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DISSERTATION

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Quantum Dissipative Dynamics and their applications

Verfasser

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The greatest words given to me which I pass down to the reader

"If you can fill the unforgiving minute, with sixty seconds' worth of distance run, Yours is the Earth and everything that's in it, And-which is more-you'll be a Man, my son!"

– Rudyard Kipling

Contents

Zusammenfassung

Diese These besteht aus verschiedenen ursprünglichen Ansätzen zu einem besseren Verständnis der (lokalen) ableitfähigen Dynamik. Diese Dynamik wird durch rundum positive Operatoren beschrieben. Die Dissertation umfasst die verschiedenen Aspekte dieses Themas.

Im ersten Teil studieren wir diese Dynamik im Rahmen der Spin-Gitter. Wir stellen Fragen und beantworten solche, wie beispielsweisedie Beziehungen zwischen Informationen-Vermehrung und der mischenden Eigenschaft dieser Dynamik zu verstehen ist. Die beiden Eigenschaften der gleichzeitigen systeminternen Dynamik sind sehr schwer zu vereinheitlichen. Erstere ist eine Eigenschaft die unabhängig von der Größe des Systems ist, während letztere sich meist ausdehnt. Die wichtigsten Ausgangspunkte sind Lokalisierungskonzepte und schnelles Mischen. Wir bieten Konstruktionen verschiedener Generatorenklassen, inspiriert von solchen Eigenschaften, und implizieren eine neue Geschwindigkeit der Informationsverbreitung, die sich in der Zeit verlangsamt.

Im zweiten Teil versuchen wir das detaillierte Gleichgewicht im Zusammenhang mit solchen großen Systemen zu verstehen. Die ursprüngliche Motivation besteht darin Beispielalgorithmen von Quantenphasen mit klassischem Nichtgleichgewichtsphasen durch die quasi adiabatische Evolution zu verbinden. Dies geschieht durch die Zuordnung des Generators der klassischen stochastischen Dynamik auf lokalen Hamiltonoperatoren.

Im dritten Teil konzentrieren wir uns auf die natürliche Entstehung der rundum positiven Operatoren in der Statistik der durch stochastische Differentialgleichungen (diskretisierten) Quanten. Es stellt sich heraus, dass Quantendynamiken natürlichere Operatoren - auch im Kontext der rein klassischen, nicht-Markovschen, Dynamik - sind.

Im letzten Teil, schauen wir nochmals die berühmte Källén-Lehmann spektrale Darstellung der Quantenfeldtheorie an, die durch das Studium der übergeordneten Hamilton injektiven Matrix Produktzustände entsteht. In der Reichweite und großen System-Größenbeschränkung können wir die niedrigen Energiespektren von diesen Hamilton lösen. Eine natürliche Darstellung der Partikel erfolgt durch die Fourier-Transformation eines Quanten-Kanals, d.h. der Transfer-Matrix. Im Sinne der Källén-Lehmann spektralen Darstellung zeigen wir, dass das Spektrum einer Interaktion freie Dynamik in der Korrelation des Grundzustands verborgen ist. Als weitere Anwendung nutzen wir diese Einsicht, um Anderson Lokalisierung des Selbstauslösers zufälliger Matrix Produktzuständezu überdenken.

Abstract

This thesis consists of various original approaches towards a better understanding of (local) dissipative dynamics. Such dynamics are described by completely positive operators. The thesis covers different aspects of this topic.

In the first part, we study such dynamics in the context of spin lattices. We then ask and answer questions such as how to understand the relations between information propagation and the mixing property of these dynamics. Both properties while being intrinsic of the dynamics are very difficult to unify. The former is a property independent of the size of system while the latter is mostly extensive. The main starting points are concepts of localization and rapid mixing. We propose constructions of different classes of generators inspired from such properties, which imply a new velocity of propagation of information which slows down in time.

In the second part, we try to understand detailed balance in the context of such large systems. The original motivation is to trivially connect quantum phases with classical non-equilibrium phases through the quasi-adiabatic evolution by mapping the generators of the classical stochastic dynamics onto local Hamiltonians. This leads us to a study and natural construction of invertible Matrix Product Operators.

In the third part, we focus on the natural emergence of completely positive operators in statistics through (discretized) quantum stochastic differential equations. It turns out that quantum dynamics are more natural operators even in the context of purely classical non-Markovian dynamics.

In the final part, we revisit the famous Källén -Lehmann spectral representation of quantum field theory through the study of parent Hamiltonians of injective Matrix Product States. In the long range and large system size limit, we can solve the low energy spectrum of these Hamiltonians. A natural representation of particles appears from the Fourier Transform of a quantum channel, i.e. the transfer matrix. In the same spirit as the the Källén-Lehmann spectral representation, we show that the spectrum of an interaction free dynamic is hidden inside the correlation of the ground state. As a side application, we use this insight to revisit Anderson's localization in the picture of random matrix product states.

Acknowledgments

These last three years were by far the most difficult years of my life. Never could I have imagined that one could lose the supervisor lottery in such a way. I wish I could thank my fellow students in Vienna and supervisor for their support, but I will not do so for obvious reasons.

I do thank the CoQuS doctoral school and my parents for the financial support allowing me to discover and better myself as a researcher and a person these last few years.

I am grateful to my sister and close friends for cheering me up during the times I felt I was set adrift alone on an iceberg. I would also like to express my gratitude to Dominic Williamson, Michael Mariën, Jutho Haegeman and James Whitfield for the brief, however insightful, discussions.

Even though I feel that this thesis did not reach the goals I had set to myself, I feel pride that I manage to get so many interesting results and ideas in such difficult circumstances. I hope that the attentive reader will enjoy the many creative ideas presented in this work. As a researcher, I hope these years of hard and challenging work will have some sort of pay-off wherever my career may take me.

Introduction

This thesis is the result of simple, yet challenging, questions concerning stochastic processes in the realm of quantum mechanics. It is surprising how a different background can give such a contrasting perspective on certain problems.

During my masters, following the death of my closest friend, I started to question the purpose of the current education of physicists. Physics, while claiming to unify the deepest laws of nature, seemed to differ in so much aspects from Mathematics. The former only seemed to be a particular case of the latter. As physicists, we seek to model the world and look for simplicity. On the other hand, Mathematics seemed to construct a much more diverse and wild world, while starting from simple building blocks.

My crisis came to an end when I was introduced to the work by Matthew Hastings [37, 38, 39, 43, 44, 45], during a short lecture given by Master thesis supervisor Marc Fannes. At the same time, I was introduced to the so-called finitely correlated states [27]. In a sense, a glimpse of hope, that my physics education had not been for nothing, was presented to me. Suddenly, I was presented to a new world where models where not necessarily simple of nature, to facilitate understanding, but also were backed by simple mathematical axioms with local physical meaning.

The following summer, I acquainted myself to the world of quantum information and tensor networks. It is surprising how much physics is contained inside seemingly complicated representations.

Upon arriving in Vienna, after an extremely brief discussion with my supervisor, I was asked to have a look at local dissipative dynamics [59, 95]. The very same day, I decided to make use of this opportunity to combine such systems with the work that revived my interest in physics and motivated me in pursuing doctoral studies. This was a starting point of many small and challenging problems which I tackled full of love and passion.

The central topic of this thesis is the study of so-called quantum dissipative dynamics. It is a basic assumption in quantum mechanics that closed system evolve under a unitary dynamic. This axiom breaks down when the system is coupled to an environment [3, 14, 97]. The dynamic loses its unitarity but preserves a more general property, complete positivity. A first consequence is that the distance between states is not preserved any more. As time passes, they are driven closer to each other. Under the assumption of ergodicity, only one state survives. In

the world of physics, this is very well understood when studying a small n-level system weakly or strongly coupled to an infinite bath in thermodynamic equilibrium. Under such assumptions, new interesting properties are derived. The first one is Markovianity. Because of the size of the bath and the weak coupling [20], interactions over small time-intervals can be averaged out, and the small systems forgets past events. A second aspect, is that transitions between the different energy levels of the small systems are reversible. The probability of transitioning from one state to the other is proportional to the probability of the reversed transition with a factor related to a gibbs state.

The study of completely positive operators opens the door to an immeasurable world of possibilities and new questions.

On the very first day of my arrival, I wrote the following two words on a paper,

Local Dissipation

Let us break it down. Local perturbations in spin Lattices propagate at a finite velocity. This finite velocity is a direct consequence of the dynamic of the system being generated by local interactions [64]. Often, in mathematical physics, a dynamics is defined to be local if such property persists. On the other hand, what is dissipation? Keeping the simple picture of our finite level-system coupled to a large environment, we can see that due to these additional interactions something will get lost in the vastness of the external bath without ever coming back.

If one knows a little about bit about classical mixing theory, one has heard about geometric bounds for mixing times [4, 23]. This vast theory attemps to estimate the second largest eigenvalue of Markov chains based on purely geometric quantities such as the maximum degree, diameter, covering numbers, etc... of graphs.

If one considers this finite group velocity of propagation in lattices as a geometrical quantity, would it not be wonderful to follow the steps of Cheeger, Stroock, Diaconis, Milman,... and figure out a way to use this physical property as a new tool for estimating mixing/dissipative properties.

This basic question was a starting point of three years of many other small questions. Each question attempts at understanding structural and spectral properties of completely positive operators from different original perspectives.

One particularly additional program that came up towards the end of the research, is the idea of unifying spectral properties of finite dimensional open system with concepts of infinitely large systems in condensed matter using the framework of Matrix Product States, which we started in chapter 4.

Outline and Summary of the results

Chapter 1: In this first chapter, we study the effect of dissipation on the loss of information. Mathematically, we do this by deriving a Lieb-Robinson bound with a time-dependent velocity, which decelerates. This problem is extremely challenging. On one hand, Lieb-Robinson bound is a "thermodynamic" property. In this context, we mean that it does not depend on the system size at all. On the other hand, the theory of mixing times does and generally needs to be extensive. Unifying both concepts, is certainly an interesting project. Starting from the intuition of rapid mixing and localization, we construct simple classes of dynamics. At the end of the chapter, we revert the question and briefly study how based on an assumption that information should propagate more slowly in dissipative systems, this should imply rapid mixing. These lead us to a definition of attractiveness of processes.

Chapter 2: In this second chapter, we are first introduced to a construction of invertible Matrix Product Operators while trying to relate quantum phases with classical dissipative phases using the mechanics of quasi-adiabatic evolution. We then present a peculiar construction of local stochastic dynamics for which the Perron-Frobenius vector has a Matrix Product Representation, which is reminiscent of detailed balance in a virtual space. Finally, we argue the importance of better understanding how Matrix Product Operators and their inverse can preserve local structures. If such class can be understood, then an interesting algorithm can be applied which evolve the Cholesky factors of positive definite Matrix Product Operators.

Chapter 3: In this third, very introductory, chapter, we study the representations of joint-probability distributions by Matrix Product States. We show that Matrix Product States arise naturally when considering some non-Markovian processes. Interestingly, all properties of such classical processes are determined by a quantum dynamic. We discuss this peculiar unravelling starting from quantum measurement theory and relating it to a non-commutative version of Girsanov theorem for the Radon-Nikodym derivative of these processes.

Chapter 4: In this final chapter, we study a peculiar limit of parent Hamiltonians of injective Matrix Product States. We show that in the long range and large system size limit, all these parent Hamiltonian turn out to have an exactly solvable low energy spectrum. Even more so, this spectrum depends solely on the properties of a quantum channel, i.e. transfer matrix of the ground state, and not on any other details of the ground-state.

As an application, we revisit the notion of (many-body)-localization with our framework. Our calculation in this chapter revealed that translational invariant Matrix Product can be interpreted as a stationary sea of particles. We consider fluctuations of the local tensors of a continuous one-parameter family of Matrix Product States. This leads to a very simple, intuitive and alternative view on the topic of localization.

Chapter 1

Mixing of Markov Processes and Loss of Information in Quantum Spin Chains

Synopsis:

One of the most fundamental properties of local dynamics in quantum spin chains is the finite velocity of propagation of information. On the other hand, generalizing unitary dynamics, we can study the effect of dissipation on the loss of information.

The fact that "something" will be lost is of course obvious. The mechanisms, however, are not. Mixing of states generally takes place on a much larger time and length scale. It is an extremely challenging problem to try and unify both concepts. In this chapter, we propose various mechanisms which make such unification possible.

We study the velocity of the propagation of information for a class of local dissipative quantum dynamics. This finite velocity is expressed by the so-called Lieb-Robinson bound. Besides the properties of the already studied dynamics, we consider an additional relation that expresses the propagation of certain subspaces. We show that for this class the velocity of propagation of information is time dependent and decays in time towards a smaller velocity. In some cases the velocity becomes zero.

A second class of example has some slight resemblance with localization. For this class, the velocity is simply "killed" off. We show specifically that for some local dynamics, local observables are propagated inside a much smaller convex subset of the total Banach space.

Based on:
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1.1 Introduction

Locality and causality are two of the most fundamental concepts in physics. Measurements are realised in a small subset of space during a small interval of time. Two measurements far away from each other should not influence each other for a long period of time. In other words, information propagates with a finite velocity. This is implied by the local Lorentz invariance or covariance. In 1972, Lieb and Robinson [64] showed that this finiteness is inherited by locality and not the property of local symmetry. This result was shown using local Hamiltonians and is known as the Lieb-Robinson bound. Since then the Lieb-Robinson bound has turned out to be a very useful tool in many areas of modern condensed matter such as the approximation of ground states [76], the study of gapped Hamiltonians and area laws [38, 39, 40], topological properties [41], and others [16, 36, 42, 70], and even linear algebra [43]. An overview of many applications such as the study of gapped systems can be found in [71]. Such a light cone property in matter has been studied experimentally [18]. Local dissipative systems have been studied in the context of quantum computing [57, 59, 95], and phase transitions [50].

Recently this bound was studied for systems with local dissipation [72, 80]. The derived bound was very similar to the original result for non-dissipative systems. In this chapter, we revisit and derive a class of local dynamics with a more refined bound. Local dissipation implies a local de-coherence of states in time. Imposing an additional constraint, we show that certain systems have a decaying velocity of propagation, i.e. the light cone closes with time. We construct a subclass and prove the new bound. We then give some examples. Next, we discuss the relationship between this concept and localization. Under other particular constraint, we prove that the mixing property of dissipation can literally kill the velocity.

1.2 Local Dissipative Dynamics

1.2.1 Propagation of Information

This section explains the connection between the Lieb-Robinson bound and propagation of information. Assume that a system is initially prepared in a state ρ. This state, then, evolves time under the dynamics Γ_t. The fact that any system can be taken to be part of a larger one automatically implies that the map Γ_t needs to be complete positive and trace-preserving. By the same argument, any change to a state of a system should be described by a complete positive trace-preserving map Φ. Any complete positive map has a special decomposition [19],

$$\Phi : \rho \to \Phi[\rho] = \sum_i \mathrm{Tr}(V_i^\dagger V_i \rho) \frac{V_i \rho V_i^\dagger}{(\mathrm{Tr}\, V_i^\dagger V_i \rho)}$$

This decomposition interprets Φ as a change to ρ due to the different measurement outcomes $\frac{V_i \rho V_i^\dagger}{\mathrm{Tr}(V_i^\dagger V_i \rho)}$ with probabilities $\mathrm{Tr}(V_i^\dagger V_i \rho)$. Any measurement can be described in a similar way using a set of matrices $\{V_i\}$ so that $\sum_i V_i^\dagger V_i = \mathbb{1}$. Let us now say that at $t = 0$, we apply some

local change,Φ_x, at some point x in space. After some time t, we make a local measurement, $V_y^\dagger$, at another point y of space. A change is detected by a difference in the probabilities of the outcome. Maximizing over all initial states gives us a bound on how local changes are being perceived at other points in space. This yields us the following,

$$|\operatorname{Tr}(V_y[.]V_y^\dagger \circ \Gamma_t \circ \Phi_x[\rho]) - \operatorname{Tr}(V_y[.]V_y^\dagger \circ \Gamma_t[\rho])| \leq \|(\Phi_x^*[.] - [.]) \circ \Gamma_t^*[V_y^\dagger V_y]\| \tag{1.1}$$

where Φ_x^*, Γ_t^* are the adjoint with respect to the Hilbert-Schmidt scalar product. The operator $\Phi_x^*[.] - [.]$ can be shown to be the generator of an unital Markovian dynamics, i.e. $\forall t \geq 0, \exp[t(\Phi_x[.] - [.])]$ is trace-preserving and complete positive. The case $\|(\Phi_x^*[.] - [.]) \circ \Gamma_t^*[V_y^\dagger V_y]\| \propto \|[B_x, .] \circ \Gamma_t^*[V_y^\dagger V_y]\|$, is bounded by the so-called the Lieb-Robinson bound stated in equation (1.2) below. However, for proving the existence of the thermodynamic limit of local Markovian dynamics, arbitrary local Lindblad generators are needed.

Obviously, for $t = 0$ we see that the upper bound in equation (1.1) is always zero. A question that is often asked is what happens for $t > 0$. The Lieb-Robinson bound shows that the change is progressive. For a fixed time, as the measurement is done farther away, it becomes harder and harder to perceive the perturbation. Measurements that detect the change at some time d, can be made at least at a distance $d + v\Delta t$ after some time Δt with the same accuracy of the results. In other words, the propagation of the information is inside a light cone. For a simple construction, we show in this chapter that due to dissipation, we can get event horizons.

1.3 Local Markovian Dynamics

Consider a D-dimensional lattice $\mathbb{Z}^D$ with a metric. At each point $x \in \mathbb{Z}^D$ of the lattice, define a d-dimensional Hilbert space $\mathcal{H}_x$ and for each finite set $\Lambda \subset \mathbb{Z}^D$ denote the product space,

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$$

Denote the algebra of all matrices acting on $\mathcal{H}_\Lambda$, i.e. the algebra of local observables of Λ, $\mathcal{A}_\Lambda$,

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x)$$

If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}_{\Lambda_1}$ can be identified with the algebra $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_1 \setminus \Lambda_2}$ and therefore $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$. Define the support of a local observable $A \in \mathcal{A}_\Lambda$ as the minimal set $X \subset \Lambda$ for which $A = A' \otimes \mathbb{1}_{\Lambda \setminus X}$.

A natural norm to consider in this type of systems is the so-called completely bounded norm. The cb-norm of an operator L is defined by,

$$\|L\|_{cb} = \sup_{n \geq 1} \|L \otimes \mathrm{id}_{\mathcal{M}_n}\|$$

As shown originally, the Lieb-Robinson velocity is proportional to the norm the local interaction. This proportionality can be made more precise using reproducing functions. We assume

there exists a non-increasing function $F : [0, \infty) \to (0, \infty)$ such that F is uniformly integrable over the lattice,

$$\|F\| := \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} F(d(x, y)) < \infty$$

and there is a constant C_μ,

$$C_\mu := \sup_{x,y \in \mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} \frac{F(d(x, y))F(d(y, z)}{F(d(x, y)} e^{-\mu(d(x,y)d(y,z)-d(x,z))}$$

such that $C_\mu\big|_{\mu=0} < \infty$.

Further information about reproducing functions can be found in [45]. We study local one-parameter dynamics $\Gamma_t : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ such that $\forall A \in \mathcal{A}_\Lambda$,

$$\frac{d}{dt}\Gamma_t(A)\big|_{t=0} = L(A) = \sum_X I_X(A)$$

with local operators $I_X : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, $\forall A \in \mathcal{A}_Y$, $I_X[A_Y] = 0$, if $X \cap Y = \emptyset$, and $\mathrm{diam}(X) \leq R < \infty$. Additionally we impose that $\forall t \geq 0$ and $\forall X$,

$$\exp[tI_X] : \mathcal{A}_X \to \mathcal{A}_X$$

is identity preserving and complete positive. Therefore L and each I_X are the Heisenberg picture equivalent of the Lindblad generator [65] of a continuous one-parameter semi-group of complete positive trace-preserving maps. Notice that the complete positivity implies $I_X + I_X^* \leq 0$.

Additionally assume there exists a $\mu > 0$, such that,

$$\|L\|_\mu = \sup_{x,y \in \mathbb{Z}^d} \sum_{Z \ni x,y} \frac{\|I_Z\|_{cb}}{F(d(x, y)} e^{\mu d(x,y)} < \infty, \text{and} \quad \lambda_{I_X + I_X^*}/2 \geq -\nu$$

where λ_T denotes the largest non-zero eigenvalue of operator T and $\nu > 0$. The existence of the thermodynamic limit of such systems was shown in [72].

Consider $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^D$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $A \in \mathcal{A}_{\Lambda_1}$ and $B \in \mathcal{A}_{\Lambda_2}$. In the case that Γ_t is automorphic, it has already been shown that there exists C', v and ξ, [64, 73, 80] such that,

$$\|[B, \exp[tL]A]\| \leq C' \|A\| \, \|B\| \, \exp\left[\frac{vt - d(A, B)}{\xi}\right] \tag{1.2}$$

These constants depend on the diameter of the support of A and B and the strength of the interaction. This bound, of course, should also always be understood for small times compared to the distance. Notice, that in the case that L has a unique fixed point, in principle by taking the right observable A, we should get something like,

$$\|[B, \exp[tL]A]\| \leq C' \|A\| \, \|B\| \exp[-t \, \mathrm{Gap}(L)]$$

This bound makes sense for all times, but does not contain any information about the distance between observables. We will reconcile and combine both properties in the following result. We show that for a certain class of Lindblad generators, the usual light-cone picture is only valid for small t. After a certain time, the dissipative character takes over and the velocity $v > 0$ becomes time dependent and decreases towards some smaller velocity which can be zero.

Define P_X to be a projector containing the Kernel of I_X, such that,

$$\forall t, \quad \max_Y \frac{\| \exp\left(t(\mathbb{1} - P_X) I_X (\mathbb{1} - P_X)\right) (\mathbb{1} - P_X)[Y] \|}{\|Y\|} \leq \exp(-\nu t) \tag{1.3}$$

This local rapid mixing property, tells us that the subspace orthogonal onto P_X is being dissipated away at a rate ν. In order to derive our desired result, we need the following ingredients,

$$P_Y I_X (\mathbb{1} - P_Y) = 0, \;\; P_Y P_X (\mathbb{1} - P_Y) = 0, \quad \forall X, Y \subset \mathbb{Z}^d \tag{1.4}$$

Notice that since projectors are self-adjoint, the second equality implies the mutual commutation relations between the projectors,

On the other hand, the first inequality does not necessarily implies that the projectors and the interaction terms commute. If this were the case, it can be easily shown that dynamics becomes ultra-local, i.e. the support of local operators does not grow under time-evolution.

Let us take a local perturbation at the origin that can be dissipated by the interaction term that overlaps with origin. Under time evolution, this perturbation is then propagated by the other interactions. However, the equations (1.4) tell us that as the perturbation spreads, it is dissipated at a faster rate.

Theorem. *Let L be a local Lindblad generator. Given that for each local interaction I_X the equations (1.4) are satisfied. Then $\forall A \in \mathcal{A}_{\Lambda_A}, B \in \mathcal{A}_{\Lambda_B}$ with $\Lambda_A \cap \Lambda_B = \emptyset$,*

$$\|[\exp(tL)A, B]\| \leq 2\frac{\|F\|}{C_\mu} \|A\| \, \|B\| \, \min\{\Lambda_A, \Lambda_B\} \left(\exp\left[\int_0^t v(\tau) d\tau \right] - 1 \right) \exp\left[-\mu d(A, B)\right] \tag{1.5}$$

with the time-dependent velocity $v(t)$,

$$v(t) = \left[\alpha + \beta e^{-\nu t}\right] \|L\|_\mu$$

with ν determined by equation (1.3),

$$\alpha = \max_X \|I_X P_X\|_{cb} / \|I_X\|_{cb} \quad \textit{and } \beta = \max_X \|I_X(\mathbb{1} - P_X)\|_{cb} / \|I_X\|_{cb}$$

Proof. The proof goes as follows. First, we find a meaningful integral representation for $[\exp(tL)A, B]$. This representation shows that in order for the two observables A and B to communicate with other, they need to be propagated by the local interactions. Second, we see

that the local interactions not only play the role of propagator but they are also dissipators. This role is expressed by the relation (1.4). If, after some propagations, the observable is being dissipated at a rate λ, under the action of another interaction term the rate becomes $\lambda + \nu$. These two ideas are then combined to give the result.

We remind the reader of the property of the exponential of a sum.

$$e^{t(A+B)} = e^{tA} + \int_0^t ds\; e^{(t-s)A} B e^{s(A+B)} \tag{1.6}$$

and

$$e^{t(A+B)} = e^{tA} + \int_0^t ds\; e^{s(A+B)} B e^{(t-s)A} \tag{1.7}$$

Originally the dynamics [71] was recursively expanded using the first property (1.6), Instead of writing down the full integral representation, a recursive inequality was used. In our case we use the second property (1.7).

Define a path of n steps, $\mathcal{P}_n$, between A and B as the set of the support of the local interactions I_{Λ_j},

$$\mathcal{P}_n = \{\Lambda_j | \Lambda_B \cap \Lambda_1 \neq \emptyset, \Lambda_1 \cap \Lambda_2 \neq \emptyset, ..., \Lambda_n \cap \Lambda_A \neq \emptyset \text{ , all other intersections are empty}\}$$

Denote $\mathcal{P}_n^k$ as the subset of $\mathcal{P}_n$ containing the first k steps of the path, where $\mathcal{P}_n^0 = \emptyset$. Define the generator containing local interaction which do not intersect with the path,

$$L^c_{\mathcal{P}_n^k} = L - \sum_{Z \cap \mathcal{P}_n^k \neq \emptyset} I_Z$$

In Lemma 1 we show,

$$\|[B, \exp(tL)A]\| \leq \sum_{n=1}^{\infty} \|L\|_\mu^n \left(\sup_{\mathcal{P}_n} \mathcal{J}_{\mathcal{P}_n}\right) \sum_{\mathcal{P}_n} \sum_{x_0 \in \Lambda_B} \prod_{\Lambda_j \in \mathcal{P}_n} \sum_{x_j \in \Lambda_j} \sum_{x_{n+1} \in \Lambda_A} F(d(x_{j-1}, x_j)) \; e^{-\mu d(x_{j-1}, x_j)}$$

with,

$$\mathcal{J}_{\mathcal{P}_n} = \Big|\Big| \int_0^t \int_0^{t-s_1} ... \int_0^{t-\sum_{j=1}^{n-1} s_j} d\mathbf{s}\; [B, .] \prod_{j=1}^{n} \left(e^{s_j L^c_{\mathcal{P}_n^{j-1}}} I_{\Lambda_j} / \|I_{\Lambda_j}\|_{cb} \right) e^{(t-\sum_{j=1}^n s_j) L^c_{\mathcal{P}_n^n}} A \Big|\Big| \tag{1.8}$$

with the ordered product $\prod_{j=1}^n C_j = C_1 C_2 ... C_n$ and the supremum is taken over all paths of length n.

Further on we show for fixed n,

$$\mathcal{J}_{\mathcal{P}_n} \leq 2\|A\|\; \|B\| \frac{\left[\max \frac{\|I_X P_X\|}{\|I_X\|_{cb}} + \max \frac{\|I_X(\mathbb{1} - P_X)\|}{\|I_X\|_{cb}} (1 - e^{-\nu t})/(\nu t)\right]^n}{n!} t^n$$

From the definition of C_μ and the triangle inequality, it can be seen that,

$$\sum_{\mathcal{P}_n}\sum_{x_0\in\Lambda_B}\prod_{\Lambda_j\in\mathcal{P}_n}\sum_{x_j\in\Lambda_j}\sum_{x_{n+1}\in\Lambda_A}F(d(x_{j-1},x_j))e^{-\mu d(x_{j-1},x_j)}$$
$$\leq \frac{\|F\|}{C_\mu}\min\{|\operatorname{supp}A|,|\operatorname{supp}B|\}\exp[-\mu d(A,B)]$$

Define the projector,

$$P_\Lambda^k=\begin{cases} P_\Lambda, \text{ if } k=0 \\ \mathbb{1}-P_\Lambda, \text{ if } k=1\end{cases}$$

where P_Λ is the projector on $\mathrm{Ker}(I_\Lambda+I_\Lambda^*)$. From condition (1.4), we can rewrite the expression,

$$\mathcal{J}_{\mathcal{P}_n}\leq 2\|A\|\,\|B\|\int_0^t\int_0^{t-s_1}\ldots\int_0^{t-\sum_{j=1}^{n-1}s_j}d\mathbf{s}\,\Big|\Big|\prod_{\Lambda_j\in\mathcal{P}_n}\left(e^{s_jL^c_{\mathcal{P}_n^{j-1}}}I_{\Lambda_j}/\|I_{\Lambda_j}\|_{cb}(P^0_{\Lambda_j}+P^1_{\Lambda_j})\right)\Big|\Big|$$

This equation consists of 2^n terms. For each string of bits of length $n-k+1$ $\vec{i}_{k,n}=\{i_k,\ldots,i_n\}\subset\{0,1\}^k$ for $k=1,\ldots,n$, define the projector

$$Q^{\vec{i}_{k,n}}_{\mathcal{P}_n^{k-1}}=\prod_{\Lambda_j\in\mathcal{P}_n\setminus\mathcal{P}_n^{k-1}}(\mathbb{1}-i_jP_{\Lambda_j})$$

From condition (1.4) we see that this is indeed a projector. Moreover using (1.4), we can rewrite our expression,

$$\mathcal{J}_{\mathcal{P}_n}\leq 2\|A\|\,\|B\|\sum_{\{i_1,\ldots,i_n\}\subset\{0,1\}^n}\mathcal{R}^{\{i_1,\ldots,i_n\}}_{\mathcal{P}_n}\mathcal{S}^{\{i_1,\ldots,i_n\}}_{\mathcal{P}_n}$$

with,

$$\mathcal{R}^{\{i_1,\ldots,i_n\}}_{\mathcal{P}_n}=\prod_{\Lambda_k\in\mathcal{P}_n}\left(\|I_{\Lambda_k}P^{i_k}_{\Lambda_k}\|_{cb}/\|I_{\Lambda_k}\|_{cb}\right)$$
$$\mathcal{S}^{\{i_1,\ldots,i_n\}}_{\mathcal{P}_n}=\int_0^t\int_0^{t-s_1}\ldots\int_0^{t-\sum_{j=1}^{n-1}s_j}d\mathbf{s}\prod_{\Lambda_k\in\mathcal{P}_n}\Big|\Big|Q^{\vec{i}_{k,n}}_{\mathcal{P}_n^{k-1}}e^{s_kQ^{\vec{i}_{k,n}}_{\mathcal{P}_n^{k-1}}L^c_{\mathcal{P}_n^{k-1}}Q^{\vec{i}_{k,n}}_{\mathcal{P}_n^{k-1}}}Q^{\vec{i}_{k,n}}_{\mathcal{P}_n^{k-1}}\Big|\Big|$$

It is a simple exercise to verify,

$$\|P\exp[P(A+B)P]P\|\leq\lim_{n\to\infty}\Big|\Big|P\exp\left[PAP/n\right]P\Big|\Big|^n\Big|\Big|P\exp\left[PBP/n\right]P\Big|\Big|^n$$

This equation and the definition of $Q^{\{i_k,\ldots,i_n\}}_{\mathcal{P}_n}$, we see that we can bound each terms in $\mathcal{S}^{\{i_1,\ldots,i_n\}}_{\mathcal{P}_n}$,

$$\Big|\Big|Q^{\{i_k,\ldots,i_n\}}_{\mathcal{P}_n^{k-1}}\exp\left[s_kQ^{\{i_k,\ldots,i_n\}}_{\mathcal{P}_n^{k-1}}L^c_{\mathcal{P}_n^{k-1}}Q^{\{i_k,\ldots,i_n\}}_{\mathcal{P}_n^{k-1}}\right]Q^{\{i_k,\ldots,i_n\}}_{\mathcal{P}_n^{k-1}}\Big|\Big|\leq\exp\left[-\nu s_k\sum_{j=k}^n i_j\right]$$

Defining new variables,

$$u_1=s_1,\;\; u_2=s_1+s_2,\;\;\ldots\;\;,u_k=\sum_{j=1}^k s_j\;\;\ldots$$

We can rewrite $\mathcal{S}_{\mathcal{P}_n}^{\{i_1,\ldots,i_n\}}$,

$$\mathcal{S}_{\mathcal{P}_n}^{\{i_1,\ldots,i_n\}} = \int_0^t \int_{u_1}^t \ldots \int_{u_1+\ldots+u_{n-1}}^t d\mathbf{u}\ \exp\left[-\sum_{j=1}^n u_j \sum_{k=j}^n i_k\right]$$

and,

$$\|I_{\Lambda_k} P_{\Lambda_k}^{i_k}\|_{cb} / \|I_{\Lambda_k}\|_{cb} \leq \begin{cases} \beta, \text{ if } i_k = 1 \\ \alpha, \text{ if } i_k = 0 \end{cases}$$

Using these two bounds $\mathcal{J}_{\mathcal{P}_n}$ becomes,

$$\begin{aligned}
\mathcal{J}_{\mathcal{P}_n} &\leq 2\|A\|\,\|B\| \int_0^t \int_{u_1}^t \ldots \int_{u_1+\ldots+u_{n-1}}^t d\mathbf{u}\ (\alpha+\beta\exp(-\nu u_1))\ldots(\alpha+\beta\exp(-\nu u_n)) \\
&= 2\|A\|\,\|B\| \frac{1}{(n-1)!} \int_0^t du_1\ \left(\alpha+\beta e^{-\nu u_1}\right)\left(\alpha(t-u_1)+\beta\left(\frac{e^{-\nu u_1}-e^{-\nu t}}{\nu}\right)\right)^{n-1} \\
&= 2\|A\|\,\|B\|\,\frac{1}{n!}\left[\alpha t+\beta\left(\frac{1-\exp(-\nu t)}{\nu}\right)\right]^n
\end{aligned}$$

The first equality is the induction hypothesis, the case $n = 1$ can be checked easily. Combining our results, we get our claim. □

Lemma 1. *For A and I_Z with $Z \cap \Lambda_A = \emptyset$,*

$$\|\ I_Z \exp(tL)A\ \| \leq \sum_{n=1}^{\infty} \|L\|_\mu^n \left(\sup_{\mathcal{P}_n} \mathcal{J}_{\mathcal{P}_n}\right) \sum_{\mathcal{P}_n} \sum_{x_0\in\Lambda_A} \prod_{\Lambda_j\in\mathcal{P}_n} \sum_{x_j\in\Lambda_j} \sum_{x_{n+1}\in Z} F(d(x_{j-1},x_j))\, e^{-\mu d(x_{j-1},x_j)}$$

Proof. The procedure is very similar to [45]. Consider the generator $L_{\cap Z}$,

$$L_{\cap Z} = \sum_{\Lambda_1\cap Z\neq\emptyset} I_{\Lambda_1}$$

Using equation (1.7) , we can then expand,

$$I_Z \exp(tL)A = \sum_{\Lambda_1\cap Z\neq\emptyset} \|I_{\Lambda_1}\|_{cb} \int_0^t I_Z \exp[s_0 L] I_{\Lambda_1} / \|I_{\Lambda_1}\|_{cb} \exp[(t-s_0)(L-L_{\cap Z})]A$$

We then continue this procedure for each term $I_{\Lambda_1}\exp[(t-s_0)(L-L_{\cap Z})]A$ for which $\Lambda_1 \cap \Lambda_A = \emptyset$. Let us say that after $k-1$ steps we have $\Lambda_k \cap \Lambda_A \neq \emptyset$, from our construction we see that we have built a path of k steps from Z to A for which $\mathcal{J}_{\mathcal{P}_k(Z,A)}$ can be bounded by equation (1.8). For each n, we can then use the definition of $\|L\|_\mu$,

$$\begin{aligned}
&\left(\sup_{\mathcal{P}_n} \mathcal{J}_{\mathcal{P}_n}\right) \sum_{\mathcal{P}_n} \prod_{\Lambda_j\in\mathcal{P}_n} \sum_{\Lambda_j} \|I_{\Lambda_j}\|_{cb} \\
&\leq \left(\sup_{\mathcal{P}_n} \mathcal{J}_{\mathcal{P}_n}\right) \sum_{\mathcal{P}_n} \|L\|_\mu^n \sup_{x_0\in\Lambda_A} \prod_{\Lambda_j\in\mathcal{P}_n} \sum_{x_j\in\Lambda_j} \sum_{x_{n+1}\in Z} F(d(x_{j-1},x_j)) e^{-\mu d(x_{j-1},x_j)}
\end{aligned}$$

□

1.4 Examples

1.4.1 Specific Subclass

We present a method to construct examples that satisfy the equations (1.4) for the case $P_X = \frac{\mathbb{1}_X}{\mathrm{Tr}(\mathbb{1}_X)}\,\mathrm{Tr}_X[.]$ and X consists of at most two sites. For this construction $\alpha = 0$ and the Lieb-Robinson velocity goes asymptotically to zero.

Consider $2n$ trace-preserving complete positive operators $\Gamma^j, \Phi^j : B(\mathcal{C}^d) \to B(\mathcal{C}^d)$ and $c_1, ..., c_n \in (0, 1]$ with $\sum_k c_k = 1$, so that Γ_j^* and Φ_j^* are not trace-preserving and $\sum_j c_j \Gamma^j \otimes \Phi^j$ only has the uniform state as fixed point. We can then easily see that interaction terms $I_{i,j}$ of the form,

$$I_{i,j}[.] = \sum_k c_k \Gamma_i^k \otimes \Phi_j^k[.] - [.]$$

satisfy the structural equation (1.4).

Let us give some concrete examples. For $A \in B(\mathcal{C}^2)$, $\lambda, t \in [-1, 1]$, consider the following trace-preserving map $\Phi(\lambda, t) : B(\mathcal{C}^2) \to B(\mathcal{C}^2)$.

$$\Phi(\lambda,t) : \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(A_{11}+A_{22}) & \frac{t}{2}(A_{11}+A_{22}) + \lambda A_{12} \\ \frac{t}{2}(A_{11}+A_{22}) + \lambda A_{21} & \frac{1}{2}(A_{11}+A_{22}) \end{pmatrix}$$

From [84], the condition $|\lambda| + |t| < 1$ implies complete positivity. We can then construct the following interaction terms,

$$I_{i,j}[.] = \frac{1}{3}\Phi_i(\lambda_1,t_1)\otimes\Phi_j(\lambda_2,t_2)[.] + \frac{1}{3}\Phi_i(\lambda_3,s)\otimes\Phi_j(\lambda_4,s)[.] + \frac{1}{3}\Phi_j(\lambda_5,r)\otimes\Phi_j(\lambda_6,-r)[.] - [.]$$

with $t_1 + r + s = t_2 - r + s = r^2 - s^2 = 0$.

As a second example consider the map, $\Psi(\lambda, t) : B(\mathcal{C}^2) \to B(\mathcal{C}^2)$.

$$\Psi(\lambda,t) : \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \to \begin{pmatrix} \frac{1+\lambda+t}{2}A_{11} + \frac{1-\lambda+t}{2}A_{22} & 0 \\ 0 & \frac{1-\lambda-t}{2}A_{11} + \frac{1+\lambda-t}{2}A_{22} \end{pmatrix}$$

with $-1 < t, \lambda < 1,\ 1 - |\lambda| - |t| > 0$. Given $-1 < r_1, r_2, s_1, s_2, t_1, t_2, u_1, u_2 < 1$, such that,

$$\det \begin{pmatrix} (1+r_1)(1+r_2) & (1+s_1)(1+s_2) & (1+t_1)(1+t_2) & (1+u_1)(1+u_2) \\ (1+r_1)(1-r_2) & (1+s_1)(1-s_2) & (1+t_1)(1-t_2) & (1+u_1)(1-u_2) \\ (1-r_1)(1+r_2) & (1-s_1)(1+s_2) & (1-t_1)(1+t_2) & (1-u_1)(1+u_2) \\ (1-r_1)(1-r_2) & (1-s_1)(1-s_2) & (1-t_1)(1-t_2) & (1-u_1)(1-u_2) \end{pmatrix} \neq 0$$

And the interactions are given by,

$$\begin{aligned} I_{i,j}[.] =& c_1\Psi_i(\lambda_1,r_1)\otimes\Psi_j(\lambda_2,r_2)[.] + c_2\Psi_i(\kappa_1,s_1)\otimes\Psi_j(\kappa_2,s_2)[.] + \\ & c_3\Psi_i(\mu_1,t_1)\otimes\Psi_j(\mu_2,-t_2)[.] + c_4\Psi_i(\nu_1,u_1)\otimes\Psi_j(\nu_2,u_2)[.] - [.] \end{aligned}$$

Let us finally remind the reader that a maximum range of the interaction is necessary, $\forall I_{i,j},\ d(i,j) \leq R < \infty$. This condition implies an exponential decay in distance. Long range interactions are possible in some cases, but mostly imply slower decays. Further discussion can be found in [45].

1.4.2 Local Covariance

In the previous section, we gave a method for the construction of non-trivial examples. These examples give rise to an approximate event horizon or in other words are quasi-ultra-local. An interesting question is how do ultra-local dynamics look like. In the case of automorphisms, we would need interaction terms to commute with each other. In this section we give an interesting class of ultra-local dynamics with non-commutating interaction terms.

We use the concept of local covariance to introduce this particular class. It can then be combined with the previous examples to construct additional quasi-ultra-local dynamics.

Given some group G with a unitary representation $V_g : \mathcal{H}_x \to \mathcal{H}_x, \forall x$. We say that I_X is local covariant if, $\forall i \in X$,

$$I_X[(\mathbb{1}_{X\setminus\{i\}} \otimes V_g)(.)(\mathbb{1}_{X\setminus\{i\}} \otimes V_g^\dagger)] = (\mathbb{1}_{X\setminus\{i\}} \otimes V_g) I_X[.](\mathbb{1}_{X\setminus\{i\}} \otimes V_g^\dagger)$$

Additionally, we say $\mathrm{Ker}(I_X + I_X^\dagger)$ is invariant with respect to the representation $\{V_g\}$ of a group G if $\forall A \in \mathrm{Ker}(I_X + I_X^\dagger), \forall i \in X$,

$$(\mathbb{1}_{X\setminus\{i\}} \otimes V_g) A (\mathbb{1}_{X\setminus\{i\}} \otimes V_g^\dagger) = A$$

in other words $\{V_g\}$ is a local gauge symmetry of $\mathrm{Ker}(I_X + I_X^\dagger)$.

For every amenable group G with invariant mean μ and representation $\{V_g\}$, we can define a projector P_X on $\mathcal{A}_X$ onto the vector space with this representation as local gauge symmetry.

$$P_X : \mathcal{A}_X \to \mathcal{A}_X : A \to \prod_{i\in X} P_i[A] \tag{1.9}$$

with,

$$P_i : A \to \sum_{g\in G} \mu(g)(V_g \otimes \mathbb{1}_{X\setminus\{i\}}) A (V_g^\dagger \otimes \mathbb{1}_{X\setminus\{i\}})$$

It can be easily checked, that equations (1.4) are satisfied when I_X are locally covariant. Additionally we can see that every interaction terms I_X commute with the projectors P_Y. This implies ultra-locality, $\forall A \in \mathcal{A}_{\Lambda_A}$,

$$\exp\left[t\sum_X I_X\right] A = \exp\left[t \sum_{X\cap\Lambda_A\neq\emptyset} I_X\right] A$$

Covariance properties of dynamics has been largely studied [46, 47, 48].

1.5 Localization

In 1958, Anderson [5] showed that random impurity potentials produce the localization of quantum wavefunctions. This lead to a better understanding of transitions between insulators and conductors. As shown earlier, when introducing local dissipation to the system, the propagation of the perturbation competes with the global relaxation of the system. This competition

does not only effectively reduce the velocity of propagation but also lead to some increasing deceleration.

By Frobenius-Perron, the adjoint of the dynamics has some positive fixed point ρ_{eq},

$$L_0^{\text{ad}}[\rho_{\text{eq}}] = 0, \;\; \rho_{\text{eq}} \geq 0, \text{Tr}\, \rho_{\text{eq}} = 1$$

Let us now perturb the dynamics at some local subset Z, be it time-independently or not,

$$\partial_t A(t) = L_0[A(t)] + \phi_Z(t)[A(t)]$$

We wish then to compare the expectation value of the undisturbed system $\langle A(t) \rangle = \langle A \rangle$ with the new one, $\langle A(t) \rangle_\phi$. This is readily derived,

$$\langle A(t) \rangle_\phi - \langle A \rangle = \int_0^t \text{Tr}\left(\rho_{\text{eq}} \phi_Z[A(s)]\right)$$

Hence the response of the systems, is determined by bounding the term inside the integral. Since $\text{Tr}(\rho_{\text{eq})} = 1$, this becomes,

$$\text{Tr}\left(\rho_{\text{eq}} \phi_Z[A(t)]\right) \leq \|\phi_Z[A(s)]\|$$

Such expression was studied for the first time by Lieb and Robinson and by reducing the problem to a Laplacian inequality it was shown that,

$$\|\phi_Z[A(s)]\| \leq C \exp\left(\frac{vt - d(Z,A)}{\xi}\right) \tag{1.10}$$

where C, v and ξ are finite constants.

In [21], it was shown that under some symmetry a class of Lindbladians could be constructed such that the bound takes the form,

$$\|\phi_Z[A(s)]\| \leq C \exp\left(\frac{\int_0^t \exp(-\mu s) v \; ds - d(Z,A)}{\xi}\right) \tag{1.11}$$

As pointed out, if one could show,

$$\max_X \frac{\|\exp\left(tL^c\right)[X] - \lim_{t\to\infty} \exp\left(tL^c\right)[X]\|}{\|X\|} \leq C' \exp(-\lambda t) \tag{1.12}$$

where $\mathcal{L}^c$ is the full generator of the dynamics minus some local interaction terms, and C' is a finite constant, the bound can be shown to be of the form,

$$\|\phi_Z[A(s)]\| \leq C'' \exp\left(\frac{vt - \lambda t - d(Z,A)}{\xi}\right) \tag{1.13}$$

This becomes particularly interesting when $\lambda \approx v$, as this imply that $A(t)$ is localized inside a fixed volume for all t. Proving the case for which an inequality such as (1.12) is satisfied is hard. We present here an alternative. We show a result similar to (1.12), by relaxing the set of (global) observables to a smaller convex subset. Under the additional property that the dynamics preserve the convex structure, a bound such as (1.20) can then be proven, and even more so localization can be implied.

1.5.1 Killing of Transport of Restricted Set of Observables

Consider some local basis $\{\sigma_k^{(j)}\}$ of the local Hilbert space $\mathbb{C}^d_{(j)}$. For the lattice $\otimes_{j\in\Lambda}\mathbb{C}^d_{(j)}$, define the convex hull of the set constructed from the local basis multiplied by some phase $\mathcal{C} = \mathrm{Conv}\left(\{e^{i\phi}\otimes_j \sigma_{k_j}^{(j)} | \phi\in[0,2\pi]\}\right)$. We say that this set is normalizable with local observables, if $\forall A\in\mathcal{A}_{\Lambda_A}$, there is some finite constant $0<C_A<\infty$, such that, $A/C_A\in\mathcal{C}$.

We say that a local dynamics preserves $\mathcal{C}$ if,

$$\forall t, X\in\mathcal{C}, \exists Y(t)\in\mathcal{C}, 0\geq r(t)\leq 1, \quad \exp(tL)[X] = r(t)Y(t) \tag{1.14}$$

As, we see further on, it is then sufficient to restrict (1.12) to the inequality,

$$\max_{X\in\mathcal{C},\|X\|=1} \frac{\|\exp(tL^c)[X] - \lim_{t\to\infty}\exp(tL^c)[X]\|}{\|X\|} \leq \exp(-\lambda t) \tag{1.15}$$

where $\mathcal{L}^c$ denotes the global generators minus any number of local interactions I_Z.

Finally, let us assume there exists some finite constant $0<C_\phi<\infty$ such that $\forall Z, \forall X\in\mathcal{C}$,

$$L_Z[X]/C_X\in\mathcal{C}, \;\; C_X\leq C_\phi \forall X$$

We say that the system is frustration free if,

$$\forall Z, \;\; \mathcal{L}_Z\circ\lim_{t\to\infty}\exp(t\mathcal{L}) = 0$$

In order to derive localization, the dynamics needs to be split into two other local ones. The first one is responsible for the spreading the perturbation, while the second is the main dissipative part which erases the perturbation. Clearly, for localization to be possible, perturbation needs to be driven towards some zero-correlated subspace. Hence, we assume that the local parts of $\mathcal{L}_1 = \sum_Z I_Z^1$ mutually commute. For the sake of the fluidity of the argument, the interaction terms of both generators are grouped inside some finite subset Z.

$$L = L_0 + L_1 = \sum_Z (I_Z^0 + I_Z^1)$$

We can now state, the theorem.

Result 2. *Let $L = L_0 + L_1 = \sum_Z (I_Z^0 + I_Z^1)$ be frustration free and irreducible local Lindblad generator satisfying (1.14) and(1.17), and $[I_Z^1, I_{Z'}^1] = 0$.Then $\forall A\in\mathcal{A}_{\Lambda_A}, B\in\mathcal{A}_{\Lambda_B}$ with $\Lambda_A\cap\Lambda_B = \emptyset$,*

$$\|[B,\exp(tL)[A]]\| \leq 2\frac{\|F\|}{C_\mu}\|A\|\,\|B\|_{cb}\,\min\{\Lambda_A,\Lambda_B\}\left(\exp[vt-\lambda t]-1\right)\exp\left[-\mu d(A,B)\right] \tag{1.16}$$

with $v = C_\phi\|L^0\|_\mu$, and,

$$\max_{X\in\mathcal{C},\|X\|=1} \frac{\|\exp(tL_1)[X] - \lim_{t\to\infty}\exp(tL_1)[X]\|}{\|X\|} \leq \exp(-\lambda t) \tag{1.17}$$

Hence, we see that the velocity and the decay rate are decoupled from one another. By varying the way between the diffusive part $\mathcal{L}^0$ an the dissipative part $\mathcal{L}^1$, it is then possible to get $\lambda > v$, from which localization follows.

As the proof of the result is extremely similar to the proof presented in earlier, we leave it out.

1.5.2 Classical Stochastic Processes

It is straightforward to embed a classical stochastic dynamics into a quantum one. For example in the case of a local stochastic generator $L = \sum_Z T_Z - I_Z$, where I_Z is the identity operator and T_Z is local stochastic. Hence, this can be embedded into a Lindbladian using the local standard basis $\{e(Z)_j\}$,

$$\mathcal{L}[.] = \sum_Z \sum_j \sqrt{\mathrm{Tr}(e(Z)_j T_Z)}\ e(Z)_j[.]e(Z)_j^\dagger - [.]_Z \tag{1.18}$$

In the case of local Hilbert spaces $\mathbb{C}_j^2$, $e(Z)_{ij} = |i\rangle\langle j|$. Additionally, we can see a clear decoupling between the vector spaces $\mathrm{span}\{\otimes\sigma_j|\sigma_j \in \{\mathbb{1}, \sigma_z\}\}$ and $\mathrm{span}\{\otimes\sigma_j|\sigma_j \in \{\mathbb{1}, \sigma_x, \sigma_y\}\}$. Hence for simplicity, let us restrict to dynamics on lattices $\otimes_{j\in\Lambda}\mathbb{C}^2$, and dynamics (1.18) restricted to observables in $\mathrm{span}\{\otimes\sigma_j|\sigma_j \in \{\mathbb{1}, \sigma_z\}\}$

Define now the local unitary $U_j = (\sigma_z + \sigma_x)/\sqrt{2}$. It is then also understood that $U_Z = \otimes_{j\in Z} U_j$. In the following definition the matrix $|A|_{ij}$ is understood as the entry-wise absolute value. The following proposition gives us a local condition which implies property (1.14).

Proposition 3. *If the matrix $|U_Z^\dagger T_Z^\dagger U_Z|_{ij}$ is sub-stochastic then the dynamics (1.18) preserves the convex set* $\mathrm{span}\{e^{i\phi} \otimes \sigma_j|\sigma_j \in \{\mathbb{1}, \sigma_z\}, \phi \in [0, 2\pi]\}$ *as defined in (1.14).*

Proof. Since,

$$\exp(A + B) = \lim_{n\to\infty} ((I + A/n)(I + B/n))^n$$

and due to the definition of (1.18), we can decompose each interaction term as $(1 - \epsilon)I + \epsilon T$. Hence, it is sufficient to show that, $\sqrt{\mathrm{Tr}(e(Z)_j T_Z)}\ e(Z)_j[.]e(Z)_j^\dagger$ satisfy the claim. It can be seen that the claim is equivalent to showing that $\forall\tau \in \{\otimes\sigma_j|\sigma_j \in \{\mathbb{1}, \sigma_z\}\}$,

$$\sum_{\sigma\in\{\otimes\sigma_j|\sigma_j\in\{\mathbb{1},\sigma_z\}\}} |\langle\sigma|T_Z|\tau\rangle| \leq 2^{|Z|}$$

which is indeed equivalent to $|U_Z^\dagger T_Z^\dagger U_Z|_{ij}$ being sub-stochastic. □

As an illustrative example, we can consider the dynamics,

$$\mathcal{L}[.] = \mathcal{L}^0[.] + \lambda\left(\frac{1}{2}\sum_j \mathrm{Tr}_j[.] - [.]\right)$$

Clearly we can see,

$$\max_{\mathcal{C},\|X\|=1} \|e^{tL^c}[X] - \lim_{t\to\infty} e^{tL^c}[X]\| \leq \lim_{n\to\infty} \|(1-\frac{t\lambda}{n})[X] + \frac{t\lambda}{2n}\sum_j \mathrm{Tr}_j[X] - P[X]]\|^n$$
$$\leq \exp(-\lambda t)$$

with $P[X] = \left(\frac{1}{2}\right)^{|\Lambda|} X$.

1.5.3 Engineering dynamics of Graph States

As a second and final illustration, we show that some Lindbladians known to generate graph states satisfy the properties (1.17) and (1.14) for $\mathcal{C} = \mathrm{span}\{e^{i\phi}\otimes\sigma_j|\sigma_j \in \{\mathbb{1},\sigma_x,\sigma_y,\sigma_z\}, \phi \in [0,2\pi]\}$. Graph states are defined as the unique eigenstate of the set of commuting observables $U_k = \sigma_{x,k}\otimes_{j\in Z}\sigma_{z,j}$, with Z some neighbourhood k. As shown in [59], defining $c_k^\alpha = \frac{1}{2}(I+U_k)\sigma_{x,\alpha}$ with $\alpha \in \{x,y\}$, the graph state can be prepared by each local Lindbladian,

$$\mathcal{L}^\alpha[.] = \sum_k c_k^{\alpha\dagger}[.]c_k^\alpha - \frac{1}{2}\{c_k^{\alpha\dagger}c_k^\alpha,.\},\ \ \alpha\in\{x,y,z\}$$

Notice that $P_k = \frac{1}{2}(I+U_k)$ is a projector. Let us therefore take $\Sigma \in \{\otimes\sigma_j|\sigma_j\in\{\mathbb{1},\sigma_x,\sigma_y,\sigma_z\}\}$. Let us fix $\alpha = x$ as the other case are proven similarly. We need then to distinguish between the cases $\sigma_{\alpha,k}\Sigma = \pm\Sigma\sigma_{\alpha,k}$ and $U_k\Sigma = \pm\Sigma U_k$ and calculate for each,

1. $\sigma_{\alpha,k}\Sigma = \Sigma\sigma_{\alpha,k},\ U_k\Sigma = \Sigma U_k$ implies that $\mathcal{L}_k^x[\Sigma] = 0$ and so $\exp(t\mathcal{L}_k^x)[\Sigma] = \Sigma$.
2. $\sigma_{\alpha,k}\Sigma = \Sigma\sigma_{\alpha,k},\ U_k\Sigma = -\Sigma U_k$ gives $\mathcal{L}_k^x[\Sigma] = -\Sigma$ and so, $\exp(t\mathcal{L}_k^x)[\Sigma] = e^{-\lambda t}\Sigma$
3. $\sigma_{\alpha,k}\Sigma = -\Sigma\sigma_{\alpha,k},\ U_k\Sigma = \Sigma U_k$ yields $\mathcal{L}_k^x[\Sigma] = -\Sigma - \Sigma\sigma_{\alpha,k}U_k\sigma_{\alpha,k}$ and so, $\exp(t\mathcal{L}_k^x)[\Sigma] = \Sigma P_k + e^{-2\lambda t}\Sigma(\mathbb{1}-P_k)$
4. $\sigma_{\alpha,k}\Sigma = -\Sigma\sigma_{\alpha,k},\ U_k\Sigma = -\Sigma U_k$ also gives $\mathcal{L}_k^x[\Sigma] = -\Sigma$ and so, $\exp(t\mathcal{L}_k^x)[\Sigma] = e^{-\lambda t}\Sigma$

Since $\sigma_{\alpha,k}U_k\sigma_{\alpha,k} \in \mathcal{C}$, all cases therefore imply (1.14). Finally in order to prove (1.17), since per construction the Lindbladian has a dimensional one kernel, case 1 cannot be satisfied for all k. The claim also follows if either case 2 or 4 are fulfilled for a single k. Hence, the only remaining point to look at, is when case 3 is satisfied at least once. So for all k, $[\Sigma, P_k] = 0$, and we can take mutual eigenvectors $|\alpha\rangle$, $\Sigma|\alpha\rangle = s_\alpha|\alpha\rangle$, $s_\alpha = \pm 1$ and $P_k|\alpha\rangle = p_{\alpha,k}|\alpha\rangle$, $p_{\alpha,k} \in \{0,1\}$. Clearly, since the graph state is per definition the unit vector which has eigenvalue 1 for all P_k, it has to be an eigenvector of Σ too. Hence for all $|\alpha\rangle$ orthogonal to the graph state, we have,

$$|\langle\alpha|\exp(t\mathcal{L}^\alpha)[\Sigma] - \lim_{t\to\infty}\exp(t\mathcal{L}^\alpha)[\Sigma]|\alpha\rangle| \leq e^{-2t}$$

Hence all case combined, we see that the decaying condition (1.17) is satisfied.

1.5.4 Exponential Clustering Theorem

In [80], A general method was given for systems with a unique invariant state and a dissipative gap. For gapped frustration free-Hamiltonian, proving correlation properties is quite easy. Similarly, we give a simple proof of the correlation properties of frustration free local Markovian dynamics with a dissipative gap.

Consider a frustration free local Markovian dynamics $L = \sum_X I_X$ with unique invariant state ρ_β. By which we mean, $L^*\rho_\beta = I_X^*\rho_\beta = 0, \forall X$. We say that L has a dissipative gap, if there is some $\lambda > 0$, such that $\forall A \in \mathcal{A}_\lambda$, with $\mathrm{Tr}(A) = 0$,

$$\| \exp(tL)A\| \leq \|A\| \exp(-\lambda t)$$

For every local Markovian dynamics, the usual Lieb-Robinson bound can be shown,

$$\|I_X \exp[tL]A]\| \leq C\|A\| \, \|I_X\| \, \exp\left[\frac{vt - d(A,X)}{\xi}\right] \tag{1.19}$$

Notice that it should be clear that for the case that dynamics are frustration-free, the following bound can be shown,

$$\|I_X^* \exp[tL^*][A\rho_\beta]\| \leq C\|A\| \, \|I_X\| \, \exp\left[\frac{vt - d(A,X)}{\xi}\right] \tag{1.20}$$

We now show that for every local observable $A \in \mathcal{A}_{\Lambda_A}, B \in \mathcal{A}_{\Lambda_B}$,

$$|\,\mathrm{Tr}(\rho_\beta AB) - \mathrm{Tr}(\rho_\beta A)\,\mathrm{Tr}(\rho_\beta B)| \leq \|A\| \, \|B\| \, \exp\left(-d(A,B)/(\xi + v/\lambda)\right)$$

Proof. Without loss of generality, we can take $\mathrm{Tr}(A) = \mathrm{Tr}(\rho_\beta B) = 0$. Notice first that from the Lieb-Robinson bound (1.20) follows,

$$\| \exp(tL^*)[B\rho_\beta] - \exp(tL^* - t \sum_{X \cap \Lambda_A \neq \emptyset} I_X^*)[B\rho_\beta]\| \leq C' \max_X \|I_X\| \, \|B\| \exp((vt - d(A,B))/\xi)$$

Indeed,

$$\begin{aligned}
&\| \exp(tL^*)[B\rho_\beta] - \exp(tL^* - t \sum_{X \cap \Lambda_A \neq \emptyset} I_X^*)[B\rho_\beta]\| \\
&= \| \int_0^t ds \exp((t-s)L^* - (t-s) \sum_{X \cap \Lambda_A \neq \emptyset} I_X^*) \sum_{X \cap \Lambda_A \neq \emptyset} I_X^* \exp(sL^*)[B\rho_\beta]\| \\
&\leq C \int_0^t ds \max_X \|I_X\| \, \|B\| \exp((vt - d(A,B))/\xi) \leq C' \max_X \|I_X\| \, \|B\| \exp((vt - d(A,B))/\xi)
\end{aligned}$$

Our claim then follows from,

$$\begin{aligned}
|\,\mathrm{Tr}(\rho_\beta BA)| &= |\,\mathrm{Tr}(\rho_\beta B \exp(tL - t \sum_{X \cap \Lambda_A \neq \emptyset} I_X)[A]) = |\,\mathrm{Tr}(A \exp(tL^* - t \sum_{X \cap \Lambda_A \neq \emptyset} I_X^*)[\rho_\beta B])| \\
&\leq \|B\| \, \| \exp(tL)A\| + \|A\| \, \| \exp(tL^*)[\rho_\beta B] - \exp(tL^* - t \sum_{X \cap \Lambda_A \neq \emptyset} I_X^*)[\rho_\beta B]\|)
\end{aligned}$$

□

It should be clear that the correlation length of systems with dynamics satisfying equation 1.14 is zero.

1.6 Reversed Theorem

There is a connection between rapid mixing and the results we presented in this chapter. By rapid mixing, we mean that local observables are mixed in the following way,

$$\|\Gamma_t[A] - \Gamma_\infty[A]\| \leq \|A\| \,\mathrm{Poly}(|\Lambda_A|) \exp(-\lambda t)$$

In this short section, we show for an effective velocity and an assumption on the dependence of the gap that for an attractive stochastic process rapid mixing ensues. The derivation contains elements of a proof of a different result, which can be found in [2]. For the sake of simplicity and clarity, we define a process to be attractive, if, it is contractive under the oscillator norm,

$$\|\Delta_k \Gamma_t[A]\| \leq \sum_j \|\Delta_j A\| \max_{O \in \mathcal{A}_{\Lambda_j}} \frac{\|\Delta_k \Gamma_t[O]\|}{\|O\|}$$

What is this oscillator norm $|||A||| = \sum_j \|\Delta_j A\|$? This norm was discussed by Werner [83] in quantum settings. However, we introduce this particular norm in a different and following way. A major distinction between quantum mechanics and probability theory, is the measurement problem. Measurements in classical physics do not affect each other, as said in mathematical terms, they commute with one another. One could choose to estimate our experiment using only such non-destructive measurements. In the case of spin-chains, a simple first proposition, are one-site measurements. The density of state is then approximated by the joint probability distribution of the local random variable X_j, which are the outcomes of the local measurements POVM's $\{V_j\}$. We saw in this chapter, that the Lieb-Robinson bound is deeply connected with the loss of distinguishability between two states.

$$\max_{\psi,\phi} \Big| \frac{\langle\psi|\Gamma_t[A]|\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\phi|\Gamma_t[A]|\phi\rangle}{\langle\phi|\phi\rangle} \Big|$$

In the spirit, of these local measurements, we could replace this mixing problem, by another one in the basis of the measurement outputs $\{X_j = x_j\}$ of rank-1 projective measurements,$V(X_j = x_j) = |x_j\rangle\langle x_j|$. We, then, optimize, previous problem over all such local measurements,

$$\max_{\{V(X_j = x_j)\}} \max_{\eta,\sigma} \Big| \Gamma_t[A](\eta) - \Gamma_t[A](\sigma) \Big|$$

We relate two configurations telescopically by evaluating the differences between local "flips".

$$\begin{aligned} A(\eta) - A(\sigma) = &(A(x_1, x_2, ..., x_N) - A(y_1, x_2, ..., x_N)) + ... \\ &+ (A(y_1, ..., y_{j-1}, x_j, x_{j+1}, ..., x_N) - A(y_1, ..., y_{j-1}, y_j, x_{j+1}, ..., x_N)) \\ &+ ... + (A(y_1, ..., y_{N-1}, x_N) - A(y_1, ..., y_N)) \end{aligned}$$

This useful lemma is then proven,

Lemma 4.

$$\max_{\{V(X_j=x_j)\}} \max_{\eta,\sigma} \Big| A(\eta) - A(\sigma) \Big| \leq \sum_j d_j^2 \|\Delta_j A\| \tag{1.21}$$

with $d_j = \operatorname{Tr} \mathbb{1}_{\Lambda_j}$ *and,*

$$\Delta_j A = \frac{\operatorname{Tr}_j[A]}{d_j} - A \tag{1.22}$$

Proof. First, we can verify that,

$$\begin{aligned} &\Big| A(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_N) - A(x_1, ..., x_{j-1}, y_j, x_{j+1}, ..., x_N \Big| \\ &\leq \|U_j A U_j^\dagger - A\| \\ &\leq d_j^2 \|\Delta_j A\| \end{aligned}$$

The first line follows by taking a local unitary which maps $|x_j\rangle \to |y_j\rangle$, the second by finding $d_j^2 - 1$ operators V_j which together with U_j form the kraus-operators of Tr_j. Using our telescopic argument and maximizing for each flip yields the lemma. □

It should now be clear to the reader how attractiveness and information propagation can be combined as two ingredients for rapid mixing. The operational meaning of the oscillator norm is an upper-bound for the total cost for distinguishing two states, which we estimated using non-destructive rank 1 projective measurements, derived from a local procedure. In this procedure, we evaluate this cost by computing the differences between configurations which only differ by 1-local unitaries. Attractiveness of the process is the reduction of the total local cost, of distinguishing the states using only 1-local measurements, with time. Since information only propagates at a finite velocity, the cost only depends on configurations which differ on a smaller sub-region. the size of this sub-region is even smaller if we assume a deceleration. Even more so, time-evolution of local observables can be approximated by an effective dynamics,

$$\tilde{\Gamma}_t\{\mathcal{B}(\alpha, r(t))\} = \exp\left(\sum_{Z \cap \mathcal{B}(\alpha,r) \neq \emptyset} I_Z + \mathcal{L}_{\partial\mathcal{B}(\alpha,r)} \right)$$

To this dynamics, one must generally add new generator on the boundary to assure ergodicity. This subregion, here the ball $\mathcal{B}(\alpha, r(t))$ at the site α with radius $r(t)$, generally grows in time due the Lieb-Robinson velocity. The rate of mixing of such effective dynamics depend on the size of the sub-region. There must, therefore, be a balance between the rate of mixing of effective dynamics and the deceleration of the velocity which imply rapid mixing.

Theorem 5. *For* $\tilde{\Gamma}_t\{\mathcal{B}(0,r)\}$ *being* Γ_t *with only local generators with support overlapping the ball around the origin and radius* r*,* $\mathcal{B}(0,r)$*. Writing,*

$$\max_{X_1 \in \mathcal{A}_{\Lambda_0}} \max_{X_2 \in \mathcal{A}_{\Lambda_{\mathcal{B}(0,r)^c}}} \frac{\|[\Gamma_t[X_1], X_2]\|}{\|X_1\|\|X_2\|} \leq f_{\textit{Lieb}}(r(t), t)$$

and,

$$\|\tilde{\Gamma}_t\{\mathcal{B}(0,r)\} - \tilde{\Gamma}_\infty\{\mathcal{B}(0,r)\}\| \leq f_{Mixing}(r(t),t)$$

The functions $f_{Mixing}(r(t),t)$ and $f_{Lieb}(r(t),t)$ are upper-bounds, such as the Lieb-Robinson bound (1.2) for the latter. Then, if $\exists \tau, r$ such that,

$$\int_{r(0)}^{r(\tau)} \left(2 f_{Mixing}(\rho(\tau),\tau) + \int_0^\tau ds \|\mathcal{L}_{\partial\mathcal{B}(0,\rho(s))}\| f_{Lieb}(\rho(s),s) \right) + \int_r^\infty d\rho f_{Lieb}(\rho(\tau),\tau) < 1 \tag{1.23}$$

Then $\exists \lambda$,

$$\max_{\{V(X_j=x_j)\}} \max_{\eta,\sigma} \left| \Gamma_t[A](\eta) - \Gamma_t[A](\sigma) \right| \leq |||A||| \exp(-\lambda t) \tag{1.24}$$

with $|||A||| = \sum_j \|\Delta_j A\|$

Proof. Using previous lemma and the attractive property of the process, it readily follows,

$$\max_{\{V(X_j=x_j)\}} \max_{\eta,\sigma} \left| \Gamma_t[A](\eta) - \Gamma_t[A](\sigma) \right| \leq |||A||| \left(\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[X]\|}{\|X\|} \right)$$

Discretize the variable $t = n\tau$ for some integer n,

$$\left(\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[X]\|}{\|X\|} \right) \leq \left(\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_\tau[X]\|}{\|X\|} \right)^n$$

Define,

$$\lambda = -\inf_{\tau>0} \frac{1}{\tau} \log \left(\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[X]\|}{\|X\|} \right)$$

Hence, if, $\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[X]\|}{\|X\|} < 1$ then the statement follows. This is where the relation (1.23) plays a role. On one hand the time evolution of the local observable O at the origin can be approximated by a local time evolution. On the other hand, this local time evolution destroys the observable O at a rate which scales with the size of the support of the local evolution.

$$\sum_j \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[O]\|}{\|O\|} = \left(\sum_{j\in\mathcal{B}(0,r)} + \sum_{j\in\mathcal{B}(0,r)^c} \right) \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[O]\|}{\|O\|}$$

The second term is estimated using the Lieb-Robinson bound,

$$\sum_{j\in\mathcal{B}(0,r)^c} \max_{O\in\mathcal{A}_{\Lambda_0}} \frac{\|\Delta_j \Gamma_t[O]\|}{\|O\|} \leq \int_{\mathcal{B}(0,r)^c} d\rho \max_{X_1\in\mathcal{A}_{\Lambda_0}} \max_{X_2\in\mathcal{A}_{\Lambda_{\mathcal{B}(0,\rho)^c}}} \frac{\|[\Gamma_t[X_1], X_2]\|}{\|X_1\|\|X_2\|}$$

Inside the ball $\mathcal{B}(0,r)$, we have to approximate the dynamics by an ergodic one with support purely on this ball.

$$\|\Gamma_t[O] - \tilde{\Gamma}_t\{\mathcal{B}(0,r)\}[O]\| \leq \int_0^t \|\mathcal{L}_{\mathcal{B}(0,r)}\| \max_{X_1\in\mathcal{A}_{\Lambda_0}} \max_{X_2\in\mathcal{A}_{\Lambda_{\mathcal{B}(0,r)^c}}} \frac{\|[\Gamma_t[X_1], X_2]\|}{\|X_1\|\|X_2\|}$$

Finally, the last ingredient is the mixing,

$$\|\Delta_j \tilde{\Gamma}\{\mathcal{B}(0,r)\}_t[O]\| \leq 2\|\tilde{\Gamma}_t\{\mathcal{B}(0,r)\} - \tilde{\Gamma}_\infty\{\mathcal{B}(0,r)\}\| \, \|O\|$$

This yields the claim. □

While we introduced the oscillator operationally, in the end it is a useful trick. As a conclusion of this section, we mention the following simple result for unital trace completely positive operators,

$$\Big|\Big|\Gamma[A] - \frac{\mathrm{Tr}\,(\Gamma[A])}{\mathrm{Tr}\,(\mathbb{1})}\Big|\Big| \leq \sum_j \|\Delta_j \Gamma[A]\|$$

This is proven telescopically. The previous discussion can then be applied again.

1.6.1 Conclusion

In this chapter, we presented a construction of a class of local dissipative dynamics. For this class, the bound reflects the dissipative character of the dynamics. The bound proven in [72] only showed the spreading of local perturbation due to the local property of the dynamics. We showed, however, that when the interactions satisfy additional structural equations (1.4), this spreading is slowed down due to the dissipative property.

Next, we studied a class of local Lindbladians which preserves a particular local convex structure. We show that in the irreducible case, a competition can emerge between the diffusive and strongly mixing part of the generator. We see that the diffusion property can be decoupled from the mixing. Localization is then implied for a finite mixing rate of the same order as the Lieb Robinson velocity. We, then, showed various constructions from classical stochastic processes, to more complex quantum dynamics with graph states as fixed points.

Finally, we discussed how deceleration of the velocity can imply rapid mixing. This turns out to be another challenging problem for the general case. We introduced, therefore, a notion of attractiveness. This property is a type of local contractivity of the dynamics under the oscillator norm. This allows to approximate the evolution of local observables by another dynamics with finite support. The mixing then scales with the size of the support, and therefore time. This lead us to a simple inequality which weights the propagation of information against this local scaling of mixing time in equation (1.23). When this is satisfied for an attractive process, rapid mixing is implied.

Chapter 2

Invertible Matrix Product Operators: Detailed Balance, Quantum Phases, RC-CR-algorithm

Synopsis:

The central topic is a study of invertible matrix product operators. The need to understand such property, emerged in various contexts. In the first part, we start from a simple mapping of a generator of local stochastic dynamics to a quantum Hamiltonian. A simple condition is then derived, which allows us to use the quasi-adiabatic evolution and so relate gapped quantum phases with non-equilibrium phases. Next, we present an ansatz for constructing local stochastic dynamics for which the Perron-Frobenius vector has a Matrix Product Representation. Finally, we discuss some consequences of (close to)-invertibility of Matrix Product Operators for simulation of time-evolution of positive definite states on the level of the Cholesky factors.

Based on:
B. Descamps, (unpublished), *e-print* arXiv: *1505.00982*, (2015).

2.1 Introduction

Local dissipative dynamics are currently of growing interest. Heavily studied examples in the classical cases, are Glauber and Kawasaki dynamics. Both satisfy detailed balance locally. The reversible property is broken when two such dynamics are combined. Such examples are also interpreted as models for classical systems coupled to bathes at different temperatures. The field of quantum dissipative engineering has been developing over the last few years. One interest is the generation of tensor network states. So far some progress has been made for pure states, but dynamics generating their thermal perturbations remain unclear, beyond commuting Hamiltonians [54, 59]. In the classical thermodynamics community, [11], tensor networks have arisen as a tool for studying non-equilibrium systems. Later, a quantum version was studied [81].

The chapter is divided into two parts. First, we show how a simple construction can relate the generator of a classical non-equilibrium dynamics with quantum Hamiltonians. The main challenge is to impose an additional constrained, shown in equation (2.5), so that the Hamiltonian is quasi-local. Then, principles such as the quasi-adiabatic evolution can be applied.

In the second part, we look at equation (2.5) from the point of view of tensor-networks. We see that we can set-up a constraint on the local tensors, so that the MPO can be inverted by another MPO.

We, then, present an peculiar construction of local dissipative dynamics satisfying a variation of detailed balance and with a MPO as fixed point.

We conclude this topic with an algorithm which depends heavily on the locality-preserving property of Matrix Product Operators and their their inverses.

2.2 Connecting non-equilibrium with quantum phases

Detailed balance describes the stochastic counterpart of time-reversal symmetry in Hamiltonian dynamics. In the case of non-equilibrium, this reversibility is broken due to some probability current J. Given some process described by the transition probability matrix $\mathcal{L}$, the detailed balance imposes the reversibility of the process. In the standard basis, $T_{ij} = \mathrm{Tr}\left(e_i^\dagger \mathcal{L}[e_j]\right)$, this equation in the form,

$$T_{ij}\pi_j = T_{ji}\pi_i + J_{ij}, \quad \forall i \neq j, T_{ij} \geq 0, \quad \forall j, \sum_i T_{ij} = 0 \tag{2.1}$$

From here on, a simple generalization of equation (2.1) and its connection with a quantum dynamics is straightforward.

The idea is to introduce a couple $(\Omega, \mathcal{L})$ consisting of a bounded operator Ω and a stochastic

dynamics, which can be associated with an Hamiltonian system, i.e.,

$$\mathcal{H} = \Omega^{1/2} \circ \mathcal{L}^\dagger \circ \Omega^{-1/2} + \frac{1}{2}\Omega^{-1/2} \circ \mathcal{J} \circ \Omega^{-1/2}, \tag{2.2}$$

$$\mathcal{J} = \mathcal{L} \circ \Omega - \Omega \circ \mathcal{L}^\dagger, \;\; \rho = \Omega[\mathbb{1}], \;\; \mathcal{L}[\rho] = 0 \tag{2.3}$$

Recall that in the classical case the fixed point ρ is diagonal in the standard basis and equal to π in equation (2.1). If this Hamiltonian is quasi-local and remains gapped under a change of a parameter, the quasi-adiabatic evolution can be applied. This is discussed at the end of the section.

Clearly, the quasi-locality, of the Hamiltonian in (2.2) is highly non-trivial. First, we illustrate these notions presented here, with simple (frustrated) Glauber dynamics.

Example I

In this first example, we illustrate how $\pi = O[\mathbb{1}]$ is the fixed point of a family of gapped stochastic processes. For a one-dimensional chain, let us have a look at,

$$O = \exp\left(\beta \sum_j Z_j Z_{j+1}\right)$$

with the Pauli-operators Z_j, with support on site $\{j\}$.

A frustrated Glauber dynamics, with O as fixed point, is given by local generators T_j so that,

$$T_j^\dagger = C_j(X_j - \mathbb{1})$$

The Pauli operator X_j is the flips the spin at site $\{j\}$, with a weight determined by C_j, which may or may not depend on the neighbouring spins. Additionally, we assume the following relationship between the positive measure O and the local interactions,

$$T_j O = O T_j^\dagger + \gamma(O Z_{j-1} X_j Z_j - O Z_{j+1} X_j Z_j)$$

First of all, let us choose $C^{(j)}$ to depend on the nearest neighbours,

$$C^{(j)} \in \mathrm{span}\{c^{i_1} \otimes c^{i_2} \otimes c^{i_3} | c^0 = \mathbb{1}, c^1 = Z\}$$

The stochasticity of the dynamics is assured if

$$\mathrm{Tr}\,(C) \geq \sum_{(i_1 i_2 i_3) \neq (000)} \left| \mathrm{Tr}\left(C[c^{i_1} \otimes c^{i_2} \otimes c^{i_3}]\right) \right|$$

We could choose C_j to be same for all sites and of the form,

$$C = \lambda_0 \mathbb{1} + \lambda_1 Z_{j-1} Z_j + \lambda_2 Z_j Z_{j+1} + \lambda_3 Z_{j-1} Z_{j+1}$$
$$\mathcal{J} = OC - O^x C^{(x)} = \gamma(Z_{j-1} X_j Z_j - X_j Z_j Z_{j+1})$$

The dynamics is, then, stochastic if,

$$\lambda_0 - \lambda_3 = -(\lambda_1 + \lambda_2)\frac{\cosh(2\beta)}{2\cosh(\beta)\sinh(\beta)},\ \ \lambda_1 - \lambda_2 = \gamma$$

Let us study the modes for the case $\lambda_2 = \lambda_3 = 0,\ \gamma < 0$. We proceed according to the coordinate Bethe Ansatz. For

$$|\psi(k)\rangle = \sum_j \exp(ikj) O^{1/2} Z_j \bigotimes_\alpha |+\rangle_\alpha$$

we find,

$$O^{-1/2}\mathcal{J}O^{-1/2}|\psi(k)\rangle = (-2)\gamma\left[\exp(ik) - \exp(-ik)\right]|\psi(k)\rangle$$
$$\mathcal{H}|\psi(k)\rangle = (-2)|\gamma|\left[\mathrm{cotanh}(2\beta) - \frac{\exp(-ik)+\exp(ik)}{2}\right]|\psi(k)\rangle$$

Notice also that $[H, O^{-1/2}\mathcal{J}O^{-1/2}] = 0$;

The proof of the gap is also straightforward. Indeed notice that H is similar to the bi-stochastic matrix $\tilde{T} = O^{1/2}\tilde{T}O^{-1/2}$

$$\tilde{T} = \sum_j \lambda_0(X_j - I) + \gamma Z_{j-1}Z_j + \frac{\gamma}{2}\left[Z_{j-1}Z_jX_j + X_jZ_jZ_{j+1}\right]$$

The bi-stochastic matrix $\tilde{T}$ can be decomposed as a sum of two other bi-stochastic matrices $\tilde{T} = \tilde{T}_1 + \tilde{T}_2$.

$$\tilde{T}_1 = \sum_j (\lambda_0 - \gamma)(X_j - I),\ \ \tilde{T}_2 = \tilde{T} - \tilde{T}_1$$

Denoting P the projector onto fixed point $\bigotimes_j |+\rangle_j$, then, the gap can be lower bounded by the constant ν, so that,

$$\max \frac{\langle\psi|(\mathbb{1}-P)\exp(t\tilde{T})(\mathbb{1}-P)|\phi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} \leq \exp(-\nu t),\ \ \forall t$$

This can be found using,

$$\begin{aligned}
\max \frac{\langle\psi|(\mathbb{1}-P)\exp(t\tilde{T})(\mathbb{1}-P)|\phi\rangle}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} &= \left\|(\mathbb{1}-P)\exp(t\tilde{T})(\mathbb{1}-P)\right\| \\
&\leq \left\|(\mathbb{1}-P)\exp\left(t\left[\frac{\tilde{T}+\tilde{T}^\dagger}{2}\right]\right)(\mathbb{1}-P)\right\| \\
&\leq \left\|(\mathbb{1}-P)\exp\left(t(\mathbb{1}-P)\left[\frac{\tilde{T}+\tilde{T}^\dagger}{2}\right](\mathbb{1}-P)\right)(\mathbb{1}-P)\right\|
\end{aligned}$$

And since,

$$(\mathbb{1}-P)\left[\frac{\tilde{T}+\tilde{T}^\dagger}{2}\right](\mathbb{1}-P) \leq (\mathbb{1}-P)\tilde{T}_1(\mathbb{1}-P) \leq (\lambda_0-\gamma)(\mathbb{1}-P)$$

Hence $\lambda_0 - \gamma$ is a lower bound for the gap. Previously, we have show that there is a mode with $\lambda_0 - \gamma$ as eigenvalue. Therefore, the gap is precisely equal to $\lambda_0 - \gamma$. As illustrated, in this example, we have seen that a non-equilibrium local stochastic dynamics, can be connected to a local gapped Hamiltonian, where the ground state is intrinsically connected to the Non-Equilibrium Steady State (NESS).

Example II

It should be clear the operator Ω in equation (2.2) needs not the be diagonal in the standard basis. In the case of the latter, then we simply get equation (2.1) back. As a next example, we can consider mixing of Ising with the uniform state,

$$\sigma = q\rho_\beta + (1-q)\frac{\mathbb{1}}{\mathrm{Tr}\,(\mathbb{1})}$$

which is implied to be a fixed point of the dynamics from a non-diagonal matrix Ω.

We can again try and find a Glauber dynamics satisfying the equation. A little algebra yields us,

$$C = \lambda_0\mathbb{1} + \lambda_1 Z_{j-1}Z_j + \lambda_2 Z_j Z_{j+1} + \lambda_3 Z_{j-1}Z_{j+1}$$
$$\lambda_3 = \lambda_1\left(\beta^2\frac{1+q}{1-q} - \alpha^2 - q\right),\ \ \lambda_2 = -\frac{1+q}{1-q}\lambda_1$$

Studying the property of the Ω is, however challenging, and it is still unclear whether the Hamiltonian $\mathcal{H}$ is quasi-local.

This brings us to the main questions of the chapter.

1. Can we find a local stochastic dynamics with some MPO as fixed point?
2. Can the (non)-equilibrium phase be related to a quantum phase, i.e. can the local stochastic dynamics be mapped onto a local quantum through equation (2.2).
3. Is the non-equilibrium/ equilibrium character of the dynamics restricted by the MPO? How can the operator Ω be related to notions such as detailed balance?

2.3 Quasi-Adiabatic Evolution

Consider some lattice Λ with a metric. At each point $x \in \Lambda$ of the lattice, define a d-dimensional Hilbert space $\mathbb{C}^d_x$ and for each finite set $\Lambda_A \subset \Lambda$ denote the product space,

$$\mathcal{H}_{\Lambda_A} = \bigotimes_{x\in\Lambda_A} \mathbb{C}^d_x$$

Denote the algebra of all matrices acting on $\mathcal{H}_{\Lambda_A}$, i.e. the algebra of local observables of Λ_A, $\mathcal{A}_{\Lambda_A}$,

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathbb{C}^d_x)$$

If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}_{\Lambda_1}$ can be identified with the algebra $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_1 \setminus \Lambda_2}$ and therefore $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$. Define the support of a local observable $A \in \mathcal{A}_\Lambda$ as the minimal set $X \subset \Lambda$ for which $A = A' \otimes \mathbb{1}_{\Lambda \setminus X}$. We can then consider local dissipative dynamics $\mathcal{L}$,

$$\mathcal{L} = \sum_Z \mathcal{L}_Z, \;\; \mathcal{L}_Z = \sum_j i[H_Z, .] + \sum_\alpha V_{\alpha Z}[.]V_{\alpha Z}^\dagger - \frac{1}{2}\{V_{\alpha Z}^\dagger V_{\alpha Z}, .\}$$

If we fix the basis to the standard basis e_{ij}, and take $V_{\alpha_Z} = \sqrt{p_{ijZ}}e_{ij}$, due to the covariant symmetry, the dynamics is stochastic on a local sub-algebra. The discussion further remains unchanged for Lindbladians, so we shall consider the general dissipative case.

Given some Hamiltonian $\mathcal{H}(\lambda)$ which depends smoothly on a parameter λ, the idea of the Quasi-Adiabatic Evolution is to construct a quasi-local generator $\mathcal{K}(q, \lambda)$, so that,

$$\forall |\psi\rangle, \;\; |\tilde{\Omega}(q,\lambda)\rangle = P \exp\left(\int_0^\lambda d_\mu \mathcal{K}(q,\mu)\right)|\psi\rangle, \;\; |\langle\tilde{\Omega}(\lambda,q)|\Omega(\lambda)\rangle| \leq C_1\|\psi\| \exp\left(-\frac{q^2}{C_2}\right)$$

and $|\Omega(\lambda)\rangle$ is the unique ground-state of the gapped Hamiltonian $\mathcal{H}(\lambda)$. Moreover the construction is only possible when the gap $\mathcal{H}(\lambda)$ does not close when varying the parameter λ.

A (gapped) quantum phase is defined as the set of ground-states (subspaces) for which the Hamiltonians can be smoothly deformed into one-another without closing the gap. The quasi-adiabatic evolution is an insightful tool when studying states inside the same phase [76].

It is of general interest to find a similar construction of local dissipative systems. If a local dissipative generator could be found, we would have a dissipative version of the adiabatic evolution used for quantum simulations.

However, when it comes down to studying properties, we could try and see if a local dissipative system, classical or quantum, can be related to local Hamiltonian dynamics. At the beginning of the section, we proposed the mapping,

$$\mathcal{H} = \frac{1}{2}\Omega^{1/2} \circ \mathcal{L}^\dagger \circ \Omega^{-1/2} + \frac{1}{2}\Omega^{-1/2} \circ \mathcal{L} \circ \Omega^{1/2}, \tag{2.4}$$

We showed in the example earlier, that even though the stochastic process may have complex eigenvalues, due to some stochastic current, the Hamiltonian can still be gapped and local. The quasi-local property of the Hamiltonian depends on $[.] \to \Omega^{1/2}[.]\Omega^{-1/2}$ and $[.] \to \Omega^{-1/2}[.]\Omega^{1/2}$ being endomorphisms on the quasi-local algebra.

In order to make this more precise, we introduce a set of super-operators $E\left(.|\Lambda_{\mathcal{A}}^{(i)}\right)$ for each local algebra. The idea is reminiscent from conditional expectations and martingale processes defined on filtrations of probability space.

Definition 6. *Given a local algebra $\mathcal{A}_{\Lambda_A}$, define the indexed set $\mathcal{F}(\mathcal{A}) = \{\Lambda_{\mathcal{A}}^{(i)}\}$,*

$$i \leq j, \; \Lambda_{\mathcal{A}}^{(i)} \subset \Lambda_{\mathcal{A}}^{(j)}, \; \Lambda_{\mathcal{A}}^{(0)} = \Lambda_A, \; \bigcup_j \Lambda_{\mathcal{A}}^{(i)} = \Lambda$$

Definition 7. *Given the couple $(\mathcal{A}_{\Lambda_A}, \mathcal{F}(\mathcal{A}))$, define the super-operators*

$$E\left(.|\Lambda_{\mathcal{A}}^{(i)}\right), \;\; \Lambda_{\mathcal{A}}^{(i)} \in \mathcal{F}(\mathcal{A})$$

so that,

1. *Independence*
$$\forall B \in \mathcal{A}_{\Lambda_A}^{(i)}, \;\; E\left(B|\Lambda_A^{(i)}\right) = 0$$

2. *Monotonicity*
$$\forall B \in \mathcal{A}_{\Lambda}, \;\; \|E\left(B|\Lambda_{\mathcal{A}}^{(i)}\right)\| \leq \|B\|$$

3. *Tower Property*
$$\forall i \geq j, \;\; E\left(E\left(.|\Lambda_{\mathcal{A}}^{(i)}\right)|\Lambda_{\mathcal{A}}^{(j)}\right) = E\left(E\left(.|\Lambda_{\mathcal{A}}^{(j)}\right)|\Lambda_{\mathcal{A}}^{(i)}\right) = E\left(.|\Lambda_{\mathcal{A}}^{(j)}\right)$$

We can now rewrite the Hamiltonian (2.4),

$$\mathcal{H} = \sum_Z \sum_i h_{Z,i}, \;\; h_{Z,i} = E\left(h_Z|\Lambda_Z^{(i+1)}\right) - E\left(h_Z|\Lambda_Z^{(i)}\right)$$
$$h_Z = \frac{1}{2}\Omega^{1/2} \circ \mathcal{L}_Z^{\dagger} \circ \Omega^{-1/2} + \frac{1}{2}\Omega^{-1/2} \circ \mathcal{L}_Z \circ \Omega^{1/2},$$

This leads us the final construction,

Construction 8. *For $\lambda \in I$, the local dissipative dynamics $\mathcal{L}(\lambda) = \sum_Z \mathcal{L}_Z$ defined on the lattice Λ and the super-operator $\Omega(\lambda)$, if,*

$$\left\|E\left(h_Z|\Lambda_Z^{(i+1)}\right) - E\left(h_Z|\Lambda_Z^{(i)}\right)\right\| \leq C\exp(-\mu_1 d(i,Z)) \tag{2.5}$$

and the filtration satisfies, $\exists \mu_2, \mu_3$, $\forall \mathcal{Z}$, $\forall i$,

$$\sum_{\Lambda_Z^{(j)} \ni i} \left|\Delta\Lambda_Z^{(j)}\right| \exp\left(-\mu_1 d(i,Z) + \mu_2 \operatorname{diam}\left(\Delta\Lambda_Z^{(j)}\right)\right) \leq \left|Z\right| \exp\left(\mu_3 \operatorname{diam}(Z)\right) \tag{2.6}$$

where, $\Delta\Lambda_Z^{(j)} = \Lambda_Z^{(j+1)} \backslash \Lambda_Z^{(j)}$.

If $\Omega^{1/2}(\Lambda)(\mathbb{1})$ is a ground-state of the gapped Hamiltonian,

$$\mathcal{H}(\lambda) = \frac{1}{2}\Omega^{1/2}(\lambda) \circ \mathcal{L}^{\dagger}(\lambda) \circ \Omega^{-1/2} + \frac{1}{2}\Omega^{-1/2} \circ \mathcal{L}(\lambda) \circ \Omega^{1/2}(\lambda)$$

then a quasi-adiabatic evolution can be constructed in the domain I.

Proof. Equations (2.5), (2.6) and the definition of $E\left(.|\Lambda_Z^{(i)}\right)$ assure that quasi-locality of the Hamiltonian $\mathcal{H}$, by which we mean that $\exists \mu, s$, so that for all sites i,

$$\sum_{\operatorname{supp}(h_{Z,j}) \ni i} \|h_{Z,j}\| |\operatorname{supp}(h_{Z,j})| \exp(\mu \operatorname{diam}(\operatorname{supp}(h_{Z,j})) \leq s < \infty \tag{2.7}$$

This implies that the Hamiltonian dynamics has a Lieb-Robinson velocity [45],

$$\forall A\mathcal{A}_{\Lambda_A}, \forall B\mathcal{A}_{\Lambda_B}, \ \ \|[\exp(-it\mathcal{H})A\exp(t\mathcal{H}), B]\| \leq C\exp\left(\frac{vt - d(A,B)}{\xi}\right)$$

Combined with the gap of the Hamiltonian, further construction of the quasi-adiabatic evolution can be found in [76]. □

Equation (2.6) imposes that we need a sufficiently "large" filtration. Equation (2.5) is very much related to the Lieb-Robinson bound itself, as we see in the following lemma,

Lemma 9. *If*

$$\|[A, \Omega^{1/2}[B]\Omega^{-1/2}]\| \leq C\exp\left(-\frac{d(A,B)}{\xi}\right) \tag{2.8}$$

Then, there exists $E(.|\Lambda_A^{(i)})$ satisfying the axioms and equation (2.5).

Proof.

$$\|[A, \Omega^{1/2}[B]\Omega^{-1/2}]\| \leq C\exp\left(-\frac{d(A,B)}{\xi}\right) \tag{2.9}$$

Let us take $E(.|\Lambda_A^{(i)})$,

$$E(.|\Lambda_A^{(i)}) = \int_{B(\Lambda_A^{(i)})^c} d\mu U A U^\dagger \tag{2.10}$$

where $d\mu$ is the Haar measure for the unitaries with support on the complement of the local ball $B(\Lambda_A, r)$. We can easily verify that equation (2.5) follows from (2.9),

$$\begin{aligned}\|E(\Omega^{1/2}[B]\Omega^{-1/2}|\Lambda^{(i+1)}) &- E(\Omega^{1/2}[B]\Omega^{-1/2}|\Lambda^{(i)})\| \\ &\leq \|\Omega^{1/2}[A]\Omega^{-1/2} - E(\Omega^{1/2}[B]\Omega^{-1/2}|\Lambda^{(i)}) \\ &\leq \max_{U\in B(\Lambda_A^{(i)})^c} \|[A, \Omega^{1/2}[U]\Omega^{-1/2}]\| \leq C\exp\left(-\frac{d(A,i)}{\xi}\right)\end{aligned}$$

□

While equation (2.9) seems to be too strong, the weaker form (2.5), given some $E(.|\Lambda_A^{(i)})$, turns out to be quite useful. In the next section, we illustrate how this appears naturally in the framework of tensor networks.

2.4 Tensor Network approach

Tensor networks appear under various contexts inside and outside of physics. Particular cases are the so-called Matrix Product States/Operators (MPS/MPO) [78] , known as finitely correlated states, tensor trains, etc... A matrix product operator is defined for a finite one-dimensional lattice of size N as,

$$O^{(N)}(\{A[L]\}) = \langle l|_\alpha A[1]_{\alpha,i_1}...A[N]_{\alpha,i_N}|r\rangle \tag{2.11}$$

with $A[L] \in B\left(\mathbb{C}^D_\alpha \otimes \mathbb{C}^D_{i_L}\right)$. This notation is of course equivalent to,

$$O^{(N)}(\{A^{(i,j)}\}) = \sum_{\vec{i},\vec{j}} \langle l|_\alpha A^{(i_1,j_1)} ... A^{(i_N,j_N)} |r\rangle |i_1\rangle\langle j_1| \otimes ... \otimes |i_N\rangle\langle j_N| \tag{2.12}$$

and here with $A^{(i_j)} \in B\left(\mathbb{C}^D\right)$.The dimensions d and D are respectively referred to as the real space and virtual/bond dimension.

2.4.1 Invertible Matrix Product Operators

When discussing the quasi-adiabatic evolution, we proposed that the operation $[.] \to \Omega^{1/2}[.]\Omega^{-1/2}$, preserves the quasi-local properties in the same spirit as Lieb-Robinson bounds (2.9)

This seems too restrictive at first. For example, such property cannot be true in the thermodynamics limit for,

$$\Omega^{1/2} = \exp\left(\beta \sum_j h_{j,j+1}\right) \tag{2.13}$$

where $h_{j,j+1}$ are some non-commuting Hamiltonians. However, operators such as (2.13) can be approximated by a Matrix Product Operator with a polynomial scaling of the bond dimension with the system size [69].

The goal of this section is to present various constructions of invertible Matrix Product Operators.

For sake of clarity, let us first shorten the vocabulary in the following definition,

Definition 10. *A superoperator O preserves local structure if $\mathcal{E}(O) : \mathcal{A} \to O[\mathcal{A}]O^{-1} \subset \mathcal{A}$ is an endomorphism on the quasi-local algebra,*

$$\forall A \in \mathcal{A}_{\Lambda_A}, \exists B \in \mathcal{A}_{\Lambda_B},\ \ \Lambda_A \subset \Lambda_A, \operatorname{diam}\Lambda_B < \infty,\ \ B = O[A]O^{-1}$$

The first lemma, introduces a condition on the local tensors of the MPO, so that equation (2.5), simplifies to,

$$\forall \mathcal{A}_A, \forall B \in \mathcal{A}_A, \exists j,\ \left\| E\left(OBO^{-1}|\Lambda_A^{(i+1)}\right) - E\left(OBO^{-1}|\Lambda_A^{(i)}\right) \right\| = 0, \forall i > j \tag{2.14}$$

Lemma 11. *For finite chain $N < \infty$, and the Matrix Operator $O^{(N)}(\{A^{(i)}\})$ as defined in (2.11), if there is another MPO $O^N(\{B^{(i)}\})$, so that $\exists n < \infty$*

$$\langle l(s-1)|C^{(i_s,j_s)} ... C^{(i_{s+n},j_{s+n})} - \frac{\delta_{i_s,j_s}}{d} \langle l(s)|C^{(i_{s+1},j_{s+1})} ... C^{(i_{s+n},j_{s+n})} = 0 \tag{2.15}$$

$$C^{(i_{s-n-1},j_{s-1})} ... C^{(i_{s-1},j_{s-1})}|r(s)\rangle - \frac{\delta_{i_s,j_s}}{d} |C^{(i_{s-n-1},j_{s-n-1})} ... C^{(i_s,j_s)}|r(s-1)\rangle = 0 \tag{2.16}$$

with boundary conditions,

$$\langle l(N-n)|C^{(i_{N-n+1},j_{s-n+1})}...C^{(i_N,j_N)}|r(N)\rangle = \delta_{(i_{N-n+1},j_{s-n+1})}...\delta_{(i_N,j_N)}, \tag{2.17}$$

$$\langle l(0)|C^{(i_1,j_1)}...C^{(i_n,j_n)}|r(n+1)\rangle = \delta_{(i_1,j_1)}...\delta_{(i_n,j_n)} \tag{2.18}$$

where we wrote,

$$C^{(i_s,j_s)} = \sum_{k_s} A^{(i_s,k_s)} \otimes B^{(k_s,j_s)}$$

and with,

$$\langle l(s)| = \langle l|\Phi(1)...\Phi(s), \;\; \Phi(N-s-1)...\Phi(N)|r(s)\rangle, \;\; \Phi(s) = \sum_{i,j} A^{(i_s,j_s)} \otimes B^{(j_s,i_s)}[s]$$

Then,

$$O^{(N)}(\{A^{(i)}\})O^{(N)}(\{B^{(i)}\}) = \mathbb{1}$$

and O preserves the local structure.

Proof. Let us use again $E(.|\Lambda_A^{(i)})$ given by,

$$E(.|\Lambda_A^{(i)}) = \int_{B(\Lambda_A^{(i)})^c} d\mu U A U^\dagger \tag{2.19}$$

Starting for an arbitrary site $j \in \Lambda$, we clearly have,

$$\begin{aligned} O_1 B O_2 = \mathbb{1} &+ \left[d \operatorname{Tr}_j E(O_1 B O_2|\Lambda_A^{(0)}) - \mathbb{1}\right] \\ &+ \sum_k d^{k+1}\left[dE(O_1 B O_2|\Lambda_j^{(k+1)}) - E(O_1 B O_2|\Lambda_j^{(k)})\right] \end{aligned}$$

Taking $O_1 = O^{(N)}(\{A^{(i)}\})$, $O_2 = O^{(N)}(\{B^{(i)}\})$, $B = \mathbb{1}$, we can see that the assumption implies,

$$d.E(O^{(N)}(\{A^{(i)}\})O^{(N)}(\{B^{(i)}\}))|\Lambda_j^{(k+1)}) - E(O^{(N)}(\{A^{(i)}\})O^{(N)}(\{B^{(i)}\}))|\Lambda_j^{(k)}) = 0, \; \forall k, j$$

Hence, $O^{(N)}(\{A^{(i)}\})O^{(N)}(\{B^{(i)}\}) = \mathbb{1}$. Similarly, we can see that O preserves the local structure. □

When considering the thermodynamics limit, one cannot help but wonder if the sequences $|l(s)\rangle$ and $|r(s)\rangle$ could converge to a fixed point of Φ for $N \to \infty$,

$$A^{(i,j)}[s] \to A^{(i,j)}, \;\; B^{(i,j)}[s] \to B^{(i,j)}, \tag{2.20}$$

$$\exists \Phi, \|\Phi\| \leq 1, \;\; \Phi = \sum_{i,j} A^{(i_j)} \otimes B^{(j,i)}, \tag{2.21}$$

$$\langle l(s)| \to \langle l(s)| \lim_{N\to\infty} \Phi^N, \;\; |r(s)\rangle \to \lim_{N\to\infty} \Phi^N |r(s)\rangle \tag{2.22}$$

If such limit exists, then we can easily set up the corollary,

Corollary 12. *Given the MPO $O(\{A^{(i,j)}\})$ as defined in (2.12), if there are sequences such as in (2.20), and if,*

$$\exists\{B^{(i,j)}\},\ \sum_{i,\alpha} A^{(m,i)}_{\alpha,k} B^{(i,n)}_{\alpha,l} = \frac{\delta_{k,l}}{d}\delta_{m,n},\ \ \|\sum_{i,j} A^{(i,j)} \otimes B^{(j,i)}\| = 1 \tag{2.23}$$

Then O preserves the local structure.

The boundness of Φ, while keeping the condition of Lemma (11) is unclear, however it is satisfied in the following special case.

Corollary 13. *Given the MPO $O(\{U\})$ as defined in (2.11), if there are sequences such as in (2.20), and if, If $U^{(\beta,i)(\alpha,j)}$ is unitary, and $U_{(\alpha,i)(\beta,j)} = U_{(\alpha,j)(\beta,i)}$ then $O\{A^{(i,j)}\}$ preserves local structures*

Proof. Without loss of generality, we can put the local observable A_{Λ_A} at the origin. For the inverse, consider $O\{B^{(i,j)}_{(\alpha,\beta)}\}$ with

$$\left(B^{(i,j)}_{(\alpha,\beta)}\right) = \frac{1}{d}\overline{U}_{(\beta,j)(\alpha,i)},$$

The first part of Lemma (12) is indeed fulfilled,

$$\sum_{i,\alpha} A^{(m,i)}_{\alpha,k} B^{(i,n)}_{\alpha,l} = \frac{1}{d}\sum_{i,\alpha} U_{(k,m)(\alpha,i)}\overline{U}_{(l,n)(\alpha,i)} = \frac{1}{d}\delta_{k,l}\delta_{m,n}$$

For the second part, we can see,

$$\Phi[.] = \sum_{i,j} A^{(i,j)}[.](B^{(j,i)})^t = \frac{1}{d}\operatorname{Tr}_d U[.]U^\dagger$$

where the partial trace is taken on the level the real space $\mathbb{C}^d$. Per construction $\Phi[.]$ is a trace-preserving unital completely positive operator (CP). Therefore $\phi \circ \phi^*$ is again unital and CP. It is easily shown that trace-preserving CP operators have largest eigenvalue 1. Hence $\|\Phi\| \leq 1$, and Lemma (12) yields the corollary. □

A special alternative worth mentioning, as for all Gibbs states of commuting Hamiltonian there exists a MPO-representation satisfying this property, for periodic boundary. The construction is inspired from the Fundamental Commutation Relation related to the Algebraic Bethe Ansatz [29].

Lemma 14. *Given the translational invariant MPO, $O(\{A^{(i,j)}\})$,*

$$O(\{A^{(i)}\}) = \operatorname{Tr}\left(A^{(i_1,j_1)}...A^{(i_N,j_N)}\right)|i_1\rangle\langle j_1| \otimes ... \otimes |i_N\rangle\langle j_N|$$

and denoting $V_{a,\alpha} = \sum_{i,j} A^{(i,j)}|i\rangle\langle j| \in \operatorname{End}\left(\mathbb{C}^D_a \otimes \mathbb{C}^d_\alpha\right)$. If there exists a projector $P_{a,b} \in \operatorname{End}\left(\mathbb{C}^D_a \otimes \mathbb{C}^D_b\right)$, $P_{a,b}\dagger = P_{a,b}$, $P_{a,b}P_{a,b} = P_{a,b}$, and $W_{b,\alpha} = \sum_{i,j} B^{(i,j)}|i\langle\rangle j| \in \operatorname{End}\left(\mathbb{C}^D_a \otimes \mathbb{C}^d_\alpha\right)$, so that

$$P_{a,b}V_{a,\alpha}W_{b,\alpha}P_{a,b} = \mathbb{1}_\alpha P_{a,b} \tag{2.24}$$

$$V_{a,\alpha_1}W_{b,\alpha_1}P(a,b)V_{a,\alpha_2}W_{b,\alpha_2} = V_{a,\alpha_1}W_{b,\alpha_1}V_{a,\alpha_2}W_{b,\alpha_2} \tag{2.25}$$

Then the MPO $O(\{B^{(i,j)}\})$, is the right-inverse,

$$O(\{A^{(i)}\})O(\{B^{(i)}\}) = \mathbb{1}$$

This is can be illustrated for Ising, and checking that equation (2.25) are satisfied for,

$$A^{(1)} = \begin{pmatrix} \exp(\beta) & 0 \\ \exp(-\beta) & 0 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 0 & \exp(-\beta) \\ 0 & \exp(\beta) \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} \exp(-\beta) & 0 \\ \exp(\beta) & 0 \end{pmatrix}, B^{(2)} = \begin{pmatrix} 0 & \exp(\beta) \\ 0 & \exp(-\beta) \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

2.4.2 An Equilibrium Ansatz for Local Stochastic Dynamics

Finding a local generator, which satisfies detailed balance for a Matrix Product Operator, is a very challenging problem.

$$\sum_Z \mathcal{L}_Z \circ \Omega = \Omega \circ \sum_Z \mathcal{L}_Z^\dagger \tag{2.26}$$

In the literature, similar problems have been studied for non-equilibtrium system, by setting up a quadratic algebra on the MPO-tensors and their boundaries [11, 81]. The solutions mostly have bond dimensions scaling with the size of the system.

There are two different strategies for tackling equations such as (2.26) with a type of MPO as starting point.

First, we could try and find a local dynamics such that the invertible MPO is the fixed point. Since the MPO is invertible the equations become local.

There is, however, in second glance an alternative approach in inspired from our discussion of invertible Matrix Product Operators. One could introduce an ancillary system $\mathcal{V}$ and define the MPO on $\mathcal{V} \otimes \Lambda_{\text{Lattice}}$. Additionally, we could try and set up a local and generalized form of detailed balance on the level of virtual space. Let us first have a look at the following equation for a local stochastic generator $T = \sum_j T_{j-1,j,j+1}$ with three-point interaction,

$$\begin{aligned} &\sum_j \mathbb{E}\left[[I],(B,B,B),(A,A,A),[I]\right]\left[T_{j-1,j,j+1}\right] \\ =&\sum_j \mathbb{E}\left[[I],(A,A,A),(B,B,B),[I]\right]\left[T^\dagger_{j-1,j,j+1}\right] \end{aligned} \tag{2.27}$$

where,

$$\begin{aligned} &\mathbb{E}[[L],(A,B,C),(D,E,F),[R]][Q] = \\ &\sum_{\vec{i},\vec{j},\vec{\alpha},\vec{\beta}} \langle L| \left(A^{(i_1,\alpha_1)} \otimes D^{(\beta_1,j_1)}\right)\left(B^{(i_2,\alpha_2)} \otimes E^{(\beta_2,j_2)}\right)\left(C^{(i_1,\alpha_1)} \otimes F^{(\beta_1,j_1)}\right)|R\rangle Q_{(\vec{\alpha}),(\vec{\beta})} \end{aligned}$$

Let us look at this equation with the Ising model. For,

$$M(\beta) = \begin{pmatrix} \exp(\beta) & \exp(-\beta) \\ \exp(-\beta) & \exp(\beta) \end{pmatrix}, \quad A^{(i,j)}(\beta) = \delta_{i,j} M(\beta) |i\rangle\langle i|$$

Then choosing the three-point Glauber dynamics,

$$T_{i-1,i,i+1} = C^{(i)}(X_i - \mathbb{1})$$

where $C^{(i)} \in \mathrm{span}\{c^{i_1} \otimes c^{i_2} \otimes c^{i_3} | c^0 = \mathbb{1}, c^1 = Z\}$. and taking $B^{(i.j)} = A^{(i,j)}(-\beta)$, we can check that equation (2.27), is equivalent to,

$$X_{(i)} C^{(i)} X_{(i)} \exp\left(2\beta(\sigma_z^{i-1}\sigma_z^i + \sigma_z^i\sigma_z^{i+1})\right) = \exp\left(2\beta(\sigma_z^{i-1}\sigma_z^i + \sigma_z^i\sigma_z^{i+1})\right) C^{(i)}$$

Hence, the state $\rho = \exp\left(2\beta \sum_i \sigma_z^{i-1}\sigma_z^i\right)$ is a fixed point of the process.

Hence, we see that we have replaced the equation (2.26), with,

$$\mathrm{Tr}_{\mathcal{V}}\left(O_1 \mathcal{L} O_2\right) = \mathrm{Tr}_{\mathcal{V}}\left(O_1^\dagger \mathcal{L} O_2\right) \tag{2.28}$$

which could be reduced to local equations such as (2.27).

We can present an additional example, with a $\mathbb{Z}_2$-invariant fixed space, as an illustration.

Example 15. *For $A^{(i)} \in \mathcal{B}\left(\mathbb{C}^2\right)$, if the following equation is satisfied,*

$$\sum_j Z_{(\alpha,\alpha)} A^{(j)}_{\alpha,\beta} = \sum_j A^{(j)}_{(\alpha,\beta)} Z_{(j,j)} Z_{(\beta,\beta)} \tag{2.29}$$

If $B^{(i)} \in \mathcal{B}\left(\mathbb{C}^2\right)$ fulfils,

$$\sum_\gamma A^{(i)}_{\alpha,\gamma} B^{(i)}_{\gamma,\beta} = \delta_{\alpha,\beta}$$

Then, if there exists a local $T = \sum_j T_{j-1,j,j+1}$ so that,

$$\mathbb{E}\left[[I],(B,B,B),(A,A,A),[I]\right][T_{j-1,j,j+1}] = \sum_j \mathbb{E}\left[[I],(A,A,A),(B,B,B),[I]\right][T^\dagger_{j-1,j,j+1}] \tag{2.30}$$

$$\mathbb{E}\left[[Z],(B,B,B),(A,A,A),[I]\right][T_{j-1,j,j+1}] = \sum_j \mathbb{E}\left[[Z],(A,A,A),(B,B,B),[I]\right][T^\dagger_{j-1,j,j+1}] \tag{2.31}$$

then for,

$$\mathcal{L} = T + (\otimes_j Z_j) T (\otimes_k Z_k),$$
$$\Omega = \sum_{i_1,..,i_N} \langle I| \left(A^{(i_1)} \otimes A^{(i_1)}\right) ... \left(A^{(i_N)} \otimes A^{(i_N)}\right) |I\rangle |i_1\rangle\langle i_1| ... |i_N\rangle\langle i_N|$$

$$\mathcal{L} \circ \Omega = \Omega \circ \mathcal{L}^\dagger$$

Hence,

$$\mathcal{L} \circ \Omega[\mathbb{1}] = 0 = \mathcal{L} \circ \Omega[\otimes_j Z_j]$$

Proof. Notice first the trivial equality,

$$|I\rangle_{1234} \propto |I\rangle_{12} \otimes |I\rangle_{34} + |Z\rangle_{12} \otimes |Z\rangle_{34} \tag{2.32}$$

and,

$$|I\rangle_{1234} \propto |I\rangle_{14} \otimes |I\rangle_{23} + |Z\rangle_{14} \otimes |Z\rangle_{23} \tag{2.33}$$

Define the Matrix Product Operators O_1 $(A \otimes B \otimes A \otimes A)$,

$$\begin{aligned}
\langle I|_{1234} O_1 |I\rangle_{1234} = \sum_{\vec{i},j} \langle I|_{1234} & \left(A^{(i_1)} \otimes B^{(i_1)} \otimes A^{(i_1)} \otimes A^{(i_1)}\right) \\
& \ldots \left(A^{(i_{j-2})} \otimes B^{(i_{j-1})} \otimes A^{(i_{j-1})} \otimes A^{(i_{j-2})}\right) \\
& \mathbb{E}((A,A,A)(B,B,B),(A,A,A),(A,A,A))[T_{j-1,j,j+1}] \\
& \ldots \left(A^{(i_{N-1})} \otimes B^{i_{N-1}} \otimes A^{(i_{N-1})} \otimes A^{(i_{N-1})}\right) \\
& \left(A^{(i_N)} \otimes B^{(i_N)} \otimes A^{(i_N)} \otimes A^{(i_N)}\right) |I\rangle_{1234} \\
\langle I|_{1234} O_2 |I\rangle_{1234} = \sum_{\vec{i},j} \langle I|_{1234} & \left(A^{(i_1)} \otimes A^{(i_1)} \otimes B^{(i_1)} \otimes A^{(i_1)}\right) \\
& \ldots \left(A^{(i_{j-2})} \otimes A^{(}i_{j-1}) \otimes B^{(i_{j-1})} \otimes A^{(i_{j-2})}\right) \\
& \mathbb{E}((A,A,A)(A,A,A),(B,B,B),(A,A,A))[T^{\dagger}_{j-1,j,j+1}] \\
& \ldots \left(A^{(i_{N-1})} \otimes A^{(i_{N-1})} \otimes B^{(i_{N-1})} \otimes A^{(i_{N-1})}\right) \\
& \left(A^{(i_N)} \otimes A^{(i_N)} \otimes B^{(i_N)} \otimes A^{(i_N)}\right) |I\rangle_{1234}
\end{aligned}$$

with,

$$\begin{aligned}
& \mathbb{E}((A,A,A)(B,B,B),(C,C,C),(A,A,A))[T]^{(i_1,i_2,i_3)}_{(j_1,j_2,j_3)} \\
& = \left(A^{(i_1)} \otimes B^{(i_1)} \otimes C^{(i_1)} \otimes A^{(j_1)}\right) \left(A^{(i_2)} \otimes B^{(i_2)} \otimes C^{(j_2)} \otimes A^{(j_2)}\right) \\
& \left(A^{(i_3)} \otimes B^{(i_3)} \otimes C^{(j_3)} \otimes A^{(j_3)}\right) T_{(i_1,i_2,i_3)(j_1,j_2,j_3)}
\end{aligned}$$

Let us compare $\langle I|_{1234} O_1 |I\rangle_{1234}$ and $\langle I|_{1234} O_2 |I\rangle_{1234}$. Using equation (2.33) and the $\mathbb{Z}_2$-symmetry (2.29), we see that the conditions (2.30,2.31), imply,

$$\langle I|_{1234} O_1 |I\rangle_{1234} = \langle I|_{1234} O_2 |I\rangle_{1234}$$

From equation (2.32) and again the $\mathbb{Z}_2$-symmetry (2.29), we can readily see,

$$\langle I|_{1234} O_1 |I\rangle_{1234} = \mathcal{L} \circ \Omega, \;\; \langle I|_{1234} O_2 |I\rangle_{1234} = \Omega \circ \mathcal{L}^{\dagger}$$

Hence, the claim follows. □

The tensor network picture gives us the opportunity to further study two lesser studied aspects of reversibility of Markov chains. First of all, we see that the reversibility can come in a somewhat-frustrated form, i.e. each local interaction do not necessarily have to be reversible themselves. This is in contrast to the vastly studied equilibrium dynamics such as Kawasaki and Glauber.

Onsager initially applied detailed balance as an equilibrium condition due to conservative forces with some potential $V(x)$,

$$e^{\beta V(x)}k(x,y) = e^{\beta V(y)}k(y,x) \tag{2.34}$$

While O is symmetric, its eigenvalues needs not to be positive. If we choose a singular value decomposition, the rate $k(x,y)$ will possible be positive. On the other hand, if O is diagonalized we might get complex potential and negative rates.

2.5 Invertibility of MPO's and Simulation of Lindbladian Dynamics

Before concluding this topic, it is interesting to point out a numerical scheme which is heavily dependent on the ability to invert Matrix Product Operators. This could be an additional motivation for studying this topic.

The scheme presented here is a novel method for computing the time-evolution under a local Lindbladian which preserves the positive definiteness. The algorithms is inspired from a QR-decompositon trick used in the time-dependent variational principle algorithm [35]. We evolve the Cholesky factor of the state, rather the state itself. The Cholesky fator is represented by a Matrix Product Operator. This allows to approximate the non-equilibrium stationary state by a positive definite Matrix Product Operator.

Consider first a time evolution determined by the Kraus Operators $\{A_l^{(j)}\}$. Starting from $A_l^{(j)}A_l^{(j)\dagger} = \mathbb{1}$, we could try and find new operators $\{A_r^{(j)}\}$ in such way that,

$$\begin{bmatrix} C_{\text{old}}A_l^{(0)} \\ \vdots \\ C_{\text{old}}A_l^{(d)} \end{bmatrix} = \begin{bmatrix} A_r^{(0)} \\ \vdots \\ A_r^{(d)} \end{bmatrix} C_{\text{new}}$$

We can easily verify,

$$\sum_j A_r^{(j)\dagger}A_r^{(j)} = \mathbb{1}, \quad \sum_j A_l^{(j)\dagger}C_{\text{old}}^\dagger C_{\text{old}}A_l^{(j)\dagger} = C_{\text{new}}^\dagger C_{\text{new}}$$

Ideally, we would like to apply the same method for Lindbladians. A first proposal would be to renormalize the tensors with some $0 < \epsilon << 1$, $A^{(0)} = \mathbb{1} + \epsilon Q$, $A^{(1)} = \sqrt{\epsilon}R$. However, each time-step introduces an additional growing error of order epsilon to the time evolution,

$$C_{\text{old}}^\dagger C_{\text{old}} + \epsilon\mathcal{L}[C_{\text{old}}^\dagger C_{\text{old}}] - C_{\text{new}}^\dagger C_{\text{new}} = \epsilon^2 Q^\dagger C_{\text{old}}^\dagger C_{\text{old}}Q$$

We can easily verify that, setting $\epsilon = t/N$

$$\|C_{\text{new}}^\dagger C_{\text{new}} - \Gamma_t[C_0^\dagger C_0]\| \leq \left(\frac{t^2}{N}\|Q\|^2 + 2\frac{t\sqrt{t}}{\sqrt{N}}\|Q\|\|R\|\right)\exp\left(\frac{t}{N}\|Q\|^2 + 2\frac{\sqrt{t}}{\sqrt{N}}\|Q\|\|R\|\right)\|C_0\|^2$$

Assume that $C \in M_D$, then,

$$\|\rho_\infty - \Gamma_t[C_0^\dagger C_0]\| \leq D^2 \exp\left(-t/\nu\right)$$

Hence for $\|\rho_\infty - \Gamma_t[C_0^\dagger C_0]\| \approx \delta$, we need $t \approx \frac{1}{\nu}\log(D^2/\delta)$ and $N = O\left(\log(1/\delta)^3/\delta^2\right)$.

However, this additional error can be easily avoided completely by solving the equations,

$$CQ_l = Q_r C + \partial C, \;\; CR_l^{(i)} = R_r^{(i)} C$$

in $R_l^{(i)}$, Q_r and ∂C, under the constraints,

$$Q_r + Q_r^\dagger + \sum_i R_l^{(i)\dagger} R_l^{(i)} = 0$$

For finite dimensional matrices, one can easily solve these equations in the two following way.

In the first case, we can calculate the QR-decomposition of the following matrix,

$$\begin{bmatrix} C_{\text{old}} \\ C_{\text{old}} \\ \epsilon C_{\text{old}} Q \\ \sqrt{\epsilon} C_{\text{old}} R \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} C_{\text{new}}$$

but rather than considering the Euclidean metric, we introduce the new metric,

$$M = \begin{bmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{bmatrix}$$

Hence if,

$$B_1^\dagger B_1 + B_3^\dagger B_2 + B_2^\dagger B_3 + B_4^\dagger B_4 = \mathbb{1}, \;\; Q + Q^\dagger + R^\dagger R = 0$$

Then,

$$C_{\text{new}}^\dagger C_{\text{new}} = C_{\text{old}}^\dagger C_{\text{old}} + \epsilon \mathcal{L}[C_{\text{old}}^\dagger C_{\text{old}}]$$

A second possibility is to directly derive a formula for Q and R from the Gram-Schmidt process which we can implement later. These equations are free from any ϵ.

Problem 16. *Given C, α, β, we want to find Q, R and ∂C so that,*

$$\begin{aligned} C + \epsilon\alpha &= \epsilon QC + \epsilon \partial C \\ \sqrt{\epsilon}\beta &= \sqrt{\epsilon} RC \\ 0 &= Q + Q^\dagger + R^\dagger R \end{aligned}$$

Proposition 17. *Write down the columns of Q and R as,*

$$Q = [q_1 \dots q_D], \quad R = [r_1 \dots r_D]$$

similarly for C, α, β. Then,

$$q_k = \mu_k - \frac{e_k}{2}\left(e_k^\dagger \mu_k + \mu_k^\dagger e_k + \rho_k^\dagger \rho_k\right)$$
$$r_k = \rho_k$$

with,

$$\mu_k = \frac{\alpha_k - \sum_{j=1}^{k-1}\left[(e_j^\dagger \alpha_k)e_j + (q_j^\dagger c_k)e_j + (e_j^\dagger c_k)q_j + (r_j^\dagger \beta_k)e_j\right]}{|C_{kk}|} \tag{2.35}$$

$$\rho_k = \frac{\beta_k - \sum_{j=1}^{k-1}(e_j^\dagger c_k)r_j}{|C_{kk}|} \tag{2.36}$$

And,

$$\partial C = \alpha - QC$$

Proof. Start from the Gram-Schmidt procedure,

$$u_k = a_k - \sum_{j=1}^{k-1}\langle u_j, a_k\rangle \frac{u_j}{\|u_j\|} \tag{2.37}$$

for the matrix $A = [a_1 \dots a_D]$,

$$A = \begin{pmatrix} C + \epsilon\alpha \\ \sqrt{\epsilon}\beta \end{pmatrix}$$

We derive a formula for the column of Q and R, by solving,

$$\left[\frac{u_1}{\|u_1\|} \dots \frac{u_D}{\|u_D\|}\right] = \begin{pmatrix} \mathbb{1} + \epsilon Q \\ \sqrt{\epsilon}R \end{pmatrix}$$

By writing down the identity in the standard basis $\mathbb{1} = [e_1 \dots e_D]$, hence we can use as ansatz for the columns u_k

$$\frac{u_k}{\|u_k\|} = \begin{pmatrix} \frac{u_k^{(1)}}{\|u_k^{(1)}\|} \\ \frac{u_k^{(2)}}{\|u_k^{(2)}\|} \end{pmatrix} = \begin{pmatrix} e_k + \epsilon q_k \\ \sqrt{\epsilon}r_k \end{pmatrix}$$

Inserting this in equation (2.37), yields us,

$$u_k^{(1)} = c_k - \sum_{j=1}^{k-1}(e_j^\dagger c_j)e_j + \epsilon\left[\alpha_k - \sum_{j=1}^{k-1}(e_j^\dagger \alpha_k)e_j - (q_j^\dagger c_k)e_j - (e_j^\dagger c_k)q_j\right]$$
$$u_k^{(2)} = \sqrt{\epsilon}\beta_k - \sqrt{\epsilon}\sum_{j=1}^{k-1}(e_j^\dagger c_k)r_j$$

Set,

$$\gamma_k = c_k - \sum_{j=1}^{k-1}(e_j^\dagger c_j)e_j$$
$$\mu_k = \alpha_k - \sum_{j=1}^{k-1}\Big[(e_j^\dagger \alpha_k)e_j + (q_j^\dagger c_k)e_j + (e_j^\dagger c_k)q_j + (r_j^\dagger \beta_k)e_j\Big]$$
$$\rho_k = \beta_k - \sum_{j=1}^{k-1}(e_j^\dagger c_k)r_j$$

Then, for $u_k^{(1)} = \gamma_k + \epsilon\mu_k$ and $u_k^{(2)} = \sqrt{\epsilon}\rho_k$,

$$\|u_k\|^2 = \|\gamma_k\|^2 + \epsilon\left(\gamma_k^\dagger\mu_k + \mu_k^\dagger\gamma_k + \rho_k^\dagger\rho_k\right)$$

Notice that since C is upper-triangular then $\gamma_k = C_{kk}e_k$,

$$q_k = \frac{\mu_k}{\|\gamma_k\|} - \frac{1}{2}e_k\left(e_k^\dagger\frac{\mu_k}{\|\gamma_k\|} + \frac{\mu_k^\dagger}{\|\gamma_k\|}e_k + \frac{\rho_k^\dagger\rho_k}{\|\gamma_k\|^2}\right)$$
$$r_k = \frac{\rho_k}{\|\gamma_k\|}$$

which yields the claim.

□

For Larger matrices, this problem becomes more complicated. However, one could propose to represent C by a Matrix Product Operator. The first step is then to calculate a MPO representations $R_r^{(j)}$ which are solutions of,

$$CR_l^{(j)} = R_r^{(j)}C \tag{2.38}$$

Following we can easily find the MPO-representation of ∂C. The time-step is then achieved from,

$$C_{new} = C + \epsilon\partial C$$

which can be done with the Time-dependent Variational Principle [35]. It is not completely certain whether this scheme could be of any use. First of all, besides the problem of solving the equation (2.38), the bond dimension increases linearly with the number of $R's$. There exists, however, examples in the literature such as in [81], where there is a single Q and two $R's$. The Cholesky decomposition of the non-equilibrium steady states seems also to play a role in extensions of the Algebraic Bethe Ansatz to non-equilibrium dynamics.

Whether, this is a sign that such scheme, which only time-evolve the Cholesky factors, could of use, should be decided by the reader and future researchers.

Chapter 3

Matrix Product States approach to non-Markovian processes

Synopsis:

Study of representations of joint-probability distributions by Matrix Product States seems to be a young topic. We show that Matrix Product States arise naturally when considering some non-Markovian processes. Interestingly, all properties of such classical processes are determined by a quantum dynamic. We discuss this peculiar unravelling starting from quantum measurement theory and relating it to a non-commutative version of Girsanov theorem for the Radon-Nikodym derivative of these processes.

Based on:
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3.1 Introduction and summary

Stochastic processes play a major role in any fields of science. Whether it is physics, biology, chemistry, finance, etc..., the effective dynamics of macro-variables are derived from coupling to some random observables.

Consider a process $(X_j)_{j\in\mathbb{N}}$ of which X_1, X_2, ..., X_n could represent observations sampled randomly from the same population at time points i = 1, 2, ... When constructing models some underlying assumption of the mechanics has to be made. The most famous example are the Autoregressive-Moving-Average model $ARMA(p,q)$ processes [98],

$$X_k = c + \sum_{j_1=1}^{p} \alpha_{j_1} X_{k-j_1} + \sum_{j_2=1}^{q} \theta_{j_2} W_{k-j_2}$$

with W_k are independently normally distributed $N(0,1)$. Notice that the transition probabilities only depend on a finite number of variables. Hence, these processes provide a useful method for describing the statistics of short-memory processes. However, as it is well known, there exists processes with long memory. Some of those have been described by Benoît Mandelbrot and are known as fractional Brownian motion [8, 67].

Clearly, for all these processes an analytically approachable form had to be chosen in order to model the desired statistical properties. At the end, given some data $(X_j)_{j\in\mathbb{N}}$, we wish to find the model with the right correlations and higher momenta. These are fully described by the characteristic function of the process,

$$f(u_1, u_2...) = E(\exp(iu_1X_1 + iu_2X_2 + ...))$$

Therefore the underlying mechanics should not be described on the level of the observations, but rather on the level of the characteristic function or the measure of the process.

In [91], Stochastic Matrix product States (sMPS) were introduced as an ansatz for studying non-equilibrium states in statistical mechanics. As shown further on, similarly to Matrix Product States (MPS) being fixed points of the Density Renormalization Group, sMPS should be seen as fixed points of Metropolis Monte-Carlo Sampling. Hence, these states represent the joint distribution of a statistical (non-Markovian) process.

The chapter is ordered as follows. In the first section, we present an introduction to Matrix Product States (MPS), their continuum version (cMPS), Quantum Measurements, and Radon-Nikodym derivatives. In the second section, we show the connection with the so-called sMPS and how it can be used to described the non-Markovian processes and their Master equations with some examples. In the last, section some miscellaneous applications can be found

3.1.1 Matrix Product States and their Continuum versions

Matrix Product States form a class of finitely correlated states used to study the low energy spectrum of local Hamiltonians in quantum spin chain. For a chain $\otimes_{j=1}^{N}\mathbb{C}^d$ of size N, they are given as follows,

$$|\psi\{A^{(j)}[k]\}\rangle = \sum_{\vec{i}} \mathrm{Tr}\left(A^{(i_1)}[1]...A^{(i_N)}[N]\right)(\sigma^+[1])^{i_1}...(\sigma^+[N])^{i_N}|0...0\rangle = \mathrm{Tr}_V(Z_N)|0...0\rangle$$

with matrix $A^{(j)}[k] \in \mathcal{M}_D$ of bound dimension D. The partial trace in the last term is in the virtual space. The following normalization condition can be chosen,

$$\sum_j A^{(j)}[k]A^{(j)}[k]^\dagger = \mathbb{1}$$

For the case $d = 2$, the operator Z_N can be rewritten as a solution of the recursion equation,

$$Z_N = Z_{N-1} + Z_{N-1}\left(A^{(0)}[N] - \mathbb{1}\right) + Z_{N-1}A^{(1)}[N] \otimes \sigma^+$$

Using notation $\Delta n = 1$, $\Delta Z_n = Z_n - Z_{n-1}$, $\Delta B_n^\dagger = \frac{1}{\sqrt{\Delta n}}\sigma^+[n]$, this equation is equivalent to,

$$Z_N = \mathbb{1} + \sum_{j=1}^{N} \Delta Z_n, \quad \Delta Z_n = Z_{N-1}Q[n]\Delta n + Z_{N-1}R[n] \otimes \Delta B_n^\dagger,$$
$$Q[n] = \frac{\mathbb{1} - A_0[N]}{\Delta_n}, \quad R[n] = \sqrt{\Delta_n}A^1[n]$$

This is a discrete quantum stochastic differential equation (QSDE). Clearly by identifying $\Delta n \leftrightarrow dt$ and $\Delta B^\dagger[n] \leftrightarrow dB^\dagger(t)$, we can find a continuum version of matrix product states,

$$|\psi\{Q[s], R[s]\}\rangle = \mathrm{Tr}\left(\exp\left(\int_0^L Q(s)ds + R(s) \otimes dB^\dagger(s)\right)\right)|\Omega\rangle = \mathrm{Tr}_V(Z_N)|0...0\rangle$$

which satisfies again the QSDE,

$$Z_t = \mathbb{1} + \int_0^t dZ_s, \quad dZ_t = Z_t Q[t]dt + Z_t R[t] \otimes dB^\dagger[t]$$

A wider variety of solutions of QSDE have been studied by Hudson and Parthasarathy [51]. We do not continue on this topic.

3.1.2 Quantum Measurement Theory

Denote $(Y, \mathcal{B}(Y))$ a Borel space representing the output events of some quantum measurement. Define a random variable $X : Y \to \mathbb{R}$ which represent the value of the outputs. Measurements are described by operators $A^{(y)}$ labeled by $y \in Y$. It is then postulated in quantum statistics that the probability of an event in $E \in \mathcal{B}(Y)$ is given,

$$P(X(y), y \in E) = \int_{y \in E} \mathrm{Tr}\left(\rho A^{(y)}A^{(y)\dagger}\right)$$

The conditional state ρ resulting from the measurement is then given by,

$$\rho_E = \int_{y \in E} \frac{A^{(y)\dagger}\rho A^{(y)}}{\mathrm{Tr}\left(\rho A^{(y)}A^{(y)\dagger}\right)}$$

From this on, we can consider a sequence of measurement at different times $t_1, ..., t_N$ with output values given by $X_{t_1}, ..., X_{t_n}$. The probability of a certain chain of outcomes $x_1, ..., x_N$ at times $t_1, ..., t_N$ is therefore given by,

$$\begin{aligned} P(X_{t_1} = x_1, ..., X_{t_n} = x_n) =& P(X_{t_n} = x_n | X_{t_{n-1}} = x_{t_{n-1}}, ..., X_{t_1} = x_{t_1}) ... \\ & P(X_{t_2} = x_2 | X_{t_1} = x_{t_1}) P(X_{t_1} = x_{t_1}) \\ & = \langle \rho | \left(A^{(x_1)} \otimes \overline{A}^{(x_1)} \right) ... \left(A^{(x_N)} \otimes \overline{A}^{(x_N)} \right) | I \rangle \end{aligned}$$

From another perspective, this measurement process can be seen as the evolution of a state in a cavity interacting with an electromagnetic field. The outputs are the values given by the detector in contact with the field. An introduction to this topic can be found in [12]. In [49], Holevo studied the representation of continuous measurement. Clearly not any arbitrary operator is eligible for a continuum limit. In order to derive the representation, Holevo studied the characteristic function of the joint probability distribution of the sequence of output. The idea is to, similarly as infinitely divisible processes in statistics also known as Levy processes, look at the characteristic function.

$$\sum_{x_1,...,x_N} \exp(i\lambda(x_1 + ... + x_n)) P(X_{t_1} = x_1, ..., X_{t_n} = x_n) = \mathrm{Tr}\left(\rho \exp\left(n\mathcal{L}(\lambda)\right)[\mathbb{1}]\right)$$

$$\phi(\lambda)[.] = \sum_x \exp(i\lambda x) A^{(y)}[.] A^{(y)\dagger}, \;\; \mathcal{L}(\lambda)[.] = \mathrm{Id} - \frac{1}{n}\left(\phi(n) - n\,\mathrm{Id}\right)[.]$$

Clearly, the limit $n \to \infty$ does not always exists, however when it does, the resulting generator $\mathcal{L}(\lambda)$ is given by a non-commutative version of the Levy-Khintchin representation theorem. One set of representations of particular interest is of the form,

$$\mathcal{L}(\lambda) = \mathcal{L}_0[.] + \mathcal{L}_1(\lambda)[.]$$

with,

$$\mathcal{L}_0[.] = Q[.] + [.]Q^\dagger + R[.]R^\dagger, \;\; Q = iH + \frac{1}{2}RR^\dagger \tag{3.1}$$

and,

$$\mathcal{L}_1(\lambda)[.] = i\lambda[.] + \sigma^2\left(R[.]R^\dagger - \frac{1}{2}\{RR^\dagger, .\} + i\lambda(R[.] + [.]R^\dagger) - \frac{1}{2}\lambda^2[.]\right)$$

We illustrate this in the following two examples,

Example 18. *For,*

$$dZ^{(1)}(t) = Z^{(1)}(t)\left(\mathcal{L}_0[.]dt + \sigma\frac{R[.] + [.]R^\dagger}{2}dB_t^x - i\sigma\frac{R[.] - [.]R^\dagger}{2}dB_t^y + \frac{1}{2}\sigma m(dB_t^x + idB_t^y)\right)$$

with solution

$$\begin{aligned} Z_t^{(1)} =& \exp\left(t\mathcal{L}_0[.] - \frac{1}{2}\sigma^t\left(\left(\frac{R[.] + [.]R^\dagger}{2} + m\right)^2 - \left(\frac{R[.] - [.]R^\dagger}{2} - m\right)^2\right) + \right. \\ & \left. \sigma\frac{R[.] + [.]R^\dagger}{2}B_t^x - i\sigma\frac{R[.] - [.]R^\dagger}{2}B_t^y + \sigma m(B_t^x + iB_t^y)\right) \end{aligned}$$

where (B_t^x, B_t^y) is a two-dimensional Brownian motion We can verify using Ito-calculus,

$$E(\mathrm{Tr}_V(\rho Z_t^{(1)})\exp(i\lambda(B_t^x + iB_t^y))) = \mathrm{Tr}\left(\rho\exp(t\mathcal{L}_0 + t\mathcal{L}_1(\lambda))[\mathbb{1}]\right),$$

We find a similar construction using 1-dimensional Brownian motion,

Example 19. *For,*

$$dZ_t^{(2)} = Z_t^{(2)}\left(\mathcal{L}_0[.]dt + \sigma\left(R[.] + [.]R^\dagger\right)dB_t + mdB_t\right) \tag{3.2}$$

$$Z_t^{(2)} = \exp\left(t\mathcal{L}_0[.] - \frac{1}{2}\sigma^2 t\left(R[.] + [.]R^\dagger + m\right)^2 + \sigma\left(R[.] + [.]R^\dagger + m\right)B_t\right)$$

where (B_t^x, B_t^y) is a two-dimensional Brownian motion We can verify using Ito-calculus,

$$E(\mathrm{Tr}_V(\rho Z_t^{(2)})\exp(i\lambda B_t)) = \mathrm{Tr}\left(\rho\exp(t\mathcal{L}_0 + t\mathcal{L}_1(\lambda))[\mathbb{1}]\right),$$

Stochastic matrix product states represent, as explained here, a sequential measurement of a cavity coupled to an electromagnetic field. The examples provided here are the continuum limits of such states.

3.1.3 Radon-Nikodym derivatives and Girsanov's theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$, that is equivalent to $\tilde{\mathcal{P}}$. Then there exist an almost surely positive random variable Z be which satisfies $\mathbb{E}Z = 1$ and for $A \in \mathcal{F}$,

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$$

The statement described above is the Radon-Nikodym theorem [74]. The random variable Z, called the Radon-Nikodym derivative, allows us to change a measure. Suppose that we have a filtration $\mathcal{F}(t)$, define for $0 \leq t \leq T$, where T is a fixed final time. Then we can define the so-called Radon-Nikodym derivative process,

$$Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$$

The main application of the change of measure, is that it allows to map a Brownian motion onto another one with a drift. This is procedure is known under the name Girsanov's theorem [30].

Theorem 20 (Girsanov)**.** *Let $B(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{F}(t)$, be the filtration generated by this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process, define,*

$$Z(t) = \exp\left(-\int_0^t \theta(u)dB(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right)$$
$$\tilde{B}(t) = B(t) + \int_0^t \Theta(u)du$$

Assume that $E\int_0^T \Theta^2(u)Z^2(u)du < \infty$. Let $\tilde{\mathbb{P}}$ be the probability measure generated by the Radon-Nikodym derivative $Z(T)$. The the process $\tilde{B}(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$

As we will see further on, continuous stochastic matrix product states allow us to map a Wiener process onto another, not necessarily Gaussian, process with correlated increment

3.2 Stochastic Matrix Product states

We finally introduce Stochastic Matrix Product States (sMPS) and a few of its continuum counterparts (csMPS).

Stochastic Product States were first introduced in [91] as possible tool for studying the partition function of spin models in statistical mechanics. The stochastic product state was of the form,

$$|p_D\rangle = \sum_{\vec{i}} \langle L|B^{(i_1)}[1]...B^{(i_N)}[N]|R\rangle |i_1...i_N\rangle \tag{3.3}$$

with the matrices B_{i_k} and vectors $\langle L|$, $\langle R|$ element-wise positive. Additionally the transfer matrices of this form $T[k] = \sum_j B^{(j)}[k]$ is a stochastic matrix.

We argue that this form is incomplete. In the last paragraph of this section, we show that for these type of states can be mapped onto a Markovian process. On the other hand, the construction proposed breaks down for the more general form, we give below.

Similarly to Matrix Product Operators (MPO), we derive Stochastic Matrix Product States from a purification method.

Consider a process $(X_j)_{j=1}^N$, $X_j \in \{y_1, ..., y_d | y_j \in \mathbb{R}\}$, with joint probability $P(X_1 = x_1, ..., P_N = x_N) = p(x_1, ..., x_n)$ Define the pure state $|\psi\rangle$,

$$|\psi\rangle = \sum_{\vec{i}} \sqrt{p(x_1, ..., x_N)} |i_1, ...\rangle_A |i_1, ...\rangle_B$$

The MPS-representation $\{B^{(i_j)}\}$ of this state can be derived. By tracing out the ancillary B, we derive the form,

$$P(X_1 = x_1, ..., X_N = x_N) = C^N \sum_{\vec{i}} \langle L|B^{(i_1)} \circ ... \circ B^{(i_N)}|R\rangle \delta(x_1, i_1) ... \delta(x_N, i_N) \tag{3.4}$$

where $B^{(i_k)}$ is completely positive. For some normalization constant C. In this case for the boundary operators, $|L\rangle = \hat{L} \otimes \mathbb{1}|I\rangle$, $|R\rangle = \hat{R} \otimes \mathbb{1}|I\rangle$, it should be taken that $\hat{L}, \hat{R} \geq 0$. In contrast to the previous form the transfer matrix is a trace preserving completely positive operator $\Gamma[k][.] = \sum_j B^{(j)}[k][.]$.

continuous-time sMPS Setting the boundaries first aside see that this form is exactly equivalent to the Quantum Measurements described earlier. We can make this more explicit with the gauge transformation $A^{(i)} \to \tilde{A}^{(i)} = X^{-1/2}\tilde{A}^{(i)}X^{1/2}$, $\rho \to \tilde{\rho} = X^{1/2}\rho X^{1/2}$.

Additionally, when rewriting,

$$P(X_1 = x_1, \dots, P_N = x_N) = Z_N(x_1, ..., x_N) \left(\frac{1}{d}\right)^N$$

We see that the sMPS functions behave as a change of measure. Let us keep this in mind and rewrite Z_N as a solution of a discrete stochastic differential equation for the case $d = 2$ and $X_j = \pm 1$. So $Z_N = \mathbb{1} + \sum_{j=1}^N \Delta Z_n$ and,

$$\Delta Z_n = Z_{n-1}\frac{(\Gamma[n] - \mathbb{1})}{\Delta n}\Delta n + Z_{n-1}\sqrt{\Delta_n}\left(A^{(-1)}[n] \otimes \overline{A}^{(-1)}[n] - A^{(1)}[n] \otimes \overline{A}^{(1)}[n]\right)\Delta B_n$$

where $\Delta B_n = \delta(x_n, +1) - \delta(x_n, -1)$. By taking the renormalization,

$$A^{(-1)} = \mathbb{1} + \Delta_n Q + \sqrt{\Delta_n}R, \;\; A^{(+1)} = \mathbb{1} + \Delta_n Q - \sqrt{\Delta_n}R$$

with $Q = iH + \frac{1}{2}R^\dagger R$, and taking limit $\Delta_n \to 0$, this leads to the form,

$$Z(t) = Z(t)\mathcal{L}_0[.]dt + Z(t)\left(R[.] + [.]R^\dagger\right)dB_t \tag{3.5}$$

derived in the Quantum Measurement section. Hence, the Radon-Nikodym satisfying this equation is a continuum version the sMPS.

Clearly other time-continuous versions are possible by considering a different product measure. Let $X_j \in \{0, 1\}$ and $p(X_j = 0) = \exp(-\lambda\epsilon)$, $p(X_j = 1) = \exp(-\lambda\epsilon)\epsilon\lambda$. Then let us write,

$$\Delta Z_n = Z_{n-1}\frac{A^{(0)}[n] \otimes \overline{A}^{(0)}[n] - \mathbb{1}}{\epsilon}\Delta n + Z_{n-1}\left(A^{(1)}[n] \otimes \overline{A}^{(1)}[n] - \mathbb{1}\right)\Delta N(n)$$

Here, we chose $\Delta n = \epsilon\delta(x_n = 0)$and $\Delta N(n) = \delta(x_j = 1)$. By taking $A^{(0)} = \mathbb{1} + i\epsilon H - \epsilon\frac{1}{2}\mu$ and $A^{(1)} = \sqrt{\mu}U$, for some unitary U, $U^\dagger U = \mathbb{1}$ and $\mu > 0$. Computing the generator $\mathcal{L}(\lambda)$ of the characteristic function yield,

$$\mathcal{L}(\lambda) = i[H, .] + \mu\left(\exp(i\lambda)U[.]U^\dagger - [.]\right)$$

In the bound dimension one case, this is nothing but a Poisson process. For higher dimension, this corresponds to a photon-counting process from the quantum measurement point of view. We do not continue on this.

sMPS as fixed points of Metropolis sampling Let us rephrase the original Metropolis Monte Carlo Algorithm in a quantum measurement based language in a some state manifold $\mathcal{M}_{\text{states}}$. This allows us a larger variety of input states, besides product state, and operations, thus extending the Markovian character of the algorithm to non-Markovianity.

The whole procedure can be resumed as follows, starting from some input state $\rho(0)$

1. at step t, pick a (local) operation $O_j[.]$
2. accept the measurement with probability $\text{Tr}\left(O_j^*[\mathbb{1}]\rho(t-1)\right)$
3. if accepted, change the state to

$$\rho(t-1) \to \rho(t) = \min_{\sigma \in \mathcal{M}_{\text{states}}} \left\| \sigma - \frac{O_j[\rho(t-1)]}{\text{Tr}\left(O_j^*[\mathbb{1}]\rho(t-1)\right)} \right\|_2$$

   if necessary, project the state back onto the manifold.

4. go back to step 1

Given some observable M, for large enough time T, the expectation $E(M)(T)$,

$$E(M)(T) = \frac{1}{T}\sum_{t=1}^{T} \text{Tr}\left(|\psi(t)\rangle\langle\psi(t)|M\right)$$

converges to the equilibrium expectation, $\lim_{T\to\infty} E(M)(T) = \text{Tr}\left(\rho_\infty M\right)$, with,

$$\rho_\infty = \sum_i \lim_{T\to\infty} \text{Tr}\left(\rho_0 \Gamma^T[A^i A^{i\dagger}]\right)|i\rangle\langle i|, \;\; \Gamma[.] = \sum_j A_j[.]A_j^\dagger$$

As an example, consider a classical 1-dimensional spin chain, described by the Hamiltonian,

$$H = \sum_j s_i s_{i+1}$$

Choose $\mathcal{M}_{\text{states}}$ first to be the manifold of product states,

$$\mathcal{M}_{\text{states}} = \{|s_1, ..., s_N\rangle | s_i = \pm 1\}$$

By taking the measurements $O(s_j) = \frac{\exp(\beta H)|s_j\rangle\langle -s_j|\exp(-\beta H)|}{\sum_{s_j}\exp(\beta H)}$, and defining the operations, and operations,

$$\tilde{O}(s_j)[.] = \sum_s \langle s_{j-1}, s_j, s_{j+1}|O|s_{j-1}, s_j, s_{j+1}\rangle\langle s_{j-1}, s_j, s_{j+1}|[.]|s_{j-1}, s_j, s_{j+1}\rangle|s_{j-1}, s_j, s_{j+1}\rangle\langle s_{j-1}, s_j, s_{j+1}|$$

the algorithm describes precisely a Glauber dynamics [68], which converges to the equilibrium state,

$$\rho_\beta = \sum_{s_1,...,s_N} \langle L|B(s_1)...B(s_N)|R\rangle|s_1, ..., s_N\rangle\langle s_1, ..., s_N|, \;\; B(s_j) = \begin{pmatrix} \exp(2\beta s_j) & 0 \\ 0 & \exp(-2\beta s_j) \end{pmatrix}$$

Let us now extend $\mathcal{M}_{\text{states}}$, to the mpo-manifold of bound dimension 2,

$$\mathcal{M}_{\text{states}} = \{\rho\{M[s_j]\}\rangle | \dim M[s_j] = 2\}, \;\; \rho\{M[s_j]\} = \sum_{\vec{s}} \langle L|M(s_1)...M(s_N)|R\rangle |\vec{s}\rangle\langle\vec{s}|$$

Clearly ρ_β is a fixed point, in the sense that,

$$\rho_\infty = \sum_j \text{Tr}\left(\rho_\beta \Gamma^T[O^{j*}[\mathbb{1}]]\right)|i\rangle\langle i|, \;\; \forall T$$

Difference between the representations Equation (3.3) can be mapped to (3.4), by taking,

$$A^{(i)}_{y,z}[k] = \sqrt{B^{(i)}_{y,z}}[k]|y\rangle\langle z|$$

Even more, in the following construction we show that we can find a Markov process that is equivalent to the ones described by (3.3) on the level of the MPS. As noted before the transfer matrix of (3.3) is a stochastic matrix. Consequently, this corresponds to the completely positive operator $\Gamma[] = \sum_{xy} T_{x,y}|x\rangle\langle y|$ which is the transfer matrix in (3.4). Consider first a Markov process (Y_j) with transfer matrix T. The joint probability distribution is easily written in smps-notation by taking $A^{(x_j)}[k] = T|j\rangle\langle j|$. Take now the process (X_j) with smps of the form (3.3). Consider its purification to MPS. Next block the sites over a length L so that the number matrices $M^{(i_1,...,i_L)} = A^{(i_1)}...A^{(i_L)}$ exceeds the Kraus-rank of the completely positive operator (CP). Next apply a change of basis in the virtual. As it is known, for any CP-map Kraus-operators are equivalent by some unitary U,

$$\Gamma[.] = \sum_j A^j[.]A^{j\dagger} = \sum_j B^j[.]B^{j\dagger} \leftrightarrow A^j = \sum_i U_{ji}B_j$$

Since we have blocked to full rank, and since the transfer matrix over this length is of the form $\Gamma[L][.] = \sum_{x,y}(T^L)_{x,y}|x\rangle\langle y|.|y\rangle\langle x|$. Take the unitary so that $M^{(i_1,...,i_L)} \to \lambda(x)\tilde{A}^{(x)} = (T^L)|x\rangle\langle x|$. Consider the equivalent class $[x] = \{[i_1,...,i_L]|M^{(i_1,...,i_L)} \propto \tilde{A}^{(x)}\}$. The new process $Y_j = \{[x]\}$ is thus a Markovian process over a rescaled time.

In some sense the MPS describes the time evolution of a cavity coupled to an electromagnetic field. At each time, some detection procedure is applied, for example Homodyne detection, fixing the real basis of the MPS, i.e. photon basis at each time slice of the e.m.-field. The change of processes is therefore nothing but a change to another commutant of the *-algebra of observables.

3.2.1 Master Equation for memory processes

Let $(X_j)_{j\geq 0}$, be a process whose joint probability distribution is described by $\mathbb{P}$. A topic of broad interest is the time evolution the marginal discribution of X_N. For any distribution, this time evolution of $P(X_N = x_N)$ with initial condition $P(X_0 = x_0)$ is given by,

$$P(X_N = x_N) = \sum_{x_1,...,x_{N-1}} P(X_N = x_n|X_{N-1} = x_{n-1},...,X_0 = x_0)...P(X_1 = x_1|x_0) \tag{3.6}$$

For a Markov process with transition matrix $P(X_j = x_j|X_{j-1} = x_{j-1}) = T_{x_j,x_{j-1}}$, this evolution can be reduced severely to the different forms,

$$P(X_N = x_N) = \sum_{x_{N-1}} T_{x_N,x_{N-1}} P(X_{N-1} = x_{N-1})$$

For $T = \mathbb{1} + \epsilon G$, with G understood as a generator of a time-continuous stochastic matrix, we can rewrite in a differential form,

$$\frac{dP(k)}{dt} = \sum_l \delta(k,l) \frac{P(X_N = k) - P(X_{N-1} = l)}{\epsilon} = \sum_{l \neq k} [G_{k,l} P(X_{N-1} = l) - G_{l,k} P(X_{N-1} = k)]$$

This is known as the Master equation.

As we can see, the number of parameters of equation (3.6) scales exponential as we try to find the time-evolution of a marginal $P(X_N = x_N)$. In a first approach, as it is done in practice, we the transition probability can be bounded up to some fixed time k. However, similarly to the Markovian case, this can be written in a simple sMPS form of bound dimension $2k$.

Example 21. *Let $k < \infty$ and $\forall n$,*

$$\begin{aligned} P(X_n = i_n|X_{n-1} = i_{n_1}, \dots, X_{n-k} = i_{n-k}, X_{n-k-1} = x_{n-k-1}, \dots, X = x_1, X_0 = x_0) \\ = P(X_n = i_n|X_{n-1} = j_{n_1}, \dots, X_{n-k} = x_k) = T_{i_{n-k},\dots,i_{n-1},i_n} \end{aligned}$$

Then, the sMPS representation is given by,

$$A^{i_n}_{y_1,\dots,y_k,z_1,\dots,z_k} = T_{y_1,\dots,y_k,z_k} \delta_{i_n = z_k} \delta_{y_2 = z_1} \dots \delta_{y_k = z_{k-1}}$$

and with boundaries,

$$|X\rangle = |I\rangle, \;\; \langle \rho|_x = P(X_0 = x)$$

As understood from the point of view of Quantum Measurement theory, the sMPS-formalism, describes smoother conditional probability. Such states describe processes whose joint probability almost, but not quite, factorizes and vice-versa [13],

$$\|p(X_1, ..., X_k, X_{k+N}, ..., X_{k+l+N}) - p(X_{k+N}, ..., X_{k+l+N})\|_1 \leq C_2 \exp(-C_1 N)$$

Thus, we can write for a compationally efficient parametrization for such memory processes. So given a process $(X_j)_{j \geq 0}$, described by the smps,

$$P(X_1 = x_1, \dots, X_N = x_N) = \mathrm{Tr}\left(\rho \hat{Z}_N(x_1, ..., x_N)\right)$$

with $Z_0 = \mathbb{1}$ and,

$$\Delta Z_N(x_1, ..., x_N) = \sum_j \hat{Z}_{n-1}(x_1, ..., x_{N-1}) \left(A^j \otimes \overline{A}^j - \mathbb{1}\right) \delta(j, x_N)$$

Then,

$$P(X_N = k) = \mathrm{Tr}\left(\sum_l \hat{T}_{k,l}\hat{P}_l\rho\right)$$

for which,

$$\sum_l \hat{T}_{k,l}\hat{P}_l = \sum_l (\overline{A}^{(k)}\overline{\otimes}A^{(k)})(\overline{A}^{(l)}\overline{\otimes}A^{(l)})\Gamma^{*N-2}$$

In the time-continuous limit this corresponds to,

$$\frac{d}{dt}P(X_t = k) = \mathrm{Tr}\left(\sum_l \hat{G}_{k,l}\hat{P}_l\rho\right)$$

and again,

$$\sum_l \hat{G}_{k,l}\hat{P}_l = \sum_l S^k \exp(tL)$$

with $S^{(k)}[.] = Q^{(k)}[.] + [.]Q^{(k)\dagger}$, $S^{(k)}[.] = R^{(k)\dagger}[.]R^{(k)}$ or sum of both

Example 22 (Non-Markovian Birth-Death processes)**.** *Let $X_j \in \mathbb{N}$, and*

$$Q^{(n)} = -\frac{1}{2}\hat{G}_{n,n} \otimes |n\rangle\langle n|, \;\; R^{(n)}_{+1} = \hat{G}_{n,n+1}|n\rangle\langle n+1|, \;\; R^{(n)}_{-1} = \hat{G}_{n,n-1}|n\rangle\langle n-1|$$

with $0 = \hat{G}_{n,n} + \hat{G}^{\dagger}_{n,n} + \hat{G}_{n+1,n}\hat{G}^{\dagger}_{n+1,n} + \hat{G}_{n+1,n}\hat{G}^{\dagger}_{n+1,n}$.

Let $S^{(n)}$ be,

$$S^{(n)} = Q^{(n)}[.] + [.]Q^{(n)\dagger} + R^{(n)\dagger}_{+1}[.]R^{(n)}_{+1} + R^{(n)\dagger}_{-1}[.]R^{(n)}_{-1}$$

For choice of boundary $\rho = \sum_n \lambda_n|n\rangle\rangle n|$ and $\dim(\hat{G}_{n,m}) = 1$, this reduces to the birth-and-death process.

3.3 Miscellaneous

A Complete Market with Finitely Correlated Increments of the Logarithm of the Stocks

As we can see, from the form of the Radon-Nikodym derivatives described in this chapter, there is some correlation between future and past increments. One of the goals of Financial Mathematics is to study the pricing of contracts so that both clients and agents cannot take advantage of each other. This theory is also known as arbitrage-pricing theory. Interestingly, our formalism allows us to define a market with a certain bias in the evolution of stocks. Yet, we see that among these markets, fairness, in the sense of "no-arbitrage", can still be found.

A Fast Introduction to Neutral-Pricing

This introduction is meant as a presentation of the ideas of arbitrage pricing and contains many holes. For a full introduction we refer to [87], [88]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability and B_t a brownian motion. Denote $\mathcal{F}(t)$ the filtration with respect to the Brownian motion. In the most simple market model two different actions are possible at each time t. We can either purchase some stock $S(t)$ or invest at some interest rate $r(t)$ in the money market. An option is a contract with a payoff at some later time T that depends on the stock $S(t)$ at different values $0 < t \leq T$. The idea of the arbitrage pricing approach is to find a portofolio $X(t)$ that replicate the options by only using two actions described above.

Assume that the stock, whose price at time t, is given by $S(t)$ satisfies the stochastic differential equation,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB_t \tag{3.7}$$

Define the discount process $D(t)$,

$$D(t) = \exp\left(-\int_0^t r(s)ds\right)$$

The randomness of the stock is described by some measure $\mathbb{P}$. However, as we will see there is a nice trick that allows us to find the replicating strategy by changing to a new measure, called the risk-neutral measure.

Definition 23. *A probability measure is said to be risk-neutral if,*

1. $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *are equivalent, and*
2. *under* $\tilde{\mathbb{P}}$*, the discounted stock price* $D(t)S(t)$ *is a martingale*

The existence of the risk neutral measure plays a central role in risk-neutral pricing as it means that there is a strategy to hedge any derivative security.

Denote $X(t)$ the value of the portofolio. The idea is to hedge the derivative security by investing at each time in $\Delta(t)$ stocks $S(t)$ and the difference in the money market with interest rate r. The differential of $X(t)$ is therefore given by,

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = rX(t)dt + \Delta(t)d(D(t)S(t))$$

We see then that the differential of the discounted portofolio,

$$d(D(t)X(t)) = \Delta(t)d(D(t)S(t))$$

The martingale representation theorem can be applied to write any martingale defined on a filtration of a Brownian motion as a stochastic integral. In the case of $D(t)S(t)$, we have $D(t)S(t) = \int_0^t \sigma D(s)S(s)dW_s$. The first fundamental theorem of asset pricing asserts that if a market model has a risk-neutral-pricing formula, then it does not admit arbitrage. Arbitrage

is a trading strategy that begins with nothing, has probability of losing money, and a strictly positive probability of making money.

Let $V(T)$ be the payoff of the option at time T. The payoffs at a time $t < T$ are called derivative securities. Therefore, we see that our model So given some contract given by the derivative security $V(T)$. Then since $D(T)S(T)$ is a martingale under $\tilde{\mathbb{P}}$,then so is $D(t)V(t)$. We can then use the martingale representation theorem to write,

$$D(t)V(t) = V(0) + \int_0^t \Gamma dB_s$$

Our portofolio will replicate the price of the derivative security, if $\Delta(t) = \frac{\Gamma(t)}{\sigma D(t)S(t)}$ and $X(0) = V(0)$. Such a market wherein every derivative security can be hedged is called a complete market.

The Correlated Model

Clearly the form of the SDE (3.2) of the stochastic matrix product state reminds a lot of geometric brownian motion. So why not define a model, where the stocks $S(t)$ are described by such SDE. According to the arbitrage pricing, our market model is then complete if $D(t)S(t)$ is a martingale under some measure $\tilde{\mathbb{P}}$ whose Radon-Nikodym derivative is also of the form (3.2). Unfortunately, it turns out that we cannot consider both $S(t)$ and $\tilde{\mathbb{P}}$ of the form (3.2) at the same time. We discuss both cases separately and derive the condition for the completeness of the market.

Case 1: Change in Evolution Stock

The idea of model is then to redefine the time evolution of the stock $S(t) = \mathrm{Tr}\left(\rho\hat{S}(t)X\right)$, with,

$$d\hat{S}(t) = \hat{S}(t)\left(\mathcal{L}_0[.]dt + \alpha dt + \sigma\left(R[.] + [.]R^\dagger\right)dB_t\right) \tag{3.8}$$

The discounted process D_tS_t will have a drift terms $\alpha - r$. This drift term can be taken care of using Girsanov's theorem.

Theorem 24. *Let,*

$$\mathcal{L}_0[X]dt + (\alpha - r)\left(R[X] + [X]R^\dagger\right) = 0 \tag{3.9}$$

with $\mathcal{L}_0[.]$ of the form (3.1). Let $S(t)$ be a process satisfying equation (3.8). For the Radon-Nikodym derivative Z_T,

$$Z(T) = \exp\left(-\frac{\alpha - r}{\sigma}B(T) - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 T\right)$$

Let $\tilde{\mathbb{P}}$ be the measure generated by Z_T is a risk-neutral measure. Then if,

$$E\left(Z_T\,\mathrm{Tr}\left(\rho\hat{S}(t)\left(R[X] + [X]R^\dagger\right)\right)^2\right) < \infty$$

$\tilde{\mathbb{P}}$ *is the risk-neutral measure*

Proof. First, we use Girsanov's theorem to shift the drift of $D(t)S(t)$ by going over to the Brownian process under $\tilde{\mathbb{P}}$,

$$d\tilde{B}(t) = dB(t) + \frac{(\alpha - r)}{\sigma}dt$$

Under the new measure $\tilde{\mathbb{P}}$, $D(t)S(t)$ is of the form,

$$dD(t)S(t) = D(t)\,\mathrm{Tr}\left[\rho\hat{S}(t)\left(\mathcal{L}_0[.]dt + (\alpha - r)\left(R[.] + [.]R^\dagger\right)dt + \sigma\left(R[.] + [.]R^\dagger\right)d\tilde{B}_t\right)X\right]$$

Under the condition of the theorem, the dt term disappears and $D(t)S(t)$ is a martingale under $\tilde{\mathbb{P}}$.

□

By the theorem, proven above our market model is therefore a complete market.

In condensed matter, the thermodynamic limit is often of interest. In this case, this corresponds to the limit $T \to \infty$. Something interesting happens in this case. Indeed for any finite $T < \infty$, we see that the boundaries have to correspond with the left and right zero-eigenvector of $\mathcal{L}_0$. In the case, the condition derived in previous theorem reduces to,

$$\left(R[X] + [X]R^\dagger\right) = 0$$

Hence the boundary is an eigenvector of both $R[.] + [.]R^\dagger$ and $\mathcal{L}_0$. This means that the bound dimension of R reduces to 1, and $Z_t = 1$. So if we consider some option that has a payoff at a finite time, then we get the usual stock again.

$$\lim_{T\to\infty} \tilde{S}(t) = S(t)$$

with,

$$dS(t) = \alpha dt + \sigma dB(t)$$

We do not continue further on this case.

Case 2: Change of Measure

The model here considers the new measure $\tilde{\mathbb{P}}$ generated by the Radon-Nikodym derivative $Z_T = \mathrm{Tr}\left(\rho\hat{Z}_T X\right)$,

$$d\hat{Z}(t) = \hat{Z}(t)\left(\mathcal{L}_0[.]dt + \left(R[.] + [.]R^\dagger\right)dB_t + m dB_t\right) \tag{3.10}$$

And the stocks $S(t)$ satisfy the usual evolution,

$$dS(t) = \alpha dt + \sigma dB(t)$$

The increments ΔB_n are now possibly correlated under this measure. First, let us make sure that the discounted stock $D(t)S(t)$ is a martingale again. Assume further on that $Z(T) > 0$.

Theorem 25. *For,*

$$\mathcal{L}_0[X] + (\alpha - r + \sigma m)X + \sigma\left(R[X] + [X]R^\dagger\right) = 0 \tag{3.11}$$

with $\mathcal{L}_0[.]$ of the form (3.1). The measure $\tilde{\mathbb{P}}$ defined under the Radon-Nikodym derivative (3.10), then,

$$d(D(t)S(t)Z_t) = \sigma D(t)S(t)\mathcal{Z}_t dB_t$$

with,

$$\mathcal{Z}(t) = \mathrm{Tr}\left(\rho\hat{Z}(t)\left(R[X] + [X]R^\dagger + (m+\sigma)X\right)\right)$$

If,

$$E(S(t)^2\mathcal{Z}(t)^2) < \infty$$

then $\tilde{\mathbb{P}}$ is a risk neutral measure

Proof. The computation is similar as the one above. Using ito calculus, we can see that,

$$E(Z_T D(t)S(t)|\mathcal{F}_s) = Z(s)D(s)S(s)$$

where $Z(s) = E(Z_T|\mathcal{F}_s)$ is the Radon-Nikodym process. From this we indeed see that $D(s)S(s)$ is a martingale under $\tilde{\mathbb{P}}$. □

Additionally, we need to show that for this model every derivative security can be hedged. Indeed, in the introduction the martingale representation theorem was used. However, $B(t)$ is not a Brownian motion anymore and not even a martingale under $\tilde{\mathbb{P}}$. Yet, we need to define a method. Notice first the following,

Lemma 26. *If M_t is a martingale under $\tilde{\mathbb{P}}$, then $Z_t M_t$ is a martingale under $\mathbb{P}$.*

Proof. Since $M(t)$ is $\mathcal{F}_t$-measurable then,

$$M_s = \tilde{E}(M_t|\mathcal{F}_s) = \frac{1}{Z_s}E(Z_t M_t|\mathcal{F}_s)$$

from which the claim follows. □

We can then show using the theorem that for the portofolio process $X(t)$,

$$d(Z(t)S(t)X(t)) = \sigma\Delta(t)S(t)\mathcal{Z}(t)X(t)dB_t$$

Defining the martingale process for the derivative security, $D(t)V(t) = \tilde{E}(D(T)V(T)|\mathcal{F}_s)$, the process $Z(t)D(t)V(t)$ is a martingale under $\mathbb{P}$. The martingale representation theorem can be used again,

$$Z(T)D(T)V(T) = V(0) + \int_0^t \Gamma(s)dB_s$$

And we can set $\Delta(t) = \frac{\Gamma_t}{\sigma S(t)\mathcal{Z}_t}$. Since the portofolio can only replicated by a unique strategy, this market model is complete.

The thermodynamic limit $T \to \infty$ can be discussed again. Indeed, again, we need the additional condition $\mathcal{L}_0[X] = 0$. the Radon-Nikodym process then reduces to,

$$\lim_{T\to\infty} Z(t) = \exp\left(-\frac{\alpha - r}{\sigma} B_t - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 t\right)$$

We summarize this in the corollary,

Corollary 27. *In the limit $T \to \infty$, a complete market model will always have in time uncorrelated increments of the logarithm of the stock.*

3.4 Conclusion

In this chapter, we discuss how Stochastic Matrix Product states can be used as a representation of the Radon-Nikodym derivative of processes defined on a filtration of another process. With this representation, we derive a simple equation for the time-evolution of the marginal distribution of the process. We showed how classical non-Markovian, classical and quantum processes can be embedded in a quantum Markovian dynamics. The properties of non-Markovian processes, such as ergodicity and mixing are therefore determined by the quantum dynamics.

Chapter 4

Signatures of Lattice Excitations in Quantum Channels: Limit of Parent Hamiltonians

Synopsis:

We prove that every injective Matrix Product State is the unique ground state of a simple hopping theory. We start by studying the low energy spectrum of parent Hamiltonians of injective Matrix Product States in a particular long range and system size limit under the validity of an asymptotic regime with low particle density. We show that in this limit a natural first quantization arises. This allows us to compute a specific type of low energy spectrum. This spectrum depends solely on the properties of a quantum channel, i.e. transfer matrix of the ground state, and not on any other details of the ground-state. We also review normal quantum channels for which the expression is more simplified. The construction possibly has some interesting uses for the study of quantum and classical Markov processes which we briefly expose. As an application, we revisit the notion of (many-body)-localization with our framework. Our calculations revealed that translational invariant Matrix Product States can be interpreted as a stationary sea of particles. As a next step rather than starting from some local Hamiltonian with random potentials, we consider fluctuations of the local tensors of a continuous one-parameter family of Matrix Product States. Localization in 1-dimension, is then understood from a simple study of spectral and mixing properties of finite dimensional quantum channels.

Based on:
B. Descamps, *New. J. Phys.*(submitted) *e-print* arXiv: *1508.07470* , (2015)

4.1 Introduction

The general study of strongly interacting quantum many body systems requires a large amount of parameters. It appears, however, that a successful description of such systems is still possible in a much smaller manifold. Matrix product states (MPS), originating from the density matrix renormalization group (DMRG) algorithm [96, 99] , have turned out to be one of these particular manifolds used to describe the ground state of these systems. Further study of the manifold laid down some ground work for studying lower energy properties [32, 33]. Some argumentation can be made for the mathematical and physical validity of the methods [34]. However, we argue at the beginning of this work that this method is heavily biased by the spectral properties of the transfer matrix of the ground state. While every numerical method has its own bias, this opens the question as to which condensed matter properties hide inside Quantum Channels, i.e. transfer matrix, or trace preserving completely positive operators.

Recalling Källén -Lehmann spectral representation in quantum field theory [52, 62], which connects 2-point correlations with free propagators, it is not completely unexpected that indeed the transfer matrix contains some information.

Apart from exactly solvable and integrable models or Lieb-Robinson bound arguments based on the structure of the dispersion relation [34], few models can be found with a clear finite-particle pictures for low excitations. In this work, we propose an interesting approach to this problem. We start from a system of size N with a dynamic described by a parent Hamiltonian of an injective matrix product state, with finite range L. We then study the limit of $N, L \to \infty$ and possible re normalisation of the transfer matrix $\Gamma(L)$ under the additional restriction that an asymptotic regime for low particle density remains valid.

Under this limit, a clear particle-picture appears, which is completely related to the transfer matrix. Even-more so these excitations do not depend on any other microscopic properties of the ground-state.

In 1992, following the seminal work by Affleck, Kennedy, Lieb and Tasaki [1], Fannes, et.al. [27] showed that Matrix Product States are the ground states of gapped local Hamiltonians. In this work, inspired by this construction and considering an additional limit, we show that not only they are the ground-states of gapped Hamiltonians, but also this Hamiltonian describes the hopping of bosons in one-dimension.

This work is divided into two distinct part. In the first part (4.2,4.3,4.4), we elaborate on the construction. The details of the main theorem (31) can be found the appendix. In the second part, we present some applications. First, we briefly describe some possible consequences and applications for the field of classical and quantum Markovian dynamics (4.5). Finally in the last part (4.6), we revisit localization using the framework and philosophy of the work presented in the first part.

4.2 From Källén -Lehmann spectral representation to the tangent plane of Matrix Product States

Gunnar Källén [52] and Harry Lehmann [62] discovered independently that the two point-correlation in quantum field theory can be related to the free propagators,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{d\mu^2}{2\pi}\rho(\mu^2)\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu^2+i\epsilon}e^{-ip(x-y)}$$

This is illustrated here for a scalar field theory, where $\rho(\mu^2)$ is a positive spectral density. For a free scalar field, one can verify that the Fourier transform of the two-point correlation function, is inversely proportional to the dispersion relation,

$$\int d^4x e^{ipx}\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \frac{i}{p^2-\mu^2+i\epsilon}$$

It is amazing to see how much information is stored simply inside the correlation function of the ground state of the system.

A first approach for studying low energy excitations in one-dimensional spin lattice makes use of the tangent plane of the ground-state MPS in the manifold [33]. For a translational invariant MPS $|\phi\rangle$,

$$|\phi\rangle = \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1}...A^{i_N}\right)|\vec{i}\rangle \tag{4.1}$$

A simple proposition for an approximation of the basis of the m-particle subspace are the states,

$$\begin{aligned} |\psi\{B_\alpha^{(i)},k_\alpha\}_{\alpha=1}^m\rangle &= \textstyle\sum_{n_1<...<n_m} e^{i\vec{k}.\vec{n}}|\phi\{B_\alpha^{(i)},n_\alpha\}\rangle \\ |\phi\{B_\alpha^{(i)},j_\alpha\}_{\alpha=1}^m\rangle &= \textstyle\sum_{\vec{i}}\mathrm{Tr}\left(A^{i_1}...A^{i_{j-1}}B^{i_{n_1}}A^{i_{j+1}}\ ...B^{n_{j_m}}A^{i_{j+m}}...A^{i_N}\right)|\vec{i}\rangle \end{aligned} \tag{4.2}$$

A sub-vector space can be considered, by choosing $B^{(i)} = A^{(i)}B$.

A particular approximation of the low energy spectrum is defined from the variational problem,

$$E_k = \min_B \sum_Z \frac{\langle\psi\{B,k\}|H_Z|\psi\{B,k\}\rangle}{\langle\psi\{B,k\}|\psi\{B,k\}\rangle},\ \ \langle\phi|\psi\{B,k\}\rangle = 0$$

In the case of frustration free Hamiltonians, such as parent Hamiltonians, $H_Z|\phi\rangle = 0, H_Z \geq 0$, both numerator and denominator scale as N. The numerator is then finite while the denominator is given by the Fourier transform of the transfer matrix (4.7),

$$E_k \leq \frac{C}{\mathrm{Tr}\left(B^\dagger\mathcal{T}_k[\Gamma][B]\right)},\ \ \mathrm{Tr}\left(\rho B\right) = 0$$

Generally maxima of the Fourier Transform of the 2-point correlation function, i.e. singular points in the Källén - Lehmann spectral representation, represent the lowest excited particles. Thus, we can argue that the tangent plane method is biased by the spectrum of the Fourier Transform of the Transfer Matrix Γ.

However, in the case that this particular ansatz is right, spectral properties of the transfer matrix, i.e. Fourier Transform, yields a certain amount of information about the spectrum of the Hamiltonian such as minima of the dispersion relation.

This particular observation leads us to the starting question of this work. Can we extract or make other predictions for the dispersion beyond this one-particle ansatz? Is it possible to construct a family of models for which the spectrum can be completely extracted from the transfer matrix?

We, indeed, manage to construct a local Hamiltonian for which excitations depend purely on a quantum channel, and of which the m-particle eigenstates can be written using the m-particle basis (4.2). The procedure is based on a limit. It will also be interesting to study some consequence when the limit slightly breaks down.

4.3 Rewriting Parent Hamiltonians

Parent Hamiltonians [27, 78] of Matrix Product States are defined as projectors onto the kernel of the vector space $\{\mathrm{Tr}\left(XA^{i_j}\ldots A^{i_{j+L}}\right)\}$. Uniqueness of the ground state is assured if the map $X \to \mathrm{Tr}\left(XA^{i_j}\ldots A^{i_{j+L}}\right)$ is injective.

In the case of our study, we notice that the following parametrization of the parent Hamiltonian, pictured in figure (4.1), is of use to us. Choose $C \in \mathcal{M}_{D^2}$, so that

$$\begin{aligned} H &= \textstyle\sum_j \left(\mathrm{Id} - h_{j,j+L}\right), \\ h_{j,j+L} &= \textstyle\sum_{\vec{\alpha},\vec{\beta}} \mathrm{Tr}\left(C\left[A^{\alpha_j}\otimes\overline{A}^{\beta_j}\right]\ldots\left[A^{\alpha_{j+L}}\otimes\overline{A}^{\beta_{j+L}}\right]\right)|\vec{\alpha}\rangle\langle\vec{\beta}| \end{aligned} \tag{4.3}$$

satisfies the parent Hamiltonian conditions. A simple choice is,

$$C^{(i_1,i_2)}_{(j_1,j_2)} = \left(\langle j_2 i_2|\Gamma^L|i_1 j_1\rangle^{i_1 i_2}_{j_1 j_2}\right)^{-1} \tag{4.4}$$

We must first point out a striking property of parent Hamiltonians which becomes apparent from equation (4.3) and figure (4.1). Let us make use of the ansatz equation (4.2) for the choice $B^{(i)}_\alpha = A^{(i)}_\alpha B$. The action of the parent Hamiltonian, as given by equations (4.3,4.4) maps the m-particles wave equations onto a subspace spanned by $1,\ldots,m$-particles states. Furthermore, and most importantly, the reader should see that the decomposition under the action, depends purely on the transfer matrix, Γ,

$$\Gamma = \sum_j \overline{A}^j \otimes A^j \tag{4.5}$$

No other information about the ground state or particular Hamiltonian details, is encoded inside the dynamical spectrum besides the transfer matrix. We will see that we can relate the minima of the dispersion with the spectrum of the transfer matrix in the vicinity of the unit circle.

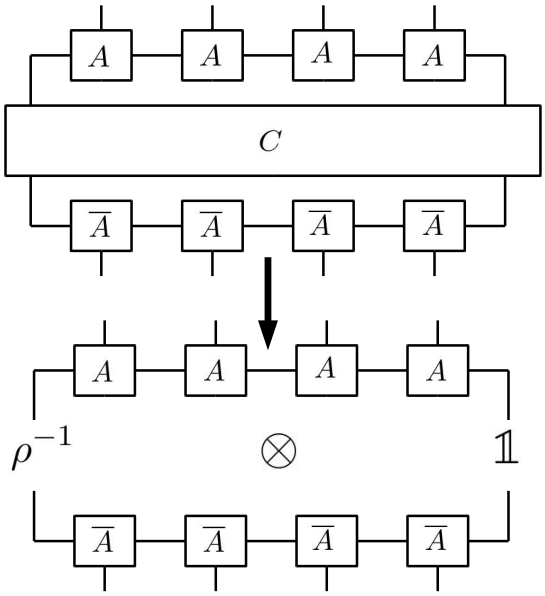

Figure 4.1: Some useful representation for the parent Hamiltonians of a Matrix Product State. The choice of the tweaking matrix C is not unique. A possible choice is given by equation (4.4). For this particular choice, the Matrix Product is an eigenvector with eigenvalue 1 of this interaction term. Furthermore, they are projectors. However, it is unlikely that some low energy spectrum can be solved generically for any of such representation. One notices that all these representations convergence towards $\rho^{-1} \otimes \mathbb{1}$ in the limit of large range and system size.

Limit, $N, L \to \infty$ Clearly the true excitations generally do not satisfy the ansatz (4.2). However, this is the case in the following limiting procedure. First of all, by a choice of gauge $A^{(i)} \to SA^{(i)}S^{-1}$, we can make sure that the transfer matrix is trace preserving with a diagonal fixed point,

$$\Gamma^*[\mathbb{1}], \;\; \Gamma[\rho] = \rho = \mathrm{diag}(\vec{\lambda}), \;\; \mathrm{Tr}(\rho) = 1$$

In the case of an injective MPS, it can be shown that the fixed point ρ is unique and non-degenerate [27, 78], even more so the spectrum always lies within the unit circle. Second, we take the thermodynamic limit $L < N \to \infty$, while varying the range of each local interaction of the parent Hamiltonian. We want a meaningful notion of quasi-particles. As we add more particles, in the generic case, the energy should increase with the number of particles. Only in special cases should bound states appear. A reasonable choice is $L/N \to 0$. It should be noted for this particular choice, there is still a notion of "local Hamiltonian" under rescaling of the metric with L, as shown in figure (4.2).

Finally, we could renormalise the transfer matrix $\Gamma(L)$. However, we would like the interaction term to forget the correction C, in equation (4.4). Indeed our choice mostly converges as $C \to \rho^{-1} \otimes \mathbb{1}$. This is not the case if the second largest eigenvalue converges towards the unit circle as $O(1/L)$. So, we must keep in mind that there are some restriction on the renormalization of the transfer.

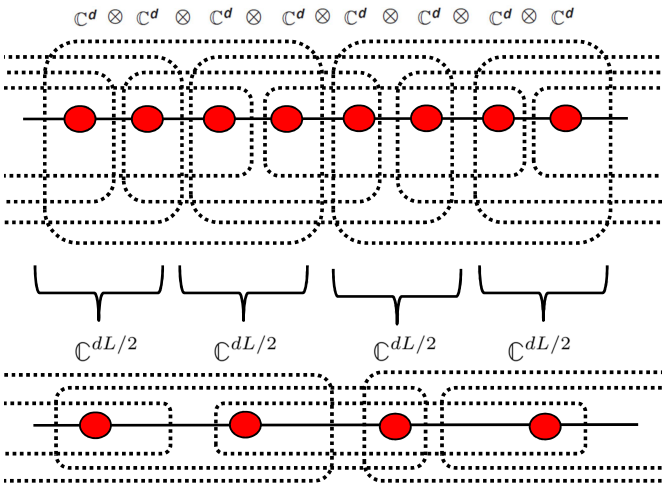

Figure 4.2: Since we take the limit $L \to \infty$, one might argue the physical relevance, as the Hamiltonian seems, at first sight, to lose its local property. By blocking sites, represented with red disks, and regrouping the interactions (dotted lines) we see however it is not the range of interactions which increases, but the dimension of the local Hilbert space (red disks in the second line). While the local Hilbert space grows in dimension, the interactions remain local, but unbounded, in the thermodynamic limit as $L, N \to \infty$.

As a side note as long as the spectrum of the renormalized super-operator stays within this $1 - O(1/L)$-circle, the martingale method remains valid and the Hamiltonian is gapped [27].

4.4 A Natural First Quantization: Hopping of Virtual Particles

4.4.1 One-particle excitations

We saw earlier that the subspace spanned by $1, \dots, m$-particles states remains stable under the dynamic. In the final result of theorem (31), we prove that they can be computed exactly within these subspaces.

In the introduction, we initially computed the energy using variational methods. As we desire to derive the whole spectrum, we seek an alternative approach. The goal is to demonstrate that for our limit, the states become exact eigenvectors of the Hamiltonian,

$$\frac{\|H|\kappa\rangle - \lambda|\kappa\rangle\|_2^2}{\langle\kappa|\kappa\rangle} \to 0 \to \text{ we set } |\kappa\rangle \text{ to be an eigenvector}$$

Let us first have a look at the limit without renormalization of the transfer matrix. We can propose the following representation for the one-particle states $|\psi\{X_a, k\}\rangle$ with momentum k,

$$|\psi\{X_a, k\} = \sum_n e^{ikn} \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1} \dots A^{i_{n-1}} A^{i_n} X_a A^{i_{n+1}} \dots A^{i_N}\right) |\vec{i}\rangle \tag{4.6}$$

In Lemma (29), we demonstrate that such states are indeed eigenstates, with energy that can calculated from an effective one-particle Hamiltonian $H^{(1)}_{\text{eff}}$. One of the central objects, contained in this effective Hamiltonian, is the Fourier Transform of the Transfer matrix $\mathcal{T}_k[\Gamma]$,

$$T_k[\Gamma] = R_\rho + e^{ik}\frac{\Gamma}{2(\mathbb{1} - e^{ik}\Gamma)} \circ R_\rho + R_\rho \circ e^{-ik}\frac{\Gamma^*}{2(\mathbb{1} - e^{-ik}\Gamma^*)} \tag{4.7}$$

This result is very evocative of the Källén -Lehmann spectral representation, [52, 62]. Hence, all information about the spectrum can be derived from the spectral analysis of $\mathcal{T}_k[\Gamma]$.

The following turns out to be extremely useful. One should notice the invariance of the one-particle states with momentum k under the gauge transformation,

$$A^{(i)}_a \to B^{(i)}_a = A^{(i)}X_a + e^{ik}Y_a A^{(i)} - A^{(i)}Y_a \tag{4.8}$$

However, this invariance breaks down when additional particles are taken into consideration. The results for the many-particle wave-function are examined in the next section. We should already announce that in the long range and large system limit, we derive the eigenstates using the philosophy of the Bethe ansatz [9]. Each eigenvector of the effective one-particle Hamiltonian, which is related to $\mathcal{T}_k[\Gamma]$, is a particle with momentum k. These particles can theb be combined in a m-particle wave function. The energy is then the sum of the individual energy of each particle,

$$H|\psi[\{X_1, \dots, X_k; k_1, \dots, k_m\}]\rangle \to \sum_j E[X_j, k_j]|\psi[\{X_1, \dots, X_k; k_1, \dots, k_m\}]\rangle \tag{4.9}$$

4.4.2 Breaking down of the Limit and Bound-States

One could try to look "beyond" this simple one-particle picture. Generally, there exists modes consisting of a superposition of scattering particles. These states consisting of more than a single particle, sometimes have a lower energy than the other stable individual particles.

The framework, we have presented here, gives us the perfect opportunity for studying this phenomenon and even more so derive how this is reflected in the spectral properties of the transfer matrix.

For this, we need to renormalize the transfer matrix $\Gamma[L]$ with the range L. The idea would be to let the ground state become "critical". As the spectrum gets closer to the boundary of the unit circle, some particles, i.e. eigenvectors of $\mathcal{T}_k[\Gamma]$, become unstable and splits into other particles.

Technically, the ansatz breaks down with an error ϵ_1, given by equation (4.26). The new excited state consists of the 1-particle state in superposition with its fusion options $\{Y_{a_1}, Y_{a_2}\} \to X_a$.

For simplicity, and as a sake of illustration, we look at normal unital trace-preserving completely positive operators,

$$\Gamma\Gamma^* = \Gamma^*\Gamma, \;\; \Gamma^*[\mathbb{1}] = \mathbb{1}, \;\; \Gamma[X_a] = \lambda_{X_a}X_a$$

The eigenvectors are orthonormal and characterized by the structure tensor f^a_{bc},

$$X_a X_b = \sum_c f^c_{ab} X_c$$

Normal unital quantum channels are particularly interesting here as the effective Hamiltonian is exactly the Fourier Transform of the Transfer matrix. The eigenvectors X_a of the transfer matrix, are finite dimensional representations of the particles with wave function $|\psi\{X_a, k\}\rangle$,

$$H|\psi\{X_a,k\}\rangle = \left(2L - 2\,\mathrm{Re}\,\frac{1}{1-e^{i[k-\phi_{X_a}]}|\lambda_{X_a}|}\right)|\psi\{X_a,k\}\rangle$$

with $\lambda_{X_a} = e^{-i\phi_{X_a}}|\lambda_{X_a}|$. Hence the phase of the eigenvalue of X_a is interpreted as its rest-momentum.

Continuing our discussion at the beginning of this short section, we let the part of the spectrum converge towards the unit circle. The rate of convergence becomes clear by comparing $\langle\psi\{\overline{X}_a,k\}|H|\psi\{X_a,k\}\rangle$ with the generic one-particle energy, $\Delta_a = 2L - 2\,\mathrm{Re}\,\frac{1}{1-e^{i\left[k-\phi_{X_a}\right]}|\lambda_{X_a}|} - \langle\psi\{\overline{X}_a,k\}|H|\psi\{X_a,k\}\rangle$,

$$\Delta_a \approx \sum_{\alpha,\beta}\frac{|f^\alpha_{a,\beta}|^2}{\sigma_{a,k}}\left|\sum_{j=1}^{L}|\lambda_{X_\alpha}|^j|\lambda_{X_\beta}|^{L-j}e^{i[k-\phi_{X_\alpha}+\phi_{X_\beta}]j}\right|^2$$

First, we see that $\Delta_a > 0$, and so decreases the energy.

When two particles have approximately the same eigenvalues, $|\lambda_{X_\alpha}| \approx |\lambda_{X_\beta}|$, and approaches the unit circle as $|1 - |\lambda_{X_\alpha}|\,| \approx \gamma(\ln L)/L$. The new contribution to the energy Δ_a peaks at $k = \phi_{X_\alpha} - \phi_{X_\beta}$ in the order of $L^{1-2\gamma}$. Additionally, the contribution is proportional to the discrepancy of the one-particle sub-space. Hence for $\gamma < 1/2$, the one-particle subspace is unstable under the dynamic. The state X_c, resulting from the fusion of X_α and X_β, has a finite probability of decaying to these two particles. The bound state $|\psi\{X_c,k\}\rangle + \sum_q c_q|\psi\{X_\alpha, k-q; X^\dagger_\beta, q\}$ emerges. This state has a minimum at $k = \phi_{X_\alpha} - \phi_{X_\beta}$.

Example We illustrate this for the transfer matrix with eigenvectors, $\mathbb{1}, \sigma^+, \sigma^-, \sigma^z$, with respective eigenvalues $1, \lambda_+, \lambda_-$ and λ_z. We choose $\lambda_+ = e^{i\phi}\lambda = \overline{\lambda}_-$, $\lambda_z = \lambda^2$. In the long range limit, the low energy spectrum is built by adding particles of the type σ^+, σ^- or σ^z to the wave function.

Choosing to renormalize the spectrum with the range L, yields a rescaling of the errors

$$\lambda(L) = e^{-\gamma\frac{\ln(L)}{L}} \to \epsilon_1(\sigma^+) = \epsilon_2(\sigma^+) \approx \frac{L^{1-2\gamma}}{\gamma\ln(L)}, \epsilon_1(\sigma^z) \approx 2\gamma\ln(L)L^{1-2\gamma}$$

For $\gamma = 1/2$, the 1-particle subspace remains stable for $|\psi\{\sigma^+,k\}\rangle$, $|\psi\{\sigma^-,k\}\rangle$, and we get the bound state,

$$|\psi^{(2)}\{Z,k\}\rangle = |\psi\{Z,k\}\rangle + \sum_q c_q^{(1)}|\psi\{\sigma^+,k-q;\sigma^-,q\} + c_q^{(2)}|\psi\{\sigma^-,k-q;\sigma^+,q\}$$

4.4.3 Solving the Low Energy Spectrum in the Long Range Limit

In 1931, Hans Bethe [9] initiated a new technique which ultimately lead to a resolution of the Heisenberg spin-chain. His method, which was later given his name, the coordinate Bethe ansatz. In his paper, Bethe introduced n-particles wavefunctions of the form,

$$\psi(\vec{k}) = \sum_{0\leq x_1\leq\ldots\leq x_n\leq N}\sum_{P\in\mathcal{S}_n} A(P)e^{i(k_{P_1}x_1+\ldots+k_{P_N}x_N)}\phi(x_1,\ldots,x_n)$$

where $\phi(x_1,\ldots,x_n)$ is a wave-function in first quantization of spins at positions $x_1,\ldots,x_n$.

Such plane-wave decomposition needs to satisfy a consistency condition,

$$A(PT_j) = S(e^{ik_{P_j}},e^{ik_{P_{j+1}}})A(P)$$

with T_j the transposition operator of j and $j+1$. Bethe's intuition is based on the idea that such many-body problems should in the most, yet non-trivial, case only depend on the scattering of two particles. Such reduction of many-particle scattering is intertwined with the Yang-Baxter equation,

$$S(u_1,u_2)S(u_3,u_1)S(u_2,u_3) = S(u_3,u_2)S(u_1,u_3)S(u_2,u_1) \tag{4.10}$$

This framework turns out to be the most natural for solving the many-particle eigenstates.

First of all, one should wonder what the dynamic actually does. Lemma's (29) offers us an elegant picture. Under the action of an interaction term, a particle will hop to another site at a distance $< L$. After hopping, this particle is split as a new superposition of other particles. This transformation is determined by the Transfer matrix and the hopping distance. Secondly, we ask how the interaction affects multiple particles simultaneously. When all particles are at the large distance, greater than L, from each other, this case never happens. This begs to question whether we should actually care about such case. We prove in Lemma (30), that, indeed, we should not do as such. For such asymptotic regime to be valid, it is imperative that the main weight of the wave-function consists of particles far apart. This is an additional restriction on the allowed density state, meaning that it should be relatively low as it can be found in more details in the statement of the Lemma.

Which particles should we choose? In the one-particle section, we learned that the states are invariant under the gauge transformation (4.8). The best choice turns out to be,

$$B_a^{(i)} = A^{(i)}X_a + e^{-ik_a}Y_aA^{(i)} - A^{(i)}Y_a,\ Y_a = (\mathrm{Id} - e^{-ik}\Gamma^*)^{-1}[X_a]$$

where $B_a^{(i)}$ should be interpreted as a particle with momentum k_a.

Generally, it turns out that we cannot find the whole emergent spectrum. However, due to the asymptotic regime and the frustration freeness of the Hamiltonian, the symmetric states emerge as eigenstates. The coefficients $A(P)$ satisfy,

$$A(T_{\alpha,\beta}P) = A(P)$$

for the transpositions $T_{\alpha,\beta}$. These states represent the particles as hopping bosons.

As a consequence of the Bethe Ansatz, the energy of a m-particle state is the sum of the individual energies,

$$E[X_{a_1}, k_1; \ldots; X_{a_n}, k_m] = \sum_j E[X_{a_j}, k_j]$$

This implies the scaling of the energy with the particle number, $m \times 2L$. Because in physics only differentials of the energy matters, a particle conservation is a consequence of this scaling.

Strangely enough, we can solve the whole energy spectrum for normal unital quantum channels. Recalling our past discussion about the one-particle spectrum, we saw that there were different particle types for each momentum. The momentum was an additional label of the particle type. This is not the case for normal unital quantum channels. Thus there are at most $D^2 - 1$ different particles, including their anti-particles. An interesting consequence is that we are able to solve the low energy spectrum exactly. Utilizing the parity symmetry, the spectrum is thus simply any symmetric and antisymmetric superposition. Hence,

$$A(T_{\alpha,\beta}P) = \pm A(P)$$

This complete results are stated formally in Theorem (31) and its corolary (32) with their proofs in this appendix, which we summarize modestly as,

Every Injective Matrix Product State is the vacuum of a Hopping Theory

4.5 An Emerging Virtual State Space for Stochastics

The main subject of this work is centralized around properties of some local Hamiltonian. Our tensorial notation of parent Hamiltonians (4.1) showed us that natural local sub-algebra's appear in the virtual space of the tensor network. Furthermore, these were related only and solely to the transfer matrix. We even showed that the low energy spectrum was exactly solvable, thereby interpreting some generators of these sub-matrix-algebras as physical particles. This point of view offers particularly interesting research directions in the field of stochastics, quantum and classical Markov processes. One could label each of these virtual particles and their position inside the network. As we showed earlier, under the action of the Hamiltonian, these particles are annihilated at the site of the action and new particles are created at the boundary of the interaction. If it turns out that such actions happens with a positive probability, then we can talk about a Markov process on this particular state space Ω. We briefly expose some

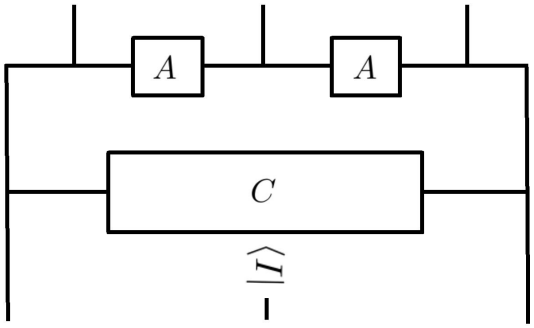

Figure 4.3: A tensorial representation of the famous 1-dimensional Glauber dynamics often studied for Ising.

applications for the Glauber dynamics, and leave the possibilities open for future work in this direction.

In 1953, an algorithm was presented for calculating a particular canonical ensemble. This method of computation grew to be known as the Metropolis-Hastings [68] algorithm. This algorithms is based on a Markov chains which takes random samples from a probability distribution. A famous use of this algorithms is for studying the gibbs state of Ising model,

$$\rho_\beta = \frac{e^{\beta H}}{\mathrm{Tr}\, e^{\beta H}}, \;\; H = \sum_j s_j s_{j+1}, \;\; s_j \in \{+1, -1\}$$

The Matrix Product State representation of the one-dimensional Ising model is given by the tensor,

$$A^{(i)}_{ab} = \delta_{i,a} A_{a,b}, \;\; A = \frac{1}{e^\beta + e^{-\beta}} \begin{pmatrix} e^\beta & e^{-\beta} \\ e^{-\beta} & e^\beta \end{pmatrix}$$

The most simple Markov chain one can think of is the so-called Glauber dynamics in the state space of spin. At each time step, one flips a spin with a probability computed from the configuration of its neighbour. It should be obvious that the generators of such dynamics are local and actually related to a parent-Hamiltonian of some Matrix Product State. Therefore, the discussion of this work should offer some interesting new perspectives on this very well-known topic. On can check that the local generator, $\mathcal{L} = \sum_j \mathcal{L}_{j-1,j,j+1}$, $\mathcal{L}_{j-1,j,j+1} = T_{j-1,j;j+1} - \mathbb{1}$ with $T_{j-1,j;j+1}$ given in figure (4.3), is the generator of such Glauber dynamics. In this case, C is given by the following equation,

$$C \circ A^2 = \mathbb{1}, \;\; (A \circ B)_{ij} = A_{ij} \dot{B}_{ij}$$

with the operation -$\circ$ being the Hadamard product. In the spirit of the main discussion, one can find that Pauli matrix Z is one-and only natural virtual particle. Therefore, we get a new state space by using the first quantization language with such particles. Writing the state,

$$\phi(Z^{(a_1)},\dots,Z^{(a_N)}) = \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1} Z^{(a_1)} \dots A^{i_N} Z^{(a_N)}\right) |\vec{i}\rangle, \;\; a_j \in \{0,1\}$$

similarly to the space of spin configurations $\sigma \in \{+1,-1\}^N$, we have the possibility to define a state space which mark the presence of a particle Z at the various sites. As promised this new state space, is closed under the dynamics, as one can verify that,

$$T_{j-1,j,j+1}\phi(Z^{(a_1)},\dots,Z^{(a_N)}) = \sum_{b_{j-1},b_{j+1}} \tau^{a_{j-1},a_j,a_{j+1}}_{b_{j-1}b_{j+1}} \phi(Z^{(a_1)},\dots,Z^{(b_{j-1})},\mathbb{1},Z^{(b_{j+1})},\dots,Z^{(a_N)})$$

and,

$$\tau^{a_{j-1},a_j,a_{j+1}}_{b_{j-1},b_{j+1}} = \frac{1}{4}\mathrm{Tr}\left(A^T Z^{a_j} A^T Z^{a_{j-1}+b_{j-1}} C Z^{a_{j+1}+b_{j+1}}\right)$$

One sees that the expression $A^T Z^{a_j} A^T$ is symmetric when $a_j = 0$ and anti-symmetric for $a_j = -1$. This implies that there will be no creation of additional particle Z, but only a hopping to the left or the right with possible annihilation if another particle is already present,

$$\tau^{a_{j-1},1,a_{j+1}}_{a_{j-1}+1,a_{j+1}} = \tau^{a_{j-1},1,a_{j+1}}_{a_{j-1},a_{j+1}+1}$$

One dimensional Glauber dynamics are known to be exactly solvable. This small calculation yields us the famous relations for the dynamical correlation function,

$$C_{n-m}(t,s) = \langle s_n(t) s_m(s)\rangle, \partial_t C_n(t,s) = -C_n(t,s) + \lambda(C_{n-1}(t,s) + C_{n+1}(t,s))$$

This first quantization, which appears naturally from this notation of the parent Hamiltonians (4.1),(4.3), presents us an interesting new state space. The stochastic dynamics, of which the tensor network is a fixed point, acts an error corrector. The virtual particles are moved around, created and annihilated a certain rate, while the whole converges towards the vacuum, i.e. the Matrix Product State without any particles added to the virtual space.

4.6 Revisiting Localization

In 1958, Anderson [5] presented the results of his study of diffusion in random lattices. He showed that for a particular form of a Hamiltonian consisting of random potentials, when the fluctuations were sufficiently large, diffusion would cease. The result first considered as a strange mathematical trickery, grew to a large field, and better understanding of insulation-conductor transition, such as Mott transition [93].

50 years later, a part of the research community focussed their attention on the so-called Tensor Networks [27, 78]. While much richer phases were discovered beyond Landau, the philosophy of local order parameter were shifted towards variation of local tensors [31]. Furthermore, Gross-Pitaevski equations have been generalized for such states [32]. Time-evolution can be understood as a transport along a Tensor-Network manifold.

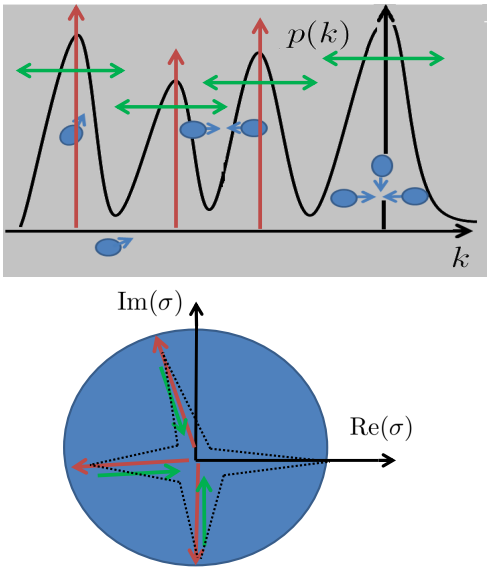

Figure 4.4: $|\mathrm{MPS}\rangle$ is a sea consisting of large number particles scattering with one-another. The distribution of the momentum has a finite number of peaks corresponding to the phase of the largest eigenvalues of the transfer-matrix. Diffusion (red) decreases the variance of the peaks, while localization increases it. On the level the eigenvalues of the transfer matrix, this corresponds to respectively pushing the eigenvalues towards unit circle and the center.

Even-though localization is a dynamical property, we show in this work that by combining both ideas presented above, the same phenomena is realised on the level of the manifold of Matrix Product States. The dynamic is exchanged with variation of the local tensors in the manifold, on top of which we consider some random fluctuations.

A very elegant intuitive interpretation of MPS leads us to this insight. In the first part of this work, we showed that the Fourier Transform of the transfer matrix of a Matrix Product, contains information about low-energy particles of a local Hamiltonian. Therefore, we propose that a translational invariant MPS can be established to be sea of scattering particles, from which the momentum-distribution has a finite number of peaks, as seen in figure (4.4),

$$|\mathrm{MPS}\rangle = \sum_{k,m} |m(0)\rangle + |(m(k) + m(-k))\rangle + |(m(k_1) + m(k_2) + m(k_3))|k_1 + k_2 + k_3 = 0\rangle \tag{4.11}$$
$$+ \ldots$$

This interpretation of MPS becomes more transparent, when trying to explain the transition from diffusion to localization as we do in section (4.8). As shown in [101], by studying the z-transform of 2-point-correlation functions we can probe the weight of various momenta. In some sense, it means that the phase of an eigenvalue is related to the momentum of some set of particles in the sea. We also argued earlier that the momentum densities $p(k)$ are related to the eigenvalues of the transfer matrix of the MPS, and this was exact for normal transfer matrices.

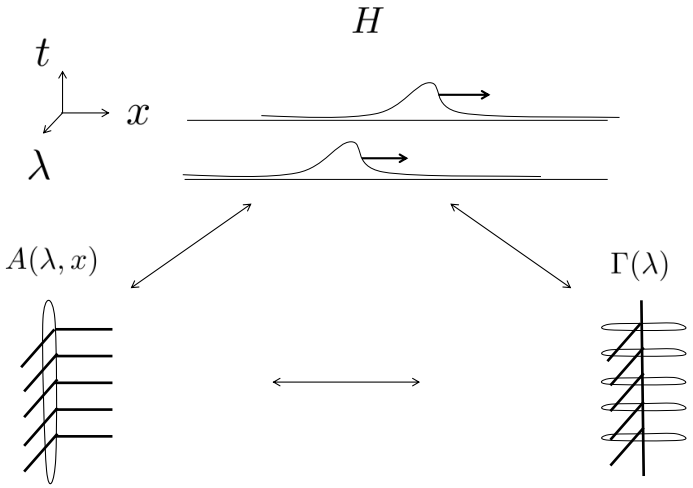

Figure 4.5: Simple illustration of the different mapping $1+1+0 \leftrightarrow 0+1+1 \leftrightarrow 1+0+1$.

4.7 Set-up

Consider a continuous one-parameter family of Matrix Product States, $\{B^i[s] = A^{(i)}[s](\lambda) + W^{(i)}[s]\}$,

$$|\psi\{B^i[s]\}\rangle = \sum_{\vec{i}} \mathrm{Tr}\left(B^{i_1}[1]...B^{i_1}[N]\right)\sigma^{+i_1}...\sigma^{+i_N}|\Omega\rangle \tag{4.12}$$

As presented earlier the parameter $\lambda \in I$ is the equivalent of the time-evolution, in the sense of quasi-adiabatic evolution [76]. The extra terms in the tensors $W^{(i)}[s]$ are taken randomly from some measure $d\mu(W^{(i)}[s])$.

We need now to make the claimed equivalence more precise. Let us start with the time evolution of the density. The creation of a particle at site 0, followed by a measurement the outcome at site n, can be calculated from the correlation function. As we see below this can be related to the transfer matrix of the MPS-family $\Gamma(\lambda) = E_\mu\left(\sum_j B^j[s] \otimes \overline{B}^{(j)}[s]\right)(\lambda)$,

$$\begin{aligned} |\psi(0,n)|^2 &\propto |\langle O_0 M_n\rangle - \langle O_0\rangle\langle M_n\rangle| \\ &\propto \tfrac{|\langle l[O]|[\Gamma^n - \Gamma^\infty]|r[M]\rangle|}{\sqrt{\langle l[O]|l[O]\rangle}\sqrt{\langle r[M]|r[M]\rangle}} \\ &\leq \|\Gamma^n(\lambda) - \Gamma^\infty(\lambda)\|_1 \end{aligned}$$

Anderson studied the time-evolution of the probability distribution, $|\psi(0,n,t)|^2$ of a particle initially at the origin and evolving under a dynamic described by a Hamiltonian $H = \tilde{H} + \sum_x W(x)$. He,[5, 93], showed the amazing result that in one and two dimension(s) for finite

magnitude of fluctuations, such distribution remains localized in the sense that,

$$\sum_n |n| \, |\psi(0,n,\lambda)|^2 < C$$

As argued above, we can now argue localization when studying the function $\Xi(\lambda, W)$,

$$\sum_n |n| \, |\psi(0,n,\lambda)|^2 \leftrightarrow \Xi(\lambda) = \frac{\sum_n |n| \, \|\Gamma^n(\lambda) - \Gamma^\infty(\lambda)\|_1}{\sum_n \|\Gamma^n(\lambda) - \Gamma^\infty(\lambda)\|_1}$$

for some measure of magnitude of the fluctuations W.

The next step, is for us to properly connect λ with time t and the diffusion. While λ maybe defined on a small interval, time always goes to infinity. Therefore, in the same spirit as the quasi-adiabatic evolution [76], time must be varied much more slowly when varying $\lambda(t)$. Additionally, the precise relation $\lambda(t)$ is fixed by the property of diffusion of the family of MPS,

$$\Xi(\lambda(t), M = 0) \leq D\sqrt{t}$$

The set-up is summarized in the table below,

Dynamical	Tensor Network
$W(x)$	$W^{(i)}[x]$
t	$\lambda(t) : \Xi(\lambda(t), W = 0) \leq D\sqrt{t}$
$U(t) = \exp(itH)$	$A^{(i)}(\lambda) + W^{(i)}[x]$
$\|\psi(0,n,t)\|^2$	$\frac{\|\Gamma^n(\lambda) - \Gamma^\infty(\lambda)\|_1}{\sum_n \|\Gamma^n(\lambda) - \Gamma^\infty(\lambda)\|_1}$

4.8 Localization

Finally, we can proceed and describe localization. While proving Anderson's localization is extremely challenging, this mapping onto a Tensor Network-picture gives a fresh simplistic view of the problem. The following explanation is illustrated in figures (4.4), and (4.5). Let us bring our insight of the translational MPS come to play. The reason for diffusion in the first place is that the leading terms of the wave function is some wave-packet. This is compatible with our picture as this means that the distribution $p(k)$ of momentum has sharp peaks around a finite number of momenta. Localization arises, when leading terms of the wave-function consists of a large number of different wave-packets with different momentum. A translational invariant MPS has zero-momentum. In the case of a sea of localized states, the distribution $p(k)$ becomes more and more peaked around the origin. Therefore no particles are scattered as, so they are all localized.

Let us translate the sea-picture, 0+1+1, for the transfer-matrix, i.e. 1+0+1. We saw in the previous section that the probability distribution $|\psi(0,n,t)|^2$ is related to the spectrum of the transfer matrix Γ.

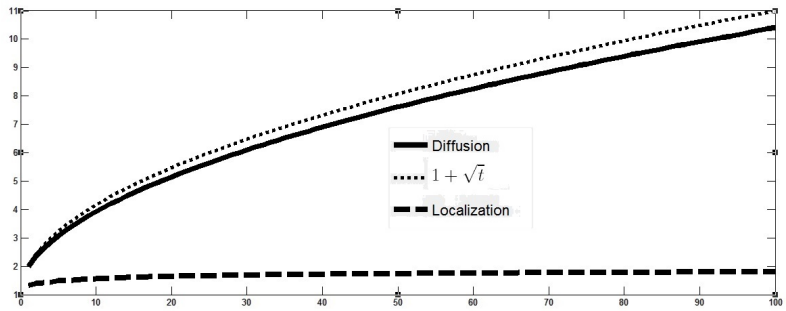

Figure 4.6: Plot of $\Xi(\lambda)$ vs t for the example (28) with $N = 100$ and $W = 1$.

The probability distribution $p(k)$ can in principle be studied through by doing some tomography and next studying the z-transform of the 2-point correlation function. Keeping this idea in mind, it is acceptable to see the phase of the eigenvalues of the transfer matrix, as the momentum of a set of particles in the MPS-sea. Clearly then diffusion arises when the eigenvalues are close to the unit circle. Next, local fluctuations increase the variance of the peaks. This, again, can be understood from the spectrum, as this means that the eigenvalues are shifted towards the origin. The spectrum of generic transfer matrices have often a star-shape. This is mathematically understood for completely positive operators, as the eigenvalues on the unit circle are a representation of some abelian group [27, 78]. This fits perfectly with the interpretation of the MPS as a stationary sea. The total momentum is zero, and so the sum of the phase must be zero, as it is for any non-trivial one-dimensional representations of abelian groups. In the extreme case of localization all eigenvalues are zero except one. remaining eigenvalue, which is 1, is the zero-momentum of the sea.

Let us illustrate this with an example,

Example 28. *Consider the local tensors,*

$$\begin{aligned} A^{(-1)}(\lambda) + W^{(-1)} &= \left(\sqrt{1-\lambda} + w_1/\sqrt{2}\right)\sigma^+, \\ A^{(+1)}(\lambda) + W^{(+1)} &= \left(\sqrt{1-\lambda} + w_1/\sqrt{2}\right)\sigma^- \\ A^{(0)}(\lambda) + W^{(0)} &= \left(\sqrt{\lambda} + w_1/2\right)\sigma^z + w_2/2\mathbb{1} \end{aligned}$$

with $w_j \sim N(0, W)$ and mutually independent for some finite variance W. Here we take $\lambda \in [1/2, 1]$, and see that AKLT belong to the family for $\lambda = 2/3$. By choosing the right function $\lambda(t) : [1, \infty] \to [1/2, 1] : t \to \frac{\sqrt{t}}{1+\sqrt{t}}$, we can plot $\Xi(\lambda, M)$ in figure (4.6). A simple calculation can verify,

$$\Xi(\lambda(t), W = 0) \leq C'(1 + \sqrt{t}), \;\; \Xi(\lambda(t), W = 1) \leq C$$

As promised, we see in figure (4.6) that our choice $\lambda(t)$ yields diffusion. Similarly to Anderson's result, a finite magnitude of the fluctuation leads to localization.

As mentioned earlier, localization is especially interesting for studying mott insulators. Hence, we need to be able to extend the result to fermions. This can of course be done for Fermionic MPS [60], by replacing $\sigma^{+i_k} \leftrightarrow \psi^{\dagger i_k}$ with fermionic creation operators. The transfer matrix may vary depending on the local observable, $\Gamma = A^0 \otimes \overline{A}^0 + A^1 \otimes \overline{A}^1$ or $\Gamma = A^0 \otimes \overline{A}^0 - A^1 \otimes \overline{A}^1$. The rest of the set-up is then the same, this time however, we need to choose transfer matrix with the largest $\Xi(\lambda, M)$.

Statements and Proofs of Lemma's and Theorem

The technical details of the limit are presented in this section. We start from a one-dimensional spin chain of size N with interactions of range L. We derive a few convergence rates for $L \to \infty$ for the low particle number eigenstates and their energy under a possible choice of renormalisation of the transfer matrix $\Gamma(L)$.

We need to compute the the action of the Hamiltonian onto a m-particle subspace. As, we see a very natural new basis appears in the virtual space. However, this basis has to be orthogonalized. We, therefore, will have to get some information about the Gram-matrix of each subspace.

First, we start our discussion with results about the one-particle states. In Lemma (29) we demonstrate that the particle have a simple finite-dimensional representation closely related to a quantum channel. Next, when combining these particles to construct multi-particle states, it can be assumed that the particles are very far from each other. Since the dynamic is frustration free, this implies that the theory is non-interactive, and the total energy is the sum of the energies of the individual particles.

Lemma 29 (One-particle energy)**.** *Given a Matrix Product State $|\phi\rangle$, the one-particle states with momentum k, $|\psi\{X_a, k\}\rangle$, given by,*

$$\begin{aligned} |\psi\{X_a, k\} &= \sum_n e^{ikn} |\phi\{X_a, n_a\}\rangle \\ |\phi\{X_a, n\}\rangle &= \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1} \dots A^{i_{n-1}} A^{i_n} X_a A^{i_{n+1}} \dots A^{i_N}\right) |\vec{i}\rangle \end{aligned} \tag{4.13}$$

are eigenvectors of the local Hamiltonian $H(L)$, (4.3) with as choice for the tweaking matrix C the relation (4.4),

$$\frac{\| H|\psi[X_a, k]\rangle - E[X_a, k]|\psi[[X_a], k]\rangle\|_2^2}{\langle\psi[\overline{X_a}, k]|\psi[X_a, k]\rangle} \to 0$$

The values $E[X_a, k]$ are the solutions of the finite-dimensional eigenvalue problem of the matrix,

$$\left(h_{(ab)}\right) = \frac{\mathrm{Tr}\left(\rho^{-1}\mathcal{T}_k[\tilde{X}_a^\dagger]\mathcal{T}_k[\tilde{X}_b]\right)}{\mathrm{Tr}\left(\tilde{X}_a^\dagger \mathcal{T}_k[\tilde{X}_a]\right)}, X_a = \sum_b U_{a,b}\tilde{X}_b,\ U^\dagger U = \mathbb{1}$$

The basis vectors $\tilde{X}_a$ are the eigenvalues of the Fourier transform of the transfer matrix (4.7) ,

$$\mathcal{T}_k[\tilde{X}_a] = \lambda_a \tilde{X}_a$$

For momentum $k = 0$, there is the additional constraint of orthogonality onto the vacuum,

$$Q_\rho^* \circ \mathcal{T}_k \circ Q_\rho[\tilde{X}_a] = \lambda_a X_a,\ Q_\rho[.] = [.] - \mathbb{1}\,\mathrm{Tr}\,[\rho .]$$

Proof. By our choices given by the equations (4.3) and (4.4), the Hamiltionan $H(L)$ is local and frustration free. Let us fix the momentum k. Our first step is to show that the one-particle subspace spanned by states of the type $|\psi\{X_a, k\}\rangle$ is closed under the action of $H(L)$, in the limit $L \to \infty$. Knowing this, we can proceed and find an orthogonal basis for this subspace, and compute the effective Hamiltonian.

A look at the choice of corrections for $H(L)$ to be frustration free, yields that in the limit $L \to \infty$ all these corrections should converge,

$$C \to \rho^{-1} \otimes \mathbb{1},\ L \to \infty$$

Using the frustration freeness of the Hamiltonia we can bound this first error,

$$\left|\left| H|\psi\{X_a, k\}\rangle - \sum_{j, j \le n \le j+L} e^{ikn} \tilde{H}_{j,j+L} |\phi\{X_a, n\}\right|\right|_2 \le \epsilon_1(L)$$

where the correction term of the interactions terms $\tilde{H}_{j,j+L}$ are given by $\rho^{-1} \otimes \mathbb{1}$. This error typically decays as $O(D^2|\lambda|^L)$, with λ the second largest eigenvalue of the transfer matrix Γ.

This simplifications leads to the actions,

$$\begin{aligned}\sum_{j, j \le n \le j+L} e^{ikn} \tilde{H}_{j,j+L} |\phi\{X_a, n\}\rangle &= 2L|\psi\{X_a, k\}\rangle - 2|\psi\{\mathcal{H}[X_a], k\}\rangle \\ &\quad - \sum_{j,\alpha,\beta} e^{ikj} |\phi\{e_\alpha, j, e_\beta, j+L\}\rangle\, \mathrm{Tr}\left(e_\alpha^\dagger \mathcal{C}\{X_a, k\}[e_\beta]\right) \qquad (4.14)\end{aligned}$$

with,

$$\begin{aligned}\mathcal{H}_k[.] &= R_{\rho^{-1}} \circ \mathcal{T}_k^{(L)}[.], \\ \mathcal{C}\{X_a, k\}[.] &= \sum_{j=1}^{L-1} eikj R_{\rho^{-1}} \circ \Gamma^j \circ \tilde{L}_{X_a} \circ \Gamma^{L-j}[.], \\ R_{\rho^{-1}}[.] &= [.]\rho^{-1},\ \ \tilde{L}_X[.] = Q_\rho[X_a Q_\rho[.]]\end{aligned}$$

An important super-operator is the Fourier Transform which is here approximated by,

$$\mathcal{T}_k^{(L)}[\Gamma] = R_\rho + \frac{1}{2}\left(\sum_{n=1}^{L-1} \Gamma^n \circ \mathcal{R}_\rho\, e^{ikn} + \mathcal{R}_\rho \circ \Gamma^{*n}\, e^{-ikn}\right)$$

The first line in the equation (4.14), is an effective actions on the finite-dimensional subspace spanned by X_a. However, the states $|\psi\{X_a, k\}\rangle$ are not necessarily orthonormal. This is sorted out by computing the Gram-matrix of the $1-$particle subspace,

$$G_k^{(1)}(X_a, X_b) = N\langle\psi[\overline{X_a}, k]|\psi[X_b, k]\rangle = \mathrm{Tr}\left(X_a^\dagger \mathcal{T}_k^{(N)}[\Gamma][X_b]\right)$$

The super-operator $\mathcal{T}_k[\Gamma]$ is hermitian and has thus orthonormal eigenvectors. Using the eigenvectors, we can construct the effective one-particle Hamiltonian presented in the statement of the Lemma. One should note that the identity particle $X_a = \mathbb{1}$ with momentum $k = 0$ is nothing but the ground-state. However, the particle is not an eigenvector for $k \neq 0$. As all one-particle states need to be orthogonal onto the vacuum state, for $k = 0$, the additional constraint for $k = 0$ has to be imposed,

$$\langle\psi|\psi[X_b, k]\rangle = \mathrm{Tr}\left(\rho X_b\right) = 0$$

We should finally discuss the error of the action,

$$\begin{aligned}
&\frac{\|H|\psi[X_a, k]\rangle - \lambda|\psi[[X_a], k]\rangle\|_2^2}{\langle\psi[\overline{X_a}, k]|\psi[X_a, k]\rangle} \\
&= \frac{N}{N\,\mathrm{Tr}\left(X_a^\dagger \mathcal{T}_k^{(N)}[\Gamma][X_a]\right)} \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2,j} \langle\phi[e_{\alpha_1}, 0, e_{\beta_1}, L]|\phi[e_{\alpha_2}, j, e_{\beta_2}, j+L]\rangle \\
&\overline{\mathrm{Tr}\left(e_{\alpha_1}^\dagger \mathcal{C}\{X_a, k\}[e_{\beta_1}]\right)}\,\mathrm{Tr}\left(e_{\alpha_2}^\dagger \mathcal{C}\{X_a, k\}[e_{\beta_2}]\right)
\end{aligned}$$

Typically, this error dies off as $O(D^2|\lambda|^L)$. As all the errors are being reduced exponentially with L without any dependence on the system size, the claim of our lemma is proven.

□

In the theory of scattering one compares the change of the wavefunction of two particles initially infinitely far from each other, with the resulting outcome after infinite time when they are again far apart. The total energy of the system, since both particles are uncorrelated in both cases, is the sum of the individual energy. This additive feature of the energy appears regularly in exactly solvable models or even integrable ones. The particles simply fill in some Fermi sea.

We prove in the next lemma, that this is also exactly what happens for parent Hamiltonians in the long range and large system size limit. The intuition is that the action of the Hamiltonian onto a m-particle wavefunction, approximately only evaluates the energy when the particles are very far apart. Clearly for such asymptotic regime to be valid, particles need enough empty space. Thus, there is a restriction on the particle number which scales with L.

Lemma 30 (Asymptotic Regime at Low Densities)**.** *For a m-particle state with particles satisfying the conditions of Lemma (29),*

$$|\psi\{X_\alpha,k_\alpha\}_{\alpha=1}^m\rangle = \sum_{n_1<\ldots<n_m} e^{i\vec{k}.\vec{n}}|\phi\{X_\alpha,n_\alpha\}\rangle$$
$$|\phi\{X_\alpha,n_\alpha\}_{\alpha=1}^m\rangle = \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1}\ldots A^{i_{n_1}}X_1A^{i_{n_1+1}}\ldots A^{i_{n_m}}X_mA^{i_{n_m+1}}\ldots A^{i_N}\right)|\vec{i}\rangle \quad (4.15)$$

the action of the Hamiltonian takes into consideration the individual particles independently from the others,

$$\begin{aligned} H|\psi\{B_\alpha^{(i)},k_\alpha\}_{\alpha=1}^m\rangle \quad &\approx 2mL|\psi\{B_\alpha^{(i)},k_\alpha\}_{\alpha=1}^m\rangle + 2|\psi\{\mathcal{H}[X_1],k_1;X_2,k_2;\ldots;X_m,k_m\}\rangle \\ &\quad + \ldots + |\psi\{X_1,k_1;\ldots;X_{m-1},k_{m-1};\mathcal{H}[X_m],k_m\}\rangle \end{aligned}$$

The error scales typically as $O\left(\sum_{n=1}^m (m-n+1)\begin{pmatrix} L \\ n\end{pmatrix} N^{-n}D^{2(n+1)}\right)$

Proof. The proof is more cumbersome in its notation than complex. As a first illustration, let us first look at what happens for 2 particles-basis states. One can verify,

$$H|\psi[X_{a_1},k_1;X_{a_2},k_2]\rangle = (4L|\psi[X_{a_1},k_1;X_{a_2},k_2]\rangle \quad (4.16)$$
$$-|\psi[\mathcal{H}_{k_1}[X_{a_1}],k_1;X_{a_2},k_2]\rangle - |\psi[X_{a_1},k_1;\mathcal{H}_{k_2}[X_{a_2}],k_2]\rangle) \quad (4.17)$$
$$-|\psi[\tau_{k_1}[X_{a_1}]X_{a_2},k]\rangle - |\psi[X_{a_1}\tau_{k_2}^*[X_{a_2}],k]\rangle \quad (4.18)$$
$$-\sum_{\alpha,\beta}\sum_{0\le j_1<j_2,|j_1-j_2|>L} e^{i[k_1j_1+k_2j_2]}|\phi[e_\alpha,j_1;e_\beta,j_1+L;X_{a_2},j_2]\rangle \,\mathrm{Tr}\left(e_\alpha^\dagger\mathcal{C}\{X_{a_1},k_1\}[e_\beta]\right) \quad (4.19)$$
$$-\sum_{\alpha,\beta}\sum_{0\le j_1<j_2,|j_1-j_2|>L} e^{i[k_1j_1+k_2j_2]}|\phi[X_{a_1},j_1;e_\alpha,j_2,e_\beta,j_2+L]\rangle \,\mathrm{Tr}\left(e_\alpha^\dagger\mathcal{C}\{X_{a_2},k_2\}[e_\beta]\right) \quad (4.20)$$
$$-\sum_{j,\alpha,\beta} e^{i(k_1+k_2)j}|\phi[e_\alpha,j,e_\beta,j+L]\rangle \,\mathrm{Tr}\left(e_\alpha^\dagger\mathcal{C}^{(2)}\{X_{a_1},k_1;X_{a_2},k_2\}[e_\beta]\right) + \epsilon_* \quad (4.21)$$

with,

$$\mathcal{C}^{(2)}\{X_{a_1},k_1;X_{a_2},k_2\}[.] = \sum_{0<j_1<j_2\le L} e^{i[k_1j_1+k_2j_2]}R_{\rho^{-1}}\circ\Gamma^{j_1}\circ\tilde{L}_{X_{a_1}}\circ\Gamma^{j_2-j_1}\circ\tilde{L}_{X_{a_2}}[.],$$
$$\tau_k[.] = \sum_{j=1}^L e^{-ikj}\Gamma^j[.]$$

The lines (4.16-4.17) are the sought results for the 2-particles. The line (4.18) results from the one-particles being transported due to the interactions and fusing with the other particle. Lines (4.19-4.20) are the errors of the one-particle approximation, we argued in the previous lemma. The last term (4.21) results from the simultaneous action of the interaction on both particles. We have already demonstrated that (4.19- and (4.20) decay exponentially with L. Hence, only

(4.18) and (4.21) are new. One should notice that when normalized, the overlap of m_1 with m_2 particle states scales as,

$$|\langle\psi\{X_\alpha,k_\alpha\}_{\alpha=1}^{m_1}|\psi\{X_\alpha,k_\alpha\}_{\alpha=1}^{m_2}\rangle| = O(N^{\frac{n_1+n_2}{2}})$$

This is elaborated in more details in the proof of the theorem. The result is similar for the state in (4.21) and taking it to be similar to a one-particle state. Looking specifically at each line. There are at most two new 1-particle states in the second line. Each new one-particle state is the result of LD^2 other terms. The argumentation is similar for (4.21). There are $D^6L(L-1)$ contribution to this one-particle state. We omitted the error terms ϵ_* which come from the local identity operation of the interactions.

$$\epsilon* = \sum_{n_1} 2(|n_1-n_2|-L)e^{i(k_1n_1+k_2n_2)} \sum_{1\leq n_2-n_1\leq L} |\phi\{X_{a_1},n_1;X_{a_2},n_2\}$$

Under normalization with $\langle\psi[X_{a_1},k_1;X_{a_2},k_2]|\psi[X_{a_1},k_1;X_{a_2},k_2]\rangle$, this error is of order $O(D^4L^2/N)$.

With this in mind, we can tackle the m-particle case and regroup the different error types together,

$$\begin{aligned}
&\Big|\Big|H|\psi\{B_\alpha^{(i)},k_\alpha\}_{\alpha=1}^m\rangle - \Big(2mL|\psi\{B_\alpha^{(i)},k_\alpha\}_{\alpha=1}^m\rangle + 2|\psi\{\mathcal{H}[X_1],k_1;X_2,k_2;\dots;X_m,k_m\}\rangle \\
&\qquad + \dots + |\psi\{X_1,k_1;\dots;X_{m-1},k_{m-1};\mathcal{H}[X_m],k_m\}\rangle\Big)\Big|\Big|_2 \\
&\leq \|\|\phi\{\mathcal{X}_{1,1}\{k_1\},X_2,k_2;\dots;X_m,k_m\}\rangle + \dots + |\psi\{X_2,k_2;\dots;X_{m-1},k_{m-1};\mathcal{X}_{1,m}\{k_m\}X_1,k_1\}\rangle\| \\
&\qquad + \|\|\zeta\{\mathcal{X}_{2,1}\{k_1,k_2\};X_3,k_3;\dots;X_m,k_m\}\rangle + \dots \\
&\qquad + |\zeta\{X_2,k_2;\dots;X_{m-2},k_{m-2};\mathcal{X}_{2,m-1}\{k_{m-1},k_m\};X_1,k_1;\}\rangle\| \\
&\quad + \dots + \|\|\zeta\{\mathcal{X}_{m-1,1}\{k_1,k_2,..,k_{m-1}\},X_mk_m\}\rangle + \dots + |\psi\{\mathcal{X}_{m-2,2}\{k_2,..,k_m\};X_1,k_1\}\rangle\| \\
&\qquad + \|\|\zeta\{\mathcal{X}_{m,1}\{k_1,k_2,..,k_m\}\rangle\| + \epsilon_*
\end{aligned}$$

The operator $\mathcal{X}_{n_1,n_2}$ consists of at most $\begin{pmatrix} L \\ n \end{pmatrix} D^{2(n+1)}$ terms. Under the states $|\zeta\rangle$, one should understand,

$$\begin{aligned}
&|\zeta\{\mathcal{X};X_1,k_1;\dots;X_n,k_n\}\rangle \\
&= \sum_{n_1+L\leq\dots\leq n_{m_1+1},\alpha,\beta} e^{i(q_1+\dots+q_{m_1})n_1}|\phi[e_\alpha,e_\beta,n_1+L,X_1,n_2,\dots,X_{m_2},n_{m_2+1}]\rangle \operatorname{Tr}\left(e_\alpha^\dagger\hat{\mathcal{X}}[e_\beta]\right)
\end{aligned}$$

with bounded super-operator $\hat{\mathcal{X}}$. Again the error ϵ_* is shown to be,

$$\epsilon* = \sum_{n_1}(|n_1-n_2|+|n_m-n_{m-1}|-2mL)e^{i(k_1n_1+k_2n_2)} \sum_{n_1<\dots<n_m,|n_1-n_m|\leq L} |\phi\{X_{a_1},n_1;X_{a_2},n_2\}$$

This yields an error of this order $O(mD^{2m}L^{m+1}/N^m)$ Similarly to our illustration the overlap of the states ψ and ζ are bounded as some power of N. Summing over all contributions and taking into account the normalization of the m-particle state yields the claim. □

This Lemma mathematically implies that the Hamiltonian decouples for this basis set. In Quantum Mechanics, the stationary physical states, i.e. eigenvectors of some Hamiltonian, are orthonormal. Thus, our final step is to orthonormalize this basis. This seems quite challenging at first. We could of course use a Gram-Schmidt procedure to orthogonalize each m-particle subspace. However, this does not solve the orthogonality between each subspace. The representation used previous lemma seems too weak to finalize the result. One should for each m-particle eigenvector take a linear superposition of not only m-particle states but also 1,2,...,m-1. Solving this problem seems very hard at first. However, nothing forbids us to still play with another representation of the particle tensor $B^{(i)}$. It turns out that this is the key for completing the construction which we summarize in the following theorem.

Theorem 31. *Every injective Matrix Product State is the ground state of a hopping theory which depends solely on the transfer matrix of the Matrix Product State. The Hamiltonian is the result of taking a long range and large system size limit of the parent Hamiltonian (4.3) with correction matrix (4.4). A set of m-particle eigenstates in the limit, are given by the symmetric super-positions,*

$$\begin{aligned}&|\psi\{X_{a_1},\dots,X_{a_m};k_1,\dots,k_m\}\rangle\\&=\sum_{P\in\mathcal{S}_n}\sum_{n_1<\dots<n_m}e^{i(k_{P(1)}n_1+\dots+k_{P(m)}n_m)}|\phi[B_{P(a_1)},n_1;\dots;B_{P(a_m)},n_{a_m}]\rangle\end{aligned}$$

summed over all permutations P. The basis states $|\phi[B_{a_1},n_1;\dots;B_{a_m},n_{a_m}]$ are given by,

$$|\phi\{B_\alpha^{(i)},j_\alpha\}_{\alpha=1}^m\rangle=\textstyle\sum_{\vec{i}}\mathrm{Tr}\left(A^{i_1}\dots A^{i_{j-1}}B^{i_{n_1}}A^{i_{j+1}}\dots B^{n_{j_m}}A^{i_{j+m}}\dots A^{i_N}\right)|\vec{i}\rangle$$

Each tensor B_{a_1} is a function of X_a and k_a,

$$B_a^{(i)}=A^{(i)}X_a+e^{-ik_a}Y_aA^{(i)}-A^{(i)}Y_a,\ Y_a=(\mathrm{Id}-e^{-ik}\Gamma^*)^{-1}[X_a]$$

The energy of each m-particle eigenvectors is the sum of the energy of the individual particle,

$$H|\psi\{X_{a_1},\dots,X_{a_m};k_1,\dots,k_m\}\rangle=\sum_j E[X_{a_j},k_j]|\psi\{X_{a_1},\dots,X_{a_m};k_1,\dots,k_m\}\rangle\rangle \quad (4.22)$$

The particle representations X_a and their respective energy are given in Lemma (29).

The states and respective energy of the theory are valid for a low-particle density which satisfies,

$$O(D^2|\lambda|^L)\to 0,\ O\left(\sum_{n=1}^m(m-n+1)\begin{pmatrix}L\\n\end{pmatrix}N^{-n}D^{2(n+1)}\right)\to 0,\ O(m!N^{-m})\to 0$$

Proof. Let us first fix m, and the particles representations $X_{a_1},\dots,X_{a_m}$ and their respective momenta $k_1,\dots,k_m$ as given by Lemma (29). One should first notice the following facts. First of all, one should see that one-particle representations with the tensor $B_a^{(i)}$ is gauge-equivalent to X_a,

$$|\psi\{X_a;k_a\}\rangle=|\psi\{B_a^{(i)};k_a\}\rangle \quad (4.23)$$

Secondly, the choice of gauge was chosen in such way that,

$$\forall k_1, k_2, a \sum_{i,j} \overline{A^{(i)}}_{j,k_1} \otimes B^{(i)}_{j,k_2;a} = \sum_{i;j} \overline{B^{(i)}}_{j,k_1;a} \otimes A^{(i)}_{j,k_2} = 0$$

It is necessary for us to compute the overlap between these new basis states $|\psi\{B^{(i)}_\alpha, k_\alpha\}^m_{\alpha=1}\rangle$,

$$|\psi\{B^{(i)}_\alpha, k_\alpha\}^m_{\alpha=1}\rangle = \sum_{n_1<\ldots<n_m} e^{i\vec{k}.\vec{n}} |\phi\{B^{(i)}_\alpha, n_\alpha\}\rangle$$
$$|\phi\{B^{(i)}_\alpha, n_\alpha\}^m_{\alpha=1}\rangle = \sum_{\vec{i}} \mathrm{Tr}\left(A^{i_1} \ldots A^{i_{j-1}} B_1^{i_{n_1}} A^{i_{j+1}} \ \ldots B_m^{n_{j_m}} A^{i_{j+m}} \ldots A^{i_N}\right) |\vec{i}\rangle$$

With some algebra, or by using the gauge equivalence (4.23), we derive for $k = k_a = k_b$,

$$\sum_i \mathrm{Tr}\left(B^{(i)\dagger}_a B^{(i)}_b\right) = \mathrm{Tr}\left(X^\dagger_a \mathcal{F}_k[X_b]\right)$$

Using these simple properties of the tensor $B^{(i)}_a$ and particle representations X_a, one easily checks,

$$\langle\psi\{B^{(i)}_{,} k_{a_j}\}^{m_1}_{j=1}\rangle|\psi\{B^{(i)}_{b_j}, k_{b_j}\}^{m_2}_{j=1}\rangle = \delta_{m,m_1}\delta_{m,m_2} N^m \prod_j^m \left[\delta_{k_j,k_{a_j}}\delta_{k_j,k_{b_j}} \mathrm{Tr}\left(X^\dagger_{a_j}\mathcal{F}_{k_j}[X_{b_j}]\right)\right] + O(N^{-1})$$

It follows that such basis states with different particle numbers are orthogonal,

$$\frac{\langle\psi\{B^{(i)}_{,} k_{a_j}\}^{m_1}_{j=1}\rangle|\psi\{B^{(i)}_{b_j}, k_{b_j}\}^{m_2}_{j=1}\rangle}{\sqrt{\langle\psi\{B^{(i)}_{,} k_{a_j}\}^{m_1}_{j=1}\rangle|\psi\{B^{(i)}_{b_j}, k_{b_j}\}^{m_1}_{j=1}\rangle\langle\psi\{B^{(i)}_{,} k_{a_j}\}^{m_2}_{j=1}\rangle|\psi\{B^{(i)}_{b_j}, k_{b_j}\}^{m_2}_{j=1}\rangle}} = O\left(\frac{N^{\min\{m_1,m_2\}}}{N^{\frac{m_1+m_2}{2}}}\right)$$

Finally, this allows us the try and find some orthonormal basis in each subspace. We fix the particles $X_{a_1}, \ldots, X_{a_m}$ with respective momentum $k_1, \ldots, k_m$ and superimpose the basis states $\langle\psi\{B^{(i)}_{a_{P(j)}}, k_{a_{P(j)}}\}^{m_1}_{j=1}|$ consisting of the permutations of the particles.

$$\begin{aligned}&|\psi\{X_{a_1}, \ldots, X_{a_m}; k_1, \ldots, k_m\}\rangle \\ &= \sum_{P\in\mathcal{S}_n} \sum_{n_1<\ldots<n_m} e^{i(k_{P(1)}n_1+\ldots+k_{P(m)}n_m)} |\phi[B_{P(a_1)}, n_1; \ldots; B_{P(a_m)}, n_{a_m}]\rangle\end{aligned}$$

We verify that this choice yields the orthogonality,

$$\begin{aligned}&\langle\psi\{X_{a_1}, \ldots, X_{a_m}; k_1, \ldots, k_m\}|\psi\{X_{b_1}, \ldots, X_{b_m}; q_1, \ldots, q_m\}\rangle \\ &\leq \frac{1}{N^m}\prod_j \delta_{a_j,b_j}\Big|\sum_{P\in S_m}\sum_{n_{P(1)}<\ldots<n_{P(m)}} e^{i\left((q_1-k_1)n_{P(1)}+\ldots+(q_m-k_m)n_{P(m)}\right)}\Big| \\ &\leq \prod_j \delta_{a_j,b_j}\delta q_j, p_j + O\left(\frac{m!}{N^m}\right)\end{aligned}$$

The sought orthonormal basis has thus been found.

One, now, needs to know the action of the Hamiltonian onto the m-particle states. Lemma's (29) and (30) can be applied.

The theorem is proven once we manage to show that the effective Hamiltonian of each m-particle subspace decouples,

$$H^{(m)}\{k_1,\dots,k_m\} = \sum_j H^{(1)}\{k_j\}$$

Our choice of orthogonal basis implies such result,

$$\langle\psi\{X_{a_1},\dots,X_{a_m};k_1,\dots,k_m\}|H|\psi\{X_{b_1},\dots,X_{b_m};q_1,\dots,q_m\}\rangle \\ = \sum_{j_1,j_2}\prod_{j\notin\{j_1,j_2\}}\delta_{a_j,b_j}\delta q_j,p_j\langle\psi\{X_{a_{j_1}};k_{j_1}\}|H|\psi\{X_{b_{j_2}};q_{j_2}\}\rangle$$

With this the theorem has been shown.

□

We showed the symmetrized states could be considered as a type of eigenstates of the Hamiltonian. Unfortunately, this is only a small part of the spectrum. From our results in the previous lemma's, we know that the n-particle spectrum is some complex superposition of the n-particle basis. However, in the case of normal unital quantum channels, this superposition turns out to be extremely simple. All n-particle energy levels are degenerate and a trivial basis can be found by simply taking the symmetric and anti-symmetric superposition of the n-particle basis.

Corollary 32. *When the transfer matrix is a normal unital quantum channel, in the long range and system size limit, the Hamiltonian (4.3) becomes exactly solvable with eigenstates*

$$|\psi\{X_{a_1},\dots,X_{a_m};k_1,\dots,k_m\}\rangle \\ = \sum_{P\in\mathcal{S}_n}\sum_{n_1<\dots<n_m} e^{i(k_{P(1)}n_1+\dots+k_{P(m)}n_m)}A(P)(X_{a_1},\dots,X_{a_m})|\phi[B_{P(a_1)},n_1;\dots;B_{P(a_m)},n_{a_m}]\rangle$$

with $A(T_{\alpha,\beta}P) = \pm A(P)$.

Normal unital trace-preserving Completely Positive Operators

In general, we need not only information about the spectral properties of the transfer matrix, but also the algebraic relation between the eigenvectors.

It turns out that normal unital trace preserving completely positive operators are an interesting class to study, as a lot of the computations, we had to endure for proving Theorem (31), simplifies.

$$\rho = \frac{\mathbb{1}}{\mathrm{Tr}(\mathbb{1})},\ \ \Gamma^*[\mathbb{1}] = \mathbb{1},\ \ \Gamma[\mathbb{1}] = \mathbb{1}$$

It is important to stress that not any spectrum is allowed. Some, sometimes heavy, restrictions are imposed due to the complete positiveness. For example, if we assume that the eigenvectors have a particular group structure

$$X_a = U_g, \;\; U_g = \oplus_j e^{\frac{2\pi i j g}{D}} \tag{4.24}$$

This example is based on the lemma that spectral vectors around unit circle are of this form (4.24).

Still, the following restrictions can be shown for $\Gamma[U_g] = |\lambda_g| e^{\frac{2\pi i \kappa_g}{D}} U_g$,

$$\forall \alpha \in \{1, \dots, 2D\}, \;\; \sum_g |\lambda_g| \cos\left(\frac{2\pi\left[\kappa_g - g\alpha\right]}{D}\right) \geq 0 \tag{4.25}$$

Writing,

$$\Gamma[.] = \sum_g \lambda_g U_g \,\mathrm{Tr}\left(U_g^\dagger [.]\right)$$

Equation (4.25) follows for the necessary and sufficient for complete positiveness [19],

$$C[\Gamma] = \sum_{\alpha,\beta} |\alpha\rangle\langle\beta| \otimes \Gamma[|\alpha\rangle\langle\beta|] \geq 0$$

This simple example illustrates that spectral properties of completely positives are not just arbitrary. There are physics restrictions. It is amazing that nonetheless, we can connect these with the spectrum of Hamiltonians, which are pretty much arbitrary.

As presented at the end of the work, the appearance of bound states, is speculated from the breaking down of the convergence towards the one-particle subspace.

$$\lim_{L\to\infty} \epsilon_1(X_a, k) = \lim_{L\to\infty} \frac{\|H|\psi\{X_a,k\}\rangle - 2(L-\sigma_{k,a})|\psi\{X_a,k\}\rangle\|_2^2}{\langle\psi\{\overline{X_a},k\}|\psi\{X_a,k\}\rangle} > 0 \tag{4.26}$$

$$\sigma_{k,a} = \mathrm{Re}\,\frac{1}{1 - e^{i[k-\phi_{X_a}]}|\lambda_{X_a}|} \tag{4.27}$$

The following conclusions can be taken. If for all momenta k, $\epsilon_1(X_a, k) \to 0$, then $|\psi\{X_a, k\}\rangle$ are eigenstates with energy $\mathrm{Re}\,1/(1 - e^{i[k-\phi_{X_a}]}|\lambda_{X_a}|)$.

In the case of bound states lying in the $2-$ and $1-$ particle, i.e. $|\psi\{X_a,k\}\rangle + \sum_q c_q |\psi\{X_\alpha, k-q; X_\beta^\dagger, q\}$, we need a convergence of the 3-particle errors,

$$\epsilon(X_\alpha, q), \epsilon(X_\beta, q) \to 0$$

Conclusions and Outlook

In this thesis, we studied various aspects of completely positive dynamics. It was interesting to see how such simple questions could lead to so many challenges and further smaller, but difficult questions.

Conclusions chapter 1: In this first chapter, we presented a construction of a class of local dissipative dynamics. For this class, the bound reflects the dissipative character of the dynamics. The bound proven in [72] only showed the spreading of local perturbation due to the local property of the dynamics. We showed, however, that when the interactions satisfy additional structural equations, this spreading is slowed down due to the dissipative property.

Next, we studied a class of local Lindbladians which preserves a particular local convex structure. We show that in the irreducible case, a competition can emerge between the diffusive and strongly mixing part of the generator. We see that the diffusion property can be decoupled from the mixing. Localization is then implied for a finite mixing rate of the same order as the Lieb Robinson velocity. We, then, showed various constructions from classical stochastic processes, to more complex quantum dynamics with graph states as fixed points.

Finally, we discussed how deceleration of the velocity can imply rapid mixing. This turns out to be another challenging problem for the general case. We introduced, therefore, a notion of attractiveness. This property is a type of local contractivity of the dynamic under the oscillator norm. This allows to approximate the evolution of local observables by another dynamic with finite support. The mixing then scales with the size of the support, and therefore time. This lead us to a simple inequality which weights the propagation of information against this local scaling of mixing time. When this is satisfied for an attractive process, rapid mixing is implied.

Conclusions chapter 2: In this second chapter, we showed how we could relate gapped quantum phases with some (non)-equilibrium phase. This is achieved by mapping the dissipative generator onto a local quantum Hamiltonian and showing the existence of a quasi-adiabatic evolution. The construction gave new insights in the tensor network language. We showed how a class of matrix product operators can be inverted.

Conclusions chapter 3: In this third chapter, we discuss how Stochastic Matrix Product states can be used as a representation of the Radon-Nikodym derivative of processes defined on a filtration of another process. With this representation, we derive a simple equation for

the time-evolution of the marginal distribution of the process. We showed how classical non-Markovian, classical and quantum processes can be embedded in a quantum Markovian dynamics. The properties of non-Markovian processes, such as ergodicity and mixing are therefore determined by the quantum dynamics.

Conclusions chapter 4: In this last chapter, we studied the spectrum of parent Hamiltonians in the long range, system size limit and under eventual renormalization of the transfer operator of the Matrix Product ground state. We showed that, in this limit, the spectrum can be solved exactly and depends purely on the properties of a quantum channel, i.e. the transfer matrix. The derived formula for the low energy spectrum was very reminiscent of the Källén -Lehmann spectral representation in quantum field theory. Particles types are represented by eigenvectors of the fourier transform of the transfer matrix (4.5). In the spirit of the Bethe ansatz, higher energy wavefunction are constructed by putting these particles in the virtual space of the ground-state with different momentum.

Secondly, we consider a renormalization of the spectrum of the transfer matrix with range L. The choice was made in such a way that the ground state becomes "critical". The consequence is then that the particles derived earlier are not necessarily stable. The first quantization breaks down when a particle can be fused from two others, of which the respective eigenvalues are close to the unit circle.

This work introduces an interesting interpretation of translational invariant Matrix Product States as nothing but stationary sea of particles scattering with one-another. This insight motivated us to understand Anderson localization on the level of Tensor Networks. The property is introduced to the Matrix Product State manifold without considering any Hamiltonian-dynamic. The local random potentials are exchanged with fluctuations of the local tensor of the network. The time-evolution is reinterpreted as transport along the manifold which moves the eigenvalues of the transfer towards the unit circle. This is shown under proper rescaling of the parameter with t to be equivalent to diffusion. Finally, it is illustrated how finite magnitude of fluctuation kills the diffusion similarly to the result proven by Anderson.

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