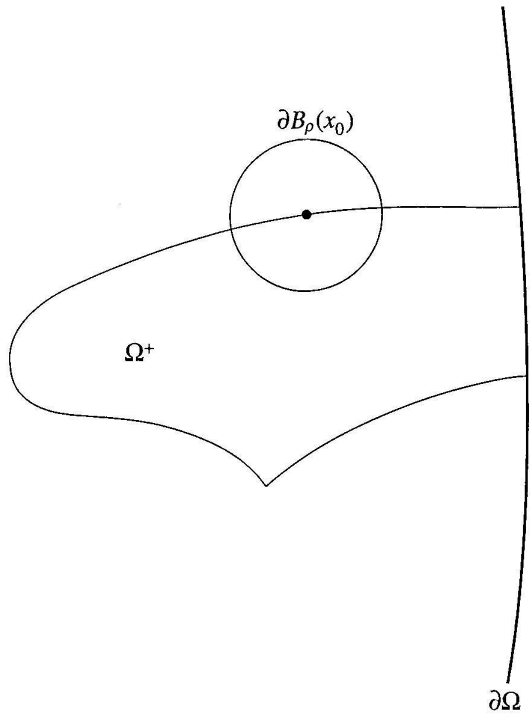

Figure 3.1:

3.3 The strong maximum principle

In this section we present the so-called strong maximum principle. In order to be able to prove it, however, we will need a preliminary result which will be presented without proof: the boundary-point lemma.

Lemma 3.2 (The boundary point lemma for balls). Suppose that

(a) $B \subset \Omega$ is a ball;

(b) $u \in C(\overline{B})$, u is a C^2-subsolution relative to L_0 or L and B;

(c) $u(x) < u(p)$ for all $x \in B$ and some $p \in \partial B$ (the boundary of the ball B), with $u(p) \geq 0$ when the coefficient c is not the zero function (i.e. when we deal with the elliptic operator L).

Let ν be an outward unit vector at p ($\nu \cdot n > 0$ and $|\nu| = 1$, where n denotes the outward unit normal to ∂B at p). Then

$$\liminf_{t \downarrow 0} \frac{u(p) - u(p - t\nu)}{t} > 0, \tag{3.6}$$

which implies that

$$\frac{\partial u}{\partial \nu}(p) := \lim_{t \downarrow 0} \frac{u(p) - u(p - t\nu)}{t} > 0 \tag{3.7}$$

whenever this one-sided directional derivative exists.

We forego the tiresome and technical proof and advise the reader to refer to[8].

We are now ready to prove the strong maximum principle.

Theorem 3.2 (The strong maximum principle)**.** *Suppose that Ω is a region (open and connected, possibly unbounded), u is a C^2-subsolution relative to L_0 or L and Ω and that $\sup_\Omega u \geq 0$ when the coefficient c is not the zero function (the case of the operator L). Then, if $\sup_\Omega u$ is attained at a point of Ω, then u is constant in Ω.*

Proof. Let $M := \sup_\Omega u$, and assume that this supremum is attained at $\hat{x} \in \Omega$. Define

$$F := \{x \in \Omega \mid u(x) = M\}, \quad G := \{x \in \Omega \mid u(x) < M\};$$

then F is closed in the metric space Ω, and not empty because $\hat{x} \in F$; the set G is open in the metric space Ω. If G is empty, the theorem is true.
Therefore, suppose that there is a point $x_0 \in G$. We shall obtain a contradiction by means of the boundary-point lemma 3.2, using the result that, because Ω is open and connected in $\mathbb{R}^n$, it is path-connected[2]. This implies existence of a continuous arc

$$\gamma := \{\xi(t) | 0 \leq t \leq 1\} \subset \Omega \text{ with } \xi(0) = x_0,\ \xi(1) = \hat{x}\},$$

as shown in Figure 3.2. Here $\xi \in C([0,1], \mathbb{R}^n)$, so that γ is compact; if Ω has a boundary, then $\operatorname{dist}(\gamma, \partial\Omega) > 0$ because $\partial\Omega$ is closed in $\mathbb{R}^n$ and disjoint from γ.

Let $\tilde{x}$ be the first point of γ at which $u(x) = M$ (here "first" means "with smallest t"), possibly $\tilde{x} = \hat{x}$. Let q be any point of γ that is strictly between x_0 and $\tilde{x}$, and is such that $|q - \tilde{x}| < \operatorname{dist}(\gamma, \partial\Omega)$ when Ω has a boundary. Now consider the ball $B := B_q(\rho)$ with $\rho = \operatorname{dist}(\gamma, \partial\Omega)$. Then

$$\rho \leq |q - \tilde{x}| < \operatorname{dist}(\gamma, \partial\Omega),$$

so that $B \subset \Omega$. Also, $B \subset G$ by construction

There exists a point $p \in F \cap \partial B$ because F is closed (possibly $p = \tilde{x}$. All the hypotheses of Lemma 3.2 hold, so that at p the outward normal derivative

$$\frac{\partial u}{\partial n}(p) = n \cdot (\nabla u)(p) > 0.$$

But since $p \in F$, it is an interior maximum point of $u \in C^1(\Omega)$. Hence $(\nabla u)(p) = 0$ and we have our contradiction. □

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[2]This is a well-known result, see for example [17]

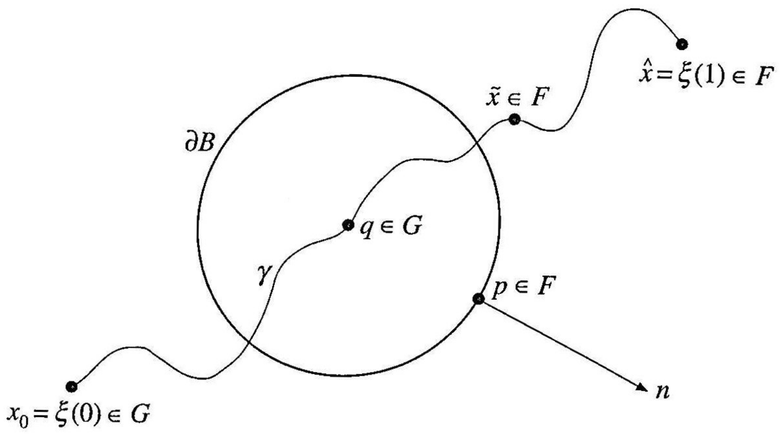

Figure 3.2:

3.4 The Hopf maximum principle

We now move on to the so-called Hopf maximum principle. First we need a definition.

Definition 3.4. *A set* Ω *has the* interior-ball property *at a point* $p \in \partial\Omega$ *if and only if there exists a ball*[3] $B_0 \subset \Omega$ *such that* $p \in \partial B_0$.

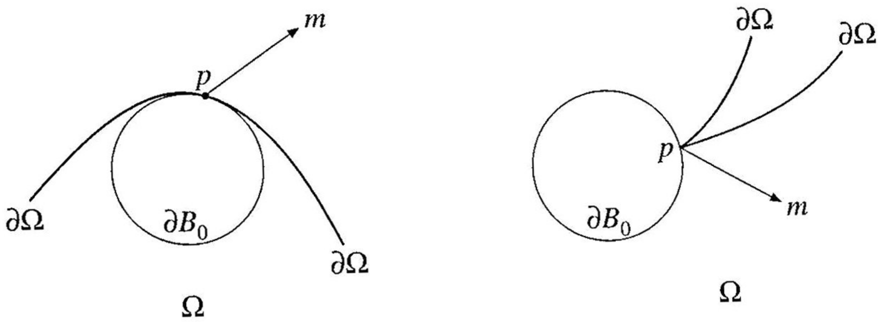

Figure 3.3: The interior-ball property

Theorem 3.3 (Hopf maximum principle)**.** *Suppose that*

[3]This notation is not to be confused with the above definition of balls; 0 is here an index not a radius.

1. *Ω is a region;*

2. *u is a C^2-subsolution relative to L_0 or L and Ω;*

3. *there is a point $p \in \partial\Omega$ such that $u \in C(\Omega \cup \{p\})$ and $u(p) = \sup_\Omega u$, with $u(p) \geq 0$ when the coefficient c is not the zero function (case of L);*

4. *Ω has the interior-ball property at p.*

Let ν be a unit vector at p, outward from an interior ball B_0 at p. Then either

$$\liminf_{t \downarrow 0} \frac{u(p) - u(p - t\nu)}{t} > 0 \tag{3.8}$$

(which implies that $(\partial u/\partial \nu)(p) > 0$ whenever this derivative exists), or u is constant in Ω.

Proof. Let x_0 be the centre of the ball B_0 and let $\rho_o := |p - x_o|$, so that $B_0 = B_{x_0}(\rho_0)$. Now consider the smaller ball $B := B_q(\frac{1}{2}\rho_0)$ with $q := \frac{1}{2}(p+x_0)$. Since $\overline{B} \subset B_0 \cup \{p\}$, we have $\overline{B} \subset \Omega \cup \{p\}$ and hence $u \in C(\overline{B})$.

If $u(x) < u(p)$ for all $x \in B$, then Lemma 3.2 implies (3.8). If $u(\hat{x}) = u(p)$ for some $\hat{x} \in B$, then $u(\hat{x}) = \sup_\Omega u$ and the strong maximum principle (Theorem 3.2) implies that u is constant in Ω. □

3.5 Serrin's Edge-Point Lemma

We will now proceed to state Serrin's edge-point lemma (without proving it). For that, we need some suitable definitions and results.

Definition 3.5. *A* neighbourhood *of a point $p \in \mathbb{R}^n$ is a set that contains p and is* open *in $\mathbb{R}^n$.*

A point $p \in \partial\Omega$ will be called an edge-point *of Ω if and only if, for some neighbourhood U of p the following are true:*

1. *$\Omega \cap U = (\Omega_1 \cap \Omega_2) \cap U$, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$ with C^2 boundaries and outward unit normals n_1 and n_2 respectively.*

2. $$p \in \partial\Omega_1 \cap \partial\Omega_2 \text{ and } -1 < n_1(p) \cdot n_2(p) < 1,$$

   as is shown in Figure 3.4

A connected set of edge-points of Ω will be called an edge *of Ω.*

Take, for example, $\Omega := \mathbb{R}^3 \setminus [0, \infty) \times [0, \infty) \times \mathbb{R}$ (i.e. the set obtained by removing two octants from $\mathbb{R}^3$). Then the origin is *not* and edge point, even though there is a corner there. This set Ω is a union of sets with smooth boundaries, rather than an intersection. Also, this set Ω has the interior-ball-property at the origin (see the previous section), whereas there is never an interior ball

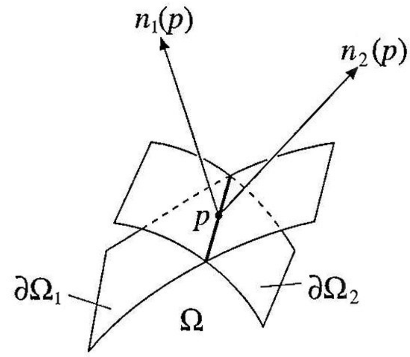

Figure 3.4: An edge-point

at an edge point. Although we have been working in an abstract setting until now for all the theorems and definitions, it's time to see an example which best suits our frame. In two dimensions, an edge point will simply be a corner. Consider $\Omega := [0, \infty) \times [0, \infty)$, i.e. the first quadrant of $\mathbb{R}^2$, and take the point of interest to be the origin again, $p = (0, 0)$. Then if we take $\Omega_1 = \mathbb{R} \times [0, \infty)$ and $\Omega_2 = [0, \infty) \times \mathbb{R}$ (then of course, $\Omega = \Omega_1 \cap \Omega_2$ and both Ω_1 and Ω_2 have C^2 boundary) and the outer normal vectors $n_1(p) = (-1, 0)^T, n_2(p) = (0, -1)^T$, it is obvious that

$$n_1(p) \cdot n_2(p) = 0,$$

so p is an edge point. It is also obvious that in this case, the origin never has the interior-ball-property.

Definition 3.6. *We will say that ϕ_1 is an* admissible function describing Ω_1 near p, *if and only if, for some neighbourhood U of p,*

1. $\phi_1 \in C^2(U)$;
2. $|\nabla\phi_1(x)| > 0$ *for all* $x \in U$;
3. *for $x \in U$, the value $\phi_1(x)$ is negative, zero or positive according as $x \in \Omega_1$, $x \in \partial\Omega_1$ or $x \notin \overline{\Omega_1}$.*

Admissible functions describing Ω_2 near p are defined accordingly.

Definition 3.7. *Let p be an edge point of Ω (with $\Omega = \Omega_1 \cap \Omega_2$ near p); let ϕ_1 and ϕ_2 be admissible functions describing Ω_1 and Ω_2 respectively, in a neighbourhood of p and let $a = (a_{ij})$, now defined on $\overline{\Omega}$, be the matrix of leading coefficients of an elliptic operator L of order two. Then the function B defined* *by*

$$B := \nabla\phi_1(x) \cdot a(x) \cdot \nabla\phi_2(x) = \sum_{i,j=1}^{n} \partial_i\phi_1(x)a_{ij}(x)\partial_j\phi_2(x), \; x \in \overline{\Omega} \cap U,$$

will be called a bluntness function *for the edge $\partial\Omega_1 \cap \partial\Omega_2$ relative to the operator L.*

The use of the word bluntness here is justified because at an edge point $B(x)$ is a weighted form of $n_1(x) \cdot n_2(x)$, and this latter varies from -1 at a perfectly sharp edge to 1 at a perfectly blunt edge. of course, both extremes are excluded by Definition 3.5.

We now state the main theorem of this section.

Theorem 3.4 (Serrin's edge-point lemma)**.** *Let u be a subsolution with respect to the operator L_0 and let Ω be a doimain. Furthermore, let $p \in \partial\Omega$ be an edge-point and suppose that $u \in C^1(\Omega \cap \{p\})$ and $u(p) > u(x)$ for all $x \in \Omega$. Also, suppose that the leading coefficients of the operator are of class C^2 in a neighbourhood U of p and that there exists a bluntness function $B(x)$ for the edge-point p which satisfies $B(p) = 0$ and the directional derivative of B at p is zero for every direction tangential to the boundary at p.*

Then either

$$\frac{\partial u}{\partial \nu}(p) > 0 \text{ or } \frac{\partial^2 u}{\partial \nu^2}(p) < 0,$$

for any outward unit vector ν at p, if these derivatives exist.

Corollary 3.1. If we have $u(p) = \max_{x\in\overline{\Omega}}$ instead of $u(p) > u(x)$ for all $x \in \Omega$, then the above statement holds unless u is constant throughout $\overline{\Omega}$.

3.6 Reformulations

All the above theorems are quite technical, so we are interested in presenting suitable reformulations to the three of the four maximum principles we discussed which will fit our proof: The weak maximum Principle, Hopf's maximum principle and the edge-point lemma.

We define

$$D := \{(x, y) \in \mathbb{R}^2 : -L/2 < x < L/2, 0 < y < \eta(x)\},$$

where L is the wave period, $\eta(x)$ is the free surface (which is a $C^2(\mathbb{R})$ function) and the four boundary points

$$\left(-\frac{L}{2}, 0\right), \quad \left(\frac{L}{2}, 0\right), \quad \left(-\frac{L}{2}, \eta\left(-\frac{L}{2}\right)\right), \text{ and } \left(\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right)$$

are edge points.

We are now ready to present these reformulations in the form of a theorem:

Theorem 3.5 (Reformulations)**.** *Let D be the domain defined above and L the elliptic operator L defined in Definition 3.2.*

(i) If $w \in C^2(D) \cap C(\overline{D})$ is a subsolution with respect to L and D and $w \geq 0$ on ∂D, then either $w > 0$ in D or $w \equiv 0$ throughout $\overline{D}$.

(ii) Let $w \in C^2(D) \cap C(\overline{D})$. Suppose that $w \geq 0$ in $\overline{D}$, $Lw \leq 0$ in D and $w = 0$ at some point $Q \in \partial D$. If D satisfies the interior ball property (as stated in section 3.2) at Q, then the outer normal derivative $\partial w/\partial \nu$ of w at Q, if it exists, satisfies the strict inequality $\partial w/\partial \nu < 0$ unless $w \equiv 0$ on $\overline{D}$.

(iii) Let T be a line normal to the top boundary $\eta(x)$ at some point Q. Choose a part of D lying on a particular side of the line T and denote it by D_0. If $w \in C^2(\overline{D_0})$ with $Lw \leq 0$ and $w \geq 0$ in D_0 and $w = 0$ at Q, then: either

$$\frac{\partial w}{\partial \mu}(Q) > 0 \ \text{or} \ \frac{\partial^2 w}{\partial \mu^2} > 0,$$

where μ is a non-tangential inward vector at Q, unless $w \equiv 0$ throughout $\overline{D_0}$

CHAPTER 4

The main result

4.1 Stagnation points

As already stated in the introduction of this thesis, the proof we present allows for stagnation points. Here, we discuss their existence in equatorial water waves.

As will be evident when reading the proof, everything works out nicely when for the horizontal velocity component u we have either $u < c$ or $u > c$, where c is the speed of wave propagation; depending on the inequality we would eventually have to change some signs. However, we need to discuss what happens when $u = c$. Recall that when reformulating the problem we found that $\psi_y = u - c$. So if we exclude the case when $u = c$, we always have that either $\psi_y > 0$ or $\psi_y < 0$, something which implies that the stream function is injective in the $y-$ direction; this, in turn, is important in order to reach conclusions about the shape of ψ. When setting $u = c$, ψ_y becomes equal to zero, which means that ψ will no longer be injective since it will have a local extremum. There is, however, a way out of this problem; in the following we discuss laminar flows and existence of stagnation points.

Consider a wave train with a *flat surface*. As always, h is the height over the flat bed, L is the wavelength, k is the wave-number ($k = \frac{2\pi}{L}$), γ is the vorticity (which is constant and non-zero) and m is the relative mass flux defined in section 2.3. The simplest solutions to the water wave problem are those exhibiting no dependence on the horizontal direction. These are called *laminar flows* - parallel shear flows with a flat free surface and such that each particle moves horizontally with a speed that depends only on the height above the flat bed - representing pure currents.

Under certain assumptions, the laminar flows give rise to genuine wave solutions to the water wave problem, see [6] for the case of flows allowing for constant vorticity and stagnation points driven by gravity, or [15] for the case of geophysical-equatorial flows with stagnation points, allowing for constant vorticity and driven by gravity and surface tension.The existence of such flows is proved by the use of the Crandall-Rabinowitz Theorem. This is a fairly technical result which is beyond the scope of this thesis. However, we will need some results from [15] which we will now quote.

The Crandall-Rabinowitz Theorem (cf. [7]) guarantees the existence of bifurcation points if certain assumptions on a map between Banach spaces are fulfilled. One of these assumptions is that the kernel of the Fréchet derivative of this map at a certain point (which is then proven to be the bifurcation point) is one-dimensional.

In [15] it is shown that in order to find bifurcation values for the map on which we wish to apply the Crandall-Rabinowitz Theorem, we have to look at the solutions of the equation

$$kn\coth(nkh)\lambda^2 + \gamma\lambda - (g - 2\omega c) - \sigma k^2 n^2 = 0,$$

for some integer $n \geq 1$, whose two solutions are

$$\lambda^n_\pm(k) = -\frac{\gamma\tanh(nkh)}{2kn} \pm \sqrt{\frac{\gamma^2\tanh^2(nkh)}{4k^2n^2} + \left(nk\sigma + \frac{g-2\omega c}{kn}\right)\tanh(nkh)}$$

Observe that $\lambda^n_\pm(k) = \lambda^1_\pm(nk)$. The objective is to find conditions on the physical parameters of the flow, which guarantee that $\lambda^1_\pm(k) \neq \lambda^n_\pm(k)$ for $n \geq 2$. Then we have the following result:

Lemma 4.1.

1. Let $\lambda(k) = \lambda^1_+(k)$.

   (a) If $\frac{\sigma}{(g-2\omega c)h^2} > \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} - \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ then the function λ is strictly increasing.

   (b) If $\frac{\sigma}{(g-2\omega c)h^2} < \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} - \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ then the function λ has a maximum at $k = 0$ and a unique local extrema, namely a local minimum at $k = k_0 > 0$. Moreover, there is a strictly decreasing sequence $(k_n)_{n\geq 2}$ such that

   $$\lambda(k) = \lambda(nk)$$

   for $k > 0$ if and only if $k = k_n$.

2. Let $\tilde{\lambda}(k) = \lambda^1_-(k)$.

   (a) If $\frac{\sigma}{(g-2\omega c)h^2} \geq \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ then the function $\tilde{\lambda}$ is strictly decreasing.

   (b) If $\frac{\sigma}{(g-2\omega c)h^2} < \frac{\gamma^2 h}{6(g-2\omega c)} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}$ then the function $\tilde{\lambda}$ has a minimum at $k = 0$ and a unique local extremum, namely a local maximum at $k = \tilde{k}_0 > 0$. Moreover, there is a strictly decreasing sequence $(\tilde{k}_n)_{n\geq 2}$ such that

   $$\tilde{\lambda}(k) = \tilde{\lambda}(nk)$$

   for $k > 0$ if and only if $k = \tilde{k}_n$.

From now on, we write $\lambda^1_\pm := \lambda_\pm$

Lemma 4.2. Let

$$\lambda_\pm = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g - 2\omega c}{k}\tanh(kh)}.$$

Then the kernel of the Freéchet derivative involved in the Crandall-Rabinowitz Theorem is one-dimensional if and only if

$$\frac{\sigma}{(g-2\omega c)h^2} \geq \frac{\gamma^2 h}{6(g-2\omega c} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}, \tag{4.1}$$

or

$$\frac{\sigma}{(g-2\omega c)h^2} < \frac{\gamma^2 h}{6(g-2\omega c} + \frac{1}{3} + \frac{\gamma}{6(g-2\omega c)}\sqrt{\gamma^2h^2 + 4(g-2\omega c)h}, \tag{4.2}$$

and $k \neq k_n$ and $k \neq \tilde{k}_n$ for all $n \geq 2$.

Theorem 4.1 (cf.[15])**.** *For every $h > 0$, $\sigma > 0$ (where σ is the surface tension), $\gamma \in \mathbb{R}$, $c < 0$, $m \in \mathbb{R}$ and k such that $k \neq k_n$, $k \neq \tilde{k}_n$ for all n, satisfying (4.1) or (4.2), there exist laminar flows with a flat free surface in water of depth h, of constant vorticity γ. The laminar flows of mass flux*

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma h\tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g - 2\omega c}{k}\tanh(kh)}$$

are exactly those whose horizontal speeds at the flat free surface equal to

$$\lambda_\pm = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{k^2\sigma + g - 2\omega c}{k}\tanh(kh)}.$$

The values of $m_\pm$ of the flux given by the above expression give rise to equatorial geophysical periodic steady capillary-gravity waves of small amplitude, with period $\frac{2\pi}{k}$ and conformal mean depth h (cf. [6], [15]), which have a smooth profile with one crest and one trough per period, monotone between consecutive crests and troughs and symmetric about any crest line.

In our case, as already stated, we neglect surface tension, so σ will equal to zero in all the above.

In terms of the stream function the laminar flow solutions are given by

$$\psi(x,y) - \frac{\gamma}{2}y + \left(\frac{m}{h} + \frac{\gamma h}{2}\right)y - m,\ x \in \mathbb{R},\ 0 \leq y \leq h,$$

with the velocity field

$$(\psi_y, -\psi_x) = \left(-\gamma y + \frac{m}{h} + \frac{\gamma h}{2}, 0\right),\ y \in \mathbb{R},\ 0 \leq y \leq h. \tag{4.3}$$

Furthermore, $\lambda = \frac{m}{h} - \frac{\gamma h}{2}$, so we can rewrite the above expression as

$$(\psi_y, -\psi_x) = (\lambda_\pm + \gamma(h-y), 0),\ x \in \mathbb{R},\ - \leq y \leq h. \tag{4.4}$$

This equation shows that for laminar flows the horizontal fluid velocity at the free surface coincides with $\lambda_\pm$. Furthermore, $\lambda_+ > 0$ and $\lambda_- < 0$, so we see that in laminar flows the stagnation points are never on the flat free surface $y = h$. Therefore, the existence of flows described in the statement of theorem 4.1 follows. The rest of the proof can be found in [15].

Next we need to address the existence of stagnation points. From equation (4.4) we see that stagnation points exist in laminar flows if and only if the equation

$$\lambda_\pm + \gamma(h-y) = 0$$

has at least one solution $y \in [0, h]$. This is equivalent to

$$\lambda_\pm(\lambda_\pm + \gamma h) \leq 0. \tag{4.5}$$

For $\gamma > 0$ it is immediate from condition (4.5) that the flow corresponding to λ_+ does not contain stagnation points, while the flow corresponding to λ_- possesses stagnation points if and only if $\lambda_- + \gamma h \geq 0$. This is equivalent to

$$\tanh(kh) \leq \frac{\gamma^2 h^2 k}{\gamma^2 h + k^2\sigma + g - 2\omega c}. \tag{4.6}$$

The case $\gamma < 0$ is similar. Namely, $\lambda_-(\lambda_- + \gamma h) > 0$ for $\gamma < 0$ and therefore the flow corresponding to λ_- does not contain stagnation points. Then the flow corresponding to λ_+ contains stagnation points if and only if $\lambda_+ + \gamma h \leq 0$. Equation (4.6) is a necessary and sufficient condition for the existence of stagnation points in the flow, although a difficult one to check.

In the case $k \to \infty$, (4.6) is false. This just implies that a flow with the wavelength $L \to 0$ has no stagnation points.

Theorem 4.2. *If the vorticity γ is such that*

$$\gamma^2 \geq \frac{4\sigma h + \sqrt{16\sigma^2 h^2 + 16h^4\sigma(g - 2\omega c)}}{2h^4}$$

there are values $k_1 \leq k_2$ with the property that (4.6) holds true whenever $k \in [k_1, k_2]$, where

$$k_1 := \frac{\gamma^2 h^2 - \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma(g - 2\omega c)}}{2\sigma},$$

$$k_2 := \frac{\gamma^2 h^2 + \sqrt{\gamma^4 h^4 - 4\sigma\gamma^2 h - 4\sigma(g - 2\omega c)}}{2\sigma}.$$

Moreover, in this situation, the stagnation points are neither on the free surface nor on the flat bed.

In our case we have already stated that we neglect surface tension ($\sigma = 0$) and so the above inequality becomes $\gamma^2 \geq 0$. Since we do not want that our flow is an irrotational one, this simply becomes $\gamma^2 > 0$. This is the reason why we do not discuss the sign of the vorticity; it can be either positive or negative.

The most important implication of the above discussion can be formulated in the form of a Lemma:

Lemma 4.3. Whenever stagnation points are present in laminar flows, the all lie on horizontal lines of the form $y = y_0$. This *stagnation line* satisfies

$$h - y_0 = \frac{\tanh(kh)}{2k} + \sqrt{\frac{\tanh^2(kh)}{4k^2} + \frac{k\sigma}{\gamma^2}\tanh(kh) + \frac{(g - 2\omega c)\tanh(kh)}{\gamma k}}. \quad (4.7)$$

Remark. Given the result of Lemma 4.3, we can write $h - y_0 > \varepsilon$ for some $\varepsilon > 0$.

From this result we can deduce that if the wave amplitude is small enough, there exists some neighbourhood around $\eta(x)$ where ψ is injective in the $y-$ direction.

4.2 Symmetry of the stream function

First of all, recall that we are only looking at one period L of the wave-train and that our system is described by equations (2.19. Finally, the wave amplitude λ is less than some $\varepsilon > 0$ which is determined by Lemma 4.3. For simplicity we choose the trough of the surface to be at $x = \pm L/2$. We define

$$D := \left\{ (x, y) \in \mathbb{R}^2 : -\frac{L}{2} \leq x \leq \frac{L}{2},\ 0 < y < \eta(x) \right\}.$$

Now fix $x_* \in (-L/2, 0]$ and consider

$$\phi_{x_*} : D_* \to D_*^R$$
$$(x, y) \mapsto (2x_* - x, y),$$

where $D_* = \{(x, y) \in \mathbb{R}^2 | -\frac{L}{2} < x < x_*, 0 < y < \eta(x)\} \subseteq D$ and $D_*^R = \phi_{x_*}(D_*)$. The line $x = x_*$ is normal to the bottom of the domain and the map ϕ_{x_*} reflects D_* to the line $x = x_*$ into some domain D_*^R.

Since $x = -\frac{L}{2}$ is the wave trough, the monotonicity of the free surface ensures the existence of some $\varepsilon > 0$ such that the function $\eta(x)$ is non-decreasing on the interval $(-\frac{L}{2}, -\frac{L}{2} + \varepsilon)$. Therefore, D_*^R will be a subset of the fluid domain D for all $x_* \in (-\frac{L}{2}, -\frac{L}{2} + \varepsilon)$. Now, as we increase the value of x_*, there will be some maximal value for which the reflected domain will be a subset of the fluid domain. Denote this value by

$$x_0 := \max\{x_* | D_*^R \subset D\} \leq 0,$$

and $D_0 := \mathrm{dom}(\phi_{x_0})$.

There are three possible cases for x_0:

I. $x_0 = 0$.

II. $(x_0, \eta(x_0))$ is the crest point.

III. The reflection maps a boundary point $\xi_0, \eta(\xi_0))$ into another boundary point; this means that ∂D_0^R is tangent to ∂D.

Theorem 4.3. *The stream function $\psi \in C^2(\overline{D})$ which is solution of the system (2.19) is symmetric.*

Proof. **Case I.**
This is the case where $x_0 = 0$. Define $w : D_0 \to \mathbb{R}$ by

$$w(x,y) = \begin{cases} \psi(-x,y) - \psi(x,y), & \text{if } \psi_y|_{\eta(x)} < 0 \\ \psi(x,y) - \psi(-x,y), & \text{if } \psi_y|_{\eta(x)} > 0 \end{cases}.$$

Obviously $w \in C^2(\overline{D_0})$. The first step is to apply the weak maximum principle. We have

$$\Delta w = \Delta\psi - \Delta\psi = 0 \text{ in } D_0.$$

We now look at the situation on the boundary. On the bottom, $w = 0$ since $\psi = m$ on $y = 0$. By L-periodicity, we obtain:

$$w\left(-\frac{L}{2}, 0\right) = \begin{cases} \psi(\frac{L}{2}, y) - \psi(-\frac{L}{2}, y) \\ \psi(-\frac{L}{2}, y) - \psi(\frac{L}{2}, y) \end{cases} = 0,$$

so $w = 0$ on $\{x = -\frac{L}{2}\}$, and the definition of x_0 in this case yields $w(0,y) = 0$ on $\{x = 0\}$. We now look at the values of w on the surface.

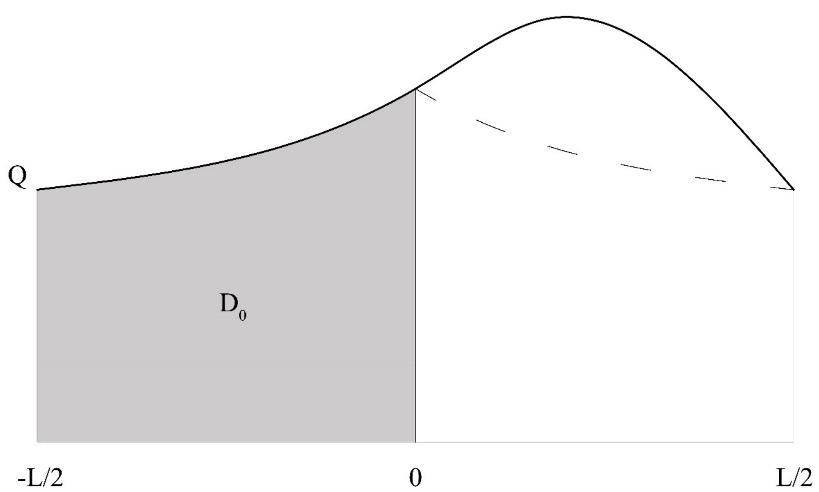

Figure 4.1: Case I

- $\psi(x, \eta(x)) = 0$ by the boundary condition of the system.
- $\psi(x, y) > 0$ between the surface and the line of stagnation $y = y_0$ if $\psi_y|_{\eta(x)} < 0$.
- $\psi(x, y) < 0$ between the surface and the line of stagnation $y = y_0$ if $\psi_y|_{\eta(x)} > 0$.

Furthermore, ψ is either decreasing or increasing so ψ_y on the surface cannot change because $\psi \in C^2(D_0)$ and $\psi_y \neq 0$ there (since there is no stagnation on the free surface).

The reflected surface D_0 is contained in $\overline{D}$ (by continuity) and since $x_0 = 0$ we have that $(-x, \eta(x)) \in \overline{D}$ for all $x \in \left(-\frac{L}{2}, 0\right)$. Therefore

$$\psi(-x, \eta(x)) \geq 0 \ \forall x \in \left(-\frac{L}{2}, 0\right),$$

as $\psi \geq 0$ within the fluid.

On the other hand, $\psi(x, \eta(x)) = 0$ for all $x \in \left(-\frac{L}{2}, 0\right)$ due to the boundary conditions of the system. Therefore, $w \geq 0$ on ∂D_0.

So we established that $\Delta w = 0$ in D_0 and $w \geq 0$ on ∂D_0. We can therefore apply the weak maximum principle to get $w \geq 0$ in D_0 unless $w \equiv 0$ on $\overline{D_0}$. In order to establish the latter case, we will use Serrin's edge-point lemma.

Choose $Q := \left(-\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right)$ and $T := \left\{= -\frac{L}{2}\right\}$. The line T is, of course, normal to Q, w satisfies $\Delta w \leq 0$ there, $w \geq 0$ in D_0 and $w\left(-\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right) = 0$ since $\psi(Q) = 0$. The plan is now to compute all partial derivatives of w up to order two and show that they are equal to zero and thus deduce that $w \equiv 0$ on $\overline{D_0}$.

- By the definition of w, it follows immediately that

  $$w_y(Q) = w_{yy}(Q) = 0.$$

- We differentiate $\psi(x, \eta(x)) = 0$ with respect to x to get:

  $$\psi_x + \psi_y \eta'(x) = 0.$$

  Now $\eta'(Q) = 0$, since Q is the through, so we get $\psi_x(Q) = 0$. Therefore,

  $$w_x(Q) = -2\psi_x(Q) = 0,$$

  due to the periodicity of ψ.

- We now differentiate w_x with respect to x:

  $$w_{xx}(Q) = \begin{cases} [\psi_{xx}(-x, y) - \psi_{xx}(x, y)]_{|Q} \\ [-\psi_{xx}(x, y) + \psi_{xx}(-x, y)_{|Q} \end{cases} = 0,$$

  again due to the L- periodicity of ψ.

- Differentiate the non-linear boundary condition $\frac{|\nabla\psi|^2}{2} + (g - 2\omega c)y = C$ with respect to x and evaluate the result on $y = \eta(x)$. First we calculate the more difficult first term (and for the moment drop the argument (x, y) for simplicity):

$$\begin{aligned}\frac{\partial}{\partial x}\left(\psi_x^2 + \psi_y^2\right) &= \frac{\partial}{\partial x}\psi_x^2 + \frac{\partial}{\partial y}\psi_x^2\eta'(x) + \frac{\partial}{\partial x}\psi_y^2 + \frac{\partial}{\partial y}\psi_y^2\eta'(x)\\ &= 2\psi_x\psi_{xx} + 2\psi_x\psi_{xy}\eta'(x) + 2\psi_y\psi_{yy}\eta'(x) + 2\psi_y\psi_{xy}.\end{aligned}$$

This is just an application of the chain rule. Now, use this to get from differentiating the non-linear boundary condition with respect to x:

$$\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{yy}\eta' + \psi_{xy}) + g\eta' - 2\omega c\eta' = 0 \text{ on } y = \eta(x).$$

Evaluation of this at Q (remember that Q is the trough and hence $\eta'(Q) = 0$), gives:

$$\psi_y(Q)\psi_{xy}(Q) = 0,$$

but since $\psi_y(Q) \neq 0$, we deduce $\psi_{xy}(Q) = 0$. This means that

$$w_{xy} = \begin{cases}[-\psi_{xy}(-x, y) - \psi_{xy}(x, y)]_{|Q} \\ [\psi_{xy}(x, y) + \psi_{xy}(x, y)]_{|Q}\end{cases} = 0.$$

Combining all the above, we see that $w(x, y) \equiv 0$ in $\overline{D_0}$ and so $\psi(x, y) = \psi(-x, y)$ in $\overline{D_0}$.

Case II.

In this case $x_0 < 0$, since by definition $x_0 \leq 0$ and $x_0 = 0$ was covered in case I. The property of x_0 ensures that D_0^R, the domain obtained by reflecting D_0 in the line $\{x = x_0\}$ by means of the transformation $(x, y) \mapsto (2x_0 - x, y)$, is contained in the fluid domain D. Since we now consider the case where $Q = (x_0, \eta(x_0))$ is the crest point, the wave profile $y = \eta(x)$ is decreasing in $[x_0, \frac{L}{2}]$. Therefore, letting $x_1 = 2x_0 + \frac{L}{2}$ and $x_2 = x_0 + \frac{L}{2}$, the reflection via $(x, y) \mapsto (2x_2 - x, y)$ of $\{(x, y) \in \mathbb{R}^2 | x_2 < x < \frac{L}{2}, 0 < y < \eta(x)\}$ in the line $\{x = x_2\}$ is also contained in D. This reflection maps $\{x = \frac{L}{2}\}$ into $\{x = x_1\}$.

For this case we define our auxiliary function a bit differently:

$$w(x, y) = \begin{cases}\psi(x, y) - \psi(2x_0 - x, y), & \text{if } x_0 \leq x \leq x_1, 0 \leq y \leq \tilde{\eta}(x), \psi_y|_{\eta(x)} < 0\\ -\psi(x, y) + \psi(2x_0 - x, y), & \text{if } x_0 \leq x \leq x_1, 0 \leq y \leq \tilde{\eta}(x), \psi_y|_{\eta(x)} > 0\\ \psi(x, y) - \psi(2x_2 - x, y), & \text{if } x_1 \leq x \leq x_2, 0 \leq y \leq \tilde{\eta}(x), \psi_y|_{\eta(x)} < 0\\ -\psi(x, y) + \psi(2x_2 - x, y), & \text{if } x_1 \leq x \leq x_2, 0 \leq y \leq \tilde{\eta}(x), \psi_y|_{\eta(x)} > 0\end{cases}$$

Here w is defined on the domain

$$\Omega_0 := \{(x, y) \in \mathbb{R}^2 | x_0 < x < x_2, 0 < y < \tilde{\eta}(x)\},$$

with

$$\tilde{\eta}(x) = \begin{cases} \eta(2x_0 - x), & \text{for } x_0 \le x \le x_1 \\ \eta(2x_2 - x), & \text{for } x_1 \le x \le x_2. \end{cases}$$

Like we did in case I, we want first to apply the weak maximum principle. Indeed, $w \in C^2(\overline{\Omega}_0)$ and $w = 0$ on the bottom $\{y = 0\}$ and on $\{x = x_0\}$ and $\{x = x_2\}$. Also, it is easy to see that $\Delta w = 0$ in Ω_0. Therefore, by the weak maximum principle, $w \ge 0$ in Ω_0 unless $w \equiv 0$ on $\overline{\Omega}_0$. Next, we wish

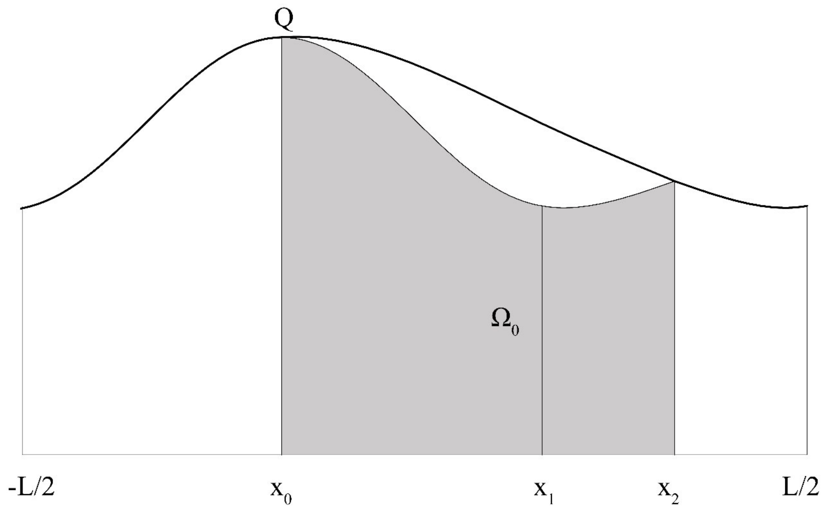

Figure 4.2: Case II

to apply the edge-point lemma. Choose $Q = (x_0, \eta(x_0))$ (the crest point) and $T = \{x = x_0\}$. Again, our task is to compute all partial derivatives up to order two to obtain $w \equiv 0$.

- By definition of w, it follows immediately that $w_y(Q) = w_{yy}(Q) = 0$.
- Differentiation of $\psi(x, \eta(x)) = 0$ with respect to x gives $\psi_x + \psi_y \eta' = 0$. Since Q is the crest, $\eta'(x_0) = 0$ and so $\psi_x(Q) = 0$. This implies $w_x(Q) = 0$.
- Differentiate w_x with respect to x again to get:

  $$\begin{aligned} w_{xx}(Q) &= [\psi_{xx}(x,y) + \psi_{xy}(x,y)\eta'(x) - \psi_{xx}(2x_0 - x, y) + \psi_{xy}(2x_0 - x, y)\eta'(x)]|_Q \\ &= \psi_{xx}(x_0, y) - \psi_{xx}(x_0, y) = 0, \end{aligned}$$

  since $\eta'(x_0) = 0$.
- Differentiation of the non-linear boundary condition with respect to x once again yields

  $$\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{yy}\eta' + \psi_{xy}) + g\eta' - 2\omega c\eta' = 0 \text{ on } y = \eta(x),$$

which implies

$$\psi_y(Q)\psi_{xy}(Q) = 0 \Rightarrow \psi_{xy} = 0,$$

since $\psi_y(Q) \neq 0$. This again leads to $w_{xy}(Q) = 0$.

Application of the edge-point lemma gives $w \equiv 0$ on $\overline{\Omega}_0$ and so

$$\psi(x,y) = \begin{cases} \psi(2x_0 - x, y), & x_0 \leq x \leq x_1 \\ \psi(2x_2 - x, y), & x_1 \leq x \leq x_2 \end{cases}$$

which proves the desired symmetry for case II.

Case III.

In this case, the reflected surface is tangent at some point of $\overline{\Omega}_0$; denote this point by $Q = (\xi_1, \eta(\xi_1))$. The auxiliary function can be taken to be the same as in case II, defined again on Ω_0 and by the same arguments as in the second case, we have to deal with two reflections. As always, we wish to show $w \equiv 0$ on $\overline{\Omega}_0$. A third application of the weak maximum principle shows that either $w \geq 0$ in Ω_0 or $w \equiv 0$ on $\overline{\Omega}_0$. For this case, we will apply the Hopf maximum principle to show that the latter holds. In order to do that, it suffices to show $w = 0$ and $\frac{\partial w}{\partial \nu} = 0$ at Q, where ν is the outer normal vector at Q.

Since $Q = (\xi_1, \eta(\xi_1))$ is the tangent point, we readily see

$$w(Q) = \pm\psi(\xi_1, \eta(\xi_1)) \mp \psi(2x_0 - \xi_1, \eta(2x_0 - \xi_1)) = 0,$$

since both points are on the surface of the fluid. Furthermore, $\overline{\Omega}_0$ satisfies an interior sphere property at Q, as a consequence of the tangency at that point. We now have to check that $\frac{\partial w}{\partial \nu} = 0$ at Q. Again, we compute the partial derivatives. Let $\xi_0 := 2x_0 - \xi_1$. We have:

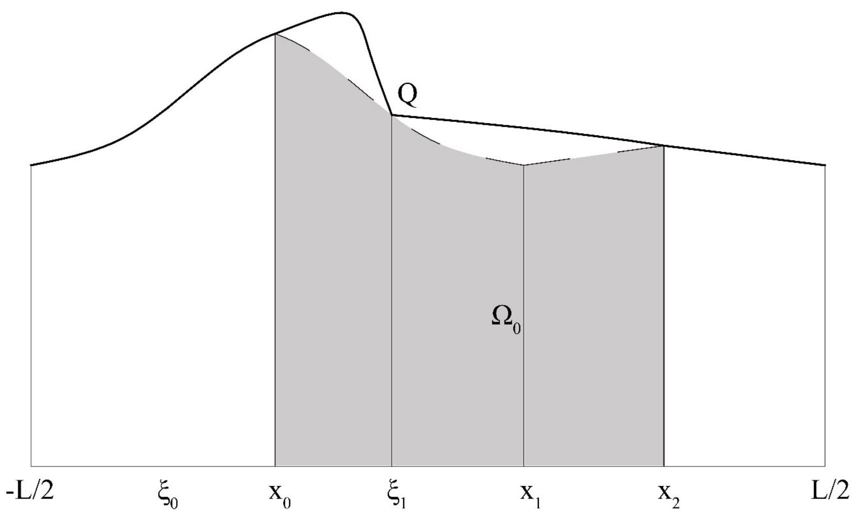

Figure 4.3: Case III

$$\eta(\xi_0) = \eta(\xi_1) \text{ and } \eta'(\xi_0) = -\eta'(\xi_1),$$

which follows from the tangency property at Q.

Differentiate $\psi(x, \eta(x))$ with respect to x to get

$$\psi_x(x, \eta(x)) + \psi_y(x, \eta(x))\eta'(x) = 0.$$

Evaluation of this equation at $x = \xi_0$ and $x = \xi_1$ gives:

$$\psi_x(\xi_0, \eta(\xi_0)) = -\psi_y(\xi_0, \eta(\xi_0))\eta'(\xi_0), \; \psi_x(\xi_1, \eta(\xi_1)) = -\psi_y(\xi_1, \eta(\xi_1))\eta'(\xi_1),$$

which implies

$$\frac{\psi_x(\xi_0, \eta(\xi_0))}{\psi_y(\xi_0, \eta(\xi_0))} = -\eta'(\xi_0) = \eta'(\xi_1) = -\frac{\psi_x(\xi_1, \eta(\xi_1))}{\psi_y(\xi_1, \eta(\xi_1))} \tag{4.8}$$

Using the non-linear boundary condition we infer:

$$|\nabla\psi|^2(\xi_0, \eta(\xi_0)) = C - 2(g - 2\omega c)\eta(\xi_0) = C - 2(g - 2\omega c)\eta(\xi_1) = |\nabla\psi|^2(\xi_1, \eta(\xi_1)).$$

This means,

$$\psi_x^2(\xi_0, \eta(\xi_0)) + \psi_y^2(\xi_0, \eta(\xi_0)) = \psi_x^2(\xi_1, \eta(\xi_1)) + \psi_y^2(\xi_1, \eta(\xi_1)). \tag{4.9}$$

Equation (4.8) implies that the quotients on the left and on the right equal some constant, say α and $-\alpha$ respectively. Plugging this result into equation (4.9), we deduce

$$\psi_x(\xi_0, \eta(\xi_0)) = \pm\psi_x(\xi_1, \eta(\xi_1))$$

and

$$\psi_y(\xi_0, \eta(\xi_0)) = \mp\psi_y(\xi_1, \eta(\xi_1)).$$

Since ψ_y must have the same sign in a neighbourhood of the surface, we see that choosing minus in the second equation is not possible. We have then,

$$\begin{aligned} \psi_x(\xi_0, \eta(\xi_0)) &= -\psi_x(\xi_1, \eta(\xi_1)) \\ \psi_y(\xi_0, \eta(\xi_0)) &= \psi_y(\xi_1, \eta(\xi_1)). \end{aligned}$$

With these results, we can now compute w_x and w_y:

$$w_x(Q) = \psi_x(\xi_1, \eta(\xi_1)) + \psi_y(\xi_1, \eta(\xi_1))\eta'(\xi_1) + \psi_x(\xi_0, \eta(\xi_0)) + \psi_y(\xi_0, \eta(\xi_0))\eta'(\xi_0)) = 0$$

$$w_y(Q) = \psi_y(\xi_1, \eta(\xi_1)) - \psi_y(\xi_0, \eta(\xi_0)) = 0$$

We can now apply the Hopf maximum principle to get $w \equiv 0$ on $\overline{\Omega}_0$ and so the stream function is again symmetric for the case III.

This completes the proof. □

Lemma 4.4. Theorem 4.3 and the injectivity of the stream function ψ imply that the wave profile described by the function $\eta(x) \in C^2(\mathbb{R})$ is symmetric between crests and troughs.

Bibliography

[1] G. K. Batchelor, *An introduction to fluid dynamics.* Cambridge University Press, 1967

[2] A. Constantin, *On the modelling of equatorial waves.* Geophys. Res. Lett., 39, 2012

[3] A. Constantin, *Nonlinear water waves with applications to wave-current interactions and tsunamis.* Society for industrial and Applied Mathematics, Philadelphia, 2011

[4] A. Constantin and J. Escher *Symmetry of steady periodic surface water waves with vorticity.* Journal of Fluid Mechanics, **498**, 171-181, 2004

[5] A. Constantin and W. Strauss, *Exact steady periodic water waves with vorticity.* Communications on Pure and Applied Mathematics, **57** (4), 481-527, 2004.

[6] A. Constantin and E. Varvaruca, *Steady periodic water waves with constant vorticity: regularity and local bifurcation.* Arch. Ration. Mech. Anal. **199** (1), 33-67, 2011.

[7] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues.* J. Funct. Anal. **8**, 321-340, 1971

[8] L. E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems.* Cambridge University press, 2000.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Springer-Verlag New York, Berlin, Heidelberg, 1st Edition, 3rd Revised Printing, 1997

[10] A. E. Gill, *Atmosphere-ocean dynamics.* Academic Press, 1982

[11] D. Henry, A. V. Matioc *On the symmetry of steady equatorial wind waves.* Nonlinear Analysis: Real World Applications, **18**, 50-56, 2014

[12] R. S. Johnson, *A modern introduction to the mathematical theory of water waves.* Cambridge University Press, 1997.

[13] J. H. LaCasce, *Atmosphere-ocean dynamics*. Lecture Notes, Dept. of Geosciences, University of Oslo, 2011.

[14] J. Lighthill, *Waves in fluids*. Cambridge University Press, 2nd Edition, 2003.

[15] C. I. Martin, *Equatorial wind waves with capillary effects and stagnation points*. Nonlinear Analysis - Theory, Methods and Applications, **96**, 1-17, 2014.

[16] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*. Springer-Verlag New York, Berlin, Heidelberg, Tokyo 2nd Edition, 1984

[17] W.A. Sutherland, *Introduction to metric and topological spaces*. Oxford Science Publications, 1975

[18] G. Tulzer, *On symmetry of steady periodic water waves with constant vorticity*. Diplomarbeit an der Universität Wien, 2011

Deutsche Zusammenfassung

In den vergangenen Jahren wurden zahlreiche Arbeiten publiziert, die die Symmetrie von verschiedenen Typen von Wasserwellen beweisen. In ([18]) wird gezeigt, dass Wellen symmetrisch sind, die zwischen Wellenbergen und –tälern monoton sind und für die die Vortizität konstant und ungleich Null ist, wobei Stagnationspunkte auftreten dürfen.

Die vorliegende Arbeit beinhaltet eingangs die Modellierung von äquatorischen Wasserwellen und eine Einführung in Maximums-Prinzipien sowie eine kurze Diskussion über Stagnationspunkte. Im Anschluss wird bewiesen, dass die in ([18]) vorgestellte Methode auch auf äquatorische Wasserwellen erweiterbar ist.

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