universität wien

DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

A Tensor Network Study of Topological Quantum Phases of Matter

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfillment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer.nat)

Wien, 2016 / Vienna, 2016

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 411
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Physik, DK: Komplexe Quantensysteme
Betreut von / Supervisor:	Univ.-Prof. Dr. Frank Verstraete

Acknowledgements

There are people without whom this thesis would not be possible and the last four years would have been harder than it was. Although it's never possible to thank enough, I'll try my best.

I want to begin with my advisor, Frank. When I started my thesis, I was struggling with finding a good research line to work on and hopped through many topics. I had a prejudice against tensor networks since it always seemed to demand numerical work, which was (and regrettably still is) not the most appealing thing to me. All of this changed, when one day in May 2013 he came up with the idea that he calls "pulling through" and here is the thesis. This one idea fortunately turned out to be very fruitful and led to many exciting projects. He is always welcoming and takes his time with us-graduate students. Beyond his usual supervision there is always a hidden supervision around in the atmosphere, from which I benefited the most. This is the great freedom that we students have, which shows a big trust to us and gives an equally big responsibility. Sometimes I failed, but I learned pursuing good ideas from his constant push "any progress?" and his drive to work hands-on with us when there is a good idea around, which made me fail less and less over time. Thanks a lot, Frank! I gained a lot from you, scientifically and personally. My only hope is that this gain has not been one-sided.

I want to continue with our group in Vienna. It was a big pleasure to work with Bernhard, Damian, Dominic, James, Martin (Schwarz and Veznik), Valentin (Murg and Zauner) and Yoju. I learned a lot from all of you and enjoyed most of the time. Thanks a lot Bernhard for the office friendship, recommending the drawing program inkscape, translating the abstract of the thesis to German, and always trying to help with any issue I was dealing with. Thanks a lot Valentin for explaining MPS to me when I was (and still am) an amateur tensor networker and telling me the numerical truths about what is possible and what is not. It was always a great privilege to have Alexandra, Canan, Christiane, Christina, Miriam, Petra, Suzanne and Verena for easing all the administrative challenges that I had to face with: You made the things easier, many many thanks!

I spent seven months at IQIM-Caltech. This visit would not be possible with John's approval and willingness to have me around. Thanks a lot John for making me feel as an insider during my visit and for creating such a great environment at IQIM with the people you gathered. I benefited a lot and I'm very excited to be a part of it in the coming months. IQIM is a great place with its amazing and inspiring people. I thank all the IQIM community and in particular: Thanks Beni, David, Fernando, Jenia, Mario, Omar, Kristan for the inspiring discussions and all I learned from you. Many thanks Sujeet and Xie for including me to your project, I learned a lot from you and enjoyed the discussions very much. Many thanks Jackie for removing all the unscientific burden about my visit by offering many great IQIM opportunities, this allowed me to purely focus on physics without any distractions. Thanks Marcia for answering all my questions with your welcoming attitude.

Here comes the collaborators and friends. This list includes Dominic, Frank, Jutho, Kristan, Matthias, Michaël (Mariën), Michael (Walter), Nick, Norbert, Sujeet and Xie. I am grateful to all of you for our fruitful discussions and the projects that we have done together. Thanks a lot Frank and Norbert for initiating together the first MPO project and everything that you taught me. Thanks Dominic for the collaboration on many projects, and for joining the project, that is second chapter of this thesis, by bringing in a formal treatment of what is called FET-injectivity and by discussions we had. Thanks Kristan for all our discussions and time we spent on Walker-Wang models and other topological stuff, and on other matters. Many thanks Michael (Walter) for the supportive supervision and collaboration starting from my master studies and continuing with the projects that we are currently doing with Dominic and Kristan. Many thanks Sujeet, I enjoyed very much all the scientific and general discussions from biology to any other topic you name. And thanks Matthias for giving me a master thesis project about 6j-symbols, I feel that this affected every scientific paper that I wrote so far, they all include these symbols in some way. Finally I want to thank Nils Carqueville in this part, although we neither wrote a paper together, nor collaborated. He impacted this thesis in a great manner by suggesting looking at the concept of weakening and higher algebras/categories when I discussed with him about some ideas to find new topologically ordered models. Thanks Nils, this led to the main ideas of the second chapter of this thesis. I also learned quite a lot from your lectures on TQFT.

I also want to thank the potential reviewers of my thesis, Nils Carqueville, Robert König and Steve Simon, for agreeing on reviewing my thesis. I very much appreciate your efforts and time for reading my thesis.

Thanks Arif, Gökhan, Okan for your friendship through many years. Thanks Ender, Özge and Özgür hoca for your invaluable friendship.

Towards the end, I'd like to thank my sister Zahire, my mother Rabia and my father Mustafa for their unconditional love and support. Each one of you has impacted me in many ways that brought me to this day. Üzerimde hakkını hiçbir zaman ödeyemeyeceğim emekleriniz çok büyük, umarım bütün bunlara layık olabilmişimdir.

Last and definitely not least, I want to thank my wife Szeyu for accompanying me in the last two years and bearing the hard but mostly enjoyable moments. What we did and are doing are amazing and would be impossible without your constant love and support.

To those who enjoy reading it and would enjoy if they had the opportunity to read it . . .

Ne bilginler geldi, neler buldular!
Mumlar gibi dünyaya ışık saldılar.
Hangisi yarıp geçti bu karanlığı?
Birer masal söyleyip uykuya kaldılar.

Hayyam

Abstract

In this thesis, the challenge of understanding the structure of quantum phases of matter, specifically the zero temperature properties, is taken using the framework of tensor networks.

A general framework for intrinsic topological order using tensor networks is introduced for the ground states of physical systems living in two space dimensions. Conditions for the existence of topological order are introduced: MPO (matrix product operator)-injectivity and pulling through. It is shown that these conditions capture the appropriate concepts of topological order emerging in a many body tensor network state: MPO-injectivity determines the correct local subspaces such that, with the use of pulling through condition, it can be extended to large regions determining the global subspace at the boundary of the given topologically ordered tensor network state. For the intrinsic topological order, MPO plays the role of determining the correct local subspaces, while in the symmetry protected case it implements the global symmetry of the system on the virtual degrees of freedom of the tensor network. Known fixed point models in the literature, such as Kitaev's quantum-double and Levin-Wen string-net models, are shown to be specific examples of the framework. This new framework goes beyond the renormalization fixed-point models and is suitable for studying topological phase transitions in the ground state space.

The framework introduced for two space dimensions is generalized to higher dimensions with the concept of topological quantum field theories (TQFTs) in mind. Tensor networks for the partition functions of lattice combinatorial TQFTs are found and conditions for topological invariance are implemented in terms of local tensors. The topological invariance conditions of these TQFTs are found to be specific cases of the TNO (tensor network operator)-injectivity and pulling through conditions in higher dimensions. TQFTs, which may exhibit various global symmetries (and hence naturally include SPT and SET phases), are renormalization fixed-point examples of the general framework. The latter includes models which are away from the fixed-points and is thus suitable to study phase transitions. Finally, Walker-Wang models are studied in detail, using the new tensor network framework. The local tensor and the TNO are expressed in terms of the local data of the relevant TQFT, TNO-injectivity and pulling through conditions are shown to be the topological invariance conditions of the TQFT in this specific model.

Zusammenfassung

Diese Arbeit behandelt die Frage nach der Struktur von Quanten-Phasen kondensierter Materie, im Speziellen die Eigenschaften am absoluten Nullpunkt, im Rahmen von Tensornetzwerken.

Diese Arbeit behandelt die Frage nach der Struktur von Quanten-Phasen kondensierter Materie, im Speziellen die Eigenschaften am absoluten Nullpunkt, im Rahmen von Tensornetzwerken. Es wird gezeigt, dass diese Bedingungen die jeweiligen Konzepte emergierender topologischer Ordnung in Vielteilchen Tensornetzwerk-Zuständen wiedergeben: MPO (Matrixproduktoperator)-Injektivität bestimmt den korrekten lokalen Teilraum, welcher mit Hilfe der pulling-through Bedingung auf große Gebiete ausgedehnt werden kann. Diese Gebiete bestimmen wiederum den globalen Teilraum am Rand des jweiligen topologisch geordneten Tensornetzwerk-Zustands. Bei intrinsischer topologischer Ordnung, wird der korrekte lokale Teilraum durch den jeweiligen MPO bestimmt, während der MPO im symmetriegeschützten Fall die globale Symmetrie des Systems in den virtuellen Freiheitsgraden des Tensornetzwerks implementiert. Es wird gezeigt, dass bekannte Fixpunkt-Modelle aus der Literatur, wie Kitaev quantum double- und Levin-Wen string-net-Modelle, Spezialfälle des eingeführten Frameworks sind. Der neue Rahmen geht über Fixpunkte hinaus und ist geeignet, topologische Phasenübergänge im Grundzustandraum zu studieren.

Vor dem Hintergrund topologischer Quantenfeldtheorien (TQFTs) wird das Framework für den zweidiminsionalen Fall auf höhere Dimensionen verallgemeinert. Zustandssummen von lattice combinatorial TQFTs werden als Tensornetzwerke formuliert und Bedinungen für topologische Invarianz werden mittels lokaler Tensoren implementiert. Die Bedinugnen für topologische Invarianz dieser TQFTs werden als Spezialfälle der TNO (Tensornetzwerk-Operator)-Injektivität und pulling-through Bedingung in höheren Dimensionen identifiziert. TQFTs, welche verschiedene globale Symmetrien aufweisen können (und damit auf natürliche Art SPT und SET Phasen beinhalten), sind Beispiele von Renormierungs-Fixpunkten des allgemeinen Frameworks. Dieses beinhaltet Modelle, welche von den Fixpunkten entfernt sind und ist somit geeignet, Phasenübergänge zu studieren. Abschließend, werden Walker-Wang Modelle im Rahmen von Tensornetzwerken detailliert behandlet. Der lokale Tensor und der TNO werden durch lokale Eigenschaften der relevanten TQFT ausgedrückt. Es wird gezeigt, dass TNO-Injektivität und pulling-through Bedingungen topologischen Invarianzbedingungen der TQFT in diesem speziellen Modell sind.

Contents

Chapter 0

Introduction

This part of the thesis is meant to be a general introduction to the whole thesis. I intend to give a bird's-eye view of the main subject of the thesis: Understanding topological order in/using tensor networks. Then, I shortly summarize the achievements of each chapter.

Topological order has been an active and exciting field of research since the work of Wegner remarking a phase transition with no local order parameter in 70's and the experimental discovery of the fractional quantum hall states in 80's. At the time, it was puzzling since it didn't fit the usual understanding of order in quantum matter that was basically the Landau's classification using symmetry breaking. The situation on the theoretical side has been more or less clarified as some exactly solvable toy models appeared, such as Dijkgraaf-Witten theories, Kitaev's quantum double models, Levin-Wen string-net models. These are systems having locally indistinguishable ground states, which are nevertheless orthogonal and drastically different globally. Furthermore, these models exhibit new kind of order in their ground states, which are accessible only through extended operators, such as Wilson loop operators, etc. They also are found to have quasi-particle excitations with anyonic statistics that goes beyond the usual bose and fermi statistics. Many others also took the challenge and scrutinized other properties of these models, such as the phase transitions, condensation of quasiparticle excitations, etc.

In late 90's, a concept called quantum error correction has been introduced, whose main aim is to come up with a scheme that has the ability to correct errors when the information (qubit) is encoded in a quantum state. Kitaev's big insight was that the ideas from topological order are intimately related to quantum error correction. He gave a simple example of this idea, now known as Toric code: A many-body local gapped hamiltonian with groundstate degeneracy on topologically nontrivial manifolds, and one requires a specific nonlocal loop operator to change a ground state to another one. He says that if one encodes logical qubits as the ground states of the Toric code on torus, i.e. the codespace is chosen to be the ground state space of the model, these logical qubits are inherently robust to local noise (errors), since the probability that a local noise implements a nontrivial operation in the codespace decays exponentially with the system size. This is impossible in thermodynamic limit, hence highly improbably for realistic systems. Hence, topologically ordered systems may imply a robust quantum memory at zero-temperature. Indeed that's the case for an interesting lattice TQFT, where the ground states are connected to each other only by extended topologically nontrivial operators of loop-type or higher dimensional surface-type.

In this thesis, we also have taken the challenge of advancing the human-kind's understanding in topologically ordered systems, and we approach the problem from the point of view of tensor network states. One reason of our choice is our belief that tensor networks are the right language of expressing the low-lying energy eigenstates of local many-body Hamiltonians. This belief indeed has turned out to be proven in some cases, such as when the system is in one space dimensions or in higher space dimensions with some certain conditions. Another reason of using tensor networks comes from the fact that topological order (or order in any broad physical sense) is indeed originated from the entanglement of the quantum state (a ground state or some excited states) that we are working with. Tensor networks is a natural way of encoding entanglement in a quantum state in a local fashion, in the sense that quantum state is determined by contraction of local tensors and may exhibit nontrivial entanglement properties.

This thesis consists of a collection of three works, each given in separate chapters:

1. Characterizing topological order with matrix product operators. [1]

   Authors: M.B. Şahinoğlu, D. Williamson, N. Bultinck, M. Mariën, J. Haegeman, N. Schuch and F. Verstraete.

   arXiv preprint arXiv:1409.2150.

2. Topological order in any dimension: A tensor network framework. [2]

   Authors: M.B. Şahinoğlu, M. Walter, D.J. Williamson.

   To appear on arXiv soon.

3. A Tensor network study of Walker-Wang models. [3]

   Authors: M.B. Şahinoğlu, K. Temme, M. Walter.

To appear on arXiv soon.

In the first chapter, we analyze the conditions for a tensor network state (TNS) to be topologically ordered in two space dimensions. We put axioms on local tensors such that the TNS constructed by one of these local tensors is a topologically ordered ground states of a local gapped many-body Hamiltonian. The characterization that we propose uses matrix product operators (MPOs) to characterize the local virtual subspaces such that a nontrivial topological order is possible to exist in the TNS. The local subspaces are characterized by MPOs (*MPO-injectivity* axiom) and these MPOs that construct a loop operator are extendible to larger and larger regions with the use of another axiom called *pulling through*. We show that given these axioms, the TNS may lead to a parent hamiltonian with degenerate ground state space and the ground states are connected to each other via topologically nontrivial MPOs. The interpretation of pulling through condition then becomes transparent: The specific position/path of the MPOs does not matter (since we can deform it anywhere on the lattice), but depending on the topological properties of the manifold on which MPO is supported, it changes one ground state to another one. This is a clear manifestation of the fact that different ground states are locally indistinguishable, however globally orthogonal. In this work, we also show that the models constructed via unitary fusion tensor categories (called Levin-Wen models) are specific examples of this framework: We explicitly show that MPO-injectivity and pulling through conditions become the pentagon equation for these specific models. We also include an outlook towards understanding the symmetry protected topological order using the new framework (which has been studied in another paper), by modifying the axioms in a way so that we can capture the global symmetry of the system. We basically modify the axiom of pulling through: When an MPO of type $g \in G$ is pulled through a local tensor, it implies a group action on the physical level of the local tensor, or in other words when a group action is acted on the physical level it implies a loop of MPO on the virtual levels. The implications of MPO-symmetry on the boundary are important: The boundary state must be invariant under the action of MPOs labeled by $g \in G$ in order for the state to be invariant under the global symmetry. There is an obstruction to realize this if we have a nontrivial SPT state in the bulk: Either the state is gapless on the edge and it is symmetric under the global symmetry action, or the states is gapped everywhere but the global symmetry is broken on the edge.

In the second chapter, we generalize the framework in the first chapter to higher dimensions. This chapter, in a general sense, contains all other chapters in it, but on a general level of discussion. We start with certain topological quantum field theories (TQFTs), called discrete lattice combinatorial TQFTs, and show that they immediately imply a tensor network structure for the partition function. These TQFTs are constructed via a single tensor, called the *simplex tensor* and the topological invariance of the partition function is implied by a finite set of local conditions, called Pachner equations. Using these conditions, we construct a local, commuting projector hamiltonian and a tensor network state representation of its ground states. We further find tensor network operators (TNOs which are surface operators) that can be pulled through the lattice. These are all implied by the topological invariance conditions of the partition function. These TQFTs are again a special case of a more general tensor network framework, where we axiomatize the TNS with the conditions called *TNO-injectivity* and *pulling through*: In higher dimensions, the TNOs are surface operators that determines the local virtual subspace and that can be deformed through the lattice (i.e., that can be pulled through the local tensor). We shortly analyze the known fixed point models and describe their tensor network construction. Indeed, intrinsic topological order, symmetry protected/enhanced topological orders are contained in this formalism, with some differences in the details of the implications of the TNO acting on the TNS. For intrinsic topological order, action of a TNO-surface operator supported on topologically trivial manifold leaves the state invariant, in constrast to the SPT/SET case in which this operator may imply an action of a group element $g \in G$ realizing some global symmetries of the system. We finally analyze the notions of extended TQFT and higher category theory, and remark their relation to TNO-symmetries on lower dimensional submanifolds.

In the third chapter, we study a particularly interesting class of $3+1$ dimensional models, called Walker-Wang models. These models are constructed via the data provided by premodular tensor categories. In case one uses a modular tensor category to construct a Walker-Wang model, one finds out that there are no extended excitations in the bulk (they are penalized by energy growing with the distance between the quasi-particles), however there are extended excitations on the boundary, imagining we put the model on a 3-manifold with boundary. This indeed resembles the typical phenomenon that one finds in topological insulators, e.g., the material is insulating in the bulk, nevertheless conducting on the boundary. Although we are being vague in this analogy, these models still can be thought of as simple toy models where a strange phenomenon occurs: there may be boundary topological order although the bulk looks physically trivial. Given this general motivations to Walker-Wang models, we investigate the tensor network state structure of ground states of these models. We explicitly find the local tensor and the locally deformable tensor network operators, which we show that are given by the data included in the relevant tensor category. This chapter can be seen as a concrete example model of the framework that we describe in Chapter 3.

Although each chapter includes its own conclusions and discussions on future directions, we conclude the thesis by offering a short summary of the findings from each chapter in a combined manner and some perspectives on potentially important future directions.

Chapter 1

Characterizing topological order with matrix product operators

One of the most striking features of gapped quantum phases that exhibit topological order is the presence of long range entanglement that cannot be detected by any local order parameter. The formalism of projected entangled-pair states is a natural framework for the parameterization of gapped ground state wavefunctions which allows one to characterize topological order in terms of the virtual symmetries of the local tensors that encode the wavefunction. In their most general form, these symmetries are represented by matrix product operators acting on the virtual level, which leads to a set of algebraic rules characterizing states with topological quantum order. This construction generalizes the concepts of G- and twisted injectivity; the corresponding matrix product operators encode all topological features of the theory and provide a complete picture of the ground state manifold on the torus. We show how the string-net models of Levin and Wen fit within this formalism, and in doing so provide a particularly intuitive interpretation of the pentagon equation for F-symbols as the pulling of matrix product operators through the string-net tensor network. Our approach paves the way to finding novel topological phases beyond string-nets, and elucidates the description of topological phases in terms of entanglement Hamiltonians and edge theories.[1]

[1]This chapter is based on "Characterizing topological order with matrix product operators." [1]
Authors: M.B. Şahinoğlu, D. Williamson, N. Bultinck, M. Mariën, J. Haegeman, N. Schuch and F. Verstraete.
arXiv preprint arXiv:1409.2150.

1.1 Introduction

Classifying phases of matter is one of the most important problems in condensed matter physics. Landau's theory of symmetry breaking [4] has been extremely successful in characterizing phases in terms of local order parameters, but it has been known since the work of Wegner [5] that topological theories do not necessarily exhibit such a local order parameter, and hence that different topological phases cannot be distinguished locally. One of the main reasons to call such phases topological is the fact that the ground state degeneracy depends on the topology of the surface on which the system is defined [6]. Since the realization that quantum Hall systems exhibit topological quantum order [7], significant effort has been put into classifying all topological phases [8–13]. A very large class of models exhibiting topological order was constructed by Levin and Wen [14], which is conjectured to provide a complete characterization of non-chiral topological theories in two dimensions.

A recent development at the interface of quantum information and condensed matter theory is the growing use of projected entangled-pair states (PEPS), and more general tensor network states [15–17]. To construct a PEPS one associates a tensor, representing a map from some virtual vector space to the local physical Hilbert space, to each site of a lattice and performs tensor contractions on the virtual space according to the graph of the lattice. The resulting quantum state can then be used as an ansatz for the ground state of a local Hamiltonian on that lattice [18–20]. There are two immediate and very important properties of PEPS. Firstly, for every PEPS there exists a local, positive-semidefinite, frustration free operator called the parent Hamiltonian whose kernel contains the PEPS. Secondly, the entanglement entropy of a region R is upper bounded by $|\partial R| \log D$ rather than the volume, where D is the virtual dimension and $|\partial R|$ is the number of virtual bonds crossing the boundary of the region. Hence PEPS are the ground states of local Hamiltonians and obey an area law (provided the bond dimension is upper bounded by a fixed constant D as the system size increases).

In this work, we propose a framework for the complete characterization of quantum order in gapped ground states using PEPS, based on the virtual symmetries of the local tensors. The PEPS network encodes the physical state in a region as a linear map from the virtual boundary space to the physical space. The symmetry dictates that this map can have no support outside the invariant subspace of the virtual symmetry action. Our framework thus starts from the consistent characterization of this invariant subspace across arbitrary lattice bipartitions in terms of local tensors that form a projection matrix product operator (MPO). This contains and generalizes the concept of G injectivity [21], where the symmetry corresponds to the tensor product action of some symmetry group G, and its extension to twisted group actions [22]. The generalized notion of MPO injectivity developed in this work provides a natural extension that applies even when no group symmetry is involved, which is required for the description of more general topological orders including the string-net models.

We first define MPO injectivity, proceed by formulating a set of algebraic conditions that have to be satisfied by valid MPOs, and then show how the ground state degeneracy and topological order is determined by those MPOs. We go on to illustrate that all ground states of the string-net models satisfy the proposed algebraic conditions, and that the key *pulling through* condition for these models is implied by the pentagon equation for the F-symbols. We conclude by providing an outlook towards possible extensions of the framework to fermionic models and higher dimensional theories, and a discussion of the potential relevance of our formalism to the development of more efficient PEPS contraction schemes.

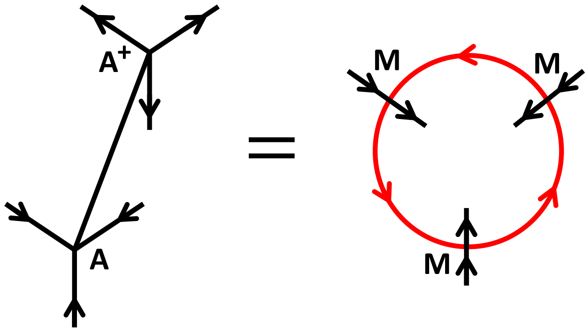
1.2 MPO injectivity

Consider a PEPS on a lattice with coordination number three, such that the PEPS tensors A can be understood as a linear map from the three D-dimensional virtual spaces to the d-dimensional physical space, $A : \mathbb{C}^D \otimes \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$. By contracting the tensors within an arbitrary region V, the PEPS network defines a map A_V from the virtual boundary space $(\mathbb{C}^D)^{\otimes|\partial V|}$ to the physical bulk $(\mathbb{C}^d)^{\otimes|V|}$. A PEPS is called MPO injective, if we can associate to every virtual link a MPO tensor $M : \mathbb{C}^D \otimes \mathbb{C}^m \to \mathbb{C}^D \otimes \mathbb{C}^m$, such that for every compact, topologically trivial region V the map A_V is injective on a subspace S_V for which the corresponding projector is given by the MPO obtained from contracting the tensors M associated to the links along the boundary.

Throughout the remainder of this work, we make use of standard tensor diagram notation, depicting each tensor as a point (or shape) with a leg emerging for each vector space it acts upon, and where a leg joining two tensors implies contraction of the associated indices. We associate a PEPS tensor A to each intersection of black lines (corresponding to the virtual spaces $\mathbb{C}^D$) and a MPO tensor M to each intersection of a black and red line (corresponding to $\mathbb{C}^m$).

1.3 Algebraic Rules for MPO injectivity

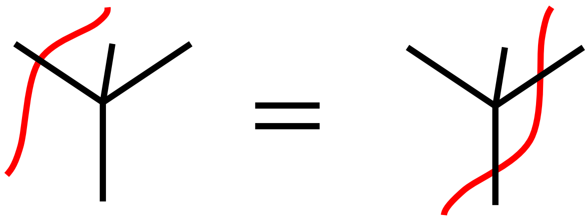
At the level of a single site (potentially after some blocking), MPO injectivity requires that the PEPS tensor A has a pseudo-inverse A^+ such that, pictorially,



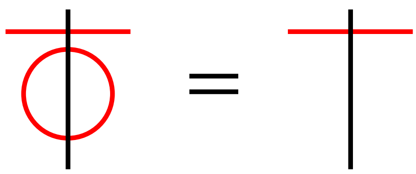
, (1.1)

We now formulate a set of local algebraic rules which imply that the PEPS built from local tensor A is MPO injective with respect to the MPO built from M, Eq. (1.1).

Firstly, the *pulling-through condition* states that the MPO can pass through the PEPS tensor A at the virtual level

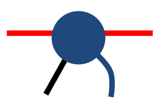

, (1.2)

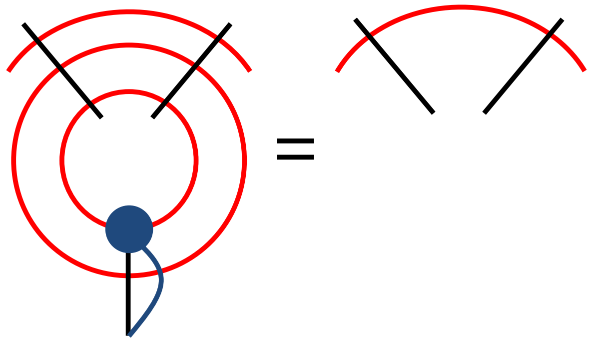
and thus acts as a generalized symmetry. By applying the pseudo inverse A^+ and Eq. (1.1), we can express this condition purely in terms of MPO tensors. We also require the following condition for an elementary MPO loop


, (1.3)

which guarantees that any closed MPO is a projector, as required by the definition of MPO injectivity. This can easily be seen by taking two elementary loops of MPOs, pulling one through the other onto a single leg (using (1.1) and (1.2)) and then applying (1.3).

We also require the existence of a tensor X,

$\mathbf{X} :=$  , (1.4) henceforth referred to as the *generalized inverse*, such that



$$. \qquad (1.5)$$

Note that there is no tensor associated to the intersection of blue and red lines.

Together with the pulling through condition [Eq. (1.2)], Eq. (1.5) implies that the injectivity of the PEPS tensors is stable when multiple tensors are concatenated, i.e. on any contiguous region V there exists a pseudo-inverse $A_V^\dagger$ which can be applied on the physical degrees of freedom in V to give access to the virtual degrees at the boundary within the subspace defined by the MPO projector. The PEPS thus satisfies the definition of MPO injectivity. These conditions further imply the existence of a local, frustration free parent Hamiltonian for which the ground state subspace in a contiguous region V corresponds exactly to the range of A_V, i.e. the concatenated PEPS tensors with an arbitrary state on the virtual boundary. This is known as the intersection property and proven, together with the stability under concatenation, in the supplementary material.

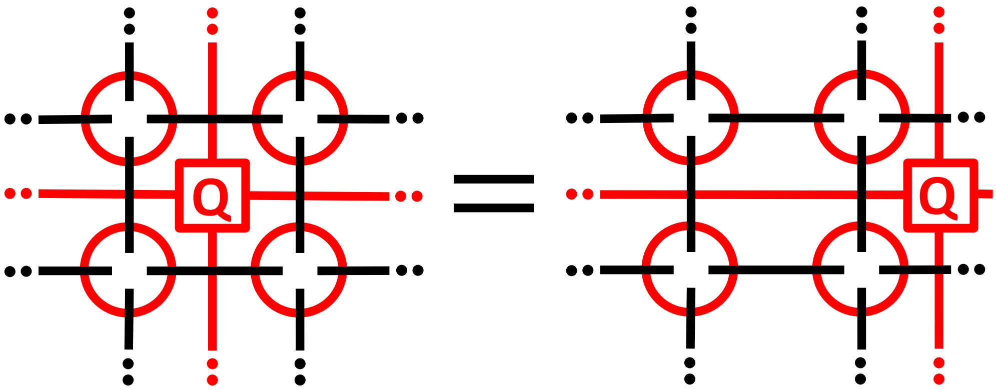
We conjecture that in two spatial dimensions all gapped ground states admitting a PEPS description can be constructed from MPO injective [Eq. (1.1)] tensors, with the MPO arising as a solution of Eq. (1.2) and Eq. (1.3) for which there exists a tensor X satisfying Eq. (1.5). Note that this includes the particular cases of injective PEPS, where the MPO is just a product operator ($m = 1$) of identities $\mathbb{1}_D$, and G-injective PEPS [21], where the MPO is $\sum_{g\in\mathsf{G}} g^{\otimes|\partial V|}$[2] and M thus takes a diagonal form with $m = |G|$ diagonal elements. In the generic case, the MPO tensors M will have a canonical block-diagonal form, in which each different block will yield an injective MPO that individually satisfies Eq. (1.2) and these MPOs together define the virtual symmetries of the local PEPS tensors. They act as virtual strings that can move freely through the lattice and are thus locally unobservable, except at open endpoints. The appearance of string operators in the characterization of two-dimensional topological models comes as no surprise [23, 24]. Indeed, the notion of MPO injectivity places the physical degrees of freedom in one-to-one correspondence with the virtual degrees of freedom in a certain subspace, allowing us to import the physical string operators to the virtual level.

[2]Here, g is a D-dimensional, semi-regular representation of the group elements of G.

Furthermore, the PEPS description is advantageous for models away from a renormalization fixed point, as we can essentially treat the virtual string operators as if we were at a fixed point, avoiding the need to explicitly broaden them over some correlation length.

1.4 Ground state subspace

As a result of the algebraic rules, the ground state subspace of a MPO injective PEPS parent Hamiltonian is spanned by a finite number of states. If the tensor network is closed on a torus, we find (see supplementary material) that the ground state subspace is spanned by states obtained from tensors, Q, that satisfy



, (1.6)

and the equivalent equation for the vertical direction, with the dots indicating periodic boundary conditions. This ensures that the physical ground states are translation invariant and the location of the closure is of no significance. The ground state degeneracy on the torus is then given by the number of linearly independent physical states arising from the solution of Eq. (1.6) that remain distinct and normalizable in the thermodynamic limit.

As this formalism was set up in order to characterize all gapped quantum phases, it naturally also includes states with discrete symmetry breaking (i.e. cat states). In that case, the different ground states can have local order parameters and the degeneracy will be independent of the topology. Throughout the remainder of this work, we focus on the more interesting case of topological phases. To determine whether any of the degeneracy is truly topological in nature, one must compare the ground state degeneracy of the tensor network on the topological manifold of interest to that arising on a topologically trivial manifold (such as the sphere). This provides a deterministic recipe for checking, for any given model, the topological dependency of the ground state subspace.

1.5 Identifying the topological order

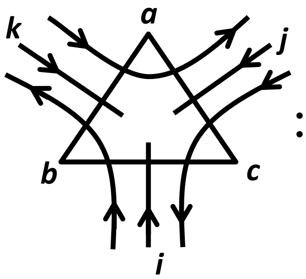
Let us now discuss how to identify the topological order in MPO injective PEPS. Since MPO injectivity is stable under concatenation, for any contiguous region the virtual indices at the boundary are supported on the invariant subspace of the MPO. This is on the one hand reflected in the low-energy excitations at the edge, which are in one-to-one correspondence with admissible boundary conditions. The edge dynamics are thus restricted to the invariant subspace of the MPO [25], which provides topological protection to the edge and allows one to infer the structure of the MPO from the edge physics. At the same time, it is reflected in the the entanglement spectrum and the corresponding entanglement Hamiltonian [26]. The entanglement spectrum is also restricted to the invariant subspace, and therefore, the entanglement Hamiltonian contains a universal term with infinite strength which restricts the system to be in the invariant subspace [27]. A consequence of this restriction (together with the MPO injectivity) is that the number of non-zero eigenvalues is equal to the dimension of the invariant subspace of the MPO, which gives rise to a topological correction to the zero Rényi entropy. In the case of RG fixed points, the correction should not depend on the Rényi index (as has been shown for string-net models [28]), which implies a corresponding topological correction to the entanglement entropy [29, 30]. Note that a non-zero correction in the zero Rényi entropy requires an MPO projector with several blocks in its canonical form, as a single blocked MPO defines a subspace S_V whose dimension scales exactly as $\log(\dim S_V) = c|\partial V| + \mathscr{O}(1/|\partial V|)$ without constant term.

The topological correction does not fully characterize the topological phase. To this end, one must obtain the modular S and T matrices, which contain the mutual and self braiding statistics of the topological excitations [31]. The fusion rules of the topological excitations can also be obtained from the S matrix via the Verlinde formula. An advantage of the MPO formalism is that it allows for an unambiguous definition of modular transformations on the ground states of a lattice system on a torus, obtained by solving Eq. (1.6). The 90° rotation can be performed directly on the ground state tensors Q_i defined in Eq. (1.6). The Dehn twist, on the other hand, corresponds to increasing the winding number of the MPO along the twisting direction by one. If one uses A^+A for the PEPS tensors then the overlap matrix of the original ground states with the rotated (twisted) ground states will only contain universal information and therefore correspond to the S (T) matrix [32–34].

The solutions of Eq. (1.6) will in general not correspond to the minimally entangled states (MES), i.e. the states that have the physical interpretation of being threaded with a definite anyon flux through one of the holes of the torus [32]. One has to find a unitary basis transformation of the ground state subspace (a basis transformation of the tensors Q) that makes T diagonal and S symmetric [35] in order to read off the topological properties of the excitations. Note that, by wrapping the projection MPO around the torus in one direction, we can construct a state with a topological flux corresponding to some Abelian anyon threaded through the hole in the orthogonal direction, since these states have maximal topological entropy 2γ (while for general topological fluxes, the correction is $2\gamma - \log(d_i^2)$ with d_i the quantum dimension). In the case that all anyons are non-Abelian, this MES clearly corresponds to the one with a trivial flux. But since this construction works for any anyon theory it is very likely that it will always lead to the MES with a trivial flux.

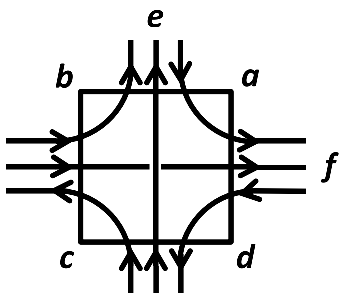
1.6 Example: String-net models

In this section, we show that the set of models described by MPO injective PEPS contains all string-net models [14]—the largest set of many-body bosonic lattice models exhibiting topological order available in the literature. The PEPS description of string-net models was constructed in Refs. [36, 37] and is summarized in the supplementary material. We start from the definition of the string-net PEPS tensor



$$:= \sqrt{v_i v_j v_k}\, G^{ijk}_{abc}\,, \tag{1.7}$$

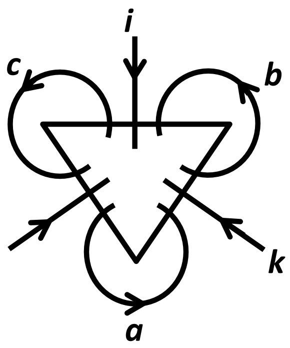
where the i, j, k legs are copied to the physical level, and define the corresponding MPO tensor



$$:= G^{ab^*e}_{cdf}\,, \tag{1.8}$$

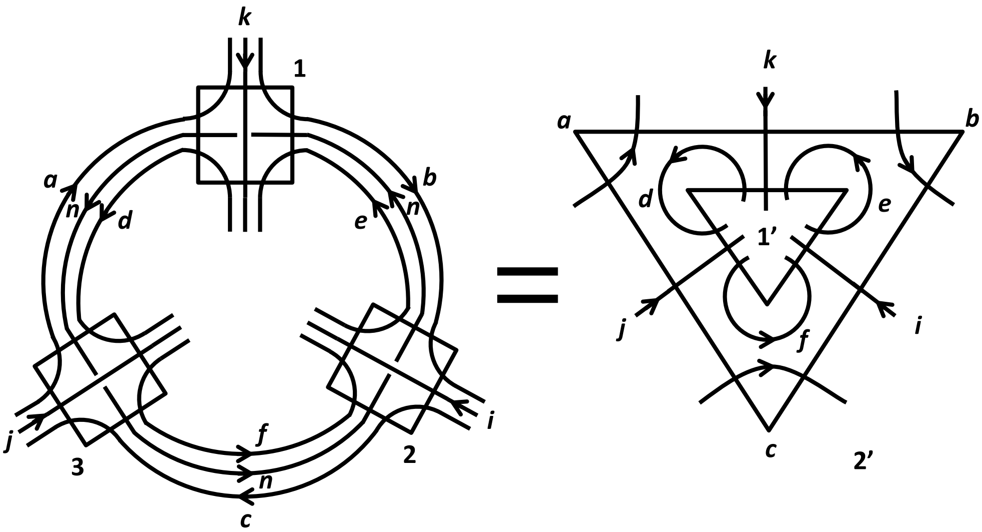
Note that we explicitly depict all tensors as 2D shapes for the string-net PEPS. The G-symbol is a symmetrized version of the F-symbol, defined in Eq. (1.A.39), which is invariant under simultaneous cyclic permutation of the upper and lower indices. These diagrams use the convention that a pair of tensor legs i, i' that are connected through the body of a tensor corresponds to a Kronecker delta on the associated indices, i.e. $T_{\{j\},i,i'} = \tilde{T}_{\{j\},i}\delta_{i,i'}$; we therefore use a single label in the pictures. In particular, the MPO tensor has a block diagonal structure because it acts diagonally on f, which therefore acts as a label for the different virtual strings that can be constructed. As a final convention, we always associate a multiplicative factor d_λ (the quantum dimension) to each term in a sum over any index λ (appearing as a closed loop in the diagrams below) [3].

The pseudo-inverse of the PEPS tensor [Eq. (1.7)] is



$$:= G^{i^*j^*k^*}_{a^*b^*c^*}, \tag{1.9}$$

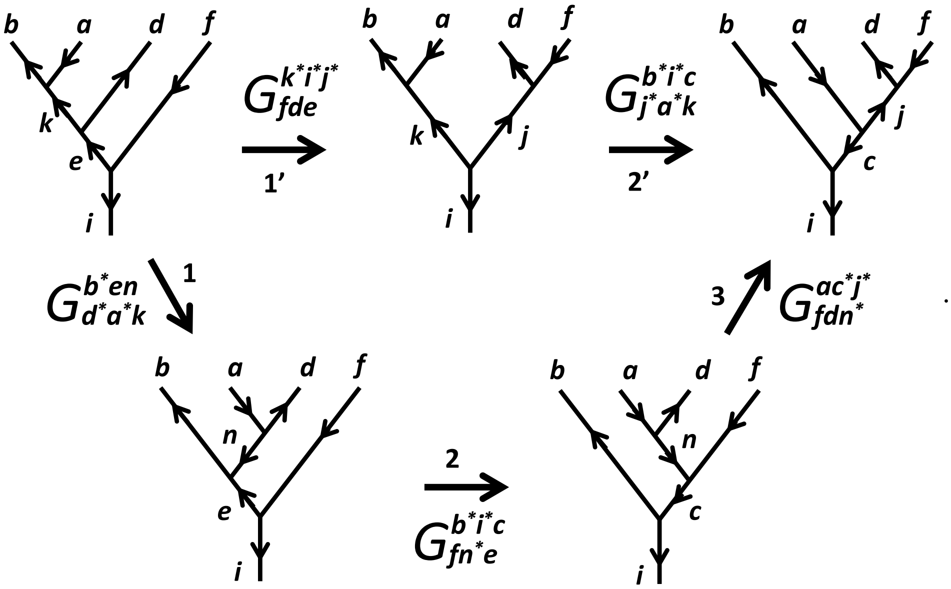
and it can readily be verified that Eq. (1.1), here given by



$$, \tag{1.10}$$

[3]This convention can be implemented locally by adding multiplicative factors to the string-net PEPS (1.7) and MPO (1.8) tensors such that every closed loop of λ gets a factor of quantum dimension d_λ. Attaching a factor of $d_\lambda^{(1-\alpha/\pi)/2}$ to every bending line of the string-net PEPS and MPO tensors, where α is the bending angle in radians. Then, for any closed loops with n bending points, i.e., polygons with n edges, we get $d_\lambda^{(n-\alpha'/\pi)/2} = d_\lambda$ because α', the total interior angle of n-polygon, is equal to $(n-2)\pi$.

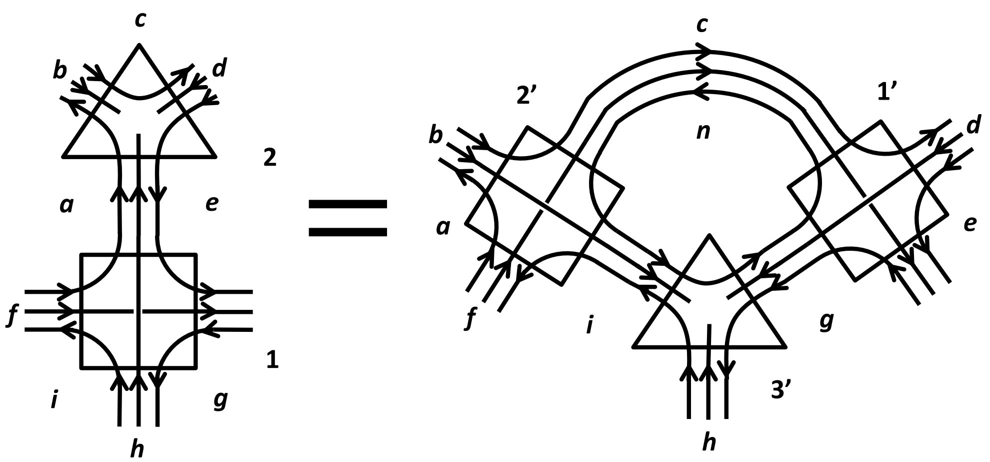
is directly equivalent to the pentagon equation [Eq. (1.A.40)] (note the sum over n and the associated quantum dimension d_n), as shown by



. (1.11)

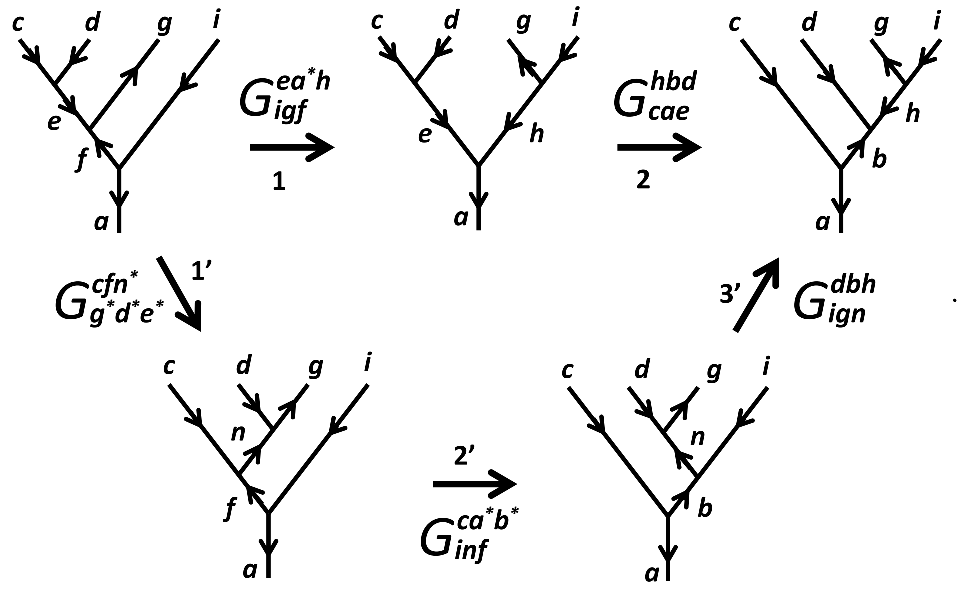
The number labeling each move in the above diagram indicates which of the tensors in Eq. (1.10) the move corresponds to. The G-symbol next to each move in Eq. (1.11) comes from the definition in Eq. (1.A.37) and is equal to the corresponding tensor in Eq. (1.10), as follows from employing several identities of the G-symbol [Eq. (1.A.38)].

The pulling through condition for the string-net PEPS and MPO tensors is represented as



, (1.12)

and is again equivalent to the pentagon equation [Eq. 1.A.40]



. (1.13)

The loop condition [Eq. (1.3)] for string-net PEPS follows from the unitarity of the F-symbols as a basis transformation. The existence of a generalized inverse $X \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^D \otimes \mathbb{C}^D$, for which Eq. (1.4) holds, follows from Eq. (1.10) (i.e. the pentagon equation) and the unitarity of the F-symbols.

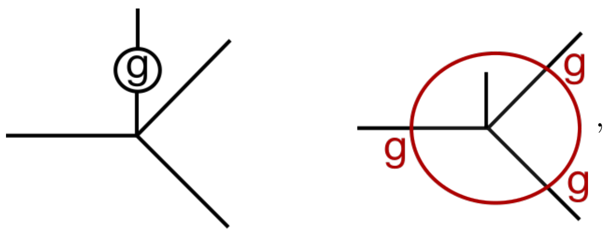
It is instructive to see that the necessary conditions for MPO injectivity in the string-net PEPS are equivalent to the the pentagon equation (1.A.40), which appears as a compatibility condition for the F-symbols [14] and is thus guaranteed to be true for any string-net model.

One can readily verify that a closed string-net MPO constructed from the tensors in Eq. (1.8), with a weighting of the normalized quantum dimension d_i/D^2 associated to the internal index forming a closed loop, is a projector for any length. Since the MPO is a projector its rank is easily obtained by calculating the trace. By examining the behavior of this rank for increasing length, the topological entanglement entropy is seen to be $\log(D^2)$, originating from the normalization factor D^{-2}.

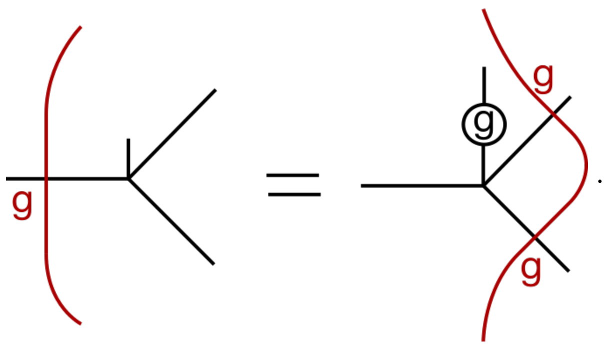
1.7 A glimpse to symmetry protected topological order via MPOs

Symmetry protected topological (SPT) order is the phenomena that occurs in quantum systems with global symmetries. The system is symmetric under the action, , $U^{\otimes N}(g)$ where $g \in G$, of a global symmetry group G acting on every site of the system of total size N. The ground state of an SPT ordered Hamiltonian is unique on closed manifolds and is invariant under the global symmetry action, but for *nontrivial* SPT order interesting phenomena occurs if we put an SPT hamiltonian on a manifold with boundary: The state is either gapless on the boundary (i.e., there is no local gapped boundary Hamiltonian for that state), or the state is gapped but breaks the global symmetry on the boundary. This result has been proven in Ref. [38] and a classification has been given. Using a gauging procedure introduced in Ref. [39], these results have been generalized to a general SPT ordered PEPS in Ref. [40] and here we only intend to give a glimpse to the main idea of that work [40]. We include here how to modify the pulling through condition, and the implications of this modification on the boundary.

The MPOs for SPT PEPS are labeled by the group elements $g \in G$, where G is the global symmetry group relevant to the SPT phase. In order to capture the global symmetry of the SPT ground state we modify the invariance under the MPO loop (which originally follows from pulling through) and pulling through axioms as follows:



, (1.14)



. (1.15)

The global symmetry on the state hence implies boundary MPO symmetries on the boundary. So,for an SPT ground state which lives on a 2-manifold with boundary (let's consider a disk for simplicity, so the boundary is a circle), the boundaries must be invariant under $||G||$ loop operators given in the form of MPOs. A finite bond dimension MPS on the boundary is possible (hence there exists a gapped parent hamiltonian), but it's always a symmetry-breaking type MPS, i.e., it's a sum of two different MPS hence not a single-block MPS and not injective. Boundary states may also be gapless, in which case one needs infinite bond dimensions in order to realize it in terms of an MPS.

SPT PEPS can be mapped to topologically ordered PEPS, by a map called *gauging*. This map has been shown to work for fixed point models, i.e., Dijkgraaf-Witten models. Gauging can be generalized to PEPS and is shown to preserve the gap of the system in Ref. [39, 40].

1.8 Conclusions

In this work, we have presented a general framework for the characterization of topological phases in two dimensions using the PEPS formalism. The key ingredient is a generalized notion of injectivity, in which the central object is a MPO fulfilling a fundamental *pulling through* condition. This encompasses the cases of normal injectivity, G-injectivity [21] and twisted injectivity [22], which can be verified directly by constructing the relevant MPOs. As a very general example, we have illustrated that all string-net models satisfy our axioms by explicitly constructing the appropriate MPOs and elucidating the correspondence between the pentagon equation and our pulling-through condition.

The characterization of topological order in terms of MPOs opens up the possibility of classifying topological phases via their MPOs, similar to results on 2D symmetry-protected topological (SPT) phases [38]. In fact, for string-net models with abelian group elements as local degrees of freedom on edges and with group multiplication as the trivalent vertex constraints, the pentagon equation reduces to a 3-cocycle condition. This close connection between the boundary symmetry MPOs of SPT states and the virtual gauge symmetry MPOs of intrinsic topological states is explicitly described by a gauging duality in the PEPS picture [39, 40].

The framework set forth in this work can be easily generalized to fermionic PEPS [41], as well as to higher dimensional systems by replacing MPOs with their higher dimensional generalization, Projected Entangled Pair Operators (PEPOs); it thus provides a systematic way to understand both topological phases of interacting fermions and exotic topological order in three dimensions such as the Haah code [42].

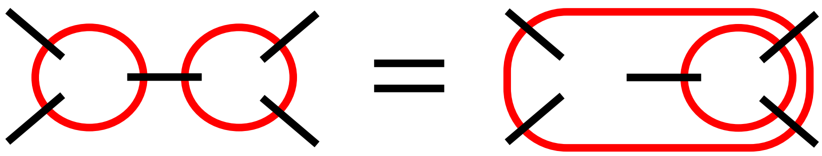
A natural question is whether our framework contains topological phases outside the string-net picture. Since the excitations of the doubled phases described by string-nets all have a Lagrangian subgroup we know that their edge modes can be gapped out [43]. Thus, to obtain phases outside string-nets we need to look at models with protected gapless edge modes (or models which do not correspond to a TQFT). Given the close connection between edge physics and the MPO, this amounts to understanding which MPOs give rise to protected gapless edge modes; indeed, in the recently discovered chiral fPEPS [44], fermionic MPOs satisfying a pulling-through condition have been identified [45].

Finally, an equation closely related to the pulling through condition [Eq. (1.2)] could yield an algorithm to bring 2D PEPS into a normal form that facilitates the calculation of physical observables. Intuitively, this is because once the algorithm has converged we find an MPO that approximates the transfer matrix of the model. Hence, contracting the whole PEPS with a physical observable reduces to contracting the PEPS in a local region around the observable and using a MPO to approximate the boundary. We leave these directions to future work.

Appendix 1: Characterizing topological order with matrix product operators

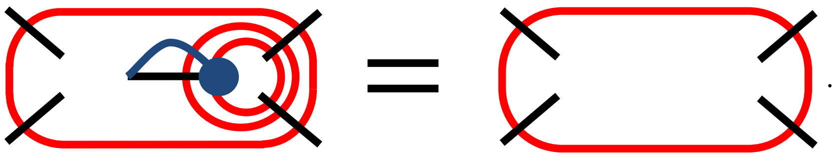
1.A.1 Stability under concatenation

We now show that the conditions we have placed on the MPO tensors [Eq. (1.2), Eq. (1.3) and Eq. (1.4)] ensure that the projector $\mathbb{1}_{S_V}$ onto the virtual subspace S_V, on which the PEPS map acts injectively, is represented by the MPO obtained from contracting the tensors M along the boundary of the region for any simply connected region of the lattice with nontrivial boundary. The pseudoinverse of the PEPS map on a larger region can be constructed by first applying a pseudoinverse to each site and then using the pulling through condition and applying the generalized inverse X, as represented by the moves in the following diagrams



(1.A.1)

and then

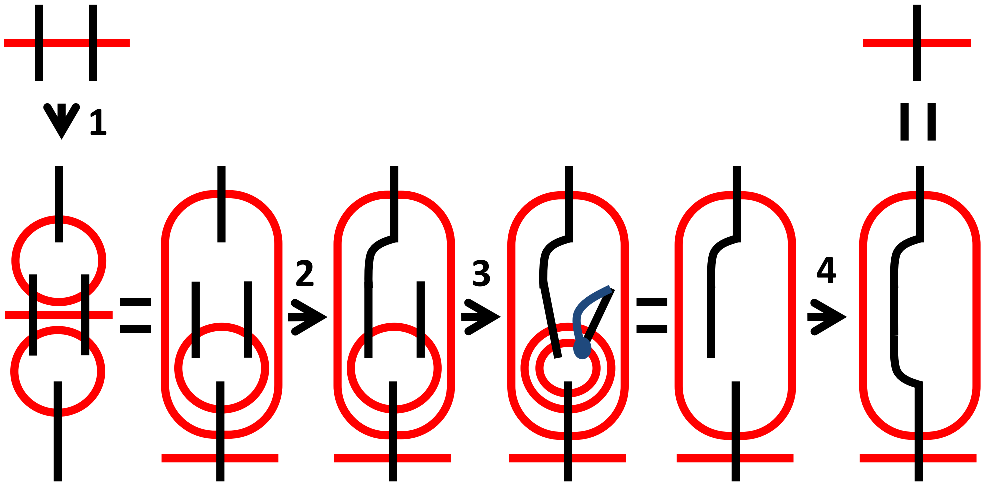


(1.A.2)

The same two moves can be used to inductively grow a region from N to $N+1$ sites. The only complication arises when the injective region encloses an elementary plaquette, this involves growing the region onto a new site with two virtual bonds in common, which is possible using a slight variation of the above process.

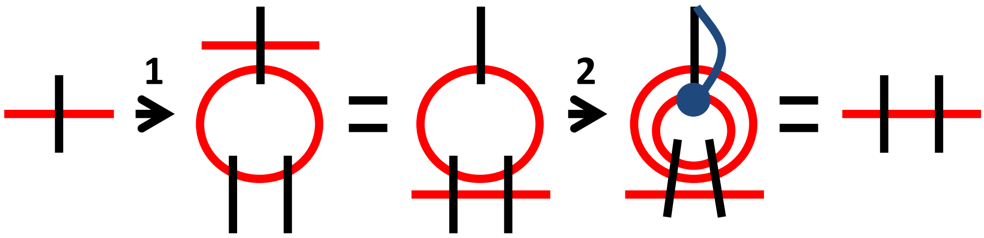
1.A.2 Renormalization group move

The pulling through condition [Eq. (1.2)] together with the generalized inverse [Eq. (1.4)] yield natural maps (acting only upon the black indices) for the addition or removal of degrees of freedom to or from a MPO. One can construct a linear map which removes a single degree of freedom from an MPO as follows



, (1.A.3)

where at step 1 we act with two MPO loops, at step 2 we contract two open indices, at step 3 we act with a generalized inverse, and at step 4 we again contract two open indices. Adding a single degree of freedom can be done similarly



, (1.A.4)

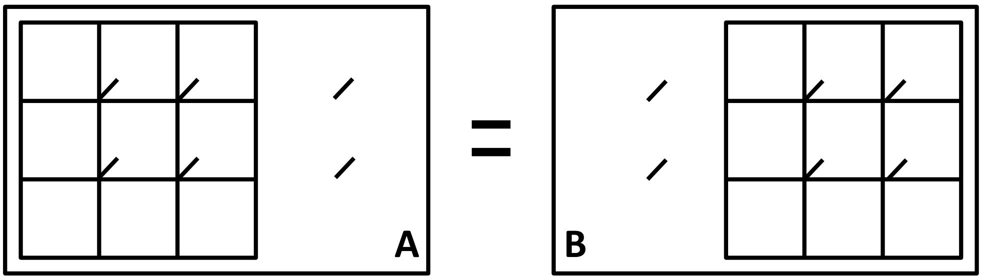
where at step 1 we act with a single MPO loop and at step 2 we apply a generalized inverse.

These moves yield linear maps between MPO injective PEPS on lattices of different sizes and allow one to define a generalized inverse acting on any number of legs of an arbitrarily large MPO loop, and to show that the pulling through condition holds for arbitrarily large MPO loops. Both the coarse-grained generalized inverse and coarse-grained pulling through condition will be utilized in the following sections.

1.A.3 Intersection

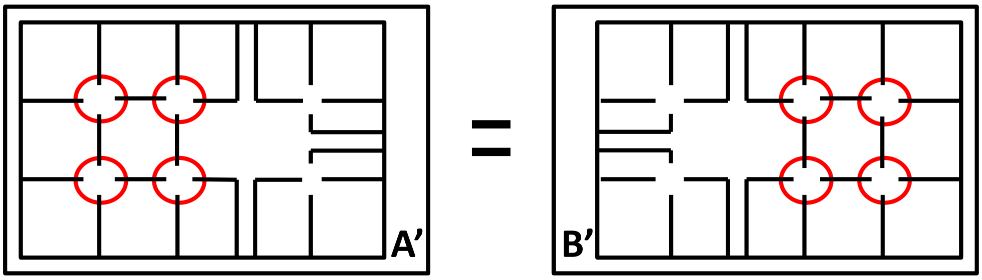
In this section we show that the MPO injective PEPS parent Hamiltonian defined on any simply connected region of the lattice with nontrivial boundary is frustration free and, furthermore, that all states within the ground subspace are given by a unique tensor network representation built from the original PEPS tensors A in the bulk with arbitrary tensors closing the network at the virtual boundary.

The parent Hamiltonian consists of a sum of 2×2 plaquette terms that each project locally onto the subspace spanned by the PEPS on that plaquette with arbitrary virtual boundary tensors. Here we consider the mutual ground state subspace of two neighbouring plaquette terms, any state within this subspace must be of the following form



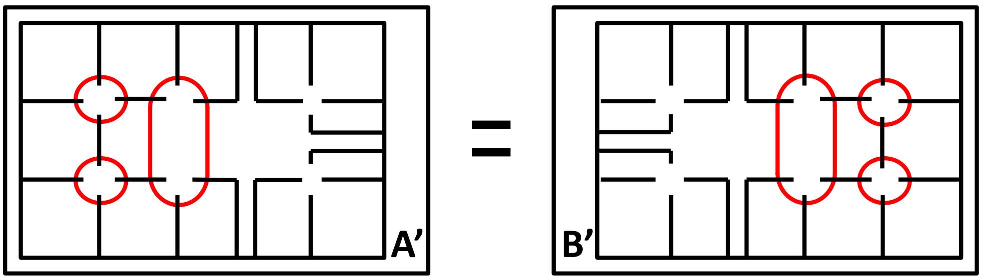
, (1.A.5)

for some boundary tensors A and B, note that we are free to choose B to be invariant under a loop of MPO on the virtual boundary. By applying the pseudoinverse to all sites we find



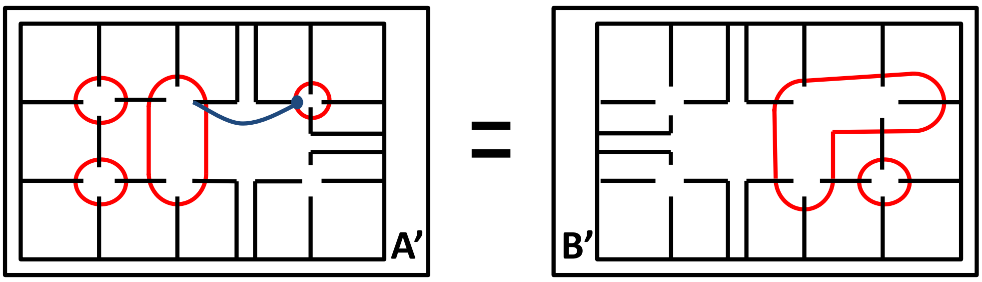
, (1.A.6)

which, after pulling through and applying a generalized inverse, leads to



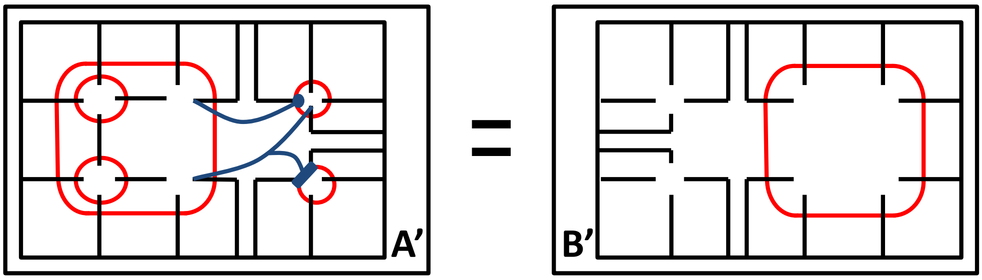
. (1.A.7)

The application of another generalized inverse yields



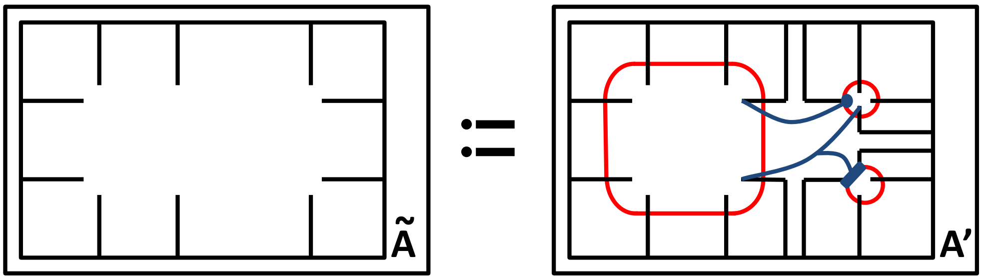
(1.A.8)

and after applying a coarse-grained generalized inverse over two legs we find



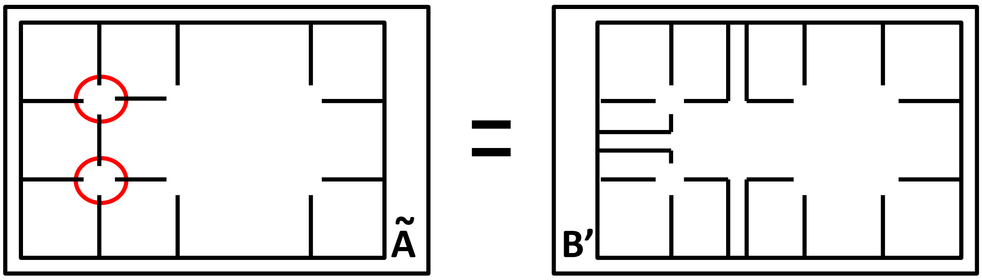
. (1.A.9)

We define a new boundary tensor



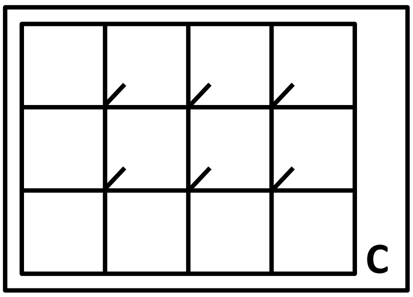
(1.A.10)

and find that B' must take the following particular form



(1.A.11)

where we have used the invariance of B under a loop of MPO on the virtual boundary of the right plaquette. Hence all states within the mutual ground space of neighboring plaquette terms are of the form



(1.A.12)

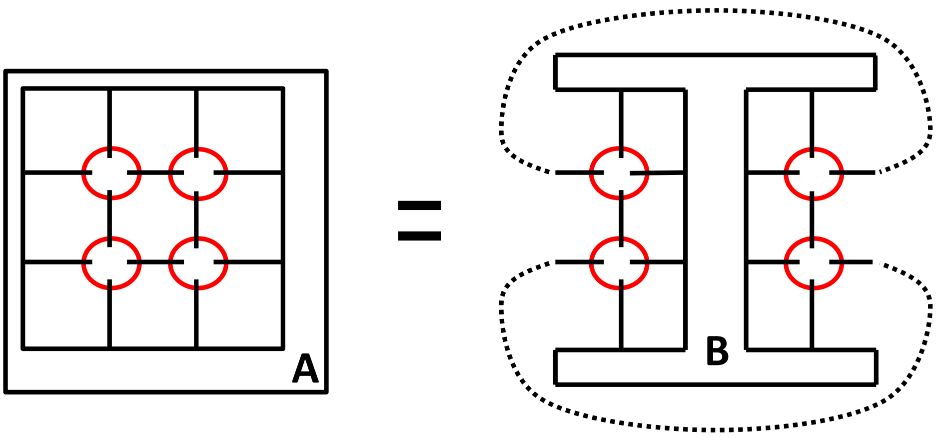
for some boundary tensor C.

It is possible to iterate this argument to show that the ground space of the parent Hamiltonian on any simply connected region with a nontrivial boundary is spanned by the PEPS on that region with an arbitrary virtual boundary tensor.

1.A.4 Closure on a torus

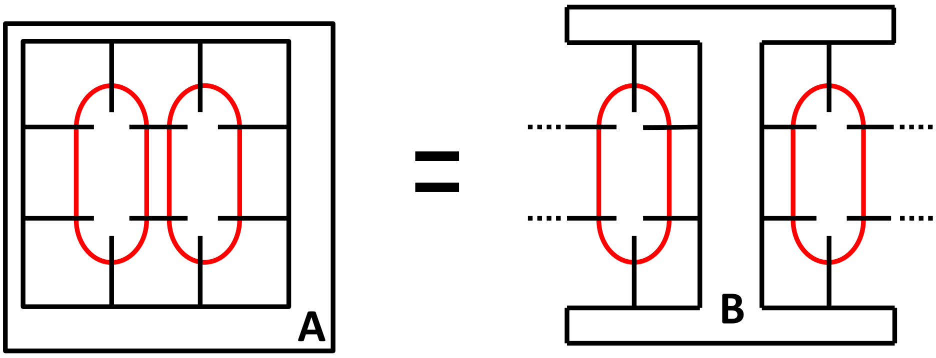
In the previous section we have shown that the ground state subspace on any simply connected region of the lattice with nontrivial boundary is spanned by a tensor network built from the PEPS tensor A in the bulk and closed by an arbitrary tensor on the virtual boundary. If we proceed to close the region on a compact manifold, the additional plaquette terms of the parent Hamiltonian now crossing the boundary will further restrict the possible form of the boundary tensor in a way that depends on the topology of the manifold.

We consider the specific case of closure on a 2×2 torus below, and note that a direct generalization of the argument to any size of torus leads to the same conclusion. By examining several different possible closures we refine the description of the boundary tensors that lead to a linearly independent set of ground states. We begin by looking at states in the intersection of two subspaces obtained from the following two different closures



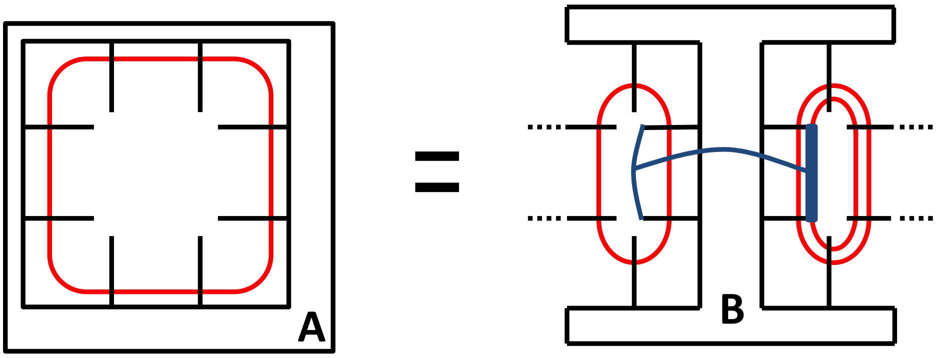
(1.A.13)

which must be of this form, for some A and B. We utilize the pulling through condition twice and apply two generalized inverses to achieve



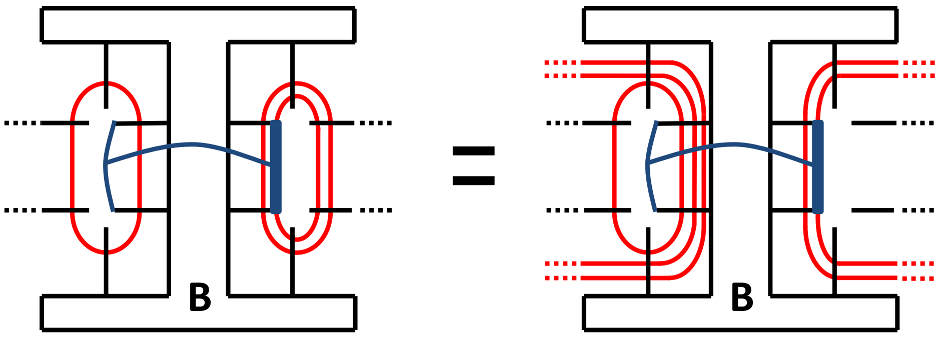
. (1.A.14)

Next we apply a coarse-grained generalized inverse over two legs and find



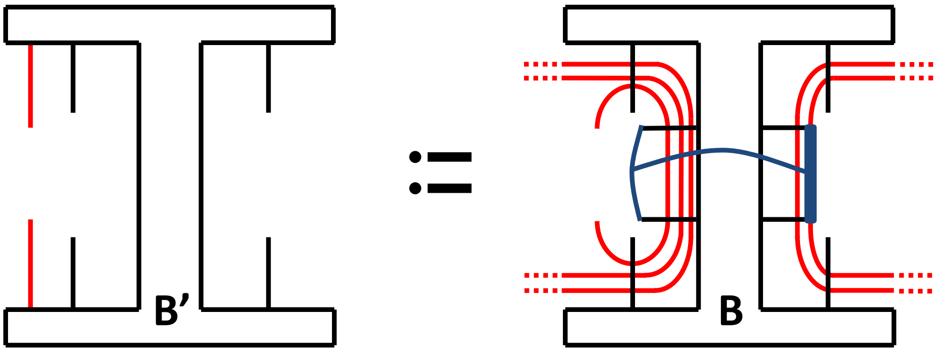
. (1.A.15)

We note the following equality, attained after using coarse-grained pulling through twice,



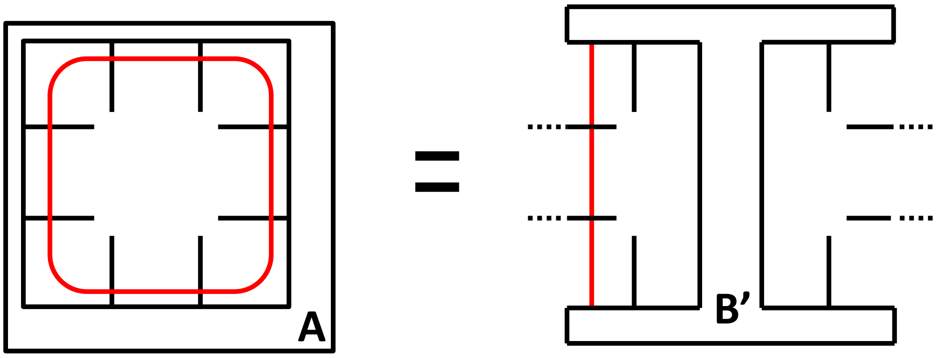
(1.A.16)

and define the tensor B',



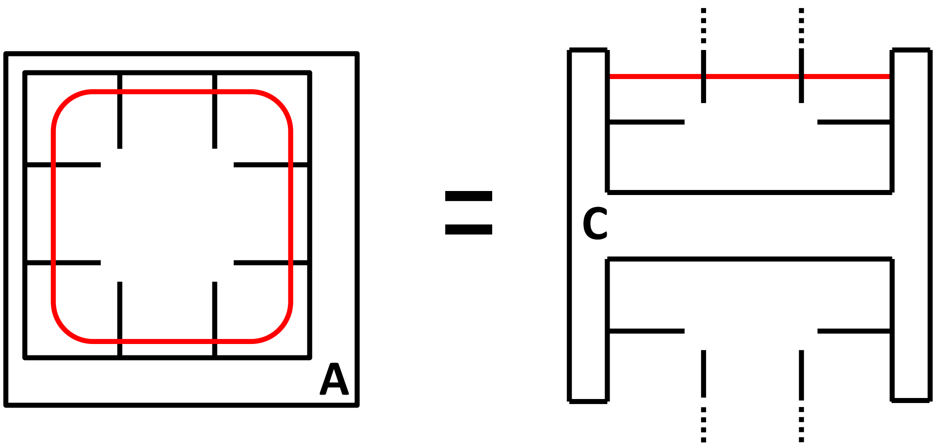
, (1.A.17)

for which we have the following equality



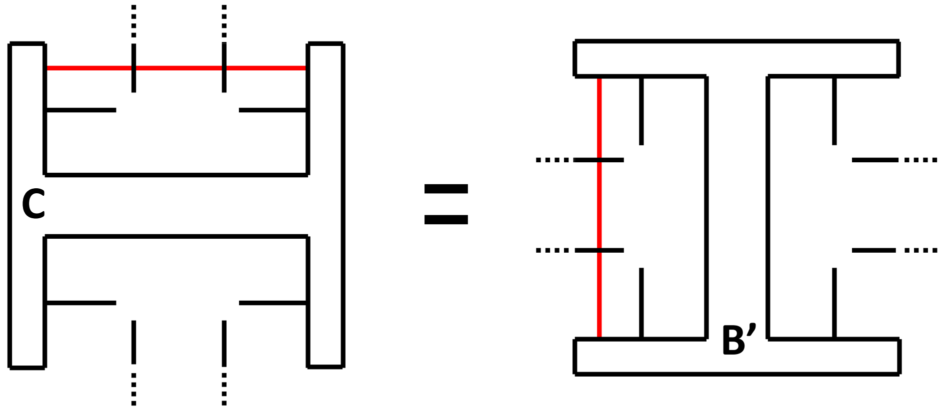
. (1.A.18)

It is possible to repeat the preceding arguments to find states in the intersection of the 90° rotated versions of the above boundary configurations. For states in the triple intersection we must have both Eq. (1.A.18) and the following



, (1.A.19)

and hence



(1.A.20)

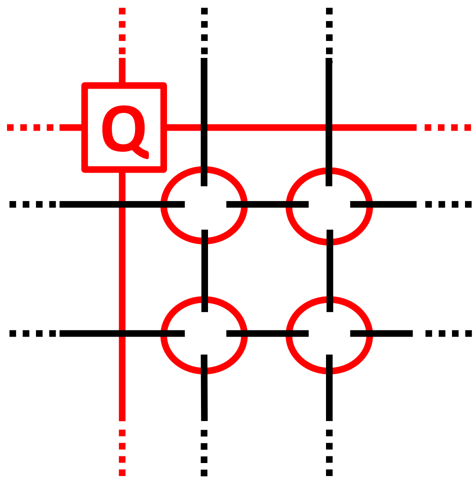
for some A, B' and C. Viewing the tensors in Eq. (1.A.20) as linear maps from the vertical to horizontal indices we have

$$C_H C_V = B_H B_V \tag{1.A.21}$$

where C_V and B_H are MPOs of length two. Writing B_H^+ for the pseudo-inverse of B_H we have that

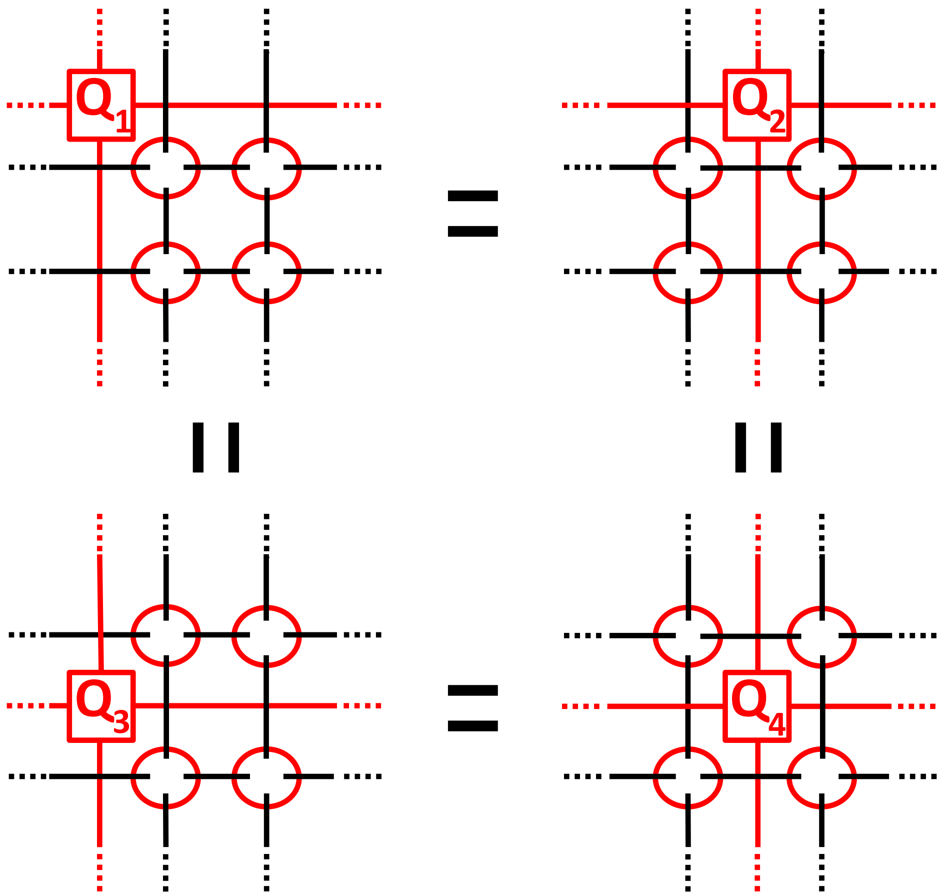
$$C_H C_V = B_H \left(B_H^+ C_H\right) C_V \tag{1.A.22}$$

and, defining $Q := B_H^+ C_H$, the equality in Eq. (1.A.20) thus ensures the boundary tensor is of the following form



. (1.A.23)

Repeating the above argument for the four different possible closures we have a set of equalities



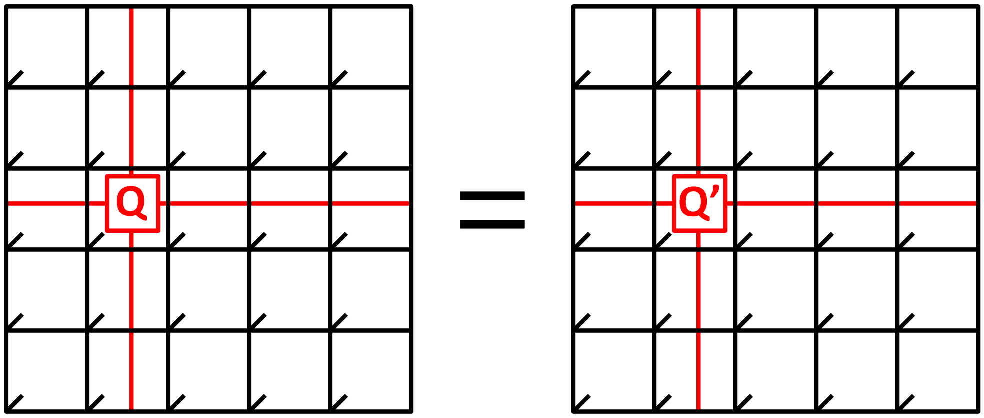
, (1.A.24)

for some possibly different boundary tensors Q_i.

1.A.5 Ground state tensors

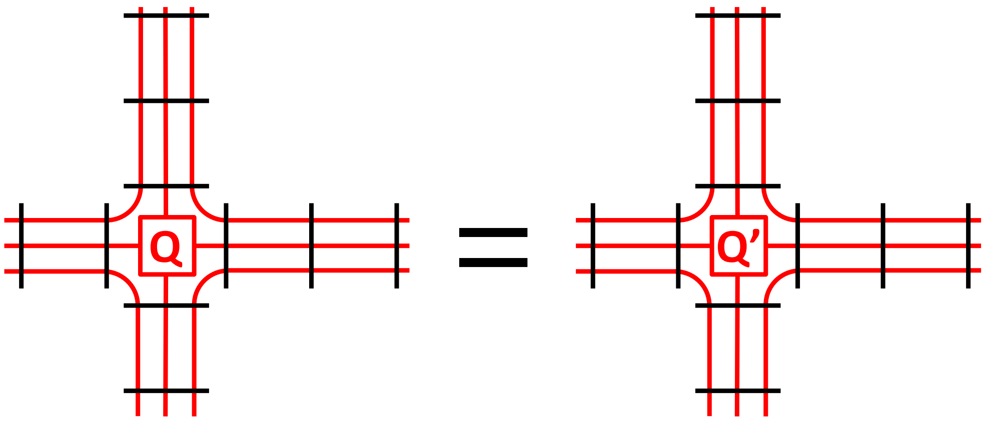
The ground state tensors Q are only defined up to transformations that do not affect the physical state. We first note that the equality of physical states for two different tensors Q

and Q' on the same plaquette,



, (1.A.25)

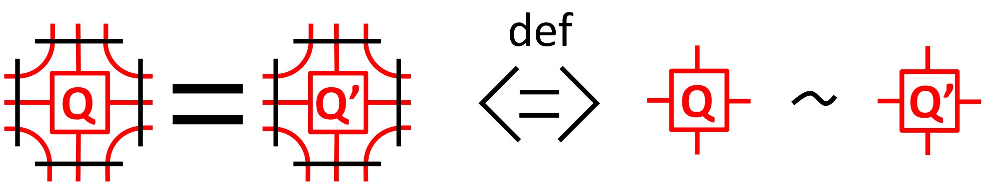
is equivalent to equality of the physical states arising from the same tensors only involving the virtual bonds they directly act upon,



, (1.A.26)

by utilizing the pseudo-inverse on the topologically trivial region not acted upon by the MPO. We are assuming periodic boundary conditions in the above two equalities and for all lattices throughout the remainder of this section.

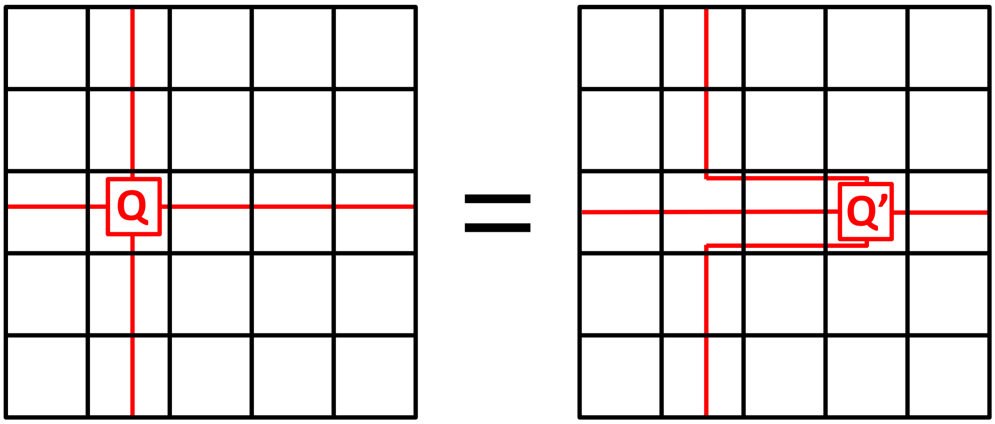
By further utilizing the RG moves we find that this is logically equivalent to equality of the states formed by Q and Q' on the smallest possible torus. Hence we use this condition [Eq. (1.A.27)] to define an equivalence relation on four index tensors, whose equivalence classes capture all tensors that lead to the same physical state,



, (1.A.27)

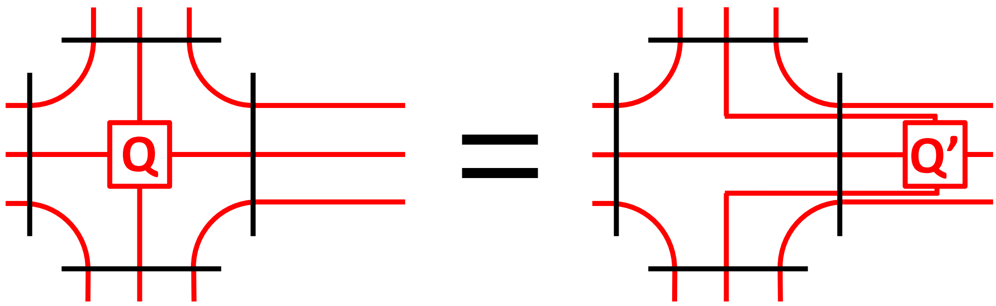
note there are periodic boundary conditions for the left equality and open boundary conditions for the right equivalence relation.

By the arguments of the previous section we know that it is possible to close the PEPS tensor network, with a possibly site dependent Q tensor, on any plaquette of the lattice to achieve the same physical state. We now compare the closures at different points, first considering Q tensors at two different locations along the same row of the dual lattice that give rise to the same physical state



(1.A.28)

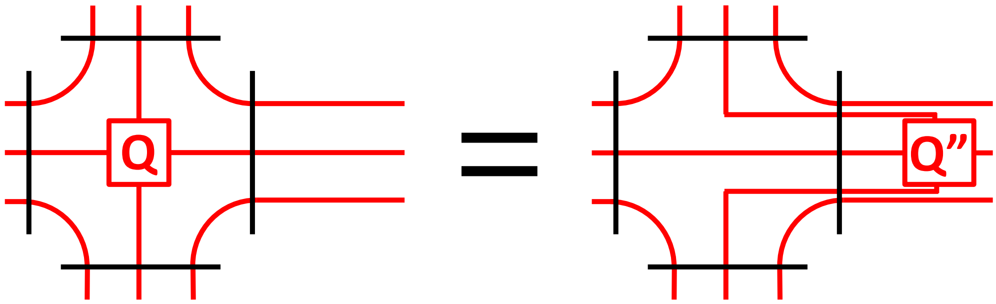
where we have pulled through the MPO such that the boundary regions match. Now by employing the pseudoinverse on the bulk, followed by RG moves, we arrive at the equation



(1.A.29)

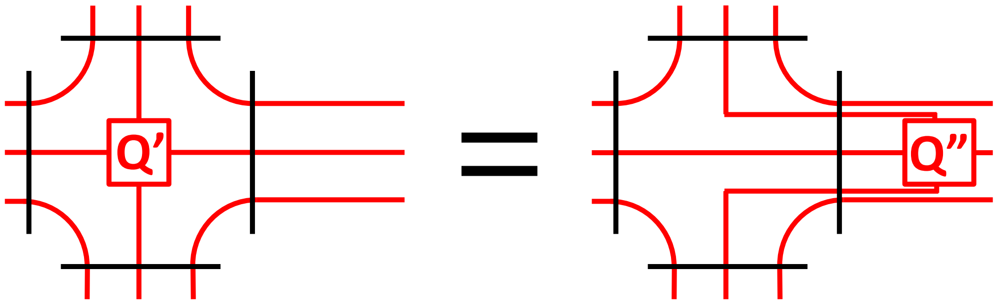
which must be satisfied by Q and Q' if they give rise to the same physical state.

Now consider a third Q'' at a different plaquette along the same row, we proceed to compare each of the original tensors Q, Q' to the new one Q" via two different maps (constructed from RG moves) to arrive at two similar conditions



(1.A.30)

and



(1.A.31)

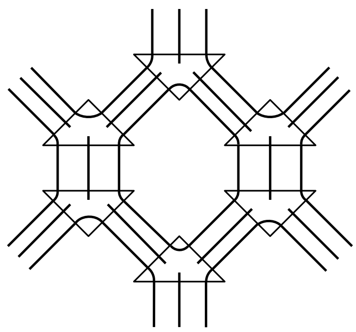
which, together, imply equality of the two physical states that arise from Q and Q' on the same plaquette of the PEPS, i.e., $Q \sim Q'$. Note, a similar argument applies to boundary conditions shifted in the vertical direction, which then implies (in combination with the horizontal) that any two tensors closing the PEPS tensor network (possibly on different plaquettes) to give the same physical state must be equivalent.

Hence on the level of equivalence classes we are searching for tensor solutions of the elementary pulling through equation [Eq. (1.6)] in both the horizontal and vertical direction and it suffices to consider a particular representative Q for each class. This ensures that the resulting tensor networks, with the same PEPS tensor on every site and a Q tensor on the virtual level, will be translation invariant. To determine the degeneracy on the torus we must then look at the dimension of the subspace spanned by physical ground states coming from all the different Q tensor solutions. Since the RG maps yield linear transformations between MPO injective PEPS on lattices of different sizes, which are invertible on the subspaces spanned by states of the form given in Eq.(1.A.27) and Eq.(1.A.25), we can be sure that the exact degeneracy does not change for any finite system size. However, it is possible that as the system grows in size any number of states within the ground state subspace may converge to a single ground state or to zero in the thermodynamic limit. Hence one must examine the stability of the subspace as the system grows.

Finally we note that these closure arguments imply that any transformation preserving the ground state subspace can implicitly be rewritten as a transformation directly upon the Q tensors, although there is no explicit formula in general.

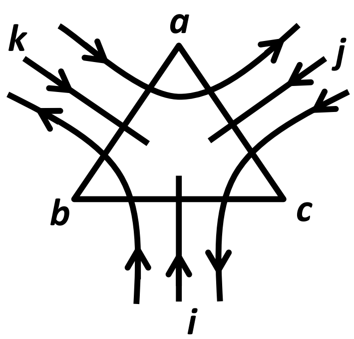
1.A.6 Tensor Network description of String-net Condensed States

As first described in [36, 37], the ground states of the string-net models have exact PEPS representations. Inside of every hexagon there is one virtual degree of freedom, these are connected to one another and to the degrees of freedom on the edges by tensors that sit on every vertex. The ground state is represented by the following tensor network



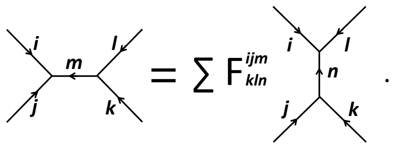
$$, \tag{1.A.32}$$

where the tensor sitting on the vertices is



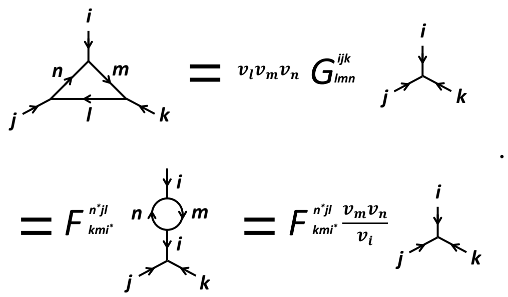
$$:= \sqrt{v_i v_j v_k}\; G^{ijk}_{abc}, \tag{1.A.33}$$

$d_s = v_s^2$ is the quantum dimension for sector s and $D = \sqrt{\sum_s d_s^2}$ is the total quantum dimension. In the tensor network description, we make the convention that every closed loop comes with the multiplicative factor $a_s = d_s/D$ and the middle legs that connect each pair of tensors are copied to physical degrees of freedom on the adjacent vertices. In the above expressions G is a six index tensor, known as the symmetric F-symbol. For the sake of completeness, we define these symbols and describe their symmetry properties which have been used in proving that the string-nets satisfy our axioms. The F-symbol is defined to be a scalar map (when the branching is multiplicity free, i.e., N_{ijk} is either 0 or 1) from one fusion path to another



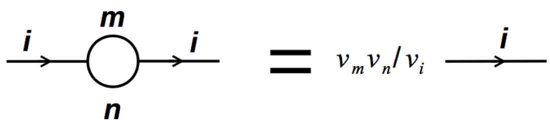
(1.A.34)

Then we define the G-symbol by the first line of the following expression



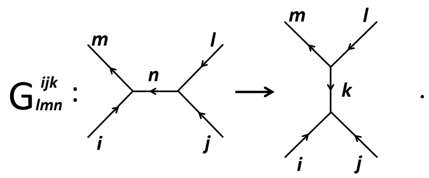
(1.A.35)

The second line of the above equation follows from the definition of the F-symbol and the fact that



(1.A.36)

regardless of arrow directions of m and n. Graphically, the G-symbol corresponds to the following scalar map



(1.A.37)

By Eq. (1.A.35), we find $G^{ijk}_{\lambda\mu\nu} := F^{\nu^* j\lambda}_{k\mu i^*}/(v_\lambda v_i)$. F and G-symbols have nice symmetry properties, referred to as tetrahedral symmetry, shown in the following equations

$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = F^{imj}_{k^*nl}\frac{v_m v_n}{v_j v_l}. \tag{1.A.38}$$

Using the above symmetry relations one finds that

$$G^{ijk}_{\lambda\mu\nu} = \frac{1}{v_k v_\nu} F^{i^* j^* k^*}_{\lambda^* \mu \nu^*} \tag{1.A.39}$$

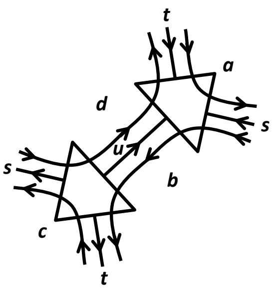
and clearly the G-symbols possess symmetry properties following from those in Eq. (1.A.38).

The pentagon equation for these G-symbols follows from the pentagon equation for the F-symbols, and is given by

$$G^{ijk}_{\lambda\mu\nu}G^{i^*j^*k^*}_{\alpha^*\beta^*\gamma^*} = \sum_n d_n G^{k\alpha^*\beta}_{n\mu^*\lambda^*}G^{j\gamma^*\alpha}_{n\lambda^*\nu^*}G^{i\beta^*\gamma}_{n\nu^*\mu^*}\ . \tag{1.A.40}$$

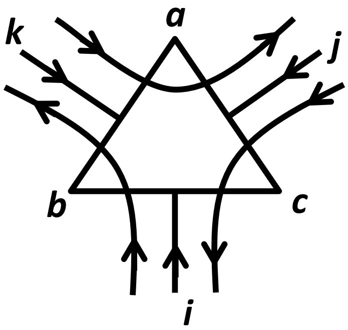
1.A.7 Modular transformations on string-net PEPS

In this section we show that the modular transformations can be performed directly on the virtual level of string-net PEPS on a torus. One can check that the tensors $Q_{stu} = v_s v_t v_u G^{b^*du}_{t^*sa}G^{d^*bu^*}_{ts^*c}$, shown in Eq. (1.A.41), satisfy all the requirements on the ground state tensor of the string-net PEPS.



(1.A.41)

In Eq. (1.A.41) the tensors are defined to be



$$:= \sqrt{v_i v_j v_k}\ G^{c^*bi}_{k^*ja}\ , \tag{1.A.42}$$

the vertical indices t must match since the PEPS is defined on a torus, and similarly for the horizontal indices s.

In general, $\{Q_{stu}\}$ will form an overcomplete set in the sense that not all Q_{stu} will lead to linearly independent ground states. By utilizing the pentagon equation one can show that the 90° rotated ground state tensor can be expressed as a linear combination of the original ground state tensors in the following way

$$S(Q_{stu}) = \sum_n F^{stu}_{s^*t^*n}Q_{tsn} \tag{1.A.43}$$

This agrees with the results of [35].

By further utilizing the specific form of the Q_{stu} tensors for the string-net PEPS [Eq.(1.A.41)] (and the pulling through condition [Eq.(1.12)]) one can express the ground state tensors of the ground states with a Dehn twist as a linear combination of the original ground state tensors via the explicit relation which we give here for completeness

$$Q_{stu}^{\text{twisted}} = \sum_n F^{stu}_{s^*t^*n}\, Q_{snt^*}\,, \tag{1.A.44}$$

where t is the label wrapping around the torus in the direction of the twist in and s is in the direction orthogonal to the twist, again agreeing with [35].

After determining the appropriate linear combinations of Q_{stu} that lead to different ground states for a particular string-net model Eq. (1.A.43) and Eq. (1.A.44) can be used to obtain the elements of the S and T matrices, respectively.

Chapter 2

Topological order in any dimension: A tensor network framework

We present a general scheme for constructing topologically ordered many-body lattice models in any space dimension using tensor networks. Our approach relies on finding 'simplex tensors' that satisfy a finite set of tensor equations. Given any such tensor, we construct a discrete topological quantum field theory (TQFT) and a local commuting projector Hamiltonian on any lattice. The ground space degeneracy of these models is a topological invariant that can be computed via the TQFT, and the ground states are locally indistinguishable when the ground space is nondegenerate on the sphere. Any ground state can be realized by a tensor network obtained by contracting simplex tensors. Our models are exact renormalization fixed points, covering a broad range of models in the literature. We identify symmetries on the virtual level of the tensor networks of our models that generalize the topological invariance properties beyond fixed point models. These symmetries are expressed in terms of extended surface operators acting on the virtual level, which can be deformed through the tensor network– called the *pulling through* property. For fixed point models, the pulling through conditions turn out to be the equations that have to be satisfied by the simplex tensors. [1]

[1]This chapter is based on "Topological order in any dimension: A tensor network framework". [2]
Authors: M.B. Şahinoğlu, M. Walter, D.J. Williamson.
To appear on arXiv soon.

2.1 Introduction

The quest to develop a general framework for the understanding of phases of matter has a rich physical and mathematical history. To a large extent Landau's theory of symmetry breaking provided such a framework. There are, however, phases beyond this classification that arise from subtle quantum correlations. For phases with strong quantum effects, particularly those with an energy gap, cooled to infinitesimal temperatures the topological features may become characteristic of the physics of the whole phase of matter.[6] In such topological phases, the ground state degeneracy is topology dependent and distinct ground states are indistinguishable to arbitrary local operators. Furthermore, the elementary excitations have a topological nature and give rise to a possibly non-abelian Aharonov-Bohm effect when one braids around another. These properties underly a close connection of topological phases to spatially local quantum codes, making these phases natural candidates for robust topological quantum computation.[23, 46] Zero-temperature quantum phases can be defined to be equivalence classes of local Hamiltonians with uniform spectral gaps which are connected by smooth paths of gapped local Hamiltonians. On one hand, simple representative exactly-solvable models with zero correlation length are often used to characterize the essential features of a quantum phase (which are preserved under adiabatic deformation). Such models can be seen as fixed points of certain real-space renormalisation procedures (e.g., blocking sites) which do not map between distinct phases. On the other hand, it is common practice to formulate a low-energy effective field theory in the continuum to describe the essential features of a phase. Such theories correspond to topological quantum field theories (TQFT) as they possess only topological degrees of freedom. In other words, TQFTs are invariant under arbitrary topology-preserving transformations (homeomorphisms),[47] which from the perspective of physics can be understood as a general real-space renormalization transformations.

These philosophies come together if one considers TQFTs formulated in a discrete combinatorial setting. There are different ways to assign TQFT data to a combinatorial description of a manifold using, for example, handlebody decompositions, surgery or knot diagrams, or triangulations. In the present context of constructing lattice models, triangulations appear most naturally due to their inherent locality and the uniformity of the basic building blocks, which are all simplices. This underlies a local approach to constructing topological invariants, so-called *state sum invariants*, and more generally topological quantum field theories by a *discrete path integral*, defined by associating a simplex tensor to each simplex in a triangulation of the euclidean spacetime manifold and summing over all degrees of freedom placed on the skeleton subject to some boundary conditions.[48–64] For a closed manifold, this path integral can be understood as a partition sum, while for manifolds with boundary it gives rise to a vector in the Hilbert space of boundary conditions. Topological invariance, or rather invariance under retriangulation, can then be characterized purely algebraically in terms of a finite number of equations on the simplex tensor, corresponding to a complete set of local lattice moves called bistellar flips or *Pachner moves*.[65]

The Turaev-Viro TQFT and their generalisations are a canonical example of such a construction in 2+1 dimensions.[52, 54, 66] Here, the simplex tensor is obtained from the F-symbols of a tensor category, which satisfies topological invariance due to the pentagon consistency equation and a unitarity constraint. There is a close relation between TQFTs in $n+1$ dimensions and n-dimensional lattice models. Specifically, it is possible to find a local commuting projection Hamiltonian whose ground space can be computed solely from the data of the TQFT. This relationship is perhaps best understood in $2+1$ dimensions, where the prototypical example is given by the Levin-Wen string net models,[14] which achieve such a Hamiltonian formulation of the TQFTs of Turaev-Viro type. In the Levin-Wen approach, the topological invariance is introduced as finding fixed points of a certain real-space renormalisation procedure which satisfies an additional recoupling condition, corresponding to the fusion rules of the underlying tensor category. The F-symbols of the category are central to the construction of the Levin-Wen models and they can also be used to construct a PEPS representation of the ground state wavefunctions.[36, 37] The appearance of category theory is no accident. While topological invariance is best understood in terms of the invariance of the simplex tensor under Pachner moves, any suitable tensor category gives rise to a tensor, namely the F-symbol of the category, that satisfies the desired invariance conditions in dimension $2+1$. Thus tensor categories constitute a rich source of models with topological invariance properties. Moreover, the anyon theory describing the emergent physical excitations in a Levin-Wen model corresponds to the center of the input category.

In recent years, it has been understood that an appropriate notion of *higher categories* should play a similar role in the construction of TQFTs in higher dimensions.[61, 57, 59, 63, 66, 67] It has also been observed that the need arises for *fully extended* theory.[68–75] A fully extended TQFT does not only produce amplitudes and states, but it assigns data to manifolds of arbitrary codimensions. This allows the construction of the global theory by glueing primitive building blocks, as in the state sum or discrete path integral formulation described above. Physically speaking, fully extended TQFTs are those theories that are "truely local".

In this work we study the general properties of discrete TQFT in the language of tensor networks. Starting from the idea of a tensor with topological invariance properties, we recount the construction of a TQFT following the state sum/discrete path integral perspective. The structure uncovered is a consequence of topological invariance alone and not the details of the algebraic input data, such as a category or group cohomology. This recasts many of the basic properties, such as a local commuting projection Hamiltonian with a topological ground state degeneracy and tensor network symmetries of the ground state, of fixed point lattice models, including Levin-Wen and Walker-Wang, as being purely topological. On the other hand, category theory is useful to finding solutions to the topological invariance conditions. While we do not address the problem of finding new solutions to the topological invariance conditions (this is already a very hard problem in (2+1)D where it is closely related to constructing and classifying UFCs and few examples are known that are not based on groups or quantum groups at roots of unity) we describe how the construction works in (3+1)D, going beyond the usual models. The general idea is that in dimension $(n+1)$ it should be related to constructing and classifying sufficiently nice n-categories.

A Hamiltonian formulation of a discrete TQFT allows one to formulate a precise connection of the phase of matter to a discrete TQFT fixed point model it contains. It also allows one to apply the notion of phase equivalence for Hamiltonians to the underlying TQFTs. In the framework of tensor networks, particularly for projected entangled pair states (PEPS), the topological order of a state is characterized by the virtual symmetries of the local tensors. PEPS allow one to consider models that are deformed away from dispersionless fixed point models, the tensor network symmetries can be extracted from the exactly solvable fixed point models and persist for deformations within the phase (they may even persist across a phase transition, necessitating the use of numerical methods). In recent work, *matrix product operator* (MPO) symmetries were found for the 2D PEPS representation of an arbitrary Levin-Wen ground state, generalizing previous work which established this for twisted quantum double models which correspond to Dijkgraaf-Witten (DW) TQFTs.[21, 1] These MPOs were constructed directly from the same input category data as the wavefunction and the symmetry condition, that these matrix product operators are freely deformable within the PEPS network, was ensured by the pentagon consistency equation for this data. In fact this symmetry turns out to be a more basic property of any discrete TQFT which emerges solely due to the topological invariance condition on the tensors (which captures the pentagon equation as a special case).

In addition to bringing together facts that exist in the literature (at varying levels of accessibility) about discrete TQFTs and their Hamiltonian formulation, we characterize the ground states as locally equivalent to a PEPS and study their topological order in terms of tensor network operator (TNO) symmetries of the tensor network. We furthermore uncover that the TNO symmetries may come in a hierarchy such that they appear, layer by layer, recursively due to compositions of tensor network operators of one dimension higher. This structure naturally corresponds to a higher category or equivalently a fully extended TQFT, which is expected for such lattice TQFTs. We define a generic class of PEPS with such a hierarchical topological symmetry and find that descending this hierarchy to the lowest level yields the tensor assigned to a single point that allows one to reconstruct a discrete TQFT. This is essentially a tensor network formulation of the cobordism hypothesis. Due to the topological invariance of the fixed point models we consider, all phases they fall into are of the recently defined topological liquid class, equivalently they can be described by a TQFT which discounts mechanisms such as fractal condensation [42, 76].

The physical picture of the mechanism for topological order in our framework for dimension $(n+1)$ corresponds to branching membrane condensation with membranes of all dimension $0 \leq m \leq (n-1)$. The tensor network approach opens up the possibility to classify the excitations of a theory directly from the TNO symmetries which corresponds to constructing the algebra of these tensor network operators that roughly corresponds to revealing out the possible boundaries of these TNO supported on a submanifold (in the original manifold where the physical model lives) with boundary. We leave this for future research.

The manuscript is laid out as follows: In section 2.2, we explain how to assign tensor network to triangulated manifolds. We formulate a topological invariance condition as a finite number of local tensor equations, whose solutions give rise to discrete TQFTs using the language of tensor networks. In section 2.3, we give the construction of a local, frustration free, commuting projector Hamiltonian (an exactly solvable model) which stabilizes the TQFT boundary vectors as its ground space. In section 2.4, we proceed to characterize the local form of the ground states as PEPS with boundary TNO symmetries on any homotopy trivial region. Then, the multiplication of TNOs on regions with boundary is considered, which gives rise to a fusion tensor networks on the boundary. In section 2.5, we collect relevant (classes of) examples from the literature and explain how they fit within the framework. Finally, in section 2.6, we argue that iterating this procedure naturally leads to a hierarchy of tensor network operators of all dimensions down to points which mirrors the fully extended structure of the discrete TQFT. A definition for a generic class of PEPS with such fully extended tensor network symmetry is proposed along with a tensor network version of the cobordism hypothesis, specifically that the full TQFT can be reconstructed by the symmetry tensor assigned to a point. We also comment on the relationship to fully extended TQFTs and higher categories, and we conclude in section 2.7.

In an attempt to keep the chapter self-contained, we use the appendices to briefly recount the tensor network formalism (section 2.A.1) and some basic combinatorial topology which is used throughout the work (section 2.A.2).

2.2 Tensor Network Topological Quantum Field Theory

2.2.1 Tensor Networks on Simplicial Complexes

Let $\mathscr{M}_n$ be an n-dimensional manifold that is triangulated, i.e., obtained by gluing together a finite number of simplices (cf. section 2.A.2). We seek to define a tensor network that does not depend on the triangulation. The basic data, central to our construction, is a tensor associated to each n-simplex with an index for each $(n-1)$-face of the simplex. This is described precisely in the following definition.

Definition 1 (Simplex Tensor)**.** A *simplex tensor* is a tensor $A_{i_0,\ldots,i_n} \in (\mathbb{C}^D)^{\otimes(n+1)}$ with one index for each of the $n+1$ faces of the standard simplex.

If Δ_n is a vertex-ordered oriented n-simplex, we define $A[\triangle_n] := A_{i_0,\ldots,i_n}$, if $\triangle_n$ is positively oriented, and $A^*_{i_0,\ldots,i_n}$ otherwise. Here, the index i_j corresponds to the degree of freedom on the j-th face of Δ_n, i.e., the one obtained by removing the j-th vertex.

In the above definition, we have restricted to degrees of freedom that live on the codimension-one faces of the simplex. This is without loss of generality, as we may always encode degrees of freedom on lower-dimensional simplices by copying them to the adjacent faces (see section 2.2.4 below). Instead of identifying complex conjugation with orientation reversal, we may also specify a pair of tensors A^+, A^-; one for each possible orientation of the simplex. The theory is readily generalized but we will not pursue this further.

This local rule for assigning a tensor to each individual simplex leads directly to a tensor network for every sufficiently nice simplicial complex. Intuitively, a tensor is placed on each n-simplex and the indices assigned to shared $(n-1)$-faces on the interior of the complex are contracted, while the indices assigned to faces on the boundary of the complex are left open. This matches the usual convention for assigning a tensor network to the Poincaré dual graph of the simplicial complex. We make this precise in the following definition:

Definition 2 (Simplicial Complex Tensor Network)**.** Let $\mathscr{M}_n$ denote a triangulated manifold of dimension n. The tensor network $A[\mathscr{M}_n] \in (\mathbb{C}^D)^{\otimes|\partial\mathscr{M}_n|}$ is defined to be

$$A[\mathscr{M}_n] := \mathrm{tr}_{\triangle_{(n-1)}\in\mathrm{int}(\mathscr{M}_n)} \left[\bigotimes_{\triangle_n \in \mathscr{M}_n} A[\triangle_n] \right] \tag{2.1}$$

where the trace, $\mathrm{tr}_{\triangle_{(n-1)}\in\mathrm{int}(\mathscr{M}_n)}$, denotes the operation of contracting each pair of indices that are mapped to a shared $(n-1)$-face of two neighbouring n-simplices.

There is a meaningful sense in which tensor networks $A[\mathscr{M}_n^1], A[\mathscr{M}_n^2]$ can be composed according to a gluing of the simplicial complexes $\mathscr{M}_n^1, \mathscr{M}_n^2$ along a common $(n-1)$-subcomplex of their boundaries:

Definition 3 (Composition)**.** Given a pair of n-dimensional triangulated manifolds $\mathscr{M}_n^1, \mathscr{M}_n^2$, together with a pair of isomorphic $(n-1)$-dimensional subcomplexes of their boundaries, collectively referred to as $\mathscr{M}_{(n-1)}$ to avoid undue detail, we define the composition of the corresponding tensor networks by

$$A[\mathscr{M}_n^1] \otimes_{\mathscr{M}_{(n-1)}} A[\mathscr{M}_n^2] := \mathrm{tr}_{\triangle_{(n-1)} \in \mathscr{M}_{(n-1)}} \left[A[\mathscr{M}_n^1] \otimes A[\mathscr{M}_n^2] \right],$$

where the trace denotes the contraction of pairs of indices that are assigned to $(n-1)$-faces that become identified in $\mathscr{M}_{(n-1)}$.

It is not hard to see that the composition of tensor networks corresponds to gluing of the corresponding simplicial complexes. That is, if we denote by $\mathscr{M}_n^1 \cup_{\mathscr{M}_{(n-1)}} \mathscr{M}_n^2$ the simplicial complex formed by identifying the copies of $\mathscr{M}_{(n-1)}$ in $\partial \mathscr{M}_n^1$, $\partial \mathscr{M}_n^2$, respectively, then definition 2 ensures that

$$A[\mathscr{M}_n^1] \otimes_{\mathscr{M}_{(n-1)}} A[\mathscr{M}_n^2] = A[\mathscr{M}_n^1 \cup_{\mathscr{M}_{(n-1)}} \mathscr{M}_n^2]. \tag{2.2}$$

Remark 4. The contractions in definition 3 and eq. (2.1) are completely natural if we think of $A[\mathscr{M}_n]$ as a linear map from the degrees of freedom placed on the negatively oriented boundary faces to the positively oriented boundary faces of the simplex. In section 2.2.3 below, we will explore this more systematically by defining the boundary Hilbert space of each $\mathscr{M}_n$ as a tensor product over Hilbert spaces for each $(n-1)$-face in the boundary complex, where we choose $\mathbb{C}^D$ or its dual space $(\mathbb{C}^D)^*$ depending on the face's orientation.

2.2.2 Topological Invariance from Pachner's Equations

The n-dimensional Pachner moves implement local retriangulations of n-dimensional simplicial complexes. This is made precise in the following definition.

Definition 5 (Pachner Move)**.** Let $\mathscr{M}_n$ be an n-dimensional simplicial complex and $\mathscr{M}_n' \subseteq \mathscr{M}_n$ a codimension-0 subcomplex isomorphic to a subcomplex of the boundary of an $(n+1)$-simplex $\triangle_{(n+1)}$, $\mathscr{M}_n' \subset \partial \triangle_{(n+1)}$ (to avoid undue detail we let $\mathscr{M}_n'$ denote both the initial subcomplex and its image under the isomorphism). Then the *Pachner move* $\mathrm{p}_{\mathscr{M}_n'}$ replaces $\mathscr{M}_n'$ with $(\partial \triangle_{(n+1)} \setminus \mathscr{M}_n')$,

$$\mathrm{p}_{\mathscr{M}_n'} : \mathscr{M}_n \mapsto (\mathscr{M}_n \setminus \mathscr{M}_n') \cup_{\partial \mathscr{M}_n'} (\partial \triangle_{(n+1)} \setminus \mathscr{M}_n').$$

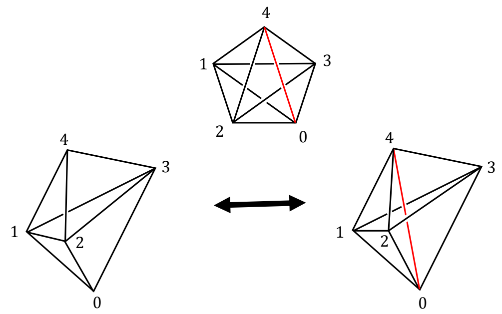


Figure 2.1 Illustration of one of the Pachner moves for $n = 3$.

As the boundaries match, $\partial\mathscr{M}'_n = \partial(\partial\triangle_{(n+1)} \setminus \mathscr{M}'_n)$, this move yields another simplicial complex.

See fig. 2.1 for an illustration. There are $\lfloor(n+2)/2\rfloor$ inequivalent Pachner moves in n-dimensions, corresponding to the different n-subcomplexes of $\partial\triangle_{(n+1)}$.

A fundamental theorem of combinatoric topology says that homeomorphism equivalence of complexes is captured by these local moves:

Theorem 6. *A pair of n-simplicial complexes are piecewise linear homeomorphic iff they are related by a sequence of Pachner moves.*

In his seminal work, Pachner extended this to prove that any two combinatoric manifolds are piecewise linear (PL) homeomorphic if and only if they are related by a sequence of local moves drawn from a finite set of operations called *shellings* and *inverse shellings*. Shellings and inverse shellings can be intuitively thought of as modifying the boundary of a combinatoric n-manifold by an $(n-1)$-dimensional Pachner move achieved by adding (removing) an n-simplex to (from) the boundary. For our purposes we require a slightly altered version which does not involve shellings, since these correspond to adding or removing tensors from the boundary of a network and we are not ensured invariance under this. The relevant theorem we need is due to Casali and it involves only n-dimensional Pachner moves within the bulk:

Theorem 7 (Casali)**.** *Let M_1^n and M_2^n be two PL n-manifolds, and let $\mathscr{M}_{M_1^n}$ (resp. $\mathscr{M}_{M_2^n}$) be a simplicial triangulation of M_1^n (resp. M_2^n) with $\partial\mathscr{M}_{M_1^n} = \partial\mathscr{M}_{M_2^n}$. Then, M_1^n and M_2^n are PL homeomorphic if and only if $\mathscr{M}_{M_1^n}$ and $\mathscr{M}_{M_2^n}$ are related by a sequence of Pachner moves.*

Note importantly that this result utilizes only n-dimensional Pachner moves within the bulk of the combinatorial manifold $\mathscr{M}_n$ that leave the boundary triangulation invariant.

The local characterisation of PL homeomorphism facilitates local constructions of combinatorial topological invariants, specifically PL homeomorphism invariants. One needs only assign some algebraic data to each n-simplex and check the invariance of this data under the finite set of Pachner moves, theorem 7 guarantees that the algebraic object assigned to a PL manifold (given a triangulation) is a PL homeomorphism invariant. This general class of invariants are called state sum invariants.[48, 77, 49–64, 67] On the level of the simplex tensors, this suggests the following definition.

Definition 8 (Pachner Move Invariant Simplex Tensor , PMIST)**.** A simplex tensor A is said to be *Pachner move invariant* if it satisfies the following set of tensor equations:

$$A[\mathscr{M}_n] = A[\partial \triangle^{\pm}_{(n+1)} \setminus \mathscr{M}_n], \qquad \forall\, \mathscr{M}_n \subset \partial \triangle^{\pm}_{(n+1)} \tag{2.3}$$

where $\triangle^{\pm}_{(n+1)}$ is a positively (negatively) oriented, vertex ordered $(n+1)$-simplex.

Note that these correspond to $\lfloor (n+2)/2 \rfloor$ sets of equations, one set for each n dimensional Pachner move.

Topological invariance under bulk homeomorphism is ensured for maps constructed from PMIST solutions as defined in the previous section.

First we consider the case of a manifold with trivial boundary which is mapped to a number by A.

Lemma 9. *Let $A[\triangle_n]$ be a PMIST. For any closed, oriented, PL n-manifold M^n and triangulation $\mathscr{M}_{M^n}$ the number $A[\mathscr{M}_{M^n}] \in \mathbb{C}$ is independent of the choice of triangulation and furthermore is a PL homeomorphism invariant. Hence we define $A[M^n] := A[\mathscr{M}_{M^n}]$ for an arbitrary triangulation $\mathscr{M}_{M^n}$.*

Proof. Theorem 7 implies that any two triangulations $\mathscr{M}_{M_1^n}, \mathscr{M}_{M_2^n}$ of PL homeomorphic manifolds M_1^n, M_2^n are related by a finite sequence of Pachner moves $\mathscr{M}_{M_2^n} = \mathrm{p}_m \cdots \mathrm{p}_1 \mathscr{M}_{M_1^n}$. Since A is a PMIST, $A[\mathscr{M}_{M_2^n}] = A[\mathrm{p}_m \cdots \mathrm{p}_1 \mathscr{M}_{M^n}] = A[\mathscr{M}_{M_1^n}]$. It follows that $A[M^n] := A[\mathscr{M}_{M^n}]$ defines a PL homeomorphism invariant. □

Note that each invariant is upper bounded by $A[M^n] \leq \|A\| \|\mathscr{M}^{\min}_{M^n}\|$, where $\|A\|$ is the norm of the tensor $A[\triangle_n]$ and $\|\mathscr{M}^{\min}_{M^n}\|$ is the minimal number of n-simplices required to triangulate M^n. Hence for a finite tensor A we have $A[M^n] < \infty$ for any oriented manifold M^n admitting a finite triangulation. Hence we have a combinatoric construction of a PL invariant for any PMIST A. In dimension ≤ 3, these invariants are in fact topological invariants since any topological manifold admits a smooth structure in dimension ≤ 3. Similarly these invariants are smooth invariants in dimension ≤ 6 where smooth and PL structure are equivalent. Note that this does not imply that such invariants will necessarily be sensitive to smooth structure. The question of finding combinatorial invariants that are sensitive to smooth structure is of great mathematical significance as they could potentially be used to prove or disprove the smooth Poincare conjecture in four dimensions, the only remaining unsolved case of the generalized Poincare conjecture.

Let us move on to consider oriented manifolds with boundary:

Lemma 10. *Let $A[\triangle_n]$ be a PMIST and M_1^n, M_2^n oriented, homeomorphic PL n-manifolds with matching, nontrivial boundaries $\partial M_1^n = \partial M_2^n \neq \emptyset$, and fix a triangulation $\mathscr{M}_{\partial M^n}$ of the boundary.*

Then, for any triangulations $\mathscr{M}_{M_1^n}, \mathscr{M}_{M_2^n}$ that restrict to the fixed boundary triangulation, we have $A[\mathscr{M}_{M_1^n}] = A[\mathscr{M}_{M_2^n}] \in (\mathbb{C}^D)^{\otimes \|\mathscr{M}_{\partial M^n}\|}$. Hence we define $A[M^n; \mathscr{M}_{\partial M^n}] := A[\mathscr{M}_{M^n}]$ for an arbitrary triangulation $\mathscr{M}_{M^n}$ extending $\mathscr{M}_{\partial M^n}$.

Proof. By theorem 7 there exists a sequence of Pachner moves p_i relating the triangulations $\mathscr{M}_{M_1^n} = \mathrm{p}_m \cdots \mathrm{p}_1 \mathscr{M}_{M_2^n}$, hence $A[\mathscr{M}_{M_1^n}] = A[\mathrm{p}_m \cdots \mathrm{p}_1 \mathscr{M}_{M_2^n}] = A[\mathscr{M}_{M_2^n}]$. □

This shows that the tensor networks associated to manifolds with nontrivial boundary are insensitive to the choice of triangulation in the bulk. We note that invariance under changing the vertex order is also a consequence of invariance under Pachner moves and does not have to be established independently [53, 58] (in fact it can be seen to follow directly from Lemma. 19). Since a manifold with nontrivial boundary can be viewed as a cobordism, these tensor networks can be thought of as linear maps assigned to cobordisms. This point of view leads us down the path towards topological quantum field theory.

2.2.3 Tensor Network Topological Quantum Field Theory

A topological quantum field theory (TQFT) is a functor from the category of cobordism to the category of vector spaces. More concretely it assigns a vector to each n-manifold (a number if the boundary is trivial), invariant under bulk homeomorphisms, and a vector space to each $(n-1)$-manifold. In this section we will realize such a functor with a tensor network map A which takes combinatoric n-manifolds to vectors and $(n-1)$-manifolds to the support vector spaces of certain projectors.

We first recount the definition of a cobordism.

Definition 11 (Cobordism)**.** An oriented (PL) *cobordism* $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$ between oriented PL $(n-1)$-manifolds $M_1^{(n-1)}$ and $M_2^{(n-1)}$ is an oriented PL n-manifold M^n satisfying $\partial M^n = M_2^{(n-1)} \sqcup \overline{M_1^{(n-1)}}$.

Note $\overline{M^n}$ denotes the oriented manifold M^n with its orientation reversed.

By lemma 10 an oriented PL cobordism $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$ of oriented PL $(n-1)$-manifolds with fixed triangulations $\mathscr{M}_{M_1^{(n-1)}}, \mathscr{M}_{M_2^{(n-1)}}$ is mapped to the linear operator

$$A[M^n; \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}] : (\mathbb{C}^D)^{\otimes \|\mathscr{M}_{M_1^{(n-1)}}\|} \to (\mathbb{C}^D)^{\otimes \|\mathscr{M}_{M_2^{(n-1)}}\|}.$$

There is a natural involution on cobordisms by reversing the orientation. On the level of the corresponding operators, definition 1 ensures that this induces the Hermitian adjoint:

Lemma 12. *Let $M^n : M_1^{(n-1)} \to M_2^{(n-1)}$ be a cobordism and $\mathscr{M}_{M_1^{(n-1)}}, \mathscr{M}_{M_2^{(n-1)}}$ triangulations. Then we have*

$$A[\overline{M^n}; \mathscr{M}_{M_1^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}] = A[M^n; \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}]^\dagger.$$

The composition of two operators with matching intermediate boundaries agrees precisely with the composition of tensor networks defined in definition 3 above:

$$\begin{aligned} &A[M^{n\prime}; \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}] A[M^n; \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}] = \\ &\qquad A[M^{n\prime}; \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}] \otimes_{\mathscr{M}_{M_2^{(n-1)}}} A[M^n; \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}]. \end{aligned}$$

There is also a natural composition operation '$\circ$' for pairs of cobordisms $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$, and $M^{n\prime} : M_2^{(n-1)} \mapsto M_3^{(n-1)}$, defined by gluing them along their matching boundary:

$$M^{n\prime} \circ M^n := M^n \cup_{M_2^{(n-1)}} M^{n\prime} : M_1^{(n-1)} \mapsto M_3^{(n-1)}$$

Note that $M^n \cup_{M_2^{(n-1)}} M^{n\prime}$ satisfies $\partial(M^n \cup_{M_2^{(n-1)}} M^{n\prime}) = M_3^{(n-1)} \sqcup \overline{M_1^{(n-1)}}$. It follows from eq. (2.2) that gluing corresponds to composition:

Lemma 13. *Given oriented, closed, PL $(n-1)$-manifolds $M_i^{(n-1)}$ with fixed triangulations $\mathscr{M}_{M_i^{(n-1)}}$, $i \in \{1,2,3\}$, and oriented PL cobordisms $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$, and $M^{n\prime} : M_2^{(n-1)} \mapsto M_3^{(n-1)}$, we have*

$$A[M^{n\prime}; \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}] A[M^n; \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}] = A[M^{n\prime} \circ M^n; \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}].$$

Proof. Pick arbitrary triangulations $\mathscr{M}_{M^n}, \mathscr{M}_{M^{n\prime}}$ satisfying $\mathscr{M}_{M^n}\big|_{\partial M^n} = \mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}$, $\mathscr{M}_{M^{n\prime}}\big|_{\partial M^{n\prime}} = \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}$ then the union of the triangulations $\mathscr{M}_{M^{n\prime} \cup_{M_2^{(n-1)}} M^n} := \mathscr{M}_{M^{n\prime}} \cup_{\mathscr{M}_{M_2^{(n-1)}}} \mathscr{M}_{M^n}$ yields a triangulation of the product of the cobordisms $M^{n\prime} \circ M^n =$

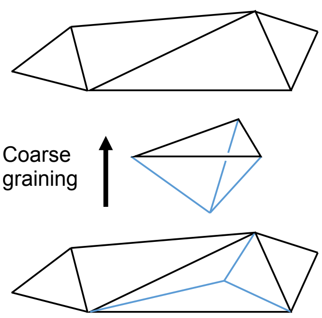


Figure 2.2 Illustration of a bulk triangulation of a cylinder interpolating between different triangulations of the boundary manifold. The corresponding TQFT operator is called the cylinder isometry, and it gives rise to a hierarchical tensor network for preparing ground states of our topological lattice models (see p. 58).

$M^{n\prime} \cup_{M_2^{(n-1)}} M^n : M_1^{(n-1)} \mapsto M_3^{(n-1)}$ which satisfies $\mathscr{M}_{M^{n\prime} \cup_{M_2^{(n-1)}} M^n}\big|_{\partial(M^{n\prime} \cup_{M_2^{(n-1)}} M^n)} = \mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}$, then by eq. (2.2) and lemma 10 we have

$$\begin{aligned} &A[M^{n\prime};\mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_2^{(n-1)}}}]A[M^n;\mathscr{M}_{M_2^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}] = A[\mathscr{M}_{M^{n\prime}}] \otimes_{\mathscr{M}_{M_2^{(n-1)}}} A[\mathscr{M}_{M^n}] \\ =&A[\mathscr{M}_{M^{n\prime} \cup_{M_2^{(n-1)}} M^n}] = A[M^{n\prime} \circ M^n;\mathscr{M}_{M_3^{(n-1)}} \sqcup \overline{\mathscr{M}_{M_1^{(n-1)}}}]. \end{aligned}$$ □

In particular, lemma 13 shows that the product of two cobordism tensor networks is independent of the intermediate boundary triangulation. We will now further extend this insensitivity to the choice of boundary triangulations, which turns out to be an interesting issue.

Definition 14 (Cylinder Isometry & Projector)**.** For a closed, oriented, PL $(n-1)$-manifold $M^{(n-1)}$ with a pair of triangulations $\mathscr{M}^1_{M^{(n-1)}}, \mathscr{M}^2_{M^{(n-1)}}$ the *cylinder isometry* is defined to be

$$P_A[\mathscr{M}^2_{M^{(n-1)}}, \mathscr{M}^1_{M^{(n-1)}}] := A[M^{(n-1)} \times [0,1]; \mathscr{M}^2_{M^{(n-1)}} \sqcup \overline{\mathscr{M}^1_{M^{(n-1)}}}],$$

where $M^{(n-1)} \times [0,1]$ is treated as an oriented n-manifold satisfying $\partial(M^{(n-1)} \times [0,1]) = M^{(n-1)} \sqcup \overline{M^{(n-1)}}$ (see fig. 2.2). An important special case is the *cylinder projector*, defined by

$$P_A[\mathscr{M}^1_{M^{(n-1)}}] := P_A[\mathscr{M}^1_{M^{(n-1)}}, \mathscr{M}^1_{M^{(n-1)}}],$$

where both triangulations agree.

The next lemma justifies the use of the name cylinder projector.

Lemma 15. *The cylinder projector satisfies* $P_A[\mathscr{M}_{M^{(n-1)}}]P_A[\mathscr{M}_{M^{(n-1)}}] = P_A[\mathscr{M}_{M^{(n-1)}}]$ *and* $P_A[\mathscr{M}_{M^{(n-1)}}]^\dagger = P_A[\mathscr{M}_{M^{(n-1)}}]$ *and hence it is a Hermitian projection operator.*

Proof. Each cylinder projector $P_A[\mathscr{M}_{M^{(n-1)}}]$ is a linear map on $(\mathbb{C}^D)^{\otimes\|\mathscr{M}_{M^{(n-1)}}\|}$ and their product satisfies $P_A[\mathscr{M}_{M^{(n-1)}}]P_A[\mathscr{M}_{M^{(n-1)}}] = A[M^{(n-1)}\times[0,1];\mathscr{M}_{M^{(n-1)}}\sqcup\overline{\mathscr{M}_{M^{(n-1)}}}]\otimes_{\mathscr{M}_{M^{(n-1)}}} A[M^{(n-1)}\times[0,1];\mathscr{M}_{M^{(n-1)}}\sqcup\overline{\mathscr{M}_{M^{(n-1)}}}] = A[M^{(n-1)}\times[0,1];\mathscr{M}_{M^{(n-1)}}\sqcup\overline{\mathscr{M}_{M^{(n-1)}}}] = P_A[\mathscr{M}_{M^{(n-1)}}]$, by an application of lemma 13. The second claim follows at once from lemma 12. □

Note that this yields a formula for the dimension of the subspace $\operatorname{supp}(P_A[\mathscr{M}_{M^{(n-1)}}])$ which is independent of the choice of triangulation and in fact topological invariant by lemma 9:

$$\dim(\operatorname{supp}(P_A[\mathscr{M}_{M^{(n-1)}}])) = \operatorname{tr}P_A[\mathscr{M}_{M^{(n-1)}}] = A[M^{(n-1)}\times S^1] < \infty,$$

where S^1 is a circle. Hence we define $\dim_A(M^{(n-1)}) := \operatorname{tr}P_A[\mathscr{M}_{M^{(n-1)}}]$ for an arbitrary triangulation $\mathscr{M}_{M^{(n-1)}}$ and note that this is necessarily a finite number if the manifold admits a finite triangulation (as argued below). While the subspace $\operatorname{supp}(P_A[\mathscr{M}_{M^{(n-1)}}])$ itself still depends on the choice of triangulation, any two triangulations are naturally isomorphic by the cylinder isometries:

Lemma 16. *The cylinder isometry* $P_A[\mathscr{M}^2_{M^{(n-1)}},\mathscr{M}^1_{M^{(n-1)}}]$ *is a partial isometry from* $\operatorname{supp}(P_A[\mathscr{M}^1_{M^{(n-1)}}])$ *onto* $\operatorname{supp}(P_A[\mathscr{M}^2_{M^{(n-1)}}])$.

Proof. It follows directly from the definitions and lemma 13 that

$$\begin{aligned}
&P_A[\mathscr{M}^1_{M^{(n-1)}},\mathscr{M}^2_{M^{(n-1)}}]P_A[\mathscr{M}^2_{M^{(n-1)}},\mathscr{M}^1_{M^{(n-1)}}]\\
=&A[M^{(n-1)}\times[0,1];\mathscr{M}^1_{M^{(n-1)}}\sqcup\overline{\mathscr{M}^2_{M^{(n-1)}}}]\otimes_{\mathscr{M}^2_{M^{(n-1)}}} A[M^{(n-1)}\times[0,1];\mathscr{M}^2_{M^{(n-1)}}\sqcup\overline{\mathscr{M}^1_{M^{(n-1)}}}]\\
=&A[M^{(n-1)}\times[0,1];\mathscr{M}^1_{M^{(n-1)}}\sqcup\overline{\mathscr{M}^1_{M^{(n-1)}}}] = P_A[\mathscr{M}^1_{M^{(n-1)}}].
\end{aligned}$$

and likewise that $P_A[\mathscr{M}^2_{M^{(n-1)}},\mathscr{M}^1_{M^{(n-1)}}]P_A[\mathscr{M}^1_{M^{(n-1)}},\mathscr{M}^2_{M^{(n-1)}}] = P_A[\mathscr{M}^2_{M^{(n-1)}}]$. □

Cobordisms defined on different boundary triangulations are related by cylinder isometries:

Lemma 17. *For an oriented, PL cobordism* $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$ *and any two pairs of boundary triangulations* $(\mathscr{M}_{M_1^{(n-1)}},\mathscr{M}_{M_2^{(n-1)}})$, & $(\mathscr{M}'_{M_1^{(n-1)}},\mathscr{M}'_{M_2^{(n-1)}})$, *we have*

$$A[M^n;\mathscr{M}_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}_{M_1^{(n-1)}}}] = P_A[\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}'_{M_2^{(n-1)}}]A[M^n;\mathscr{M}'_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}'_{M_1^{(n-1)}}}]P_A[\mathscr{M}'_{M_1^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}].$$

Proof. By lemma 13: $P_A[\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}'_{M_2^{(n-1)}}]A[M^n;\mathscr{M}'_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}'_{M_1^{(n-1)}}}]P_A[\mathscr{M}'_{M_1^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}]$ $=A[M_2^{(n-1)}\times[0,1];\mathscr{M}_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}'_{M_2^{(n-1)}}}]A[M^n;\mathscr{M}'_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}'_{M_1^{(n-1)}}}]A[M_1^{(n-1)}\times[0,1];\mathscr{M}'_{M_1^{(n-1)}}\sqcup$ $\overline{\mathscr{M}_{M_1^{(n-1)}}}]=A[(M_2^{(n-1)}\times[0,1])\circ M^n\circ(M_1^{(n-1)}\times[0,1]);\mathscr{M}_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}_{M_1^{(n-1)}}}]=A[M^n;\mathscr{M}_{M_2^{(n-1)}}\sqcup$ $\overline{\mathscr{M}_{M_1^{(n-1)}}}]$ □

The important special case $(\mathscr{M}_{M_1^{(n-1)}},\mathscr{M}_{M_2^{(n-1)}})=(\mathscr{M}'_{M_1^{(n-1)}},\mathscr{M}'_{M_2^{(n-1)}})$ implies that all linear cobordism maps act within the support subspaces of the cylinder projectors.

We are now in a position to verify that our construction defines in fact a functor from the cobordism category of triangulated PL $(n-1)$-manifolds to the category of vector spaces and hence defines a TQFT: For any $(n-1)$-manifold $M^{(n-1)}$ with triangulation $\mathscr{M}_{M^{(n-1)}}$, we define a vector space

$$A[\mathscr{M}_{M^{(n-1)}}]:=\mathrm{supp}(P_A[\mathscr{M}_{M^{(n-1)}}])$$

and for any cobordism $M^n\colon M_1^{(n-1)}\to M_2^{(n-1)}$ with boundary triangulations $\mathscr{M}_{M_1^{(n-1)}}$ and $\mathscr{M}_{M_2^{(n-1)}}$ we set

$$A[M^n;\mathscr{M}_{M_2^{(n-1)}}\sqcup\overline{\mathscr{M}_{M_1^{(n-1)}}}]=A[\mathscr{M}_{M^n}]:A[\mathscr{M}_{M_1^{(n-1)}}]\to A[\mathscr{M}_{M_2^{(n-1)}}],$$

where $\mathscr{M}_{M^n}$ is an arbitrary triangulation extending the fixed boundary triangulations. We now show that A satisfies the discrete version of Atiyah's axioms for a TQFT:[47]

- **Atiyah 1:** A is functorial with respect to orientation preserving PL homeomorphisms.

  Proof. For any orientation preserving PL homeomorphism of closed $(n-1)$-manifolds $f:M_1^{(n-1)}\to M_2^{(n-1)}$ with triangulations $\mathscr{M}_{M_1^{(n-1)}},\mathscr{M}_{M_2^{(n-1)}}$ respectively the cylinder isometry $P_A[\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}]$ yields an isomorphism between the vector spaces $A[\mathscr{M}_{M_1^{(n-1)}}]$ and $A[\mathscr{M}_{M_2^{(n-1)}}]$ (which are embedded within different ambient vector spaces when $\|\mathscr{M}_{M_1^{(n-1)}}\|\neq\|\mathscr{M}_{M_2^{(n-1)}}\|$). Furthermore, for a second OP PL homeomorphism $g:M_2^{(n-1)}\to M_3^{(n-1)}$ given triangulations $\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}_{M_3^{(n-1)}}$ we have that $P_A[\mathscr{M}_{M_3^{(n-1)}},\mathscr{M}_{M_2^{(n-1)}}]P_A[\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}]=P_A[\mathscr{M}_{M_3^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}]$. Hence defining $A[f]:=P_A[\mathscr{M}_{M_2^{(n-1)}},\mathscr{M}_{M_1^{(n-1)}}]$ fulfills the $(n-1)$-dimensional requirement, $A[g]A[f]=A[g\circ f]$.
  Now let $M^n:M_1^{(n-1)}\to M_2^{(n-1)},M^{n\prime}:M_1^{(n-1)\prime}\to M_2^{(n-1)\prime}$ be a pair of cobordisms with fixed boundary triangulations $M_i^{(n-1)},M_i^{(n-1)\prime}$. For an extension of an orientation preserving PL homeomorphism from the boundaries to the bulk $F:M^n\to M^{n\prime}$ let

$f_i : M_i^{(n-1)} \to M_i^{(n-1)'}$ denote the restriction of F to the relevant boundary components. We have that $A[M_1^n]A[f_2] = A[f_1]A[M_2^n]$, in other words the 2-morphism associated to F is the identity. $\square$

- **Atiyah 2:** A is involutory, i.e, $A[\overline{\mathscr{M}_{M^{(n-1)}}}] = A[\mathscr{M}_{M^{(n-1)}}]^*$.

  Proof. Since we exclusively work with vector spaces isomorphic to $\mathbb{C}^{\otimes N}$ there is no real distinction between the vector space and its dual. The lemma is simply due to our convention that we treat components of the boundary with an orientation that matches that induced by the bulk as output (kets) and those with opposite orientation as inputs (bra) and hence reversing the orientation of an $(n-1)$-manifold switches it between input and output. This yields a natural pairing of manifolds with a common oriented boundary, i.e., $\partial M_1^n = \partial M_2^n$, fixing some triangulation of the boundary $\mathscr{M}_{(n-1)}$ one can define $(M_1^n, M_2^n) := A[M_1^n, \mathscr{M}_{(n-1)}] \otimes_{\mathscr{M}_{(n-1)}} A[\overline{M_2^n}, \mathscr{M}_{(n-1)}] = A[M_1^n \cup_{\partial M_1^n} \overline{M_2^n}] \in \mathbb{C}$ which is the topological invariant amplitude of the glued manifold $M_1^n \cup_{\partial M_1^n} \overline{M_2^n}$. $\square$

- **Atiyah 3:** A is multiplicative with respect to composition of cobordisms.

  We have proved this in lemma 13.

- **Atiyah 4:** $A[\emptyset^{(n-1)}] = \mathbb{C}$, treating $\emptyset$ as an $(n-1)$ manifold, $A[\emptyset^n] = 1$, treating $\emptyset$ as an n manifold, and $A[M^{(n-1)} \times [0,1]; \mathscr{M}_{M^{(n-1)}} \sqcup \overline{\mathscr{M}_{M^{(n-1)}}}] = \mathbb{1}_{A[\mathscr{M}_{M^{(n-1)}}]}$.

  The purpose of the fourth axiom is to exclude the case of trivial (identically zero) invariants. Let us briefly explain these conditions, since $A[\emptyset^{(n-1)}] = A[\emptyset^{(n-1)} \cup \emptyset^{(n-1)}] = A[\emptyset^{(n-1)}] \otimes A[\emptyset^{(n-1)}]$, $A[\emptyset^{(n-1)}]$ is idempotent under $\otimes$ and hence is $\mathbb{C}$ or 0, to discount trivial theories we require the former. Similarly $A[\emptyset^n] = A[\emptyset^n]A[\emptyset^n]$ and hence $A[\emptyset^n]$ is 1 or 0, and we require the former. Finally, $A[M^{(n-1)} \times [0,1]; \mathscr{M}_{M^{(n-1)}} \sqcup \overline{\mathscr{M}_{M^{(n-1)}}}] = \mathbb{1}_{A[\mathscr{M}_{M^{(n-1)}}]}$ follows directly from lemma 15.

- **Atiyah 5:** Orientation reversal induces Hermitian conjugation.

  We have proved this in lemma 12. We note that this forces the topological invariant assigned to a closed manifold to be equal to the complex conjugate of its orientation reversed counterpart, $A[M^n] = A[\overline{M^n}]^*$. Finally it forces the natural pairing induced by the action of orientation reversal to match the Hermitian inner product of the target vector spaces. That is for $\partial M_1^n = \partial M_1^n = N^{(n-1)}$ with an arbitrary triangulation $\mathscr{M}_{N^{(n-1)}}$ we have $\langle A[M_2^n; \mathscr{M}_{N^{(n-1)}}] | A[M_1^n; \mathscr{M}_{N^{(n-1)}}] \rangle = A[M_1^n; \mathscr{M}_{N^{(n-1)}}] \otimes_{N^{(n-1)}} A[M_2^n; \mathscr{M}_{N^{(n-1)}}]^* = A[M_1^n; \mathscr{M}_{N^{(n-1)}}] \otimes_{N^{(n-1)}} A[\overline{M_2^n}; \mathscr{M}_{N^{(n-1)}}] = A[M_1^n \cup_{N^{(n-1)}} \overline{M_2^n}] = (M_1^n, M_2^n)$

and hence a state overlap corresponds to a manifold pairing which equals the topological invariant amplitude of the glued manifold.

One may observe that our construction of a discrete TQFT retains dependence on the choice of boundary triangulations $\mathscr{M}_{M^{(n-1)}}$ and they merely determine an embedding of the topological vector space within some larger vector space $A[M^{(n-1)}] \subseteq (\mathbb{C}^D)^{\otimes \|\mathscr{M}_{M^{(n-1)}}\|}$. One can completely remove the triangulation dependence from the construction by taking an equivalence relation under cylinder isometries $P_A[\mathscr{M}^2_{M^{(n-1)}}, \mathscr{M}^1_{M^{(n-1)}}]$ (e.g., [75, 71]). Doing so associates a single vector space $A[M^{(n-1)}] \cong \mathbb{C}^{\dim_A(M^{(n-1)})}$ to each closed $(n-1)$-manifold $M^{(n-1)}$ and to each cobordism $M^n : M_1^{(n-1)} \mapsto M_2^{(n-1)}$ a linear map $A[M^n] : A[M_1^{(n-1)}] \to A[M_2^{(n-1)}]$. The matrix entries of each $A[M^n]$ can be written explicitly if one fixes a triangulation $\mathscr{M}_{M^{(n-1)}}$ of each $(n-1)$-manifold and a basis for each subspace $\mathrm{supp}(P_A[\mathscr{M}_{M^{(n-1)}}])$.

These equivalence classes still capture all the topological properties of our tensor construction, but miss some of the physics that lies within the structure of the tensor networks describing topological states on different lattices. In the next section we will study exactly solvable models on concrete lattices whose ground spaces are described by our topological tensor networks, demonstrating the significance of the information associated to a fixed boundary triangulation.

2.2.4 Structure of the Tensor Solutions to the Pachner Move Equations

We now discuss some additional structure that needs to be considered in looking for the most general tensor solutions which satisfy the Pachner move constraints. The extra detail we explain here does not substantially influence any of the arguments made in the remainder of the manuscript.

First we note that direct sum and tensor product of PMIST solutions induces a sum and product, respectively, of the resulting topological invariants. That is, given PMIST solutions $A[\triangle_n], A'[\triangle_n]$ both $(A \oplus A')[\triangle_n] = A[\triangle_n] \oplus A'[\triangle_n]$ and $(A \otimes A')[\triangle_n] = A[\triangle_n] \otimes A'[\triangle_n]$ are PMIST solutions which satisfy $(A \oplus A')[M^n] = A[M^n] + A'[M^n]$ and $(A \otimes A')[M^n] = A[M^n]A'[M^n]$ for any closed manifold M^n.

The PMIST invariants form a commutative *rig* (semiring) under these addition and multiplication operations, to see this first note both $\otimes$ and $\oplus$ are commutative and associative and the multiplication is distributive over the addition. The invariant given by the 0 tensor $A[\triangle_n] = 0$, which assigns 0 to any closed n-manifold, yields the additive identity element. The invariant corresponding to the trivial theory, which assigns 1 to any closed n-manifold, yields the multiplicative identity. This corresponds to taking $A[\triangle_n]$ to be a tensor product of the same state (which may be 1-dimensional i.e. a complex number). Note that the GHZ symmetry breaking type theories correspond to a direct sum of trivial theories.

There cannot be multiplicative inverses in general since the invariant assigned to a manifold of the form $M^{(n-1)} \times S^1$ gives the ground state degeneracy of the theory on the manifold $M^{(n-1)}$ (a natural number) which can be larger than 1. There is some interest in studying the subset of 'invertible' theories which are defined to be those which assign modulus one numbers to any closed manifold. In particular this implies such theories have a unique ground state on any closed spatial manifold. The invertible theories include the symmetry protected phases and those with a graviational anomaly. For the invertible theories complex conjugation of the PMIST tensor yields the multiplicative inverse theory.

Thus far we have avoided providing an explicit definition of which theories are considered to be equivalent as this is a somewhat subtle issue. It is somewhat implicit in our discussion that two theories are considered equivalent if they assign the same number to all closed n-manifolds, in particular this would imply that two equivalent theories have matching ground state degeneracy on any spatial manifold. Depending on one's point of view this may be merely a necessary condition and one may need to look at classifying Wilson loops and excitations to fully specify the physics of the theory. These are not unrelated concepts however, for example in 3D if one knows the effect of modular transformations upon the ground space of different topologies one can recover the invariant assigned to any closed 3-manifold by taking the appropriate matrix element of a product of modular transformations corresponding to the Dehn surgery that produces the desired 3-manifold.

On the other hand it is unclear whether we need to define a physical theory for arbitrary n-manifolds as those which are considered in physical systems are of the form $M^{(n-1)} \times [0,1]$. Such theories defined only on this restricted set of manifolds are commonly referred to as $(n-1)+\varepsilon$ TQFTs, and may fail to be defined on closed n-manifolds such as $M^{(n-1)} \times S^1$ since they yield infinity. However within the scope of models we examine everything is finite and we expect by using modular transformations one can construct all closed n-manifolds invariants.

Let us discuss the equivalence of theories a little further, first note there are some purely local equivalence relations on the individual tensors $A[\triangle_n]$ the simplest of which is to apply an invertible matrix M (unitary if we specify the action of orientation reversal to be Hermitian) to each tensor index associated to a positively oriented face, and M^{-1} to each negatively oriented face which will cancel out upon contracting neighboring tensors. One possibility our framework opens is to consider a pair of theories equivalent if their corresponding Hamiltonians lie within the same gapped phase of matter, this in particular ensures the theories have matching ground state degeneracy on different topologies, although it is unclear precisely how this corresponds to other notions of equivalence.

A fundamental step in the classification of inequivalent theories is understanding whether the equivalence classes of PMIST solutions fall into discrete families. Ocneanu rigidity implies that the solutions in 3D (corresponding to the F-symbols of fusion categories) fall into discrete families. Kitaev's proof [9] of the rigidity theorem (referred to as Theorem E.16) appears to imply that the PMIST solutions fall into discrete families in any spacetime dimension, so long as one restricts the degrees of freedom to live on the edges of the tensors. It is an important task to extend this proof to the most general tensor solutions we consider (or to discover if it does not extend which would make the classification problem far more intractable).

In looking for PMIST solutions $A[\triangle_n]$ one may reduce the number of variables within the tensor and equations by considering rotation invariant tensors, however this is known to exclude some solutions and hence we do not consider enforcing such a symmetry. We also note in many examples the approach is taken to find solutions to some distinguished Pachner moves such as $k \leftrightarrow k$, $k-1 \leftrightarrow k+1$ or $k \leftrightarrow k+1$, with an additional invertibility (on a subspace) constraint on the tensor solutions. The constraint ensures the remaining Pachner moves are satisfied since one can use the distinguished moves to reduce the equality to the invertibility constraint (e.g. F-symbols).

Although we have formulated our tensors to have indices associated to codimension-1 faces of a simplex, one can easily embed degrees of freedom on subsimplices of arbitrary codimension within this index. This is done by allowing the tensor to depend on a degree of freedom associated to a subsimplex which is in a GHZ state (delta condition) with a copy of the information in tensor product with the index on each codimension-1 faces that intersect the subsimplex. The form of a tensor with degrees of freedom on all subsimplices is as follows

$$A[\triangle_n] = \sum_{\{s_{\triangle_k}\}} (A[\triangle_n])_{\{s_{\triangle_k}\}} \bigotimes_{\triangle_{(n-1)}\in\triangle_n} \bigotimes_{\triangle_k\in\triangle_{(n-1)}} |s_{\triangle_k}\rangle_{\triangle_{(n-1)},\triangle_k} \tag{2.4}$$

where $0 \leq k < n$ and $\{s_{\triangle_k}\}$ is some configuration of spin variables on all subsimplices $\triangle_k \in \triangle_n$. To our knowledge all interesting models depend on degrees of freedom of codimension at least 2.

To find the most general PMIST solutions when working with degrees of freedom on lower dimensional subsimplices $\triangle_k$, $k < n$, one needs to associate variable dependent weights $w(s_{\triangle_k})$ to those subsimplices. For codimension-1 faces these weights can be introduced by a diagonal matrix acting on the index associated to a codimension-1 face within the interior of a simplicial complex, this matrix can be absorbed into the adjacent tensors by taking the square root, or using a convention that it appears on all positively (or negatively) oriented faces. However, since for $k \leq (n-2)$ many n-simplices may meet at a single lower dimensional k-subsimplex one needs a convention to introduce these weights. This can be achieved with a diagonal matrix on one of the indices associated to a k-subsimplex in the interior of a complex. Note if one fixes a lattice triangulation with a fixed coordination number of n-simplices at each subsimplex, such as the dual to a hexagonal lattice, one can absorb the weights into the n-simplex tensors surrounding a subsimplex by dividing the weights in the matrix evenly and distributing this to an index on each of the surrounding n-simplex tensors.

With such a convention enforced while taking the tensor trace one finds the following formula for the tensor network assigned to an n dimensional simplicial complex K with PMIST $A[\triangle_n]$ and weight function w

$$\begin{aligned}&\mathrm{tr}_{\triangle_{(n-1)}\in \mathrm{int}(K)}\left[\bigotimes_{\triangle_n\in K} A[\triangle_n]\right]\\&=\sum_{\{s_{\triangle_k}\}_{\triangle_k\in K}}\prod_{\triangle_n\in K}(A[\triangle_n])_{\{s_{\triangle_k}\}_{\triangle_k\in\triangle_n}}\prod_{\substack{\triangle_k\in K\\ k<n}} w(s_{\triangle_k})^{\gamma(\triangle_k)}\bigotimes_{\triangle_{(n-1)}\in\partial K}\bigotimes_{\triangle_k\in\triangle_{(n-1)}}|s_{\triangle_k}\rangle_{\triangle_{(n-1)},\triangle_k},\end{aligned} \tag{2.5}$$

where $\gamma(\triangle_k) = 1$ or $1/2$, depending on whether the k-simplex is in the interior or in the boundary of K. This adequately incorporates the effect of the weight function w and it is readily verified that the TQFT axioms still hold. However, we note that suitable care has to be taken when gluing along manifolds with corners, such as in the construction of the Hamiltonian vertex term. We will not comment on this further but refer to [78] for the details.

2.3 Exactly Solvable Lattice Models

In the previous section we established that a single tensor satisfying a local set of topological invariance conditions defines a TQFT on oriented manifolds with triangulated boundaries. We made extensive use of the bulk topological invariance condition to establish the required arguments. In this section we will demonstrate that given a single tensor satisfying the local topological invariance we can go further and build a lattice model on any triangulated $(n-1)$-manifold $\mathscr{M}_{M^{(n-1)}}$. The lattice model is given by a frustration free, commuting projector Hamiltonian which stabilizes the ground space $A[\mathscr{M}_{M^{(n-1)}}]$. Again the key to our arguments will be the bulk topological invariance of the tensor network built from $A[\triangle_n]$ which leads to conceptually simple proofs of the basic properties outlined above.

Figure 2.3 Simplicial complexes defining the vertex terms $H_v = A[M_v]$ in $2+1$ and $3+1$ dimensions.

A lattice spin model for each PMIST $A[\triangle_n]$ is defined on a vertex ordered triangulation of an oriented $(n-1)$-manifold $\mathscr{M}_{M^{(n-1)}}$ with a spin $i_{\triangle_{(n-1)}}$ of dimension D on each $(n-1)$-simplex $\triangle_{(n-1)} \in \mathscr{M}_{M^{(n-1)}}$. The Hamiltonian is a sum of local terms $H_{\mathscr{M}_{M^{(n-1)}}} = \sum_{v \in \mathscr{M}_{M^{(n-1)}}} (\mathbb{1} - H_v)$, each term H_v acts on the spins sitting on $(n-1)$-simplices adjacent to the vertex $v \in \mathscr{M}_{M^{(n-1)}}$. More precisely H_v acts on the simplices in the $(n-1)$ dimensional star of v, $\triangle_{(n-1)} \in \mathrm{st}\, v$. The local Hamiltonian term H_v is constructed concretely by introducing an auxiliary copy of vertex v, labeled v', and mapping the simplicial complex $v' * \mathrm{st}(v)$ (the cone over the star of v) to the tensor network $A[v' * \mathrm{st}(v)]$. Note the boundary of this complex consists of two copies of $\mathrm{st}(v)$, since $\partial(v' * \mathrm{st}(v)) = \{v, v'\} * \mathrm{lk}(v)$ and $v * \mathrm{lk}(v) = \mathrm{st}(v)$.

Definition 18. Let $M_v = v' * \mathrm{st}(v)$. The *vertex term* H_v is defined by the tensor network $A[M_v]$, viewed as an operator with open indices on $\mathrm{st}(v)$ as input and those on $\mathrm{st}(v')$ as output (see fig. 2.3).

An aspect we have glossed over in this definition is how v' should be included into the vertex ordering. Since we want the ordering of $\mathrm{st}(v)$ to match that of $\mathrm{st}(v')$ we define v' to have the same ordering as v with respect to any other vertex $w \in \mathscr{M}_{M^{(n-1)}}$ which leaves two possibilities, $v < v'$ and $v > v'$, we denote the corresponding Hamiltonian terms $H_v^{\uparrow}$ and $H_v^{\downarrow}$ respectively. One can verify $(H_v^{\downarrow})^{\dagger} = H_v^{\uparrow}$ since $(v')^{\uparrow} * \mathrm{st}(v)$ is converted to $(v')^{\downarrow} * \mathrm{st}(v)$ by flipping the direction of the edge $[v, v']$, which implements a conjugate transpose on the tensor network $A[v' * \mathrm{st}(v)]$.

Let us show that the choice of v' does not effect the definition of H_v by proving $H_v^{\uparrow} = H_v^{\downarrow}$, which also implies that the local Hamiltonian terms are Hermitian.

Lemma 19. $H_v^{\uparrow} = H_v^{\downarrow}$.

Proof. We first point out that $H_v^{\downarrow} H_v^{\uparrow} = A[\{(v')^{\downarrow}, (v'')^{\downarrow}\} * \mathrm{st}(v)]$ (treated as a map from the $\mathrm{st}(v')$ $(n-1)$-surface to $\mathrm{st}(v)$ then to $\mathrm{st}(v'')$, where $v'' > v'$). Together with $(H_v^{\downarrow})^{\dagger} = H_v^{\uparrow}$ this implies $A[\{(v')^{\downarrow}, (v'')^{\downarrow}\} * \mathrm{st}(v)] = A[\{(v')^{\downarrow}, (v'')^{\downarrow}\} * \mathrm{st}(v)]^{\dagger} = A[\{(v'')^{\downarrow}, (v')^{\downarrow}\} * \mathrm{st}(v)]$ where $\mathrm{st}(v'')$ is now the input and $\mathrm{st}(v')$ the output. We also have $A[\{(v')^{\downarrow}, (v'')^{\downarrow}\} * \mathrm{st}(v)] = A[v'' * \mathrm{st}(v')]$ (treating v' as input) and $A[\{(v'')^{\downarrow}, (v')^{\downarrow}\} * \mathrm{st}(v)] = A[v' * \mathrm{st}(v'')]$ (treating v'' as input)

since each pair of these simplicial complexes is related by a sequence of $\| \mathrm{st}(v)\|_{(n-1)}$ n-Pachner moves.
Hence we have that $H_v^{\uparrow} = A[v'' * \mathrm{st}(v')] = A[v' * \mathrm{st}(v'')] = H_v^{\downarrow}$ showing that both choices $\{\uparrow, \downarrow\}$ are equivalent. □

Alternatively, we may directly observe that $(v')^{\uparrow} * \mathrm{st}(v)$ and $(v')^{\downarrow} * \mathrm{st}(v)$ are PL homeomorphic and therefore induce the same operator by lemma 10.

Corollary 20. *As previously mentioned, the above lemma implies that H_v is Hermitian. Furthermore the argument shows that H_v is a projection since $H_v^2 = A[\{(v')^{\downarrow}, (v'')^{\downarrow}\} * \mathrm{st}(v)] = A[v'' * \mathrm{st}(v')] = H_v$.*

We can employ similar reasoning to demonstrate that the Hamiltonian terms commute

Lemma 21. $[H_v, H_u] = 0$ *for all v and u.*

Proof. The only nontrivial case of the above commutator is when $v \neq u$ are neighboring vertices (connected by an edge $[u,v] \in \mathscr{M}_{M^{(n-1)}}$). The region on which both terms act is $\mathrm{st}(u) \cap \mathrm{st}(v) = \mathrm{st}([u,v])$ and we have $H_v H_u = A[(v' * \mathrm{st}(v)) \cup_{\mathrm{st}\,[u',v]} (u' * \mathrm{st}(u))]$ where the input copy $\mathrm{st}(v)$ in $v' * \mathrm{st}(v)$ is glued along $\mathrm{st}([u',v])$ to the output copy $\mathrm{st}(u')$ in $u' * \mathrm{st}(u)$, and analogously $H_u H_v = A[(u' * \mathrm{st}(u)) \cup_{\mathrm{st}([u,v'])} (v' * \mathrm{st}(v))]$. These two tensor networks differ only in that $(v' * \mathrm{st}([u',v])) \cup_{\mathrm{st}([u',v])} (u' * \mathrm{st}([u,v]))$ appears in the first whereas $(u' * \mathrm{st}([u,v'])) \cup_{\mathrm{st}([u,v'])} (v' * \mathrm{st}([u,v]))$ appears in the second, note these complexes have the same boundary. One can transform the first of these complexes to the second by a sequence of $\| \mathrm{st}([u,v])\|_{(n-1)}$ n-Pachner moves, hence $A[(v' * \mathrm{st}(v)) \cup_{\mathrm{st}([u',v])} (u' * \mathrm{st}(u))] = A[(u' * \mathrm{st}(u)) \cup_{\mathrm{st}([u,v'])} (v' * \mathrm{st}(v))]$ implying that the Hamiltonian terms commute. □

Thus far we have shown the Hamiltonian is a sum of local commuting projector terms that each act on the $(n-1)$-simplices around a vertex $v \in \mathscr{M}_{M^{(n-1)}}$.

We proceed to demonstrate that the ground space of this Hamiltonian is given by the image of the cylinder projector $P_A[\mathscr{M}_{M^{(n-1)}}]$.

Lemma 22. *Assume that $A[M^{(n-1)} \times S^1] > 0$. Then $H_{\mathscr{M}_{M^{(n-1)}}}$ is frustration free and the ground space is equal to* $\ker(H_{\mathscr{M}_{M^{(n-1)}}}) = \mathrm{im}(P_A[\mathscr{M}_{M^{(n-1)}}])$.

Proof. By the preceding lemmas, $H_{\mathscr{M}_{M^{(n-1)}}}$ is a sum of projectors that commute pairwise, and therefore frustration free. Thus the ground state projector is $\prod_{v \in \mathscr{M}_{M^{(n-1)}}} H_v$ (if nonzero). But multiplying the projectors H_v amounts to gluing together the manifolds $M_v = v' * \mathrm{st}(v)$, which results in a triangulation $\mathscr{M}'_{M^{(n-1)} \times [0,1]}$ of $M^{(n-1)} \times [0,1]$. While this triangulation depends on the order of multiplication, the resulting TQFT operators does not; it is precisely the cylinder projector $P_A[\mathscr{M}_{M^{(n-1)}}]$. By assumption, its trace is nonzero, and so it is indeed the ground state projector. □

This argument also establishes that the dimension of the ground space is equal to $A[M^{(n-1)} \times S^1]$, which is a topological invariant independent of the triangulation $\mathscr{M}_{M^{(n-1)}}$. We now show that as a consequence, the ground space for *any* system size can be parameterised by a tensor that we call Q which has a constant number of indices, upper bounded by the size of a minimal triangulation of $M^{(n-1)}$.

Definition 23 (Ground state tensor Q)**.** To construct Q we fix some (minimal) triangulation $\mathscr{M}^{\min}_{M^{(n-1)}}$ and a basis $|j\rangle$ for $\mathrm{im}(P_A[\mathscr{M}^{\min}_{M^{(n-1)}}])$. Then Q^j is defined by the coefficients of the vectors $|j\rangle$ in a tensor product basis for the spins $i_{\triangle_{(n-1)}}$, where $\triangle_{(n-1)} \in \mathscr{M}^{\min}_{M^{(n-1)}}$, in the following way $|j\rangle = \sum_{i_1,\ldots,i_N} Q^j_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle$ where $N := \|\mathscr{M}^{\min}_{M^{(n-1)}}\|$.

Now, for any system on a triangulation $\mathscr{M}_{M^{(n-1)\prime}}$ of a manifold $M^{(n-1)\prime}$ that is PL homeomorphic to $M^{(n-1)}$, the ground space of $H_{\mathscr{M}_{M^{(n-1)\prime}}}$ has basis $P_A[\mathscr{M}_{M^{(n-1)\prime}}, \mathscr{M}^{\min}_{M^{(n-1)}}] \otimes_{\mathscr{M}^{\min}_{M^{(n-1)}}} Q^j$, where $P_A[\mathscr{M}_{M^{(n-1)\prime}}, \mathscr{M}^{\min}_{M^{(n-1)}}]$ is the cylinder isometry, since $A[M^{(n-1)\prime} \times S^1] = A[M^{(n-1)} \times S^1]$.

Hence we have demonstrated that any $(n-1)$-D ground state can be written as a fixed n-D tensor network contracted with a fixed size ground state tensor from a finite set. We note that this tensor network naturally has the hierarchical form of a MERA, as it is obtained from an increasingly coarse bulk triangulation interpolating between the given boundary triangulation and the minimal one, $\mathscr{M}^{\min}_{M^{(n-1)}}$ (see fig. 2.2).

2.4 Tensor Operator Symmetry and Topological Order

In this section we explore the properties of a PMIST $A[\triangle_n]$ from the point of view of PEPS, particularly exploring its injectivity structure and virtual symmetries. We will show that such tensors give rise to PEPS that fit into a higher dimensional extension of the recently developed MPO-injectivity framework. One can construct the PEPS from the tensor $A[\triangle_n]$ on a triangulated manifold $\mathscr{M}_{M^{(n-1)}}$ by introducing an auxiliary vertex v and taking the tensor network $A[v * \mathscr{M}_{M^{(n-1)}}]$. Our focus on the simple tensor network $A[v * \mathscr{M}_{M^{(n-1)}}]$ is justified by showing a cleaning lemma that implies for any simply connected, codimension-0 submanifold $N^{(n-1)} \subseteq M^{(n-1)}$ with triangulation $\mathscr{M}_{N^{(n-1)}} \subseteq \mathscr{M}_{M^{(n-1)}}$ the support of the reduced density matrix of the ground space in region $N^{(n-1)}$ is spanned by states of the form $A[v * \mathscr{M}_{N^{(n-1)}}]\,|\psi)$, where $A[v * \mathscr{M}_{N^{(n-1)}}]$ is treated as a map from the virtual indices on $\triangle_{(n-1)} \in v * \partial \mathscr{M}_{N^{(n-1)}}$ to physical indices on $\triangle_{(n-1)} \in \mathscr{M}_{M^{(n-1)}}$ and $|\psi)$ is an arbitrary boundary tensor (state) on the virtual level. This implies that the tensor network spanning the ground space can be brought into a form which looks locally like the simple PEPS, but

Figure 2.4 *Left:* Illustration of the PEPS tensor. *Right:* Illustration of the TNO tensor.

Figure 2.5 *Left:* Illustration of TNO-injectivity. *Right:* Illustration of TNO-symmetry on the virtual level.

with the addition of some topological objects on the virtual level which are free to move through the lattice, extending the findings of the MPO-injectivity framework in 2D. Note that to systematically construct these topological objects would require a notion of Morse theory for tensor networks which specifies the topological object to be added to the tensor network as it is grown on the lattice $\mathscr{M}_{M^{(n-1)}}$, by concatenation of tensors, past the critical point of a Morse function. We leave this direction to future work.

Definition 24. Given a PMIST $A[\triangle_n]$ and a physical lattice $\mathscr{M}_{M^{(n-1)}}$ the PEPS wavefunction is defined to be $\mathscr{A}[\mathscr{M}_{M^{(n-1)}}] := A[v * \mathscr{M}_{M^{(n-1)}}]$ where v is an auxiliary vertex $v \notin \mathscr{M}_{M^{(n-1)}}$.

In the proceeding propositions we show that a PMIST PEPS is TNO-injective (fig. 2.5, left), which we define in analogy to MPO-injectivity as follows: A single (possibly after some real space blocking) PEPS tensor (fig. 2.4, left) is injective on the support subspace of a TNO (a single TNO fig. 2.4, right) acting on the virtual indices, the TNO is a symmetry on the virtual level of the PEPS and furthermore there exists a tensor that allows the injectivity subspace to be extended to larger regions. All of these properties essentially follow by considering appropriate simplicial complexes, related by Pachner moves, and then applying the tensor network map A to these and exploiting its invariance under Pachner moves.

Proposition 25. *For a physical site $\triangle^p_{(n-1)}$ the PEPS tensor $\mathscr{A}[\triangle^p_{(n-1)}]$, treated as a linear map from the n virtual indices on $\triangle_{(n-1)} \in v * (\partial \triangle^p_{(n-1)})$ to a single physical index on $\triangle^p_{(n-1)}$, is injective on the subspace* $\mathrm{supp}(A[v' * (v * \partial \triangle^p_{(n-1)})])$. *A generalized inverse is given by $\mathscr{A}[\triangle^p_{(n-1)}]^+ := A[v' * \triangle^p_{(n-1)}]$ treated as a linear map from the physical index on $\triangle^p_{(n-1)}$ to n auxiliary indices on $\triangle_{(n-1)} \in v' * (\partial \triangle^p_{(n-1)})$.*

Proof. This is true because the complex $(v' * \triangle^p_{(n-1)}) \cup_{\triangle^p_{(n-1)}} (v * \triangle^p_{(n-1)})$ is related to $v' * (v * \partial\triangle^p_{(n-1)})$ by a single Pachner move. Hence $A[v' * \triangle^p_{(n-1)}] \otimes_{\triangle^p_{(n-1)}} \mathscr{A}[\triangle^p_{(n-1)}] = A[v' * (v * \partial\triangle^p_{(n-1)})]$ which is a Hermitian projector by corollary 20. □

Furthermore, since we have specified the action of orientation reversal to be conjugation we have that $A[v' * \triangle^p_{(n-1)}] = A[v * \triangle^p_{(n-1)}]^\dagger$ and hence the PEPS tensor $\mathscr{A}[\triangle^p_{(n-1)}]$ is an isometry.

Now we will show that the TNO described above is a symmetry of the virtual level of the PMIST PEPS, by this we mean that the TNO acting on the virtual level of the PEPS can be freely deformed (fig. 2.5, right). To make this precise, note that the TNO can be defined on any region $\mathscr{M}_{W^{(n-2)}} \subseteq \mathscr{M}_{M^{(n-1)}}$ that is a homotopy $(n-2)$-sphere (bounding an $(n-1)$-ball $N^{(n-1)} \subseteq M^{(n-1)}$, $\partial N^{(n-1)} = W^{(n-2)}$) by $A[v' * (v * \mathscr{M}_{W^{(n-2)}})]$ (generalizing the case of $\partial\triangle^p_{(n-1)}$). Then this TNO acting on the virtual indices of the PEPS is defined as follows:

Definition 26. A PEPS on $\mathscr{M}_{M^{(n-1)}}$ with a virtual symmetry acting on a homotopy $(n-2)$-sphere $W^{(n-2)}$ that bounds a homotopy $(n-1)$-ball $\mathscr{M}_{N^{(n-1)}} \subset \mathscr{M}_{M^{(n-1)}}$, $\partial N^{(n-1)} = W^{(n-2)}$ is given by $\mathscr{A}[\mathscr{M}_{M^{(n-1)}}; \mathscr{M}_{W^{(n-2)}}] := A[v' * \mathscr{M}_{M^{(n-1)} \setminus N^{(n-1)}}] \otimes_{v' * \mathscr{M}_{W^{(n-2)}}} A[v' * (v * \mathscr{M}_{W^{(n-2)}})] \otimes_{v * \mathscr{M}_{W^{(n-2)}}} A[v * \mathscr{M}_{N^{(n-1)}}]$.

The next proposition demonstrates that for any deformation $N^{(n-1)} \mapsto N^{(n-1)'}$ such that $\mathscr{M}_{N^{(n-1)'}} \subseteq \mathscr{M}_{M^{(n-1)}}$, $W^{(n-2)'} := \partial N^{(n-1)'}$ we have $\mathscr{A}[\mathscr{M}_{M^{(n-1)}}; \mathscr{M}_{W^{(n-2)}}] = \mathscr{A}[\mathscr{M}_{M^{(n-1)}}; \mathscr{M}_{W^{(n-2)'}}]$.

Proposition 27. *For a subset of the boundary of a single site $K \subseteq \partial\triangle^p_{(n-1)}$ we have $\mathscr{A}[\triangle^p_{(n-1)}] \otimes_{v*K} A[v' * (v * K)] = \mathscr{A}[\triangle^p_{(n-1)}] \otimes_{v' * (\partial\triangle^p_{(n-1)} \setminus K)} A[v' * (v * (\partial\triangle^p_{(n-1)} \setminus K))]$ corresponding to a TNO moving through the virtual level of the PEPS tensor.*

Proof. Consider the complex $(v * \triangle^p_{(n-1)}) \cup_{v*K} (v' * (v * K))$ which is related to $(v' * \triangle^p_{(n-1)}) \cup_{v' * (\partial\triangle^p_{(n-1)} \setminus K)} (v * (v' * (\partial\triangle^p_{(n-1)} \setminus K)))$ by a Pachner move. Hence $\mathscr{A}[\triangle^p_{(n-1)}] \otimes_{v*K} A[v' * (v * K)] = A[(v * \triangle^p_{(n-1)}) \cup_{v*K} (v' * (v * K))] = A[(v' * \triangle^p_{(n-1)}) \cup_{v' * (\partial\triangle^p_{(n-1)} \setminus K)} (v * (v' * (\partial\triangle^p_{(n-1)} \setminus K)))] = \mathscr{A}[\triangle^p_{(n-1)}] \otimes_{v' * (\partial\triangle^p_{(n-1)} \setminus K)} A[v' * (v * (\partial\triangle^p_{(n-1)} \setminus K))]$. □

We have demonstrated that the TNO symmetries on the virtual level of a PEPS are freely deformable, for the special case of $K = \partial\triangle^p_{(n-1)}$ this implies that a closed homotopy $(n-2)$-sphere TNO is a symmetry of a single PEPS tensor. Writing this out explicitly, we have $\mathscr{A}[\triangle^p_{(n-1)}] \otimes_{v*\partial\triangle^p_{(n-1)}} A[v' * (v * \partial\triangle^p_{(n-1)})] = \mathscr{A}[\triangle^p_{(n-1)}]$. This can be seen to imply that the TNO $A[v' * (v * \partial\triangle^p_{(n-1)})]$ is a projector by applying the generalized inverse $\mathscr{A}[\triangle^p_{(n-1)}]^+$ to the physical index on $\triangle^p_{(n-1)}$ and using proposition 25 (one can also prove this directly by applying the argument used for lemma 19).

We will now show that this implies a generalized version of the 'trivial loops' condition of MPO-injectivity by deforming the boundary symmetry such that it acts on a single index.

Corollary 28. *For any $\triangle_{(n-2)} \in \partial\triangle^{p}_{(n-1)}$ we have $\mathscr{A}[\triangle^{p}_{(n-1)}] \otimes_{v*\triangle_{(n-2)}} A[v' * (v * \triangle_{(n-2)})]$ $\otimes_{v'*\partial(v*\triangle_{(n-2)})} A[\triangle'_n] = \mathscr{A}[\triangle^{p}_{(n-1)}]$.*

Proof. This follows by first deforming the TNO tensor $A[v' * (v * \triangle_{(n-2)})]$ through the PEPS tensor, leaving the PEPS tensor surrounded by an $(n-2)$-ball TNO which is a symmetry by proposition 27. Writing this out, we have $\mathscr{A}[\triangle^{p}_{(n-1)}] \otimes_{v*\triangle_{(n-2)}} A[v' * (v * \triangle_{(n-2)})] \otimes_{v'*\partial(v*\triangle_{(n-2)})} A[\triangle'_n] = \mathscr{A}[\triangle^{p}_{(n-1)}] \otimes_{v'*\partial\triangle^{p}_{(n-1)}} \Big(A[v * (v' * (\partial\triangle^{p}_{(n-1)} \setminus \triangle_{(n-2)}))]$
$\otimes_{(v'*\partial(v*\triangle_{(n-2)}))\setminus(v'*\triangle_{(n-2)})} A[\triangle'_n]\Big) = \mathscr{A}[\triangle^{p}_{(n-1)}] \otimes_{v'*\partial\triangle^{p}_{(n-1)}} A[v * (v' * \partial\triangle^{p}_{(n-1)})] =$
$\mathscr{A}[\triangle^{p}_{(n-1)}]$. □

Hence the TNO on any homotopy $(n-2)$-sphere $W^{(n-2)}$ (bounding an $(n-1)$-ball) s.t. $\mathscr{M}_{W^{(n-2)}} \subset \mathscr{M}_{M^{(n-1)}}$ is a symmetry of the PEPS network $\mathscr{A}[\mathscr{M}_{M^{(n-1)}}; \mathscr{M}_{W^{(n-2)}}] = \mathscr{A}[\mathscr{M}_{M^{(n-1)}}]$.

In the next proposition we demonstrate that the generalised inverse for the PEPS tensor can be grown to larger regions that are homotopy $(n-1)$-balls and that the injectivity subspace of the PEPS map on such regions is still given by the support subspace of the TNO acting on the boundary homotopy $(n-2)$-sphere. This can again be achieved by considering only local objects that yield an induction step.

Proposition 29. *For a physical site $\triangle^{p}_{(n-1)}$ with a component of the boundary $K \subset \partial\triangle^{p}_{(n-1)}$ lying on the edge of the region that has been inverted, acting with the generalized inverse of the PEPS tensor $\mathscr{A}[\triangle^{p}_{(n-1)}]^{+}$ and tracing out the virtual indices on $v * K$, that have been acted on by the TNO from the boundary of the inverted region, with the corresponding indices of $\mathscr{A}[\triangle^{p}_{(n-1)}]^{+}$ leaves only the TNO symmetry acting on the region $\partial\triangle^{p}_{(n-1)} \setminus K$.*

Proof. First note that since we are growing a homotopy $(n-1)$-ball at each step we only need consider a boundary that is an $(n-2)$-sphere with local patches that look like $(n-2)$-balls this is used implicitly in the following argument. The inverse acting on the PEPS tensor at site $\triangle^{p}_{(n-1)}$ on the boundary, together with a TNO acting on the indices in $v * K$ is given by $A[v'' * \triangle^{p}_{(n-1)}] \otimes_{\triangle^{p}_{(n-1)} \cup_K (v'*K)} \Big(\mathscr{A}[\triangle^{p}_{(n-1)}] \otimes_{v*K} A[v' * (v * K)]\Big) =$ $A\Big[\Big((v'' * \triangle^{p}_{(n-1)}) \cup_{\triangle^{p}_{(n-1)}} (v * \triangle^{p}_{(n-1)})\Big) \cup_{\{v,v'\}*K} (v' * (v * K))\Big] = A[v' * (v * (\partial\triangle^{p}_{(n-1)} \setminus K))]$ since the complexes within the second and third PEPS maps are related by a Pachner move, note the gluing identifies v'' with v'. □

Propositions 27 and 29 imply that the PEPS map on any homotopy $(n-1)$-ball $\mathscr{M}_{N^{(n-1)}} \subseteq \mathscr{M}_{M^{(n-1)}}$ is an isometry on the support subspace of a TNO acting on the boundary $\mathscr{M}_{\partial N^{(n-1)}}$

as i.e. $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger \mathscr{A}[\mathscr{M}_{N^{(n-1)}}] = A[\{v,v'\} * \mathscr{M}_{N^{(n-1)}}] = A[(v*v') * \mathscr{M}_{\partial N^{(n-1)}}]$. Furthermore $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}; \mathscr{M}_{\partial N^{(n-1)}}] = \mathscr{A}[\mathscr{M}_{N^{(n-1)}}]$ by proposition 27 and corollary 28 and similarly $A[(v*v') * \mathscr{M}_{\partial N^{(n-1)}}]\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger = \mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger$, note one could prove these equalities directly by following a sequence of Pachner moves. We note that these equalities also hold (with the orientation reversed PEPS map rather than the Hermitian conjugate) when no action of orientation reversal is assumed, hence $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]$ has a generalised reflexive inverse for any homotopy $(n-1)$-ball $N^{(n-1)}$.
When the Hermitian action of orientation reversal is assumed, $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger$ is the unique pseudoinverse to $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]$, due to the reflexive generalised inverse property already shown together with the fact that the projectors $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger \mathscr{A}[\mathscr{M}_{N^{(n-1)}}]$ and $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]^\dagger$ are Hermitian.

We spend the remainder of this section proving an intersection property which shows that all ground states of $H_{\mathscr{M}_{M^{(n-1)}}}$ locally look like $\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]$ on regions $N^{(n-1)}$ homotopic to an $(n-1)$-ball. This demonstrates that the topological objects which carry global information about the groundstate are locally undetectable.

Lemma 30. *For any region $N^{(n-1)} \subseteq M^{(n-1)}$ homotopic to an $(n-1)$-ball the support of the reduced density matrix of the ground space in that region $\mathrm{supp}(\rho^0_{\mathscr{M}_{N^{(n-1)}}})$ is spanned by the set of states $\{\mathscr{A}[\mathscr{M}_{N^{(n-1)}}]|\psi\rangle\}_\psi$ where $|\psi\rangle$ is an arbitrary boundary state on the virtual degrees of freedom along $v * \mathscr{M}_{\partial N^{(n-1)}}$.*

Proof. Note that the above is trivially true for any region $\mathrm{st}(v)$, $v \in \mathscr{M}_{M^{(n-1)}}$ as $\mathrm{supp}(\rho^0_{\mathrm{st}(v)}) = \mathrm{supp}(H_v) = \mathrm{supp}(A[v' * \mathrm{st}(v)]) = \mathrm{im}(\mathscr{A}[\mathrm{st}(v)])$. For the induction step we consider a region homotopic to an $(n-1)$-ball of the form $K = \bigcup_u \mathrm{st}(u)$ where the hypothesis implies $\mathrm{supp}(\rho^0_K) = \mathrm{im}(\mathscr{A}[K])$ and we consider growing K to $K \cup \mathrm{st}(v)$ for some $v \in \partial K$. Note that we assume $K \cup \mathrm{st}(v)$ is again a homotopy $(n-1)$-ball as we only claim the result for this restricted case.

The induction hypothesis implies that any state $|\psi\rangle \in \mathrm{supp}(\rho^0_{K\cup\mathrm{st}(v)})$ has two equivalent PEPS descriptions $|\psi\rangle = \mathscr{A}[K] \otimes_{v*\partial K} B[\mathrm{st}(v) \setminus \Lambda; \partial K] = \mathscr{A}[\mathrm{st}(v)] \otimes_{v*\mathrm{lk}(v)} B[K \setminus \Lambda; \mathrm{lk}(v)]$ where $\Lambda := \mathrm{st}(v) \cap K$ and $B[X;Y]$ is an arbitrary boundary tensor with physical indices on $\triangle_{(n-1)} \in X$ and virtual indices on $\triangle_{(n-2)} \in Y$. Note that each boundary tensor $B[X;Y]$ is only defined on the support subspace of the PEPS it bounds $\mathrm{supp}(\mathscr{A}[\partial^* Y]) = \mathrm{im}(A[v' * (v*Y)])$, where $\partial(\partial^* Y) = Y$, hence we assume throughout that $A[v' * (v*Y)] \otimes_{v*Y} B[X;Y] = B[X;Y]$. To show that $|\psi\rangle$ is of the form $\mathscr{A}[K \cup \mathrm{st}(v)] \otimes_{v*\partial(K\cup\mathrm{st}(v))} B[\emptyset; \partial(K \cup \mathrm{st}(v))]$ we first apply the pseudoinverse on region $\mathrm{st}(v)$ to the two equivalent descriptions of $|\psi\rangle$ as follows

$$A[v' * \mathrm{st}(v)] \otimes_{\mathrm{st}(v)} (\mathscr{A}[\mathrm{st}(v)] \otimes_{v*\mathrm{lk}(v)} B[K \setminus \Lambda; \mathrm{lk}(v)]) = A[v' * \mathrm{st}(v)] \otimes_{\mathrm{st}(v)} (\mathscr{A}[K] \otimes_{v*\partial K} B[\mathrm{st}(v) \setminus \Lambda; \partial K]),$$
$$A[v' * (v * \mathrm{lk}(v))] \otimes_{v*\mathrm{lk}(v)} B[K \setminus \Lambda; \mathrm{lk}(v)] = (A[v' * (\mathrm{st}(v) \setminus \Lambda)] \otimes_{\mathrm{st}(v)\setminus\Lambda} B[\mathrm{st}(v) \setminus \Lambda; \partial K])$$

$$\otimes_{(v*\partial K)\cup(v'*\lambda_1)} (A[v'*\Lambda] \otimes_\Lambda \mathscr{A}[K]),$$
$$B[K\setminus\Lambda;\mathrm{lk}(v)] = (A[v'*(\mathrm{st}(v)\setminus\Lambda)] \otimes_{\mathrm{st}(v)\setminus\Lambda} B[\mathrm{st}(v)\setminus\Lambda;\partial K])$$
$$\otimes_{(v*\partial K)\cup(v'*\lambda_1)} (A[v'*(v*\partial\Lambda)] \otimes_{v*\lambda_2} A[v*(K\setminus\Lambda)])$$

where we have defined $\lambda_1 = \partial K \cap \mathrm{st}(v)$ and $\lambda_2 = \mathrm{lk}(v) \cap K$. Note that the second equality follows from the invariance of the boundary under the TNO and the third from $\mathscr{A}[K] = A[v*(K\setminus\Lambda)] \otimes_{v*\lambda_2} A[v*\Lambda]$. Proposition 27 implies $A[v*(K\setminus\Lambda)] \otimes_{v*\lambda_2} A[v'*(v*\lambda_2)] = A[v'*(K\setminus\Lambda)] \otimes_{v'*(\partial K\setminus\lambda_1)} A[v'*(v*(\partial K\setminus\lambda_1))]$ which describes a TNO symmetry moving through a region of the PEPS homotopic to an $(n-1)$-ball. Hence $A[v'*(v*\partial\Lambda)] \otimes_{v*\lambda_2} A[v*(K\setminus\Lambda)] = A[v'*(v*(\partial K))] \otimes_{v'*(\partial K\setminus\lambda_1)} A[v'*(K\setminus\Lambda)]$ which leads to

$$B[K\setminus\Lambda;\mathrm{lk}(v)] = \big(A[v'*(\mathrm{st}(v)\setminus\Lambda)] \otimes_{(\mathrm{st}(v)\setminus\Lambda)\cup(v'*\lambda_1)} (B[\mathrm{st}(v)\setminus\Lambda;\partial K] \otimes_{v*\partial K} A[v'*(v*\partial\Lambda)])\big) \otimes_{v'*(\partial K\setminus\lambda_1)} \mathscr{A}[K\setminus\Lambda]$$

note the resulting boundary $B[\emptyset;\partial(K\cup\mathrm{st}(v))] := A[v'*(\mathrm{st}(v)\setminus\Lambda)] \otimes_{(\mathrm{st}(v)\setminus\Lambda)\cup(v'*\lambda_1)} (B[\mathrm{st}(v)\setminus\Lambda;\partial K] \otimes_{v*\partial K} A[v'*(v*\partial\Lambda)])$ has no physical indices. Applying the PEPS map $\mathscr{A}[\mathrm{st}(v)]$ to both sides yields the desired result

$$|\psi\rangle = \mathscr{A}[\mathrm{st}(v)] \otimes_{v*\mathrm{lk}(v)} B[K\setminus\Lambda;\mathrm{lk}(v)] = \mathscr{A}[K\cup\mathrm{st}(v)] \otimes_{v*\partial(K\cup\mathrm{st}(v))} B[\emptyset;\partial(K\cup\mathrm{st}(v))].$$

□

2.5 Examples

2.5.1 Trivial Theories

The most trivial examples are constructed by taking any state $|\psi\rangle$ and constructing the tensor product $A[\triangle_n] := \bigotimes_{\triangle_{(n-1)}\in\partial\triangle_n} |\psi\rangle^{\sigma(\triangle_{(n-1)})}$ where $\sigma(\triangle_{(n-1)}) = *$ (complex conjugation) if the orientation of $\triangle_{(n-1)}$ induced by $\triangle_n$ is negative and is trivial otherwise. Such a theory is completely trivial as any tensor network yields a tensor product vector, in particular the tensor network assigned to any complex depends only upon the boundary $A[M^n;K] = \bigotimes_{\triangle_{(n-1)}\in K} |\psi\rangle^{\sigma(\triangle_{(n-1)})}$ where K is some triangulation of the boundary ∂M^n. This is an even stronger condition than Pachner move invariance and hence these trivial theories fall within our framework.

2.5.2 Symmetry Breaking (GHZ) Type Theories

A slightly less trivial example, with almost no topological dependence, is given by GHZ (delta) tensors which gives rise to a symmetry breaking theory. In this case $A[\triangle_n] := \delta(s_{\triangle^0_{(n-1)}},\ldots,s_{\triangle^n_{(n-1)}})$ where the $s_{\triangle^i_{(n-1)}}$ are spins of the same dimension, each one assigned to a simplex $\triangle^0_{(n-1)},\ldots,\triangle^n_{(n-1)} \in \partial\triangle_n$, and $\delta(i_0,\ldots,i_n) = 1$ if $i_0 = i_1 = \cdots = i_n$ and 0 otherwise. It is simple to check that contracting two delta tensors (along any number of indices) yields another delta tensor, with this identity in hand one immediately sees that the tensor network on any connected complex depends only upon the boundary $A[M^n;K] = \delta(\triangle^0_{(n-1)},\ldots,\triangle^{\|K\|}_{(n-1)})$ where $K = \bigcup_i \triangle^i_{(n-1)}$ is some triangulation of the boundary ∂M^n. Again this is a stronger condition than Pachner move invariance and consequently these symmetry breaking type theories fall within our framework. Note that for these theories the ground states are not unique on closed manifolds as the action of a single spin unitary that is diagonal in the basis of the delta tensors is locally undetectable, but may lead to a number of orthogonal ground states equal to the dimension of the spins on each connected spatial component.

2.5.3 Dijkgraaf-Witten (Twisted Quantum Double) Theories

A truly non trivial family of examples is given by discrete Dijkgraaf-Witten theories, which take as input data a spacetime dimension n, a finite group G and an n^{th} cohomology class $[\omega] \in H^n(G,U(1))$. We first outline the 'untwisted' case, also referred to as quantum double models, where the cohomology class is trivial as it exemplifies the basic structure on top of which we will later add a twist. The basic idea of the theory is to take a superposition over all flat G-connections on each manifold. The local tensor is given by $A[\triangle_n] := \sum_{g_{v_0},\ldots,g_{v_n}} \bigotimes_{\triangle_{(n-1)}\in\partial\triangle_n} \bigotimes_{e\in\triangle_{(n-1)}} |g_{v_e^-} g^{-1}_{v_e^+}\rangle_{\triangle_{(n-1)},e}$ where the vertex label $v_e^+ > v_e^-$. One can easily see that for any configuration of group variables on the vertices $\{g_v\}$ the product of edge variables around a triangle $\triangle_2$ is trivial $g_{e_1}^{\sigma(e_1)} g_{e_2}^{\sigma(e_2)} g_{e_3}^{\sigma(e_3)} = 1$, where $g_e = g_{v_e^-} g^{-1}_{v_e^+}$, edge e is oriented from v_e^- to v_e^+ and $\sigma(e) = 1$ if the orientation of e matches that induced by $\triangle_2$ and -1 otherwise. With this definition a tensor network on any complex K yields a superposition over flat G-connections on ∂K with weight proportional to the number of flat bulk G-connections consistent with the boundary. In particular this implies the tensors satisfy the Pachner moves.

To add a cohomological twist to the model we add a phase $f_{\triangle_n}$ to each local tensor $A[\triangle_n]$. Note that due to the flatness conditions there are only n free group variables, for instance one can take the edge variables $g_{e_1},\ldots,g_{e_n}$ along any path from the lowest to highest numbered vertex which passes through each other vertex once. Hence the phase is a function of the form $f_{\triangle_n}(g_{v_0}g_{v_1}^{-1},\ldots,g_{v_{n-1}}g_{v_n}^{-1})$, which will generically not give rise to a tensor satisfying the Pachner moves. The condition which ensures Pachner move invariance is precisely that $f_{\triangle_n}$ is an n-cocycle, to formulate this we define the following coboundary map on functions of n group variables $\delta_n\omega(g_0,\ldots,g_n) := \prod_{i=1}^{n}\omega(g_0,\ldots,g_{i-1}g_i,\ldots,g_n)^{(-1)^i}\omega(g_0,\ldots,g_{n-1})^{(-1)^{n+1}}$. The condition that ω is a cocycle is then $\delta_n\omega = 1$, to make the correspondence with the Pachner moves exact we must include a conjugation depending upon the orientation of the simplex to which the phase is assigned $f_{\triangle_n}(g_{v_0}g_{v_1}^{-1},\ldots,g_{v_{n-1}}g_{v_n}^{-1}) = \omega^{\sigma(\triangle_n)}(g_{v_0}g_{v_1}^{-1},\ldots,g_{v_{n-1}}g_{v_n}^{-1})$, where $\sigma(\triangle_n) = +1$ if $\triangle_n$ is positively oriented and -1 otherwise. Hence the local tensors are given by $A[\triangle_n] := \sum_{g_{v_0},\ldots,g_{v_n}}\omega^{\sigma(\triangle_n)}(g_{v_0}g_{v_1}^{-1},\ldots,g_{v_{n-1}}g_{v_n}^{-1})\bigotimes_{\triangle_{(n-1)}\in\partial\triangle_n}\bigotimes_{e\in\triangle_{(n-1)}}|g_{v_e^-}g_{v_e^+}^{-1}\rangle_{\triangle_{(n-1)},e}$ and they satisfy the Pachner move invariance condition due to the flatness and cocycle conditions. To construct the n^{th} group cohomology $H^n(G,U(1))$ one takes an equivalence over n-coboundaries β defined as the image of δ_{n-1}. To see further why only the cohomology class $[\omega]$ is relevant to the topological order of the model rather than the explicit representative, note that if we have $\omega' = \omega\,\delta_{n-1}f$ each term in the product $\delta_{n-1}f$ is only a function of the group variables on the edges of some $\triangle_{(n-1)} \in \partial\triangle_n$ and hence can be realized as a local unitary on the tensor index corresponding to that $\triangle_{(n-1)}$. For a shared $(n-1)$-simplex of neighbouring n-simplices the same phase factor will occur with opposite orientations and hence cancel each other, this shows that all closed manifold path integrals are the same for both theories and that all states are related by single site local unitaries.

2.5.4 Symmetry Protected Models

A closely related family of examples are the SPT models, which only have a limited topological dependence as they are invertible TQFTs. A TQFT is invertible if the path integral on any closed spacetime manifold is a complex number of modulus 1, in particular this implies that there is a unique ground state of the model on any spatial manifold since $\dim_A(M^{(n-1)}) = A[M^{(n-1)}\times S^1] = 1$. The local tensors are given by $A[\triangle_n] := \sum_{g_{v_0},\ldots,g_{v_n}}\omega^{\sigma(\triangle_n)}(g_{v_0}g_{v_1}^{-1},\ldots,g_{v_{n-1}}g_{v_n}^{-1})\bigotimes_{\triangle_{(n-1)}\in\partial\triangle_n}\bigotimes_{v\in\triangle_{(n-1)}}|g_v\rangle_{\triangle_{(n-1)},v}$ and the SPT models they give rise to can be thought of as cohomologically twisted trivial models since for $\omega = 1$ we recover a trivial product theory. Again it follows from the n-cocycle condition on ω that these tensors satisfy the Pachner move invariance condition. In the Hamiltonian framework for classifying gapped phases connected by gapped smooth paths of local Hamiltonians, all SPT phases are considered equivalent to the trivial phase unless a global symmetry restriction is enforced in the classification. Hence the SPT models can be thought of as only being nontrivial in so much as they exhibit interesting transformations under the relevant symmetry. This is very concrete in the tensor network constructions of SPT phases as the cocycle conditions imply a group action to some region of the bulk gives rise to an anomalous TNO group action on the boundary. As explained in section 2.6, by following the boundary reduction of these anomalous symmetries down to points one recovers the cocycle that labels the theory. The precise connection of the SPT models to DW models is realized concretely by applying a gauging map to an SPT phase specified by (n, G, ω) which takes it to a DW model with the same data. Furthermore the gauging opertor is gap preserving and hence two SPT models lying within the same phase is a sufficient condition for the gauged models to also lie in the same phase.

2.5.5 Levin-Wen (Turaev-Viro) models

The next family of (2+1)D examples known as Levin-Wen string net models or Turaev-Viro TQFTs form the prototypical example which the current framework is intended to generalise. The input is a unitary fusion category, which can be specified by some finite set of labels $\{a\}$, fusion multiplicities $\mathscr{N}(a \times b \to c)$ which are 0 for disallowed fusions, and a set of $F^{ab,m;\mu\nu}_{cd,n;\gamma\delta}$ symbols which specify the associator map between two different fusion paths $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$ with outcome d and fusion degeneracies $\mu\nu, \gamma\delta$ respectively. The F symbols are taken to be 0 outside the subspace of allowed fusions and hence F contains the fusion information also. The associator maps satisfy a consistency condition called the pentagon equation which precisely corresponds to the local tensors $A[\triangle_n] := \sum_{\substack{a_{01}, a_{02} \cdots, a_{23} \\ \mu_0, \ldots, \mu_3}} F^{a_{01} a_{12}, a_{02}; \mu_3 \mu_1}_{a_{03} a_{23}, a_{13}; \mu_2 \mu_0} \bigotimes_{\triangle_2 \in \partial \triangle_3} |\mu_{\triangle_2}\rangle_{\triangle_2} \bigotimes_{e \in \triangle_2} |a_e\rangle_{\triangle_2, e}$ satisfying a 3-2 Pachner move invariance condition. In this case as the variables live on codimension-2 complexes and to allow the most general solutions we have to allow a weight tensor assigned to each edge of the complex on which the tensor network lives which adds a factor quantum dimension d_{a_e} (cf. the discussion in section 2.2.4). The 1-4 Pachner move invariance condition is usually formulated as an invertibility condition on the F-symbols which implies two tensors give the identity (at least on the relevant subspace) after applying a 3-2 move to 4 contracted tensors which leaves a single tensor as desired. It is known that the emergent anyons of a string net model correspond to the center of input category, while the matrix product operator symmetries of the PEPS wavefunction correspond to the input category. This is resolved by a recent construction which yields an algebra of matrix product operators whose central idempotents seem to contain the same information as the center of the input category. [79]

2.5.6 Walker-Wang (Crane-Yetter) models

Next we briefly discuss the Walker-Wang (WW) models (Crane, Yetter (CY) and Kauffman TQFTs) which take as input a unitary braided fusion (or sometimes a premodular tensor category), which has the extra structure of a braiding rule R^{ab}_c that is consistent with the fusion (i.e. R and F together satisfy the hexagon equation) this extra structure allows one to consider braided anyon diagrams analogous to the unbraided string net diagrams, but which are inherently (3+1)D. The modular WW models essentially realize some modular anyon theory coupled to a guage theory based on a finite Abelian group G in some possibly nontrivial way. Due to the modularity condition the theory within the bulk has no extended excitation and most interesting properties of the model, such as nontrivial anyonic exciatitons, are confined to the boundary. This is expected as the modular portion of the theory realizes some generalisation of the CY TQFT which was shown to satisfy what the authors call the 'blob' property which implies the tensor network on any connected complex depends only upon the boundary. CY gave an explicit expression for the local tensor and proved that it satisfies the blob property which is a stronger condition than Pachner move invariance, we do not reproduce the definition and proof here. As explained in WW the fact that these models depend mostly upon the boundary is by design, since they can be understood as a lifting of the non fully extended Witten Reshetikhin Turaev (WRT) TQFT in 3D spacetime to a fully extended 4D theory which admits a state sum (lattice model) description. WW go on to explain how the invariant assigned to a 4-manifold that is bounded by a given 3-manifold is equivalent to the invariant assigned to the same 3-manifold by WRT which requires some additional structure on the manifold that roughly specifies an extension to a 4D bulk.

2.5.7 Generalized Toric Codes

All the nontrivial examples considered thus far have their fundamental degrees of freedom on the edges of a complex, there is no reason that this has to be the case other than that the theories were based on an underlying structure with a pairwise composition operation (multiplication, fusion). We now consider the generalized toric codes defined on qubits living on the $0 < k < d$-cells of some generic d-cell complex (Poincaré dual to a simplicial complex) the Hamiltonian for such a model has terms enforcing a $\mathbb{Z}_2$ constraint on k-simplices meeting at a $(k-1)$-simplex and fluctuation terms on $(k+1)$-simplices that flip the value of the qubits on the surrounding k-faces. To fit such a model into our framework it is more convenient to work on the dual simplicial complex, hence qubits live on $(d-k)$ simplices, constraints around $(d-k+1)$ simplices, and fluctuation about $(d-k-1)$ simplices. The spacetime dimension is $n = d+1$ and hence the local tensor for such a theory is given by

$$A[\triangle_n] := \sum_{g_{\triangle^0_{(d-k-1)}},\ldots,g_{\triangle^N_{(d-k-1)}}} \bigotimes_{\triangle_{(n-1)}\in\partial\triangle_n} \bigotimes_{\triangle_{(d-k)}\in\triangle_{(n-1)}} \left| \sum_{\triangle_{(d-k-1)}\in\partial\triangle_{(d-k)}} g_{\triangle_{(d-k-1)}} \right\rangle_{\triangle_{(n-1)},\triangle_{(d-k)}}$$

where $N := \binom{n+1}{d-k+1}$ where the summation is mod-2. Note that this can be adapted to arbitrary Abelian groups by including an appropriate $\pm$ factor depending upon the relative orientation of the $\triangle_{(d-k-1)} \in \triangle_{(d-k)}$. The argument for Pachner move invariance of these tensors is in direct analogy with that for the flat G-connections. Note that one can also add a cohomology twist in the form of a $U(1)$ valued function of allowed group variable configurations $\{g\}_{\triangle_n}$ on the $(d-k)$ subsimplices of an n-simplex, written schematically as $\omega(\{g\}_{\triangle_n})$, that satisfies the condition $\prod_{\triangle_n\in\partial\triangle_{(n+1)}} \omega^{\sigma(\triangle_n)}(\{g\}_{\triangle_n}) = 1$ for any allowed configuration $\{g\}$ (this is enough to guarantee Pachner move invariance of the twisted tensor).

2.5.8 Mackaay-Type Theories

Mackaay has defined a general class of manifold invariants in four dimensions by a state sum approach.[61] The input to his construction is a spherical 2-category. Roughly speaking, a 2-category can be understood as a generalization of a tensor category where the F-symbols do no longer satisfy the pentagon equation. Instead, the two sides of the pentagon equation are related by a higher-order symbol, called the *pentagonator*. The pentagonator in turn satisfies a number of consistency equations. The "categorification" of the pentagon equation has a precise counterpart in the language of TQFT: When we go up from a (2+1)D to a (3+1)D TQFT, we replace the 2-3 and 1-4 Pachner equations (which are induced by the 4-simplex) by a simplex tensor; the latter in turn is invariant under the Pachner moves induced by the 5-simplex (see section 2.2.2). It can be verified that the pentagonator of a spherical 2-category does indeed give rise to a simplex tensor, so that our construction extends Mackaay's state sum invariant to a discrete TQFT and corresponding lattice models.

The class of models constructed from spherical 2-categories covers many of the previously mentioned lattice models in 3+1 dimensions. For example, Walker-Wang models are obtained by considering 2-categories with a single object, while Dijkgraaf-Witten theories correspond to 2-categories with trivial morphisms. A new class of models that goes beyond the preceding can be obtained from the 2-category introduced in [62], which leads to group degrees of freedom on *both* edges and vertices (cf. [58, 80, 81]).

We note that appropriate notions of higher categories are expected more generally to play a fundamental role in topological order in higher dimensions; [68, 69, 75, 67] we briefly comment on this idea in section 2.6.1 below.

2.6 Fully Extended Topological Tensor Network Symmetry

In the previous sections we have seen that the n-dimensional Pachner moves can take on different meanings depending on the dimension, in dimension n they imply topological invariance of the partition function while in dimension $(n-1)$ they form operators that implement $(n-1)$-dimensional Pachner moves on the ground states and imply a symmetry of freely deformable TNOs on the virtual indices of the PEPS for the ground state. In this section we continue to study the meaning of the n-dimensional Pachner on lower dimensional objects, we find that, similar to the codimension-1 case, in codimension-k the $(n-k)$-dimensional Pachner moves on an $(n-k)$-dimensional tensor network can be implemented by applying a product of k tensors to the bulk. Furthermore on the boundary things become even more interesting for codimension-2 and above tensor networks as there is a meaningful sense in which these objects can be composed corresponding to the k-dimensional Pachner moves, such composition rules hold exactly for closed $(n-k)$-manifolds but hold only up to some boundary tensor network in general revealing the fully extended nature of the corresponding TQFT.

First let us define the tensor networks that we assign to lower dimensional submanifolds.

Definition 31. A tensor network for a codimesion-k manifold $M^{(n-k)}$ with triangulation $\mathscr{M}_{M^{(n-k)}}$ is defined as follows $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}] := A[\triangle_{(k-1)} * \mathscr{M}_{M^{(n-k)}}]$ for an auxiliary simplex $\triangle_{(k-1)} \notin \mathscr{M}_{M^{(n-k)}}$.

Note that each tensor $\mathsf{A}_k[\triangle_{(n-k)}]$, for $\triangle_{(n-k)} \in \mathscr{M}_{M^{(n-k)}}$, in the network $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]$ has k external indices on $\triangle_{(n-1)} \in \triangle_{(n-k)} * \partial\triangle_{(k-1)}$ and $(n-k+1)$ internal indices on $\triangle_{(n-1)} \in \partial\triangle_{(n-k)} * \triangle_{(k-1)}$ connected to neighbouring tensors in the network.

The following proposition addresses the effect of a Pachner move on the submanifold tensor network operators.

Proposition 32. *For $K \subset \mathscr{M}_{M^{(n-k)}}$ external sites satisfying $K \subset \partial\triangle_{(n-k+1)}$, and $J \subset \partial\triangle_{(k-1)}$ auxiliary $(k-2)$-simplices we can implement a Pachner move on $\mathscr{M}_{M^{(n-k)}}$ as follows $A[J * \triangle_{(n-k+1)}] \otimes_{K*J} \mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}] = A[J^c * \triangle_{(n-k+1)}] \otimes_{K^c*J^c} \mathsf{A}_k[\mathrm{p}_K\mathscr{M}_{M^{(n-k)}}]$ where $J^c := \partial\triangle_{(k-1)} \setminus J$ and $K^c := \partial\triangle_{(n-k+1)} \setminus K$.*

Proof. Consider the n-Pachner move relating $(J * \triangle_{(n-k+1)}) \cup_{K*J} (K * \triangle_{(k-1)}) \mapsto (J^c * \triangle_{(n-k+1)}) \cup_{K^c*J^c} (K^c * \triangle_{(k-1)})$, hence $A[J * \triangle_{(n-k+1)}] \otimes_{K*J} A[K * \triangle_{(k-1)}] = A[J^c * \triangle_{(n-k+1)}] \otimes_{K^c*J^c} A[K^c * \triangle_{(k-1)}]$ from which the result follows. □

The special case $J = \partial\triangle_{(k-1)}$ implies that the tensor network $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]$ is Pachner move invariant up to some auxiliary operators as follows $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}] \otimes_{K*\partial\triangle_{(k-1)}} A[\partial\triangle_{(k-1)} * \triangle_{(n-k+1)}] = \mathsf{A}_k[\mathrm{p}_K\mathscr{M}_{M^{(n-k)}}]$.

Let us clarify the statement for the codimension-1 case, since $\partial\triangle_0 = \emptyset$, it goes as follows $A[\triangle_n] \otimes_K \mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}] = \mathsf{A}_k[\mathrm{p}_K\mathscr{M}_{M^{(n-k)}}]$ or $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}] = A[\triangle_n] \otimes_{K^c} \mathsf{A}_k[\mathrm{p}_K\mathscr{M}_{M^{(n-k)}}]$, the proof is essentially the same, but dropping any mention to J. Note that this implies that all models within our framework fall under the newly defined class of topological liquids since arbitrary retriangulations of the lattice on which the state is defined (including growing the lattice) can be induced by local linear transformations $A[\triangle_n]$ which could be extended to unitaries, although it may be necessary to include ancillary qudits.

We will move on to consider composition of the submanifold tensor networks, codimension-k tensor networks are composed according to some triangulated $(k-1)$-manifold $\mathscr{M}_{N^{(k-1)}}$ which matches the usual notion of multiplication in codimension-2.

Definition 33. Given an auxiliary triangulated $(k-1)$-manifold $\mathscr{M}_{N^{(k-1)}}$ we define the composition of $\|\mathscr{M}_{N^{(k-1)}}\|$ copies of the tensor network $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]$ as follows $\prod_{\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}])$ $:= A[\mathscr{M}_{N^{(k-1)}} * \mathscr{M}_{M^{(n-k)}}]$.

The condition that codimension-2 operators are projectors corresponds to one dimensional Pachner move invariance of the auxiliary complex $\mathscr{M}_{N^{(k-1)}}$ (discussed below), and this generalizes to invariance of the auxiliary complex under k-dimensional Pachner moves for operators of codimension-k. We will show this invariance holds exactly for closed submanifolds, but in general holds only up to boundary operators which can be thought of as implementing the k-Pachner moves on the boundary.

Proposition 34. *For a triangulated $(n-k)$-homotopy ball $\mathscr{M}_{M^{(n-k)}}$ with a homotopy $(n-k-1)$-ball subcomplex of the boundary $J \subset \partial\mathscr{M}_{M^{(n-k)}}$ and an auxiliary triangulated $(k-1)$-manifold $\mathscr{M}_{N^{(k-1)}}$ with subcomplex $K \subseteq \mathscr{M}_{N^{(k-1)}}$ satisfying $K \subset \partial\triangle_k$ we have the following $A[J * \triangle_k] \otimes_{K*J} \prod_{\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]) = A[J^c * \triangle_k] \otimes_{K^c*J^c} \prod_{\mathrm{p}_K\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}])$ where $K^c := \partial\triangle_k \setminus K$ and $J^c := \partial\mathscr{M}_{M^{(n-k)}} \setminus J$.*

Note that above describes a tensor network operator on the boundary $\partial\mathscr{M}_{M^{(n-k)}}$ moving through the bulk of a product of tensor networks on $\mathscr{M}_{M^{(n-k)}}$ to implement a pachner move on the virtual level, which can be thought of as a generalized multiplication rule.

There are two notable special cases of the above, firstly one could consider $\mathscr{M}_{M^{(n-k)}}$ a homotopy $(n-k)$-sphere rather than ball in this case there is no boundary and one has the equality $\prod_{\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]) = \prod_{\mathrm{p}_K\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}])$ with no boundary operators. Secondly if $J = \partial\mathscr{M}_{M^{(n-k)}}$ is a homotopy $(n-k-1)$-sphere rather than ball one has

$A[\partial\mathscr{M}_{M^{(n-k)}} * \triangle_k] \otimes_{K*\partial\mathscr{M}_{M^{(n-k)}}} \prod_{\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]) = \prod_{\mathrm{p}_K\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}])$ and similarly $\prod_{\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]) = A[\partial\mathscr{M}_{M^{(n-k)}} * \triangle_k] \otimes_{K^c*\partial\mathscr{M}_{M^{(n-k)}}} \prod_{\mathrm{p}_K\mathscr{M}_{N^{(k-1)}}}(\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}])$. The proofs for these cases follow from essentially the same argument as the main case.

Proof. Note that it suffices to consider the case where $\mathscr{M}_{M^{(n-k)}} = \triangle_{(n-k)}$ since $\mathscr{M}_{M^{(n-k)}}$ is a homotopy ball. Hence once we show the boundary operator can be moved through a single $(n-k)$-simplex and has the correct effect on the boundary, the result extends to $\mathscr{M}_{M^{(n-k)}}$ as the boundary operator can be freely moved through the $(n-k-1)$-homotopy trivial bulk. It also suffices to consider $\mathscr{M}_{N^{(k-1)}} = K \subset \triangle_k$ since the region of $\mathscr{M}_{N^{(k-1)}}$ not involved in the boundary $(k-1)$-Pachner move is irrelevant.

Now we have the n-Pachner move $(J * \triangle_k) \cup_{K*J} (K * \triangle_{(n-k)}) \mapsto (J^c * \triangle_k) \cup_{K^c*J^c} (K^c * \triangle_{(n-k)})$ which implies $A[J * \triangle_k] \otimes_{K*J} A[K * \triangle_{(n-k)}] = A[J^c * \triangle_k] \otimes_{K^c*J^c} A[K * \triangle_{(n-k)}]$ from which the result follows. $\square$

The proof of proposition 34 demonstrates that the codimension-k tensor network $\mathsf{A}_k[\mathscr{M}_{M^{(n-k)}}]$ can be thought of as fusing k codimension-$(k-1)$ tensor networks $\mathsf{A}_{(k-1)}[\mathscr{M}_{M^{(n-k+1)}}]$ along a common boundary which can be freely deformed.

For the case of codimension-2 submanifold tensor network operators, the boundary structure induced by multiplication is one dimensional and the boundary Pachner move (reducing two line segments to one) corresponds to the operators being projections. More precisely given a triangulation of a codimension-2 submanifold $\mathscr{M}_{M^{(n-2)}}$ the tensor network $\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]$ is an operator. Furthermore assuming $M^{(n-2)}$ is a homotopy $(n-2)$-sphere then $\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]$ is a projector i.e. $\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] \otimes_{v'*\mathscr{M}_{M^{(n-2)}}} \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] = \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]$, whereas if $M^{(n-2)}$ is a homotopy $(n-2)$-ball then $\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]$ is almost a projector but up to a boundary operator as follows $\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] \otimes_{v'*\mathscr{M}_{M^{(n-2)}}} \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] = \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] \otimes_{\partial\mathscr{M}_{M^{(n-2)}}*[v,v'']} A[\partial\mathscr{M}_{M^{(n-2)}} * \triangle_2]$ or similarly $(\mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}] \otimes_{v'*\mathscr{M}_{M^{(n-2)}}} \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]) \otimes_{\partial\mathscr{M}_{M^{(n-2)}}*([v,v'] \cup_{v'} [v',v''])} A[\partial\mathscr{M}_{M^{(n-2)}} * \triangle_2] = \mathsf{A}_2[\mathscr{M}_{M^{(n-2)}}]$.

Another important special case of the reduction is at the lowest level of codimension-n submanifolds, i.e. points. To a point v we assign the tensor $\mathsf{A}_n[v] := A[\triangle_{(n-1)} * v]$ now for an auxiliary $(n-1)$-complex $K \subset \partial\triangle_n$ we have $\prod_K(\mathsf{A}_n[v]) = A[\triangle_n] \otimes_{K^c} \prod_{K^c}(\mathsf{A}_n[v])$ where $K^c := \partial\triangle_n \setminus K$. This provides a way to read the tensor $A[\triangle_n]$ which fully determines the theory directly from local manipulation of the tensors. This seems to be a tensor network realisation of the phenomenon that the data assigned to a point by a fully extended TQFT uniquely determines the resultant theory.

This leads us to propose a definition for a new class of PEPS for an arbitrary dimension which aim to describe models within the same phase as a fully extended TQFT.

Definition 35 (Fully Extended Topologically (FET)-Injective PEPS)**.** Given a system with physical sites on the d-faces of a d-dimensional complex C the PEPS $T[v*C]$, contracted on the dual graph of $v*C$ is FET-injective if the following conditions hold

1. *TNO-injectivity:* For $c_d \in C$ the PEPS tensor $T[v*c_d]$ treated as a map from the virtual indices on $c_d \in v*\partial c_d$ to the physical index on c_d has a generalized inverse $T[v'*c_d]^+$ such that $T[v'*c_d]^+ T[v*c_d] = \mathsf{A}_2[\partial c_d]$ a TNO on the $(d-1)$D submanifold ∂c_d.

2. *TNO symmetry:* The TNO is a local symmetry of the PEPS i.e. for $K \subseteq \partial c_d$ a homotopy $(d-1)$-ball (sphere) one has $T[v*c_d] \otimes_{v*K} \mathsf{A}_2[K] = T[v*c_d] \otimes_{v*K^c} \mathsf{A}_2[K^c]$ for $K^c := \partial c_d \setminus K$.

3. *Extended Inverse:* There exists a tensor X, we dub the extended inverse, which allows the injectivity subspace to be consistently grown to larger regions, i.e. $K \subset \partial c_d$ a homotopy $(d-1)$-ball $\exists X[c_d;K]$, a tensor, s.t. $X[c_d;K] \otimes_{c_d \cup (v'*K)} (T[v*c_d] \otimes_{v*K} \mathsf{A}_2[K]) = \mathsf{A}_2[K^c]$.

4. *Fully Extended Structure:* For a complex $C_{(d+1-k)}$ in dimension $(d+1-k)$ we have a tensor network $\mathsf{A}_k[C_{(d+1-k)}]$ with k external indices per tensor in the network which can be composed according to an auxiliary simplicial $(k-1)$-complex $\mathscr{M}_{(k-1)}$ and for $K \subseteq \mathscr{M}_{(k-1)}$ s.t. $K \subset \partial \triangle_k$ and $C_{(d+1-k)}$ a homotopy $(d+1-k)$-sphere they satisfy $\prod_{\mathscr{M}_{(k-1)}} (\mathsf{A}_k[C_{(d+1-k)}]) = \prod_{\mathrm{p}_K \mathscr{M}_{(k-1)}} (\mathsf{A}_k[C_{(d+1-k)}])$.
   Furthermore for $C_{(d+1-k)}$ a homotopy $(d+1-k)$-ball we have a similar equation but up to a boundary tensor network of dimension $(d-k)$ as follows $\mathsf{A}_{(k+1)}[\partial C_{(d+1-k)}] \otimes_{\mathscr{M}_{(k-1)}*\partial C_{(d+1-k)}} \prod_{\mathscr{M}_{(k-1)}} (\mathsf{A}_k[C_{(d+1-k)}]) = \prod_{\mathrm{p}_K \mathscr{M}_{(k-1)}} (\mathsf{A}_k[C_{(d+1-k)}])$.

Note that in the above definition we have shifted our convention and labeled the spacial dimension by d hence in our usual convention $n = (d+1)$ and by codimension k we mean $(n-k) = (d+1-k)$. The above also allows for arbitrary spatial lattices dual to some d-complex C (in particular hypercubic lattices). There are many minor modifications to the above definition one may want to consider, such as incorporating a '*trivial* $(d-1)$*-spheres*' condition rather than allowing homotopy spheres in the TNO symmetry condition, there are also many possible slightly different ways to formulate the extended inverse condition.

The first three conditions alone capture a direct generalisation of the MPO-injectivity framework to arbitrary dimension, which we call *TNO-injectivity*, with these conditions alone one can derive many consequences analogous to those of MPO-injectivity such as the fact that the TNO is a projector, a cleaning (intersection) lemma for the tensor network on regions homotopic to a d-ball and that the boundary tensors look like the TNO on $(d-1)$-homotopy balls. Note that this pushes the possibility of arbitary boundary tensors on the virtual level down to codimension-2 which is precisely the result derived in the framework of MPO-injectivity, although in that case codimension-2 objects correspond to points whereas in higher dimension they may be extended objects. The addition of the condition 4 allows one to extract a piece of topological data in the form of some tensor $A[\triangle_{(d+1)}]$ from the composition rules of the 0-dimensional boundary tensor networks $\mathsf{A}_{(d+1)}[v]$. Note that the tensor $A[\triangle_{(d+1)}]$ will only be defined up to some tensor product of invertible matrices acting on each of the indices, since the tensors $\mathsf{A}_{(d+1)}[v]$ always come in pairs connected by an internal index which is invariant under adding a trivial product of invertible matrices MM^{-1} (this internal index is the one on which $A[\triangle_{(d+1)}]$ appears). Furthermore, by comparing distinct paths for implementing the same retriangulation of the auxiliary complex for the composition of the $\mathsf{A}_{(d+1)}[v]$ that differ by a $(d+2)$-simplex one finds a consistency condition on $A[\triangle_{(d+1)}]$ which implies it is invariant under the $(d+1)$-Pachner moves and hence is a PMIST.

More concretely, given a $(d+1)$-complex $K \subset \partial\triangle_{(d+2)}$ we take $J \subset \partial K$ as the auxiliary complex, then there are two ways to deform J through $\partial\triangle_{(d+2)}$ (one $(d+1)$-simplex at a time) to reach $(\partial K \setminus J)$, one which traverses the $(d+1)$-simplices of K and the other which traverses those of $(\partial\triangle_{(d+2)} \setminus K)$. Moving J through a $(d+1)$-simplex $\triangle_{(d+1)}$ corresponds to a d-Pachner $\mathrm{p}_{J\cap\partial\triangle_{(d+1)}}$ move on the auxiliary complex which gives rise to a factor $A[\triangle_{(d+1)}]$ as follows $\prod_J(\mathsf{A}_{(d+1)}[v]) = A[\triangle_{(d+1)}] \otimes_{(\mathrm{p}_{J\cap\partial\triangle_{(d+1)}}J)} \prod_{(\mathrm{p}_{J\cap\partial\triangle_{(d+1)}}J)}(\mathsf{A}_{(d+1)}[v])$. Whence the path through K gives rise to the product $A[K]$ and similarly the path through $(\partial K \setminus J)$ gives rise to the product $A[\partial K \setminus J]$ and we have $A[K] \otimes_{J^c} \prod_{J^c}(\mathsf{A}_{(d+1)}[v]) = A[\partial K \setminus J] \otimes_{J^c} \prod_{J^c}(\mathsf{A}_{(d+1)}[v])$, where $J^c := \partial K \setminus J$, from which we infer $A[K] = A[\partial K \setminus J]$ (possibly after restriction to the relevant subspace) implying $A[\triangle_{(d+1)}]$ is a PMIST.

There is a further addition to the conditions above we did not include which may be relevant, let us call it *5.* (Fully Extended Injectivity) which requires the tensor network associated to each codimension satisfy conditions *1-3* w.r.t. a new TNO of the appropriate dimension. This yields a new family of tensor network operators associated to submanifolds of arbitrary codimension, one could go further and require a version of condition *4* for these TNOs leading to new families of tensor networks at each codimension although we do not pursue this direction in the current work. With condition *5* one can use a version of the cleaning lemma to show that boundary tensors of any codimension k must locally look like the tensor network $\mathsf{A}_k[K]$ on homotopy $(d+1-k)$-balls K which allows one to show that arbitrary boundary tensors only occur in codimension-$(d+1)$ rather than in codimension-2, and hence are not extended objects and can contain only some finite amount of topological information. A special case of this extension where the codimension-d (1D) tensor networs are required to be injective rather than just block diagonal (TNO injective in dimension 1) which can be argued to yield a unique ground state similar to the 2D case. This prescription describes the trivial phase, and in the presence of global symmetries G of the tensor network which are represented by codimension-2 symmetry TNO on the virtual level this captures SPT phases. If in addition we have a version of condition 4 for the local multiplication of the symmetry TNOs (which form a representation of the symmetry group on homotopy-$(d-1)$ spheres) we can follow similar arguments to those presented in the paragraph above to find a tensor $A^G[\triangle_{(d+1)}]$ which contains the topological information of the theory. Due to the additional injectivity assumption on the 1D tensor network the internal index of $\mathsf{A}_{(d+1)}[v]$ is diagonal (trivial) and hence $A^G[\triangle_{(d+1)}]$ is defined up to a phase rather than a matrix and the PMIST equation becomes a $(d+1)$-cocycle equation. This yields the well known characterisation of SPT for this class of models in terms of the cohomology group $H^{(d+1)}(G,U(1))$.

We also note that this definition could be adapted to only extend down to some codimension rather than all the way down to points, this definition clearly does not capture the full structure present in a topological fixed point model but may be unrestrictive enough to realise some interesting non-fully extended theories, we leave this direction to future work.

Let us comment on the special case of $(d+1)=3$, since FET-injectivity is only mildly stronger than the MPO-injectivity framework in this case. In particular the fully extended structure is essentially equivalent to assuming the projection MPO for the PEPS can be brought into canonical (block diagonal) form. For SPT in $(2+1)$D it corresponds to having an injective MPO representation of the symmetry group G on the boundary of the PEPS. Working with essentially these assumptions Chen et. al. made arguments equivalent to the reduction procedure for the MPO representation of G to define the third cohomology of said representation, which they argued classified the SPT phase of the model considered. In higher dimensions it is not clear how restricted the FET-injective class is as there are few results concerning the structure of ground states in dimension greater than $(1+1)$. On one hand it is sufficient to recover the cohomology classification of SPT and DW models in any dimension and the F-symbol determining a string net model in $(2+1)$D, furthermore we conjecture it is sufficient to capture any fully extended topological field theory on the lattice. Note that the reduction procedure for SPT outlined above is a tensor network version of the SPT classification due to Else et. al. [82]

On the other hand having such TNO symmetries can be very hard to enforce in practice, and are certainly not generic in the class of all PEPS (generic PEPS are simply injective). Also note that the topological label obtained by the reduction procedure is only sure to be a true label of a phase close to an exact fixed point model since one may apply local filtering operations to the physical level of a PEPS which add a penalty to growing the topological objects which fluctuate to give a groundstate nontrivial topological order. Identifying the topological symmetries (particularly the extended structure) may in fact lead to new methods which work only within the subset of PEPS with the appropriate symmetry to study phase transitions that may be induced by such local filtering, this has been successfully carried out in $(2+1)$D by Haegeman et. al. [83]

Finally we would like to point out that piecewise linear TQFT can become much more complicated in 4D since the PL category no longer matches the purely topological category and at the same time TNO symmetries of the boundary can exhibit new behavior such as polynomially decaying correlations which are impossible with MPO symmetries.

2.6.1 Fully Extended TQFT and Higher Category Theory

In this last section we outline a rough formulation of the simplicial approach to n-categories since it most closely mirrors the construction of discrete fully extended TQFTs we consider although there are multiple different definitions which are also presumably relevant.

Let us first provide an overview of the basic concepts of fully extended topological field theory and higher category theory, and why there is a very natural correspondence. By definition a TQFT associates a numerical invariant to each closed n-manifold but there is additional structure that allows one to cut a closed n-manifold to open up a boundary to which the theory associates a state. This leads one to associate a vector space to each closed $(n-1)$-manifold. Extended TQFTs push this philosophy further with additional structure that allows one to cut a closed $(n-1)$-manifold by associating some algebraic object to $(n-2)$-manifolds (for example a 2-vector space). A k-extended TQFT possesses sufficient additional structure to allow one to continue cutting closed manifolds to reduce to boundary objects down to codimension-k, i.e. some algebraic data (e.g. an m-vector space) is assigned to each closed manifold of codimension-$m \leq k$ (a normal TQFT is 1-extended by definition). Fully extended TQFTs take this philosophy as far as it goes with a structure that allows one to continue cutting manifolds to reduce to algebraic objects assigned to those manifolds of arbitrary codimension eventually terminating at 0-dimensional points. The algebraic data assigned to a point then suffices to reconstruct the full theory.

It is generally understood that there is an iff relation between state sum invariants and fully extended TQFTs in that one can construct a state sum model from a fully extended TQFT and vice versa, this point of view is supported by the arguments in this manuscript. Non-fully extended theories can be thought of as possessing an anomaly which prevents the construction a lattice model for the theory (i.e. fixing a cutoff). For example a chiral model is difficult to realize on a lattice due to a doubling of energy levels induced by periodicity of the Brillouin zone. In some (perhaps all) cases such an anomaly can be resolved by considering the boundary of a higher dimensional fully extended theory (which has a state sum formulation) with an anomaly inflow to the boundary which exactly cancels that of the desired boundary theory. The canonical example of this construction is given by the (possibly chiral) 3D 2-extended Witten-Reshetikhin-Turaev TQFT which can be realized as the boundary of the 4D fully extended Crane-Yetter-Kauffman TQFT (Walker-Wang model).

On the other hand, higher categories are more naturally thought of in terms of building up rather than reducing down. One can start by considering a set of objects which naturally have a 0-dimensional structure. A category has the additional structure of morphisms between objects which satisfy a composition rule, this has a natural 1-dimensional structure with objects as points and morphisms as edges between them. A 2-category relaxes the composition of two morphisms from an equality to something called a 2-morphism which fill in the (triangular) faces of a 2-dimensional structure. In this context equality is lifted to a consistency condition on the 2-morphisms enforcing associativity as an equality. Similarly a 3-category relaxes the associativity of 2-morphisms from an equality to a 3-morphism which fill in the 3-simplices of a 3-dimensional structure. In this case the consistency condition corresponds to a pentagon equation or equivalently the 3D Pachner moves.

This pattern continues for higher categories for arbitrary n, an n-category consists of objects and morphisms which compose up to 2-morphisms which are associative up to 3-morphisms and so on up to $(n-1)$-morphisms which satisfy equations mirroring the $(n-1)$D Pachner moves up to an n-morphism and finally these n-morphisms satisfy a consistency equation corresponding to the nD Pachner moves with a strict equality. Hence the set of such n-morphisms naturally give rise to PMIST solutions, if one can write them down as a concrete tensor.

Rephrasing this in a more relevant setting, one can think of 0-dimensional objects as states, then the 1-dimensional data naturally correspond to evolutions of these states. If one has a composition law for the evolutions, reducing a pair down to a single evolution (a 1D Pachner move), this data naturally lives on 2-simplices. An associator which relates two different orders of composition of three evolutions (2D Pachner move) is naturally associated to a 3-simplex. There is a completely analogous, although less familiar, notion of a 2-associator which relates two equivalent paths for changing the order of composition of four evolutions via associators (3D Pachner move) associated to a 4-simplex. Again this pattern can be continued up to an arbitrary n where the $(n-1)$-associator relates a product of k and $(n+2-k)$, $(n-2)$-associators (and nD Pachner move) and is naturally associated to an $(n+1)$-simplex. This can either be cut off at some dimension m by enforcing strict equality of an mD Pachner move equation thus yielding a PMIST solution. Alternatively this structure can be iterated ad infinitum leading to the idea of an ∞-category, however for the finite dimensional models we consider this appears to be unnecessary.

Hopefully it is quite clear from the description thus far how one would naturally map from a k-category to a k-extended n-dimensional TQFT. One essentially starts from the bottom of the category and the top of the TQFT and assigns some element of $\mathbb{C}$ to each closed n-manifold, a vector space to each closed $(n-1)$-manfiold, a category to each closed $(n-2)$-manifold and so forth, down to a k-category to each closed $(n-1-k)$-manifold.

The general structure discussed throughout this section can be seen quite concretely in our tensor network framework. Given a PMIST solution the tensor network on any closed n-manifold yields a topological invariant in $\mathbb{C}$ and the projection tensor network operator on any closed $(n-1)$-manifold specifies a vector space. The fully extended structure begins to emerge when one considers cutting the tensor networks associated to closed manifolds, firstly an n-manifold M^n with a nontrivial boundary yields a vector within the vector space assigned to ∂M^n. It becomes more interesting for $(n-1)$-manifolds with nontrivial boundary as this gives rise to open virtual indices of the tensor network along the boundary, tensor product of such tensor networks yields a tensor product of the boundary indices and corresponds to the 0-dimensional structure of a set of points arising from the 0-dimensional boundary complex. Descending further one finds the composition rules which hold exactly for closed $(n-2)$-manifold TNOs hold only up to a fusion tensor network operator on the closed $(n-3)$-manifold boundary. From the point of view of the 1-dimensional boundary complex the boundary fusion operators implement the 1D Pachner moves giving rise to a 1D structure. This pattern continues and one sees that for $(n-k)$-manifold tensor networks with nontrivial boundary there is a $(k-1)$-dimensional boundary complex and the tensor networks satisfy the $(k-1)$D Pachner moves up to a $(k-2)$-associator given by a tensor network on the closed boundary $(k-1)$-manifold. Hence the correspondence between the tensor network and a higher category is seen most clearly by considering the boundary complex, i.e., for closed $(n-k)$-manifolds one has a k-category which extends to a $(k+1)$-category once one introduces a nontrivial boundary. Similar to the $(n-1)$D case, for arbitrary codimension one can consider manifolds of arbitrary topology that are consistent with a given boundary which may give rise to different boundary behaviour, however the techniques developed in this work only fully capture the case of trivial bulk topology.

To extract the physically relevant information from a topological theory one needs a method to construct and classify the elementary excitations. It is well known in 3D Levin-Wen (Turaev-Viro) models the elementary excitations correspond to the Drinfeld double, or center, of the input category and we expect that a similar phenomenon holds for the higher dimensional models that fall within our framework. Recently it has been discovered how to construct the elementary excitations directly within $(2+1)$D topological PEPS (in particular for PMIST solutions) and how this mirrors the more abstract categorical double construction. [79] This proposal can be extended to higher dimensions for PMIST networks by utilizing the tensor network operators of all codimension. This would in principle allow one to construct the excitations directly from the PMIST solutions without having to consider more abstract constructions.

Let us take a final quick detour back to the notion of anomalies, we note that one can consider the anomaly inflow of a boundary tensor network to be the boundary operators that occur when one enacts a relevant transformation (such as symmetry, or a Pachner move) upon the bulk. Hence if some boundary tensors satisfy the $(n-1)$D Pachner moves up to some anomolous factor which exactly corresponds to the boundary operator induced by a PMIST bulk, then the anomaly can be canceled by the TQFT corresponding to that bulk solution.

2.7 Conclusions

In this work, we have laid out the framework for constructing and understanding exactly solvable topological lattice models in terms of discrete TQFTs in any dimension. It is conceptually simple in the sense that models correspond to solutions to a finite set of local tensor equations but general enough to capture the physical picture of branching membranes of all codimension condensing. This picture is expected to describe phases whose topological order corresponds to n-categories and captures all canonical examples including LW models, WW models, generalized toric codes, DW theories and their 2-group generalization. Tensors satisfying retriangulation invariance form the fundamental structure that underlies any $(n+1)$D topological model since it is an if and only if condition for models formulated in terms of local data whose discrete path integral is topologically (PL) invariant and can be defined on arbitrary triangulations of the $(n+1)$-ball.

An important aspect of this work was the many interpretations of the topological invariance condition on the tensors that define a model. From topological invariance in codimension-0, linear transformations implementing retriangulations in codimension-1, Wilson membrane symmetry tensor network operators on the virtual level in comdinesion-2 and a hierarchy of boundary fusion operators for codimension-3 and higher. This is very suggestive of the fully extended structure expected of any discrete TQFT, equally a k-category assigned to codimension-$(k-2)$. It remains to make this connection fully rigorous and fill in the detail of a tensor network approach to constructing excitations. The natural reduction procedure may also be of some practical use, as tensor network numerical methods are significantly more advanced for $(1+1)$D than any higher dimension and furthermore the virtual symmetry constraints satisfied by topological models may be difficult to maintain with naive numerics, hence an approach which explicitly utilizes the topological symmetries while reducing to 1D calculations appears promising, and has proved useful in recent results for $(2+1)$D. [83]

An interesting consequence of our lattice model construction is that it allows one to use the natural equivalence relation defining gapped phases of Hamiltonians as a criteria for physical equivalence of discrete TQFTs that fall within these phases. This may be relevant in addressing subtle differences in the definitions of phases from the Hamiltonian verses Lagrangian point of view. On one hand a necessary condition for the equivalence of TQFTs is the equality of the path integral on all closed spacetime manifolds, at the same time it seems only manifolds of the form $M^n \times [0,1]$ are relevant in the Hamiltonian approach. TQFTs that are only defined on manifolds of this form are called $n+\varepsilon$ TQFTs, but as all models we consider have finite dimensional ground spaces one can take the trace to construct manifolds of the form $M^n \times S^1$. At the intersection of these two points of view lies the sufficient condition of equality of ground state degeneracies on all spatial topologies since they can be expressed as path integrals on manifolds of the form $M^n \times S^1$. The inclusion of S and T matrix data specify the effect of modular transformations within the ground space, which suffice to reconstruct the path integral on an arbitrary spacetime manifold if one constructs it via surgery on a link in S^3.

Our approach demonstrates the flexibility of tensor network states as those we consider have a natural holographic dimension, since only the physical lattice (triangulation) is fixed the dependence on the extra dimension is purely topological. In particular this allows for a topological version of the ER=EPR proposal with tensor network TQFTs. Extending the framework to allow conformal dependence of the holographic dimension could be relevant for the study of conformal field theories and may uncover a connection to the MERA.

The use of tensor network symmetries to study phases of matter is expected to apply beyond the setting of models described by TQFT, in particular the Haah code and chiral fermionic phases have been found to posses nontrivial virtual symmetries. The case of $(3+1)$D and higher is interesting and more complex as the TNO symmetries can exhibit polynomial decay of correlation, which are impossible for MPOs.

An immediate extension of the framework is to fermionic theories which requires additional structure (corresponding to a discrete framing) on the triangulated manifold upon which the theory is defined. This allows one to consider topological invariance up to some phases that cancel with those specified by the additional manifold structure, which can be thought of as an anomaly inflow from an almost trivial theory in one higher dimension. This may be a specific case of the more general situation in which one looks for boundary tensors that satisfy a constraint such as Pachner move invariance or symmetry invariance up to tensors arising from the bulk which can be thought of as an anomaly inflow. Such extensions become particularly relevant starting in dimension $(3+1)$ since null vectors begin to appear in the manifold pairing of any unitary TQFT. Four dimension is also the lowest where piecewise linear and smooth structures are no longer synonymous with a pure topological structure which necessitates the use of nonunitary (fermionic or otherwise anomolous) TQFTs for the potential of sensitivity to smooth structures. Four dimensional gauge theories in the continuum have become central tools for the study of smooth structure, particularly for proving negative, nonexistence results as a counterpart to positive existence results from Kirby calculus, it would be very interesting to see finite condensed matter implications of these theories, particularly those of Seiberg-Witten and Donaldson-Witten.

Appendix 2: Topological order in any dimension: A tensor network framework

2.A.1 Tensor Network Formalism

In this appendix we briefly review the basic ideas and results arising from the works on tensor networks, with a focus on results related to topological order. The central concept underlying tensor networks is that the coefficients of a state (map etc.) with some natural locality structure can be decomposed into local tensors that are contracted only with their neighbours according to the locality inherent to the problem.

The simplest example is a matrix product state which involves 3-index tensors $A^i_{a,b}$ which can be thought of as a list of matrices. These tensors are contracted according to the locality of a 1D line with open or periodic boundary conditions to yield a state

$$|\psi\rangle = \sum_{i_1,\dots i_N} \mathrm{tr}(A^{i_1}\cdots A^{i_N})\,|i_0,\dots i_N\rangle$$

where the matrices are multiplied in the usual way and the trace corresponds to periodic boundary conditions, while no trace would correspond to open boundary conditions.

More generally one can take a graph G without self edges but possibly with some edges that start at a vertex but do not end at another vertex, we will call such edges external and group them into a subset $E_X \subset G_e$. To G let us assign some bond dimension d_e (a finite natural number) to each edge and then a tensor $(A_v)_{i_1,\dots,i_{N_e(v)}} \in \mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_{N_e(v)}}$ to each vertex $v\in G$, where $N_e(v)$ is the number of edges around a given vertex, and we consider ordering them from 1 to $N_e(v)$ in some way. Then these tensors can be contracted according to G to yield a state as follows $|\mathrm{tr}_{e\in G_e\setminus E_X}[\bigotimes_v A_v]\rangle := \sum_{\{i_e\}}\bigotimes_v (A_v)_{i_{e_1},\dots,i_{e_{N_e(v)}}}\,|\bigcup_{e\in E_X} i_e\rangle$. Note one may consider grouping the external indices into input $I\subseteq E_X$ and output $O\subseteq E_X$ then the tensor network corresponds to a map $\sum_{\{i_e\}}\bigotimes_v (A_v)_{i_{e_1},\dots,i_{e_{N_e(v)}}}\,|\bigcup_{e\in O} i_e\rangle\langle\bigcup_{e\in I} i_e|$.

Often a graphical notation is employed where a tensor is drawn as a vertex (or polygon) together with its adjacent edges which are identified with its indices and a tensor contraction over a pair of compatible indices is depicted as a pair of vertices (tensors) with some edges connected corresponding to the indices to be contracted.

A particularly relevant class of tensor networks called PEPS, which generalize matrix product states to arbitrary dimension, are associated to a physical lattice L by taking the graph G formed by attaching an extra external edge to each vertex (corresponding to a physical site) of L with bond dimension d_e equal to the physical dimension of the site and constructing the tensor network state $|\mathrm{tr}_{e\in L}[\bigotimes_{v\in G} A_v]\rangle$.

The MERA is another class of physically motivated tensor network states which have an internal structure roughly corresponding to a discretized AdS space in a holographic extra dimension. These states can be constructed by taking a graph G corresponding to a discretization of AdS cut at some radius, the edges crossing the cut become the external physical indices, and forming the tensor network state $|\mathrm{tr}_{e\in G\setminus E_X}[\bigotimes_{v\in G} A_v]\rangle$. Interestingly the construction in this work also has a holographic dependence, which is to be expected in the context of TQFT, however the dependence on this holographic dimension in our tensor networks is of a purely topological nature which is fundamentally very different from the holographic dependence of the MERA.

If we imagine cutting out some simply connected region from a physical lattice $R \subset L$ we have a subset of the external indices corresponding to those crossing the cut which we will group into I and then the usual extra indices attached to each vertex in the PEPS construction which are grouped into O. Then the PEPS assigned to this region is interpreted as a map $A_R := \sum_{\{i_e\}} \bigotimes_v (A_v)_{i_{e_1},\dots,i_{e_{N_e(v)}}} |\bigcup_{e\in O} i_e\rangle\,(\bigcup_{e\in I} i_e|$ where the curved bracket emphasizes that the inputs are considered to be virtual boundary states. Since the volume of the interior grows faster than the area of its boundary (considering Euclidean space) the PEPS map is generically injective after grouping several sites together. It was shown that this property is stable on larger regions and leads to a unique ground state on any topology. Furthermore it was shown to be incompatible with nontrivial Wilson loop operators, and hence such generic injective PEPS do not accurately capture the basic properties of topological states. This has been addressed by considering PEPS that are injective on a subspace which is stable when grown to larger regions, first in the framework of G-injective PEPS [21] then twisted injective PEPS [22] and most recently and generally in MPO-injective PEPS which capture all doubled phases in (2+1)D. [1]

Furthermore, strong results about the canonical form of matrix product states imply that the topological MPOs decompose into irreducible MPOs that form an algebra and give rise to something analogous to fusion tensors. These MPOs together with fusion tensors can be used to construct an asatz for excitations that closely corresponds to the construction of the Drinfeld double of the algebra. [79]

2.A.2 Basic Combinatorial Topology

This appendix covers some elementary constructions of combinatorial topology which are used throughout the body of the chapter. Our use of the language of combinatorial topology is not particularly deep, and we mostly use it in place of drawing n-dimensional pictures (which is considerably harder).

- An n-simplex $\triangle_n = [v_0 \dots v_n]$ is defined to be the convex hull of a set of $(n+1)$ points $v_0, \dots, v_n \in \mathbb{R}^{m \geq n}$ such that the vectors $\{v_i - v_0 | \; i \neq 0\}$ are linearly independent. For our purposes only the combinatorial aspects of an n-simplex are relevant, in particular the fact that the convex hull of any subset of $0 \leq (j+1) \leq n$ vertices $v_{i_0}, \dots, v_{i_j}$ is itself a j-simplex $[v_{i_0} \dots v_{i_j}]$ referred to as a face or subsimplex of $\triangle_n$. Hence there are $\binom{n+1}{j+1}$ j-subsimplices within an n-simplex. The orientation of a simplex can be defined as $\sigma(\triangle_n) := \mathrm{sgn}(\det(v_1 - v_0, \dots, v_n - v_0))$ and satisfies $\sigma([v_0 \dots v_n]) = \mathrm{sgn}(\pi)\sigma([v_{\pi(0)} \dots v_{\pi(n)}])$ for any permutation π.

- A pair of n-simplices $\triangle_n, \triangle'_n$ can be glued along an $(n-1)$-face by identifying an $(n-1)$-subsimplex (and its subcomplexes) of each n-simplex. Throughout the chapter, we work with *pure* n-dimensional simplicial complexes $\mathscr{M}_n$ which are defined to be a union of m-simplices ($\forall m \leq n$) such that the following properties hold

  - Any subsimplex $\triangle_q$ of a simplex $\triangle_p \subseteq \mathscr{M}_n$ is also contained in $\mathscr{M}_n$.
  - The intersection $\triangle_q \cap \triangle_p$ for $\triangle_q, \triangle_p \subseteq \mathscr{M}_n$ is a subsimplex of both $\triangle_q$ and $\triangle_p$.
  - Any q-simplex $\triangle_q \subset \mathscr{M}_n$ (for $q < n$) is a subsimplex of some $\triangle_n \subset \mathscr{M}_n$.

  We usually consider the vertices of a simplicial complex $\mathscr{M}_n$ to be ordered, which induces a direction on each edge from the lesser to the greater adjacent vertex. The directed edges satisfy a no-cycles condition since one can never travel in a closed cycle when following the direction along a path of edges since the vertices one visits are monotonically increasing.

- The m-skeleton of a simplicial complex $\mathscr{M}_n$ is defined to be the union of all its j-subsimplices $\forall j \leq m$. We use $\|\mathscr{M}_n\|_m$ to denote the number of m-subsimplices of $\mathscr{M}_n$, and set $\|\mathscr{M}_n\| = \|\mathscr{M}_n\|_n$ by default.

- The underlying polyhedra of a simplicial complex $\mathscr{M}_n$ is the union of all simplices within the complex, denoted by $|\mathscr{M}_n|$.

- The boundary of a simplicial complex $\mathscr{M}_n$ is the $(n-1)$-simplicial complex composed of all $\triangle_{(n-1)} \subset \mathscr{M}_n$ that are a subsimplex of precisely one n-simplex $\triangle_n \subset \mathscr{M}_n$. Note that the boundary of a boundary is zero.

- Given a collection of simplices $D \subseteq \mathscr{M}_n$:

  - The closure of D, denoted $\mathrm{cl}(D)$, is the minimal simplicial subcomplex of $\mathscr{M}_n$ containing all the simplices in D.
  - The star of D is denoted $\mathrm{st}(D)$ and is given by the set of simplices in $\mathscr{M}_n$ which have any subsimplex contained in D.
  - The link of D, denoted $\mathrm{lk}(D)$, is defined to be $(\mathrm{cl}\ \mathrm{st}(D) \setminus \mathrm{st}\ \mathrm{cl}(D))$.

  Note that we have identified a complex with its closure when convenient throughout the body of the text.

- A triangulation of a topological manifold M^n is a simplicial complex we denote as $\mathscr{M}_{M^n}$ together with a homeomorphism $\phi : |\mathscr{M}_{M^n}| \to M^n$.

- A piecewise linear manifold is a topological manifold with an atlas of coordinate charts such that the transition functions between charts are piecewise linear. Not all topological manifolds admit a PL structure, and those which do may admit infinitely many inequivalent PL structures. A piecewise linear triangulation of a topological manifold M^n is a simplicial complex $\mathscr{M}_{M^n}$ together with a homeomorphism $\phi : |\mathscr{M}_{M^n}| \to M^n$, satisfying the extra constraint that the link of any vertex $\mathrm{lk}(v)$, $v \in \mathscr{M}_{M^n}$ is homeomorphic to a PL $(n-1)$-sphere (not merely a homotopy sphere). Such simplicial complexes are called combinatoric manifolds or simplicial manifolds. One can easily construct a triangulation that is not PL by taking two (or more) suspensions of a triangulated Poincaré sphere. Note that it is unclear to what extent many of the arguments in the body of the work depend upon having a PL triangulation, however we have assumed this throughout as some results we rely upon at various points make this assumption.

- A smooth manifold is a topological manifold with an atlas of coordinate charts such that the transition functions are smooth. Every smooth manifold admits a unique (up to PL homeomorphism) PL triangulation but the converse does not hold, there are PL manifolds which cannot be smoothed and there are PL manifolds which admit multiple inequivalent smooth structures (e.g. Milnor's exotic 7-spheres).

- Diff, PL, Top are the categories with (smooth, PL, topological) n-manifolds as objects and (smooth, PL, continuous) maps between them as morphisms. Let us briefly and loosely outline the relation between these fundamental manifold categories:

- Diff$\simeq$PL for dimension≤ 6. The inclusion Diff$\subseteq$PL is neither injective nor surjective in general, but for dimension$= 7$ it is surjective.
- Top$\simeq$PL for dimension≤ 3. Hence Diff$\simeq$PL$\simeq$Top for dimension≤ 3. The inclusion PL$\subseteq$Top is neither injective nor surjective in general.
- Triangulations and PL triangulations are equivalent for dimension≤ 4.

The second point hints at the difficulty inherent to finding the most general combinatoric topological invariants and models in dimension≥ 4. There is strong evidence one must consider models beyond bosonic lattice models to find the most general discrete 4D TQFTs.

- We now move on to describe some basic constructions used throughout the work.
  - The join of two simplices $\triangle_n = [v_0 \dots v_n], \triangle_m = [v_{n+1}, \dots v_{n+m+1}]$ is the simplex $\triangle_n * \triangle_m = [v_0 \dots v_{n+m+1}] \simeq \triangle_{n+m+1}$.
  - The join of two simplicial complexes $\mathscr{M}_n, \mathscr{M}_m$ is defined as the simplicial complex $\mathscr{M}_n * \mathscr{M}_m$ consisting of all $\triangle_i * \triangle_j$, $\forall \triangle_i \in \mathscr{M}_n, \forall \triangle_j \in \mathscr{M}_m$ (note this includes joins with the empty simplex $\emptyset * \triangle_j = \triangle_j$).
  - The cone of a simplicial complex $\mathscr{M}_n$ is given by $v * \mathscr{M}_n$ for an additional vertex $v \notin \mathscr{M}_n$.
  - The suspension of a simplicial complex $\mathscr{M}_n$ is given by $\{v, v'\} * \mathscr{M}_n$ for additional vertices $v, v' \notin \mathscr{M}_n$.

  Note the join is associative and commutative (possibly up to orientation reversal). There is a simple relation for any simplex $\triangle_i \in \mathscr{M}_n$ given by $\mathrm{st}\,\triangle_i = \triangle_i * \mathrm{lk}(\triangle_i)$.

- A bistellar flip on any k-simplex $\triangle_k \in \mathscr{M}_n$ is constructed from an auxiliary $(n-k)$-simplex $\triangle_{(n-k)} \notin \mathscr{M}_n$ by taking $(\mathscr{M}_n \setminus \mathrm{st}\,\triangle_k) \cup_{\mathrm{lk}(\triangle_k)} (\partial \triangle_k * \triangle_{(n-k)})$ with the identification $\mathrm{lk}(\triangle_k) \simeq \partial \triangle_k * \partial \triangle_{(n-k)}$. Note that this is equivalent to definition 5 of Pachner moves given in the body of the text, and the term bistellar flip predates the term Pachner move.

- The Poincaré dual of an n-dimensional simplicial complex is an n-dimensional generic cell complex with an $(n-k)$-cell for each k-simplex of the simplicial complex. To construct this dual one can start by associating a vertex v^i to each simplex $\triangle_n^i \in \mathscr{M}_n$ then iteratively adding a j-cell c_j for $j = 1, \dots, n$ for each $(n-j)$-simplex $\triangle_{(n-j)} \in \mathscr{M}_n$. For $\triangle^i_{(n-j)} \in \mathscr{M}_n$ we add a j-cell c^i_j with k_l $(j-l)$ faces, where k_l is the number of

$(n-j+l)$-simplices that intersect $\triangle_{(n-j)}$ in $\mathscr{M}_n$. Each $(j-l)$ face of c^i_j is glued to the $(j-l)$-cell that is dual to the corresponding $(n-j+l)$-simplex intersecting $\triangle_{(n-j)}$.

Chapter 3

A tensor network study of Walker-Wang models

We present a natural tensor network state representation of the ground states of Walker-Wang models. Walker-Wang models have been proposed in 2011 as local, commuting Hamiltonians in $3+1$ dimensions, which have been conjectured to be low-energy effective TQFT descriptions of topological insulators. We show that a single local tensor describes the ground state of Walker-Wang models, and a tensor network operator (TNO) encodes the properties of the topological order. We prove TNO-injectivity and pulling through equations by showing that they are equivalent to $2 \to 4$ and $3 \to 3$ Pachner equations, which are the consistency conditions that have to be satisfied for topologically ordered ground states. TNOs are surface operators supported on two-dimensional manifolds whose one dimensional terminations at the boundaries give information about boundary topological order. [1]

[1]This chapter is based on "A tensor network study of Walker-Wang models". [3]
Authors: M.B. Şahinoğlu, K. Temme, M. Walter.
To appear on arXiv soon.

3.1 Introduction

One of the main problems in condensed matter and high energy physics is to characterize and classify all possible macroscopic low energy behaviour that may arise from a local, microscopic Hamiltonian. A promising approach to this goal has been made by defining simple quantum field theories that have topological properties, called topological quantum field theories (TQFT) [47, 48, 52]. These theories are defined via a local data from which the partition function and ground states can be constructed. These are the coarsest, but still interesting, physical theories and are believed to capture the zero-temperature physics of possible physical theories. [23, 9, 14]

Another approach to the problem has been proposed with the motivations coming from quantum information. Tensor network states (TNS) have been proposed as states that describe the low-energy eigenstates of local microscopic Hamiltonians. In one dimension, it has been proven that, TNS can describe the ground states faithfully [18], and in higher dimensions there has been an overwhelming evidence that TNS is successful in efficiently describing the low-energy eigenstates of physical systems. [19, 21] A TNS is constructed via contracting a local tensor, which is the local data to be specified according to the model we are trying to understand. In two space dimensions, it has been shown that a finite bond-dimension TNS can describe topologically ordered ground states. [36, 37, 21, 22, 1] In the scheme proposed in [1], there are matrix product operators (MPOs) that encode the topological order in the TNS, with which one can map different ground states to each other. MPOs are line/loop operators acting on the virtual level of the tensor network state, such that the action is trivial or nontrivial depending on the topological properties of the line/loop on which the MPO is supported. Specifically, we can understand Turaev-Viro TQFTs [52] (or Levin-Wen string-net models [14]) in tensor network language: Ground state TNSs are the states that TQFT assigns to 2d closed manifolds and the MPOs are the Wilson line/loop operators.

In this work, we study the so-called Walker-Wang models [84] in $3+1$ dimensions using the framework of tensor networks. These models are constructed via premodular categories and originated from Crane-Yetter TQFTs. [57] They are speculated to describe the low energy physics of topological insulators, since in some cases they describe a state which is topologically trivial in the bulk but nontrivial on the boundary, i.e., it is possible to create quasi-particle excitation pairs on the boundary with the same energy penalty independent of the distance between particle-antiparticle pairs, although within the bulk, quasi-particle pairs are penalized with the distance between the pairs. Hence, it is a system that behaves like an insulator in the bulk but more like a conductor on the boundary. For a more detailed physical analysis of these models refer to Refs. [85–87]. We study these models by finding explicit tensor network states for their ground states. We find that there is one local tensor

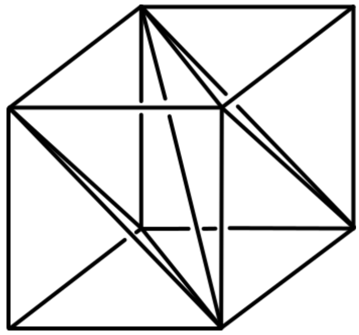


Figure 3.1 A triangulation of a cube

T by which we can describe one of the ground states. The local tensor T is nothing but the local data that is provided by the Crane-Yetter TQFT. We further find that the local virtual subspace is determined by the so called *TNO-injectivity*, i.e., the local virtual degrees of freedom are projected onto a subspace which is determined by a closed surface operator, that can be described as a tensor network of operators. These tensor network operators are deformable through the lattice, which is an important property that we call *pulling through*. Topologically nontrivial TNOs map ground states to each other and give us a basis for the possible different ground states.

The organization is as follows: First, we describe Walker-Wang models from a slightly different point of view compared to the original work (although Walker and Wang describe the way that we proceed in words). We start with triangulations, write down the local Hamiltonian, then map the lattice and Hamiltonian to the ones as presented in the original work of Walker and Wang. [84] Then, we find the tensor network state description of the ground states, i.e., we find the local tensor A and the TNO T. After that, we continue with showing the properties of the local tensor A and the TNO T, in particular we argue that TNO-injectivity and pulling through equations are satisfied since they correspond to Pachner equations that must be fulfilled by the local data of the TQFT. We conclude with closing remarks about boundary topological order and some open questions.

3.2 Description of Walker-Wang models

In this section we follow the description of Walker and Wang to construct the models. [84] Given a 3-manifold M, we first triangulate it: We take the cubic lattice and triangulate each cube as shown in Fig.3.1. Any triangulation is equivalent for our purposes, however we choose a particular one for the purpose of a consistent visualization of the model on a fixed regular lattice. The local degrees of freedom of the many-body Hamiltonian model, which

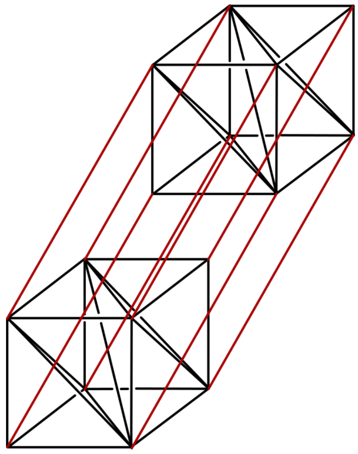


Figure 3.2 Evolution of the unit cube. The input and output 3-dimensional cubes are connected via an evolution operator in 3+1 dimensions.

Figure 3.3 The pictorial description of the Hamiltonian vertex term.

can take values $i \in \{0, \dots N\}$, live on the triangular (2-dimensional) surfaces of the lattice. We define the ground state space of the model (yet to be written) as the fixed points of the evolution operator that is a $3+1$ dimensional projector map from the same lattice to itself, as shown for the unit cube in Fig.3.2. The evolution operator is realized by triangulating the evolution in $3+1$ dimensions and assigning a tensor T to every 4-simplex. The Hamiltonian is defined to be the map that lifts the lattice to itself only on one vertex, and is written in terms of a contraction of tensors T. In Fig.3.3, we show how the local Hamiltonian term on the vertex v acts, pictorially: It maps the input triangles with all black edges to output triangles with black and gray edges. The degrees of freedom on the triangles with a red edge are summed over. If we look at the Hamiltonian term partially on one tetrahedron, it consists of one T-tensor which is pictorially shown in Fig.3.4.

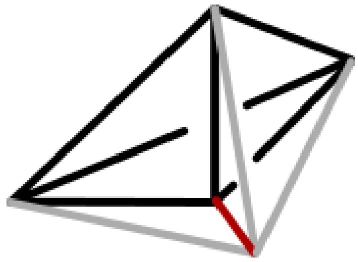


Figure 3.4 The unit T-tensor from which we construct the Hamiltonian.

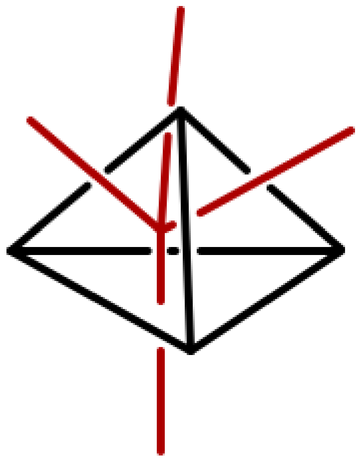


Figure 3.5 Dualizing a tetrahedron.

Until now, we formulated our model on a triangulated lattice, which is the way that is described in words by Walker and Wang. We further dualize the lattice by going from triangulated lattice to the lattice with string diagrams. We map k-cells to $3-k$-cells to obtain the new lattice. For example from a tetrahedron, we obtain 4 strings meeting at a vertex, i.e., see Fig.3.5.

Hence, the degrees of freedom on the triangles are now on the edges of the new lattice. Our new lattice is a 3-dimensional lattice constructed via strings that always intersect at 4-leg vertices. A cell in the dual lattice can be seen in Fig.3.6.

The local vertex (0-cell) Hamiltonian term in the triangulated lattice, becomes now a cell (3-cell) term in the dual lattice. This operator can be seen in the dual lattice as inserting loops on the plaquettes, where the operator acts on the whole cell as one single term. (See Fig.3.7). The inserted loops on the plaquettes next to a cell, then are annihilated by using local modifications, i.e., T-tensors, into the original cell as shown in Fig. 3.8. This is how we write down local Hamiltonian terms using the simplex tensor T.

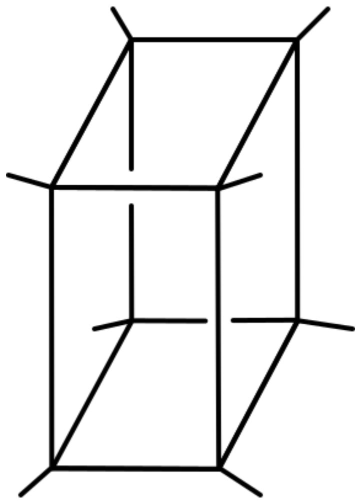


Figure 3.6 A cell in the dual lattice.

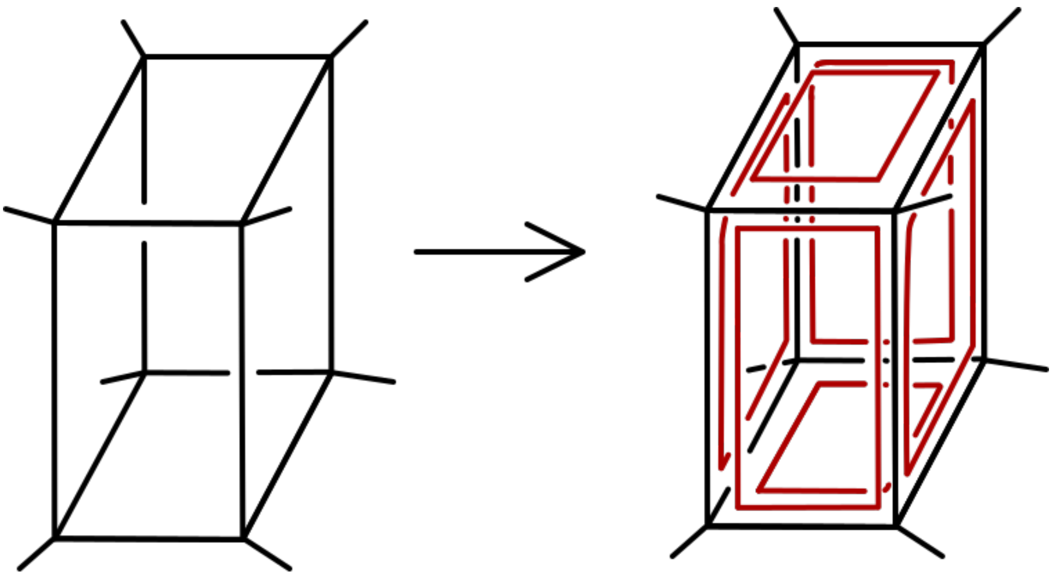


Figure 3.7 The first step that constructs the cell term in the Hamiltonian: Introducing loops on plaquettes of the same cell.

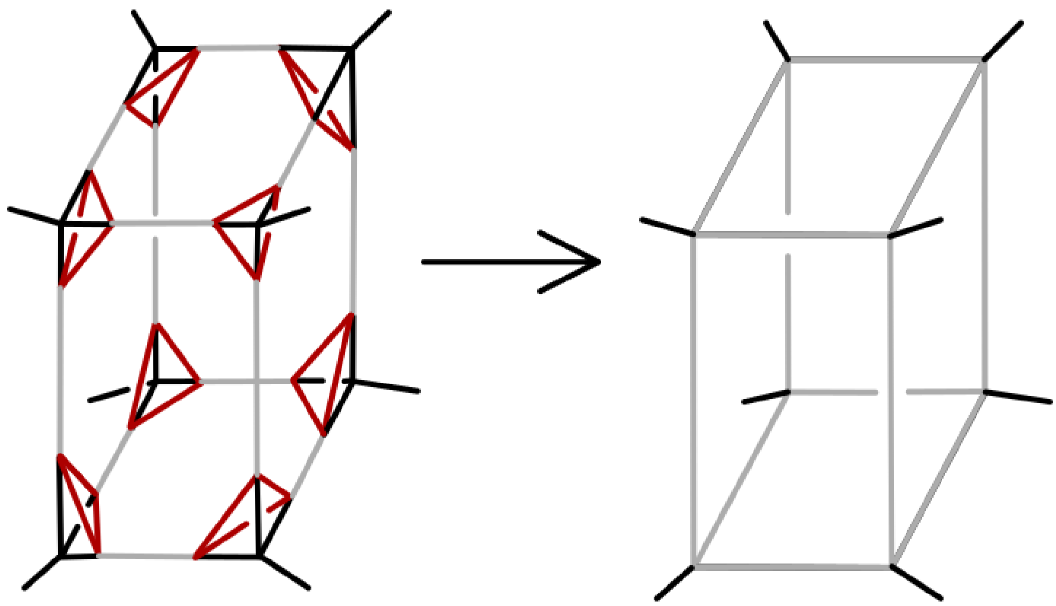


Figure 3.8 The second step of constructing the Hamiltonian: Annihilating the loops using the T-tensor.

Note that this Hamiltonian term is in principle a cell term, although it starts with introducing loops into the plaquettes. However, for Walker-Wang models, the cell term decomposes into a sum of the plaquette terms, where the plaquette terms are defined in the way that is pictorially described in the Walker-Wang paper. Before describing how they write down each plaquette term, we need one more modification to do in our lattice. For the purposes of Walker-Wang model, we split each 4-leg vertex into two 3-leg vertices, as shown in Fig.3.9.

Let's now describe how Walker and Wang define their plaquette term. We first assign labels to the edges adjacent to the vertices which are around the plaquette p_{xy}, where x, y

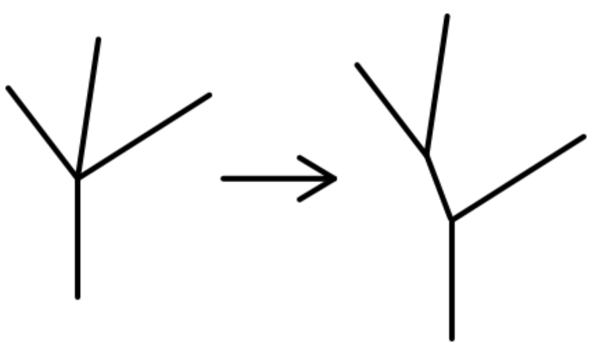
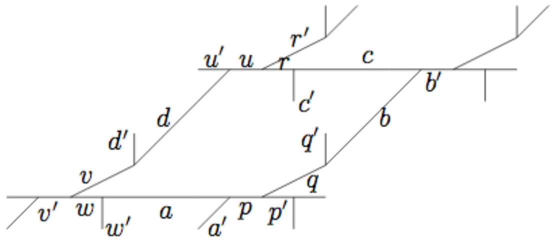


Figure 3.9 Splitting a 4-leg vertex into two 3-leg vertices.

are the coordinates of the plaquette with some fixed coordinate value in z-direction. The plaquette looks like



Then, we introduce a loop labeled by s in the plaquette and dissolve it into the lattice by using F- and R-moves. This process determines the local Hamiltonian term for plaquettes lying on xy-plane. The Hamiltonian term is described in the same way for every plaquette, but the terms differ according to where we apply R-moves, which depends on from which perspective we are looking at the lattice. This is fixed from the beginning, hence there are different looking Hamiltonian terms for plaquettes on xy-, yz- and xz-planes, whose descriptions are the same in essence. Exact mathematical expression of the Hamiltonian term is not needed and indeed redundant for our discussion, hence we direct the reader to the original work of Walker and Wang. In order to derive the tensor network state description, we only need the process of deriving the Hamiltonian term which has been described above.

3.3 Tensor network state description

In this section, we give a tensor network state description of the ground states of the Walker-Wang models. Given the fact that, the Hamiltonian is a local commuting projector, we obtain a ground state by applying each local Hamiltonian term on the product state $|00\ldots0\rangle$, in the same way as we do while defining the local Hamiltonian term, i.e., in Fig.3.7 and Fig.3.8. In the dual lattice, applying Hamiltonian terms correspond to introducing loops on the plaquettes next to the five spins in the product state $|00000\rangle$ (shown as dotted edges), which are partially shown in Fig.3.10.

Local Hamiltonian terms imply an action on the five spins which are initiated at the product state $|00000\rangle$. As a result of applying local Hamiltonian terms we get a new state, which is a linear combination of states in the computational basis. The explicit steps of obtaining this action are shown in Fig.3.11. Now let's write down what each step means mathematically, i.e., in terms of the quantum dimensions d_is (and $v_i = \sqrt{d_i}$ of the label i), F- and the R-symbols of the premodular tensor category (or simply the given input data).

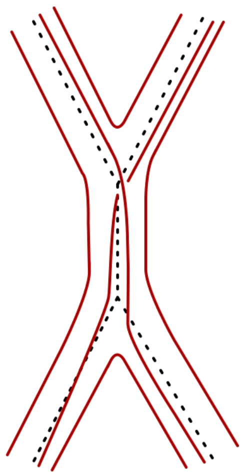
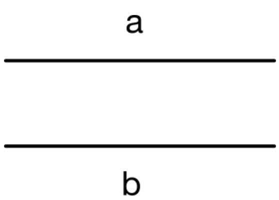
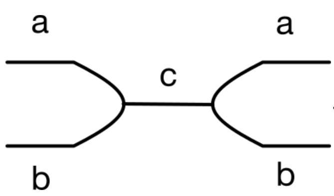


Figure 3.10 Dotted lines symbolize the $|0\rangle$ state on the edge. The red lines carry the labels $i = 0, \ldots N$, and comes from the action of the local Hamiltonian terms.

- **Step-1:** We fuse two parallel lines to one line in order to get the physical degrees of freedom. This step uses a special type of F-symbol, which is a fraction of quantum dimensions as shown in Eq.(3.1).



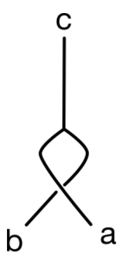
$$= \frac{d_c}{d_a d_b}$$



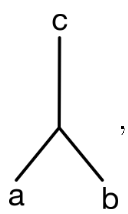
. (3.1)

- **Step-2:** In this step, we essentially repeat step-1 on lines labeled by e and f in Fig.3.11.

- **Step-3:** In this step, we do two operations: We first perform an R-move:



$$= R^c_{ab}$$



, (3.2)

and then two 3-1 F-moves (we use G-symbols, which are the symmetric F-symbols):

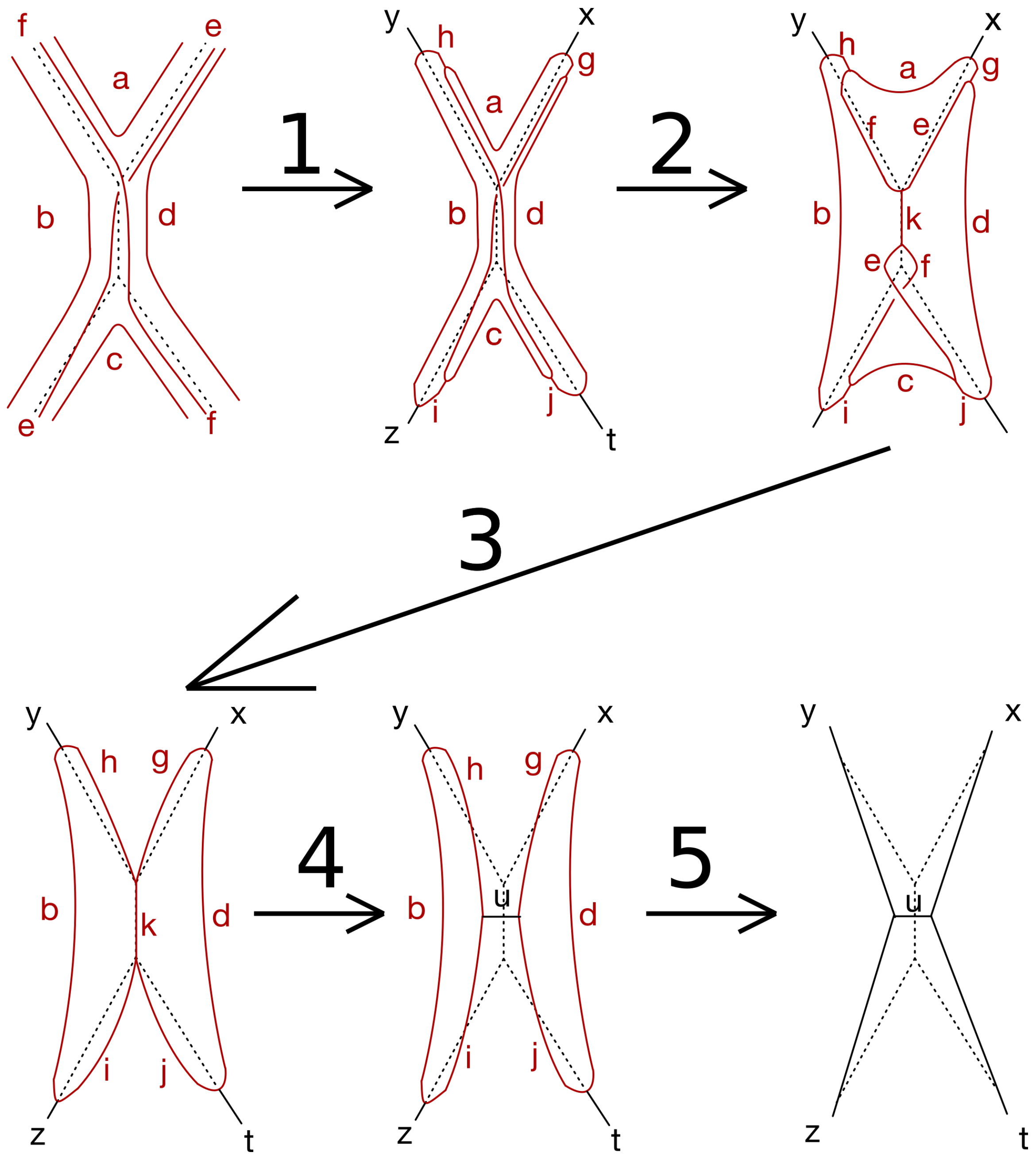
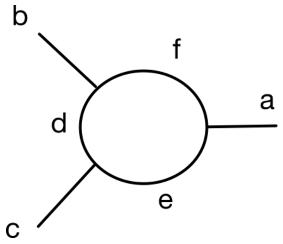
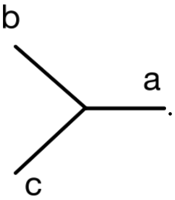


Figure 3.11 The steps to derive the local tensor of a Walker-Wang model.

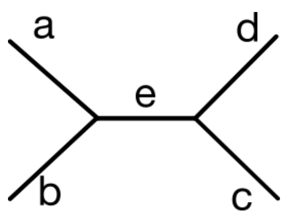


$= v_d v_e v_f G^{abc}_{def}$

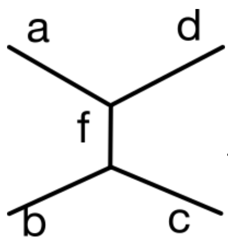


. (3.3)

- **Step-4:** We perform a 2-2 F-move:



$= v_e v_f G^{abe}_{cdf}$



. (3.4)

- **Step-5:** We apply two $3-1$ F-moves as in Step-3, and obtain purely physical legs.

As described above, we obtain one of the ground states by applying the local commuting projector Hamiltonian to the product state $|00\ldots0\rangle$. We find the local tensor by zooming in this operation on a particular vertex: The indices labeled red are the virtual degrees of freedom (i.e., they are contracted) and the indices labeled black are the physical degrees of freedom of the local tensor that constructs the tensor network state. The local tensor is given in Fig.3.12, where each individual component is explicitly shown in Fig.3.13.

As a result, the local tensor A is given by:

$$A^{xyztu}_{abcdef,ghij} = \sum_k d_k \sqrt{v_y v_x v_t v_z} v_g v_h v_i v_j v_u G^{hkg}_{eaf} G^{ijk}_{efc} R^k_{fe} G^{uyz}_{bih} G^{utx}_{dgj} G^{ghk}_{iju} . \tag{3.5}$$

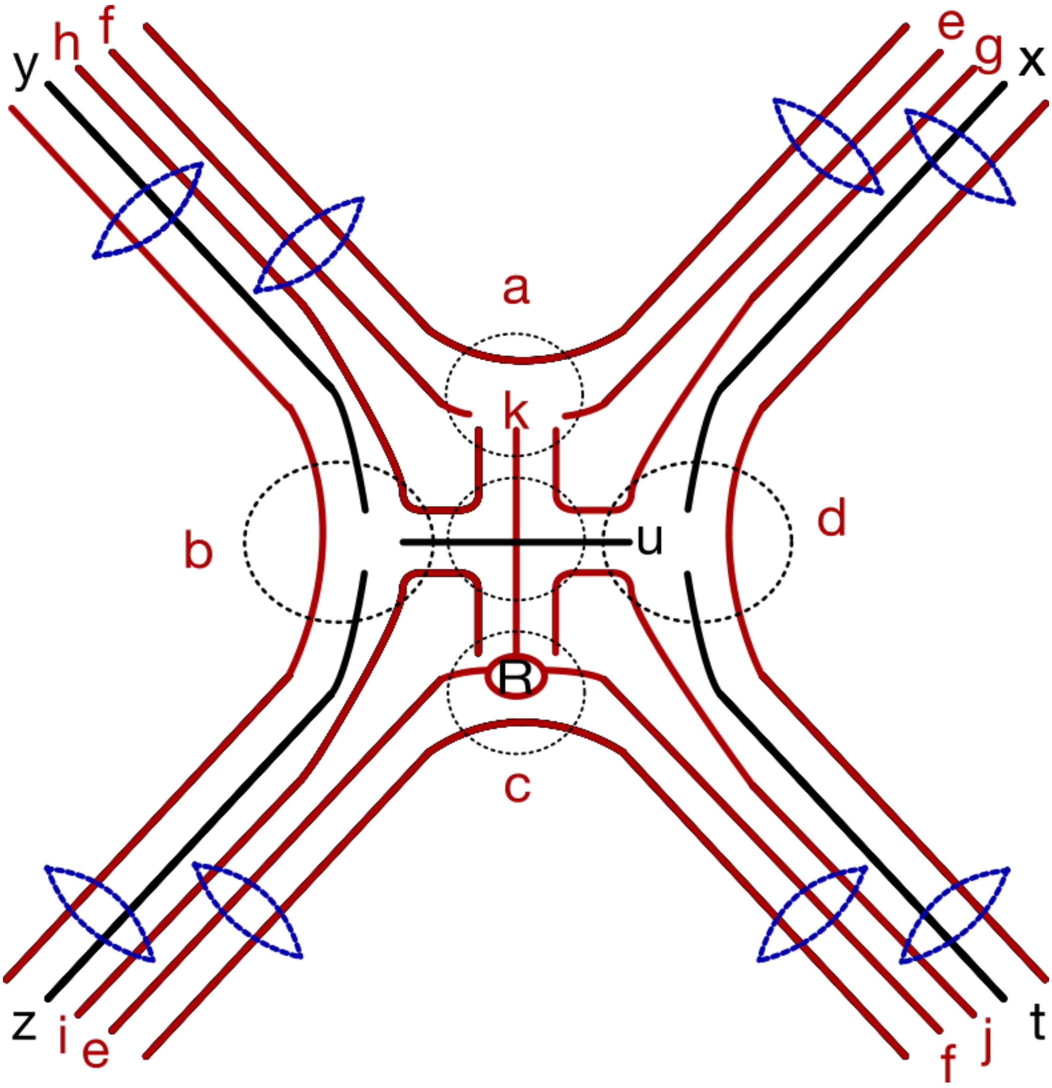


Figure 3.12 The bulk local tensor of the ground state of a Walker-Wang model.

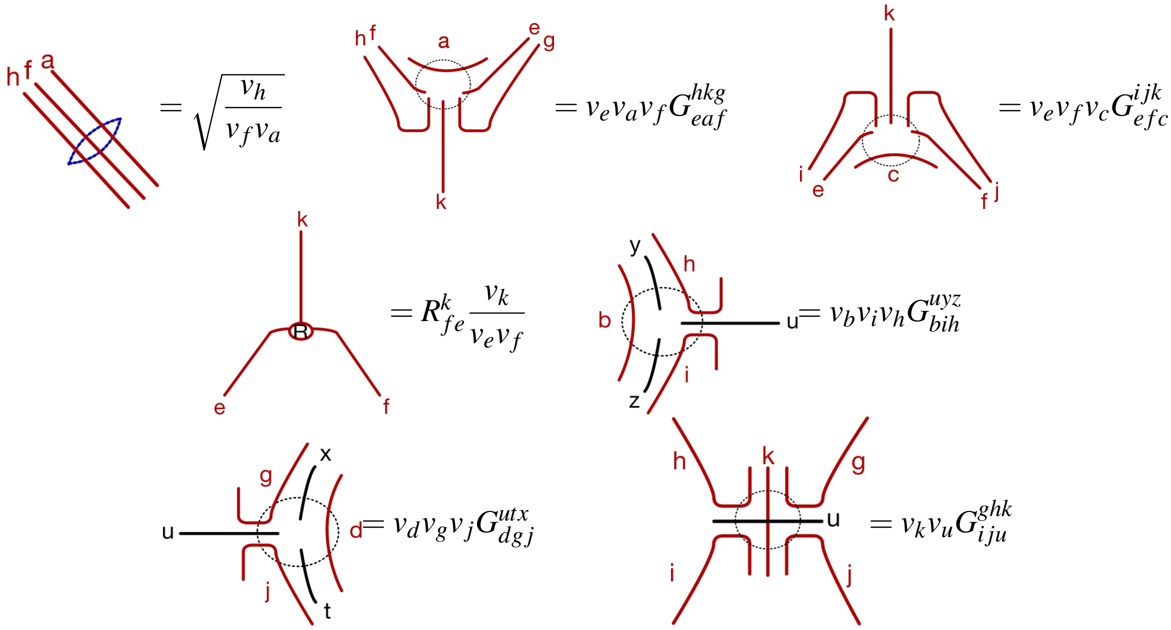
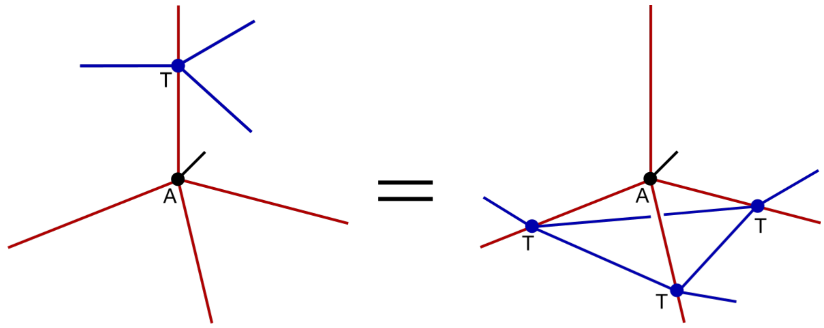


Figure 3.13 Individual tensor components of the local tensor of a Walker-Wang model.

3.4 Tensor network operators and their properties

In the previous section, we found a particular tensor network state description for a ground state of a given Walker-Wang model (by F- and R-symbols). Other ground states can be obtained from this particular ground state when a topologically nontrivial operator is applied on it. This section is devoted to finding that operator: Our goal is to find tensor network operators (TNO) that can be deformed through the tensor network state. This deformability condition is a local condition which we call *pulling through* in the tensor network language. It has been schematically shown in Eq.(3.6), where A is the local tensor of the tensor network state and T is the tensor network operator acting on some of the virtual legs of the local tensor. On the left hand side, a TNO acts on one virtual leg of the local tensor, while on the right hand side three TNOs act on the other three virtual legs of the local tensor: TNO is pulled from one side to the other side of the local tensor, i.e., its shape is locally deformed.



(3.6)

Above is a general picture for tensor network states and deformable tensor network operators in $3+1$ dimensions. For Walker-Wang models, we already know the local tensor (see previous section). Hence, we proceed with finding the TNO that can be pulled through and show that for Walker-Wang models the pulling through condition (3.6) is nothing but the 2-4 Pachner equation that has to be satisfied by $3+1$-dimensional discrete TQFTs.

A detailed analysis reveals out that each move in the 2-4 equation corresponds to either the local tensor A or the TNO T. Local tensor A corresponds to the 4-1 Pachner move, and TNOs correspond to either 2-3 or 3-2 Pachner moves. This is indeed how we construct the local tensor and TNOs, then we automatically know that the particular pulling through equation 3.6 is nothing but the $2-4$ Pachner equation. This is also the case for other pulling through equations (where one pulls two TNOs from one side, to other two TNOs on the other side of the local tensor), and TNO-injectivity, where one corresponds to the 3-3 Pachner equation and the other corresponds to 2-4 Pachner equation. The invariance of the local tensor under the closed tensor network operator around the local tensor corresponds to the 1-5 Pachner equation.

Given this knowledge, let's construct the TNO from a 3-2 Pachner move. The move we want to realize is shown in Fig.3.14. The data we have however requires splitting the 4-valent vertices into 3-valent vertices, which has been performed in Fig.3.15, where we see the 3-2 Pachner move with 3-valent vertices. Finally we perform the 3-2 Pachner move. step by step, by using more elementary moves that are called F-move and R-move, as we did in the previous section in order to find the local tensor for the ground state of the Walker-Wang models. A detailed schematic picture of these steps are shown in Fig.3.16.

- **Step-1:** In this step, we apply a 3-1 F-move (We use G-symbols, which are the symmetric F-symbols), i.e., Eq.(3.4). We get the following contribution:

$$v_h v_j v_i G^{kml}_{hji}. \tag{3.7}$$

- **Step-2:** We apply the operation in Eq.(3.1) and get the contribution:

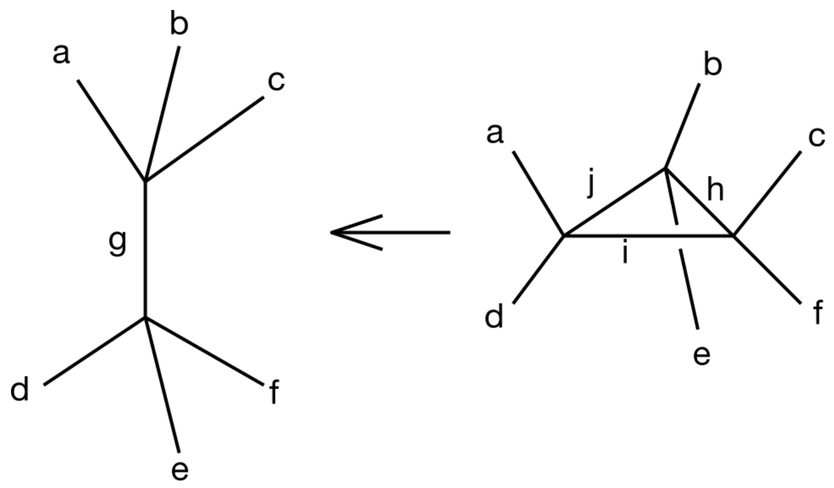


Figure 3.14 $3-2$ Pachner move in the dual picture.

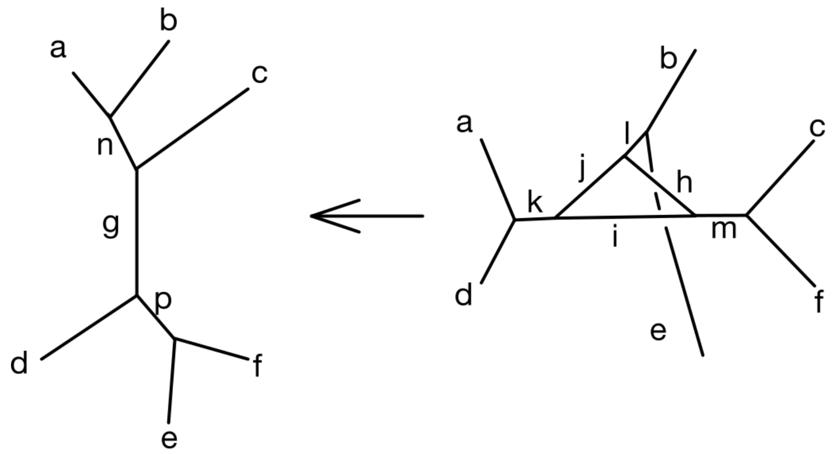


Figure 3.15 $3-2$ Pachner move when the 4-valent vertices are split into 3-valent vertices.

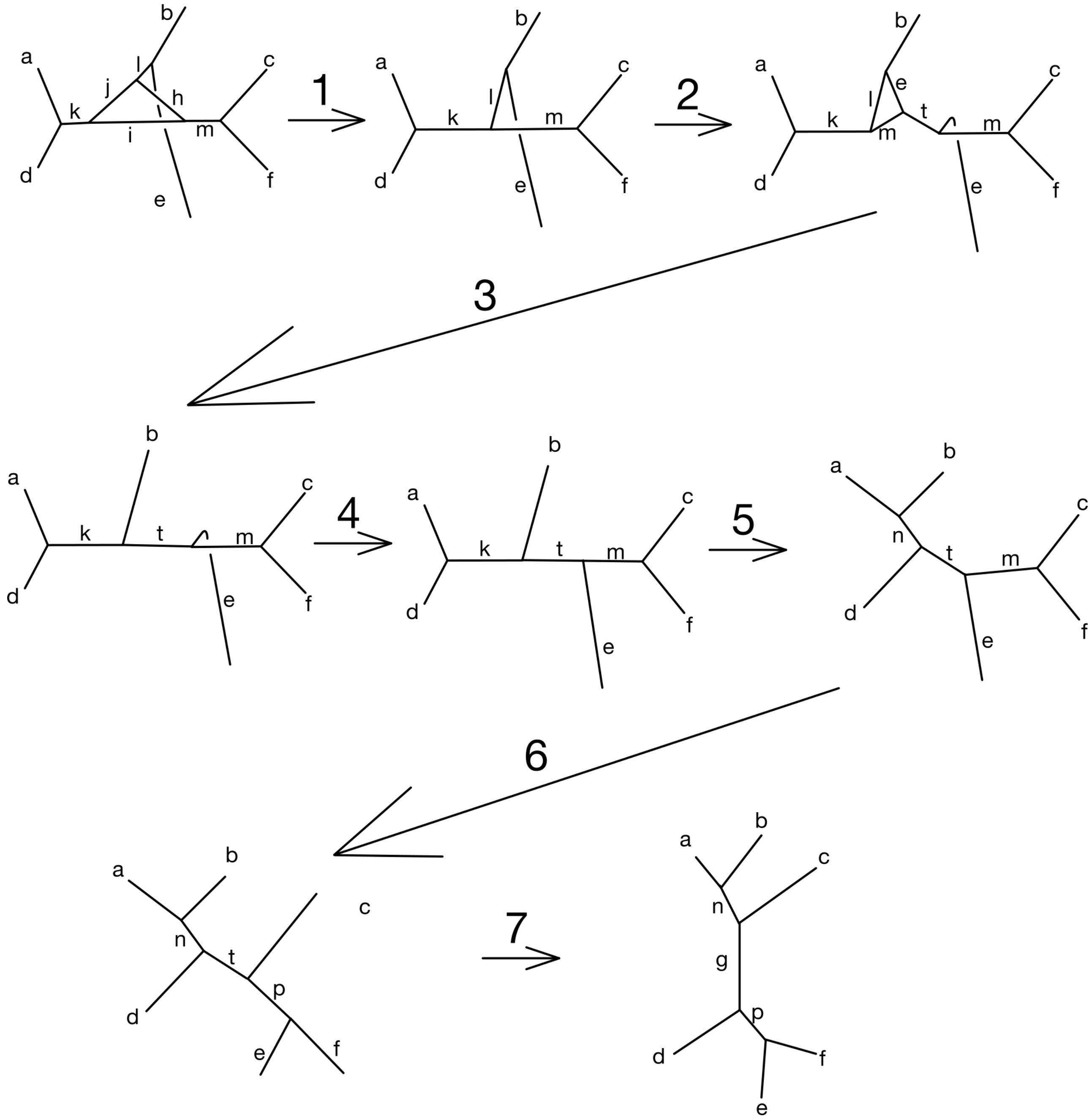


Figure 3.16 The elementary steps to derive $3-2$ Pachner move.

$$\frac{v_t}{v_e v_m} \tag{3.8}$$

- **Step-3:** We apply a 3-1 F-move as in Step-1 and get:

$$v_e v_l v_m G^{ktb}_{elm}. \tag{3.9}$$

- **Step-4:** We perform an R-move and get:

$$R^t_{me}. \tag{3.10}$$

- **Step-5:** We apply a 2-2 F-move and get:

$$v_k v_n G^{adk}_{tbn}. \tag{3.11}$$

- **Step-6:** We apply another 2-2 F-move and get:

$$v_m v_p G^{tem}_{fcp}. \tag{3.12}$$

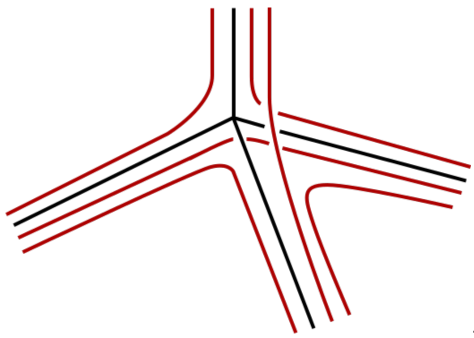
- **Step-7:** We apply the last 2-2 F-move and get:

$$v_t v_g G^{ndt}_{pcg}. \tag{3.13}$$

When we gather all these factors, we get the expression of the TNO T in terms of G- and R-symbols:

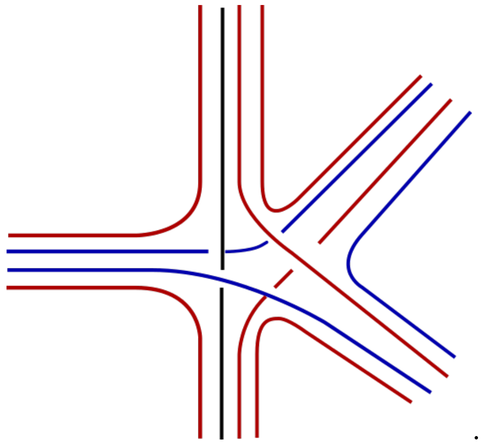
$$T^{abc,def,npg}_{hij,klm} = \sum_t d_t v_h v_j v_i v_k v_l v_m v_n v_p v_g G^{kml}_{hji} G^{ktb}_{elm} R^t_{me} G^{adk}_{tbn} G^{tem}_{fcp} G^{ndt}_{pcg}. \tag{3.14}$$

Schematically, before splitting the 4-valent vertices, the local tensor looks like:

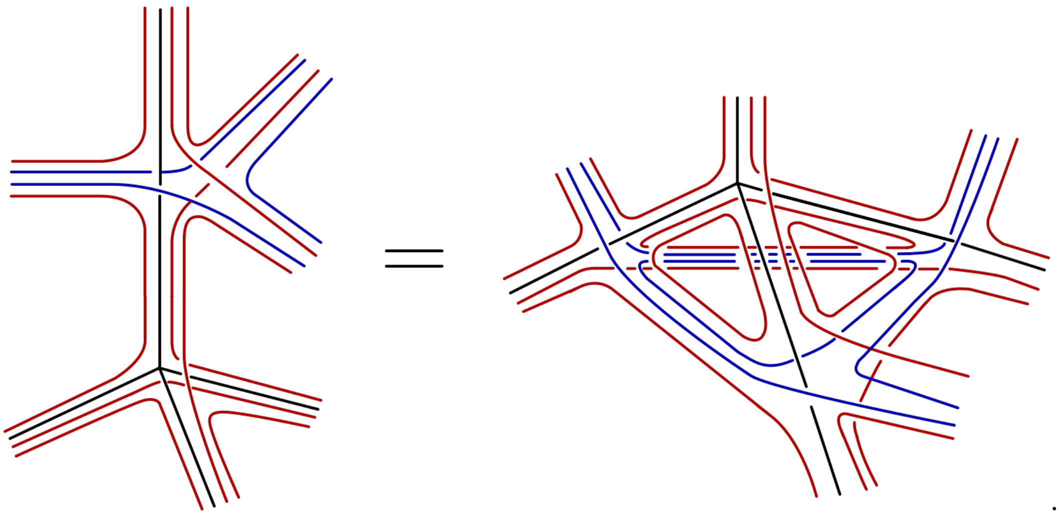


,

and the TNO looks like:



Pulling one TNO through the local tensor into three TNOs on the other side is then shown by the following equation:



Until now, we found the TNS and TNO (that can be deformed) in the bulk. If we put the model on a manifold without boundary, then these are all the tensor network data about the ground states of the model. We can generalize our findings to manifolds with boundaries: Imagine we put our tensor network on a 3-manifold with boundary. We have to close our tensor network with a tensor network state supported on the 2-manifold which is the boundary of the 3-manifold under consideration. We then may understand the physics of the boundaries by studying the properties of this boundary 2d-tensor network state. There is indeed more than one possible boundary state, and they may not all be expressible in term of a finite bond-dimension tensor network state. For those boundary states that are expressible in terms of a finite-bond dimension 2d tensor network state, the boundary may in principle be gapped or gapless. In either case, possible 1d-MPO terminations of the bulk 2d-TNO tells much about the boundary topological order. These boundary MPOs may even not be a termination of any nontrivial bulk TNO. Given these aspects about boundary states, we develop examples of them for Walker-Wang (and other models in $2+1$ and $3+1$ dimensions) and study the physical features of these states in the future parts of the work, yet to be written.

3.5 Conclusions

In this work, we started a first step of studying Walker-Wang models from the point of view of tensor network states and operators. We present a finite bond dimension tensor network state representation of ground states of the Walker-Wang models. Given the data used in a particular Walker-Wang model, i.e., F- and R-symbols, we showed that the local tensor and the tensor network operators that can be pulled through the lattice are given in terms of these data. In particular for Walker-Wang models, we also showed a correspondence between the notions of topological order in tensor network states to the notions of topological invariance in discrete lattice TQFTs: The tensor network conditions for topological order, i.e., TNO-injectivity and pulling through (which may lead to a trivial or nontrivial topological order) are equivalent to Pachner equations that have to be satisfied for the topological invariance of the partition function in a TQFT.

Ongoing work includes the study of possible boundary theories by finding the 2d boundary tensor network states which are invariant under the action of the 2d-TNO. In order to understand the topological order that is carried by the boundary state, it is important to reveal the MPOs (which may or may not be a termination of the bulk 2d-TNO) that can be pulled through the boundary tensor network state. Another important subject to study is the bulk and boundary excitations in these models, which depends on the nature of the input data of the model. It is important to understand these properties in terms of tensor network language, which originally depend on the features of the category that we use to construct these models.

Chapter 4

Summary and Outlook

We conclude the thesis with a short summary of achievements and an outlook towards future directions.

In the first chapter, which is based on the preprint [1], we characterized topologically ordered tensor network states in two space dimensions using matrix product operators. We introduced the novel concepts which we call MPO-injectivity and pulling-through condition with which fixed point models, e.g., Levin-Wen models, as well as models away from renormalization fixed points can be understood. This part of the thesis already led to some other preprints [40, 79], which are natural follow-ups of this work. In [40], the tensor network axioms of intrinsic topological order have been generalized to systems with global symmetries, e.g., symmetry protected topological order. In [79], the concept of matrix product operator algebras has been introduced which led to a natural way of understanding the excitations in topologically ordered tensor network states. In addition to these, the work in the first chapter enabled other coworkers to study topological phase transitions [83], since the framework is also valid away from renormalization fixed point and the virtual line/loop operators are not smeared out which makes the task accessible via tensor network techniques. Studying topological phase transitions in full generality is the most important future direction. There are several approaches: One [83] may approach the problem numerically by perturbing the local tensor and examining the expectation values of extended observables. Whenever a sudden jump is observed, we know that a phase transition occurs. A more detailed analysis also reveals out which anyons are condensed in the ground state by this procedure. Another way [88] to approach this problem is to place ground states of different phases on the same lattice, next to each other such that they are divided via an interface. One then writes down a set of consistency equations such that through the interface we have a local gapped Hamiltonian (indeed a local commuting projector Hamiltonian). It has been studied that some anyons are condensed on one side of the interface and can not penetrate to the other side and vice versa. One can even go further and place ground states of three different phases and study interfaces between every two of them, and the intersection point of these interfaces. This way can indeed be tackled analytically and we are working on it from the viewpoint of tensor networks.

In the second chapter, which is based on a recent work [2], we extend the characterization in the first chapter to any spacetime dimension. Discrete combinatorial TQFTs have been formalized in terms of tensor networks, the properties of the local tensor and tensor network operator have been extensively studied. As in the first chapter, tensor network framework allows us to go beyond the fixed-point models (i.e., TQFTs), and to axiomatize topological order for higher dimensional tensor network states by what we call TNO-injectivity and pulling-through conditions. We further relate the tensor network operators in lower dimensional submanifolds to the concept of higher categories and extended TQFTs. Promising future directions are indeed very diverse. The first and most obvious one is to come up with a new and interesting solution of the consistency conditions in $3+1$ dimensions. This may be tackled in two ways: One may try to solve the tensor equations by brute force and the help of computers. Although many experts/non-experts think that this is extremely hard to do, it is plausible if one tackles the problem with small-rank tensors (when the local Hilbert space dimension is low). Another way to approach the problem is to come up with a 3-category, we assign the objects (0-morphism) and 1-,2-,3-morphisms to 0-,1-,2-,3-cells on the lattice respectively. The tensor equations are then automatically satisfied. As hard as finding a new solution to consistency equations, is extracting the physical properties of the model that corresponds to the new solution. Understanding the physics of the model corresponds to an understanding of quasi-particle, quasi-loop (and higher dimensional) excitations of the model. This can be achieved via studying loop and surface operators, and analogously using tensor network operators and their local algebras.

In the third chapter, which is based on another recent work [3], we study Walker-Wang models [84] from the perspective of tensor networks. This is an explicit example model of the models discussed in the second chapter. The tensor network state is constructed using the data of the Crane-Yetter type TQFTs, and tensor network operators that are deformable on the lattice have been found. By construction, the TNO-injectivity and pulling through conditions are implied by Pachner equations, as expected. Important future directions include finding excitations in the bulk and on the boundary, understanding the braiding properties of these excitations and ultimately figuring out more physical models from these toy models so that a theoretical insight for topological insulators can be gained.

All in all, this thesis presented a tensor network framework for topologically ordered systems in any dimension and studied examples of this framework in $2+1$ and $3+1$

dimensions. On top of the understanding it provides, the thesis left us with many more open questions which hopefully will lead to more progress in the way we understand nature.

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