MASTERARBEIT

Classification of lattice walks in the quarter plane

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Abstract Deutsch

Diese Masterarbeit handelt von Gitterwegen. Gitterwege sind gerichtete Polygonzüge mit ganzzahligen Eckpunkten, wobei nur eine fixe Menge an Schritten zugelassen ist. Bei uns werden dies meist eine Teilmenge der sogenannten einfachen Schritte, die den Himmelsrichtungen N, NO, O, SO, S, SW, W und NW entsprechen, sein. Oft stellt man auch Einschränkungen an das Gebiet, das der Weg nicht verlassen darf, wie etwa die Halbebene, die Viertelebene oder den Keil zwischen einer Diagonale und der x-Achse. Wir werden uns hier überwiegend auf die algebraischen Eigenschaften der erzeugenden Funktionen eines Gitterwegs konzentrieren. Da diese bei der ganzen Ebene und der Halbebene einfach zu klassifizieren sind, betrachten wir hier meistens Wege mit einfachen Schritten in der Viertelebene, die erst unlängst vollständig klassifiziert wurden. Fr viele Schrittmengen arbeitet diese Klassifikation mit einer Korrespondenz zwischen der Schrittmenge und einer Gruppe. Nur bei den so genannten Gessel-Wegen schlägt diese Methode fehl. In den letzten Jahren entstanden allerdings drei unterschiedliche Beweise fr die Algebraizität der Gessel-Wege, die wir etwas genauer beleuchten werden. Zu guter Letzt werde ich auch noch einen Ausblick über Gitterwege mit anderen Schritten oder in anderen Regionen sowie über Gitterwege in drei Dimensionen geben.

Abstract English

This master-thesis is about lattice walks. Lattice walks are directed polygonal chains with integer edges, where only a fixed set of steps is allowed. In our case this will be mostly a subset of the so called simple steps, which correspond to the directions N, NE, E, SE, S, SW, W and NW. Often we also impose some conditions on the region which the walk is not allowed to leave, for example the half-plane, the quarter-plane or a wedge between a diagonal and the x-axis. We will focus mostly on the algebraic properties of the generating function of a walk. Since they are rather simple for walks in the plane and halfplane, we will consider walks with small steps in the quarter plane, which have recently been classified completly. For most step sets this classification works with a correspondence between the step set and a group. Only for the so called Gessel walks this method fails. In recent years three different proofs for the algebraicity of the Gessel walks came up. We will have a closer look at these proofs. Finally I will also give an overwiev over lattice walks with other step sets or in other regions as well as lattice walks in three dimensions.

Contents

1 Introduction

A lattice path is in all generality a directed polygonal line in the discrete Cartesian plane $\mathbb{Z} \times \mathbb{Z}$. Usually, from each point, there is a finite set of movements we are allowed to take. Often we will also make assumptions on the region which the walk is not allowed to leave. In our setting, this will be mostly the quarter plane.
Lattice walks appear in many fields of mathematics and computer science. They are a classical topic in combinatorics. Many combinatorical objects (for example trees, permutations, maps, Young tableaux, ...) can be encoded by a lattice walk and because of that lattice walks enumeration has many applications.
There is also a close relation between lattice walks and formal languages. A famous example are correctly nested strings of opening and closing parentheses. We require to have the same number of opening and closing parentheses and that in each substring the number of opening parentheses is not less than the number of closing parentheses (this condition ensures that there no closing parenthesis appears before its corresponding opening parenthesis appeared). We can encode opening parentheses with a NE-step and a closing parenthesis with a SE-step. Hence, correctly nested strings of paretheses correspond to walks with step set {NE, SE} that start in $(0, 0)$, end on the x-axis and never cross the x-axis. This type of lattice paths is called *Dyck-paths* (named after the German mathematician Walther von Dyck). Dyck-paths also appear in many other settings, for example, binary trees can be encoded as Dyck-paths as well.
Also in probability theory and statistics lattice paths will appear. Lattice paths describe the evolution of sums of independent discrete random variables, for example the gain in coin-tossing games. In queueing models and birth-death-processes lattice paths find their applications, too. Also, Brownian motions can be described by lattice walks.
In the following, we will focus mostly on the algebraic properties of the generating function assigned to a lattice walk in the quarter plane. We will see that this generating function is always algebraic for walks in the entire plane or walks in the half-plane, while generating functions of walks in the quarter plane tend to have a more complicated behaviour.

In recent years the generating functions of lattice walks with small steps in the quarter plane have been completely classified. In this master thesis I am going to give a summary of this classification.
In the next chapter I will introduce some basic notations and concepts. The third chapter will be devoted to classifiying lattice walks with small steps in the quarter plane. We will encounter a correspondence between walks and groups that turns out to be very helpful for this classification. There are 79 non-equivalent, non-simple walks, among them 23 with a finite group and 56 with an infinite group. It turns out that those with a finite group have a D-finite or even algebraic generating function, whereas the ones with an infinite group have a generating function that is not D-finite. The fourth chapter is all about Gessel walks. They are the only model where the previous techniques turned out to be fruitless. We will see three proofs for the fact that the generating function of Gessel walks is algebraic, which are very different in their nature. The fifth chapter is a tabularly overview, whereas in the sixth chapter we are going to discuss some new approaches and techniques that may turn out to be useful for proving algebraicity or D-finiteness results of generating functions. The seventh and last chapter will be some kind of excursion to generalizations. After the classification of lattice paths in the quarter plane has been completed there are already some efforts made to generalize there results to three-dimensional walks in the octant, but the sheer number of walks makes this task rather difficult. I will also briefly

discuss other regions of interest, for example walks under a diagonal or walks in a stripe.

2 Preliminaries

There are many equivalent ways to write down a step. Instead of writing the direction of a step we will usually write its coordinates or use x's and y's to describe it. For example we write $(1, -1)$ or $x\overline{y}$ for a SE step. Here and in the following $\overline{y}$ stands for $\frac{1}{y}$.
For a walk with fixed step set S we denote the number of walks that end in the point (i, j) after n steps with $f(i, j, n)$. The expression

$$F(x, y, t) = \sum_{i,j,n \geq 0} x^i y^j t^n$$

denotes the generating function of this walk.

We use bold letters or numbers to denote vectors in $\mathbb{R}^d$, for example $\mathbf{x} = (x_1, \ldots, x_d)$ or $\mathbf{0} = (0, \ldots, 0)$. Similarly we will use this multi-index-notation to denote order relations $\mathbf{u} \leq \mathbf{v}$ if $u_i \leq v_i$ for all $i = 1, \ldots, d$, monomials $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \ldots x_d^{a_d}$, scalar products $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_d v_d$ or other concepts.

Definition 2.1. *We call a power series $f(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]$*

- *rational, if it can be written as the quotient of two polynomials, i.e. there are $P(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$, $Q(\mathbf{x}) \neq 0$ with $f(\mathbf{x}) = \frac{P(\mathbf{x})}{Q(\mathbf{x})}$.*
- *algebraic, if it fulfills a nontrivial polynomial equation, i.e. there is a polynomial $P(x_1, \ldots, x_d, x_{d+1}) \in \mathbb{K}[x_1, \ldots, x_d, x_{d+1}], P \neq 0$ such that $P(x_1, \ldots, x_d, f(\mathbf{x})) = 0$. A power series that is not algebraic is called transcendental.*
- *D-finite, if for all $i \leq d$ the series f satisfies a nontrivial linear differential equation in x_i with coefficients in $\mathbb{K}[x_1, \ldots, x_d]$. Said differently, the $K(x_1, \ldots, x_d)$- vector space spanned by the partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_d$ is finite dimensional. Series that are not D-finite are called hypertranscendental.*

It is clear that each rational function is algebraic and each algebraic function is D-finite.

2.1 Walks in the plane

Walks with step set $S \subseteq \{$N, NE, E, SE, S, SW, W, NW$\}$ in the entire plane are an easy case to consider. Their generation function is always rational.
Proof: Let $f(i, j, n)$ be the number of walks starting in $(0, 0)$ and ending in (i, j) after n steps. We have the recursion

$$f(i, j, n) = \sum_{\alpha,\beta} c_{\alpha,\beta} f(i + \alpha, j + \beta, n - 1)$$

for all $n > 0$ and the initial conditions $f(i, j, 0) = 0$ for $(i, j) \neq (0, 0)$ and $f(0, 0, 0) = 1$. The parameters α and β are in $\{-1, 0, 1\}$ and the constants $c_{\alpha,\beta}$ are integers and depend on the step set. Multiplying with $x^i y^j z^n$ and summation over all values of i, j and n gives us

$$F(x, y, z) = \sum_{i,j,n \geq 0} f(x, y, z) x^i y^j z^n = \sum_{i,j,n \geq 0} \sum_{\alpha,\beta} c_{\alpha,\beta} f(i + \alpha, j + \beta, n - 1) x^i y^j z^n.$$

We can rewrite the right hand side as

$$\sum_{\alpha,\beta} c_{\alpha,\beta} x^{-\alpha} y^{-\beta} z \sum_{i,j,n\geq 0} x^i y^j z^n + P(x,y,z)$$

where P is a polynomial in x, y and z which occurs because of some corrections made such that all sums start at $(0,0,0)$. We can rewrite the above equation as

$$F(x,y,z)(xy - R(x,y,z)) = xyP(x,y,z)$$

where

$$R(x,y,z) = xy \sum_{\alpha,\beta} c_{\alpha,\beta} x^{-\alpha} y^{-\beta} z$$

is also a polynomial in x, y and z. Hence, our generating function is rational. $\square$

Remark: The same holds for walks with other integer step sets or in higher dimensions for walks in the entire hypercubic lattice $\mathbb{Z}^d$. Since we have no restriction on the region in which the walk should lie in, walks in the entire plane are a rather easy thing to consider. The situation will become more delicate if we restrict the walks to the half-plane (which always have a algebraic generating function) or to the quarter plane where the two constraints on the walk will give rise to more difficult and interesting problems. For walks in the quarter plane the generating function will show a more complicated behavior. There are step sets for which the generation function will not be algebraic, there are even step sets where it is not even D-finite (for example, quarter plane walks with step set {NE, NW, SE} have a non-D-finite generating function, see [26]).

2.2 Recurrences with constant coefficients

We are going to study a d-dimensional sequence $(f_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ in the field $\mathbb{K}$ which is defined by the linear recurrence

$$f_{\mathbf{n}} = \begin{cases} \varphi(\mathbf{n}) & \text{for } \mathbf{n} \in \mathbb{N}^d \setminus (\mathbf{s} + \mathbb{N}^d) \\ \sum_{\mathbf{t}\in H} c_{\mathbf{t}} f_{\mathbf{n}+\mathbf{t}} & \text{for } \mathbf{n} \in \mathbf{s} + \mathbb{N}^d \end{cases} \qquad (2.1).$$

The set $H \subseteq \mathbb{Z}^d$ is a finite set of shifts that occur in the recurrence and $\mathbf{s} \in \mathbb{N}^d$ is the starting point of the recurrence. We require $\mathbf{s} + H \subseteq \mathbb{N}^d$ for this recurrence to be well defined. The values of f in the shifted quadrant $\mathbf{s}+\mathbb{N}^d$ can be computed via the recurrence relation, while all other values need to be given by the initial conditions denoted by the function $\varphi : \mathbb{N}^d \setminus (\mathbf{s}+\mathbb{N}^d) \to \mathbb{K}$. In the following we are going to assume that the coefficients $c_{\mathbf{t}}$ are constants lying in $\mathbb{K}$.
To make sure that this way of defining a sequence gives us a unique solution we will need to pose a few further conditions which we are going to see in the following theorem

Definition 2.2. *A vector $\mathbf{w} \in \mathbb{R}^d$ with $\mathbb{Q}$-linearly independent components is called a weight vector. It induces a total order $<_{\mathbf{w}}$ on $\mathbb{Z}^d$ via $\mathbf{a} <_{\mathbf{w}} \mathbf{b}$ if $\mathbf{w}\cdot\mathbf{a} < \mathbf{w}\cdot\mathbf{b}$. This order can be generalized to the sets of monomials $\mathbf{x}^{\mathbf{n}} \in \mathbb{K}[[\mathbf{x}]]$ in the following sense: $\mathbf{x}^{\mathbf{a}} <_{\mathbf{w}} \mathbf{x}^{\mathbf{b}}$ if $\mathbf{w}\cdot\mathbf{a} < \mathbf{w}\cdot\mathbf{b}$.*

Theorem 2.3. *If there exists a weight vector $\mathbf{w} \in \mathbb{R}^d$ with positive components and $\mathbf{w}\cdot\mathbf{t} < 0$ for all shifts $\mathbf{t} \in H$, then there exists a unique solution for the recurrence (2.1).*

For the proof we will need a few lemmas and definitions.

Definition 2.4. *For $H \subseteq \mathbb{Z}^d$ and $\mathbf{p}, \mathbf{q} \in \mathbb{N}^d$ define $\mathbf{p} \prec_H \mathbf{q} \Leftrightarrow \mathbf{p} \in \mathbf{q} + \mathbf{h} \subseteq \mathbb{N}^d$ for some $\mathbf{h} \in H$. The transitive closure $\prec_H^+$ of $\prec_H$ in $\mathbb{N}^d$ is the dependency relation corresponding to H.*

Lemma 2.5. *Let $\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{N}^d$. Then $\mathbf{p} \prec_h^+ \mathbf{q}$ and $\mathbf{u} \prec_H^+ \mathbf{v}$ implies that $\mathbf{p} + \mathbf{u} \prec_H^+ \mathbf{q} + \mathbf{v}$.*

Proof. Clearly $\prec_H$ is invariant under translation: if $\mathbf{p} \prec_H \mathbf{q}$, then also $\mathbf{p} + \mathbf{r} \prec_H \mathbf{q} + \mathbf{r}$. Hence $\prec_H^+$ is invariant under translation, too, and we have $\mathbf{p} \prec_h^+ \mathbf{q} \Rightarrow \mathbf{p} + \mathbf{u} \prec_h^+ \mathbf{q} + \mathbf{u}$ and $\mathbf{u} \prec_H^+ \mathbf{v} \Rightarrow \mathbf{q} + \mathbf{u} \prec_H^+ \mathbf{q} + \mathbf{v}$. Since $\prec_H^+$ is transitive it follows that $\mathbf{p} + \mathbf{u} \prec_H^+ \mathbf{q} + \mathbf{v}$. □

Lemma 2.6. *Let $H \subseteq \mathbb{Z}^d$ be a finite set and let $\prec_H^+$ be the corresponding dependency relation. If there exists a vector $\mathbf{v} \in \mathbb{N}^d$, $\mathbf{v} > 0$ such that $\mathbf{v} \cdot \mathbf{h} < 0$ for all $\mathbf{h} \in H$ then the dependency relation $\prec_H^+$ can be extended to an ordering of $\mathbb{N}^d$ of order type ω.*

Note: If an order$(X, <_X)$ is of order type ω it is order isomorphic to $(\mathbb{N}, <)$, the set of natural numbers with the canonical ordering, i.e. there exists a $f : X \to \mathbb{N}$ bijective such that $x <_X y \Rightarrow f(x) < f(y)$ and $m < n \Rightarrow f^{-1}(m) <_X f^{-1}(n)$.

Proof. Let $\mathbf{v} \in \mathbb{N}^d, \mathbf{v} > 0$ be such that $\mathbf{v} \cdot \mathbf{h} < 0$ for all $\mathbf{h} \in H$ and let $<_L$ be any linear ordering of $\mathbb{N}^d$ (for example the lexicographic ordering). Now define a new linear ordering $<_{L_\mathbf{v}}$ of $\mathbb{N}^d$ by

$$\mathbf{p} <_{L_\mathbf{v}} \mathbf{q} \Leftrightarrow \mathbf{v} \cdot \mathbf{p} < \mathbf{v} \cdot \mathbf{q} \text{ or } (\mathbf{v} \cdot \mathbf{p} = \mathbf{v} \cdot \mathbf{q} \text{ and } \mathbf{p} <_L \mathbf{q}).$$

The equation $\mathbf{v} \cdot \mathbf{x} = k$ has only finitely many solutions $x \in \mathbf{N}^d$ for $k \in \mathbb{N}$ arbitrary, since this equation implies $0 \leq x_i \leq \frac{k}{v_i}$ for all i, which only has finitely many solutions $x_i \in \mathbb{N}$. Therefore $<_{L_\mathbf{v}}$ is an order of order type ω and since $\mathbf{p} \prec_H^+ \mathbf{q}$ implies $\mathbf{v} \cdot \mathbf{p} < \mathbf{v} \cdot \mathbf{q}$ the dependence relation $\prec_H^+$ can be embedded into $<_{L_\mathbf{v}}$. □

Remark: To be precisely, the two properties of the previous lemma are also equivalent. There are even some further conditions being equivalent to these two, see for example [14]

Proof of the Theorem. Write $H = \{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$. From the previous lemma we know that there exists a well ordering $<_H$ fulfilling the dependency relation $\prec_H^+$. Let $\mathbf{p} : \mathbb{N} \to \mathbb{N}^d$ be a bijection with the property $i < j \Leftrightarrow \mathbf{p}_i <_H \mathbf{p}_j$ (2.2).
The sequence $(a_\mathbf{n})$ fulfills the recurrence and its initial conditions if and only if there exists a function $f : \mathbb{N} \to \mathbb{K}$ defined by $f(i) = a_{\mathbf{p}_i}$ fulfills

$$f(i) = \begin{cases} \Phi(f \circ \mathbf{p}^{-1}(\mathbf{p}_i + \mathbf{h}_1), f \circ \mathbf{p}^{-1}(\mathbf{p}_i + \mathbf{h}_2), \ldots, f \circ \mathbf{p}^{-1}(\mathbf{p}_i + \mathbf{h}_k) & \text{for } \mathbf{p_i} \geq \mathbf{s} \\ \varphi(\mathbf{p}_i) & \text{for } \mathbf{p_i} \not\geq \mathbf{s} \end{cases} \tag{2.3}$$

where Φ is given by $a_\mathbf{n} = \Phi(a_{\mathbf{n}+\mathbf{h}_1}, a_{\mathbf{n}+\mathbf{h}_2}, \ldots, a_{\mathbf{n}+\mathbf{h}_k})$.
We are going to show by induction on i that (2.3) defines a unique series $f(i)$. The unique solution of(2.1) can then be determined by $a_\mathbf{n} = f \circ \mathbf{p}^{-1}(\mathbf{n})$.

Step 0: We will show that equation (2.3) determines $f(0)$ by showing $\mathbf{p}_0 \not\geq \mathbf{s}$ (in this case $f(0)$ is given by $\varphi(\mathbf{p}_0)$. If $\mathbf{p}_0 \geq \mathbf{s}$ then $\mathbf{p}_0 + H \subseteq \mathbb{N}^d$ (since $\mathbf{s} + H \subseteq \mathbb{N}^d$ by assumption). Said differently, $\mathbf{p}_0 + \mathbf{h}_j \prec_H \mathbf{p}_0$ and therefore $\mathbf{p}_0 + \mathbf{h}_j <_H \mathbf{p}_0$ for $1 \leq j \leq k$. But since H is nonempty and $0 \notin H$ this gives us a contradiction to property (2.2). Hence $\mathbf{p}_0 \not\geq \mathbf{s}$ and equation (2.3) determines $f(0) = \varphi(\mathbf{p}_0)$ uniquely.

Step i ($i > 0$): Suppose that $f(0), \ldots, f(i-1)$ are determined uniquely by (2.3). If $\mathbf{p}_i \not\geq \mathbf{s}$ then we have that $f(i) = \varphi(\mathbf{p}_i)$. Otherwise $\mathbf{p}_i + \mathbf{h}_j \prec_H \mathbf{p}_i$ holds for all $1 \leq j \leq k$ and hence $\mathbf{p}_i + \mathbf{h}_j <_H \mathbf{p}_i$. Then $\mathbf{p}^{-1}(\mathbf{p}_i + \mathbf{h}_j) < i$ holds according to (2.2), i.e. the values of $f \circ \mathbf{p}^{-1}(\mathbf{p}_i + \mathbf{h}_j)$ are already known and $f(i)$ is determined uniquely by the first equation of (2.3). $\square$

At first glance the condition of the theorem might seem rather restrictive, but in fact it is not restrictive at all. Let G be a finite nonempty subset of $\mathbb{Z}^d$ and consider the linear relation

$$\sum_{\mathbf{g} \in G} b_{\mathbf{g}} a_{\mathbf{n}+\mathbf{g}}$$

where $b_g \in K$ are nonzero constants. For any $\mathbf{g} \in G$ we can rewrite this as

$$a_{\mathbf{n}} = -\sum_{\mathbf{g}' \in G \setminus \{\mathbf{g}\}} \frac{b_{\mathbf{g}'}}{b_{\mathbf{g}}} a_{\mathbf{n}+\mathbf{g}-\mathbf{g}'} = \sum_{\mathbf{h} \in H_{\mathbf{g}}} c_{\mathbf{h}} a_{\mathbf{n}+\mathbf{h}}$$

where $H_{\mathbf{g}} = \{\mathbf{g}' - \mathbf{g} : \mathbf{g}' \in G \setminus \{\mathbf{g}\}\}$ and $c_{\mathbf{h}} = -\frac{b_{\mathbf{g}'}}{b_{\mathbf{g}}}$. Hence we have $|G|$ ways of rewriting the recurrence. The following proposition implies that there is always at least one way of rewriting the recurrence such that the condition of the theorem is fulfilled and we can compute $a_{\mathbf{n}}$ starting from suitable initial conditions.

Proposition 2.7. *Let $G \subseteq \mathbb{Z}^d$ a finite nonempty set. Then there exists an element $\mathbf{g} \in G$ such that for $H_{\mathbf{g}} := \{\mathbf{g}' - \mathbf{g} : \mathbf{g}' \in G, \mathbf{g}' \neq \mathbf{g}\}$ exists a vector $\mathbf{v} \in \mathbb{N}^d, \mathbf{v} > 0$ such that $\mathbf{v} \cdot \mathbf{h} < 0$ for all $\mathbf{h} \in H_{\mathbf{g}}$.*

Proof. Let $\mathbf{g}$ be the largest element in G with respect to the lexicographic ordering of $\mathbb{Z}^d$. Then for all $\mathbf{g}' \neq \mathbf{g}$ there exists an $i \in \{1, \ldots, d\}$ such that $g'_j = g_j$ for all $j < i$ and $g'_i < g_i$. Define M to be

$$M := \max_{1 \leq j \leq d, \mathbf{g}' \in G} |g'_j - g_j|$$

and let $\mathbf{v}$ be the vector $((1+M)^d, \ldots, (1+M)^2, (1+M))$. Consider now $\mathbf{g}' \neq \mathbf{g}$. Because of $g'_j = g_j$ for all $j < i$ and $g'_i < g_i$ we get the following estimate

$$\mathbf{v} \cdot (\mathbf{g}' - \mathbf{g}) = \sum_{j=i}^{d} v_j (g'_j - g_j) \leq -v_i + M \sum_{j=i+1}^{d} v_j = -(1+M) < 0.$$

$\square$

Remark: There can be more than one fitting choices of $\mathbf{g}$. Consider for example the recurrence

$$a_{m,n} = a_{m,n+1} + a_{m+1,n}.$$

Here the set $H = \{(1,0), (0,1)\}$ does not satisfy the condition of the theorem. But both equivalent formulations

$$a_{m,n} = a_{m,n-1} - a_{m+1,n-1}$$

and

$$a_{m,n} = a_{m-1,n} - a_{m-1,n+1}$$

satisfy the condition. Note that we need $\mathbf{s} \geq (0,1)$ in the former case and $\mathbf{s} \geq (1,0)$ in the latter. That means that we need different sets of initial values for these two recurrences: $(a_{m,0})_{m\geq 0}$ for the former and $(a_{0,n})_{n\geq 0}$ for the latter.

Definition 2.8. *The apex of the recurrence (2.1) is defined to be the vector* $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{Z}^d$ *with* $p_i = \max\{t_i : \mathbf{t} \in H \cup \{\mathbf{0}\}\}$.

Example: Let us consider the shift set $H = \{(-2,0), (0,-1), (-1,1)\}$. A recurrence with this step set has starting point $s = (2,1)$ and the apex of H is $\mathbf{p} = (0,1)$. We can find a weight vector that fulfills the condition of the theorem, for example we can take $\mathbf{w} = (\sqrt{2}, 1)$. Since all shift vectors $\mathbf{t} \in H$ lie below the line perpendicular to $\mathbf{w}$ passing through $\mathbf{s}$, the condition $\mathbf{w} \cdot \mathbf{t} < 0$ for all $\mathbf{t} \in H$ is fulfilled.

We can transform our recurrence relation

$$f_{\mathbf{n}} = \begin{cases} \varphi(\mathbf{n}) & \text{for } \mathbf{n} \in \mathbb{N}^d \setminus (\mathbf{s} + \mathbb{N}^d) \\ \sum_{\mathbf{t} \in H} c_{\mathbf{t}} f_{\mathbf{n}+\mathbf{t}} & \text{for } \mathbf{n} \in \mathbf{s} + \mathbb{N}^d \end{cases}.$$

into a functional equation satisfied by the generating function $F_{\mathbf{s}}(\mathbf{x})$. Multiplying the recurrence by $\mathbf{x}^{\mathbf{n}-\mathbf{s}}$ and summing over all $\mathbf{n} \geq \mathbf{s}$ gives us

$$\begin{aligned} F_{\mathbf{s}}(\mathbf{x}) &= \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \sum_{\mathbf{n} \geq \mathbf{s}} a_{\mathbf{n}+\mathbf{h}} \mathbf{x}^{\mathbf{n}-\mathbf{s}} = \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{-\mathbf{h}} \sum_{\mathbf{n} \geq \mathbf{s}+\mathbf{h}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}} \\ &= \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{-\mathbf{h}} (F_{\mathbf{s}}(\mathbf{x}) + P_{\mathbf{h}}(\mathbf{x}) - M_{\mathbf{h}}(\mathbf{x})) \quad (2.4) \end{aligned}$$

where

$$P_{\mathbf{h}}(\mathbf{x}) = \sum_{\mathbf{n} \ngeq \mathbf{s}, \mathbf{n} \geq \mathbf{s}+\mathbf{h}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}} = \sum_{\mathbf{n} \ngeq \mathbf{s}, \mathbf{n} \geq \mathbf{s}+\mathbf{h}} \varphi(\mathbf{n}) \mathbf{x}^{\mathbf{n}-\mathbf{s}}$$

and

$$M_{\mathbf{h}}(\mathbf{x}) = \sum_{\mathbf{n} \geq \mathbf{s}, \mathbf{n} \ngeq \mathbf{s}+\mathbf{h}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}}.$$

We can easily compute the entire generating function $F(\mathbf{x})$ from $F_{\mathbf{s}}(\mathbf{x})$ because of the following relation between those series:

$$F(\mathbf{x}) = \sum_{\mathbf{n} \geq \mathbf{0}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} = \mathbf{x}^{\mathbf{s}} \left(\sum_{\mathbf{n} \geq \mathbf{s}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}+\mathbf{s}} + \sum_{\mathbf{n} \ngeq \mathbf{s}, \mathbf{n} \geq \mathbf{0}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}+\mathbf{s}} \right) = \mathbf{x}^{s} (F_{\mathbf{s}}(\mathbf{x}) + P_{-\mathbf{s}}(\mathbf{x})).$$

Rewrite now (2.4) as

$$\left(1 - \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{-\mathbf{h}}\right) F_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{-\mathbf{h}} (P_{\mathbf{h}}(\mathbf{x}) - M_{\mathbf{h}}(\mathbf{x})).$$

To get rid of the denominators the notion of the apex $\mathbf{p}$ comes in handy. Multiplying the above equation with $\mathbf{x}^{\mathbf{p}}$ gives us

$$Q(\mathbf{x}) F_{\mathbf{s}}(\mathbf{x}) = K(\mathbf{x}) - U(\mathbf{x}) \quad (2.5)$$

where

$$Q(\mathbf{x}) = \mathbf{x}^{\mathbf{p}} - \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}}$$

$$K(\mathbf{x}) = \sum_{\mathbf{h}\in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}} P_{\mathbf{h}}(\mathbf{x})$$

$$U(\mathbf{x}) = \sum_{\mathbf{h}\in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}} M_{\mathbf{h}}(\mathbf{x})$$

and the series $P_{\mathbf{h}}$ and $M_{\mathbf{h}}$ are given by their definition above.
The definition of the apex implies that $Q(\mathbf{x})$ is a polynomial in $\mathbf{x}$, called the *characteristic polynomial* or the *kernel* of the recurrence. The coefficients of $Q(\mathbf{x})$ and $K(\mathbf{x})$ are given directly via the coefficients of the recursion or the initial conditions respectively. The coefficients of $U(\mathbf{x})$, however, can be computed via the recurrence (2.1) but are not explicitly known. Because of this, $K(\mathbf{x})$ is called the *known initial function* while $U(\mathbf{x})$ is called the *unknown initial function.*
The functional equation (2.5) is equivalent to our recurrence and hence defines $a_{\mathbf{n}}$ for all $\mathbf{n} \geq \mathbf{s}$ and therefore also $F_{\mathbf{s}}(\mathbf{x})$ completely. At first glance there seem to be not only one, but two unknown functions, $F_{\mathbf{s}}(\mathbf{x})$ and $U(\mathbf{x})$. But if $U(\mathbf{x})$ is known, $F_{\mathbf{s}}(\mathbf{x})$ can be computed via

$$F_{\mathbf{s}}(\mathbf{x}) = \frac{K(\mathbf{x}) - U(\mathbf{x})}{Q(\mathbf{x})}.$$

The rationality, algebraicity or D-finiteness of $F_{\mathbf{s}}$ is closely intertwined with the properties of K and U. We will exactify this in the next few theorems but first we need to introduce the definition of the section of a power series.

Definition 2.9. *Let $F(x_1, \ldots, x_d) = \sum_{\mathbf{n}\geq 0} a_{n_1,\ldots,n_d} \mathbf{x}^{\mathbf{n}}$ be a formal power series in d variables. A section of F is a formal power series obtained by fixing some of the indices in the coefficients of F.*

For example the series

$$\sum_{n_2,\ldots,n_d \geq 0} a_{26,n_2,\ldots,n_d} x_2^{n_2} \ldots x_d^{n_d}$$

or

$$\sum_{n_3,\ldots,n_{d-1} \geq 0} a_{1,1,n_3,\ldots,n_{d-1},1} x_3^{n_3} \ldots x_{d-1}^{n_{d-1}}$$

are sections of F. The series F itself and also coefficients of F like $a_{5,0,\ldots,0}$ are considered to be sections as well.

The terminology of sections of formal power series was introduced by Lipshitz. It is not difficult to prove that all sections of rational power series are again rational, and similarly for algebraic or D-finite series.

Proposition 2.10. *Let $F_{\mathbf{s}}(\mathbf{x})$ be the generating function of the unique solution of the recurrence (2.1). Then $F_{\mathbf{s}}$ is rational (respectively algebraic or D-finite) if and only if both its known and unknown initial function $K(x)$ and $U(x)$ are rational (respectively algebraic or D-finite).*

Proof. Assume that $K(x)$ and $U(x)$ are rational. Then the relation

$$F_{\mathbf{s}}(\mathbf{x}) = \frac{K(\mathbf{x}) - U(\mathbf{x})}{Q(\mathbf{x})}$$

implies that $F_{\mathbf{s}}(\mathbf{x})$ is rational, since $Q(x)$ is known to be a polynomial. The same holds for algebraic and D-finite series since the families of algebraic and D-finite series are closed under sums, products and the division by polynomials as well.
Conversely, we have that for any $\mathbf{h} \in H$ the series $M_{\mathbf{h}}$ defined as above by

$$M_{\mathbf{h}}(\mathbf{x}) = \sum_{\mathbf{n} \geq \mathbf{s}, \mathbf{n} \not\geq \mathbf{s}+\mathbf{h}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}}$$

are sections of $F_{\mathbf{s}}$ and hence rational since we assumed $F_{\mathbf{s}}$ to be rational. The series

$$U(\mathbf{x}) = \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}} M_{\mathbf{h}}(\mathbf{x})$$

is a finite linear combination of series M_h and therefore also rational. Finally, the relation $K(\mathbf{x}) = F_{\mathbf{s}}(\mathbf{x})Q(\mathbf{x}) + U(\mathbf{x})$ implies that $K(\mathbf{x})$ is rational, too. The same argument works for algebraic respectively D-finite series as well. □

Remark: The known initial function $K(x)$ can be expressed as a linear combination of sections of the full generating function $F(\mathbf{x})$ but unlike the series $U(\mathbf{x})$ it can not be expressed as a linear combination of sections of $F_{\mathbf{s}}(\mathbf{x})$.

If we know certain properties of the apex of the step set, we can get even more results.

Theorem 2.11. *Assume that the step set H has apex $\mathbf{p} = 0$. Then the generating function of the unique solution $F_{\mathbf{s}}(\mathbf{x})$ is rational if and only if the known initial function $K(\mathbf{x})$ is rational.*

Proof. If the apex is zero it follows that $\mathbf{h} \leq \mathbf{0}$ for all $\mathbf{h} \in H$. Hence $\mathbf{s}+\mathbf{h} \geq \mathbf{s}$ and $M_{\mathbf{h}} = 0$. Therefore $U(\mathbf{x}) = 0$ which gives us

$$F_{\mathbf{s}}(\mathbf{x}) = \frac{K(\mathbf{x})}{Q(\mathbf{x})}.$$

Since $Q(\mathbf{x})$ is a polynomial we get that $F_{\mathbf{s}}(\mathbf{x})$ is rational if and only if $K(\mathbf{x})$ is rational. □

We also get a similar result for algebraic functions.

Theorem 2.12. *Let $\mathbb{K} = \mathbb{C}$ and suppose that the apex of H has at most one positive coordinate. Then the generating function $F_{\mathbf{s}}(\mathbf{x})$ of the unique solution of the recurrence is algebraic if and only if the known initial function $K(\mathbf{x})$ is algebraic.*

The proof of this theorem works with the so called *kernel method*, a method that is used in various other settings. The main idea of the kernel method is to express one variable in a recurrence as a power series of other variables. Before we are going to prove this, we will consider an example that illustrates the main idea of the proof.

Example (Dyck paths): Consider walks with step set $S = \{(1,1),(1,-1)\}$ that start in the origin and do not touch the horizontal axis again once they have left the origin. For this walks we get the recursion

$$a_{m,n} = \begin{cases} a_{m-1,n-1} + a_{m-1,n+1} & \text{for } m,n \geq 1 \\ \delta_{(m,n),(0,0)} & \text{for } m = 0 \text{ or } n = 0 \end{cases}.$$

The starting point is $\mathbf{s} = (1,1)$ and the set of shifts is $H = \{(-1,-1),(-1,1)\}$ and has apex $\mathbf{p} = (0,1)$. We want to consider

$$F_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{h}\in H}\sum_{\mathbf{n}\geq\mathbf{s}} a_{\mathbf{n}+\mathbf{h}}\mathbf{x}^{\mathbf{n}-\mathbf{s}} = \sum_{\mathbf{h}\in H}\sum_{\mathbf{n}\geq(1,1)} a_{\mathbf{n}+\mathbf{h}}x^{m-1}y^{n-1}.$$

We have that

$$Q(x,y) = \mathbf{x}^{\mathbf{p}} - \sum_{\mathbf{h}\in H} c_{\mathbf{h}}\mathbf{x}^{\mathbf{p}-\mathbf{h}} = y - x - x^2y.$$

Using $P_{(-1,-1)}(x,y) = x^{-1}y^{-1}$ and $P_{(-1,1)}(x,y) = 0$ respectively $M_{(-1,-1)}(x,y) = 0$ and $M_{(-1,1)}(x,y) = \sum_{m\geq 0} a_{m,1}x^{m-1}$ to compute $K(x,y)$ or respectively $U(x,y)$ we get

$$K(\mathbf{x}) = \sum_{\mathbf{h}\in H} c_{\mathbf{h}}\mathbf{x}^{\mathbf{p}-\mathbf{h}}P_{\mathbf{h}}(\mathbf{x}) = xy^2 \cdot x^{-1}y^{-1} + 0 = y$$

and

$$U(\mathbf{x}) = \sum_{\mathbf{h}\in H} c_{\mathbf{h}}\mathbf{x}^{\mathbf{p}-\mathbf{h}}M_{\mathbf{h}}(\mathbf{x}) = x\sum_{m\geq 0} a_{m,1}x^{m-1} + 0 = \sum_{m\geq 0} a_{m,1}x^m.$$

Note that $U(x,y)$ only depends on x, hence we will write $U(x)$ for it. The relation

$$F_{\mathbf{s}}(\mathbf{x}) = \frac{K(\mathbf{x}) - U(\mathbf{x})}{Q(\mathbf{x})}$$

we derived in the previous section becomes here

$$(y - x - xy^2)F_{\mathbf{s}}(x,y) = y - U(x). \quad (2.6)$$

Let now $\xi(x)$ be the formal power series in x defined by $Q(x,\xi(x)) = 0$, i.e. defined by the equation

$$\xi(x) - x - x\xi(x)^2 = 0.$$

Solving this equation gives us

$$\xi_{1/2}(x) = \frac{1 \pm \sqrt{1-4x^2}}{2x}.$$

Considering these expressions at $x = 0$ we can exclude the solution with plus, hence

$$\xi(x) = \frac{1 - \sqrt{1-4x^2}}{2x}.$$

Inserting $\xi(x)$ instead of y in (2.6) we get

$$0 \cdot F_{\mathbf{s}}(x,y) = \xi(x) - U(x)$$

and hence

$$U(x) = \xi(x) = \frac{1 - \sqrt{1-4x^2}}{2x}.$$

This gives us that

$$F_{\mathbf{s}} = \left(y - \frac{1 - \sqrt{1-4x^2}}{2x}\right) \cdot \frac{1}{y - x - xy^2}$$

which is an algebraic power series in x and y.

Proof of the theorem. If $F_{\mathbf{s}}$ is algebraic, then also k is algebraic because of the previous proposition.
The other direction is a bit more difficult. If $\mathbf{p} = 0$ the proof works similarly to the proof of the previous theorem on rational series. Assume now that $\mathbf{p}$ has exactly one positive coordinate. Without loss of generality we can assume that $p_1 = \cdots = p_{d-1} = 0$ and $p_d > 0$. Then

$$\begin{aligned} U(\mathbf{x}) &= \sum_{\mathbf{h}\in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}} \sum_{\mathbf{n}\geq\mathbf{s},\mathbf{n}\not\geq\mathbf{s}+\mathbf{h}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}} \\ &= \sum_{\mathbf{h}\in H, h_d>0} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}} \sum_{n_d=s_d}^{s_d+h_d-1} \sum_{(n_1,\ldots,n_{d-1})\geq(s_1,\ldots,s_{d-1})} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}-\mathbf{s}}. \end{aligned}$$

From this we see that $U(\mathbf{x})$ is a polynomial in x_d of degree less or equal $p_d - 1$. The functional equation of the recurrence then becomes

$$U(\mathbf{x}) = K(\mathbf{x}) - Q(\mathbf{x})F_{\mathbf{s}}(\mathbf{x})$$

where

$$Q(\mathbf{x}) = x_d^{p_d} - \sum_{\mathbf{h}\in H} c_{\mathbf{h}} \mathbf{x}^{\mathbf{p}-\mathbf{h}}$$

is a polynomial with degree in x_d greater or equal p_d. We want to show that $Q(\mathbf{x})$ (considered as a polynomial in x_d) has at least p_d roots $\xi_i(x_1, \ldots, x_{d-1})$ (counted with multiplicities) such that

$$\xi_i(0, \ldots, 0) = 0 \quad (2.7).$$

If the above statement is true we can substitute x_d with $\xi_i(x_1, \ldots, x_{d-1})$ in the functional equation and obtailn, if ξ_i is a root of Q with multiplicity m, that

$$U(\xi_i) = K(\xi_i) \quad U'(\xi_i) = K'(\xi_i) \quad \ldots \quad U^{(m-1)}(\xi_i) = K^{(m-1)}(\xi_i)$$

where all derivatives are with respect to x_d. Condition (2.7) ensures that above equations hold as equations for convergent power series in a neighbourhood of the origin, which is needed for this substitution to be legitimately.
The p_d roots of Q deliver p_d equations for the polynomial U which has degree $\leq p_d - 1$. Hence we can reconstruct U via Hermite interpolation (or via Lagrange interpolation if all zeros of Q have multiplicity 1). Because the ξ_i are zeros of a polynomial and hence algebraic in $x_1, \ldots, x_{d-1}$ we get that $U(\mathbf{x})$ and hence $F_{\mathbf{s}}(\mathbf{x})$ is algebraic if $K(\mathbf{x})$ is.
It remains to show the existence of the roots ξ_i. Consider

$$Q(0, \ldots, 0, x_d) = x_d^{p_d} - \sum_{\mathbf{h}\in H, h_1=\cdots=h_{d-1}=0} c_{\mathbf{h}} x_d^{p_d-h_d}.$$

Because $\{x \in \mathbb{R}^d : x \geq 0\} \cap H = \emptyset$ we have that $h_d < 0$ for all $\mathbf{h} \in H$ such that $h_1 = \cdots = h_{d-1} = 0$. Hence $p_d - h - d > p_d$ for all such $\mathbf{h}$. This implies that $x_d = 0$ is a root of $Q(0, \ldots, 0, x_d)$ with multiplicity p_d. Hence at least p_d roots of Q satisfy the condition (2.7). □

Corollary 2.13. *All walks in the quarter plane or in the half plane $y \geq 0$ with steps (i, j) such that $i \geq 0$ for all i have an algebraic generating function.*

Proof. For such a step set, the second entry of the apex is 0, hence the apex has at most one positive entry and we can apply the previous theorem. □

Corollary 2.14. *Walks in the half line* $\mathbb{N}$ *with integer steps have an algebraic generating function.*

Proof. Let $n_1, \ldots, n_k$ with $n_i \in \mathbb{Z}$ for $i = 1, \ldots, k$ the step set of our walk. We add an artifical second component to our steps such that we have a new step set $S = \{(n_1, 1) \ldots, (n_k, 1)\}$. This new walk is a walk in the quarter plane since the y component will always be positive by definition of the new step set and we are not allowed to leave the half plane $x \geq 0$. Projecting down to the x-axis gives us our original walk. The new step set has apex $(p_1, 0)$ where p_1 is either a positive integer or zero. Hence we have an apex with at most one positive component and can apply the previous result. □

Remark: The same result holds for weighted walks in $\mathbb{N}$. (without proof)

Corollary 2.15. *Walks in the half plane with integer steps always have an algebraic function.*

Proof (sketch): Consider walks in the half plane $y \geq 0$. They can be viewed as weighted walks (in y) with weights in $\mathbb{Z}(x)$. As seen above, they always have an algebraic generating function. □

3 Walks in the quarter plane

3.1 The number of non-equivalent non-trivial walks

Considering walks with steps N, NE, E, SE, S, SW, W and NW in the quarterplane, there are only finitely many models to study, to be precisely 2^8. Some of them are trivial, for example $S = \emptyset$ or $S = \{\mathrm{W}\}$. Among the remaining cases it can happen that one of the constraints given by the quarterplane is automatically fulfilled (for example, walks with step set $S = \{\mathrm{N}, \mathrm{NE}, \mathrm{SE}, \mathrm{S}\}$ will always be on the nonnegative side of the y-axis, we only have to consider if they are above the x-axis). These walks are equivalent to a walk in the halfplane and we already know that walks in the halfplane always have a algebraic generating function.
We will show that out of these $2^8 = 256$ models there are only 138 true two-constrained walks, which are thus worth considering in more detail. Among these remaining problems some are equivalent up to an x/y symmetry. We will show that there are 79 essentially different true two-constrained walks.

A step (i, j) is defined to be x-positive if $i > 0$. We define x-negative, y-positive and y-negative steps analogous. There are some reasons that reduce a walk in the quarter plane with step set S to a simpler problem: If S contains no x-positive steps, we can ignore the x-negative steps in S, since they will not be used in a quarter planes walk. Hence, the problem is reduced to counting walks with vertical steps on a vertical half-line. The solution of this problem is always algebraic. If $S = \emptyset$, $S = \{y\}$ or $S = \{\overline{y}\}$ it is even rational. If S contains no y-positive step, the problem also has an algebraic solution because of symmetry.
If S contains no x-negative step all walks with step set S starting in $(0, 0)$ lie in the halfplane $i \geq 0$. We only have to check if the y-coordinates of the points of our walks are nonnegative, too. Hence our walk with step set S in the quarterplane is equivalent the walk with step set S in the halfplane $j \geq 0$. Hence its generating function is algebraic. Due to symmetry the same holds for walks with no y-negative step.

Therefore we can restrict ourselves to sets S containing x-positive, x-negative, y-positive and y-negative steps. We can count their number by an inclusion-exclusion argument; there are 161 such sets. More precisely, the polynomial that counts them is

$$\begin{aligned} P_1(z) &= (1+z)^8 - 4(1+z)^5 + 2(1+z)^2 + 4(1+z)^3 - 4(1+z) + 1 \\ &= z^8 + 8z^7 + 28z^6 + 52z^5 + 50z^4 + 20z^3 + 2z^2. \end{aligned}$$

One of the 4 terms $(1+z)^5$ counts the sets with no x-positive steps, one term $(1+z)^3$ those with no x-positive and no y-positive steps, and the terms $(1+z)^2$ count the steps with no x-positive nor x-negative or no y-positive nor y-negative steps respectively, and so on. The step sets we have discarded are either trivial or can be solved using the kernel method.
Among the remaining 161 sets, some do not contain any steps with both coordinates nonnegative. For such sets the only walk in the quarter plane is the empty walk. Such sets are subsets of $\{\overline{x}, \overline{y}, \overline{xy}, x\overline{y}, \overline{x}y\}$. But since we assumed that S contains a x-positive and a y-positive step, the steps $x\overline{y}$ and $\overline{x}y$ have to belong to S. The other three steps may or may not belong to S, giving us $2^3 = 8$ step sets we can exclude. So we only have 153 sets to consider and our new generating polynomial is

$$P_2(z) = P_1(z) - z^2(1+z)^3 = z^8 + 8z^7 + 28z^6 + 51z^5 + 47z^4 + 17z^3 + z^2.$$

Another reason that simplifies our problem is when one constraint of the walk to be in the quarter plane implies the other. If all walks with step set S that end in a point with nonnegative x-coordinate automatically end in a point with nonnegative y-coordinate, we say that the x-condition implies the y-condition. In this case, the steps $\overline{y}$ and $x\overline{y}$ do not belong to S. Since we assumed that there is a y-negative step in S. we have $\overline{xy} \in S$. But then x can not belong to S because the some walks with a nonnegative final abscissa would have a negative ordinate, for example a x-step followed by an $\overline{xy}$ step. Thus we have that $S \subseteq \{\overline{x}, y, xy, \overline{x}y, \overline{xy}\}$ and S has to contain $\overline{xy}$ and xy, too, because we also need at least one x-positive step. Observe that those sets lie above the first diagonal. Conversely, for any such super-diagonal set, the x-condition forces the y-condition. The polynomial that counts the super-diagonal sets is $z^2(1+z)^3$. Because of symmetry we also need not consider sets where the y-condition implies the x-condition. A similar argument shows that these are the sub-diagonal sets. Hence, the polynomial counting the number of non-simple walks is

$$P_3(z) = P_2(z) - 2z^2(1 + z^3) = z^8 + 8z^7 + 28z^6 + 49z^5 + 41z^4 + 11z^3,$$

or said differently, there are 138 walks with non-simple step set.

Symmetries

Among the remaining walks there are some walks which are equivalent up to symmetry. Among the eight symmetries of the square, only the x/y symmetry leaves the quarter plane fixed. Hence two step sets obtained from each other by this symmetry can be considered to be equivalent. So we want to know how many of the 138 sets S have this x/y symmetry. We repeat the previous counting argument but count only walks with this symmetry and obtain the polynomials

$$P_1^{\text{sym}} = (1 + z)^2(1 + z^2)^3 - 2(1 + z)(1 + z^2) + 1,$$

$$P_2^{\text{sym}} = P_1^{\text{sym}} - z^2(1 + z)(1 + z^2),$$

$$P_3^{\text{sym}} = P_2^{\text{sym}} - z^2 = z^8 + 2z^7 + 4z^6 + 5z^5 + 5z^4 + 3z^3.$$

Here, for instance P_1^{sym} counts the number of walks that contain x-positive, x-negative, y-positive and y-negative steps and have a step set with a x/y-symmetry. Hence, the polynomial that counts the number of non-equivalent models that are neither trivial, nor equivalent to a 1-constraint-problem is

$$\frac{1}{2}(P_3 + P_3^{\text{sym}}) = z^8 + 5z^7 + 16z^6 + 27z^5 + 23z^4 + 7z^3.$$

Evaluating at $z = 1$ gives us that there are 79 such models.

3.2 The group of a walk

Let S be a step set that contains x-positive, x-negative, y-positive and y-negative steps (this is the case for the 79 sets we want to consider). Let $S(x, y)$ be the generating polynomial of S, i.e.

$$S(x, y) = \sum_{(i,j)\in S} x^i y^j.$$

This is a Laurent polynomial in x and y. We can write $S(x,y)$ as

$$S(x,y) = A_{-1}(x)\overline{y} + A_0(x) + A_1(x)y = B_{-1}(y)\overline{x} + B_0(y) + B_1(y)x$$

where $\overline{x}$ denotes $\frac{1}{x}$. Since we assumed that S contains x-positive, x-negative, y-positive and y-negative steps we get that $A_{-1} \neq 0$, $A_1 \neq 0$, $B_{-1} \neq 0$ and $B_1 \neq 0$. The Laurent polynomial $S(x,y)$ remains invariant under the transformations

$$\Phi : (x,y) \mapsto \left(\overline{x}\frac{B_{-1}(y)}{B_1(y)}, y\right)$$

and

$$\Psi : (x,y) \mapsto \left(x, \overline{y}\frac{A_{-1}(x)}{A_1(x)}\right).$$

An easy computation shows that Φ and Ψ are involutions and hence birational transformations.
Denote by $G(S)$ the group generated by Φ and Ψ (we will only write G instead $G(S)$ if it is clear which step set we are talking about). We have that $G \cong D_n$ where D_n is the dihedral group with $2n$ elements, where $n \in \mathbb{N} \cup \{\infty\}$. For any $g \in G$ holds $S(x,y) = S(g(x,y))$. The sign of an element $g \in G$ is defined to be 1 if g can be written as an even number of generators and -1 otherwise.

Remark: It is possible that there exist rational transformations of (x,y) that leave $S(x,y)$ unchanged but are not elements of G. Consider for example the step set $S = \{\text{N, E, S, W}\}$. Then the transformation $(x,y) \mapsto (y,x)$ leaves $S(x,y)$ unchanged, but the orbit of (x,y) under G is

$$(x,y) \to (\overline{x},y) \to (\overline{x},\overline{y}) \to (\overline{x},\overline{y}) \to (x,y).$$

Examples: 1) Let S be a step set that remains invariant under the reflection along the vertical line. Then we have that $S(x,y) = S(\overline{x},y)$ which means that $B_1(y) = B_{-1}(y)$ and $A_i(x) = A_i(\overline{x})$ for $i = 1, 0, -1$. The orbit of (x,y) under the action of G then is

$$(x,y) \xrightarrow{\Phi} (\overline{x},y) \xrightarrow{\Psi} (\overline{x},R(x)y) \xrightarrow{\Phi} (x,R(x)\overline{y}) \xrightarrow{\Psi} (x,y)$$

where $R(x) = \frac{A_{-1}(x)}{A_1(x)}$. Hence G is of order 4.
2) Let $S = \{\text{N, SE, W}\} = \{\overline{x}, y, x\overline{y}\}$. Then $S(x,y) = \overline{x} + y + x\overline{y}$ and $A_{-1}(x) = x, A_1(x) = 1, B_{-1}(y) = 1$ and $B_1(y) = \overline{y}$. We get that $\Phi(x,y) = (\overline{x}y, y)$ and $\Psi(x,y) = (x, x\overline{y})$. The transformations Φ and Ψ generate a group of order 6, which we see from considering the orbit of (x,y):

$$(x,y) \xrightarrow{\Phi} (\overline{x}y,y) \xrightarrow{\Psi} (\overline{x}y,\overline{x}) \xrightarrow{\Phi} (\overline{y},\overline{x}) \xrightarrow{\Psi} (\overline{y},x\overline{y})\overset{\Phi}{}(x,x\overline{y}) \xrightarrow{\Psi} (x,y).$$

3) Now let $S = \{\text{NE, S, W}\} = \{\overline{x}, \overline{y}, xy\}$. This is the same step set as before after a rotation about 90 degrees. We have that $A_{-1}(x) = 1, A_1(x) = x, B_{-1}(y) = 1$ and $B_1(y) = y$. This gives us that $\Phi(x,y) = (\overline{xy}, y)$ and $\Psi(x,y) = (x, \overline{xy})$. The group generated by Φ and Ψ has again order 6 and the orbit of (x,y) is

$$(x,y) \xrightarrow{\Phi} (\overline{xy},y) \xrightarrow{\Psi} (\overline{xy},x) \xrightarrow{\Phi} (y,x) \xrightarrow{\Psi} (y,\overline{xy}) \xrightarrow{\Phi} (x,\overline{xy}) \xrightarrow{\Psi} (x,y).$$

That the groups of example 2 and 3 are isomorphic is not a coincidence, as the following Lemma shows.

Lemma 3.1. *Let S and $\tilde{S}$ be two step sets that differ by one of the eight symmetries of the square. Then $G(S) \cong G(\tilde{S})$.*

Proof. The symmetries of the square are generated by two reflections: the reflection Δ along the first diagonal and the reflection V along the vertical line. Hence it suffices to prove the claim for $\tilde{S} = \Delta(S)$ and $\tilde{S} = V(S)$. Denote with Φ and Ψ the transformations corresponding to S and with $\tilde{\Phi}$ and $\tilde{\Psi}$ the transformations corresponding to $\tilde{S}$.
First consider $\tilde{S} = \Delta(S)$. Then we have that $\tilde{A}_i(x) = B_i(y)$ and $\tilde{B}_i(y) = A_i(x)$ for $I = 1, 0, -1$. Define δ as $\delta : (x, y) \mapsto (y, x)$.The transformation δ is an involution and an easy computation shows that $\tilde{\Phi} = \delta \circ \Phi \circ \delta$ and $\tilde{\Psi} = \delta \circ \Psi \circ \delta$. Hence $G(\tilde{S})$ and $G(S)$ are conjugated by δ and therefore isomorphic.
Now let $\tilde{S} = V(S)$. In this case $\tilde{A}_i(x) = A_i(x)$ and $\tilde{B}_i(y) = B_{-i}(y)$ for $i = 1, 0, -1$. Let $v : (x, y) \mapsto (\overline{x}, y)$. Then we get that $\tilde{\Phi} = v \circ \Phi \circ v$ and $\tilde{\Psi} = v \circ \Psi \circ v$. Because $G(\tilde{S})$ and $G(S)$ are conjugated by v we get that they are isomorphic. □

This group of a walk also appears in Fayolle's, Iasnogorodski's and Malyshev's book [16]. The construction of the group that was given there might seem a bit more complicated at first glance, but is also more insightful, since it shows that this group originates from a Galois group.

Define

$$\tilde{K}(x, y) = xyK(x, y) = xy \sum_{(i,j)\in S} x^i y^j$$

or, if we are dealing with a weighted walk

$$\tilde{K}(x, y) = xy \sum_{(i,j)\in S} p_{i,j} x^i y^j$$

where $p_{i,j}$ is the probability of a step in direction (i, j). If the walk contains x-positive and x-negative as well as y-positive and y-negative steps this polynomial will be quadratic and irreducible in y over the field $\mathbb{C}(x)$ of rational functions in x. Denote by $\mathbb{C}(x)[y(x)]$ the vector space over $\mathbb{C}(x)$ generated by 1 and $y(x)$, where $y(x)$ is a zero of $\tilde{K}$. This vector space is a field and a field extension of $\mathbb{C}(x)$ of degree 2. Each element in $\mathbb{C}(x)[y(x)]$ can be written uniquely as $u(x) + v(x)y(x)$, where $u(x), v(x) \in \mathbb{C}(x)$. Identify $\mathbb{C}(x)[y(x)]$ with $\mathbb{C}(x)[T]/\tilde{K}(x, T)$. Analogously we define $\mathbb{C}(y)[x(y)]$ and $\mathbb{C}(y)[T]/\tilde{K}(T, y)$. Denote by $\mathbb{C}(x, y)$ the field of rational functions in (x, y) over $\mathbb{C}$. Because $\tilde{K}$ is irreducible is the quotient ring of $\mathbb{C}(x, y)$ a field, called $\mathbb{C}_{\tilde{K}}(x, y)$.

Proposition 3.2. *The fields $\mathbb{C}(x)[T]/\tilde{K}(x, T)$ and $\mathbb{C}(y)[T]/\tilde{K}(T, y)$ are isomorphic to $\mathbb{C}_{\tilde{K}}(x, y)$.*

Proof. For all $p \in \mathbb{C}_{\tilde{K}}(x, y)$ exists a unique pair $u(x), V(x) \in \mathbb{C}(x)$ such that $p \equiv u(x) + v(x)y(x) \mod Q$. The isomorphism is then given by

$$i_x : \{u(x) + v(x)T\} \to \{u(x) + v(x)y(x)\}$$

where the brackets denote the corresponding equivalence class.
Analogously we define $i_y : \{u(y) + v(y)T\} \to \{u(y) + v(y)x(y)\}$. □

We have that

$$\mathbb{C}_{\tilde{K}}(x, y) \cong \mathbb{C}(x)[T]/\tilde{K}(x, T) \cong \mathbb{C}(y)[T]/\tilde{K}(T, y) \cong \mathbb{C}(x)[y(x)] \cong \mathbb{C}(y)[x(y)].$$

The Galois group of $\mathbb{C}(x)[y(x)]$ (respectively $\mathbb{C}(y)[x(y)]$) is cyclic of order 2. Its generator is Ψ (respectively Φ) such that

$$\Psi(u(x)) = u(x)\forall u(x) \in \mathbb{C}(x) \quad \Psi(y(x)) = \frac{a_0(x)}{y(x)a_2(x)} = -\frac{a_1(x)}{a_2(x)} - y(x)$$
$$\Phi(w(y)) = w(y)\forall w(y) \in \mathbb{C}(y) \quad \Phi(x(y)) = \frac{b_0(y)}{x(y)b_2(y)} = -\frac{b_1(y)}{b_2(y)} - x(y)$$

where

$$\tilde{K}(x,y) = a_2(x)y^2 + a_1(x)y + a_0(x) = b_2(y)x^2 + b_1(y)x + b_0(y).$$

There are two automorphisms $\tilde{\Psi}, \tilde{\Phi}$ of $\mathbb{C}_{\tilde{K}}(x,y)$ such that $\tilde{\Psi} = i_x \circ \Psi \circ i_x^{-1}$ and $\tilde{\Phi} = i_y \circ \Phi \circ i_y^{-1}$. We have that

$$\tilde{\Psi}(f(x,y)) = f(x, \Psi(y)) \bmod Q \quad \forall f \in \mathbb{C}(x,y)$$
$$\tilde{\Phi}(g(x,y)) = g(\Phi(x), y) \bmod Q \quad \forall g \in \mathbb{C}(x,y).$$

Definition 3.3. *The group G of a walk is defined to be the group of automorphisms of $\mathbb{C}_{\tilde{K}}(x,y)$ generated by $\tilde{\Psi}$ and $\tilde{\Phi}$. It only depends on the step set (and in the case of a weighted walk also on the transition probabilities $p_{i,j}$).*

We will write Ψ and Φ instead of $\tilde{\Phi}$ and $\tilde{\Psi}$ since usually the context will tell us, where Ψ and Φ live.

There are some explicit conditions for the walk to have a group of certain order.

Lemma 3.4. *The group of a walk is of order 4 if and only if*

$$\det \begin{pmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix} = 0. \quad (3.1)$$

Proof. Define $\delta = \Phi\Psi$. Since the group is of order 4 we have that $\delta^2 = \mathrm{Id}$ which is equivalent to $\Phi\Psi = \Psi\Phi$. This is equivalent to

$$\Psi\Phi(x) = \Phi(x) \quad \text{and} \quad \Phi\Psi(y) = \Psi(y)$$

since $\Psi(x) =$ and $\Phi(y) = y$. Thus $\Phi(x)$ is left-invariant under Ψ (respectively $\Psi(y)$ under Φ). Hence $\Phi(x) \in \mathbb{C}(x)$ and $\Psi(y) \in \mathbb{C}(y)$.
Thus Ψ and Φ are conformal automorphisms on $\mathbb{C}_x$ and $\mathbb{C}_y$ and thus fractional linear transforms of the form

$$\Phi(x) = \frac{rx + s}{tx - r} \quad \Psi(y) = \frac{\tilde{r}y + \tilde{s}}{\tilde{t}y - \tilde{r}}$$

where all coefficients lie in $\mathbb{C}$. We have that

$$\begin{aligned} \Phi(x) = \frac{rx+s}{tx-r} &\Leftrightarrow tx\Phi(x) = r(x + \Phi(x)) + s \\ &\Leftrightarrow 1, x + \Phi(x), x\Phi(x) \text{ are linearly dependent over } \mathbb{C} \\ &\Leftrightarrow 1, -\frac{b_1(y)}{b_2(y)}, \frac{b_0(y)}{b_2(y)} \text{ are linearly dependent over } \mathbb{C} \\ &\Leftrightarrow b_2(y), b_1(y), b_0(y) \text{ are linearly dependent over } \mathbb{C} \end{aligned}$$

Rewriting the linear dependence of the polynomials in the last line of this chain of equivalences as a determinant gives us the claim. □

Remark: Previously we had that the group of a (unweighted) walk has cardinality 4 if the walk exhibits a vertical (or horizontal) symmetry. In this case the first and the last column of the matrix are the same and thus its determinant is zero.

Lemma 3.5. *The group G of a walk has order 6 if and only if*

$$\det\begin{pmatrix} \Delta_{23} & \Delta_{33} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & -\Delta_{23} & \Delta_{12} & -\Delta_{22} \\ \Delta_{22} & \Delta_{32} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & -\Delta_{22} & \Delta_{11} & -\Delta_{21} \end{pmatrix} = 0$$

where Δ_{ij} is obtained from (3.1) if we leave out the i-th line and the j-th column.

Proof. Similar to the previous lemma. □

Theorem 3.6. *Out of the 79 walks we want to consider there are 23 models that have a finite associated group. Among them, there are*

- *16 that have a vertical symmetry and hence a group isomorphic to D_2 (order 4)*
- *5 that have a group isomorphic to D_3 (order 6)*
- *2 that have a group isomorphic to D_4 (order 8)*

The remaining 56 walks all have an infinite associated group.

Proof. Because Φ and Ψ are involutions, we only have to consider the order of $\Psi \circ \Phi$. If this order is finite and equal to n, the corresponding group is $G \cong D_n$. In the previous examples we already saw that groups with a vertical symmetry have D_4 as corresponding group. For the other walks with groups of finite order it can be easily computed that $\Psi \circ \Phi$ has finite order.
The proofs that $G(S)$ is infinite for all the other step sets are more difficult. Bousquet-Mélou and Mishna presented two strategies in [12]. It depends on S which of them can be used (the first one works of $i + j \geq 0$ for all $i, j \in S$, the second in the remaining cases).

1. The valuation argument

Let z be an indeterminate and x and y Laurent series in z with coefficients in $\mathbb{Q}$ with valuation a respectively b. We assume the trailing coefficients $[z^a]x$ respectively $[z^b]y$ to be positive.
Define x' by $\Phi(x, y) = (x', y)$. Then the trailing coefficient of x' (and y) is positive and the evaluation of x' (and y) depends only on a and b:

$$\phi(a, b) := (\mathrm{val}(x'), \mathrm{val}(y)) = \begin{cases} \left(-a + b(v_{-1}^{(y)} - v_1^{(y)}), b\right) & \text{for } b \geq 0 \\ \left(-a + b(d_{-1}^{(y)} - d_1^{(y)}), b\right) & \text{for } b \leq 0 \end{cases}$$

where $v_i^{(y)}$ respectively $d_i^{(y)}$ denotes the valuation respectively the degree in y of $B_i(y)$ for $I = \pm 1$.
Analogously we define y' by $\Psi(x, y) = (x, y')$. This is well defined and the valuation of x and y' again only depend on a and b:

$$\psi(a, b) := (\mathrm{val}(x), \mathrm{val}(y')) = \begin{cases} \left(a, -b + a(v_{-1}^{(x)} - v_1^{(x)})\right) & \text{for } a \geq 0 \\ \left(a, -b + a(d_{-1}^{(x)} - d_1^{(x)})\right) & \text{for } a \leq 0 \end{cases}$$

where $v_i^{(x)}$ or $d_i^{(x)}$ respectively denotes the valuation respectively the degree of $A_i(x)$ in x for $i = \pm 1$.
To show that G is infinite it suffices to show that G', the group generated by ϕ and ψ infinite. This can be shown by finding $(a,b) \in \mathbb{Z}^2$ such that the orbit of (a,b) under the action of G' is infinite. Let S be a step set with $i + j \geq 0$ for all $i, j \in S$. Then we have that $A_{-1}(x) = x$ and $B_{-1}(y) = y$, hence $v_{-1}^{(x)} = d_{-1}^{(x)} = v_{-1}^{(y)} = d_{-1}^{(y)}$. We also have that $v_1^{(x)} = v_1^{(y)} = -1$ since S contains both $\overline{x}y$ and $x\overline{y}$. Hence ϕ and ψ are given by

$$\phi(a,b) = \begin{cases} (-a+2b, b) & \text{for } b \geq 0 \\ (-a + b(1 - d_1^{(y)}), b) & \text{for } b \leq 0 \end{cases}$$

and

$$\psi(a,b) = \begin{cases} (a, 2a - b)) & \text{for } a \geq 0 \\ (a, -b + a(1 - d_1^{(x)})) & \text{for } a \leq 0 \end{cases}$$

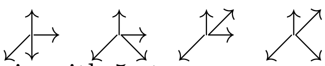
With induction on n can be shown that

$$(\psi \circ \phi)^n(1,2) = (2n+1, 2n+2) \quad \text{and} \quad \phi \circ (\psi \circ \phi)^n(1,2) = (2n+3, 2n+2).$$

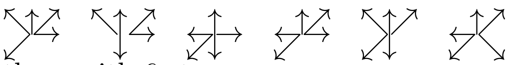
Since all pairs have positive entries we do not need to know $d_1^{(x)}$ and $d_1^{(y)}$. Hence the orbit of $(1,2)$ under the action of ϕ and ψ is infinite, and thus also G' and G.
In the remaining 51 cases this method did turned out to be successful since there could no point (a,b) with infinite orbit under G' be found. But there is another argument for these cases that takes fixed points of $\theta = \psi \circ \phi$ into consideration.

2. The fixed point argument

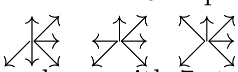
We do not have to consider all of the remaining 51 models, since walks that differ only by one of the eight symmetries of the square have the same associated group. If we take this into consideration, we are left with 14 models:
four of them with 4 steps



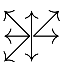
six with 5 steps



three with 6 steps



and one with 7 steps



Suppose that $\Theta = \Psi \circ \Phi$ is well defined in a neighbourhood of $(a,b) \in \mathbb{C}^2$ and (a,b) is a fixed point of Θ. Then a and b are algebraic over $\mathbb{Q}$. Write Θ as $\Theta = (\Theta_1, \Theta_2)$ where Θ_i are the coordinates of Θ. Then $\Theta_i(x,y)$ is a rational function in x and y. Locally around (a,b) the function Θ can by Taylor series expansion be represented as

$$\Theta(a+u, b+v) = (a,b) + (u,v)J_{a,b} + O(u^2) + O(v^2) + O(uv)$$

where $J_{a,b}$ is the Jacobian matrix of Θ in (a,b):

$$J_{a,b} = \begin{pmatrix} \frac{\partial \Theta_1}{\partial x}(a,b) & \frac{\partial \Theta_2}{\partial x}(a,b) \\ \frac{\partial \Theta_1}{\partial y}(a,b) & \frac{\partial \Theta_2}{\partial y}(a,b) \end{pmatrix}.$$

Iteration yields for $m \geq 1$:

$$\Theta^m(a+u, b+v) = (a,b) + (u,v)J_{a,b}^m + O(u^2) + O(v^2) + O(uv).$$

Suppose that $G(S)$ is finite of order $2n$, then Θ has order n. This implies that $\Theta^n(a+u, b+v) = (a,b) + (u,v)$ hence $J_{a,b}^n$. In particular, all eigenvalues of $J_{a,b}$ are roots of unity. The strategy to prove that $G(S)$ is infinite is finding a fixed point (a,b) of Θ and computing the characteristic polynomial $\chi(X)$ for the Jacobian matrix $J_{a,b}$. We have that $\chi(X) \in \mathbb{Q}(a,b)[X]$. To decide whether the zeros of χ are roots of unity, we eliminate a and b from the equation $\chi(X) = 0$ and obtain a polynomial $\overline{\chi}(X) \in \mathbb{Q}[X]$ that vanishes for all eigenvalues of $J_{a,b}$. If none of its factors is cyclotomic then $G(S)$ is infinite. Because all cyclotomic polynomials of fixed degree are known one can easily check if our polynomial is cyclotomic.
We will illustrate how this strategy works for one of the above step sets.

Example: Consider the step set $S = \{x, y, \overline{x}, \overline{xy}\}$. Then we have that $\Theta(x,y) = \Psi \circ \Phi(x,y) = (\overline{xy}, x + \overline{y})$. Each pair (a,b) with $a^4 + a^3 = 1$ and $b = \frac{1}{a^2}$ is a fixed point of Θ. Consider such a pair. Then

$$J_{a,b} = \begin{pmatrix} -1 & 1 \\ -a^3 & -a^4 \end{pmatrix}$$

and

$$\chi(X) = \det(XI - J_{a,b}) = X^2 + X(1 + a^4) + a^3 + a^4.$$

Let X be a zero of χ. Eliminate a (because $a^4 + a^3 = 1$ this is possible) and obtain

$$\overline{\chi}(X) = X^8 + 9X^7 + 31X^6 + 62X^5 + 77X^4 + 62X^3 + 31X^2 + 9X + 1 = 0.$$

The polynomial $\overline{\chi}$ is irreducible and different from all cyclotomic polynomials of degree 8. Hence its zeros can not be roots of unity and we get that $J_{a,b}^n = I$ for all $n \geq 1$. This implies that $G(S)$ is infinite.

The same strategy also works for the other thirteen remaining models. For each model exists a condition for (a,b) to be a fixed point and a polynomial $\overline{\chi}(X) \in \mathbb{Q}[X]$ that vanishes on all eigenvaules of $J_{a,b}$. Then can be checked that none of its factors is cyclotomic.

Remark: This strategy does not work in the five cases of the valuation method. Three of these models do not have any fixed point of Θ. In the other two cases there exists a fixed point but we have that $J_{a,b}^6 = I$.

3.3 Some tools and techniques

Let S be one of the 79 step sets we want to consider. Denote with $\mathcal{Q}$ the set of walks with steps in S starting in $(0,0)$ and remaining in the first quadrant. Let $q(i,j,n)$ be the number of such walks with length n that end in position (i,j) and $Q(x,y,t)$ the associated generating function, i.e.

$$Q(x,y,t) = \sum_{i,j,n \geq 0} q(i,j,n) x^i y^j t^n.$$

This is a formal power series in t with coefficients in $\mathbb{Q}[x,y]$.

A functional equation for Q

Lemma 3.7. *As a power series in t the generation function $Q(x,y,t) = Q(x,y)$ of the walks with step set S in the quarter plane starting in $(0,0)$ is determined by the following functional equation*

$$K(x,y)xyQ(x,y) = xy - txA_{-1}(x)Q(x,0) - tyB_1(y)Q(0,y) + t\varepsilon Q(0,0)$$

where

$$K(x,y) = 1 - tS(x,y) = 1 - t\sum_{i,j\in S} x^i y^j$$

is the so called kernel of the equation and $A_{-1}(x)$ and $B_1(y)$ are the coefficients of $\overline{x}$ respectively $\overline{y}$ in $S(x,y)$ (as in the previous section) and ε is 1 if $(-1,-1) \in S$ and zero otherwise.

Proof. The idea of the proof is to construct the walk step by step. We start with the empty walk and always focus on the new step at the end of the walk. The empty walk has weight 1. Adding a step to a walk has generating function $tS(x,y)Q(x,y)$. But we have take into account that some of these walks will leave the quadrant. This will happen if we add a y-negative step to a point with ordinate 0 or an x-negative step to a point with abscissa 0. The walks with ordinate 0 are counted by $Q(x,0)$, while the walks with abscissa 0 are counted by $Q(0,y)$. Hence we have to subtract $t\overline{y}A_{-1}(x)Q(x,0)$ and $t\overline{x}B_{-1}(y)Q(0,y)$ from the total number of walks. But if $(-1,-1)$ is in the step set, we subtracted those walks twice, once with the x-negative walks and once with the y-negative walks. We have to correct this mistake by adding $t\varepsilon\overline{xy}Q(0,0)$. This inclusion-exclusion-argument gives us

$$Q(x,y) = 1 + tS(x,y)Q(x,y) - t\overline{y}A_{-1}(x)Q(x,0) - t\overline{x}B_{-1}Q(0,y) + \varepsilon t\overline{xy}Q(0,0).$$

Multiplying with xy and using the definition of $K(x,y)$ gives us the claim of the Lemma. The fact that this functional equation determines $Q(x,y,t)$ completely (as a power series in t) comes from the fact that the coefficient of t^n in $Q(x,y,t)$ can be computed inductively via this equation. This is closely related to the fact that the walks in $\mathcal{Q}$ can be described by a recursion. □

Orbit sums

In the previous section we saw that all transformations $g \in G(S)$ leave the polynomial $S(x,y)$ unchanged. Hence they also leave the kernel $K(x,y) = 1 - tS(x,y)$ unchanged. We can rewrite the equation from lemma 3.7

$$K(x,y)xyQ(x,y) = xy - F(x) - G(y) + t\varepsilon Q(0,0) \quad (3.2)$$

where $F(x) = txA_{-1}(x)Q(x,0)$ and $G(y) = tyB_{-1}(y)Q(0,y)$. Replace (x,y) by $\Phi(x,y) = (x',y)$ and obtain

$$K(x,y)x'yQ(x',y) = x'y - F(x') - G(y) + t\varepsilon Q(0,0) \quad (3.3).$$

Subtracting (3.3) from (3.2) gives us

$$K(x,y)(xyQ(x,y) - x'yQ(x',y)) = xy - x'y - F(x) + F(x').$$

In the above equation we have eliminated $G(y)$. We can repeat this process with $(x', y') = \Psi(x', y)$ to obtain

$$K(x,y)(xyQ(x,y)-x'yQ(x',y)+x'y'Q(x',y')) = xy-x'y+x'y'-F(x)-G(y')+t\varepsilon Q(0,0).$$

Now $F(x')$ has vanished. If G is finite of order $2n$ we can repeat this process until we reach $(\Psi \circ \Phi)^n(x, y) = (x, y)$ again. Said differently, we are considering the alternating sums of the functional equations of the orbit of (x, y). All unknown functions on the right hand side vanish and we obtain

Proposition 3.8. *Suppose that $G(S)$ is finite. Then we have*

$$\sum_{g\in G} \operatorname{sign}(g)(xyQ(x,y,t)) = \frac{1}{K(x,y,t)} \sum_{g\in G} \operatorname{sign}(g)g(xy) \quad (3.4)$$

where $g(A(x, y)) := A(g(x, y))$ for $g \in G$.

The right hand side is a rational function in x, y and t. We will see later that this identity implies that 19 of the 23 walks with a finite associated group have a D-finite generating function.

The four remaining models are Gessel walks (step set $S = \{x, \overline{x}, xy, \overline{xy}\}$), walks with step set $S_1 = \{\overline{x}, \overline{y}, xy\}$, $S_2 = \{x, y, \overline{xy}\}$ or $S_1 \cup S_2$. Note that all of these step sets have an x/y-symmetry, which implies that (y, x) lies in the orbit of (x, y). More precisely, the orbit of (x, y) is for all of the above walks

$$(x,y) \xrightarrow{\Phi} (\overline{xy},y) \xrightarrow{\Psi} (\overline{xy},x) \xrightarrow{\Phi} (y,x) \xrightarrow{\Psi} (y,\overline{xy}) \xrightarrow{\Phi} (x,\overline{xy}) \xrightarrow{\Psi} (x,y).$$

If $g(x, y) = (y, x)$ then $\operatorname{sign}(g) = -1$. Hence the right hand side of the equation in the proposition vanishes for these models and the equation becomes

$$xyQ(x,y) - \overline{x}Q(\overline{xy},y) + \overline{y}Q(\overline{xy},x) = xyQ(y,x) - \overline{x}Q(y,\overline{xy}) + \overline{y}Q(x,\overline{xy}).$$

But this equation already follows from the obvious relation $Q(x, y) = Q(y, x)$, hence we don't get any new information. But we can solve three of these models by considering sums over the half-orbit. Because $A_{-1}(x) = B_{-1}(x)$ the identity for the half-orbit summation is

$$xyQ(x,y) - \overline{x}Q(\overline{xy},y) + \overline{y}Q(\overline{xy},x) = \frac{xy - \overline{x} + \overline{y} + 2txA_{-1}(x)Q(x,0) + t\varepsilon Q(0,0)}{K(x,y)}.$$

Remark: For Gessel walks one obtains from the proposition

$$\sum_{g\in G} \operatorname{sign}(g)g(xyQ(x,y)) = 0.$$

But there is no symmetry explaining this result.

The roots of the kernel

Lemma 3.9. *Let $Y_1(x)$ and $Y_2(x)$ be the roots of the kernel $K(x,y) = 1 - t\sum_{(i,j)\in S} x^i y^j$, where $K(x, y)$ is considered as a polynomial in y. They are Laurent series in t with coefficients in $\mathbb{Q}(x)$:*

$$Y_1(x) = \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)} \quad \textit{and} \quad Y_2(x) = \frac{1 - tA_0(x) + \sqrt{\Delta(x)}}{2tA_1(x)}$$

where

$$\Delta(x) = (1 - tA_0(x))^2 - 4tA_{-1}A_1(x).$$

The valuations of Y_1 and Y_2 in t is 1 respectively -1. Furthermore is $K(x,y)$ a power series in t with coefficients in $\mathbb{Q}[x,y,\overline{x},\overline{y}]$ and the coefficient of y^j in this series is

$$\frac{1}{K(x,y)} = \frac{1}{\sqrt{\Delta(x)}}\left(\frac{1}{1-\overline{y}Y_1(x)} - \frac{1}{1-y/Y_2(x)} - 1\right). \quad (3.5)$$

Proof. The equation $K(x,y) = 0$ can be rewritten as

$$y = t(A_{-1}(x) + yA_0(x) + y^2A_1(x)).$$

This quadratic equation has the above solutions Y_1 and Y_2. Since $\Delta(x) = 1 + O(t)$ the series Y_2 has valuation -1 in t. Its first term is $\frac{1}{tA_1(x)}$. The equation

$$Y_1(x) - Y_2(x) = \frac{1}{tA_1(x)} - \frac{A_0(x)}{A_1(x)}$$

shows that for $n \geq 1$ the coefficient of t^n in $Y_2(x)$ is a Laurent polynomial in x (this is not true in general, for example for the coefficients of t^0 or t^{-1}).
The identity (3.5) follows by partial fraction expansion in y. The series Y_1 and $\frac{1}{Y_2}$ both have valuation 1 in t. Hence the expansion of y in $\frac{1}{K(x,y)}$ is

$$[y^j]\frac{1}{K(x,y)} = \begin{cases} \frac{Y_1(x)^{-j}}{\sqrt{\Delta(x)}} & j \leq 0 \\ \frac{Y_2(x)^{-j}}{\sqrt{\Delta(x)}} & j \geq 0 \end{cases}. \quad (3.6)$$

□

Canonical factorization of the discriminant $\Delta(x)$

Consider the kernel as a polynomial in y. Its discriminant is a Laurent polynomial in x:

$$\Delta(x) = (1 - tA_0(x))^2 - 4tA_{-1}(x)A_1(x).$$

Say Δ has valuation δ and degree d in x. Then this Laurent polynomial has $\delta + d$ roots $X_i = X_i(t)$ for $1 \leq 1 \leq \delta + d$. Exactly δ of them, let's say $X_1, \ldots, X_\delta$ are finite and vanish at $t = 0$. We write

$$\Delta(x) = \Delta_0\Delta_-(x)\Delta_+(x)$$

where

$$\Delta_-(\overline{x}) = \Delta_-(\overline{x}, t) = \prod_{i=1}^{\delta}(1 - \overline{x}X_i)$$

$$\Delta_+(x) = \Delta_+(x,t) = \prod_{i=\delta+1}^{\delta+d}\left(1 - \frac{x}{X_i}\right)$$

and

$$\delta_0 = \delta_0(t) = (-1)^\delta \frac{[x^\delta]\Delta(x)}{\prod_{i=1}^{d} X_i} = (-1)^d[x^d]\Delta(x)\prod_{I=\delta+1}^{\delta+d} X_i.$$

We see that Δ_0 (respectively $\Delta_-(x)$ and $\Delta_+(x)$) are formal power series in t with coefficients in $\mathbb{C}$ (respectively $\mathbb{C}[\overline{x}]$ and $\mathbb{C}[x]$) and constant term 1.

D-finiteness via orbit sums

Now we are going to show that 19 of the 23 walks with a finite associated group can be solved via their orbit sum. Among these 19 step sets 16 have a vertical symmetry and 3 do not. We will take a closer look at these three models and we will find a closed form expression for the number of walks ending in a certain position.

Proposition 3.10. *For the 23 walks with finite group (expect for walks with step set $S = \{\overline{x}, \overline{y}, xy\}, S = \{x, y, \overline{xy}\}, S = \{x, y, \overline{x}, \overline{y}, xy, \overline{xy}\}$ and $S = \{x, \overline{x}, xy, \overline{xy}\}$) holds: The rational function*

$$R(x, y, t) = \frac{1}{K(x, y, t)} \sum_{g \in G} \operatorname{sign}(g) g(xy)$$

is a power series in t with coefficients in $\mathbb{Q}(x)[y, \overline{y}]$, i.e. the coefficients are Laurent-polynomials in y with coefficients in $\mathbb{Q}(x)$. The positive part in y of $R(x, y, t)$, called $R^+(x, y, t)$, is a power series in t with coefficients in $\mathbb{Q}[x, \overline{x}, y]$. Extracting the positive part in x of $R^+(x, y, t)$ gives $xyQ(x, y, t)$. Summarized

$$xyQ(x, y, t) = [x^{>0}][y^{>0}]R(x, y, t).$$

In particular, $Q(x, y, t)$ is D-finite. The number of walks with n steps ending in (i, j) is

$$q(i, j, i) = [x^{i+1} y^{j+1}] \left(\sum_{g \in G} \operatorname{sign}(g) g(xy) \right) S(x, y)^n$$

where

$$S(x, y) = \sum_{(a,b) \in S} x^a y^b$$

is the characteristic polynomial of the step set.

Proof. We start with the 16 walks with an associated group of order four. They all have a vertical symmetry and hence $K(x, y) = K(\overline{x}, y)$. As we saw in the examples before the orbit of (x, y)

$$(x, y) \xrightarrow{\Phi} (\overline{x}, y) \xrightarrow{\Psi} (\overline{x}, C(x)y) \xrightarrow{\Phi} (x, C(x)\overline{y}) \xrightarrow{\Psi} (x, y)$$

where $C(x) = \frac{A_{-1}(x)}{A_1(x)}$. The orbit sum of Prop 3.8 is

$$xyQ(x, y) - \overline{x}yQ(\overline{x}, y) + \overline{xy}C(x)Q(\overline{x}, C(x)\overline{y}) + x\overline{y}C(x)Q(x, C(x)\overline{y}) = R(x, y).$$

Both sides of the identity are series in t with coefficients in $\mathbb{Q}[x, \overline{x}, y]$. Since only the first two parts contribute terms with positive exponents in y, by extracting the part positive in y we obtain

$$xyQ(x, y) - \overline{x}yQ(\overline{x}, y) = R^+(x, y).$$

From the expression on the left hand side we see that $R^+(x, y)$ has coefficients in $\mathbb{Q}[x, \overline{x}, y]$. If we are extracting the positive part in x then we obtain $xyQ(x, y) = [x^>][y^>]R(x, y, t)$ because the second term of the left hand side does not contribute.
Consider now the cases $S = \{\overline{x}, y, x\overline{y}\}$, $S = \{x, \overline{x}, x\overline{y}\overline{x}y\}$ and $S = \{x, \overline{x}, y, \overline{y}x\overline{y}, \overline{x}y\}$. For each of these sets the orbit of (x, y) consists of pairs of the form $(x^a y^b, x^c y^d)$ with integers $a, b, c, d \in \mathbb{Z}$. From this follows that $R(x, y)$ is a series in t with coefficients in $\mathbb{Q}[x, \overline{x}, y, \overline{y}]$.

If we extract the part positive in x and y of the orbit sum of proposition 3.8, then in each of these cases on the left hand side only $xyQ(x,y)$ remains and $xyQ(x,y) = [x^>][y^>]R(x,y,t)$ follows.
That $Q(x,y,t)$ is D-finite follows from the following proposition and the expression for $q(,i,j,t)$ follows via coefficient extraction. □

Proposition 3.11. *If $F(x,y,t)$ is a power series in t with coefficients lying in $\mathbb{C}(x)[y,\overline{y}]$ then $[y^>]F(x,y,t)$ is algebraic over $C(x,y,t)$. If the latter series has coefficients in $\mathbb{C}[x,\overline{x},y]$ its positive part in x, i.e. $[x^>][y^>]F(x,y,t)$, is D-finite in x,y and t (without proof).*

The models $S = \{\overline{x}, y, x\overline{y}\}$ and $S = \{x, \overline{x}, y, \overline{y}, x\overline{y}, \overline{x}y\}$

Consider first $S = \{\overline{x}, y, x\overline{y}\}$. A walk with steps in S that remains in the first quadrant has in each of its prefixes more north (y) steps than southeast ($x\overline{y}$) steps and more southeast than west ($\overline{x}$) steps. Because of this properties these walks are in correspondence with Young tableaux of height at most three. The first line tells us when to take a north step, the second when to take a southeast step and the third when to go west. For example the tableaux

1	2	5	9
3	6	7	
4	8		

corresponds to the walk

N-N-SE-W-N-SE-SE-W-N

About the enumeration of Young tableaux are several results known. In particular the number of tableaux of a particular given form (and hence also the number of walks ending in a given position) can be written in a closed form by the hook length formula. It is known that the total number of tableaux of size n with height at most three is given by the n-th Motzkin number. Motzkin numbers can be computed by the recurrence

$$M_{n+1} = M_n + \sum_{i=0}^{n-1} M_i M_{n-i-1} = \frac{2n+3}{n+3} M_n + \frac{3n}{n+3} M_{n-1}.$$

Proposition 3.12. *The generating function of the walks with step set $S = \{\overline{x}, y, x\overline{y}\}$ in the quarter plane is the nonnegative part (in x and y) of a rational series in t with coefficients in $\mathbb{Q}[x,\overline{x},y,\overline{y}]$:*

$$Q(x,y,t) = [x^{\geq}][y^{\geq}]\tilde{R}(x,y,t)$$

with

$$\tilde{R} = \frac{(1-\overline{xy})(1-\overline{x}^2 y)(1-x\overline{y}^2)}{1-t(\overline{x}ty + x\overline{y})}.$$

Especially we get that $Q(x,y)$ is D-finite. The number of walks of length $n = 3m+2i+j$ ending in (i,j) is given by

$$q(i,j,n) = \frac{(i+1)(j+1)(i+j+2)(3m+2i+j)!}{m!(m+i+1)!(m+i+j+2)!}.$$

The representation $n = 3m + 2i + j$ occurs because before each SE-step has to be a N-step, and for each W-step has to be both a N- and a SE-step (in total we have $i + j + m$ north steps, $j + m$ southeast steps and m west steps). In particular we see that

$$q(0,0,3m) = \frac{2(3m)!}{m!(m+1)!(m+2)!} \sim \sqrt{3}\frac{3^{3m}}{\pi m^4}.$$

Hence $Q(0,0,t)$ and thus also $Q(x,y,t)$ are transcendental. But we have that $Q(x,\frac{1}{x},t)$ is algebraic of degree two:

$$Q(x,\frac{1}{x},t) = \frac{1 - t\overline{x} - \sqrt{1 - 2\overline{x}t + t^2\overline{x}^2 - 4t^2x}}{2xt^2}.$$

In particular the total number of walks with n steps in the first quadrant is the n-th Motzkin number:

$$Q(1,1,t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2} = \sum_{n\geq 0} t^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}.$$

Proof. We considered the orbit of (x,y) in second example at the beginning of chapter 3.2. The first result follows from Prop 3.10 with $\tilde{R}(x,y) = R(x,y)/(xy)$. The coefficient of $x^iy^jt^n$ in $\tilde{R}(x,y,t)$ can be extracted via

$$[x^iy^j](\overline{x} + y + x\overline{y})^n = \frac{(3m+2i+j)!}{m!(m+i)!(m+i+j)!}$$

if $n = 3m + 2i + j$.
The algebraicity of $Q(x,\overline{x})$ can be shown by considering the alternate sum of the three functional equations of lemma 3.7 that are obtained by replacing (x,y) by replacing it with the first three elements of its orbit. In these three elements y only occurs with nonnegative exponent. We obtain

$$K(x,y)(xyQ(x,y) - \overline{x}y^2Q(\overline{x}y,y) - \overline{x}^2yQ(\overline{x}y,\overline{x})) = xy - \overline{x}y^2 + \overline{x}^2y - tx^2Q(x,0) - t\overline{x}Q(0,\overline{x}).$$

We consider two special values of y. First, replace y by $\overline{x}$. The second and the third term on the left hand side cancel and it remains

$$K(x,\overline{x})Q(x,\overline{x}) = 1 - tx^2Q(x,0) - t\overline{x}Q(0,\overline{x}).$$

Secondly, replace y by the root $Y_0(x)$ of the kernel. Recall that $Y_0(x) = \frac{1-tA_0(x)-\sqrt{\Delta(x)}}{2tA_1(x)}$. Because $Y_0(x)$ has evaluation 1 in t this substitution is well defined. The left hand side vanishes and it remains

$$0 = xY_0(x) - \overline{x}Y_0(x)^2 + \overline{x}^2Y_0(x) - tx^2Q(x,0) - t\overline{x}Q(0,\overline{x}).$$

If we combine these two equations we get

$$K(x,\overline{x})Q(x,\overline{x}) = 1 - xY_0(x) + \overline{x}Y_0(x^2) - \overline{x}Y_0(x).$$

From this follows that $Q(x,\overline{x},t)$ is algebraic and has the form stated in the proposition. □

The case $S = \{$N, E, S, W, SE, NW$\}$ is similar since the orbit of (x,y) is of the same form. The proof can be done analogously as before. Here Motzkin-Numbers occur, too.

Proposition 3.13. *The generating function of walks with step set $S = \{N, E, S, W, SE, NW\}$ in the first quadrant is the in x and y nonnegative part of a rational function:*

$$Q(x, y, t) = [x^{\geq}, y^{\geq}]\tilde{R}(x, y, t)$$

with

$$\tilde{R}(x, y, t) = \frac{(1 - \overline{x}\overline{y})(1 - \overline{x}^2 y)(1 - x\overline{y}^2)}{1 - t(x + y + \overline{x} + \overline{y} + x\overline{y} + \overline{x}y)}.$$

In particular we have that $Q(x, y, t)$ is D-finite. The specialization $Q(x, \overline{x}, t)$ is algebraic of degree two:

$$Q(x, \overline{x}, t) = \frac{1 - tx - t\overline{x} - \sqrt{(1 - t(x + \overline{x}))^2 - 4t(1 + x)(1 - \overline{x})}}{2t^2(1 + x)(1 + \overline{x})}.$$

The total number of walks with n steps in the first quadrant is 2^n times the n-th Motzkin number:

$$Q(1, 1, t) = \frac{1 - 2t - \sqrt{(1 + 2t)(1 - 6t)}}{8t} = \sum_{n\geq 0} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^n}{k + 1} \binom{n}{2k} \binom{2k}{k}.$$

3.4 Half orbit sums

In this section we are going to consider the three models $S_1 = \{\overline{x}, \overline{y}, xy\}$, $S_2 = \{x, y, \overline{x}\overline{y}\}$ and $S_3 = S_1 \cup S_2$. Due to their x/y-symmetry their orbit sum vanishes. Using half orbit sums we are going to show that all of these three models have an algebraic generating function. The result for S_1 was proven by Bousqeut-Mélou in [11], S_2 by Mishna in [26] and S_3 by both of them in [12].
For each of these models we have according to proposition 3.8 that

$$xyQ(x, y) - \overline{x}Q(\overline{x}\overline{y}, y) + \overline{y}Q(\overline{x}\overline{y}, x) = \frac{xy - \overline{x} + \overline{y} + 2txA_{-1}(x)Q(x, 0) + t\varepsilon Q(0, 0)}{K(x, y)}.$$

We want to extract the coefficient of y^0 from this equation. On the left hand side only the second term contributes and its contribution is $\overline{x}Q_d(\overline{x})$ where $Q_d(x) = Q_d(x, t)$ is the generating function of walks ending on the diagonal, i.e.

$$Q_d(x, t) = \sum_{n,i\geq 0} t^n x^i q(i, i, n).$$

The coefficient of y^0 on the right hand side can be computed via (3.6) and we get

$$-\overline{x}Q_d(\overline{x}) = \frac{1}{\sqrt{\Delta(x)}} \left(xY_0(x) - \overline{x} + \frac{1}{Y_1} - 2txA_{-1}(x)Q(x, 0) + t\varepsilon Q(0, 0) \right)$$

where $\Delta(x) = (1 - tA_0(x))^2 - 4t2A_{-1}(x)A_1(x)$. If we use the expression for Y_1 from lemma 3.9, we get

$$Y_1(x) = \frac{1 - tA_0(x) - \sqrt{\Delta(x)}}{2tA_1(x)}$$

and the fact that $Y_1Y_2 = \overline{x}$ we can rewrite the above as

$$\frac{x}{tA_1(x)} - \overline{x}Q_d(\overline{x}) = \frac{1}{\sqrt{\Delta(x)}} \left(\frac{x(1 - tA_0(x))}{tA_1(x)} - \overline{x} - 2txA_{-1}(x)Q(x, 0) + t\varepsilon Q(0, 0) \right).$$

Recall the canonical factorization of $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\overline{x})$ and multiply the equation with $A_{-1}\sqrt{\Delta_-(\overline{x})}$ to obtain

$$\sqrt{\Delta_-(\overline{x})}\left(\frac{x}{t} - \overline{x}A_{-1}(x)Q_d(\overline{x})\right) =$$

$$\frac{1}{\sqrt{\Delta_0\Delta_+(x)}}\left(-\overline{x}A_1(x) - 2txA_{-1}(x)A_1(x)Q(x,0) + t\varepsilon A_1(x)Q(0,0)\right). \quad (3.7)$$

Each term in this equation is a Laurent series in t with coefficients in $\mathbb{Q}[x,\overline{x}]$. On the left hand side are only few positive x-powers while on the right hand side there are only few negative x-powers. We will extract the positive and negative parts in x. From this we will obtain algebraic expressions for $Q_d(x)$ and $Q(x,0)$. From now on we will consider each of the three models separately.

3.4.1 Case 1: $S = \{\overline{x}, \overline{y}, xy\}$

In this case we have that $A_{-1}(x) = 1, A_0(x) = \overline{x}, A_1(x) = x$ and $\varepsilon = 0$. Furthermore $\Delta(x) = (1-t\overline{x})^2 - 4t^2x$. The curve $\Delta(x,t) = 0$ has a rational parametrization in the sense of the series $W \equiv W(t)$ defined as the only power series in t that fulfills $W = t(2+W^3)$. Replace t by $\frac{W}{2+W^3}$ in $\Delta(x)$ to obtain the canonical factorization $\Delta(x) = \Delta_0\Delta_+(x)\Delta_-(\overline{x})$ with

$$\Delta_= \frac{4t^2}{W^2} \quad \Delta_+(x) = 1 - xW^2 \quad \text{and} \quad \Delta_-(x) = 1 - \frac{W(W^3+4)}{4x} + \frac{W^2}{4x^2}.$$

Extracting the positive part in x form (3.7) gives us

$$\frac{x}{t} = -\frac{(2t^2x^2Q(x,0) - x + 2t)W}{2t^2\sqrt{1-xW^2}} + \frac{W}{t}.$$

From this we obtain an expression for $Q(x,0)$ by W. Extracting the non-positive part in x from (3.7) gives us

$$\sqrt{1 - \frac{W(W^3+4)}{4x} + \frac{W^2}{4x^2}}\left(\frac{x}{t} - Q_d(\overline{x})\right) - \frac{x}{t} = -\frac{W}{t}.$$

This gives us an expression for $Q_d(\overline{x})$.

Proposition 3.14. *Let $W \equiv W(t)$ be the series defined above. Then the generating function of quarter plane walks with step set $\{W,\ S,\ NE\}$ ending in the x-axis is*

$$Q(x,0,t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1-xW^2}\right).$$

The generating function of the walks ending in $(i,0)$ is

$$[x^i]Q(x,0,t) = \frac{W^{2i+1}}{2\cdot 4^i t}\left(C_i - \frac{C_{i+1}W^3}{4}\right)$$

where $C_i = \binom{2i}{i}\frac{1}{i+1}$ denotes the i-th Catalan number. Using Lagrange's inversion formula we get that the number of walks of length $n = 3m+2i$ is

$$q(i,0,3m+2i) = \frac{4^m(2i+1)}{(m+i+1)(2m+2i+1)}\binom{2i}{i}\binom{3m+2i}{m}.$$

The generating function of walks ending on the diagonal is

$$Q_d(x,t) = \frac{W - \overline{x}}{t\sqrt{1 - xW(1 + W^3/4) + x^2W^2/4}} + \frac{1}{xt}.$$

Remark: The function $Q(0,0)$ is algebraic of degree 3, while $Q(x,0)$ is algebraic of degree 6 and $Q(x,y)$, which can be expressed by $Q(x,0)$ and $Q(0,y) = Q(y,0)$ via the functional equation, is algebraic of degree 12.

3.4.2 Case 2: $S = \{x, y, \overline{xy}\}$

This step set is obtained by reversing the previous step set. In particular, we have that the series $Q(0,0)$ counting the number of walks starting and ending in $(0,0)$ is the same for both models.
In this case we have that $A_{-1}(x) = \overline{x}, A_0(x) = x, A_1(x) = 1$ and $\varepsilon = 1$. The discriminant is given by $\Delta(x) = (1-tx)^2 - 4t^2\overline{x}$. It is obtained from replacing x with $\overline{x}$ in the previous discriminant. The canonical factorization of $\Delta(x)$ is

$$\Delta_0 = \frac{4t}{W^2} \quad \Delta_+(x) = 1 - \frac{W(W^3+4)}{4}x + \frac{W^2}{4}x^2 \quad \Delta_-(x) = 1 - \overline{x}W^2$$

where W again is the only power series in t fulfilling $W = t(2+W^3)$. Extract the coefficient of x^0 from (3.7) to obtain

$$\frac{W^2}{2t} = \frac{W(W^4 + 4W + 8tQ(0,0)}{16t}.$$

From this we obtain an expression for $Q(0,0)$. Now we extract the nonnegative part in x from (3.7) and get

$$\frac{x}{t} - \frac{W^2}{2t} = -\frac{(2xtQ(x,0) - xt^2Q(0,0) + t - x^2 + x^3t)W}{2xt^2\sqrt{1 - xW(W^3+4)/4 + x^2W^2/4}} + \frac{W}{2xt}.$$

This gives us an expression for $Q(x,0)$. Finally we extract the negative part in x from (3.7) and obtain

$$\left(\frac{x}{t} - \frac{Q_d(\overline{x}}{x}\right)\sqrt{1 - \frac{W^2}{x}} - \frac{x}{t} + \frac{W^2}{2t} = -\frac{W}{2xt}.$$

From this we get an expression for $Q_d(\overline{x})$.

Proposition 3.15. *Let $W \equiv W(t)$ be as above. Then the generating function of the walks in the quarter plane with steps N, E and SW ending in the x-axis is*

$$Q(x,0,t) = \frac{W(4-W^3)}{16t} - \frac{t - x^2 + tx^3}{2xt} - \frac{2x^2 - xW - W\sqrt{1 - xW(W^3+4)/4 + x^2W^2/4}}{2xtW}.$$

The generating function of walks ending in the diagonal is

$$Q_d(x,t) = \frac{xW(x+W) - 2}{2tx^2\sqrt{1 - xW^2}} + \frac{1}{tx^2}.$$

In particular, we have that the generating function of the walks ending in (i,i) is

$$[x^i]Q_d(x,t) = \frac{W^2 2i+1}{4^{i+1}t(i+2)}\binom{2i}{i}(2i + 4 - (2i+1)W$$

With the help of Lagrange's inversion formula we get that the number of such walks with length $n = 3m + 2i$ is

$$q(i, i, 3m + 2i) = \frac{4^m (i+1)^2}{(m+i+1)(2m+2i+1)} \binom{2i}{i} \binom{3m+2i}{m}.$$

Remark: The series $Q(0,0)$ is algebraic of degree 3, while $Q(x,0)$ is algebraic of degree 6. The series $Q(x,y)$ can be expressed in terms of $Q(x,0)$ and $Q(0,y) = Q(y,0)$ and is algebraic of degree 12.

3.4.3 Case 3: $S = \{x, y, \overline{x}, \overline{y}, xy, \overline{xy}\}$

In this case we have that $A_{-1}(x) = 1 + \overline{x}, A_0(x) = x + \overline{x}, A_1(x) = 1 + x$ and $\varepsilon = 1$. The discriminant is given by $\Delta(x) = (1 - t(x + \overline{x}))^2 - 4t^2(1+x)(1+\overline{x})$. It is symmetric in x and $\overline{x}$. Two of its zeros, say X_1 and X_2 have evaluation 1 in t, the other two are given by $\frac{1}{X_1}$ and $\frac{1}{X_2}$. Consider the two elementary symmetric functions of X_1 and X_2 (these are the coefficients of $\Delta_-(\overline{x})$. This leads us to the power series $Z \equiv Z(t)$ that fulfills

$$Z = t\frac{1 - 2Z + 6Z^2 + 2Z^3 + z^4}{(1-Z)^2}$$

and has no constant term. Replace t by Z in $\Delta(x)$ and obtain the canonical factorization

$$\Delta_0 = \frac{t^2}{Z^2} \quad \Delta_+(x) = 1 - 2Z\frac{1+Z^2}{(1-Z)^2}x + Z^2x^2 \quad \Delta_-(\overline{x}) = \Delta_+(\overline{x}).$$

As before we obtain by extracting from (3.7) the coefficient of x^0 an expression for $Q(0,0)$:

$$Q(0,0) = \frac{Z(1 - 2Z - Z^2)}{t(1-Z)^2}.$$

If we extract the positive and the negative part from (3.7) we obtain expressions for $Q(x,0)$ and $Q_d(\overline{x})$.

Proposition 3.16. *Let $Z \equiv Z(t)$ as above and let*

$$\Delta_+(x) = 1 - 2Z\frac{1-Z^2}{(1-Z)^2}x + Z^2x^2.$$

Then the generating function of walks in the quarter plane with step set N, S, E, W, SE, NW ending in the x-axis is given by

$$Q(x,0,t) = \frac{(Z(1-Z) + 2xZ - (1-Z)x^2)\sqrt{\Delta_+(x)}}{2txZ(1-Z)(1+x)^2}$$

$$-\frac{Z(1-Z)^2 + Z(Z^3 + 4Z^2 - 5Z + Z)x - (1 - 2Z + 7Z^2 - 4Z^3)x^2 + x^3Z(1-Z)^2}{2txZ(1-Z)^2(1+x)^2}.$$

The generating function of the walks ending on the diagonal is

$$Q_d(x,t) = \frac{1 - Z - 2xZ + x^2Z(Z-1)}{tx(1+x)(Z-1)\sqrt{\Delta_+(x)}} + \frac{1}{tx(1+x)}.$$

Remark: The series $Q(0,0)$ is algebraic of degree 4 and $Q(x,0)$ is algebraic of degree 8. The series $Q(x,y)$ can be expressed by $Q(x,0)$ and $Q(0,y) = Q(y,0)$ via the functional equation and is algebraic of degree 16. But we have that $Q(1,1)$ is algebraic of only degree 4 and fulfills

$$Q(1+tQ)(1+2tQ+2t^2Q^2) = \frac{1}{1-6t}.$$

Also in this case, Motzkin numbers appear again.

Corollary 3.17. *Let $N \equiv N(t)$ be the only power series in t satisfying*

$$N = t(1+2N+4N^2).$$

Up to a factor t this series is the generating function of $2^n M_n$ where M_n is the n-th Motzkin number:

$$N = \sum_{n\geq 0} t^{n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^n}{k+1} \binom{n}{2k} \binom{2k}{k}.$$

Then the generating function of all walks in the quarter plane with step set N, S, E, W, SE, NW is given by

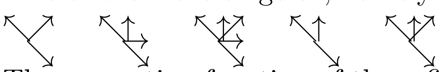
$$Q(1,1,t) = \frac{1}{2t}\left(\sqrt{\frac{1+2N}{1-2N}} - 1\right)$$

and the generating function of the walk ending in the origin is

$$Q(0,0,t) = \frac{(1+4N)^{3/2}}{2Nt} - \frac{1}{2t^2} - \frac{2}{t}.$$

3.5 Walks with an infinite group

So far, we have classified all lattice walks with a finite group (except Gessel walks, which we will consider in the next section). But what about the 56 walks with an infinite group? Five of them are singular, namely



The generating function of these five walks was proven to be not D-finite by Melczer and Mishna in [25] via singularity analysis. For the 51 remaining walks we have an result by Kurkova and Raschel:

Theorem 3.18. *For each of the 51 nonsingular walks with a infinite group the set $\left(0, \frac{1}{|S|}\right)$, where $|S|$ is the cardinality of the step set, splits into a subset H and $\left(0, \frac{1}{|S|}\right) \setminus H$. Both of them are dense in $\left(0, \frac{1}{|S|}\right)$ and fulfill*

1. *The functions $x \mapsto Q(x,0,t)$ and $y \mapsto Q(0,y,t)$ are D-finite for $t \in H$*

2. *The functions $x \mapsto Q(x,0,t)$ and $y \mapsto Q(0,y,t)$ are not D-finite for $t \in \left(0, \frac{1}{|S|}\right) \setminus H$*

From the second part of theorem 3.18 follows that $(x,y,t) \mapsto Q(x,y,t)$ is not D-finite. The main idea of the proof is to lift the kernel equation $K(x,y)$ to a Riemannian surface of genus 1 and then study its branches there (the full proof can be seen in [22]). We will encounter a similar idea when we study the proof of the algebraicity of Gessel walks by Bostan, Kurkova and Raschel in the next chapter.

4 Gessel walks

Gessel walks are walks in the halfplane starting at the origin with step set W, E, NE, SW. They are the only step set for which the approach via the group of a walk did not turn out to be successful. Denote

$$G(x, y, t) = \sum_{i,j,n \geq 0} g(i, j, n) x^i y^j t^n$$

the generating function of the Gessel walks. Gessel considered such walks with endpoint $(i, j) = (0, 0)$. Their counting sequence starts with

$$1, 0, 2, 0, 11, 0, 85, 0, 782, 0, 8004, 0, \ldots$$

He observed that the first terms of this sequence admit a closed form expression using hypergeometric series. This led to the Gessel conjecture, a statement that remained open for years but was finally proven true. The first proof by Bostan and Kauers in 2008 (see [6]) was computer aided. In 2014 Bostan, Kurkova and Raschel gave a proof of Gessel's conjecture without computer aid that uses higher complex analysis and properties of the Weierstraß-$\wp$-function (see [8]), while in early 2015 Bouquet-Mélou came up with a proof that uses some kind of generalization of the kernel method (see [9]). We will have a look at all three proofs, since they are very different in nature.
By a linear transform Gessel walks can be interpreted as simple walks, i.e. walks with step set N, E, S, W in the 135°-cone. Simple walks in other cones are well studied. For example, the number of simple excursions with n steps in the halfplane is given by $\binom{2n+1}{n} C_n$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The number of simple excursions with n steps in the quarter plane is $C_n C_{n+1}$ and $C_n C_{n+2} - C_{n+1}^2$ for the 45°-cone.

4.1 Bostan's and Kauers' computer aided proof

Theorem 4.1. *(Gessel conjecture) For the series $G(0, 0, t)$ holds*

$$G(0, 0, t) = {}_3F_2\left(\frac{5}{6}, \frac{1}{2}, 1; \frac{5}{3}, 2; 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^2 n$$

where

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} z^n$$

is the hypergeometric series in z with parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$. The expression $(a)_n = a(a + 1) \cdot \cdots \cdot (a + n - 1)$ denotes the (rising) Pochhammer-Symbol.

This result implies that $G(0, 0, t)$ is D-finite. It even holds that $G(0, 0, t)$ is algebraic because of the alternative representation

$${}_3F_2\left(\frac{5}{6}, \frac{1}{2}, 1; \frac{5}{3}, 2; 16t^2\right) = \frac{1}{t^2}\left(\frac{1}{2}{}_2F_1(-1/6, -1/2; 2/3; 16t^2) - \frac{1}{2}\right).$$

This series is algebraic due to Schwarz's classification of algebraic ${}_2F_1$'s [29], but for a long time it was overlooked that the parameters $((-1/6, -1/2; 2/3)$ fit case III of Schwarz's tables.

Corollary 4.2. *$G(0,0,t)$ is algebraic.*

Proof. The idea is to find a polynomial $P(T,t) \in \mathbb{Q}[T,t]$ that admits

$$g(t) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(16t)^n$$

as root. With the help of the theorem this implies that $P(T,t^2)$ is an annihilating polynomial for $G(0,0,t)$. Such a polynomial can be guessed starting from the first few, say 100, terms of the series $g(t)$ with the help of a computer.
By the implicit function theorem this polynomial admits a root $r(t) \in \mathbb{Q}[[t]]$ with $r(0) = 1$. Because $P(T,0) = T - 1$ has a single root in $\mathbb{C}$, the series $r(t)$ it the unique root of P. Since $r(t)$ is algebraic and hence also D-finite, its coefficients satisfy a recurrence with polynomial coefficients, namely

$$(n+2)(3n+5)g_{n+1} - 4(6n+5)(2n+1)g_n = 0, \quad g_0 = 1.$$

The solution of this recurrence is $g_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}16^n$. Hence $g(t)$ and $r(t)$ coincide and thus $g(t)$ is a solution of P, hence algebraic. □

The statement of the corollary can be generalized to

Theorem 4.3. *$G(x,y,t)$ is algebraic.*

In contrast to the counting sequence of excursions $g(0,0,n)$ the series $g(i,j,n)$ is not hypergeometric for arbitrary $(i,j) \in \mathbb{N}^2$. Moreover, there seems to exist no closed formula at all for $g(i,j,n)$ for arbitrary (i,j).
The proof of this theorem uses an approach based on automated guessing. The annihilating polynomials that occur there are far too large to be printed out, they are even too large to be processed efficiently by many standard computer algebra systems.

Illustration of the main ideas with Krewera's walks

In [6] Bostan and Kauers first considered Krewera's walks (step set NE, S, W) to illustrate the main techniques used in their proof. We already saw in the previous section that their generating function is algebraic. As we will see later, the proof for Gessel walks is similar to the proof for Krewera walks, but some additional difficulties occur.

Denote $f(i,j,n)$ the number of Krewera's walks with n steps starting in the origin and ending in (i,j) and $F(x,y,t)$ their generating function. Then the sequence $f(i,j,n)$ satisfies the following recurrence

$$f(i,j,n+1) = f(i+1,j,n) + f(i,j+1,n) + f(i-1,j-1,n)$$

for $n,i,j \geq 0$. Together with the boundary conditions $f(-1,0,n) = f(0,-1,n) = 0$ for $n \geq 0$ and $f(i,j,0) = \delta_{i,j,0}$ the recurrence gives us the functional equation

$$F(x,y,t) = 1 + \left(\frac{1}{x} + \frac{1}{y} + xy\right) tF(x,y,t) - \frac{1}{y}tF(x,0,t) - \frac{1}{x}tF(t,0,y).$$

Using the fact that $F(0,y,t)$ and $F(y,0,t)$ are equal because of the x/y symmetry of the step set and multplying with xy this equation becomes

$$((x+y+x^2y^2)t - xy)F(x,y,t)F(x,y,t) = xtF(x,0,t) + yF(y,0,t) - xy \quad (4.1).$$

Now we apply the kernel method and use the substitution

$$y \to Y(x,t) = \frac{x - t - \sqrt{-4t^2x^3 + x^2 - 2tx + t^2}}{2tx^2}$$
$$= t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \frac{3x^3+1}{x^3}t^4 + \cdots \in \mathbb{Q}[x,\overline{x}][[t]].$$

This substitution is legitimate because $Y(x,t)$ has positive valuation and puts the left hand side of (4.1) to zero. Hence $U = F(x,0,t)$ is a solution of the reduced kernel equation

$$U(x,t) = \frac{Y(x,t)}{t} - \frac{Y(x,t)}{x}U(Y(x,t),t). \quad (4.2)$$

The most important feature of the equation (4.2) is that its unique solution in $\mathbb{Q}[[x,t]]$ is $U = F(x,0,t)$. This is a consequence of the following

Lemma 4.4. *Let $A, B, Y \in \mathbb{Q}[x,\overline{x}][[t]]$ with $ord_tB > 0$ and $ord_tY > 0$. Then there exists at most one $U \in \mathbb{Q}[[x,t]]$ such that*

$$U(x,t) = A(x,t) + B(x,t) \cdot U(Y(x,t),t).$$

Proof. Because of linearity we only have to show that the only solution in $\mathbb{Q}[[x,t]]$ of the homogeneous equation $U(x,t) = B(x,t) \cdot U(Y(x,t),t)$ is the trivial solution $U = 0$. Suppose U is non-zero. Then the valuation of $B(x,t) \cdot U(Y(x,t),t)$ is at least equal to $\mathrm{ord}_tB + \mathrm{ord}_tU$ and hence strict greater than the valuation of $U(x,t)$, a contradiction. $\square$

Now we are ready to prove

Theorem 4.5. *$F(x,y,t)$ is algebraic.*

Proof. The computer-assisted proof consists of two steps.

1. Guess an algebraic equation for the series $F(x,0,t)$ by inspecting its initial terms.
2. Prove that the equation guessed in the first step admits exactly one solution in $\mathbb{Q}[[x,t]]$, denoted by $F_{\text{cand}}(x,0,t)$, and that the power series $U = F_{\text{cand}}(x,0,t)$ satisfies the reduced kernel equation (K_{red}).

If this is accomplished, the fact that $U = F(x,0,t)$ satisfies (k_{red}) and the above lemma (with $A(x,t) = Y(x,t)/t$ and $B(x,t) = -Y(x,t)/x$) give us that $F_{\text{cand}}(x,0,t)$ and $F(x,0,t)$ coincide.
Hence the series $F(x,0,t)$ satisfies the guessed equation and is thus algebraic. Because $Y(x,t)$ is algebraic, too, and algebraic power series are closed under addition, multiplication and inversion we get that $F(x,y,t)$ is algebraic using equation (4.1). $\square$

Guessing

It is possible to determine potential equations that a power series may satisfy using its first few terms, for example by making an ansatz with underdetermined coefficients. In practice Gaussian elimination or algorithms based on Hermite-Padé approximation are used. Computing such candidate equations is called *automated guessing*.
If we know sufficiently many terms of the series, automated guessing will find an equation

whenever there is one. But this method has two possible drawbacks. First, it may return false equation (although this virtually never happens in practice). Second, the precision required to recover the equation is very and can take very long unless specific software and efficient algorithms are used.
The guessed polynomial for $F(x,0,t)$ that Bostan and Kauers obtained is

$$\begin{aligned} P(T,x,t) = {} & (16x^3t^4 + 108t^4 - 72xt^3 + 8x^2t^2 - 2t + x) \\ & + (96x^2t^5 - 48x^3t^4 - 144t^4 + 104xt^3 - 16x^2t^2 + 2t - x)T \\ & + (48x^4t^6 + 192xt^6 - 264x^2t^5 + 64x^3t^4 + 32t^4 - 32xt^3 + 9x^2t^2)T^2 \\ & + (192x^3t^7 + 128t^7 - 96x^4t^6 - 192xt^6 + 128x2t^5 - 32x^3t^4)T^3 \\ & + (48x^5t^8 + 192x^2t^8 - 192x3t^7 + 56x^4t^6)T^4 \\ & + (96x^4t^9 - 48x^5t^8)T^5 + 16x^6t^{10}T^6. \end{aligned}$$

Now the correctness of the guessed polynomial needs to be verified.

Proving

We are going to verify the two statements from step (2) in the proof of theorem 4.5.

First we show the existence and uniqueness of a root of P. Since $P(1,x,0) = 0$ and $\frac{\partial P}{\partial T}(1,x,0) = -x$, the implicit function theorem gives us that P admits a unique root $F_{\text{cand}}(x,0,t) \in \mathbb{Q}((x))[[t]]$. From this follows that P has at most one root in $\mathbb{Q}[[x,t]]$ and if it exists it belongs to $\mathbb{Q}[x,\overline{x}][[t]]$.
For proving the existence of a root the implicit function theorem does not work since $P(1,0,0) = 0$ and $\frac{\partial P}{\partial T}(1,0,0) = 0$. Instead this can be done by an argument using the fact that the polynomial $P(T,x,t)$ defines a curve of genus zero over $\mathbb{Q}(x)$ and can thus be rationally parametrized (see [6] for more details).

We also need to show that $F_{\text{cand}}(x,0,t)$ is compatible with the reduced kernel equation (4.2). This can be done by proving that the power series $S(x,t) \in \mathbb{Q}[x,\overline{x}][[t]]$ defined by

$$S(x,t) := \frac{Y(x,t)}{t} - \frac{Y(x,t)}{x} F_{\text{cand}}(Y(x,t),0,t)$$

is a root of the polynomial $P(T,x,t)$ and using the fact that the only root of P in $\mathbb{Q}[x,\overline{x}][[t]]$, namely $F_{\text{cand}}(x,0,t)$. Hence $S(x,t)$ and $F_{\text{cand}}(x,0,t)$ have to coincide and thus $F_{\text{cand}}(x,0,t)$ satisfies the equation $(\mathrm{K}_{\text{red}})$.
An annihilating polynomial of $S(x,t)$ can be computed by using the following

Lemma 4.6. *Let $\mathbb{K}$ be a field and $P, Q \in K[T,x,t]$ be annihilating polynomials of the power series $A, B \in \mathbb{K}[x,\overline{x}][[t]]$. Then the following hold*

1. *For every rational series $p \in \mathbb{K}(x,t)$ the series pA is algebraic and is a root of $p^{deg_T^P} P(T/p,x,t)$.*
2. *The series $A \pm B$ is algebraic and it is a root of $res_z(P(z,x,t), Q(\pm(T-z),x,t))$.*
3. *The product AB is algebraic and is a root of $res_z(P(z,t,x), z^{deg_T Q} Q(T/z,t,x))$*
4. *If $ord_x B > 0$, then the composition $A(B(x,t),t)$ is algebraic and is a root of $res_z(P(T,z,t), Q(z,x,t))$.*

Using these computations we get that an annihilating polynomial for $S(x,t)$ is $P(T,x,t)^2$, which proves that $S(x,t)$ is a root of $P(T,x,t)$.

Consequences

Setting $x = 0$ in P gives us that the generating series $F(0,0,t)$ of Kreweras excursions is a root of the polynomial $64t^6T^3 + 16t^3T^2 + T - 72t^3T + 54t^3 - 1$. From an argument similar to the one used in corollary 4.2 follows that the coefficients a_n of $F(0,0,t)$ satisfy the recursion

$$(n+6)(2n+9)a_{n+3} - 54(n+2)(n+1)a_n = 0 \quad a_0 = 1, a_1 = 0, a_2 = 0.$$

Solving this recursion gives us that the series $F(0,0,t)$ is algebraic and hypergeometric and has the closed form

$$F(0,0,t) = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, 1; \frac{3}{2}, 2; 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Proof that the generating function of Gessel walks is algebraic

For the proof that the generating function $G(x,y,t)$ of Gessel walks is algebraic one can use more or less the same proof as before. The main difference is that in the computation the intermediate expressions will become very big and can only be handled by special purpose software. There are also some arguments in the proof that are slightly different because of some complications.
Let $g(i,j,n)$ denote the number of Gessel walks of length n that end in $(i,j) \in \mathbb{N}^2$. They satisfy the recurrence

$$g(i,j,n+1) = g(i-1,j-1,n) + g(i+1,j+1,n) + g(i-1,j,n) + g(i+1,j,n)$$

for $n,i,j \geq 0$. Together with the boundary conditions we obtain the generating function

$$G(x,y,t) = \sum_{i,j,n \geq 0} g(i,j,n) x^i y^j t^n.$$

We want to prove that this series is algebraic. It satisfies the equation

$$((1+y+x^2y+x^2y^2)t - xy)G(x,y,t) = (1+y)tG(0,y,t) + tG(x,0,t) - tG(0,0,t) - xy. \quad (4.3)$$

Now we can apply the kernel method. But in this case we lack the x/y symmetry we had before. Hence there are two different ways to put the left hand side to zero, using the two substitutions

$$\begin{aligned} y \to Y(x,t) :&= -(tx^2 - x + t + \sqrt{(tx^2 - x + t^2)^2 - 4t^2x^2})/(2tx^2) \\ &= \frac{1}{x}t + \frac{x^2+1}{x^2}t^2 + \frac{x^4+3x^2+1}{x^3}t^3 + \dots \end{aligned}$$

and

$$\begin{aligned} x \to X(y,t) :&= (y - \sqrt{y(y - 4t^2(y+1)^2)})/(2ty(y+1)) \\ &= \frac{y+1}{y}t + \frac{(y+1)^3}{y^2}t^3 + \frac{2(y+1)^5}{y^3}t^5 + \dots \end{aligned}$$

They give us the two equations (4.4):

$$G(x,0,t) = \frac{Y(x,t)}{t} + G(0,0,t) - (1+Y(x,t))G(0,Y(x,t),t),$$
$$(1+y)G(0,y,t) = \frac{X(y,t)y}{t} + G(0,0,t) - G(X(y,t),0,t).$$

Observe that the first equation is free of x while the second is free of y. If we rename y to x in the second equation, then all terms belong to $Q[x,\overline{x}][[t]]$. Let us rewrite

$$G(x,0,t) = G(0,0,t) + xU(x,t)$$

and

$$G(0,x,t) = G(0,0,t) + xV(x,t)$$

for certain power series $U, V \in \mathbb{Q}[[x,t]]$. Using this the above equations are equivalent to (4.5):

$$xU(x,t) = \frac{xY(x,t)}{t} - (1+Y(x,t))G(0,0,t) - Y(x,t)(1+Y(x,t))V(Y(x,t),t),$$
$$(1+x)xV(x,t) = \frac{xX(x,t)}{t} - (1+x)G(0,0,t) - X(x,t)U(X(x,t),t).$$

These two equations correspond to the equation (4.2) in the section about Kreweras walks. But the situation here is a bit more complicated. First, we have two equations in two unknown power series U, V. This difference originates from the lack of symmetry of the Gessel step set with respect to the main diagonal. Second the two equation still contain the term $G(0,0,t)$, while it was not present in (4.2). This is because the Gessel step set contains the SW step while the Kreweras step set does not. But this occurrence is not really problematic. For the other difference we need an adaption of lemma 4.4.

Lemma 4.7. *Let $A_1, A_2, B_1, B_2, Y_1, Y_2 \in \mathbb{Q}[x,\overline{x}][[t]]$ with $ord_t B_i > 0$ and $ord_t Y_i > 0$ for $i = 1, 2$. Then there exists at most one pair $(U_1, U_2) \in \mathbb{Q}[[x,t]]^2$ with*

$$U_1(x,t) = A_1(x,t) + B_1(x,t) \cdot U_2(Y_1(x,t),t)),$$

$$U_2(x,t) = A_2(x,t) + B_2(x,t) \cdot U_1(Y_2(x,t),t)).$$

Proof. By linearity it is enough to show that the trivial solution $(U_1, U_2) = (0,0)$ is the only solution $(U_1, U_2) \in \mathbb{Q}[[x,t]]^2$ of the homogeneous system

$$U_1(x,t) = B_1(x,t) \cdot U_2(Y_1(x,t),t)),$$

$$U_2(x,t) = B_2(x,t) \cdot U_1(Y_2(x,t),t)).$$

Assume that U_1 and U_2 are both nonzero. Then the valuation of $B_1(x,t) \cdot U_2(Y_1(x,t),t)$ is strictly greater than the valuation of U_2. Also, the valuation of $B_2(x,t) \cdot U_1(Y_2(x,t),t)$ is greater than the valuation of $U_1(x,t)$. Putting this together we get $\text{ord}_t U_1 > \text{ord}_t U_2 > \text{ord}_t U_1$, a contradiction. Hence one of U_1 and U_2 is zero and the system implies that the other unknown is zero, too. $\square$

By a slightly adaption the lemma can be refined such as that there is only one triple of power series (U, V, G) with $U, V \in \mathbb{Q}[[x,t]]$ and $G \in \mathbb{Q}[[t]]$ (free of x) which satisfies (4.5). Now we can continue with the same procedure as before:

1. Guess algebraic equations for $U(x,t)$ and $V(x,t)$ by inspecting the initial terms of $G(x,0,t)$ and $G(0,x,t)$.

2. Prove that each of the two guessed equations has a unique solution in $\mathbb{Q}[[x,t]]$, denoted by $U_{\text{cand}}(x,0,t)$ and $V_{\text{cand}}(x,0,t)$ and that the power series $U_{\text{cand}}(x,0,t)$ and $V_{\text{cand}}(x,0,t)$ satisfy the two equations in (4.5).

Once this is done lemma 4.7 gives us that these candidate series are equal to U and V. Hence these series as well as $G(x,0,t)$ and $G(0,y,t)$ are algebraic, which implies by equation (4.3) that $G(x,y,t)$ is algebraic, too.

Guessing

Since the differential equations for $G(x,0,t)$ and $G(0,y,t)$ seemed to be very big, Bostan and Kauers tried a modular approach: they set x and y to specific values $x_0, y_0 = 1,2,3,\ldots$ and additionally kept the coefficients reduced modulo several fixed primes to avoid large numbers. Modulo a fixed prime p and considering the first 1000 terms of $G(x,0,t)$ and $G(0,y,t)$ they used a automated guessing scheme based on the Becker-Labahn (FFT-based) algorithm for computing Hermite-Padé approximations. They made the following observations:

- For any choice of p and x_0 there are several differential operators in $\mathbb{Z}_p[t]\langle D_t\rangle$ of order 14 and with coefficients of degree at most 43 which seem to annihilate $G(x_0,0,t)$ in $\mathbb{Z}_p[[t]]$.
- For any choice of p and y_0 there are several differential operators in $\mathbb{Z}_p[t]\langle D_t\rangle$ of order 15 and with coefficients of degree at most 34 which seem to annihilate $G(0,y_0,t)$ in $\mathbb{Z}_p[[t]]$.

Here and in the following D_t stands for the differential operator $\frac{d}{dt}$ and $R[t]\langle D_t\rangle$ denotes the Weyl algebra of differential operators with coefficients in the ring R.

Then they tried to apply an interpolation mechanism in order to reconstruct the two candidate operators that will annihilate $G(x,0,t)$ and $G(0,y,t)$ in $\mathbb{Q}[x][[t]]$ or $\mathbb{Q}[y][[t]]$ respectively, from the various choices of x_0, y_0 and p. But it turned out that an unreasonably large number of interpolation points $x_0, y_0 = 1,2,3,\ldots$ was needed, which suggested a large degree of the operators with respect to x or y, and the computation was aborted.

Their next attempt was to find candidate operators of smaller total size by trading order against degree. They considered the series $G(x_0,0,t)$ modulo p and tried to find the least order operator $\mathcal{L}^{(p)}_{x_0,0} \in \mathbb{Z}_p[t]\langle D_t\rangle$ that annihilates it. This can be achieved by taking several candidate operators and computing their greatest common right divisor in the rational Weyl algebra $\mathbb{Z}_p(t)\langle D_t\rangle$. Similarly they wanted to find the least order operator $\mathcal{L}^{(p)}_{0,y_0} \in \mathbb{Z}_p[t]\langle D_t\rangle$ annihilating the series $G(0,y_0,t)$.

Using several evaluation points $x_0, y_0 = 1,2,3,\ldots$ and several primes p it was possible to reconstruct two candidates $\mathcal{L}_{x,0} \in \mathbb{Q}[x,t]\langle D_t\rangle$ and $\mathcal{L}_{0,y} \in \mathbb{Q}[y,t]\langle D_t\rangle$ with reasonable degrees in x and y. These two operators are posted on Bostan and Kauers' website to their article [7]. The operator $\mathcal{L}_{x,0}$ has order 11, degree 96 in t and degree 78 in x. Its longest coefficient has 61 decimal digits. The operator $\mathcal{L}_{0,y}$ also has order 11. Its degree with respect to t is 68 and only 28 with respect to y. Its longest integer coefficient has 51 decimal digits.

The exceptionally small size of $\mathcal{L}_{x,0}$ and $\mathcal{L}_{0,y}$ (compared to the intermediate expressions) speaks in favour of their correctness. The fact that $\mathcal{L}_{x,0}$ and $\mathcal{L}_{0,y}$ satisfy

$$\mathcal{L}_{x,0}(G(x,0,t)) \equiv 0 \mod t^{1000} \quad \text{and} \quad \mathcal{L}_{0,y}(G(0,y,t)) \equiv 0 \mod t^{1000}$$

also provides some empirical evidence for their correctness.

As a next step Bostan and Kauers searched for possible potential polynomial equations satisfied by the power series $U, V \in \mathbb{Q}[[x,t]]$ defined by $G(x,0,t) = G(0,0,t) + xU(x,t)$ and $G(0,y,t) = G(0,0,t) + yV(y,t)$. Starting from 1200 terms and using guessing techniques based on fast modular Hermite-Padé approximation they found two polynomials $P_1(T,x,t) \in \mathbb{Q}[T,x,t]$ and $P_2(T,y,t) \in \mathbb{Q}[T,y,t]$ which fulfill

$$P_1(U(x,t),x,t) \equiv 0 \mod t^{1200} \quad \text{and} \quad P_2(V(y,t),y,t) \equiv 0 \mod t^{1200}.$$

These polynomials can also be viewed on their website [7]. The polynomial P_1 has degrees 24, 32 and 44 in T, x and t and its coefficients have at most 21 decimal digits. The polynomial P_2 has degrees 24, 56 and 46 in T, y and t and its coefficients have at most 27 decimal digits. If they both were spelled out explicitly they would together fill about thirty pages, while the operators $\mathcal{L}_{x,0}$ and $\mathcal{L}_{0,y}$ together fill over 500 pages.

Bostan and Kauers did a few heuristic test with the polynomials P_1 and P_2. Since they appeared plausible they went on with proving that these polynomials are indeed valid.

Proving

Let $P_1 \in \mathbb{Q}[T,x,t]$ and $P_2 \in [T,y,t]$ the polynomials guessed in the previous section. We will see that these polynomials admit unique power series solutions $U_{\text{cand}}(x,t)$ and $V_{\text{cand}}(x,t)$ and that these power series fulfill the reduced kernel equations (4.5).

The proof of the existence is here much more complicated than in the case of Kreweras' walks since the implicit function theorem does not apply to these polynomials and an existence proof using a suitable rational parametrization is not possible either, since the polynomials define curves of positive genus and therefore do not allow a rational parametrization. The proof Bostan and Kauers is rather lengthy and technical, hence I will only sketch it. The full proof can be read in [6]. It involves a theorem of McDonald to obtain the existence of a series solution

$$\sum_{p,q \in \mathbb{Q}} c_{p,q} x^p t^q$$

with $c_{p,q} = 0$ for all (p,q) outside a certain halfplane $H \subseteq \mathbb{Q}^2$. Then they computed a system of bivariate recurrences with polynomial coefficients that the coefficients $c_{p,q}$ have to satisfy and using the form of these recurrences they showed that the coefficients $c_{p,q}$ of any solution can be nonzero only in a finite union of cones $v + \mathbb{N}u + \mathbb{N}w$ with vertices $v \in \mathbb{Q}^2$ and base vectors $u, w \in \mathbb{Q}^2$ that can be computed explicitly. By applying McDonald's generalization of Puiseux' algorithm they determined the first coefficients of series solutions to an accuracy such that all further coefficients belong to an translate of H that does not contain any vertices. Since one of these partial solutions contained no terms with fractional powers it was possible to deduce that the entire series contains no expression with fractional powers. Considering u and w implied that there were no term with negative exponents either, so the only remaining possibility was that the solution is actually a power series.

Now it remains to show that these solutions U_{cand} and V_{cand} are compatible with the reduced kernel equation (4.5). Since $X(Y(x,t),t) = x$ the substitution $x \to Y(x,t)$ turns the second equation of the system into the first. Hence it suffices to prove only the second equation

$$(1+x)xV_{\text{cand}}(x,0,t) = \frac{X(x,t)x}{t} - (1+x)G(0,0,t) - X(x,t)U_{\text{cand}}(X(x,t),0,t).$$

Denote $G_1(x,t) := G(0,0,t) + xU_{\text{cand}}$ and $G_2(x,t) := G(0,0,t) + xV_{\text{cand}}(x,0,t)$. Then the above equation is equivalent to

$$(1+x)G_2(x,t) - G(0,0,t) = \frac{xX(x,t)}{t} - G_1(X(x,t),t).$$

By corollary 4.2 and lemma 4.6 the power series

$$(1+x)G_2(x,t) - G(0,0,t) \quad \text{and} \quad \frac{xX(x,t)}{t} - G_1(X(x,t),t)$$

are both algebraic and their minimal polynomials can be computed (but the required resultant computations are so big that specific software is needed). After determining a suitable number of initial terms of both series it can be observed that they match. Hence the above equation holds and the proof of theorem 4.3 is complete. □

Consequences

The fact that $G(x,y,t)$ has some consequences that are of combinatorical interest.

Corollary 4.8. *The following series are algebraic:*

- *The generating function $G(1,1,t)$ of Gessel walks with arbitrary endpoint.*
- *The generating functions $G(1,0,t)$ and $G(0,1,t)$ of Gessel walks ending on the x-axis or y-axis respectively.*

All these series and $G(0,0,t)$ can be expressed in terms of nested radicals, for example

$$G(1,1,t) = \frac{1}{6t}\left(-3 + \sqrt{3}\sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2U(t)} - U(t)^2 + 3}}\right)$$

where $U(t) = \sqrt{1 + 4\sqrt[3]{t(4t+1)^2/(4t-1)^4}}$.

The proof of theorem 4.3 did not give us an minimal polynomial of $G(x,y,t)$. But from the sizes of the minimal polynomials of $G(x,0,t)$ and $G(0,y,t)$ which are known explicitly it can be deduced that the minimal polynomial $p(T,x,y,t)$ of $G(x,y,t)$ has degrees 72, 263, 287 and 141 with respect to T,x,y and t. Thus it consists of more than 750 million terms.

Corollary 4.9. *For every fixed (i,j) the series $G_{i,j}(t) := \sum_{n=0}^{\infty} g(i,j,n)t^n$ is algebraic.*

Proof. Since

$$G_{i,j}(t) = \frac{1}{i!j!}\left(\frac{d^i}{dx^i}\frac{d^j}{dy^j}G(x,y,t)\right)|_{x=y=0}$$

and being algebraic is preserved under differentiation and evaluation the claim follows. □

The next result is of computational interest.

Corollary 4.10. *For fixed i and j the number $G(i,j,n)$ can be computed with $O(n)$ arithmetic operations.*
For fixed x and y the coefficient $\langle t^n\rangle\, G(x,y,t)$ can be computed with $O(n)$ arithmetic operations.

Proof. Since the coefficient sequence $G(i,j,n)$ is P-finite with respect to n by the previous corollary it satisfies an uniform recurrence with respect to n. Together with appropriate initial conditions this recurrence allows the computation of $g(i_0,j_0,n)$ in linear time.
The proof for the second statement works similar. □

4.2 Bostans, Kurkovas and Raschels proof

In their paper [8] Bostan, Kurkova and Raschel proved the following two statements about Gessel walks

(A) For all $n \geq 0$ we have that

$$g(0,0,2n) = 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n}$$

where $g(i,j,n)$ denoted the number of Gessel walks of length n ending in (i,j).
(B) The generating function

$$G(x,y,t) = \sum_{i,j,n\geq 0} g(i,j,n)x^iy^jt^n$$

is algebraic.

Their idea was using the fact that a function is algebraic if and only if it has finitely many branches. By explicitly constructing these branches Bostan, Kurkova and Raschel could prove the algebraicity.

Fix $t \in \left(0,\frac{1}{4}\right)$. To prove (A) and (B) consider the generating functions $G(x,0,t)$ and $G(y,0,t)$ and the functional equation

$$K(x,y,t)G(x,y,t) = K(x,0,t)G(x,0,t) + K(0,y,t)G(0,y,t) - K(0,0,t)G(0,0,t) - xy$$

where

$$K(x,y,t) = xyt\left(\sum_{(i,j)\in S} x^iy^j - \frac{1}{t}\right) = xyt\left(xy + x + \frac{1}{x} + \frac{1}{xy} - \frac{1}{t}\right)$$

is the kernel of the walk. This functional equation is defined for $|x| < 1$ and $|y| < 1$. We want to construct all branches of these functions. To do this we consider meromorphic continuations of $x \mapsto G(x,0,t)$ and $y \mapsto G(0,y,t)$ along an arbitrary path in the complex plane (not only in their natural domains of definition $\{x : |x| < 1\}$ and $\{y : |y| < 1\}$). Consider the elliptic curve defined by the zeros of the kernel

$$T_t = \{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : K(x, y, t) = 0 \quad (4.6)$$

and the universal covering of T_t. This is the complex plane $\mathbb{C}_\omega$ with a new variable ω. The functions $x \mapsto G(x, 0, t)$ and $y \mapsto G(0, y, t)$ can on their domain of definition be lifted to T_t and further to the corresponding regions of the universal covering, namely to $\{\omega \in \mathbb{C}_\omega : |x(\omega)| < 1\}$ and $\{\omega \in \mathbb{C}_\omega : |y(\omega)| < 1\}$. They are vertical stripes.
We thus have three levels of the lifting:

- The first level are the complex planes $\mathbb{C}_x$ and $\mathbb{C}_y$ where $x \mapsto G(x, 0, t)$ and $y \mapsto G(0, y, t)$ are defined.
- On the second level x and y are not independent anymore. The second level is given by T_t.
- The third and top level is the universal covering of T_t.

The key point of the argument Bostan, Kurkova and Raschel used is defining the lifted function $r_x(\omega) := K(x(\omega), 0, t)G(x(\omega), 0, t)$. Then

$$r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega) \quad (4.7)$$

holds for all $\omega \in \mathbb{C}_\omega$ where the shift vector ω_3 and the function f_x are explicitly known. A similar result holds for $G(0, y, t)$. Equation (4.7) has many consequences:

- According to (4.7) r_x can be extended from its domain of definition (which is a vertical stripe) to the entire of $\mathbb{C}_\omega$. By projection on $\mathbb{C}_x$ we obtain all branches of $G(x, 0, t)$.
- There is only an finite number of branches which gives us the algebraicity of $G(x, 0, t)$ and $G(0, y, t)$. With the functional equation we finally get the algebraicity of $G(x, y, t)$.
- The poles of r_x form a two-dimensional lattice and its residues are periodic. Hence r_x is and elliptic function (all poles have order 1). From this an explicit expression of r_x in terms of ζ-functions can be obtained. By projecting down to $\mathbb{C}_x$ we obtain a new expression for $G(x, 0, t)$ (and analogously for $G(0, y, t)$ and $G(x, y, t)$).
- Evaluating $G(x, 0, t)$ at $x = 0$ and using some simplifications gives us problem (A):

$$g(0, 0, 2n)) = 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n}.$$

Meromorphic continuation of the generating functions

First we are going to consider the branch points. For the sake of simplicity and since t is supposed to be fixed we are going to drop t in the notation and write T instead of T_t and $G(x, y)$ instead of $G(x, y, t)$ if there is no confusion. The kernel $K(x, y)$ defines a polynomial of degree two in x and y. Hence the algebraic function $X(y)$ defined by $K(X(y), y) =$ has two branches and for branching points, called $y_1, \ldots, y_4$. They are the zeros of the discriminant of the equation $K(x, y) = 0$ in x:

$$\tilde{d}(y) = (-y)^2 - 4t(y^2 + y)(y + 1).$$

We have that $y_1 = 0, y_4 = \infty = \frac{1}{y_1}$ and $y_2 = \frac{1-8t^2-\sqrt{1-16t^2}}{8t^2}, y_3 = \frac{1-8t^2+\sqrt{1-16t^2}}{8t^2} = \frac{1}{y_2}$. They are ordered such that $y_1 < y_2 < y_3 < y_4$. Because there are four different branching points the Riemannian surface $X(y)$ is a torus T^y. Similar results hold for $Y(x)$ defined by $K(x, Y(x))$. the branching points are the zeros of

$$d(x) = (tx^2 - x + t)^2 - 4t^2x^2 \quad (4.8).$$

These zeros are real and $x_1 = \frac{1+2t-\sqrt{1+4t}}{2t}, x_2 = \frac{1-2t-\sqrt{1-4t}}{2t}, x_3 = \frac{1}{x_2}$ and $x_4 = \frac{1}{x_1}$. Again they are ordered such that $x_1 < x_2 < x_3 < x_4$. The Riemannian surface $Y(x)$ is also a torus T^x. Since T^x and T^y are equivalent we are going to consider only one Riemannian surface T with two coverings $x, y : T \to S$.

Next we are going to look at the universal covering. The torus T is isomorphic to $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ where ω_1 and $\omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. They can be interpreted as the fundamental parallelogram spanned by ω_1 and ω_2 glued together at its edges. The periods ω_1 and ω_2 are (up to unimodular transformation) unique. In our case they are given by

$$\omega_1 = i\int_{x_1}^{x_2} \frac{dx}{\sqrt{-d(x)}} \quad \text{and} \quad \omega_2 = i\int_{x_2}^{x_3} \frac{dx}{\sqrt{d(x)}}.$$

The universal covering of T has the form $(\mathbb{C}, \lambda)$ where $\mathbb{C}$ is the complex plane, which is the union of infinitely many fundamental parallelograms

$$\Pi_{m,n} = \omega_1[m, m+1) + \omega_2[n, n+1)$$

glued together and $\lambda : \mathbb{C} \to T$ is a non-branching covering map. For arbitrary $\omega \in \mathbb{C}$ with $\lambda\omega = s \in T$ is $x(\lambda\omega) = x(s)$ and $y(\lambda\omega) = y(s)$. The uniformation formulas are

$$x(\omega) = x_4 + \frac{d'(x_4)}{\wp(\omega)d''(x_4)/6} \quad (4.9)$$

$$y(\omega) = \frac{1}{2a(x(\omega))}\left(-b(x(\omega) + \frac{d'(x_4)\wp(\omega)}{2(\wp(\omega) - d''(x_4)/6)^2}\right) \quad (4.10)$$

where $a(x) = tx^2$, $b(x) = tx^2 - x + t$, d is defined as above and $\wp$ is the Weierstraß elliptic function with periods ω_1 and ω_2, i.e.

$$\wp(\omega) = \frac{1}{\omega^2} + \sum_{\substack{\gamma=m\omega_1+n\omega_2 \\ m,n\in\mathbb{Z}^2\setminus(0,0)}} \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}.$$

Let us write $x(\lambda\omega) = x(\omega)$ and $y(\lambda\omega) = y(\omega)$. According to (4.9) and (4.10) these are elliptic functions on $\mathbb{C}$ with periods ω_1 and ω_2. They satisfy

$$K(x(\omega), y(\omega)) = 0 \quad \forall\omega \in \mathbb{C} \quad (4.11).$$

Since each parallelogram $\Pi_{m,n}$ represents a torus, the function $x(\omega)$ and respectively $y(\omega)$ attains each value of $\mathbb{C} \cup \{\infty\}$ twice (aside the branch points $x_1, \dots, x_4$ respectively $y_1, \dots, y_4$). The points $\omega_{x_i} \in \Pi_{0,0}$ where $x(\omega_{x_i}) = x_i$ are

$$\omega_{x_1} = \frac{\omega_2}{2} \quad \omega_{x_2} = \frac{\omega_1 + \omega_2}{2} \quad \omega_{x_3} = \frac{\omega_1}{2} \quad \omega_{x_4} = 0.$$

The points $\omega_{y_i} \in \Pi_{0,0}$ where $y(\omega_{y_i}) = y_i$ are shifts of ω_{x_i} with $\frac{\omega_3}{2}$, i.e. $\omega_{y_i} = \omega_{x_i} + \frac{\omega_3}{2}$ where ω_3 is given by

$$\omega_3 = \int_{-\infty}^{x_1} \frac{dx}{\sqrt{d(x)}}.$$

For Gessel walks for all $t \in \left(0, \frac{1}{4}\right)$ holds: $\frac{\omega_3}{\omega_2} = \frac{3}{4}$ (4.12).

Galois Automorphisms

As we saw in section 3.2, the functions

$$\xi(x,y) = \left(x, \frac{1}{x^2 y}\right) \quad \text{and} \quad \eta(x,y) = \left(\frac{1}{xy}, y\right) \quad (4.13)$$

leave the quantity $\sum_{(i,j)\in S} x^i y^j$ invariant and span the group of the walk. We have that

$$\xi^2 = \eta^2 = \mathrm{id}(4.14).$$

The automorphisms ξ and η are defined on $\mathbb{C}^2 = \mathbb{C}_x \times \mathbb{C}_y$.
Now we want to lift them on the second level, the torus T. Each point $s \in T$ has two "coordinates" $(x(s), y(s))$. By construction, they fulfil $K(x(s), y(s)) = 0$. For $s \in T$ arbitrary exists a unique s' (respectively s'') with $x(s) = x(s')$ (respectively $y(s) = y(s'')$). The values $x(s)$ and $x(s')$ are the two zeros of the equation $K(x, y(s)) = 0$. The automorphism $\xi : T \to T$ is defined by the identity $\xi s = s'$ and is a Galois-automorphism. The automorphism $\eta : T \to T$ is defined analogously by $\eta s = s''$. The formulas (4.13) and (4.14) imply that for arbitrary $s \in T$ the following hold:

$$x(\xi s) = x(s) \quad y(\xi s) = \frac{1}{x^2(s)y(s)} \quad x(\eta s) = \frac{1}{y(s)x(s)} \quad y(\eta s) = y(s) \quad \xi^2(s) = \eta^2(s) = s$$

and $\xi s = s$ if and only if $x(s) = x_i, i \in \{1, \dots 4\}$ (analogously for $\eta s = s$ if and only if $y(s) = y_i$).
There are many ways to lift ξ and η to the torus T. Here we want that ω_{x_2} and ω_{y_2} are their fixed points, i.e.

$$\xi\omega = -\omega + \omega_1 + \omega_2 \quad \text{and} \quad \eta\omega = -\omega + \omega_1 + \omega_2 + \omega_3 \forall \omega \in \mathbb{C}.$$

Recall that $\omega_1 + \omega_2 = 2\omega_{x_2}$ and $\omega_1 + \omega_2 + \omega_3 = 2\omega_{y_2}$. Hence we have that

$$x(\xi\omega) = x(\omega) \quad \text{and} \quad y(\eta\omega) = y(\omega) \quad \forall \omega \in \mathbb{C}.$$

Lifting of the generating functions to the universal covering

The domains

$$\{\omega \in \mathbb{C} : |x(\omega) < 1\} \quad \text{and} \quad \{\omega \in \mathbb{C} : |y(\omega) < 1\}$$

consist of infinitely many curvilinear strips that differ by translation with a multiple of ω_2. The strips that lie in $\bigcup_{n\in\mathbb{Z}} \Pi_{0,n}$ respectively $\bigcup_{n\in\mathbb{Z}} \Pi_{0,n} + \frac{\omega_3}{2}$ are called Δ_x respectively Δ_y. The domain Δ_x (respectively Δ_y) is bounded by vertical lines. The functions $G(x(\omega), 0)$ respectively $G(0, y(\omega))$ are well defined on Δ_x respectively Δ_y. Let

$$r_x(\omega) = K(x(\omega), 0)G(x(\omega), 0) \quad \forall \omega \in \Delta_x$$
$$r_y(\omega) = K(0, y(\omega))G(0, y(\omega)) \quad \forall \omega \in \Delta_y.$$

The domain $\Delta_x \cap \Delta_y$ is a nonempty open strip. From the functional equation and (4.11) follows that

$$r_x(\omega) + r_y(\omega) - K(0,0)G(0,0) - x(\omega)y(\omega) = 0 \quad \forall \omega \in \Delta_x \cap \Delta_y. \quad (4.15)$$

Meromorphic continuation of the generationg function on the universal covering

Let Δ be the union of Δ_x and Δ_y. Because of (4.15) the functions r_x and r_y can be continued meromorphically on Δ:

$$r_x(\omega) = -r_y(\omega) + K(0,0)G(0,0) + x(\omega)y(\omega) \quad \forall \omega \in \Delta_x$$

$$r_y(\omega) = -r_x(\omega) + K(0,0)G(0,0) + x(\omega)y(\omega) \quad \forall \omega \in \Delta_y.$$

To continue these functions from Δ to the entire complex plane $\mathbb{C}$ we use that $\cup_{n\in\mathbb{Z}}(\Delta + n\omega_3) = \mathbb{C}$. Let

$$f_x(\omega) = y(\omega)[x(-\omega + 2\omega_{y_2}) - x(\omega)] \quad (4.16)$$

$$f_y(\omega) = x(\omega)[y(-\omega + 2\omega_{x_2})y(\omega)] \quad (4.17).$$

Then the following results hold:

Lemma 4.11. *The functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically to $\mathbb{C}$. For ω in $\mathbb{C}$ arbitrary holds*

$$\begin{aligned}
&r_x(\omega - \omega_3) = r_x(\omega) + f_x(\omega) \\
&r_y(\omega + \omega_3) = r_y(\omega) + f_y(\omega) && (4.18) \\
&r_x(\omega) + r_y(\omega) - K(0,0)G(0,0) - x(\omega)y(\omega) = 0 && (4.19) \\
&r_x(\xi\omega = r_x(\omega) \quad \text{and} \quad r_y(\eta\omega) = r_y(\omega) && (4.20) \\
&r_x(\omega + \omega_1) = r_x(\omega) \quad \text{and} \quad r_y(\omega + \omega_1) = r_y(\omega). && (4.21)
\end{aligned}$$

Theorem 4.12. *We have that*

$$G(0,0) = \frac{\zeta_{1,3}\left(\frac{\omega_2}{4}\right) - 3\zeta_{1,3}\left(\frac{\omega_2}{2}\right) + 2\zeta_{1,3}\left(\frac{3\omega_2}{4}\right) + 3\zeta_{1,3}\left(\omega_2\right) - 5\zeta_{1,3}\left(\frac{5\omega_2}{4}\right) + 2\zeta_{1,3}\left(\frac{3\omega_2}{2}\right)}{2t^2}$$

where $\zeta_{1,3}$ is the Weierstraß-ζ-function with periods ω_1 and $3\omega_2$.

Theorem 4.13. *For all $\omega \in \mathbb{C}$ holds*

$$\begin{aligned}
r_y(\omega) = c &+ \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{\omega_2}{8}\right) - \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{3\omega_2}{8}\right) + \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{11\omega_2}{8}\right) \\
&- \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{13\omega_2}{8}\right) - \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{15\omega_2}{8}\right) + \frac{1}{t}\zeta_{1,3}\left(\omega - \frac{17\omega_2}{8}\right) \\
&- \frac{1}{t}\zeta_{1,3}\left(\omega - \frac{21\omega_2}{8}\right) + \frac{1}{2t}\zeta_{1,3}\left(\omega - \frac{23\omega_2}{8}\right)
\end{aligned}$$

where c is a constant.

The constant c can be computed explicitly: $c = tG(0,0) - \hat{\zeta}_{1,3}\left(\frac{7\omega_2}{8}\right)$ where $\hat{\zeta}_{1,3}$ denotes the above sum of eight ζ-functions.
A similar expression holds for $r_x(\omega)$ with a different constant. Later we will use theorem 4.12 and 4.13 to obtain explicit expressions for $G(x,0,t)$ and $G(0,y,t)$. With the functional equation we then obtain an expression for $G(x,y,t)$. Before we prove these theorems we will need a few preliminary results about the poles of f_y.

Lemma 4.14. *In the fundamental parallelogram $\omega_1[0,1)+\omega_2[0,1)$ the function f_y has poles in $\frac{\omega_2}{8}, \frac{3\omega_2}{8}, \frac{5\omega_2}{8}$ and $\frac{7\omega_2}{8}$. They are simple poles with residues $-\frac{1}{2t}, \frac{1}{2t}, \frac{1}{2t}$ and $-\frac{1}{2t}$ respectively.*

Lemma 4.15. *In the fundamental parallelogram $\omega_1[0,1)+\omega_2[0,1)$ are the only poles of $x(\omega)$ (of order 1) $\frac{\omega_2}{8}$ and $\frac{7\omega_2}{8}$ and the only zeros (of order 1) are $\frac{3\omega_2}{8}$ and $\frac{5\omega_2}{8}$. The only pole of $y(\omega)$ (of order 2) is $\frac{3\omega_2}{8}$ and its only zero (of order 2) is $\frac{7\omega_2}{8}$.*

Proof of Lemma 4.14: With the definition of $f_y(\omega)$ in (4.16) and (4.17) and the uniformation formulas (4.9) and (4.10) we get $f_y(\omega) = \frac{1}{2t}\frac{x'(\omega)}{x(\omega)}$. Hence, if $x(\omega)$ has a simple zero respectively simple pole at ω_0 then $f_y(\omega)$ has a simple pole at ω_0 with residue $\frac{1}{2t}$ respectively $-\frac{1}{2t}$. Lemma 4.14 then follows from Lemma 4.15. □

Lemma 4.16. *The function r_y is elliptic with periods ω_1 and $3\omega_2$.*

Proof. According to (4.21) r_y is meromorphic and ω_1-periodic. Lemma 4.11 gives us that

$$r_y(\omega+4\omega_3) - r_y(\omega) = f_y(\omega) + f_y(\omega+\omega_3) + f_y(\omega+2\omega_3) + f_y(\omega+3\omega_3) \quad \forall \omega \in \mathbb{C}.$$

According to Lemma 4.14 and (4.12) the elliptic function $\theta(\omega) = \sum_{k=0}^{3} f_y(\omega + k\omega_3)$ has no poles in $\mathbb{C}$, hence it must be constant and $r_y(\omega+4\omega_3) = r(\omega) + c$ for all $\omega \in \mathbb{C}$. In particular, we have that $r_y(\omega_{y_2} - 4\omega_3) + 2c = r_y(\omega_{y_2} + 4\omega_3)$. From (4.20) follows that $r_y(\omega_{y_2} - 4\omega_3) = r_y(\omega_{y_2} + 4\omega_3)$ which implies that $c = 0$. It follows that $r_y(\omega)$ is $4\omega_3 = 3\omega_2$-periodic, hence elliptic with periods ω_1 and $3\omega_2$. □

Proof of theorem 4.13: Since r_y is elliptic with periods ω_1 and $3\omega_2$ by the previous lemma and since every elliptic fucntion is characterized by its poles in the fundamental parallelogram it suffices to consider r_y in $\omega_1[-\frac{1}{2},\frac{1}{2}) + \omega_2[-\frac{5}{2},\frac{1}{2})$. Using a result about the main parts at poles (see [23], theorem 6) we get that a pole d of r_y fulfills $\mathcal{N}_d^- \neq \emptyset$ where

$$\mathcal{N}_d^- = \left\{ n \in \mathbb{N} : d + n\omega_3 \text{ is pole of } f_y \text{ with } -\frac{5\omega_2}{2} < \operatorname{Re}(d+n\omega_3) < \frac{\omega_2}{2} \right\}.$$

Recall that $\frac{\omega_2}{2}$ is real since $\operatorname{Re}\omega_{x_1} = \frac{\omega_2}{2}$. If a point d fulfills $\mathcal{N}_d^- \neq \emptyset$ then r_y has main part $R_{d,y}$ where

$$R_{d,y}(\omega) = \sum_{n \in \mathcal{N}_d^-} -\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right) F_{d+n\omega_3,y}(\omega + n\omega_3) \quad (4.22)$$

where $F_{d+n\omega_3,y}$ denotes the main part of f_y at the pole $d+n\omega_3$. Hence we have to find the poles in $\omega_1\left[\frac{1}{2},\frac{1}{2}\right) + \omega_2\left[-\frac{5}{2},\frac{1}{2}\right)$. According to lemma 4.14 they are $P = \{\frac{3\omega_2}{8} - \frac{2k\omega_2}{8} : k \in \{0,\dots 11\}\}$. Hence $\mathcal{N}_d^- = \{n \in \mathbb{N} : d + n\omega \in P\}$. Then the points d in the fundamental parallelogram for which $\mathcal{N}_d^- \neq \emptyset$ lie in P. Let's consider them individually:

For $k \in \{0,1,2\}$ are the points $d = \frac{3\omega_2}{8}, \frac{\omega_2}{8}, -\frac{\omega_2}{8} \in P$ such that $\mathcal{N}_d^- = \{0\}$. Hence there is only one term in formula (4.22), namely $R_{d,y}(\omega) = -F_{d,y}(\omega)$. Lemma 4.14 then implies that r_y has a pole of order 1 in d with residues $-\frac{1}{2t}, \frac{1}{2t}, \frac{1}{2t}$ respectively.

For $k \in \{3,4,5\}$ the corresponding points $d = -\frac{3\omega_2}{8}, -\frac{5\omega_2}{8}, -\frac{7\omega_2}{8} \in P$ are such that $\mathcal{N}_d^- = \{0,1\}$. A similar argument as above gives us that among them $d = -\frac{3\omega_2}{8}$ is the only pole of r_y. It has residue $-\frac{1}{t}$.

For $k \in \{6,7,8\}$ and $d = -\frac{9\omega_2}{8}, -\frac{11\omega_2}{8}, -\frac{13\omega_2}{8}$ we have that $\mathcal{N}_d^- = \{0,1,2\}$. All these points are simple poles of r_y with residues $-\frac{1}{2t}, -\frac{1}{2t}$ and $\frac{1}{2t}$ respectively.

For $k \in \{9,10,11\}$ the points d are $-\frac{15\omega_2}{8}, -\frac{17\omega_2}{8}$ and $-\frac{19\omega_2}{8}$ with $\mathcal{N}_d^- = \{0,1,2,3\}$. For them, all main parts are zero, hence these points are no poles.

To conclude the proof we use the following fact about elliptic functions: For any elliptic function f with periods $\tilde{\omega}, \hat{\omega}$ that has in the fundamental parallelogram $[0,\tilde{\omega}) + [0,\hat{\omega})$ only poles $\tilde{\omega}_1, \ldots, \tilde{\omega}_p$ of order one with residues $r_1, \ldots, r_p$ there exists a constant c such that f can be written as

$$f(\omega) = c + \sum_{\ell=1}^{p} r_\ell \zeta(\omega - \tilde{\omega}_\ell).$$

Using this we get the claim. □

Proof of theorem 4.12: Equation (4.19) gives us $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + K(0,0)G(0,0)$. We compute $K(0,0)G(0,0)$ using $r_y(\omega_0^y) = K(0, y(\omega_0^y))G(0, y(\omega_0^y))$ where $\omega_0^y \in \Delta_y$ such that $y(\omega_0^y) = 0$. Lemma 4.15 gives us an unique solution for ω_0^y, namely $v\omega_0^y = \frac{7\omega_2}{8}$. Hence $r_x(\omega) = x(\omega)y(\omega) - r_y(\omega) + r_y(\frac{7\omega_2}{8})$. Let's substitute $\omega := \frac{5\omega_2}{8}$ in this equation. The point $\frac{5\omega_2}{8}$ is a zero of $x(\omega)$ lying in Δ_x, such that

$$r_x\left(\frac{5\omega_2}{8}\right) = K\left(x\left(\frac{5\omega_2}{8}\right), 0\right) G\left(x\left(\frac{5\omega_2}{8}\right), 0\right) = K(0,0)G(0,0) = tG(0,0).$$

This point is not a pole of $y(\omega)$, hence $x(\frac{5\omega_2}{8})y(\frac{5\omega_2}{8}) = 0$. This gives us that

$$tG(0,0) = r_y\left(\frac{7\omega_2}{8}\right) - r_x\left(\frac{7\omega_2}{8}\right) \quad (4.23).$$

Using theorem 4.13 and (4.23) we can write $G(0,0)$ as a sum of sixteen $\zeta_{1,3}$-Weierstraß-functions (evaluated at a rational multiple of ω_2). By using the fact that $\zeta_{1,3}$ is an odd function and some properties of ζ-functions we can do some simplifications in (4.23) to obtain the claim of theorem 4.12. □

Proof of Gessel's conjecture (Problem A)

Our strategy is to use theorem 4.12, which gives us $G(0,0)$ as a linear combination of Weierstraß functions (the individual terms are transcendental functions) and rearrange them in such a way that we have a linear combination of algebraic hypergeometric series.

Sketch of the proof: More precisely, we have

$$G(0,0) = \frac{1}{2t}\left({}_2F_1\left(-\frac{1}{2}, -\frac{1}{6}; \frac{2}{3}; 16t^2\right) - 1\right).$$

Or, in view of theorem 4.12, Gessel's conjecture is equivalent to

$$F_1 - 3F_2 + 2F_3 + 3F_4 - 5F_5 + 2F_6 = G - 1 \quad (4.24)$$

where $G =_2 F_1\left(-\frac{1}{2}, -\frac{1}{6}; \frac{2}{3}; 16t^2\right)$ and $F_k = \zeta_{1,3}(\frac{k\omega_2}{4})$ for $1 \le k \le 6$. Define $V_{i,j,k} := F_i + F_j - F_k$. Then the left hand side of (4.24) becomes $4V_{1,4,5} - V_{2,4,6} - V_{1,5,6} - 2V_{1,2,3}$.

To prove (4.24) we are going to prove the identities

$$V_{1,4,5} = \frac{2G+H}{3} - \frac{K}{2} \quad (4.25) \qquad V_{2,4,6} = \frac{2G+H}{3} - K \quad (4.26)$$
$$V_{1,5,6} = \frac{J+1}{2} \quad (4.27) \qquad V_{1,2,3} = \frac{2G+2H-J-2K+1}{4} \quad (4.28)$$

where G is defined as above. The functions H, J and K are auxiliary functions defined as $H := {}_2F_1(-\frac{1}{2}, \frac{1}{6}; \frac{1}{3}; 16t^2)$, $J := (G-K)^2$ and $K = tG' = 4t^2{}_2F_1(\frac{1}{2}, \frac{5}{6}; \frac{5}{3}; 16t^2)$. Gessel's conjecture then follows from (4.25)-(4.28). Summation of these four equations gives us

$$4V_{1,4,5} - V_{2,4,6} - V_{1,5,6} - 2V_{1,2,3} = G - 1.$$

It remains to show the equations (4.25)-(4.28). To show this we show that their valuations at $t = \left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}$ are equal. This choice is inspired by the Darboux-covering of hypergeometric series of Schwarz type $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$. The series G, H, J, K belong to this class. Using a corollary of the Frobenius-Stickelberger-Identity, which states that if $\alpha+\beta+\gamma = 0$ then $(\zeta(\alpha)+\zeta(\beta)+\zeta(\gamma))^2 = \wp(\alpha)+\wp(\beta)+\wp(\gamma)$, gives us that $V_{i,j,k} = \sqrt{T_i + T_j + T_k}$ if $k = i+j$ and $T_\ell = \wp_{1,3}(\frac{\ell\omega_2}{4})$. Using some properties of $\wp$- and ζ-functions $T_\ell\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right)$ can be computed for $1 \leq k \leq 6$ and thus also $V_{1,4,5}, V_{2,4,6}, V_{1,5,6}, V_{1,2,3}$ at $t = \left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}$. Then the equations (4.25)-(4.28) can be verified at $T = \left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}$.

Some preliminary results

Let ζ and $\wp$ be the elliptic functions with periods ω_1 and ω_2 and $\zeta_{1,3}$ and $\wp_{1,3}$ those with periods ω_1 and $3\omega_2$. An elliptic function can be characterized by its periods or by its invariants. Denote with g_2 and g_3 the invariants of $\wp$. They are such that $\wp'(\omega) = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3$ for all *omega* $\in \mathbb{C}$. They can be computed explicitly via the following Lemma.

Lemma 4.17. *The invariants g_2 and g_3 are given by*

$$g_2 = \frac{4}{3}(1-16t^2+16t^4), \quad g_3 = -\frac{8}{27}(1-8t^2)(1-16t^2+8t^4) \quad (4.29).$$

(without proof)

The invariants $g_2^{1,3}$ and $g_3^{1,3}$ of $\wp_{1,3}$ are defined analogously. To compute them, define R to be the unique positive zero of

$$X^4 - 2g_2X^2 + 8g_3 - \frac{g_2^2}{3} = 0 \quad (4.30).$$

With equations (*33) and (*34) we have that $R(t) = 2 + 16t^2 + 48t^4 + O(t^6)$ locally at zero. The following lemmas will give us $g_2^{1,3}$ and $g_3^{1,3}$ as well as $T_\ell = \wp_{1,3}(\frac{\ell\omega_2}{4})$ in terms of R.

Lemma 4.18. *We have that*

$$T_4 = \wp_{1,3}(\omega_2) = \frac{R}{6}, \quad g_2^{1,3} = -\frac{g_2}{9} + \frac{10R^2}{27}, \quad g_2^{1,3} = -\frac{35R^3}{729} + \frac{7g_2R}{243} - \frac{g_3}{27}.$$

(without proof)

Lemma 4.19. *The following results hold:*

- $T_1 = \wp_{1,3}(\frac{\omega_2}{4})$ *is the unique solution of*

$$X^3 - \left(\frac{R}{3} + \frac{1+4t^2}{3}\right)X^2 + \left(\frac{R(1+4t^2)}{9} + \frac{R^2}{9} + \frac{g_2}{18}\right)X$$
$$+ \left(\frac{23R^2}{2916} - \frac{R^2(1-4t^2}{108} + \frac{g_3}{27} - \frac{19Rg_2}{972}\right) = 0 \quad (4.31).$$

  with $T_1(t) = \frac{1}{3} + \frac{4t^2}{3} - 4t^4 - 56t^6 + O(t^8)$ *near zero.*

- $T_2 = \wp_{1,3}(\frac{2\omega_2}{4})$ *is* $T_2 = \frac{R+1-8t^2}{6} - \frac{T_6}{2}$.

- $T_3 = \wp_{1,3}(\frac{3\omega_2}{4})$ *is another solution of (4.31) such that* $T_3(t) = \frac{1}{3} - \frac{8t^2}{3} - 8t^4 - 60t^6 + O(t^8)$ *near zero.*

- $T_5 = \wp_{1,3}(\frac{5\omega_2}{4})$ *is another solution of (4.31) such that* $T_3(t) = \frac{1}{3} - \frac{8t^2}{3} - 8t^4 - 64t^6 + O(t^8)$ *near zero.*

- $T_6 = \wp_{1,3}(\frac{6\omega_2}{4})$ *is* $T_6 = \frac{R+1-8t-\sqrt{3R^2-4R(1+8t^2)+4(1-8t^2)^2-6g_2}}{9}$.

(without proof)

Next we need explicit expressions for T_ℓ evaluated at $t = \left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}$ to continue with the proof of Gessel's conjecture.

Lemma 4.20. *The following equations hold*

$$T_1\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = \frac{4x^4+28x^3+30x^2+10x+1}{3(4x+1)^3} + \frac{2x(x+1)(2x+1)}{(4x+1)^{5/2}},$$
$$T_2\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = \frac{4x^4+16x^3+12x^2+4x+1}{3(4x+1)^3},$$
$$T_3\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = \frac{4x^4+4x^3+4x+1}{3(4x+1)^3},$$
$$T_4\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = \frac{(2x^2-2x-1)^2}{3(4x+1)^3},$$
$$T_5\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = \frac{4x^4+28x^3+30x^2+10x+1}{3(4x+1)^3} - \frac{2x(x+1)(2x+1)}{(4x+1)^{5/2}},$$
$$T_6\left(\left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}\right) = -\frac{8x^4+8x^3-4x-1}{3(4x+1)^3}.$$

Proof. These equations are consequences of lemma 4.19. Starting with R we replace t by $\varphi(x) = \left(\frac{x(x+1)^3}{(4x+1)^3}\right)^{1/2}$ in equation (4.30). Then we factor the result and identify the corresponding minimal polynomial of $R(\varphi(x))$ in $\mathbb{Q}(x)[T]$. To do so, we use the local expansion of $R(\varphi(x))$ near zero, which is $R(\varphi(x)) = 2 - 16x + O(x^2)$. The minimal polynomial has degree one, which proves

$$R(\varphi(x)) = \frac{2(2x^2 - 2x - 1)^2}{(4x-1)^3}.$$

Using that $T_4 = \frac{R}{6}$ we get the expression for $T_4(\varphi(x))$. Replacing t by $\varphi(x)$ in lemma 4.19 gives us the expression for $T_6(\varphi(x))$ and similarly an expression for $T_2(\varphi(x))$ is obtained. An annihilating polynomial for $T_1(\varphi(x)), T_3(\varphi(x))$ and $T_5(\varphi(x))$ can be found using lemma 4.19. This polynomial in $\mathbb{Q}(x)[T]$ is product of a quadratic and a linear factor. Using the local expansions $\frac{1}{3} + \frac{4}{3}x - 12x^2 + 80x^3 + O(x^4), \frac{1}{3} - \frac{8}{3}x + 16x^2 + 84x^3 + O(x^4)$ and $\frac{1}{3} - \frac{8}{3}x - 16x^2 + 88x^3 + O(x^4)$ the above formulas can be deduced. □

From the definitions of $V_{i,j,k}$ and the previous lemmas follows

Corollary 4.21. *The algebraic functions $V_{1,4,5}, V_{2,4,6}, V_{1,5,6}$ and $V_{1,2,3}$ satisfy*

$$V_{1,4,5}(\varphi(x)) = \frac{2x^2+4x+1}{(4x+1)^{3/2}}, \qquad V_{2,4,6}(\varphi(x)) = \frac{2x+1}{(4x+1)^{3/2}},$$
$$V_{1,5,6}(\varphi(x)) = \frac{2x+1}{4x+1}, \qquad V_{1,2,3}(\varphi(x)) = \frac{x}{4x+1} + \frac{(x+1)(2x+1)}{(4x+a)^{3/2}}.$$

To finish the proof of Gessel's conjecture we need to show equations (4.25)-(4.28) (where $V_{i,j,k}$ were expressed in terms of G, H, K and J). The starting point is that the functions $G =_2 F_1(-\frac{1}{2}, -\frac{1}{6}; -\frac{2}{3}; 16t^2)$ and $H =_2 F_1(-\frac{1}{2}, \frac{1}{6}; \frac{1}{3}; 16t^2)$ are algebraic and fulfill the equations from the following lemma.

Lemma 4.22. *We have that*

$$G(\varphi(x)) = \frac{4x^2+8x+1}{(4x+1)^{3/2}}, \quad H(\varphi(x)) = \frac{4x^2+2x+1}{(4x+1)^{3/2}},$$
$$K(\varphi(x)) = \frac{4x(x+1)}{(4x+1)^{3/2}}, \quad J(\varphi(x)) = \frac{1}{4x+1}$$

(without proof)

With corollary 4.21 and lemma 4.22 we can prove equations (4.25)-(4.28) evaluated at $z = \varphi(x)$. For example we have that equation (4.25) evaluated at $t = \varphi(x)$ is

$$\frac{2x^2+4x+1}{(4x+1)^{3/2}} = \frac{2}{3}\frac{4x^2+8x+1}{(4x+1)^{3/2}} + \frac{1}{3}\frac{4x^2+2x+1}{(4x+1)^{3/2}} - \frac{1}{2}\frac{4x(x+1)}{(4x+1)^{3/2}}.$$

The proof of the formula for the Gessel excursions is now complete, i.e. $G(0,0,t)$ is algebraic. Next we will see that $G(x,y,t)$ is algebraic as well.

Proof of (B)

First we will show that $G(0, y)$ is algebraic in y. To do so we will need some properties about elliptic functions, the can be found for example in [1].

Property 1: For $\tilde{\omega}_1, \ldots, \tilde{\omega}_p \in \mathbb{C}$ fixed we have that $f(\omega) := c + \sum_{1\leq \ell \leq p} r_\ell \zeta(\omega - \tilde{\omega}_\ell)$ is elliptic if and only if $\sum_{\ell=1}^{p} = 0$.

The sum of the residues in theorem 4.13 is zero, hence $r_y(\omega)$ is by the above property an algebraic function of $\wp_{1,3}(\omega)$. According to the following property we have that $\wp_{1,3}$ is algebraic in $\wp(\omega)$ and according to (4.9) and (4.10) we have that $\wp(\omega)$ is algebraic in $y(\omega)$.

Property 2: Let $p \in \mathbb{Z}^+$. Then the Weierstraß elliptic function with periods $\overline{\omega}, \frac{\hat{\omega}}{p}$ can be written as

$$\wp(\omega) + \sum_{1\leq \ell \leq p-1} \wp\left(\omega + \frac{\ell\hat{\omega}}{p}\right) - \wp\left(\frac{\ell\hat{\omega}}{p}\right) \quad \forall \omega \in \mathbb{C}.$$

Analogously we get that $Q(x, 0)$ is algebraic. With the help of the functional equation follows that $G(x, y, t)$ is algebraic in x and y. Now it only remains to show that it is algebraic in t as well.

Proof that $G(x, y, t)$ is algebraic in x, y and t

First we will show that $G(0, y, t)$ is algebraic as an function in y and t. To do so, consider the representation of r_y from theorem 4.13 and apply property 3 to get

$$\wp_{1,3}\left(\omega - \frac{k\omega_2}{8}\right) = \wp_{1,3}(\omega) - \wp_{1,3}\left(\frac{k\omega_2}{8}\right) + \frac{1}{2}\frac{\wp_{1,3}'(\omega) + \wp_{1,3}'(\frac{k\omega_2}{8})}{\wp_{1,3}(\omega) + \wp_{1,3}(\frac{k\omega_2}{8})}.$$

Property 3: For all $\omega, \tilde{\omega} \in \mathbb{C}$ we have that

$$\zeta(\omega + \tilde{\omega}) = \zeta(\omega) + \zeta(\tilde{\omega}) + \frac{1}{2}\frac{\wp'(\omega) - \wp'(\tilde{\omega})}{\wp(\omega) - \wp(\tilde{\omega})}$$

$$\wp(\omega + \tilde{\omega}) = -\wp(\omega) - \wp(\tilde{\omega}) - \frac{1}{4}\left(\frac{\wp'(\omega) - \wp'(\tilde{\omega})}{\wp(\omega) - \wp(\tilde{\omega})}\right)^2$$

Consider the weighted sum of the eight above identities (for the eight "good" values for k) and obtain

$$r_y(\omega) = U_1(\omega) + U_2(\omega) + U_3(\omega)$$

where $U_1(\omega)$ is the weighted sum of the eight functions $\zeta_{1,3}(\omega)$, U_2 is the sum of the constants c and the weighted sum of the functions $\zeta_{1,3}(\frac{k\omega}{8})$ and U_3 is the weighted sum of

$$\frac{\wp_{1,3}'(\omega) + \wp_{1,3}'(\frac{k\omega_2}{8})}{\wp_{1,3}(\omega) + \wp_{1,3}(\frac{k\omega_2}{8})} \quad (4.32).$$

Since the sum of the residues in theorem 4.13 is zero, the coefficient of $\zeta_{1,3}(0)$ is zero and hence $U_1(\omega) \equiv 0$. That U_2 is algebraic follows from a similar proof with regrouping and using the Frobenius-Stickelberger-Equation and the addition theorems for ζ-functions.
It remains to show that there is a nonzero polynomial P such that $P(U_3(\omega), y(\omega)) = 0$, i.e. $U_3(\omega)$ is algebraic. To achieve this we show that each term of 4.32 satisfies the above equation (with a different polynomial).

Lemma 4.23. *For $k \in \mathbb{Z}$ arbitrary and $\ell \in \mathbb{Z}^+$ arbitrary we have that $\wp^{(\ell)}\left(\frac{k\omega_2}{8}\right)$ and $\wp_{1,3}^{(\ell)}\left(\frac{k\omega_2}{8}\right)$ are algebraic functions of t or infinite. (without proof)*

Lemma 4.23 then implies that $\wp_{1,3}(\frac{k\omega_2}{8})$ and $\wp'_{1,3}(\frac{k\omega_2}{8})$ are algebraic in t. Property 2 gives us that $\wp_{1,3}(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in t. The same holds for $\wp'_{1,3}(\omega)$ (this follows from the above fact and the differential equation for Weierstraßfunctions). Analogously we get that $G(x, 0)$ is algebraic. With the functional equation for G we can finally deduce that $G(x, y, t)$ is algebraic in x, y and t.

4.3 Bousquet-Mélou's proof

In March 2015, Bousquet-Mélou came up with a third proof for Gessel's Conjecture (see BM-ele). This proof is constructive and the tools used there can also be generalized to other models. It relies on a generalization of the kernel method and remains on the level of formal power series and polynomial equations.

Theorem 4.24. *The generating function $Q(x, y, t)$ of the Gessel walks in the quarter plane is algebraic over $\mathbb{Q}(x, y, t)$ and of degree 72. The specialization $Q(0, 0, t)$ has degree 8. It can be written as*

$$Q(0,0,t) = \frac{32Z^3(3+3Z-3Z^2+z^3)}{(1+z)(Z^3+3)^3}$$

where $Z = \sqrt{T}$ and T is the unique power series in t with constant term 1 that satisfies

$$T = 1 + 256t^2 \frac{T^3}{(T+3)^3}.$$

The series $Q(xt, 0, t)$ is an even series in t with coefficients in $\mathbb{Q}[x]$. It is cubic in t and can be written as

$$Q(xt,0,t) = \frac{16T(U+UT-2T)M(U,Z)}{(1-t)(T+3)^3(U+Z)(U^2-9T+8TU+T^2-TU^2)}$$

where

$$\begin{aligned} M(U,Z) =&(T-1)^2U^3 + Z(T-1)(T-16Z-1)U^2 \\ &- TU(T^2+16ZT-82T-16Z+17) - ZT(T^2-18T+128Z+81) \end{aligned}$$

and U is the unique power series in t with constant term 1 (and coefficients in $\mathbb{Q}[x]$) that satisfies

$$16T^2(U^2-T) = x(U+UT-2T)(U^2-9t+8TU+T^2-TU^2).$$

The series $Q(0, y, t)$ is cubic over $\mathbb{Q}(Z, y)$ and can be written as

$$Q(0,y,t) = \frac{16VZ(3+V+T-VT)N(V,Z)}{(T-1)(T+3)^3(1+V)^2(1+Z+V-VZ)^2}$$

where

$$N(V,Z) = (Z-1)^2(T+3)V^3+(T-1)(T+2Z-7)V^2+(T-1)(T-2Z-7)V+(Z+1)^2(T+3)$$

and V is the only series in t with constant term zero that satisfies

$$1-T+3V+VT = yV^2(3+V+T-VT).$$

Via step-by-step construction of the walk we obtain

$$xyK(x,y)Q(x,y) = xy - t(Q(x,0) - Q(0,0)) - t(1+y)Q(0,y) \quad (4.33)$$

where

$$K(x,y) = 1 - t(\overline{x} + \overline{xy} + x + xy)$$

is the kernel of the walk. As we already saw in section 3.2, K remains invariant under $\Phi(x,y) = (\overline{xy}, y)$ and $\Psi(x,y) = (x, \overline{x}^2\overline{y})$. These transformations generate a group of order 8.
Viewed as a Polynomial in y, the kernel $K(x,y)$ has two zeros

$$Y_0(x) = \frac{1 - t(x + \overline{x} - \sqrt{(1 - t(x+\overline{x})^2 - 4t^2}}{2tx} = \overline{x}t + O(t^2)$$

and

$$Y_0(x) = \frac{1 - t(x + \overline{x} + \sqrt{(1 - t(x+\overline{x})^2 - 4t^2}}{2tx} = \frac{\overline{x}}{t} + (1 + \overline{x}^2) - \overline{x}t + O(t^2).$$

The expressions $xY_i(x)$ are symmetric in x and $\overline{x}$, i.e.

$$\overline{x}Y_i(\overline{x}) = xY_i(x).$$

We also have that the elementary symmetric functions $Y_0 + Y_1 = -1 - \frac{\overline{x}}{t} - \overline{x}^2$ and $Y_0Y_1 = \overline{x}^2$ are polynomials in $\overline{x}$. This property will play an important role in the proof. In the following lemma we will see how to extract the constant term from a symmetric polynomial.

Lemma 4.25. *Let $P(u,v)$ a symmetric polynomial in u and v. Then $P(Y_0, Y_1)$ is a polynomial in $\overline{x}$ with constant term $P(0,-1)$.*

Proof. Since every symmetrical polynomial in u and v is a polynomial in $u+v$ and uv and since $Y_0 + Y_1$ and Y_0Y_1 are both polynomials in $\overline{x}$ the first statement follows. It suffices to check the second statement only for polynomials of the form $P(u,v) = u^m v^n + u^n v^m$ because of linearity. If $\min(m,n) > 0$ then $P(Y_0, Y_1)$ has a factor $\overline{x}^2$ and hence its constant term is zero. Otherwise $P(u,v) = u^n + v^n$ and then can be shown by induction on n that the constant term is 2 if $n = 0$ and $(-1)^n$ for $n > 0$, which proves the claim. $\square$

Consider the orbit of (x, Y_0) under the action of the group G:

$$\begin{aligned}
&(x, Y_0) \overset{\Phi}{\leftrightarrow} (xY_1, Y_0) \overset{\Psi}{\leftrightarrow} (xY_1, x^2Y_0) \overset{\Phi}{\leftrightarrow} (\overline{x}, x^2Y_1) \overset{\Psi}{\leftrightarrow} \\
&(\overline{x}, x^2Y_1) \overset{\Phi}{\leftrightarrow} (xY_0, x^2Y_1) \overset{\Psi}{\leftrightarrow} (xY_0, Y_1) \overset{\Phi}{\leftrightarrow} (x, Y_1) \overset{\Psi}{\leftrightarrow} (x, Y_0).
\end{aligned}$$

According to the construction each of these pairs (x', y') annihilates the kernel $K(x,y)$. If the series $Q(x', y')$ is well-defined we can deduce from (4.33) that

$$R(x') + S(y') = x'y'$$

where

$$R(x) = t(Q(x,0) - Q(0,0)) \quad \text{and} \quad S(y) = t(1+y)Q(0,y).$$

Since Y_0 is a power series in t we can substitute (x, Y_0) and $(\overline{x}, x^2Y_0)$ for (x,y) in $Q(x,y)$. Because xY_0 is symmetric in x and $\overline{x}$, we can obtain these pairs from another by replacing x by $\overline{x}$. For the Gessel step set, each monom $x^i y^j t^n$ in $Q(x,y,t)$ satisfies $n+i-j \geq \frac{n}{2}$. Since

$Y_0 = \Theta(t)$ and $Y_1 = \Theta(\frac{1}{t})$ this implies that (xY_0, Y_1) and (xY_0, x^2Y_1) can be substituted for (x, y) in $Q(x, y)$. Hence we obtain four equations

$$R(x) + S(Y_0) = xY_0 \qquad (4.34)$$
$$R(xY_0) + S(Y_1) = \overline{x} \qquad (4.35)$$
$$R(\overline{x}) + S(x^2Y_0) = xY_0 \qquad (4.36)$$
$$R(xY_0) + S(x^2Y_1) = x \qquad (4.37)$$

From this equations we want to obtain an equation that relates $R(x)$ and $R(\overline{x})$. We add the first two equations and subtract the last two. From this we obtain

$$R(x) - S(x^2Y_0) - S(x^2Y_1) + x = R(\overline{x} - S(Y_0) - S(Y_1) + \overline{x}.$$

Lemma 4.25 tells us that the right hand side is a series in t with coefficients in $\mathbb{Q}[\overline{x}]$. The left hand side is obtained by replacing x by $\overline{x}$. Hence both sides are independent of x and thus equal a constant term which equals $S(0) + S(-1)$ according to Lemma 4.25. Since $S(y)$ is a multiple of $(1+y)$, this constant is $S(0)$. Hence we obtain a new set of equations

$$S(Y_0) + S(Y_1) = R(\overline{x}) + \overline{x} + S(0).$$

Combining this with (4.34) and (4.36) we get

$$S(Y_1) - xY_1 = R(x) + R(\overline{x} + 2\overline{x} + S(0). \quad (4.38)$$

For the second symmetric function of Y_0 and Y_1 we get by multiplying (4.34) and (4.38) that

$$(S(Y_0) - xY_0)(S(Y_1) - xY_1) = -R(x)(R(x) + R(\overline{x}) + 2\overline{x} - \frac{1}{t} + x + S(0)).$$

Extracting the nonnegative part in x gives us the nonnegative part of $R(x)R(\overline{x})$:

$$1 + (x - \frac{1}{t})S(0) = -R(x^2) - [x^{\geq}]R(x)R(\overline{x}) - (2\overline{x} - \frac{1}{t} + x + S(0))R(x).$$

Extracting the constant term in x gives us

$$1 - \frac{S(0)}{t} = -[x^0]R(x)R(\overline{x}) - 2R'(0).$$

Since $R(x)R(\overline{x})$ is symmetric in x and $\overline{x}$ it can be reconstructed from $[x^{\geq}]R(x)R(\overline{x})$

$$\begin{aligned} R(x)R(\overline{x}) =& [x^{\geq}]R(x)R(\overline{x}) + [x^{\leq}]R(x)R(\overline{x}) + [x^0]R(x)R(\overline{x}) \\ =& -R(x^2) - (2\overline{x} - \frac{1}{t} + x + S(0))R(x) - R(\overline{x})^2 \\ & - (2x - \frac{1}{t} - \overline{1}t + \overline{x} + S(0))R(\overline{x}) - 1 - (\overline{x} + x - \frac{1}{t}S(0) + 2R'(0)). \end{aligned}$$

This can be rewritten as

$$R(x)^2 + R(x)R(\overline{x}) + R(\overline{x})^2 + (2\overline{x} - \frac{1}{t} + x + S(0))R(x) + (2x - \frac{1}{t} + \overline{x} + S(0))R(x)$$
$$= 2R'(0) - (\overline{x} + x - \frac{1}{t})S(0) - 1. \quad (4.39)$$

Next, we want to deduce an equation for $R(x)$ only. But we can't extract the positive part from the above equation explicitly (because of the "hybrid term" $R(x)R(\overline{x})$). Multiply (4.34) with $R(x) - R(\overline{x} + \overline{x} + x$ to separate series in x from series in $\overline{x}$ and obtain

$$P(x) = P(\overline{x}) \quad (4.40)$$

where

$$\begin{aligned} P(x) =& R(x)^3 + (S(0) + 3\overline{x} - 1/t)R(x)^2 + (2\overline{x}^2 - \overline{x}/t + x/t - x^2 - 2R'(0) + (2\overline{x} - 1/t)S(0))R(x) \\ & - x^2 S(0) + x(2R'(0) + S(0)/t - 1). \quad (4.41) \end{aligned}$$

Since $R(x)$ is a multiple of x, each term in the expansion of P has an x-exponent of at least -1. But (4.40) implies that $P(x)$ is an symmetric Laurent-polynomial in x of degree 1 and valuation -1. Hence

$$P(x) = [x^0]P(x) + (x + \overline{x})[x]P(x).$$

By expanding (4.41) in x at 0 we obtain

$$P(x) = 2(x + \overline{x})R'(0) + R'(0)(2S(0) - 1/t) + R''(0).$$

Returning to (4.41) we get

$$\begin{aligned} & R(x)^3 + (S(0) + 3\overline{x} - 1/t)R(x)^2 + (2\overline{x}^2 - \overline{x}/t + x/t - x^2 - 2R'(0) + (2\overline{x} - 1)S(0)/t)R(x) \\ & = R''(0) + R'(0)(2S(0) + 2\overline{x} - 1/t) + xS(0)(x - \overline{x}) + x. \quad (4.42) \end{aligned}$$

Hence $R(x)$ fulfills a cubic equation over $\mathbb{Q}(t, x, S(0), R'(0), R''(0))$:

$$\tilde{P}(R(x), S(0), R'(0), R''(0), t, x) = 0,$$

where

$$\begin{aligned} \tilde{P}(x_0, x_1, x_2, x_3, t, x) =& x_0^3 + (x_1 + 3\overline{x} - 1/t) + (2\overline{x}^2 - \overline{x}/t + x/t - x^2 - 2x_2 + (2\overline{x} - 1/t)x_1)x_0 \\ & - x_3 - x_2(2x_1 + 2\overline{x} - 1/t) - xx_1(x - 1/t) - x. \quad (4.43) \end{aligned}$$

Hence, if $S(0), R'(0)$ and $R''(0)$ are algebraic over $\mathbb{Q}(t)$ then $R(x)$ is algebraic over $\mathbb{Q}(x, t)$.

The generalized quadratic method

We consider an equation of the form

$$\tilde{P}(R(x), A_1, \ldots, A_n, t, x) = 0 \quad (4.44)$$

where $\tilde{P}(x_0, x_1, \ldots, x_n, t, x)$ is a polynomial with rational coefficients, $R(x) = R(x, t)$ is a formal power series with coefficients in $\mathbb{Q}[x]$ and $A_1, \ldots A_n$ are auxiliary series depending only on t. The idea is to find a power series $X(t) \equiv X$ which fulfills

$$\frac{\partial \tilde{P}}{\partial x_0}(R(x), A_1, \ldots, A_n, t, X) = 0. \quad (4.45)$$

Differentiating (4.44) with respect to x gives us that each such series fulfills

$$\frac{\partial \tilde{P}}{\partial x}(R(x), A_1, \ldots, A_n, t, X) = 0. \quad (4.46)$$

With the help of equations (4.44) and (4.45) we obtain relations relating the $k+2$ unknown series $R(X), A_1, \dots, A_n$ and X. If we are able to find k different series $X_1, \dots, X_k$ that fulfill (4.45), we obtain $3k$ equations in the $3k$ unknowns $R(X_1), \dots, R(X_k), A_1, \dots, A_k, X_1, \dots, X_k$. If there is no redundancy among them we are able to solve the system and obtain that each of these $3k$ series is algebraic over $\mathbb{Q}(t)$.

Now we want to apply this strategy to our problem (4.42). Here we have $A_1 = S(0), A_2 = R'(0)$ and $A_3 = R''(0)$. Equation (4.45) becomes

$$\begin{aligned} R(X)^2 + 2\left(S(0) + \frac{3}{X} - \frac{1}{t}\right) R(X) + \frac{2}{X^2} - \frac{1}{tX} \\ - \frac{1}{tX} + \frac{X}{t} - X^2 - 2R'(0) - \left(\frac{2}{X} - \frac{1}{t}\right) S(0) = 0. \quad (4.47) \end{aligned}$$

The series $R(x)$ and $S(0)$ are multiples of t. Multiply above equation with tX^2 and obtain

$$X(1-X)(1+X) = t\tilde{P}(Q(X,0), R'(0), Q(0,0), t, x).$$

Hence there are exactly three series in t, called X_0, X_1 and X_2 that annihilate (4.47). Their constant terms are $0, 1$ and -1. We obtain

$$tx^2 \left(\frac{\partial \tilde{P}}{\partial x_0} + x^2 \frac{\partial \tilde{P}}{\partial x} \right) = (1-x)(1+x)(2tx^2 + 2t - x)(x_0 x + x_1 x + 1). \quad (4.48)$$

Since the series X_i cancel both partial derivatives of $\tilde{P}$, it follows that $X_1 = 1$, $X_2 = -1$ and X_0 is the only power series in t which annihilates $2tX + 2t - X$.The fourth factor on the right hand side ca not vanish for $x_0 = R(x)$ and $x_1 = S(x)$ (because of the factor t in these series).
Let $D(x)$ be the discriminant of $\tilde{P}(x_0, S(0), R'(0), R''(0), t, x)$ with respect to x_0. We have that $D(X_i) = 0$ for $i = 0, 1, 2$. Thus we obtain three polynomial equations for $S(0), R'(0), R''(0), t$ and X_0. Furthermore, $D(x)$ is symmetric in x and $\overline{x}$ and is thus a polynomial in $s = x + \overline{x}$. We can alternatively say that this polynomial vanishes at $s = \pm 2$ and $s = \frac{1}{2t}$. Elimination in the system of the three equations obtained via D gives us algebraic equations for $S(0), R'(0)$ and $R''(0)$. They are of degree 8,4 and 8 respectively. With them we can derive an algebraic equation for $R(x)$ and using the functional equation for the recurrence we also obtain an algebraic equation for $Q(x, y)$.

Rational parametrizations

To avoid using large polynomials we are going to use rational parametrizations. The equation fulfilled by $R'(0)$ is

$$\begin{aligned} 729t^6 R'(0)^4 + 243t^4(4t^2+1)R'(0)^3 - 27t^2(14t^4 + 19t^2 - 1)R'(0)^2 \\ - (20t^2 - 1)(7t^2 - 6t + 1)(7t^2 + 6t + 1)R'(0) - t^2(343t^2 - 37t^2 + 1) = 0. \end{aligned}$$

It has genus zero and can be parametrized if we find a power series $T = T(t)$ with constant term 1 that fulfils

$$T = 1 + 256t^2 \frac{T^3}{(T+3)^3}.$$

Then we have that

$$R'(0) = \frac{(T-1)(21 - 6T + T^2)}{(T+3)^3}.$$

The equation fulfilled by $S(0)$ is

$$\begin{aligned}
&27t^7S(0)^8+108t^6S(0)^7+189t^5S(0)^6+189^t4S(0)^5-9t^3(32t^4+28t^2-13)S(0)^4\\
&-9t^2(64t^4+56t^2-5)S(0)^3-2t(256t^6-312t^4+156t^2-5)S(0)^2\\
&-(32t^2-1)(4t^2-6t+1)(4t^2+6t+1)S(0)-t(256t^6+576t^4-48t^2+1)=0.
\end{aligned}$$

Because $S(0)=tQ(0,0)$ we obtain an equation for $Q(0,0)$ that only contains even powers of t. If we replace t^2 by its rational expression in T, the equation of degree 8 in $Q(0,0)$ factors in a term of degree 6 and in a quadratic term. If we insert the first few coefficients of $Q(0,0)$ we see that the quadratic term vanishes. Hence $Q(0,0)$ has degree 2 over $\mathbb{Q}(T)$ and can be written as $Z=\sqrt{T}$ like in theorem 4.24:

$$Q(0,0)=\frac{32Z^3(3+3Z-3Z^2+Z^3)}{(1+Z)(Z^3+3)^3}.$$

The power series $R''(0)$ also has a rational expression in Z:

$$R''(0)=2t[x^2]Q(x,0)=1024t\frac{Z^3(Z-1)(1+2Z+7Z^2-z^4-2Z^5+Z^6)}{(1+Z9(3+Z^2)^6}.$$

Return to equation (4.42) which is satisfied by $R(x)=t(Q(x,0)-Q(0,0))$. If we inject in it the above expressions for $S(0)=tQ(0,0)$, $R'(0)$ and $R''(0)$ we obtain a cubic equation for $Q(x,0)$ over $\mathbb{Q}(t,Z,x)$. Instead of $Q(x,0)$ consider now $Q(xt,0)$ since it is an even series in t. Using

$$T=1+256t^2\frac{T^3}{(T+3)^3}$$

to express t^2 in terms of $T=z^2$ we see that this power series is cubic over $\mathbb{Q}(Z,x)$.
We now can construct a equation for $Q(0,y)$ via the kernel equation(4.34). For xt instead of x this equation is

$$xt(Q(xt,0)-Q(0,0))+t(1+y)Q(0,y)=xyt$$

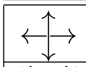
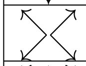
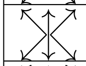
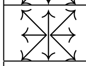
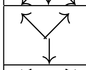
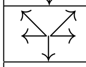
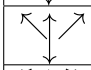
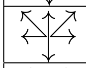
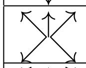
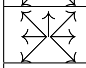
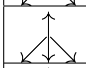
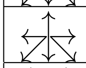
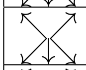
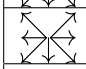
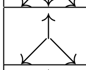
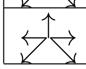
with $y=Y_0(xt)$. Hence the equation for $Q(xt,0)$ gives us a cubic equation for $Q(0,y)$. Eliminating x between this equation and $K(xt,0)=0$ gives us a cubic equation for $Q(0,y)$ over $\mathbb{Q}(Z,y)$.
Instead of giving explicit equations for $Q(xt,0)$ and $Q(0,y)$ they can be parametrized since they have genus zero. The parametrization is obtained via computing the parametrization for a few values of Z and reconstructing the parametrization from them. This way the parametrization of theorem 4.24 is obtained.

The generalized quadratic method also works for proving the algebraicity for some other step sets. It also turned out to be useful for proving the algebraicity of walks with multiple steps. Such walks arise after the projection of a 3D walk in the first octant.

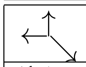
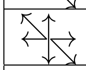
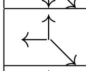
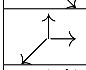
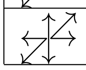
5 Tables

The tables below list the 79 non-equivalent non-trivial walks in the quarter plane ordered by the cardinality of their group. Any models not in the table either differ from one of the models below by an x/y-symmetry or have an algebraic generating function.

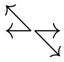
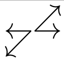
The following 16 walks have a group isomorphic to D_2 of order 4. They all have a D-finite generating function

step set	$G(S)$	properties of the GF
	$(x,y),(\overline{x},y),(\overline{x},\overline{y}),(x,\overline{y})$	D-finite, not algebraic
		D-finite, not algebraic
		D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y),(\overline{x},\frac{\overline{y}}{x+\overline{x}}),(x,\frac{\overline{y}}{x+\overline{x}})$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y),(\overline{x},\frac{\overline{y}}{x+1+\overline{x}}),(x,\frac{\overline{y}}{x+1+\overline{x}})$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y)(\overline{x},\overline{y}\frac{x+\overline{x}}{x+1+\overline{x}}),(x,\overline{y}\frac{x+\overline{x}}{x+1+\overline{x}})$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y),(\overline{x},\overline{y}(x+1+\overline{x})),(x,\overline{y}(x+1+\overline{x}))$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y),(\overline{x},\overline{y}\frac{x+1+\overline{x}}{x+\overline{x}}),(x,\overline{y}\frac{x+1+\overline{x}}{x+\overline{x}})$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{x},y),(\overline{x},\overline{y}(x+\overline{x})),(x,\overline{y}(x+\overline{x}))$	D-finite, not algebraic
		D-finite, not algebraic

The following five walks have a group isomorphic to D_3 (order 6). All of them have a D-finite generating function. The last three even have an algebraic generating function.

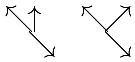
step set	$G(S)$	properties of the GF
	$(x,y),(\overline{x}y,y),(\overline{x}y,\overline{x}),(\overline{y},\overline{x}),(\overline{y},\overline{y}x),(x,\overline{y}x)$	D-finite, not algebraic
		D-finite, not algebraic
	$(x,y),(\overline{xy},y),(\overline{xy},x),(y,x),(y,\overline{xy}),(x,\overline{xy})$	algebraic
		algebraic
		algebraic

There are two walks with group isomorphic to D_4 (order 8). They both have a D-finite generating function. One of them has even an algebraic generating function.

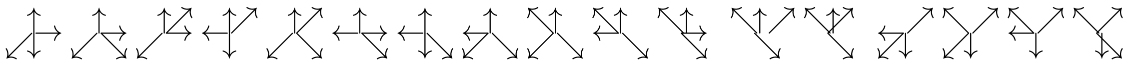
step set	$G(S)$	properties of the GF
	$(x,y),(y\overline{x},y),(y\overline{x},y\overline{x}^2),(\overline{x},y\overline{x}^2),$ $(\overline{x},\overline{y}),(x\overline{y},\overline{y}),(x\overline{y},x^2\overline{y}),(x,\overline{y}x^2)$	D-finite, not algebraic
	$(x,y),(\overline{xy},y),(\overline{xy},x^2y),(\overline{x},x^2y),$ $(\overline{x},\overline{y}),(xy,\overline{y}),(xy,\overline{x}^2\overline{y}),(x,\overline{yx}^2)$	algebraic

The following 56 step sets, ordered by their cardinality, are associated with a infinite group. Their generating function is not D-finite.

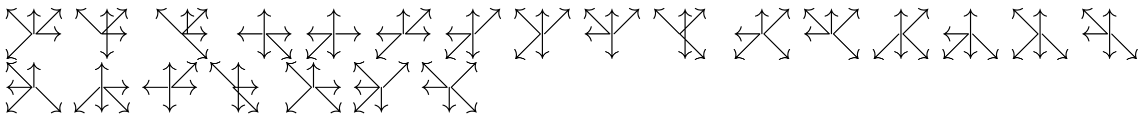
Step sets of cardinality 3:



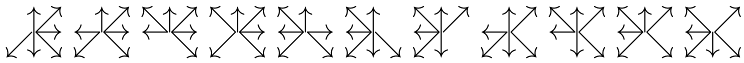
Step sets of cardinality 4:



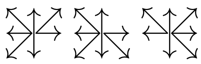
Step sets of cardinality 5:



Step sets of cardinality 6:



Step sets of cardinality 7:



6 Other techniques

In this section we are going to see a few techniques my thesis advisor and I discussed that might turn out to be useful for proving algebraicity or D-finiteness results.

6.1 Division of formal power series

Remember that in chapter 2.2 we wrote the functional equation for the generating function as

$$K(\mathbf{x}) = Q(\mathbf{x})F_{\mathbf{s}}(\mathbf{x}) + U(\mathbf{x})$$

where $Q(\mathbf{x})$ was the kernel of the recurrence, $F_{\mathbf{s}} = \sum_{\mathbf{n}\in\mathbf{s}+\mathbb{N}^d} f_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ was the "interesting" part of the generating function and $K(\mathbf{x})$ and $U(\mathbf{x})$ were the known respectively unknown initial function. This equation can be interpreted as Euclidean division with remainder: the series $K(\mathbf{x})$ is divided by the polynomial $Q(\mathbf{x})$ with quotient $F_{\mathbf{s}}(\mathbf{x})$ and remainder $U(\mathbf{x})$. The advantage of this interpretation is that there are some results on the division of power series, the so called division theorems.
Since in our setting we have more than one variable we need to fix an monomial order. We will take the weight vector $\mathbf{w}$ from theorem 2.3, such that $\mathbf{x}^{\mathbf{p}}$, where $\mathbf{p}$ is the apex of the recurrence, is the inital monom of Q. We had that $\mathbf{w}\cdot\mathbf{t} < 0$ for all $\mathbf{t}\in H$ which implies $\mathbf{x}^{\mathbf{p}} <_{\mathbf{w}} \mathbf{x}^{\mathbf{p}-\mathbf{t}}$ for all $t \in H$. Hence $\mathbf{x}^{\mathbf{p}}$ is $<_{\mathbf{w}}$-minimal in supp(Q). We know that $U(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]^{\ngeq\mathbf{p}}$, which matches exactly the condition imposed on the remainder in Euclidean division.

Example: The division of a power series $P(\mathbf{x})$ by a monom $\mathbf{x}^{\mathbf{n}}$, $\mathbf{n}\in\mathbb{N}^d$ is equivalent to the direct sum decomposition

$$\mathbb{K}[[\mathbf{x}]] = \mathbf{x}^{\mathbf{n}}\mathbb{K}[[\mathbf{x}]] \oplus [[\mathbf{x}]]^{\ngeq\mathbf{n}}$$

where the identity is considered as an identity of vector spaces. We obtain that $P(\mathbf{x}) = \mathbf{x}^n F(\mathbf{x}) + R(\mathbf{x})$ where the remainder $R(\mathbf{x})$ has to fulfill the condition supp$(R) \subseteq \mathbb{K}[[\mathbf{x}]]^{\nleq\mathbf{n}}$. We have that $\mathbb{K}[[\mathbf{x}]]^{\nleq\mathbf{n}} \cong \mathbb{K}[[\mathbf{x}]]/\langle\mathbf{x}^{\mathbf{n}}\rangle$ (again as vector spaces).

We can generalize this concept to power series $A(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]$ with initial monom $\mathbf{x}^{\mathbf{n}}$ with respect to a fixed monomial order $<_{\mathbf{w}}$. We obtain that

$$\mathbb{K}[[\mathbf{x}]] = A(\mathbf{x})\mathbb{K}[[\mathbf{x}]] \oplus \mathbb{K}[[\mathbf{x}]]^{\ngeq\mathbf{n}}.$$

Consider the mapping $u : \mathbb{K}[[\mathbf{x}]] \times \mathbb{K}[[\mathbf{x}]]^{\ngeq\mathbf{n}} \to \mathbb{K}[[\mathbf{x}]]$ given by $(B, C) \mapsto AB + C$. This is an $\mathbb{K}$-linear isomorphism. We can split it in $u = v + w$ such that $v(B, C) = \mathbf{x}^{\mathbf{n}}B + C$ and $w(B, C) = (A - \mathbf{x}^{\mathbf{n}})B$. Note that w does not depend on C. The mapping v is a K-linear isomorphism because C lies in $\mathbb{K}[[\mathbf{x}]]^{\ngeq\mathbf{n}}$. Hence u is an isomorphism if and only if the geometric series $\sum_{k=0}^{\infty}(-v^{-1}w)^k$ converges, i.e. if the limit $\sum_{k=0}^{\infty}(-v^{-1}w)^k(B, C)$ exists as formal power series. This limit exists because the order of the summands tends to infinity. Let $(B_{k+1}, C_{k+1}) = (-v^{-1}w)(B_k, C_k)$ with $B_{k+1} \neq 0$. Then the initial monom of B_{k+1} is strictly bigger (with respect to $<_{\mathbf{w}}$) than the initial monom of B_k. The initial monoms of $C_k, C_{k+1}, ...$ grow, too. In our case, where the division is obtained from a recurrence, we are not necessarily interested in carrying out the division explicitly. We are much rather interested in the properties of the generating function.

Definition 6.1. *Suppose $\mathbb{K}$ is a complete valued field (for example $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or$\mathbb{Q}_p$). A power series $A \in \mathbb{K}[[\boldsymbol{x}]]$ is called convergent if it defines a analytic function near $(0, \dots, 0)$. The ring of all convergent power series is denoted by $\mathbb{K}\{\boldsymbol{x}\}$.*

The Weierstraß Division Theorem and its generalization by Grauert, Hironaka and Galligo to ideals of convergent power series gives a sufficient condition for $F_{\mathbf{s}}(\mathbf{x})$ to be convergent. We will only consider the case of division by one power series.

Theorem 6.2. *Let K be a complete valued field and $A(\mathbf{x}) \in \mathbb{K}\{\mathbf{x}\}$ a convergent power series. Let $\mathbf{x}^{\mathbf{n}}$ be the initial monom of $A(\mathbf{x})$ with respect to some fixed monomial ordering on $\mathbb{N}^d$. Then*

$$\mathbb{K}\{\mathbf{x}\} = A(\mathbf{x})\mathbb{K}\{\mathbf{x}\} \oplus \mathbb{K}\{\mathbf{x}\}^{\not\geq n}.$$

Proof. For a power series $F \in \mathbb{K}[[\mathbf{x}]]$ and a positive real number $r > 0$ we define $|F(\mathbf{x})|_r = -\sum_{\mathbf{n} \in \mathbb{N}^d} |f_n| r^{|\mathbf{n}|}$. We have that $F(\mathbf{x}) \in \mathbb{K}\{\mathbf{x}\}$ if and only if there exists an r such that $|F(\mathbf{x})| < \infty$. The space of all power series $\mathbb{K}\{\mathbf{x}\}_r$ with $|F(\mathbf{x})|_r < \infty$ is a Banach space. Consider as before the map $u = v + w : \mathbb{K}\{\mathbf{x}\} \times K\{\mathbf{x}\}^{\not\geq \mathbf{n}} \to \mathbb{K}\{\mathbf{x}\}$ For sufficiently small r this mapping can be restricted to $\mathbb{K}\{\mathbf{x}\}_r \times \mathbb{K}\{\mathbf{x}\}_r^{\not\geq \mathbf{n}}$ and $\mathbb{K}\{\mathbf{x}\}_r$. The convergence of the geometric series $\sum_{k=0}^{\infty} (-v^{-1}w)^k$ follows, if we show that the restriction of $v^{-1}w$ has operator norm less than 1 for sufficiently small r. This follows from the fact that the initial monom of a series is the highest among the monoms in its expansion. Thus $F_{\mathbf{s}}(\mathbf{x})$ is convergent if the initial conditions $\mathbb{K}(\mathbf{x}) \in K[[\mathbf{x}]]$ are a convergent series. □

Theorem 6.3. *(Lafon-Hironaka division theorem) Let $A(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]^{alg}$, where $\mathbb{K}[[\mathbf{x}]]^{alg} \subseteq \mathbb{K}[[\mathbf{x}]]$ is the subalgebra of algebraic power series, and $\mathbf{x}^{\mathbf{n}}$ the initial monom of $A(\mathbf{x})$ with respect to some monomial order where $\mathbf{n} = (0, \dots, 0, n_k, 0 \dots, 0)$. Then*

$$\mathbb{K}[[\mathbf{x}]]^{alg} = A(\mathbf{x})\mathbb{K}[[\mathbf{x}]]^{alg} \oplus (\mathbb{K}[[\mathbf{x}]]^{alg})^{\not\geq \mathbf{n}}.$$

(without proof)

Note that the condition on $\mathbf{n}$ can not be dropped. For example, dividing xy by $xy - x^3 - y^3 + x^2y^2$ with initial monom xy gives a transcendent remainder.
The theorem implies that if $F_{\mathbf{s}}(\mathbf{x})$ and $U(\mathbf{x})$ are algebraic in the division if $K(\mathbf{x})$ is algebraic and the initial monom of $Q(\mathbf{x})$ has only one variable, i.e. the apex has only one positive component. We already saw this in theorem 2.12 before.

Let's see what happens if we apply the concept of polynomial division to Gessel walks. The recurrence is

$$f(i, j, n) = f(i+1, j, n-1) + f(i-1, j, n-1) + f(i+1, j+1, n-1) + f(i-1, j-1, n-1)$$

and the characteristic polynomial is

$$Q(x, y, t) = xy - (yt - x^2yt + t + x^2y^2t) = xy - t(1+y)(1+x^2y).$$

Since the apex is $(1, 1, 0)$ we can not apply the theorem directly. But we can rewrite the recurrence as

$$f(i, j, n) = f(i-1, j-1, n+1) - f(i, j-1, n) - f(i-2, j-1, n) - f(i-2, j-2, n).$$

Now the apex is $(0,0,1)$ and the characteristic polynomial has changed its sign. We can take $(2,2,0)$ as starting point of this recurrence and $\mathbf{w}=(1,1,1)$ as weight vector since it fulfills $\mathbf{w}\cdot\mathbf{h}<0$ for all shifts $\mathbf{h}$. Calculation gives us that

$$\begin{aligned}
&F_{\mathbf{s}}(x,y,t)=t^2+3xt^3+12t^4+xyt^3+3yt^4+\dots\\
&K(x,y,t)=t^3-xyt^2+yt^3+3xt^4-2x^2yt^3-8xyt^4-2xy^2t^4+\dots\\
&U(x,y,t)=0.
\end{aligned}$$

If we are able to show that $K(x,y,t)$ is algebraic, then $F_{\mathbf{s}}(\mathbf{x})$ is also algebraic and the full generating function $F(\mathbf{x})$ of Gessel walks is algebraic as well.

6.2 Diagonals

Another idea that might be useful for proving algebraicity results is using results about diagonals of power series. There are many known facts about diagonals, for example it is known that any diagonal of a D-finite power series is D-finite again. The idea is now to find an antidiagonalization of our power series and then use these results. This idea is motivated by a concept from algebraic geometry and resolutions of singularities, the so called blow-up. For the sake of completeness I will mention the definition and a few results about blow-ups although it is not necessarily needed to understand the section about diagonals.

Definition 6.4. *Let $Z\subseteq M=\mathbb{R}^n$ be a submanifold, $a\in M\setminus Z$ a point and $\varrho: M\to Z$ be a local retraction (i.e. ϱ is differentiable and $\varrho^2=\varrho$). Denote by g_a the line through a and $\varrho(a)$, considered as a point in the projectivized normal bundle $\mathbb{P}(N(M,Z))$. We obtain a differentiable mapping $\varrho: M\setminus Z\to\mathbb{P}(N(M,Z),a)\mapsto g_a$ with graph $\Gamma\subseteq(M\setminus Z)\times\mathbb{P}(N(M,Z))$. Let $\overline{\Gamma}$ be its closure and $\pi:\overline{\Gamma}\to M,(a,\ell)\mapsto a$ the canonical projection. Then $\overline{\Gamma}$ together with π is the blow-up of M with center Z.*

Remark: In most cases, the center of the blow-up is $Z=\{0\}$. Then we have that $N(M,Z)=\bigcup_{a\in Z}T_aM/T_AZ=T_0M=T_0\mathbb{R}^n=\mathbb{R}^n$, where T_aM is the tangential space to M in the point a. Hence $\mathbb{P}(N(M,Z))=\mathbb{P}^{n-1}(\mathbb{R})$.

Theorem 6.5. *(a) Let $Z=\{0\}\subseteq M=\mathbb{R}^n$ with blow-up $\tilde{M}\subseteq\mathbb{R}^n\times\mathbb{P}^{n-1}(\mathbb{R})$. Then $\tilde{M}$ is determined in local coordinates $((x-1,\dots,x_n),(u_1:\dots:u_n))$ via the equations*

$$x_iu_j-x_ju_i=0$$

for all $i\neq j$ and

$$\pi:\tilde{M}\to M,((x_1,\dots,x_n),(u_1:\dots:u_n))\mapsto(x_1,\dots,x_n).$$

(b) Let $Z=0$ and let the canonical affine coordinates on $\mathbb{P}^{n-1}(\mathbb{R})$ given by the open subsets $U_j=\{u\in\mathbb{P}^{n-1}(\mathbb{R}),u_j\neq 0\}\cong\mathbb{R}^{n-1}$. Then the j-th chart expression of π is given by

$$\pi_j:\mathbb{R}^n\to\mathbb{R}^n,(x_1,\dots,x_n)\to(x_1x_j,\dots,x_{j-1}x_j,x_j,x_{j+1}x_j,\dots,x_nX_j).$$

(without proof)

Example: Let $X = V(x^3 - y^2)$ and $Z = 0$. This algebraic curve has a singularity at $(0,0)$. Let us consider its blow-up, given by the charts

$$\pi_1(V(x^3 - y^2)) = V(x^3 - x^2y^2) = V(x^2) \cap V(x - y^2)$$

$$\pi_2(V(x^3 - y^2)) = V(x^3y^3 - y^2) = V(y^2) \cap V(x^3y - 1).$$

They are a parabola and a hyperbola respectively and thus smooth. The singularity has vanished in the blow-up. In general, the singularity of a curve not always vanishes after a blow-up, but its order decreases. Thus there exist a chain of blow-ups resolving the singularity.

Let us now apply this concept of "making things nicer by going to higher dimensions" to power series and diagonals.

Definition 6.6. *Let $\mathbb{K}$ be a field of characteristic zero and $f(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n}$ a power series. The primitive diagonal $I_{1,2}$ of the power series f is defined as*

$$I_{1,2}(f) = \sum a_{i_1,i_1,i_3,\dots,i_n} x_1^{i_1} x_3^{i_3} \dots x_n^{i_n}.$$

The other primitive diagonals I_{ij} are defined similar. A general diagonal is any composition of some $I_{ij}'s$ and the complete diagonal of f is

$$I_{1,2}I_{2,3}\dots I_{n-1,n}(f) = \sum_{i\in\mathbb{N}} a_{i,\dots,i} x^i.$$

Often the complete diagonal is called diagonal.

We will show that any diagonal of a D-finite function is D-finite. If we are dealing with a series in 2 variables over the field $K = \mathbb{C}$ we even have that the diagonal of a rational power series $f(x,y)$ is algebraic since we can write

$$Df = \frac{1}{2\pi i} \int_{|\zeta|=\xi} f\left(\zeta, \frac{t}{\zeta}\right) \frac{d\zeta}{\zeta}$$

for ξ and $|t|$ small. This representation can easily be checked by Residue Theorem. But this argument does not work for $n > 3$. In general, the diagonal of a rational power series over a field with characteristic zero need not be algebraic.
But in characteristic $p > 0$ we have that the diagonal of any rational function is algebraic. The proof of this can be seen in [17].

Let now $\text{char}\mathbb{K} = 0$ and $f \in \mathbb{K}[[x_1,\dots,x_n]]$ be D-finite, i.e. f fulfills a system of linear partial differential equations of the form

$$\left(a_{in_i}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i} + a_{in_i-1}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i-1} + \dots + a_{i0}(x)\right) f = 0 \quad \text{for } i = 1,\dots n.$$

Let s be a new variable and define

$$F(s, x_1, x_3, \dots, x_n) := \frac{1}{s} f(s, \frac{x_1}{s}, x_3, \dots, x_n).$$

The function F is not a power series in $s, x_1, x_3, \dots, x_n$ but an element of the $K[s, x_1, x_3, \dots, x_n]$-module M of all

$$G = \sum_{j\in\mathbb{Z}, i_2,\dots,i_n\in\mathbb{N}, j+i_2\geq k} a_{ji_2\dots i_n} s^j x_1^{i_2} x_3^{i_3} \dots x_n^{i_n}$$

for a k depending on G. Let $\mathcal{D}$ be the ring of all linear partial differential operators in $\frac{\partial}{\partial s}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_n}$ with coefficients in $\mathbb{K}[s, x_1, x_3, \dots, x_n]$. Then M is a $\mathcal{D}$-module. The coefficient of $\frac{1}{s}$ in F is $I_{1,2}(f)$.

Lemma 6.7. *If $0 \neq p \in \mathbb{K}[s, x_1, x_3, \dots, x_n]$ and $G \in M$ fulfills $pG = 0$ then $G = 0$.*

Proof. For k suitable we have that $s^k G \in \mathbb{K}[[s, \frac{x_1}{s}, x_3, \dots, x_n]]$. Substitute $x_1 = su$ where u is a new variable. From this we obtain that

$$0 = p(s, su, x_3, \dots, x_n) s^k G(s, u, x_3, \dots, x_n) \in \mathbb{K}[[s, u, x_3, \dots, x_n]].$$

The claim follows because multiplication with s and substitution $x_1 = su$ are both one to one. $\square$

Lemma 6.8. *F is D-finite (in $s, x_1, x_3, \dots, x_n$).*

Proof. This follows from the fact that f is D-finite and the chain rule. $\square$

Hence there exist nonzero linear partial differential operators with polynomial coefficients

$$A\left(s, x_1, x_3, \dots, x_n; \frac{\partial}{\partial s}\right) = L(s, x_1, x_3, \dots, x_n)\left(\frac{\partial}{\partial s}\right)^m + \text{low-order-terms in} \frac{\partial}{\partial s}$$

and

$$B_i\left(s, x_1, x_3, \dots, x_n; \frac{\partial}{\partial x_i}\right) = L_i(s, x_1, x_3, \dots, x_n)\left(\frac{\partial}{\partial x_i}\right)^{m_i} + \text{low-order-terms in} \frac{\partial}{\partial x_i}$$

such that $AF = 0$ and $B_i F = 0$ for $i = 1, 3, \dots, n$.

Lemma 6.9. *There exist nonzero linear partial differential operators $P_i(x_1, x_3, \dots, x_n; \frac{\partial}{\partial s}, \frac{\partial}{\partial x_i}$ for $i = 1, 3, \dots, n$ with coefficients in $K[x_1, x_3, \dots, x_n]$ and P_i contains only derivatives of the form $\left(\frac{\partial}{\partial s}\right)^\beta, \left(\frac{\partial}{\partial x_i}\right)^\gamma$ such that*

$$P\left(x_1, x_3, \dots, x_n; \frac{\partial}{\partial s}, \frac{\partial}{\partial x_i}\right) F = 0$$

for $i = 1, 3, \dots, n$. (without proof, the proof can be found in [24]).

Theorem 6.10. *If $f \in K[[x_1, \dots x_n]]$ is D-finite and I is an arbitrary diagonal, then $I(f)$ is D-finite (without proof, the proof can be found in [24]).*

Now we want to apply this result to prove the D-finiteness of generating functions of lattice walks. The idea is to find an rationalization A of G such that $I(A) = G$ where I is any diagonal and then prove that A is D-finite. From the theorem above then follows that G is D-finite as well.
Of course, at first glance it might seem counterintuitive to go to a higher dimension and

then prove there that something is D-finite. But as we already saw in the motivational example with the blow-ups, things can get nicer and easier in higher dimension. The singularity we had in dimension 2 disappeared after the blow-up and going to dimension 3.

Definition 6.11. *An antidiagonalization of a power series $f \in K[[x_1, \ldots, x_n]]$ is a power series $A \in K[[x_1, \ldots, x_m]]$, $m > n$ together with a diagonal I such that $I(A) = f$.*

Note that antidiagonalization of a power series f is not unique. For example, let $f(x) = \sum_{i\geq 0} x^i$ be the geometric series. Then

$$A_1(x, y) = \sum_{i\in\mathbb{N}} x^i y^j$$

is an antidiagonalization of f, because $I_{1,2}(A_1(x, y)) = f(x)$, but also

$$A_2(x, y) = \sum_{i,j\in\mathbb{N}} (i - j + 1)x^i y^j$$

is an antidiagonalization of f, since also $I_{1,2}(A_2(x, y)) = f(x)$.

Example: Consider the walk with $S = \{\rightarrow, \uparrow\}$. Then $f(i, j, n) = \binom{n}{i}$ if $n = i + j$ and $f(i, j, n) = 0$ else. We have that

$$F(x, y, t) = \sum_{n\geq 0}\sum_{i+j=n} \binom{n}{i} x^i y^j t^n.$$

Now we want to find a power series G with $\Delta G = F$. This can be achieved by taking for example $\Delta = I_{12}I_{34}I_{56}$ and

$$G = \sum_{a_2,a_4,a_5,a_6\geq 0}\ \sum_{a_1,a_3\geq 0, a_1+a_3=a_5} \binom{a_1 + a_3}{a_1} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6}.$$

The power series G can be rewritten as

$$\sum_{a_2,a_4,a_5,a_6\geq 0} (x_1 + x_3)^{a_5} x_2^{a_2} x_4^{a_4} x_5^{a_5} x_6^{a_6} = \frac{1}{1 - (x_1 + x_3)x_5} \frac{1}{1 - x_2} \frac{1}{1 - x_4} \frac{1}{1 - x_6}$$

which clearly is D-finite (and even rational). Thus $\Delta G = F$ is D-finite as well.
Of course, in this rather easy example one could show directly that F is rational without using the antidiagonalization and rewriting F as a geometric series. But in cases where F is more complicated this method might actually be helpful.

Diagonals also appear in various other proofs connected with rationality or irrationality, the most famous example is probably the irrationality of $\zeta(3)$, where the transcendental series

$$f(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k} t^n \in \mathbb{Z}[[t]]$$

plays an important role (see for example [28]).

7 Final notes and comments

In the last few years the classification of lattice paths with small steps in the quarter plane has finally been completed. However, there are many more interesting questions in lattice path walks, some topics I did not cover in this thesis and there are still a few open questions in this field. This chapter is some kind of overview on these topics and questions.

First of all, it is natural to consider not only small steps but any kind of steps with integer coordinates. Some of these cases can be already covered with what we did here: if the apex of the step set has at most one positive coordinate, the generating function is algebraic. In [13] a simple example of a walk with a non-D-finite generating function is mentioned: the knight's walk.They are walks starting in $(1,1)$ with steps in $\{(-1,2),(2,-1)\}$ that stay in the first quadrant. The steps correspond to the knight's moves on a chessboard. In [4] there are also some examples of walks with bigger steps. It is still possible to write down a functional equation for the series $Q(x,y,t)$, but the kernel has now degree larger than 2 in x or y.

Another natural generalization is considering walks in other regions than the quarter plane. When dealing with walks with integer steps below a rational diagonal we can apply a rational transform, mapping the diagonal to the vertical axis and are back in the case of walks with larger integer steps in the quarter planes. But there are also some other regions of interest. In [21] the authors discuss walks in a horizontal stripe between the two lines $y = 0$ and $y = C$ for a constant C and their interpretation as the *gambler's ruin problem*, where two players gamble against each other until one of them goes bankrupt. It may also be of interest to keep track not only of the number of steps, but also of the number of contacts with the diagonal, as seen in [27].

Furthermore, we can also consider generalizations to higher dimensions and study walks with simple steps in three or n dimensions. In [5] Bostan and Kauers studied three-dimensional walks in the first octant with at most five steps empircally. They tried to find some criteria for their generating function to be D-finite, i.e. certain symmetries of the step set.
Already in the three dimensional case, the large number of different walks causes difficulties. In three dimensions we have 2^{26} different step sets, in n dimensions 2^{3^n-1}. Taking into account that some step sets only differ by a symmetry of the octant or only result in the empty walks, we are left with 11074225 essentially different step sets. This number is obtained via a similar inclusion-exclusion argument as in the 2D-case, for details see [3]. In this paper the authors tried to classify walks in 3D with the same techniques as in the two-dimensional case. Again there can be a group associated with each step set, in the three-dimensional case this group has three generators. Among walks with not more than six steps the highest cardinality of a finite group that appeared was 48, the other groups are conjectured to be infinite and the corresponding generating functions are conjectured to be not D-finite. In some of the cases with a finite group the kernel method turns out to be helpful again. Other cases can be tackled with the so called *Hadamard decomposition*, where a walk in $\mathbb{Z}^3$ is reduced to a pair of walks in $\mathbb{Z}$ and $\mathbb{Z}^2$. There are 19 cases left, where the group of the walk is infinite but the nature of the generating function remains unknown.
The following diagram summarizes what is known about the classification of three dimensional walks with less or equal six steps

$$\begin{tikzcd}
 & & \text{models with } \leq 6 \text{ steps } (20804) \arrow[dl, "|G|<\infty"'] \arrow[dr, "|G|=\infty?"] & \\
 & 170 \arrow[dl, "\text{orbit sum } \neq 0"'] \arrow[d, "\text{orbit sum } =0"] & & 20634 \arrow[d] \\
108 \arrow[d, "\text{kernel method}"'] & 62 \arrow[d, "\text{Hadamard}"'] \arrow[dr, "\text{not Hadamard}"] & & \text{not D-finite?} \\
\text{D-finite} & \text{D-finite} & \text{unknown} &
\end{tikzcd}$$

There are also some interesting questions within the setting of lattice walks with small steps in the quarter plane. We often obtained hypergeometric expressions for certain numbers of walks, for example the number of Gessel walks ending in the origin is

$$q(0,0,2n) = 16^n \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}.$$

Are there any kind of combinatorical explanations or correspondences for such results? For some models, there are combinatorical correspondences known, for example, the walks with step set {W,N,SE} correspond to Young tableaux of height three or the walks with step set {E, W, NW, SE} stand in bijection with pairs of non-intersecting Dyck-paths.
Another interesting question is if there exists a proper subset $A \subsetneqq K[[\mathbf{x}]]$ of the set of all power series such that all generating functions of lattice path walks lie in A. Clearly, the set of D-finite functions is included in A. There is the concept of D-D-finite power series, where the coefficients of the differential equations are not polynomials (as in the D-finite case), but D-finite series. Of course, this concept can be iterated, but it is unknown with what set we will end up and if we already gain all kinds of power series with this process.

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