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Chapter 1

Introduction

1.1 About the Thesis

There are two main goals in this thesis:

We want to give an overview of methods for pricing barrier options in certain models, including the Black-Scholes model, the Heston model and furthermore a model-free approach.

The second aim is to give an overview of the mathematical foundation to the approach of approximating the Black-Scholes formulas for an up-and-out call (UOC) option in the Heston model, given in the paper by Griebisch and Pilz in [4].

In subsection 1.2, we define and explain the most important technical terms, which we need throughout the following chapters, including vanilla options, strike price, expiration date and corresponding pay-off formulas, as well as the introduction to barrier options, i.e. different types like knock-out and knock-in, single - and double barriers and corresponding pay-off formulas. This chapter aims to give a financial interpretation and some general information on barrier options and their properties .

In the second chapter we introduce the classical Black-Scholes model (BSM), its assumptions and the most important results in a mathematically precise way. Subsection 2.1 aims to briefly derive the pricing formula, which we use to price the first two types of barrier options in the following two subsections. The assumptions in the BSM - those of a complete market without arbitrage - guarantee us the existence of a unique risk neutral measure, under which

we can give unique prices of contingent claims as discounted expected values of their pay-off formulas at expiration date.

In subsection 2.2 we price a UOC option in the Black-Scholes framework. Like mentioned above, we use the pricing formula for this. So, in other words, we calculate the discounted expected value of the UOC's pay-off. The calculation is straight forward and results in an explicit pricing formula for the UOC.

In section 2.3 we price the remaining three types of barrier options and we start with the pricing of a down-and-out call, which is done analogously to section 2.2.

For the remaining two options we use some general relation between barrier - and vanilla options. For an up-and-in call (UIC), this means, we can replicate the pay-off of a vanilla call with adding the pay-offs of an UIC and a UOC, so in the absence of arbitrage the price of an UIC is equal to the price difference of a vanilla call and a UOC.

With a similar relation, we calculate the price of the remaining down-and-in call (DIC).

The third chapter is about pricing of barrier options in the Heston framework.

In 3.1 we briefly discuss the most unrealistic features of the BSM as a motivation to introduce stochastic volatility.

In subsection 3.2 we give the definition of a general stochastic volatility model, where stock price and volatility are modeled with a general diffusion process and furthermore, we point out the additional feature which renders this market structure incomplete. The problem we face in this set up is, that we don't have a unique risk neutral measure like in the BSM, but several measures, which depend on the market price of volatility risk. This is a function of volatility and the underlying and has to be determined exogenously.

This leads to subsection 3.3, where we introduce the Heston model. We specify the volatility process as a Cox-Ingersoll-Ross square root process and assume a specific form for the market price of volatility risk. Having specified this function, there exists a unique risk neutral measure and we derive the stock price - and volatility dynamics under it.

Since there are no closed formula solutions for barrier option prices in the Heston framework, in 3.4, we give an overview of methods used to approximate those.

In 3.4.1 we give the foundation for the finite difference approach, i.e. we show the PDE with the corresponding boundary conditions.

In 3.4.2 we focus on a method to approximate the price of an UOC option within the Heston model, using the same pricing formula, which we also use in the second chapter within Black-Scholes model.

This approach starts in a Heston model framework under some risk neutral measure, derived in 3.3.

Three cases are considered, namely the case with zero interest rate and zero correlation, arbitrary interest rate and zero correlation and the last case considers arbitrary interest rate and ρ - correlation.

In the end of this subsection, we get an explicit formula for the price of an UOC option for the first case and approximation formulas for the remaining two cases.

The derivation is structured in four steps:

In the first step, we rewrite the stock price process in a way that it incorporates the variance dynamics. Then, we condition on the sigma algebra generated by the variance path up to exercise time. In other words, we treat the variance as a deterministic function of time, which is governed by the independence lemma, described in the appendix A.2.6.

The pricing formula, which we used in section 2.2 and 2.3, applies here as well and can be divided into two parts: in order to calculate the discounted expected value of the UOC option's pay-off, we first calculate the expected value of the pay-off conditioned on the variance path, called inner expectation and afterwards, we take the expected value of the result.

To compute the inner expectation, we need to derive the density function of the stock price and its maximum process, which is done in step 2, which is done for each case individually. For case A we get an exact formula for the density and for case B and C approximation formulas.

In the third step we use these densities for each case to calculate the inner expectations. We divide this step as follows:

according to the independence lemma, we calculate the (unconditional) expected value of the pay-off, treating the variance as deterministic. This is content of step 3.1. By the implication of the independence lemma, we get the conditional expected value by replacing the deterministic variance function by our initial variance process, which is done in step 3.2.

In the last step, we calculate the outer expectation, which means, we calculate the expected value of the resulting formulas from step 3.2. Those contain the time integrated variance as random variable, therefore we have to find the corresponding density function, which can be calculated by Fourier transformation of the characteristic function of the time integrated variance.

The derivation is done in appendix A.2.6.

For case A we get an explicit pricing formula and for cases B and C we get

approximation formulas, since the densities are approximated as mentioned above.

In chapter 4 we consider a completely different approach, which is known in the literature as *robust hedging*. Assuming that call option prices are given for a continuum of strike prices, allows us to build a hedge, by buying and selling amounts of the underlying and call options in order to replicate the pay-off of a barrier option. By no arbitrage arguments, the price of the barrier option is equal to the time zero value of the hedge.

In chapter 4.1 we motivate this approach, followed by assumptions and the important results in 4.2. Then, in 4.3, we derive upper and lower bounds for a UIC option (in 4.3.1) and for a UOC option (in 4.3.2), as well as their corresponding hedges.

1.2 About Barrier Options

Vanilla Option, Strike Price, Expiration Date, Pay-Off

A European style (vanilla) put/call option, is the right to sell/buy a fixed amount of underlying assets for fixed price and time.

This fixed price is usually called strike price and is denoted with K and the time when the option is exercised is called exercise time or expiration date, denoted with T .

The value of a call at expiration date C_T is given by

$$C_T = (S_T - K)^+ = (S_T - K)\mathbb{I}_{\{S_T > K\}},$$

where S_T denotes the underlying stock price at time T and $\mathbb{I}_{\{S_T > K\}}$ the indicator function, which takes values 1 if $S_T > K$ and 0 if $S_T \leq K$.

The pay-off of a put option is given by

$$C_T = (K - S_T)^+ = (K - S_T)\mathbb{I}_{\{S_T < K\}}.$$

Barrier Option

A barrier option is an exotic option, for which its value at exercise time doesn't only depend on the value of the underlying at exercise time, but

also on the path of the stock price. This means, there is the additional requirement that the stock price has to stay below/above a predetermined level, called barrier level, in order not to knock out and lose its value/exceed this level, in order to be activated.

We will explain this more precisely when we classify different types of barrier options.

Usually the barrier level is denoted with a B .

Knock-out

Is a vanilla option with the requirement that the stock price has to stay below or above the barrier level. We distinguish between Up-and-Out Call/Put and Down-and-Out Call/Put.

Up-and-Out Call/Put (UOC/UOP) and Pay-Off:

The stock price starts below B and must not exceed this level, in order not to lose its value. An equivalent formulation is, that the maximum of the stock price path must not exceed B .

So the value of a UOC option at exercise time can be described by:

$$UOC_T = (S_T - K)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B\}} = (S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}},$$

and for a UOP option its pay-off is given by:

$$UOP_T = (K - S_T)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B\}} = (K - S_T) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T < K\}}.$$

Down-and-Out Put/Call:

The stock price starts above the Barrier level, and the minimum of the stock price path has to stay above B .

For the pay-off of a DOC at time T , we get

$$DOC_T = (S_T - K)^+ \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B\}} = (S_T - K) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}},$$

and for a DOP it is given by

$$DOP_T = (K - S_T)^+ \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B\}} = (K - S_T) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T < K\}}.$$

Knock-In

The other type of Barrier Options are knock-in options, where the price of the underlying has to exceed a certain limit, in order to be activated. In other words, the value of the option is zero until the barrier level is reached and behaves like a vanilla option from that point.

We distinguish again if the Barrier is above or below the stock price and if the stock price has to exceed from below or above or equivalently, if the maximum of the stock price path has to exceed B or the minimum has to go below B , respectively.

Up-and-In Call/Put (UIC/UIP), Pay-Off

The stock price starts below B and must exceed this level, in order not to lose its value/the maximum of the stock price path has to exceed B . So the value at expiration date can be described by

$$UIC = (S_T - K)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B\}} = (S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B, S_T > K\}},$$

for a call and for put it is given by

$$UIP = (K - S_T)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B\}} = (K - S_T) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B, S_T < K\}}.$$

Down-and-In Put/Call

Here, the stock price starts above B and the minimum of its path has to be below B :

$$DIC = (S_T - K)^+ \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t < B\}} = (S_T - K) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t < B, S_T > K\}},$$

for a call and for put we have

$$DIP = (K - S_T)^+ \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t < B\}} = (K - S_T) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t < B, S_T < K\}}.$$

Single vs Double Barriers

So far we have introduced single barrier options, this means there is only one barrier level B .

A double barrier option is a barrier option with two predetermined barriers B_1 and B_2 , ($B_1 < B_2$).

There are various knock in/out combinations on the stock price path requirements.

For example, in a double knock out option, the stock price path has to stay within the barrier interval $[B_1, B_2]$, with $B_1 < K < B_2$, in order to be valid. In the following, we will focus on single barrier options only.

General Information on Barrier Options

In general, there are two main exercise types, i.e. European-style and American-style. In the first, the option holder can exercise his option only on execution date, whereas in the latter, the holder can exercise his option at all times up to exercise time.

Barrier options are a special type of financial derivatives and can be counted to the class of exotic option. We will focus only on European-style exercise type.

Since the additional requirement has to be fulfilled, it is clear that all these types of barrier options are riskier than vanilla options and therefore should be cheaper as well.

Mostly they are traded on OTC-markets, but also more liquidly on foreign exchange markets, (see [12]).

Chapter 2

Pricing of Barrier Options in the Black-Scholes Model

2.1 The Black-Scholes Model and the Valuation Formula

We follow the assumptions of Black and Scholes, given in [3].

We assume that our market is frictionless, i.e. there are no transaction costs, furthermore, we can lend and borrow from the money market with the same interest rate r , which is known and continuously compounded over $[0, T]$, so the discount factor at time t is given by e^{-rt} . Borrowing or lending can also be interpreted as buying or selling a risk-free asset.

In addition, there is a risky asset, which can be interpreted as a - in our case dividend-free - stock, described below.

Short-selling is permitted and any amount of the risky and risk-free asset can be bought or sold.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The stock price process $\{S_t\}$ is modeled by the geometric Brownian motion and is described by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (2.1)$$

where $\{W_t\}$ is a $\mathbb{P}$ -Brownian motion, μ and σ are constants, which influence the trend and the volatility of the stock price process [26, p. 148].

The filtration generated by $\{S_t\}$ is denoted by $\mathbb{F} := \mathbb{F}^S$ and it holds that

$\mathbb{F}^S = \mathbb{F}^W$, where $\mathbb{F}^W$ denotes the standard filtration generated by the Brownian motion $\{W_t\}$. The details are captured in appendix A.1.2.

The Black-Scholes Model is arbitrage-free, with an appropriate choice of admissible trading strategies. The first fundamental theorem of mathematical finance proves, that there exists an equivalent martingale measure (EMM) or risk neutral measure, respectively, such that the value of a contingent claim can be given as the discounted conditional expectation under the EMM [26, p. 231].

Since one can show that this EMM is unique [22, p. 121], the second fundamental theorem tells us, that the Black-Scholes model is complete [26, p. 232].

This means, a European-style financial derivative can be priced uniquely as the expected value under the unique EMM of the derivative's discounted pay-off at expiration date.

The construction of this EMM is captured in Girsanov's theorem and is derived in appendix A.1.1.

We call $\mathbb{P}$ the real world measure and $\bar{\mathbb{P}}$ shall denote the EMM.

In appendix A.1.1.2, we show that, under the EMM $\bar{\mathbb{P}}$, (2.1) can be rewritten into

$$dS_t = rS_t dt + \sigma S_t d\bar{W}_t, \quad (2.2)$$

with $\bar{\mathbb{P}}$ -Brownian motion $\{\bar{W}_t\}$, which can be solved with Ito's formula, given in appendix A.2.2 and has the following solution:

$$S_t = S_0 e^{\sigma \bar{W}_t + (r - \frac{1}{2}\sigma^2)t} \quad \forall t \in [0, T], \quad (2.3)$$

where S_0 denotes the initial value of our stock at time $t = 0$.

As we said above, we can give the value of any contingent claim $X(S_T)$ at time t , as its discounted conditional expected value, i.e

$$X(S_t) = \mathbb{E}_{\bar{\mathbb{P}}}[e^{-r(T-t)} X(S_T) | \mathcal{F}_t].$$

So if V_T denotes the pay-off of a European-style derivative security at execution time T , we can describe its value or price, respectively, at time t , as

$$V_t = \mathbb{E}_{\bar{\mathbb{P}}}[e^{-r(T-t)} V_T | \mathcal{F}_t],$$

and for $t = 0$, we get the *Pricing Formula*

$$V_0 = \mathbb{E}_{\mathbb{P}}[e^{-rT}V_T], \quad (2.4)$$

where V_0 denotes the price of the derivative security.

We will use this formula in the next section, in order to derive prices of barrier options.

2.2 Pricing of an Up-and Out Call Option

We follow the approach in [26, p. 295-307] and [22, p. 235-238].

The pay-off of an up-and-out call (UOC) option is given by

$$UOC_T = V_T = (S_T - K)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B\}} = (S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}}$$

where K denotes the strike price and B the barrier level of the UOC.

Now we use the pricing formula (2.4) and we get

$$UOC = V_0 = \mathbb{E}_{\mathbb{P}}[e^{-rT}(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}}].$$

If we plug in the solution to (2.2), given by (2.3) and resolve the bracket, it follows

$$UOC = S_0 \mathbb{E}_{\mathbb{P}} \left[e^{\sigma \bar{W}_T - \frac{1}{2} \sigma^2 T} \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right] - e^{-rT} K \mathbb{E}_{\mathbb{P}} [\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}}]$$

and we set

$$I_1 = \mathbb{E}_{\mathbb{P}} \left[e^{\sigma \bar{W}_T - \frac{1}{2} \sigma^2 T} \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right]$$

$$I_2 = \mathbb{E}_{\mathbb{P}} \left[\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right],$$

so, for the UOC price, we get

$$UOC = S_0 I_1 - e^{-rT} K I_2.$$

and we calculate I_1 and I_2 separately in the following and start with the

easier one, i.e. I_2 and using this result, we derive I_1 .

Rewriting S_t and $\max_{0 \leq t \leq T} S_t$ for I_2 :

We set $\widetilde{W}_t = \overline{W}_t + \alpha t$, with $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$, $t \in [0, T]$ and $\widetilde{M}_T = \max_{0 \leq t \leq T} \widetilde{W}_t$, this implies

$$S_t = S_0 e^{\sigma \widetilde{W}_t} \quad \forall t$$

and for the maximum of the stock price process, we get

$$\max_{0 \leq t \leq T} S_t = S_0 \max_{0 \leq t \leq T} e^{\sigma \widetilde{W}_t} = S_0 e^{\sigma \max_{0 \leq t \leq T} \widetilde{W}_t} = S_0 e^{\sigma \widetilde{M}_T}.$$

The set of the indicator function can be rewritten in the following way:

$$\left\{ \max_{0 \leq t \leq T} S_t < B, S_T > K \right\} = \left\{ S_0 e^{\sigma \widetilde{M}_T} < B, S_0 e^{\sigma \widetilde{W}_T} > K \right\} = \left\{ \widetilde{M}_T < b, \widetilde{W}_T > k \right\},$$

where $b = \frac{1}{\sigma} \ln \frac{B}{S_0}$ and $k = \frac{1}{\sigma} \ln \frac{K}{S_0}$.

Therefore, we can rewrite the indicator function in I_2 , as

$$I_2 = \mathbb{E}_{\overline{\mathbb{P}}} [\mathbb{I}_{\{\widetilde{M}_T < b, \widetilde{W}_T > k\}}].$$

In order to calculate this expected value, we have to find the joint density $\overline{f}_{\widetilde{M}_T, \widetilde{W}_T}$ of $(\widetilde{M}_T, \widetilde{W}_T)$ under $\overline{\mathbb{P}}$, since

$$\mathbb{E}_{\overline{\mathbb{P}}} [\mathbb{I}_{\{\widetilde{M}_T < b, \widetilde{W}_T > k\}}] = \overline{\mathbb{P}}\{\widetilde{M}_T < b, \widetilde{W}_T > k\} = \int_{\{\widetilde{M}_T < b, \widetilde{W}_T > k\}} \overline{f}_{\widetilde{M}_T, \widetilde{W}_T} d\overline{\mathbb{P}}.$$

Density of $(\widetilde{M}_T, \widetilde{W}_T)$:

Note that, since $\widetilde{W}_0 = 0$, $\widetilde{M}_T \geq 0$ and $\widetilde{W}_T \leq \widetilde{M}_T$ the pair $(\widetilde{M}_T, \widetilde{W}_T)$ can take values in the area $\{(x, y) : x \geq 0, y \leq x\}$.

The next theorem and corresponding proof are from [26, p. 296].

Theorem:

The joint density function $\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}$ of $(\widetilde{M}_T, \widetilde{W}_T)$ is given by

$$\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) = \frac{2(2x - y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x - y)^2}, \text{ for } x \geq 0 \text{ and } y \leq x$$

Proof:

The proof will consist of several steps:

First we change the risk neutral measure $\bar{\mathbb{P}}$ to measure $\widetilde{\mathbb{P}}$, via Girsanov's theorem, under which $\{\widetilde{W}_t\}$ is a drift-less $\widetilde{\mathbb{P}}$ -Brownian motion.

In the next step, we derive the joint density $\widetilde{f}_{(\widetilde{M}_T, \widetilde{W}_T)}$ of the pair $(\widetilde{M}_T, \widetilde{W}_T)$, under $\widetilde{\mathbb{P}}$, using the reflection principle of the Brownian motion, given in appendix A.1.3, and in the last step, we derive the joint density $\bar{f}_{(\widetilde{M}_T, \widetilde{W}_T)}$ of $(\widetilde{M}_T, \widetilde{W}_T)$ under $\bar{\mathbb{P}}$.

1. Changing the measure:

Recall that $\widetilde{W}_t = \overline{W}_t + \alpha t$ and set

$$Z_T = e^{-\alpha \overline{W}_T - \frac{1}{2}\alpha^2 T},$$

then, defining $\widetilde{\mathbb{P}}$ as

$$\widetilde{\mathbb{P}}(A) = \int_A Z_T d\bar{\mathbb{P}}, \quad A \in \mathcal{F},$$

from Girsanov's theorem¹ (see appendix A.1.1.1) follows, that $\{\widetilde{W}_t\}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$.

2. Finding $\widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}$:

Let x and y be real numbers, with $y \leq x$ and $x \geq 0$, furthermore let τ_x be the first passage time, defined as

$$\tau_x = \min\{t \geq 0 : W_t = x\}.$$

The reflection principle [26, p.111-112], tells us that

$$\widetilde{\mathbb{P}}\{\tau_x \leq T, \widetilde{W}_T \leq y\} = \widetilde{\mathbb{P}}\{\widetilde{W}_T \geq 2x - y\}.$$

¹Trivially, the Novikov condition is fulfilled in this case, allowing us to use Girsanov's theorem.

We can express the fact, that the path of the Brownian motion exceeds point x in the interval $[0, T]$ at least once at some time point, equivalently with requiring, that its maximum in $[0, T]$ exceeds x .
Mathematically speaking:

$$\tau_x \leq T \text{ iff } \widetilde{M}_T \geq x.$$

We use this relation to rewrite the reflection principle into

$$\widetilde{\mathbb{P}}\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\} = \widetilde{\mathbb{P}}\{\widetilde{W}_T \geq 2x - y\}. \quad (2.5)$$

Since $\widetilde{W}_T$ is normally distributed with zero mean and T variance [26], we can rewrite the right hand side of (2.5) into

$$\widetilde{\mathbb{P}}\{\widetilde{W}_T \geq 2x - y\} = \frac{1}{\sqrt{2\pi T}} \int_{2x-y}^{\infty} e^{-\frac{1}{2T}z^2} dz \quad (2.6)$$

The left hand side of (2.5) can be rewritten into

$$\widetilde{\mathbb{P}}\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\} = \int_x^{\infty} \int_{-\infty}^y \widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(m, w) dw dm \quad (2.7)$$

Combining (2.6), (2.7) and partially differentiating both sides with respect to x and y , we get

$$\begin{aligned} \frac{\partial}{\partial y \partial x} \int_x^{\infty} \int_{-\infty}^y \widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(m, w) dw dm &= \frac{\partial}{\partial y \partial x} \frac{1}{\sqrt{2\pi T}} \int_{2x-y}^{\infty} e^{-\frac{1}{2T}z^2} dz \\ \Leftrightarrow \frac{\partial}{\partial y} \int_{-\infty}^y -\widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, w) dw &= -\frac{\partial}{\partial y} \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2} \\ \Leftrightarrow -\widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) &= -\frac{2(2x-y)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2}. \end{aligned}$$

Hence,

$$\widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) = \frac{2(2x-y)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2}.$$

3. Finding $\overline{f}_{\widetilde{M}_T, \widetilde{W}_T}$:

We want to find $\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}$, by deriving $\bar{\mathbb{P}}\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\}$ with respect to x and y , using the relation between $\bar{\mathbb{P}}$ and $\tilde{\mathbb{P}}$:

$$\begin{aligned} \bar{\mathbb{P}}\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\} &= \mathbb{E}_{\bar{\mathbb{P}}} \left[\mathbb{I}_{\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\}} \right] = \int_{\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\}} \bar{f}_{\widetilde{M}_T, \widetilde{W}_T} d\bar{\mathbb{P}} \\ &= \int_{\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\}} \frac{1}{Z_T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T} d\tilde{\mathbb{P}} = \int_x^\infty \int_{-\infty}^y e^{\alpha w - \frac{1}{2}\alpha^2 T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(m, w) dw dm \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\partial}{\partial y \partial x} \bar{\mathbb{P}}\{\widetilde{M}_T \geq x, \widetilde{W}_T \leq y\} \\ &= \frac{\partial}{\partial y \partial x} \int_x^\infty \int_{-\infty}^y \bar{f}_{\widetilde{M}_T, \widetilde{W}_T}(m, w) dw dm \\ &= \frac{\partial}{\partial y} \int_{-\infty}^y -\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, w) dw \\ &= -\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) \end{aligned} \tag{2.8}$$

For the right side, we get

$$\begin{aligned} &\frac{\partial}{\partial y \partial x} \int_x^\infty \int_{-\infty}^y e^{\alpha w - \frac{1}{2}\alpha^2 T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(m, w) dw dm \\ &= \frac{\partial}{\partial y} \int_{-\infty}^y -e^{\alpha w - \frac{1}{2}\alpha^2 T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, w) dw \\ &= -e^{\alpha y - \frac{1}{2}\alpha^2 T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) \end{aligned} \tag{2.9}$$

So, combining (2.8) and (2.9) and plugging in for $\tilde{f}$, we get

$$\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) = e^{\alpha y - \frac{1}{2}\alpha^2 T} \tilde{f}_{\widetilde{M}_T, \widetilde{W}_T}(x, y) = \frac{2(2x - y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x - y)^2}$$

for all x and y , with $x \geq 0$ and $y \leq x$. $\square$

Calculation of I_2 :

Knowing the density $\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}$, we can rewrite

$$I_2 = \int_{\{\widetilde{M}_T \leq b, \widetilde{W}_T > k\}} \bar{f}_{\widetilde{M}_T, \widetilde{W}_T} d\bar{\mathbb{P}} \quad (2.10)$$

into a double (Riemann-) integral, but in order to do this, we have to figure out the integration limits first:

Adding the bounds k and b to the requirement $x \geq 0$ and $y \leq x$, we look at two cases:

If $k < 0$, we get $\{(x, y) : k \leq y \leq x, 0 \leq x \leq b\}$ and
if $k \geq 0$, we get $\{(x, y) : k \leq y \leq x \leq b\}$.

We can capture both cases by the set

$$\{(x, y) : k \leq y \leq x, y^+ \leq x \leq b\},$$

where $y^+ = y$ if $y \geq 0$ and $y^+ = 0$ else.

So the double integral of (2.10) reads

$$I_2 = \int_k^b \int_{y^+}^b \frac{2(2x - y)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x - y)^2} dx dy.$$

Solving the first integral and rearranging terms, we get

$$\begin{aligned} I_2 &= - \int_k^b \frac{1}{\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x - y)^2} \Big|_{x=y^+}^{x=b} dy \\ &= - \int_k^b \frac{1}{\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b - y)^2} dy + \int_k^b \frac{1}{\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}y^2} dy \\ &= - \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta_1 + \delta_1 y - \frac{1}{2T}y^2} dy + \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta_2 + \delta_2 y - \frac{1}{2T}y^2} dy, \end{aligned}$$

where $\beta_1 = -\frac{1}{2}\alpha^2 T - \frac{2b^2}{T}$, $\delta_1 = \alpha + \frac{2b}{T}$ and $\beta_2 = -\frac{1}{2}\alpha^2 T$, $\delta_2 = \alpha$.

Furthermore, we get

$$\begin{aligned}
& -\frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta_1 + \delta_1 y - \frac{1}{2T} y^2} dy + \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta_2 + \delta_2 y - \frac{1}{2T} y^2} dy, \\
& = -e^{\beta_1 + \frac{1}{2}\delta_1^2 T} \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-\frac{1}{2}\left(\frac{y-\delta_1 T}{\sqrt{T}}\right)^2} dy + e^{\beta_2 + \frac{1}{2}\delta_2^2 T} \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-\frac{1}{2}\left(\frac{y-\delta_2 T}{\sqrt{T}}\right)^2} dy
\end{aligned}$$

Substituting $z_1 = \frac{y-\delta_1 T}{\sqrt{T}}$, $z_2 = \frac{y-\delta_2 T}{\sqrt{T}}$ and $dy = \sqrt{T} dz_i$, $i = 1, 2$, leads to

$$\begin{aligned}
& -e^{\beta_1 + \frac{1}{2}\delta_1^2 T} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T}}(k-\delta_1 T)}^{\frac{1}{\sqrt{T}}(b-\delta_1 T)} e^{-\frac{1}{2}z_1^2} dz_1 + e^{\beta_2 + \frac{1}{2}\delta_2^2 T} \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T}}(k-\delta_2 T)}^{\frac{1}{\sqrt{T}}(b-\delta_2 T)} e^{-\frac{1}{2}z_2^2} dz_2 \\
& = -e^{\beta_1 + \frac{1}{2}\delta_1^2 T} \left[N\left(\frac{b-\delta_1 T}{\sqrt{T}}\right) - N\left(\frac{k-\delta_1 T}{\sqrt{T}}\right) \right] \\
& \quad + e^{\beta_2 + \frac{1}{2}\delta_2^2 T} \left[N\left(\frac{b-\delta_2 T}{\sqrt{T}}\right) - N\left(\frac{k-\delta_2 T}{\sqrt{T}}\right) \right], \tag{2.11}
\end{aligned}$$

where $N(x)$ denotes the standard cumulative normal distribution up to point x .

First we resolve the factors before the brackets in (2.11).

Using the definitions of α , b , β_1 and δ_1 for the first factor, we get

$$\begin{aligned}
e^{\beta_1 + \frac{1}{2}(\alpha + \frac{2b}{T})^2 T} & = e^{\beta_1 + \frac{1}{2}\alpha^2 T + \frac{2b^2}{T} + 2b\alpha} = e^{2b\alpha} \\
& = e^{\frac{2}{\sigma} \ln \frac{B}{S_0} \frac{1}{\sigma} (r - \frac{1}{2}\sigma^2)} = e^{\ln\left(\frac{B}{S_0}\right) \frac{2r}{\sigma^2} - 1} \\
& = \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} \tag{2.12}
\end{aligned}$$

Analogously we can resolve the second factor and we get

$$e^{\beta_2 + \frac{1}{2}\alpha^2 T} = e^0 = 1. \tag{2.13}$$

Now we plug in α , b and δ_1 into the cumulative normal distribution and for the first brackets in (2.11) we get

$$N\left(\frac{1}{\sqrt{T}}(b - \delta_1 T)\right) - N\left(\frac{1}{\sqrt{T}}(k - \delta_1 T)\right) =$$

$$\begin{aligned}
& N\left(\frac{1}{\sqrt{T}}\left(\frac{1}{\sigma}\ln\frac{B}{S_0}-\left(\alpha+\frac{2b}{T}\right)T\right)\right)-N\left(\frac{1}{\sqrt{T}}\left(\frac{1}{\sigma}\ln\frac{K}{S_0}-\left(\alpha+\frac{2b}{T}\right)T\right)\right) \\
&= N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B}{S_0}-\left(r-\frac{1}{2}\sigma^2\right)T-2\ln\frac{B}{S_0}\right]\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{K}{S_0}-\left(r-\frac{1}{2}\sigma^2\right)T+2\ln\frac{S_0}{B}\right]\right) \\
&= N\left(\frac{1}{\sigma\sqrt{T}}\left[-\ln\frac{B}{S_0}-\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left[-\ln\frac{B^2}{KS_0}-\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \\
&= 1-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B}{S_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-\left(1-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)\right) \\
&\quad = N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad \quad -N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B}{S_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \tag{2.14}
\end{aligned}$$

Analogously we rewrite the second term and hence

$$\begin{aligned}
& N\left(\frac{b-\delta_2T}{\sqrt{T}}\right)-N\left(\frac{k-\delta_2T}{\sqrt{T}}\right) \\
&= N\left(\frac{1}{\sigma\sqrt{T}}\left[-\ln\frac{S_0}{B}-\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left[-\ln\frac{S_0}{K}-\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \\
&= 1-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{B}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-\left(1-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)\right) \\
&\quad = N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad \quad -N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{B}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right) \tag{2.15}
\end{aligned}$$

Finally, using (2.12), (2.13), (2.14) and (2.15) in (2.11), we can compute I_2 :

$$\begin{aligned}
I_2 &= -\left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2}-1\right)}\left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B}{S_0}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)\right] \\
&\quad +N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right)-N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{B}+\left(r-\frac{1}{2}\sigma^2\right)T\right]\right).
\end{aligned}$$

Calculation of I_1 :

Instead of calculating

$$I_1 = \int_{\{\widetilde{M}_T \leq b, \widetilde{W}_T > k\}} e^{\sigma \widetilde{W}_T - \frac{1}{2} \sigma^2 T} \widetilde{f}_{\widetilde{M}_T, \widetilde{W}_T} d\widetilde{\mathbb{P}}$$

and transform it into a double integral, like we did for I_2 , we will change the measure by applying Girsanov's theorem and end up in the same situation as for I_2 .

We start with the initial definition of S_t :

$$S_t = S_0 e^{\sigma \overline{W}_t + (r - \frac{1}{2} \sigma^2) t}.$$

Next, we set

$$\begin{aligned} Z_T &= e^{\sigma \overline{W}_T - \frac{1}{2} \sigma^2 T} \\ \widehat{W}_t &= \overline{W}_t - \sigma t \end{aligned}$$

Then, applying Girsanov's theorem, defining our new measure as

$$\widehat{\mathbb{P}}(A) = \int_A Z_T d\overline{\mathbb{P}}, \quad A \in \mathcal{F},$$

$\{\widehat{W}_t\}$ becomes a $\widehat{\mathbb{P}}$ -Brownian motion.

So we can rewrite I_1 in the following way:

$$\mathbb{E}_{\overline{\mathbb{P}}} \left[e^{-rT} S_T \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right] = \mathbb{E}_{\widehat{\mathbb{P}}} \left[e^{-rT} \frac{1}{Z_T} S_T \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right].$$

Since

$$e^{-rT} \frac{1}{Z_T} S_T = e^{-rT} e^{-\sigma \overline{W}_T + \frac{1}{2} \sigma^2 T} S_0 e^{\sigma \overline{W}_T + (r - \frac{1}{2} \sigma^2) T} = S_0,$$

we get

$$\begin{aligned} & \mathbb{E}_{\widehat{\mathbb{P}}} \left[e^{-rT} \frac{1}{Z_T} S_T \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right] \\ &= S_0 \mathbb{E}_{\widehat{\mathbb{P}}} \left[\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right]. \end{aligned} \tag{2.16}$$

Rewriting S_t in terms of $\widehat{W}_t$, gives us

$$S_t = S_0 e^{\sigma \overline{W}_t + (r - \frac{1}{2} \sigma^2) t} = S_0 e^{\sigma [\widehat{W}_t + \sigma t] + (r - \frac{1}{2} \sigma^2) t} = S_0 e^{\sigma \widehat{W}_t + (r + \frac{1}{2} \sigma^2) t}$$

Similar like we did for I_2 , we define $\tilde{W}_t = \widehat{W}_t + \lambda t$, with $\lambda = \frac{1}{\sigma}(r + \frac{1}{2}\sigma^2)$ and $\tilde{M}_T = \max_{0 \leq t \leq T} \tilde{W}_t$, this implies

$$S_t = S_0 e^{\sigma \tilde{W}_t} \text{ for all } t \text{ and } \max_{0 \leq t \leq T} S_t = e^{\sigma \tilde{M}_T}.$$

Now we can rewrite the set of the indicator function, like we did it for I_2 :

$$\left\{ \max_{0 \leq t \leq T} S_t < B, S_T > K \right\} = \{ \tilde{M}_T < b, \tilde{W}_T > k \},$$

for b and k like above.

Therefore, (2.16) can be rewritten into

$$\mathbb{E}_{\widehat{\mathbb{P}}} \left[\mathbb{I}_{\left\{ \max_{0 \leq t \leq T} S_t < B, S_T > K \right\}} \right] = \mathbb{E}_{\widehat{\mathbb{P}}} \left[\mathbb{I}_{\{ \tilde{M}_T < b, \tilde{W}_T > k \}} \right].$$

We observe, that this can be solved exactly like I_2 , replacing the $\overline{\mathbb{P}}$ -Brownian motion $\overline{W}_t$ with the $\widehat{\mathbb{P}}$ -Brownian motion $\widehat{W}_t$ and replacing the drift term α in I_2 with λ in I_1 .

Hence,

$$\begin{aligned} I_1 = & -e^{\beta_3 + \frac{1}{2}\delta_3^2 T} \left[N\left(\frac{b - \delta_3 T}{\sqrt{T}}\right) - N\left(\frac{k - \delta_4 T}{\sqrt{T}}\right) \right] \\ & + e^{\beta_4 + \frac{1}{2}\delta_4^2 T} \left[N\left(\frac{b - \delta_4 T}{\sqrt{T}}\right) - N\left(\frac{k - \delta_4 T}{\sqrt{T}}\right) \right], \end{aligned} \quad (2.17)$$

where $\beta_3 = -\frac{1}{2}\lambda^2 T - \frac{2b^2}{T}$, $\delta_3 = \lambda + \frac{2b}{T}$ and $\beta_4 = -\frac{1}{2}\lambda^2 T$, $\delta_4 = \lambda$.

So plugging in, calculating the factors before the brackets in (2.17) and rearranging terms, like in (2.11), we get

$$\begin{aligned} I_1 = & -\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+2} \left[N\left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T \right] \right) - N\left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + \left(r + \frac{1}{2}\sigma^2\right)T \right] \right) \right] \\ & + \left[N\left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T \right] \right) - N\left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + \left(r + \frac{1}{2}\sigma^2\right)T \right] \right) \right]. \end{aligned}$$

So to summarize, we get the following

Price of the UOC option:

$$UOC = S_0 I_1 - e^{-rT} K I_2$$

$$\begin{aligned} &= -B \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2} + 1} \left[N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{B^2}{K S_0} + \left(r + \frac{1}{2} \sigma^2 \right) T \right] \right) - N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{B}{S_0} + \left(r + \frac{1}{2} \sigma^2 \right) T \right] \right) \right] \\ &\quad + S_0 \left[N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2} \sigma^2 \right) T \right] \right) - N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{S_0}{B} + \left(r + \frac{1}{2} \sigma^2 \right) T \right] \right) \right] \\ &\quad + e^{-rT} K \left(\frac{B}{S_0} \right)^{\left(\frac{2r}{\sigma^2} - 1 \right)} \left[N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{B^2}{K S_0} + \left(r - \frac{1}{2} \sigma^2 \right) T \right] \right) - N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{B}{S_0} + \left(r - \frac{1}{2} \sigma^2 \right) T \right] \right) \right] \\ &\quad - e^{-rT} K \left[N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2} \sigma^2 \right) T \right] \right) - N \left(\frac{1}{\sigma \sqrt{T}} \left[\ln \frac{S_0}{B} + \left(r - \frac{1}{2} \sigma^2 \right) T \right] \right) \right]. \end{aligned}$$

2.3 Pricing of DOC-, UIC-, DIC Options

Now we want to derive pricing formulas for the remaining barrier options, which we listed in the introduction, i.e. DOC, UIC and DIC.

As we have seen in the first chapter, the pay-off formulas are given by

$$DOC_T = (S_T - K) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}$$

$$UIC_T = (S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B, S_T > K\}}$$

$$DIC_T = (S_T - K) \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t < B, S_T > K\}}$$

As we see, there does not change a lot in the pay-offs from case to case, with respect to the UOC pay-off, so their prices can be derived quite similar to the one of the UOC.

First, we will derive the price of a DOC, using the same approach as above.

The main difference to the UOC is, that we are dealing here with the minimum of the stock price process.

Price of a DOC Option

First of all we assume that $K \geq B$.

Given the pay-off of the DOC option DOC_T , we again use the pricing formula (2.4) in order to get:

$$\begin{aligned} DOC &= \mathbb{E}_{\mathbb{P}}[e^{-rT}(S_T - K)\mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}] \\ &= S_0 \mathbb{E}_{\mathbb{P}}[e^{\sigma \bar{W}_T - \frac{1}{2}\sigma^2 T} \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}] - e^{-rT} K \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}] \\ &= S_0 I_1 - e^{-rT} K I_2 \end{aligned} \quad (2.18)$$

and we calculate I_1 and I_2 separately, starting again with I_2 .

Again, we call $\widetilde{W}_t = \bar{W}_t + \alpha t$, with $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$ and now $\widetilde{m}_T = \min_{0 \leq t \leq T} \widetilde{W}_t$, so

$$S_T = S_0 e^{\sigma \widetilde{W}_T} \text{ and } \min_{0 \leq t \leq T} S_t = S_0 e^{\sigma \widetilde{m}_T}.$$

Furthermore

$$\{\min_{0 \leq t \leq T} S_t > B, S_T > K\} = \{\widetilde{m}_T > b, \widetilde{W}_T > k\},$$

where $b = \frac{1}{\sigma} \ln \frac{B}{S_0}$ and $k = \frac{1}{\sigma} \ln \frac{K}{S_0}$.

In order to calculate

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{\widetilde{m}_T > b, \widetilde{W}_T > k\}}]$$

we need the joint density again, but this time of the pair $(\widetilde{m}_T, \widetilde{W}_T)$ under $\mathbb{P}$, which is slightly different from $\bar{f}_{\widetilde{m}_T, \widetilde{W}_T}$, but the derivation is similar, so we will do it briefly.

Density of $(\widetilde{m}_T, \widetilde{W}_T)$:

First we derive the density $\widetilde{f}_{\widetilde{m}_T, \widetilde{W}_T}$ of a Brownian motion without drift $(\widetilde{m}_T, \widetilde{W}_T)$

We use the same notation as above and will not repeat the measure change.

The key difference comes from the "reverse" reflection principle, which says

$$\tilde{\mathbb{P}}\{\tilde{m}_T \leq x, \tilde{W}_T \geq y\} = \tilde{\mathbb{P}}\{\tilde{W}_T \leq 2x - y\}, \quad (2.19)$$

for $x \leq 0$ and $y \geq x$.

Proof: Let $x \leq 0$ and $x \leq y$. Then $-x \geq 0$ and $-x \geq -y$, therefore

$$\begin{aligned} \{\tilde{m}_T \leq x, \tilde{W}_T \geq y\} &= \{-\tilde{m}_T \geq -x, -\tilde{W}_T \leq -y\} \\ &= \{\max_{0 \leq t \leq T} (-\tilde{W}_t) \geq -x, -\tilde{W}_T \leq -y\}. \end{aligned}$$

Since $-\tilde{W}_t$ is a Brownian motion as well, we get that $\tilde{M}_T$ has the same distribution as $\max_{0 \leq t \leq T} \tilde{W}_t$ and so

$$\begin{aligned} \tilde{\mathbb{P}}\{\tilde{m}_T \leq x, \tilde{W}_T \geq y\} &= \tilde{\mathbb{P}}\{\tilde{M}_T \geq -x, -\tilde{W}_T \leq -y\} \\ &= \tilde{\mathbb{P}}\{-\tilde{W}_T \geq -2x + y\} = \tilde{\mathbb{P}}\{\tilde{W}_T \leq 2x - y\}, \end{aligned}$$

where the second equality in the first line is the reflection principle from (2.5). $\square$

From (2.19), we get

$$\int_{-\infty}^x \int_y^{\infty} \tilde{f}_{\tilde{M}_T, \tilde{W}_T}(m, w) dw dm = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2x-y} e^{-\frac{1}{2T}z^2} dz, \quad (2.20)$$

deriving both sides again, with respect to x and y , we get

$$\frac{\partial}{\partial y \partial x} \int_{-\infty}^x \int_y^{\infty} \tilde{f}_{\tilde{M}_T, \tilde{W}_T}(m, w) dw dm = -\tilde{f}_{\tilde{M}_T, \tilde{W}_T}(x, y)$$

for the left side of equation (2.20) and for the right side, we get

$$\frac{\partial}{\partial y \partial x} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2x-y} e^{-\frac{1}{2T}z^2} dz = \frac{\partial}{\partial y} \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2} = \frac{2(2x-y)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2}$$

Hence,

$$\tilde{f}_{\tilde{M}_T, \tilde{W}_T}(x, y) = -\frac{2(2x-y)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2} = \frac{2(y-2x)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2x-y)^2}.$$

Since the only thing that is different comes from the factor $\frac{2(y-2x)}{T\sqrt{2\pi T}}$, (compared to $\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}$) and this factor doesn't influence the result for $\bar{f}_{\widetilde{M}_T, \widetilde{W}_T}$ how we derived it in chapter 2.2, we can adopt it for $\bar{f}_{\widetilde{m}_T, \widetilde{W}_T}$ and we get

$$\bar{f}_{\widetilde{m}_T, \widetilde{W}_T}(x, y) = \frac{2(y-2x)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x-y)^2}, \text{ for } x \leq y \text{ and } x \leq 0.$$

So, in order to rewrite

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{\widetilde{m}_T > b, \widetilde{W}_T > k\}}] = \int_{\{\widetilde{m}_T \geq b, \widetilde{W}_T > k\}} \bar{f}_{\widetilde{m}_T, \widetilde{W}_T} d\mathbb{P}.$$

into a double integral, we have to find the integration limits (which differ from the ones of the double integral of I_2 in chapter 2.2).

Since $\widetilde{m}_T$ has to stay above b and $\widetilde{W}_T$ above k , the area is given by $\{(x, y) : b \leq x \leq y, k \leq y\}$, therefore

$$I_2 = \int_k^\infty \int_b^y \frac{2(y-2x)}{T\sqrt{2\pi T}} e^{\alpha y - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2x-y)^2} dx dy$$

Similar as before, we calculate these integrals and we get

$$\begin{aligned} I_2 &= -e^{-\frac{1}{2}\alpha_2 T + \frac{1}{2}\alpha^2 T} \frac{1}{\sqrt{2\pi T}} \int_{\frac{k-\alpha T}{\sqrt{T}}}^\infty e^{-\frac{1}{2}z^2} \sqrt{T} dz \\ &+ e^{-\frac{1}{2}\alpha_2 T - \frac{2b^2}{T} + \frac{(2b-\alpha T)^2}{2T}} \frac{1}{\sqrt{2\pi T}} \int_{\frac{k-\alpha T-2b}{\sqrt{T}}}^\infty e^{-\frac{1}{2}z^2} \sqrt{T} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}z^2} dz - c \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\alpha T+2b-k}{\sqrt{T}}} e^{-\frac{1}{2}z^2} dz \\ &= N\left(\frac{\alpha T - k}{\sqrt{T}}\right) - c N\left(\frac{\alpha T + 2b - k}{\sqrt{T}}\right), \end{aligned} \tag{2.21}$$

where

$$c = e^{-\frac{1}{2}\alpha_2 T - \frac{2b^2}{T} + \frac{(2b-\alpha T)^2}{2T}}.$$

Like in chapter 2.2 we can plug in for c and resolve and we get the same result as in (2.11), i.e.

$$c = e^{2\alpha b} = \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1}.$$

Plugging in α , b and c in (1.21), gives us

$$I_2 = N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) - \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right).$$

We will solve I_1 the same way as in chapter 2.2. that's why we do it very briefly and skip all the details regarding the measure change. So

$$\begin{aligned} I_1 &= \mathbb{E}_{\mathbb{P}}\left[e^{\sigma\bar{W}_T - \frac{1}{2}\sigma^2 T} \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}\right] \\ &= \mathbb{E}_{\hat{\mathbb{P}}}\left[\frac{1}{Z_T} e^{\sigma\bar{W}_T - \frac{1}{2}\sigma^2 T} \mathbb{I}_{\{\min_{0 \leq t \leq T} S_t > B, S_T > K\}}\right] \\ &= \mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{I}_{\{\min_{0 \leq t \leq T} S_t \geq B, S_T > K\}}\right], \end{aligned}$$

where $\hat{\mathbb{P}}$ and Z_T are defined as in the previous section.

Rewriting S_t in terms of $\widehat{W}_t$, gives us

$$S_t = S_0 e^{\sigma\bar{W}_t + (r - \frac{1}{2}\sigma^2)t} = S_0 e^{\sigma\widehat{W}_t + (r + \frac{1}{2}\sigma^2)t}$$

and $\tilde{W}_t = \widehat{W}_t + \lambda t$, with $\lambda = \frac{1}{\sigma}(r + \frac{1}{2}\sigma^2)$ and $\tilde{m}_T = \min_{0 \leq t \leq T} \tilde{W}_t$, this implies

$$S_t = S_0 e^{\sigma\tilde{W}_t} \text{ for all } t \text{ and } \min_{0 \leq t \leq T} S_t = e^{\sigma\tilde{m}_T}.$$

Therefore, we get

$$I_1 = \mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{I}_{\{\tilde{m}_T \geq b, \tilde{W}_T > k\}}\right]$$

and we know this expected value from I_2 , we just have to replace α with λ , so

$$I_1 = N\left(\frac{\lambda T - k}{\sqrt{T}}\right) - dN\left(\frac{\lambda T + 2b - k}{\sqrt{T}}\right), \quad (2.22)$$

where $d = e^{2\lambda b} = \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+2}$.
Hence,

$$I_1 = N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - \left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+2} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right).$$

Finally, if we plug in for I_1 and I_2 in (2.22), we can compute the

Price of the DOC Option:

$$\begin{aligned} DOC &= S_0 N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - B\left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) \\ &\quad - e^{-rT} K N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) + e^{-rT} K \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2}-1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right). \end{aligned}$$

Pricing of UIC and DIC:

We could derive the prices of an UIC and a DIC option the same way as for the UOC and DOC, but we will use an easy relation to determine those prices. First we start with the

UIC Price:

The following relation holds:

$$\begin{aligned} &(S_T - K)\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t \leq B, S_T > K\}} + (S_T - K)\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B, S_T > K\}} \\ &= (S_T - K)\left[\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t \leq B, S_T > K\}} + \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t > B, S_T > K\}}\right] \\ &= (S_T - K)\left[\mathbb{I}_{\{\max_{0 \leq t \leq T} S_t \leq B, S_T > K\} \cup \{\max_{0 \leq t \leq T} S_t > B, S_T > K\}}\right] \\ &= (S_T - K)\mathbb{I}_{\{S_T > K\}} = C_T, \end{aligned}$$

where C_T denotes the pay-off of a vanilla call option.

So,

$$UOC_T + UIC_T = C_T \quad (2.23)$$

Taking the discounted expectation under $\bar{\mathbb{P}}$ on both sides of equation (2.23), we get

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{P}}}[e^{-rT}(UOC_T + UIC_T)] &= \mathbb{E}_{\bar{\mathbb{P}}}[e^{-rT}UOC_T] + \mathbb{E}_{\bar{\mathbb{P}}}[e^{-rT}UIC_T] \\ &= UOC + UIC = \mathbb{E}_{\bar{\mathbb{P}}}[e^{-rT}C_T] = C, \end{aligned}$$

hence

$$C = UOC + UIC, \quad (2.24)$$

where UIC and C denote the prices of a UIC and a vanilla call option, respectively.

The price of a call option is given by the well-known Black-Scholes formula (see e.g. in [22], [26] or any other financial mathematics book):

$$C = S_0 N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - e^{-rT} K N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right).$$

Since one can separate the UOC price into two parts, i.e.

$$\begin{aligned} UOC &= S_0 N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - e^{-rT} K N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \\ &\quad - B\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+1} \left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B}{S_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) \right] \\ &\quad + e^{-rT} K \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2}-1\right)} \left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) - N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{B}{S_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \right] \\ &\quad + e^{-rT} K N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{B} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) + S_0 N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln \frac{S_0}{B} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) \\ &= C \end{aligned}$$

$$\begin{aligned}
& -B \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}+1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) \right] \\
& + e^{-rT} K \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}-1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) \right] \\
& + e^{-rT} K N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r - \frac{1}{2}\sigma^2)T \right] \right) + S_0 N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r + \frac{1}{2}\sigma^2)T \right] \right),
\end{aligned}$$

we can use equation (2.24) and resolve for the price of the UIC, we get

$$UIC = C - UOC, \quad (2.25)$$

therefore

$$\begin{aligned}
& UIC = C - C \\
& - \left[-B \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}+1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) \right] \right. \\
& + e^{-rT} K \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}-1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) \right] \\
& \left. + e^{-rT} K N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r - \frac{1}{2}\sigma^2)T \right] \right) + S_0 N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r + \frac{1}{2}\sigma^2)T \right] \right) \right]
\end{aligned}$$

Hence, we get the

Price of the UIC Option:

$$\begin{aligned}
UIC &= B \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}+1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r + \frac{1}{2}\sigma^2)T \right] \right) \right] \\
& - e^{-rT} K \left(\frac{B}{S_0} \right)^{\frac{2r}{\sigma^2}-1} \left[N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B^2}{KS_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) - N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{B}{S_0} + (r - \frac{1}{2}\sigma^2)T \right] \right) \right] \\
& - e^{-rT} K N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r - \frac{1}{2}\sigma^2)T \right] \right) - S_0 N \left(\frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{B} + (r + \frac{1}{2}\sigma^2)T \right] \right).
\end{aligned}$$

DIC Price:

Similar, we get the price of an DIC option, using the relation

$$C = DOC + DIC, \quad (2.26)$$

which can be derived analogously as for *UIC*.

Again we can separate the DOC price into two parts:

$$\begin{aligned}
DOC &= S_0 N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) - e^{-rT} K N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad + e^{-rT} K \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} - 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad - B \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} + 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) \\
&= C \\
&\quad + e^{-rT} K \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} - 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad - B \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} + 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right).
\end{aligned}$$

Resolving (2.26) for DIC and plugging in, we get

$$\begin{aligned}
DIC &= C - DOC \\
&= B \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} + 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]\right) \\
&\quad - e^{-rT} K \left(\frac{B}{S_0}\right)^{\left(\frac{2r}{\sigma^2} - 1\right)} N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{B^2}{KS_0} + \left(r - \frac{1}{2}\sigma^2\right)T\right]\right).
\end{aligned}$$

Remark:

- The fact, that we can separate the DOC- and UOC prices into the price of a vanilla call minus a positive rest, can be interpreted as pricing a knock out call option, as a vanilla call and subtracting the risk premium, that comes from the additional barrier requirement.
- Like we said before, it doesn't make sense for the UOC to require $B \leq K$, since the option has to be knocked out to be in the money, but for the DOC we can choose $B \geq K$, which would be even more restrictive, since the option can be knocked out although the stock price path is in the money at all times.

Chapter 3

Pricing of Barrier Options in the Heston Model

3.1 Shortcomings of the Black-Scholes Model

It is well known that the Black-Scholes Model is based on some unrealistic assumptions, which are all those of a perfect market, amongst others. In real life there are transaction costs, the lending rate is different from the borrowing rate and changes in time and might depend on $\omega \in \Omega$ as well. Furthermore, assuming log-normal distributed stock-returns is not realistic as shown in various empirical studies.

Another well known, crucially wrong assumption, is that volatility of the stock price is constant and known. See e.g. in [22].

In fact, it may seem reasonable to assume that volatility depends on events that influence the stock price.

We use this fact as a motivation to describe volatility as a stochastic process, in addition to the stock price process.

3.2 General Stochastic Volatility Models

We follow [22] and define a general stochastic volatility model, where the volatility of the stock price is described by a general diffusion process.

Although, we want to derive prices for barrier options in the Heston model later on, we use the general definition to point out, where exactly the incompleteness of the model comes into play.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The stock price $\{S_t\}$ and its volatility

process $\{\sigma_t\}$ are given by

$$dS_t = \mu(S_t, t)S_t dt + \sigma_t S_t dW_t^1 \quad (3.1)$$

$$d\sigma_t = \alpha_1(\sigma_t, t)dt + \alpha_2(\sigma_t, t)dW_t^2 \quad (3.2)$$

where $\{W_t^i\}$, $i = 1, 2$ are correlated (standard) Brownian motions, with covariation given by

$$[W^1, W^2]_t = \int_0^t \rho_u du.$$

(see A.1.4 for a general definition of covariation stochastic processes) or in differential notation

$$dW_t^1 dW_t^2 := d[W^1, W^2]_t = \rho_t dt$$

for $t \in [0, T]$.

Let $\{\mathcal{F}_t\}$ be a filtration for $\{v_t\}$ and $\{S_t\}$. The correlation $\{\rho_t\}$ shall denote an $\mathcal{F}_t$ -adapted process with $\rho_t \in (-1, 1)$ for all $t \in [0, T]$ and μ and α_i are some functions of the stock price and the volatility at time t , respectively. Finally, we assume a constant interest rate r , like in chapter 2.

Our next goal is to find any measure $\bar{\mathbb{P}}$ equivalent to $\mathbb{P}$, s.t. the discounted stock price process becomes a martingale under $\bar{\mathbb{P}}$.

Following [22, p.267], we decompose W_t^2 as follows:

$$W_t^2 = \int_0^t \rho_u dW_u^1 + \int_0^t \sqrt{1 - \rho_u^2} dW_u^*$$

where $\{W_t^*\}$ is a standard BM on $(\Omega, \mathbb{F}, \mathbb{P})$, independent of $\{W_t^1\}$.

Therefore $(W_t^1, W_t^*) = \mathcal{W}_t$ can be interpreted as a 2-dim. BM on $(\Omega, \mathbb{F}, \mathbb{P})$.

Let $\Theta = (\Theta^1, \Theta^2)$ be an adapted process.

To get rid of the drift term in (3.1) under our new measure, we already know how to chose Θ^1 , namely

$$\Theta_t^1 = \frac{\mu - r}{\sigma_t}.$$

Furthermore, since we want Θ to hold that

$$\mathbb{E} \left[\frac{1}{2} \int_0^T \|\Theta_t\|^2 dt \right] < \infty,$$

in order to be able to apply Girsanov's theorem and this holds if

$$\mathbb{E} \left[\frac{1}{2} \int_0^T \left(\frac{\mu - r}{\sigma_t} \right)^2 dt \right] < \infty$$

and

$$\mathbb{E} \left[\frac{1}{2} \int_0^T (\Theta_t^2) dt \right] < \infty.$$

which is the only assumption we make on Θ^2 .

Next define process $\{Z_t\}$ as

$$Z_t = \exp \left\{ - \int_0^t \Theta_u \times dW_u - \frac{1}{2} \int_0^t \|\Theta_u\|^2 du \right\}. \quad (3.3)$$

and by Girsanov's theorem in appendix A.2.1, we get that, under measure $\bar{\mathbb{P}}$, defined as

$$\bar{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \quad A \in \mathcal{F} \quad (3.4)$$

the 2-dimensional process

$$\bar{W}_t = \left(\bar{W}_t^1 = W_t^1 + \int_0^t \frac{\mu - r}{\sigma_u} du, \bar{W}_t^* = W_t^* + \int_0^t \Theta_u^2 du \right)$$

is a two-dimensional $\bar{\mathbb{P}}$ -B.M.

Clearly $\{\bar{W}_t^1\}$ and $\{\bar{W}_t^*\}$ are also two one-dim. $\bar{\mathbb{P}}$ -B.Ms and for $\{W_t^2\}$ we get

$$\begin{aligned} W_t^2 &= \int_0^t \rho_u (d\bar{W}_u^1 - \frac{\mu - r}{\sigma_u} du) + \int_0^t \sqrt{1 - \rho_u^2} (d\bar{W}_u^* - \Theta_u^2 du) \\ &= \int_0^t \rho_u d\bar{W}_u^1 + \int_0^t \sqrt{1 - \rho_u^2} d\bar{W}_u^* - \int_0^t \left(\rho_u \frac{\mu - r}{\sigma_u} + \sqrt{1 - \rho_u^2} \Theta_u^2 \right) du. \end{aligned}$$

Now setting

$$\begin{aligned} \lambda_t &:= \rho_t \frac{\mu - r}{\sigma_t} + \sqrt{1 - \rho_t^2} \Theta_t^2 \\ \bar{W}_t^2 &= W_t^2 + \int_0^t \lambda_u du \end{aligned}$$

we get that $\{\bar{W}_t^2\}$ is a (one-dim.) $\bar{\mathbb{P}}$ -B.M. as well.

Expressing (3.1) and (3.2) with $\{\bar{W}_t^i\}$, $i = 1, 2$, we get

$$dS_t = rS_t dt + \sigma_t S_t d\bar{W}_t^1$$

$$d\sigma_t = (\alpha_1 - \alpha_2 \lambda_t) \sigma_t dt + \alpha_2 \sigma_t d\bar{W}_t^2,$$

Hence, choosing a process $\{\lambda_t\}$ defines us $\{\Theta_t^2\}$ and clearly

$$\mathbb{E} \left[\frac{1}{2} \int_0^T \lambda_t^2 dt \right] < \infty$$

implies

$$\mathbb{E} \left[\frac{1}{2} \int_0^T (\Theta_t^2)^2 dt \right] < \infty,$$

This means that every process $\{\lambda_t\}$ which fulfills the above condition defines our Radon-Nikodym derivative Z_T with corresponding measure $\bar{\mathbb{P}} = \bar{\mathbb{P}}_\lambda$, under which the discounted stock price process is a martingale.

Usually $\{\lambda_t\}$ is described by

$$\lambda_t = \lambda(\sigma, t)$$

for a (sufficient regular) function λ of the variance process and is called *market price of vol. risk*. It has to be determined exogenously and "its choice is motivated by practical considerations", [22, p.291].

Therefore, using the same valuation methods as in the previous chapter, i.e. give the arbitrage price of a contingent claim as the discounted expected value under an EMM, leads to different prices, which depend on the particular choice of λ .

Hence, to actually price barrier options in a stochastic volatility model, we have to be more precise about the structure of the volatility process and the "right" choice of the corresponding market price of volatility risk.

Several models were developed in the course of time and they mostly differ in the diffusion process, describing volatility.

The Heston model is one of the most popular ones and is widely used by practitioners in order to price vanilla- and exotic options, (see [13]). For

this reason, we will use it to analyse some methods of how to price barrier options under this specific type of stochastic volatility.

Remark:

We set

$$\Lambda(v, t) = \alpha_2 \lambda(v, t), \tag{3.5}$$

since Heston specifies Λ instead of λ in [14] and gives a motivation for his choice.

3.3 The Heston Model

Following [13], let $\{S_t\}$ be the stock price process and $\{v_t\}$ the volatility process of the stock on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \quad (3.6)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \quad (3.7)$$

where κ , θ and $\sigma > 0$ are constants and the correlation between W_t^1 and W_t^2 is described by the differential of the covariation $dW_t^1 dW_t^2 = \rho dt$, with $\rho \in (-1, 1)$.

The variance process $\{v_t\}$ belongs to the class of square root processes and is called "CIR" (from Cox-Ingersol-Ross), who used it for modeling interest rate behavior. A Further name is "Feller" process (after the mathematician William Feller).

Remark: (Ornstein-Uhlenbeck process)

Instead of (3.7) Heston (in [14]) proposed the so called Ornstein-Uhlenbeck process, given by

$$d\sqrt{v_t} = -\beta\sqrt{v_t}dt + \delta dW_t^2 \quad (3.8)$$

to model the volatility.

We show that this form is equivalent to (3.7) by applying Ito's Lemma, in appendix A.2.2, for $f(t, x) = x^2$, where x is the value of $\sqrt{v_t} = X_t$ at time t .

From Ito's Lemma it holds, that

$$df(t, x) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial^2 x} (dX)^2.$$

So plugging in for f , its derivatives and X we get

$$dv_t = 0dt + 2\sqrt{v_t}d\sqrt{v_t} + \frac{1}{2}2(d\sqrt{v_t})^2.$$

From (3.8), we have

$$d\sqrt{v_t} = -\beta\sqrt{v_t}dt + \delta dW_t^2$$

and

$$(d\sqrt{v_t})^2 = (-\beta\sqrt{v_t}dt + \delta dW_t^2)^2 = \delta^2 dW_t^2 dW_t^2 = \delta^2 dt,$$

since $dt dt = dt dW_t^2 = 0$ and $dW_t^2 dW_t^2$ is the differential of the quadratic variation $[W, W]_t$ of a standard Brownian motion (see appendix A.1.4 for a general definition of quadratic variation) and it holds that $[W, W]_t = t$ a.s. (See also appendix A.1.4 or [26] for the proof, respectively).

Hence,

$$\begin{aligned} dv_t &= 2\sqrt{v_t}(-\beta\sqrt{v_t}dt + \delta dW_t^2) + \delta^2 dt \\ &= (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dW_t^2. \end{aligned} \quad (3.9)$$

Setting $2\beta = \kappa$, $\delta^2/2\beta = \theta$ and $2\delta = \sigma$, we get our initial equation (3.7). But there is a subtle disadvantage to this approach:

Problem: From (3.9) we see that there are only two parameters δ and β defining the volatility process and furthermore, δ occurs even in both terms before the dt and dW_t^2 part. This leads to less flexibility for calibrating the process and more importantly, the structure of the parameters prevents the process to fulfill the Feller condition (see right below in the next paragraph). That is the reason why we start with the CIR process defined in (3.7).

Properties of the CIR Process

The *Feller condition* is defined as

$$2\kappa\theta \geq \sigma^2$$

on the parameters κ , θ and σ from (3.7) and prevents process $\{v_t\}$ to take the value zero, i.e. $v_t > 0$ for all $t \in [0, T]$. See e.g. [17]

In [9] there is a collection of interesting results concerning our CIR process: the moment generating function is derived and shown, that this process follows a non-central chi squared distribution.

Furthermore it is shown, that all moments of the integrated process

$$Y_t = \int_0^t v_u du,$$

exist, i.e.

$$\mathbb{E}[Y_t^r] < \infty \quad (3.10)$$

for $r \in \mathbb{R}$. See [9, p.18] for details and proof.

The CIR process belongs to the class of diffusion processes, therefore its sample paths are continuous but nowhere differentiable. See e.g. [12] or [17].

In the following we will always assume that the Feller condition is fulfilled, since later on we will have expressions are not defined if the variance takes the value zero.

Market Risk and Risk-Neutrality

Heston proposed

$$\Lambda(v, t) = \gamma v_t \quad (3.11)$$

for (3.5), where γ is some constant, with $\gamma \neq -2\beta$ (note that Heston used the Ornstein-Uhlenbeck process (3.8)).

An economic motivation for this particular choice of Λ , can be found in [14, p.329].

We rewrite Λ in terms of the market price of volatility risk λ , so using (3.5), we get

$$\Lambda(v, t) = \gamma v_t = \lambda(v, t) \sigma \sqrt{v_t},$$

hence, $\lambda(v, t) = \bar{\lambda} \sqrt{v_t}$, for some constant $\bar{\lambda}$, with $\gamma = \bar{\lambda} \sigma$.

After the method in the previous subsection about the measure change in the general stochastic volatility model, we get an EMM $\bar{\mathbb{P}}$ for the corresponding $\lambda = \bar{\lambda} \sqrt{v_t}$ if we show that

$$\mathbb{E} \left[\frac{1}{2} \int_0^T \lambda^2 dt \right] < \infty,$$

which follows immediately from (3.10).

Hence, defining the Radon-Nikodin derivative Z_t as in (3.3) with corresponding EMM $\bar{\mathbb{P}}$ as in (3.4), we get the following $\bar{\mathbb{P}}$ -Brownian motions $\{\bar{W}_t^i\}$, $i = 1, 2$:

$$\begin{aligned} d\bar{W}_t^1 &= dW_t^1 + \nu_t dt \\ d\bar{W}_t^2 &= dW_t^2 + \bar{\lambda} \sqrt{v_t} dt, \end{aligned}$$

where $\{\nu_t\}$ is given by

$$\nu_t = \frac{\mu - r}{\sqrt{v_t}}.$$

Therefore (3.6) can be written as

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\bar{W}_t^1 \quad (3.12)$$

and if we plug in for (3.7), we get

$$\begin{aligned} dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}[d\bar{W}_t^2 - \bar{\lambda}\sqrt{v_t}dt] \\ &= (\kappa(\theta - v_t) - \sigma\bar{\lambda}v_t)dt + \sigma\sqrt{v_t}d\bar{W}_t^2 \\ &= (\kappa + \bar{\lambda}\sigma)\left(\frac{\kappa\theta}{\kappa + \bar{\lambda}\sigma} - v_t\right)dt + \sigma\sqrt{v_t}d\bar{W}_t^2 \end{aligned}$$

finally, set $\bar{\kappa} = (\kappa + \bar{\lambda}\sigma)$ and $\bar{\theta} = \frac{\kappa\theta}{\kappa + \bar{\lambda}\sigma}$, in order to get v_t back in the differential form of the CIR process:

$$dv_t = \bar{\kappa}(\bar{\theta} - v_t)dt + \sigma\sqrt{v_t}d\bar{W}_t^2. \quad (3.13)$$

The only assumption on the parameters κ , $\bar{\lambda}$ and σ is that they are chosen such that $\kappa + \bar{\lambda}\sigma \neq 0$.

3.4 Methods and Approaches for the Pricing of Barrier Options in Heston's Stochastic Volatility Model

There exist closed formula solutions for the price of (European) vanilla options, but there are no such solutions for barrier option prices. [11, p.2]

Therefore, several numerical methods were developed in the course of time. We follow [13] and distinguish between *Simulation Methods* on the one hand and *Finite Difference Methods* on the other, in order to categorize the main approaches of evaluating barrier options.

A third approach, which cannot be counted as a proper numerical method, tries to relate Black-Scholes formulas for barrier option prices, to approximations to closed formula expressions for these prices in the Heston model. As Griebisch puts it: " for practitioners in the financial industry, it may be important to understand what makes a stochastic volatility price different from the Black-Scholes price that often serves as the basic hedging tool.

Therefore, a recent focus in the literature has been the development of approximate closed-form option pricing formulas using the main features of the Black-Scholes model.” [13, p.1-2]

In the next section we talk about finite difference methods first and set the ground scene for these type of approaches. But since it is more related to numerically solving PDE’s, we will just give an overview and want to focus more on the approach in approximating pricing formulas, given by Griebisch in [13].

3.4.1 Finite Difference Methods

Motivated by Black and Scholes and their PDE-approach in [3], it is possible to derive an analogue for the Heston model and even for the general stochastic volatility model, presented in section 3.2.

We will present the PDE, which every stock dependent derivative security in the Heston model has to satisfy, but we will skip the derivation, since it is beyond the scope of this thesis. A detailed derivation of this PDE in the general stochastic volatility model, with subsequent application on the Heston model, can be found in [20].

Following Heston in [14], in the absence of arbitrage, the value of any asset $U(S, v, t)$ has to fulfill the following PDE

$$\begin{aligned} \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} \\ + [\kappa(\theta - v_t) - \Lambda(S, v, t)]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0, \end{aligned} \quad (3.14)$$

where $\Lambda(S, v, t)$ is given by (3.11) and κ and θ are the constants from (3.9).

If we plug in for Λ in the brackets in (3.14), we get

$$[\kappa(\theta - v_t) - \Lambda(S, v, t)] = [\kappa(\theta - v_t) - \bar{\lambda}\sigma v_t]$$

and we can rewrite this in terms of $\bar{\kappa}$ and $\bar{\theta}$ like we did in the previous section for dv_t in (3.13), hence

$$[\kappa(\theta - v_t) - \bar{\lambda}\sigma v_t] = [\bar{\kappa}(\bar{\theta} - v_t)].$$

So using the measure change to $\bar{\mathbb{P}}$ in chapter 3.3, reduces the PDE in (3.13) to

$$\begin{aligned} \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma v\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} \\ + \bar{\kappa}(\bar{\theta} - v_t)\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0, \end{aligned} \quad (3.15)$$

In order to numerically solve (3.14) or (3.15), one has to specify the financial derivative $U(S)$ and its boundary and terminal conditions.

So in the case of an UOC option, we set

$$U(S, v, 0) = (S - K)^+$$

with domain $0 \leq S \leq B$, $0 < v < \infty$, with boundary conditions

$$U(0, v, \tau) = 0$$

$$U(B, v, \tau) = 0$$

$$\lim_{v \rightarrow \infty} U(S, v, \tau) = 0$$

with time to maturity $\tau = T - t$.

This function together with its boundary conditions and the corresponding PDE in (3.15) is the basis where most PDE-approaches start. (After [7]).

3.4.2 Approximation Approach

Like mentioned above, in this subsection we want to discuss an approach to approximate the Black-Scholes formulas for barrier options from chapter 2, following [13].

This method, uses probabilistic methods similar to chapter 2 and can be regarded as an extension of it. Again, the price is given by the discounted expected value of the barrier options' pay-off under the chosen risk neutral measure. In the following approach, we consider a UOC option only.

We start with the stock price - and volatility dynamics, described by (3.12)

and (3.13), i.e. we already use the rewritten form of (3.6) and (3.9), respectively, under the risk neutral measure $\bar{\mathbb{P}}$. For simplicity we write $\mathbb{P}$ instead of $\bar{\mathbb{P}}$, W^i instead of $\bar{W}^i$, for $i = 1, 2$ and furthermore, we write κ and θ instead of $\bar{\kappa}$, $\bar{\theta}$.

So the stock price - and volatility dynamics are given by

$$dS_t = rS_t dt + \sqrt{v_t}S_t dW_t^1 \quad (3.16)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2, \quad (3.17)$$

for fixed T and $t \in [0, T]$ and positive constants κ, σ, θ and r , where $\{W_t^i\}$, $i = 1, 2$, are both standard Brownian motions under the risk neutral measure $\mathbb{P}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, (with real world measure $\mathbb{Q}$).

The correlation between the Brownian motions is given by

$$dW_t^1 dW_t^2 = \rho dt$$

with $\rho \in [-1, 1]$.

Let $\{\mathcal{F}_t\}$ be the filtration of $\{S_t\}$ and $\{v_t\}$.

Pay-off of an UOC Option:

We will focus on the pricing of an UOC option, with pay-off given in section 1.2 and 2.2, i.e.

$$UOC_T = (S_T - K)^+ \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B\}} = (S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}}.$$

We distinguish between three cases and then follow four steps to get an exact pricing formula for the first case and approximations to the true prices for the last two cases.

Three Cases:

(A): $\rho = 0$ and $r = 0$

(B): $\rho = 0$ and arbitrary r

(C): $\rho \in [-1, 1]$ and arbitrary r

Four Steps:

1. Condition on the variance paths up to exercise time T .
2. Find the common density of the Brownian motion and its maximum for all cases.
3. Using the density, compute the price by calculating the expected value of the UOC option's pay-off conditioned on the variance paths.
4. Calculate the outer expectation.

1. Conditioning:

We start with rewriting (3.17) into its integral form:

$$v_t = v_0 + \int_0^t \kappa(\theta - v_s) ds + \int_0^t \sigma \sqrt{v_s} dW_s^2,$$

where v_0 is the (given) starting value of $\{v_t\}$. We reformulate this into

$$\int_0^t \sqrt{v_s} dW_s^2 = \frac{1}{\sigma} (v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s ds). \quad (3.18)$$

Now, we want to express W_t^1 (from the stock price dynamics) as the sum of W_t^2 and another standard Brownian motion $\tilde{W}_t$, independent of W_t^2 , for all $t \in [0, T]$.

For every t it holds, that W_t^i , $i = 1, 2$, are normal distributed random variables, with zero mean and t variance. Hence, the variance-covariance matrix V of the pair $(W_t^2, W_t^1)^T$ is given by

$$V = \mathbb{V}[(W^2, W^1)^T] = \begin{pmatrix} \mathbb{V}[W^2] & \text{Cov}[W^2, W^1] \\ \text{Cov}[W^2, W^1] & \mathbb{V}[W^1] \end{pmatrix} = \begin{pmatrix} t & \rho t \\ \rho t & t \end{pmatrix}$$

We use the Cholesky decomposition of V , see appendix A.2.3, i.e.

$$V = \begin{pmatrix} \sqrt{t} & 0 \\ \rho\sqrt{t} & \sqrt{t-\rho^2t} \end{pmatrix} \begin{pmatrix} \sqrt{t} & \rho\sqrt{t} \\ 0 & \sqrt{t-\rho^2t} \end{pmatrix} = RR^T,$$

then, there exists some standard normally distributed random vector $(\phi^1, \phi^2)^T$, s.t.

$$\begin{pmatrix} W_t^2 \\ W_t^1 \end{pmatrix} = R \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}.$$

Hence,

$$W_t^2 = \sqrt{t}\phi^1 \quad (3.19)$$

$$\begin{aligned} W_t^1 &= \rho\sqrt{t}\phi^1 + \sqrt{t-\rho^2t}\phi^2 \\ &= \rho\sqrt{t}\phi^1 + \sqrt{1-\rho^2}\sqrt{t}\phi^2 \end{aligned} \quad (3.20)$$

and if we define $\phi^2 := \frac{1}{\sqrt{t}}W_t$ and plug in for ϕ^1 of (3.19) in (3.20), we get

$$W_t^1 = \rho W_t^2 + \sqrt{1-\rho^2}W_t, \quad t \in [0, T]. \quad (3.21)$$

Following [26], the general solution of the SDE in (3.16) for the stock price S_t , is given by

$$\begin{aligned} S_t &= S_0 \exp \left\{ \int_0^t \left(r - \frac{1}{2}(\sqrt{v_s})^2 \right) ds + \int_0^t \sqrt{v_s} dW_s^1 \right\} \\ &= S_0 \exp \left\{ rt - \frac{1}{2} \int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s^1 \right\}. \end{aligned}$$

Using (3.21) for dW_s^1 , i.e.

$$dW_s^1 = d\left[\rho W_s^2 + \sqrt{1-\rho^2}W_s\right] = \rho dW_s^2 + \sqrt{1-\rho^2}dW_s,$$

we get

$$S_t = S_0 \exp \left\{ rt - \frac{1}{2} \int_0^t v_s ds + \rho \int_0^t \sqrt{v_s} dW_s^2 + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} dW_s \right\} \quad (3.22)$$

Now we use equation (3.18) and multiply both sides with ρ and replace $\int_0^t \sqrt{v_s} dW_s^2$ by the right hand side of equation (3.18), hence

$$S_t = S_0 \exp \left\{ rt - \frac{1}{2} \int_0^t v_s ds + \frac{\rho}{\sigma} (v_t - v_0 - \kappa \theta t + \kappa \int_0^t v_s ds) + \sqrt{1 - \rho^2} \int_0^t \sqrt{v_s} dW_s \right\}. \quad (3.23)$$

Before we technically start with the conditioning, we substitute the drift part of (3.23) by γ_t :

$$\gamma_t = rt - \frac{1}{2} \int_0^t v_s ds + \frac{\rho}{\sigma} (v_t - v_0 - \kappa \theta t + \kappa \int_0^t v_s ds) \quad (3.24)$$

and furthermore, the stochastic integral term by R_t :

$$R_t = \sqrt{1 - \rho^2} \int_0^t \sqrt{v_s} dW_s. \quad (3.25)$$

Similar to chapter 2, we set

$$\tilde{R}_t = R_t + \gamma(t) \quad (3.26)$$

$$\tilde{M}_t = \max_{0 \leq s \leq t} \{R_s + \gamma(s)\}. \quad (3.27)$$

So we can rewrite the pay-off of the stock price process the following way:

$$(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} = (S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} \quad (3.28)$$

where $b = \ln \frac{B}{S_0}$ and $k = \ln \frac{K}{S_0}$.

Now, we define the filtration of the variance process $\{\mathcal{F}_t^v\}$, as $\mathcal{F}_t^v = \sigma(v_s : 0 \leq s \leq t)$, $t \in [0, T]$.

In the following, we need the "conditional" probability measure $\mathbb{P}^v$, defined in appendix A.2.6, i.e. for any function $h : \{W(\omega) : \omega \in \Omega\} \times \{v(\omega) : \omega \in \Omega\} \rightarrow \mathbb{R}$, we have

$$\mathbb{P}^v \{h(W, v) < a\} := \mathbb{P} \{h(W, \tilde{v}) < a\}, \quad a \in \mathbb{R}$$

where $\tilde{v} = \{\tilde{v}_t\}$ is deterministic and can be considered as a non-random dummy variable. The reason behind all that, is the independence lemma, given in A.2.6.

Pay-off and Pricing Formula:

Like in chapter 2, we use the pricing formula (2.4), i.e.

$$UOC = e^{-rT} \mathbb{E} \left[(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} \right],$$

since clearly $\mathcal{F}_0 \subseteq \mathcal{F}_T^v$, we can use the tower property of conditional expectations, see e.g. [26], and we get

$$\begin{aligned} UOC &= e^{-rT} \mathbb{E} \left[(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} | \mathcal{F}_0 \right] \\ &= e^{-rT} \mathbb{E} \left[\mathbb{E} \left[(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} | \mathcal{F}_T^v \right] | \mathcal{F}_0 \right] \end{aligned}$$

Hence, we can split the pricing process into two parts, i.e. calculating the conditional expectation, which we call the *inner expectation*,

$$\mathbb{E} \left[(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} | \mathcal{F}_T^v \right],$$

which is - by definition of the conditional expectation - a random variable and as a second part, we take the expected value, called *outer expectation*, of the result.

Because of (3.28) we can write the inner expectation in the following way:

$$\mathbb{E} \left[(S_T - K) \mathbb{I}_{\{\max_{0 \leq t \leq T} S_t < B, S_T > K\}} | \mathcal{F}_T^v \right] = \mathbb{E} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} | \mathcal{F}_T^v \right] =: E^v$$

As a result of the independence lemma in appendix A.2.6, we can calculate E^v in two steps, i.e. first calculate

$$\mathbb{E}_{\mathbb{P}^v} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} \right] =: E_{\mathbb{P}^v}^v \quad (3.29)$$

and secondly, replacing the dummy variance function $\{\tilde{v}_t\}$ with the initial variance process $\{v_t\}$, in order to get E^v .

In order to calculate $E_{\mathbb{P}^v}^v$, we have to find the joint density of the pair $(\widetilde{M}_T, \widetilde{R}_T)$ under $\mathbb{P}^v$. This is content of the second step.

2. Calculating Densities.

The derivation of the densities will be similar to chapter 2. This means, at first, we find the joint density of the drift-less pair (M_t, R_t) , where $M_t = \max R_s$ and this result holds for all three cases.

Then, for each case individually, we find the exact density of $(\widetilde{M}_T, \widetilde{R}_T)$ for case A and approximation formulas for the densities for cases B and C.

To find the joint density of (M_t, R_t) , we follow the same steps as in the second chapter, therefore we will do it rather briefly.

Density of (M_t, R_t) :

Again, the key relation is given by a reflection principle, but since R_t is not a Brownian motion, we can not use the same as in chapter 2 (or A.1.3). Nevertheless, there is a version for R_t and we refer to the appendix of [13], where this is explained in detail. The result applied to our case of (M_t, R_t) is

$$\mathbb{P}^v\{M_t \geq x, R_t \leq y\} = \mathbb{P}^v\{R_t \geq 2x - y\} \quad (3.30)$$

for $x > 0$ and $y \leq x$.

Theorem 3.1:

The joint density function f_{M_t, R_t} of the pair (M_t, R_t) , is given by

$$f_{M_t, R_t}(x, y) = \frac{2(2x - y)}{\nu_t^3 \bar{\rho}^3 \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(2x - y)^2}{\bar{\rho}^2 \nu_t^2}} \quad (3.31)$$

for $x > 0$ and $y \leq x$, where

$$\nu_t^2 = \int_0^t v_s ds \text{ and } \bar{\rho} = \sqrt{1 - \rho^2} \quad (3.32)$$

Proof:

Since, by definition, we have

$$\mathbb{P}^v\{M_t \geq x, R_t \leq y\} = \int_x^\infty \int_{-\infty}^y f_{M_t, R_t}(m, w) dm dw$$

and we know from (A.28) in appendix A.2.6, that R_t is normally distributed (under $\mathbb{P}^v$), therefore

$$\mathbb{P}^v\{R_t \geq 2x - y\} = \frac{1}{\nu_t \bar{\rho} \sqrt{2\pi}} \int_{2x-y}^\infty e^{-\frac{1}{2} \frac{z^2}{\bar{\rho}^2 \nu_t^2}} dz.$$

Hence, from the extended version of the reflection principle in (3.30), we get

$$\int_x^\infty \int_{-\infty}^y f_{M_t, R_t}(m, w) dm dw = \frac{1}{\nu_t \bar{\rho} \sqrt{2\pi}} \int_{2x-y}^\infty e^{-\frac{1}{2} \frac{z^2}{\bar{\rho}^2 \nu_t^2}} dz$$

and partial differentiation with respect to x gives us

$$-\int_{-\infty}^y f_{M_t, R_t}(x, w) dw = -\frac{2}{\nu_t \bar{\rho} \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(2x-y)^2}{\bar{\rho}^2 \nu_t^2}},$$

then with respect to y we get

$$-f_{M_t, R_t}(x, y) = -\frac{2(2x-y)}{\nu_t^3 \bar{\rho}^3 \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(2x-y)^2}{\bar{\rho}^2 \nu_t^2}}.$$

□

Now we treat each case individually, in order to get the joint distribution of $(\widetilde{M}_T, \widetilde{R}_T)$.

Case A: ($\rho = 0 = r$)

We will show, that the joint distribution of the pair $(\widetilde{M}_T, \widetilde{R}_T)$ is given by

$$f_{\widetilde{M}_T, \widetilde{R}_T}(x, y) = \frac{2(2x-y)}{\nu_T^3 \sqrt{2\pi}} e^{-\frac{1}{2}y - \frac{1}{8}\nu_T^2 - \frac{1}{2} \frac{(2x-y)^2}{\nu_T^2}} \quad \text{for } x > 0 \text{ and } y \leq x.$$

Proof:

Since $\rho = 0$, for $\bar{\rho}$ in (3.32), we get $\bar{\rho} = 1$ and therefore R_t from (3.25) reads

$$R_t = \int_0^t \sqrt{v_s} dW_s,$$

in differential notation, this is

$$dR_t = \sqrt{v_t} dW_t.$$

For the drift $\gamma(t)$ in (3.24), we get

$$\gamma(t) = -\frac{1}{2} \int_0^t v_s ds,$$

since $r = 0$. Equivalently

$$d\gamma(t) = -\frac{1}{2} v_t dt.$$

Hence, for $\widetilde{R}_t$ from (3.26), we get

$$\widetilde{R}_t = R_t + \gamma(t) = \int_0^t \sqrt{v_s} dW_s - \frac{1}{2} \int_0^t v_s ds,$$

with differential

$$d\widetilde{R}_t = \sqrt{v_t} dW_t - \frac{1}{2} v_t dt.$$

We can use Girsanov's theorem again, like we did it in the second chapter:

Define process $\{\widetilde{W}_t\}$ as

$$d\widetilde{W}_t = dW_t - \frac{1}{2} \frac{v_t}{\sqrt{v_t}} dt = dW_t - \frac{1}{2} \sqrt{v_t} dt = dW_t + \Theta_t dt,$$

where $\Theta_t = -\frac{1}{2} \sqrt{v_t}$.

Hence,

$$d\widetilde{R}_t = \sqrt{v_t} d\widetilde{W}_t.$$

Furthermore, define

$$Z_t = \exp \left\{ - \int_0^t \Theta_s dW_s - \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} = \exp \left\{ - \int_0^t \Theta_s d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\}$$

and after Girsanov's theorem¹ (see appendix A.1.1.1), we define our new probability measure as

$$\tilde{\mathbb{P}}(A) = \int_A Z_T d\mathbb{P}^v, \quad A \in \mathcal{F}_T^v,$$

and $\{\tilde{W}_t\}$ becomes a $\tilde{\mathbb{P}}$ - Brownian motion, without drift.

So, using the density from Theorem 3.1 and plugging in for $Z_T = \exp\{\frac{1}{2}\tilde{R}_T + \frac{1}{8}\nu_T^2\}$, we get

$$\begin{aligned} & \mathbb{P}^v\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\} \\ &= \mathbb{E}^{\mathbb{P}^v}\left[\mathbb{I}_{\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\}}\right] = \mathbb{E}_{\tilde{\mathbb{P}}}\left[Z_T^{-1}\mathbb{I}_{\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\}}\right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}\left[\exp\left\{-\frac{1}{2}\tilde{R}_T - \frac{1}{8}\nu_T^2\right\}\mathbb{I}_{\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\}}\right] \\ &= \int_{-\infty}^y \int_{-\infty}^x \exp\left\{-\frac{1}{2}w - \frac{1}{8}\nu_T^2\right\} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(m, w) dm dw \\ &= \int_{-\infty}^y \int_{-\infty}^x \frac{2(2m-w)}{\nu_T \bar{\rho} \sqrt{2\pi}} e^{-\frac{1}{2}w - \frac{1}{8}\nu_T^2 - \frac{1}{2}\frac{(2m-w)^2}{\nu_T^2}} dm dw. \end{aligned} \quad (3.33)$$

Since

$$\frac{\partial}{\partial y \partial x} \mathbb{P}^v\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\} = f_{\tilde{M}_T, \tilde{R}_T}(x, y),$$

differentiating (3.33), analogously to the calculations in (2.8) and (2.9), gives us the resulting density for $(\tilde{M}_T, \tilde{R}_T)$ (under $\mathbb{P}^v$). $\square$

Case B: ($\rho = 0, r \neq 0$)

For case B, we get an approximation to the joint distribution of the pair $(\tilde{M}_T, \tilde{R}_T)$, given by

$$f_{\tilde{M}_T, \tilde{R}_T}(x, y) \approx \frac{2(2x-y)}{\nu_T^3 \sqrt{2\pi}} e^{A(T) + (\frac{rT}{\nu_T^2} - \frac{1}{2})y - \frac{1}{2}\frac{(2x-y)^2}{\nu_T^2}} \quad \text{for } x > 0, y \leq x, \quad (3.34)$$

where

$$A(T) = \frac{1}{2}r^2 \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} - \frac{1}{2}r^2 \bar{\nu}_T^2 - \frac{1}{2}rT + \frac{1}{8}\nu_T^2$$

¹Again, the Novikov condition is fulfilled because of (3.10)

with ν_T^2 defined as in (3.32) and

$$\bar{\nu}_T^2 = \int_0^T \frac{1}{v_s} ds.$$

Proof:

Since $r \neq 0$, we get a different drift term $\gamma(t)$ (3.24), i.e.

$$\gamma(t) = rt - \frac{1}{2} \int_0^t v_s ds,$$

with differential

$$d\gamma(t) = d\left(rt - \frac{1}{2} \int_0^t v_s ds\right) = rdt - \frac{1}{2} v_t dt = \left(r - \frac{1}{2} v_t\right) dt = \gamma'(t) dt$$

So γ is still differentiable, which allows us to change the measure analogously to case A, the only difference is that we get a different Θ , namely

$$\Theta_t = \frac{r}{\sqrt{v_t}} - \frac{1}{2} \sqrt{v_t} \quad (3.35)$$

and therefore our exponential martingale Z_t is given by

$$\begin{aligned} Z_t &= \exp \left\{ - \int_0^t \Theta_s d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} \\ &= \exp \left\{ - \int_0^t \left(\frac{r}{\sqrt{v_s}} - \frac{1}{2} \sqrt{v_s} \right) d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} \\ &= \exp \left\{ \frac{1}{2} \int_0^t \sqrt{v_s} d\widetilde{W}_s - r \int_0^t \frac{1}{\sqrt{v_s}} d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} \\ &= \exp \left\{ \frac{1}{2} \widetilde{R}_t - r \widetilde{U}_t + \alpha(t) \right\} \end{aligned}$$

where

$$\begin{aligned} \widetilde{R}_t &= \int_0^t \sqrt{v_s} d\widetilde{W}_s \\ \widetilde{U}_t &= \int_0^t \frac{1}{\sqrt{v_s}} d\widetilde{W}_s \end{aligned}$$

$$\alpha(t) = \frac{1}{2} \int_0^t \Theta_s^2 ds.$$

Plugging in the right hand side of equation (3.35) for Θ in $\alpha(t)$, we get

$$\alpha(t) = \frac{1}{2} \int_0^t \left(\frac{r}{\sqrt{v_s}} - \frac{1}{2} \sqrt{v_s} \right)^2 ds = \frac{1}{2} r^2 \int_0^t \frac{1}{v_s} ds - \frac{1}{2} r \int_0^t ds + \frac{1}{8} \int_0^t v_s ds$$

So

$$\alpha(t) = \frac{1}{2} r^2 \bar{\nu}_t^2 - \frac{1}{2} r t + \frac{1}{8} \nu_t^2, \quad (3.36)$$

where ν_t^2 is defined as above in (3.32) and

$$\bar{\nu}_t^2 = \int_0^t \frac{1}{v_s} ds.$$

Hence, following the calculation of case A², we get

$$\begin{aligned} \mathbb{P}^v \{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \} &= \mathbb{E}_{\widetilde{\mathbb{P}}} \left[Z_T^{-1} \mathbb{I}_{\{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \}} \right] \\ &= \mathbb{E}_{\widetilde{\mathbb{P}}} \left[\exp \left\{ r \widetilde{U}_T - \frac{1}{2} \widetilde{R}_T - \alpha(T) \right\} \mathbb{I}_{\{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \}} \right] \end{aligned} \quad (3.37)$$

The problem we are facing now is, that we have an additional random variable $\widetilde{U}_T$ in (3.37), i.e. in order to solve this expected value, we require the common density of the triple $(\widetilde{U}_T, \widetilde{M}_T, \widetilde{R}_T)$ and the method of using the extended reflection principle, as in case A, fails here.

Therefore, at first we try to express $\widetilde{U}_T$ in terms of $\widetilde{R}_T$ and an independent, normally distributed random variable U_T and secondly, we will make the assumption that U_T and $\widetilde{M}_T$ are independent, or how Griebisch puts it in [13, p.11]: "... but ignoring the dependence of $\widetilde{M}_T$ and U_T ...". The intuition is, that $\widetilde{M}_T$ depends on the path of $\widetilde{R}_T$ only and therefore shouldn't contribute much information to the value of U_T .

This is also the reason, why the resulting formula will be an approximation to the density of $(\widetilde{M}_T, \widetilde{R}_T)$.

²Again, the Novikov condition is fulfilled because of (3.10) and furthermore $\bar{\nu}_T^2$ does not make any trouble, since $v_t > 0$ for all t . Therefore Girsanov's theorem can be applied here like in case (A).

Decomposition of $\tilde{U}_T$:

The pair $(\tilde{R}_T, \tilde{U}_T)$ is normally distributed with zero mean and variance-covariance matrix

$$\Sigma := \begin{pmatrix} \nu_T^2 & T \\ T & \bar{\nu}_T^2 \end{pmatrix},$$

where $Cov(\tilde{R}_T, \tilde{U}_T) = Cov(\tilde{U}_T, \tilde{R}_T) = T$.

The proof is given in appendix A.2.4.3.

An equivalent notation is:

$$(\tilde{R}_T, \tilde{U}_T) \sim N((0, 0), \Sigma).$$

Now we use the Cholesky decomposition of Σ , i.e.

$$\begin{aligned} \Sigma &= R R^T \\ &= \begin{pmatrix} \nu_T^2 & T \\ T & \bar{\nu}_T^2 \end{pmatrix} = \begin{pmatrix} \nu_T & 0 \\ \frac{T}{\nu_T} & \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} \end{pmatrix} \begin{pmatrix} \nu_T & \frac{T}{\nu_T} \\ 0 & \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} \end{pmatrix}. \end{aligned}$$

Again, using appendix A.2.4.3, there exists a random vector

$$(\phi_1, \phi_2) \sim N\left((0, 0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

such that

$$\begin{pmatrix} \tilde{R}_T \\ \tilde{U}_T \end{pmatrix} = R \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \nu_T & 0 \\ \frac{T}{\nu_T} & \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \nu_T \phi_1 \\ \frac{T}{\nu_T} \phi_1 + \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} \phi_2 \end{pmatrix}.$$

Solving the first equation for ϕ_1 and plugging into the second and furthermore, calling $\phi_2 = U_T$, we get the decomposition of $\tilde{U}_T$:

$$\tilde{U}_T = \frac{T}{\nu_T^2} \tilde{R}_T + \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}} U_T.$$

We rewrite this into:

$$\tilde{U}_T = a_1 \tilde{R}_T + a_2 U_T, \quad (3.38)$$

where

$$a_1 = \frac{T}{\nu_T^2}, \quad a_2 = \sqrt{\nu_T^2 - \frac{T^2}{\nu_T^2}} \quad (3.39)$$

with standard normal distributed random variable U_T , independent of $\tilde{R}_T$.

If we replace $\tilde{U}_T$ in (3.37) by its decomposition (3.38), we get

$$\begin{aligned} & \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ r(a_1 \tilde{R}_T + a_2 U_T) - \frac{1}{2} \tilde{R}_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ (ra_1 - \frac{1}{2}) \tilde{R}_T + a_2 r U_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right]. \end{aligned} \quad (3.40)$$

Since $\tilde{R}_T$ and U_T are independent and we assume $\tilde{M}_T$ and U_T to be independent as well, like mentioned above, one can approximate (3.40) by

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ (ra_1 - \frac{1}{2}) \tilde{R}_T + a_2 r U_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right] \\ & \approx \mathbb{E}_{\tilde{\mathbb{P}}} [e^{a_2 r U_T}] \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ (ra_1 - \frac{1}{2}) \tilde{R}_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right] \\ &= e^{\frac{1}{2} a_2^2 r^2} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ (ra_1 - \frac{1}{2}) \tilde{R}_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right], \end{aligned} \quad (3.41)$$

since

$$\mathbb{E}_{\tilde{\mathbb{P}}} [e^{a_2 r U_T}] = e^{\frac{1}{2} a_2^2 r^2},$$

as shown in appendix A.2.4.

Rewriting the expected value in (3.41) in terms of a double integral, we get

$$e^{\frac{1}{2} a_2^2 r^2} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ (ra_1 - \frac{1}{2}) \tilde{R}_T - \alpha(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right],$$

$$= e^{\frac{1}{2}a_2^2r^2} \int_{-\infty}^y \int_{-\infty}^x \exp \left\{ (ra_1 - \frac{1}{2})w - \alpha(T) \right\} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(m, w) dm dw$$

So, to summarize, we have

$$\begin{aligned} & \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ & \approx e^{\frac{1}{2}a_2^2r^2} \int_{-\infty}^y \int_{-\infty}^x \exp \left\{ (ra_1 - \frac{1}{2})w - \alpha(T) \right\} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(m, w) dm dw. \end{aligned} \quad (3.42)$$

Like in case A, we get the approximation formula for the density $\tilde{f}_{\tilde{M}_T, \tilde{R}_T}$ of $(\tilde{M}_T, \tilde{R}_T)$, by deriving (3.42) with respect to x and y , i.e.

$$\begin{aligned} f_{\tilde{M}_T, \tilde{R}_T}(x, y) &= \frac{\partial^2}{\partial y \partial x} \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ &\approx e^{\frac{1}{2}a_2^2r^2 - \alpha(T) + (ra_1 - \frac{1}{2})y} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(x, y) = e^{A(T) + (ra_1 - \frac{1}{2})y} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(x, y), \end{aligned}$$

where

$$A(T) = \frac{1}{2}a_2^2r^2 - \alpha(T), \quad (3.43)$$

$\alpha(T)$ given in (3.36) and $a_{1,2}$ given in (3.39).

Now, plugging in for $\tilde{f}_{\tilde{M}_T, \tilde{R}_T}$, from Theorem 3.1, we finally get

$$f_{\tilde{M}_T, \tilde{R}_T}(x, y) \approx \frac{2(2x - y)}{\nu_T^3 \sqrt{2\pi}} e^{A(T) + (ra_1 - \frac{1}{2})y - \frac{1}{2} \frac{(2x - y)^2}{\nu_T^2}} \quad (3.44)$$

□

Case C: ($\rho \neq 0, r \neq 0$)

So far, we could easily change the probability measure and apply Girsanov's theorem to get rid of the drift term in $\tilde{R}_T$, in both previous cases. The reason was, that the drift-term γ was differentiable with respect to t or put differently, $d\gamma(t)$ was of order dt .

Now we face the problem that v_t is part of the drift term, which is not of order dt .

To see this recall that γ was given by

$$\gamma(t) = rt - \frac{1}{2} \int_0^t v_s ds + \frac{\rho}{\sigma} (v_t - v_0 - \kappa \theta t + \kappa \int_0^t v_s ds)$$

and in differential form, we get

$$d\gamma(t) = rdt - \frac{1}{2}v_t dt + \frac{\rho}{\sigma}(dv_t - \kappa\theta dt + \kappa v_t dt),$$

rearranging terms and plugging in the definition of dv_t from (3.17), gives us

$$\begin{aligned} d\gamma(t) &= (r - \frac{1}{2}v_t - \frac{\rho}{\sigma}\kappa\theta + \frac{\rho}{\sigma}\kappa v_t)dt + \frac{\rho}{\sigma}(\kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2) \\ &= (r - \frac{1}{2}v_t - \frac{\rho}{\sigma}\kappa\theta + \frac{\rho}{\sigma}\kappa v_t + \frac{\rho}{\sigma}\kappa(\theta - v_t))dt + \rho\sqrt{v_t}dW_t^2. \end{aligned}$$

So the differential of $\tilde{R}_t$ reads

$$d\tilde{R}_t = \bar{\rho}\sqrt{v_T}dW_t + \rho\sqrt{v_T}dW_t^2 + (r - \frac{1}{2}v_t - \frac{\rho}{\sigma}\kappa\theta + \frac{\rho}{\sigma}\kappa v_t + \frac{\rho}{\sigma}\kappa(\theta - v_T))dt$$

and hence, Girsanov's theorem cannot be applied here, (at least not directly).

We fix this problem by approximating v_t , by a differentiable function $\bar{v}_t$, with $v_0 = \bar{v}_0$ and $v_T = \bar{v}_T$.³

In this case, the differential of the drift can be written as

$$\begin{aligned} d\gamma(t) &= rdt - \frac{1}{2}v_t dt + \frac{\rho}{\sigma}(d\bar{v}_t - \kappa\theta dt + \kappa v_t dt) \\ &= \left[r - \frac{1}{2}v_t + \frac{\rho}{\sigma}\left(\frac{d\bar{v}_t}{dt} - \kappa\theta + \kappa v_t\right) \right] dt \\ &= \left[\left(r - \frac{\rho}{\sigma}\kappa\theta\right) + \left(\frac{\rho}{\sigma}\kappa - \frac{1}{2}\right)v_t + \frac{\rho}{\sigma}\bar{v}_t' \right] dt, \end{aligned}$$

where $d\bar{v}_t/dt = \bar{v}_t'$.

Again we change the measure analogously to case A, this time with the following Θ :

$$\Theta_t = \frac{\gamma(t)}{\bar{\rho}\sqrt{v_t}}$$

³Note: We are still in the first step of the independence lemma from A.2.6. This means that v_t , $t \in [0, T]$ is not stochastic but one particular path of the set of all paths of the variance process. Since all those paths are continuous but not differentiable, we can apply e.g. the Stone-Weierstrass theorem, which states that every real-valued continuous function on a closed interval can be approximated by polynomials.

$$\begin{aligned}
&= \frac{1}{\bar{\rho}\sqrt{v_t}} \left[\left(r - \frac{\rho}{\sigma}\kappa\theta \right) + \left(\frac{\rho}{\sigma}\kappa - \frac{1}{2} \right) v_t + \frac{\rho}{\sigma} \bar{v}'_t \right] \\
&= \left(r - \frac{\rho}{\sigma}\kappa\theta \right) \frac{1}{\bar{\rho}\sqrt{v_t}} + \left(\frac{\rho}{\sigma}\kappa - \frac{1}{2} \right) \frac{v_t}{\bar{\rho}\sqrt{v_t}} + \frac{\rho}{\sigma} \frac{\bar{v}'_t}{\bar{\rho}\sqrt{v_t}} \\
&= c_1 \frac{1}{\bar{\rho}\sqrt{v_t}} + c_2 \frac{\sqrt{v_t}}{\bar{\rho}} + c_3 \frac{\bar{v}'_t}{\bar{\rho}\sqrt{v_t}}
\end{aligned}$$

where

$$c_1 = \left(r - \frac{\rho}{\sigma}\kappa\theta \right), \quad c_2 = \left(\frac{\rho}{\sigma}\kappa - \frac{1}{2} \right) \text{ and } c_3 = \frac{\rho}{\sigma}. \quad (3.45)$$

Hence, the exponential Brownian motion Z_t is of the form

$$\begin{aligned}
Z_t &= \exp \left\{ - \int_0^t \Theta_s d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} \\
&= \exp \left\{ - \frac{c_1}{\bar{\rho}} \int_0^t \frac{1}{\sqrt{v_s}} d\widetilde{W}_s - \frac{c_2}{\bar{\rho}} \int_0^t \sqrt{v_s} d\widetilde{W}_s \right. \\
&\quad \left. - \frac{c_3}{\bar{\rho}} \int_0^t \frac{\bar{v}'_s}{\sqrt{v_s}} d\widetilde{W}_s + \frac{1}{2} \int_0^t \Theta_s^2 ds \right\} \\
&= \exp \left\{ - \frac{c_1}{\bar{\rho}^2} \widetilde{U}_t^1 - \frac{c_2}{\bar{\rho}^2} \widetilde{R}_t - \frac{c_3}{\bar{\rho}^2} \widetilde{U}_t^2 + B(t) \right\},
\end{aligned}$$

where

$$\widetilde{U}_t^1 = \bar{\rho} \int_0^t \frac{1}{\sqrt{v_s}} d\widetilde{W}_s, \quad \widetilde{R}_t = \bar{\rho} \int_0^t \sqrt{v_s} d\widetilde{W}_s \text{ and } \widetilde{U}_t^2 = \bar{\rho} \int_0^t \frac{\bar{v}'_s}{\sqrt{v_s}} d\widetilde{W}_s \quad (3.46)$$

and

$$B(t) = \frac{1}{2} \int_0^t \Theta_s^2 ds$$

which is

$$\begin{aligned}
B(t) &= \frac{1}{2\bar{\rho}^2} \left(c_1^2 \int_0^t \frac{1}{v_s} ds + c_2^2 \int_0^t v_s ds + c_3^2 \int_0^t \frac{(\bar{v}'_s)^2}{v_s} ds \right. \\
&\quad \left. + 2c_1c_2 \int_0^t ds + 2c_1c_3 \int_0^t \frac{\bar{v}'_s}{v_s} ds + 2c_2c_3 \int_0^t \bar{v}'_s ds \right)
\end{aligned}$$

or in short notation,

$$B(t) = \frac{1}{2\bar{\rho}^2}(c_1^2\bar{\nu}_t^2 + c_2^2\nu_t^2 + c_3^2\underline{\nu}_t^2 + 2c_1c_2t + 2c_1c_3\hat{\nu}_t^2 + 2c_2c_3(\bar{\nu}_t - \nu_0)), \quad (3.47)$$

with

$$\bar{\nu}_t^2 = \int_0^t \frac{1}{v_s} ds, \quad \nu_t^2 = \int_0^t v_s ds, \quad \underline{\nu}_t^2 = \int_0^t \frac{(\bar{v}_s')^2}{v_s} ds, \quad \hat{\nu}_t^2 = \int_0^t \frac{\bar{v}_s'}{v_s} ds. \quad (3.48)$$

Like in the cases before, we write

$$\begin{aligned} \mathbb{P}^v \{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \} &= \mathbb{E}_{\widetilde{\mathbb{P}}} \left[Z_T^{-1} \mathbb{I}_{\{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \}} \right] \\ &= \mathbb{E}_{\widetilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}} \widetilde{U}_T^1 + \frac{c_2}{\bar{\rho}} \widetilde{R}_T + \frac{c_3}{\bar{\rho}} \widetilde{U}_T^2 - B(T) \right\} \mathbb{I}_{\{ \widetilde{M}_T \leq x, \widetilde{R}_T \leq y \}} \right] \end{aligned} \quad (3.49)$$

and in order to get the joint density of $(\widetilde{M}_T, \widetilde{R}_T)$, we have to find the common density of the vector $(\widetilde{U}_T^1, \widetilde{U}_T^2, \widetilde{M}_T, \widetilde{R}_T)$.

From here, we follow the same steps as in case B, i.e. we express the random variables $\widetilde{U}_T^i$, $i = 1, 2$, in terms of $\widetilde{R}_T$ and independent standard normally distributed random variables U_T^i , $i = 1, 2$ and again, we ignore the dependence between $\widetilde{U}_T^i$ and $\widetilde{M}_T$.

Decomposition of $\widetilde{U}_T^1$ and $\widetilde{U}_T^2$:

Case $\widetilde{U}_T^1$:

The decomposition of $\widetilde{U}_T^1$ works exactly like in case B (although, now we have $\bar{\rho} \neq 1$), therefore

$$\widetilde{U}_T^1 = a_1 \widetilde{R}_T + a_2 U_T^1, \quad (3.50)$$

where

$$a_1 = \frac{T}{\nu_T^2} \text{ and } a_2 = \bar{\rho} \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}}, \quad (3.51)$$

with standard normal distributed random variable U_T^1 , independent of $\widetilde{R}_T$.

Case $\tilde{U}_T^2$:

For $\tilde{U}_T^2$, we consider the pair $(\tilde{R}_T, \tilde{U}_T^2)$, with distribution given by

$$(\tilde{R}_T, \tilde{U}_T^2) \sim N \left((0, 0), \begin{pmatrix} \bar{\rho}^2 \nu_T^2 & \bar{\rho}^2 (\bar{v}_T - v_0) \\ \bar{\rho}^2 (\bar{v}_T - v_0) & \bar{\rho}^2 \nu_T^2 \end{pmatrix} =: \Sigma \right),$$

which is shown in appendix A.2.4.4.

The Cholesky decomposition of Σ , is given by

$$\begin{aligned} \Sigma &= R R^T \\ &= \begin{pmatrix} \bar{\rho} \nu_T & 0 \\ \bar{\rho} \frac{\bar{v}_T - v_0}{\nu_T} & \bar{\rho} \sqrt{\nu_T^2 - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}} \end{pmatrix} \begin{pmatrix} \bar{\rho} \nu_T & \bar{\rho} \frac{\bar{v}_T - v_0}{\nu_T} \\ 0 & \bar{\rho} \sqrt{\nu_T^2 - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}} \end{pmatrix}. \end{aligned}$$

So after A.2.3, there exists a random vector

$$(\phi_1, \phi_2) \sim N \left((0, 0), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

s.t.

$$\begin{aligned} \begin{pmatrix} \tilde{R}_T \\ \tilde{U}_T^2 \end{pmatrix} &= R \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \bar{\rho} \nu_T & 0 \\ \bar{\rho} \frac{\bar{v}_T - v_0}{\nu_T} & \bar{\rho} \sqrt{\nu_T^2 - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\rho} \nu_T \phi_1 \\ \bar{\rho} \frac{\bar{v}_T - v_0}{\nu_T} \phi_1 + \bar{\rho} \sqrt{\nu_T^2 - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}} \phi_2 \end{pmatrix}. \end{aligned}$$

Solving the first equation for ϕ_1 and plugging into the second and furthermore, calling $\phi_2 = U_T^2$, we get the decomposition of $\tilde{U}_T^2$:

$$\tilde{U}_T^2 = \frac{\bar{v}_T - v_0}{\nu_T^2} \tilde{R}_T + \bar{\rho} \sqrt{\nu_T^2 - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}} U_T^2.$$

We rewrite this into

$$\tilde{U}_T^2 = a_3 \tilde{R}_T + a_4 U_T^2, \quad (3.52)$$

where

$$a_3 = \frac{\bar{v}_T - v_0}{\nu_T^2} \text{ and } a_4 = \bar{\rho} \sqrt{\frac{\nu_T^2}{\nu_T^2} - \frac{(\bar{v}_T - v_0)^2}{\nu_T^2}}, \quad (3.53)$$

with standard normal distributed random variable U_T^2 , independent of $\tilde{R}_T$.

If we replace $\tilde{U}_T^1$ from (3.50) and $\tilde{U}_T^2$ from (3.52) in (3.49), we get

$$\begin{aligned} & \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}^2} \tilde{U}_T^1 + \frac{c_2}{\bar{\rho}^2} \tilde{R}_T + \frac{c_3}{\bar{\rho}^2} \tilde{U}_T^2 - B(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}^2} (a_1 \tilde{R}_T + a_2 U_T^1) + \frac{c_2}{\bar{\rho}^2} \tilde{R}_T \right. \right. \\ &\quad \left. \left. + \frac{c_3}{\bar{\rho}^2} (a_3 \tilde{R}_T + a_4 U_T^2) - B(T) \right\} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} \tilde{R}_T \right\} \right. \\ &\quad \left. \cdot \exp \left\{ \frac{c_1}{\bar{\rho}^2} a_2 U_T^1 + \frac{c_3}{\bar{\rho}^2} a_4 U_T^2 \right\} \exp \{ -B(t) \} \mathbb{I}_{\{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \}} \right]. \end{aligned} \quad (3.54)$$

Like mentioned above, U_T^1 and U_T^2 are independent of $\tilde{R}_T$ and we assume their independence of $\tilde{M}_T$ and since from appendix A.2.4.5 we have

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}^2} a_2 U_T^1 + \frac{c_3}{\bar{\rho}^2} a_4 U_T^2 \right\} \right] \\ &= \exp \left\{ \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_5) \right\} \\ &:= e^{\tilde{B}(T)} \end{aligned} \quad (3.55)$$

where

$$a_5 = \bar{\rho}^2 \left(\hat{\nu}_T^2 - \frac{T}{\nu_T^2} (\bar{v}_T - v_0) \right) \quad (3.56)$$

and

$$\tilde{B}(T) = \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_5). \quad (3.57)$$

Therefore, we can approximate (3.54) by

$$\begin{aligned} & \approx e^{\tilde{B}(T)-B(T)} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} \tilde{R}_T \right\} \mathbb{I}_{\{\tilde{M}_T \leq x, \tilde{R}_T \leq y\}} \right] \\ & = e^{\tilde{B}(T)-B(T)} \int_{-\infty}^y \int_{-\infty}^x \exp \left\{ \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} w \right\} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(m, w) dm dw \end{aligned}$$

So, to summarize, we have

$$\begin{aligned} & \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ & \approx e^{\tilde{B}(T)-B(T)} \\ & \cdot \int_{-\infty}^y \int_{-\infty}^x \exp \left\{ \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} w \right\} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(m, w) dm dw \end{aligned} \quad (3.58)$$

Now, like in the cases A and B, we get the density $f_{\tilde{M}_T, \tilde{R}_T}$ of $(\tilde{M}_T, \tilde{R}_T)$, by deriving (3.58) with respect to x and y , hence

$$\begin{aligned} f_{\tilde{M}_T, \tilde{R}_T}(x, y) & = \frac{\partial^2}{\partial y \partial x} \mathbb{P}^v \{ \tilde{M}_T \leq x, \tilde{R}_T \leq y \} \\ & \approx e^{\tilde{B}(T)-B(T)} e^{\frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} y} \tilde{f}_{\tilde{M}_T, \tilde{R}_T}(x, y). \end{aligned}$$

Finally, plugging in for the driftless density $\tilde{f}_{\tilde{M}_T, \tilde{R}_T}$ of (3.31), we get

$$f_{\tilde{M}_T, \tilde{R}_T}(x, y) \approx \frac{2(2x-y)}{\bar{\rho}^3 \nu_T^3 \sqrt{2\pi}} e^{\frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} y + \bar{B}(T) - \frac{1}{2} \frac{(2x-y)^2}{\bar{\rho}^2 \nu_T^2}} \quad (3.59)$$

for $x > 0$ and $y \leq x$, where

$$\bar{B}(T) = \tilde{B}(T) - B(T),$$

with $\tilde{B}(T)$ is given in (3.57) and $B(T)$ given in (3.47), i.e.

$$\tilde{B}(T) = \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_5)$$

$$B(T) = \frac{1}{2\bar{\rho}^2} (c_1^2 \bar{\nu}_T^2 + c_2^2 \nu_T^2 + c_3^2 \nu_T^2 + 2c_1 c_2 T + 2c_1 c_3 \hat{\nu}_T^2 + 2c_2 c_3 (\bar{\nu}_T - \nu_0))$$

and the corresponding $a_{1,2}$ given in (3.51), $a_{3,4}$ in (3.53), a_5 in (3.56), $c_{1,2,3}$ in (3.45) and ν_T^2 , $\bar{\nu}_T^2$, $\underline{\nu}_T^2$ and $\hat{\nu}_T^2$ in (3.48).

General Form of the Density:

We will summarize the density function for all three cases as follows. Define

$$F = \begin{cases} -\frac{1}{2} & \text{for Case A} \\ ra_1 - \frac{1}{2} & \text{for Case B} \\ \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} & \text{for Case C} \end{cases}$$

and

$$G = \begin{cases} -\frac{1}{8}\nu_T^2 & \text{for Case A} \\ A(T) & \text{for Case B} \\ \bar{B}(T) & \text{for Case C} \end{cases}$$

and hence, the joint density of $(\widetilde{M}_T, \widetilde{R}_T)$, is of the form

$$f_{\widetilde{M}_T, \widetilde{R}_T}(x, y) = \frac{2(2x - y)}{\bar{\rho}^3 \nu_T^3 \sqrt{2\pi}} e^{Fy + G - \frac{1}{2} \frac{(2x - y)^2}{\bar{\rho}^2 \nu_T^2}} \text{ for } x \geq 0, y \leq x. \quad (3.60)$$

In the next step, we calculate E^v using the general form of the density in (3.60).

3. Calculating the Inner Expectation E^v

Like mentioned before, by the independence lemma and appendix 3.5, we calculate first $E_{\mathbb{P}^v}^v$, where the variance is treated as deterministic and then in the second step, we replace it with the our initial variance process $\{v_t\}$, in order to get E^v .

3.1 Calculating $E_{\mathbb{P}^v}^v$:

Recall, that $E_{\mathbb{P}^v}^v$ was given by

$$E_{\mathbb{P}^v}^v = \mathbb{E}_{\mathbb{P}^v} \left[(S_0 e^{\widetilde{R}_T} - K) \mathbb{I}_{\{\widetilde{M}_T < b, \widetilde{R}_T > k\}} \right].$$

From chapter 2, we know the area in order to compute the double integral in order to calculate the expected value, i.e.

$$\begin{aligned} E_{\mathbb{P}^v}^v &= \int_k^b \int_{w^+}^b (S_0 e^y - K) f_{\widetilde{M}_T, \widetilde{R}_T}(x, y) dx dy \\ &= - \int_k^b \int_{w^+}^b (S_0 e^y - K) \left(-\frac{2(2x-y)}{\bar{\rho}^3 \nu_T^3 \sqrt{2\pi}} \right) e^{Fy+G-\frac{1}{2}\frac{(2x-y)^2}{\bar{\rho}^2 \nu_T^2}} dx dy. \end{aligned} \quad (3.61)$$

Next, we use the fact that

$$\frac{\partial}{\partial x} e^{Fy+G-\frac{1}{2}\frac{(2x-y)^2}{\bar{\rho}^2 \nu_T^2}} = -\frac{2(2x-y)}{\bar{\rho}^2 \nu_T^2} e^{Fy+G-\frac{1}{2}\frac{(2x-y)^2}{\bar{\rho}^2 \nu_T^2}},$$

therefore, we can solve the inner integral in (3.61) and we get

$$\begin{aligned} E_{\mathbb{P}^v}^v &= \frac{1}{\bar{\rho} \nu_T \sqrt{2\pi}} \int_k^b (S_0 e^y - K) e^{Fy+G-\frac{1}{2}\frac{(2y-y)^2}{\bar{\rho}^2 \nu_T^2}} dy \\ &\quad - \frac{1}{\bar{\rho} \nu_T \sqrt{2\pi}} \int_k^b (S_0 e^y - K) e^{Fy+G-\frac{1}{2}\frac{(2b-y)^2}{\bar{\rho}^2 \nu_T^2}} dy \\ &= S_0 I_{0,1} - K I_{0,0} - S_0 I_{1,1} + K I_{1,0}, \end{aligned} \quad (3.62)$$

where

$$I_{0,x} = \frac{1}{\bar{\rho} \nu_T \sqrt{2\pi}} e^G \int_k^b e^{-\frac{1}{2}\left[-2(F+x)y + \frac{y^2}{\bar{\rho}^2 \nu_T^2}\right]} dy \quad (3.63)$$

$$I_{1,x} = \frac{1}{\bar{\rho} \nu_T \sqrt{2\pi}} e^{G-\frac{2b^2}{\bar{\rho}^2 \nu_T^2}} \int_k^b e^{-\frac{1}{2}\left(\left(-2(F+x)-\frac{4b}{\bar{\rho}^2 \nu_T^2}\right)y + \frac{y^2}{\bar{\rho}^2 \nu_T^2}\right)} dy \quad (3.64)$$

for $x \in \{0, 1\}$.

For any f and g , it holds

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_k^b e^{-\frac{1}{2}(fy+gy^2)} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_k^b e^{-\frac{1}{2}\left(\frac{f^2}{4g} + fy + gy^2\right) + \frac{1}{8}\frac{f^2}{g}} dy = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{8}\frac{f^2}{g}} \int_k^b e^{-\frac{1}{2}\left(\frac{f}{2\sqrt{g}} + \sqrt{g}y\right)^2} dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{8}\frac{f^2}{g}} \int_k^b e^{-\frac{1}{2}g(\frac{f}{2g}+y)^2} dy = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{8}\frac{f^2}{g}} \int_k^b e^{-\frac{1}{2}(\frac{\frac{f}{2g}+y}{\sqrt{1/g}})^2} dy \\
&= e^{\frac{1}{8}\frac{f^2}{g}} \frac{1}{\sqrt{g}} \frac{1}{\sqrt{2\pi}} \int_{\frac{\frac{f}{2g}+k}{\sqrt{1/g}}}^{\frac{\frac{f}{2g}+b}{\sqrt{1/g}}} e^{-\frac{1}{2}z^2} dz \\
&= \frac{e^{\frac{1}{8}\frac{f^2}{g}}}{\sqrt{g}} \left[N\left(\frac{\frac{f}{2g}+b}{\sqrt{1/g}}\right) - N\left(\frac{\frac{f}{2g}+k}{\sqrt{1/g}}\right) \right] \\
&= \frac{e^{\frac{1}{8}\frac{f^2}{g}}}{\sqrt{g}} \left[N\left(\sqrt{g}\left(\frac{f}{2g}+b\right)\right) - N\left(\sqrt{g}\left(\frac{f}{2g}+k\right)\right) \right],
\end{aligned}$$

using $N(z) = 1 - N(-z)$, we get

$$= \frac{e^{\frac{1}{8}\frac{f^2}{g}}}{\sqrt{g}} \left[N\left(\sqrt{g}\left(-\frac{f}{2g}-k\right)\right) - N\left(\sqrt{g}\left(-\frac{f}{2g}-b\right)\right) \right].$$

To summarize, we get

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_k^b e^{-\frac{1}{2}(fy+gy^2)} dy \\
&= \frac{e^{\frac{f^2}{8g}}}{\sqrt{g}} \left[N\left(\sqrt{g}\left(-\frac{f}{2g}-k\right)\right) - N\left(\sqrt{g}\left(-\frac{f}{2g}-b\right)\right) \right]. \tag{3.65}
\end{aligned}$$

Now, for $I_{0,x}$ in (3.63), we plug in for $f = -2(F+x)$ and $g = \frac{1}{\bar{\rho}^2 \nu_T^2}$ and use the result in (3.65), therefore

$$\begin{aligned}
I_{0,x} &= \frac{e^G}{\bar{\rho} \nu_T} \frac{1}{\sqrt{2\pi}} \int_k^b e^{-\frac{1}{2}\left[-2(F+x)y + \frac{y^2}{\bar{\rho}^2 \nu_T^2}\right]} dy \\
&= \frac{e^G}{\bar{\rho} \nu_T} \frac{1}{\sqrt{\frac{1}{\bar{\rho}^2 \nu_T^2}}} \exp\left\{\frac{1}{8} \frac{(-2(F+x))^2}{\frac{1}{\bar{\rho}^2 \nu_T^2}}\right\} \\
&\cdot \left[N\left(\sqrt{\frac{1}{\bar{\rho}^2 \nu_T^2}} \left(-\frac{(-2(F+x))}{2\frac{1}{\bar{\rho}^2 \nu_T^2}} - k\right)\right) - N\left(\sqrt{\frac{1}{\bar{\rho}^2 \nu_T^2}} \left(-\frac{-2(F+x)}{2\frac{1}{\bar{\rho}^2 \nu_T^2}} - b\right)\right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \exp \left\{ G + \frac{1}{2}(F+x)^2 \bar{\rho}^2 \nu_T^2 \right\} \\
&\cdot \left[N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + \ln \frac{S_0}{K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + \ln \frac{S_0}{B}}{\bar{\rho} \nu_T} \right) \right].
\end{aligned}$$

For $I_{1,x}$ in (3.63), we plug in $f = -2(F+x) - \frac{4b}{\bar{\rho}^2 \nu_T^2}$ and $g = \frac{1}{\bar{\rho}^2 \nu_T^2}$ and use (3.65) again, hence

$$\begin{aligned}
I_{1,x} &= \frac{1}{\bar{\rho} \nu_T} e^{G - \frac{2b^2}{\bar{\rho}^2 \nu_T^2}} \frac{1}{\sqrt{2\pi}} \int_k^b e^{-\frac{1}{2} \left(\left(-2(F+x) - \frac{4b}{\bar{\rho}^2 \nu_T^2} \right) y + \frac{y^2}{\bar{\rho}^2 \nu_T^2} \right)} dy \\
&= \frac{1}{\bar{\rho} \nu_T} \exp \left\{ G - \frac{2b^2}{\bar{\rho}^2 \nu_T^2} \right\} \frac{1}{\sqrt{\frac{1}{\bar{\rho}^2 \nu_T^2}}} \exp \left\{ \frac{1}{8} \frac{\left(-2(F+x) - \frac{4b}{\bar{\rho}^2 \nu_T^2} \right)^2}{\frac{1}{\bar{\rho}^2 \nu_T^2}} \right\} \\
&\cdot \left[N \left(\frac{\left((F+x) + \frac{2b}{\bar{\rho}^2 \nu_T^2} \right) \bar{\rho}^2 \nu_T^2 - k}{\bar{\rho} \nu_T} \right) - N \left(\frac{\left((F+x) + \frac{2b}{\bar{\rho}^2 \nu_T^2} \right) \bar{\rho}^2 \nu_T^2 - b}{\bar{\rho} \nu_T} \right) \right] \\
&= \exp \left\{ G - \frac{2b^2}{\bar{\rho}^2 \nu_T^2} \right\} \exp \left\{ \frac{1}{2} (F+x)^2 \bar{\rho}^2 \nu_T^2 + 2b(F+x) + \frac{2b^2}{\bar{\rho}^2 \nu_T^2} \right\} \\
&\cdot \left[N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + 2b - k}{\bar{\rho} \nu_T} \right) - N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + 2b - b}{\bar{\rho} \nu_T} \right) \right] \\
&= \exp \left\{ \frac{1}{2} (F+x)^2 \bar{\rho}^2 \nu_T^2 + 2b(F+x) + G \right\} \\
&\cdot \left[N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + \ln \frac{B^2}{S_0 K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{(F+x) \bar{\rho}^2 \nu_T^2 + \ln \frac{B}{S_0}}{\bar{\rho} \nu_T} \right) \right].
\end{aligned}$$

So, to summarize, for $I_{i,x}$, $i = 0, 1$ we get

$$I_{0,x} = \exp \left\{ G + \frac{1}{2} (F+x)^2 \bar{\rho}^2 \nu_T^2 \right\}$$

$$\cdot \left[N \left(\frac{(F+x)\bar{\rho}^2\nu_T^2 + \ln \frac{S_0}{K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{(F+x)\bar{\rho}^2\nu_T^2 + \ln \frac{S_0}{B}}{\bar{\rho}\nu_T} \right) \right] \quad (3.66)$$

$$\begin{aligned} I_{1,x} &= \exp \left\{ \frac{1}{2}(F+x)^2\bar{\rho}^2\nu_T^2 + 2b(F+x) + G \right\} \\ &\cdot \left[N \left(\frac{(F+x)\bar{\rho}^2\nu_T^2 + \ln \frac{B^2}{S_0K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{(F+x)\bar{\rho}^2\nu_T^2 + \ln \frac{B}{S_0}}{\bar{\rho}\nu_T} \right) \right], \end{aligned} \quad (3.67)$$

for $x = 0, 1$.

Now, we come to the case by case distinction and we plug in for F and G .

Case A: ($\rho = 0 = r$)

In this case, we have $F = -\frac{1}{2}$, $G = -\frac{1}{8}\nu_T^2$ and $\bar{\rho} = 1$.

For the factor of $I_{0,x}$ in (3.66), we get

$$e^{-\frac{1}{8}\nu_T^2 + \frac{1}{2}(-\frac{1}{2}+x)^2\nu_T^2} = 1,$$

for $x = 0, 1$.

Hence, $I_{0,x}$ is of the form

$$I_{0,0} = N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \quad (3.68)$$

$$I_{0,1} = N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right). \quad (3.69)$$

For the factor of $I_{1,x}$ in (3.67), we get

$$e^{-\frac{1}{8}\nu_T^2 + \frac{1}{2}(-\frac{1}{2})^2\nu_T^2 - b} = e^{-b} = \frac{S_0}{B}$$

for $x = 0$ and

$$e^{-\frac{1}{8}\nu_T^2 + \frac{1}{2}(-\frac{1}{2})^2\nu_T^2 + b} = e^b = \frac{B}{S_0}$$

for $x = 1$.

Hence $I_{1,0}$ and $I_{1,1}$ are of the form

$$I_{1,0} = \frac{S_0}{B} \left[N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right] \quad (3.70)$$

$$I_{1,1} = \frac{B}{S_0} \left[N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right]. \quad (3.71)$$

Case B: ($\rho = 0$ and $r \neq 0$)

Here, we have $F = ra_1 - \frac{1}{2}$, $G = A(T)$ and $\bar{\rho} = 1$, where $a_1 = \frac{T}{\nu_T^2}$, given in (3.39).

Recall, $A(T)$ was given in (3.43), by

$$A(T) = \frac{1}{2}a_2^2 r^2 - \alpha(T),$$

with $a_2 = \sqrt{\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}}$ and $\alpha(T) = \frac{1}{2}\nu_T^2 r^2 - \frac{1}{2}rT + \frac{1}{8}\nu_T^2$ (see (3.36)), therefore we get

$$\begin{aligned} A(T) &= \frac{1}{2}(\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2})r^2 - \frac{1}{2}\nu_T^2 r^2 + \frac{1}{2}rT - \frac{1}{8}\nu_T^2 \\ &= -\frac{1}{8}\nu_T^2 - \frac{r^2 T^2}{2\nu_T^2} + \frac{1}{2}rT \end{aligned}$$

So, for the factor of $I_{0,x}$ in (3.66), we get

$$e^{-\frac{1}{8}\nu_T^2 + \frac{rT}{2} - \frac{r^2 T^2}{2\nu_T^2} + \frac{1}{2}(\frac{rT}{\nu_T^2} - \frac{1}{2} + x)^2 \nu_T^2}, \quad (3.72)$$

hence, for $x = 0$, we get

$$e^{-\frac{1}{8}\nu_T^2 + \frac{rT}{2} - \frac{r^2 T^2}{2\nu_T^2} + \frac{r^2 T^2}{2\nu_T^2} - \frac{rT}{2} + \frac{1}{8}\nu_T^2} = e^0 = 1 \quad (3.73)$$

and for $x = 1$, we get

$$e^{-\frac{1}{8}\nu_T^2 + \frac{rT}{2} - \frac{r^2 T^2}{2\nu_T^2} + \frac{1}{2}(\frac{rT}{\nu_T^2} - \frac{1}{2} + 1)^2 \nu_T^2}$$

$$= e^{0 + \frac{1}{2} \cdot 2 \left(\frac{rT}{\nu_T^2} - \frac{1}{2} \right) \nu_T^2 + \frac{1}{2} \nu_T^2} = e^{rT}$$

Therefore, we get the following $I_{0,0}$ and $I_{0,1}$:

$$I_{0,0} = \left[N \left(\frac{rT - \frac{1}{2} \nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{rT - \frac{1}{2} \nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \right] \quad (3.74)$$

$$I_{0,1} = e^{rT} \left[N \left(\frac{rT + \frac{1}{2} \nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{rT + \frac{1}{2} \nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \right]. \quad (3.75)$$

We rewrite the factors of $I_{1,x}$ into

$$\begin{aligned} & \exp \left\{ \frac{1}{2} (F+x)^2 \nu_T^2 + 2b(F+x) + G \right\} \\ &= \exp \left\{ \frac{1}{2} (F+x)^2 \nu_T^2 + G \right\} \exp \{ 2b(F+x) \} \end{aligned} \quad (3.76)$$

For $x = 0$, we get

$$\begin{aligned} & \exp \left\{ \frac{1}{2} F^2 \nu_T^2 + G \right\} \exp \{ 2bF \} = \exp \{ 2bF \} = \exp \left\{ 2b \left(\frac{rT}{\nu_T^2} - \frac{1}{2} \right) \right\} \\ &= \exp \left\{ \frac{2brT}{\nu_T^2} - b \right\} = \exp \{ -b \} \exp \left\{ \frac{2brT}{\nu_T^2} \right\} \\ &= \left(\frac{S_0}{B} \right) \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2}} = \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2} - 1} \end{aligned}$$

and for $x = 1$, we get

$$\begin{aligned} & \exp \left\{ \frac{1}{2} (F+1)^2 \nu_T^2 + G \right\} \exp \{ 2b(F+1) \} \\ &= \exp \{ rT \} \exp \left\{ 2b \left(\frac{rT}{\nu_T^2} + \frac{1}{2} \right) \right\} = \exp \{ rT \} \exp \{ 2b \} \exp \left\{ \frac{2brT}{\nu_T^2} \right\} \\ &= e^{rT} \left(\frac{B}{S_0} \right) \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2}} = e^{rT} \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2} + 1}. \end{aligned}$$

Therefore, $I_{1,0}$ and $I_{1,1}$ are of the form

$$I_{1,0} = \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2} - 1} \cdot \left[N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right] \quad (3.77)$$

$$I_{1,1} = e^{rT} \left(\frac{B}{S_0} \right)^{\frac{2rT}{\nu_T^2} + 1} \cdot \left[N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right]. \quad (3.78)$$

Case C: ($\rho \neq 0$ and $r \neq 0$)

In this case, $F = \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2}$ and $G = \bar{B}(T)$. First we will recall the definitions of the a_i and $\bar{B}(T)$. (See (3.47), (3.57), (3.51), (3.53) and (3.56) for their derivations).

Since $B(T)$ and $\tilde{B}(T)$ were given by

$$B(T) = \frac{1}{2\rho^2} (c_1^2 \bar{\nu}_T^2 + c_2^2 \nu_T^2 + c_3^2 \underline{\nu}_T^2 + 2c_1 c_2 T + 2c_1 c_3 \hat{\nu}_T^2 + 2c_2 c_3 (v_T - v_0))$$

and

$$\begin{aligned} \tilde{B}(T) &= \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_5) \\ &= \frac{1}{2\bar{\rho}^4} \left(c_1^2 \bar{\rho}^2 (\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}) + c_3^2 \bar{\rho}^2 (\underline{\nu}_T^2 - \frac{(v_T - v_0)^2}{\nu_T^2}) + 2c_1 c_3 \bar{\rho}^2 (\hat{\nu}_T^2 - \frac{T}{\nu_T^2} (v_T - v_0)) \right) \\ &= \frac{1}{2\bar{\rho}^2} \left(c_1^2 (\bar{\nu}_T^2 - \frac{T^2}{\nu_T^2}) + c_3^2 (\underline{\nu}_T^2 - \frac{(v_T - v_0)^2}{\nu_T^2}) + 2c_1 c_3 (\hat{\nu}_T^2 - \frac{T}{\nu_T^2} (v_T - v_0)) \right) \\ &= \frac{c_1^2 \bar{\nu}_T^2}{2\bar{\rho}^2} + \frac{c_3^2 \underline{\nu}_T^2}{2\bar{\rho}^2} + \frac{c_1 c_3 \hat{\nu}_T^2}{\bar{\rho}^2} - \frac{1}{2\bar{\rho}^2} \left(c_1^2 \frac{T^2}{\nu_T^2} + c_3^2 \frac{(v_T - v_0)^2}{\nu_T^2} + 2c_1 c_3 \frac{T(v_T - v_0)}{\nu_T^2} \right), \end{aligned}$$

for $\bar{B}(T)$, we get

$$\bar{B}(T) = \tilde{B}(T) - B(T)$$

$$\begin{aligned}
&= \frac{c_1^2 \nu_T^2}{2\bar{\rho}^2} + \frac{c_3^2 \nu_T^2}{2\bar{\rho}^2} + \frac{c_1 c_3 \hat{\nu}_T^2}{\bar{\rho}^2} - \frac{1}{2\bar{\rho}^2} \left(c_1^2 \frac{T^2}{\nu_T^2} + c_3^2 \frac{(v_T - v_0)^2}{\nu_T^2} + 2c_1 c_3 \frac{T(v_T - v_0)}{\nu_T^2} \right) \\
&\quad - \left(\frac{1}{2\bar{\rho}^2} (c_1^2 \nu_T^2 + c_2^2 \nu_T^2 + c_3^2 \nu_T^2 + 2c_1 c_2 T + 2c_1 c_3 \hat{\nu}_T^2 + 2c_2 c_3 (v_T - v_0)) \right) \\
&= -\frac{1}{2\bar{\rho}^2} \left(c_1^2 \frac{T^2}{\nu_T^2} + c_3^2 \frac{(v_T - v_0)^2}{\nu_T^2} + 2c_1 c_3 \frac{T(v_T - v_0)}{\nu_T^2} + c_2^2 \nu_T^2 + 2c_2 c_3 (v_T - v_0) + 2c_1 c_2 T \right),
\end{aligned}$$

where c_1 , c_2 and c_3 are given in (3.45).

For F , we get

$$\begin{aligned}
F &= \frac{(c_1 a_1 + c_2 + c_3 a_3)}{\bar{\rho}^2} \\
&= \frac{1}{\bar{\rho}^2} \left(c_1 \frac{T}{\nu_T^2} + c_2 + c_3 \frac{(v_T - v_0)}{\nu_T^2} \right).
\end{aligned}$$

Hence, for the factor of $I_{0,x}$ in (3.66), we get

$$\begin{aligned}
&\exp \left\{ \frac{1}{2} (F + x)^2 \bar{\rho}^2 \nu_T^2 + G \right\} \\
&= \exp \left\{ \frac{1}{2} \left(\frac{1}{\bar{\rho}^2} \left(c_1 \frac{T}{\nu_T^2} + c_2 + c_3 \frac{(v_T - v_0)}{\nu_T^2} \right) + x \right)^2 \bar{\rho}^2 \nu_T^2 + G \right\} \\
&= \exp \left\{ \frac{\nu_T^2}{2\bar{\rho}^2} \left(c_1 \frac{T}{\nu_T^2} + c_2 + c_3 \frac{(v_T - v_0)}{\nu_T^2} + x \right)^2 + G \right\}.
\end{aligned}$$

and for $x = 0$, we get

$$\begin{aligned}
&\exp \left\{ \frac{\nu_T^2}{2\bar{\rho}^2} \left(c_1 \frac{T}{\nu_T^2} + c_2 + c_3 \frac{(v_T - v_0)}{\nu_T^2} \right)^2 + G \right\} \\
&= \exp \left\{ \frac{1}{2\bar{\rho}^2} \left(c_1^2 \frac{T^2}{\nu_T^2} + c_3^2 \frac{(v_T - v_0)^2}{\nu_T^2} + 2c_1 c_3 \frac{T(v_T - v_0)}{\nu_T^2} + c_2^2 \nu_T^2 + 2c_2 c_3 (v_T - v_0) + 2c_1 c_2 T \right) \right\} \\
&\cdot \exp \left\{ -\frac{1}{2\bar{\rho}^2} \left(c_1^2 \frac{T^2}{\nu_T^2} + c_3^2 \frac{(v_T - v_0)^2}{\nu_T^2} + 2c_1 c_3 \frac{T(v_T - v_0)}{\nu_T^2} + c_2^2 \nu_T^2 + 2c_2 c_3 (v_T - v_0) + 2c_1 c_2 T \right) \right\} \\
&= e^0 = 1,
\end{aligned}$$

since $G = \overline{B}(T)$.

Therefore,

$$\exp \left\{ \frac{1}{2} F^2 \bar{\rho}^2 \nu_T^2 + G \right\} = 1. \quad (3.79)$$

Using this result, the factor for $x = 1$ reads

$$\begin{aligned} & \exp \left\{ \frac{1}{2} (F + 1)^2 \bar{\rho}^2 \nu_T^2 + G \right\} \\ &= \exp \left\{ \frac{1}{2} F^2 \bar{\rho}^2 \nu_T^2 + G \right\} \exp \left\{ F \bar{\rho}^2 \nu_T^2 + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \\ &= \exp \left\{ \frac{1}{\bar{\rho}^2} \left(c_1 \frac{T}{\nu_T^2} + c_2 + c_3 \frac{(v_T - v_0)}{\nu_T^2} \right) \bar{\rho}^2 \nu_T^2 + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \\ &= \exp \left\{ c_1 T + c_2 \nu_T^2 + c_3 (v_T - v_0) + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \\ &= \exp \left\{ h + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\}, \end{aligned}$$

where

$$h = c_1 T + c_2 \nu_T^2 + c_3 (v_T - v_0). \quad (3.80)$$

Hence, $I_{0,0}$ and $I_{0,1}$ are given by

$$I_{0,0} = N \left(\frac{h + \ln \frac{S_0}{K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{h + \ln \frac{S_0}{B}}{\bar{\rho} \nu_T} \right) \quad (3.81)$$

$$\begin{aligned} & I_{0,1} = \exp \left\{ h + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \\ & \cdot \left[N \left(\frac{h + \bar{\rho}^2 \nu_T^2 + \ln \frac{S_0}{K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{h + \bar{\rho}^2 \nu_T^2 + \ln \frac{S_0}{B}}{\bar{\rho} \nu_T} \right) \right]. \end{aligned} \quad (3.82)$$

We rewrote the factor of $I_{1,x}$ in (3.67) into (3.76), i.e.

$$\exp \left\{ \frac{1}{2} (F + x)^2 \bar{\rho}^2 \nu_T^2 + G \right\} \exp \{ 2b(F + x) \},$$

so, for $x = 0$, we use (3.79) and the definitions of F , G and b , hence

$$\begin{aligned} & \exp \{2bF\} \\ &= \exp \left\{ \frac{2h}{\bar{\rho}^2 \nu_T^2} \ln \left(\frac{B}{S_0} \right) \right\} \\ &= \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2 \nu_T^2}}, \end{aligned}$$

where h is given in (3.80.).

In the same way we get the factor for $x = 1$, hence

$$\begin{aligned} & \exp \left\{ \frac{1}{2}(F+1)^2 \nu_T^2 + G \right\} \exp \{2b(F+1)\} \\ &= \exp \left\{ h + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \left(\frac{B}{S_0} \right)^{2(F+1)} \\ &= \exp \left\{ h + \frac{1}{2} \bar{\rho}^2 \nu_T^2 \right\} \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2 \nu_T^2} + 2}. \end{aligned}$$

Therefore, $I_{1,0}$ and $I_{1,1}$ are of the form

$$I_{1,0} = \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2 \nu_T^2}} \left[N \left(\frac{h + \ln \frac{B^2}{S_0 K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{h + \ln \frac{B}{S_0}}{\bar{\rho} \nu_T} \right) \right] \quad (3.83)$$

$$\begin{aligned} I_{1,1} &= e^{h + \frac{1}{2} \bar{\rho}^2 \nu_T^2} \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2 \nu_T^2} + 2} \\ &\cdot \left[N \left(\frac{h + \bar{\rho}^2 \nu_T^2 + \ln \frac{B^2}{S_0 K}}{\bar{\rho} \nu_T} \right) - N \left(\frac{h + \bar{\rho}^2 \nu_T^2 + \ln \frac{B}{S_0}}{\bar{\rho} \nu_T} \right) \right]. \end{aligned} \quad (3.84)$$

3.2. Computing E^v :

Remember that so far, by calculating $E_{\mathbb{P}^v}^v$, we have done the first (and hardest) step in calculating E^v . In order to actually get E^v , we just have to replace our dummy variance by the initial variance process. (Again, see

appendix A.2.6 for details).

In addition, this subsection is like a summary of step 3.1. Since using the notation of $\mathbb{P}^v$ instead of writing $\tilde{v}_t$ for the dummy, the substitution for the variance process will not be visible in the formula of (3.62) and the corresponding $I_{x,y}$ in equations (3.68), (3.69), (3.70), (3.71) for case A; (3.74), (3.75), (3.77), (3.78) for case B and (3.81), (3.82), (3.83), (3.84) for case C, since we used the abbreviations $E_{\mathbb{P}^v}^v$ and E^v , but remember, that they were given by

$$E_{\mathbb{P}^v}^v = \mathbb{E}_{\mathbb{P}^v} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} \right]$$

and

$$E^v = \mathbb{E}_{\mathbb{P}} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} | \mathcal{F}_T^v \right]$$

So in E^v we are back to our initial measure $\mathbb{P}$ with random variance v (and therefore random ν). Keeping this in mind we can take these equations or the $I_{x,y}$ directly.

Using (3.62) - which can be used for all three cases - and replacing the dummy variance by $\{v_t\}$, we get

$$E^v = S_0 I_{0,1} - K I_{0,0} - S_0 I_{1,1} + K I_{1,0},$$

where $I_{x,y}$ differ from case to case:

Case A:

$$I_{0,0} = N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \quad (3.85)$$

$$I_{0,1} = N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \quad (3.86)$$

$$I_{1,0} = \frac{S_0}{B} \left[N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{-\frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right] \quad (3.87)$$

$$I_{1,1} = \frac{B}{S_0} \left[N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{\frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right]. \quad (3.88)$$

Case B:

$$I_{0,0} = \left[N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \right] \quad (3.89)$$

$$I_{0,1} = e^{rT} \left[N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{S_0}{K}}{\nu_T} \right) - N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{S_0}{B}}{\nu_T} \right) \right] \quad (3.90)$$

$$I_{1,0} = \left(\frac{B}{S_0} \right)^{\frac{2brT}{\nu_T^2} - 1} \cdot \left[N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{rT - \frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right] \quad (3.91)$$

$$I_{1,1} = e^{rT} \left(\frac{B}{S_0} \right)^{\frac{2brT}{\nu_T^2} + 1} \cdot \left[N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{rT + \frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right]. \quad (3.92)$$

Case C:

$$I_{0,0} = N \left(\frac{h + \ln \frac{S_0}{K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{h + \ln \frac{S_0}{B}}{\bar{\rho}\nu_T} \right) \quad (3.93)$$

$$I_{0,1} = \exp \left\{ h + \frac{1}{2}\bar{\rho}^2\nu_T^2 \right\} \cdot \left[N \left(\frac{h + \bar{\rho}^2\nu_T^2 + \ln \frac{S_0}{K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{h + \bar{\rho}^2\nu_T^2 + \ln \frac{S_0}{B}}{\bar{\rho}\nu_T} \right) \right]. \quad (3.94)$$

$$I_{1,0} = \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2\nu_T^2}} \left[N \left(\frac{h + \ln \frac{B^2}{S_0 K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{h + \ln \frac{B}{S_0}}{\bar{\rho}\nu_T} \right) \right] \quad (3.95)$$

$$I_{1,1} = e^{h + \frac{1}{2}\bar{\rho}^2\nu_T^2} \left(\frac{B}{S_0} \right)^{\frac{2h}{\bar{\rho}^2\nu_T^2} + 2} \cdot \left[N \left(\frac{h + \bar{\rho}^2\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\bar{\rho}\nu_T} \right) - N \left(\frac{h + \bar{\rho}^2\nu_T^2 + \ln \frac{B}{S_0}}{\bar{\rho}\nu_T} \right) \right], \quad (3.96)$$

where h is given by (3.80), as

$$h = c_1 T + c_2 \nu_T^2 + c_3 (v_T - v_0)$$

and the c_i are given in (3.45).

Now, we are finally ready to compute the price of the UOC, by taking the expectation of the inner (conditional) expectation E^v , i.e.

$$UOC = e^{-rT} \mathbb{E}[E^v].$$

This is content of the next subsection.

4. Calculating the Outer Expectation:

All that is left to do in order to compute the price of an UOC option, is to calculate the outer expectation given by

$$UOC = e^{-rT} \mathbb{E}_{\mathbb{P}}[S_0 I_{0,1} - K I_{0,0} - S_0 I_{1,1} + K I_{1,0}].$$

In order to do this, we need the density $f_{\nu_T^2}$ of ν_T^2 under $\mathbb{P}$ for cases A and B and the joint density $f_{\nu_T^2, v_T}$ of (v_T, ν_T^2) for case C, which is explained in the following.

Case A:

A powerful tool to calculate a density function for a random variable is to compute the Fourier inverse of its characteristic function. From probability theory we know that the characteristic function of a random variable always exists (see e.g. [19]). Since $I_{0,x}$ and $I_{1,x}$ contain the time integrated variance ν_T^2 as random variable, we can calculate its characteristic function and we get an explicit formula for it (actually for all three cases).

Like mentioned, the density function $f_{\nu_T^2}$, is given by the Fourier inverse

$$f_{\nu_T^2}(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \mathcal{R}(e^{-ixu} \phi_{\nu_T^2}(u)) du, \quad (3.97)$$

where $\mathcal{R}$ denotes the real part of $e^{-ixu} \phi_{\nu_T^2}(u)$, (which is a complex number for every $u \in \mathbb{R}$) and $\phi_{\nu_T^2}$ denotes the characteristic function of ν_T^2 , which

is of the form

$$\phi_{\nu_T}(u) = e^{A(u,T)+B_1(u,T)v_0}, \quad u \in \mathbb{R},$$

where

$$A(u, T) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2} \ln \left[\frac{2d}{de_+ + \kappa e_-} \right]$$

$$B_1(u, T) = \frac{2iue^-}{\kappa e_- + de_+},$$

with

$$e_{\pm} = (1 \pm e^{-dT})$$

and

$$d = \sqrt{\kappa^2 - 2iu\sigma^2}.$$

The derivation is given in the appendix A.2.5, (see equations (A.20) and (A.21)).

Price of the UOC Option

This enables us to compute the price of the UOC option, given by

$$UOC = e^{-rT} \int_{\mathbb{R}_+} (S_0 I_{1,1}(y) - K I_{1,0}(y) - S_0 I_{2,1}(y) + K I_{2,0}(y)) f_{\nu_T^2}(y) dy \quad (3.98)$$

where $I_{0,x}$ and $I_{1,x}$, $x = 0, 1$ are given in (3.85), (3.86) and (3.87), (3.88).

Case B:

In this case, the density $f_{\nu_T^2}$ is the same as in case A, but recall, since we derived an approximation formula to the density $f_{\widetilde{M}_T, \widetilde{R}_T}$ of the random vector $(\widetilde{M}_T, \widetilde{R}_T)$, we only get an approximation to the true price of the UOC, hence

$$UOC \approx e^{-rT} \int_{\mathbb{R}_+} (S_0 I_{1,1}(y) - K I_{1,0}(y) - S_0 I_{2,1}(y) + K I_{2,0}(y)) f_{\nu_T^2}(y) dy, \quad (3.99)$$

with corresponding $I_{x,y}$ from (3.89), (3.90), (3.91) and (3.92).

Case C:

In this case, we have an additional random variable in the $I_{x,y}, v_T$. Recall, e.g. $I_{1,1}$ is given by

$$I_{1,1} = e^{rT} \left(\frac{B}{S_0} \right)^{\frac{2brT}{\nu_T^2} + 1} \left[N \left(\frac{h + \frac{1}{2}\nu_T^2 + \ln \frac{B^2}{S_0 K}}{\nu_T} \right) - N \left(\frac{h + \frac{1}{2}\nu_T^2 + \ln \frac{B}{S_0}}{\nu_T} \right) \right]$$

where v_T is hidden in h , given by

$$h = c_1 T + c_2 \nu_T^2 + c_3 (v_T - v_0).$$

This means, in order to compute the price in the same way as we did in case B, we need to find the joint density $f_{\nu_T^2, v_T}$ of the pair (ν_T^2, v_T) , which can be done like for $f_{\nu_T^2}$: we derive the characteristic function $\phi_{\nu_T^2, v_T}$ of the pair (ν_T^2, v_T) and give the density as the Fourier inverse of $\phi_{\nu_T^2, v_T}$.

So, the joint density $f_{\nu_T^2, v_T}$ of the pair (ν_T^2, v_T) , is given by the (double) Fourier inverse

$$f_{\nu_T^2, v_T}(x, y) = \frac{1}{(2\pi)^2} \int \int_{\mathbb{R}^2} e^{-iux - iwy} \phi_{\nu_T^2, v_T}(u, w) du dw, \quad (3.100)$$

where $\phi_{\nu_T^2, v_T}$ denotes the bivariate characteristic function of (ν_T^2, v_T) and is given by

$$\phi_{\nu_T, \nu_T}(u, w) = e^{A(u, w, T) + B_1(u, w, T)v_0}, \quad u, w \in \mathbb{R}$$

where

$$A(u, w, T) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2} \ln \frac{2d}{(\kappa - iw\sigma^2)e_- + de_+}$$

and

$$B_1(u, w, T) = \frac{(2iu - \kappa iw)e_- + diwe_+}{(\kappa - iw\sigma^2)e_- + de_+},$$

with

$$e_{\pm} = (1 \pm e^{-dT})$$

and

$$d = \sqrt{\kappa^2 - 2iu\sigma^2}$$

The derivation is given in the appendix A.2.5, (see equations (A.24) and (A.25)).

Price of the UOC Option:

Like in case B, we had a an approximation formula to the density $f_{\widetilde{M}_T, \widetilde{R}_T}$, therefore, we get the following approximation to the UOC price:

$$\begin{aligned} UOC \approx \\ e^{-rT} \int_{\mathbb{R}_+^2} (S_0 I_{1,1}(x, y) - K I_{1,0}(x, y) - S_0 I_{2,1}(x, y) + K I_{2,0}(x, y)) f_{\nu_T^2, v_T}(x, y) dx dy \end{aligned} \quad (3.101)$$

where $I_{i,j}$, $i, j \in \{0, 1\}$ are given in (3.93), (3.94) and (3.95), (3.96).

Remark:

The densities and the resulting approximation formulas for our UOC option prices can be calculated numerically. This involves, given the characteristic functions, to compute the (inverse) Fourier transformation in (3.97) (for case A and B) and (3.100) (for case C) in order to get the densities, which can be done e.g. by fast Fourier transformation with low computation times. (See numbers below).

The second step involves calculating the one dimensional integrals (3.98) and (3.99) (for case A and B resp.) and the two-dim. integral in (3.101) (for case C), for which there are plenty of efficient methods to do so. (Matlab offers a number of solver functions, e.g. *integral2*).

In the numerical analysis in [13, p.22-29] UOC prices are calculated with our approximation method (AM) for all three cases for barriers in the interval [105, 145], as well as strikes in the interval [80, 100], using a set of parameters, which fulfill the Feller Condition. As reference method, a finite difference (FD) method was considered, derived by Foulon and In't Hout in [11] and is used as benchmark model for cases B and C. (Recall, in case

A we have an exact formula given by (3.98)).

For case A the results show, that absolute differences between the prices of both methods stay below 0.0015. Mathematically speaking

$$|P_{AM}(B, K) - P_{FD}(B, K)| \leq 0.0015 \text{ for } B \in [105, 145], K \in [80, 100],$$

where $P_{AM}(B, K)$ and $P_{FD}(B, K)$ denote the prices calculated with the AM and FD method respectively, for barrier B and strike K .

Relative price differences stay below 0.04%, i.e.

$$|P_{AM}(B, K)/P_{FD}(B, K) - 1| \leq 0.04\%$$

for all B and K in the above intervals and it can be observed that the highest deviations occur for low barrier- and high strike level. The computation times for the FD scheme is about 150 seconds and the AM does not exceed 1.1 seconds.

In case B, where a non-zero r is chosen, absolute price differences stay below 0.025% of the initial spot price S_0 (which was set to 100 in the parameter setup) and relative price differences stay below 2.5%, with highest deviations for barriers and strikes in the same region as in case A. The computation times also stay the same as in case A.

For the most general case C, with an additional correlation parameter of $\rho = -0.5$, the absolute price differences stay below 0,25% of the initial spot price and relative price differences stay below 8% and reach those for low barriers close to the initial spot. The region where the percentages are high is neglectable though, since prices there are virtually zero (due to high probability of knocking out). Computation times for our AM stay below 8 seconds and depend mostly on the computation of the density in (3.100). For details and exact numbers and visualizations, we refer to [13, p.22-29].

Chapter 4

Pricing of Barrier Options: A Model-free Approach

4.1 Motivation

So far, our approach, concerning the pricing of financial derivatives, was to propose a model for the underlying stock price and using Girsanov's theorem to eliminate the market price of risk and use the resulting risk neutral measure as pricing tool.

But there is still a crucial problem with this approach: We do not know how precise our proposed models describe reality. So we face the problem of model risk and model-misspecifications might lead to wrong derivative prices.

In practice the option price formula is not used to calculate option prices itself, but for calibration purposes. This means, the model parameters are determined by observations in real markets. For example instead of proposing a (constant) volatility as an input parameter and calculate the price of a call option via its Black-Scholes formula, the observed call option price is used to determine the volatility (called implied volatility), which can be used in turn, to calculate prices of exotic options. For example, the price of an UOC using formula 2.23 with this calculated implied volatility as an input parameter.

Nevertheless, this method leads to problems as well, since options with different strike prices or maturities lead to different implied volatilities (well known as volatility smile). So, the question remains, which strike price/maturity to use, in order to calculate the "right" implied volatility to price the exotic option.

These ambiguities were the motivation for the development of the model-free approach:

Since nowadays, (vanilla) options are liquidly traded assets, we take their prices as given, for a whole spectrum of strike prices (with common, fixed maturity). Then, by investing in options and in the underlying, we replicate the pay-off of an exotic option. By no arbitrage arguments, the price of the exotic option must be equal to the prices of the options used for hedging. A huge advantage is, that there is no dynamic hedging involved in this approach, but a semi-static hedging. This means, that we trade only in a few time points (in our particular case, three at most).

Another crucial part is that we do not propose a model for the underlying stock price. So the resulting price (or price range, resp.) holds for a whole class of models. A model whose prices fit in this range is called *robust* and this whole approach therefore *robust hedging*. (After [15] and [5]).

4.2 Assumptions and important Results

4.2.1 Assumptions

Following [15], let $T > 0$ denote a fixed exercise time and K denote the strike price of a vanilla call option. Let S be the forward price of the asset. The price of a call option with underlying S and strike K is denoted with $C(K)$ and has pay-off $(S_T - K)^+$.

Let B be a barrier level and we define the first hitting time for the forward to cross the barrier, as

$$H_B = \inf\{t_0 > 0 : S_{t_0} \geq B\},$$

then the pay-off of an UOC for example, can be written as

$$UOC_T = (S_T - K)^+ \mathbb{I}_{\{H_B > T\}}. \quad (4.1)$$

We assume that call options with corresponding prices $C(K)$ are given as differentiable function of strike price K , in our case it will be enough to consider all $K \in (0, B)$.

Since we are interested in unique prices or price ranges, we have to ban arbitrage, or, equivalently, consider the class of all arbitrage-free models.

We assume that there are no transaction costs and also allow for buying and short selling any amounts of the underlying at any time. Our replicating strategies will consist of investing in call - or put options and buying or selling the underlying at certain time points.

The only assumptions we make on the price process of the underlying, is right-continuity of its paths. (Note, we do not assume any distribution of the stock returns, like in the previous chapters).

Last but not least, we work in zero interest rate environment. Adding non-zero interest rates complicates things in the sense, that we have a non-constant barrier level.

In section (4.3) we will focus on finding replicating strategies for UIC and UOC options. Since barrier options are path dependent, we can only find sub- and super-replicating strategies, which means that the pay-off of the barrier options is always lower, higher respectively, than the value of our portfolio. Therefore the price of a BO ends up lying in an interval.([15, p.5]).

4.2.2 Important Results

(See [5] and [15]).

Instead of proposing a distribution for our underlying process, like we did in the previous approaches, we can deduce it from given option prices. This means, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the probability law μ of S_T (see Appendix A.4 for more information), is the solution to

$$C(K) = \mathbb{E}_\mu[(S_T - K)^+] = \int_{\mathbb{R}} (x - K)^+ \mu(dx). \quad (4.2)$$

In chapter 4.3 we will need some simple results concerning digital options with pay-off $\mathbb{I}_{\{S_T \geq B\}}$.

First of all, if the call option price, interpreted as a function of strike K , is differentiable, then, using (4.2), we get

$$\begin{aligned} C'(K) &= \frac{d}{dK} \int_{\mathbb{R}} (x - K)^+ \mu(dx) = \frac{d}{dK} \int_{x \geq K} (x - K) \mu(dx) = \\ &= \int_{x \geq K} \frac{d}{dK} (x - K) \mu(dx) = - \int_{x \geq K} \mu(dx) = -\mu[K, \infty), \end{aligned}$$

hence,

$$\mu[K, \infty) = -C'(K). \quad (4.3)$$

Furthermore, we have

$$\begin{aligned}\mathbb{P}\{S_T \geq B\} &= \mathbb{E}_\mu[\mathbb{I}_{\{S_T \geq B\}}] = \int_{\mathbb{R}} \mathbb{I}_{\{x \geq B\}}(x) \mu(dx) = \int_{x \geq B} \mu(dx) \\ &= \int_{x \geq B} \mu(dx) = \mu[B, \infty).\end{aligned}$$

So, using (4.3), we get

$$\mathbb{P}\{S_T \geq B\} = \mu[B, \infty) = -C'(B). \quad (4.4)$$

Best lower/upper Bound:

We want to briefly discuss the procedure, how to prove that a certain lower/upper bound for the arbitrage price of a financial derivative is indeed the best possible lower/upper bound, without going too much into detail, because the methods are based on some deeper results of robust hedging theory and is beyond our scope.

Let a financial derivative on an underlying forward price S (e.g. an exotic option) be defined by its pay-off $f(S_T)$, with corresponding arbitrage-free price $\mathcal{P}(f(S_T))$. Furthermore, let U be an upper bound for this price, i.e.

$$\mathcal{P}(f(S_T)) \leq U$$

Our process S here, can be interpreted as a variable out of the class of all arbitrage-free models $\mathcal{M}$.

The idea how to show that our bound U is the lowest upper bound for a hole class of models is quite standard:

First of all it has to hold, that

$$\mathcal{P}(f(S_T)) \leq U, \forall S \in \mathcal{M}$$

and furthermore

$$\exists S^* \in \mathcal{M} : \mathcal{P}(f(S_T)) = U. \quad (4.5)$$

To actually show the existence of such a process is the difficult part and its construction generally uses some particular solution of the Skorokhod Embedding Problem (SEP) (see chapter 3 in [5]), which aims to find a stopping time for a given process and a given law, such that the stopped process at this time possesses this law. There are several approaches to tackle that problem and the construction of a process fulfilling (4.5) depends on the derivative (its pay-off, resp.) and on the (sub/super-) replicating hedge and corresponding to that, the solutions to the SEP- which fits best to the particular case- can be applied for the construction.

4.3 Replicating Strategies and Price Ranges of Barrier Options

In the following subsection, we consider an UIC option and give a sub- and super-replicating strategy and derive the corresponding price bounds. In section 4.3.2, we consider the case of a UOC and use relations (2.23) and (2.24) to derive the sub- and super-replicating strategy and price bounds from the results of the UIC in subsection 4.3.1.

4.3.1 UIC Options

For the UIC Option with pay-off

$$UIC_T = (S_T - K)^+ \mathbb{I}_{\{H_B \leq T\}}$$

we set $K < B$, since for $K \geq B$ the UIC is automatically knocked-in and therefore has the pay-off and price of a vanilla call option with strike K .

Upper Bound:

For $K < B$, we have the following super-replicating strategy:

$$(S_T - K)^+ \mathbb{I}_{\{H_B \leq T\}} \leq \frac{B - K}{B - a} (S_T - a)^+ + \frac{a - K}{B - a} (S_{H_B} - S_T) \mathbb{I}_{\{H_B \leq T\}} \quad (4.6)$$

for $a \in [K, B)$.

Proof:

We distinguish the cases $H_B > T$ and $H_B \leq T$ first.

$H_B > T$: The left hand side of (4.6) is equal to zero in this case and the second term of the right hand side vanishes, since the indicator function is zero. Hence, for all $a \in [K, B)$

$$0 \leq \frac{B-K}{B-a}(S_T - a)^+.$$

If $S_T \leq a$ the right hand side and the inequality is correct (in that case it is an equality). For $S_T > a$ the right hand side is positive, since all factors are positive and the inequality is correct as well.

$H_B \leq T$: In that case, we get

$$(S_T - K)^+ \leq \frac{B-K}{B-a}(S_T - a)^+ + \frac{a-K}{B-a}(S_{H_B} - S_T)$$

Firstly, we consider the easiest case, where $S_T \leq K$, hence

$$0 \leq \frac{a-K}{B-a}(S_{H_B} - S_T)$$

this is true, since $S_{H_B} \geq B > K > S_T$, therefore $(S_{H_B} - S_T)$ is positive.

Let $S_T > K$. In this case we get

$$S_T - K \leq \frac{B-K}{B-a}(S_T - a)^+ + \frac{a-K}{B-a}(S_{H_B} - S_T)$$

If $S_T \leq a$, we get

$$S_T - K \leq \frac{a-K}{B-a}(S_{H_B} - S_T).$$

Since

$$S_T - K \leq a - K$$

this holds if we show that

$$a - K \leq (a - K) \frac{S_{H_B} - S_T}{B - a}$$

which is equivalent to

$$1 \leq \frac{S_{H_B} - S_T}{B - a}$$

or

$$B - a \leq S_{H_B} - S_T.$$

This is true, since $B \leq S_{H_B}$ and $S_T \leq a$.

If $S_T > a$, we get

$$S_T - K \leq \frac{B - K}{B - a}(S_T - a) + \frac{a - K}{B - a}(S_{H_B} - S_T).$$

which is equivalent to

$$\begin{aligned} (S_T - K)(B - a) &\leq (B - K)(S_T - a) + (a - K)(S_{H_B} - S_T) \\ S_TB - KB - S_Ta + aK &\leq S_TB - S_TK - aB + aK + S_{H_B}a - S_{H_B}K - S_Ta + KS_T \\ -KB &\leq -aB + S_{H_B}(a - K) \\ B(a - K) &\leq S_{H_B}(a - K) \end{aligned}$$

which is true for all $a \in (S_T, B)$. $\square$

The corresponding upper bound for the UIC is given by:

$$UIC \leq (B - K) \inf_{a \in [K, B]} \frac{C(a)}{B - a} \quad (4.7)$$

and the infimum is attained for some $a^* \in [K, B)$.

Proof:

Taking expectations on both sides of (4.6), leads to

$$UIC \leq \frac{B - K}{B - a} \mathbb{E}(S_T - a)^+ + \frac{a - K}{B - a} \mathbb{E}(S_{H_B} - S_T).$$

Since,

$$\mathbb{E}(S_{H_B} - S_T) = S_0 - S_0 = 0,$$

we get

$$UIC \leq \frac{B - K}{B - a} C(a), \quad a \in [K, B)$$

When optimizing over all $a \in [K, B)$, the minimum is attained for some $a^* \in [K, B)$, since, first of all, the right interval bound B is not interesting for us, since for $a \nearrow B$, we get

$$\frac{C(a)}{B - a} \rightarrow \infty.$$

C is continuous on $[K, B]$ and we can assume that option prices are positive on this interval, we don't get any problems at point $C(B)$, such as $C(B) = 0$.

Therefore we can choose a $B^- = B - \epsilon$ for some $\epsilon > 0$ and consider the interval $[K, B^-]$.

Since $C(a)/(B - a)$ is a continuous function on this interval, such an $a^* \in [K, B^-]$ exists and we get

$$UIC \leq (B - K) \inf_{a \in [K, B]} \frac{C(a)}{B - a} = (B - K) \frac{C(a^*)}{B - a^*}$$

□

Lower Bound:

For $K < B$, when no assumptions are made on S , we have the following (trivial) sub-replicating strategy:

$$(S_T - K)^+ \mathbb{I}_{\{H_B \leq T\}} \geq (S_T - K)^+ \mathbb{I}_{\{S_T \geq B\}} \quad (4.8)$$

Proof:

This inequality holds since

$$\{S_T \geq B\} \subseteq \{H_B \leq T\} \quad (4.9)$$

which is true, since the set on the right hand side contains all paths, where the underlying has crossed the barrier level at time H_B , all paths with $S_T \geq B$ are included. □

The corresponding price bound is given by

$$UIC \geq C(B) + (B - K)\mu[B, \infty) \quad (4.10)$$

Proof:

First, we split the right hand side of (4.8) the following way:

$$(S_T - K)^+ \mathbb{I}_{\{S_T \geq B\}} = (S_T - B + B - K)^+ \mathbb{I}_{\{S_T \geq B\}}$$

$$\begin{aligned}
&= ((S_T - B)^+ + (B - K))\mathbb{I}_{\{S_T \geq B\}} \\
&= (S_T - B)^+\mathbb{I}_{\{S_T \geq B\}} + (B - K)\mathbb{I}_{\{S_T \geq B\}} \\
&= (S_T - B)^+ + (B - K)\mathbb{I}_{\{S_T \geq B\}},
\end{aligned}$$

therefore

$$(S_T - K)^+\mathbb{I}_{\{S_T \geq B\}} = (S_T - B)^+ + (B - K)\mathbb{I}_{\{S_T \geq B\}}.$$

Taking expectations and using (4.4), we get

$$\begin{aligned}
UIC &\geq \mathbb{E}[(S_T - B)^+] + (B - K)\mathbb{E}[\mathbb{I}_{\{S_T \geq B\}}] \\
&= C(B) + (B - K)\mu[B, \infty)
\end{aligned}$$

□

If we allow S to be continuous, we can even improve this bound, using the following sub-replicating strategy:

$$\begin{aligned}
(S_T - K)^+\mathbb{I}_{\{H_B \leq T\}} &\geq (S_T - K)^+ - \frac{B - K}{B - a}[(S_T - a)^+ - (S_T - B)^+] \\
&\quad + (B - K)\mathbb{I}_{\{S_T \geq B\}} + \frac{a - K}{B - a}(S_{H_B \wedge T} - B),
\end{aligned} \tag{4.11}$$

for $a \in [K, B)$.

Proof:

Case $H_B > T$: In this case we get

$$0 \geq (S_T - K)^+ - \frac{B - K}{B - a}[(S_T - a)^+ - (S_T - B)^+] + \frac{a - K}{B - a}(S_T - B).$$

If $K \geq S_T$, we get

$$0 \geq \frac{a - K}{B - a}(S_T - B),$$

which is true since $a < B$ and $S_T < B$.

If $a \geq S_T > K$, we get

$$0 \geq S_T - K + \frac{a - K}{B - a}(S_T - B).$$

$$\begin{aligned}
0 &\geq (B - a)(S_T - K) + (a - K)(S_T - B) \\
&= S_TB + aK - KB - S_Ta + S_Ta - KS_T - aB + KB \\
&= S_TB + aK - S_TK - aB = (S_T - a)(B - K)
\end{aligned}$$

which is true by our assumption that $S_T \leq a$.

Finally, let $a < S_T$, then

$$0 \geq S_T - K - \frac{B - K}{B - a}[(S_T - a) - (S_T - B)^+] + \frac{a - K}{B - a}(S_T - B).$$

Since $S_T > B$ is not possible, (because we assumed $H_B > T$), we get

$$0 \geq S_T - K - \frac{B - K}{B - a}(S_T - a) + \frac{a - K}{B - a}(S_T - B)$$

and reformulating this, leads to

$$\begin{aligned}
0 &\geq (S_T - K)(B - a) - (B - K)(S_T - a) + (a - K)(S_T - B) \\
&= S_TB - KB - aS_T + aK - S_TB + KS_T + aB - aK + aS_T - S_TK - aB + KB = 0
\end{aligned}$$

Hence, we have equality for $a \in [K, S_T]$.

Case $H_B \leq T$: Here, we get

$$\begin{aligned}
(S_T - K)^+ &\geq (S_T - K)^+ - \frac{B - K}{B - a}[(S_T - a)^+ - (S_T - B)^+] \\
&\quad + (B - K)\mathbb{I}_{\{S_T \geq B\}} + \frac{a - K}{B - a}(S_{H_B} - B)
\end{aligned}$$

Since S is continuous, we have $S_{H_B} = B$, hence

$$\frac{a - K}{B - a}(S_{H_B} - B) = 0,$$

therefore

$$0 \geq -\frac{B - K}{B - a}[(S_T - a)^+ - (S_T - B)^+] + (B - K)\mathbb{I}_{\{S_T \geq B\}}$$

If $a \geq S_T$, $S_T < B$, therefore we have equality in that case.

If $a < S_T$ and $S_T > B$, we get

$$0 \geq -\frac{B-K}{B-a}[S_T - a - S_T + B] + B - K = 0$$

if $S_T \leq B$, we have

$$0 \geq -\frac{B-K}{B-a}(S_T - a)$$

which is a strict inequality for $a \in [K, S_T)$. $\square$

For the corresponding price bound, we get

$$UIC \geq C(K) + (B-K)\mu[B, \infty) - \inf_{a < B} \left\{ \frac{(B-K)(C(a) - C(B))}{B-a} + \frac{(a-K)(B-S_0)}{B-a} \right\}$$

where the infimum is attained by some $a^* \in [K, B)$.

Proof:

Analogously to the upper bound above -

by taking expectations on both sides (4.8) and also the same argument as above, proves the existence of $a^* \in [K, B)$. $\square$

Remark: Like mentioned in (4.2.2), in order to prove that a boundary is the best possible, one has to find a process and show that it attains the bound. For both boundaries such processes can be constructed. Further information can be found in [5].

4.3.2 UOC Options

For the UOC option with pay-off $(S_T - K)^+ \mathbb{I}_{\{H_B > T\}}$, we use relation (2.23), i.e.

$$(S_T - K)^+ \mathbb{I}_{\{H_B \leq T\}} + (S_T - K)^+ \mathbb{I}_{\{H_B > T\}} = (S_T - K)^+ \quad (4.12)$$

and corresponding prices, by taking expectation:

$$UIC + UOC = C. \quad (4.13)$$

This relation allows us to derive the sub- and super-replicating strategies and their corresponding price boundaries quite easily.

Upper Bound:

Without continuity assumptions on S , we get the following (trivial) super-replicating strategy:

$$(S_T - K)^+ \mathbb{I}_{\{H_B > T\}} \leq (S_T - K)^+ \mathbb{I}_{\{S_T < B\}},$$

with corresponding price bound

$$UOC \leq C(K) - C(B) - (B - K)\mu[B, \infty) \quad (4.14)$$

If S is continuous, we get the following super-rep. strategy:

$$\begin{aligned} (S_T - K)^+ \mathbb{I}_{\{H_B > T\}} &\leq \frac{(a - K)}{B - a} (B - S_{H_B \wedge T}) \\ &+ \frac{(B - K)}{B - a} [(S_T - a)^+ - (S_T - B)^+] - (B - K) \mathbb{I}_{\{S_T \geq B\}} \end{aligned} \quad (4.15)$$

with corresponding price bound

$$UOC \leq \frac{(a^* - K)(B - S_0)}{B - a^*} - \frac{B - K}{B - a^*} (C(B) - C(a^*)) - (B - K)\mu[B, \infty) \quad (4.16)$$

Proof:

For S without any assumptions, for the super-rep. strategy, we take complements on (4.9) in order to get the relation

$$\{H_B > T\} \subseteq \{S_T < B\}.$$

For UIC we plug in (4.10) and using (4.12), we get

$$C(K) - UOC \geq C(B) + (B - K)\mu[B, \infty)$$

solving for UOC gives us (4.14).

If S is continuous, use (4.12) in (4.11) and solve for $(S_T - K)^+ \mathbb{I}_{\{H_B > T\}}$ in order to get (4.15).

For (4.16), we can use (4.13) in the corresponding UIC price bound and

solve for UOC. □

Lower Bound:

For every $a \in [K, B)$ a sub-replicating strategy is given by

$$(S_T - K)^+ \mathbb{I}_{\{H_B > T\}} \geq (S_T - K)^+ - \frac{B - K}{B - a} (S_T - a)^+ - \frac{a - K}{B - K} (S_{H_B} - S_T) \mathbb{I}_{\{H_B \leq T\}} \quad (4.17)$$

with best possible lower price bound

$$UOC \geq C(K) - (B - K) \inf_{a \in [K, B)} \frac{C(a)}{B - a}, \quad (4.18)$$

furthermore, the infimum is attained by some $a^* \in (K, B)$.

Proof:

For (4.17), take the result of (4.12), plug in (4.6) and solve for $(S_T - K)^+ \mathbb{I}_{\{H_B > T\}}$.

For (4.18), take (4.13), plug into (4.7) and solve for UOC .

The existence of a^* can be shown analogously to the upper bound of the UIC in the section before. □

Remark:

Like for the case of an UIC, it is possible to construct a process, which actually attains the (lower/)upper price bound. See again [5].

Appendices

Appendix A

Relevant Material for all Chapters

A.1 Appendix for Chapter 2

A.1.1 Girsanov's Theorem

(After [26]).

Let $t \in [0, T]$ and $\{W_t\}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$. Let $\{\Theta_t\}$ be an adapted stochastic process and define

$$Z_t = \exp \left\{ - \int_0^t \Theta_s dW_s - \frac{1}{2} \int_0^t \Theta_s^2 ds \right\}$$

$$\overline{W}_t = W_t + \int_0^t \Theta_s ds, \quad t \in [0, T]$$

and assume that

$$\mathbb{E} \left[\int_0^T \Theta_s^2 Z_s^2 du \right] < \infty$$

Then, it follows that $\mathbb{E}[Z_T] = 1$ and under the probability measure

$$\overline{\mathbb{P}}(A) := \int_A Z_T d\mathbb{P}, \quad A \in \mathcal{F}$$

$\{\overline{W}_t\}$ is a $\overline{\mathbb{P}}$ -Brownian Motion.

$\bar{\mathbb{P}}$ is called *equivalent Martingale measure* (EMM).

A.1.1.1 Novikov Condition

It suffices to show that the Novikov condition holds, i.e.

$$\mathbb{E} \left[\frac{1}{2} \int_0^T \Theta_s^2 du \right] < \infty,$$

as condition for Girsanov's theorem. See [26, p.250].

A.1.1.2 The discounted Stock Price under the EMM

Let the stock price process $\{S_t\}$ be given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and the discount factor $\{D_t\}$ by

$$dD_t = -r D_t dt.$$

We want to apply Girsanov's theorem to the discounted stock price process $D_t S_t$ and using Ito's Lemma in A.2.2, its differential is given by

$$\begin{aligned} d(D_t S_t) &= dD_t S_t + D_t dS_t + dD_t dS_t \\ &= -r D_t dt S_t + D_t (\mu S_t dt + \sigma S_t dW_t) - r D_t dt (\mu S_t dt + \sigma S_t dW_t) \\ &= -r D_t S_t dt + \mu D_t S_t dt + \sigma D_t S_t dW_t \\ &= (\mu - r) D_t S_t dt + \sigma D_t S_t dW_t = \sigma D_t S_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right) \end{aligned}$$

since $dt dt = dW_t dt = 0$, hence

$$d(D_t S_t) = \sigma D_t S_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right)$$

We want to get rid of the dt -drift term of the last equation, therefore we use Girsanov's theorem¹ by defining

$$\bar{W}_t := W_t + \frac{\mu - r}{\sigma} t$$

¹Novikov's condition is clearly fulfilled here.

$$\begin{aligned}
Z_T &= \exp \left\{ - \int_0^T \left(\frac{\mu - r}{\sigma} \right) dW_t - \frac{1}{2} \int_0^T \left(\frac{\mu - r}{\sigma} \right)^2 dt \right\} \\
&= \exp \left\{ - \left(\frac{\mu - r}{\sigma} \right) W_T - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 T \right\}
\end{aligned}$$

and the new probability measure $\bar{\mathbb{P}}$, by

$$\bar{\mathbb{P}}(A) := \int_A Z_T d\mathbb{P}, \quad A \in \mathcal{F},$$

then $\{\bar{W}_t\}$ is a drift-less Brownian motion under $\bar{\mathbb{P}}$.

Applying this to $d(D_t S_t)$, we get

$$d(D_t S_t) = \sigma D_t S_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right) = \sigma D_t S_t d\bar{W}_t$$

Finally, for our undiscounted process $\{S_t\}$, we get

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t d \left(\bar{W}_t - \frac{\mu - r}{\sigma} t \right) \\
dS_t &= r S_t dt + \sigma S_t d\bar{W}_t.
\end{aligned}$$

A.1.2 Filtrations and Usual Conditions

Definition and Filtered Probability Space. (After [23, p.98])

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration $\{\mathcal{F}_t\} = \mathbb{F}$ is an increasing family of σ -algebras, such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$ and all $t \in [0, T]$.

We call $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space and since we want that $\mathcal{F}_T = \mathcal{F}$, we just write $(\Omega, \mathbb{F}, \mathbb{P})$.

Generated Filtration. (After [23, p.98, p.99])

The natural filtration of a stochastic process $\{X_t\}$, $t \in [0, T]$, is given by

$$\mathcal{F}_t^X = \sigma\{X_u : u \leq t\} = \sigma\{X_u(B)^{-1} : u \leq t, B \in \mathcal{B}\}, \quad t \in [0, T]$$

where $\mathcal{B}$ denotes the Borel σ -algebra on $\mathbb{R}$.

Furthermore, $\{X_t\}$ is called $\mathbb{F}$ -adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \in [0, T]$.

Usual Hypothesis (After [23, p.117])

Filtration $\mathbb{F}$ fulfills the "usual hypothesis" or "usual conditions" with respect to $\mathbb{P}$, if:

1. $\mathcal{F}_0$ contains all null-sets of $\mathcal{F}$, i.e.

$$\mathcal{N} = \{N \in \mathcal{F} : \mathbb{P}(N) = 0\} \subseteq \mathcal{F}_0$$

2. the filtration is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}, \quad t \in [0, T].$$

The motivation behind this hypothesis is of technical nature:

We want to exclude cases, where for two random variables X and Y , it holds that $X = Y$ almost surely, where X is $\mathcal{F}_t$ -measurable, but Y is not. Furthermore, we want that if X is $\mathcal{F}_t$ -measurable, then X is also $\mathcal{F}_s$ -measurable for all $s \leq t$, which is assured by 2.

Let $\widetilde{\mathbb{F}}^W$ be the natural filtration of a Brownian motion $\{W_t\}$. We set

$$\mathcal{F}_t^W = \sigma\{\widetilde{\mathcal{F}}_t^W \cup \mathcal{N}\}$$

for every t and we call $\mathbb{F}^W = \{\mathcal{F}_t^W\}$ the standard filtration of $\{W_t\}$, then it holds that $\mathbb{F}^W$ satisfies the usual conditions. (See [23, p.118] for a proof).

Filtration generated by the Stock Price Process

Finally, we consider our stock price process $\{S_t\}$ of chapter 2. In [22, p.103] it is shown, that $\mathbb{F} = \mathbb{F}^S = \mathbb{F}^W$. This means, that the only information which we need in our model, stems from the stock price process and furthermore that only the Brownian motion part in the geometric Brownian motion is responsible for this information flow.

To sum up, we work on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and we assume that $\mathbb{F} = \mathbb{F}^W$, where $\mathbb{F}^W$ is the standard filtration of the Brownian motion $\{W_t\}$.

Equivalently, we can start on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with standard filtration $\mathbb{F} := \mathbb{F}^W$, generated by the Brownian motion $\{W_t\}$, with $\mathcal{F}_T = \mathcal{F}$, which is how we start in chapter 2.

A.1.3 Reflection Principle

Let $\{W_t\}$ be a drift-less Brownian motion with standard filtration $\mathbb{F}$ and let $r > 0$.

A random variable $\tau_r : \Omega \rightarrow [0, T]$ is called first passage time, if $\tau_r = \min\{t \geq 0 : W_t = r\}$. If W_t doesn't reach r in the interval $[0, T]$, we set $\tau_r = \infty$.

Then the *reflection principle* holds, that is

$$\tilde{\mathbb{P}}\{\tau_x \leq T, W_T \leq y\} = \tilde{\mathbb{P}}\{W_T \geq 2x - y\}.$$

A derivation can be found in [26, p.111-112].

A.1.4 Quadratic Variation and Covariation

(After [16, p.83])

Let $\{X_t\}$, $t \in [0, T]$, be a stochastic process and δ_n a partition of the interval $[0, t]$, i.e.

$$\delta_n = \{0 = t_0 < t_1 < \dots < t_n = t\}.$$

and

$$\|\delta_n\| = \max_{j=0,1,\dots,n-1} (t_{j+1} - t_j)$$

Then the quadratic variation, denoted by $[X, X]_t$ is defined as

$$[X, X]_t = p - \lim_{\|\delta_n\| \rightarrow 0} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})^2$$

where $p - \lim$ denotes the convergence in probability.

Process $\{X_t\}$ has finite quadratic variation if $[X, X]_t$ is bounded a.s. for all $t \in [0, T]$.

Let $\{Y_t\}$ be another process, then similarly to the quadratic variation, the covariation between $\{X_t\}$ and $\{Y_t\}$ is defined as

$$[X, Y]_t = p - \lim_{\|\delta_n\| \rightarrow 0} \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k})$$

Quadratic Variation of Brownian Motion
(After [26, p.102])

Let $\{W_t\}$ be a Brownian motion. Then $[W, W]_t = t$ for all $t \in [0, T]$ a.s.
(See [26, p.103] for a proof.)

A.2 Appendix for Chapter 3

A.2.1 Multidimensional Version of Girsanov's Theorem

(After [26, p.224]).

Let $W(t) = (W_1(t), \dots, W_n(t))$ be a n -dim. Brownian motion and $\Theta(t) = (\Theta_1(t), \dots, \Theta_n(t))$ be an n -dimensional adapted process. For $t \in [0, T]$, define

$$Z_t = \exp \left\{ - \int_0^t \Theta(u) dW(u) - \frac{1}{2} \int_0^t \|\Theta(u)\|^2 du \right\}$$

$$\overline{W}(t) = W(t) + \int_0^t \Theta(u) du$$

and assume that

$$\mathbb{E} \left[\int_0^T \|\Theta(t)\|^2 Z(t)^2 dt \right] < \infty,$$

where the norm $\|\cdot\|$ denotes the usual Euclidian norm and $\Theta(u)dW(u)$ the usual inner product, given by

$$\Theta(t)dW(t) = \sum_{i=1}^n \Theta_i(t)dW_i(t), \quad t \in [0, T].$$

Then $\mathbb{E}[Z_T] = 1$ and under measure $\overline{\mathbb{P}}$, given by

$$\overline{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \quad A \in \mathcal{F}$$

process $\{\overline{W}(t)\}$ is an n -dimensional Brownian motion.

A.2.1.1: (Novikov Condition for the Multidimensional Version)

See [17].

Like in the 1-dimensional case, the Novikov condition can be extended to the multi-dimensional case:

$$\mathbb{E} \left[\int_0^T \|\Theta_u\|^2 du \right] < \infty.$$

A.2.2 Cholesky Decomposition

After [1, p.152-153]

For our purposes it is enough to show the two-dimensional case of the Cholesky decomposition.

Let V be a 2×2 -Matrix, given by

$$V = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where a_{ij} are real numbers, with $a_{12} = a_{21}$ and $a_{11} > 0$.

We want to decompose V , into

$$V = RR^T$$

where R is an upper triangular matrix, given by

$$R = \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix}$$

and R^T denotes its transpose.

So, we for the elements in R , we get

$$\begin{aligned} \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix}^T &= \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \\ &= \begin{pmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2 + r_3^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{aligned}$$

therefore, we get

$$r_1 = \sqrt{a_{11}}, r_2 = \frac{a_{12}}{\sqrt{a_{11}}} \text{ and } r_3 = \sqrt{a_{22} - \frac{a_{12}^2}{a_{11}}}.$$

Hence

$$R = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{a_{12}}{\sqrt{a_{11}}} & \sqrt{a_{22} - \frac{a_{12}^2}{a_{11}}} \end{pmatrix}.$$

Let $X = (X_1, X_2) \sim N\left(\mu = (\mu_1, \mu_2), V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right)$, with $\sigma_1, \sigma_2 > 0$ and $\sigma_1\sigma_2 \neq \sigma_{12}$, be a random vector on $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a standard normal distributed vector $\phi = (\phi_1, \phi_2)$, s.t.

$$X = R\phi + \mu$$

where R is the matrix from the Cholesky decomposition of V . Furthermore ϕ_1 is independent of ϕ_2 .

The existence is easy to show, since we can construct our standard normally distributed random variables the following way:

Plugging in the corresponding coefficients of V , for R we get

$$R = \begin{pmatrix} \sigma_1 & 0 \\ \frac{\sigma_{12}}{\sigma_1} & \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} \end{pmatrix}$$

Since the determinant of V is not zero its inverse exists, therefore the inverse of R exists as well and is given by

$$R^{-1} = \begin{pmatrix} 1/\sigma_1 & 0 \\ -\sigma_{12}/\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} & 1/\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} \end{pmatrix}.$$

Hence, we can construct our vector ϕ as

$$\begin{aligned} \phi &= R^{-1}(X - \mu) \\ &= \begin{pmatrix} 1/\sigma_1 & 0 \\ -\sigma_{12}/\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} & 1/\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} \end{pmatrix} \left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right), \end{aligned}$$

therefore,

$$\phi_1 = \frac{1}{\sigma_1}X_1 - \frac{\mu_1}{\sigma_1}$$

and

$$\phi_2 = -\frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}}(X_1 - \mu_1) + \frac{X_2 - \mu_2}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}}.$$

Since (X_1, X_2) is bivariate normally distributed and then every linear transformation of it is normally distributed as well. Therefore ϕ_1 and ϕ_2 are normally distributed as well, with mean and variance given by:

$$\mathbb{E}[\phi_1] = \frac{1}{\sigma_1}\mathbb{E}[X_1] - \frac{\mu_1}{\sigma_1} = 0$$

$$\mathbb{E}[\phi_2] = -\frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}}(\mathbb{E}[X_1] - \mu_1) + \frac{\mathbb{E}[X_2] - \mu_2}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} = 0$$

and

$$\mathbb{V}[\phi_1] = \frac{1}{\sigma_1^2}\mathbb{V}[X_1] = \frac{1}{\sigma_1^2}\sigma_1^2 = 1$$

$$\begin{aligned} \mathbb{V}[\phi_2] &= \left(-\frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \right)^2 \mathbb{V}[X_1] + \frac{\mathbb{V}[X_2]}{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} \\ &\quad - 2 \frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \frac{1}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \text{COV}(X_1, X_2) \\ &= \frac{\sigma_{12}^2 \sigma_1^2}{\sigma_1^4 \left(\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \right)} + \frac{\sigma_2^2}{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} - 2 \frac{\sigma_{12}^2}{\sigma_1^2 \left(\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \right)} \\ &= \frac{\sigma_2^2}{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}} - \frac{\sigma_{12}^2}{\sigma_1^2 \left(\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \right)} \\ &= \frac{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2}{\sigma_1^2 \left(\frac{\sigma_2^2 \sigma_1^2 - \sigma_{12}^2}{\sigma_1^2} \right)} = 1. \end{aligned}$$

For the independence it is enough to show that ϕ_1 and ϕ_2 are uncorrelated, since their distribution coincides. It holds, that

$$\text{Cov}(\phi_1, \phi_2)$$

$$\begin{aligned}
&= Cov \left(\frac{1}{\sigma_1} X_1 - \frac{\mu_1}{\sigma_1}, -\frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} (X_1 - \mu_1) + \frac{X_2 - \mu_2}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \right) \\
&= Cov \left(\frac{1}{\sigma_1} X_1, -\frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} X_1 + \frac{X_2}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \right) \\
&= -\frac{1}{\sigma_1} \frac{\sigma_{12}}{\sigma_1^2 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} Cov(X_1, X_1) + \frac{1}{\sigma_1} \frac{Cov(X_1, X_2)}{\sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} \\
&= -\frac{\sigma_{12}}{\sigma_1 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} + \frac{\sigma_{12}}{\sigma_1 \sqrt{\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}}} = 0.
\end{aligned}$$

A.2.3 Moment Generating Function (MGF) of Normal Distribution

After [26, p.90]

Let $X \sim N(\mu, \sigma^2)$ and $Y = e^X$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The MGF of X is defined, for all $u \in \mathbb{R}$, as

$$\phi(u) = \mathbb{E}[e^{uX}]$$

Since X is normally distributed, its density is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \mathbb{R}$$

So, for ϕ we get

$$\begin{aligned}
\phi(u) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ux} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx \\
&= e^{\mu u + \frac{1}{2} \sigma^2 u^2} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-(\mu+\sigma^2 u))^2}{\sigma^2}} dx
\end{aligned}$$

$$= e^{\mu u + \frac{1}{2}\sigma^2 u},$$

since $\frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}\frac{(x-(\mu+\sigma^2 u))^2}{\sigma^2}\}$ is the density of a $N(\mu+\sigma^2 u, \sigma^2)$ -distributed random variable and hence,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-(\mu+\sigma^2 u))^2}{\sigma^2}} dx = 1.$$

Therefore

$$\mathbb{E}[e^{uX}] = e^{\mu u + \frac{1}{2}\sigma^2 u}.$$

A.2.4.1 (Distribution of an Ito-Process)

Let $\{W_t\}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, with standard filtration $\mathbb{F}$ and $\{X_t\}$ be a stochastic process of the form

$$X_t = \int_0^t \Delta(s) dW_s, \quad t \in [0, T],$$

where $\{\Delta(s)\}$ is a non random function in s .

Then, from [26, p.149], we know that

$$X_t \sim N\left(0, \int_0^t \Delta_s^2 ds\right).$$

A.2.4.2 (Multivariate Characteristic Function)

After [19].

Let $X = (X_1, \dots, X_n)^T \sim N(\mu = (\mu_1, \dots, \mu_n)^T, \Sigma)$, where Σ denotes the (positive semi-definite) variance-covariance Matrix of X .

Then the characteristic function of X , defined by

$$\phi_X(u) = \mathbb{E} \left[\exp \left\{ i \sum_{k=1}^n u_k X_k \right\} \right],$$

is given by

$$\phi_X(u) = \exp \left\{ i u^T \mu - \frac{1}{2} u^T \Sigma u \right\},$$

where $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$

For the MGF $M_X(u)$ of X , which we obtain by $M_X(u) = \phi_X(-iu) = \phi_X(-iu_1, \dots, -iu_n)$, we get

$$\mathbb{E} \left[\exp \left\{ \sum_{k=1}^n u_k X_k \right\} \right] = \exp \left\{ u^T \mu + \frac{1}{2} u^T \Sigma u \right\} \quad (\text{A.1})$$

and it holds

$$X = (X_1, \dots, X_n)^T \sim N(\mu = (\mu_1, \dots, \mu_n)^T, \Sigma)$$

if and only if its characteristic function is of the form of ϕ_X above, the MGF is of the form (A.1), respectively. [19]

A.2.4.3 (Distribution of $(\tilde{U}_T, \tilde{R}_T)$)

The pair $(\tilde{U}_T, \tilde{R}_T)$ is normally distributed with zero mean and variance-covariance matrix

$$\begin{pmatrix} \tilde{\nu}_T^2 & T \\ T & \nu_T^2 \end{pmatrix},$$

where $\text{Cov}(\tilde{U}_T, \tilde{R}_T) = T$.

Proof:

For $(\tilde{U}_T, \tilde{R}_T)$ and $(u_1, u_2) \in \mathbb{R}^2$, by definition of the characteristic function of a random vector, we have that

$$\phi_{(\tilde{U}_T, \tilde{R}_T)}(u_1, u_2) = \mathbb{E}_{\tilde{\mathbb{P}}}[e^{iu_1 \tilde{U}_T + iu_2 \tilde{R}_T}]$$

$$\begin{aligned}
&= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ iu_1 \int_0^T \frac{1}{\sqrt{v_t}} d\widetilde{W}_t + iu_2 \int_0^T \sqrt{v_t} d\widetilde{W}_t \right\} \right] \\
&= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ i \int_0^T \left(\frac{u_1}{\sqrt{v_t}} + u_2 \sqrt{v_t} \right) d\widetilde{W}_t \right\} \right] \tag{A.2}
\end{aligned}$$

and from appendix A.2.4.2, we know the solution to (A.2), i.e.

$$\begin{aligned}
&= \exp \left\{ \frac{1}{2} i^2 \int_0^T \left(\frac{u_1}{\sqrt{v_t}} + u_2 \sqrt{v_t} \right)^2 dt \right\} \\
&= \exp \left\{ -\frac{1}{2} \left(u_1^2 \int_0^T \frac{1}{v_t} dt + 2u_1 u_2 \int_0^T dt + u_2^2 \int_0^T v_t dt \right) \right\}, \\
&= \exp \left\{ -\frac{1}{2} \left(u_1^2 \bar{\nu}_T^2 + 2u_1 u_2 T + u_2^2 \nu_T^2 \right) \right\}.
\end{aligned}$$

So to summarize, we have

$$(\tilde{U}_T, \tilde{R}_T) \sim N \left((0, 0), \begin{pmatrix} \bar{\nu}_T^2 & T \\ T & \nu_T^2 \end{pmatrix} \right)$$

or equivalently

$$(\tilde{R}_T, \tilde{U}_T) \sim N \left((0, 0), \begin{pmatrix} \nu_T^2 & T \\ T & \bar{\nu}_T^2 \end{pmatrix} =: \Sigma \right)$$

□

A.2.4.4 (Distribution of $(\tilde{R}_T, \tilde{U}_T^2)$)

For $\tilde{U}_T^2$, we consider the pair $(\tilde{R}_T, \tilde{U}_T^2)$, with distribution given by

$$(\tilde{R}_T, \tilde{U}_T^2) \sim N \left((0, 0), \Sigma = \begin{pmatrix} \bar{\rho}^2 \nu_T^2 & \bar{\rho}^2 (\bar{v}_T - v_0) \\ \bar{\rho}^2 (\bar{v}_T - v_0) & \bar{\rho}^2 \bar{\nu}_T^2 \end{pmatrix} \right).$$

Proof:

The characteristic function of $(\tilde{R}_T, \tilde{U}_T^2)$ reads

$$\phi_{(\tilde{R}_T, \tilde{U}_T^2)}(u_1, u_2) = \mathbb{E}_{\tilde{\mathbb{P}}} [e^{iu_1 \tilde{R}_T + iu_2 \tilde{U}_T^2}]$$

$$\begin{aligned}
&= \mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ i\bar{\rho} \int_0^T (u_1 \sqrt{v_t} + u_2 \frac{\bar{v}'_t}{\sqrt{v_t}}) d\tilde{W}_t \right\} \right] \\
&= \exp \left\{ -\frac{1}{2} \bar{\rho}^2 \int_0^T \left(u_1 \sqrt{v_t} + u_2 \frac{\bar{v}'_t}{\sqrt{v_t}} \right)^2 dt \right\} \\
&= \exp \left\{ -\frac{1}{2} \bar{\rho}^2 (u_1^2 \int_0^T v_t dt + 2u_1 u_2 \int_0^T \bar{v}'_t dt + u_2^2 \int_0^T \frac{(\bar{v}'_t)^2}{v_t} dt) \right\} \\
&= \exp \left\{ -\frac{1}{2} (u_1^2 \bar{\rho}^2 \nu_T^2 + 2u_1 u_2 \bar{\rho}^2 (\bar{v}_t - v_0) + u_2^2 \bar{\rho}^2 \nu_T^2) \right\}
\end{aligned}$$

□

A.2.4.5 (Proof of Equation (3.55))

Let $X = (\frac{c_1 a_2}{\bar{\rho}^2} U_T^1 + \frac{c_3 a_4}{\bar{\rho}^2} U_T^2)$, then X is multivariate normally distributed, since U_T^i are (standard) normally distributed, with mean $\mu = (0, 0)$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \frac{c_1^2 a_2^2}{\bar{\rho}^4} & \frac{c_1 c_3 a_2 a_4}{\bar{\rho}^4} \text{Cov}(U_T^1, U_T^2) \\ \frac{c_1 c_3 a_2 a_4}{\bar{\rho}^4} \text{Cov}(U_T^1, U_T^2) & \frac{c_3^2 a_4^2}{\bar{\rho}^4} \end{pmatrix}.$$

We want to show that equation (3.55) holds, i.e.

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}^2} a_2 U_T^1 + \frac{c_3}{\bar{\rho}^2} a_4 U_T^2 \right\} \right] = \exp \left\{ \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_5) \right\}.$$

using (4.19) for $(u_1, u_2) = (1, 1)$.

So, plugging in for X , u , μ and Σ , we get

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[\exp \left\{ \frac{c_1}{\bar{\rho}^2} a_2 U_T^1 + \frac{c_3}{\bar{\rho}^2} a_4 U_T^2 \right\} \right] = \exp \left\{ \frac{1}{2\bar{\rho}^4} (c_1^2 a_2^2 + c_3^2 a_4^2 + 2c_1 c_3 a_2 a_4 \text{Cov}(U_T^1, U_T^2)) \right\}.$$

So, it remains to show, that

$$\text{Cov}(U_T^1, U_T^2) = \frac{a_5}{a_2 a_4},$$

where $a_5 = \bar{\rho}^2 \left(\hat{\nu}_T^2 - \frac{T}{\nu_T^2} (v_t - v_0) \right)$, given in (3.56).

Since $U_T^1 = \frac{1}{a_2} \tilde{U}_T^1 - \frac{a_1}{a_2} \tilde{R}_T$ and $U_T^2 = \frac{1}{a_4} \tilde{U}_T^2 - \frac{a_3}{a_4} \tilde{R}_T$, we get

$$\text{Cov}(U_T^1, U_T^2) = \text{Cov}\left(\frac{1}{a_2} \tilde{U}_T^1 - \frac{a_1}{a_2} \tilde{R}_T, \frac{1}{a_4} \tilde{U}_T^2 - \frac{a_3}{a_4} \tilde{R}_T\right)$$

$$\begin{aligned}
&= \frac{1}{a_2 a_4} \left(\text{Cov}(\tilde{U}_T^1, \tilde{U}_T^2) - a_3 \text{Cov}(\tilde{U}_T^1, \tilde{R}_T) - a_1 \text{Cov}(\tilde{U}_T^2, \tilde{R}_T) + a_1 a_3 \mathbb{V}[\tilde{R}_T] \right) \\
&= \frac{1}{a_2 a_4} \left(\text{Cov}(\tilde{U}_T^1, \tilde{U}_T^2) - \frac{(v_T - v_0)}{\nu_T^2} \bar{\rho}^2 T - \frac{T}{\nu_T^2} \bar{\rho}^2 (\bar{v}_T - v_0) + \frac{T}{\nu_T^2} \frac{(v_T - v_0)}{\nu_T^2} \bar{\rho}^2 \nu_T^2 \right) \\
&= \frac{1}{a_2 a_4} \left(\text{Cov}(\tilde{U}_T^1, \tilde{U}_T^2) - \frac{(v_T - v_0)}{\nu_T^2} \bar{\rho}^2 T \right).
\end{aligned}$$

In the last step, we show that

$$\text{Cov}(\tilde{U}_T^1, \tilde{U}_T^2) = \bar{\rho}^2 \hat{\nu}_T^2 \quad (\text{A.3})$$

This can be done by using the characteristic function $\phi_{(\tilde{U}_T^1, \tilde{U}_T^2)}$ of $(\tilde{U}_T^1, \tilde{U}_T^2)$, given by

$$\begin{aligned}
\phi_{(\tilde{U}_T^1, \tilde{U}_T^2)}(u_1, u_2) &= \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ i \bar{\rho} \int_0^T \left(\frac{u_1}{\sqrt{v_t}} + u_2 \frac{\bar{v}_t'}{\sqrt{v_t}} \right) d\tilde{W}_t \right\} \right] \\
&= \exp \left\{ -\frac{1}{2} \bar{\rho}^2 \int_0^T \left(\frac{u_1}{\sqrt{v_t}} + u_2 \frac{\bar{v}_t'}{\sqrt{v_t}} \right)^2 dt \right\} \\
&= \exp \left\{ -\frac{1}{2} (u_1^2 \bar{\rho}^2 \hat{\nu}_T^2 + 2u_1 u_2 \bar{\rho}^2 \hat{\nu}_T^2 + u_2^2 \bar{\rho}^2 \underline{\nu}_T^2) \right\},
\end{aligned}$$

which shows (A.3) □

A.2.4 Characteristic Functions for Cases A+B and C

(After [10], [21] and [18]).

We want to derive the characteristic function of ν_T^2 for cases A and B and afterwards the (bivariate) characteristic function of the pair (ν_T^2, v_T) for case C.

Before we do that, we need some preliminaries:

Let $\{X_t\}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with values in an open subset $D \subset \mathbb{R}^d$, described by

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

with d -dimensional $\{\mathcal{F}_t\}$ -Brownian motion $\{W_t\}$, drift vector $a : D \rightarrow \mathbb{R}^d$ and volatility matrix $b : D \rightarrow \mathbb{R}^{d \times d}$.

Process $\{X_t\}$ is called *affine* diffusion process, if a and b are of the form:

$$a = A_0 + A_1 X_t$$

$$bb^T = B_0 + \sum_{i=1}^d (B_1)_i (X_t)_i$$

with $A_0 \in \mathbb{R}^d$, $A_1, B_0 \in \mathbb{R}^{d \times d}$ and $B_1 \in \mathbb{R}^{d \times d \times d}$.

The characteristic function (CF) of X_t (for fixed t) is defined by

$$\phi_{X_t}(u) = \mathbb{E} \left[e^{iu^T X_t} \right], \quad u \in \mathbb{R}^d$$

and the *conditional* CF (CCF) for $X_{t+\tau}$, for $\tau > 0$ is given by

$$\phi_{X_{t+\tau}}(u|\mathcal{F}_t) = \mathbb{E} \left[e^{iu^T X_{t+\tau}} | \mathcal{F}_t \right], \quad u \in \mathbb{R}^d$$

Theorem: For an affine diffusion process $\{X_t\}$ the CCF is given by

$$\phi_{X_t}(u|\mathcal{F}_t) = \mathbb{E} \left[e^{iu^T X_t} | \mathcal{F}_t \right] = e^{A(u,\tau) + B(u,\tau)^T X_t}, \quad u \in \mathbb{R}^d$$

where $A \in \mathbb{C}$ and $B \in \mathbb{C}^n$ are the solutions to the followig *Riccati equations*:

$$\frac{\partial}{\partial \tau} B(u, \tau) = A_1^T B(u, \tau) + \frac{1}{2} B(u, \tau)^T B_1 B(u, \tau), \quad B(u, 0) = iu \quad (\text{A.4})$$

$$\frac{\partial}{\partial \tau} A(u, \tau) = A_0^T A(u, \tau) + \frac{1}{2} A(u, \tau)^T B_0 A(u, \tau), \quad A(u, 0) = 0 \quad (\text{A.5})$$

A Proof can be found in [10].

Remark: There are several methods of solving such Riccati equations and depend also on the dimension d , one considers. In [21] those approaches are discussed in detail.

We are only interested in small dimensions (≤ 3). In fact in our cases, there are closed form solutions to the Riccati equations, which we will see in the following application of the theory above, for the CCF of the extended Heston Model.

Application: Let $\{X_t\}$ be the logarithmic stock price process ($X_t = \ln S_t$), where the appropriate form of S_t is given in (3.21) and $\{v_t\}$ its volatility process given in (3.17). Furthermore, let $\{Y_t\}$ be the time integrated variance process, given by $\nu_t = \int_0^t v_s ds$ for $t \in [0, T]$. The SDE's are given by

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2, \quad v_0 \in \mathbb{R} \quad (\text{A.6})$$

$$dX_t = (r - \frac{1}{2}v_t)dt + \rho\sqrt{v_t}dW_t^2 + \sqrt{1 - \rho^2}\sqrt{v_t}dW_t, \quad X_0 = \ln S_0 \quad (\text{A.7})$$

$$d\nu_t = v_t dt, \quad \nu_0 = 0. \quad (\text{A.8})$$

This model is affine with dimension $d = 3$, therefore the CCF of $X_{t+\tau}$ is of the form

$$\phi(u|\mathcal{F}_t) = e^{A(u,\tau) + B_1(u,\tau)v_t + B_2(u,\tau)X_t + B_3(u,\tau)\nu_t}, \quad u \in \mathbb{R}^3. \quad (\text{A.9})$$

where

$$A(u, \tau) = iu_2 r \tau + \frac{\kappa\theta}{\sigma^2} \left[(\xi - d)\tau - 2 \ln \frac{I(u)e^{-d\tau} - 1}{I(u) - 1} \right] \quad (\text{A.10})$$

$$B_1(u, \tau) = \left[\frac{\xi + d}{\sigma^2} \frac{I(u)e^{-d\tau} - H(u)}{I(u)e^{-d\tau} - 1} \right] \quad (\text{A.11})$$

$$B_2(u, \tau) = iu_2 \quad (\text{A.12})$$

$$B_3(u, \tau) = iu_3, \quad (\text{A.13})$$

where

$$I(u) = \frac{iu_1\sigma^2 - (\xi - d)}{iu_1\sigma^2 - (\xi + d)} \quad (\text{A.14})$$

$$H(u) = \frac{\xi - d}{\xi + d} \quad (\text{A.15})$$

with

$$\xi = \xi(u_2) = \kappa - i\sigma\rho u_2 \quad (\text{A.16})$$

$$d = d(u_2, u_3) = \sqrt{\xi(u_2)^2 - 2\sigma^2(iu_3 - iu_2/2 - u_2^2/2)}. \quad (\text{A.17})$$

(See [10] for more details).

One can proof this also with a guess and verify method, i.e. we suppose

that the CCF ϕ is of the form in (A.9), then we show that the coefficients A and B_i , $i = 1, 2, 3$ fulfill their corresponding Ricatti equations in (A.10) and (A.11)-(A.13).

Characteristic Function of ν_T : (For cases (A) and (B))

In order to get the characteristic function ϕ_{ν_T} , given by

$$\phi_{\nu_T}(s) = \mathbb{E} [e^{is\nu_T}] , s \in \mathbb{R}.$$

We use the application above, but ignoring the contributions to the joint CCF of processes (A.6) and (A.7), i.e. we set $u_1 = u_2 = 0$ and $u = u_3$. Furthermore, we set $t = 0$ and $\tau = T$, then $\nu_0 = 0$ and using (A.9), we get

$$\phi_{v_T, X_T, \nu_T}((0, 0, u)) = e^{A((0,0,u),T) + B_1((0,0,u),T)v_0 + B_2((0,0,u),\tau)X_0 + B_3((0,0,u),\tau)\nu_0},$$

which can be reduced to

$$\phi_{\nu_T}(u) = e^{A(u,T) + B_1(u,T)v_0} , u \in \mathbb{R}$$

since from (A.12) and (A.13), $B_2 = 0$ and since $\nu_0 = 0$, B_3 vanishes as well.

The solutions to A and B_1 are then given by (A.18), (A.19) respectively:

$$A(u, T) = \frac{\kappa\theta}{\sigma^2} \left[(\kappa - d)T - 2 \ln \frac{I(u)e^{-dT} - 1}{I(u) - 1} \right] \quad (\text{A.18})$$

where I and d are given by

$$I(u) = \frac{\kappa - d}{\kappa + d}$$

$$d = \sqrt{\kappa^2 - 2iu\sigma^2}$$

and

$$B_1(u, T) = \left[\frac{\kappa + d}{\sigma^2} \frac{I(u)e^{-dT} - H(u)}{I(u)e^{-dT} - 1} \right] \quad (\text{A.19})$$

with

$$H(u) = \frac{\kappa - d}{\kappa + d}.$$

To show that this is consistent with the result of Griebisch in [GR,p.19], we

reformulate (A.18) and (A.19):

Since

$$\begin{aligned} \frac{I(u)e^{-dT} - 1}{I(u) - 1} &= \frac{\frac{\kappa-d}{\kappa+d}e^{-dT} - 1}{\frac{\kappa-d}{\kappa+d} - 1} = \frac{(\kappa-d)e^{-dT} - (\kappa+d)}{\kappa-d - (\kappa+d)} \\ &= \frac{\kappa(e^{-dT} - 1) - d(e^{-dT} - 1)}{-2d} = \frac{\kappa(1 - e^{-dT}) + d(1 + e^{-dT})}{2d} \\ &= \frac{de_+ + \kappa e_-}{2d} \end{aligned}$$

where $e_{\pm} = (1 \pm e^{-dT})$ and

$$-\ln \left[\frac{de_+ + \kappa e_-}{2d} \right] = \ln \left[\frac{2d}{de_+ + \kappa e_-} \right],$$

we get

$$A(u, T) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2} \ln \left[\frac{2d}{de_+ + \kappa e_-} \right]. \quad (\text{A.20})$$

For B_1 we reformulate

$$\begin{aligned} \frac{I(u)e^{-dT} - H(u)}{I(u)e^{-dT} - 1} &= \frac{\frac{\kappa-d}{\kappa+d}e^{-dT} - \frac{\kappa-d}{\kappa+d}}{\frac{\kappa-d}{\kappa+d}e^{-dT} - 1} \\ &= \frac{(\kappa-d)e^{-dT} - (\kappa-d)}{(\kappa-d)e^{-dT} - (\kappa+d)}, \end{aligned}$$

then

$$\begin{aligned} B_1 &= \frac{\kappa + d}{\sigma^2} \frac{(\kappa-d)e^{-dT} - (\kappa+d)}{(\kappa-d)e^{-dT} - (\kappa+d)} = \frac{1}{\sigma^2} \frac{(\kappa^2 - d^2)e^{-dT} - (\kappa^2 - d^2)}{(\kappa-d)e^{-dT} - (\kappa+d)} \\ &= \frac{1}{\sigma^2} \frac{(\kappa^2 - \kappa^2 + 2iu\sigma^2)e^{-dT} - (\kappa^2 - \kappa^2 + 2iu\sigma^2)}{\kappa(e^{-dT} - 1) - d(e^{-dT} - 1)} = \\ &= \frac{2iue^{-dT} - 2iu}{\kappa(e^{-dT} - 1) - d(e^{-dT} - 1)} = \frac{2iu(1 - e^{-dT})}{\kappa(1 - e^{-dT}) + d(1 + e^{-dT})} \\ &= \frac{2iue^-}{\kappa e_- + de_+}. \end{aligned}$$

Therefore

$$B_1 = \frac{2iue^-}{\kappa e_- + de_+}. \quad (\text{A.21})$$

Characteristic Function of ν_T and v_T : (For case (C))

In order to get the characteristic function ϕ_{v_T, ν_T} , given by

$$\phi_{v_T, \nu_T}(u, w) = \mathbb{E} [e^{iuv_T + iw\nu_T}] , u, w \in \mathbb{R}.$$

We again use the application above, but ignoring the contributions to the joint CCF of the process (1.4), so in this case, we only set $u_2 = 0$ and $u = u_3$, $w = u_1$. Again, we set $t = 0$ and $\tau = T$, then $\nu_0 = 0$ and using (1.6), we get

$$\phi_{v_T, X_T, \nu_T}((w, 0, u)) = e^{A((w, 0, u), T) + B_1((w, 0, u), T)v_0 + B_2((w, 0, u), \tau)X_0 + B_3((w, 0, u), \tau)\nu_0}$$

$$\phi_{v_T, \nu_T}(u, w) = e^{A(u, w, T) + B_1(u, w, T)v_0} , u, w \in \mathbb{R}$$

where B_2 and B_3 disappear like in the latter case.

The solutions to A and B_1 are then given by (A.22), (A.23) and their reformulations are given in (A.24) and (A.25):

$$A(u, w, T) = \frac{\kappa\theta}{\sigma^2} \left[(\kappa - d)T - 2 \ln \frac{I(u, w)e^{-dT} - 1}{I(u, w) - 1} \right] \quad (\text{A.22})$$

where I and d are given by

$$I(u, w) = \frac{iw\sigma^2 - (\kappa - d)}{iw\sigma^2 - (\kappa + d)}$$

$$d = \sqrt{\kappa^2 - 2iu\sigma^2}$$

and

$$B_1(u, w, T) = \left[\frac{\kappa + d}{\sigma^2} \frac{I(u, w)e^{-dT} - H(u)}{I(u, w)e^{-dT} - 1} \right] \quad (\text{A.23})$$

with

$$H(u) = \frac{\kappa - d}{\kappa + d}$$

Again, we show that this is consistent with the result of Griebisch in [GR,p.19], we reformulate (A.22) and (A.23).

Since

$$\frac{I(u, w)e^{-dT} - 1}{I(u, w) - 1} = \frac{\frac{iw\sigma^2 - (\kappa - d)}{iw\sigma^2 - (\kappa + d)} e^{-dT} - 1}{\frac{iw\sigma^2 - (\kappa - d)}{iw\sigma^2 - (\kappa + d)} - 1}$$

$$\begin{aligned}
&= \frac{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))}{iw\sigma^2 - (\kappa - d) - (iw\sigma^2 - (\kappa + d))} \\
&= \frac{(\kappa - iw\sigma^2)e_- + de_+}{2d}
\end{aligned}$$

where $e_{\pm} = (1 \pm e^{-dT})$.

Hence, A reads

$$A(u, w, T) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2} \ln \frac{2d}{(\kappa - iw\sigma^2)e_- + de_+} \quad (\text{A.24})$$

For B_1 , we reformulate

$$\begin{aligned}
\frac{I(u, w)e^{-dT} - H(u)}{I(u, w)e^{-dT} - 1} &= \frac{\frac{iw\sigma^2 - (\kappa - d)}{iw\sigma^2 - (\kappa + d)}e^{-dT} - \frac{\kappa - d}{\kappa + d}}{\frac{iw\sigma^2 - (\kappa - d)}{iw\sigma^2 - (\kappa + d)}e^{-dT} - 1} \\
&= \frac{(iw\sigma^2 - (\kappa - d))e^{-dT} - \frac{(\kappa - d)(iw\sigma^2 - (\kappa + d))}{\kappa + d}}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))}
\end{aligned}$$

So for B_1 we get

$$\begin{aligned}
B_1(u, w, T) &= \frac{\kappa + d}{\sigma^2} \frac{(iw\sigma^2 - (\kappa - d))e^{-dT} - \frac{(\kappa - d)(iw\sigma^2 - (\kappa + d))}{\kappa + d}}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))} \\
B_1(u, w, T) &= \frac{1}{\sigma^2} \frac{(\kappa + d)(iw\sigma^2 - (\kappa - d))e^{-dT} - (\kappa - d)(iw\sigma^2 - (\kappa + d))}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))} \\
&= \frac{1}{\sigma^2} \frac{(\kappa + d)iw\sigma^2 e^{-dT} - (\kappa^2 - d^2)e^{-dT} - (\kappa - d)iw\sigma^2 + (\kappa^2 - d^2)}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))} \\
&= \frac{1}{\sigma^2} \frac{(\kappa + d)iw\sigma^2 e^{-dT} - 2iu\sigma^2 e^{-dT} - (\kappa - d)iw\sigma^2 + 2iu\sigma^2}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))} \\
&= \frac{(\kappa + d)iw e^{-dT} - 2iue^{-dT} - (\kappa - d)iw + 2iu}{(iw\sigma^2 - (\kappa - d))e^{-dT} - (iw\sigma^2 - (\kappa + d))} \\
&= \frac{\kappa iwe^{-dT} - \kappa iw - 2iue^{-dT} + 2iu + diwe^{-dT} + diw}{(\kappa - iw\sigma^2)e_- + de_+} \\
&= \frac{(2iu - \kappa iw)e_- + diwe_+}{(\kappa - iw\sigma^2)e_- + de_+} \quad (\text{A.25})
\end{aligned}$$

A.2.5 Independence Lemma v.s. Conditional Probability

Independence Lemma:

(After [8]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space, and let $\mathcal{G}$ be a sub- σ -algebra of $\mathcal{F}$. Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces and $X : \Omega \rightarrow E_1$ and $Y : \Omega \rightarrow E_2$ two random variables, where X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$ -measurable.

Let $g : E_1 \times E_2 \rightarrow \mathbb{R}$ be bounded and measurable.

Define

$$f(y) = \mathbb{E}[g(X, y)]$$

for some $y \in Y = \{Y(\omega) : \omega \in \Omega\}$.

Then

$$f(Y) = \mathbb{E}[g(X, Y) | \mathcal{G}].$$

A proof can be found in [8].

On our probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider processes $\{W_t\}$ and $\{v_t\}$, $t \in [0, T]$, as random variables $W : \Omega \rightarrow \mathcal{C}^0[0, T]$ and $v : \Omega \rightarrow \mathcal{C}^0[0, T]$, with measurable space $(\mathcal{C}^0[0, T], \mathcal{B}_{\mathcal{C}^0[0, T]})$, where $\mathcal{C}^0[0, T]$ denotes the set of continuous functions $f : [0, T] \rightarrow \mathbb{R}$ on the interval $[0, T]$ and $\mathcal{B}_{\mathcal{C}^0[0, T]}$ the corresponding Borel- σ -algebra. This makes sense, since all paths of the Brownian motion and the variance are continuous and are therefore contained in $\mathcal{C}^0[0, T]$.

Recall, the filtration of the variance process $\{\mathcal{F}_t^v\}$ is given by $\mathcal{F}_t^v = \sigma(v_s : 0 \leq s \leq t)$, $t \in [0, T]$.

Since W_t is independent of W_t^2 , which generates $\mathcal{F}_t^v$, W_t is independent of $\mathcal{F}_t^v$ for all $t \in [0, T]$. Therefore the random variable W is independent of $\mathcal{F}^v = \mathcal{F}_T^v$ and v is $\mathcal{F}^v$ -measurable.

Our overall goal is to calculate E^v , given by

$$E^v = \mathbb{E} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} | \mathcal{F}_T^v \right].$$

With respect to the independence lemma, we define $g : \mathcal{C}^0[0, T] \times \mathcal{C}^0[0, T] \rightarrow$

$\mathbb{R}$ as

$$g(v, W) = \left(S_0 \exp \left\{ \bar{\rho} \int_0^T \sqrt{v_t} dW_t + \gamma_T \right\} - K \right) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}}^2$$

Since g is Borel-measurable by construction, we still have to show that it is bounded in order to apply the lemma above.

Recall, that our stock price at exercise time S_T is given by

$$S_T = S_0 \exp \left\{ \bar{\rho} \int_0^T \sqrt{v_t} dW_t + \gamma_T \right\}.$$

Whenever we have $S_T > B$, the pay-off of the UOC is zero. Therefore a natural bound for g is given by

$$g(v, W) \leq S_0 B - K$$

for all pairs $(v, W) \in \mathcal{C}^0[0, T] \times \mathcal{C}^0[0, T]$.

Next, define

$$f(\tilde{v}) = \mathbb{E}[g(\tilde{v}, W)]$$

for some fixed (or dummy) path, $\tilde{v} \in \{v(\omega) : \omega \in \Omega\}$.

Therefore, using the result of the independence lemma, we get

$$f(v) = \mathbb{E}[g(v, W) | \mathcal{F}_T^v] = E^v. \quad (\text{A.26})$$

To show what this means in our case, we plug in for g in $f(\tilde{v})$, hence

$$f(\tilde{v}) = \mathbb{E}[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}}], \quad (\text{A.27})$$

where

$$\tilde{\tilde{R}}_T = \tilde{\gamma}_T + \tilde{R}_T$$

and

$$\tilde{\tilde{M}}_T = \max_{t \leq T} \{\tilde{\gamma}_t + \tilde{R}_t\},$$

with

$$\tilde{R}_t = \bar{\rho} \int_0^t \sqrt{\tilde{v}_s} dW_s$$

²Our emphasis here is on the exponential in the brackets, not on the indicator function, which is the reason why we do not write out $\tilde{M}_T$ and $\tilde{R}_T$ in $\mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}}$

and

$$\tilde{\gamma}_T = rT - \frac{1}{2} \int_0^T \tilde{v}_t dt + \frac{\rho}{\sigma} \left(\tilde{v}_T - \tilde{v}_0 - \kappa \theta T + \kappa \int_0^T \tilde{v}_t dt \right).$$

Now, the expectation in (A.27) is way easier to compute than the conditional expectation in (A.26), which is the reason why we apply the independence lemma here. To do this, we find the density function of the pair $(\tilde{M}_T, \tilde{R}_T)$ which is done in step 2 and then calculate the expectation as integral of the pay-off times the density. Once we have a resulting formula for $f(\tilde{v})$, we replace the dummy function $\{\tilde{v}_t\}$ by the initial variance process $\{v_t\}$ in order to get the conditional expectation in (A.26).

Notation:

Instead of rewriting the random variables $\tilde{R}_T$ and $\tilde{M}_T$ into $\tilde{\tilde{R}}_T$ and $\tilde{\tilde{M}}_T$, whenever we want to express that v_t is deterministic, we introduce the notation of "measure" $\mathbb{P}^v$, given by

$$\mathbb{P}^v \left\{ \tilde{R}_T < a \right\} := \mathbb{P} \left\{ \tilde{\tilde{R}}_T < a \right\}, \quad a \in \mathbb{R}$$

and we call $\mathbb{P}^v$ *conditional probability measure*.

In this notation, we have

$$\mathbb{E} \left[(S_0 e^{\tilde{R}_T} - K) \mathbb{I}_{\{\tilde{M}_T < b, \tilde{R}_T > k\}} \right] = \mathbb{E}^{\mathbb{P}^v} \left[(S_0 e^{\tilde{\tilde{R}}_T} - K) \mathbb{I}_{\{\tilde{\tilde{M}}_T < b, \tilde{\tilde{R}}_T > k\}} \right].$$

So, whenever we calculate something under $\mathbb{P}^v$, this means that we actually calculate it under $\mathbb{P}$ but treat the variance as deterministic wherever it appears.

Distribution of $\tilde{R}_T$:

Using the result in appendix A.2.4.1 or [26, p.149], we have

$$\tilde{R}_T \sim N \left(0, \bar{\rho}^2 \int_0^T \tilde{v}_t dt \right), \quad (\text{A.28})$$

under $\mathbb{P}$, which after our notation above, can be written as

$$R_T \sim N \left(0, \bar{\rho}^2 \int_0^T v_t dt \right)$$

under $\mathbb{P}^v$.

A.3 Appendix for Chapter 4

A.3.1 Probability Law

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}$ a random variable. Then, the *probability law* μ of X is defined by

$$\mu(B) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\} , \quad B \in \mathcal{B},$$

where $\mathcal{B}$ denotes the Borel- σ -algebra on $\mathbb{R}$.

It is easy to show, that μ is a measure on $(\mathbb{R}, \mathcal{B})$.

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Abstract

The aim of this thesis is to give an overview of pricing methods for barrier options in different models and furthermore, derive price boundaries in a model-free approach.

At first the Black-Scholes model is considered and prices of four standard types of single barrier options are derived within this model.

Next, the Heston model is considered, allowing the volatility of the underlying to follow its own stochastic dynamics.

There, special emphasis is put on an approach to derive semi-analytic formulas as approximations to barrier option prices (for one type of single barrier), with the aim to give a precise mathematical derivation of this approach.

Finally the model-free approach is considered, where no specific dynamics of the underlying is assumed (nor of its volatility). Using no-arbitrage arguments, upper and lower price boundaries for the two types of single barrier options are derived.

Abstract

Das Ziel dieser Arbeit ist es einen Überblick über die Methoden zur Bewertung von Barrier Optionen in verschiedenen Modellen zu geben und darüberhinaus noch einen modellfreien Ansatz zu betrachten und in diesem Preisgrenzen herzuleiten.

Als erstes wird das klassische Black-Scholes Modell betrachtet und darin Preisformeln für vier Typen von Barrier Optionen mit einem Barrier-Level (single Barrier) hergeleitet .

Als nächstes wird das Heston Modell betrachtet, indem die Volatilität des Basiswertes einer eigenen stochastischen Dynamik folgt.

Besondere Aufmerksamkeit wurde einer Methode geschenkt, um semi-analytische Formeln als Approximationen für Barrier Optionspreise herzuleiten. Der Fokus hier war auf eine mathematisch präzise Herleitung dieser Formeln gelegt.

Zum Schluss wird noch ein modellfreier Ansatz betrachtet, in dem weder der Basiswert, noch seine Volatilität einer bestimmten stochastischen Dynamik folgt. Mit No-Arbitrage Argumenten werden obere- und untere Schranken für zwei Typen von single Barrier Optionen hergeleitet.