DISSERTATION

Titel der Dissertation

Some Topics in Mathematical Water Wave Theory

Verfasser

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Abstract

This thesis comprises selected publications that are representative of the directions of research that were explored during the PhD studies. It is dedicated to the mathematical study of water waves. Throughout, we model water as an inviscid, incompressible fluid under the influence of gravity. Making a traveling wave ansatz, we will be studying different aspects of periodic water waves in finite depth.

The first paper is devoted to the relation between the pressure at the flat bed and the wave profile. We obtain an exact relation that allows for a recovery formula in terms of Fourier series.

The second and the third paper deal with the connection between symmetry of waves and traveling wave solutions. We show that a solution to the governing equations with the property that the horizontal velocity component at the free surface as well as the free surface itself are symmetric with respect to a common, time-dependent axis of symmetry necessary defines a traveling wave. A similar result holds true if we require the horizontal velocity component at the flat bed to be symmetric and the acceleration at the flat bed to be antisymmetric at any instant of time.

In the fourth paper, we are concerned with stability properties of periodic gravity waves in finite depth. Taking advantage of a one-dimensional reformulation of the governing equations as a pseudo-differential equation that admits a variational structure, we show that certain solutions are linearly stable. The proof relies upon estimates for the Hilbert transform on the strip, a nonlinear coordinate change in the infinite dimensional part of the Lagrangian and spectral properties of matrices of operators.

Contents

Introduction

The governing equations for free-boundary fluid flows are of particular difficulty. They consist of a system of time-dependent nonlinear partial differential equations, the Euler equations, posed on an a priorily unknown domain, together with nonlinear boundary conditions. We will be dealing with a model that includes the influence of gravity, but neglects surface tension effects, since we are interested in water waves in scales of meters. In the study of this problem, which is of great interest both in applied and in pure mathematics, several simplifying approaches have been considered. Apart from modeling equations, like the famous Korteweg-de-Vries of the Camassa-Holm equation, which rely upon asymptotic smallness of parameters, the traveling wave ansatz proved to be a useful and physically applicable choice. Keeping all the nonlinear effects of the initial problem, the time dependence is removed by assuming that the solution is given by a fixed wave profile that is translated in time. These wave solutions can be observed in real life, for instance in coastal regions before wave-breaking, and are called *swells* in the periodic case, cf. Figure 1.

Once existence of traveling wave solutions is guaranteed, for instance via bi-

Figure 1: Periodic wave motion on the Grundlsee, Austria 2015

furcation theory, there are several natural questions, both physically and mathematically motivated, that could be asked about swells. Is there a comparably simple relation between the pressure, the flow field and the free surface in the traveling wave case? How much information about a wave is minimally needed to uniquely determine all its remaining physical quantities? Are the wave profile and the underlying flow always symmetric? How stable are those waves with respect to small perturbations? In the following, we will analyze some of these questions using mathematical methods, ranging from complex analysis, over classical theorems in PDEs to calculus of variations and spectral theory.

For the prediction of wave motions in field experiments, pressure data are used to infer properties of the wave surface. Using in situ techniques at the bottom of the sea and assuming the *hydrostatic approximation*, the elevation of the free surface can be estimated. This approach is rather crude, since it only holds true for waves of small amplitude. If we take into account vertical variations in the flow field of the wave, we obtain, at a linear level of approximation, the so called *transfer function formula*. Although more accurate, this formula misses nonlinear effects, which become prominent for waves of large amplitude. In the first paper of this thesis, we describe a recovery formula for the free surface in terms of the pressure at the bottom of the sea, that takes into account all nonlinear effects and therefore provides an exact representation of the wave profile as a Fourier series.

All known solution to the periodic water wave problem are symmetric. It therefore seems likely that there only exist symmetric solutions. This intuition is confirmed by a series of theorems assuming that the solutions have a characteristic feature that can be observed easily and then deduce symmetry. The sharpest of these results up today states that once the wave profile of a periodic traveling wave is monotonic between successive crest and troughs, then the wave has to be symmetric. The proof makes use of the so called *moving plane method*. Also the converse question is of interest: If the solution is symmetric at any instant of time with respect to a time-dependent axis of symmetry, is it true that the solution automatically defines a traveling wave? This relation between traveling waves and symmetry was pursued in [4] and answered affirmatively. However, the proof relied upon the assumption of symmetry for the free surface, flow field and the pressure throughout the fluid domain, which makes it hard to check the premises experimentally. In the second and third paper, we propose two reduced sets of assumptions that can be checked in an experiment, which suffice to ensure that the solution defines a traveling wave. Namely, in [5], we show that a symmetric wave profile together with a symmetric horizontal velocity component at the free surface at any instant of time guarantee a traveling wave solution, while in [6], we prove that symmetry of the horizontal velocity component at the flat bed together with antisymmetry of the acceleration at the flat bed suffice to deduce a traveling wave solution. The proof in the first case is based upon maximum principles for elliptic PDEs and the proof in the second one is based upon properties of holomorphic functions. These two results relate to the first paper insofar, as information about the two dimensional fluid domain is encoded in parts of its one dimensional boundary.

In fact, this is true for the whole equation. We refer to [3, 2], where the authors derived a reformulation of the governing equations as a one dimensional pseudo-differential equation on the circle. The nonlocal operator involved is similar to the well-known Hilbert transform, which relates the real part to the imaginary part of the boundary value of a function holomorphic in the half plane. This reformulation allows for a variational structure, as the equations are the Euler-Lagrange equations for suitably defined functional and thus allows to study linear stability properties of potential solutions.
In general, water waves are expected to be unstable. The famous Benjamin-Feir or sideband instability, cf. [1], is an example of an instability result for nonlinear water waves. On the other hand, modeling equations for nonlinear water waves relying upon, say, an long-wave asymptotic expansion, exhibit very strong stability properties. Thus, it is likely that also the full problem possesses

some notion of stability for small depths or large periods respectively. Indeed, we will prove that the smaller the mean depth of a wave the more stable the wave. To be more precise, we can show that the trivial solution to the irrotatinoal water wave problem is linearly stable provided that

$$c^2 \geq gd,$$

where c is the constant wave speed of the traveling wave, d is its mean depth and g is the constant acceleration due to gravity. Our prove relies upon spectral properties of matrices of operators, a change of coordinates in the infinite dimensional part of the Lagrangian and estimates for the Hilbert transform on a strip.

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Recovery of the wave profile for irrotational periodic water waves from pressure measurements

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Abstract

Using a conformal hodograph transform and the governing equations for periodic traveling irrotational water waves, we obtain a relation between the bottom pressure and the free surface, expressed in terms of Fourier coefficients. This permits us to perform a formal asymptotic analysis that yields an approximate solution for the free surface provided the pressure at the flat bed is known.

1 Introduction

Information about water waves, such as the shape of the free surface, are often obtained in field experiments by pressure measurements at the bottom, cf. [9]. For periodic waves, the two simplest and most widely used relations between the pressure at the bed and the free surface are the so called *hydrostatic approximation*, cf. [10],

$$\eta(x) = \frac{1}{g}\mathfrak{p}(x), \qquad x \in \mathbb{R}, \tag{1}$$

where η is the free surface, g is the gravitation of the earth and $\mathfrak{p}$ is the the dynamic pressure at the flat bed, and the *transfer-function formula*

$$\eta(x) = \frac{\cosh(dy)}{g}\mathfrak{p}(x) - d, \qquad x \in \mathbb{R}, \tag{2}$$

where d is the average water depth and $y = 2\pi/L$ is the wavenumber corresponding to the wavelength $L > 0$. The latter is derived in the context of waves of small amplitude in finite depth, cf. [7]. However, these expressions do not take into account non-linear effects and sometimes differ considerably from the measured behavior of the wave, cf. [12]. Non-linear relations are, for instance, obtained in [10].

A different approach, described in [2, 1], is based on complexifying the velocity potential and taking advantage of the properties of holomorphic functions. We will follow [4], relying on the Fourier transform in hodograph coordinates. Adopting this approach, we will obtain an analogous relation for the Fourier coefficients of a periodic traveling wave. Namely, we will show that the parametric representation

$$\begin{cases} x(q) = & \dfrac{d}{M}q + 2\displaystyle\sum_{n=1}^{\infty}\left\{\dfrac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}}\right\}^{\wedge}(n)\dfrac{\cosh(nM)}{n}\sin(nq) \\ \eta(q) = & 2\displaystyle\sum_{n=1}^{\infty}\left\{\dfrac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}}\right\}^{\wedge}(n)\dfrac{\sinh(nM)}{n}\cos(nq) \end{cases}$$

for the wave profile holds true.

2 Preliminaries

We are concerned with the study of two-dimensional, periodic gravity water waves with zero vorticity. Let $(u(X,Y,t), v(X,Y,t))$ be the velocity field of the flow over the flat bed $\{Y=-d\}$ and beneath the free surface $\{Y=\eta(X,t)\}$, where $d>0$ is the average depth of the fluid. Because $Y=0$ is the mean water level for all times, we have that

$$\langle\eta\rangle := \int_0^L \eta(X,t)\,dX = 0 \tag{3}$$

where $L>0$ is the wave period. Let us derive the governing equations for the water wave problem. We approximate water as an incompressible, inviscid fluid of constant density $\rho \approx 10^3$ kg/m^3, cf. [3], so that the equation of mass conservation takes the form

$$u_X + v_Y = 0 \tag{4}$$

and Euler's equations become

$$\begin{cases} u_t + uu_X + vu_y = -P_X \\ u_t + uv_X + vv_Y = -P_Y - g \end{cases} \tag{5}$$

where $P=P(X,Y,t)$ is the pressure and $g\approx 9.81$ m/s^2 is the gravity of the Earth.
The motion of the fluid beneath the surface is decoupled from the motion of the air above. This is expressed in the dynamic boundary condition

$$P = P_{atm} \text{ on } Y=\eta(X,t) \tag{6}$$

where P_{atm} is the constant atmospheric pressure. Furthermore, we obtain the kinematic boundary condition from the fact that the same particles always form the free surface:

$$v = \eta_t + u\eta_X \text{ for } Y=\eta(X,t). \tag{7}$$

The fluid cannot penetrate flat bed, which means that

$$v = 0 \text{ on } Y=-d \tag{8}$$

We are dealing with flows with zero vorticity by assuming that

$$u_Y = v_X \tag{9}$$

throughout the fluid.

All u, v, P and η are L-periodic functions in the x-variable.

Let us look for *traveling wave* solutions to the above equations. Assume that there exists a $c > 0$, the *wave speed*, such that u, v, P depend on $(X - ct)$ and Y, while η only depends on $(X - ct)$. Moreover, there is only a single crest and a single trough per period, the profile is decreasing from crest to trough and the functions η, u and P are symmetric, whereas v is antisymmetric about the crest line.
We may reformulate the governing equations in the variables $x := X - ct$ and $y := Y$ as

$$\begin{cases} u_x + u_y = 0 & \text{for } (x, y) \in \Omega \\ (u - c)u_x + vu_y = -P_x & \text{for } (x, y) \in \Omega \\ (u - c)v_x + vv_y = -P_y - g & \text{for } (x, y) \in \Omega \\ u_y = u_x & \text{for } (x, y) \in \Omega \\ P = P_{atm} & \text{for } y = \eta(x) \\ v = (u - c)\eta_x & \text{for } y = \eta(x) \\ v = 0 & \text{for } y = -d \end{cases} \tag{10}$$

where we have set $\Omega := \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}$.
In the moving frame, Bernoulli's Law takes the form

$$\tilde{P} + g(y + d) + \frac{(u - c)^2 + v^2}{2} = Q \tag{11}$$

for $(x, y) \in \Omega$ and for some constant Q, called the *hydraulic head*, and where

$$\tilde{P}(x, y) = P(x, y) - P_{atm} \text{ for } (x, y) \in \Omega \tag{12}$$

is the normalized relative pressure.
We may take advantage of (4) to define the *stream function* $\psi(x, y)$ as a solution to

$$\begin{cases} \psi_y = u - c \\ \psi_x = -v \end{cases} \tag{13}$$

It follows from (9) that ψ is a harmonic function in Ω and the boundary conditions (7) and (8) imply that the stream function is constant on the flat bed and on the free surface. We normalize ψ such that $\psi(x, \eta(x)) = 0$ and $\psi(x, -d) = m$, so we can explicitly write it as

$$\psi(x, y) = m + \int_{-d}^{y} \{u(x, s) - c\}\, ds. \tag{14}$$

The constant m can be interpreted as the relative mass flux. Clearly, by definition, ψ is also L-periodic in the x-variable. The governing equations (10) are equivalent to the following system

$$\begin{cases} \Delta\psi = 0 & \text{for } (x, y) \in \Omega \\ \frac{|\nabla\psi|^2}{2} + g(y + d) = Q & \text{for } y = \eta(x) \\ \psi = 0 & \text{for } y = \eta(x) \\ \psi = m & \text{for } y = -d \end{cases} \tag{15}$$

For a non-trivial solution to (15), it follows $m \neq 0$, cf. [11, 13]. With out loss of generality, assume that $m > 0$, so that

$$\psi_y = u - c < 0 \text{ in } \bar{\Omega} \tag{16}$$

by the strong maximum principle, cf. [8].
Let us define the mean current κ on the flat bed via

$$\kappa := \frac{1}{L} \int_0^L u(x, -d)\, dx. \tag{17}$$

It follows from (16) that $\kappa < c$. Note that we allow a uniform underlying current ($\kappa \neq 0$), whereas [2] is devoted to the case of no underlying current.

Without loss of generality, we assume in the following that the wave length is normalized to $L = 2\pi$. Furthermore, we assume that the wave crest is at $(0, \eta(0))$ and the wave troughs are at $(\pm\pi, \eta(\pm\pi))$ respectively.
We introduce the velocity potential ϕ as a solution to the system

$$\begin{cases} \phi_x = u - c \\ \phi_y = v \end{cases} \tag{18}$$

with the normalization $\phi(0, y) = 0$ for $-d < y < \eta(0)$. Thanks to condition (9) we can write ϕ explicitly as

$$\phi(x, y) = \int_0^x \{u(r, -d) - c\}\, dr + \int_{-d}^y v(x, s)\, ds \tag{19}$$

From the definition of the mean current (17) it follows that the function $\phi(x, y) + (c - \kappa)x$ is 2π-periodic in x. The function ϕ is an odd function in x and $\phi(0, y) = 0$ for $-d \leq y \leq \eta(0)$ because of the boundary condition (8).

The stream function and the velocity potential permit us to introduce new variables. Define the hodograph transform $\mathcal{H} : \bar{\Omega} \to \bar{\mathcal{S}}$ by

$$\mathcal{H}(x, y) = \begin{pmatrix} q \\ p \end{pmatrix} := \frac{1}{\tilde{c}} \begin{pmatrix} -\phi(x, y) \\ -\psi(x, y) \end{pmatrix} \tag{20}$$

where $\tilde{c} := c - \kappa$ and, setting $M := \frac{m}{\tilde{c}}$,

$$\mathcal{S} = \left\{(q, p) \in \mathbb{R}^2 : -M < p < 0\right\} \tag{21}$$

is a strip in the q, p-plane. The function $\mathcal{H}$ is a smooth bijection, cf. [5] and also [6] . We can now transform the free boundary problem (15) into Laplace's equation with a non-linear boundary term for the height function $h : \bar{\mathcal{S}} \to \mathbb{R}$

$$h(q, p) = y + d. \tag{22}$$

The transformed problem looks as follows:

$$\begin{cases} \Delta_{(q,p)} h = 0 & \text{for } (q, p) \in \mathcal{S} \\ h = 0 & \text{for } p = -M \\ 2(Q - gh)(h_q^2 + h_p^2) = \tilde{c}^2 & \text{for } p = 0 \end{cases} \tag{23}$$

The function h is even, non-negative and 2π-periodic in q. Moreover, h is bounded for all $(q,p) \in \bar{\mathcal{S}}$:

$$0 \leq h(q,p) \leq \sup_{x \in [0,2\pi]} \eta(x) + d = \eta(0) + d. \tag{24}$$

The following changes of coordinates for the derivative can be verified easily:

$$\begin{cases} \partial_q = h_p \partial_x + h_q \partial_y \\ \partial_p = -h_q \partial_x + h_p \partial_y \end{cases} \tag{25}$$

and conversely

$$\begin{cases} \partial_x = \dfrac{1}{\tilde{c}} \left[(c-u)\partial_q + v\partial_p \right] \\ \partial_y = \dfrac{1}{\tilde{c}} \left[-v\partial_q + (c-u)\partial_p \right]. \end{cases} \tag{26}$$

Also we have

$$\begin{cases} h_q = -\tilde{c} \dfrac{v}{(u-c)^2 + v^2} & = -\dfrac{\partial x}{\partial p} = \dfrac{\partial y}{\partial q} \\ h_p = \tilde{c} \dfrac{(c-u)}{(u-c)^2 + v^2} & = \dfrac{\partial x}{\partial q} = \dfrac{\partial y}{\partial p}. \end{cases} \tag{27}$$

3 Recovery of the wave profile

3.1 Exact parametric representation

If we rewrite Bernoulli's Law in the hodograph coordinates, we obtain

$$\frac{\tilde{c}^2}{2(h_q^2 + h_p^2)} + gh + \tilde{P} = Q, \quad (q,p) \in \bar{\mathcal{S}}. \tag{28}$$

Evaluating the above equation at $p = -M$ and using (23), we have

$$\frac{\tilde{c}^2}{h_p^2(q,-M)} = 2Q - 2\mathfrak{p}(q), \quad q \in \mathbb{R} \tag{29}$$

where

$$\mathfrak{p}(q) := \tilde{P}(q,-M), \quad q \in \mathbb{R} \tag{30}$$

is the normalized relative pressure at the flat bed in hodograph coordinates. From (16) and (29) we infer that

$$h_p(q,-M) = \frac{\tilde{c}}{\sqrt{2Q - 2\mathfrak{p}(q)}}, \quad q \in \mathbb{R}. \tag{31}$$

Since h is 2π-periodic and bounded from above by $\eta(0) + d$, we can compute its Fourier coefficients with respect to q:

$$\hat{h}(n,p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(q,p) e^{-inq} \, dq, \quad n \in \mathbb{Z} \tag{32}$$

The fact that h is a harmonic function in $\mathcal{S}$ translates to the following sequence of differential equations

$$\hat{h}_{pp}(n,p) = n^2\hat{h}(n,p), \quad n \in \mathbb{Z} \tag{33}$$

which have the general solution

$$\hat{h}(n,p) = \begin{cases} A(n)e^{np} + B(n)e^{-np} & \text{if } n \neq 0 \\ A(0)p + B(0) & \text{if } n = 0 \end{cases} \tag{34}$$

for $-M \leq p \leq 0$ and some sequences $A, B : \mathbb{Z} \to \mathbb{C}$ which have to be determined. We can now make use of the boundary condition for $p = -M$ to obtain that $B(n) = -A(n)e^{-2nM}$ if $n \in \mathbb{Z}^*$, while $B(0) = MA(0)$ and consequently

$$\hat{h}(n,p) = \begin{cases} 2A(n)e^{-nM}\sinh n(p+M) & \text{if } n \neq 0 \\ A(0)(p+M) & \text{if } n = 0 \end{cases} \tag{35}$$

for $-M \leq p \leq 0$. Differentiating relation (35) with respect to p at $p = -M$ we obtain

$$\hat{h}_p(n,-M) = \begin{cases} 2A(n)e^{-nM}n & \text{if } n \neq 0 \\ A(0) & \text{if } n = 0 \end{cases} \tag{36}$$

Since both functions in (31) are bounded and 2π-periodic, we can compute their Fourier coefficients

$$\hat{h}_p(n,-M) = \left\{ \frac{\tilde{c}}{\sqrt{2Q - 2\mathfrak{p}(q)}} \right\}^\wedge (n), \quad n \in \mathbb{Z} \tag{37}$$

Comparing (36) and (37) we infer that

$$\left\{ \frac{\tilde{c}}{\sqrt{2Q - 2\mathfrak{p}(q)}} \right\}^\wedge (n) = \begin{cases} 2A(n)e^{-nM}n & \text{if } n \neq 0 \\ A(0) & \text{if } n = 0 \end{cases} \tag{38}$$

and consequently, due to (35) ,

$$\hat{h}(n,p) = \left\{ \frac{\tilde{c}}{\sqrt{2Q - 2\mathfrak{p}(q)}} \right\}^\wedge (n)\frac{\sinh\left(n(p+M)\right)}{n} \tag{39}$$

for $-M \leq p \leq 0$ and $n \in \mathbb{Z}$. In the case of $n = 0$, the right hand side has to be understood as the limit of $\hat{h}(n,p)$ as $n \to 0$.

The free surface in hodograph coordinates takes the form

$$\eta(q) = h(q,0) - d$$

The inverse of the hodograph transform is given in terms of the height function and a function corresponding to the x-variable by

$$x = \theta(q,p), \quad y = h(q,p) - d, \tag{40}$$

where $\theta : \bar{\mathcal{S}} \to \mathbb{R}$ is a smooth function. Using $h_p = \theta_q$, which follows from (**??**), and equation (39) we deduce that

$$\{\theta_q\}^\wedge (n,p) = \left\{ \frac{\tilde{c}}{\sqrt{2Q - 2\mathfrak{p}(q)}} \right\}^\wedge (n)\cosh(n(p+M)) \tag{41}$$

Since $\theta_q = h_p \geq 0$, it follows that $\theta(.,p)$ is monotone. Finally, if we put together (35) and (41) , we obtain the following parametric representation of the free surface.

$$\begin{cases} x(q) = & \dfrac{d}{M}q + 2\displaystyle\sum_{n=1}^{\infty}\left\{\frac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}}\right\}^{\wedge}(n)\frac{\cosh(nM)}{n}\sin(nq) \\ \eta(q) = & 2\displaystyle\sum_{n=1}^{\infty}\left\{\frac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}}\right\}^{\wedge}(n)\frac{\sinh(nM)}{n}\cos(nq) \end{cases}$$

where we have used that

$$\left\{\frac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}}\right\}^{\wedge}(0) = \frac{1}{M}\hat{h}(0,0) = \frac{d}{M}. \tag{42}$$

3.2 Asymptotic expansion

Let us try a formal asymptotic expansion of order two for the pressure at the flat bed. (Of course this can be done up to arbitrarily high orders, but the expressions get nastier and nastier.) In terms of the small amplitude parameter ε, we have

$$\mathfrak{p}(q) = p_0 + 2\varepsilon p_1\cos(q) + 2\varepsilon^2 p_2\cos(2q) + \mathcal{O}(\varepsilon^3), \tag{43}$$

for $q \in \mathbb{R}$. We have for $\tilde{\mathfrak{p}}(q) := \mathfrak{p}(q)/Q$, using the multinomial theorem

$$\begin{aligned}\frac{1}{\sqrt{1-\tilde{\mathfrak{p}}(q)}} &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\tilde{p}_0 + 2\varepsilon\tilde{p}_1\cos(q) + 2\varepsilon^2\tilde{p}_2\cos(2q) + \mathcal{O}(\varepsilon^3)\right)^n \\ &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\tilde{p}_0 + 2\varepsilon\tilde{p}_1\cos(q) + 2\varepsilon^2\tilde{p}_2\cos(2q))^n + \mathcal{O}(\varepsilon^3) \\ &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\sum_{k_0+k_1+k_2=n}\binom{n}{k_0,k_1,k_2}\tilde{p}_0^{k_0}(2\varepsilon\tilde{p}_1\cos(q))^{k_1}(2\varepsilon^2\tilde{p}_2\cos(2q))^{k_2} + \mathcal{O}(\varepsilon^3)\end{aligned}$$

In order to be strictly less than ε^3, the admissible triples for (k_0,k_1,k_2) in the sum are $(n,0,0),(n-1,1,0),(n-1,0,1)$ and $(n-2,2,0)$. This implies that the above expression is equal to

$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\Big[\tilde{p}_0^n + n\tilde{p}_0^{n-1}(2\varepsilon\tilde{p}_1\cos(q)) + n\tilde{p}_0^{n-1}(2\varepsilon^2\tilde{p}_2\cos(2q)) \\ &+\binom{n}{n-2}\tilde{p}_0^{n-2}(2\varepsilon\tilde{p}_1\cos(q))^2\Big] + \mathcal{O}(\varepsilon^3)\end{aligned}$$

Regrouping and inserting $\tilde{p}_i = \frac{p_i}{Q}, i = 0,1,2$ we find

$$\begin{aligned}\frac{\tilde{c}}{\sqrt{2Q-2\mathfrak{p}(q)}} =& \frac{\tilde{c}}{\sqrt{2}}\Bigg[\left(\frac{1}{\sqrt{Q+p_0}} + \frac{3\varepsilon^2 p_1^2}{4\sqrt{(Q+p_0)^5}}\right) + \left(-\frac{\varepsilon p_1}{2\sqrt{(Q+p_0)^3}}\right)\cos(q) \\ &+ \left(\frac{3\varepsilon^2 p_1^2}{2\sqrt{(Q+p_0)^5}} - \frac{\varepsilon^2 p_2^2}{\sqrt{(Q+p_0)^3}}\right)\cos(2q)\Bigg] + \mathcal{O}(\varepsilon^3).\end{aligned}$$

The above expression can now be used to determine the Fourier series of the height function h up to order ε^2.

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Symmetric irrotational water waves are traveling waves

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Abstract

We show that a spatially periodic solution to the irrotational two-dimensional gravity water wave problem, with the property that the horizontal velocity component at the surface, as well the wave profile are symmetric, necessarily defines a traveling wave. The proof makes use of maximum principles for harmonic functions and structural properties of the governing equations for nonlinear water waves.

4 Introduction

Periodic traveling water waves are the main regular manifestations of water flows with a free surface. There is a vast recent research literature devoted to the symmetry of these waves [13, 12, 15, 16, 6, 5], which provide sufficient conditions for symmetry. The sharpest results state that symmetry is ensured as soon as the wave profile is monotone between successive crests and troughs, under some relatively mild conditions on the underlying flows. In particular, monotonicity of the wave profile between successive crests and troughs for irrotational waves - the celebrated Stokes waves, see the discussion in [2]- guarantees symmetry of the wave profile. The close connection between symmetry and steady state is further emphasized by the considerations in [7], where the authors show that if, in a two-dimensional free-surface water flow over a flat bed, the free surface and the velocity field as well as the pressure admit at any instant an axis of symmetry, then the flow is that of a traveling wave. The physical relevance of this nice mathematical result is slightly impeded by the impractical nature of its hypotheses. Our aim is to provide a new result that has the advantage of a considerably reduced set of hypotheses in the setting of irrotational flows. More specifically, we will show that under the assumptions that the wave profile as well as the horizontal velocity component of the flow at the free surface are both symmetric about a common (time-dependent) vertical axis, the flow necessarily defines a traveling wave.

It seems reasonable to expect that our hypotheses can be checked in an experiment that only involves measurement at the free surface. Actually, there are two possibilities to detect ocean waves: Remote sensing and *in situ*. For *in situ* aspects, we refer to the discussion in [1, 4, 10]. Remote Sensing is the only available technique to observe waves in inaccessible regions. By means of the reflection and refraction of electromagnetic radiation emitted from a satellite it is possible to gather data about the sea surface. For a detailed discussion of remote sensing techniques and their physical background, we refer to [14, 11, 8].

5 The governing equations

We are concerned with the study of two-dimensional water waves propagating over a flat bed under the influence of gravity. In such a closed system, mass can neither be created nor destroyed, reflected by

$$u_x + v_y = 0, \tag{44}$$

the *equation of mass conservation*. The balance of momentum within a fluid, *Euler's equations*, become

$$\left.\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned}\right\} \tag{45}$$

where $P = P(x, y, t)$ is the pressure and $g \approx 9.81$ m/s^2 is the gravity of the Earth.
The motion of the fluid beneath the surface is decoupled from the motion of the air above. This is expressed in the dynamic boundary condition

$$P = P_{atm} \quad \text{on} \quad y = \eta(x,t), \tag{46}$$

where P_{atm} is the constant atmospheric pressure. Furthermore, we obtain the kinematic boundary condition from the fact that the same particles always form the free surface:

$$v = \eta_t + u\eta_x \quad \text{for} \quad y = \eta(x,t). \tag{47}$$

The fluid cannot penetrate flat bed, which means that

$$v = 0 \quad \text{on} \quad y = -d. \tag{48}$$

We are dealing with flows with zero vorticity by assuming that

$$u_y = v_x \tag{49}$$

throughout the fluid. The relations (44)-(49) constitute the governing equations for the irrotational water wave problem. For a detailed derivation and discussion, we refer to [3, 9]. In the following, we will be dealing with periodic wave solutions with wavelength L, therefore assuming that u, v, P and η are smooth and L-periodic in x.

6 Main result

We may take advantage of (44) to define the *stream function* $\psi(x, y; t)$ as a solution to

$$\left.\begin{aligned} \psi_y(x,y,t) &= u(x,y,t), \\ \psi_x(x,y,t) &= -v(x,y,t), \end{aligned}\right\} \tag{50}$$

at any instant of time. It follows from (49) that ψ is a harmonic function in Ω. We may write ψ explicitly as a line integral

$$\psi(x,y,t) = \int_{(x_0,y_0)}^{(x,y)} u\,dy - v\,dx,$$

where the path of integration can be any curve in the fluid domain joining a fixed reference point (x_0, y_0) and (x, y). This is nothing but the volume flux through that curve. One readily verifies, using (48), that ψ is L-periodic in the x-coordinate and that $\psi(x, -d, t) = \psi_0(t)$ in view of (48).

Theorem 6.1. *Let (u, v, P, η) be an x-periodic solution to the irrotational water wave problem (44)-(49), with the property that η and the horizontal component u at the surface are both symmetric about $x = c(t)$ at any instant of time. Then the solution defines a traveling wave.*

Proof. The function

$$w(x, y, t) := \psi(x, y, t) - \psi(2c(t) - x, y, t) \tag{51}$$

is harmonic, L-periodic and, by (48), zero on the flat bed.
We will show, using maximum principles, that w is identically zero in Ω provided that the free surface and the horizontal velocity component at the surface are symmetric about $c(t)$. That is to say, we assume

$$\eta(x, t) = \eta(2c(t) - x, t) \tag{52}$$

as well as

$$u(x, \eta(x, t), t) = u(2c(t) - x, \eta(x, t), t) \tag{53}$$

for all $x \in [c(t) - L/2, c(t) + L/2]$.
Let us use the following abbreviations:

$$\begin{aligned} \eta^+ &:= \eta(x, t), \quad & \eta^- &:= \eta(2c(t) - x, t), \\ u^+ &:= u(x, \eta(x, t), t), \quad & u^- &:= u(2c(t) - x, \eta(x, t), t), \\ v^+ &:= v(x, \eta(x, t), t), \quad & v^- &:= v(2c(t) - x, \eta(x, t), t). \end{aligned}$$

Clearly, the periodicity and harmonicity of w prevents the maximum from being attained solely in the interior of the fluid domain. If both the maximum and the minimum could be found at the flat bed, where $w = 0$, w would already be identically zero. So suppose, without loss of generality, that the maximum of w is attained at the free surface. Say, at $(x_0, \eta(x_0, t))$. This means that the tangential derivative at $(x_0, \eta(x_0, t))$ is zero, which reads as

$$\nabla w \cdot (1, \eta_x) = -v^+ - v^- + \eta_x^+(u^+ - u^-) = 0,$$

by the definition of ψ in (50). But then $v^+ = -v^-$ at the maximum, by assumption (53). By Hopf's Lemma, the derivative in the normal direction at the maximum would be strictly positive, that is

$$\eta_x^+(v^+ + v^-) + u^+ - u^- > 0$$

at the maximum, a contradiction.
This means that $\psi(x, y, t) = \psi(2c(t) - x, y, t)$ for all $(x, y) \in \Omega$. Taking derivatives, in view of (50), we have

$$u(x, y, t) = u(2c(t) - x, y, t) \tag{54}$$

$$v(x, y, t) = -v(2c(t) - x, y, t) \tag{55}$$

for all $(x, y) \in \Omega$.

To prove symmetry of the pressure, we proceed in a similar fashion as for the stream function. Define the function

$$z(x, y, t) := P(x, y, t) - P(2c(t) - x, y, t),$$

which is L-periodic in x and equal zero at the free surface by (46). We will use the fact that the pressure is a super-harmonic function in Ω, namely that

$$\begin{aligned} \Delta P &= -\partial_x(u_t + uu_x + vu_y) - \partial_y(v_t + uv_x + vv_y + g) && (56) \\ &= -2(u_x^2 + u_y^2), && (57) \end{aligned}$$

where we have used equations (45), (44) and (49). This implies, by means of (54), that $\Delta z = 0$. Combining the boundary condition (48) with the second Euler equation (45), we deduce that $P_y(x, -d, t) = -g$ and hence $z_y(x, -d, t) = 0$. But z_y is the derivative of z in the normal direction at the flat bed, which means that neither a maximum nor a minimum can be attained at the flat bed. Since $z = 0$ at the surface, it follows that $z \equiv 0$. Therefore

$$P(x, y, t) = P(2c(t) - x, y, t) \tag{58}$$

for all $(x, y) \in \Omega$.
We can now conclude by Theorem 4.1 in [**?**]: symmetry of u, P, η and antisymmetry of v throughout the fluid domain imply a traveling wave. $\square$

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On the symmetry of spatially periodic two-dimensional water waves

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Disc. Cont. Dyn. Syst. A

Abstract

We show that a spatially periodic solution to the irrotational two-dimensional gravity water wave problem, with the property that the horizontal velocity component at the flat bed is symmetric, while the acceleration at the flat bed is anti-symmetric with respect to a common axis of symmetry, necessarily constitutes a traveling wave. The proof makes use complex variables and structural properties of the governing equations for nonlinear water waves.

7 Introduction

Questions related to the symmetry of the wave profile as well as the underlying flow arise naturally in the study of spatially-periodic gravity water waves. For recent discussions of qualitative properties of solutions that guarantee symmetry of the wave, we refer to [6], [7], [11], [13], [14] and [15]. The sharpest results state that symmetry is ensured as soon as the wave profile is monotone between successive crests and troughs, under some relatively mild conditions on the underlying flows. In particular, monotonicity of the wave profile between successive crests and troughs for irrotational waves - the celebrated Stokes waves present this distinctive feature, see the discussion in [3]- guarantees symmetry of the wave profile.

Symmetry properties of the underlying flow and the free surface are closely related to steady state solutions. In [8] the authors prove that a solution to the two-dimensional gravity water wave problem over a flat bed with the property that the free surface and the velocity field as well as the pressure admit at any instant of time an axis of symmetry necessarily defines a traveling wave. Since the symmetry assumptions in [8] require information about the velocity field and the pressure in the whole fluid domain, it is difficult to check them experimentally. We will present a reduced set of assumptions that guarantee symmetry in the fluid domain and that only involve measurements at the flat bed. Actually, we will show that a solution with the property that the horizontal velocity component at the flat bed is symmetric, while the acceleration at the flat bed is anti-symmetric with respect to a common axis of symmetry, necessarily constitutes a traveling wave.

Information about the flow at the flat bed is obtained via *in situ* measurements. For instance, pressure measurements at the flat bed can be used to recover the wave profile and to estimate the wave height, cf. [1], [2], [5] and [10].

8 The governing equations

The governing equations for an ideal, incompressible, homogeneous fluid in two space dimensions under the influence of gravity consist of the *equation of mass conservation*

$$u_x + v_y = 0, \tag{59}$$

coupled to *Euler's equations*

$$\left.\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned}\right\} \tag{60}$$

where $(u, v) = (u(x, y, t), v(x, y, t))$ is the velocity field, $P = P(x, y, t)$ is the pressure and $g \approx 9.81$ m/s^2 is the acceleration of gravity at the surface of the Earth.

In the study of two-dimensional water waves, we additionally require that there exists a free surface and that the motion of the fluid beneath the surface is independent from the motion of the air above, which is expressed in the kinematic boundary condition

$$v = \eta_t + u\eta_x \quad \text{for} \quad y = \eta(x, t), \tag{61}$$

and the dynamic boundary condition

$$P = P_{atm} \quad \text{on} \quad y = \eta(x, t), \tag{62}$$

where P_{atm} is the constant atmospheric pressure. We are concerned with waves propagating over a flat bed of mean depth d. The fluid is allowed to slip at the flat bed, but cannot penetrate it:

$$v = 0 \quad \text{on} \quad y = -d. \tag{63}$$

Additionally, we impose that the flow is irrotational, assuming that the vorticity is zero throughout the fluid:

$$u_y = v_x. \tag{64}$$

The relations (59)-(64) constitute the governing equations for the irrotational water wave problem. For a detailed derivation and discussion, we refer to [4] and [9]. In the following, we will be dealing with periodic wave solutions with wavelength L, therefore assuming that u, v, P and η are smooth and L-periodic in x. The a priori unknown fluid domain at time t is therefore given by

$$\Omega = \Omega(t) = \{(x, y) \in \mathbb{R}^2 : -L < x < L, -d < y < \eta(x, t)\}.$$

9 Main result

We will show that symmetry properties at the flat bed imply that the flow is that beneath a traveling wave.

Theorem 9.1. *Let (u, v, P, η) be a smooth x-periodic solution to the irrotational water wave problem (59)-(64), with the property that the horizontal velocity component $u(x, -d, t)$ at the flat bed is symmetric about $x = \lambda(t)$, while the acceleration $\vec{a}(x, -d, t)$ at the flat bed is antisymmetric about $x = \lambda(t)$ at any instant of time. Then the solution defines a traveling wave.*

The first assumption reads as

$$u(x,-d,t) = u(2\lambda(t) - x, -d, t) \tag{65}$$

at any instant of time. In order to compute the acceleration, let $t \mapsto (X(t), Y(t))$ be the path of a fluid particle. The velocity relates to the flow field via $(\dot{X}(t), \dot{Y}(t)) = (u(X(t), Y(t), t), v(X(t), Y(t), t))$, while the acceleration is given by

$$\vec{a} = (\ddot{X}, \ddot{Y}) = (u_t + uu_x + vu_y, v_t + uv_x + vv_y). \tag{66}$$

In view of (66), (65) and (63), the second assumption reads

$$u_t(x,-d,t) = -u_t(2\lambda(t) - x, -d, t), \tag{67}$$

since the function uu_x is anti-symmetric about $x = \lambda(t)$.

Remark 9.2. Note that there is, in general, no a priori connection between the horizontal velocity component u and its time derivative u_t. Taking the derivative in (65) with respect to t, we get

$$u_t(x,-d,t) = u_t(2\lambda(t) - x, -d, t) + 2\dot{\lambda}(t)u_x(2\lambda(t) - x, -d, t),$$

which does not immediately relate to the assumed symmetry of the velocity component u due to the unknown character of $\dot{\lambda}$.

In the proof of Theorem (9.1) we will use the following result, proved in [8]:

Lemma 9.3. *Only traveling wave solutions solution to the governing equations (59)-(64) have the property that*

$$\begin{aligned}
u(x,y,t) &= u(2\lambda(t) - x, y, t),\\
v(x,y,t) &= -v(2\lambda(t) - x, y, t),\\
P(x,y,t) &= P(2\lambda(t) - x, y, t),\\
\eta(x,t) &= \eta(2\lambda(t) - x, t)
\end{aligned}$$

throughout the fluid domain.

The proof of Theorem (9.1) is organized as follows. First, we will show that the horizontal velocity component u is symmetric, while the vertical velocity component v is anti-symmetric in some sub-domain of $\tilde{\Omega} \subseteq \Omega$. Then we will show that a similar relation holds true for u_t and v_t, while the pressure is symmetric in $\tilde{\Omega}$, which will permit us to infer by the dynamic boundary condition (62) that necessarily the free surface is symmetric. By Lemma (9.3), we then conclude that the flow defines a traveling wave solution.

Proof. Let us show that the velocity field (u, v) has a symmetry property in a sub-domain of Ω. The complex velocity field $F(x,y,t) := u(x,y,t) - \mathrm{i}v(x,y,t)$ defines, by (59) and (64), at any instant of time a holomorphic function on the fluid domain Ω. Since F is real-valued at the flat bed in view of (63), we may apply the standard Schwartz reflection principle, cf. [12], to define

$$F(z,t) = \begin{cases} F(z,t) & \text{if } z \in \Omega \\ (F(z^*,t))^* & \text{if } z \in \Omega^* \end{cases}, \tag{68}$$

which is a holomorphic function on $\Omega \cup \Omega^*$ at any instant of time. Here $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$ denotes complex conjugation.
Now let $R_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (2\lambda - x, y)$ be the reflection with respect to the axis $\{x = \lambda\}$ and define $\Omega_\lambda := R_\lambda(\Omega)$, the reflected fluid domain. Note that in general $\Omega_\lambda \neq \Omega$, unless the free surface η is symmetric about λ. We define the reflected complex velocity field as $F^\lambda(z, t) := R_\lambda[(F(z, t))^*] = u(2\lambda(t) - x, y, t) + iv(2\lambda(t) - x, y, t)$, which is a holomorphic function on Ω_λ at any instant of time. Similar to (68), F^λ can be extended to a holomorphic function on $\Omega_\lambda \cup \Omega_\lambda^*$ at any instant of time. The function $F(z, t) - F^\lambda(z, t)$ is holomorphic on the domain $(\Omega \cup \Omega^*) \cap (\Omega_\lambda \cup \Omega_\lambda^*)$ and identically zero on the real axis by (63) and our first assumption (65). Therefore, by the Identity Theorem for holomorphic functions, cf. [12], it follows that $F(z, t) = F^\lambda(z, t)$ for all $z \in (\Omega \cup \Omega^*) \cap (\Omega_\lambda \cup \Omega_\lambda^*)$, which readily implies that

$$u(x, y, t) = u(2\lambda(t) - x, y, t) \tag{69}$$

$$v(x, y, t) = -v(2\lambda(t) - x, y, t) \tag{70}$$

for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time.
Following the same line of reasoning for $F_t(x, y, t) = u_t(x, y, t) - \mathrm{i}v_t(x, y, t)$, which is holomorphic on Ω, $F_t^\lambda(z, t) = R_\lambda[(F_t(z, t))^*]$ and $F_t(z, t) + F_t^\lambda(z, t)$ by taking advantage of our second assumption (67), we deduce that

$$u_t(x, y, t) = -u_t(2\lambda(t) - x, y, t) \tag{71}$$

$$v_t(x, y, t) = -v_t(2\lambda(t) - x, y, t) \tag{72}$$

for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time. To prove symmetry of the pressure on the domain $\Omega \cap \Omega_\lambda$, observe that by the symmetry conditions (71), (72), (69), (70) and the first Euler equation in (60) we have

$$P_x(x, y, t) = -P_x(2\lambda(t) - x, y, t) \tag{73}$$

for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time. Integrating (73) with respect to x we obtain $P(x, y, t) = P(2\lambda(t) - x, y, t) + \alpha(y, t)$ for some function α. Evaluating the expression at $x = \lambda$, it follows that $\alpha \equiv 0$ and that

$$P(x, y, t) = P(2\lambda(t) - x, y, t) \tag{74}$$

for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time.
So far, we have proved that the velocity field (u, v) as well as the pressure P satisfy a symmetry condition on the domain $\Omega \cap \Omega_\lambda$. In order to apply Lemma (9.3), we have to show that η is symmetric with respect to λ and hence $\Omega = \Omega_\lambda$. We observe that the pressure is a super-harmonic function in Ω:

$$\begin{aligned}\Delta P &= -\partial_x(u_t + uu_x + vu_y) - \partial_y(v_t + uv_x + vv_y + g)\\ &= -2(u_x^2 + u_y^2),\end{aligned}$$

where we have used equations (60), (59) and (64). Assume by contradiction that the free surface is not symmetric about λ. Then there exists a boundary point $(x_0, \eta(x_0))$ that is mapped into the interior of Ω under the reflection R_λ, that is to say $(2\lambda(t) - x_0, \eta(x_0)) \in \Omega$. By the boundary condition (62) and the symmetry of the pressure (74), it follows that $P = P_{atm}$ at an interior point. Since $P_{atm} = \min_{\bar{\Omega}} P$, cf. [4], this would contradict the super-harmonicity of P. Hence the wave profile is symmetric and we can now conclude that the flow defines a traveling wave. $\square$

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Linear stability for irrotational water waves

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submitted

Abstract

We will prove linear stability for solutions to the periodic irrotational gravity water wave problem in finite depth that satisfy certain physical assumptions. The assumptions involve the mean depth of the wave, the mean velocity, the hydraulic head and Plotnikov's potential as well as the wave number.

10 Introduction

Traveling wave solutions to the fully nonlinear water wave model are supposed to be unstable. There is a vast literature on instability results for free boundary flows. Maybe most notably, in [7] it was shown that irrotational water waves in a small-amplitude regime exhibit the so called *sideband instability*. This means that the periodic waves under consideration are unstable with respect to perturbation of a different period, cf. also [8]. For some classes of exact solutions, different kinds of instabilities are proved. To only mention a few, instability has been proved for Gerstner's wave [21, 29], equatorial waves [16, 20, 24, 23] and edge waves [26, 27].

On the other hand, traveling wave solutions to model equations for different water wave models show rather strong stability properties. Amongst others, (orbital) stability has been proved for periodic wave solutions to equations of (generalized) Korteweg-de-Vries type [32, 36], the Camassa-Holm equation [30] and the BBM equation [5]. In the case of solitary traveling wave solution, stability results have been obtained in [2, 1] for long wave models, in [3] for internal waves and in [6] for the Korteweg-de-Vries equation. For general theorems on stability, we refer to [22]. Those model equations have in common that they rely upon a combination of small-amplitude, shallow-water and long-wave asymptotical approximation respectively. It is therefore natural to expect that also the full problem exhibits some, presumably weak, notion of stability for physically small depths or physically bounded amplitudes.

In the following, we will be dealing with periodic traveling wave solutions to the irrotational free-boundary problem for the Euler equations under the influence of gravity in finite depth, cf. [15, 28, 35]. For a discussion of linear stability for the Hamiltonian system in the velocity potential formulation for irrotaional waves, we refer to [31, 42]. We will neglect surface tension effects. Under the additional assumption that surface tension is present, which changes the boundary conditions, stability and energetic stability for the capillary-gravity water wave problem can be shown, cf. [14] for the periodic and [9, 33] for the solitary case. A similar result can be shown for flows where the free surface offers some

resistance to stretching and bending [10]. This confirms the well-known intuition that surface tension positively enhances stabilizing effects. We also refer to [18] for waves with non-zero vorticity.
The structure of the paper is as follows. In section 2, we present a reformulation of the governing equations as a one dimensional pseudo-differential equation coupled to a scalar constraint, which was obtained in [19, 17], while in section 3 we prove some estimates related to the Hilbert transform on a strip. These estimates are similar to the ones in [39, 40, 41]. Section 4 is devoted to the study of linear stability. Working on a product space, we have to consider matrices of operators and show that their spectrum is positive. This will be achieved by considering the diagonal elements separately and using a standard theorem on matrices of operators, cf. Appendix. Our analysis for the infinite dimensional part relies upon a transform derived in [37] in the context of solitary waves, which was successfully applied in [11, 12]. For a generalization of the theory to Bernoulli free-boundary value problems, cf. [38].

11 Preliminaries

Let $L^2_{2\pi}(\mathbb{R})$ be the space of 2π-periodic square integrable functions. We denote by

$$H^s_{2\pi}(\mathbb{R}) := \left\{ f \in L^2_{2\pi}(\mathbb{R}) : \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{f}_n|^2 < \infty \right\}$$

the s-order Sobolev space, where the Fourier coefficients are defined by $\hat{f}_n := \langle f, \exp(-\mathrm{i}n\cdot)\rangle_{L^2}$. The mean value of a function $f \in L^2_{2\pi}(\mathbb{R})$ is denoted as

$$[f]_{2\pi} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{75}$$

In particular, $[c]_{2\pi} = c$ for any constant $c \in \mathbb{R}$. Define

$$H^s_{2\pi,\circ}(\mathbb{R}) := \{ f \in H^s_{2\pi}(\mathbb{R}) : [f]_{2\pi} = 0 \},$$

the space of H^s-functions with zero mean.
We will be concerned with solutions $v := w + h$ to the following one-dimensional pseudo-differential equation. For a detailed derivation, we refer to [17, 19]. Here, considering the irrotational case, we have set $\gamma = 0$ to obtain the equation

$$\left(\frac{Q-2gh}{g}\right)\mathcal{C}_{kh}(w') = \frac{w}{k} + w\mathcal{C}_{kh}(w') + \mathcal{C}_{kh}(ww') - [w\mathcal{C}_{kh}(w')]_{2\pi}, \tag{76}$$

coupled to the constraint

$$\left(\frac{m}{kh}\right)^2 = \left[(Q-2gv)\Big((v')^2 + \mathcal{G}_{kh}(v)^2\Big)\right]_{2\pi}. \tag{77}$$

The constant Q represents the hydraulic head of the flow, m is the mass flux and k is the wave number. The wave number is related to the period L of the wave via $k = \frac{2\pi}{L}$. The constant h is the so-called *conformal mean depth*, as defined in [17, 19]. In our case, for irrotational water waves, the conformal mean depth coincides with the "mean depth" as defined in [4]:

$$h = \frac{m}{c}, \tag{78}$$

where c is the mean horizontal velocity of the traveling wave solution.
The operator $\mathcal{C}_{kh}$ is a Fourier multiplier on $L^2_{2\pi,\circ}(\mathbb{R})$ with symbol $\{\mathrm{symb}(\mathcal{C}_{kh})\}_n = -\mathrm{i}\coth(khn)$, called the *Hilbert transform for a strip*, cf. section 11. Given a 2π-periodic solution to system (76)-(77), we can recover the free surface of the wave as

$$\mathcal{S} = \left\{ \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x) \right) \, , \, x \in [-\pi, \pi] \right\}. \tag{79}$$

The free surface associated to a non-singular solution to (76)-(77) is given by the graph of a function if and only if

$$\frac{1}{k} + \mathcal{C}_{kh}(w')(x) > 0 \tag{80}$$

for all $x \in [-\pi, \pi]$. If (80) holds true, denoting $I(x) = x/k + \mathcal{C}_{kh}(w)(x)$, we can write the free surface in terms of the parametrization v as a graph

$$\eta(x) := v(I^{-1}(x)).$$

Note that I is a smooth bijection from $[-\pi, \pi]$ onto $[0, L]$. This allows us to relate the mean depth to the the conformal mean depth h and the function $w := v - h$:

$$\begin{aligned} d = \frac{1}{L}\int_0^L \eta(x)\, dx &= \frac{1}{L}\int_0^L v(I^{-1}(x))\, dx \\ &= \frac{1}{L}\int_{-\pi}^{\pi} v(\xi) \left(\frac{1}{k} + \mathcal{C}_{kh}(w')(\xi) \right), d\xi \\ &= \frac{2\pi h}{kL} + \frac{2\pi}{L}[w\mathcal{C}_{kh}(w')]_{2\pi} \\ &= h + k[w\mathcal{C}_{kh}(w')]_{2\pi}, \end{aligned}$$

where we have transformed the integral according to $x = I(\xi)$ and used the fact that $v = w + h$ with $[w]_{2\pi} = 0$. This means that

$$[w\mathcal{C}_{kh}(w')]_{2\pi} = \frac{d-h}{k}. \tag{81}$$

In [17, 19], it is shown that any non-singular solution to (76)-(77) also satisfies the following set of equations

$$\left(\frac{m}{kh}\right)^2 = (Q - 2gv)\big((v')^2 + \mathcal{G}_{kh}(v)^2\big) \tag{82}$$

$$[v]_{2\pi} = h \tag{83}$$

$$v(x) > 0 \text{ for } x \in [-\pi, \pi] \tag{84}$$

$$\text{the mapping } x \mapsto \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x) \right) \text{ is injective on } \mathbb{R} \tag{85}$$

$$(v'(x))^2 + \mathcal{G}_{kh}(v)(x)^2 \neq 0 \text{ for all } x \in [-\pi, \pi] \tag{86}$$

In fact, the above equations are even equivalent to (76)-(77). We remark that any solution to equation (76) that describes a regularly parametrized surface necessarily satisfies the inequality

$$\frac{Q}{2g} > w + h \tag{87}$$

as it can be seen from (82).

The advantage of the equations (76)-(77) as compared to (82) is that a they are the Euler-Lagrange equations of a functional. Namely, separating the mean $h = [v]_{2\pi}$ from the function $v = w + h$, any stationary point of the functional

$$\mathcal{J}(w,h) = \int_{-\pi}^{\pi} \left(Qv - gv^2\right)\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) + \frac{m^2}{kh}\,dx \tag{88}$$

in the space $H^1_{2\pi,\circ}(\mathbb{R}) \times \mathbb{R}$ is a solution to equation (76) that satisfies constraint (77).

Remark 11.1. For waves of infinite depth, an equation similar to (76) for a parametrization of the wave profile can be derived:

$$\mu\mathcal{C}(w') = w + w\mathcal{C}(w') + \mathcal{C}(ww'), \tag{89}$$

where the wave number $k = 1$ and

$$\mathcal{C}(w)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{w(y)}{x-y}\,dy$$

is the Hilbert transform, cf. [13, 40, 41]. In fact, (89) can be viewed, at least formally, as the limiting equation of (76) as $d \to \infty$. A Lagrangian for equation (89) is given by

$$\mathcal{J}_\infty(w) = \int_{-\pi}^{\pi} w\mathcal{C}(w') - \frac{w^2}{\mu}\big(1 + \mathcal{C}(w')\big)\,dx. \tag{90}$$

The Hilbert transform on the strip and point-wise estimates

In this section, we collect some properties and lemmata related to the Hilbert transform on the strip and on point-wise positive operators. This will prove to be useful in the subsequent analysis of stability for the functional (88).

Consider the Dirichlet problem on a strip

$$\mathcal{S}_D = \{(x,y) \in \mathbb{R}^2 : -D < y < 0\}$$

for some boundary value $u \in L^2_{2\pi}(\mathbb{R})$:

$$\begin{cases} \Delta U = 0 \quad \text{in } \mathcal{S}_D \\ U(x,-D) = 0 \quad \text{for } x \in \mathbb{R} \\ U(x,0) = u(x) \quad \text{for } x \in \mathbb{R}. \end{cases} \tag{91}$$

Expanding the boundary condition in a Fourier series,

$$u(x) = \sum_{n\in\mathbb{Z}} \hat{u}_n e^{inx}, \tag{92}$$

the solution to system (91) is given by

$$U(x,y) = \sum_{n\in\mathbb{Z}} \frac{\sinh\big(n(y+D)\big)}{\sinh(nD)}\hat{u}_n e^{inx}. \tag{93}$$

Henceforth, we denote complex variables as $z = x + \mathrm{i}y$ and the complex conjugate as $z^* = x - \mathrm{i}y$. For any function $u \in L^2_{2\pi}(\mathbb{R})$ with Fourier series expansion given by (92), the expression

$$\mathbf{R}_D[u](z) := \left(\frac{D - \mathrm{i}z}{D}\right)\hat{u}_0 - \sum_{n\in\mathbb{Z}^*} \frac{e^{-nD}}{\sinh(nD)}\hat{u}_n e^{inz} \tag{94}$$

defines a 2π-periodic function holomorphic in the strip $\mathcal{S}_D$. Its real and imaginary part at $y = 0$ are related by the *Hilbert transform for the strip* $\mathcal{S}_D$, an operator $\mathcal{C}_D : L^2_{2\pi,\circ}(\mathbb{R}) \to L^2_{2\pi,\circ}(\mathbb{R})$, defined for an L^2-function

$$u(x) = \sum_{n\in\mathbb{Z}} \hat{u}_n e^{\mathrm{i}nx}$$

by

$$\mathcal{C}_D[u](x) := -\sum_{n\in\mathbb{Z}} \mathrm{i}\coth(Dn)\hat{u}_n e^{\mathrm{i}nx} \tag{95}$$

Namely, the real and imaginary part are related as:

$$\mathbf{R}_D[u](x - i\varepsilon) \to u - \mathrm{i}\left(\frac{\hat{u}_0}{D}x + \mathcal{C}_D(u - \hat{u}_0)\right) \tag{96}$$

in $L^2_{2\pi}(\mathbb{R}, \mathbb{C})$ as $\varepsilon \searrow 0$. We will call $\mathbf{R}_D[u]$ the *holomorpic extension* of u to the strip $\mathcal{S}_D$.
To simplify notation, we will write

$$\mathbf{R}[u](x) := \lim_{\varepsilon\searrow 0} \mathbf{R}_D[u](x - i\varepsilon), \tag{97}$$

if the width of the strip is known in the context.

Lemma 11.2. *Let $u, v \in L^2_{2\pi,\circ}(\mathbb{R})$ and let $\mathcal{C}_D : L^2_{2\pi,\circ}(\mathbb{R}) \to L^2_{2\pi,\circ}(\mathbb{R})$ be the Hilbert transform for the strip $\mathcal{S}_D$. The equality*

$$\mathcal{C}_D\big(u\mathcal{C}_D(v) + v\mathcal{C}_D(u)\big) = \mathcal{C}_D(u)\mathcal{C}_D(v) - uv \tag{98}$$

holds true.

Proof. Expanding u and v as Fourier series

$$u(x) = \sum_{n\in\mathbb{Z}^*} \hat{u}_n e^{inx}, \quad v(x) = \sum_{n\in\mathbb{Z}^*} \hat{v}_n e^{inx}$$

and using the identity

$$\coth(x + y) = \frac{1 + \coth(x)\coth(y)}{\coth(x) + \coth(y)},$$

for all $x, y \in \mathbb{R}$, we find

$$
\begin{aligned}
\mathcal{C}_D\big(u\mathcal{C}_D(v)+v\mathcal{C}_D(u)\big) &= \mathcal{C}_D\left(\sum_{n\in\mathbb{Z}^*}\left(\sum_{k\in\mathbb{Z}}(-\mathrm{i})\hat{u}_k\hat{v}_{n-k}\Big(\coth\big((n-k)D\big)+\coth\big(kD\big)\Big)\right)e^{inx}\right)\\
&= -\sum_{n\in\mathbb{Z}^*}\left(\sum_{k\in\mathbb{Z}}\hat{u}_k\hat{v}_{n-k}\coth(nD)\Big(\coth\big((n-k)D\big)+\coth\big(kD\big)\Big)\right)e^{inx}\\
&= -\sum_{n\in\mathbb{Z}^*}\left(\sum_{k\in\mathbb{Z}}\hat{u}_k\hat{v}_{n-k}\Big(1+\coth\big(kD\big)\coth\big((n-k)D\big)\Big)\right)e^{inx}\\
&= \mathcal{C}_D(u)\mathcal{C}_D(v)-uv
\end{aligned}
$$

as desired. ☐

Lemma 11.3. *Let $u\in H^1_{2\pi,\circ}(\mathbb{R})$. For almost every $x\in\mathbb{R}$ the integral representation*

$$
\mathcal{C}_D(u)(x)=\text{p.v.}\frac{1}{2D}\int_{\mathbb{R}}\left\{\tanh\left(\frac{\pi y}{2D}\right)+\coth\left(\frac{\pi(x-y)}{2D}\right)\right\}u(y)\,dy \tag{99}
$$

holds true.

Proof. A proof for $u\in C^{1,\alpha}_{2\pi,\circ}(\mathbb{R})$ can be found in [17].The proof for $u\in H^1_{2\pi,\circ}(\mathbb{R})$ is similar. ☐

To simplify notation, we define

$$
\mathcal{K}_D(x,y):=\tanh\left(\frac{\pi y}{2D}\right)+\coth\left(\frac{\pi(x-y)}{2D}\right). \tag{100}
$$

Lemma 11.4. *Let $u\in H^1_{2\pi}(\mathbb{R})$. For almost every $x\in\mathbb{R}$ the inequality*

$$
\big(u\mathcal{C}_D(u')-\mathcal{C}_D(uu')\big)(x)\geq 0 \tag{101}
$$

holds true.

Proof. We expand the left-hand side according to Lemma 11.3 and obtain for almost every $x\in\mathbb{R}$:

$$
\begin{aligned}
(u\mathcal{C}_D(u')-\mathcal{C}_D(uu'))(x) &= \text{p.v.}\frac{1}{2D}\int_{\mathbb{R}}\mathcal{K}_D(x,y)(u(x)-u(y))u'(y)\,dy\\
&= -\text{p.v.}\frac{1}{4D}\int_{\mathbb{R}}\mathcal{K}_D(x,y)\frac{d}{dy}\left(\int_x^y u'(s)\,ds\right)^2 dy \qquad (102)\\
&= \text{p.v.}\frac{1}{4D}\int_{\mathbb{R}}\frac{\partial\mathcal{K}_D}{\partial y}\big(u(y)-u(x)\big)^2\,dy\geq 0,
\end{aligned}
$$

where we have used the weak form of the Fundamental Theorem of Calculus, cf. [25], in the second equality and the fact that

$$
\lim_{\varepsilon\to 0+}\left\{\Big[\mathcal{K}_D(u(y)-u(x))\Big]_{x-\frac{1}{\varepsilon}}^{x-\varepsilon}+\Big[\mathcal{K}_D(u(y)-u(x))\Big]_{x+\varepsilon}^{x+\frac{1}{\varepsilon}}\right\}=0
$$

for the third equality. Finally, we observed that

$$\frac{\partial \mathcal{K}_D}{\partial y}(x,y) = \frac{\pi}{2D}\left\{\operatorname{sech}^2\left(\frac{\pi y}{2D}\right) + \operatorname{csch}^2\left(\frac{\pi(x-y)}{2D}\right)\right\} \geq 0$$

to deduce non-negativity. □

We denote by

$$L^{2,+}_{2\pi}(\mathbb{R}) := \{u \in L^2_{2\pi}(\mathbb{R}) : u(x) \geq 0 \text{ for a.e. } x \in [-\pi,\pi]\} \tag{103}$$

the convex cone of non-negative periodic L^2-functions.

Definition 11.5. A possibly unbounded operator $T : L^2_{2\pi}(\mathbb{R}) \to L^2_{2\pi}(\mathbb{R})$ with domain $\mathcal{D}(T)$ is called *point-wise positive* if

$$u \in L^{2,+}_{2\pi}(\mathbb{R}) \cap \mathcal{D}(T) \implies T(u) \in L^{2,+}_{2\pi}(\mathbb{R}). \tag{104}$$

To simplify notation, we define the Fourier multiplier $\mathcal{G}_D : L^2_{2\pi}(\mathbb{R}) \to L^2_{2\pi}(\mathbb{R})$, called the *Dirichlet-to-Neumann operator for the strip $\mathcal{S}_D$*, as

$$\mathcal{G}_D\left(\sum_{n\in\mathbb{Z}} \hat{u}_n e^{inx}\right) = \sum_{n\in\mathbb{Z}} n\coth(Dn)\hat{u}_n e^{inx}. \tag{105}$$

It is related to the Hilbert transform for a strip by

$$\mathcal{G}_D[u] = \frac{[u]_{2\pi}}{D} + \mathcal{C}_D[u']. \tag{106}$$

Lemma 11.6. *The operator $S_D := (\partial^2 + \mathcal{G}_D^2) : L^2_{2\pi}(\mathbb{R}) \to L^2_{2\pi}(\mathbb{R})$ with domain $\mathcal{D}(S_D) = H^2_{2\pi}(\mathbb{R})$ is point-wise positive, provided that*

$$2D \geq \sqrt{3}\pi\coth\left(\frac{\sqrt{3}\pi}{D}\right). \tag{107}$$

Proof. The operator S_D being a Fourier multiplier with symbol

$$\left\{\operatorname{symb}(S_D)\right\}_n = \frac{n^2}{\sinh(nD)^2},$$

it suffices to prove that the 2π-periodic function

$$\hat{S}_D(x) := \sum_{n\in\mathbb{Z}} \frac{n^2}{\sinh(nD)^2} e^{inx}$$

belongs to $L^{2,+}_{2\pi}(\mathbb{R})$, since, according to the Convolution Theorem for 2π-periodic functions, it then represents an a.e. non-negative integral kernel for the operator S_D. Keeping in mind that $\left\{\operatorname{symb}(S_D)\right\}_0 = \frac{1}{D^2}$ and that $\left\{\operatorname{symb}(S_D)\right\}_{(-n)} = \left\{\operatorname{symb}(S_D)\right\}_n$, we estimate

$$
\begin{aligned}
\inf_{x\in[-\pi,\pi]} \hat{S}_D(x) &= \inf_{x\in[-\pi,\pi]} \left(\frac{1}{D^2} + 2\sum_{n=1}^{\infty} \frac{n^2}{\sinh(nD)^2}\cos(nx) \right) \\
&\geq \frac{1}{D^2} - 2\sum_{n=1}^{\infty} \frac{n^2}{\sinh(nD)^2} \geq \frac{1}{D^2} - 2\sum_{n=1}^{\infty} \frac{n^2}{(nD)^2 + \frac{(nD)^4}{3}} \\
&= \frac{1}{D^2} - \frac{\sqrt{3}\pi}{D^3} \sum_{n=1}^{\infty} \frac{\frac{2\sqrt{3}\pi}{D}}{\frac{3\pi^2}{D^2} + \pi^2 n^2} \\
&= \frac{1}{D^2} - \frac{\sqrt{3}\pi}{D^3} \left(\coth\left(\frac{\sqrt{3}\pi}{D}\right) - \frac{D}{\pi\sqrt{3}} \right) \\
&= \frac{2}{D^2} - \frac{\sqrt{3}\pi}{D^3} \coth\left(\frac{\sqrt{3}\pi}{D}\right) \geq 0,
\end{aligned}
$$

where we have used the Taylor series expansion of $\sinh(x)^2$:

$$
\sinh(x)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n!} \geq x^2 + \frac{x^4}{3},
$$

the partial fraction decomposition of coth:

$$
\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 + x^2}
$$

and assumption (107). This proves the claim. □

12 Stability Analysis

In this section, we will investigate stability properties of critical points of the functional (88).

Definition 12.1. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ be a C^2-functional. A solution u to the Euler-Lagrange equation $\delta\mathcal{J}(u) = 0$ is called *linearly stable* if

$$
\delta^2\mathcal{J}(u,u)\cdot(\phi,\phi) \geq 0
$$

for all $\phi \in \mathcal{H}$.

The second partial variation of (88) with respect to w for $\phi \in H^1_{2\pi}(\mathbb{R})$ is given by

$$
\frac{\delta^2\mathcal{J}}{\delta w^2}(w,h)(\phi,\phi) = 2\int_{-\pi}^{\pi} (Q-2gv)\phi\mathcal{C}_{kh}(\phi') - g\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)\phi^2\,dx \tag{108}
$$

as a straight forward computation shows.
The mixed partial variations for $(\phi,\nu) \in H^1_{2\pi}(\mathbb{R}) \times \mathbb{R}$ are given by

$$
\begin{aligned}
\frac{\delta^2\mathcal{J}}{\delta w\delta h}(w,h)(\phi,\nu) = \nu\int_{-\pi}^{\pi} &(Q-2gv)\mathcal{C}_{kh}(\phi') - 2g\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)\phi \\
&- 2k(Qv - gv^2)\big(\phi'' + \mathcal{C}^2_{kh}(\phi'')\big)\,dx,
\end{aligned} \tag{109}
$$

where we have used the fact that $\partial/\partial h(\mathcal{C}_{kh}(f')) = -kf'' - k\mathcal{C}_{kh}^2(f'')$.
In order to compute the second derivative of $\mathcal{J}$ with respect to h, we first observe that

$$\begin{aligned}
\frac{\partial^2}{\partial h^2}\mathcal{C}_{kh}(f') &= \frac{\partial}{\partial h}\big(- kf'' - k\mathcal{C}_{kh}^2(f'')\big) \\
&= -\frac{\partial}{\partial h}\sum_{n\in\mathbb{Z}^*} kn^2 \coth(nhk)^2 \hat{f}_n e^{inx} \\
&= -\sum_{n\in\mathbb{Z}^*} 2k^2n^3 \coth(nhk)\big(1 - \coth(nhk)^2\big)\hat{f}_n e^{inx} \\
&= 2k^2\big(\mathcal{C}_{kh}(f''') + \mathcal{C}_{kh}^3(f''')\big).
\end{aligned}$$

With this in mind, we readily compute

$$\begin{aligned}
\frac{\delta^2\mathcal{J}}{\delta h^2}(w,h)(\nu,\nu) = 2\nu^2 \int_{-\pi}^{\pi} & k^2(Qv - gv^2)\big(\mathcal{C}_{kh}(w''') + \mathcal{C}_{kh}^3(w''')\big) \\
& - k(Q-2gv)\big(w'' + \mathcal{C}_{kh}^2(w'')\big) - g\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right) + \frac{m^2}{kh^3}\, dx.
\end{aligned} \tag{110}$$

Putting this together, for $(\phi,\nu) \in L^2_{2\pi,\circ}(\mathbb{R}) \times \mathbb{R}$, the second variation of $\mathcal{J}$ is then given by

$$\delta^2\mathcal{J}(w,h)\big((\phi,\nu),(\phi,\nu)\big) = \frac{\delta^2\mathcal{J}}{\delta w^2}(w,h)(\phi,\phi) + 2\frac{\delta^2\mathcal{J}}{\delta w\delta h}(w,h)(\phi,\nu) + \frac{\delta^2\mathcal{J}}{\delta h^2}(w,h)(\nu,\nu). \tag{111}$$

Before we turn to the investigation of stability for general solutions, let us have a look at the form $\delta^2\mathcal{J}(w,h)(.,.)$ at the trivial solution $w = 0$ in detail. We claim that the trivial flow is linearly stable if the relation

$$m^2 \geq gh^3 \tag{112}$$

holds true. First we note, that for the trivial solution, the hydraulic head Q and the conformal mean depth h are related by

$$Q = 2gh + \left(\frac{m}{h}\right)^2. \tag{113}$$

Since $\big(\delta^2\mathcal{J}/\delta w\delta h\big)(0,h)(\phi,\nu) = 0$ for $\phi \in H^1_{2\pi,\circ}(\mathbb{R})$, the second variation reduces to

$$\delta^2\mathcal{J}(0,h)((\phi,\nu),(\phi,\nu)) = 2\int_{-\pi}^{\pi}(Q-2gh)\phi\mathcal{C}_{kh}(\phi') - \frac{g}{k}\phi^2 + \left(\frac{m^2}{kh^3}-\frac{g}{k}\right)\nu^2 dx$$
$$= 2\int_{-\pi}^{\pi}\left(\frac{m}{h}\right)^2\phi\mathcal{C}_{kh}(\phi') - \frac{g}{k}\phi^2 + \left(\frac{m^2}{kh^3}-\frac{g}{k}\right)\nu^2$$
$$+ \left(\frac{m}{h}\right)^2\nu\mathcal{C}_{kh}(\phi') + \left(\frac{m}{h}\right)^2\frac{\nu}{kh}\phi - 2\frac{g}{k}\nu\phi\,dx$$
$$= 2\int_{-\pi}^{\pi}\left(\frac{m}{h}\right)^2(\phi+\nu)\left(\mathcal{C}_{kh}(\phi')+\frac{\nu}{kh}\right) - \frac{g}{k}(\phi+\nu)^2\,dx$$
$$= 2\int_{-\pi}^{\pi}\theta\left(\left(\frac{m}{h}\right)^2\mathcal{G}_{kh}(\theta) - \frac{g}{k}\theta\right)dx \geq 0, \tag{114}$$

by assumption (112) and the definitions (95) and (105). We have set $\theta := \phi+\nu \in H^1_{2\pi}(\mathbb{R})$ and used the fact that $[\phi]_{2\pi} = 0$ as well as relation (113) in the second equality. We can use (81) and (78) to rewrite condition (112) to

$$c^2 \geq gd. \tag{115}$$

Remark 12.2. Note that condition (115) excludes, by the bifurcation theory in [17], the branches of Stokes waves that bifurcate from the trivial solution from our analysis.

We contrast this result to waves of infinite depth. The second derivative of $\mathcal{J}_\infty$ at the trivial solution $w = 0$ is given by

$$\delta^2\mathcal{J}_\infty(0)(\phi,\phi) = \int_{-\pi}^{\pi}\left(\mathcal{C}(\phi') - \frac{\phi}{\mu}\right)\phi\,dx. \tag{116}$$

This form cannot be positive/negative definite for any choice $\mu \geq 0$, since the spectrum of $(\mathcal{C}\partial - 1/\mu)$ on the space $H^1_{2\pi}(\mathbb{R})$ always contains at least the negative eigenvalue $-1/\mu$.
On the contrary, the positivity of $\delta^2\mathcal{J}(0,h)$ is due to a spectral gap for the eigenvalues of the operator $\mathcal{G}_{kh}$, which is not present for the operator $\mathcal{C}\partial$, cf.Figure 1.

Let us now go back to the general case. We rewrite (111) as

$$\delta^2\mathcal{J}(w,h)\big((\phi,\nu),(\phi,\nu)\big) = \left\langle \begin{pmatrix}\phi\\ \nu\end{pmatrix}, \mathcal{L}\begin{pmatrix}\phi\\ \nu\end{pmatrix}\right\rangle_{L^2_{2\pi,\circ}(\mathbb{R})\times\mathbb{R}}, \tag{117}$$

where $\langle(\phi_1,\nu_1),(\phi_2,\nu_2)\rangle_{L^2_{2\pi,\circ}(\mathbb{R})\times\mathbb{R}} := \langle\phi_1,\phi_2\rangle_{L^2_{2\pi}(\mathbb{R})} + \nu_1\nu_2$ and $\mathcal{L}$ is a matrix of operators given by

$$\mathcal{L} = \begin{pmatrix} L_{ww} & L_{wh} \\ L^*_{wh} & L_{hh}\end{pmatrix},$$

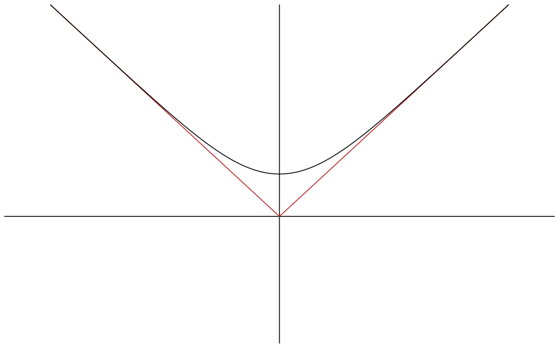

Figure 2: The spectral gap for $xcoth(Dx)$ as compared to $|x|$.

$$L_{ww} : L^2_{2\pi,\circ}(\mathbb{R}) \to L^2_{2\pi,\circ}(\mathbb{R}),$$

$$L_{ww}(\phi) = (Q - 2gv)\mathcal{C}_{kh}(\phi') - g\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right)\phi + 3g[w\phi]_{2\pi},$$

$$L_{wh} : \mathbb{R} \to L^2_{2\pi,\circ}(\mathbb{R}),$$

$$L_{wh}(\nu) = -\big(4g\mathcal{C}_{kh}(w') + 2k(1 + \mathcal{C}^2_{kh})\partial^2(Qv - gv^2)\big)\nu$$

$$L^*_{wh} : L^2_{2\pi,\circ}(\mathbb{R}) \to \mathbb{R},$$

$$L^*_{wh}(\phi) = -\int_{-\pi}^{\pi} 4g\mathcal{C}_{kh}(w')\phi + 2k(Qv - gv^2)(\phi'' + \mathcal{C}^2_{kh}(\phi''))\,dx,$$

$$L_{hh} : \mathbb{R} \to \mathbb{R},$$

$$\begin{aligned} L_{hh}(\nu) = 2\nu \int_{-\pi}^{\pi} & k^2(Qv - gv^2)\big(\mathcal{C}_{kh}(w''') + \mathcal{C}^3_{kh}(w''')\big) \\ & - k(Q - 2gv)\big(w'' + \mathcal{C}^2_{kh}(w'')\big) - g\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right) + \frac{m^2}{kh^3}\,dx. \end{aligned}$$

Remark 12.3. The slight change of L_{ww} compared to (108) by $3g[w\phi]_{2\pi}$ makes the operator well defined as an mapping $L^2_{2\pi,\circ}(\mathbb{R}) \to L^2_{2\pi,\circ}(\mathbb{R})$ and does not alter the form (111).

The operator $\mathcal{L} : L^2_{2\pi,\circ}(\mathbb{R}) \times \mathbb{R} \to L^2_{2\pi,\circ}(\mathbb{R}) \times \mathbb{R}$ is self-adjoint with domain $\mathcal{D}(\mathcal{L}) = H^2_{2\pi}(\mathbb{R}) \times \mathbb{R}$.
Linear stability in the sense of definition 12.1 is guaranteed provided that $\sigma(\mathcal{L}) \subset (0, \infty)$. In order to ensure positivity of $\mathcal{L}$, we will impose conditions of the diagonal operators L_{ww} and L_{hh}.

12.1 Positivity for the operator L_{ww}

We adapt a transform introduced in [37] in the context of solitary wave that simplifies the variation $\delta^2\mathcal{J}/\delta w^2$ to provide a condition for which the operator L_{ww} is positive. To do so, we will rewrite the integral (108) in terms of the

holomorphic extension of the solution w and the function ϕ.
We define

$$W(z) = \frac{1}{k} + \mathrm{i}\mathbf{R}_D[w'](z), \tag{118}$$

which represents a holomorphic function on the strip $\mathcal{S}_D$ by (94). For $x \in [-\pi, \pi]$, we find

$$W(x) = \frac{1}{k} + \mathcal{C}_{kh}(w')(x) + \mathrm{i}w'(x). \tag{119}$$

For further computations, we will show that $W(z) \neq 0$ for all $z \in \mathcal{S}_D$. If $w = 0$ is the trivial solution, we have $W \equiv \frac{1}{k} > 0$. So assume that $w \not\equiv 0$. Since $\Im\{W\}(x - \mathrm{i}D) \equiv 0$ by the definition of $\mathbf{R}_D$ in (94), we also have that $\Im\{W\}_x(x - \mathrm{i}D) \equiv 0$. However, $\Im\{W\}_x(x - \mathrm{i}D) = \Re\{W\}_y(x - \mathrm{i}D) = 0$ so that according to Hopf's maximum principle, the harmonic function $\Re\{W\}$ cannot attain neither a maximum, nor a minimum on the line $\{y = -D\}$. Assuming that the free surface is given by the graph of function (80), we deduce that $1/k + \mathcal{C}_{kh}(w') = \Re\{W\}(x) > 0$ and hence, by the minimum principle for harmonic functions, $\Re\{W\}(x, y) > 0$ for all $(x, y) \in \mathcal{S}_D$. In particular, the holomorphic function W has no zeros in $\mathcal{S}_D$.
We can now reformulate equation (82) in terms of W:

$$(Q - 2gv)|W|^2 = \mu. \tag{120}$$

Taking the derivative of (120), we have

$$(Q - 2gv)2\Re\{W'W^*\} - 2gw'|W|^2 = 0,$$

or, using equation (120) to express $(Q - 2gv)$:

$$gw'|W|^2 = \mu\Re\left\{\frac{W'}{W}\right\}. \tag{121}$$

Definition 12.4. Let $u \in L^2_{2\pi}(\mathbb{R})$ and let $\mathbf{R}[u]$ be its holomorphic extension as defined in (94). For any fixed solution W to problem (120), the *Plotnikov transform* of u is defined as

$$\begin{aligned}\mathcal{P}[u](x) &:= \Re\{W(x)\mathbf{R}[u](x)\} \\ &= u(x)\left(\frac{1}{k} + \mathcal{C}_{kh}(w')(x)\right) + w'(x)\frac{\hat{u}_0}{kh}x + w'(x)\mathcal{C}_{kh}(u - \hat{u}_0)(x)\end{aligned} \tag{122}$$

Remark 12.5. For functions $u \in L^2_{2\pi,\circ}(\mathbb{R})$, Plotnikov's transform simplifies to

$$\mathcal{P}[u] = u\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right) + w'\mathcal{C}_{kh}(u), \tag{123}$$

which implies that $[\mathcal{P}[u]]_{2\pi} = 0$. Henceforth, we will only be interested in this special case, since we want to transform $\delta^2\mathcal{J}/\delta w^2$, which operates on $L^2_{2\pi,\circ}(\mathbb{R})$.

Before we rewrite the integral (108), we collect some properties of the transform $\mathcal{P}$. We find that for any $u \in L^2_{2\pi,\circ}(\mathbb{R})$

$$
\begin{aligned}
\mathbf{R}[\mathcal{P}[u]] &= u\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right) + w'\mathcal{C}_{kh}(u) \\
&\quad - \mathrm{i}\left(\mathcal{C}_{kh}\left(\frac{u}{k}\right) + \mathcal{C}_{kh}\left(u\mathcal{C}_{kh}(w')\right) + \mathcal{C}_{kh}\left(w'\mathcal{C}_{kh}(u)\right)\right) \\
&= u\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right) + w'\mathcal{C}_{kh}(u) - \mathrm{i}\left(\frac{\mathcal{C}_{kh}(u)}{k} + \mathcal{C}_{kh}(u)\mathcal{C}_{kh}(w') - uw'\right) \\
&= W\mathbf{R}[u],
\end{aligned}
\tag{124}
$$

where we have used identity (98) and the fact that $u, w' \in L^2_{2\pi,\circ}(\mathbb{R})$. This implies that Plotnikov's transform is a homeomorphism $\mathcal{P} : L^2_{2\pi,\circ}(\mathbb{R}) \to L^2_{2\pi,\circ}(\mathbb{R})$ with inverse given by

$$
\mathcal{P}^{-1}[v] = \Re\left\{\frac{\mathbf{R}[v]}{W}\right\}. \tag{125}
$$

For further computations, we also remark that for any $u \in L^2_{2\pi,\circ}(\mathbb{R})$:

$$
\mathbf{R}[\mathcal{P}[u]'] = \mathbf{R}[\mathcal{P}[u]]' = (W\mathbf{R}[u])' = W\mathbf{R}[u'] + W'\mathbf{R}[u] \tag{126}
$$

by linearity of the holomorphic extension $\mathbf{R}$ and relation (124).

We will also need the following result in the computations:

Lemma 12.6. *Let $F, G : \mathcal{S}_D \to \mathbb{C}$ be 2π-periodic holomorphic functions with the property that $\Re\{F\}(x - \mathrm{i}D) = \Re\{G\}(x - \mathrm{i}D) = 0$. Then*

$$
\int_{-\pi}^{\pi} \Im\{F^*G\}(x)\,dx = 2\int_{-\pi}^{\pi} \Re\{F\}(x)\Im\{G\}(x)\,dx.
$$

Proof. Integrating the holomorphic function FG along the rectangle $(0, -\pi) \to (0, \pi) \to (-D, \pi) \to (-D, -\pi)$, using periodicity, the claim follows from Cauchy's Integral Theorem by comparison of imaginary parts. □

Now we are ready to prove the following

Proposition 12.7. *Let $\phi \in L^2_{2\pi,\circ}(\mathbb{R})$ and let W be a solution to (120). Then Plotnikov's transform rewrites the integral form (108) according to*

$$
\frac{\delta^2 \mathcal{J}}{\delta w^2}\left(\mathcal{P}[\phi], \mathcal{P}[\phi]\right) = 2\mu \int_{-\pi}^{\pi} \{\phi\mathcal{C}_{kh}(\phi') - \mathcal{Q}\phi^2\}\,dx, \tag{127}
$$

where

$$
\mathcal{Q} := \Im\left\{\frac{W'}{W}\right\} + |W|^2 \frac{g}{\mu}\left(\frac{1}{k} + \mathcal{C}_{kh}(w')\right). \tag{128}
$$

Proof. We first express the second partial variation $\delta\mathcal{J}/\delta w^2$ in terms of the holomorphic extension operator $\mathbf{R}$ in order to use the formulae (124), (125) and

(126). Integrating by parts the expression $(Q-2gv)\big(\phi\mathcal{C}_{kh}(\phi')\big)$ we obtain:

$$\frac{\delta^2\mathcal{J}}{\delta w^2}(\phi,\phi) = 2\int_{-\pi}^{\pi}(Q-2gv)\phi\mathcal{C}_{kh}(\phi') - g\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)\phi^2\,dx$$

$$= \int_{-\pi}^{\pi}(Q-2gv)\big(\phi\mathcal{C}_{kh}(\phi')-\phi'\mathcal{C}_{kh}(\phi)\big)\,dx$$

$$+\,2g\int_{-\pi}^{\pi}\phi\left\{w'\mathcal{C}_{kh}(\phi)-\phi\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)\right\}\,dx.$$

By the definition of W in (119), the definition of the complexification $\mathbf{R}$ in (97) and by equation (120), we find

$$\frac{\delta^2\mathcal{J}}{\delta w^2}(\phi,\phi) = \int_{-\pi}^{\pi}(2gv-Q)\Im\{\mathbf{R}[\phi]^*\mathbf{R}[\phi]\}\,dx - 2g\int_{-\pi}^{\pi}\phi\Re\{\mathbf{R}[\phi]^*W\}\,dx$$

$$= -\int_{-\pi}^{\pi}\frac{\mu}{|W|^2}\Im\{\mathbf{R}[\phi]^*\mathbf{R}[\phi]\}\,dx - 2g\int_{-\pi}^{\pi}\phi\Re\{\mathbf{R}[\phi]^*W\}\,dx$$

$$= -\mu\int_{-\pi}^{\pi}\Im\left\{\left(\frac{\mathbf{R}[\phi]}{W}\right)^*\left(\frac{\mathbf{R}[\phi']}{W}\right)\right\}\,dx - 2g\int_{-\pi}^{\pi}\phi|W|^2\Re\left\{\left(\frac{\mathbf{R}[\phi]}{W}\right)^*\right\}\,dx.$$

We apply Lemma (12.6) to the holomorphic functions $F=\mathbf{R}[\phi]/W$ and $G=\mathbf{R}[\phi']/W$ (remembering that W has no zeros in $\mathcal{S}_D$) in order to further simplify the integral:

$$\frac{\delta^2\mathcal{J}}{\delta w^2}(\phi,\phi) = -2\mu\int_{-\pi}^{\pi}\Re\left\{\frac{\mathbf{R}[\phi]}{W}\right\}\Im\left\{\frac{\mathbf{R}[\phi']}{W}\right\}\,dx - 2g\int_{-\pi}^{\pi}\phi|W|^2\Re\left\{\left(\frac{\mathbf{R}[\phi]}{W}\right)\right\}\,dx$$

$$= 2\int_{-\pi}^{\pi}\Re\left\{\frac{\mathbf{R}[\phi]}{W}\right\}\left(\mu\Re\left\{\frac{\mathrm{i}\mathbf{R}[\phi']}{W}\right\} - g|W|^2\phi\right).$$

Now the second variation is reformulated, we are ready to invoke Plotnikov's transform. Taking advantage of (124) and (126), we arrive at

$$\frac{\delta^2\mathcal{J}}{\delta w^2}(\mathcal{P}[\phi],\mathcal{P}[\phi]) = 2\int_{-\pi}^{\pi}\Re\left\{\frac{\mathbf{R}[\mathcal{P}[\phi]]}{W}\right\}\left(\mu\Re\left\{\frac{\mathrm{i}\mathbf{R}[\mathcal{P}[\phi]']}{W}\right\} - g|W|^2\mathcal{P}[\phi]\right)$$

$$= 2\int_{-\pi}^{\pi}\phi\left(\mu\mathcal{C}_{kh}(\phi') + \mu\Re\left\{\mathrm{i}\frac{W'}{W}\mathbf{R}[\phi]\right\}\right.$$

$$\left. -g|W|^2\left\{\phi\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)+w'\mathcal{C}_{kh}(\phi)\right\}\right)\,dx$$

$$= 2\int_{-\pi}^{\pi}\phi\left(\mu\mathcal{C}_{kh}(\phi') + \mu\Re\left\{\frac{W'}{W}(\mathrm{i}\mathbf{R}[\phi]-\mathcal{C}_{kh}(\phi))\right\}\right.$$

$$\left. -g|W|^2\phi\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)\right)\,dx$$

$$= 2\int_{-\pi}^{\pi}\mu\phi\mathcal{C}_{kh}(\phi') - \left\{\mu\Im\left\{\frac{W'}{W}\right\} + |W|^2\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)\right\}\phi^2\,dx$$

□

Remark 12.8. The modification of Plotnikov's transform for periodic waves of finite depth relies upon the separation of the mean h from v in the equations (76) and (77) so that we can take variations in $L^2_{2\pi,\circ}(\mathbb{R})$. In the original paper [37], the argument for solitary waves is based upon decay properties of the solution as $x \to \infty$, while for periodic waves in infinite depth, decay properties of the solutions as $y \to -\infty$ are employed, cf. [11, 12, 38]. For periodic waves in finite depth, the calculations would fail for general $\phi \in L^2_{2\pi}$.

The following theorem is now an immediate consequence of Proposition 12.7 together with the spectral properties of the operator $\mathcal{C}_{kh}\partial$:

Theorem 12.9. *Let w be a solution to equation (76) with the property that*

$$\mathcal{Q}(x) \leq \coth(kh) \tag{129}$$

for all $x \in [-\pi, \pi]$, then the operator L_{ww} is positive.

It would be interesting to give a physical interpretation of the potential $\mathcal{Q}$. This could help in understanding the stability condition (129).

12.2 Positivity for the operator L_{hh}

Let us look for a condition that ensures the positivity of $\delta^2\mathcal{J}/\delta h^2$ or L_{hh} respectively. We will find that, surprisingly, there is only little difference to the case of the trivial solution, cf. (115):

Proposition 12.10. *Let (w, h) be a solution to equation (76) with the property that*

$$c^2 \geq gd \quad \text{and} \quad 2kh \geq \sqrt{3}\pi \coth\left(\frac{\sqrt{3}\pi}{kh}\right) \tag{130}$$

Then L_{hh} is positive.

Proof. We use the symmetry of the operator $\mathcal{C}_{kh}\partial$ to rewrite (110) as

$$\begin{aligned}\frac{\delta\mathcal{J}}{\delta h^2} = 2\int_{-\pi}^{\pi} k^2\Big\{(Q-2gh)\mathcal{C}_{kh}(w') - 2g\mathcal{C}_{kh}(ww')\Big\}\Big(w'' + \mathcal{C}^2_{kh}(w'')\Big)\\ + 2gkw\Big(w'' + \mathcal{C}^2_{kh}(w'')\Big) + \left(\frac{m^2}{kh^3} - \frac{g}{k}\right)\,dx\end{aligned}$$

Now we use equation (76) to replace $(Q-2gh)\mathcal{C}_{kh}(w')$:

$$\begin{aligned}\frac{\delta\mathcal{J}}{\delta h^2} = 2\int_{-\pi}^{\pi} k^2\Big\{gw\mathcal{C}_{kh}(w') - g\mathcal{C}_{kh}(ww')\Big\}\Big(w'' + \mathcal{C}^2_{kh}(w'')\Big)\\ + 3gkw\Big(w'' + \mathcal{C}^2_{kh}(w'')\Big) + \left(\frac{m^2}{kh^3} - \frac{g}{k}\right)\,dx\end{aligned}$$

By relation (106), we can write:

$$\begin{aligned}\frac{\delta\mathcal{J}}{\delta h^2} = 2\int_{-\pi}^{\pi} gk^2\Big\{w\mathcal{C}_{kh}(w') - \mathcal{C}_{kh}(ww')\Big\}\Big(v'' + \mathcal{G}^2_{kh}(v)\Big)\\ - \frac{g}{h}\Big\{w\mathcal{C}_{kh}(w') - \mathcal{C}_{kh}(ww')\Big\} + 3gkw\Big(w'' + \mathcal{C}^2_{kh}(w'')\Big) + \left(\frac{m^2}{kh^3} - \frac{g}{k}\right)\,dx.\end{aligned}$$

We use relation (81) and the fact that $[\mathcal{C}_{kh}(ww')]_{2\pi} = 0$ to find

$$\begin{aligned}\frac{\delta\mathcal{J}}{\delta h^2} &= 2\int_{-\pi}^{\pi} gk^2\Big\{w\mathcal{C}_{kh}(w') - \mathcal{C}_{kh}(ww')\Big\}\Big(v'' + \mathcal{G}_{kh}^2(v)\Big) \\ &\qquad + 3gkw\Big(w'' + \mathcal{C}_{kh}^2(w'')\Big) + \left(\frac{m^2}{kh^3} - \frac{g}{k} - \frac{g(d-h)}{kh}\right)dx \\ &\geq \frac{4\pi}{kh^3}\big(m^2 - gh^3 - g(d-h)h^2\big) = \frac{4\pi c}{km}\big(c^2 - gd\big) \geq 0\end{aligned}$$

where for the first inequality we have used the fact that

$$\int_{-\pi}^{\pi} 3gkw\Big(w'' + \mathcal{C}_{kh}^2(w'')\Big)\,dx = 3gk\sum_{n\in\mathbb{Z}^*} n^2\big(\coth(hkn)^2 - 1\big)|\hat{w}_n|^2 \geq 0$$

together with Lemma 11.4 in combination with Lemma 11.6 for the function $v > 0$. Keeping in mind that $h = \frac{m}{c}$, cf. (78), the claim follows by assumption (130). $\square$

12.3 Positivity of the operator $\mathcal{L}$

Now we are ready to give a proof of our main result:

Theorem 12.11. *Let (w, h) be a solution to equation (76) with the property that*

$$c^2 \geq gd \quad \text{and} \quad 2kh \geq \sqrt{3}\pi\coth\left(\frac{\sqrt{3}\pi}{kh}\right)$$

as well as

$$\mathcal{Q} \leq \coth(kh).$$

Then the solution is linearly stable.

Proof. We will show that $\sigma(\mathcal{L}) \subset (0,\infty)$. We already know that $\sigma(L_{ww}) \cup \sigma(L_{hh}) \subset (0,\infty)$ and hence it suffices to show that $\sigma(\mathcal{L}) \cap \big(\rho(L_{ww}) \cup \rho(L_{hh})\big) \subset (0,\infty)$. To this end, we apply Lemma (13.3) to the $\mathbb{R}$-characteristic of $\mathcal{L}$. We find that $\lambda \in \sigma(\mathcal{L}) \cap \big(\rho(L_{ww}) \cup \rho(L_{hh})\big)$ if an and only if $\lambda \in \mathbb{C}$ is a solution to the equation

$$\lambda - L_{hh}(1) - L_{wh}^*\mathcal{R}(L_{ww},\lambda)L_{wh}(1) = 0. \tag{131}$$

Setting

$$V := \mathcal{R}(L_{ww},\lambda)L_{wh}(1) \in L^2_{2\pi}(\mathbb{R}),$$

we can rewrite (131) as

$$\lambda - L_{hh}(1) - \big\langle (L_{ww} - \lambda)V, V\big\rangle = 0,$$

or equivalently as

$$\lambda = \frac{L_{hh}(1) + \langle L_{ww}V, V\rangle}{1 + \|V\|^2} \geq 0,$$

by the positivity of L_{hh} and L_{ww}. So if there exists a solution to (131) at all, it necessarily has to be non-negative. This proves the claim. $\square$

Remark 12.12. Investigating the second variation $\delta^2\mathcal{J}(0,h)(.,.)$ at the trivial solution according to Theorem 12.11, we find that the restriction that is put upon $\delta^2\mathcal{J}/\delta w^2(0,h)$ is superfluous. Indeed, all the information that guarantees stability is already included in the expression $\delta\mathcal{J}/\delta h^2(0,h)$. However, since the trivial solution is independent upon the wave number k, we cannot expect to find a similarly simple behavior at a general solution.

13 Appendix: Matrices of Operators

In this section, following [34], we will prove a lemma about matrices of operators. Throughout, we will make these general assumptions:

Assumption 13.1. Let E and F be Banach spaces and assume that

1. The operator A with domain $\mathcal{D}(A)$ has non-empty resolvent set $\rho(A)$ in E,
2. The operator D with domain $\mathcal{D}(D)$ has non-empty resolvent set $\rho(D)$ in F,
3. The operator B with domain $\mathcal{D}(B)$ is relatively D-bounded, i.e. $\mathcal{D}(D) \subset \mathcal{D}(B)$ and $B\mathcal{R}(\lambda, D) \in \mathcal{L}(F, E)$ for $\lambda \in \rho(D)$,
4. The operator C with domain $\mathcal{D}(C)$ is relatively A-bounded, i.e. $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $C\mathcal{R}(\lambda, A) \in \mathcal{L}(E, F)$ for $\lambda \in \rho(A)$,
5. The operator $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(D)$ is closed in $E \times F$.

In order to relate the spectrum of the matrix of operators $\mathcal{A}$ to the spectra of its entries, we are looking for an analog of the characteristic polynomial of a matrix.

Definition 13.2. Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix of operators satisfying condition (13.1) and let $\lambda \in \rho(A) \cap \rho(D)$. The operators

$$\Delta_E(\lambda) := \lambda - A - B\mathcal{R}(\lambda, D)C, \tag{132}$$

$$\Delta_F(\lambda) := \lambda - D - C\mathcal{R}(\lambda, A)B \tag{133}$$

are called *E-characteristic* and *F-characteristic* for the matrix of operators $\mathcal{A}$ respectively.

Now it is easy to relate the spectral properties of the matrix $\mathcal{A}$ on the set $\rho(A) \cap \rho(D)$ to its E-characteristic and F-characteristic respectively:

Lemma 13.3. *Consider a matrix of operators $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acting on the Banach space $E \times F$ for which the assumptions (13.1) hold true. For any $\lambda \in \rho(A) \cap \rho(D)$ the following assertions are equivalent:*

1. $\lambda \in \sigma(\mathcal{A})$

2. $0 \in \sigma(\Delta_E(\lambda))$

3. $0 \in \sigma(\Delta_F(\lambda))$

The prove can be found in [34].

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Zusammenfassung in Deutsch

Diese Arbeit umfasst ausgewählte Publikationen, die während der Zeit des Doktoratsstudiums an der Universität Wien entstanden sind. Sie beschäftigen sich mit der mathematischen Analyse von Wasserwellen. Wasser wird dabei als ein inkompressibles Fluid unter dem Einfluss der Schwerkraft modelliert, wobei viskose Effekte vernachlässigt werden. Das Hauptaugenmerk liegt dabei auf periodischen Wasserwellen in endlicher Tiefe.

Die erste Arbeit handelt vom Zusammenhang zwischen dem Druck am Grund und dem Profil der Welle. Wir leiten exakte Relationen zwischen diesen beiden her und erhalten eine Darstellungsformel für die Oberfläche als Fourierreihe.

Die zweite und dritte Arbeit beleuchten den Zusammenhang zwischen Symmetrie und Wanderwellenlösungen. Wir zeigen, dass eine Lösung mit der Eigenschaft, dass sowohl die horizontale Komponente des Geschwindigkeitsfeldes an der Oberfläche als auch das Wellenprofil selbst symmetrisch um eine gemeinsame, zeitabhängige Symmetrieachse sind, durch eine Wanderwelle gegeben ist. Ein ähnliches Resultat kann für die horizontale Komponente des Geschwindigkeitsfeldes am Grund und eine antisymmetrische Beschleunigung gezeigt werden.

In der vierten Arbeit behandeln wir Stabilitätseigenschaften periodischer Wellen unter Berücksichtigung der Gravitation in endlicher Tiefe. Wir verwenden hierzu eine Umformulierung der Gleichungen als eine Pseudodifferentialgleichung, die wir als Euler-Lagrange Gleichung eines nichtlinearen Funktionals erhalten. Für die zweite Ableitung dieses Funktionals zeigen wir, dass sie positiv semidefinit ist und damit die entsprechenden Lösungen der Gleichungen linear stabil sind. Der Beweis macht Gebrauch von Abschätzungen für die Hilberttransformation auf einem Streifen, einem nichtlinearen Koordinatenwechsel im unendlichdimensionalen Teil des Lagrangefunktionals und von Spektraleigenschaften von Operatormatrizen.

Curriculum Vitae

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Nov. 2013 – present	Ph.D. studies in Mathematics, University of Vienna Advisor: Adrian Constantin
Oct. 2012 – Oct 2013	Master Studies in Mathematics, University of Vienna Diploma thesis: On some planar Hamiltonian systems Advisor: Adrian Constantin Graduation with distinction
Oct. 2011 – Oct. 2012	Bachelor Studies in Mathematics, University of Vienna Graduation with distinction
June 2011	High school graduation with distinction

List of publications

1. *On the Cauchy problem for planar autonomous Hamiltonian systems with a discontinuous right-hand side*, Monatsh. Mathem. 173(4), 2014
2. *Recovery of the wave profile for irrotational periodic water waves from pressure measurements*, Nonl. Anal. Real World Appl. 22:219–224, 2015
3. *Symmetric irrotational water waves are traveling waves*, J. Diff. Equ. 259(10):5271-5275, 2015
4. *On the trigonometric polynomial Ansatz for gravity water waves*, Appl. Anal., 2015
5. *On symmetric water waves with constant vorticity*, J. Nonl. Math. Phys. 22(4): 494–498, 2015
6. *On the symmetry of spatially periodic two-dimensional water waves*, acc. for publ. in Disc. Cont. Dyn. Sys. A

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German:	Mother tongue
English:	Fluent, written and spoken
French:	Basic, written and spoken