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For all the wonderful people in my life, especially for Nils, whose unwavering support and encouragement helped me a great deal in writing this thesis, and whose patience survived even my best efforts to exhaust all possible deadlines literally to the last minute. I yield.

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1 Introduction

Symmetries appear in several forms throughout mathematics and its applications. Consider for example a square embedded into the plane such that it is centered around the origin. If we rotate it by $\frac{\pi}{2}$ degree it still looks the same so intuitively speaking this rotation should be a symmetry of our square. We then notice that applying this rotation two or more times still doesn't change the square so iterated rotations by $\frac{\pi}{2}$ should be symmetries as well. More generally we should be able to *compose* symmetries. Clearly doing nothing is a symmetry. We can also rotate in the opposite direction, so symmetries should have inverses.

These observations are captured by the concept of a *group*. However the group structure only remembers how the symmetries of our square behave relative to each other and knows nothing about its specifics. Given an abstract group G one can then ask how it can be concretely realized as symmetries of some 'object' x . This amounts to finding a subgroup in the automorphism group of x that is isomorphic to G . More generally one can ask for a homomorphism $G \rightarrow \text{Aut}(x)$. The systematic study of these realizations or *representations* is known as the representation theory of G . Classically the objects x are vector spaces but we could think of anything else here.

A bit more precisely let $\mathcal{C}$ be any category and $x \in \text{ob}(\mathcal{C})$. From G construct a category with only a single object and endomorphisms on that object precisely the elements of G . Composition of morphisms is the group multiplication. This category is known as the *delooping* $\mathbf{B}G$ of G . A representation of G on x is then defined to be a functor $\mathbf{B}G \rightarrow \mathcal{C}$ that sends the single object of $\mathbf{B}G$ to x . In the case where indeed $\mathcal{C} = \text{Vect}$ this is just a classical linear representation of G on x . The advantage of this viewpoint is that it is immediately generalizable to the case where G is allowed to be a groupoid. Intuitively this would mean that we have now many groups acting on many objects simultaneously and also ways to translate between these actions.

If we want to consider only one structure acting on one object at a time we can generalize in another way. We can replace $\mathcal{C}$ by higher categories and G by higher versions of groups. Let's start by going up one dimension. G is then replaced with a monoidal category that has inverses in some precise sense. These are known as *2-groups*. $\mathcal{C}$ is replaced by a bi- or 2-category. A representation of G in $\mathcal{C}$ is then a (now 2-)functor from $\mathbf{B}G \rightarrow \mathcal{C}$.

The presence of 2-cells allows us to also compare different symmetries, so we think of the 2-cells as 'symmetries of symmetries'. For example if we think of the objects in $\mathcal{C}$ as physical theories and of the 1-morphisms as consisting of data telling us how to change between two theories then there might be many different ways to change from one theory to another. The 2-cells then allow us to compare these different ways of changing the theory.

Similarly one can also allow even higher structures. The next step would be tri- or Gray-categories¹. Roughly, the procedure of increasing the dimension of the categories involved is known as categorification. So what we outlined so far is the categorification of group representations.

The content of this work is to provide classification theorems for certain degenerate versions of categorified group representations. The degeneracy comes from letting ‘low dimensional’ cells of the representation all be identities and having interesting information only in the higher cells. If the 1-up to k cells are trivial in this sense then we speak of k -degenerate representations.

Let G be an ordinary group and $\mathcal{C}$ a 2-category. It is a well known result that up to equivalence 1-degenerate representations in this setup can be classified by the second cohomology group of G with coefficients in some abelian group extracted from $\mathcal{C}$. A similar statement holds if $\mathcal{C}$ is a Gray-category and we look at 2-degenerate representations. In this case the third cohomology group makes an appearance.

To the best of the author’s knowledge nobody is able to give a hands-on definition of weak 4- or higher categories but we conjecture that the above statement holds in all dimensions, i.e. that $(n - 1)$ -degenerate representations of G in an n -category are up to equivalence classified by the n -th cohomology group of G (with certain coefficients). Taking a closer look at how the cohomology groups appear we can infer that we should think of the formulas defining the differentials of the group cohomology in the following way: Take a functor from G into some higher category and consider its coherence equations for the top cells. These live in many dimensions but now squash them into a single dimension and obtain said formulas.

As we lower the degeneracy of the functors involved things become increasingly complicated. Our main result is that for a fixed 2-group G and a Gray-category $\mathcal{C}$ we will be able to give a classification of 1-degenerate representations of G on some fixed object $x \in \mathcal{C}$ in terms of the *second 2-cohomology group of G* (with certain coefficients coming from $\mathcal{C}$). A considerable amount of this thesis will be spent on categorifying the cohomology of a group to the 2-cohomology of a 2-group as this is a new construction that hasn’t appeared in the literature yet (to the best of the author’s knowledge).

1. In section 2 we will introduce the basic concepts. To set them in perspective we formulate them for categories of arbitrary dimension.
2. In section 3 we will provide some results about 2-groups in a way that suits our purposes.
3. In section 4 we will consider the case where the target category is 2-dimensional and the representation is 1-degenerate.

¹Gray-categories are a certain strict version of tricategories.

4. In order to categorify cohomology we will need to be able to take higher analogues of kernels and quotients of groups. The necessary definitions and an existence theorem for kernels is provided in section 6. The existence theorem for quotients is a bit involved. Before we can give the fully weak version we will construct a strict version of quotients of categories in section 5. In section 7 we will use double categories and the results about strict quotients to obtain the existence theorem for fully weak quotients of ‘abelian’ 2-groups.
5. In section 8 we will then construct the 2-cohomology 2-groups of a 2-group with coefficients in an ‘abelian’ 2-group.
6. Finally in section 9 we will make use of the concepts developed so far to obtain our main classification theorem.

2 General Preliminaries

2.1 Prerequisites

The reader is assumed to be familiar with the basics of category theory including the definition of categories, functors, natural transformations, monoidal categories, bicategories and their basic properties. An intuitive understanding of n -categories for $n \geq 3$ would be helpful as well.

Furthermore we will require the basics of homological algebra, hardly exceeding the definition of a cochain complex and its cohomology and the definition of the cohomology of a monoid (or group) with coefficients in an abelian group.

The standard reference for basic category theory is [6]. However there can be found many introductions to the subject and lecture notes online. A good place for reference is also [5]. An introductory exposition on bicategories can be found in [7]. For tri- and in particular Gray-categories [1] is an excellent reference. In section 7 we will also need some facts about double categories. We will use [2] as our main reference.

2.2 Notation and Conventions

With an n -category we generally refer to a weak n -category, $n \in \mathbb{N}$. However in low dimensions we will restrict our attention to certain strict ones that are provided by the various coherence theorems (strict 2-categories in dimension 2 and Grey-categories in dimension 3). Because of said theorems this will not really impose any loss of generality.

Given an n -category we will refer to n as its *dimension*. If the dimension can be determined from the context or its specific value doesn’t matter we will refer to all higher categories just as categories and leave out the n . The same holds for functors, natural transformations and so on.

For simplicity functors will always be assumed to be strictly unital.

We will not distinguish between sets and classes (or even higher “containers”) since for our purposes this differentiation is just a technical nuisance.² The reader (rightfully) infuriated by this lack of rigor may replace sets with classes (or higher analogues thereof) when appropriate throughout the text.

2.3 Generalities in arbitrary dimensions

Let G be an m -category and $m \leq n$. We can canonically regard G as an n -category G by adding identities on top.

If G is a monoidal m -category we can build the *delooping* $\mathbf{B}G$ of G . It is an $(m + 1)$ -category obtained from G by shifting all cells of G up one dimension and adding a new 0-cell. The monoidal product becomes the composition of 1-morphisms in $\mathbf{B}G$.

Conversely, by fixing an object $a \in G$ we obtain a monoidal $(m - 1)$ -category $\Omega_a G := G(a, a)$. If G has only one object we will just write ΩG . Note that since $\Omega_a G$ is monoidal it has a distinguished object, namely the monoidal unit. By $\Omega_a^k G$ we denote the monoidal category obtained from iterating this procedure k times with the monoidal unit as base point after the first iteration.

For monoidal G we have $\Omega \mathbf{B}G = G$ and if G only has one 0-cell then also $\mathbf{B}\Omega G = G$. (Of course “=” has to be understood in some appropriate sense.) We can use Ω also to *define* monoidal (higher) categories as the ones that are of the form $\Omega G'$ for some G' . More generally G is called k -tuply monoidal if it is of the form $G = \Omega^k G'$ for some G' . If G is k -tuply monoidal we can apply $\mathbf{B}$ at least up to k times to G .

By applying Ω repeatedly more and more of the various compositions on the different levels of G get “mashed” together and the resulting monoidal structure becomes more and more symmetric. This phenomenon is summarized in the so called *periodic table* of category theory (see for example [7, p. xx]).

This means that the k monoidal products in a k -tuply monoidal category all agree and the single remaining product gets more and more “commutative” as k increases.³

A p -tuply monoidal functor between p -tuply monoidal categories is a monoidal one that preserves the symmetry in a suitable sense.

Definition 2.1. As above let G be an m -category, $\mathcal{C}$ an n -category and $m \leq n$. A *representation* of G in $\mathcal{C}$ is a functor $\rho : G \longrightarrow \mathcal{C}$.

²That does not mean that this distinction is not relevant for other purposes, think for example of the adjoint functor theorem.

³Again, they agree only in a weak sense but can be reconstructed from any single one together with the data of the symmetry.

Such a ρ consists of a lot of data. First there is some prescription how to map all the cells of G to cells of the same dimension in $\mathcal{C}$. But instead of strictly respecting the various compositions of G , ρ comes equipped with additional ‘comparison’ cells that replace equalities in the equations that have to hold for this prescription. We describe them inductively.

Assume we know what the k -constraints are for an $(n-1)$ -functor between $(n-1)$ -categories ($k = 1, \dots, n-1$). Locally (i.e. on the Hom -($n-1$)-categories) ρ is given by $(n-1)$ -functors each of which comes with their own 1- to $(n-1)$ -constraints. All of them together form the 1-constraints of ρ . They know nothing about the lowest-level composition in G and $\mathcal{C}$, the one induced (on all levels) from the composition of 1-cells. There are comparison cells for this composition of dimension 2 to n . We refer to them as the 2-constraints of ρ .

They have to satisfy coherence of their own, namely some kind of associativity (recall that we assume strict unitality so we need no additional constraints for the units). Again it doesn’t hold on the nose but rather there are cells of dimension 3 to n that replace equalities and relate the different ways of “bracketing” the 2-constraints. We call these cells the 3-constraints of ρ . The coherence for them is realized via the 4-constraints of ρ and so on up to n .

Definition 2.2. Let ρ be as above. It is called *k-degenerate on $x \in \text{ob}(\mathcal{C})$* if it maps all 0-cells to x and its 1- to k -constraints are all given by identities.

Definition 2.3. The *representation category of G in $\mathcal{C}$* is the functor category $[G, \mathcal{C}]$ and denoted by $\text{Rep}_{\mathcal{C}}(G)$.

The *category of k-degenerate representations of G on $x \in \mathcal{C}$* , $\text{Rep}_{x, \mathcal{C}}^k(G)$, is the full subcategory on k -degenerate representations on x . If x or $\mathcal{C}$ is clear from the context we may also leave these subscripts out.

Definition 2.4. If G is p -tuply monoidal then a *representation of G on $x \in \mathcal{C}$* is a p -tuply monoidal functor $\rho : G \rightarrow \Omega_x^p \mathcal{C}$. Any such functor can be canonically regarded as a functor $\mathbf{B}^p G \rightarrow \Omega$. It is called *k-degenerate* if it is k -degenerate under this identification.

Remark 2.5. p -tuply monoidal functors $G \rightarrow \Omega_x^p \mathcal{C}$ can be identified with functors $\mathbf{B}^p G \rightarrow \Omega$ that map the base point of $\mathbf{B}^p G$ to x and where the j - to $(j+p-2)$ -cells of the j -constraints are given by identities for all j . A similar statement holds for natural transformations, modifications etc. There however we have to be a bit careful which cells are assumed to be (weakly) invertible.

In particular for $p = 1$ the second condition is empty and we can identify monoidal functors $G \rightarrow \Omega_x \mathcal{C}$ with functors $\mathbf{B}G \rightarrow \Omega$ that map the basepoint of $\mathbf{B}G$ to x .

3 Categorical Groups

In the following monoidal categories will tacitly assumed to be strict. Also monoidal functors will tacitly assumed to be strictly unital. Let $f, g : A \rightrightarrows B$ be monoidal functors between monoidal categories. Then by our assumption we have $f(I) = I = g(I)$ (where I denotes the monoidal unit). For any monoidal natural transformation $\omega : f \rightarrow g$ we will require strict unitality as well, i.e. $\omega(I) = 1_I$.

3.1 Groupoidal Categories and 2-Groups

Lemma 3.1. *Let $(G, \otimes, I)$ be a monoidal groupoid. The following pieces of data (for all objects of G) canonically determine each other:*

- (i) *Left duals with invertible units and counits.*
- (ii) *Right duals with invertible units and counits.*
- (iii) *Left and right duals such that*

$$\cup = (\cap)^{-1} \quad \text{and} \quad \cap = (\cup)^{-1}. \quad (3.1)$$

The resulting autonomous category is pivotal.

Definition 3.2. A *groupoidal category* is a monoidal category G together with any of the above structures. The dual of an object $X \in G$ is denoted X^* and called its tensor inverse.

Remark 3.3. Since all units and counits are invertible we have $X \otimes X^* \cong I \cong X^* \otimes X$. Note however that it doesn't make sense to ask if these isomorphisms are natural.

To do so we would first have to specify the source and target functors of such an isomorphism. But the assignment $x \mapsto x \otimes x^*$ does not canonically give rise to a functor $G \rightarrow G$. The problem is that while $x \mapsto (x, x^*)$ can be made into a functor $G \rightarrow G \times G^{op, rev}$ we have $\otimes : G \times G \rightarrow G$. So they can not be composed.

This suggests that groupoidal categories are not quite the right notion of categorified groups yet.

Definition 3.4. A *2-group* or *groupoidal groupoid* is a monoidal groupoid with left and right duals such that the units and counits satisfy (3.1). Equivalently it is a groupoidal category that happens to be a groupoid, i.e. all morphisms are invertible.

Proposition 3.5. *In a 2-group G taking tensor inverses canonically induces a functor $i : G \rightarrow G$ given on objects by $i(x) = x^*$ and on morphisms by $i(\alpha) = (\alpha^{-1})^*$. Furthermore we can assume $i^2 = 1_G$ and there are natural isomorphisms $l(x) : i(x) \otimes x \rightarrow I$ and $r(x) : x \otimes i(x) \rightarrow I$.*

Remark 3.6. In particular this means that if we want to check the equality of two morphisms

$$\begin{array}{c} \uparrow \\ \bullet \\ \alpha \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \beta \end{array} \quad (3.2)$$

we can equivalently verify that

$$\begin{array}{c} \uparrow \\ \bullet \\ \alpha \end{array} \begin{array}{c} \uparrow \\ \bullet \\ i(\beta) \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \end{array} \quad (3.3)$$

holds. Similarly we can also use a cup instead of a cap.

Proof. If we apply i to a composit of morphisms the order of composition gets reversed twice (once by taking duals and once by taking inverses) and therefore stays the same. Thus i is a functor. Since taking double duals is monoidally isomorphic to the identity in any pivotal category and this isomorphism is not visible in the graphical calculus G is equivalent to a 2-group where it equals the identity. We can thus assume this property henceforth. Also taking vertical inverses squares to the identity and therefore we can assume $i^2 = 1_G$. The isomorphisms l and r are given by caps,

$$l(x) = \begin{array}{c} \uparrow \\ x \end{array} \quad \text{and} \quad r(x) = \begin{array}{c} \uparrow \\ x \end{array} . \quad (3.4)$$

We have

$$\begin{array}{c} \uparrow \\ \bullet \\ \alpha \end{array} \begin{array}{c} \uparrow \\ \bullet \\ i(\alpha) \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \alpha \end{array} \begin{array}{c} \uparrow \\ \bullet \\ \alpha^{-1} \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \alpha^{-1} \end{array} \begin{array}{c} \uparrow \\ \bullet \\ \alpha \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \end{array} \quad (3.5)$$

which shows naturality of r . Naturality of l follows accordingly. $\square$

Remark 3.7. Due to Proposition 3.5 a 2-group can also be defined as a group weakly internal to $\mathbf{Grpd}$, the 2-category of groupoids, functors and natural transformations. (Where the bialgebra part is strict, corresponding to our assumption that all monoidal categories are strict).

Note that groupoidal categories can not be defined as groups weakly internal to $\mathbf{Cat}$ because of the problem pointed out in Remark 3.3.

Remark 3.8. For any 2-group G the functor i together with suitable constraints can be regarded as monoidal functor $G \rightarrow G^{rev}$. It follows from the coherence theorem for pivotal categories that G is equivalent to a 2-group

G' with the property that $i : G' \longrightarrow G'^{rev}$ is even strictly monoidal. In particular for objects $x, y \in G'$ we have $i(x \otimes y) = i(y) \otimes i(x)$.

We will henceforth tacitly assume this property for all 2-groups.

Definition 3.9. 2-groups, monoidal functors and monoidal natural transformations form a 2-category which we will denote by 2Grp .

3.2 Braided 2-groups and Abelian 2-groups

Definition 3.10. A *braided 2-group* (A, b) is a 2-group A together with a braiding b .

Lemma 3.11. Let (A, b) be a braided 2-group. Then the inversion functor $i : A \longrightarrow A$ from Proposition 3.5 together with the constraints $\nu(x, y) := b(i(x), i(y))$ is monoidal. It is braided if and only if A is symmetric monoidal. In this case it is also symmetric.

Proof. Naturality of ν and that it satisfies the relevant equation for monoidality of i follow directly from the axioms for the braiding. For objects x, y we compute

$$\nu(x, y) \circ b(i(x), i(y)) = \begin{array}{c} \text{Diagram: A braid with two strands. The top strand is labeled } i(x) \text{ and the bottom strand is labeled } i(y). \text{ They cross, with } i(x) \text{ over } i(y). \\ x \quad y \end{array} \quad (3.6)$$

and

$$i(b(x, y)) \circ \nu(x, y) = \begin{array}{c} \text{Diagram: A braid with two strands. The top strand is labeled } x \text{ and the bottom strand is labeled } y. \text{ They cross, with } x \text{ over } y. \\ x \quad y \end{array} \quad (3.7)$$

Equality holds if and only if b is a symmetry. $\square$

Lemma 3.12. Let A be a braided 2-group. Then the transformations l and r from Proposition 3.5 are monoidal.

Proof. In string diagram notation monoidality of r translates to

$$\begin{array}{c} \text{Diagram: A complex string diagram with four strands labeled } x, x, y, y \text{ from left to right. The first } x \text{ strand goes up and over the others. The second } x \text{ strand goes up and over the } y \text{ strands. The } y \text{ strands cross each other.} \\ x \quad x \quad y \quad y \end{array} = \begin{array}{c} \text{Diagram: Two separate arcs. The first arc connects the first } x \text{ and } x \text{ strands. The second arc connects the first } y \text{ and } y \text{ strands.} \\ x \quad x \quad y \quad y \end{array} \quad (3.8)$$

which is clearly true. The respective statement for l follows analogously. $\square$

Definition 3.13. A 2-group A is called *abelian* if it is symmetric monoidal.

Definition 3.14. Abelian 2-groups, symmetric monoidal functors and (symmetric) monoidal natural transformations form a category which we will denote by Ab2Grp .

Remark 3.15. From now on in abelian 2-groups we will denote the monoidal product as $+$ and the unit as 0 . The inversion i is denoted by $-$. An expression of the form $x - y$ is to be understood as $x + (-y)$. Due to Remark 3.8 we have $-(x + y) = -y - x$. In particular from this and Proposition 3.5 it follows that $-(x - y) = y - x$.

Remark 3.16. Every abelian 2-group A is compactly closed (by definition). This means in particular that for any $x, y \in \text{ob}(A)$ we have a natural bijection

$$A(a + b, c) \xrightarrow{\cong} A(a, c - b). \quad (3.9)$$

It is given in the diagrammatic calculus by bending strings up like this:

$$(3.10)$$

Its inverse bends strings down again. Similarly we could also bend strings up and down on the left side.

For a tuple of objects $x = (x_1, \dots, x_n)$ and any permutation $\sigma \in \mathcal{S}_n$ there is a unique isomorphism

$$b_{x,\sigma} : x_1 \otimes \dots \otimes x_n \xrightarrow{\cong} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)} =: x' \quad (3.11)$$

consisting only of symmetries. Precomposition with $b_{x,\sigma}$ gives for every object $y \in \text{ob}(A)$ a natural bijection

$$b_{x,\sigma}^* : A(x', y) \xrightarrow{\cong} A(x, y). \quad (3.12)$$

Remark 3.17. Consider two monoidal categories G, H with duals and a monoidal functor $f : G \rightarrow H$. Let $x \in \text{ob}(G)$ be an object with dual x^* and denote their evaluation and coevaluation by ε respectively μ . Then $(f(x), f(x^*), f(\varepsilon), f(\mu))$ is again an adjoint pair and thus $f(x^*) \cong f(x)^*$.

Let now G, H be 2-groups. Then these isomorphisms give rise to a natural transformation $f \circ i \cong i \circ f$. Let now G, H be furthermore abelian and f symmetric monoidal. Then this transformation is also monoidal.

Denote the constraints of f by γ . We have

$$\gamma(x, y) : f(x) + f(y) \xrightarrow{\cong} f(x + y). \quad (3.13)$$

By composing with the isomorphism from above we also get

$$f(x) - f(y) \xrightarrow{\cong} f(x - y) \quad (3.14)$$

naturally and monoidally. We will notationally make no distinction between this composed isomorphism and the constraint of f and just denote it by γ as well.

3.3 Functor Categories

For any two categories $G, \mathcal{C}$ the functor category from G to $\mathcal{C}$ is denoted by $[G, \mathcal{C}]$. The subcategory of strictly unital functors and transformations we denote by $[G, \mathcal{C}]_*$.

Remark 3.18. The full subcategory $[G, \mathcal{C}]_{const}$ of constant functors will be identified with $\mathcal{C}$. In particular we may use the same symbol for a constant functor and its value in $\mathcal{C}$, and similarly for natural transformations.

Lemma 3.19. *Let G be a category and A a monoidal category. The point-wise multiplication of functors $f, h \in [G, A]_*$ gives a monoidal structure on $[G, A]_*$. If A is braided or symmetric or a 2-group then so is $[G, A]_*$ by carrying over these structures pointwise.*

Let now G, f and h be monoidal (G as a category and f, h as functors). If A is braided then we can choose constraints for $f + h$ such that it is again monoidal.

Let now also G, f and h be braided monoidal. Then $f + h$ is braided monoidal if and only if A is symmetric monoidal.

If A, G are braided monoidal denote by $(A, G)_* \subseteq [A, G]_*$ the subcategory of (strictly unital) braided monoidal functors and braided monoidal natural transformations. If A is symmetric monoidal then $(A, G)_*$ inherits the structure of a symmetric monoidal category.

Proof. That the product of two functors is again a functor follows from the interchange law. That all the structures carry over pointwise is obvious. If G, f, h are monoidal and A braided monoidal then define constraints $\gamma_{f \otimes h}$

for $f \otimes h$ by

$$\gamma_{f \otimes h}(x, y) := \begin{array}{c} \begin{array}{cc} \gamma_f(x, y) & \gamma_h(x, y) \end{array} \\ \begin{array}{c} \text{Diagram: A string diagram for } \gamma_{f \otimes h}(x, y). \text{ It consists of four vertical strands. The left two strands are labeled } f(x) \text{ and } h(x) \text{ at the bottom, and } f(y) \text{ and } h(y) \text{ at the top. The right two strands are labeled } f(y) \text{ and } h(y) \text{ at the bottom, and } f(x) \text{ and } h(x) \text{ at the top. The strands cross each other. The crossing is labeled } \gamma_f(x, y) \text{ on the left and } \gamma_h(x, y) \text{ on the right.} \end{array} \end{array} \quad (3.15)$$

where $x, y \in \text{ob}(G)$ and γ_f, γ_h denote the constraints of f and h respectively. The necessary coherence is easy to check with string diagrams. If G, f, h are braided monoidal then in order for $f \otimes h$ to be braided we must have

$$\begin{array}{c} \begin{array}{c} \begin{array}{cc} \gamma_f(y, x) & \gamma_h(y, x) \end{array} \\ \text{Diagram: A string diagram for } \gamma_{f \otimes h}(y, x). \text{ It consists of four vertical strands. The left two strands are labeled } f(x) \text{ and } h(x) \text{ at the bottom, and } f(y) \text{ and } h(y) \text{ at the top. The right two strands are labeled } f(y) \text{ and } h(y) \text{ at the bottom, and } f(x) \text{ and } h(x) \text{ at the top. The strands cross each other. The crossing is labeled } \gamma_f(y, x) \text{ on the left and } \gamma_h(y, x) \text{ on the right.} \end{array} \\ \text{Diagram: A string diagram for } \gamma_{f \otimes h}(x, y). \text{ It consists of four vertical strands. The left two strands are labeled } f(x) \text{ and } h(x) \text{ at the bottom, and } f(y) \text{ and } h(y) \text{ at the top. The right two strands are labeled } f(y) \text{ and } h(y) \text{ at the bottom, and } f(x) \text{ and } h(x) \text{ at the top. The strands cross each other. The crossing is labeled } \gamma_f(x, y) \text{ on the left and } \gamma_h(x, y) \text{ on the right.} \end{array} \\ \text{Diagram: A string diagram for } \gamma_{f \otimes h}(y, x). \text{ It consists of four vertical strands. The left two strands are labeled } f(x) \text{ and } h(x) \text{ at the bottom, and } f(y) \text{ and } h(y) \text{ at the top. The right two strands are labeled } f(y) \text{ and } h(y) \text{ at the bottom, and } f(x) \text{ and } h(x) \text{ at the top. The strands cross each other. The crossing is labeled } \gamma_f(y, x) \text{ on the left and } \gamma_h(y, x) \text{ on the right.} \end{array} \\ \text{Diagram: A string diagram for } \gamma_{f \otimes h}(x, y). \text{ It consists of four vertical strands. The left two strands are labeled } f(x) \text{ and } h(x) \text{ at the bottom, and } f(y) \text{ and } h(y) \text{ at the top. The right two strands are labeled } f(y) \text{ and } h(y) \text{ at the bottom, and } f(x) \text{ and } h(x) \text{ at the top. The strands cross each other. The crossing is labeled } \gamma_f(x, y) \text{ on the left and } \gamma_h(x, y) \text{ on the right.} \end{array} \end{array} \quad (3.16)$$

For the last equation we used the symmetry of f and h . Clearly the above equation holds for all x, y if and only if the braiding of A is symmetric. The last claim follows from the other ones. $\square$

Remark 3.20. From this and Remark 3.16 it follows that if G is braided and A is symmetric then $(G, A)_*$ is compactly closed. In particular for any two $f, h \in (G, A)_*$ we can identify braided monoidal natural transformations $f \rightarrow h$ with ones $f - h \rightarrow 0$.

Furthermore if $f_1, \dots, f_n, h \in (G, A)_*$ then we can identify braided monoidal natural transformations $\sum_i f_i \rightarrow H$ with ones $\sum_i f_{\sigma(i)} \rightarrow H$ for any per-

mutation $\sigma \in \mathcal{S}^n$.

For a category G , G^n denotes the n -fold cartesian product of G with itself.

Lemma 3.21. *Let G be a monoidal category, let A be a braided 2-group and let $n \in \mathbb{N}$. For every $k \in \{0, \dots, n+1\}$ there is a strictly monoidal and braided functor*

$$T_k : [G^n, A]_* \longrightarrow [G^{n+1}, A]_* \quad (3.17)$$

defined as follows.

- (i) *Let $f \in [G^n, A]_*$ and let $(\alpha_1, \dots, \alpha_{n+1})$ be a morphism in G^{n+1} . We set*

$$T_k(f)(\alpha_1, \dots, \alpha_{n+1}) := f(\alpha_1, \dots, \alpha_k \otimes \alpha_{k+1}, \dots, \alpha_{n+1}). \quad (3.18)$$

(This also defines it on objects of G^{n+1} .) for $k = 0$ the above formula is to be understood as

$$T_0(f)(\alpha_1, \dots, \alpha_{n+1}) = f(\alpha_2, \dots, \alpha_{n+1}) \quad (3.19)$$

and for $k = n+1$ as

$$T_0(f)(\alpha_1, \dots, \alpha_{n+1}) = f(\alpha_1, \dots, \alpha_n). \quad (3.20)$$

- (ii) *Let $\omega : f \longrightarrow f'$ be a morphism in $[G^n, A]_*$ i.e. a natural transformation. Let $(x_1, \dots, x_{n+1})$ be an object of G^{n+1} . Similarly as above we define $T_k(\omega) : T_k(f) \longrightarrow T_k(f')$ by*

$$T_k(\omega)(x_1, \dots, x_n) := \omega(x_1, \dots, x_k \otimes x_{k+1}, \dots, x_{n+1}). \quad (3.21)$$

The remark about the cases $k = 0$ and $k = n+1$ applies here as well.

The proof of this is straightforward.

4 Representations in 2-Categories

4.1 Preliminaries

With a 2-category we will always mean a strict one. In particular monoidal categories are assumed to be strict throughout. Let G and $\mathcal{C}$ be 2-categories and denote by $\otimes$ the composition functors in G and $\mathcal{C}$.

Definition 4.1. A 2-functor $\rho : G \longrightarrow \mathcal{C}$ consists of the following data.

1. A function $\rho_0 : \text{ob}(G) \longrightarrow \text{ob}(\mathcal{C})$,
2. for all $a, b \in \text{ob}(G)$ a functor $\rho_{a,b} : G(a, b) \longrightarrow \mathcal{C}(\rho(a), \rho(b))$ (we will usually drop the subscripts for ρ),

3. for all $a, b, c \in \text{ob}(G)$ a natural isomorphism $\chi_{a,b,c}$ given in components by

$$\chi_{a,b,c}(f, g) : \rho(f) \otimes \rho(g) \longrightarrow \rho(f \otimes g) \quad (4.1)$$

where $f : b \longrightarrow c$ and $g : a \longrightarrow b$. Again we will leave out the subscripts for χ from now on. These χ are the 2-constraints of ρ . Since there are no other constraints around we will also just call them *the* constraints of ρ .

This data is subject to the following axioms.

1. Strict unitality: For all objects a and morphisms $a \xrightarrow{f} b$ of G we have

$$\begin{aligned} \rho(1_a) &= 1_{\rho(a)}, \\ \chi(f, 1_a) &= 1_{\rho(f)} \end{aligned} \quad (4.2)$$

and similar equations if the arguments of χ are permuted.

2. The diagram

$$\begin{array}{ccc} \rho(f) \otimes \rho(g) \otimes \rho(h) & \xrightarrow{\chi(f,g) \otimes 1} & \rho(f \otimes g) \otimes \rho(h) \\ \downarrow 1 \otimes \chi(g \otimes h) & & \downarrow \chi(f \otimes g, h) \\ \rho(f) \otimes \rho(g \otimes h) & \xrightarrow{\chi(f, g \otimes h)} & \rho(f \otimes g \otimes h) \end{array} \quad (4.3)$$

commutes for all composable 1-cells f, g, h in G .

We are interested in the special case where G is monoidal (i.e. has only one object as a 2-category). If we want to represent G in $\mathcal{C}$ on some object x by Definition 2.4 we have to consider monoidal functors $\rho : G \longrightarrow \Omega_x \mathcal{C}$.

A monoidal functor between monoidal categories is a 2-functor between their deloopings. More concretely:

Definition 4.2. Let G, H be monoidal categories with monoidal units denoted by I . A monoidal functor $\rho : G \longrightarrow H$ consists of the following data.

1. A functor ρ of the underlying categories,
2. a natural isomorphism χ given in components by

$$\chi(f, g) : \rho(f) \otimes \rho(g) \longrightarrow \rho(f \otimes g). \quad (4.4)$$

We refer to χ as the constraint isomorphism of ρ .

This data is subject to the following axioms.

1. Strict unitality: For all objects f of G we have

$$\begin{aligned}\rho(I) &= I, \\ \chi(f, I) &= 1_{\rho(f)}\end{aligned}\tag{4.5}$$

and a similar equation if the arguments of χ are permuted.

2. χ satisfies associativity coherence as in (4.3) where f, g, h are now objects of G .

Thus indeed as predicted by Remark 2.5 the notions of monoidal functor $G \rightarrow \Omega_x \mathcal{C}$ and of functor $\mathbf{B}G \rightarrow \mathcal{C}$ coincide in dimension 2. (By definition if you want.)

According to the same remark however the notions of natural transformations are different. The reason is that under the monoidal viewpoint there is not even the option of changing the image of the base point (in fact from the monoidal point of view there doesn't exist one) so it is unnatural to ask for comparison cells (of dimension 0 – these would be the ones that correspond to the 1-cells of the transformation after delooping).

Definition 4.3. Let G and $\mathcal{C}$ be 2-categories and $\rho, \rho' : G \rightrightarrows \mathcal{C}$ functors between them (with constraints χ and χ'). A natural transformation $\theta : \rho \rightarrow \rho'$ consists of the following data.

1. For each object a in G a 1-cell $\rho(a) \xrightarrow{\theta(a)} \rho'(a)$.
2. For all objects a, b in G a natural transformation $\theta_{a,b}$ given in components by

$$\begin{array}{ccc} \rho(a) & \xrightarrow{\rho(f)} & \rho(b) \\ \theta(a) \downarrow & \swarrow \theta_{a,b}(f) & \downarrow \theta(b) \\ \rho'(a) & \xrightarrow{\rho'(f)} & \rho'(b) \end{array} \tag{4.6}$$

where f is any morphism $a \rightarrow b$ in G . We will usually drop the subscripts for θ .

This data is subject to the following axioms.

1. Strict unitality: For all objects a of G we have $\theta(1_a) = 1_{\theta(a)}$.

2. The diagram

$$\begin{array}{ccc}
\rho(f) \otimes \rho(g) \otimes \theta(a) & \xrightarrow{1 \otimes \theta(g)} & \rho(f) \otimes \theta(b) \otimes \rho'(g) \\
\downarrow \chi(f,g) \otimes 1 & & \downarrow \theta(f) \otimes 1 \\
& & \theta(c) \otimes \rho'(f) \otimes \rho'(g) \\
& & \downarrow \chi(f,g) \\
\rho(f \otimes g) \otimes \theta(a) & \xrightarrow{\theta(f \otimes g)} & \theta(c) \otimes \rho'(f \otimes g)
\end{array} \quad (4.7)$$

commutes for all objects $a, b, c \in \text{ob}(G)$ and morphisms $a \xrightarrow{g} b$ and $b \xrightarrow{f} c$.

Definition 4.4. Let G and H be monoidal categories with units denoted by I and $\rho, \rho' : G \rightrightarrows \mathcal{C}$ monoidal functors between them (with constraints χ and χ'). A monoidal natural transformation $\theta : \rho \rightarrow \rho'$ consists of the following data.

1. A natural transformation $\theta : \rho \rightarrow \rho'$.

This data is subject to the following axioms.

1. Strict unitality: $\theta(I) = 1_I$.
2. The diagram

$$\begin{array}{ccc}
\rho(f) \otimes \rho(g) & \xrightarrow{\theta(f) \otimes \theta(g)} & \rho'(f) \otimes \rho'(g) \\
\downarrow \chi(f,g) & & \downarrow \chi'(f,g) \\
\rho(f \otimes g) & \xrightarrow{\theta(f \otimes g)} & \rho'(f \otimes g)
\end{array} \quad (4.8)$$

commutes for all objects f, g of G .

Let G be a monoidal category, $\mathcal{C}$ a 2-category, $x \in \text{ob}(\mathcal{C})$ and denote the basepoint of $\mathbf{B}G$ by $\bullet$. Let $\rho, \rho' : G \rightrightarrows \Omega_x \mathcal{C}$ be monoidal functors and $\theta : \rho \rightarrow \rho'$ an invertible monoidal natural transformation between them. If ρ and ρ' are regarded as functors $\mathbf{B}G \rightrightarrows \mathcal{C}$ we can regard θ as a natural transformation $\tilde{\theta}$ between those with $\tilde{\theta}(\bullet) = 1_\bullet$. The 1-cells of θ become the constraints of $\tilde{\theta}$. Note that with $\rho(\bullet) = 1_\bullet$ (4.7) reduces to (4.8).

Conversely any $\theta : \rho \rightarrow \rho'$ in the delooped setting that satisfies $\theta(\bullet) = 1_\bullet$ can be canonically regarded as monoidal natural transformation $\tilde{\theta}$ between the monoidal counterparts of ρ and ρ' . This is done by using the constraints of θ as 1-cells for $\tilde{\theta}$. It is then automatically invertible since the constraints are. This is in accordance with Remark 2.5.

4.2 1-Degenerate Representations of Monoids

We are now trying to give classification theorems for certain special cases of representations. To warm up we consider one of the simplest cases that still contains some interesting information.

Lemma 4.5. *Let H be a monoidal category with monoidal unit I . Then both compositions on $H(I, I)$ agree and $H(I, I)$ with either one of them forms a commutative monoid.*

Proofs can be found for example in [4] or [7]. The reader is also invited to try it on his/her own since it makes for an interesting exercise.

Let G be a monoid. We can view it as a monoidal category that has only identity morphisms. Thus if $\mathcal{C}$ is a 2-category and $x \in \text{ob}(\mathcal{C})$ we can represent G on x . Let ρ be 1-degenerate representation of G on $x \in \text{ob}(\mathcal{C})$. Recall Definition 2.4 and Definition 4.2. Since ρ is 1-degenerate its underlying functor is trivial. What remains are the $\chi(f, g) : \rho(f) \otimes \rho(g) \rightarrow \rho(f \otimes g)$. With the underlying functor being trivial the naturality condition of χ becomes empty.

Furthermore all the cells $\chi(f, g)$ live in $\Omega_x^2 \mathcal{C}$ which by Lemma 4.5 is a commutative monoid (c.f. ??). Also the $\chi(f, g)$'s are all invertible and thus lie in the group of units

$$A := (\Omega_x^2 \mathcal{C})^\times \quad (4.9)$$

of $\Omega_x^2 \mathcal{C}$. We denote composition in A additively. We can identify ρ with a function $G \times G \rightarrow A$.

Exactly what functions $\chi \in C^1$ define a 1-degenerate representation of G on x ? The only additional properties χ has to satisfy is the strict unitality (4.2) and the associativity coherence (4.3) with $f, g, h \in G$. The first condition amounts to χ being *normed* as a function $G \times G \rightarrow A$, i.e. it satisfies $\chi(1, g) = 0 = \chi(f, 1)$ for all elements f, g of G . We denote the abelian group of all normed functions $G \times G \rightarrow A$ by $[G^2, A]_*$. More generally a function $G^n \rightarrow A$ is called normed if its value is 0 if any of its arguments is the identity. Such functions are denoted by $[G^n, A]_*$.

By Lemma 4.5 both compositions of A agree. Furthermore 1_{1_x} is the unit of A . Thus (4.3) reduces to

$$\chi(f, g) + \chi(fg, h) = \chi(g, h) + \chi(f, gh) \quad (4.10)$$

or equivalently

$$\chi(g, h) - \chi(fg, h) + \chi(f, gh) - \chi(f, g) = 0. \quad (4.11)$$

But this is exactly the 2-cocycle condition for the normed cochain complex of G with coefficients in A .

Let G be a monoid and $\mathcal{C}$ a 2-category. Denote the normed cochain complex of G with coefficients in $A := \Omega_x^2 \mathcal{C}$ by $C_\bullet^\bullet(G; A) = (C^n, d^n)_{n \in \mathbb{Z}}$. Here $C^n = [G^n, A]_*$ and $d^n : C^n \rightarrow C^{n+1}$ is given by the usual formula.

Lemma 4.6. *There is a bijection between 1-degenerate representations of G on $x \in \mathcal{C}$ and the 2-cocycles $Z_*^2(G; A)$ of $C_*^\bullet(G; A)$.*

However $\text{Rep}_x^1(G)$ is not just a set but a category. So let us also take its morphisms (= monoidal natural transformations) into account. Consider two representations $\rho, \rho' \in \text{Rep}_x^1(G)$ and a monoidal isomorphism $\theta : \rho \longrightarrow \rho'$.

Similarly as above we can view θ as a normed function $G \longrightarrow A$. Now (4.8) reduces to

$$\chi'(f, g) + \theta(g) + \theta(f) = \theta(fg) + \chi(f, g) \quad (4.12)$$

or equivalently

$$\chi(f, g) - \chi'(f, g) = \theta(g) - \theta(fg) + \theta(f) = d^1\theta(f, g). \quad (4.13)$$

So ρ is isomorphic to ρ' in $\text{Rep}_x^1(G)$ if and only if they differ by a coboundary as elements of $Z_*^2(G; A)$. We record this as a proposition.

Let G be a monoid and $\mathcal{C}$ a 2-category. Denote the n -th normed cohomology group of G with coefficients in $A := \Omega_x^2 \mathcal{C}$ by $H_*^n(G; A)$.

Proposition 4.7. *There is a bijection between isomorphism classes of $\text{Rep}_x^1(G)$ and $H_*^2(G; A)$.*

The above statement can be refined by not only keeping track of which representations are isomorphic but also *how* they are.

Consider a set X together with an equivalence relation R on X . R tells us if two elements of X are equivalent or not but contains no information on how they relate to each other. The presence of R allows us to build the quotient set X/R which consists of equivalence classes.

On the other hand if we have a quiver with vertex set X then the arrows give us more information on how points relate to each other. We can obtain a relation on X by setting $x \sim y$ iff there exists an arrow $x \longrightarrow y$, $x, y \in X$. If we want this relation to be symmetric every arrow should come with a dual that goes in the other direction. If we want it to be reflexive every x should be equipped with a distinguished arrow $x \longrightarrow x$ and if we want it to be transitive we should have some notion of composition. A groupoid with object set X clearly does the job.

If X is not only a set but instead a monoid then the relation R should be compatible with the monoid multiplication and with units so that we can form a quotient object. If X is even a group then such an R automatically respects inverses.

Let now C, D be two abelian groups together with a morphism $C \xrightarrow{d} D$. Instead of immediately taking the quotient group D/d we first build an intermediate 2-group $\mathcal{G}(d)$ from this data in the following way.

1. The object set of $\mathcal{G}(d)$ is D .
2. The morphism set of $\mathcal{G}(d)$ is given by $\tilde{C}$. An element of $\tilde{C}$ is a pair (g, x) where $g \in C$ and $x \in D$.
3. The source and target maps s, t are defined by

$$s(g, x) = x, \quad t(g, x) = x - d(x) \quad (4.14)$$

so that $\mathcal{G}(d)(x, y) = \{g \in C : d(g) = x - y\}$.

4. Composition of morphisms is given by

$$(h, y) \circ (g, x) := (h + g, x). \quad (4.15)$$

5. The monoidal product is given by

$$(g, x) \otimes (h, y) := (g + h, x + y). \quad (4.16)$$

It is not hard to check that this indeed gives a 2-group which even has strict inverses and is strictly abelian.

Lemma 4.8. *The Grothendieck group $[\mathcal{G}(d)]$ of $\mathcal{G}(d)$ is isomorphic to D/d .*

Proof. This holds basically by construction. The objects D of $\mathcal{G}(d)$ form themselves a strict group under the monoidal product and the projection $\pi : D \rightarrow [\mathcal{G}(d)]$ is a group homomorphism. Denote the projection $D \rightarrow D/d$ by π' . We have

$$\begin{aligned} \pi'(x) = 0 &\iff x = d(g) \text{ for some } g \in C \\ &\iff x \xrightarrow{\cong} 0 \text{ in } \mathcal{G}(d) \text{ (via } (g, x)) \iff \pi(x) = 0 \end{aligned} \quad (4.17)$$

and the claim follows. $\square$

Now we can state the refinement of Proposition 4.7. Let G be a monoid, $\mathcal{C}$ a 2-category, $x \in \text{ob}(\mathcal{C})$ and $A := \Omega_x^2 \mathcal{C}$. Consider the first differential $C^1 \xrightarrow{d^1} Z_*^2(G; A)$ of the normed cochain complex of G with coefficients in A . Furthermore let $\text{Rep}_x^1(G)^\times$ be the groupoid of invertible morphisms of $\text{Rep}_x^1(G)$.

Theorem 4.9. *There is an isomorphism(!) of categories $\text{Rep}_x^1(G)^\times \cong \mathcal{G}(d^1)$.*

Proof. We have already seen that we can identify 1-degenerate representations ρ of G on x with $Z_*^2(G; A)$ by mapping ρ to its constraints χ . This gives the isomorphism on objects. We have also seen that a monoidal isomorphism $\theta : \rho \rightarrow \rho'$ amounts to a 1-cochain with $d^1(\theta) = \chi - \chi'$ so we can define it on morphisms by this correspondence. That it is indeed functorial and an isomorphism is easy to see. $\square$

Corollary 4.10. *We can endow $\text{Rep}_x^1(G)$ with the structure of a strict abelian 2-group and we can endow equivalence classes of representations with the structure of a group. The resulting group is isomorphic to $H_*^2(G, A)$ and to $[\mathcal{G}(d^1)]$.*

Proof. This immediately follows from Theorem 4.9, Proposition 4.7 and Lemma 4.8. $\square$

5 Categorifying Quotient constructions

Our goal in this section is to give a rigorous notion of free categories and quotient categories. Here we will do this in a strict way in the sense that our constructions will satisfy strict universal properties. Using these we will then be able to construct ‘the’ 2-cokernel (see subsection 6.3) of a morphism of abelian 2-groups (see section 3) in subsection 7.2.

The free category over a quiver is a very well known standard construction. Quotients of categories probably already appear somewhere in the literature, however the author is not aware of any such exposition.

The main difficulty when constructing quotients of categories is a new phenomenon that arises in the categorified case, best illustrated by an example. Consider the following category with exactly two non trivial morphisms:

$$\bullet \longleftarrow \bullet \qquad \bullet \longleftarrow \bullet \tag{5.1}$$

If we identify the two middle points we should end up with something like this:

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \tag{5.2}$$

Now there is no canonical choice for the composition of the two arrows since they are not composable in the original category. To remedy this problem we build the free category on the quiver in (5.2), respecting all given relations.

We won’t have to identify objects when constructing 2-cokernels but for the sake of completeness and because of the lack of literature on this subject (that the author knows of) we deal with the general case here.

5.1 Categories from Generators and Relations

5.1.1 Quivers and Words

Let $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ be a quiver with object set Γ_0 , arrow set Γ_1 and source and target maps $s, t : \Gamma_1 \rightrightarrows \Gamma_0$. For $a, b \in \Gamma_0$ we write $\Gamma(a, b)$ for the set of arrows with source a and target b .

For any set X a word in X is a finite list of elements of X . By $\mathcal{W}^*(X)$ we denote the set of all non-empty words in X . The concatenation of two words w and w' is denoted by ww' . We define $\mathcal{W}(\Gamma) = \mathcal{W}^*(\Gamma_1 \cup \Gamma_0)$ and

think of the elements of $\Gamma_0 \subseteq \mathcal{W}(\Gamma)$ as formal units. If viewed in this way we may also write 1_a for a .

We extend s and t to maps $\mathcal{W}(\Gamma) \rightrightarrows \Gamma_0$ in two steps.

1. For $a \in \Gamma_0$ we set $s(1_a) = t(1_a) = a$.
2. For $w = (w_n \dots w_1) \in \mathcal{W}(\Gamma)$ we set $s(w) = s(w_1)$ and $t(w) = t(w_n)$.

For $a, b \in \Gamma_0$, $\mathcal{W}(\Gamma)(a, b)$ denotes the subset of $\mathcal{W}(\Gamma)$ given by words with source a and target b . A word $w = (w_n \dots w_1)$ is called *well composed* if $t(w_i) = s(w_{i+1})$ for all i . The subset $\mathcal{W}_c(\Gamma) \subseteq \mathcal{W}(\Gamma)$ is given by all well composed words.

A map of quivers $\varphi : \Gamma \rightarrow \Delta$ is a pair of functions $\varphi_0 : \Gamma_0 \rightarrow \Delta_0$ and $\varphi_1 : \Gamma_1 \rightarrow \Delta_1$ that respects the source and target maps. Quivers together with maps of quivers form a category $\mathbf{Quiv}$. Any such φ canonically extends to maps $\varphi : \mathcal{W}(\Gamma) \rightarrow \mathcal{W}(\Delta)$ and $\varphi : \mathcal{W}_c(\Gamma) \rightarrow \mathcal{W}_c(\Delta)$ by applying it componentwise to words.

Remark 5.1. Note that in particular we have $\varphi(1_a) = 1_{\varphi_0(a)}$ for $a \in \Gamma_0$. Also note that the extension to well composed words is well defined since φ respects sources and targets.

5.1.2 Relations on Quivers

Definitions 5.2. A *quiver with (well composed) relations* (Γ, R) is a quiver Γ together with a relation R on $\mathcal{W}(\Gamma)$ (resp. $\mathcal{W}_c(\Gamma)$). Given a quiver with well composed relations, R can have the following properties:

- It is called *compatible* if whenever $w \sim v$ then $s(w) = s(v)$ and $t(w) = t(v)$.
- It is called *unital* if it contains all possible relations of the form $w \sim w1_a$ and $w \sim 1_a w$, for $a \in \Gamma_0$.
- It is called *closed* if $w \sim v$ and $w' \sim v'$ imply $ww' \sim vv'$ whenever $t(w') = s(w)$ and $t(v') = s(v)$.
- It is called *complete* if it is a compatible unital and closed equivalence relation.

A map $\varphi : (\Gamma, R) \rightarrow (\Delta, S)$ of quivers with relations is map of the underlying quivers that respects the given relations, i.e. whenever $w \sim_R v$ then $\varphi(w) \sim_S \varphi(v)$. We obtain the category $\mathbf{RelQuiv}$ of quivers with relations. Inside $\mathbf{RelQuiv}$ sits the full subcategory of quivers with complete relations $\mathbf{Rel}_c\mathbf{Quiv}$.

We will need the following lemma in the proof of Lemma 5.11.

Lemma 5.3. *Let $\varphi : (\Gamma, R) \longrightarrow (\Delta, S)$ be a map of quivers with relations where S is a unital and closed equivalence relation. Define a relation Φ on $\mathcal{W}_c(\Gamma)$ by $w \sim_\Phi w' \iff \varphi(w) \sim_S \varphi(w')$. Then Φ is a unital and closed equivalence relation.*

If T is any relation on Γ then φ respects T if and only if $T \subseteq \Phi$.

Remark 5.4. Φ is a weakened version of the *fiber relation* of φ which relates two elements if and only if they are equal under φ . Here equality is replaced by S -equivalence.

Proof. Clearly Φ is an equivalence relation. Let $w \in \mathcal{W}_c(\Gamma)$ with $s(w) = a$. Then $\varphi(w1_a) = \varphi(w)\varphi(1_a) = \varphi(w)1_{\varphi(a)} \sim_S \varphi(w)$, i.e. $w1_a \sim_\Phi w$. The first and second equality hold by the definition of the extension of φ to $\mathcal{W}_c(\Gamma)$, the relation because S is unital. The same argument works if $t(w) = a$ and thus Φ is unital. Similarly it follows that Φ is closed. The last statement is just a reformulation of the definition of Φ . $\square$

5.1.3 Completion of Relations

Our next goal is given any quiver with relations (Γ, R) to build its completion $(\bar{\Gamma}, \bar{R})$ where $\bar{R}$ is a complete relation on $\mathcal{W}_c(\bar{\Gamma})$. We proceed in four steps.

Step 1 There is an induced equivalence relation R_0 on Γ_0 generated by the following relations: Whenever $(w_n \dots w_1) \sim_R (v_m \dots v_1)$ then

- $s(w_1) \sim_{R_0} s(v_1)$,
- $t(w_n) \sim_{R_0} t(v_m)$,
- $t(w_i) \sim_{R_0} s(w_{i+1})$ for all i .
- $t(v_i) \sim_{R_0} s(v_{i+1})$ for all i .

We obtain a new quiver $\bar{\Gamma}$ by quotienting out R_0 :

- $\bar{\Gamma}_0$ is given by Γ_0/R_0 .
- $\bar{\Gamma}_1$ is given by all the words $[w]$ where $w \in \Gamma_1$ and $[w]$ is obtained from w by replacing all occurrences of formal units with their respective equivalence classes.
- The new source map is defined as $s([w]) = \overline{s(w)}$ where $\overline{s(w)}$ denotes the equivalence class of $s(w)$ under R_0 . The target is defined analogously. Note that they are well defined.

R canonically induces a set of relations R' on $\bar{\Gamma}$: Let R' be given by all pairs $([w], [v])$, such that $(w, v) \in R$.

Lemma 5.5. *R' is a compatible set of relations on $\mathcal{W}_c(\bar{\Gamma})$.*

Proof. This holds by construction. $\square$

Step 2 We set $R^* = R' \cup \{(w, w1_a), (w, 1_bw) \mid w \in \mathcal{W}_c\bar{\Gamma}, a, b \in \bar{\Gamma}_0, s(w) = a, t(w) = b\}$.

Lemma 5.6. R^* is a unital and compatible relation on $\mathcal{W}_c\bar{\Gamma}$.

Proof. The unitality part is clear from the construction. That all the words related by R^* are well composed and that their sources and targets match also follows directly from the definition of R^* . $\square$

Step 3 We set $\tilde{R} = (R^*)_{eq}$, the equivalence closure of R^* in $\mathcal{W}_c\bar{\Gamma}$.

Lemma 5.7. $\tilde{R}$ is a unital and compatible equivalence relation on $\mathcal{W}_c\bar{\Gamma}$.

Proof. It is an equivalence relation on $\mathcal{W}_c\bar{\Gamma}$ by construction. Unitality is preserved since $R^* \subseteq \tilde{R}$. To see that compatibility is preserved note that we pass from R^* to $\tilde{R}$ in three steps:

1. We add all relations of the form $w \sim w$.
2. We add all relations of the form $w' \sim w$ when $w \sim_{R^*} w'$ is given.
3. We add all relations of the form $w \sim w'$ when there exist $w = w_1, w_2, \dots, w_n = w'$ such that $w_i \sim_{R^*} w_{i+1}$ for all i .

Each of these steps clearly preserves compatibility. $\square$

Step 4 We set $\bar{R} = \{(w_n \dots w_1, w'_n \dots w'_1) \mid n \in \mathbb{N}, (w_i, w'_i) \in \tilde{R}, t(w_i) = t(w'_i) = s(w_{i+1}) = s(w'_{i+1}) \forall i\}$.

Lemma 5.8. $\bar{R}$ is a complete relation on $\mathcal{W}_c\bar{\Gamma}$.

Proof. Closedness follows by construction. Composability of all the words related by $\bar{R}$ follows directly from the definition. Compatibility is also clear. Unitality again follows from $\tilde{R} \subseteq \bar{R}$. Reflexivity follows by taking $w_i = w'_i$. This is possible since $\tilde{R}$ is reflexive. Similarly we see that $\bar{R}$ is symmetric and transitive. $\square$

Remark 5.9. Note that the order in which the above steps are taken is important as otherwise we would have to apply certain steps more than once.

Remark 5.10. The first step differs from the others in that it changes the underlying quiver. Given (Γ, R) with R compatible we have $\bar{\Gamma} = \Gamma$ and $\bar{R}$ is the smallest unital and closed equivalence relation that contains R . With this it follows immediately that $\bar{\bar{\Gamma}} = \bar{\Gamma}$ and $\bar{\bar{R}} = \bar{R}$. Thus the name “completion” is justified.

5.1.4 Completion as a Free Functor

A general principle is that free functors are left adjoint to forgetful functors. This also holds true here.

First we have to make the process of completion into a functor. Let $\varphi : (\Gamma, R) \longrightarrow (\Delta, S)$ be a map of quivers with relations. We define a map $\overline{\varphi} : (\overline{\Gamma}, \overline{R}) \longrightarrow (\overline{\Delta}, \overline{S})$ as follows.

- For $a \in \Gamma_0$ we let $\overline{a}$ be its equivalence class in $\overline{\Gamma}_0$ (and similarly for Δ). We now set $\overline{\varphi}_0(\overline{a}) = \overline{\varphi_0(a)}$.
- On arrows we set $\overline{\varphi}_1(f) = \overline{\varphi_1(f)}$.

Lemma 5.11. *This construction gives a well defined map of quivers with complete relations. We obtain a functor*

$$P : \text{RelQuiv} \longrightarrow \text{Rel}_c\text{Quiv}$$

that sends a quiver with relations to its completion.

Proof. $\overline{\varphi}_0$ is well defined: Given b such that $\overline{a} = \overline{b}$ we have to be in one of the four possible situations at the beginning of Step 1. Since φ respects relations the same holds true in Δ for $\varphi(a)$ and $\varphi(b)$ and thus they are equal in $\overline{\Delta}_0$.

$\overline{\varphi}_1$ is well defined since $\Gamma_1 = \overline{\Gamma}_1$. To see that it respects sources and targets note that

$$s(\overline{\varphi}_1(f)) = \overline{s(\varphi_1(f))} = \overline{\varphi_0(s(f))} = \overline{\varphi}_0(s(f))$$

where sources are to be taken in the appropriate quivers. The same argument works for the target map.

$\overline{\varphi}$ respects the relation $\overline{R}$: Recall the definition of the relation R' from Step 1. By Remark 5.10 $\overline{R}$ is the smallest unital and closed equivalence relation that contains R' . By Lemma 5.3 it therefore suffices to show that $\overline{\varphi}$ respects R' . But this is certainly true since φ respects R .

Functoriality is immediate from the definition. $\square$

Our second ingredient is the forgetful functor

$$V : \text{Rel}_c\text{Quiv} \longrightarrow \text{RelQuiv} \tag{5.3}$$

. Finally there is a canonical projection map $\pi_{(\Gamma, R)} : (\Gamma, R) \longrightarrow (\overline{\Gamma}, \overline{R})$ given on objects by $a \longmapsto \overline{a}$ and on arrows by $f \longmapsto f$. Obviously π respects the relations.

Proposition 5.12. *(P, V, π, id) is an adjunction with unit π and counit id .*

Proof. By Remark 5.10 applying P to a quiver with complete relation does nothing so PV is the identity functor 1 . Therefore $\text{id} : PV \rightarrow 1$ is a natural transformation. It is also straightforward to check that π is a natural transformation $1 \rightarrow VP$. The snake identities reduce to $P\pi = 1_P$ and $\pi_V = 1_V$ which hold trivially. $\square$

5.1.5 The Free Category on a Quiver with Relations

After these preparations we are now ready to tackle the construction of free categories on some given data.

Let (Γ, R) be a quiver with relations with completion $(\bar{\Gamma}, \bar{R})$. The *free category* $\mathcal{F}(\Gamma, R)$ on (Γ, R) is constructed as follows. Note that $\bar{R}$ restricts to equivalence relations $\bar{R}_{a,b}$ on all the sets $\mathcal{W}_c(\bar{\Gamma})(a, b)$, $a, b \in \bar{\Gamma}_0$.

- We set $\text{obj}(\mathcal{F}(\Gamma, R)) = \bar{\Gamma}_0$.
- For $a, b \in \bar{\Gamma}_0$ we set $\mathcal{F}(\Gamma, R)(a, b) = \mathcal{W}_c(\bar{\Gamma})(a, b) / \bar{R}_{a,b}$.
- Composition is concatenation modulo $\bar{R}$.
- Units are given by equivalence classes of formal units.

Proposition 5.13. *This construction gives a well defined category $\mathcal{F}(\Gamma, R)$.*

Proof. That composition is well defined follows from closedness of $\bar{R}$. It is associative since concatenation is. That the units are unital with respect to composition follows from unitality of $\bar{R}$. $\square$

We now show that taking the free category defines a functor

$$\mathcal{F} : \text{RelQuiv} \rightarrow \text{Cat}. \quad (5.4)$$

Since we already know that passing from a quiver with relations to its completion is functorial we can assume it to be complete. Thus let $\varphi : (\Gamma, R) \rightarrow (\Delta, S)$ be a map of quivers with complete relations. Define $\mathcal{F}\varphi$ as follows.

- Since $\text{obj}(\mathcal{F}(\Gamma, R)) = \Gamma_0$ (and similarly for Δ) we can set $\mathcal{F}\varphi(a) = \varphi_0(a)$ on objects a .
- Any morphism in $\mathcal{F}(\Gamma, R)$ is given by an equivalence class $\bar{w}$ where $w \in \mathcal{W}_c(\Gamma)$. We set $\mathcal{F}\varphi(\bar{w}) = \bar{\varphi}_1(w)$.

Lemma 5.14. *This construction gives a well defined functor*

$$\mathcal{F}\varphi : \mathcal{F}(\Gamma, R) \rightarrow \mathcal{F}(\Delta, S). \quad (5.5)$$

Altogether we obtain a functor $\mathcal{F} : \text{Rel}_c\text{Quiv} \rightarrow \text{Cat}$ where Cat stands for the (1-)category of (small) categories and functors.

Proof. Clearly $\mathcal{F}\varphi$ is well defined on objects. On morphisms this follows since φ respects the relations. Functoriality is easy to see. $\square$

5.1.6 The Free Category as a Free Functor

We have a forgetful functor

$$U : \mathbf{Cat} \longrightarrow \mathbf{Rel}_c\mathbf{Quiv} \quad (5.6)$$

sending a category $\mathcal{C}$ to a quiver Γ together with relations $R_{\mathcal{C}}$ given as follows.

- The vertices of Γ are the objects of $\mathcal{C}$.
- for $a \neq b \in \Gamma_0$, $\Gamma(a, b)$ is given by $\text{Hom}_{\mathcal{C}}(a, b)$. $\Gamma(a, a)$ is given by $\text{Hom}_{\mathcal{C}}(a, a) - \{1_a\}$.
- $(w_n \dots w_1) \sim (w'_m \dots w'_1)$ whenever $w_n \circ \dots \circ w_1 = w'_m \circ \dots \circ w'_1$ in $\mathcal{C}$. On the right hand side formal units are replaced with actual ones in $\mathcal{C}$.

Note that $R_{\mathcal{C}}$ is complete. U sends functors to their underlying maps of quivers. These clearly respect the relations $R_{\mathcal{C}}$.

Our goal now is to describe the “free category functor” as left adjoint to U .

There are canonical maps $\eta_{(\Gamma, R)}$ given by

$$\begin{aligned} \eta_{(\Gamma, R)} : (\Gamma, R) &\longrightarrow \mathbf{UF}(\Gamma, R) \\ f &\longmapsto [f] \end{aligned}$$

on arrows and by the identity on objects. Here $[f]$ denotes the equivalence class of the word (f) with a single letter f .

Furthermore we have maps $\varepsilon_{\mathcal{C}}$ given by

$$\begin{aligned} \varepsilon_{\mathcal{C}} : \mathbf{FUC} &\longrightarrow \mathcal{C} \\ [f_n \dots f_1] &\longmapsto f_n \circ \dots \circ f_1 \end{aligned}$$

on morphisms and by the identity on objects.

Lemma 5.15. *ε is well defined and an isomorphism. Both η and ε are natural.*

Proof. Clearly ε is an isomorphism. Let $[f_n \dots f_1] = [f'_m \dots f'_1]$. This translates to $(f_n \dots f_1) \sim_{R_{\mathcal{C}}} (f'_m \dots f'_1)$ in $\mathbf{UC}$ which means precisely that $f_n \circ \dots \circ f_1 = f'_m \circ \dots \circ f'_1$ in $\mathcal{C}$. The inverse is given by $f \longmapsto [f]$.

The naturality statement is straightforward to check. $\square$

Proposition 5.16. *$(\mathcal{F}, U, \eta, \varepsilon)$ is an adjunction with unit η and counit ε .*

Proof. Only the snake identities are left to check. For any $\mathcal{C}$ and f a morphism in $\mathcal{C}$ we have

$$f \xrightarrow{\eta_{\mathcal{U}\mathcal{C}}} [f] \xrightarrow{\mathcal{U}\varepsilon_{\mathcal{C}}} f.$$

This shows the first snake identity. Similarly for any (Γ, R) and $[f_n \dots f_1]$ a morphism in $\mathcal{F}(\Gamma, R)$ we have

$$[f_n \dots f_1] \xrightarrow{\mathcal{F}\eta_{(\Gamma, R)}} [[f_n \dots f_1]] \xrightarrow{\varepsilon_{\mathcal{F}(\Gamma, R)}} [f_n \dots f_1],$$

which shows the second one. $\square$

Corollary 5.17. *By combining Proposition 5.16 with Proposition 5.12 we obtain an adjunction*

$$(\mathcal{F} : \text{RelQuiv} \longrightarrow \text{Cat}) \dashv (\mathcal{U} : \text{Cat} \longrightarrow \text{RelQuiv}).$$

5.2 Quotients of Categories by Relations

5.2.1 Categories with relations

Definition 5.18. A *category with relations* $(\mathcal{C}, R)$ is a category $\mathcal{C}$ together with a relation R on $\text{Hom}(\mathcal{C})$, the set of morphisms of $\mathcal{C}$. A map of categories with relations $\phi : (\mathcal{C}, R) \longrightarrow (\mathcal{D}, S)$ is a functor that additionally satisfies $\phi f \sim_S \phi g$ whenever $f \sim_R g$. We obtain the category RelCat of (small) categories with relations and morphisms between them.

We have a forgetful functor

$$\mathcal{U}' : \text{RelCat} \longrightarrow \text{RelQuiv}$$

sending $(\mathcal{C}, R)$ to its underlying quiver (minus units) together with the relation $R_{\mathcal{C}} \cup R$.

Also there is a “forgetful” functor

$$\mathcal{J} : \text{Cat} \longrightarrow \text{RelCat}$$

that just sends $\mathcal{C}$ to $(\mathcal{C}, \emptyset)$.

5.2.2 Universal Property of Quotients

Informally speaking, given $(\mathcal{C}, R)$ the quotient $\mathcal{C}/R$ should come with a map $\pi : \mathcal{C} \longrightarrow \mathcal{C}/R$ that satisfies the following universal property: For any $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ such that $f \sim g$ implies $\phi f = \phi g$ there is a unique $\bar{\phi} : \mathcal{C}/R \longrightarrow \mathcal{D}$ with $\bar{\phi}\pi = \phi$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi} & \mathcal{C}/R \\ \phi \downarrow & \swarrow \exists! \bar{\phi} & \\ \mathcal{D} & & \end{array} \quad (5.7)$$

Here a functor $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ with the above property is equivalently a map of categories with relations $(\mathcal{C}, R) \longrightarrow (\mathcal{D}, \emptyset) = J\mathcal{D}$. Of course taking quotients should be functorial, let's denote this functor by $\mathcal{E}$. Now π can be seen as a map $(\mathcal{C}, R) \longrightarrow J\mathcal{E}(\mathcal{C}, R)$. Furthermore from this viewpoint $J\bar{\phi}$ completes the above diagram.

In other words: *Taking quotients should be left adjoint to J with the unit of the adjunction given by π .*

5.2.3 Construction

Recall the “free category functor” $\mathcal{F}$ from Corollary 5.17 and set $\mathcal{E} = \mathcal{F}U'$.

Theorem 5.19. *There are natural transformations $\pi : 1 \longrightarrow J\mathcal{E}$ and $\varepsilon : \mathcal{E}J \longrightarrow 1$ such that $(\mathcal{E}, J, \pi, \varepsilon)$ is an adjunction with unit π and counit ε . The latter is an isomorphism.*

So $\mathcal{E}$ is our quotient functor. This means that the quotient of $(\mathcal{C}, R)$ is obtained by first taking the underlying quiver of $\mathcal{C}$, secondly adding relations induced from R and the category structure of $\mathcal{C}$ and finally taking the free category over the resulting quiver with relations.

Proof. We begin by constructing ε . We have $\mathcal{E}J = \mathcal{E}\mathcal{F}U' = \mathcal{F}U$. This suggests using the counit from the adjunction in Proposition 5.16. So we again define ε by

$$\begin{aligned} \varepsilon_{\mathcal{C}} : \mathcal{F}U\mathcal{C} &\longrightarrow \mathcal{C} \\ [f_n \dots f_1] &\longmapsto f_n \circ \dots \circ f_1 \end{aligned}$$

(and to be the identity on objects). It was already proven in Proposition 5.16 that this is a well defined natural isomorphism.

Let's denote the category $\mathcal{E}(\mathcal{C}, R)$ by $\mathcal{C}/R$. We then define π as follows.

$$\begin{aligned} \pi_{(\mathcal{C}, R)} : (\mathcal{C}, R) &\longrightarrow (\mathcal{C}/R, \emptyset) \\ a &\longmapsto \bar{a} \\ f &\longmapsto [f] \end{aligned}$$

maps both objects a and morphisms f to their equivalence classes. That $\pi_{(\mathcal{C}, R)}$ is a functor follows from the fact that the relation on $U'(\mathcal{C}, R)$ contains $R_{\mathcal{C}}$. If $f \sim_R g$ in $\mathcal{C}$ then $f = g$ in $\mathcal{C}/R$ so π is a map of categories with relations.

The snake identities follow as in Proposition 5.16. □

6 2-Universal Objects

We will now give a formal frame to categorify the concept of kernels and quotients of groups to 2-groups. Rather than giving a general definition

of categorified (co-)limits we will develop only the parts of the theory that are relevant to us. Furthermore we will only consider 2-categories (i.e strict ones). Due to the coherence theorem for bicategories this is not really a restriction.

So let $\mathcal{B}$ be a 2-category. Horizontal composition will be denoted by $\otimes$. If it causes no confusion we may denote both vertical and horizontal composition just as concatenation. We also may identify any 1-cell f notationally with its identity i.e. we may just write f for 1_f .

6.1 2-Zero Objects

Definition 6.1. An object t of $\mathcal{B}$ is called *2-terminal* if for every object x in $\mathcal{B}$ there exists a morphism $x \rightarrow t$ and for any two such maps there exists a unique isomorphism between them.

A *2-initial object* is defined dually.

A *2-zero object* is one that is both 2-terminal and 2-initial.

Lemma 6.2. Denote by $\underline{\mathcal{B}}$ the category formed by the objects and 1-morphisms of $\mathcal{B}$. Any object that is terminal (initial) in $\mathcal{B}$ (in the sense that it is terminal (initial) in $\underline{\mathcal{B}}$) is 2-terminal (2-initial) in $\mathcal{B}$. In particular this also holds true for zero-objects.

Proof. This is immediately clear. □

Remark 6.3. One could give an accordingly weakened uniqueness theorem for 2-zero objects and so on. However the categories we are interested in will have true zero objects in the sense of ordinary category theory so we won't need the weakened version. They should rather give the reader a taste of what is to come.

Note that as in ordinary category theory in the presence of a strict zero object in $\mathcal{B}$ for any two objects x, y there is the *zero map* $x \rightarrow y$ given by factoring through the zero object.

We will denote both zero objects and zero maps by 0.

6.2 2-Kernels

6.2.1 Coherent 2-Kernels

We now provide a generalized version of *2-kernels*.

Definition 6.4. Let the 2-category $\mathcal{B}$ have a zero object. Consider morphisms $X \xrightarrow{d} Y \xrightarrow{e} Z$ together with an isomorphism $\kappa : ed \rightarrow 0$. Then $K \xrightarrow{\iota} X$ is called *coherent 2-kernel* or *bikernel of d with respect to (e, κ)* and denoted by $\ker_e d$ if it satisfies the following properties.

1. There is a *coherent* isomorphism $\omega : d\iota \xrightarrow{\cong} 0$ i.e. one such that $e\omega = \kappa\iota$.
2. Whenever there is some $A \xrightarrow{f} X$ together with a coherent isomorphism $\theta : df \xrightarrow{\cong} 0$ (coherence for θ as in 1.) then there exists $A \xrightarrow{\Phi} K$ together with a coherent isomorphism $\phi : \iota\Phi \xrightarrow{\cong} f$ and the pair (Φ, ϕ) is *essentially unique*.

$$\begin{array}{ccccc}
 K & \xrightarrow{\iota} & X & \xrightarrow{d} & Y & \xrightarrow{e} & Z \\
 & \nwarrow \Phi & \uparrow f & & & & \\
 & & A & & & &
 \end{array} \tag{6.1}$$

Here ϕ being coherent means that the diagram

$$\begin{array}{ccc}
 d\iota\Phi & \xrightarrow{d\phi} & df \\
 & \searrow \omega\Phi & \downarrow \theta \\
 & & 0
 \end{array} \tag{6.2}$$

commutes. Essential uniqueness of the pair (Φ, ϕ) means that if there is another pair (Ψ, ψ) satisfying the same conditions as (Φ, ϕ) then there exists a unique coherent isomorphism $\sigma : \Phi \xrightarrow{\cong} \Psi$. Written out the condition on (Ψ, ψ) reads

$$\begin{array}{ccc}
 d\iota\Psi & \xrightarrow{d\psi} & df \\
 & \searrow \omega\Psi & \downarrow \theta \\
 & & 0
 \end{array} . \tag{6.3}$$

The coherence for σ is given by

$$\begin{array}{ccc}
 \iota\Phi & \xrightarrow{\iota\sigma} & \iota\Psi \\
 & \searrow \phi & \downarrow \psi \\
 & & f
 \end{array} . \tag{6.4}$$

Note that for $Z = 0$ (and thus also e and κ trivial) one gets back the usual notion of a 2-kernel.

Lemma 6.5. *Assume the situation of Definition 6.4. If $K \xrightarrow{\iota} X$ has the following properties then it is a coherent 2-kernel of d with respect to (e, κ) .*

1. *There is a coherent isomorphism $\omega : d\iota \xrightarrow{\cong} 0$.*

2. It is a coherent 2-monomorphism with respect to (d, ω) : Whenever there are $\Phi, \Psi : A \rightrightarrows K$ and a coherent $\tau : \iota\Phi \xrightarrow{\cong} \iota\Psi$ in the sense that

$$\begin{array}{ccc} d\iota\Phi & \xrightarrow{d\tau} & d\iota\Psi \\ & \searrow \omega\Phi & \downarrow \omega\Psi \\ & & 0 \end{array} \quad (6.5)$$

commutes then there exists a unique $\sigma : \Phi \xrightarrow{\cong} \Psi$ such that $\iota\sigma = \tau$.

3. Whenever there is some $A \xrightarrow{f} X$ together with a coherent isomorphism $\theta : df \xrightarrow{\cong} 0$ then there exists a unique $A \xrightarrow{\Phi} K$ such that $\iota\Phi = f$ and $\omega\Phi = \theta$.

Proof. Let f be as above. Then there exists a Φ such that $\iota\Phi = f$. Note that $\phi = \text{id}$ is coherent by the assumption $\omega\Phi = \theta$ and so the existence part is satisfied (with the pair (Φ, id)). Let now (Ψ, ψ) be given as in Definition 6.4. We have

$$\psi : \iota\Psi \xrightarrow{\cong} f = \iota\Phi. \quad (6.6)$$

By setting $\tau = \psi$ and inserting $\omega\Phi = \theta$, (6.5) reduces to (6.3) and we see that ψ is coherent. Thus since ι is a coherent 2-monomorphism with respect to (d, ω) we obtain $\sigma : \Phi \xrightarrow{\cong} \Psi$ with $\iota\sigma = \psi$. Note that (6.4) now also just reads $\iota\sigma = \psi$ and thus is fulfilled. From this we also see that σ is unique with this property. Finally note that if $\sigma : \Phi \xrightarrow{\cong} \Psi$ and $\sigma' : \Phi \xrightarrow{\cong} \Psi'$ are coherent then so is $\sigma'\sigma^{-1} : \Psi \xrightarrow{\cong} \Psi'$. $\square$

6.2.2 Coherent 2-Kernels in 2Grp

Recall the definition of 2Grp from Definition 3.9. This 2-category has a zero object given by the category with only one object and only one morphism (and the unique monoidal product on it).

Lemma 6.6. *Let X, Y be 2-groups and consider a monoidal functor $X \xrightarrow{d} Y$ and a strictly unital natural transformation $d \xrightarrow{\omega} 0$. Then for any object $x \in X$ we have $\omega(x \otimes i(x)) = d(\curvearrowright)$. Similar statements hold for the other cups and caps.*

Proof. Naturality and strict unitality of ω and strict unitality of d imply the commutativity of the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{0} & 0 \\ d(\curvearrowright) \uparrow & & \uparrow 0 \\ d(x \otimes i(x)) & \xrightarrow{\omega(x \otimes i(x))} & 0 \end{array} \quad (6.7)$$

which gives the statement. $\square$

Proposition 6.7. *2Grp has all coherent 2-kernels.*

Proof. Let d, e, κ be given as in Definition 6.4 and denote the constraint isomorphism of d by δ . We define a 2-group K as follows.

1. The objects of K are given by pairs (χ, ω_χ) where χ is an object of X and ω_χ is an isomorphism $d\chi \xrightarrow{\cong} 0$ such that $e\omega_\chi = \kappa(\chi)$. If χ is of the form $\chi = \eta \otimes i(\eta)$ or $\chi = i(\eta) \otimes \eta$ then we only allow $\omega_\chi = d(\curvearrowright)$. This condition comes from the fact that we tacitly assume everything to be strictly unital (c.f. Lemma 6.6).
2. A morphism $\alpha : (\chi, \omega_\chi) \longrightarrow (\eta, \omega_\eta)$ in K is a morphism $\alpha : \chi \longrightarrow \eta$ in X (by abuse of notation) such that

$$\begin{array}{ccc} d\chi & \xrightarrow{d\alpha} & d\eta \\ & \searrow \omega_\chi & \downarrow \omega_\eta \\ & & 0 \end{array} \quad (6.8)$$

commutes. Composition in K is given by composition in X .

3. The monoidal product is given by

$$(\chi, \omega_\chi) \otimes (\eta, \omega_\eta) := (\chi \otimes \eta, (\omega_\chi \otimes \omega_\eta) \circ \delta(\chi, \eta)^{-1}).$$

and the monoidal unit by $(0, 0)$.

4. Evaluation and coevaluation for the duals are given by the respective morphisms in X .

We now show that the duals are well defined and that K indeed is a 2-group.

That it is a category is clear. To see that it is monoidal we have to check that the above product is strictly unital and associative (recall our assumption that all monoidal categories are strict). This follows from the coherence of δ and the fact that X is strictly monoidal.

That the evaluation and coevaluation defined above are indeed morphisms in K follows from the requirement on ω_χ if χ is of the form $\eta \otimes i(\eta)$ or $i(\eta) \otimes \eta$ and Lemma 6.6. That they satisfy the necessary equations is clear since they do so in X . Finally K is a groupoid since X is and it follows that K is a 2-group.

Next we define $K \xrightarrow{\iota} X$ by $(\chi, \omega_\chi) \longmapsto \chi$ on objects and analogously on morphisms. Note that this functor is strictly monoidal. The isomorphism $\omega : d\iota \xrightarrow{\cong} 0$ is given by $\omega(\chi, \omega_\chi) := \omega_\chi$.

To show that $K \xrightarrow{\iota} X$ together with ω is a coherent 2-kernel with respect to (e, κ) we verify that the conditions from Lemma 6.5 are satisfied.

1. Naturality of ω follows from the assumption on the morphisms of K . For monoidality of ω , after inserting objects (χ, ω_χ) and (η, ω_η) , we have to check that the diagram

$$\begin{array}{ccc}
d(\chi \otimes \eta) & \xrightarrow{(\omega_\chi \otimes \omega_\eta) \circ \delta(\chi, \eta)^{-1}} & 0 \\
\delta(\chi, \eta) \uparrow & & \uparrow = \\
d\chi \otimes d\eta & \xrightarrow{\omega_\chi \otimes \omega_\eta} & 0
\end{array} \quad (6.9)$$

commutes. But this holds by construction. That ω is coherent is immediate from the definition.

2. Let τ, Φ, Ψ be as in Lemma 6.5. For any object $a \in A$ set $\sigma(a) = \tau(a)$. Because of the coherence for τ all the components $\sigma(a)$ are actually morphism in K and thus we have a natural isomorphism $\sigma : \Phi \longrightarrow \Psi$. It is monoidal since τ is. Trivially it satisfies $\iota\sigma = \tau$. Thus ι is a coherent 2-monomorphism with respect to (d, ω) .
3. Let now (f, θ) be as in Lemma 6.5. Define Φ by $a \mapsto (f(a), \theta(a))$ on objects $a \in A$ and by $\nu \mapsto f(\nu)$ on morphisms. Naturality and coherence of θ imply that Φ is well defined on objects. That it is well defined on morphisms also follows from the naturality of θ . $\iota\Phi = f$ and $\omega\Phi = \theta$ hold by construction.

It is left to show that we can endow Φ with a suitable constraint isomorphism ϕ such that it becomes a monoidal functor. Denote the monoidal constraint of f by c and set $\phi(a, b) = c(a, b)$ for all objects $a, b \in A$. To see that ϕ is well defined we have to show that

$$\begin{array}{ccc}
d(f(a) \otimes f(b)) & \xrightarrow{\delta(f(a), f(b))^{-1}} & df(a) \otimes df(b) \\
dc(a, b) \downarrow & & \downarrow \theta(a) \otimes \theta(b) \\
df(a \otimes b) & \xrightarrow{\theta(a \otimes b)} & 0
\end{array} \quad (6.10)$$

or equivalently

$$\begin{array}{ccc}
& & df(a) \otimes df(b) \\
dc(a, b) \circ \delta(f(a), f(b)) \swarrow & & \downarrow \theta(a) \otimes \theta(b) \\
df(a \otimes b) & \xrightarrow{\theta(a \otimes b)} & 0
\end{array} \quad (6.11)$$

commutes. But the diagonal arrow is precisely the constraint isomorphism of df and thus the diagram commutes because θ is monoidal.

Finally we have to check that ϕ satisfies the coherence condition for constraints of monoidal functors. But this follows directly from the fact that c has this property.

This completes the proof. $\square$

6.3 2-Cokernels

6.3.1 Dualizing Definitions

2-Cokernels are defined dually to 2-kernels. We won't need the coherent variant here.

Definition 6.8. Let $\mathcal{B}$ be 2-category with a zero object. Consider morphisms $Y \xrightarrow{d} X$. Then $X \xrightarrow{\pi} Q$ is called *2-cokernel of d* if it satisfies the following properties.

1. There is an isomorphism $\omega : \pi d \xrightarrow{\cong} 0$.
2. Whenever there is some $X \xrightarrow{f} A$ together with an isomorphism $\theta : fd \xrightarrow{\cong} 0$ then there exists $Q \xrightarrow{\Phi} A$ together with a coherent isomorphism $\phi : \Phi\pi \xrightarrow{\cong} f$ and the pair (Φ, ϕ) is *essentially unique*.

$$\begin{array}{ccccc} Y & \xrightarrow{d} & X & \xrightarrow{\pi} & Q \\ & & f \downarrow & \nearrow \Phi & \\ & & A & & \end{array} \quad (6.12)$$

Here ϕ being coherent means that the diagram

$$\begin{array}{ccc} \Phi\pi d & \xrightarrow{\phi d} & fd \\ & \searrow \Phi\omega & \downarrow \theta \\ & & 0 \end{array} \quad (6.13)$$

commutes. Essential uniqueness of the pair (Φ, ϕ) means that if there is another pair (Ψ, ψ) satisfying the same conditions as (Φ, ϕ) then there exists a unique coherent isomorphism $\sigma : \Phi \xrightarrow{\cong} \Psi$. The coherence for σ is given by

$$\begin{array}{ccc} \Phi\pi & \xrightarrow{\sigma\pi} & \Psi\pi \\ & \searrow \phi & \downarrow \psi \\ & & f \end{array} . \quad (6.14)$$

By using Lemma 6.5 together with the duality principle we automatically obtain the following Lemma.

Lemma 6.9. *Assume the situation of Definition 6.8. If $X \xrightarrow{\pi} Q$ has the following properties then it is a 2-cokernel of d .*

1. *There is a coherent isomorphism $\omega : \pi d \xrightarrow{\cong} 0$.*

2. It is a coherent 2-epimorphism or bimonomorphism with respect to (d, ω) : Whenever there are $\Phi, \Psi : Q \rightrightarrows A$ and a coherent $\tau : \Phi\pi \xrightarrow{\cong} \Psi\pi$ in the sense that

$$\begin{array}{ccc} \Phi\pi d & \xrightarrow{\tau d} & \Psi\pi d \\ & \searrow \Phi\omega & \downarrow \Psi\omega \\ & & 0 \end{array} \quad (6.15)$$

commutes then there exists a unique $\sigma : \Phi \xrightarrow{\cong} \Psi$ such that $\sigma\pi = \tau$.

3. Whenever there is some $Y \xrightarrow{f} A$ together with a coherent isomorphism $\theta : fd \xrightarrow{\cong} 0$ then there exists a unique $Q \xrightarrow{\Phi} A$ such that $\Phi\pi = f$ and $\Phi\omega = \theta$.

6.3.2 2-Cokernels in 2Grp

Proposition 6.10. *Ab2Grp has all 2-cokernels.*

We will proof this by giving an explicit construction in subsection 7.2.

7 Double Categories

A nice and concise introduction to double categories with all the definitions we are going to use can be found in [2]. We will follow this paper concerning notation and conventions for double categories. The necessary definitions will be referenced when they appear.

7.1 From Morphisms of 2-Groups

Let G and H be two abelian 2-groups and $G \xrightarrow{d} H$ a symmetric monoidal functor between them. Denote the constraints of d by δ . We construct a symmetric monoidal double category (see [2, Definition 2.1, p. 2]) $\mathcal{G}(d)$ from this data as follows.

1. The object category $\mathbb{D}_0$ is given by H .
2. The morphism category $\mathbb{D}_1$ is given as follows. An object of $\mathbb{D}_1$ is a quadruple (θ, Π, x, y) where θ is an object of G , $x, y \in \text{ob}(H)$ and $d(\theta) \xrightarrow{\Pi} x - y$ is a morphism in H .

A morphism $(\theta, \Pi, x, y) \rightarrow (\theta', \Pi', x', y')$ is a triple (m, α, β) with $\theta \xrightarrow{m} \theta'$ in G and $x \xrightarrow{\alpha} x', y \xrightarrow{\beta} y'$ in H such that

$$\begin{array}{ccc} d\theta & \xrightarrow{\Pi} & x - y \\ dm \downarrow & & \downarrow \alpha - \beta \\ d\theta' & \xrightarrow{\Pi'} & x' - y' \end{array} \quad (7.1)$$

commutes. As an equality of string diagrams in G this reads

$$(7.2)$$

Composition in $\mathbb{D}_1$ is defined component wise:

$$(m', \alpha', \beta') \circ (m, \alpha, \beta) := (m' \circ m, \alpha' \circ \alpha, \beta' \circ \beta). \quad (7.3)$$

It follows from the interchange law in H that it is well defined. Identities are given by triples consisting only of identities.

3. The unit functor $U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ is given on objects by

$$U(x) := (0, \text{cup}, x, x) \quad (7.4)$$

and on morphisms by

$$U(\alpha) := (0, \alpha, \alpha). \quad (7.5)$$

To see that this is well defined use the naturality of the cups. Clearly U is a functor.

4. The source and target functors $S, T : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ are given on objects by

$$S(\theta, \Pi, x, y) = x, \quad T(\theta', \Pi, x, y) = y \quad (7.6)$$

and on morphisms by

$$S(m, \alpha, \beta) = \alpha, \quad T(m, \alpha, \beta) = \beta. \quad (7.7)$$

It's easy to see that these are indeed functors. Note that we have

$$S \circ U = 1_{\mathbb{D}_0} = T \circ U. \quad (7.8)$$

Instead of

$$x \xrightarrow{(\theta, \Pi, x, y)} y \quad (7.9)$$

we may also just unambiguously write

$$x \xrightarrow{(\theta, \Pi)} y. \quad (7.10)$$

5. Let $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ denote the (strict) pullback of $\mathbb{D}_1 \xrightarrow{T} \mathbb{D}_0 \xleftarrow{S} \mathbb{D}_1$. It is a subcategory of the product $\mathbb{D}_1 \times \mathbb{D}_1$.

The composition functor $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ is given as follows. On objects we set

$$(\theta, \Pi, y, z) \odot (\theta', \Pi', x, y) := (\theta + \theta', \Pi \odot \Pi', x, z) \quad (7.11)$$

where $\Pi \odot \Pi'$ is given by the composition

$$\begin{array}{ccc} d(\theta + \theta') \xrightarrow{\delta^{-1}(\theta, \theta')} d(\theta) + d(\theta') \xrightarrow{\Pi + \Pi'} y - z + x - y & & \\ & \searrow \Pi \odot \Pi' & \downarrow b(y-z, x-y) \\ & & x - y + y - z \\ & & \downarrow 1+ \quad \curvearrowright \quad +1 \\ & & x - z \end{array} \quad (7.12)$$

As a string diagram this reads

$$\Pi \odot \Pi' = \begin{array}{c} \begin{array}{c} \text{Diagram with two vertical lines crossing twice, forming a figure-eight shape. The top part is a loop. The bottom part has two dots labeled \Pi and \Pi'. Below these dots is a dot labeled \delta^{-1} with an arrow pointing up to it.} \end{array} \end{array} \quad (7.13)$$

On morphisms we define

$$(m, \beta, \gamma) \odot (m', \alpha, \beta) := (m + m', \alpha, \gamma). \quad (7.14)$$

Lemma 7.1. *The composition $\odot$ is indeed a functor and satisfies*

$$S(M \odot M') = S(M'), \quad T(M \odot M') = T(M) \quad (7.15)$$

for morphisms M, M' in $\mathbb{D}_1$ (and thus also for objects).

Proof. The statement about the source and target functors is clear.

To show that $\odot$ is a functor we have show that it respects the internal units, source and target maps and composition of $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ and $\mathbb{D}_1$. The part about identities is easily verified.

To see that it respects source and target consider morphisms $(m, \beta, \gamma) : (\theta, \Pi, y, z) \longrightarrow (\omega, \Lambda, y', z')$ and $(m', \alpha, \beta) : (\theta', \Pi', x, y) \longrightarrow (\omega', \Lambda', x', y')$.

We have to check that

$$\begin{array}{ccc}
d(\theta + \theta') & \xrightarrow{\Pi \odot \Pi'} & x - z \\
d(m+m') \downarrow & & \downarrow \alpha - \gamma \\
d(\omega + \omega') & \xrightarrow{\Lambda \odot \Lambda'} & x' - z'
\end{array} \quad (7.16)$$

commutes. Writing this out and inserting some arrows we get

$$\begin{array}{ccccccc}
d(\theta + \theta') & \xrightarrow{\delta^{-1}} & d\theta + d\theta & \xrightarrow{\Pi + \Pi'} & y - z + x - y & \longrightarrow & x - z \\
d(m+m') \downarrow & & \downarrow dm + dm' & & \downarrow \alpha - \beta + \beta - \gamma & & \downarrow \alpha - \gamma \\
d(\omega + \omega') & \xrightarrow{\delta^{-1}} & d\omega + d\omega' & \xrightarrow{\Lambda + \Lambda'} & y' - z' + x' - y' & \longrightarrow & x' - z'
\end{array} \quad (7.17)$$

The first square commutes because δ is natural. The second one commutes since after applying the interchange law it is the sum of two commutative squares, hence commutative. For the third one we use the naturality of the symmetry and the caps.

To see that $\odot$ respects composition consider four morphisms M, M', N, N' of $\mathbb{D}_1$. Since we have already shown that sources and targets are preserved it suffices to consider only their first components. A simple application of the interchange law in G then gives the claim. $\square$

Lemma 7.2. *$\mathcal{G}(d)$ is a strict double category. In more detail this means that for all composable morphisms M, N, P (and thus also objects) in $\mathbb{D}_1$ we have*

$$(M \odot N) \odot P = M \odot (N \odot P). \quad (7.18)$$

Furthermore for all appropriate morphisms α (and thus also objects) in H we have

$$U(\alpha) \odot M = M \quad (7.19)$$

and

$$M \odot U(\alpha) = M. \quad (7.20)$$

Proof. The statements about U are easy to check. For morphisms they are completely obvious, for α, m objects just insert (7.4) into (7.12) and look at the resulting string diagrams. (Note that $\delta(\theta, 0) = 1_{d\theta}$.)

Now for associativity. Again we only need to check it for objects. The

corresponding equation of string diagrams is given by

$$(7.21)$$

The coherence for δ shows that this is indeed true. $\square$

We will now use the monoidal structures of G and H to introduce a monoidal structure (see [2, Definition 2.9, p. 5]) on the double category $\mathcal{G}(d)$ associated to the morphism d . To this end we define double functors

$$\otimes : \mathcal{G}(d) \times \mathcal{G}(d) \longrightarrow \mathcal{G}(d) \quad (7.22)$$

and

$$I : * \longrightarrow \mathcal{G}(d). \quad (7.23)$$

Here $*$ denotes the trivial double category. I just picks out the elements $0 \in \mathbb{D}_0 = H$ and $(0, 0, 0, 0) \in \mathbb{D}_1$ and trivially satisfies the necessary axioms. $\otimes$ is given in the following way.

1. On the object category $\mathbb{D}_0 = H$ it is just the tensor functor of H .
2. On $\mathbb{D}_1$ we set

$$(\theta, \Pi, x, y) \otimes (\theta', \Pi', x', y') := (\theta + \theta', \Pi \otimes \Pi', x + x', y + y') \quad (7.24)$$

where $\Pi \otimes \Pi'$ is given by

$$\begin{array}{ccc}
 d(\theta + \theta') & \xrightarrow{\delta^{-1}(\theta, \theta')} d(\theta) + d(\theta') & \xrightarrow{\Pi + \Pi'} x - y + x' - y' \\
 & \searrow \Pi \otimes \Pi' & \downarrow 1+b(-y, x')+1 \\
 & & x + x' - y - y' \\
 & & \downarrow 1+b(-y, -y') \\
 & & x + x' - (y + y')
 \end{array} \quad (7.25)$$

As a string diagram this looks as follows.

$$\Pi \otimes \Pi' = \text{Diagram} \quad (7.26)$$

Morphisms are multiplied componentwise:

$$(m, \alpha, \beta) \otimes (m', \alpha', \beta') := (m + m', \alpha + \alpha', \beta + \beta'). \quad (7.27)$$

A diagram similar to (7.17) shows that this is well defined.

It is easy to see that we have $S \circ \otimes = \otimes \circ S$ and $T \circ \otimes = \otimes \circ T$ and that $\otimes$ strictly preserves units. This follows from the respective properties of the tensor functors of H and G , the coherence for δ and the coherence theorem for symmetric monoidal categories.

To establish $\otimes$ as a double functor it is left to give a natural isomorphism $\varphi : \odot \circ (\otimes \times \otimes) \longrightarrow \otimes \circ (\odot \times \odot)$ realizing the interchange law whose components are of the form $(m, 1, 1)$ (m a morphism in G) and satisfy the usual coherence for constraints of monoidal functors (with respect to $\odot$).

Let $\Theta = (\theta, \Pi, y, z)$, $\Theta' = (\theta', \Pi', x, y)$, $\Omega = (\omega, \Lambda, y', z')$, $\Omega' = (\omega', \Lambda', x', y') \in \text{ob}(D_1)$. We define

$$\varphi(\Theta, \Omega, \Theta', \Omega') = (1_\theta + b(\omega, \theta') + 1_{\omega'}, 1_{x+x'}, 1_{z+z'}). \quad (7.28)$$

To see that φ is well defined we have to check (7.1). We compute

$$\begin{aligned}
 & (\Theta \otimes \Omega) \odot (\Theta' \otimes \Omega') \\
 &= (\theta + \omega + \theta' + \omega', (\Pi \otimes \Lambda) \odot (\Pi' \otimes \Lambda'), x + x', z + z').
 \end{aligned} \quad (7.29)$$

and

$$\begin{aligned}
& (\Theta \odot \Theta') \otimes (\Omega \odot \Omega') \\
&= (\theta + \theta' + \omega + \omega', (\Pi \odot \Pi') \otimes (\Lambda \odot \Lambda'), x + x', z + z')
\end{aligned} \tag{7.30}$$

In string diagrams:

$$(\Pi \otimes \Lambda) \odot (\Pi' \otimes \Lambda') = \tag{7.31}$$

$d(\theta + \omega + \theta' + \omega')$

and

$$(\Pi \odot \Pi') \otimes (\Lambda \odot \Lambda') = \begin{array}{c} \begin{array}{cccc} x & x' & z' & z \end{array} \\ \begin{array}{c} \text{Diagram with four strands } x, x', z', z \text{ and two crossings. The strands } x \text{ and } x' \text{ cross, and } z' \text{ and } z \text{ cross. Below the crossings are two loops, each with a dot labeled } \Pi \text{ and } \Pi' \text{ respectively. The strands } \Pi \text{ and } \Pi' \text{ cross, and } \Lambda \text{ and } \Lambda' \text{ cross. Below these crossings are two loops, each with a dot labeled } \delta^{-1} \text{ respectively. The strands } \delta^{-1} \text{ cross, and } d(\theta + \theta' + \omega + \omega') \text{ is the bottom strand.} \end{array} \end{array} \quad (7.32)$$

The monoidality and symmetry of d together with the naturality of δ imply

$$\begin{array}{c} \begin{array}{cccc} d(\theta) & d(\theta') & d(\omega) & d(\omega') \end{array} \\ \begin{array}{c} \text{Diagram with four strands } d(\theta), d(\theta'), d(\omega), d(\omega') \text{ and two crossings. The strands } d(\theta) \text{ and } d(\theta') \text{ cross, and } d(\omega) \text{ and } d(\omega') \text{ cross. Below the crossings are two loops, each with a dot labeled } \delta^{-1} \text{ respectively. The strands } \delta^{-1} \text{ cross, and } d(\theta + \theta' + \omega + \omega') \text{ is the bottom strand.} \end{array} \end{array} = \begin{array}{c} \begin{array}{cccc} d(\theta) & d(\theta') & d(\omega) & d(\omega') \end{array} \\ \begin{array}{c} \text{Diagram with four strands } d(\theta), d(\theta'), d(\omega), d(\omega') \text{ and two crossings. The strands } d(\theta) \text{ and } d(\theta') \text{ cross, and } d(\omega) \text{ and } d(\omega') \text{ cross. Below the crossings are two loops, each with a dot labeled } \delta^{-1} \text{ respectively. The strands } \delta^{-1} \text{ cross, and } d(\theta + \omega + \theta' + \omega') \text{ is the bottom strand.} \end{array} \end{array} \quad (7.33)$$

Thus composing (7.32) with φ gives precisely (7.31) which means that (7.1) is satisfied. Therefore φ is well defined. Naturality is immediate from

the definition. That φ satisfies coherence follows from the coherence theorem for symmetric monoidal categories.

Next we want to lift the symmetric structure from G and H to $\mathcal{G}(d)$. For this we have to give a vertical natural isomorphism $s : \otimes \circ \tau \longrightarrow \otimes$ where τ is the twist endomorphism on $\mathcal{G}(d) \times \mathcal{G}(d)$, interchanging the two arguments (see [2, Definition 2.9, p. 5]). On the $\mathbb{D}_0$ -level we set

$$s_0(x, y) := b_H(x, y) \quad (7.34)$$

and on the $\mathbb{D}_1$ -level

$$s_1((\theta, \Pi, x, y), (\theta', \Pi', x', y')) := (b_G(\theta, \theta'), b_H(x, x'), b_H(y + y')). \quad (7.35)$$

(We will leave the sub indices out usually.) Here b_H and b_G denote the respective symmetries of H and G .

Lemma 7.3. *s_1 is well defined and s_0 as well as s_1 are natural transformations. Furthermore they define a (symmetric) braiding on $\mathcal{G}(d)$, i.e. the following axioms are satisfied.*

$$1. S(s_1(\Theta', \Theta)) = s_0(S(\Theta'), S(\Theta)) \text{ and } S(s_1(\Theta', \Theta)) = s_0(S(\Theta'), S(\Theta)).$$

2.

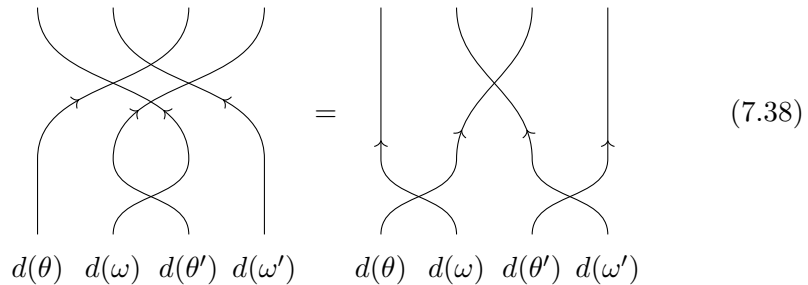
$$\begin{aligned} s(\Theta' \odot \Theta, \Omega' \odot \Omega) \circ \varphi(\Theta, \Omega, \Theta', \Omega') \\ = \varphi(\Omega, \Theta, \Omega', \Theta') \circ (s(\Theta, \Omega) \odot s(\Theta', \Omega')) \end{aligned} \quad (7.36)$$

3.

$$\begin{aligned} s((0, \text{ } \smile \text{ }, x + x', x + x'), (0, \text{ } \smile \text{ }, x' + x, x' + x)) \\ = (0, b(x, x'), b(x', x)) \end{aligned} \quad (7.37)$$

Proof. In order for s_1 to be well defined we need (7.1) to commute. This follows from the fact that d is symmetric monoidal. Naturality of s_0 and s_1 follows from the naturality of the symmetry in G respectively H .

All of the above axioms are fairly easy to see. For example as a string diagram in G the second one reads



$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (7.38)$$

$d(\theta) \quad d(\omega) \quad d(\theta') \quad d(\omega')$ $d(\theta) \quad d(\omega) \quad d(\theta') \quad d(\omega')$

which is clearly true. □

7.2 The Quotient of $\mathcal{G}(d)$

We will now construct the *quotient category* $[\mathcal{G}(d)]$ of the double category $\mathcal{G}(d)$ induced by the morphisms of abelian 2-groups $G \xrightarrow{d} H$. The symmetric monoidal structure on $\mathcal{G}(d)$ will allow us to endow the quotient with the structure of a symmetric monoidal category. Due to the presence of inverses in $\mathcal{G}(d)$ we will even get an abelian 2-group.

Informally speaking we want to take the quotient of H by the relations induced from d and the construction of $\mathcal{G}(d)$ was an intermediate step to this end.

We think of the objects of $\mathbb{D}_1$ as relations between objects of H and of the morphisms of $\mathbb{D}_1$ as relations between morphisms of H . It turns out that if we collapse all the relations to identities then the resulting object neither satisfies the universal property of a strict nor that of a weak quotient. It is too strict to be weak since we collapsed all the relations strictly and it is too weak to be strict since the Π 's in a quadruple $(\theta, \Pi, x, y) \in \text{ob}(\mathbb{D}_1)$ are only isomorphisms and not identities.

The correct thing to do is to turn the relations between objects into morphisms and only factor out the relations between morphisms. That means that as morphisms between two objects of the quotient we will have the vertical *and* horizontal morphisms between them in $\mathcal{G}(d)$. Every

$$\begin{array}{ccc} x & \xrightarrow{(\theta, \Pi)} & x' \\ \alpha \downarrow & \Downarrow M & \downarrow \beta \\ y & \xrightarrow{(\omega, \Lambda)} & y' \end{array} \quad (7.39)$$

will give a relation that forces this square to commute.

The result (Lemma 7.11) will be an abelian 2-group $[\mathcal{G}(d)]$ that satisfies the universal property of a weak cokernel of d . This procedure will now be described in detail.

Throughout this subsection, if not stated otherwise, $x, y, \dots$ will always denote objects of H , $\alpha, \beta, \dots$ will denote morphism in H , $\Theta = (\theta, \Pi, x, x'), \Omega = (\omega, \Lambda, y, y'), \dots$ will denote objects of $\mathbb{D}_1$ and $M = (m, \alpha, \beta), \dots$ will denote morphisms of $\mathbb{D}_1$.

We take some intermediate steps in order to construct $[\mathcal{G}(d)]$. We first build a quiver $\bar{H}$. We set

$$\bar{H}_0 := \text{ob}(H). \quad (7.40)$$

and for $x, y \in \bar{H}_0$

$$\bar{H}(x, y) := H(x, y) \sqcup \mathcal{G}(d)_{\text{hor}}(x, y). \quad (7.41)$$

Here $\mathcal{G}(d)_{hor}$ is the set of all horizontal 1-morphisms from x to y , i.e. all quadruples (θ, Π, x, y) where $\theta \in \text{ob}(G)$ and $\Pi : d(\theta) \xrightarrow{\cong} x - y$ in H .

Next we introduce the following relations on H .

1. Whenever there is a morphism

$$\begin{array}{ccc} x & \xrightarrow{(\theta, \Pi)} & x' \\ \alpha \downarrow & \Downarrow M & \downarrow \beta \\ y & \xrightarrow{(\omega, \Lambda)} & y' \end{array} \quad (7.42)$$

of $\mathbb{D}_1$ the relation

$$\beta \circ (\theta, \Pi, x', x) \sim (\omega, \Lambda, y', y) \circ \alpha. \quad (7.43)$$

2. All relations R_H from H induced from composition, i.e. $\alpha \circ \beta \sim \gamma$ if $\alpha \circ \beta = \gamma$ in H (if they are composable).
3. Similarly all relations from $\mathcal{G}(d)$ induced from $\odot$ i.e. $\Theta \circ \Theta' \sim \Omega$ if $\Theta \odot \Theta' = \Omega$ in $\mathcal{G}(d)$ (if they are composable)
4. Identities of H and $\mathcal{G}(d)$ are identified with formal identities.

We call the above set of relations R and form the free category $\mathcal{F}(\bar{H}, R)$ over $(\bar{H}, R)$ (see section 5).

Definition 7.4. The *Grothendieck category* $[\mathcal{G}(d)]$ associated to the double category $\mathcal{G}(d)$ is given by the free category

$$[\mathcal{G}(d)] := \mathcal{F}(\bar{H}, R) \quad (7.44)$$

over the quiver with relations $(\bar{H}, R)$.

Remark 7.5. The ‘identity’

$$(\bar{H}, \emptyset) \longrightarrow (\bar{H}, R) \quad (7.45)$$

induces a full and surjective projection functor

$$P : \mathcal{F}(\bar{H}) = \mathcal{F}(\bar{H}, \emptyset) \longrightarrow [\mathcal{G}(d)] \quad (7.46)$$

We can thus view morphisms in $\mathcal{F}(\bar{H}, R)$ as equivalence classes of morphisms in $\mathcal{F}(\bar{H})$. We will indicate this by denoting the morphisms of the latter category by morphisms of the first one with brackets around them.

Furthermore the inclusion

$$(UH, R_H) \longrightarrow (\bar{H}, R) \quad (7.47)$$

(where U is the forgetful functor) gives a functor

$$\pi : H \longrightarrow [\mathcal{G}(d)] \quad (7.48)$$

which is the identity on objects and maps a morphism of H to its equivalence class in $[\mathcal{G}(d)]$.

Remark 7.6. Note that this definition would work for any double category. In the general case however we would only use invertible horizontal morphisms (in a suitable sense) to define the relations. Here this is automatically given. If we would allow all horizontal 1-morphisms as relations we would probably obtain a 2-cokernel of d in the 2-category of (abelian) 2-groups with *lax* (symmetric) monoidal transformations.

Before we can carry over the symmetric monoidal structure of $\mathcal{G}(d)$ to $[\mathcal{G}(d)]$ we will first need to deal with some technicalities.

Lemma 7.7. *For every morphism in $\mathcal{F}(\bar{H})$ of the form $\alpha \circ \Theta$ there is an equivalent one of the form $\hat{\Theta} \circ \hat{\alpha}$, i.e. there exists*

$$\begin{array}{ccc} x & \xrightarrow{\Theta} & x' \\ \hat{\alpha} \downarrow & \Downarrow \Delta & \downarrow \alpha \\ \hat{y} & \xrightarrow{\hat{\Theta}} & y' \end{array} \quad (7.49)$$

Proof. We set

$$\hat{y} := d(\theta) + y', \quad (7.50)$$

$$\hat{\Theta} := (\theta, 1 + \text{cup}, y, y') \quad (7.51)$$

and give $\hat{\alpha}$ in string diagram notation by

$$\hat{\alpha} := \Pi^{-1} \cdot \text{cup} \cdot \alpha \quad (7.52)$$

where $\Theta = (\theta, \Pi, x, x')$. The relation Δ can then be taken to be

$$\Delta := (1_\theta, \hat{\alpha}, \alpha). \quad (7.53)$$

That Δ is well defined follows from

$$\begin{aligned} (\hat{\alpha} - \alpha) \circ \Pi &= \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \quad = \quad \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \\ &= \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \quad = \quad \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \\ &= \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \quad = \quad \Pi^{-1} \cdot \text{cup} \cdot \alpha - \Pi^{-1} \cdot \text{cup} \cdot \alpha \end{aligned} \quad (7.54)$$

□

Corollary 7.8. *Every morphism of $\mathcal{F}(\bar{H})$ is equivalent to one of the form $\alpha \circ \Theta$.*

Proof. Morphisms in $\mathcal{F}(\bar{H})$ are well composed words in the generators (and formal identities) and composition is given by concatenation.

By Lemma 7.7 for every $\alpha \circ \Theta$ there are $\hat{\alpha}$ and $\hat{\Theta}$ such that $\alpha \circ \Theta \sim \hat{\Theta} \circ \hat{\alpha}$. Hence, after taking equivalence classes they are equal. This allows us to iteratively interchange α 's and Θ 's to collect all the α 's on the left side. $\square$

Lemma 7.9. *We can endow $[\mathcal{G}(d)]$ with the structure of a monoidal category that is canonically induced from the monoidal structure of $\mathcal{G}(d)$.*

Proof. Given $[\alpha \circ \Theta]$ and $[\beta \circ \Omega]$ we define

$$[\alpha \circ \Theta] \otimes [\beta \circ \Omega] := [(\alpha + \beta) \circ (\Theta \otimes \Omega)]. \quad (7.55)$$

First we have to show that this is well defined. Let α', Θ' be such that $\alpha' \circ \Theta' \sim \alpha \circ \Theta$. Then we have

$$\hat{\Theta}' \circ \hat{\alpha}' \sim \alpha \circ \Theta. \quad (7.56)$$

From the definition of R it follows that this is the case if and only if there exists

$$\begin{array}{ccc} x & \xrightarrow{\Theta} & x' \\ \hat{\alpha}' \downarrow & \Downarrow M & \downarrow \alpha \\ \hat{y} & \xrightarrow{\hat{\Theta}'} & y' \end{array} \quad (7.57)$$

But then since also

$$\begin{array}{ccc} z & \xrightarrow{\Omega} & z' \\ \hat{\beta} \downarrow & \Downarrow \Delta & \downarrow \beta \\ \hat{w} & \xrightarrow{\hat{\Omega}} & w' \end{array} \quad (7.58)$$

we have

$$\begin{array}{ccc} x + z & \xrightarrow{\Theta \otimes \Omega} & x' + z' \\ \hat{\alpha}' + \hat{\beta} \downarrow & \Downarrow M \otimes \Delta & \downarrow \alpha + \beta \\ \hat{y} + \hat{w} & \xrightarrow{\hat{\Theta}' \otimes \hat{\Omega}} & y' + w' \end{array} \quad (7.59)$$

and thus

$$\begin{aligned} [\alpha' \circ \Theta'] \otimes [\beta \circ \Omega] &= [(\alpha' \otimes \beta) \circ (\Theta' \otimes \Omega)] = [(\hat{\alpha}' \otimes \hat{\beta}) \circ (\hat{\Theta}' \otimes \hat{\Omega})] \\ &\stackrel{(7.59)}{=} [(\alpha \otimes \beta) \circ (\Theta \otimes \Omega)] = [\alpha \circ \Theta] \otimes [\beta \circ \Omega] \end{aligned} \quad (7.60)$$

Similarly we can also perturb $\beta \circ \Omega$.

Associativity, unitality and the interchange law hold since they do (up to equivalence) in $\mathcal{G}(d)$. $\square$

From this definition it follows immediately that $\pi : H \longrightarrow [\mathcal{G}(d)]$ from Remark 7.5 is strictly monoidal.

Recall the construction of the symmetry s on $\mathcal{G}(d)$ from (7.34) and (7.35).

Lemma 7.10. *We can endow $[\mathcal{G}(d)]$ with a symmetry $\bar{s}$ that is canonically induced from the symmetry on $\mathcal{G}(d)$.*

Proof. We define

$$\bar{s}(x, y) := [s_0(x, y)]. \quad (7.61)$$

In order to show naturality of $\bar{s}$ we have to find a relation filling the following square.

$$\begin{array}{ccc} x + y & \xrightarrow{(\alpha \circ \Theta) \otimes (\beta \circ \Omega)} & x'' + y'' \\ s_0(x, y) \downarrow & & \downarrow s_0(x'', y'') \\ y + x & \xrightarrow{(\beta \circ \Omega) \otimes (\alpha \circ \Theta)} & y'' + x'' \end{array} \quad (7.62)$$

Inserting (7.55) we can calculate in $\mathcal{F}(\bar{H})$ as follows.

$$\begin{aligned} s_0(x'', y'') \circ (\alpha \otimes \beta) \circ (\Theta \otimes \Omega) &\sim (\beta \otimes \alpha) \circ s_0(x', y') \circ (\Theta \otimes \Omega) \\ &\sim (\alpha \otimes \beta) \circ (\Omega \otimes \Theta) \circ s_0(x, y). \end{aligned} \quad (7.63)$$

For the first equivalence we used the naturality of s and for the second one the fact that we have

$$\begin{array}{ccc} x + y & \xrightarrow{(\Theta \otimes \Omega)} & x' + y' \\ s_0(x, y) \downarrow & \Downarrow M & \downarrow s_0(x', y') \\ y + x & \xrightarrow{(\Omega \otimes \Theta)} & y' + x' \end{array} \quad (7.64)$$

where $M = s_1(\Theta, \Omega)$.

Coherence and strict unitality of $\bar{s}$ carry over directly from s_0 . $\square$

Lemma 7.11. *$[\mathcal{G}(d)]$ is an abelian 2-group.*

Proof. It is only left to show that we have duals such that evaluation and co-evaluation satisfy the conditions from Lemma 3.1. Since the objects of $[\mathcal{G}(d)]$ are just the objects of H we can take as duals the ones already present in H . As cups and caps we then take equivalence classes of cups and caps in H . Because $\sim$ respects the relations of H the necessary equations automatically hold in $[\mathcal{G}(d)]$. $\square$

With this we are now ready to state the main theorem of this section. Recall the functor $\pi : H \longrightarrow [\mathcal{G}(d)]$ from Remark 7.5.

Theorem 7.12. *Let $G \xrightarrow{d} H$ be a morphisms of abelian 2-groups with constraint δ . Then the Grothendieck category $[\mathcal{G}(d)]$ of the double category $\mathcal{G}(d)$ associated to d naturally carries the structure of an abelian 2-group. Furthermore $H \xrightarrow{\pi} [\mathcal{G}(d)]$ satisfies the universal property of a 2-cokernel of d from Definition 6.8.*

Proof. Only⁴ the universal property is left to show. We start by giving a symmetric monoidal natural isomorphism $\lambda : \pi \circ d \xrightarrow{\cong} 0$. For θ an object in G its component $\lambda(\theta)$ is given by the equivalence class of the following horizontal 1-morphism of $\mathcal{G}(d)$.

$$\lambda(\theta) := [(\theta, 1_{d(\theta)}, d(\theta), 0)]. \quad (7.65)$$

We start by showing naturality of λ . For each $\theta \xrightarrow{m} \omega$ in G we need to find a relation

$$\begin{array}{ccc} d(\theta) & \xrightarrow{\lambda(\theta)} & 0 \\ d(m) \downarrow & \Downarrow M & \parallel \\ d(\omega) & \xrightarrow{\lambda(\omega)} & 0 \end{array} . \quad (7.66)$$

Taking $M = (m, d(m), 0)$ does the job. For monoidality we need

$$\begin{array}{ccc} d(\theta) + d(\omega) & \xrightarrow{\lambda(\theta) \times \lambda(\omega)} & 0 \\ \delta \downarrow & \Downarrow M & \parallel \\ d(\theta + \omega) & \xrightarrow{\lambda(\theta + \omega)} & 0 \end{array} . \quad (7.67)$$

Here we take $M = (1_{\theta + \omega}, \delta, 0)$. Symmetry follows similarly from

$$\begin{array}{ccc} d(\theta + \omega) & \xrightarrow{\lambda(\theta + \omega)} & 0 \\ d(b_G(\theta, \omega)) \downarrow & \Downarrow M & \parallel \\ d(\omega + \theta) & \xrightarrow{\lambda(\omega + \theta)} & 0 \end{array} \quad (7.68)$$

with $M = (b_G(\theta, \omega), d(b_G(\theta, \omega)), 0)$.

To see that it satisfies the universal property from Definition 6.8 we show that the assumptions of Lemma 6.9 are satisfied. What was called ω there is now our $\lambda : \pi \circ d \xrightarrow{\cong} 0$.

Since we already have λ the next thing to check is that π is a coherent 2-epimorphism with respect to (d, λ) . Let $\Phi, \Psi : [\mathcal{G}(d)] \rightrightarrows A$ be given,

⁴As we shall see this will require quite a bit of work.

where A is some abelian 2-group. Consider $\tau : \Phi \circ \pi \xrightarrow{\cong} \Psi \circ \pi$ such that for all θ

$$\begin{array}{ccc} \Phi(\pi(d(\theta))) & \xrightarrow{\tau(d(\theta))} & \Psi(\pi(d(\theta))) \\ & \searrow \Phi(\lambda(\theta)) & \downarrow \Psi(\lambda(\theta)) \\ & & 0 \end{array} \quad (7.69)$$

commutes. We are looking for a $\sigma : \Phi \xrightarrow{\cong} \Psi$ such that $\sigma(\pi(x)) = \tau(x)$ for all x . But since π is surjective on objects (the identity even) such a σ is unique if it exists. To show existence we define σ by

$$\sigma(x) := \tau(x). \quad (7.70)$$

First we have to show that this is a natural transformation. The problem is that in $[\mathcal{G}(d)]$ we have morphisms (namely the ones coming from horizontal morphisms of $\mathcal{G}(d)$) that didn't appear in H . Naturality of σ means that the following diagram commutes (for all appropriate objects and morphisms).

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\Phi([\alpha \circ \Theta])} & y \\ d(\sigma(x)) \downarrow & & \downarrow \sigma(y) \\ \Psi(x) & \xrightarrow{\Psi([\alpha \circ \Theta])} & \Psi(y) \end{array} \quad (7.71)$$

We can also write this as

$$\begin{array}{ccccc} \Phi(x) & \xrightarrow{\Phi([\Theta])} & \Phi(x') & \xrightarrow{\Phi([\alpha])} & \Phi(y) \\ \tau(x) \downarrow & & \downarrow \tau(x') & & \downarrow \tau(y) \\ \Psi(x) & \xrightarrow{\Psi([\Theta])} & \Psi(x') & \xrightarrow{\Psi([\alpha])} & \Psi(y) \end{array} \quad (7.72)$$

The right square commutes since $\Phi([\alpha])$ is just a different expression for $\Phi(\pi(\alpha))$ and $\tau : \Phi \circ \pi \longrightarrow \Psi \circ \pi$ is natural. So let's concentrate on the left one.

For every $\Theta = (\theta, \Pi, x, x')$ in $\mathcal{G}(d)$ we define

$$\tilde{\Theta} := (\theta, \Pi, x - x', 0) \quad (7.73)$$

so that we have

$$[\hat{\Theta}] = [\Theta] \quad \begin{array}{c} \bullet \\ \uparrow \\ x \end{array} \quad \begin{array}{c} \downarrow \\ x' \end{array} \quad (7.74)$$

in $[\mathcal{G}(d)]$ (c.f. Remark 3.16). Now we have

$$\begin{array}{ccc} d(\theta) & \xrightarrow{\lambda(\theta)} & 0 \\ \Pi \downarrow & \Downarrow M & \parallel \\ x - x' & \xrightarrow{\bar{\Theta}} & 0 \end{array} \quad (7.75)$$

in $\mathcal{G}(d)^5$ with $M = (1_\theta, \Pi, 0)$ and thus an equality

$$[\widetilde{\Theta}] = \lambda(\theta)[\Pi]^{-1} \quad (7.76)$$

in $[\mathcal{G}(d)]$. Writing this as a string diagram and combining it with (7.73) we obtain

$$[\Theta] \bullet \text{ (vertical line with dot) } = \text{ (string diagram with a loop labeled } [\Pi]^{-1} \text{ and a dot labeled } \lambda(\theta) \text{) } \quad (7.77)$$

Using (7.52) we see that we can write this also as

$$[\Theta] = (\lambda(\theta) + 1) \circ \widehat{1_{x'}}. \quad (7.78)$$

Coming back to the left square of (7.72) we thus get

$$\begin{array}{ccccc} \Phi(x) & \xrightarrow{\Phi(\widehat{1_{x'}})} & \Phi(d(\theta) + x') & \xrightarrow{\Phi(\lambda(\theta)+1)} & \Phi(x') \\ \tau(x) \downarrow & & \downarrow \tau(d(\theta)+x') & & \downarrow \tau(x') \\ \Psi(x) & \xrightarrow{\Psi(\widehat{1_{x'}})} & \Psi(d\theta + x') & \xrightarrow{\Psi(\lambda(\theta)+1)} & \Psi(x') \end{array} \quad (7.79)$$

The left square here commutes because $\widehat{1_{x'}}$ lies in the image of π . So we consider the right one. Denote the constraints of Φ and Ψ by ϕ and ψ respectively. We then have

$$\begin{array}{ccccc} & & \Phi(\lambda(\theta)+1)_{\Phi(x')} & & \\ & \searrow & & \swarrow & \\ \Phi(d(\theta)) + \Phi(x') & \xrightarrow{\phi} & \Phi(d(\theta) + x') & \xrightarrow{\Phi(\lambda(\theta)+1)} & \Phi(x') \\ \tau(d(\theta))+\tau(x') \downarrow & & \downarrow \tau(d(\theta)+x') & & \downarrow \tau(x') \\ \Psi(d(\theta)) + \Psi(x') & \xrightarrow{\psi} & \Psi(d\theta + x') & \xrightarrow{\Psi(\lambda(\theta)+1)} & \Psi(x') \\ & \swarrow & & \searrow & \\ & & \Psi(\lambda(\theta)+1)_{\Psi(x')} & & \end{array} \quad (7.80)$$

⁵To be precise, the $\lambda(\theta)$ in the above diagram that lives in $\mathcal{G}(d)$ is a representative of the actual $\lambda(\theta) \in [\mathcal{G}(d)]$.

The upper and lower triangle commute by the naturality and strict unitality of ϕ and ψ . The left square commutes by the monoidality of τ . Furthermore the outer square is the sum (=tensor product) of two squares one of which commutes trivially and the other one because of the assumption (7.69) on τ . With only the right square in the above diagram left unchecked it must now also commute. This shows that σ is a natural transformation.

Next up on our bucket list is the monoidality of σ . This is a direct consequence of the monoidality of τ however. It only requires τ to work well together with the constraints ϕ, ψ and for this it doesn't matter if we view the $x, x', \dots$ as objects of H or as objects of $[\mathcal{G}(d)]$. In fact we already used this to show that the left square in the above diagram commutes.

So π is a coherent 2-epimorphism with respect to (d, λ) . It is left to show that it satisfies the third condition from Lemma 6.9. Consider a morphism of abelian 2-groups $X \xrightarrow{f} A$ together with an isomorphism $\mu : f \circ d \xrightarrow{\cong} 0$.

$$\begin{array}{ccccc} Y & \xrightarrow{d} & X & \xrightarrow{\pi} & Q \\ & & f \downarrow & \swarrow \Phi & \\ & & A & & \end{array} \quad (7.81)$$

We have to show that there is a unique Φ as above such that the triangle commutes and we have $\Phi(\lambda) = \mu$.

We first show uniqueness of Φ . On objects this is clear. On morphisms note that on $[\alpha]$ in the image of $H \xrightarrow{\pi} [\mathcal{G}(d)]$ Φ has to be equal to $f(\alpha)$. For a $[\Theta]$ we have the decomposition (7.77). Since the values of Φ are already prescribed on the λ 's it follows that Φ is unique if it exists.

Now for existence. Denote the constraints of f by γ . On objects we define Φ by

$$\Phi(x) := f(x) \quad (7.82)$$

and on morphisms

$$\Phi([\alpha]) := f(\alpha) \quad (7.83)$$

and

$$\Phi([\Theta]) := \begin{array}{c} \mu(\theta) \circ \\ \bullet \\ f(\Pi)^{-1} \\ \bullet \\ \gamma \\ \bullet \\ \text{hook} \end{array} \quad (7.84)$$

To see that this is well defined (and by the universal property of $\mathcal{F}(\bar{H})$ thus automatically a (strictly unital) functor) consider morphisms $[\beta \circ \Theta], [\Omega \circ \alpha]$

and

$$\begin{array}{ccc} x & \xrightarrow{\Theta} & x' \\ \alpha \downarrow & \Downarrow M & \downarrow \beta \\ y & \xrightarrow{\Omega} & y' \end{array} \quad (7.85)$$

We then have the following commutative diagram in A .

$$\begin{array}{ccccccc} 0 & \xleftarrow{\mu} & 0 & \xrightarrow{f(\Pi)} & 0 & \xleftarrow{\gamma} & 0 \\ \parallel & & \downarrow f(d(m)) & & \downarrow f(\alpha-\beta) & & \downarrow f(\alpha)-f(\beta) \\ 0 & \xleftarrow{\mu} & 0 & \xrightarrow{f(\Lambda)} & 0 & \xleftarrow{\gamma} & 0 \end{array} \quad (7.86)$$

As an equality of string diagrams this reads

$$\begin{array}{c} \mu(\omega) \circ \\ \uparrow \\ f(\Lambda)^{-1} \bullet \\ \uparrow \\ \gamma \\ \uparrow \quad \downarrow \\ f(\alpha) \bullet \quad \bullet -f(\beta) \end{array} = \begin{array}{c} \mu(\theta) \circ \\ \uparrow \\ f(\Pi)^{-1} \bullet \\ \uparrow \\ \gamma \\ \uparrow \quad \downarrow \end{array} \quad (7.87)$$

or after composing with a cup and $f(\beta)$

$$\begin{array}{c} \mu(\omega) \circ \\ \uparrow \\ f(\Lambda)^{-1} \bullet \\ \uparrow \\ \gamma \\ \uparrow \quad \downarrow \\ f(\alpha) \bullet \end{array} = \begin{array}{c} \mu(\theta) \circ \\ \uparrow \\ f(\Pi)^{-1} \bullet \\ \uparrow \\ \gamma \\ \uparrow \quad \downarrow \\ f(\beta) \bullet \end{array} \quad (7.88)$$

But this is now precisely

$$f([\Omega \circ \alpha]) = f([\beta \circ \Theta]). \quad (7.89)$$

So Φ is a well defined functor. In order to make it into a symmetric monoidal one we just use the constraints γ of f . That they get along with the symmetry of $[\mathcal{G}(d)]$ is easy to see. For monoidality we see (with the help of the interchange law) that we can check the relevant diagram for α 's and

Θ 's separately. For the α 's it is trivial. Fix Θ and use (7.77) (and the interchange law as well as the definition of Φ) to reduce the problem to the commutativity of

$$\begin{array}{ccc} f(x+y) & \xrightarrow{\mu(x+y)} & 0 \\ \gamma \uparrow & & \parallel \\ f(x) + f(y) & \xrightarrow{\mu(x)+\mu(y)} & 0 \end{array} \quad (7.90)$$

for all x, y . But this follows now from the monoidality of μ .

Finally we have to verify that Φ satisfies $\Phi(\lambda) = \mu$. This follows directly from inserting in the definition and using the strict monoidality of f and γ .

With this our quotient constructing endeavours finally come to an end. $\square$

7.3 Horizontal Bicategory of a Double category

To every double category $\mathbb{D}$ there can be associated a bicategory $\mathbb{B}(\mathbb{D})$ by setting all the vertical morphisms (i.e. the ones in $\mathbb{D}_0$) to the identity. The object set of $\mathbb{B}(\mathbb{D})$ is then given by the objects of $\mathbb{D}_0$ and the local Hom-categories by subcategories of $\mathbb{D}_1$ where source and target are fixed. [Reference]

In our case we deal with strict double categories and thus obtain 2-categories from them. In particular consider the double category $\mathcal{G}(d)$ associated to a morphism of 2-groups $X \xrightarrow{d} Y$. Here we obtain as $\mathbb{B}(\mathcal{G}(d))$ the following 2-category.

1. Objects are given by objects of $\mathbb{D}_0 = H$.
2. For $x, y \in H$ the local Hom-category $\mathbb{B}(\mathcal{G}(d))(x, y)$ is given by the subcategory of $\mathbb{D}_1$ that has as objects all Quadruples (θ, Π, x, y) (for x and y fixed) where θ is an object of G , $x, y \in \text{ob}(H)$ and $d(\theta) \xrightarrow{\Pi} x - y$ is a morphism in H .

A morphism $(\theta, \Pi, x, y) \xrightarrow{m} (\theta', \Pi', x, y)$ is an $\theta \xrightarrow{m} \theta'$ in G such that

$$\begin{array}{ccc} d\theta & \xrightarrow{\Pi} & x - y \\ dm \downarrow & \nearrow \Pi' & \\ d\theta' & & \end{array} \quad (7.91)$$

commutes.

The composition functor is given by the restriction of $\odot$.

Note that the horizontal bicategory of $\mathcal{G}(d)$ consists basically of all the relations induced from d . On the other hand the Grothendieck category $[\mathcal{G}(d)]$ from Definition 7.4 tries to forget them by factoring them out. While we

saw in the construction of $\mathcal{G}(d)$ that some of the relations remain in $[\mathcal{G}(d)]$ as morphisms, $\mathbb{B}(\mathcal{G}(d))(x, y)$ on the other hand knows very little about the morphisms of $[\mathcal{G}(d)]$ since it knows nothing about the vertical morphisms of $\mathcal{G}(d)$.

Remark 7.13. One could also use the symmetric monoidal structure on $\mathcal{G}(d)$ to turn $\mathbb{B}(\mathcal{G}(d))(x, y)$ into a symmetric monoidal bicategory. For more details on this see [2].

8 Categorical Group (Co-)Homology

8.1 Categorical Cochain Complexes

Definition 8.1. A 2-cochain complex $C^\bullet$ consists of the following data. For every $n \in \mathbb{Z}$

1. an abelian 2-group C^n ,
2. a symmetric monoidal functor $d^n : C^n \rightarrow C^{n+1}$ called the *n-th differential*,
3. a symmetric monoidal natural isomorphism $\omega^n : d^n \circ d^{n-1} \xrightarrow{\cong} 0$ called the *n-th constraint*,

subject to the following axiom.

For every $n \in \mathbb{Z}$, ω^n is coherent with respect to (d^{n+1}, ω^{n+1}) i.e.

$$d^{n+1} \otimes \omega^n = \omega^{n+1} \otimes d^{n-1}. \quad (8.1)$$

Here $\otimes$ denotes the horizontal composition of natural transformations and d^{n+1}, d^n are really the identities on the respective functors.

8.1.1 The Cohomology of a 2-Cochain Complex

We now categorify the construction of Cohomology groups of a cochain complex to that of homology 2-groups of a 2-cochain complex.

Let $C^\bullet = (C^n, d^n, \omega^n)_{n \in \mathbb{Z}}$ be a 2-cochain complex with differentials d^n and constraints ω^n .

By Proposition 6.7 all coherent 2-kernels exist in 2Grp. This allows us to give the following definition.

Definition 8.2. The *n-th 2-group of cocycles* of $C^\bullet$ is defined to be “the” coherent kernel of d^n with respect to d^{n+1} . It is denoted by $Z^n \xrightarrow{\iota^n} C^n$. Note that actually there may be many 2-groups satisfying the universal property in question but we choose the one constructed in the proof of Proposition 6.7.

By the universal property of Z^{n+1} , $C^n \xrightarrow{d^n} C^{n+1}$ factors to $C^n \xrightarrow{\partial^n} Z^{n+1}$. By Proposition 6.10 all 2-cokernels exist in Ab2Grp.

Definition 8.3. The n -th cohomology 2-group of $\mathcal{C}^\bullet$ is “the” 2-cokernel of $C^n \xrightarrow{\partial^n} Z^n$. We use the one constructed in subsection 7.2.

8.2 The 2-Cohomology of a 2-Group

Let G be a monoidal category and A an abelian 2-group. The *normed 2-cochain complex of G with coefficients in A* is defined as follows. (That it actually is a 2-cochain complex is proven below.)

Let $n \in \mathbb{Z}$. For $n < 0$ let $C_n = 0$ be given by the trivial abelian 2-group and $d^n = 0$ by the trivial monoidal functor. Also the natural isomorphisms ω^n are given by the trivial ones in this case. For $n \geq 0$ we set $C^n = [G^n, A]_*$. Next, recall the construction of the T_k ’s from Lemma 3.21. We define

$$d^n := \sum_{k=0}^{n+1} i^k \circ T_k \quad (8.2)$$

More concretely d^n is given by

$$d^n(\chi)(\alpha_1, \dots, \alpha_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \chi(\alpha_1, \dots, \alpha_k \otimes \alpha_{k+1}, \dots, \alpha_{n+1}) \quad (8.3)$$

for objects $\chi \in C^n$ and $\alpha_k \in G$ and similarly if χ or the α_k ’s are replaced by morphisms in the respective categories.

Now consider a composition $d^n \circ d^{n-1}$. From the classical theory of group cohomology we know that in the resulting sum every term appears exactly twice, once with positive and once with negative sign. We can thus define ω^n to consist only of caps (appropriate orientation) that connect these two summands respectively. Note that these caps may cross each other. By the coherence theorem for symmetric monoidal categories there is only one way to do so.

Theorem 8.4. *The above construction indeed defines a 2-cochain complex.*

Proof. That the C^n ’s are abelian 2-groups and that the differentials are symmetric monoidal was shown in subsection 3.3.

Monoidality and symmetricity of the ω^n ’s follows from Lemma 3.11.

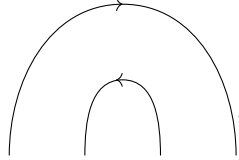
It is left to check (8.1). Consider a triple composition $d^{n+1} \circ d^n \circ d^{n-1}$. It is not hard to see that every term appears exactly four times and with alternating orientations. So the relative picture for a fixed label looks (without loss of generality) like

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \quad (8.4)$$

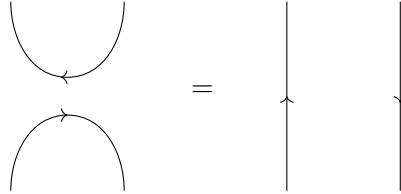
Both $d^{n+1} \otimes \omega^n$ and $\omega^{n+1} \otimes d^{n-1}$ again only consist of an arrangement of caps. Now there are two ways to connect these strands with caps,


(8.5)

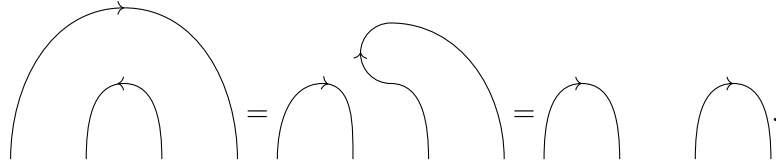
and


(8.6)

But using


(8.7)

we see that they are in fact equal:


(8.8)

Thus we must have

$$d^{n+1} \otimes \omega^n = \omega^{n+1} \otimes d^{n-1}. \quad (8.9)$$

□

9 Representations in Gray-Categories

9.1 Preliminaries

Given a Gray-category $\mathcal{C}$, $\Omega_x^2 \mathcal{C}$ forms a braided (strict) monoidal category in the following way.

1. The category structure is directly inherited from $\mathcal{C}$. The monoidal product is the horizontal composition of 2-cells in $\mathcal{C}$.
2. For 1-cells f, g, f', g' and 2-cells $f \xrightarrow{\alpha} f'$, $g \xrightarrow{\beta} g'$ the Gray-structure provides us with isomorphisms

$$\begin{array}{ccc}
 fg & \xrightarrow{1\beta} & fg' \\
 \alpha 1 \downarrow & \cong \swarrow & \downarrow \alpha 1 \\
 f'g & \xrightarrow{1\beta} & f'g'
 \end{array}
 \quad (9.1)$$

If $f = g = f' = g' = 1_x$ then $1f = f$ and $g1 = g$ and one can show that b defines a braiding on $\Omega_x^2\mathcal{C}$. For more details on this construction see for example [3].

Let G be a monoidal category with unit I and $\mathcal{C}$ a Gray-category. We denote the monoidal products of G and $\Omega_x\mathcal{C}$ by $\otimes$ but for readability we will also just write it as concatenation. For any 2-category we denote horizontal composition by $*$.

Definition 9.1. A monoidal functor $\rho : G \longrightarrow \Omega_x\mathcal{C}$ consists of the following data.

1. A functor $\rho : G \longrightarrow \Omega_x\mathcal{C}$ (by abuse of notation) where G is regarded as 2-category by adding trivial cells on top.
2. An adjoint equivalence $\chi : \otimes \circ (\rho \times \rho) \xrightarrow{\cong} \rho \circ \otimes$, thus for all $f \xrightarrow{\alpha} g$, $f' \xrightarrow{\beta} g'$ in G cells

$$\begin{array}{ccc}
\rho(f)\rho(g) & \xrightarrow{\chi(f,g)} & \rho(fg) \\
\rho(\alpha)\rho(\beta) \downarrow & \swarrow \chi(\alpha,\beta) & \downarrow \rho(\alpha\beta) \\
\rho(f')\rho(g') & \xrightarrow{\chi(f',g')} & \rho(f'g')
\end{array} \tag{9.2}$$

(together with specified adjunction data for the $\chi(f, g)$'s).

3. An invertible modification ω given in components by

$$\begin{array}{ccc}
\rho(f)\rho(g)\rho(h) & \xrightarrow{\chi(f,g)^1} & \rho(fg)\rho(h) \\
1\chi(g,h) \downarrow & \swarrow \omega(f,g,h) & \downarrow \chi(fg,h) \cdot \\
\rho(f)\rho(gh) & \xrightarrow{\chi(f,gh)} & \rho(fgh)
\end{array} \tag{9.3}$$

This data is subject to the following axioms.

1. Strict unitality: For all objects f, g of G and morphisms α, β we have

$$\begin{aligned}
\rho(I) &= 1_x, \\
\rho(1_I) &= 1_{1_x}, \\
\chi(f, I) &= 1_{\rho(f)}, \\
\chi(\alpha, 1_I) &= 1_{\rho(\alpha)}, \\
\omega(f, g, I) &= 1_{\chi(f,g)}
\end{aligned} \tag{9.4}$$

and similar equations if the arguments of χ or ω are permuted.

2. The modification ω satisfies the coherence condition

$$\begin{array}{ccc}
 \rho^3 & \xrightarrow{\chi^1} & \rho^2 \xrightarrow{\chi} \rho \\
 \chi^{11} \uparrow & \searrow \omega^1 & \uparrow \chi^1 \searrow \omega \uparrow \chi \\
 \rho^4 & \xrightarrow{1_{\chi^1}} & \rho^3 \xrightarrow{1_{\chi}} \rho^2 \\
 & \searrow 1_{1\chi} & \downarrow 1_{\omega} \nearrow 1_{\chi} \\
 & & \rho^3
 \end{array} = \begin{array}{ccc}
 \rho^3 & \xrightarrow{\chi^1} & \rho^2 \xrightarrow{\chi} \rho \\
 \chi^{11} \uparrow & \searrow 1_{\chi} & \downarrow \omega \nearrow \chi \uparrow \chi \\
 \rho^4 & \xrightarrow{b} & \rho^2 \xrightarrow{\omega} \rho^2 \\
 & \searrow 1_{1\chi} & \downarrow \chi^1 \nearrow 1_{\chi} \\
 & & \rho^3
 \end{array} . \quad (9.5)$$

Definition 9.2. Let ρ, ρ' be two monoidal functors $G \rightrightarrows \Omega_x \mathcal{C}$ with constraints χ, ω and χ', ω' .

A monoidal natural transformation $\theta : \rho \longrightarrow \rho'$ consists of the following data.

1. A natural transformation $\theta : \rho \longrightarrow \rho'$ (by abuse of notation).
2. An invertible modification Π given in components by

$$\begin{array}{ccccc}
 & & \rho'(f)\rho(g) & & \\
 & \nearrow \theta(f)1 & & \searrow 1\theta(g) & \\
 \rho(f)\rho(g) & & \Downarrow \Pi(f, g) & & \rho'(f)\rho'(g) \\
 \chi(f, g) \downarrow & & & & \downarrow \chi'(f, g) \\
 \rho(fg) & \xrightarrow{\theta(fg)} & & & \rho'(fg)
 \end{array} \quad (9.6)$$

This data is subject to the following axioms.

1. Strict unitality: For all objects f we have

$$\begin{aligned}
 \theta(I) &= 1_{1_x}, \\
 \theta(1_I) &= 1_{1_{1_x}}, \\
 \Pi(f, I) &= 1_{\theta(f)}
 \end{aligned} \quad (9.7)$$

and a similar equation if the arguments of Π are permuted.

2. Π satisfies the following coherence condition.

(9.8)

Definition 9.3. Let ρ, ρ' be two monoidal functors $G \rightrightarrows \Omega_x \mathcal{C}$ with constraints χ, ω and χ', ω' . Let furthermore θ, θ' be two natural transformations $\rho \rightrightarrows \rho'$ with constraints Π, Π' .

A *monoidal modification* $m : \theta \longrightarrow \theta'$ consists of the following data.

1. A modification $m : \theta \longrightarrow \theta'$ (by abuse of notation).

This data is subject to the following axioms.

1. Strict unitality: $m(I) = 1_{1_x}$
2. m satisfies the following coherence condition.

(9.9)

One could also give analogous definitions where G is allowed to be a monoidal 2-category. Similar as in the 2-dimensional case it is then possible to compare the above definitions to the corresponding ones in the delooped setting (c.f. Remark 2.5 and the paragraph after Definition 4.4).

For fixed monoidal 2-categories the monoidal functors, monoidal natural transformations and monoidal modifications, each of them strictly unital, form a 2-category, the *normed functor 2-category* $[G, \mathcal{C}]_*$. The various compositions in $[G, \mathcal{C}]_*$ are given as follows. Fix functors ρ, ρ', ρ'' .

1. Let $(\theta, \Pi), (\theta', \Pi') : \rho \rightrightarrows \rho'$. Let $(\omega, \Lambda) = (\theta, \Pi) * (\theta', \Pi')$ be their horizontal composite. Then we have component wise

$$\omega(g) = \theta'(g) * \theta(g) \quad (9.10)$$

for objects g of G . For a 1-morphism $f \xrightarrow{\alpha} g$, $\omega(\alpha)$ is given by the pasting diagram

$$\begin{array}{ccc} \rho(f) & \xrightarrow{\rho(\alpha)} & \rho(g) \\ \theta(f) \downarrow & \swarrow \theta(\alpha) & \downarrow \theta(g) \\ \rho'(f) & \longrightarrow & \rho'(g) \\ \theta'(f) \downarrow & \swarrow \theta'(\alpha) & \downarrow \theta'(g) \\ \rho''(f) & \xrightarrow{\rho''(\alpha)} & \rho''(g) \end{array} \quad (9.11)$$

Finally Λ is obtained from

$$\begin{array}{ccccc} & & \rho''\rho & & \\ & \theta'1 \nearrow & & \searrow 1\theta & \\ & \rho'\rho & \Uparrow b & \rho''\rho' & \\ \theta 1 \nearrow & & & & \searrow 1\theta' \\ \rho\rho^2 & \Downarrow \Pi & \rho'^2 & \Downarrow \Pi' & \rho''^2 \\ \downarrow \chi & & \downarrow \chi' & & \downarrow \chi'' \\ & \xrightarrow{\theta} & \rho' & \xrightarrow{\theta'} & \rho'' \end{array} . \quad (9.12)$$

2. Vertical and horizontal composition of modifications is given by the respective operations in $\Omega_{\mathcal{C}}$.

It is a straightforward task to check the various axioms that have to be satisfied.

9.2 2-Degenerate Representations of Groups

Theorem 9.4. *Let G be a monoid, $\mathcal{C}$ a Gray-category and set $A := \Omega_x^3 \mathcal{C}^\times$. Then A is an abelian group and there is a one to one correspondence between equivalence classes of 2-degenerate representations of G on x in $\mathcal{C}$ and the third cohomology group $H^3(G; A)$ of G with coefficients in A .*

Proof. The proof is very similar to the 2-dimensional case (c.f. Proposition 4.7) and left as an exercise to the reader. $\square$

9.3 1-Degenerate Representations of Monoidal Categories

Let G be a monoidal category, let $\mathcal{C}$ be a Gray-category and $x \in \mathcal{C}$. Consider the braided 2-group $A := \Omega_x^2 \mathcal{C}^\times$.

Assume now that A is actually symmetric monoidal and recall the definition of the 2-cochain complex $C_\bullet^*(G; A) = (C^n, d^n, \omega^n)_{n \in \mathbb{Z}}$ from subsection 8.2. We have $C^n = [G^n, A]_*$ and d^n is given by the same formula as in the non categorified case. The ω^n 's are isomorphisms $d^n \circ d^{n-1} \xrightarrow{\omega^n} 0$. Recall that by the universal property Definition 6.4 of $Z_\bullet^n$ every d^n induces a map $\partial^n : C^n \longrightarrow Z_\bullet^{n+1}(G; A)$.

Recall also the construction of $\mathcal{G}(\partial^1)$ from subsection 7.1. It is a double category with object category $\mathbb{D}_0 = Z_\bullet^2$. Its category of morphisms $\mathbb{D}_1$ is constructed from C^1 and encodes the information how objects and morphisms of $Z_\bullet^2$ are related by the relation of differing by an image of ∂^1 . It is an intermediate step to building the quotient 2-group Z^2/∂^1 . In subsection 7.3 we constructed its horizontal 2-category $\mathbb{B}(\mathcal{G}(d))$. Recall also the Definition 2.3 of the representation (2-)category.

Theorem 9.5. *There is an isomorphism(!) of 2-groupoids*

$$\Phi : \text{Rep}_{x, \mathcal{C}}^1(G)^\times \xrightarrow{\cong} \mathbb{B}(\mathcal{G}(\partial^1)). \quad (9.13)$$

We split the proof of this theorem into several partial results.

Lemma 9.6. *There is a bijection of object sets*

$$\Phi_0 : \text{ob}(\text{Rep}_{x, \mathcal{C}}^1(G)^\times) \xrightarrow{\cong} \text{ob}(\mathbb{B}(\mathcal{G}(\partial^1))). \quad (9.14)$$

Proof. Recall subsection 7.3. We have

$$\text{ob}(\mathbb{B}(\mathcal{G}(\partial^1))) = \text{ob}(\mathcal{G}(\partial^1)) = \text{ob}(Z_\bullet^2). \quad (9.15)$$

Recall the construction of $Z_\bullet^2$ from section 8 as coherent 2-kernel of d^2 with respect to (d^3, ω^3) .

Objects of $\text{Rep}_{x, \mathcal{C}}^1(G)^\times$ are given by 1-degenerate representations ρ of G on $x \in \mathcal{C}$. Consider Definition 9.1. Since $\rho(\alpha) = \rho(\beta) = 1_{1_x}$ (9.2) simplifies to

$$\chi(f, g) \xrightarrow{\chi(\alpha, \beta)} \chi(f', g'). \quad (9.16)$$

That χ is an adjoint equivalence ensures that all its components lie in A . From the coherence for χ as a natural transformation it follows that this defines a functor $\chi : G^2 \longrightarrow A$.

The components of ω are all invertible thus lie in A . The coherence for being a modification gives us the naturality of ω as a transformation $\omega : F_1 \longrightarrow F_2$ where $F_1, F_2 \in C^3 = [G^3, A]_*$ are built from χ and the

monoidal products in the following way. For morphisms α, β, γ in G (and thus also objects) we have

$$F_1(\alpha, \beta, \gamma) = \chi(\alpha\beta, \gamma) + \chi(\alpha, \beta) \quad (9.17)$$

and

$$F_2(\alpha, \beta, \gamma) = \chi(\alpha, \beta\gamma) + \chi(\beta, \gamma). \quad (9.18)$$

(Compare for (9.3).) The coherence property (9.5) translates to the following equality of string diagrams in A .

$$(9.19)$$

In order for things to work out we will have to switch to $-\omega$. We thus set $\vartheta := -\omega$. Bringing the right hand side to the left via Remark 3.6 and then adding a $-$ on both sides (i.e. applying the inversion i) we get

$$(9.20)$$

We now want to construct a $\bar{\vartheta} : d^2(\chi) \xrightarrow{\cong} 0$ from ϑ . To accomplish this we use Remark 3.16 and Remark 3.20 and define

$$\begin{array}{c} \boxed{\bar{\vartheta}(f, g, h)} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \end{array} := \begin{array}{c} \boxed{\vartheta(f, g, h)} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \end{array} \cdot \quad (9.21)$$

Conversely we then have

$$\begin{array}{c} \downarrow \quad \downarrow \\ \boxed{\vartheta(f, g, h)} \\ \uparrow \quad \uparrow \end{array} = \begin{array}{c} \boxed{\bar{\vartheta}(f, g, h)} \\ \downarrow \quad \downarrow \end{array} \cdot \quad (9.22)$$

Inserting this into the left hand side of (9.20), ‘pulling’ all the $\bar{\vartheta}$ to the top and then reordering them if needed we obtain a diagram of the form

$$\begin{array}{c} \boxed{\bar{\vartheta}(g, h, k)} \quad \boxed{-\bar{\vartheta}(fg, h, k)} \quad \boxed{\bar{\vartheta}(f, gh, k)} \quad \boxed{-\bar{\vartheta}(f, g, hk)} \quad \boxed{\bar{\vartheta}(f, g, h)} \\ \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \\ \boxed{\mathbb{D}} \\ \uparrow \quad \uparrow \quad \uparrow \quad \downarrow \quad \downarrow \quad \downarrow \end{array} \quad (9.23)$$

where $\mathbb{D}$ can be composed of only cups and braidings. The top part of (9.23) is precisely $d^3(\bar{\vartheta})$. Note that the particular choices of ϑ and $\bar{\vartheta}$ ensured that the top part has the right source. By (9.20) (9.23) has to be equal to a collection of caps. By composing with $\mathbb{D}^{-1}$ from the bottom we thus get

$$d^3(\bar{\vartheta}) = \mathcal{E} \quad (9.24)$$

where $\mathcal{E}$ is some morphism that can be composed of only caps and braidings and has as source $d^3(d^2(\chi))$. But in the proof of Theorem 8.4 we saw that

any such morphism must be equal to $\omega^3(\chi)$. Thus the pair $(\chi, \bar{\vartheta})$ defines an object of Z_*^2 .

Clearly all of the above steps can be reversed to obtain a 1-degenerate representation from an element of Z_*^2 . This gives the bijection on objects. $\square$

Lemma 9.7. *Fix 1-degenerate representations $\rho, \rho' \in \text{ob}(\text{Rep}_{x,C}^1(G)^\times)$ with constraints $(\chi, \omega), (\chi', \omega')$ respectively. Then there is an isomorphism of categories*

$$\Phi_1 : \text{Rep}_{x,C}^1(G)^\times(\rho, \rho') \xrightarrow{\cong} \mathbb{B}(\mathcal{G}(\partial^1))(\Phi_0(\rho), \Phi_0(\rho')). \quad (9.25)$$

Proof. Again, recall subsection 7.3 where we constructed $\mathbb{B}(\mathcal{G}(d))$.

An object of $\text{Rep}_{x,C}^1(G)^\times(\rho, \rho')$ is given by a pair (θ, Π) as in Definition 9.2.

First we want to consider θ as normed functor $G \rightarrow A$. Let $g, g' \in \text{ob}(G)$ and $g \xrightarrow{\alpha} g'$ be a morphism in G . Because our representations are 1-degenerate the components $\theta(g), \theta(g')$ are just objects in A and the component $\theta(\alpha)$ is a map $\theta(g) \xrightarrow{\theta(\alpha)} \theta(g')$. That this assignment gives a functor follows from the coherence for the $\theta(\alpha)$'s. This functor is normed since θ is strictly unital. So we can view θ as object of $C^1 = [G, A]_*$.

Next we want to build from Π a morphism $d^1(\theta) \rightarrow \Phi_0(\rho) - \Phi_0(\rho')$. Now $\Phi_0(\rho)$ is given by a pair $(\chi, \bar{\vartheta})$ where (χ, ω) are the constraints of ρ and $\bar{\vartheta}$ is obtained from ω as in the proof of the previous Lemma. Similarly $\Phi_0(\rho')$ is given by a pair $(\chi', \bar{\vartheta}')$. Because of 1-degeneracy of ρ and ρ' (9.6) reduces to

$$\chi'(f, g) + \theta(g) + \theta(f) \xrightarrow{\Pi(f, g)} \theta(fg) + \chi(f, g). \quad (9.26)$$

By changing sides for $\chi'(f, g)$ and $\theta(fg)$, reordering the terms and letting f, g vary we obtain a natural transformation $d^1(\theta) \xrightarrow{P} \chi - \chi'$. (c.f. Remark 3.16 and Remark 3.20.) Naturality of P is a direct consequence of the fact that Π is a modification. So P is a morphism in C_*^2 . What we need however is $\partial^1 \theta \xrightarrow{P} (\chi, \bar{\vartheta}) - (\chi', \bar{\vartheta}')$ in Z_*^2 . Note that $\partial^1 \theta$ is given by $(\theta, \omega^2(\theta))$. We thus have to check the commutativity of

$$\begin{array}{ccc} d^2(d^1(\theta)) & \xrightarrow{d^2(P)} & d^2(\chi - \chi') \\ \omega^2(\theta) \downarrow & & \downarrow \delta^{-1} \\ 0 & \xleftarrow{(\bar{\vartheta} - \bar{\vartheta}')} & d^2(\chi) - d^2(\chi') \end{array} \quad (9.27)$$

where δ is the appropriate constraint. To see this consider (9.8). As a string

diagram it reads

Similarly as in the proof of Lemma 9.6 by bringing the right hand side to the left, inserting P and then moving things around one can infer from this (9.27).

This allows us now to define

$$\Phi_1((\theta, \Pi)) := (\theta, \Phi_0(\rho), \Phi_0(\rho'), P). \quad (9.29)$$

Again all of the steps we went through are reversible and thus this is a bijection.

Next we give Φ_1 on morphisms. Fix two objects (θ, Π) and (θ', Π') in $\text{Rep}_{x, \mathcal{C}}^1(G)^\times(\rho, \rho')$. A morphism $(\theta, \Pi) \longrightarrow (\theta', \Pi')$ consists of a modification m as in Definition 9.3. Because of 1-degeneracy it defines a natural transformation $(\theta \longrightarrow \theta')$ where θ, θ' are viewed as functors $G \rightrightarrows A$. The strict unitality of m as a modification implies its strict unitality as a natural transformation. By applying similar steps as above (9.9) becomes just (7.91) and again this is reversible. This gives Φ_1 on morphisms. That Φ_1 is a functor is straightforward. $\square$

It is left to show that Φ respects the composition functors.

Proof of Theorem 9.5. Fix representations ρ, ρ', ρ'' in $\text{Rep}_{x, \mathcal{C}}^1$. We deal with horizontal composition of 1-morphisms first. Let $(\theta, \Pi) : \rho \longrightarrow \rho'$ and $(\theta', \Pi') : \rho' \longrightarrow \rho''$. Their composition is given by $(\theta + \theta', \Lambda)$ where Λ is given as in (9.12). Write Λ as a string diagram. Bending all the strings down and reordering them (c.f. Remark 3.16 and Remark 3.20) gives precisely (7.12).

On 2-cells horizontal composition is the same in both categories. $\square$

10 Anhang

10.1 Zusammenfassung

Ziel der vorliegenden Arbeit ist eine systematische Untersuchung ‘hherer’ Darstellungen von Gruppen und hherdimensionalen algebraischen Objekten, sogenannten *2-Gruppen*.

Klassisch ist eine (lineare) Darstellung einer Gruppe G auf einem Vektorraum V gegeben durch einen Gruppenhomomorphismus $G \rightarrow \mathrm{GL}(V)$. Dies lsst sich auch als Funktor $\mathbf{B}G \rightarrow \mathbf{Vect}$ interpretieren, wobei $\mathbf{Vect}$ die Kategorie der Vektorräume (ber einem fixierten Krper) und linearen Abbildungen bezeichnet. $\mathbf{B}G$ ist hierbei die *Delooping-Kategorie* von G , die nur ein Objekt besitzt, dessen Endomorphismen die Gruppenelemente sind und deren Komposition durch die Gruppenmultiplikation gegeben ist.

Von diesem Gesichtspunkt aus gesehen bieten sich sofort mehrere Verallgemeinerungen des Begriffs einer Darstellung an. Es knnen zum Beispiel sowohl G als auch $\mathbf{Vect}$ durch beliebige Kategorien ersetzt werden. Eine andere Mglichkeit zur Verallgemeinerung ist gegeben indem, G und $\mathbf{Vect}$ durch hherdimensionale Strukturen ersetzt werden, sprich hhere Kategorien.

Wir wollen uns auf den Fall konzentrieren in dem G nur ein Object besitzt und somit eine monoidale Struktur trgt. Wir werden fr G maximal Dimension 2 zulassen. In der Zielkategorie wollen wir Dimension 3 nicht berschreiten. Unter diesen Voraussetzungen werden wir in der Lage sein eine vollstndige Klassifikation gewisser *degenerierter* Darstellungen in Termen der Gruppenkohomologie zu geben. Sptestens in Dimension 3 wird hier gewhnliche Gruppenkohomologie nicht mehr ausreichen. Wir werden deshalb eine kategorifizierte Variante derselbigen entwickeln. Diese ist nach dem besten Wissen des Authors in der Literatur bisher noch nicht aufgetaucht.

10.2 Abstract

The aim of this work is the systematic investigation of ‘higher’ representations of groups and higher dimensional algebraic objects, known as *2-groups*.

Classically a (linear) representation of a group G on a vector space V is given by a group homomorphism $G \rightarrow \mathrm{GL}(V)$. This can also be interpreted as a functor $\mathbf{B}G \rightarrow \mathbf{Vect}$, where $\mathbf{Vect}$ is the category of vector spaces (over some field) and linear maps. $\mathbf{B}G$ is the *delooping category* of G which has only one object and whose endomorphisms are the group elements. Composition on $\mathbf{B}G$ is given by the group multiplication.

From this point of view there are immediately several directions to generalize the notion of representation. For example we could replace both G and $\mathbf{Vect}$ with arbitrary categories. Another possibility is to replace them by higher dimensional structure, i.e. higher categories.

We want to focus on the case where G has only one object and thus carries a monoidal structure. We will allow G to have dimension not greater than 2 and the target category to have dimension not greater than 3. It turns out that certain *degenerate* Representations can be classified in terms of the group cohomology of G . However we will need a categorified version of the latter. Since this has to the author's best knowledge not appeared in the literature yet we will spend a considerable portion of this work on rigorously developing what we call 2-group cohomology.

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