



universität
wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

VOLATILITY KNOCK OUT OPTIONS A PRICING GUIDE

verfasst von / submitted by

Stefan Müller, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements
for the degree of

Master of Science, MSc

Wien, 2016 / Vienna 2016

Studienkennzahl lt. Studienblatt /
degree programme code as it appears
on the student record sheet:

A 066 920

Studienrichtung lt. Studienblatt /
degree programme as it appears
on the student record sheet:

Masterstudium Quantitative Economics,
Management and Finance

Betreut von / Supervisor:

Univ.-Prof. Dr. Nikolaus Hautsch

Abstract

For risk managers and hedgefund managers likewise it is important to have a suited list of eligible financial products in order to be prepared for the increasingly difficult financial environment. Consequently, newly created exotic derivatives stream into the markets, which requires the market participants to first understand the inherited risks of complex exotics before including them to their possible investment vehicles. This thesis analyses the innovative Volatility Knock Out options and all necessary prerequisites, like the Vanna Volga method for the foreign exchange market, discretization methods for simulation and numerical integration for the Heston model. The Volatility Knock Out options exhibit interesting sensitivities and can be combined into a trading strategy for extracting the volatility risk premium.

Zusammenfassung

In einer immer komplexer werdenden Finanzwelt ist es für Risikomanager und Hedgefonds Manager von äußerst wichtiger Bedeutung über die passenden Finanzprodukte zu verfügen. Dies führt dazu, dass immer neue exotische Finanzprodukte entwickelt werden. Damit jedoch diese auch von den agierenden Managern verwendet werden können, müssen sie nachweislich die inkludierten Risiken in den exotischen Derivaten verstehen. Diese Masterarbeit analysiert die neue Sorte von exotischen Finanzprodukten namens Volatilität Knock Out Optionen. Darüber hinaus werden die notwendigen Methoden der Preisfindung, wie die Vanna Volga Methode, das Heston Model oder die numerische Integration, erörtert. Wie sich zeigt, weisen die Volatilität Knock Out Optionen äußerst interessante Parametersensitivitäten auf und eignen sich darüber hinaus auch zur Vereinnahmung der Volatilitäts Prämie.

Contents

1	Introduction	1
2	The Volatility Knock Out option	3
2.1	Definition Volatility Knock Out option	3
2.2	Realized volatility in the past	4
2.3	Motivation	5
3	Basics of financial Calculus	7
3.1	Brownian Motion	7
3.2	Martingale Property	8
3.3	Stochastic Calculus and Ito's Formula	8
3.4	Cameron-Martin-Girsanov Theorem	11
4	The Black-Scholes model	14
4.1	The Black Scholes framework and theory	14
4.1.1	Stock Price SDE	15
4.1.2	Bond Price SDE	15
4.2	The Black-Scholes Call Price	16
4.3	The Volatility Smile	18
5	The Heston Model	20
5.1	Definition Heston Model	20
5.2	BS-like Call Price	23
5.2.1	The Heston PDE	24
5.3	The Little Heston Trap	27
5.4	Numerical Integration Scheme	28
5.4.1	Gauss-Laguerre Quadrature	29
6	Foreign Exchange Market Data	30
6.1	Market Conventions	30
6.1.1	Market Quotes	31
6.1.2	Delta types	32
6.2	The Vanna Volga Method	35
7	Calibration	42
7.1	Loss Functions	42
7.1.1	General approach	43
7.1.2	MSE loss function	43

7.1.3 RSME loss function	44
7.1.4 IVMSE loss function	44
7.1.5 Christoffersen loss function	45
7.2 Estimation verification	45
7.2.1 Nelder-Mead	46
7.2.2 Differential Evolution	48
7.2.3 Results	49
8 Simulation	53
8.1 Monte Carlo simulation for the Heston model	53
8.1.1 Euler scheme	54
8.1.2 Milstein scheme	56
8.1.3 Exotics simulation	57
9 Volatility Knock Out option analysis	60
9.1 Volatility premium strategy	64
10 Conclusion	68
Bibliography	68
Curriculum Vitae	73

List of Figures

2.1	Historical realized volatility analysis	5
4.1	Volatility smile for FX options	18
4.2	Risk neutral probability density function	19
5.1	Variable effects on implied Volatility	22
5.2	Discontinuity of the Integrands	28
6.1	FX convention table	34
6.2	Foreign exchange volatility market data	34
6.3	Comparison of first and second order interpolated smile against market smile in EURUSD.	38
6.4	Constructed implied volatility surface	40
7.1	Calibration results	50
7.2	Calibration Fits - 2Y Maturity	51
8.1	Simulated sample paths for underlying and variance of EUR- USD exchange rate.	57
9.1	Histogram of realized volatility	61
9.2	Expected payoff for a Volatility Knock Out option	62
9.3	Expected delta and vega effects	62
9.4	Mark to market of a Volatility Knock Out option after 1 year .	63
9.5	Knock out probabilities for a Volatility Knock Out option after 1 year	64
9.6	Foreign exchange rate from 2010 until 2015. EURUSD Curncy	65
9.7	Accumulated inflows by trading activity	66
9.8	Accumulated outflows by trading activity	66
9.9	Total return graph of trading strategy	67
9.10	Realized - implied volatility spread	67

1 Introduction

For several years it is becoming increasingly difficult to keep up with the speed of innovations of the financial market and financial calculus likewise. The previous years are branded by newly created exotic financial products (exotics) as well as innovative and complex methods to measure and quantify these derivatives. This trend towards exotics is driven by a rising demand for highly specific cash flows to meet the market participants' individual risk profiles. One of these products are "Volatility Knock Out options", which are the topic of this thesis.

The reasoning behind this academic writing is neither to derive new valuation methods nor to come up with a new product idea but, on the contrary, to gather available material from a vast amount of academic writings, scientific papers and literature to give some guidance for "Volatility Knock-Out options" pricing on the foreign exchange market. Starting from the beginning, the pricing process may lead to various problems along the way and to keep an overview is not an easy task.

Even though this thesis will offer a full working manual to price a "Volatility Knock Out option", each individual step might be improved by a different academic approach. Nevertheless this is not the purpose of this thesis and even though higher accuracy perhaps could be achieved, the methods introduced in this paper are of sufficient accuracy for research purposes of the new product.

The procedure in which this product is implemented will be based on the foreign exchange market, but the interested reader will have no difficulties to apply similar methods to the equity or the commodity market. More precisely we will investigate a "Volatility Knock Out option" on the exchange rate between the Euro and the US Dollar. However a simulation for interest rates are not within the capabilities of the chosen model and thus readers will be referred to the LIBOR market model.

This thesis will start with an introduction of the "Volatility Knock Out option" in chapter 2, followed by the basics of financial calculus in chapter 3. Many crucial and required results that will be needed throughout the thesis will be introduced and derived. After that, the model by (Black and Scholes, 1973) and its shortcomings will be discussed in chapter 4. Chapter 5 then introduces the model by (Heston, 1993). Chapter 6 is about the fundamentals of the foreign exchange market and a guiding on how to extract volatility information of a currency pair. And finally Chapter 7 will deal with the calibration process for the Heston model. A chapter about the Monte Carlo simulation and necessary discretization methods follows. The ending of this thesis will be an analysis of the "Volatility Knock Out option", which will be followed by the conclusion.

2 The Volatility Knock Out option

The Volatility Knock Out option (short VoKo) is a rather new financial product in the market and closely resembles a standard option defined in (Hull, 2009). According to the classical definition of an option, a put or call option gives the contract's buyer the right to sell or buy a specific quantity of an underlying to a predefined date at a price negotiated in advance. The innovation of VoKos compared to a basic option is its additional dependency on the underlying's volatility, which changes the dynamics of the product fundamentally.

Since the product is new to the market, there does not yet exist a liquid market and hence all contracts depend on individually negotiated, private contracts traded over the counter (OTC).

2.1 Definition Volatility Knock Out option

The contract is closely related to a standard vanilla European option, such that the pay-out function does not deviate at all and is the same as the classical definition. The difference is the restriction on the underlying's realized volatility formulated as an additionally knock-out (knock-in) condition. Formally the definition is as follows:

$$Payout = \Pi_{EuropeanOption} * \mathbb{1}_{\sigma_{RV} < \sigma_{Budget}} \quad (2.1)$$

and

$$\begin{aligned}
 \Pi_{EuropeanPutOption} &= \max(Strike - Underlying, 0) \\
 \Pi_{EuropeanCallOption} &= \max(Underlying - Strike, 0) \\
 \sigma_{Budget} &= \text{a arbitrary volatility strike level} \\
 \sigma_{RV} &= \sqrt{252} * \sqrt{\frac{1}{N} * \sum_{i=1}^N x_i^2} \dots \text{realized volatility} \\
 X_i &= \text{i-th fixing of the relevant underlying} \\
 x_i &= \ln(X_{i+1}/X_i) \\
 N &= \text{number of spot fixings} - 1
 \end{aligned}$$

1 illustrates the indicator function.

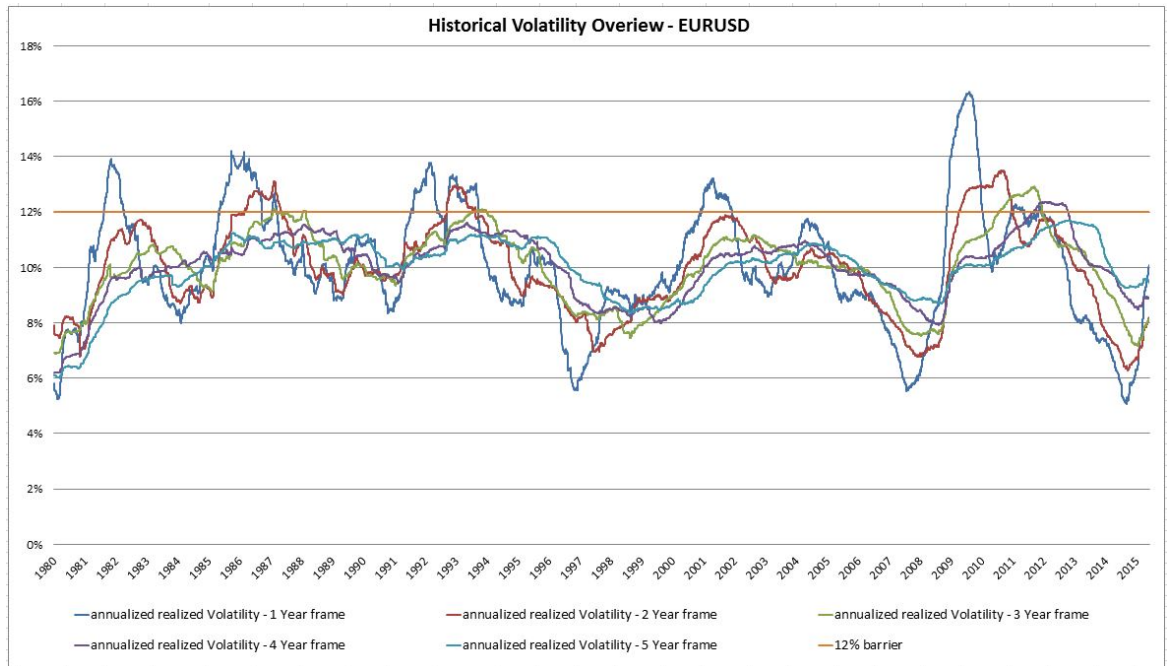
Since the VoKo is an European option with a restriction, the fair premium for this OTC contract has to be bounded from above by the price for the vanilla option. On the other hand it is also obvious, that compared to the European option, the investor who is willing to enter this contract expects a certain discount due to the additional probability of getting knocked out and not receiving a payment at all. This leads to the basic summary that the fair premium of a VoKo, independently from the volatility level, is bounded between 0 and the vanilla price. This basic statement seems to be intuitively true without limitations.

2.2 Realized volatility in the past

First we conduct an analysis of the realized volatility for the history of the EUR-USD exchange rate. The graphical analysis shows that most of the time the realized volatility for different time horizons remained below 12%. Obviously we compare the annualized realized volatility to account for different windows, because no meaningful comparison would be feasible otherwise.

Apart from the year 2008, we observe a volatility below 14 %. We also have calculated "knock out" probabilities based on this data in the figure (2.1b). For example, setting the volatility budget constraint (volatility knock out barrier) above 17% would have resulted in 0 knockouts and hence only a marginal discount compared to the vanilla option premium can be ex-

pected. On the other hand lowering the restriction below 5% almost surely results in a knockout and thus no reasonable price would be paid for a contract with this knock out barrier. These dynamics of the product seem very intuitive, nevertheless to get better insights further research will be conducted in this thesis.



(a) Historical annualized realized volatility graph

12%	1Y	2Y	3Y	4Y	5Y	since 2000	1Y	2Y	3Y	4Y	5Y
Below Barrier	7504	8235	8694	8945	9239		3341	3574	3650	3727	4021
Total	9239	9239	9239	9239	9239		4021	4021	4021	4021	4021
% below	81,22%	89,13%	94,10%	96,82%	100,00%		83,09%	88,88%	90,77%	92,69%	100,00%
min	5,078%	6,276%	6,895%	6,194%	6,004%		5,078%	6,276%	7,152%	7,910%	8,652%
max	16,362%	13,505%	12,922%	12,376%	11,685%		16,362%	13,505%	12,922%	12,376%	11,685%
average	10,047%	9,979%	10,032%	10,030%	9,993%		9,967%	9,968%	10,087%	10,129%	10,103%
stddev	2,169%	1,663%	1,345%	1,184%	1,107%		2,425%	1,841%	1,387%	1,060%	0,834%

(b) Historical annualized realized volatility table

Figure 2.1: Historical realized volatility analysis

Source: Own calculations

2.3 Motivation

One might raise the question of the necessity of such a payout profile or if market participants do really require such exotic cash flow profiles. But it

happens to be the case that there actually is a demand for this product. In times of decreasing yields on fixed income securities and simultaneously increasing capital demands by regulators for risky asset classes like equity, institutional and private investors likewise are gravitating for high yielding alternatives.

The supposed solution often seems to be absolute return funds. Their claim is to be able to achieve a specific return, given in percent, independently from any directional risk. One way they try to achieve this is by taking advantage of disruptions in volatility markets and at the same time hedging other risk factors cost efficiently. This is one of the reasons why VoKos do exist. Since volatility knock out barriers are calibrated on the implied volatility of the exchange rate, but at maturity the barrier is compared to realized volatility, this product adds another possibility to gain from potential mismatches of the implied realized volatility market.

Furthermore the product might also work for soft hedging. Given that most asset classes show a negative volatility skew, we can therefore conduct, that if we combine the realized volatility barrier with a put option, we end up with a cheaper hedge, compared to a plain vanilla put option, for rather steady decreases in the underlying. Since a very steep and turbulent fall of the exchange rate might lead to a knock out of our barrier we would not be hedged for tail events, nevertheless we find ourselves hedged against soft weakening of the currency in a very cost efficient way.

We now review some of the most important results of mathematical finance, which will be used to derive the Black Scholes and the Heston model.

3 Basics of financial Calculus

This chapter gives a brief review of concepts from mathematical disciplines that will be essential for further derivations. All calculations found in this chapter are based on existing literature like (Biagini et al., 2008), (Klebaner et al., 2005) and further sources that will be mentioned accordingly.

3.1 Brownian Motion

A Brownian Motion is a stochastic process $W_t : 0 \leq t \leq 1$ on a probability space $(\Omega, \mathbb{F}, P)$ adapted to $\mathbb{F}_t$ with the following properties:

- $W_0 = 0$,
- W_t has independent increments, for any $0 < t_1 < \dots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$ are independent,
- for any $0 < s < t$, the independent increment $W_t - W_s$ is a normal random variable with variance $\mathbb{E}[(W_t - W_s)^2] = t - s$. Furthermore, W_t is normally distributed with mean 0 and variance t , thus $W_t \sim \mathbb{N}(0, t)$ -distributed
- With probability 1 W_t is continuous

A filtration $\mathbb{F}_{t, t \geq 0}$ is a family of sub-sigma algebras of some sigma-algebra $\mathbb{F}$ that if $s < t$ it follows that $\mathbb{F}_s \subset \mathbb{F}_t$. Saying a process is adapted to a filtration means that the process X_t is $\mathbb{F}_t$ measurable for any t . A more detailed description and the mathematical proofs can be found in (Biagini et al., 2008).

Brownian motions are the basic tool any financial simulation is based on. It helps to artificially produce probable paths for any underlying and hence should be part of the basic repertoire of any financial engineer.

3.2 Martingale Property

Let $(\Omega, \mathbb{F}, P)$ be a probability space. A martingale sequence of length n is a chain $X_1, X_2, \dots, X_n$ of random variables and corresponding sub σ -fields, $\mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_n$ that satisfy the following relations:

1. Each X_i is an integrable random variable which is measurable with respect to the σ -field $\mathbb{F}_i$
2. The σ -field $\mathbb{F}_i$ are increasing i.e. σ -field $\mathbb{F}_i \subset \mathbb{F}_{i+1}$ for every i .
3. For every $i \in [1, 2, \dots, n - 1]$, we have the relation

$$X_i = E[X_{i+1} | \mathbb{F}_i]$$

Simply put, a martingale is the exact mathematical definition for a 'fair game', such that the best estimate for future outcomes of a game in the mathematical sense, is the starting value. At any point in time i , given the filtration $\mathbb{F}_i$, the best estimate for the expected value of X_i is X_i .

This is a crucial concept for calculating expected payoffs and hence a fundamental part of option valuation.

3.3 Stochastic Calculus and Ito's Formula

Assuming we model the risky asset according to a Brownian motion B_t . Furthermore an investor is only able to buy and sell shares of this asset to given points in time $0 = t_0 < \dots < t_n$. At each point t_k the investor decides on a number of shares X_k he has to hold at least until t_{k+1} . At the end of the trading periods t_n his total wealth is equal to:

$$\sum_{i=1}^n X_{i-1} (B_{t_i} - B_{t_{i-1}})$$

Considering now that we decrease the distance between the time points we measure the wealth before. Basically the sum then converges to a normal integral. Due to the time dependency and the random movements driven by a Brownian motion, it is considerably different to the usual integration. The following mathematical concept is crucial for different classes

of stochastic processes and is considered to be the basis of the stochastic differential equation theory more closely derived in (Pelsser, 2000). An intuitive description can also be found in (Mbele, 2004) or (Klebaner et al., 2005)

A $\mathbb{F}$ -adapted process $X = (X_t)_{t \in [0, T]}$ is called *step process* or *elementary process* on the time interval $[0, T]$ if there exists a partition $P = 0 = t_0 < \dots < t_n = T$ such that $X_t \equiv x_k$ is constant for $t_k \leq t < t_{k+1}$; $k = 0, \dots, n-1$.

We define the space of all square integrable real-valued $\mathbb{F}$ -adapted processes X such that the square integral from a to b is finite. We will further define the class of all functions where $E[\int_a^b X^2(t)dt]$ is finite as $\mathbb{S}_2(a, b)$.

Let X be an adapted square-integrable step process, then the *Ito Integral* of the step process for interval $[0, T]$ is defined as following

$$\int_0^T X dW = \sum_{k=0}^{n-1} X_k (W_{t_{k+1}} - W_{t_k}) \quad (3.1)$$

Given a process $X \in \mathbb{S}_2(0, T)$ then there exists a elementary process $X_n \in \mathbb{S}_2(0, T)$ such that X is the limit of X_n . A more detailed formulation is as following:

$$\lim_{n \rightarrow \infty} \int_0^T |X - X_n|^2 dt = 0 \text{ a.s.} \quad (3.2)$$

So we've given a definition for an integral of an elementary process in equation (2.1) and have stated, that for every process $X \in \mathbb{S}_2$ there exists a step process which converges at it's limit to the process X . This is called the *Ito Integral*.

$$\int_0^T X dW := \lim_{n \rightarrow \infty} \int_0^T X_n dW \quad (3.3)$$

As an example we apply the *Ito Integral* on a simple process to show how it is used.

$\int dW$ is the most basic process we can apply this fundamental result of financial mathematics on. As long as an Ito integral resembles the ordinary calculus rules, we assume it to be W , which exactly is the case as shown

below.

$$\begin{aligned}\int_a^b dW &= \sum_{t_i} W(t_{i+1}) - W(t_i) \\ &= W(b) - W(a)\end{aligned}$$

The next example to show an Ito integral is $\int W dW$. First we have to check if W is square-integrable.

$$E \left[\int_0^t W^2(s) ds \right] = \int_0^t E[W^2(s)] ds = \int_0^t s ds,$$

The integral of any constant over an finite interval is clearly finite.

If ordinary rules apply once more, we expect the integral to be $\int_0^t W dW = \frac{1}{2} W^2(t)$. However this is not feasible, because a square-able and adapted process has to be a martingale, but by taking the expectations of the previous result, we end up with a value of $\frac{1}{2}t$. A subtraction of $t/2$ to come up with $\int W dW = \frac{1}{2} W^2 - \frac{t}{2}$, surprisingly yields the correct solution. Even though the approach seems naive, it happens to be right and can be shown by a cumbersome approximation via elementary processes, which is shown in (Klebaner et al., 2005). Therefore a basic change in variables from $\int W dW$ to $\frac{1}{2} \int dW^2$ is not possible.

A central theorem of stochastic calculus provides a method to efficiently derive stochastic integrals. But first we define an *Ito process*. Any adapted stochastic process X is called an *Ito process* if there exists a function F , with $\int_0^T |F| dt < \infty$ and G in $\mathbb{S}_2$ such that for for all $0 \leq s \leq r \leq T$, we can say

$$X(r) = X(s) + \int_s^r F(t) dt + \int_s^r G(t) dW \quad (3.4)$$

is an Ito process and is also written as $dX = F dt + G dW$.

Accordingly *Ito's Formula*, as a main pillar for stochastic calculus, is defined as: For an Ito process X with $dX = F dt + G dW$, let $u(t, x) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ be a function with continuous partial derivative $\frac{\partial u}{\partial t}$ and continuous second derivatives $\frac{\partial^2 u}{\partial x^2}$. Then $Y(t) = u(t, X(t))$ is an Ito process and

$$dY = \frac{\partial u}{\partial t}(t, X(t)) dt + \frac{\partial u}{\partial x}(t, X(t)) dX + \frac{1}{2} G^2(t) \frac{\partial^2 u}{\partial x^2}(t, X(t)) dt \quad (3.5)$$

that is,

$$dY = \frac{\partial u}{\partial t}(t, X(t))dt + \frac{\partial u}{\partial x}(t, X(t))Fdt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, X(t))G^2dt + \frac{\partial u}{\partial x}(t, X(t))GdW \quad (3.6)$$

To come back to the example from above, we are applying Ito's formula on $\int W dW$. Let the Ito process be $X(t) = W(t)$, by setting F equal to zero, G equal to 1 and using the function $u(t, x) = x^2/2$. Then the previous problem becomes far less difficult.

$$\begin{aligned} dY &= \frac{\partial u}{\partial t}(t, X(t))dt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, X(t))G^2dt + \frac{\partial u}{\partial x}(t, X(t))GdW \\ dY &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dW + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}dt \\ \frac{1}{2}dW^2 &= WdW + \frac{1}{2}dt \\ \frac{1}{2} \int dW^2 &= \int WdW + \frac{1}{2} \int dt \\ \int_0^t W(s)dW &= \frac{1}{2}W^2(t) - \frac{t}{2} \end{aligned}$$

To illustrate the idea of stochastic differential equations, we once again assume $X(t) = W(t)$, but change $u(t, x)$ to $u(t, x) = e^{\lambda x - \lambda^2 \frac{t}{2}}$. Ito's Formula then leads to:

$$e^{\lambda x - \lambda^2 \frac{t}{2}} = \left(-\frac{\lambda^2}{2} e^{\lambda x - \lambda^2 \frac{t}{2}} + \frac{\lambda^2}{2} e^{\lambda x - \lambda^2 \frac{t}{2}} \right) dt + \lambda e^{\lambda x - \lambda^2 \frac{t}{2}} dW$$

Since the first and second term distinguish we end up with the solution of the stochastic process $Y_t = e^{\lambda W_t - \lambda^2 \frac{t}{2}}$:

$$\begin{aligned} dY &= \lambda Y dW \\ Y_0 &= 1 \end{aligned}$$

This idea is an important corner stone of modern mathematical finance, and will be used many times throughout the whole thesis.

3.4 Cameron-Martin-Girsanov Theorem

In general most stochastic processes are not martingales under the real world probability measure. Girsanov's Theorem is used to change a probability measure such that the given stochastic process becomes a martin-

gale. This is very important to further handle the stochastic equations and is a crucial step used several times in the Black Scholes model and the Heston model respectively. This section follows (Mbele, 2004) and (Pelsser, 2000).

Given a market of several bonds with price processes $B_1(t), \dots, B_n(t)$ which are modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the processes $B_i(t)$ following an Ito process

$$dB_i = \mu(t, w)dt + \sigma(t, w)dW$$

A *numeraire* then is an strictly non-negative asset B which is used to discount other asset price processes. In finance numeraires are often savings accounts or bonds appreciating with the risk-free rate. Given a specific numeraire we can express all assets B_i in relative prices and thus being able to compare them pairwise.

An *equivalent martingale measure* (with respect to B) is a probability measure $\mathbb{Q}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F})$ such that

- $\mathbb{Q}$ and $\mathbb{P}$ have the same null-set
- the discounted price process $B'_i = B_i/B$, $i = 1, \dots, n$ are martingales under $\mathbb{Q}$

Change of Numeraire

If $\mathbb{Q}$ and $\mathbb{Q}^*$ are two equivalent martingale measures associated to the numeraires B and B^* , then for all A in $\mathbb{F}$ we have

$$\mathbb{Q}(A) = \int_A \rho(t) d\mathbb{Q}^*$$

where $\rho(t) = \frac{B(T)/B(t)}{B^*(T)/B^*(t)}$ is a random variable called *Radon–Nikodym derivative* and is denoted as $\frac{d\mathbb{Q}}{d\mathbb{Q}^*}$.

Girsanov's Theorem

Let $f(t)$ be a square integrable stochastic process adapted to $\mathbb{F}_\approx$. Then

$$W_t^* = W_t - \int_0^t f_s ds$$

is a martingale with respect to a equivalent martingale measure $\mathbb{Q}^*$ given by

$$d\mathbb{Q}^* = \rho(t)d\mathbb{Q} = \exp\left(\int_0^t f(s)dW_s - \frac{1}{2}\int_0^t f(s)^2 ds\right) d\mathbb{Q}$$

Hence the Girsanov's Theorem enables us to find a probability measure, so that a certain process, becomes a martingale. As mentioned before this is necessary for the Ito process and combined it creates the possibility to efficiently handle stochastic models like the Black Scholes model.

The presented results are part of every curriculum of finance and should serve as a quick reminder for the following derivations of the Black Scholes framework, which will be conducted in the next chapter. Also the Heston model heavily relies on the Ito calculus and the measurement change by Girsanov.

4 The Black-Scholes model

After reviewing selected theorems and fundamental ideas of mathematical finance, we now move to the basis of most pricing theories. This chapter begins with the famous and well known model by (Black and Scholes, 1973) which, despite all shortcomings and innovations of other models, is even nowadays undoubtedly of great importance. The suggested framework still serves as the starting point for many pricing methods of derivatives in the commodity, foreign exchange, rates and equity market. Despite the model's restrictive assumptions revised in (Hull, 2009), the significance of the Black Scholes equation is undeniable and should be part of every derivatives pricing lecture. This chapter is mostly based on (Rouah, 2011b) and (Hull, 2009).

4.1 The Black Scholes framework and theory

The existence of two assets is assumed in the Black Scholes framework: S is the risky asset, the stock and B is the risk-less bond. The differential equations describing the price of a bond at a specific point in time t , is defined as followed:

$$dB_t = r_t B_t dt$$

where r is the constant instantaneous interest rate for borrowing and lending funds. The stochastic differential equation for the risky asset can be written as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (4.1)$$

where $\mu \in \mathbb{R}$ is a constant mean return and $\sigma > 0$ is a constant volatility. $(W_t)_{t \geq 0}$ is a standard Brownian motion.

4.1.1 Stock Price SDE

With Ito's Formula applied on the logarithm of equation (4.1), the SDE of $\ln S_t$ is:

$$d \ln S_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \quad (4.2)$$

Integrating from 0 to t receive the total relative change leads to:

$$\int_0^t d \ln S_u = \int_0^t \left(\mu - \frac{1}{2} \sigma^2 \right) du + \sigma \int_0^t dW_u$$

such that

$$\ln S_t - \ln S_0 = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t$$

since $W_0 = 0$. Therefore the solution of the stock price's SDE equals:

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (4.3)$$

with W_t being a Wiener process which is normally distributed such that $W_t \sim N(0, t)$ and as a consequence it follows that the random variable $\ln S_t$ is drawn from a random distribution with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t$ and variance $\sigma^2 t$. This implies that the process S_t is log-normally distributed with mean $S_0 e^{\mu t}$ and variance $S_0^2 e^{2\mu t} (e^{\sigma^2 t} - 1)$. Furthermore the integration from t to T instead of 0 to T is straight forward and similar to the stated above.

4.1.2 Bond Price SDE

By applying Ito's Lemma again to the riskless bond, we end up with:

$$d \ln B_t = r_t dt \quad (4.4)$$

Integration from 0 to t yields:

$$\ln B_t - \ln B_0 = \int_0^t r_u du.$$

If $B_0 = 1$ and a constant interest rate the stochastic differential equation sums up to

$$B_t = B_0 \exp \left(\int_0^t r_u du \right) = e^{rt} \quad (4.5)$$

In the Black-Scholes framework the stock price discounted by the bonds, which works as the numeraire, $\tilde{S}_t = \frac{S_t}{B_t}$ should be a martingale. Thus we need to change the measurement to $\mathbb{Q}$, where $W_t^{\mathbb{Q}} = W_t + \frac{\mu - r}{\sigma}t$ gives us:

$$\begin{aligned} E^{\mathbb{Q}}[\tilde{S}|\mathbb{F}_s] &= \tilde{S}_0 \exp(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{Q}}) \exp(\frac{1}{2}\sigma^2(t-s)) \\ &= \tilde{S}_0 \exp(-\frac{1}{2}\sigma^2 s + \sigma W_s^{\mathbb{Q}}) \\ &= \tilde{S}_s \end{aligned}$$

which shows that $\tilde{S}_t$ is a $\mathbb{Q}$ -martingale. This leads to the implication, that the expected value of the stock price S_T at any point in time $t \leq T$ is S_t , or more formally: $E^{\mathbb{Q}}[S_T|\mathbb{F}_t] = S_t$.

4.2 The Black-Scholes Call Price

Given the bond's and stock's SDE in addition to the measure under which the stock price is a martingale, we can formulate and later on derive the fair premium of an European call option. Assuming a constant interest rate r , the fair premium of a European call option on a non-dividend paying stock, with Strike K and maturity $\tau = T - t$ is

$$C(S_t, K, T) = e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)^+|\mathbb{F}_t], \quad (4.6)$$

where $(S_T - K)^+$ means $\max(S_T - K, 0)$. Therefore the formula is giving the expected intrinsic value of the option at maturity discounted to the day of valuation, taken into account all information available at the specific point in time t . Remembering that the discounted European call price equals a martingale under measurement $\mathbb{Q}$ we can see that:

$$C(S_t, K, T) = e^{-r\tau} \int_K^{\infty} (S_T - K) dF(S_T) \quad (4.7)$$

$$= e^{-r\tau} \int_K^{\infty} S_T dF(S_T) - e^{-r\tau} K \int_K^{\infty} dF(S_T) \quad (4.8)$$

Since the terminal stock price S_T under measurement $\mathbb{Q}$ and a point in time t follows a lognormal distribution with mean $\ln S_t + (r - \frac{\sigma^2}{2})\tau$ and variance $\sigma^2\tau$ with $\tau = T - t$. The integrals can be formulated as a conditional expectation

with $S_T > K$:

$$\begin{aligned}
 \int_K^\infty S_T dF(S_T) &= E^{\mathbb{Q}}[S_T | S_T > K] \\
 &= \exp\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2\tau}{2}\right) \\
 &\quad \Phi\left(\frac{-\ln K + \ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) \\
 &= S_t e^{r\tau} \Phi(d_1)
 \end{aligned}$$

with $d_1 = \left(\frac{-\ln K + \ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)$. More detailed derivations are explained in (Hogg and Klugman, 2009).

The second integral can be transformed in the same manner and holds that:

$$\begin{aligned}
 e^{-r\tau} K \int_K^\infty dF(S_T) &= e^{-r\tau} K [1 - F(K)] \\
 &= e^{-r\tau} K \left[1 - \Phi\left(\frac{\ln K - \ln S_t - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] \\
 &= e^{-r\tau} K [1 - \Phi(-d_2)] \\
 &= e^{-r\tau} K \Phi(d_2)
 \end{aligned}$$

with $d_2 = d_1 - \sigma\sqrt{\tau}$. Substituting the results for the two integrals into equation (4.8) we arrive at the well known formula from Black and Scholes to price European Call options.

$$C(S_t, K, T) = e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)^+ | \mathbb{F}_t] = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \quad (4.9)$$

with $d_1 = \left(\frac{-\ln K + \ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

The prices for a put option is derived by

$$P + S_0 = C + K e^{-rT} \quad (4.10)$$

This relationship is known as the put-call parity and enables to translate call into put prices and vice versa.

4.3 The Volatility Smile

Within the pricing framework of the Black Scholes equation the definition of an option price requires five input parameters. The interest rate r , the underlying's spot price S_t , the strike K , the time to maturity $\tau = T - t$ and the volatility σ . The interest rate and the underlying are exogenously defined and hence can not be changed by the calculation agent. The time to maturity and the strike price on the other hand are negotiable but once well defined not changeable anymore.

Furthermore given market values of liquid, traded options and accepting the arbitrage free structure of the market, there exists one unique value for the volatility to match the market price frictionless. This value is called the implied volatility, since the option price implies this specific level of volatility.

The Black Scholes model suggests a constant volatility, however by calculating the implied volatilities for the market values, we see that the implied volatility is variable.

Plotting the implied volatility against a set of strike prices for a specific maturity is called 'volatility smile'. Adding a third dimension to the graph by plotting different maturities on the additional axis, creates the 'volatility surface'.

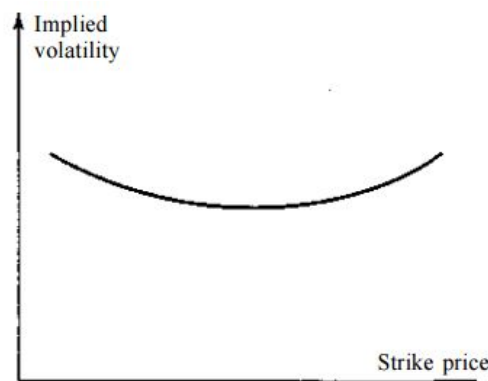


Figure 4.1: Volatility smile for FX options

Source:(Hull, 2009)

The reason why the volatility is not constant among different strikes is because of the underlying concept of the risk-neutral lognormal distribution

of returns. Since the asset's volatility tends to change if the market environment becomes increasingly turbulent and furthermore the price path is not free from price jumps, the concept of log normality is not flawless and at best solely an approximation of reality. The density function in figure (4.2) with higher curtosis is supposed to be closer to real price behavior and thus driver for the non flat implied volatility.

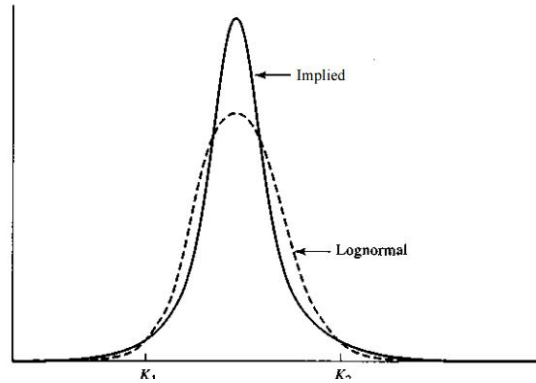


Figure 4.2: Risk neutral probability density function

Source: (Hull, 2009)

These problems of non efficient assumptions of volatilities, can be partly addressed by using correct implied volatility but for pricing vol-dependent options the Black Scholes model would be forced beyond its limitations. Hence, we substitute this with a more complex model which can reproduce the implied volatility surface in a more sophisticated manner.

5 The Heston Model

As shown in the previous chapter the Black Scholes model is only partly capable of dealing with variable volatilities. Nevertheless for many purposes the Black Scholes framework does the trick. However, to price volatility dependent structures which are influenced by the whole volatility surface, a switch to a more complex model is necessary. Hence we substitute the Black Scholes model with the Heston model which is a representative of the class of stochastic volatility models.

Stochastic volatility models neither use a constant volatility as an input factor, nor a deterministic function to calculate time dependent volatility, but model the variability of the variance according to a stochastic differential equation. The SDE of the stochastic volatility part is fitted to the observable implied volatility surface and hence mathematical engineers can simulate volatility dependent payoffs by randomly drawing realizations of the volatility path. Prices for variance or volatility swaps can be found in this manner.

The Heston model is the used stochastic volatility model in this thesis due to its wide acceptance by market participants. One of the reasons might be the Heston model's easy tractability and it's robustness with calibration. This chapter is widely based on (Rouah, 2013), (Mikhailov and Nögel, 2004) and (Gatheral, 2011).

5.1 Definition Heston Model

The dynamics of the processes are as following:

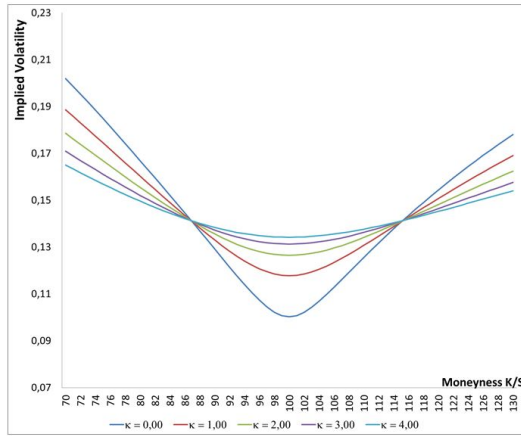
$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)} \\ dv_t &= \kappa(v_t - \theta) dt + \xi \sqrt{v_t} dW_t^{(2)} \end{aligned} \tag{5.1}$$

and

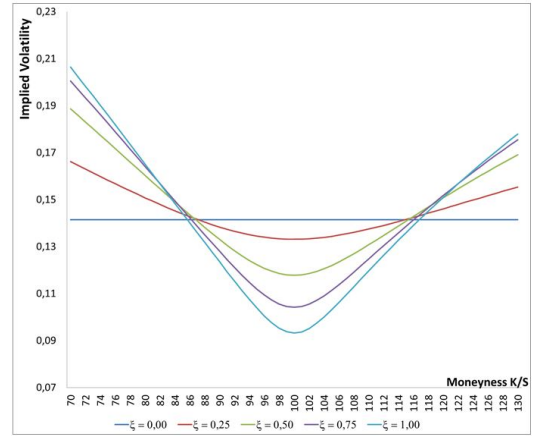
$$\langle W_t^{(1)} W_t^{(2)} \rangle = \rho,$$

where $-1 < \rho < 1$ is the correlation between the Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$, $\kappa > 0$ is the mean-reversion rate, $\theta > 0$ is the long-term variance and $\xi > 0$ is the volatility of volatility. Furthermore $v_0 > 0$ is the initial variance, which is the process's start value.

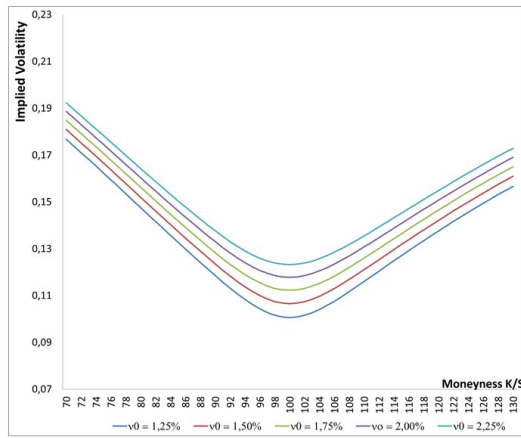
The underlying stock price in (5.1) moves accordingly to (4.1), which is the standard Black Scholes equation. From the second equation in (5.1) we see the reason why the Heston model belongs to the class of stochastic volatility models. Even though the volatility is not directly modeled, the variance follows a square root mean process developed by (Cox et al., 1985). To get a better grasp of the models input variables, a sensitivity analysis for each of the model's inputs is conducted. This gives a better intuition on the model's behavior and the impact of the input parameter illustrated by the change of the volatility smile for different levels of inputs. Looking at figures (5.1a) to (5.1e) three different effects can be discovered: the vol-smile level shift by the variables v_0 and θ , the curvature defined by κ and ξ and thirdly the rotation or skew influenced by the correlation coefficient ρ .



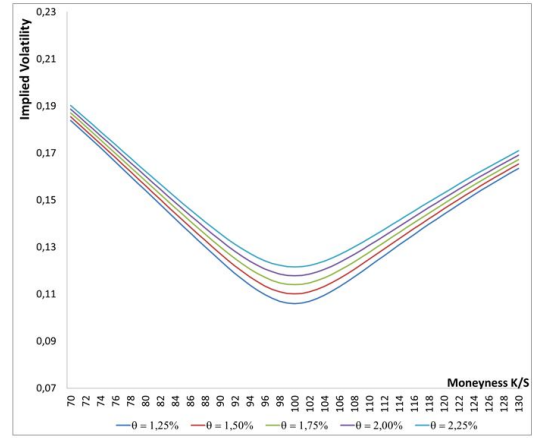
(a) Parameter: Kappa



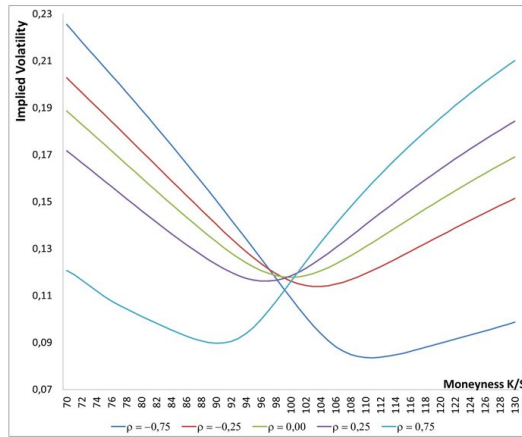
(b) Parameter: Xi



(c) Parameter: V0



(d) Parameter: Theta



(e) Parameter: Rho

Figure 5.1: Variable effects on implied Volatility

Stock price and variance itself follow the processes in (5.1) under a real world measure $\mathbb{P}$. To be able to do pricing according to the Black Scholes

model, a change to the risk-neutral measurement is needed. By applying the Girsanov's theorem, the risk neutral log price process is

$$d \ln S_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \quad (5.2)$$

equal to the process described in (Black and Scholes, 1973). Risk neutrality for the variance process is achieved by introducing a function $\lambda(S_t, v_t, t)$ for the drift of dv_t . The function $\lambda(S_t, v_t, t)$ describes the volatility risk premium as described in (Heston, 1993) or in (Woodard, 2011). Implementing the function leads to

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \xi \sqrt{v_t} \left(W_{2,t} + \frac{\lambda(S_t, v_t, t)}{\xi \sqrt{v_t}} \right)$$

In Heston's model the volatility risk premium is set to be proportional to the variance, thus $\lambda(S_t, v_t, t) := \lambda v_t$ and furthermore

$$dv_t = \kappa^*(\theta^* - v_t)dt + \xi \sqrt{v_t} d\tilde{W}_{2,t} \quad (5.3)$$

with $\kappa^* = \kappa + \lambda$, $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$ being the parameters for the risk neutral version of 5.1. Without loss of generality, we can set $\lambda = 0$ and get $\kappa^* = \kappa$ and $\theta^* = \theta$.

5.2 BS-like Call Price

One of the biggest advantages of the Heston model over other stochastic volatility models is the possibility to formulate a nearly closed form solution also called "Black-Scholes-like" equation by various researchers or article authors. A European call option price at time t with no dividends, an interest rate r , with spot S_t , time to maturity $T - t = \tau$ and strike K equals the discounted expected value of the option's payoff under the risk neutral measurement $\mathbb{Q}$. It is the same approach as in the Black Scholes framework and thus the name "BS-like" price is reasonable.

$$\begin{aligned} C(K) &= e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)^+] \\ &= e^{-r\tau} E^{\mathbb{Q}}[(S_T - K) \mathbb{1}_{S_T > K}] \\ &= e^{-r\tau} E^{\mathbb{Q}}[S_T \mathbb{1}_{S_T > K}] - K e^{-r\tau} E^{\mathbb{Q}}[\mathbb{1}_{S_T > K}] \end{aligned} \quad (5.4)$$

The second expectation in the last line of 5.4 equals under measurement $\mathbb{Q}$ the probability that $S_t = e^{x_t}$ is greater than K at point T , hence the call expires in-the-money under measurement $\mathbb{Q}$. Consequently $E^{\mathbb{Q}}[\mathbb{1}_{S_T > K}]$ can be reformulated to:

$$E^{\mathbb{Q}}[\mathbb{1}_{S_T > K}] = \mathbb{Q}(S_T > K) = \mathbb{Q}(\ln S_T > \ln K) = P_2$$

The first expected value in equation (5.4) $e^{-r\tau} E^{\mathbb{Q}}[S_T \mathbb{1}_{S_T > K}]$ is formulated into a probability by an additional usage of a measurement change from $\mathbb{Q}$ to $\mathbb{Q}^S$. Recalling the Radon-Nikodym derivative.

$$\frac{\partial \mathbb{Q}}{\partial \mathbb{Q}^S} = \frac{B_T/B_t}{S_T/S_t} = \frac{E^{\mathbb{Q}}[e^{x_T}]}{e^{x_T}} \quad (5.5)$$

and

$$B_t = \exp\left(\int_0^t r du\right) = e^{rt}$$

The term $E^{\mathbb{Q}}[e^{x_T}]$ can be used since under $\mathbb{Q}$ assets grow at the risk-free rate r . Therefore the first expression of the last equation in (5.4) can be reformulated as followed:

$$\begin{aligned} e^{-r(T-t)} E^{\mathbb{Q}}[S_T \mathbb{1}_{S_T > K}] &= S_t E^{\mathbb{Q}}\left[\frac{S_T/S_t}{B_T/B_t} \mathbb{1}_{S_T > K}\right] = S_t E^{\mathbb{Q}^S}\left[\frac{S_T/S_t}{B_T/B_t} \mathbb{1}_{S_T > K} \frac{\partial \mathbb{Q}}{\partial \mathbb{Q}^S}\right] \\ &= S_t E^{\mathbb{Q}^S}[\mathbb{1}_{S_T > K}] = S_t \mathbb{Q}^S(S_T > K) = S_t P_1 \end{aligned} \quad (5.6)$$

Thus equation (5.4) can be logically expressed correctly in one single equation similar to the Black Scholes formula.

$$C(K) = S_t \mathbb{Q}^S(S_t > K) - K e^{-r\tau} \mathbb{Q}(S_T > K) = S_t P_1 - K e^{-r\tau} P_2 \quad (5.7)$$

5.2.1 The Heston PDE

This subsection will guide through the derivation of the Heston's partial derivative equation (PDE) which will be necessary to calculate P_1 and P_2 . The following subsection follows a similar approach of a self financing portfolio like in the Black Scholes model. Nevertheless a further derivative is needed in the Heston framework to hedge the stochastic volatility and therefore transforms it into a riskless portfolio.

We consider a portfolio with one single option V , Δ units of the underlying S_t and φ of an additional option to hedge the volatility.

$$\Pi = V + \Delta S + \varphi U$$

Consider the given portfolio is self-financing then the following equation also holds:

$$d\Pi = dV + \Delta dS + \varphi dU \quad (5.8)$$

Similar to the derivations in the Black Scholes framework, to find the hedging portfolio, Ito's lemma is applied.

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} dt + \frac{1}{2} v \xi^2 \frac{\partial^2 V}{\partial v^2} dt + \xi \rho v S \frac{\partial^2 V}{\partial S \partial v} dt \quad (5.9)$$

Replicating equation (5.9) in terms of U and substituting it into (5.8) provides:

$$\begin{aligned} d\Pi &= dV + \Delta dS + \varphi dU \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \xi v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} \right] dt \\ &\quad + \varphi \left[\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \xi v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 U}{\partial v^2} \right] dt \\ &\quad + \left[\frac{\partial V}{\partial S} + \varphi \frac{\partial U}{\partial S} + \Delta \right] dS + \left[\frac{\partial V}{\partial v} + \varphi \frac{\partial U}{\partial v} \right] dv \end{aligned} \quad (5.10)$$

For the portfolio to be insensitive to movements of the underlying and the volatility, the last two brackets in the last equation have to be equal to zero. Hence, we can conduct that

$$\varphi = -\frac{\partial V}{\partial v} / \frac{\partial U}{\partial v}, \quad \Delta = -\varphi \frac{\partial U}{\partial S} - \frac{\partial V}{\partial S} \quad (5.11)$$

holds. Furthermore to be arbitrage free, the portfolio has to earn the risk-free rate r , otherwise by construction of the model framework a free lunch opportunity would be possible. Because of that the following also applies:

$$d\Pi = r(V + \Delta S + \varphi U) dt \quad (5.12)$$

Now setting equation (5.12) equal with (5.10) and including (5.11) gives:

$$\begin{aligned} & \frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\xi vS \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\xi^2 v \frac{\partial^2 V}{\partial v^2} \right] - rV + rS \frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} \\ &= \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\xi vS \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\xi^2 v \frac{\partial^2 U}{\partial v^2} \right] - rU + rS \frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}} \end{aligned} \quad (5.13)$$

Equation (5.13) is nicely divided in terms of U and V . Due to that both sides can be expressed in a function like in (Heston, 1993)

$$f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$$

Combined with (5.13), an application of the chain and product rule and the substitution of $\lambda(S, v, t) = \lambda v$ yields the Heston PDE, where $x = \ln S$

$$\begin{aligned} & \frac{\partial U}{\partial t} + \frac{1}{2}v \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}v\right) \frac{\partial U}{\partial x} + \rho\xi v \frac{\partial^2 U}{\partial v \partial x} \\ & + \frac{1}{2}\xi^2 v \frac{\partial^2 U}{\partial v^2} - rU + [\kappa(\theta - v) - \lambda v] \frac{\partial U}{\partial v} = 0 \end{aligned} \quad (5.14)$$

From this point on the Heston PDE can be combined with the final part of equation (5.4) to reproduce the results in (Heston, 1993).

Given the characteristic functions the inversion theorem by Gil-Pelaez (Gil-Pelaez, 1951) the in the money probability can be retrieved by:

$$P_j = \Pr(\ln S_T > \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-i\phi \ln K} f_j(\phi; x; v)}{i\phi} \right] d\phi \quad (5.15)$$

Heston himself postulated that the characteristic functions of the logarithm of the stock price at maturity $x_T = \ln S_T$ are of a log linear form and thus:

$$f_j(\phi; x_t; v_t) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t), \quad (5.16)$$

with i being the imaginary unit, $\tau = T - t$, furthermore C_j and D_j are coefficients and additionally the index $j = 1, 2$ refers to the probabilities P_1 and P_2 like in (Rouah, 2013)

$$\begin{aligned}
 C_j(\tau, \phi) &= ri\phi\tau + \frac{a}{\xi^2} \left[(b_j - \rho\xi i\phi + d_j)\tau - 2 \ln \left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j} \right) \right] \\
 D_j(\tau, \phi) &= \frac{b_j - \rho\xi i\phi + d_j}{\xi^2} \left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \right) \\
 a &= \kappa\theta \\
 b_1 &= \kappa + \lambda - \rho\xi \\
 b_2 &= \kappa + \lambda
 \end{aligned} \tag{5.17}$$

With (5.17) all necessary equations are available such that an option price can be calculated numerically.

5.3 The Little Heston Trap

As widely discussed in the literature the integrand derived by Heston may be well behaved, such that a numerical integration is without any issues, but may also be problematic for different parameters. This has been stated in papers from (Albrecher et al., 2006) or (Kahl and Jäckel, 2005)

Albrecher found an equivalent formulation of the coefficients C_j and D_j which results in fewer and less critical numerical integration problems, which is why the use is highly recommended. By multiplication of D_j with the term $\exp(-d_j\tau)$ the following result can be formulated.

$$D_j = \frac{b_j - \rho\xi\phi + d_j}{\xi^2} \left(\frac{1 - e^{-d_j\tau}}{1 - c_j e^{-d_j\tau}} \right) \tag{5.18}$$

where

$$c_j = \frac{b_j - \rho\xi i\phi - d_j}{b_j - \rho\xi i\phi + d_j}$$

and

$$C_j = ri\phi\tau + \frac{\kappa\theta}{\xi^2} \left[(b_j - \rho\xi i\phi - d_j)\tau - 2 \ln \left(\frac{1 - c_j e^{-d_j\tau}}{1 - c_j} \right) \right] \tag{5.19}$$

The adaption required by Albrecher's replacements are minor and lead to significant improvements to the original equation as can be seen in figure (5.2). What becomes clearly visible, is that an integration along the main branch of the complex logarithm of equation (5.15) may lead to discontinuities which is represented by the red line. Adjusting the function $f_j(\phi; x_t; v_t)$ according to the formulation in (5.18) and (5.19) will mitigate several prob-

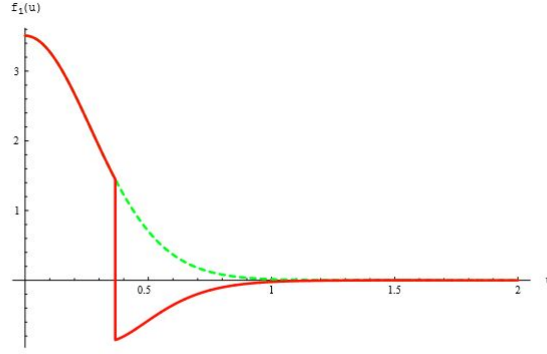


Figure 1: The function f_1 defined in (4). Underlying: $dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_S(t)$ with $S = 1$ and $\mu = 0$. Variance: $dV_t = \kappa(\theta - V_t)dt + \omega\sqrt{V_t}dW_V(t)$ with $V_0 = 0.2$, $\kappa = 1$, $\theta = 0.2$, $\omega = 0.5$ and $\rho = 0.3$, time to maturity: $\tau = 30$, strike: $K = 1$, red curve: implementation using only the principal branche, green dashed curve: adjusted curve using formula (32)

Figure 5.2: Discontinuity of the Integrands

Source:(Kahl and Jäckel, 2005)

lems concerning discontinuities and hence will be beneficial for further numerical integrations.

5.4 Numerical Integration Scheme

Since the calculation of the Heston call price demands an integral's solution and it is not analytically solvable because anti-derivatives can't be found, a Quadrature rule is required for approximation. A Quadrature approximates an integral in the following elementary way:

$$\int_a^b f(x)dx = \sum_{j=1}^N w_j f(x_j) \quad (5.20)$$

where $(w_1, \dots, w_j, \dots)$ are called weights or coefficients and $(x_1, \dots, x_j, \dots)$ are called nodes or abscissas.

In general two classes of Quadrature rules are commonly known, which have influence on the values of x_j and w_j . The Newton-Cotes on the one hand with equidistant nodes is easily computed and implemented, but lack complexity to efficiently approximate highly oscillating integrals with little computation time. The Gauss Quadrature on the other hand allows a successful approximation in fewer computation steps and thus yields a better performance but sacrifices simplicity and intuition due to a more complex

derivation rule. A detailed description about Quadrature's can be found in (Cohen, 2011)

5.4.1 Gauss-Laguerre Quadrature

The Gauss-Laguerre is primary designed for evaluating integrals in the domain $(0, \infty)$ and thus is perfectly suited for the Heston calculation. The abscissas can be retrieved by calculating the roots of the polynomial $L_N(x)$ of order N :

$$L_N(x) = \sum_{k=0}^N \frac{(-1)^k}{k!} \binom{N}{k} x^k$$

The weights are defined as:

$$w_j = \frac{(n!)^2 e^{x_j}}{x_j |L'_N(x_j)|^2} \quad \text{for } j = 1, 2, \dots, N$$

with

$$L'_N(x_j) = \sum_{k=1}^N \frac{(-1)^k}{(k-1)!} \binom{N}{k} x_j^{k-1} \quad \text{for } j = 1, 2, \dots, N$$

After the first calculation of the abscissas and weights, they can be stored and do not need any further calculations during the integral evaluation steps.

Given some Heston parameters, this chapter shows how to calculate a European call option price numerically. To find a meaningful price for any exotic structure, we now have to calibrate the model's parameters to the observable volatility surface. Since this thesis is looking at a "Volatility Knock Out option" on the Euro-Dollar exchange rate, the following chapter will explain how to extract useful volatility information from the foreign exchange market.

6 Foreign Exchange Market Data

This chapter reviews how to deal with the foreign exchange market and how to extract the necessary volatility information for a specific currency pair required for the calibration process of the Heston parameter set.

Compared to other asset classes, the foreign exchange market (FX market) is to be considered special in various ways. One of the reasons for this specialty is owing to the market's duality. Depending on the market participant's point of view, the foreign currency can be either of the currencies of a single currency pair.

Despite the situation in equity markets, where it is obvious that the numeraire will be the currency the stock is denominated in, it is not as clear for the foreign exchange market. Given the exchange rate between the Euro and the US Dollar, it can be defined as either 1 Euro for X US Dollar or vice versa. Hence it is not obvious which currency will act as the lead numeraire.

For example, given the situation of a euro-zone based newspaper it is reasonable to print all exchange rates in the way that it gives the amount of the foreign currency a reader gets for 1 euro, but this also holds true for users in other countries. Since the beginning of the foreign exchange market, a market convention got established to avoid potential misconceptions among traders.

After introducing the most common and necessary market conventions stated in (Clark, 2011) and (Castagna, 2010), the Vanna Volga method to construct volatility surfaces is reviewed based on (Castagna et al., 2007).

6.1 Market Conventions

The heuristic rule for the market convention (even though counter examples exist) states that the following hierarchy of the major currencies is applicable. With a currency being above (stronger) another currency, it

normally denominates as the leading currency. For example the exchange rate between USD and JPY is denominated as USDJPY which is defined as 1 US dollar for X quantities of Japanese yen.

$$EUR > GBP > AUD > NZD > USD > CAD > CHF > JPY$$

Further rules and explanations can be found in (Clark, 2011).

Considering an European option on a foreign currency, the buyer of the contract has the right, but not the obligation to buy a defined amount of a foreign currency to a pre-set exchange rate. For example a EURUSD option might give the contract owner the right to exchange 1.000.000 USD for 1.000.000 EUR. Hence a FX option combines a put and a call option and is either called EURUSD call or USDEUR put. For option prices in the foreign exchange market a market player can follow different market quoting styles.

6.1.1 Market Quotes

We start from the Garman Kohlhagen model, which comes as a natural extension to the Black Scholes formula stated in (Clark, 2011).

$$\omega e^{-r^d T} [FN(\omega d_1) - KN(\omega d_2)] \quad (6.1)$$

with

$$d_{1,2} = \frac{\ln(S_0/K) + (r^d - r^f \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$F = S_0 e^{(-r^f + r^d)T}$$

and ω being either 1 for a call, or -1 for a put option. Equation (6.1) sets the exchange rate of K Units of domestic currency for 1 unit of foreign currency, measured in the domestic numeraire and thus it is logically defined as $V_{d/f}$ (value of domestic per foreign). The value notion of domestic per foreign, is also the first of the four most common market quote stylings for options.

- domestic per foreign (d/f)
- percentage foreign (%f)
- percentage domestic (%d)

- foreign per domestic (f/d)

For a transformation to a %f notation, the domestic per foreign quote d/f has to be divided by the spot exchange rate. Thus we transform the domestic rate into a foreign one and end up with a %f quote.

$$V_{\%f} = \frac{V_{d/f}}{S_0}. \quad (6.2)$$

If the information of the option should be displayed in percentage by domestic currency, the formula needs to be divided by K . Since the Strike price sets the exchange rate, the division yields the expected result.

$$V_{\%d} = \frac{V_{d/f}}{K} \quad (6.3)$$

Flipping the quote form d/f to f/d is done by combining (6.2) and (6.3)

$$V_{f/d} = \frac{V_{d/f}}{S_0 K}. \quad (6.4)$$

All of the above mentioned market quotes express the same option in a different way and which of the above is used depends on the market convention.

6.1.2 Delta types

One further specialty of the foreign exchange market is, that available listed options are solely described by the implied volatility and the Delta value. The first derivative of an option's price with respect to the exchange rate, gives the option's Delta. Basically, it states, how much the option's value changes if the underlying exchange rate moves. This is crucial for derivative traders since it is used to hedge open short positions. Obviously for different notations for the option's value, different Delta styles follow as shown in (Clark, 2011).

- Pips spot Delta

The pips spot Delta measures the change of the option's present value, if the spot, expressed in d/f , changes.

$$\Delta_{S;pips} = \frac{\partial V_d}{\partial S_0} = \omega e^{-r^f T} N(\omega d_1) \quad (6.5)$$

- Percentage spot Delta

The percentage spot Delta gives the change of the PV of the option (in %) to changes of the spot (in %).

$$\Delta_{S;\%} = \frac{\partial V_{d;pips}/S_0}{\partial(\ln S_0)} = \frac{\partial V_{d;pips}}{\partial S_0} - \frac{V_{d;pips}}{S_0} = \Delta_{S;pips} - V_{\%f} \quad (6.6)$$

- Pips forward Delta

This quotation gives the change of the value at maturity with respect to a change of the underlying's forward.

$$\Delta_{F;pips} = e^{r^f T} \Delta_{S;pips} = \omega N(\omega d_1) \quad (6.7)$$

- Percentage forward Delta

The change of the option's future value to a forward change is measured by the percentage forward Delta which is also premium adjusted.

$$\Delta_{F;\%} = \frac{\partial V_{d;pips}/S_0}{\partial(\ln F_{0,T})} = \omega \frac{K}{F_{0,T}} N(\omega d_2) \quad (6.8)$$

All previous quotation styles except from the premium adjusted forward Delta have the Strike parameter K solely inside the normal distribution, so that the normal distribution can be inverted and the corresponding Strike is available in a closed solution. To obtain the Strike with the premium adjusted forward Delta, a numerical root-finding algorithm, like the Bisection (Burden and Faires, 1985) has to be applied, which is crucial for the construction of the volatility surface upcoming in the next section.

In the foreign exchange market we distinguish between spot and forward Delta and furthermore between regular and percentage or premium-adjusted Delta. Which Delta convention we have to use for a currency pair can unfortunately not be answered systematically. Though it is mostly common to use forward Delta measures for options with maturity longer than one year and spot Delta otherwise. The premium adjusted Delta is mostly used if the option's premium currency equals the foreign currency, but counter examples exist and thus a consequent check is advisable. However a list of selected currency pairs is provided in figure (6.1).

Knowing which Delta type is applicable we are able to interpret market data, that we can find in data portals like Bloomberg or Reuters.

Currency pair	Premium ccy	delta convention
EUR-USD	USD	regular
USD-JPY	USD	premium-adjusted
EUR-JPY	EUR	premium-adjusted
USD-CHF	USD	premium-adjusted
EUR-CHF	EUR	premium-adjusted
GBP-USD	USD	regular
EUR-GBP	EUR	premium-adjusted
AUD-USD	USD	regular
AUD-JPY	AUD	premium-adjusted
USD-CAD	USD	premium-adjusted
USD-BRL	USD	premium-adjusted
USD-MXN	USD	premium-adjusted

Figure 6.1: FX convention table

Source:(Reiswich and Uwe, 2012)



Figure 6.2: Foreign exchange volatility market data

Source: Bloomberg

Figure (6.2) shows how a market participant can extract volatility data for a specific currency pair. Given rather high trading volume, the liquidity might be sufficient to get reliable 10- Δ butterflies and 10- Δ risk reversal data, but since this is not the case for any currency pair, the Vanna Volga method solely relies on the 25- Δ option strategies since these are quoted independently of available liquidity.

6.2 The Vanna Volga Method

Considering the equity market, a volatility smile is explicitly given by the implied volatility values, but the FX market is different. Hereby it is common to quote a currency pair volatility solely with three different option strategies. The at-the-money straddle (6.2), the 25- Δ risk-reversal (6.2) and the vega-weighted 25- Δ butterfly (6.2) is used to define the volatility smile (Bisesti et al., 2005). For currency pairs with higher liquidity options strategies with different Δ -levels might be available. Nevertheless to be able to calibrate the Heston model also for currency pairs with lower liquidity (e.g. USDMYR - US Dollar vs Malaysian Ringgit), the Vanna Volga method mentioned in (Castagna et al., 2007) is solely based on:

- At-the-money straddles consist of a Call Option and a Put Option with the same maturity and the Strike set at the corresponding forward rate.
- A 25- Δ Risk-Reversal is defined as the out of the money European Call option less the out of the money European Put option, both with the same maturity and struck such that the Strike prices imply the same Deltas.
- 25- Δ (vega weighted) butterflies consist in total out of four options. The long position includes a ATM straddle, and a out of the money put and call option with 25- Δ are being sold. The ratio between the options held long and short is chosen such that the overall position is Delta and Vega neutral. (DeRosa, 2011)

The Vanna Volga method for volatility smile surface construction inter- and extrapolates volatilities for the whole range of Strikes from the volatilities of the 25 Δ Put, 25 Δ Call and the ATM-volatility and hence we have to extract the vanilla options out of the previously mentioned option strategies. Reformulating the straddle, butterfly and risk reversal gives the volatilities by:

$$\sigma_{ATM} = \sigma_{Straddle} \quad (6.9)$$

$$\sigma_{25-\Delta P} = \sigma_{ATM} - \frac{1}{2}\sigma_{RR} + \sigma_{BF} \quad (6.10)$$

$$\sigma_{25-\Delta C} = \sigma_{ATM} + \frac{1}{2}\sigma_{RR} + \sigma_{BF} \quad (6.11)$$

The idea behind the Vanna Volga method is to define a replication portfolio which renders an option insensitive up to the second order. One implicit assumption is that, in Black Scholes no vol-smile exists, but volatility itself is stochastic, which is good enough for practical reasons. By applying Itos lemma and denoting the quantity of the option with Strike K_i as x_i we get according to (Castagna et al., 2007) for the replication portfolio:

$$\begin{aligned}
 dC^{BS}(t; K) - \Delta_t dS_t - \sum_{i=3}^3 x_i dC_i^{BS}(t) \\
 = \left[\frac{\partial C^{BS}(t; K)}{\partial t} - \sum_{i=3}^3 x_i \frac{\partial dC_i^{BS}(t)}{\partial t} \right] dt \\
 + \left[\frac{\partial C^{BS}(t; K)}{\partial S} - \Delta_t - \sum_{i=3}^3 x_i \frac{\partial dC_i^{BS}(t)}{\partial S} \right] dS_t \\
 + \left[\frac{\partial C^{BS}(t; K)}{\partial \sigma} - \sum_{i=3}^3 x_i \frac{\partial dC_i^{BS}(t)}{\partial \sigma} \right] d\sigma_t \\
 + \frac{1}{2} \left[\frac{\partial^2 C^{BS}(t; K)}{\partial S^2} - \sum_{i=3}^3 x_i \frac{\partial^2 dC_i^{BS}(t)}{\partial S^2} \right] (dS_t)^2 \\
 + \frac{1}{2} \left[\frac{\partial^2 C^{BS}(t; K)}{\partial \sigma^2} - \sum_{i=3}^3 x_i \frac{\partial^2 dC_i^{BS}(t)}{\partial \sigma^2} \right] (d\sigma_t)^2 \\
 + \left[\frac{\partial^2 C^{BS}(t; K)}{\partial S \partial \sigma} - \sum_{i=3}^3 x_i \frac{\partial^2 dC_i^{BS}(t)}{\partial S \partial \sigma} \right] dS_t d\sigma_t
 \end{aligned} \tag{6.12}$$

Setting the weights x_1, x_2 and x_3 such that the stochastic brackets vanish, we construct a portfolio with a long Δ position of the underlying a short position in the calls with Strikes K_1, K_2 and K_3 . In (Castagna et al., 2007) it is mentioned that this approach is not flawless by a theoretical point of view, but empirical calculations have shown good results.

Calculation of the VV weights

The explicit weights in (6.12) are calculated by solving the system of equations in (6.13) and thus finding the replication portfolio that has the same

sensitivities to vega, $\partial \text{Vega} / \partial \sigma$ and $\partial \text{Vega} / \partial S$.

$$\begin{aligned}\frac{\partial C^{BS}}{\partial \sigma}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial C^{BS}}{\partial \sigma}(K_i) \\ \frac{\partial^2 C^{BS}}{\partial \sigma^2}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial^2 C^{BS}}{\partial \sigma^2}(K_i) \\ \frac{\partial^2 C^{BS}}{\partial \sigma \partial S_0}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial^2 C^{BS}}{\partial \sigma \partial S_0}(K_i)\end{aligned}\tag{6.13}$$

Following the notation of (Bisesti et al., 2005), the vega value of the option with Strike K is written as $\mathcal{V}(K)$ and defined as:

$$\begin{aligned}\mathcal{V}(K) &= \frac{\partial C^{BS}}{\partial \sigma}(K) = S_0 e^{-r^f T} \sqrt{T} \varphi(d_1(K)) \\ d_1(K) &= \frac{\ln \frac{S_0}{K} + (r^d - r^f + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \\ \varphi(x) &= \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}\end{aligned}\tag{6.14}$$

and furthermore (5.12) is:

$$\begin{aligned}\frac{\partial^2 C^{BS}}{\partial \sigma^2}(K) &= \frac{\mathcal{V}(K)}{\sigma} d_1(K) d_2(K) \\ \frac{\partial^2 C^{BS}}{\partial \sigma \partial S_0}(K) &= -\frac{\mathcal{V}(K)}{S_0 \sigma \sqrt{T} d_2(K)} \\ d_2(K) &= d_1(K) - \sigma \sqrt{T}\end{aligned}\tag{6.15}$$

The system of equations can be written as

$$A \begin{pmatrix} x_1(K) \\ x_2(K) \\ x_3(K) \end{pmatrix} = B\tag{6.16}$$

and assuming $K_1 < K_2 < K_3$, which is true for typical market parameters, the $\det(A)$ is strictly positive and due to the rule of Cramer, a unique

solution for the system of equations exists. Simple algebra provides:

$$\begin{aligned} x_1(K) &= \frac{\mathcal{V}(K) \ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\mathcal{V}(K_1) \ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}} \\ x_2(K) &= \frac{\mathcal{V}(K) \ln \frac{K}{K_1} \ln \frac{K_3}{K}}{\mathcal{V}(K_2) \ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}} \\ x_3(K) &= \frac{\mathcal{V}(K) \ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\mathcal{V}(K_3) \ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}} \end{aligned} \quad (6.17)$$

Given the closed solution for the weights, the Vanna-Volga (VV) price can be defined, which is in line with the market prices of the three options.

$$C(K) = C^{BS}(K) + \sum_{i=1}^3 x_i(K) [C^{MKT}(K_i) - C^{BS}(K_i)] \quad (6.18)$$

With (5.17) and (5.16) a straight forward approximation for the volatility of $C(K)$, is given by a first order taylor expansion in σ .

$$\sigma(K) = \sigma_1(K) \approx \frac{\ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}} \sigma_{25\Delta p} + \frac{\ln \frac{K}{K_1} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}} \sigma_{ATM} + \frac{\ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\ln \frac{K_3}{K_1} \ln \frac{K_2}{K_2}} \sigma_{25\Delta c} \quad (6.19)$$

The implied volatility for a arbitrary Strike can thus be approximated by a linear combination of the market observable volatilities. As observable in figure (5.3) the interpolation is superb in the range between K_1 and K_3 but decreases in its accuracy in the tails.

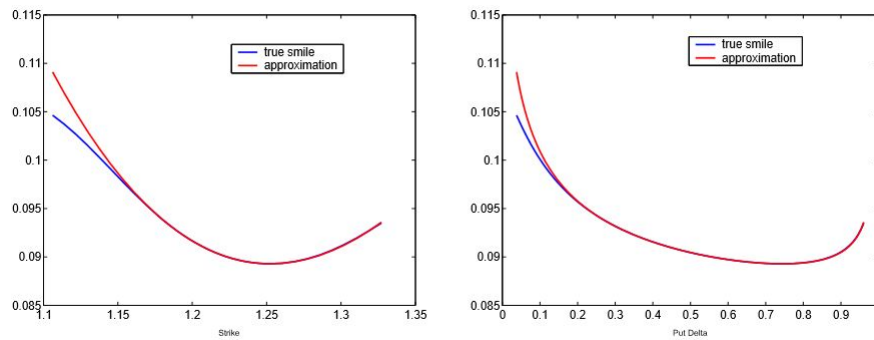


Figure 6.3: Comparison of first and second order interpolated smile against market smile in EURUSD.

Source: (Bisesti et al., 2005)

A second order Taylor expansion addresses the problem of overshooting tails successfully, but opening the issue of not defined square roots in the approximation. Nevertheless the fit increases compared to the approximation of first order.

$$\sigma(K) = \sigma_2(K) \approx \sigma_2 + \frac{-\sigma_2 + \sqrt{\sigma_2^2 + d_1(K)d_2(K)(2\sigma_2^2 D_1(K) + D_2(K))}}{d_1(K)d_2(K)} \quad (6.20)$$

with the additional definitions as:

$$\begin{aligned} D_1(K) &= \sigma_1(K) - \sigma_2 \\ D_2(K) &= \frac{\ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}} d_1(K_1) d_2(K_1) (\sigma_1 - \sigma_2)^2 \\ &\quad + \frac{\ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}} d_1(K_3) d_2(K_3) (\sigma_3 - \sigma_2)^2 \end{aligned}$$

With the right Delta value interpretation and the mechanics from the Vanna Volga method we can conveniently construct a complete volatility surface.

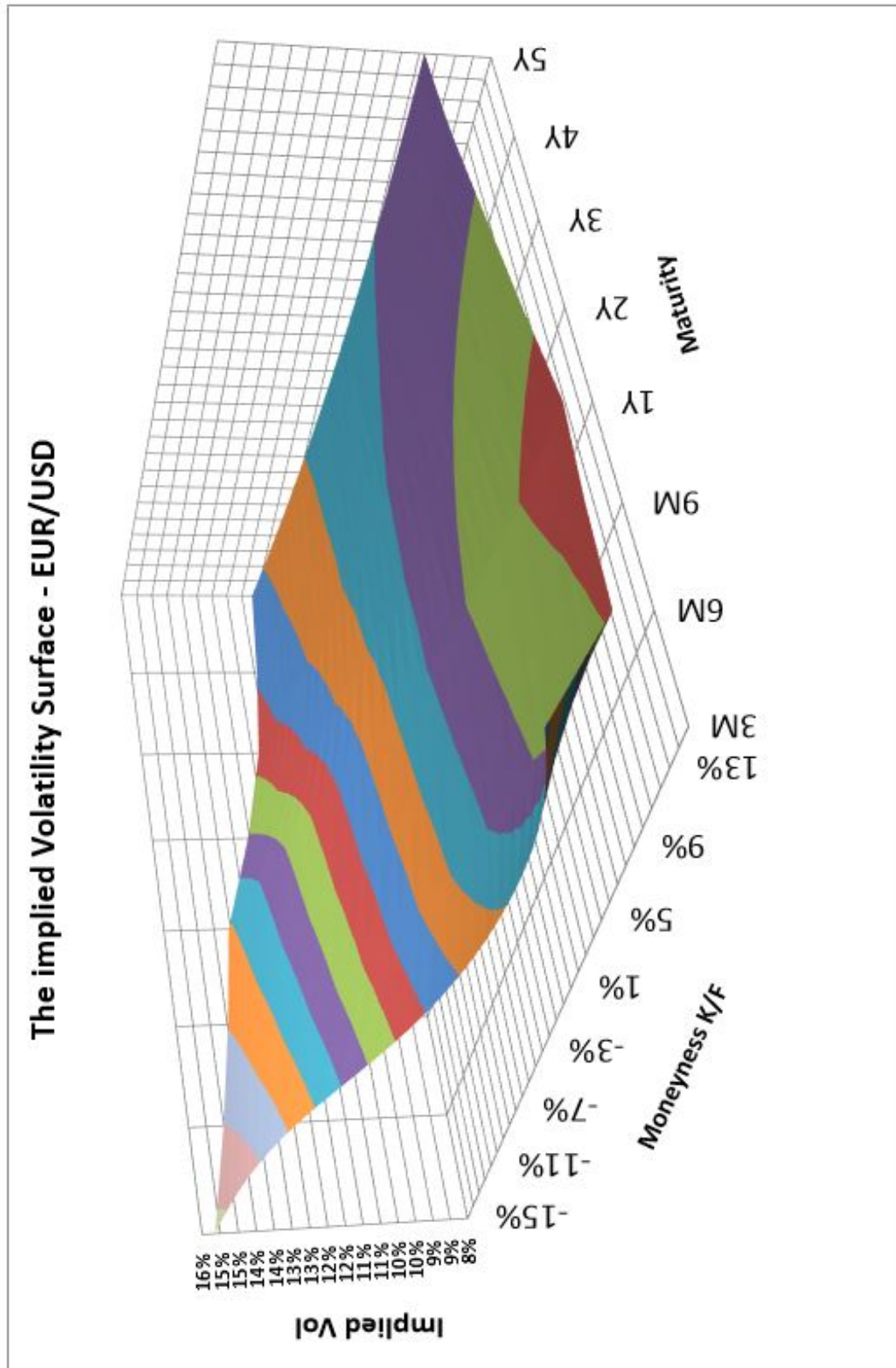


Figure 6.4: Constructed implied volatility surface
Source: Own Calculations

For newcomers the foreign exchange market can offer various obstacles due to non systematical conventions and rule based quotations. This makes it surprisingly difficult to access specific knowledge if not used to the framework of the foreign exchange market. However this chapter hopefully provides some guiding for information extraction of currency pairs, such that interested readers can from now on extract a volatility surface for any currency pair and transform it now to suit their further purposes. Within this thesis we apply the volatility surface from figure (5.4) to the derivations of chapter 4 to find a specific Heston parameter set, which suits real market data. This is done by a process called calibration which will be subject of the following chapter.

7 Calibration

Given the Black Scholes-like pricing formula in (5.4) and the numerical integration method of Gaussian Quadrature in chapter 5 we are able to calculate an option call price given a Heston parameter set, specified in (5.1).

Assuming now that the Heston model's dynamics represent the real evolution of the underlying and the volatility, then the real parameter can be estimated. The method established to achieve this is called calibration and it refers to a process that fits the parameter set to the previously extracted volatility surface data in a way that differences between the market data and the model's output will be minimized.

7.1 Loss Functions

A well established way of doing so is to minimize a function that measures a total error by comparing model specific data to quoted market data. We call such a function, which maps a parameter set to a sum of total error, a loss function.

If we minimize this function with respect to the input parameters, we end up with the Heston parameters that are best suited for simulating a financial product. To find a global minimum of a multivariate loss function turns out to be a complex task and various different minimizing algorithms have been modified and developed to efficiently fit the Heston parameters to a volatility surface, as described in (Mikhailov and Nögel, 2004) or (Rouah, 2013).

The different ways of estimating a parameter set for a model framework have in common that a restricted minimizing algorithm is applied to minimize the loss function. The restriction described in equation (7.1) ensure a

economical meaningful interpretation for the Heston framework.

$$\begin{aligned}
\kappa &> 0 & \theta &> 0 \\
v_0 &> 0 & \xi &> 0 \\
\rho &\in [-1, +1]
\end{aligned} \tag{7.1}$$

7.1.1 General approach

Given a set of maturities $\tau_t (t = 1, \dots, n)$ and a set of strikes $K_k (k = 1, \dots, m)$, each combination (τ_t, K_k) gives a valid combination for an option and thus has a quoted volatility IV_{tk} and market price C_{tk} . For any arbitrary parameter set Θ we can derive the closed Heston call price as described in chapter 5. We denote the Heston price as C_{tk}^Θ and the implied volatility of the corresponding Heston price as IV_{tk}^Θ .

In (Rouah, 2013) different possible ways to formulate the loss function are mentioned and they can be categorized into functions that use market prices as input factors and on the other hand implied volatility dependent loss functions.

7.1.2 MSE loss function

For example a calibrated parameter set $\hat{\Theta}$ can be obtained by minimizing the mean error sum of squares loss function with respect to Θ .

$$\frac{1}{N} \sum_{t,k} w_{tk} (C_{tk} - C_{tk}^\Theta)^2$$

where w_{tk} is a weight attached to the option and N is the total sum of quoted prices.

A possible shortcoming of the mean error sum of squares loss function, is due to the fact that out-of-the-money options tribute less to the estimation than in-the-money options. The same is applicable for options close to expiry compared to long maturity options.

7.1.3 RSME loss function

The second loss function is the relative mean error sum of squares. This estimates are obtained with by minimizing

$$\frac{1}{N} \sum_{t,k} w_{tk} \frac{(C_{tk} - C_{tk}^{\Theta})^2}{C_{tk}}$$

A similar problem but in reverse order occurs by using the relative mean error sum of squares. The fact that C_{tk} is the denominator, the wings account for a big part of the loss function and thus force the estimate to fit the out-of-the-money options better at cost of at-the-money and in-the-money options. A remedy to a synchronous contribution of the wings is to use in-the-money-options only, such that if the Strike is below the spot level a call option and otherwise a put option is used. Nevertheless the overall under- and over-contribution can be mitigated by selecting specific weights in the loss function, but the right weights are a more subjective choice than an objective rule and hence will not be applied in this thesis.

7.1.4 IVMSE loss function

The second category of loss functions minimizes the estimation errors, by, similarly as in the MSE and the RMSE, defining the error terms as squared differences, but the differences rather reflect differences in implied volatilities and not pricing differences. Since options in the FX market are mostly quoted in terms of implied volatility it is straight-forward to measure a model's fitting by the distance between the volatilities. Hence a way to find a parameter estimation is to minimize the implied volatility mean error sum of squares (IVMSE)

$$\frac{1}{N} \sum_{t,k} w_{t,k} (IV_{t,k} - IV_{t,k}^{\Theta})^2 \quad (7.2)$$

At this point it is important to mention that even though the estimation through equation (6.2) seems to be the best approach to calibrate a model to market data, it requires a vast amount of computational effort, since retrieving the implied volatility can not be done analytically. First a Heston option price C_{tk}^{Θ} has to be determined and given that, a root finding algorithm like the Bisection method has to be applied to find the corresponding

Black Scholes conform implied volatility.

7.1.5 Christoffersen loss function

A way to bypass the bisection completely is by using the loss function introduced by (Christoffersen and Jacobs, 2004), which approximates the IVMSE. The described loss function used the Black Scholes vega as a weighting for the terms of the MSE.

$$\frac{1}{N} \sum_{t,k} \frac{(C_{tk} - C_{tk}^{\Theta})^2}{BSVega_{tk}^2} \quad (7.3)$$

where

$$BSVega_{tk} = S \exp(-q\tau_t) n(d_{tk}) \sqrt{\tau_t}$$

with

$$d_{tk} = \frac{\log(S/K_k) + (r - q + IV_{tk}^2/2)\tau_t}{IV_{tk}\sqrt{\tau_t}}$$

and $n(x) = \exp(-x^2/2)/\sqrt{2\pi}$ being the standard normal density. The function of (Christoffersen and Jacobs, 2004) enables a fast implementation at small cost of effectivity for the calibration.

In fact given the research from (Bakshi et al., 1997), (Bams et al., 2009) and (Mikhailov and Nögel, 2004) no consensus can be found which loss function works best for parameter estimation.

7.2 Estimation verification

Eventually a specific loss function performs better for a certain currency pair but on the other hand results in a worse estimation of parameters for another pair of currencies. Hence we can not state that some loss functions are superior compared to others and a total comparison is required for each estimation process. Anyways, independently of the choice of loss function, an algorithm to estimate the parameters based on the corresponding loss function is required.

Hence we implement two different optimization algorithms. The Nelder-Mead introduced in (Nelder and Mead, 1965) and the Differential Evolu-

tion algorithm from (Storn and Price, 1997), are going to be explained in the next subsections and then are compared to the built-in excel solver, which surprisingly good performance is mentioned in (Mikhailov and Nögel, 2004).

7.2.1 Nelder-Mead

The Downhill-Simplex, also the so called Nelder-Mead algorithm, is a numerical method designed to find a minimum or maximum of a objective function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ created and described by (Nelder and Mead, 1965). By definition a simplex is a n-dimensional geometric figure and the algorithm creates iteratively a decreasing series of simplicies to end up with an optimal solution for $\min f(x)$. For this purpose four different routines are available: *reflection, expansion, contraction, shrink*, which are used to move along the function surface to the minimum in the region.

Each of the routines are subject to a scalar factor usually set to

$$[\alpha, \beta, \gamma, \delta] = [1, 2, 1/2, 1/2]$$

Further define the centroid as

$$\bar{x} = (x_0 + x_1 + \dots + x_n)/n$$

The optimization procedure's iteration consists of the following six steps and are fully described in the article from (Gao and Han, 2012).

- Step 1 - Sort. Calculate the objective function at the vertices and sort them such that (7.4) holds.

$$f(x_0) \leq f(x_1) \leq \dots \leq f(x_n) \quad (7.4)$$

- Step 2 - Reflection. Compute the reflection point x_r defined as

$$x_r = \bar{x} + \alpha(\bar{x} - x_n)$$

if $f(x_1) \leq f(x_r) \leq f(x_n)$ then replace x_n with x_r and return to step 1. Otherwise continue with step 3.

- Step 3 - Expansion. If $f(x_r) \leq f(x_0)$ then the expansion point x_e is calculated

$$x_e = \bar{x} + \beta(x_r - \bar{x})$$

In case $f(x_e) \leq f(x_r)$ replace x_n with the expansion point x_e and return to step 1. For $f(x_e) > f(x_r)$, replace x_n with the reflection point x_r and also return to step 1. For $f(x_r) > f(x_0)$ continue to step 4.

- Step 4 - Outside Contraction. The outside contraction point x_{oc} is evaluated by:

$$x_{oc} = \bar{x} + \gamma(x_r - \bar{x})$$

If $f(x_{n-1}) \leq f(x_r) < f(x_n)$ and $f(x_{oc}) < f(x_n)$ then replace x_n with x_{oc} and return to step 1, otherwise proceed with step 5.

- Step 5 - Inside Contraction. In the case of $f(x_r) \geq f(x_n)$ then the inside contraction point

$$x_{ic} = \bar{x} + \gamma(x_n - \bar{x})$$

is replaced with x_n if $f(x_{ic}) < f(x_n)$ holds. Considering a successful replacement a new iteration step begins with step 1, otherwise the algorithm continues to step 6.

- Step 6 - Shrink. For $i = 1$ to n , the points x_i are replaced by $x_i + \delta(x_0 - x_i)$

The Nelder-Mead stops after a preset number of iteration or if the absolute distance between the best and the worst point $|f(x_0) - f(x_n)|$ is below a tolerance value, which means that the simplex itself has become such a small size, it will not change the end result significantly anymore.

Although the Nelder-Mead does not ensure global convergence, it generally performs well and is often the chosen algorithm for real world optimization problems. It is also the basis for the *fmincon* function in MATLAB, which is widely used among practitioners. Nevertheless for objective functions with many local minimums, the algorithm will not necessarily ensure to converge to the global minimum and thus might lead to a non optimal parameter set. Hence this has to be taken into consideration before using this optimizing algorithm.

7.2.2 Differential Evolution

The Differential Evolution algorithm introduced by (Storn and Price, 1997) has been applied successfully to the Heston model by (Vollrath and Wendland, 2009). They found that the Differential Evolution approach is an efficient way to identify the global minimum for the Heston parameters, but requires a higher computational effort.

The algorithm is described stepwise in (Rouah, 2013) or (Vollrath and Wendland, 2009).

- Step 1 - First create a matrix for the so called population. It is a matrix of size $5 \times N_p$ with randomized Parameter values. Remark that the random numbers may not violate the restrictions of (7.1) and furthermore upper bounds are chosen to ensure reasonable meaning.
- Step 2 - Mutation phase. For each member of the population a new member is created by

$$y_i = x_{r1} + F(x_{r2} - x_{r3})$$

where F is a constant mutation factor and x_{r1}, x_{r2} and x_{r3} are different members than x_i .

- Step 3 - Recombination phase. A new parameter set u_i is constructed by applying the rules to each element j .

$$u_{ij} = \begin{cases} y_{ij} & \text{if } U_{ij} \leq CR \text{ or } j = R \\ x_{ij} & \text{otherwise} \end{cases}$$

U_{ij} is a uniform random number, CR is the constant cross-factor and R a randomly chosen integer between 1 and 5.

- Step 4 - Selection. For each member x_i of the population, the corresponding parameter set u_i is evaluated by the objective function. If u_i is smaller x_i it is replaced. After evaluating the whole population the next generation begins with step 2.

7.2.3 Results

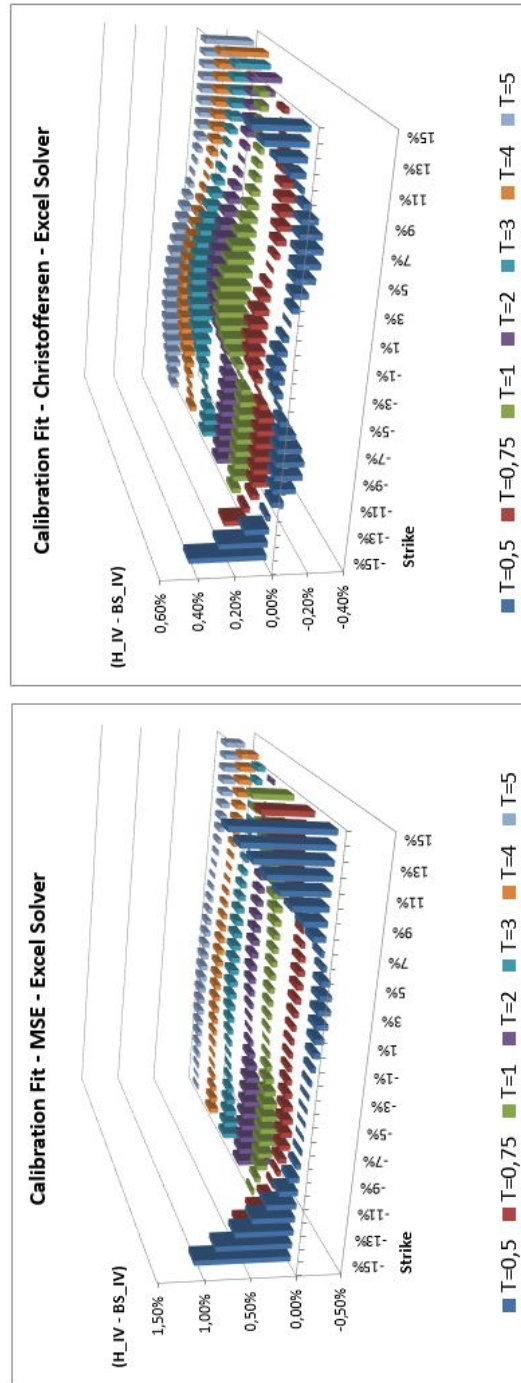
We implemented the algorithms mentioned above and can find similar results as (Mikhailov and Nögel, 2004).

The built-in excel solver delivers astonishingly efficient results. Also the Nelder-Mead algorithm delivers favorable results in estimating the parameter set. Even though neither of both methods ensure a convergence to the global minimum, with sufficiently good starting parameters, the estimation yields great calibration results, so that a successful simulation of the exchange rate can be conducted in the next chapter. We calibrated the Heston parameter to the Euro-Dollar volatility surface based on the end of day data of the 10.11.2015. The starting values for the calibration process were as followed:

$$\begin{aligned}\kappa &= 3 \\ \theta &= 0.01 \\ \xi &= 0.5 \\ v_0 &= 0.02 \\ \rho &= -0.50\end{aligned}$$

Calibration Fit													
EUR-USD 10.11.2015	Parameter	MSE			RSME			IVMSE			Christoffersen		
		NM	DE	ES	NM	DE	ES	NM	DE	ES	NM	DE	ES
	Kappa	3,2330	3,1138	3,8737	3,2461	3,1101	3,6371	3,1085	3,0366	2,4966	3,0098	2,7926	2,5463
	Theta	0,0138	0,0110	0,0112	0,0106	0,0104	0,0112	0,0104	0,0101	0,0113	0,0103	0,0104	0,0113
	Xi	0,6539	0,4307	0,4763	0,4853	0,5173	0,4559	0,5170	0,4935	0,2907	0,4775	0,4807	0,2952
	V0	0,0104	0,0158	0,0179	0,0144	0,0157	0,0177	0,0157	0,0148	0,0151	0,0153	0,0157	0,0152
	Rho	-0,4672	-0,5365	-0,5061	-0,4997	-0,4944	-0,5070	-0,5074	-0,5198	-0,5959	-0,4764	-0,5372	-0,5934
Calibration Fit $(H_{IV}-BS_{IV})^2$		0,00780241	0,00752858	0,00103110	0,00768767	0,00973562	0,00103651	0,01022188	0,01424220	0,00033188	0,00899190	0,01013374	0,00033116

(a) Calibration Fit



(c) Calibration fit for Christoffersen

(b) Calibration fit for MSE

Figure 7.1: Calibration results

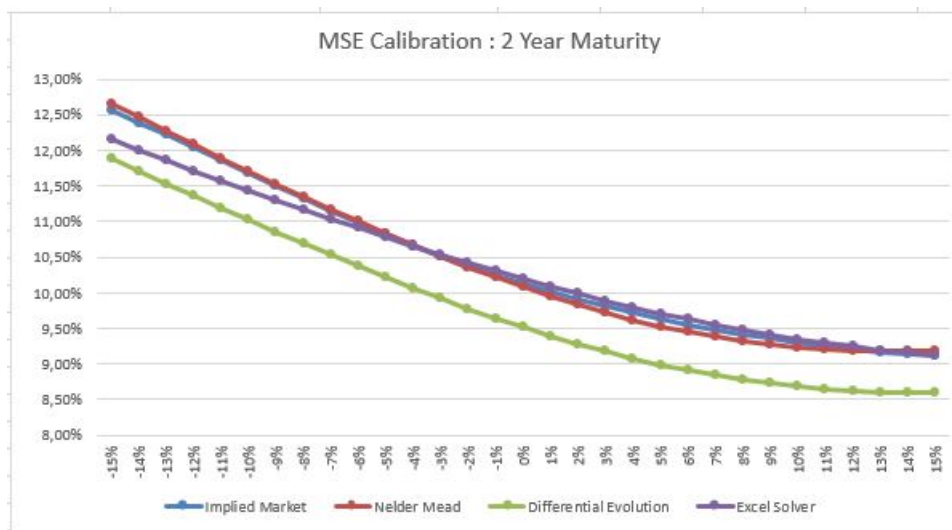


Figure 7.2: Calibration Fits - 2Y Maturity

What we can observe is a drastic misfit of the short term wings visible in (7.1b). The misfit is described as $(H_{IV} - BS_{IV})$, which refers to the difference between the Black Scholes market implied volatility and the fitted Heston model translated to the Black Scholes implied volatility. This is also well documented in (Janek et al., 2011). The Heston model is not capable of reproducing short term skews without limitations and hence this has to be considered in possible scenarios. If we force the calibration to fit the wings, the parameters ξ and κ are greatly increased and thus imitating a jump diffusion process even though a non continuous movement is not foreseen by the models definition.

It is important to keep the model's abilities in mind and to recall in the simulation part that short term skew simulations are not reliable enough to base a meaningful valuation on.

Given the calibration results, depending on which calibration method the reader wants to implement, either the Christoffersen loss function or the MSE loss function seems preferable. There does not exist a specific set of rules to achieve the best possible fit of a model and hence can be considered more art than science. Due to that a financial engineer has to try several approaches and can not be sure about the outcome up front.

In (7.1a) we can see that the Differential Evolution algorithm tends to underestimate the Volatility surface. For more accurate results a higher pop-

ulation and generation number can be used to have a better and more detailed search along the surface but also further increases the computational efforts and thus is not preferable compared to the Nelder-Mead algorithm.

8 Simulation

After the calibration of the stochastic differential equations for the exchange rate and the corresponding stochastic volatility, the Heston model is finally set up. As we have seen in chapter 5 we could now price European call or put options by solving the Black Scholes like pricing formula numerically. However similar semi analytic solutions can not be found for more complex structures. Hence a fully numerical approach has to be chosen.

The Monte Carlo method is one of the most powerful techniques in financial engineering and if done correctly can enable the practitioner to find prices even for the most exotic payoffs. To do so, we draw sample paths of the known stochastic processes and the law of great numbers ensures that the mean estimator converges to the real mean, given the stochastic process is defined correctly. This helps the financial engineer to price any given payout by simply drawing many sample paths and calculating the payout in question for each run.

The Monte Carlo technique furthermore is very wide spread among risk managers to find Value at Risk measures.

This chapter is widely based on (Glasserman, 2003), (Rouah, 2011a), (Gatheral, 2006) and (Kahl and Jäckel, 2006).

8.1 Monte Carlo simulation for the Heston model

Given the definition of the Heston processes in (5.1) and the fitted parameters of chapter 6, we have to implement simulation schemes to be able to draw sample paths.

The Euler scheme is one of the most intuitive ways to do so. Additionally we will have a look at the improvements of the Milstein scheme. Even though both techniques have minor shortcomings we have chosen to implement

these due to the tractability and their intuitive idea.

Returning to the equation of the Heston model the underlying and its variance is modeled by the following stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)} \\ dv_t &= \kappa(v_t - \theta) dt + \xi \sqrt{v_t} dW_t^{(2)} \end{aligned} \quad (8.1)$$

with $E[dW^{(1)}dW^{(2)}] = \rho dt$.

The Euler scheme as well as the Milstein scheme tries to approximate the continuous processes with discrete time steps and hence these methods are also known as discretizations.

We now review the derivation of the correct discretization.

8.1.1 Euler scheme

Note that the stock price in the Heston model does not depend directly on the time variable and thus can alternatively be written as:

$$dX_t = \mu_t dt + \sigma_t dW_t$$

and in Integral form

$$X_{t+dt} = X_t + \int_t^{t+dt} \mu_s ds + \int_t^{t+dt} \sigma_s dW_s \quad (8.2)$$

The intuitive discretization also called Euler Scheme uses the left-point rule to approximate the integrals by the product of the known value on the left side of the integral with the constant integration domain dt which is spaced equally so that

$$\int_t^{t+dt} \mu(X_u, u) du \approx \mu(X_t, t) \int_t^{t+dt} du = \mu(X_t, t) dt.$$

For the second part of equation (8.2) it follows that

$$\int_t^{t+dt} \xi(X_u, u) du \approx \xi(X_t, t) dt$$

$$\int_t^{t+dt} dW_u = \xi(X_t, t)(W_{t+dt} - W_t) = \xi(X_t, t)\sqrt{dt}Z$$

The last part of the equation indicates, that $W_{t+dt} - W_t$ is the same in distribution as $\sqrt{dt}Z$ if Z is a standard normal distributed variable. As a result the equation (8.2) can be approximately described as

$$X_{t+dt} = X_t + \mu(X_t, t)dt + \xi(X_t, t)\sqrt{dt}Z \quad (8.3)$$

Applying the Euler Scheme as described in (8.2) and (8.3) the equation of the variance can be rewritten as following

$$v_{t+dt} = v_t + \int_t^{t+dt} \kappa(\theta - v_u)du + \int_t^{t+dt} \xi\sqrt{v_u}dW_u$$

and thus

$$v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\sqrt{dt}Z_v \quad (8.4)$$

is true. Creating a variance time series with this discretization method can lead to meaningless negative variance levels. To avoid problems created by negative variance additionally the reflection scheme is applied. In case of a negative value in a sample path the reflection scheme changes the sign of the variance and hence ensures always meaningful values for the variance. For the Stock price process the stochastic differential equation can be expressed as

$$S_{t+dt} = S_t \exp((\mu - \frac{1}{2}v_t)dt + \sqrt{v_t}\sqrt{dt}Z_s) \quad (8.5)$$

The variables Z_s and Z_v are constructed by correlating two standard normal variables Z_1, Z_2 with the correlation coefficient ρ . Define $Z_v = Z_1$ and $Z_s = \rho Z_1 + \sqrt{1 - \rho^2}Z_2$.

The Euler discretization scheme in (8.5) and (8.4) gives a first possibility to draw sample paths from the stochastic differential equations. However to increase the accuracy of the discretization the Milstein scheme mentioned in (Kloeden and Platen, 1992) is implemented additionally.

8.1.2 Milstein scheme

The idea is to improve the efficiency of the simulation by expanding the coefficients via Ito's lemma, thus the coefficients SDEs are:

$$\begin{aligned} d\mu_t &= (\mu'_t\mu_t + \frac{1}{2}\mu''_t\xi_t^2)dt + (\mu'_t\xi_t)dW_t \\ d\xi_t &= (\xi'_t\mu_t + \frac{1}{2}\xi''_t\xi_t^2)dt + (\xi'_t\xi_t)dW_t \end{aligned}$$

or written in integral form:

$$\begin{aligned} \mu_{t+dt} &= \mu_t + \int_t^{t+dt} (\mu'_u\mu_u + \frac{1}{2}\mu''_u\xi_u^2)du + \int_t^{t+dt} (\mu'_u\xi_u)dW_u \\ \xi_{t+dt} &= \xi_t + \int_t^{t+dt} (\xi'_u\mu_u + \frac{1}{2}\xi''_u\xi_u^2)du + \int_t^{t+dt} (\xi'_u\xi_u)dW_u \end{aligned}$$

Substituting this in equation (8.2) and omitting integrals of higher order produces

$$X_{t+dt} = X_t + \mu_t \int_t^{t+dt} ds + \xi_t \int_t^{t+dt} dW_s + \int_t^{t+dt} \int_t^s (\xi'_u\xi_u)dW_u dW_s \quad (8.6)$$

Euler discretization applied on the last term gives

$$\begin{aligned} \int_t^{t+dt} \int_t^s (\xi'_u\xi_u)dW_u dW_s &\approx \xi'_t\xi_t \int_t^{t+dt} \int_t^s dW_u dW_s = \xi'_t\xi_t \int_t^{t+dt} (W_s - W_t)dW_s \\ &= \xi'_t\xi_t \left(\int_t^{t+dt} W_s dW_s - W_t W_{t+dt} + W_t^2 \right) \\ &= \frac{1}{2}\xi'_t\xi_t [(W_{t+dt} - W_t)^2 - dt] \end{aligned}$$

For the last step in the equation above once more Ito's lemma has to be applied. We ultimately substitute the last term $W_{t+dt} - W_t$ with the in distribution equal term $\sqrt{dt}Z$ and thus end up with the general Milstein scheme

$$X_{t+dt} = X_t + \mu_t dt + \xi\sqrt{dt}Z + \frac{1}{2}\xi'_t\xi_t dt(Z^2 - 1) \quad (8.7)$$

Finally (8.7) is applied to definition of the Heston variance and stock price (5.1) results in:

$$v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\sqrt{dt}Z_V + \frac{1}{4}\xi^2 dt(Z_V^2 - 1) \quad (8.8)$$

and

$$S_{t+dt} = S_t + \mu S_t dt + \sqrt{v_t} \sqrt{dt} S_t Z_S + \frac{1}{2} v_t S_t dt (Z_S^2 - 1) \quad (8.9)$$

With the calibrated parameter set (7.1a) we simulated a sample path for the underlying and the variance process. The sample path is created for 1 year

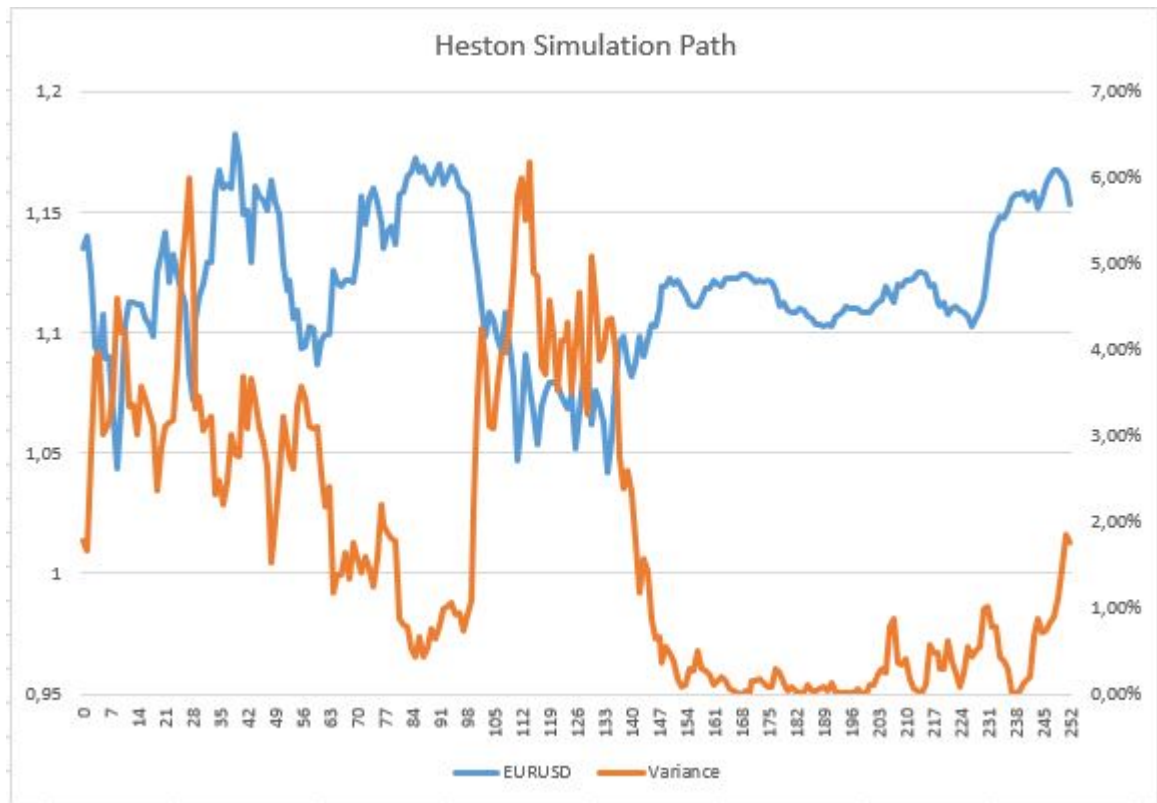


Figure 8.1: Simulated sample paths for underlying and variance of EUR-USD exchange rate.

by simulating 252 business days and the impact of the negative correlation can be observed clearly by comparing the blue and orange line.

8.1.3 Exotics simulation

In fact the Milstein scheme seems to outperform the Euler scheme and therefore we mostly use this discretization method. Since we are now able to create a single sample path for the exchange rate and the corresponding volatility, we are able to calculate a derivative's payout based on this single pair of paths. The following pseudo code illustrates how to find the price of an option with 1 year to maturity estimated with a Monte Carlo method.

```

Option_value = 0 'resetting the Option value to 0

for i = 1 to n 'n is the number of simulations

    s_t = s0
    'reset underlying to the starting spot(exchange rate)
    v_t = v0
    'reset variance to the starting spot variance

    for j = 1 to 252
    '252 is the number of business days in one year

        s_t(j) = Milstein(s_t)
        v_t(j) = Milstein(v_t)
    'Milstein(s_t/v_t) is the function for the Milstein
    'discretization scheme, to simulate one further step.

    next j

    Option_value = Option_value + Payoff(s_t,v_t)
    'Payoff is the function that determines the exotic
    'derivatives intrinsic value at maturity.
    'In this line the total intrinsic of all simulation
    'rounds will be aggregated.

next i

    Option_value = Option_value / n
    'Since the arithmetic average is an unbiased estimator for
    'the real expected value, we can calculate the options value
    'as the average intrinsic value.

```

We draw a total of n sample paths for variance and exchange rate. For each sample path, we create one discrete time step for each business day. At the end of the single path simulation, we accumulate the option values

for all sample paths given some payoff formula. The final estimate then is the mean of the payoffs.

The challenges with simulation of exotic payoffs is mostly within the payoff function. It might be easy to implement payoffs like an european call option - $\max(0, S - K)$, but on the other hand can be more challenging for american options for example.

This chapter introduced the necessary methods for simulating the Heston price and variance process given a calibrated parameter set, we found in the previous chapter.

9 Volatility Knock Out option analysis

After revising several different methodologies, theoretical concepts and numerical techniques, we have gathered all necessary puzzle pieces to successfully analyse the Volatility Knock Out option. As mentioned in chapter 2, this kind of product is rather new and hence not yet subject to any academic papers.

The inclusion of a new product in the investment universe of a portfolio manager requires detailed knowledge about the imbedded risks and the individual sensitivities.

Hence we priced a Volatility Knockout put option on the EUR-USD exchange rate based on the data of the 10.11.2016 and conducted several analyses.

To begin with, we simulate paths for the underlying and the variance process for 1 year (252 steps) by using equations (8.9) and (8.8) and afterwards calculate the realized volatility. The realized volatility is defined as:

$$\text{realized volatility} = \text{RV} = \sum_{i=1}^n \log\left(\frac{S_i}{S_{i-1}}\right)^2$$

As demonstrated in figure (9.1) the majority of the simulated paths show realized volatilities below 15%.

For further insights we calculated the expected payoffs of this product for different realized volatility barrier levels. Additionally we included the delta and vega sensitivity of the option.

- The delta will be simulated by shifting the underlying in the starting point by 1%.

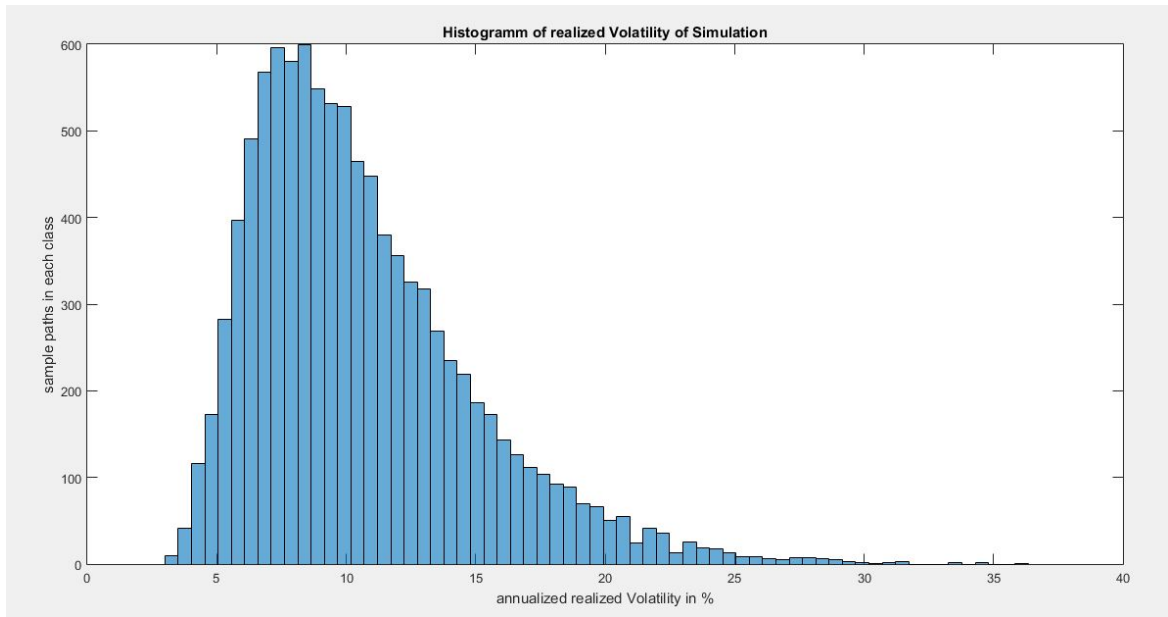


Figure 9.1: Histogram of realized volatility

Source: Own calculations

- The vega component is more difficult to handle. Since the implied volatility is expressed by the five parameters of the Heston model, it is not possible to apply a simple shift to express the vega of an option. Thus we have to manipulate the volatility surface before the calibration by moving the complete surface by 1% and afterwards recalibrating to estimate the new parameters.
- To omit the Monte Carlo error of estimation, we use the same standard normal random numbers for the sample paths, which ultimately gives us three different values for the volatility knock out put options with specific volatility barriers.

What we can clearly observe is the steep increase in payoffs between the volatility levels of 10% and 20%, which is closely linked to the big number of sample paths with realized volatility in this range. Compared to the histogram in figure (9.1), where we can see a high percentage of the historical distribution's mass in the region between 10% and 15%, a quickly decreasing quantity of sample paths with higher volatility can be analyzed. Nevertheless these paths have a heavy impact on the profitability of the product, due to the negative correlation between volatility and exchange rate.

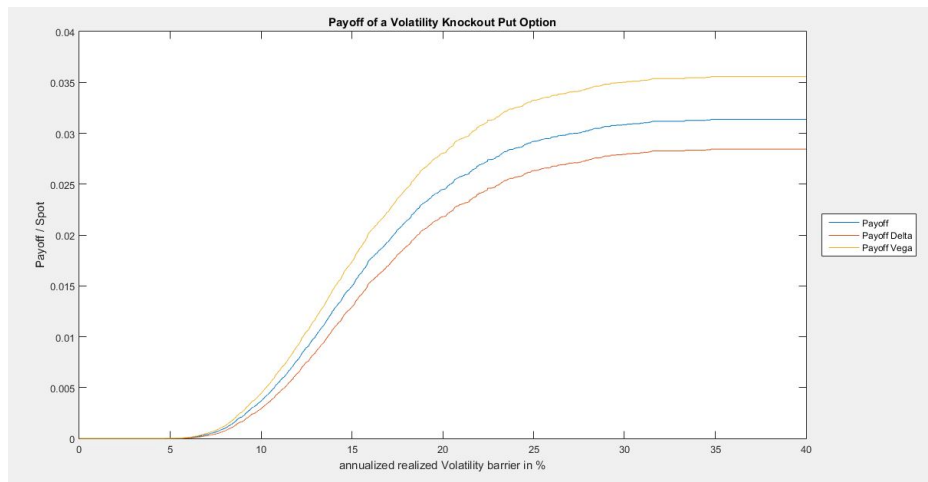


Figure 9.2: Expected payoff for a Volatility Knock Out option

Strike = 1.1 , Maturity = 1Y

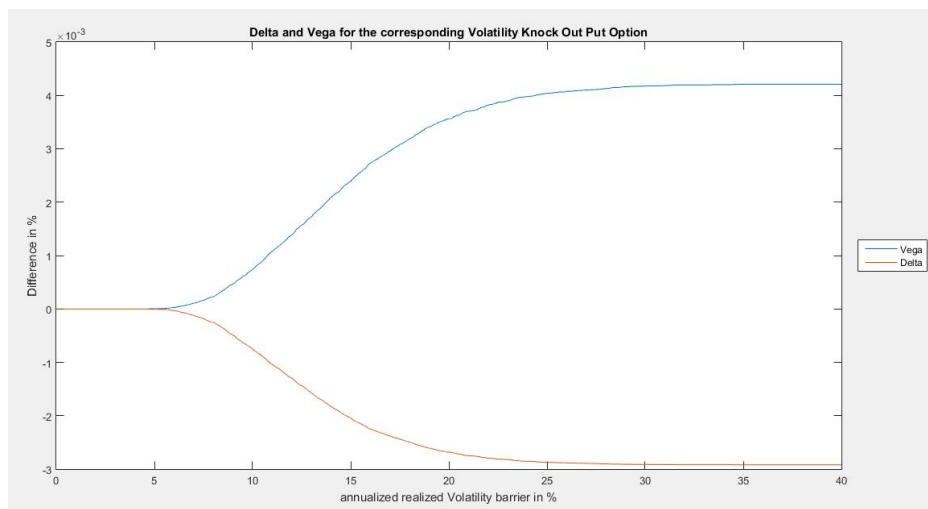


Figure 9.3: Expected delta and vega effects

Strike = 1.1 , Maturity = 1Y

The negative correlation is the reason for this effect since a negative correlation coefficient means that with increasing volatility the underlying's movement will be in the opposite direction which is already shown by the calculations in figure (9.1).

With the assumption of an overall increased volatility environment for the vega, the negative correlation may lead to better payouts at expiry and on the other hand, a positive shift of 1% of the exchange rate pushes the option further out of the money and thus it becomes less likely to expire with

a positive intrinsic option value.

We now look at a Volatility Knock Out option with 1 year time to maturity and plot the mark-to-market after 1 year.

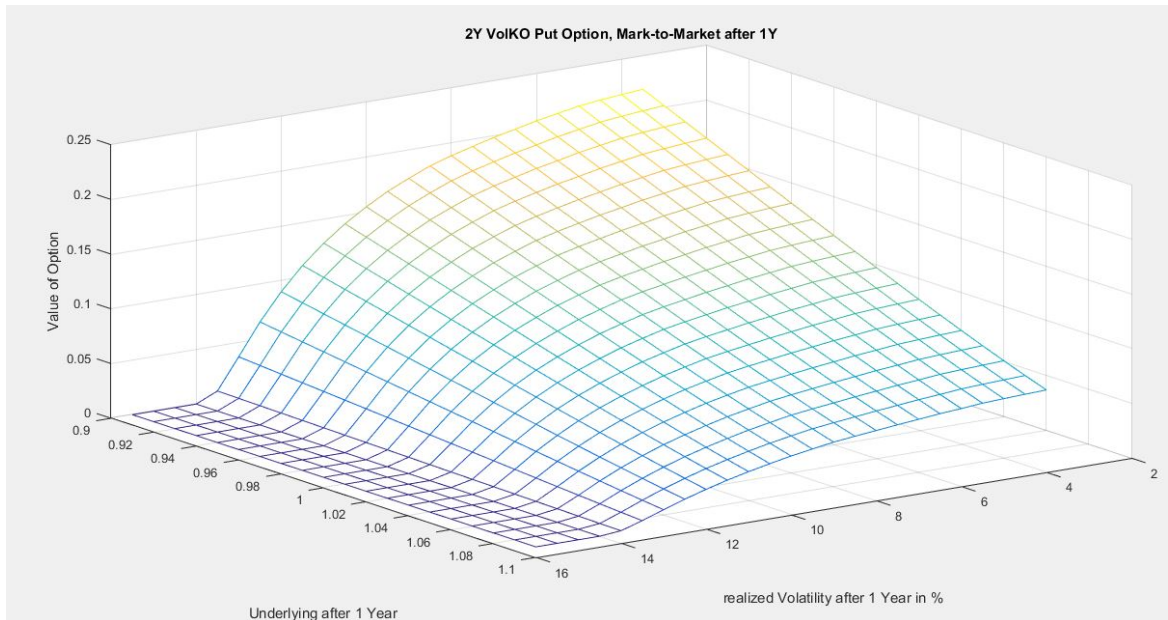


Figure 9.4: Mark to market of a Volatility Knock Out option after 1 year

Strike = 1.2, Volatility Barrier = 10%, Maturity = 2Y, MtM after 1Y

The figure (9.5) gives additional insights into the product's dynamics. Along the axis of the realized volatility after 1 year we see a non linear connection. With decreasing realized volatility the chances of getting knocked out decrease and the product converges to the unconditioned put option's value. The probability of getting knocked out is displayed in the next figure (9.5).

This way we can simulate and calculate all necessary sensitivities, show the most important dynamics of the product and furthermore gain valuable information about the behavior, such that a risk management can integrate this product into the eligible investment universe for portfolio managers. Hence we enable derivatives traders to realize complex trading strategies for extracting the volatility risk premium or use it for specific hedging purposes.

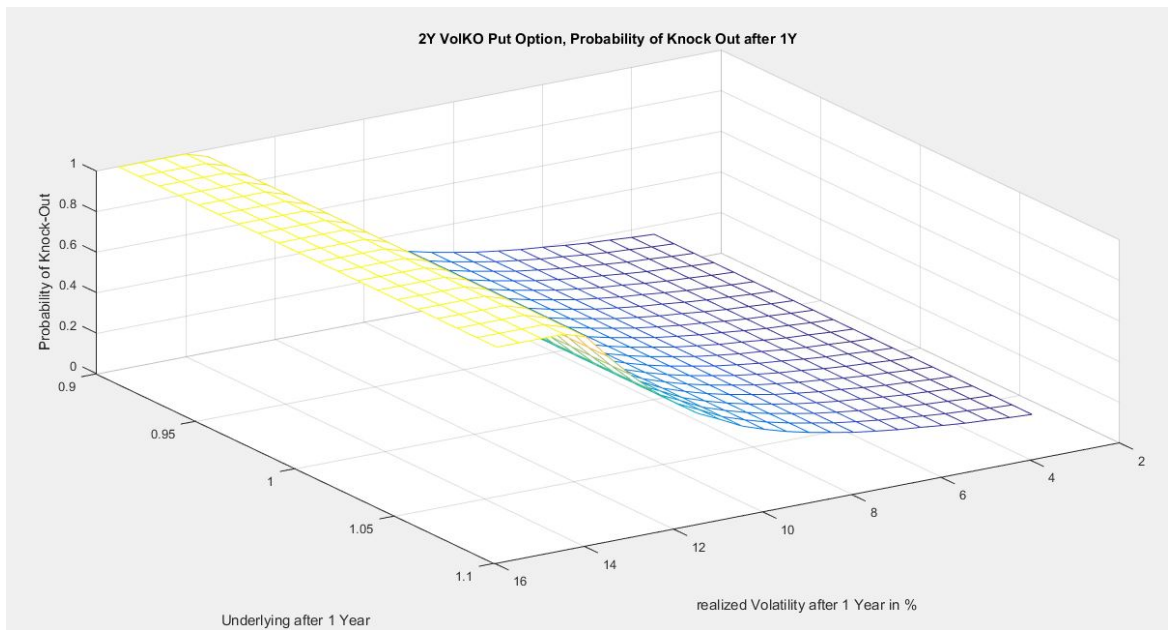


Figure 9.5: Knock out probabilities for a Volatility Knock Out option after 1 year

Strike = 1.2, Volatility Barrier = 10%, Maturity = 2Y, Probability of Knock-Out after 1Y

9.1 Volatility premium strategy

We have investigated the Volatility Knock Out option and concluded with a method to price those new, exotic products. The question that has not been answered yet is, "what for?". Since the product's price heavily depends on the volatility barrier, we can construct a trading strategy to extract the implied - realized volatility premium.

The idea is to enter a long position in a "1 year at the money Volatility Knock Out option" with a realized volatility barrier of the implied at the money volatility and simultaneously enter a short position in a similar vanilla put contract. The trading idea is as following:

- Scenario 1: realized volatility is below implied volatility barrier**
 Since we are both long and short a vanilla put contract, the cashflow at maturity is independent of the direction of the movement netted and we can collect the difference in prices between the VoKo and the vanilla.
- Scenario 2: volatility barrier got breached and the exchange rate**

strengthened

The volatility barrier was broken and hence the VoKo is terminated. Nevertheless since the exchange rate strengthened, the vanilla put option also matures with an intrinsic value of zero and thus we also collect the price difference between the VoKo and the vanilla option.

- **Scenario 3: volatility barrier got breached and the exchange rate weakened**

The last scenario is the harmful outcome for the trading strategy. The vanilla put option matures with a positive value and since our long position which should offset the potential negative cashflow is knocked out, the strategy has a negative performance in this scenario.

To show this strategy, we traded every month according to the description above from 2010 until end of 2015. The following figure (9.6) shows the movement of the fx rate in this period.

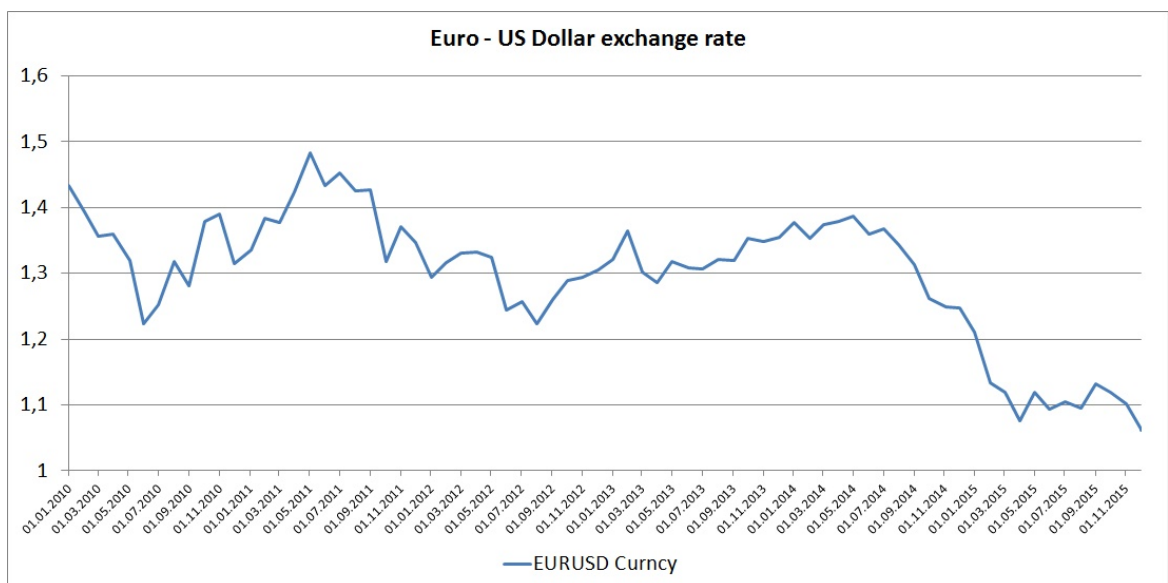


Figure 9.6: Foreign exchange rate from 2010 until 2015. EURUSD Currency

Source: Bloomberg data with own illustration

The next two graphs illustrate the different parts of the strategy. On the one hand, we have a steady income flow by buying cheaper options and selling the more expensive ones as can be seen in figure (9.7). On the other hand the figure (9.8) illustrates the accumulated outflows given that scenario 3 will happen.



Figure 9.7: Accumulated inflows by trading activity

Source: Own calculations



Figure 9.8: Accumulated outflows by trading activity

Source: Own calculations

Combined in a total return graph we have:

We can see a steady increment of the strategy's value from 2010 until the beginning of 2015 and then severe losses. This is mostly due to the drastic fall of the foreign exchange rate and also due to the very low implied volatility barrier we can see in figure (9.10).

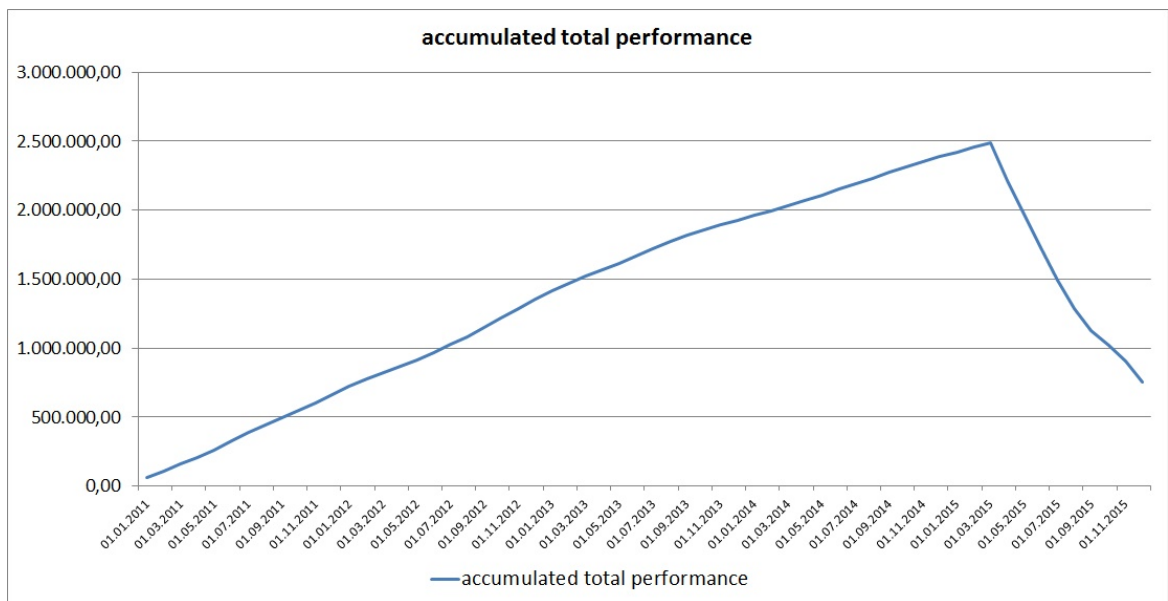


Figure 9.9: Total return graph of trading strategy

Source: Own calculations

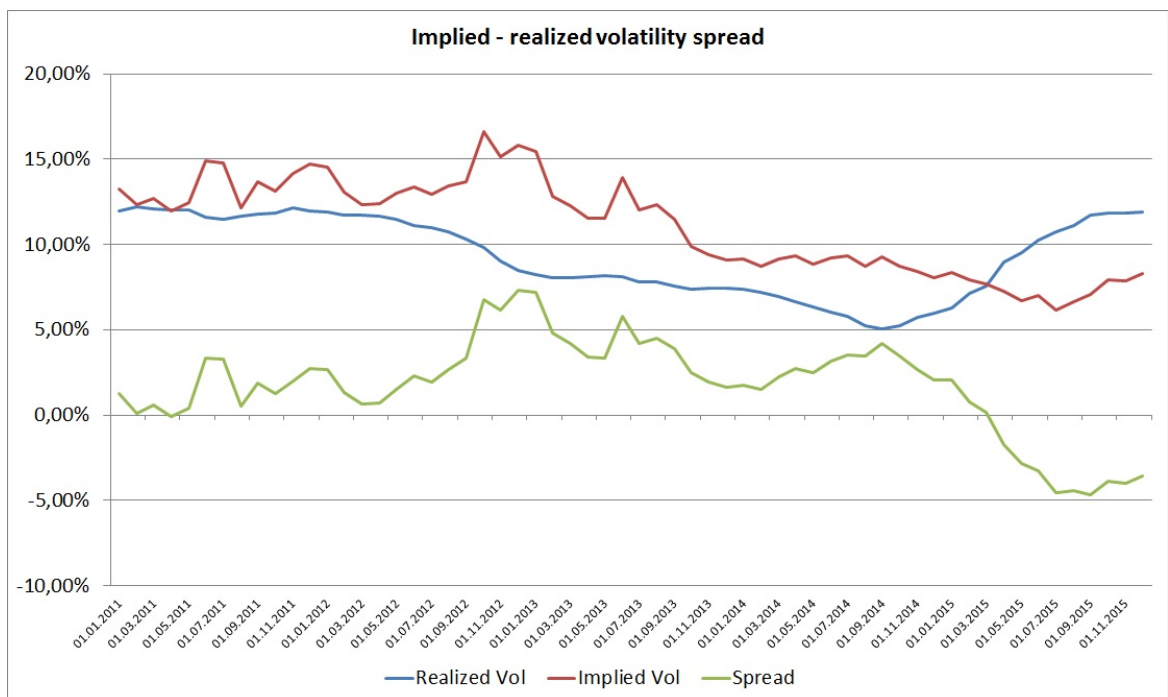


Figure 9.10: Realized - implied volatility spread

Source: Bloomberg data with own illustration

To omit such unfavorable drawdowns, we suggest that the execution of this strategy is restricted to an implied volatility above a certain barrier.

10 Conclusion

The findings we have produced, may help to get a good understanding of the exotic product and it's possibilities to enrich an investment universe. For any reasonable investor it is crucial to have a deep understanding of the products they invest in and hence, we provided a guide to develop this within this thesis.

On the way to find a pricing we introduced many methods in various fields that have been necessary and because of this the overall performance might be improved manifold. From challenging the Heston model by substitute it with the Double Heston model up to stating different calibration methods or implementing another discretization algorithm many improvements are imaginable. Whatsoever the different approaches do not guarantee a better overall pricing mechanism.

What we have found is that the negative movement of the underlying is not always beneficial for Volatility Knock Out put options, since to the negative correlation between underlying and volatility a knock out due to a breach of the volatility budget is more likely. Furthermore we have found that the product is capable of harvesting the volatility premium as we have seen in the implementation of the trading strategy. At this point we want to add, that it is most likely not a pure volatility premium, but rather a combination with the skew risk premium.

A further analysis of the trading strategy might lead to an lucrative investment opportunity. What we could observe however is, that a systematic approach without further restrictions can lead to substantial losses and thus we plead rather for a discretionary implementation of such a strategy than a automatized algorithm. We believe that in near future the Volatility Knock Out option will be on the list of eligible products for many alternative investment funds. Since the product, if used in a favorable market environment, offers the possibility for an investor to great return opportunities on the one hand or cost efficient ways of hedging on the other hand.

Bibliography

- Albrecher, H., Mayer, P., Schoutens, W., and Tistaert, J. (2006). The little heston trap.
- Bakshi, G., Cao, C., and Chen, Z. (1997). Empirical performance of alternative option pricing models. *The Journal of finance*, 52(5):2003–2049.
- Bams, D., Lehnert, T., and Wolff, C. C. (2009). Loss functions in option valuation: a framework for selection. *Management Science*, 55(5):853–862.
- Biagini, F., Hu, Y., Øksendal, B., and Zhang, T. (2008). *Stochastic calculus for fractional Brownian motion and applications*. Springer Science & Business Media.
- Bisesti, L., Castagna, A., and Mercurio, F. (2005). Consistent pricing and hedging of an fx options book. *The Kyoto Economic Review*, pages 65–83.
- Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *The journal of political economy*, pages 637–654.
- Burden, R. L. and Faires, J. D. (1985). 2.1 the bisection algorithm. *Numerical Analysis*. Prindle, Weber & Schmidt, Boston, MA., pp. x, 676.
- Castagna, A. (2010). *FX options and smile risk*. John Wiley & Sons.
- Castagna, A., Mercurio, F., and Tarenghi, M. (2007). Smile-consistent cms adjustment in closed form: Introducing the vanna-volga approach. Technical report, Working paper.
- Christoffersen, P. and Jacobs, K. (2004). The importance of the loss function in option valuation. *Journal of Financial Economics*, 72(2):291–318.

- Clark, I. J. (2011). *Foreign exchange option pricing: a practitioner's guide*. John Wiley & Sons.
- Cohen, H. (2011). *Numerical approximation methods*. Springer.
- Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, pages 385–407.
- DeRosa, D. F. (2011). *Options on foreign exchange*, volume 437. John Wiley & Sons.
- Gao, F. and Han, L. (2012). Implementing the nelder-mead simplex algorithm with adaptive parameters. *Computational Optimization and Applications*, 51(1):259–277.
- Gatheral, J. (2006). *The implied volatility surface: A practitioner's guide*.
- Gatheral, J. (2011). *The volatility surface: a practitioner's guide*, volume 357. John Wiley & Sons.
- Gil-Pelaez, J. (1951). Note on the inversion theorem. *Biometrika*, 38(3-4):481–482.
- Glasserman, P. (2003). *Monte Carlo methods in financial engineering*, volume 53. Springer Science & Business Media.
- Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343.
- Hogg, R. V. and Klugman, S. A. (2009). *Loss distributions*, volume 249. John Wiley & Sons.
- Hull, J. (2009). *Optionen, futures und andere derivate*, volume 1. Pearson Deutschland GmbH.
- Janek, A., Kluge, T., Weron, R., and Wystup, U. (2011). Fx smile in the heston model. In *Statistical Tools for Finance and Insurance*, pages 133–162. Springer.
- Kahl, C. and Jäckel, P. (2005). Not-so-complex logarithms in the heston model. *Wilmott magazine*, 19(9):94–103.

- Kahl, C. and Jäckel, P. (2006). Fast strong approximation monte carlo schemes for stochastic volatility models. *Quantitative Finance*, 6(6):513–536.
- Klebaner, F. C. et al. (2005). *Introduction to stochastic calculus with applications*, volume 57. World Scientific.
- Kloeden, P. E. and Platen, E. (1992). Stochastic differential equations. In *Numerical Solution of Stochastic Differential Equations*, pages 103–160. Springer.
- Mbele, B. M. L. D. (2004). Libor and swap market models for pricing interest rate derivatives monte carlo simulation.
- Mikhailov, S. and Nögel, U. (2004). *Heston’s stochastic volatility model: Implementation, calibration and some extensions*. John Wiley and Sons.
- Nelder, J. A. and Mead, R. (1965). A simplex method for function minimization. *The computer journal*, 7(4):308–313.
- Pelsser, A. (2000). *Efficient methods for valuing interest rate derivatives*. Springer Science & Business Media.
- Reiswich, D. and Uwe, W. (2012). Fx volatility smile construction. *Wilmott*, 2012(60):58–69.
- Rouah, F. D. (2011a). Euler and milstein discretization. *FRouah.com*.
- Rouah, F. D. (2011b). Four derivations of the black scholes pde.
- Rouah, F. D. (2013). *The Heston model and its extensions in MATLAB and C#*. John Wiley & Sons.
- Storn, R. and Price, K. (1997). Differential evolution—a simple and efficient heuristic for global optimization over continuous spaces. *Journal of global optimization*, 11(4):341–359.
- Vollrath, I. and Wendland, J. (2009). Calibration of interest rate and option models using differential evolution. *Available at SSRN 1367502*.
- Woodard, J. (2011). *Options and the Volatility Risk Premium*. Pearson Education.

Curriculum Vitae

Müller Stefan BSc

Haidmannsgasse 2/15 – 1150, Vienna – Austria – Date of Birth: 06.07.1989
+43 676 974 7222 – mueller.stefan@outlook.com

EDUCATION

University of Vienna <i>MSc. in Quantitative Economics, Management and Finance</i> Master Thesis: “Volatility Knock Out Options”	Vienna, Austria 09/2012 – 05/2016
Vienna University of Economics and Business <i>BSc. in Economics</i> Bachelor Thesis: “Efficient Market Hypothesis in the course of time”	Vienna, Austria Expected 2016
University of Vienna <i>BSc. in theoretical Mathematics, Specialization in Financial Mathematics</i> Bachelor Thesis: “Calibration of implied volatility and calculation of barrier options” “Variance reduction techniques”	Vienna, Austria 10/2008 – 06/2012
Universidad de Valencia <i>Undergraduate Exchange Program</i>	Valencia, Spain 09/2014 – 01/2015
Global Association of Risk Professionals (GARP) <i>FRM Level I Candidate</i>	Vienna, Austria 05/2016

PROFESSIONAL EXPERIENCE

UNIQA Capital Markets <i>Tactical Asset Allocation Manager</i> - Development and implementation of allocation models for the tactical asset allocation for Austria’s biggest insurance company. - Idea generation and numerical modeling of exotic derivatives and trading ideas for the Platinum I SICAV - Diversified Risk Premium Fund. - Backtesting of trading ideas and overlay strategies to cost-efficiently increase risk/return profiles. - Development of reporting tools based on existing database solutions to improve management’s information basis.	Vienna, Austria 03/2015 – today
Raiffeisenbank International <i>Freelancer (20-25h / week) – Liquidity Risk Management & Investment Finance</i> - Maintenance and improving of liquidity models for network units of the bank. - Improved existing and developed and programmed additional VBA based Excel tools. - Performed quantitative analysis related to loan structuring. - Supported the team in modeling and structuring of interest payments on EIB loans.	Vienna, Austria 07/2012 – 08/2014
Allianz Insurance AG <i>Internship (20h / week)</i> Testing and maintaining the IT system for insurance offers.	Vienna, Austria 02/2012 – 09/2012
Financial market authority Austria <i>Internship (20h / week)</i> - Insurance control – Solvency II assessments. - Evaluation of the implementation process for the Solvency II requirements.	Vienna, Austria 09/2011 – 12/2011
Santander Consumer Bank <i>Internship</i>	Vienna, Austria 07/2011 – 08/2011
Schülerhilfe <i>Tutor for mathematic lessons</i>	Vienna, Austria 02/2008 – 08/2010

SKILLS

Languages:	German (native) – English (fluent) – Spanish (basic)
Computing:	MS Office (proficient) – VBA (proficient) – SQL (proficient) – MATLAB (proficient) – R (basic) – Bloomberg Terminal – BLAN Scripting language

INTERESTS & OTHER ACTIVITIES

- Participant at the win² - Zukunftskonferenz 2014 & 2013
- M&A Workshop at the Vienna University of Economics and Business organized by Morgan Stanley
- Traveling – Sports – Economics – Politics
- Marathon: Valencia (2014), Vienna (2015)

Müller Stefan - mueller.stefan@outlook.com - +43 676 974 7222 - Haidmannsgasse 2/15, 1150 Vienna