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An alternative proof to the Bichteler-Dellacherie
theorem

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Introduction

In this thesis we want to give an alternative proof to the Bichteler-Dellacherie theorem, which basically states that integration with respect to a process X is possible if and only if the process X is a semimartingale, i.e. the sum of a local martingale and a process of finite variation. Moreover we will try to establish a new equivalence to semimartingales by using Riemann sums and Riemann integrators.

Applying the Doob-Meyer decomposition theorem to the Bichteler-Dellacherie theorem we see that a proper integration theory for a bounded process X can be found if and only if X is locally the difference of two submartingales. Comparing to the deterministic case, where one can integrate with respect to a function f if and only if it is the difference between two increasing functions, we want to adapt this result on to the stochastic case to prove the Bichteler-Dellacherie theorem.

As a consequence we will see that a càdlàg adapted process $(X_t)_{t \in [0,1]}$ is a semimartingale if and only if it is a Riemann integrator, i.e. if for every adapted bounded continuous process H the sums

$$\sum_{i=0}^{2^n-1} H_{\frac{i}{2^n}}(X_{\frac{i+1}{2^n}} - X_{\frac{i}{2^n}})$$

converge in probability as n increases to ∞ . Therefore we get the conclusion that semimartingales are in a way the stochastic equivalent to processes of finite variation. This is in contrast to the fact that there are continuous processes which are not semimartingales such that the above property holds for all integrands H with $H_t = f(t, X_t)$, where f is a bounded continuous function. However, every continuous deterministic Riemann integrator X with processes H being of the form $H_t = f(X_t)$ is a function of finite variation. Further remarks can be found in [10] and [18].

The Bichteler-Dellacherie theorem was first published in [11] and independently in [4] and [5].

The most popular proofs for the Bichteler-Dellacherie theorem apply functional analysis and measure changing techniques as in [6]. A more modern version of this argument is given in [13]. In particular one uses a version of the Hahn-Banach separation theorem to construct an equivalent measure Q under which the good integrator X is a quasimartingale. Applying Rao's theorem together with the Doob-Meyer decomposition theorem one concludes that X is a Q -semimartingale and finally Girsanov's theorem implies that X is a P -semimartingale.

A different proof using orthogonal decomposition is used in [9]. However we shall give the proof of the desired theorem following [1] in great detail.

This thesis is organized as follows:

In Chapter 1, we will introduce the stochastic integrals for simple processes as well as good integrators and will provide some examples.

Chapter 2 is dedicated to introduce the quasimartingales together with Rao's theorem and its proof.

In Chapter 3 we will collect some ingredients to prove an important characterization of good integrators, which will help us to proof the Bichteler-Dellacherie theorem.

Chapter 4 recalls the Doob-Meyer decomposition theorem as well as a proof to it, following [3].

Finally, in Chapter 5, we are ready to give, after gathering some last facts, the full proof of the Bichteler-Dellacherie theorem.

The concluding Chapter 6 then gives the alternative characterization of semimartingales via Riemann-integrators using the Bichteler-Dellacherie theorem.

The reader is assumed to have basic knowledge about measure theory, probability theory, basic stochastic analysis as well as functional analysis .

I want to thank my supervisor Professor Mathias Beiglböck for giving me a chance to take a closer look to the Bichteler-Dellacherie theorem and I am very grateful for the helpful discussions we had. Furthermore, I want to express my gratitude to my family for helping me not only financially but also for being there for me at any time. Last but not least I want to thank all my friends, who supported me during my studies.

1 Good integrators

We will always require a finite time horizon T which without loss of generality gets the value 1. Furthermore $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ should be a filtered probability space. First of all we are going to recall some basic definitions. Then we want shortly to introduce the stochastic integral for simple processes, since this will be enough for our purposes.

Definition 1.1. A filtered probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ or more precisely the filtration $\mathcal{G}_t$ is said to satisfy the usual conditions if the following two requirements hold true:

- (i) $\mathcal{G}_0$ contains all P -null sets of the σ -algebra $\mathcal{G}$, i.e. if $N \in \mathcal{G}$ and $P(N) = 0$ then it follows that $N \in \mathcal{G}_0$.
- (ii) The filtration $\mathcal{G}_t$ is right continuous, i.e. $\mathcal{G}_t = \bigcap_{\varepsilon > 0} \mathcal{G}_{t+\varepsilon}$.

Moreover recall that a process X is said to be càdlàg if it has almost surely sample paths which are right continuous and have left limits. The term càdlàg is an abbreviation for the french terminology “continue à droite, limite à gauche”.

Definition 1.2. A simple process (or simple integrand) is a stochastic process $H = (H_t)_{t \in (0,1]}$ of the form

$$H_t = \sum_{i=1}^n \xi_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t),$$

where n is a finite number, $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{n+1} \leq 1$ is an increasing sequence of stopping times and ξ_i are bounded $\mathcal{G}_{\tau_i}$ -adapted random variables. We will denote the vector space of simple processes by $\mathcal{S}$, which is equipped with the supremum norm, i.e.

$$\|H\|_\infty := \sup_{t \in [0,1]} \|H_t\|_{L^\infty(P)}.$$

Definition 1.3. Given a càdlàg, adapted and real-valued stochastic process $X = (X_t)_{t \in [0,1]}$ and a simple process $H \in \mathcal{S}$ as in Definition 1.2 we may well-define the (Itô)-integral as an operator $\mathcal{I}_X : \mathcal{S} \rightarrow L^0(P)$ such that

$$\mathcal{I}_X(H) := \sum_{i=1}^n \xi_i (X_{\tau_{i+1}} - X_{\tau_i}),$$

where $L^0(P)$ is the space of all random variables with the metrizable topology of convergence in probability.

By interpreting the above defined Itô-integral as a linear operator, we are able to define the so-called good integrators.

Definition 1.4. A process X is called a good integrator if $\mathcal{I}_X : \mathcal{S} \rightarrow L^0(P)$ is continuous, i.e. if $H^n \in \mathcal{S}$ and $\|H^n\|_\infty \rightarrow 0$ it follows that $\mathcal{I}_X(H^n)$ goes to 0 in probability as $n \rightarrow \infty$.

As it will turn out, good integrators possess a nice property, which will become one of the main steps for proving the Bichteler-Dellacherie theorem.

This definition yields that the integrands have to be adapted. If we would exclude the presumption of adaptedness we would have to consider all simple processes adapted to the filtration $\mathcal{G}_t := \mathcal{G}_1, 0 \leq t \leq 1$ (i.e. to the constant filtration). Furthermore the Bichteler-Dellacherie theorem would imply that every $\mathcal{G}_t$ -good integrator has paths of finite variation since any $\mathcal{G}_t$ -local martingale would be constant in that case. As conclusion we therefore see that if we tolerate integrands which are not adapted, we are left with a small class of integrators.

At this point let us recall two basic concepts of martingale theory, since we will need them later on.

Definition 1.5. A family of real-valued random variables $(X_t)_{t \in [0,1]}$ is uniformly integrable if, given $\varepsilon > 0 \exists K \in [0, \infty)$ such that for all t

$$E[|X_t|; |X_t| > K] < \varepsilon,$$

where $E[X; K] = E[X \mathbb{1}_F] = \int_F X(\omega) P(d\omega)$. Furthermore an adapted process $(X_t)_{t \in [0,1]}$ is said to be of class D if the family of random variables X_τ is uniformly integrable, where τ ranges over all stopping times.

Definition 1.6. Under the assumptions of the usual conditions 1.1, an adapted càdlàg process X is said to be a local martingale if there is an increasing sequence of stopping times τ_n such that $\lim_n \tau_n = \infty$ almost surely and the process $X_{t \wedge \tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a uniformly integrable martingale for every n .

Now we want to take a closer look on the good integrator property and deal with some examples. If X is a process and there exists a sequence of stopping times (τ_n) with $\lim_n \tau_n = \infty$ almost surely, such that X^{τ_n} is a good integrator, then X is a good integrator as well. More specifically if X is locally a good integrator, then it is already a good integrator. For this purposes let us recall that a property is said to hold locally if there is a sequence of stopping times (τ_n) with $\lim_n \tau_n = \infty$ almost surely such that $X^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ has the property for every $n \geq 1$.

The following theorem provides us with additional examples of good integrators.

Theorem 1.7.

- (i) Every adapted càdlàg process X with paths of finite variation is a good integrator.
- (ii) Every square integrable martingale X is a good integrator.
- (iii) A càdlàg, locally square integrable local martingale is a good integrator.

Proof. (i) Suppose X has a finite total variation. Then it is immediate that

$$|\mathcal{I}_X(H^n)| \leq \underbrace{\|H^n\|_\infty}_{\rightarrow 0} \underbrace{\int_0^T d|X_s|}_{< \infty} \rightarrow 0$$

(ii) Without loss of generality let $X_0 = 0$ and let $H^n \in \mathcal{S}$, i.e. $H_t^n = \sum_{i=1}^n \xi_i^n \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t)$ such that $\|H^n\|_\infty \rightarrow 0$. Then

$$\begin{aligned} E[(\mathcal{I}_X(H^n))^2] &= E\left[\left\{\sum_{i=1}^n \xi_i^n (X_{\tau_{i+1}} - X_{\tau_i})\right\}^2\right] \\ &= E\left[\sum_{i=1}^n (\xi_i^n)^2 (X_{\tau_{i+1}} - X_{\tau_i})^2\right] \quad (\text{orthogonality of the increments}) \\ &\leq \|H^n\|_\infty^2 E\left[\sum_{i=1}^n (X_{\tau_{i+1}} - X_{\tau_i})^2\right] \\ &= \|H^n\|_\infty^2 E\left[\sum_{i=1}^n (X_{\tau_{i+1}}^2 - X_{\tau_i}^2)\right] \quad (X \text{ is a martingale}) \\ &= \|H^n\|_\infty^2 (E[X_{\tau_{n+1}}^2] - E[X_0^2]) \leq \underbrace{\|H^n\|_\infty^2}_{\rightarrow 0} \underbrace{E[X_\infty^2]}_{< \infty}. \end{aligned}$$

So $\mathcal{I}_X(H^n) \rightarrow 0$ in L^2 , therefore in probability and we may conclude that X is a good integrator.

(iii) Apply (ii) together with the remark before that every locally good integrator is a good integrator. □

In fact one could also prove that any (local) martingale is a good integrator. For a simple proof of this fact we refer to [7]. The reverse statement is of key importance, since this is the result of the desired theorem.

2 Quasimartingales

Before introducing the quasimartingales, let us collect some facts about functions of finite variation. Let f be a real valued and right continuous function which takes values in $[0, \infty)$. Furthermore let $\Lambda = \{t_0, t_1, \dots, t_n\}$ be a subdivision of the interval $[0, t]$ such that $0 = t_0 < t_1 < \dots < t_n = t$. The number $|\Lambda| = \sup_i |t_{i+1} - t_i|$ is the mesh of the subdivision. Additionally define

$$X_t^\Lambda = \sum_i |f(t_{i+1}) - f(t_i)|.$$

For another subdivision Λ' that is a refinement of Λ , i.e. every point of Λ is contained in Λ' , we get by the triangular inequality that $X_t^\Lambda \leq X_t^{\Lambda'}$. The function f is said to be of finite variation if for every t

$$X_t = \sup_\Lambda X_t^\Lambda < \infty.$$

The function $t \rightarrow X_t$ is called total variation of f . As a conclusion we get

Proposition 2.1. *A function f is of finite variation if and only if it is the difference of two increasing functions.*

Proof. ($\Rightarrow$) Take the functions $\frac{X+f}{2}$ and $\frac{X-f}{2}$ which are easily shown to be positive and increasing. Since f is the difference between those two functions, we are done with this direction.

($\Leftarrow$) If f and g are two functions of finite variation, so is $f - g$. If f is increasing and $0 = t_0 < t_1 < \dots < t_n = t$ then $\sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| = |f(t) - f(0)|$, so f is of finite variation. Similar one can do this for the function g to conclude the proof. $\square$

This result has an equivalent theorem in stochastic analysis. For this we have to introduce quasimartingales.

2.1 Definition and Properties

Definition 2.2. Assume a partition of $[0, 1]$, i.e. a finite collection of points $\pi = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ is given and $X_{t_i} \in L^1$ for $t_i \in \pi$. Then we define

$$C(X, \pi) = \sum_{t_i \in \pi} |E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}]|.$$

The mean variation of X along π is defined as

$$MV(X, \pi) = E[C(X, \pi)].$$

Finally the mean variation of X is defined to be the supremum over all partitions π , i.e.

$$MV(X) = \sup_{\pi} MV(X, \pi).$$

Before finally defining a quasimartingale, let us first observe that the mean variation of a process along a given partition is an increasing function.

Proposition 2.3. *The mean variation of X along π is an increasing function of π , that means*

$$MV(X, \pi) \leq MV(X, \pi')$$

if π' is a refining partition of π .

Proof. Let s and t be two partition points of π . For simplicity assume we only add one more partition point r between s and t so that π' is a refining partition of π , i.e.

$$\pi = \{s, t\}, \quad \pi' = \{s, r, t\}.$$

To simplify our notation let

$$A = E[X_r - X_t | \mathcal{G}_s], A' = E[X_r - X_t | \mathcal{G}_r] \text{ and } B = E[X_s - X_r | \mathcal{G}_s] = B'.$$

By the well-known tower property of the conditional expectation we get

$$A = E[A' | \mathcal{G}_s].$$

Now using the conditional Jensen inequality we obtain

$$|A| = |E[A' | \mathcal{G}_s]| \leq E[|A'| | \mathcal{G}_s],$$

and therefore

$$E[|A|] \leq E[|A'|].$$

Combining everything and applying the expectation again (to obtain the mean variation), we conclude that

$$\begin{aligned} MV(X, \pi) &= E[|E[X_s - X_t | \mathcal{G}_s]|] \\ &= E[|E[X_s - X_r + X_r - X_t | \mathcal{G}_s]|] \\ &= E[|E[X_s - X_r | \mathcal{G}_s] + E[X_r - X_t | \mathcal{G}_s]|] \\ &\leq E[|E[X_s - X_r | \mathcal{G}_s]| + |E[X_r - X_t | \mathcal{G}_r]|] = MV(X, \pi'). \end{aligned}$$

□

Definition 2.4. An adapted càdlàg process X is a quasimartingale if $E[|X_t|] < \infty$ for each $t \in [0, 1]$ and if $MV(X) < \infty$.

An easy example for a quasimartingale is given by the next proposition.

Proposition 2.5. *Any martingale, submartingale or supermartingale X is a quasimartingale. In particular we have that*

$$MV(X) = |E[X_0 - X_t]|.$$

Proof. Let us suppose that X is a supermartingale. We can replace X with $-X$ afterwards to get the desired result for submartingales. Since for a supermartingale we have that

$$E[X_{t_{i+1}}|\mathcal{G}_{t_i}] \leq X_{t_i} \Leftrightarrow E[X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}] \geq 0,$$

we get that

$$C(X, \pi) = \sum_{t_i \in \pi} |E[X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}]| = \sum_{t_i \in \pi} E[X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}]$$

and therefore we see

$$\begin{aligned} MV(X, \pi) &= E\left[\sum_{t_i \in \pi} E[X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}]\right] \\ &= \sum_{t_i \in \pi} E[E[X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}]] \\ &= \sum_{t_i \in \pi} E[X_{t_i} - X_{t_{i+1}}] = E[X_0 - X_t]. \end{aligned}$$

Finally we get that the mean variation of X along any partition π is finite. In particular, X is a quasimartingale. $\square$

It turns out that the quasimartingales, unlike the sub or supermartingales, are closed under taking linear combination and hence form a vector space.

Proposition 2.6. *The space of quasimartingales is closed under taking linear combinations. In particular,*

(i) *if X is an integrable process and $\lambda \in \mathbb{R}$ we have*

$$MV(\lambda X) = |\lambda| MV(X),$$

(ii) *if X and Y are integrable processes, then*

$$MV(X + Y) \leq MV(X) + MV(Y).$$

Proof. (i) This follows by the following easy computation:

$$\begin{aligned} MV(\lambda X, \pi) &= E\left[\sum_{t_i \in \pi} |E[\lambda(X_{t_i} - X_{t_{i+1}})|\mathcal{G}_{t_i}]|\right] \\ &= |\lambda| E\left[\sum_{t_i \in \pi} |X_{t_i} - X_{t_{i+1}}|\mathcal{G}_{t_i}|\right], \end{aligned}$$

so if X is a quasimartingale, $MV(\lambda X)$ is finite and therefore λX is also a quasimartingale.

- (ii) If we substitute X by $X + Y$ in the definition of the mean variation of X , we just need to apply the inequality

$$|E[X_{t_i} - X_{t_{i+1}} + Y_{t_i} - Y_{t_{i+1}} | \mathcal{G}_{t_i}]| \leq |E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}]| + |E[Y_{t_i} - Y_{t_{i+1}} | \mathcal{G}_{t_i}]|$$

to conclude that if X and Y are quasimartingales, $X + Y$ is again a quasimartingale. $\square$

Concluding let us recall the definition of a seminorm. A map $p : V \rightarrow \mathbb{R}$ defined on a vector space V is called a seminorm if it satisfies the following two conditions:

- (i) $p(\lambda x) = |\lambda| p(x)$
- (ii) $p(x + y) \leq p(x) + p(y)$.

So as an implication of Proposition 2.6 we get that $\{MV(\cdot)\}$ gives us a family of seminorms on the space of quasimartingales.

If we want to deal with the mean variation of a stopped process, the following lemma will be useful:

Lemma 2.7. *Let X be a bounded process. Given a partition π and a stopping time τ define $\tau^+ := \inf\{t \in \pi : t \geq \tau\}$. Then*

$$MV(X^{\tau^+}, \pi) = E \left[\sum_{t_i \in \pi} \mathbb{1}_{\{t_i < \tau\}} |E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]| \right]$$

and

$$|MV(X^{\tau^+}, \pi) - MV(X^\tau, \pi)| \leq 2\|X\|_\infty.$$

Proof. For the first characterisation we observe that for each $t_i \in \pi$

$$E[X_{t_i}^{\tau^+} - X_{t_{i+1}}^{\tau^+} | \mathcal{G}_{t_i}] = E[(X_{t_i} - X_{t_{i+1}}) \mathbb{1}_{\{t_i < \tau\}} | \mathcal{G}_{t_i}] = \mathbb{1}_{\{t_i < \tau\}} E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}],$$

since on $\mathbb{1}_{\{t_i \geq \tau\}}$, $X_{t_i}^{\tau^+} - X_{t_{i+1}}^{\tau^+} = 0$. Using conditional Jensen inequality as well as the basic Jensen inequality and some elementary manipulations, we see for some given processes

X' and X'' that

$$\begin{aligned}
|MV(X', \pi) - MV(X'', \pi)| &= \left| E \left[\sum_{t_i \in \pi} \left| E[(X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) | \mathcal{G}_{t_i}] \right| \right] \right| \\
&\stackrel{\text{cond. Jensen}}{\leq} \left| E \left[\sum_{t_i \in \pi} E \left[\left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| | \mathcal{G}_{t_i} \right] \right] \right| \\
&= \left| \sum_{t_i \in \pi} E \left[E \left[\left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| | \mathcal{G}_{t_i} \right] \right] \right| \\
&= \left| \sum_{t_i \in \pi} E \left[\left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| \right] \right| \\
&\stackrel{\text{triang. ineq.}}{\leq} \sum_{t_i \in \pi} \left| E \left[\left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| \right] \right| \\
&\stackrel{\text{Jensen}}{\leq} \sum_{t_i \in \pi} E \left[\left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| \right] \\
&= E \left[\sum_{t_i \in \pi} \left| (X'_{t_i} - X'_{t_{i+1}}) - (X''_{t_i} - X''_{t_{i+1}}) \right| \right]
\end{aligned}$$

Applying this to $X' = X^\tau$ and $X'' = X^{\tau^+}$ completes the proof as the only possibly non-zero term in the above sum is the one for which $\tau \in [t_i, t_{i+1})$. $\square$

2.2 Rao's theorem

Assume a given process X is only defined on $[0, 1)$. Then we can extend it to $[0, 1]$ by setting $X_1 = 0$.

The stochastic analogue of Proposition 2.1 is provided by the following theorem, which is usually known as Rao's theorem.

Theorem 2.8. *Let X be a process defined on $[0, 1)$, then X is a quasimartingale if and only if X has a decomposition $X = Y - Z$ where Y and Z are positive and right continuous supermartingales.*

Proof. ($\Rightarrow$) For given $s \geq 0$ let $\Lambda(s)$ be the set of finite subdivisions of $[s, 1]$. Set for every $\pi \in \Lambda(s)$

$$Y_s^\pi = E[C(X, \pi)^+ | \mathcal{G}_s], Z_s^\pi = E[C(X, \pi)^- | \mathcal{G}_s],$$

where $C(X, \pi)^\pm = \sum_{t_i \in \pi} E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}]^\pm$. Let $\prec$ denote the ordering on $\Lambda(s)$, i.e. $\sigma \prec \tau$ means that the subdivision τ consists of more division points than σ . Suppose $\sigma \prec \tau$ for $\sigma, \tau \in \Lambda(s)$. We claim that $Y_s^\sigma \leq Y_s^\tau$ almost surely. Indeed, let $\sigma = (t_0, t_1, \dots, t_n)$. It suffices to check what happens if we add one subdivision point t to σ . Therefore we have three possibilities:

- (i) t is added before t_0 ,
- (ii) t is added after t_n or
- (iii) t is between t_i and t_{i+1} .

Let us only consider the third case, since the first two situations are clear. Let

$$A = E[X_{t_i} - X_t | \mathcal{G}_{t_i}], B = E[X_t - X_{t_{i+1}} | \mathcal{G}_t] \text{ and}$$

$$C = E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}].$$

Then we get

$$\begin{aligned} C &= E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}] \\ &= E[X_{t_i} - X_t + X_t - X_{t_{i+1}} | \mathcal{G}_{t_i}] \\ &= E[X_{t_i} - X_t | \mathcal{G}_{t_i}] + E[X_t - X_{t_{i+1}} | \mathcal{G}_{t_i}] \\ &= A + E[B | \mathcal{G}_{t_i}], \end{aligned}$$

where the last equality holds by the tower property of the conditional expectation, i.e. we get with $\mathcal{G}_{t_i} \subset \mathcal{G}_t$ that

$$E[B | \mathcal{G}_{t_i}] = E[E[X_t - X_{t_{i+1}} | \mathcal{G}_t] | \mathcal{G}_{t_i}] = E[X_t - X_{t_{i+1}} | \mathcal{G}_{t_i}]$$

By applying the conditional Jensen inequality we obtain now

$$C^+ \leq A^+ + E[B | \mathcal{G}_{t_i}]^+ \leq A^+ + E[B^+ | \mathcal{G}_{t_i}].$$

Finally, conditioning on $\mathcal{G}_s$ and using again the tower property we end up with

$$E[C^+ | \mathcal{G}_s] \leq E[A^+ | \mathcal{G}_s] + E[B^+ | \mathcal{G}_s].$$

Combining everything we conclude that $Y_s^\sigma \leq Y_s^\tau$.

Since $E[Y_s^\tau]$ is bounded by $MV(X)$ we could take the limits in L^1 along the directed ordered set $\Lambda(s)$ and define

$$\hat{Y}_s = \lim_{\tau} Y_s^\tau \text{ and } \hat{Z}_s = \lim_{\tau} Z_s^\tau.$$

Then taking a subdivision with $t_0 = s$ and $t_{n+1} = 1$, we see that

$$\begin{aligned}
Y_s^\tau - Z_s^\tau &= E[C(X, \tau)^+ - C(X, \tau)^- | \mathcal{G}_s] \\
&= E[C(X, \tau) | \mathcal{G}_s] \\
&= E\left[\sum_{t_i \in \tau} E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}] | \mathcal{G}_s\right] \\
&= \sum_{t_i \in \tau} E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_s] \quad (\text{tower property and linearity}) \\
&= E[X_{t_0} - X_{t_1} + X_{t_1} - X_{t_2} + \cdots + X_{t_n} - X_{t_{n+1}} | \mathcal{G}_s] \\
&= E[X_s - X_1 | \mathcal{G}_s] = X_s,
\end{aligned}$$

since $X_1 = 0$. Hence $\hat{Y}_s - \hat{Z}_s = X_s$. Moreover if $s < t$ we get that $\hat{Y}_s \geq E[\hat{Y}_t | \mathcal{G}_s]$ and $\hat{Z}_s \geq E[\hat{Z}_t | \mathcal{G}_s]$, which we get by the following computation:

$$\begin{aligned}
E[\hat{Y}_t | \mathcal{G}_s] &= E[\lim_{\tau} Y_t^\tau | \mathcal{G}_s] \\
&\leq \lim_{\tau} E[Y_t^\tau | \mathcal{G}_s] \\
&= \lim_{\tau} E[E[C(X, \pi)^+ | \mathcal{G}_t] | \mathcal{G}_s] \\
&= \lim_{\tau} E[C(X, \pi)^+ | \mathcal{G}_s] \\
&= \lim_{\tau} Y_s^\tau = \hat{Y}_s.
\end{aligned}$$

Therefore we define the right continuous process $Y_t := \hat{Y}_{t+}$ and $Z_t := \hat{Z}_{t+}$ where the right limits are taken through the rationals. Then Y and Z are positive supermartingales and $Y_s - Z_s = X_s$.

($\Leftarrow$) Let $X = Y - Z$ where Y and Z are positive supermartingales. Then for a partition π of $[0, t]$ we have

$$\begin{aligned}
E\left[\sum_{t_i \in \pi} |E[X_{t_i} - X_{t_{i+1}} | \mathcal{G}_{t_i}]|\right] &\leq E\left[\sum_{t_i \in \pi} E[Y_{t_i} - Y_{t_{i+1}} | \mathcal{G}_{t_i}]\right] + E\left[\sum_{t_i \in \pi} E[Z_{t_i} - Z_{t_{i+1}} | \mathcal{G}_{t_i}]\right] \\
&= \sum_{t_i \in \pi} (E[Y_{t_i}] - E[Y_{t_{i+1}}] + E[Z_{t_i}] - E[Z_{t_{i+1}}]) \\
&= E[Y_0] + E[Z_0] - (E[Y_t] + E[Z_t]),
\end{aligned}$$

therefore X is a quasimartingale on $[0, t] \forall t \geq 0$. □

If we replace Y and Z by $-Y$ and $-Z$, Rao's theorem 2.8 yields that a quasimartingale can be written as a decomposition into the difference of two submartingales as it is often stated.

3 Alternative characterization of good integrators

Now we want to derive the nice property, that a good integrator is locally the difference of two càdlàg submartingales. For this let us slightly redefine what it means for a property to hold locally.

Definition 3.1. A process X defined on $[0, 1]$ fulfils a property locally if for every $\varepsilon > 0$ there is a $[0, 1] \cup \{\infty\}$ -valued stopping time τ such that $X^\tau = X_{t \wedge \tau}$ satisfies that property and $P(\tau = \infty) \geq 1 - \varepsilon$. Note that by definition $\mathcal{G}_\infty = \mathcal{G}_1$.

Remark 3.2. If we have a property which is preserved under stopping and we start with a sequence of stopping times $(\tau_n)_n$ such that $P(\tau_n < \infty) \rightarrow 0$ and the property holds for X^{τ_n} , we are able to create an increasing such sequence by passing to a subsequence n_i such that

$$P(\tau_{n_i} < \infty) \leq 2^{-i}$$

and then choose $\sigma_k := \inf_{i \geq k} \tau_{n_i}$. The probability of σ_k being finite indeed tends to zero since

$$P(\sigma_k < \infty) \leq 2^{-(k-1)} \rightarrow 0.$$

Let us recall that D_n is the n -th dyadic partition of $[0, 1]$, i.e.

$$D_n = \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, \dots, 1\right\}.$$

As pointed out in Section 1, we have that $(H \cdot X)_t := \mathcal{I}_{X_t}(H)$.

Theorem 3.3. Let $X = (X_t)_{t \in [0, 1]}$ be a bounded, càdlàg and adapted process. If X is a good integrator then it is locally the difference of two càdlàg submartingales.

To motivate the proof let a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ be given and consider its Riemann-Stieltjes integral

$$g \mapsto \int g(t) df(t).$$

This integral is continuous on the space of piecewise constant functions $g : [0, 1] \rightarrow \mathbb{R}$ equipped with the supremum norm. It follows that f has finite total variation since

$$g^n := \sum_{t_i \in D_n} \mathbb{1}_{(t_i, t_{i+1}]} \operatorname{sgn}(f(t_{i+1}) - f(t_i))$$

is uniformly bounded and

$$\int_0^1 g^n df = \sum_{t_i \in D_n} |f(t_{i+1}) - f(t_i)|$$

converges to the total variation of f .

We will skip the proof at this point and give it at the end of this section.

Definition 3.4. A family $E \subseteq L^0(P)$ is bounded if for each $\varepsilon > 0$ there exists a constant C such that

$$P(|X| \geq C) \leq \varepsilon \quad \forall X \in E.$$

With the next theorem we just want to recall that a linear operator is continuous if and only if it is bounded. Since this is a quite standard functional analytic statement, we will skip the proof and refer the reader to [17].

Theorem 3.5. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are topological vector spaces and $\mathcal{I} : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear operator. Then we obtain for the properties

- (i) $\mathcal{I}$ is continuous.
- (ii) $\mathcal{I}$ is bounded.
- (iii) If $\lim_n x_n = 0$, then $\{\mathcal{I}(x_n) : n = 1, 2, \dots\}$ is bounded.
- (iv) If $\lim_n x_n = 0$, then $\lim_n \mathcal{I}(x_n) = 0$.

the following chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii).$$

Additionally if $\mathcal{X}$ is metrizable, then

$$(iii) \Rightarrow (iv) \Rightarrow (i)$$

hold and therefore the properties stated above are equivalent.

In particular we get that a linear operator from a normed space into $L^0(P)$ is continuous if and only if it is bounded. So if X is for example a good integrator, $\mathcal{I}_X$ is per definition continuous and therefore by Theorem 3.5 bounded in $L^0(P)$. We will need the simple result stated in the following lemma.

Lemma 3.6. Let $X = (X_t)_{t \in [0,1]}$ be a bounded càdlàg and adapted good integrator. Then $\forall \varepsilon \exists$ a constant C and a sequence of stopping times $(\tau_n)_n$ which are $[0, 1] \cup \{\infty\}$ valued such that $P(\tau_n = \infty) \geq 1 - \varepsilon$ and $MV(X^{\tau_n}, D_n) \leq C$.

Proof. As pointed out before, since X is a good integrator, it is bounded in $L^0(P)$, i.e. for $\varepsilon > 0$ there is a constant $C > 0$ such that for every simple process H with $\|H\|_\infty \leq 1$ we get that

$$P((H \cdot X)_1 \geq C - 2\|X\|_\infty) \leq \varepsilon.$$

Now let us define the following sequence of simple processes (H^n) and a sequence of stopping times (τ_n) :

$$H^n := \sum_{t_i \in D_n} \mathbb{1}_{(t_i, t_{i+1}]} \operatorname{sgn} (E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]),$$

$$\tau_n := \inf\{t \in D_n : (H^n \cdot X)_t \geq C - 2\|X\|_\infty\}.$$

Now we see, restricted on the set $\{\tau_n < \infty\}$,

$$(H^n \mathbb{1}_{(0, \tau_n]}) \cdot X = (H^n \cdot X)^{\tau_n} \text{ satisfies } (H^n \cdot X)_1^{\tau_n} \geq C - 2\|X\|_\infty,$$

and therefore $P(\tau_n = \infty) \geq 1 - \varepsilon$. Additionally, since the increments $X_t - X_s$ of X are bounded by $2\|X\|_\infty$, we have that $C \geq (H^n \cdot X)_1^{\tau_n}$ holds. Using Lemma 2.7 we see that

$$\begin{aligned} C &\geq E[(H^n \cdot X)_1^{\tau_n}] \\ &= E\left[\sum_{t_i \in D_n} \mathbb{1}_{\{t_i < \tau_n\}} \operatorname{sgn}(E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]) (X_{t_{i+1}} - X_{t_i})\right] \\ &= E\left[\sum_{t_i \in D_n} \mathbb{1}_{\{t_i < \tau_n\}} |E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]|\right] \\ &= MV(X^{\tau_n}, D_n), \end{aligned}$$

where the second equality holds since

$$\begin{aligned} E[\operatorname{sgn}(E[A | \mathcal{G}]) \cdot A] &= E[E[\underbrace{\operatorname{sgn}(E[A | \mathcal{G}])}_{\in \mathcal{G}} \cdot A | \mathcal{G}]] \\ &= E[\operatorname{sgn}(E[A | \mathcal{G}]) E[A | \mathcal{G}]] \\ &= E[|E[A | \mathcal{G}]|] \end{aligned}$$

□

Before gathering the last ingredient to provide the proof of Theorem 3.3, let us present two technical results turning out to be very useful. The first one is a famous result by Mazur.

Lemma 3.7. *Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions on a probability space $(\Omega, \mathcal{G}, P)$, such that $\sup_{n \geq 1} \|f_n\|_2 < \infty$. Then there exist functions $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n \geq 1}$ converges in $\|\cdot\|_{L^2(P)}$ and almost surely.*

Proof. Let $\mathcal{H}$ be the Hilbert space generated by the given functions $(f_n)_{n \geq 1}$. For $n \geq 1$ let K_n be the strong closure of $\operatorname{conv}(f_n, f_{n+1}, \dots)$ which coincides with the weak closure by convexity. Note that the closure is taken with respect to the L^2 -norm. Applying the Banach-Alaoglu-theorem we get that the K_n are weakly compact and since by definition $K_n \subseteq K_{n+1}$ for every n we obtain

$$\bigcap_{n \geq 1} K_n \neq \emptyset$$

by the finite intersection property of compact sets. Therefore there exist a sequence $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$ such that

$$\lim_n g_n = g \text{ in } L^2,$$

where $g \in \bigcap_{n \geq 1} K_n$. If necessary we pass to a subsequence to get the almost sure convergence. $\square$

The second theorem which we want to remind of is Egorov's theorem. Since it is a quite standard result in measure theory, the proof will be skipped and we refer the reader for example to [19].

Theorem 3.8. *Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions on M , where M is a separable metric space on some measure space $(X, \mathcal{A}, \mu)$. Suppose there is a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $(f_n)_{n \geq 1}$ converges μ -almost everywhere on A to some limit function f . Then for every $\varepsilon > 0$ there exists $B \subset A, B \in \mathcal{A}$ such that $\mu(B) < \varepsilon$ and $(f_n)_{n \geq 1}$ converges uniformly to f on $A \setminus B$.*

Recall that given a bounded, càdlàg and adapted good integrator $X = (X_t)_{t \in [0,1]}$ and applying Lemma 3.6 we get $MV(X^{\tau_n}, D_n) \leq C$. If we assume now that

$$MV(X^{\tau_n}, D_k) \leq C \quad \forall k \leq n,$$

it would be nice to define a “limiting stopping time” τ of the sequence of stopping times $(\tau_n)_n$, such that $MV(X^\tau, D_k) \leq C \quad \forall k$ which would prove that X^τ is by definition a quasimartingale. Moreover it would be desirable if τ is “as large as τ_n ” but also that $\tau \leq \tau_{n_k}$ for some subsequence n_k . Since this is not possible, we can weaken this assumption to

$$\mathbb{1}_{[0, \tau]} \leq \mathbb{1}_{[0, \tau_{n_k}]}$$

by passing on to convex combinations, so it is compatible with τ being “big”. This motivation gives us the following important lemma.

Lemma 3.9. *Assume $(\tau_n)_n$ to be a sequence of $[0, 1] \cup \{\infty\}$ -valued stopping times such that $P(\tau_n = \infty) \geq 1 - \varepsilon$ holds for $n \geq 1, \varepsilon > 0$. Then there is a stopping time τ such that $P(\tau = \infty) \geq 1 - 3\varepsilon$ and such that for every $n \geq 1$ there are convex weights $\mu_n^n, \dots, \mu_{N_n}^n$ satisfying*

$$\mathbb{1}_{[0, \tau]} \leq 2 \sum_{k=n}^{N_n} \mu_k^n \mathbb{1}_{[0, \tau_k]}.$$

The constant 2 in the above estimation of $\mathbb{1}_{[0, \tau]}$ can actually be replaced by $1 + \delta$ for a $\delta > 0$. In this case one for sure can find a stopping time τ such that

$$P(\tau = \infty) \geq 1 - \eta\varepsilon \quad \text{for } \eta > \frac{1}{1 - (1 + \delta)^{-1}}.$$

Proof. Let the random variables X_n be defined as $X_n = \mathbb{1}_{\{\tau_n = \infty\}} \in L^2(P)$ for $n \geq 1$. Then we get for each n some convex weights $\mu_n^n, \dots, \mu_{N_n}^n$ such that

$$Y_n := \mu_n^n X_n + \dots + \mu_{N_n}^n X_{N_n}$$

converges to a random variable $X \in L^2(P)$, which is just an application of Lemma 3.7. Since $\mu_n^n, \dots, \mu_{N_n}^n$ are convex weights for every $n \geq 1$, we have that $\mu_n^n + \dots + \mu_{N_n}^n = 1$

and therefore $X \leq 1$ by definition of the random variable X . Furthermore we may deduce by our assumption $P(\tau_n = \infty) \geq 1 - \varepsilon$ that $E[X] \geq 1 - \varepsilon$ and therefore get that $P(X < \frac{2}{3}) < 3\varepsilon$.

From the fact that $P(\lim_m Y_m \geq \frac{2}{3}) > 1 - 3\varepsilon$ and by Theorem 3.8 we find a set A with $P(A) \geq 1 - 3\varepsilon$ such that $Y_n \geq \frac{1}{2}$ on A for every $n \geq n_0 \in \mathbb{N}$, where we assume n_0 to be equal to 1. To get the stopping time we are looking for, we define

$$\tau = \inf_{n \geq 1} \inf \left\{ t : \mu_n^n \mathbb{1}_{[0, \tau_n]}(t) + \dots + \mu_n^{N_n} \mathbb{1}_{[0, \tau_{N_n}]}(t) < \frac{1}{2} \right\}.$$

With this definition the desired formula follows and additionally since $A \subseteq \{\tau = \infty\}$ we get $P(\tau = \infty) \geq 1 - 3\varepsilon$. $\square$

Now we are finally able to prove Theorem 3.3.

Proof. Given some $\varepsilon > 0$ we pick a constant C and stopping times $(\tau_n)_n$ and τ like we chose them in Lemma 3.6 and 3.9. For a fixed $n \geq 1$ we apply the formula in Lemma 3.9 to obtain immediately

$$\begin{aligned} E \left[\sum_{t_i \in D_n} \mathbb{1}_{\{t_i < \tau\}} |E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]| \right] &\leq 2E \left[\sum_{t_i \in D_n} \sum_{k=n}^{N_n} \mu_k^n \mathbb{1}_{\{t_i < \tau_k\}} |E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]| \right] \\ &= 2 \sum_{t_i \in D_n} \sum_{k=n}^{N_n} \mu_k^n E[\mathbb{1}_{\{t_i < \tau_k\}} |E[X_{t_{i+1}} - X_{t_i} | \mathcal{G}_{t_i}]|] \\ &\leq 2 \sum_{k=n}^{N_n} \mu_k^n (MV(X^{\tau_k}, D_n) + 2\|X\|_\infty) \\ &\leq 2C + 4\|X\|_\infty, \end{aligned}$$

where we used lemma 2.7. With $n \rightarrow \infty$ we obtain that

$$MV(X^\tau) \leq 2C + 6\|X\|_\infty.$$

Therefore X is locally a quasimartingale and after applying Rao's theorem 2.8, X is locally the difference of two submartingales. $\square$

4 Doob-Meyer Theorem

In this section we want to establish the Doob-Meyer theorem, since it is necessary to give a complete proof to the Bichteler-Dellacherie theorem.

In the beginning let us recall the definition of a predictable process.

Definition 4.1. A stochastic process is called predictable if it is measurable with respect to the predictable σ -algebra where the predictable σ -algebra on $\mathbb{R}^+ \times \Omega$ is generated by the left continuous and adapted processes.

Theorem 4.2. *Let $X = (X_t)_{t \in [0,1]}$ be a càdlàg submartingale of class D . Then X can be uniquely decomposed as*

$$X = M + A$$

where M is a martingale and A is a predictable increasing process starting at 0.

Proof. The proof of the uniqueness is quite standard and we will therefore skip it and refer the reader to [8].

Starting with the process $X^n = (X_t)_{t \in D_n}$, take its discrete time Doob decomposition, meaning that we define A^n and M^n by

$$\begin{aligned} A_0^n &:= 0, \\ A_t^n - A_{t-1/2^n}^n &:= E[X_t - X_{t-1/2^n} | \mathcal{G}_{t-1/2^n}] \text{ and} \\ M_t^n &:= X_t - A_t^n. \end{aligned}$$

Therefore the process $(M_t^n)_{t \in D_n}$ is a martingale and $(A_t^n)_{t \in D_n}$ is predictable with respect to the filtration $(\mathcal{G}_t)_{t \in D_n}$. To get the continuous-time decomposition, we clearly want to get it as a limit point of the processes $M^n, A^n, n \geq 1$. In order to get the desired limit, we want to apply lemma the following Komlos type lemma.

Lemma 4.3. *Let $(f_n)_{n \geq 1}$ be a uniformly integrable sequence of functions on a probability space $(\Omega, \mathcal{G}, P)$. Then there are functions $g_n \in \text{conv}(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n \geq 1}$ converges in $\|\cdot\|_{L^1(\Omega)}$.*

Proof. For $i, n \in \mathbb{N}$ define $f_n^{(i)} := f_n \mathbb{1}_{\{|f_n| \leq i\}}$, so we get $f_n^{(i)} \in L^2(\Omega)$. Our goal now is to show that for every convex weights $\lambda_n^n, \dots, \lambda_{N_n}^n$ there are functions

$$\lambda_n^n f_n^{(i)} + \dots + \lambda_{N_n}^n f_{N_n}^{(i)}$$

such that they converge in $L^2(\Omega) \forall i \in \mathbb{N}$. First we use Lemma 3.7 to get convex weights $\lambda_n^n, \dots, \lambda_{N_n}^n$ such that $(\lambda_n^n f_n^{(1)} + \dots + \lambda_{N_n}^n f_{N_n}^{(1)})_{n \geq 1}$ converges. Then we apply the lemma to the sequence $(\lambda_n^n f_n^{(2)} + \dots + \lambda_{N_n}^n f_{N_n}^{(2)})_{n \geq 1}$ to get convex weights which work for the first two sequences. By repeating inductively one obtains a sequence of convex weights which work for the first m sequences and by a diagonalization argument we are able to conclude the claim.

Now applying the uniform integrability we see that

$$\lim_{i \rightarrow \infty} \|f_n^{(i)} - f_n\|_1 = 0$$

uniformly. So we obtain

$$\lim_{i \rightarrow \infty} \|(\lambda_n^n f_n^{(i)} + \dots + \lambda_{N_n}^n f_{N_n}^{(i)}) - (\lambda_n^n f_n + \dots + \lambda_{N_n}^n f_{N_n})\|_1 = 0,$$

again uniformly. Therefore $(\lambda_n^n f_n + \dots + \lambda_{N_n}^n f_{N_n})_{n \geq 1}$ is a Cauchy sequence in $L^1(\Omega)$ which finishes the proof. $\square$

Now we have to show, that $(M_1^n)_{n \geq 1}$ is uniformly integrable, which is the goal of the next lemma.

Lemma 4.4. *The sequence $(M_1^n)_{n \geq 1}$ is uniformly integrable.*

Proof. Substracting the uniformly integrable martingale $(E[X_1 | \mathcal{G}_t])_{t \in [0,1]}$ from X_t we are allowed to assume that $X_1 = 0$ and $X_t \leq 0 \forall t$. Therefore we get that $M_1^n = -A_1^n$ and for every $(\mathcal{G}_t)_{t \in D_n}$ -stopping time τ we see that

$$X_\tau^n = -E[A_1^n | \mathcal{G}_\tau] + A_\tau^n.$$

We want to prove now that $(A_1^n)_{n \geq 1}$ is uniformly integrable and hence also $(M_1^n)_{n \geq 1}$. For this let $c > 0$ and $n \geq 1$ and define

$$\tau_n(c) := \inf \left\{ \frac{j-1}{2^n} : A_{j2^{-n}}^n > c \right\} \wedge 1,$$

which is a stopping time since A^n is a predictable process. Since $A_{\tau_n(c)}^n \leq c$ and $\{A_1^n > c\} = \{\tau_n(c) < 1\}$ we get that

$$X_{\tau_n(c)}^n = -E[A_1^n | \mathcal{G}_{\tau_n(c)}] + A_{\tau_n(c)}^n \leq -E[A_1^n | \mathcal{G}_{\tau_n(c)}] + c.$$

Therefore we get the following inequality:

$$\begin{aligned} E[A_1^n \mathbb{1}_{\{A_1^n > c\}}] &= E[A_1^n \mathbb{1}_{\{\tau_n(c) < 1\}}] \\ &= E[E[A_1^n | \mathcal{G}_{\tau_n(c)}] \mathbb{1}_{\{\tau_n(c) < 1\}}] \\ &= E[(A_{\tau_n(c)}^n - X_{\tau_n(c)}) \mathbb{1}_{\{\tau_n(c) < 1\}}] \\ &\leq cE[\mathbb{1}_{\{\tau_n(c) < 1\}}] - E[X_{\tau_n(c)} \mathbb{1}_{\{\tau_n(c) < 1\}}] \\ &= cP(\tau_n(c) < 1) - E[X_{\tau_n(c)} \mathbb{1}_{\{\tau_n(c) < 1\}}]. \end{aligned}$$

Furthermore we see easily that $\{\tau_n(c) < 1\} \subseteq \{\tau_n(\frac{c}{2}) < 1\}$ and hence we may conclude

that

$$\begin{aligned}
-E[X_{\tau_n(\frac{c}{2})} \mathbb{1}_{\{\tau_n(\frac{c}{2}) < 1\}}] &= E[(A_1^n - A_{\tau_n(\frac{c}{2})}^n) \mathbb{1}_{\{\tau_n(\frac{c}{2}) < 1\}}] \\
&\geq E[(A_1^n - A_{\tau_n(c)}^n) \mathbb{1}_{\{\tau_n(c) < 1\}}] \\
&\geq \frac{c}{2} P(\tau_n(c) < 1).
\end{aligned}$$

We may change the previous inequality so that we get

$$cP(\tau_n(c) < 1) \leq -2E[X_{\tau_n(\frac{c}{2})} \mathbb{1}_{\{\tau_n(\frac{c}{2}) < 1\}}].$$

Finally we see that

$$\begin{aligned}
E[A_1^n \mathbb{1}_{\{A_1^n > c\}}] &\leq cP(\tau_n(c) < 1) - E[X_{\tau_n(c)} \mathbb{1}_{\{\tau_n(c) < 1\}}] \\
&\leq -2E[X_{\tau_n(\frac{c}{2})} \mathbb{1}_{\{\tau_n(\frac{c}{2}) < 1\}}] + E[X_{\tau_n(c)} \mathbb{1}_{\{\tau_n(c) < 1\}}].
\end{aligned}$$

Since

$$P(\tau_n(c) < 1) = P(A_1^n > c) \leq \frac{E[A_1^n]}{c} = -\frac{E[M_1^n]}{c} = -\frac{E[X_0]}{c},$$

where we have used the Markov inequality. Hence we have that

$$\lim_{c \rightarrow \infty} P(\tau_n(c) < 1) = 0$$

uniformly in n , which together with the fact that X is of class D implies that $(A_1^n)_{n \geq 1}$ is uniformly integrable and so is $(M_1^n)_{n \geq 1} = (X_1 - A_1^n)_{n \geq 1}$. $\square$

Since we have proven the uniform integrability of the sequence $(M_1^n)_{n \geq 1}$ we are able to use Lemma 4.3 to construct the limits.

For every n we extend M^n to a càdlàg martingale on $[0, 1]$ by setting

$$M_t^n := E[M_1^n | \mathcal{G}_t].$$

By Lemma 4.4 we can apply Lemma 4.3 to say that there is a $M \in L^1(\Omega)$ and convex weights $\lambda_n^n, \dots, \lambda_{N_n}^n$ such that for

$$\mathcal{M}^n := \lambda_n^n M^n + \dots + \lambda_{N_n}^n M^{N_n}$$

we have $\lim_{n \rightarrow \infty} \mathcal{M}_1^n = M$ in $L^1(\Omega)$. If we apply then the conditional Jensen inequality we get that

$$\lim_{n \rightarrow \infty} \mathcal{M}_t^n = M_t := E[M | \mathcal{G}_t]$$

for every $t \in [0, 1]$. Similarly we extend A^n to $[0, 1]$ by setting

$$A^n := \sum_{t \in D_n} A_t^n \mathbb{1}_{(t-1/2^n, t]} \text{ and define}$$

$$\mathcal{A}^n := \lambda_n^n A^n + \dots + \lambda_{N_n}^n A^{N_n}$$

where $(\lambda_i^n)_{i \in \{n, \dots, N_n\}}$ are the same convex weights as in the definition of $\mathcal{M}^n$. Then the càdlàg process

$$A_t := X_t - M_t \text{ for } 0 \leq t \leq 1$$

satisfies $\forall t \in D$

$$\mathcal{A}_t^n = X_t - \mathcal{M}_t^n \rightarrow X_t - M_t = A_t \text{ in } L^1(\Omega).$$

If we pass to a subsequence we get the almost sure convergence. Therefore the process A is almost surely increasing on D and so by right continuity also on $[0, 1]$.

In the end we still have to prove that the constructed process A is indeed predictable. Since both A^n and $\mathcal{A}^n$ are left continuous and adapted, they are predictable. Our goal now is to show

$$\limsup_n \mathcal{A}_t^n(\omega) = A_t(\omega) \text{ for a.e. } \omega \text{ and } \forall t \in [0, 1].$$

By right continuity we see that

$$\limsup_n \mathcal{A}_t^n \leq A_t \text{ } \forall t \text{ and } \lim_n \mathcal{A}_t^n = A_t \text{ if } A \text{ is continuous at } t.$$

Furthermore since A is càdlàg, there are only countably many jump times. So it is enough to prove the property for every stopping time τ , i.e.

$$\limsup_n \mathcal{A}_\tau^n(\omega) = A_\tau(\omega)$$

Since $\mathcal{A}_\tau^n \leq \mathcal{A}_1^n \rightarrow A_1$ in $L^1(\Omega)$ we can apply Fatou's Lemma to get

$$\liminf_n E[A_\tau^n] \leq \limsup_n E[\mathcal{A}_\tau^n] \leq E[\limsup_n \mathcal{A}_\tau^n] \leq E[A_\tau].$$

By this chain of inequalities it is enough to prove $\lim_n E[A_\tau^n] = E[A_\tau]$, since we then get a chain of equalities and therefore the desired result. So for $n \geq 1$ let

$$\rho_n := \inf\{t \in D_n | t \geq \tau\}$$

be a sequence of stopping times. We get that $A_\tau^n = A_{\rho_n}^n$ and $\rho_n \downarrow \tau$ as $n \rightarrow \infty$. Since X is of class D we finally get

$$\begin{aligned} \lim_n E[A_\tau^n] &= \lim_n E[A_{\rho_n}^n] \\ &= \lim_n E[X_{\rho_n}] - E[M_{\rho_n}] \text{ (by discrete Doob-Meyer)} \\ &= E[X_\tau] - E[M_0] \text{ (X is of class D)} \\ &= E[A_\tau]. \end{aligned}$$

□

5 The proof of the Bichteler-Dellacherie theorem

This section is finally dedicated to provide the proof of the Bichteler-Dellacherie theorem. Before explaining it in complete detail, let us collect some last important facts.

The first claim asserts that if a càdlàg process X is locally a local martingale, then it is already a local martingale.

Proposition 5.1. *Let X be a càdlàg process and τ_n an increasing sequence of stopping times such that $\lim_{n \rightarrow \infty} \tau_n = \infty$ almost surely. Furthermore we require that $X^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a local martingale for every n . Then X itself is a local martingale.*

Proof. By the assumption we know that the process $M^n = X^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a local martingale for every n . For a fixed n there exists a reducing increasing sequence $S^{n,k}$ such that $\lim_{k \rightarrow \infty} S^{n,k} = \infty$ almost surely. Now choose for every n a $k = k(n)$ such that

$$P(S^{n,k(n)} < \tau_n \wedge n) < \frac{1}{2^n}.$$

Then we obtain that $\lim_{n \rightarrow \infty} S^{n,k(n)} = \infty$ almost surely as well as $S^{n,k(n)} \wedge \tau_n$ reduces X for every n . If we set $U^m = \max(S^{1,k(1)} \wedge \tau_1, \dots, S^{m,k(m)} \wedge \tau_m)$ we see that U^m also reduces X since the maximum of reducing sequences is again reducing. Furthermore U^m is increasing and $\lim_{m \rightarrow \infty} U_m = \infty$ almost surely. In total we obtain that X is a local martingale as required. $\square$

A similar assertion can be made in the case of semimartingales as we will see below. Before doing that recall how a semimartingale is defined.

Definition 5.2. An adapted càdlàg process X is called a semimartingale if there exist processes M and A with $M_0 = A_0 = 0$ such that

$$X_t = X_0 + M_t + A_t,$$

where M is a local martingale and A a process of finite variation.

Proposition 5.3. *Let X be a process which is locally a semimartingale, i.e. there is an increasing sequence of stopping times τ_n with $\lim_n \tau_n = \infty$ almost surely such that $X^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a semimartingale for every $n \geq 1$. Then X is a semimartingale itself.*

Proof. If $X = (X_t)_{t \in [0,1]}$ is locally a semimartingale, then there is by Remark 3.2 an increasing sequence $(\tau_n)_n$ of stopping times such that $P(\tau_n < \infty) \rightarrow 0$. Since X is locally a semimartingale there are processes M_n and A_n such that

$$X^{\tau_n} = M_n + A_n.$$

where M_n is a local martingale and A_n a process of finite variation. Since the sequence of stopping times is increasing (i.e. $\min\{\tau_n, \tau_{n+1}\} = \tau_n$) we see that

$$X^{\tau_n} = (X^{\tau_{n+1}})^{\tau_n} \text{ and therefore}$$

$$M_n^{\tau_n} + A_n^{\tau_n} = M_{n+1}^{\tau_n} + A_{n+1}^{\tau_n}.$$

We can decompose X as

$$\begin{aligned} X &= X^{\tau_1} + (X^{\tau_2} - X^{\tau_1}) + (X^{\tau_3} - X^{\tau_2}) + \dots \\ &= (M_1^{\tau_1} + (M_2^{\tau_2} - M_2^{\tau_1}) + \dots) + (A_1^{\tau_1} + (A_2^{\tau_2} - A_2^{\tau_1}) + \dots) \\ &=: M + A, \end{aligned}$$

where for each pair (t, ω) only one term in each sum is non-zero. A is of finite variation and since M is locally a local martingale, it is by Proposition 5.1 a local martingale itself and therefore X is a semimartingale. $\square$

The last useful result involves again the definition of a process being of class D .

Lemma 5.4. *A càdlàg submartingale $X = (X_t)_{t \in [0,1]}$ is locally of class D , i.e. locally the set $\{X^\tau | \tau \text{ stopping time}\}$ is uniformly integrable.*

Proof. Let $\sigma_n := \inf\{t \in [0,1] | |X_t| \geq n\}$ be a sequence of stopping times. For an arbitrary stopping time τ we see that

$$|X_\tau^{\sigma_n}| \leq n + |X_{1 \wedge \sigma_n}|.$$

By the optional stopping theorem $X_{1 \wedge \sigma_n}$ is integrable which proves that

$$\{X_\tau^{\sigma_n} | \tau \text{ stopping time}\}$$

is uniformly integrable. $\square$

Theorem 5.5 (Bichteler-Dellacherie). *Let $(X_t)_{t \in [0,1]}$ be a càdlàg adapted process. If $\mathcal{I}_X : \mathcal{S} \rightarrow L^0(P)$ is continuous then X can be written as a sum of a càdlàg local martingale and a càdlàg adapted process of finite variation.*

Proof. At first we claim that the process X can be decomposed as a sum of two adapted processes, where one is of finite variation and the other one is locally bounded. Since X is càdlàg, we may well-define

$$\Delta X_t := X_t - X_{t-} \text{ and}$$

$$J_t := \sum_{0 < s \leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s| \geq 1\}}.$$

Now we see that the process J has finite variation and is adapted and that $X - J$ has bounded jumps and therefore $X = J + (X - J)$ is the decomposition we were interested in. Furthermore J is a càdlàg good integrator by Theorem 1.7 (i) as well as $X - J$. By localizing and Proposition 5.3 we may assume without loss of generality that X is bounded. By Theorem 3.3 X is locally the difference of two càdlàg submartingales. By Lemma 5.4 and Theorem 4.2 X is locally a local semimartingale. Finally, applying Proposition 5.3 twice we obtain that X is a semimartingale. $\square$

6 Equivalence of semimartingales and Riemann-integrators

The goal of this section is to show that a Riemann integrator is a good integrator and therefore by Theorem 5.5 a semimartingale. To begin with let us state the following definition.

Definition 6.1. The space of elementary integrands contains all processes H such that

$$H = \sum_{i=1}^k \xi_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]},$$

where (t_i) is a sequence of deterministic times with $0 \leq t_1 < \dots < t_{k+1} = 1$ and ξ_i is bounded and $\mathcal{G}_{t_i}$ measurable.

It is easy to see that in Definition 1.4 of good integrators the space $\mathcal{S}$ of simple integrands can be replaced by the subset of elementary integrands. In particular, we will prove this fact in a stronger form in Lemma 6.2.

Now let $\mathcal{E}_{D_n}$ be the space of all processes H such that

$$H = \sum_{i=0}^{2^n-1} \xi_i \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]},$$

where each ξ_i is again bounded and $\mathcal{G}_{\frac{i-1}{2^n}}$ measurable. Additionally set $H^0 = 0$ and define

$$\mathcal{E}_D := \bigcup_{n \geq 1} \mathcal{E}_{D_n}.$$

With this setup we want to prove an interesting and helpful result, which says that a càdlàg adapted process X is a good integrator if and only if $\lim \mathcal{I}_X(H^n) = 0$ in probability whenever $\lim \|H^n\|_\infty = 0$ and $H^n \in \mathcal{E}_{D_n}$ for every n . So we want to prove the following, slightly reformulated lemma:

Lemma 6.2. *Let X be an adapted process which is right continuous in probability. Then $\mathcal{I}_X : \mathcal{S} \rightarrow L^0(P)$ is continuous if and only if its restriction to $\mathcal{E}_D$ is continuous.*

Proof. We want to show that if the operator $\mathcal{I}_X$ is bounded in $\mathcal{E}_D$, then it is also bounded on $\mathcal{S}$. Then Theorem 3.5 finishes the proof.

Let $\varepsilon > 0$ and pick a constant $C > 0$ such that

$$P(|\mathcal{I}_X(K)| > C) < \varepsilon$$

for each process $K \in \mathcal{E}_D$ with $\|K\|_\infty \leq 1$. Furthermore let H be a simple integrand, i.e.

$$H_t = \sum_{i=1}^n \xi_i(\omega) \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t)$$

with $\|H\|_\infty \leq 1$ and define the stopping times

$$\sigma_i^n := 1 \wedge \frac{i+2}{2^n} \text{ on } \left\{ \frac{i}{2^n} < \tau_i \leq \frac{i+1}{2^n} \right\}.$$

Now since the stopping times have values in $\mathcal{E}_{D_n}$ and $\tau_i + \frac{1}{2^n} \leq \sigma_i^n$, while ξ_i is $\mathcal{G}_{\tau_i}$ measurable, the process

$$K^n := \sum_{i=1}^k \xi_i(\omega) \mathbb{1}_{(\sigma_i^n, \sigma_{i+1}^n]} \in \mathcal{E}_{D_n}.$$

Finally, since $\mathcal{I}_X(K^n)$ converges to $\mathcal{I}_X(H)$ in probability and by taking n large enough it follows that

$$P(|\mathcal{I}_X(H)| > C) \leq P(|\mathcal{I}_X(H) - \mathcal{I}_X(K^n)| > C) + P(|\mathcal{I}_X(K^n)| > C) < 2\varepsilon.$$

Therefore, since C is independent of $H \in \mathcal{S}$, the operator $\mathcal{I}_X$ is bounded on $\mathcal{S}$. $\square$

Before stating the desired result of this chapter, let us state the following definition.

Definition 6.3. A càdlàg adapted process $X = (X_t)_{t \in [0,1]}$ is called a Riemann integrator if for every bounded adapted continuous process H the sequence

$$\sum_{i=0}^{2^n-1} H_{\frac{i}{2^n}} (X_{\frac{i+1}{2^n}} - X_{\frac{i}{2^n}})$$

converges in probability as n increases to ∞ .

To give a proper proof of the new characterization of semimartingales, let us consider the space $L^\infty(\Omega, C^0([0,1]))$ of all bounded continuous processes $(H_t)_{t \in [0,1]}$, which is a Banach space. Furthermore we will equip it with the supremum norm, which we already defined in section 1, i.e.

$$\|H\|_\infty = \sup_{t \in [0,1]} \|H_t\|_{L^\infty(P)}.$$

Then A should be the subspace formed by adapted processes. It follows that A is a closed subspace and therefore a Banach space with the induced norm. In the end define the linear continuous operator $\mathcal{I}_X^n : A \rightarrow L^0(P)$ by

$$\mathcal{I}_X^n(K) := \mathcal{I}_X(K^{D_n}) \text{ with } K^{D_n} := \sum_{i=0}^{2^n-1} K_{\frac{i}{2^n}} \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}.$$

So X is a Riemann integrator if for every $K \in A$ we have that $\mathcal{I}_X^n(K)$ converges in probability as n increases to ∞ , which we see easy by Definition 6.3. With the Banach-Steinhaus theorem, which is also known as the uniform boundness principle, see for example [17], we get the following lemma:

Lemma 6.4. *If X is a Riemann integrator, then for each $\varepsilon > 0$ there is a constant $C > 0$ such that $P(\mathcal{I}_X^n(K) \geq C) \leq \varepsilon \forall n \geq 1$ and all continuous adapted processes K such that $\|K\|_\infty \leq 1$.*

Remember that a sequence of processes $(G^n)_{n \geq 1}$ converges to a process G uniformly on compacty in probability (u.c.p.) if for every $t > 0$ we have that $\sup_{0 \leq s \leq t} |G_s^n - G_s|$ converges to 0 in probability. In the end let us recall the dominated convergence theorem for stochastic integrals, which is basically the stochastic counterpart to the usual dominated convergence theorem known from Lebesgue integration theory.

Theorem 6.5. *Let X be a semimartingale and let G^n and G be predictable processes. If*

$$\lim_n G_t^n(\omega) = G_t(\omega) \quad \forall t \text{ almost surely}$$

and if there exists a process H that is integrable with respect to X such that $|G^n| \leq H$ for any $n \in \mathbb{N}$, then also

$$\lim_{n \rightarrow \infty} (G^n \cdot X)_t = (G \cdot X) \text{ in u.c.p.}$$

For a proof of this standard result, we want to refer for example to [13]. So we get the next important consequence, which is the desired result of this section:

Theorem 6.6. *A process $X = (X_t)_{t \in [0,1]}$ is a semimartingale if and only if X is a Riemann integrator.*

Proof. $(\Rightarrow)$ Suppose the process X is a semimartingale. Then the stochastic dominated convergence theorem implies that the random variables

$$\sum_{\tau_i \in \pi_n} H_{\tau_i} (X_{\tau_{i+1}} - X_{\tau_i})$$

converge in probability to $\mathcal{I}_X(H)$ (resp. $\mathcal{I}_X(H_-)$) as $n \rightarrow \infty$ for every left-continuous (resp. càdlàg) process H . Therefore we see that semimartingales can be described via Riemann sums.

$(\Leftarrow)$ Let $H \in \mathcal{E}_{D_n}$, i.e.

$$H = \sum_{i=0}^{2^n-1} \xi_i \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]},$$

such that $\|H\|_\infty \leq 1$. Then define a process K by the following procedure:

- (i) K is equal to ξ_i at the time $t = \frac{i}{2^n}$ for $0 \leq i \leq 2^n - 1$,
- (ii) $K = 0$ at time 1 and
- (iii) we extend it to $t \in [0, 1]$ by affine interpolation.

Then K is a continuous adapted process with $\|K\|_\infty \leq 1$ and $K^{D_n} = H$. Since $\mathcal{I}_X(H) = \mathcal{I}_X(K^{D_n}) = \mathcal{I}_X^n(K)$ we see that $\mathcal{I}_X$ is bounded on $\mathcal{E}_D$ by Lemma 6.4. Finally by Lemma 6.2 X is a good integrator and therefore by Theorem 5.5 a semimartingale, which completes the proof. □

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Abstract

In this thesis we give an alternative proof to the Bichteler-Dellacherie Theorem, which states that a process is a good integrator if and only if it is a semimartingale, i.e. the sum of a local martingale and a finite variation process. A process is a good integrator if the corresponding Itô-operator is continuous.

As a consequence we obtain a new characterization of semimartingales via Riemann sums and Riemann integrators.

The Bichteler-Dellacherie theorem was first published by P. Meyer and K. Bichteler in 1979 independently.

Zusammenfassung

In dieser Arbeit werden wir den Satz von Bichteler-Dellacherie auf eine neue Art beweisen. Der Satz sagt aus, dass ein Prozess ein guter Integrand ist genau dann wenn er ein Semimartingal ist, das heißt, die Summe eines lokalen Martingals und eines Prozesses mit endlicher Variation. Ein Prozess ist ein guter Integrand wenn der zugehörige Itô-Operator stetig ist.

Als Korollar werden wir zeigen, dass Semimartingale mittels Riemann-Summen und Riemann-Integranden charakterisiert werden können.

Der Satz von Bichteler-Dellacherie wurde 1979 unabhängig von P.Meyer und K. Bichteler das erste Mal veröffentlicht.

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