MASTERARBEIT

Titel der Masterarbeit

DEHN PRESENTATIONS OF RELATIVELY HYPERBOLIC GROUPS

verfasst von

Mag. rer. nat. Manuel Raphael Urbina Moreano, BSc.

angestrebter akademischer Grad

Master of Science (MSc.)

Studienkennzahl:	A 066 821
Studienrichtung:	Masterstudium Mathematik
Betreuer:	Univ.-Prof. Goulnara Arzhantseva, PhD

Wien, am 05. 08. 2015

Dedicated to Melanie and Noah.

Contents

List of Figures

Abstract / Kurzbeschreibung

Abstract

The purpose of this master thesis is to fix the notion of a relative Dehn presentation for a group relative to a collection of subgroups and to generalize a well-known theorem for hyperbolic groups to the case of relatively hyperbolic groups. More precisely, it is known that a finitely generated group is hyperbolic if and only if it admits a finite Dehn presentation. We generalize this result by showing that a countable group is relatively hyperbolic if and only if it admits a finite relative Dehn presentation. Moreover, we introduce the notion of ends for a relative pair $(G, \mathbb{P})$, where G is a group and $\mathbb{P}$ is a collection of subgroups of G.

Kurzbeschreibung

Die Aufgabe dieser Masterarbeit ist die Festlegung des Begriffes der relativen Dehn Präsentation für eine Gruppe relativ zu einer Kollektion von Untergruppen und die Verallgemeinerung eines wohl bekannten Theorems für hyperbolische Gruppen für den Fall relativ hyperbolischer Gruppen. Um dies zu präzisieren, es ist bekannt, dass eine endlich erzeugte Gruppe dann und nur dann hyperbolisch ist wenn sie eine endliche Dehn Präsentation besitzt. Wir verallgemeinern dieses Resultat indem wir beweisen, dass eine abzählbare Gruppe dann und nur dann relativ hyperbolisch ist wenn sie eine endliche relative Dehn Präsentation besitzt. Weiters führen wir den Begriff der Enden für ein relatives Paar $(G, \mathbb{P})$ ein, wobei G eine Gruppe und $\mathbb{P}$ eine Kollektion von Untergruppen von G ist.

Declaration

I do declare that I unassistedly wrote this thesis without additional help other than supervision and that I did not use other than quoted sources and knowledge from lectures.

Vienna, August 5, 2015

Manuel Raphael Urbina Moreano

Erklärung

Ich erkläre, dass ich diese Arbeit selbständig und abgesehen von der vorgesehen Betreuung ohne fremde Hilfe verfasst habe und dass ich keine außer den zitierten Quellen und Wissen aus Lehrveranstaltungen verwendet habe.

Wien, 5. August, 2015

Manuel Raphael Urbina Moreano

Curriculum vitae

Personal Data

Name:	Manuel Raphael Urbina Morenao
Date of Birth:	07.11.1988
Place of Birth:	Grieskirchen, Upper Austria, Austria
Nationality:	Austria
Current Residence:	Vienna, Austria
Current Email:	manuel.urbina@gmx.at

Education

Graduation in diploma studies :	Mathematics and Biology education, University of Vienna, 22.08.2014, diploma exam with distinction
Graduation in undergraduate studies :	Mathematics, University of Vienna, 06.11.2013, with distinction
Undergraduate studies:	Mathematics, University of Vienna, March 2012 - November 2013
Diploma studies:	Mathematics and Biology education, University of Vienna, since March 2010
Diploma studies:	Human medicine, Medical university of Vienna, October 2008 - January 2010
High school diploma:	Katholisches Privatgymnasium Dachsberg, Prambachkirchen, Matura 21.06.2007
High school:	Katholisches Privatgymnasium Dachsberg, Prambachkirchen, September 1999 - June 2007
Elementary school:	Volksschule Michaelnbach, Michaelnbach, September 1995 - June 1999

Military Service

Military Service :	Hörsching Barracks and Wels Barracks, Upper Austria, September 2007 - March 2008

Working Experience

High school teacher (Mathematics, Biology) :	Europagymnasium Auhof, Linz, since September 2014
Teaching assistant:	University of Vienna, Department for Mathematics, March 2013 - June 2013 and March 2014 - June 2014
Draftsman (*'technischer Zeichner'*) :	KUVAG, Neumarkt, July 2011 - September 2011
Course instructor:	IFL, Wels, since April 2010

Acknowledgement

At first place I want to thank my supervisor *Goulnara Arzhantseva* for suggesting to me such a beautiful topic which forces me getting myself deeper into the fascinating field of (hyperbolic) geometry and its astonishing applications to group theory. It was and will always be a joy to read and work upon this topic and related questions.

At last I need to thank *Haken*, *Tool* and *Pain of Salvation* for their incredible music getting me through nights of excessive drawing, working, writing and thinking.

Vienna, August 5, 2015

Manuel Raphael Urbina Moreano

1. Introduction

The theory of (word-) hyperbolic groups plays a central role in geometric group theory and is well-studied by many authors (see e.g. [4] for an elementary discussion of some related topics). An elementary way to introduce the notion of *hyperbolic groups* is with the help of their Cayley graphs (a geodesic metric space which naturally describes the geometry of a finitely generated group) and its geodesic triangles. More generally, one says that a geodesic metric space Y is hyperbolic if geodesic triangles are *slim*, meaning that any of its side is contained in the δ-neighbourhood of the other two sides for some constant $\delta \geq 0$ depending only on the space Y. Since hyperbolicity is a quasi-isometric invariant and different Cayley graphs of a finitely generated group with respect to different finite generating sets are quasi-isometric, we get the well-defined notion of a *hyperbolic group*. It is often convinient to think of hyperbolic spaces as thickened versions of metric trees (see [4] p. 399) and in some sense this can be made quite precise through the notion of *asymptotic cones* of metric spaces (although we do not touch this direction in this thesis). One can think of this in the following intuitive way: No matter where you start to look at your hyperbolic Cayley graph, at infinity you will see an $\mathbb{R}$-tree (a 0-hyperbolic geodesic metric space).

Moreover, there is also a combinatorial characterisation of hyperbolic groups using group presentations by generators and relators. First of all one can prove that finitely generated hyperbolic groups are necessarily finitely presented. Moreover, one can say even more: Let G be a finitely generated group and $R := \{u_1v_1^{-1}, \ldots, u_nv_n^{-1}\}$ be a set of words in a (finite) alphabet X (a finite generating set for G) with the following properties: For each i, u_i is a strictly longer word than v_i, $u_i = v_i$ as elements in G and any word w, which represents the identity in G contains some u_i as a subword. In this case, the set R is called a *Dehn set* and a presentation $\langle X \mid R\rangle$ is called a *Dehn presentation* for G if R is a Dehn set. Given a Dehn presentation of G, can (almost instantly) write down an algorithm, so-called *Dehn's algorithm*, to solve the *word problem* for G (in linear time). The main statement is now that a finitely generated group is hyperbolic if and only if it admits a finite Dehn presentation (see Theorem 2.6 in [4]).

The purpose of this thesis is to generalize this theorem, that is, this combinatorial characterization of hyperbolic groups to so-called *relatively hyperbolic groups*. Intuitively speaking, relatively hyperbolic groups can be thought of as "almost" hyperbolic groups, i.e. spaces which would be hyperbolic if we forget about some non-hyperbolic subspaces (subgroups). There are many different equivalent definitions for a group to be hyperbolic relative to a collection of subgroups. For instance, in [11], six different definitions have been analyzed and under some mild assumptions all of these are equivalent.

To generalize the main theorem (Dehn presentations for hyperbolic groups) we use the notion of *relative (group) presentations* which has been developed by Osin (see [16]). We will introduce our notion of a *relative Dehn presentation* and prove an analogous theorem for relatively hyperbolic groups (see Theorem 4.1.7 and its succeeding corollaries) in the sense of Osin, which seems to be the most general one. However, this does not yield a *finite* relative Dehn presentation. Nonetheless, if one restricts to the definitions given in [11], where under some mild assumptions all become equivalent, one can prove that the so obtained relative Dehn set is indeed *finite* (see Theorem 4.2.16).

We give a short outline of this thesis:

In section 2, we give an easy warm-up with basic notions and properties of metric spaces, groups, graphs, and hyperbolic spaces. In particular, we prove the theorem about the correspondence of Dehn presentations and hyperbolic groups in full detail (see Theorem 2.5.21).

In section 3, we focus on the notion of a *relative pair* which is simply a group G and a specified collection of subgroups. We recall the notion of relative presentations [16], a relative analogue of a Cayley graph, called *coned-off Cayley graph*, [9] and a short outline on ends in such non-proper spaces.

In section 4, we prove our main theorem, which is a generalization of Theorem 2.5.21 for hyperbolic groups to the case of relatively hyperbolic groups. We end our discussion with explicit examples and (well-known) properties of hyperbolic groups and its analogues for relatively hyperbolic groups.

2. Preliminaries

We start this thesis with preliminary notions from group theory, graph theory, metric spaces, and hyperbolic spaces. The main purpose of this section is to fix notation.

2.1. Metric spaces.

2.1. Metric spaces. We begin with the basic properties of metric spaces, i.e. sets where we are able to measure the distance between two points in an appropriate way.

2.1.1. *Definition.* Let X be a set. A *metric* on X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in X$:

(1) $d(x, y) \geq 0$.
(2) $d(x, y) = 0$ if and only if $x = y$.
(3) $d(x, y) = d(y, x)$.
(4) $d(x, y) \leq d(x, z) + d(z, y)$.

We refer to the last property as the *triangle inequality.* For $x, y \in X$, we call $d(x, y)$ the *distance* between x and y.

A *metric space* is a pair (X, d), where X is a set and d is metric on X. A metric space is said to be *complete* if every Cauchy sequence in it converges. The *open ball of radius r and center $x \in X$*, where $r > 0$, is defined as the set $B(x, r) := \{y \in X \mid d(x, y) < r\}$. The *closed ball of radius r and center $x \in X$* is defined as the set $\bar{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$.

For $\varepsilon \geq 0$ and a subset $A \subseteq X$ the *ε-neighbourhood* of A, denoted by $N_\varepsilon(A)$ is the set of points $y \in X$ such that there exists $a \in A$ with $d(y, a) \leq \varepsilon$.

2.1.2. *Remark.* If no confusion can arise we shall follow the common practice of writing X instead of (X, d) and we speak of the metric space X.

2.1.3. *Definition.* An *isometric embedding* from a metric space (X, d_X) to a metric space (Y, d_Y) is a function $f\colon X \to Y$ such that

$$d_X(x, y) = d_Y(f(x), f(y)) \quad \text{for all } x, y \in X.$$

Note that an isometric embedding is necessarily injective. If, in addition, f is bijective then it is called an *isometry* and, in this case, we say that (X, d_X) and (Y, d_Y) are *isometric*. Note that the inverse f^{-1} is also an isometric embedding. Indeed, let $v, w \in Y$ then

$$d_Y(v, w) = d_Y((f \circ f^{-1})(v), (f \circ f^{-1})(w)) = d_X(f^{-1}(v), f^{-1}(w)).$$

The set of all isometries from a metric space (X, d) to itself will be denoted $\mathrm{Isom}(X, d)$ or more briefly, $\mathrm{Isom}(X)$. It is easily shown that $\mathrm{Isom}(X)$ forms a group (see next subsection).

2.1.4. *Definition.* Let (X, d) be a metric space and $x, y \in X$. A *geodesic* (*path*) joining x and y is an isometric embedding of a closed interval $[0, l] \subset \mathbb{R}$ with the induced metric to X where $c(0) = x$ and $c(l) = y$. In particular, $l = d(x, y)$. The *image* of a geodesic is called a *geodesic segment* with endpoints x and y and is denoted by $[x, y]$. Note that the notation $[x, y]$ is a bit ambiguos since geodesic segments are *not* unique in general. Nonetheless, we will use it to denote *some* geodesic segment between x and y.

(X, d) is said to be a *geodesic* (*metric*) *space* if every two points in X can be joined by a geodesic.

Fix $k > 0$. A map $c\colon [a,b] \to X$ is said to be a $k-$*local geodesic* if $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [a,b]$ with $|t-t'| \leq k$.

2.1.5. *Definition.* A *curve* in a metric space X is a *continuous* map $c\colon [a,b] \to X$. As above we say that c *joins* the point $c(a)$ to the point $c(b)$.

The *length* $l(c)$ of a curve as above is defined as

$$l(c) = \sup\{\sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}) \mid a = t_0 \leq t_1 \leq \cdots \leq t_n = b, n \in \mathbb{N}\}.$$

A curve is *rectifiable* or *has finite length* if $l(c)$ is *finite*.

If we want to study the *large-scale geometry*, i.e. in some sense we want to 'forget' about the local structure of a metric space, we require the following

2.1.6. *Definition.* Let $f\colon X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y), $b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}_{\geq 1}$.

Then f is called a (c,b)-*quasi-isometric embedding* if

$$\frac{1}{c} d_X(x,x') - b \leq d_Y(f(x), f(x')) \leq c d_X(x,x') + b \quad \text{for all } x, x' \in X.$$

Moreover, f is said to have a *quasi-dense image* if there is a constant $d \in \mathbb{R}_{>0}$ such that for all $y \in Y$ there exists $x \in X$ satisfying

$$d_Y(f(x), y) \leq d.$$

We call a constant d as above a *quasi-dense constant* for f. At last, f is called a (c,b)-*quasi-isometry* (between X and Y) if f is a (c,b)-quasi-isometric embedding and f has a quasi-dense image. In this case, we call X and Y *quasi-isometric* and write $X \sim_{QI} Y$. Note that this indeed defines a equivalence relation on the class of metric spaces, where reflexivity is trivial, transitivity rather clear but symmetry needs the axiom of choice to construct a quasi-isometry from Y to X, a so-called *quasi-inverse*.

2.1.7. *Lemma.* Let ϕ and ψ be two maps between metric spaces (X, d) and (Y, d'). Suppose that ϕ and ψ agree *up to a bounded distance*, i.e. there exists a constant $k \geq 0$ such that $d'(\phi(x), \psi(x)) \leq k$ for all $x \in X$. Then ϕ is a quasi-isometric embedding (has quasi-dense image) if and only if ψ is (does). In particular, ϕ is a quasi-isometry if and only if ψ is.

Proof. We only show the 'if'-part, since the other implication is proven by simply interchanging ψ and ϕ. Suppose ψ is a (c,b)-quasi-isometric embedding. Then by the triangle inequality we obtain for all $x, x' \in X$

$$\frac{1}{c} d(x,x') - b \leq d'(\psi(x), \psi(x')) \leq \underbrace{d'(\psi(x), \phi(x))}_{\leq k} + d'(\phi(x), \phi(x')) + \underbrace{d'(\phi(x'), \psi(x'))}_{\leq k},$$

and similarly

$$d'(\phi(x), \phi(x')) \leq d'(\phi(x), \psi(x)) + d'(\psi(x), \psi(x')) + d'(\psi(x'), \phi(x')) \leq$$
$$\leq cd(x,x') + b + 2k.$$

Thus ϕ is a $(c, b+2k)$-quasi-isometric embedding.

Suppose now that ψ has quasi-dense image with constant b. Let $y \in Y$. Then there exists $x \in X$ such that $d'(\psi(x), y) \leq b$. By the triangle inequality, we obtain

$$d'(\phi(x), y) \leq d'(\phi(x), \psi(x)) + d'(\psi(x), y) \leq k + b.$$

Thus ϕ has a quasi-dense image with constant $b + k$. □

2.2. **Group theory.** The following subsection mainly follows [2, 10] and [14].

2.2.1. *Definition.* A *binary operation* '$\circ$' on a set G assigns to any two elements $a, b \in G$ an element of G denoted by $a \circ b$. In other words

$$\begin{aligned} \circ \colon G \times G &\to G \\ (a, b) &\mapsto a \circ b \end{aligned}$$

A set G with a binary operation is called a *group* if the following holds:

(1) the operation is *associative*, i.e., $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$,
(2) there exists an element $e \in G$, called the *identity element*, such that $a \circ e = e \circ a = a$ for all $a \in G$,
(3) for each $a \in G$ there exists $b \in G$, called the *inverse* of a, such that $a \circ b = b \circ a = e$.

A group G is called *commutative* or *abelian* if in addition $a \circ b = b \circ a$ holds for all $a, b \in G$.

In the sequel we will omit "$\circ$" for notational convenience and simply write ab instead of $a \circ b$.

2.2.2. *Remark.* Clearly the identity element is *unique*. If confusion may arise we will specify *the* identity element of a group G by e_G. Similary for each $a \in G$ the inverse is *unique* and we will denote it by a^{-1}.

2.2.3. *Definition.* The product of n elements all equal to $g \in G$ is denoted by g^n. We define $g^0 = e$ and $g^m = (g^{-1})^{-m}$ for negative integers m. If $g^n = e$ for some $n \geq 1$, then the smallest n with this property is called the *order of the element g*. In this case we call g a *torsion element*. If $g^n \neq e$ for all n, we say that g has *infinite order*. We call a group G a *torsion group* if all its elements have finite order.

2.2.4. *Definition.* A non-empty subset H of a group G is called a *subgroup* of G if for any $a, b \in H$ the elements ab and a^{-1} also lie in H. In that case we write $H \leq G$. Note that trivially the intersection of subgroups is a again a subgroup.

Let $S \subseteq G$. Then $\langle S \rangle$ denotes the smallest subgroup of G containing S, i.e.

$$\langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H.$$

There is a more "concrete" way to describe the subgroup $\langle S \rangle$. In this sense let $L := \{s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n} | s_i \in S, \epsilon_i = \pm 1, n \in \mathbb{N}\}$. The set L clearly contains S and is obviously a subgroup hence $\langle S \rangle \subseteq L$, since $\langle S \rangle$ is the intersection of *all* such subgroups. Conversely, each subgroup H which contains S must contain elements of the form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}$ as described above, i.e. $L \subseteq H$ for each such H. Therefore $L = \langle S \rangle$.

A subset $S \subseteq G$ is called a *generating set* of G if $\langle S\rangle = G$ and G is called *finitely generated* if it is generated by a *finite* set S.

2.2.5. *Definition.* Let H be a subgroup of a group G. The sets $gH := \{gh|h \in H\}$, where $g \in G$, are called *left cosets* of H. *Right cosets* Hg are defined similarly. By definition, $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$. This directly implies that any two different left (resp. right) cosets have *empty intersection.* Hence the set of left (resp. right) cosets form a partition of G. In other words, each $g \in G$ lies in *exactly* one left (resp. right) coset of H. We denote the set of left cosets by G/H.

The correspondence $xH \leftrightarrow Hx^{-1}$ is one-to-one, therefore the cardinality of the set of left cosets of H and the set of right cosets of H coincides. This cardinality is called the *index* of H.

2.2.6. *Definition.* The product of two subsets A, B of a group G is defined as $AB := \{ab \mid a \in A, b \in B\}$. Note that for a subgroup H it holds that $HH = H$ and that $\{g\}H$ coincides with gH as defined above. Moreover, we say that a subgroup H of G is *normal*, writing $H \trianglelefteq G$, if $gH = Hg$ for every $g \in G$. Note that H is normal if and only if $gHg^{-1} = H$ for every $g \in G$. In this case, the product of any two left (resp. right) cosets is again a coset:

$$g_1H \cdot g_2H = g_1(Hg_2)H = g_1(g_2H)H = g_1g_2H.$$

It is easily shown that G/H with this *product* forms a group, called the *factor group.* Indeed G/H forms a group with this product if and only if $H \trianglelefteq G$. Note that the intersection of normal subgroups is again a normal subgroup.

2.2.7. *Definition.* We say that an element $a \in G$ is *conjugate* to an element b if $a = gbg^{-1}$ for some $g \in G$. Similarly, we say that a subgroup A is *conjugate* to a subgroup B if $A = gBg^{-1}$ for some $g \in G$.

2.2.8. *Definition.* Let R be a subset of a group G. The *normal closure* of R, denoted by $\langle\langle R\rangle\rangle$, is defined as the smallest normal subgroup of G containing R, i.e.

$$\langle\langle R\rangle\rangle := \bigcap_{\substack{N \trianglelefteq G \\ R \subseteq N}} N$$

Similar to above there is a more "concrete" way to describe $\langle\langle R\rangle\rangle$ if R is non-empty. Indeed, let $L := \{\prod_{i=1}^{k} g_i r_i^{\epsilon_i} g_i^{-1} | g_i \in G, r_i \in R, \epsilon_i = \pm 1, k \geq 0\}$. Clearly L is a subgroup and contains all conjugates by construction, thus L is normal. Trivially it contains R, hence $\langle\langle R\rangle\rangle \subseteq L$. Conversely, each normal subgroup N which contains R must contain conjugates of the form $g_i r_i^{\epsilon_i} g_i^{-1}$ and thus $L \subseteq N$. In summary $\langle\langle R\rangle\rangle = L$.

Our notation of the normal closure is a bit ambiguous since we have not specified the group G. Nonetheless, in most cases it will be clear which group is meant, otherwise, we will use $\langle\langle R\rangle\rangle_G$.

2.2.9. *Definition.* Let G and H be two groups. A function $\phi\colon G \to H$ is called a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b) \text{ for each } a, b \in G.$$

Clearly the *image* $\operatorname{im}(\phi) := \phi(G)$ is a subgroup of H and the *kernel* $\ker(\phi) := \phi^{-1}(e_H)$ is a normal subgroup of G. We call ϕ a *mono-*, *epi-* or *isomorphism* if it is *injective, surjective* or *bijective* respectively. If ϕ is an isomorphism we call G and H *isomorphic* and write $G \cong H$.

Suppose we have given a homomorphism ϕ as above. Let $g, h \in G$ and suppose $\phi(g) = \phi(h)$. Thus $\phi(gh^{-1}) = e_H$, i.e. $gh^{-1} \in \ker(\phi)$ which is equivalent to $g\ker(\phi) = h\ker(\phi)$. This implies that the mapping $g\ker(\phi) \mapsto \phi(g)$ is well-defined and moreover injective. Trivially it is a homomorphism, thus we have proven the following

2.2.10. *Lemma.* (see Theorem 3.4 in [2]) Let $\phi\colon G \to H$ be homomorphism of groups G and H. Then

$$G/\ker(\phi) \cong \operatorname{im}(\phi).$$

2.2.11. *Definition.* An *action* of a group G on a metric space (X, d) *by isometries* is a homomorphism $\Phi\colon G \to \operatorname{Isom}(X)$. We shall usually suppress Φ and set $g \cdot x := (\Phi(g))(x)$ and write $g \cdot Y$ for the image of a subset $Y \subseteq X$.

An action Φ is said to be *faithful* if Φ is injective. The *orbit* of $x \in X$ is defined as the set $G \cdot x := \{g \cdot x \mid g \in G\}$ and the *stabilizer* of $x \in X$ is defined as the set $G_x := \{g \in G \mid g \cdot x = x\}$. Note that G_x is a subgroup of G for each $x \in X$ and that orbits are either disjoint or equal, i.e. the set of orbits partition X.

An action Φ is said to be *free* if every stabilizer is trivial. Clearly a free action is faithful. An action is said to be *transitive* if for some, equivalently every, $x \in X$ it holds that $G \cdot x = X$ and it is said to be *almost transitive* if there are *finitely many* orbits.

We now introduce one of the most important groups.

2.2.12. *Definition.* Let X be a subset of a group F. Then F is a *free group with basis* X if for *any* group G and *any* map $\phi\colon X \to G$, there exists a *unique* extension of ϕ to a homomorphism $\phi^*\colon F \to G$.

We refer to the above-mentioned as the *universal property* of the free group F.

2.2.13. *Remark.* Clearly, X generates the group F in the above definition. Indeed, let $X \to \langle X \rangle$ be the identity on X and extend it to a homomorphism $F \to \langle X \rangle$ hence to a homomorphism $F \to F$ with image $\langle X \rangle$. Since the identity mapping $F \to F$ also extends the identity mapping X and by *uniqueness* of extensions we get that $F = \langle X \rangle$.

2.2.14. *Remark.* Clearly $e_F \notin X$. Moreover, suppose that we have $x, y \in X$ which are inverse to each other, i.e. $xy = e_F$. Clearly this yields a contradiction, since the group G and the map $\phi\colon X \to G$ in the above definition are *arbitrary.* More precisely, if G is a group with at least two elements we could map x to the identity e_G and y to a non-trivial element. A similar argument shows that any product $x_1x_2\cdots x_n$ with $x_i \in X$ and $n \geq 1$ *can't* be the identity e_F. If we write $X^{-1} = \{x^{-1}|x \in X\}$ one has the same statement for products of the form $x_1x_2\cdots x_n$ where $x_ix_{i+1} \neq e_F$ and $x_i \in X \sqcup X^{-1}$.

The above considerations contain quite the whole idea behind the *construction* of a free group for a given *set* X.

Let therefore X be an arbitrary *set* and denote by $X^{-1} = \{x^{-1}|x \in X\}$, where x^{-1} denotes a new *symbol* corresponding to the element x. We assume that $X \cap X^{-1} = \emptyset$ and that the expression $(x^{-1})^{-1}$ denotes the element x. The set $X^{\pm} = X \cup X^{-1}$ is called an *alphabet* and its elements are called *letters.* A *word* is a finite sequence of letters written in the form $x_1x_2\ldots x_n$, $n \neq 0$, $x_i \in X^{\pm}$. For $n = 0$ we have the *empty word.* We call n the *length* of the word w and denote it by $|w|$. A *subword* of a word is any subsequence of *consecutive* letters.

Let now $W(X)$ be the set of all words in the alphabet $X^{\pm}$. Given two words v, w, we define their *product* by *concatenation of words* as the word vw. So far $W(X)$ is a monoid, called the *free monoid with basis* X.

Two words v, w are called *equivalent* if there exists a finite sequence of words $v = f_1, f_2, \ldots, f_k = w$ such that each f_{i+1} can be obtained from f_i by insertion or deletion of subwords of the form xx^{-1}, where $x \in X^{\pm}$. Trivially this defines an equivalence relation on $W(X)$. A word w is called *freely reduced* if it does not contain subwords of the form xx^{-1}, where $x \in X^{\pm}$. It is an easy exercise to show that any class $[w]$ contains a *unique* freely reduced word (see Proposition 3.3 in [2]). This implies that $F(X) := W(X)/\sim$, where $\sim$ denotes the equivalence relation defined above, is indeed a group and moreover a free group with basis X, hence

2.2.15. *Theorem.* (see Theorem 3.2 in [2]) For any set X there exists a free group with the basis X.

The universal property can be used to prove the following

2.2.16. *Lemma.* (see Exercise 3.4 in [2]) Any free F group with basis X is isomorphic to the above construced free group $F(X)$.

Although it is a little bit ambiguous we will omit the brackets for elements (i.e. equivalence classes) of $F(X)$ and write w instead of $[w]$. The following is trivial but quite important.

2.2.17. *Theorem.* (see Theorem 3.14 in [2]) Any arbitrary group is a factor group of an apropriate free group.

Proof. Let G be a group generated by Y. By the universal property for the identity map on Y we get a homomorphism $F(Y) \to G$ which is surjective since Y generates G. $\square$

2.2.18. *Definition.* Let G be a group generated by Y and consider $\phi\colon F(Y) \to G$ as in the theorem above, i.e. $\phi(y) = y$ for $y \in Y$. If w is a word that maps to the identity in G under ϕ, i.e. $\phi(w) = e_G$, then w is called a *relation.*

For example, if a and b generate $\mathbb{Z} \times \mathbb{Z}$ then $bb^3a^{-2}b^{-2}a^2b^{-2}$ is a relation.

A subset $R \subseteq \ker(\phi) \subseteq F(Y)$ is a *set of defining relations* or a *set of relators* if $\ker(\phi) = \langle\langle R \rangle\rangle$. Hence each element in the kernel can be expressed as a finite product of conjugates of the elements of R and their inverses and since ϕ is surjective we get that

$$F(Y)/\langle\langle R \rangle\rangle \cong G.$$

The above is called a *presentation* of G and one writes

$$\langle Y \mid R \rangle := F(Y)/\langle\langle R \rangle\rangle.$$

We say that G is *finitely presented* if Y is *finite* (i.e. G is finitely generated) and R is *finite*. Note that there exist finitely generated groups which are *not* finitely presented.

Suppose that we have a homomorphism $\phi\colon G \to H$ and let $N \trianglelefteq G$. If, in addition, $N \subseteq \ker(\phi)$ then we can define a homomorphism $\bar{\phi}\colon G/N \to H$ by the rule $\bar{\phi}(gN) := \phi(g)$. This is well-defined since for $gN = hN$ we have $g^{-1}h \in N \subseteq \ker(\phi)$, i.e. $\phi(g) = \phi(h)$. If $R \subseteq G$ and $\langle\langle R \rangle\rangle = N$, it is clear that we only have to check that $\phi(r) = e_H$, for $r \in R$, to get a homomorphism as above. Hence if $\phi\colon X \to H$ is a map from a set X to a group H and for a word $r = x_1 x_2 \dots x_n$ in the alphabet $X^{\pm}$ we set $\phi(r) = \phi(x_1)\phi(x_2)\dots\phi(x_n)$, assuming that $\phi(x^{-1}) = (\phi(x))^{-1}$ for $x \in X$, we obtain

2.2.19. *Theorem.* (see Theorem 5.7 in [2]) Let G be a group presented by generators and defining relations $\langle X \mid R \rangle$ and let H be another group. *Every* map $\phi\colon X \to H$ such that $\phi(r) = e_H$ for all $r \in R$ can be extended to a homomorphism $G \to H$.

We can easily reformulate the above theorem to the following

2.2.20. *Theorem.* (see Theorem 5.8 in [2]) Let G and H be groups represented by generators and defining relations $\langle X \mid R \rangle$ and $\langle X' \mid R' \rangle$ respectively. Then *every* map $\phi\colon X \to X'$ with the property that all the words $\phi(r)$, for $r \in R$ lie in the normal closure of R' in $F(X')$ can be extended to a homomorphism $G \to H$.

We now introduce the *free product* of groups, which can be thought of as a group generated by copies of the groups involved where there are no "relations" between the groups in question.

2.2.21. *Definition.* Let $(G_i)_{i \in I}$ be a family of groups. Set $A := \bigsqcup_{i \in I} G_i$ (we can work with a disjoint union since we can shift to isomorphic copies if necessary). Let $\sim$ be the equivalence relation of $W(A)$ generated by

$$ve_i w \sim vw \quad \text{where } e_i \text{ is the unit in } G_i \text{ for some } i \in I$$

$$vabw \sim vcw \quad \text{whenever } a, b, c \in G_i \text{ with } c = ab \text{ for some } i \in I.$$

for all $v, w \in W(A)$. It is straightforward to check that the quotient $W(A)/\sim$ is a group. It is called the *free product* of the groups G_i, $i \in I$ and is denoted by

$$*_{i\in I} G_i.$$

For A as above a word $w = a_1 a_2 \dots a_n \in W(A)$, with $a_j \in G_{i_j}$ say, is said to be *reduced* if $i_{j+1} \neq i_j$ for $1 \leq j \leq n-1$ and if a_j is *not* the unit element of G_{i_j}. In other words, w can't be *shorten* by the above equivalence relations.

Similar to the case of free groups one can show that any element (i.e. equivalence class) in the free product is represented by a *unique* reduced word in $W(A)$ (see Proposition 1 in [10], p. 18). As above we shall slightly abuse notation and write w instead of $[w]$. As a corollary, one obtains that for each $i_o \in I$ the canonical homomorphism

$$\iota_0 \colon G_{i_0} \to *_{i\in I} G_i$$

is injective (see Corollary 2 in [10], p.18). We can also characterize the free product by the following universal property:

Let G be a group and let $(G_i)_{i\in I}$ be a family of groups and let $(h_i \colon G_i \to G)_{i\in I}$ be a family of homomorphisms. Then there exists a *unique* homomorphism $h \colon *_{i\in I} G_i \to G$ such that the diagram

$$\begin{tikzcd} G_{i_0} \arrow[d, "\iota_0"'] \arrow[dr, "h_{i_0}"] & \\ *_{i\in I} G_i \arrow[r, "h"] & G \end{tikzcd}$$

is *commutative* for each i_0, i.e. $h \circ \iota_0 = h_{i_0}$. Hence, h extends each h_i.

2.3. **Graph theory.** There are many different ways to define graphs. The key point is if one wants to allow multiple edges and loops or not. Wel start with the definition given in [4]:

2.3.1. *Definition.* A *(combinatorial) graph* X is defined as a quadruple (V, E, α, ω), where V is a (possibly infinite but in any case non-empty) set, E is a (possibly infinite) set, $\alpha \colon E \to V$ and $\omega \colon E \to V$ are two functions.

V is the *set of vertices* of X and its elements are called *vertices*. Similar E is the *set of edges* of X and its elements are called *edges*. The two functions α and ω are the *endpoint maps*.

For an edge $e \in E$ we call $\alpha(e)$ the *initial vertex* of e and $\omega(e)$ the *terminal vertex* of e. In a picture involving the edge e this will be indicated by an arrow pointing towards $\omega(e)$ and with its "tail" at the vertex $\alpha(e)$, where vertices will be drawn as points. Two vertices v, w are *adjacent* if there exists an edge e *between* them, i.e. $\alpha(e) = v$ and $\omega(e) = w$ or vice versa.

The above definition is probably the most general one. Another possible way to define a graph (in a more restrictive way) is as follows:

Let $X = (V, E)$, where V is a set as above and E is a set of 2-element subsets of V. In other words between two vertices there is *at most one edge* and loops (i.e. edges with $\alpha(e) = \omega(e)$) are permitted. Such graphs are often called *simple* graphs. In this case we don't draw an arrow between two vertices but just a simple segment. We refer to such graphs as being *undirected*.

In most cases which we will encounter there will be *at most one* edge e between two vertices v, w with $\alpha(e) = v$ and $\omega(e) = w$. Hence we can interpret the set E as a subset of $V \times V$ and α as the projection onto the first entry and ω as the projection onto the second entry.

If we have to deal with two or more different graphs we will write VX for V and EX for E to emphasize that we talk about the vertex and edge set respectively of the graph X.

2.3.2. *Definition.* Let v be a vertex. The set $N(v)$, the *set of neighbours of* v, denotes the set of vertices adjacent to v. A graph is called *locally finite* if $N(v)$ is *finite* for each vertex v.

2.3.3. *Definition.* A *walk* is a sequence of vertices $(v_1, v_2, \dots, v_n)$, $n \geq 1$, such that v_i and v_{i+1} are adjacent for $1 \leq i \leq n-1$. Note that we have not specified which edges we "walk along". If we want to emphasize this occasion we write $(v_1, e_1, v_2, e_2, \dots, e_{n-1}, v_n)$, where v_i is a vertex and e_i is an edge between v_i and v_{i+1}. A *path* is a walk where all vertices are *distinct*. We say that a walk (or path) $(v_1, v_2, \dots, v_n)$ *connects* two vertices v, w if $v = v_1$ and $w = v_n$ or vice versa. Conversely, two vertices v, w *can be connected by a walk* (*path*) if such a walk (path) exists. Note that every walk from v_1 to v_n contains a path connecting v_1 to v_n.

A *cycle* (or *edge-loop*) is a walk $(v_1, v_2, \dots, v_n)$, $n \geq 3$, such that $v_1 = v_n$. A *circuit* is a cycle where all vertices are *distinct* excluding $v_1 = v_n$. In such a case we speak of a cycle (circuit) *at* v_1. Note that for a vertex v each cycle at v contains a circuit at v. Moreover, we can view a cycle (or a circuit) as a concatenation of incident edges of the graph.

The *length* of a walk, path, cycle or circuit is defined as the *number of edges* traversed. Hence a path $(v_1, v_2, \dots, v_n)$ has length $n - 1$.

For two vertices v, w define $d_X(v, w)$ as the *minimal length* of a path conntecting v and w.

A subset $C \subseteq V$ is said to be *connected* if any two vertices of C can be connected by a path which stays entirely in C. Maximal subsets (w.r.t inclusion) subject to being connected are called *components* of a graph X. The graph X is said to be *connected* if V is connected. In this case (V, d_X) is a metric space and we call d_X the *combinatorial metric* of X.

2.3.4. *Remark.* Since the metric of (V, d_X) (see above) only takes integer values the obtained metric space is discrete. Nonetheless, it seems natural to view an edge e and *its* two vertices $\alpha(e)$ and $\omega(e)$ as an isometric copy of a compact (unit) interval (since this is the way one "draws" this edge).

If we identify each edge with the interval $[0, 1]$ and an appropriate equivalence relation on the disjoint union of V and the family of these intervals we obtain a "satisfying" metric space as a quotient into which (V, d_X) embeds isometrically, sometimes called the *geometric realization* of a graph X (in [4] this is called a *metric graph*). We refer to [4], p.7 for details. We will refer to this construction as the *1-complex* associated to the graph X or simply say the *metric graph* X. It is sometimes convenient to identify an edge not only with the unit interval but with an interval of the form $[0, a]$ for some $a \in (0, \infty)$. We encode this matter with a map

$$\lambda \colon E \to (0, \infty).$$

In the sequel the set $\lambda(E)$ will be *finite*, i.e. we consider only finitely many *different* intervals. In this case our 1-complex is still a *complete geodesic space*.

2.4. **Graphs and groups.** Many interesting examples of metric graphs or 1-complexes are given by the following construction.

2.4.1. *Definition.* The *directed Cayley graph* $\Gamma(G, S)$ of a group G and a subset $S \subseteq G$ is defined as follows:

Its vertex set is in one-to-one correspondence with the elements of G and there is an edge, *labelled* s, of length one joining g to gs for each $g \in G$ and $s \in S$.

In the above notation we have $V = G$, $E = \{(g, s) \mid g \in G, s \in S\}$, $\alpha((g, s)) = g$, $\omega((g, s)) = gs$ and λ is the constant function 1.

The group G acts by left multiplication on this graph, where we have a transitive and free action on the vertices.

Clearly, the component of $\Gamma(G, S)$ containing S coincides with $\langle S \rangle$. In particular $\Gamma(G, S)$ is connected if and only if S generates G. Moreover $\Gamma(G, S)$ is locally finite if and only if S is finite.

Note that when we speak of the Cayley graph of a group we *always* talk about the associated 1-complex.

The Cayley graph of a group is transitive, i.e. the "neighbourhood" of any two vertices "look similar". In the following figure let $x, y \in G$ be elements of order $\neq 2$, i.e. $x \neq x^{-1}, y \neq y^{-1}$ and $z \in G$ with order 2, i.e. $z = z^{-1}$. Then for an element $g \in G$ the Cayley graph w.r.t $S = \{x, y, z\}$ looks locally around g like it is shown in the figure below.

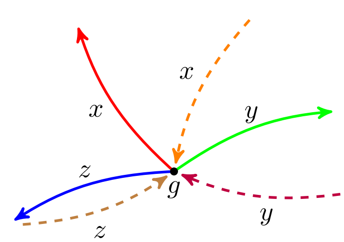

Figure 1. The local structure of a Cayley graph with G and S as above.

2.4.2. *Remark.* In the sequel we consider the set S as being *symmetrized*, i.e. if $s \in S$ then $s^{-1} \in S$. Note that if S is a finite set than in general $S^{\pm}$ does *not* have an even number of elements. Indeed if $S = \{s\}$ and s has order 2 than $S^{\pm} = S$, since $s = s^{-1}$.

Since we assume S to be symmetrized we have an edge $e := (g, s)$ and an edge $\tilde{e} := (gs, s^{-1})$ which are both edges between the vertices g and gs with $\alpha(e) = \omega(\tilde{e})$

and $\omega(e) = \alpha(\tilde{e})$. If we identify these edges we can view $\Gamma(G, S)$ as an undirected graph, simplifying the picture of the graph.

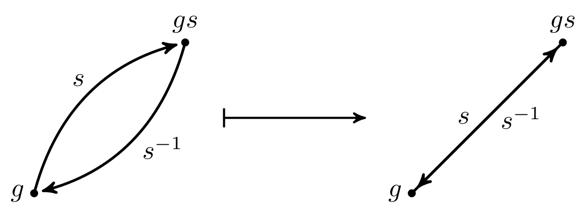

FIGURE 2. Edges in a Cayley graph with symmetrized generating set.

2.4.3. *Remark.* There is an one-to-one correspondence of walks connecting vertices g, h (we start with g) and words in S representing $g^{-1}h$.

Indeed, let $g^{-1}h = s_1 s_2 \cdots s_n$. We start at the vertex g and walk along the edge with label s_1. Note that there is *exactly* one such edge (in the literature this is sometimes called *regularity* w.r.t the labeling). We arrive at the vertex gs_1. Proceeding in this manner, we finally stop at the vertex $gs_1 s_2 \cdots s_n$. But by assumption this is the vertex h. If we have a different word representing $g^{-1}h$, than in the above procedure we would walk along *different* edges and in particular we would obtain a *different* walk.

Conversely, if we have a walk connecting g and h, say $(g = g_1, g_2, \ldots, g_n = h)$. There is precisely one proper edge to walk along from g to g_2. We record the label of this edge, say s_1. In particular, $g_2 = gs_1$. Proceeding in this manner, we obtain $h = gs_1 s_2 \cdots s_{n-1}$. In particular, $g^{-1}h = s_1 s_2 \cdots s_{n-1}$. As above different walks induce different words representing $g^{-1}h$.

In particular, *relators* and *relations* correspond to *cycles* (at the identity) in the Cayley graph.

It is clear that the "shape" of a Cayley graph $\Gamma(G, S)$ strongly depends on S. More precisely, if S and T are two generating sets for G, how are $\Gamma(G, S)$ and $\Gamma(G, T)$ related to each other? Indeed, if both generating sets are finite the question is rather easy to answer. However we omit a proof for the following lemms since we will prove a similar more general statement.

2.4.4. *Lemma.* Let G be a group and let S and T be two *finite* generating sets of G. Then

$$\Gamma(G, S) \sim_{QI} \Gamma(G, T).$$

We note that for the above proof it suffices that S (considered as a set of vertices) is contained in a ball of finite radius in $\Gamma(G, T)$ and similarly for T.

2.4.5. *Definition.* Let G be a group with a generating set S. The *word metric* of G w.r.t S is defined as follows:

Let $g, h \in G$. Then their distance $d_S(g,h)$ is defined to be the *shortest* word in the pre-image of $g^{-1}h$ under the canonical projection $F(S) \to G$. Note that this metric coincides with the (combinatorial) metric given on the vertices of the Cayley graph $\Gamma(G,S)$. Trivially $(G,d_S) \sim_{QI} \Gamma(G,S)$ hence for two finite generating sets S and T we have

$$(G,d_S) \sim_{QI} (G,d_T).$$

2.4.6. *Remark.* The above quasi-isometries are defined with the help of the *identity* on G. In general one should not hope that a homomorphism between two groups is a quasi-isometric embedding or vice versa. Indeed, we have the following

2.4.7. *Lemma.* (cf. Exercise 8.20 (1) in [4]) Let $\phi\colon G \to H$ be a homomorphism between finitely generated groups. Then:

(1) If ϕ is a quasi-isometric embedding then $\ker(\phi)$ is *finite*.
(2) ϕ has quasi-dense image if and only if $H/\mathrm{im}(\phi)$ is *finite*.
(3) ϕ is a quasi-isometry if and only if $\ker(\phi)$ and $H/\mathrm{im}(\phi)$ is *finite*.

Proof. Let ϕ be a (c,b)-quasi-isometric embedding and let S and T be finite generating sets for G and H respectively. Then for $g,h \in \ker(\phi)$ we have

$$\frac{1}{c}d_S(g,h) - b \leq d_T(e_H, e_H) = 0.$$

This implies that $\ker(\phi)$, as a set of vertices, is contained in a ball of finite radius in $\Gamma(G,S)$. But this graph is locally finite, i.e. every ball has only finitely many vertices. This proves (1).

Suppose now that ϕ has a quasi-dense image with b a quasi-dense constant for ϕ. If ϕ is surjective or H is finite there is nothing to prove. Moreover, we set $A := \mathrm{im}(\phi)$. Let $g \in G$ and set $B := B(\phi(g), b)$. Note that B has only *finitely* many vertices. We will prove that $A \cdot B = H$ which directly implies that A has finite index in H. Trivially $A \cdot B \subseteq H$. Let therefore $h \in H$. Then by assumption there exists $\tilde{g} \in G$ such that

$$d_T(h, \phi(\tilde{g})) \leq b.$$

Since group elements act as isometries and ϕ is a homomorphism we obtain

$$d_T(\phi(g\tilde{g}^{-1})h, \phi(g)) \leq b,$$

thus $\phi(g\tilde{g}^{-1})h \in B$ which implies $h \in A \cdot B$.

Suppose conversely that $H/\mathrm{im}(\phi)$ is finite, where we again set $A := \mathrm{im}(\phi)$. Let $\{e = x_1, x_2, \ldots x_b\}$ be a set of representatives of *right* cosets of A. We consider the so-called *factor graph* of A and H. Its vertex set is the set of cosets and two cosets Ax_i and Ax_j are *adjacent* if and only if there are adjacent vertices x, y in $\Gamma(H,T)$ with $x \in Ax_i$ and $y \in Ax_j$. Even more generally (when there are infinitely many cosets) it is easily shown that the connectness and local finiteness of $\Gamma(H,T)$ imply that the factor graph is connected and locally finite. Note that the orbits of the action of A on the $\Gamma(H,T)$ coincide with *right* cosets of A (that is why we are working with right cosets). In particular let $h \in H$ and consider Ah. Take a path in the factor graph from this coset to the coset A. This path has length at most b. Suppose this path starts with an edge from Ah to Ax_2. By definition there exists ah and $\tilde{a}x_2$ in $\Gamma(H,T)$, which are adjacent. But by the action of a^{-1} we obtain that h is adjacent with $a^{-1}\tilde{a}x_2 \in Ax_2$. Proceeding in this manner, we are able to lift the path from the factor graph to a path in $\Gamma(H,T)$. In particular this proves the existence of some $\phi(g) \in A$ such that $d_T(h, \phi(g)) \leq b$. This proves (2).

Finally let $\ker(\phi)$ and $H/\mathrm{im}(\phi)$ be finite. By (2) we only have to prove that ϕ is a quasi-isometric embedding. As above let S be a finite generating set for G, thus $\phi(S)$ is a finite generating set for $\phi(G)$. In particular $\phi(S)$ is contained in some ball of finite radius r around e_H in $\Gamma(H,T)$. Let b be a quasi-dense constant for ϕ provided by (2). Set $a := \max\{2b+1, r\}$ and define $B := B(e_H, a)$. Since $\phi(S) \subseteq B$ we have that $\langle \phi^{-1}(B) \rangle = G$. By assumption the kernel of ϕ is *finite* thus $\tilde{S} := \phi^{-1}(B)$ is a *finite* generating set for G. Let $g \in G$ and suppose $d_T(e_H, \phi(g)) = n$, i.e. $\phi(g) = t_1 t_2 \cdots t_n$ with $t_i \in T$ for all i. The corresponding geodesic path consists of $n+1$ vertices, denoted $e_H = v_0, v_1, \ldots, v_n = \phi(g)$. For each v_i we can choose $g_i \in G$ such that $d_T(v_i, \phi(g_i)) \leq b$, where we set $g_0 = e_G$ and $g_n = g$. By the triangle inequality we have

$$d_T(\phi(g_i), \phi(g_{i+1}) \leq 2b+1,$$

which implies

$$d_T(e_H, \phi(g_i^{-1} g_{i+1})) \leq a,$$

thus $s_{i+1} := g_i^{-1} g_{i+1} \in \tilde{S}$. Hence we can write g as a word in $\tilde{S}$ in the following way:

$$g = g_{n-1} g_{n-1}^{-1} g_n = \cdots = (g_0 g_1)(g_1^{-1} g_2) \cdots (g_{n-1}^{-1} g_n) = s_1 \cdots s_n.$$

Since this is *some* word representing g we obtain

$$d_{\tilde{S}}(e_G, g) \leq n = d_T(e_H, \phi(g)).$$

Since word metrics are invariant under left multiplication of the corresponding group we have

$$d_{\tilde{S}}(g,h) \leq d_T(\phi(g), \phi(h)),$$

for $g, h \in G$. Conversely let $d_{\tilde{S}}(g,h) = m$, i.e. $g^{-1}h = s_1 s_2 \cdots s_m$ for some $s_i \in \tilde{S}$. Set $g_i := s_1 \cdots s_i$ for $i \geq 1$. Finally set $\alpha := \max\{d_T(e_H, \phi(s) \mid s \in \tilde{S}\}$. We obtain that

$$d_T(\phi(g_{i-1}), \phi(g_i)) = d_T(e_H, \phi(g_{i-1}^{-1} g_i)) = d_T(e_H, \phi(s_i)) \leq \alpha.$$

Together with the traingle inequality we have

$$\begin{gathered} d_T(\phi(g), \phi(h)) = d_T(e_H, \phi(g^{-1}h)) \leq \\ \leq d_T(e_H, \phi(g_1)) + d_T(\phi(g_1), \phi(g_2)) + \cdots + d_T(\phi(g_{n-1}), \phi(\underbrace{g^{-1}h}_{=g_n})) \leq \\ \leq \alpha n = \alpha d_{\tilde{S}}(g,h). \end{gathered}$$

Note that for this last inequality $d_T(\phi(g), \phi(h)) \leq \alpha d_{\tilde{S}}(g,h)$ we did *not* need our assumptions except the fact that G is finitely generated or more generally that $\phi(S)$ is bounded in $\Gamma(H,T)$ (cf. Lemma 8.18 in [4], p.140). □

The above lemma shows that a quasi-dense constant gives an upper bound on the index of $\mathrm{im}(\phi)$. As a direct consequence we can state

2.4.8. *Corollary.* (cf Proposition 11.41 in [14]) Let H be a finite index subgroup of G, both of which are finitely generated. Then H and G are quasi-isometric.

2.5. **Hyperbolic spaces.** We recall basic definitions and statements about hyperbolic spaces. We mainly follow [4].

2.5.1. *Definition.* Let $\delta \geq 0$. A *geodesic triangle* in a metric space is said to be *δ-slim* if each of its sides is contained in the δ-neighbourhood of the union of the other two sides.

A geodesic space X is said to be *δ-hyperbolic* if every geodesic triangle is δ-slim. The space X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

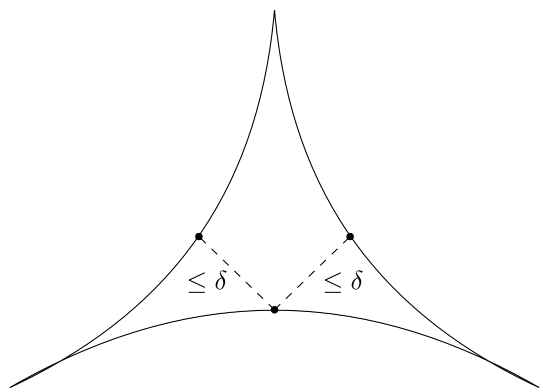

FIGURE 3. A slim triangle
(cf. figure 20 in [18] p.242)

2.5.2. *Theorem.* (see Theorem 1.9 in [4], p.402) Let X and Y be geodesic metric spaces and let $f\colon X \to Y$ be a quasi-isometric embedding. If Y is hyperbolic then X is hyperbolic. In particular, hyperbolicity is preserved under quasi-isometries

2.5.3. *Remark.* Although the above theorem sounds quite obvious its proof is *not.* One way to see this is the fact that the definition of hyperbolicity relies on a *local property*, where on the other hand quasi-isometries (or more generally quasi-isometric embeddings) "forget" about the local structure and focus on the global shape of a space. In particular, this shows that slim triangles (local data) have global consequences.

The proof of Theorem 2.5.2 relies on the fact that in hyperbolic spaces *quasi-geodesics* stay Hausdorff close to geodesics, where quasi-geodesics are just quasi-isometric embeddings of intervals.

2.5.4. *Definition.* A (λ, ε)-*quasi-geodesic* in a metric space X is a (λ, ε)-quasi isometric embedding $c\colon I \to X$, where I is a (bounded or unbounded) interval of the real line or else the intersection of $\mathbb{Z}$ with such an interval.

If $I = [a, b]$ then $c(a)$ and $c(b)$ are called the *endpoints* of c. If $I = [0, \infty)$ then c is called a *quasi-geodesic-ray.*

As mentioned above we have the following

2.5.5. *Theorem.* (see Theorem 1.7 in [4], p.401) For all $\delta > 0, \lambda \geq 1, \varepsilon \geq 0$ there exists a constant $R = R(\delta, \lambda, \varepsilon)$ such that:

If X is a δ-hyperbolic space, c is a (λ, ε)-quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is less than R, i.e.

$$\inf\{\alpha \mid [p, q] \subseteq N_\alpha(\mathrm{im}(c)) \text{ and } \mathrm{im}(c) \subseteq N_\alpha([p, q])\} \leq R.$$

2.5.6. *Remark.* Note that the above Theorem is completely false in non-hyperbolic spaces. Consider $\mathbb{R}^2$ with the euclidean metric. The triangle defined by the points $(0, 0), (4\delta, 0)$ and $(0, 4\delta)$ shows that $\mathbb{R}^2$ cannot be hyperbolic for any $\delta > 0$. (However one can easily define a metric such that $\mathbb{R}^2$ becomes an $\mathbb{R}$-tree, i.e. a simply connected, 0-hyperbolic space). Indeed, since any triangle $\mathbb{R}^n$ (for $n \geq 2$) is contained in a plane this shows that $\mathbb{R}^n$ is *not* hyperbolic for $n \geq 2$. Moreover, if one considers half-circles parametrized by arclength one easily verifies that these are quasi-geodesics. More precisely, for $r > 0$ the map $c\colon [0, r\pi] \to \mathbb{R}^2$, sending t to $c(t) = (r \cos(t/r), r \sin(t/r))$, is a $(\pi, 0)$-quasi-geodesic with endpoints $(-r, 0)$ and $(r, 0)$. But clearly there exists no constant such that these half-circles stay Hausdorff close to *the* geodesic segment joining their endpoints.

However, the above theorem makes the proof of Theorem 2.5.2 quite easy. Moreover, Theorem 2.5.2 allows to define *hyperbolic groups* in the following way.

2.5.7. *Definition.* A finitely generated group is *hyperbolic* if its Cayley graph (w.r.t some finite generating set) is a δ-hyperbolic metric space for some $\delta \geq 0$.

The proof of the following theorem is quite technical but has an important consequence for our group theoretic considerations.

2.5.8. *Theorem.* (see Theorem 1.13 in [4], p.405) Let X be a δ-hyperbolic space and let $c\colon [a, b] \to X$ be a k-local geodesic, where $k > 8\delta$. Then:

(1) $\mathrm{im}(c)$ is contained in the 2δ-neighbourhood of *any* geodesic segment $[c(a), c(b)]$ connecting its endpoints,
(2) $[c(a), c(b)]$ is contained in the 3δ-neighbourhood of $\mathrm{im}(c)$, and
(3) c is a (c, b)-quasi-geodesic, where $b = 2\delta$ and $c = \frac{k+4\delta}{k-4\delta}$.

2.5.9. *Corollary.* (see Corollary 1.14 in [4], p.407) If X is a δ-hyperbolic space and $c\colon [a, b] \to X$ is a k-local geodesic for some $k > 8\delta$, then either c is constant (i.e. $a = b$) or else $c(a) \neq c(b)$.

In particular, the above corollary shows that if $k > 8\delta$ then there are *no* non-trivial closed k-local geodesics (cycles) in a δ-hyperbolic space. In other words, every "proper" cycle *must* contain a piece which is of length at most 8δ but it is *not* geodesic. In the sequel we are interested in metric graphs which are hyperbolic hence we make the above mentioned more precise in the following way.

2.5.10. *Lemma.* (see Lemma 2.6 in [4], p.417) Let X be a δ-hyperbolic metric graph whose edges all have integer lengths and $\delta > 0$ is an integer. Given any non-trivial locally-injective loop $c\colon [0, 1] \to X$, i.e. $c(0) = c(1)$, where $c(0)$ is a vertex, one can find $s, t \in [0, 1]$ such that $c(s)$ and $c(t)$ are vertices of X, $d(c(s), c(t)) \leq l(c|_{[s,t]}) - 1$ (*the loop contains a non-geodesic piece*) and $d(c(s), c(t)) + l(c|_{[s,t]}) \leq 16\delta$.

2.5.11. *Remark.* It is quite crucial that the above lemma holds not only for locally finite graphs.

We now introduce an equivalent notion of hyperbolicity which will play an essential role in the sequel (see [4], p.414 - 421). For this we introduce (combinatorial) complexes. There is a vast literature on this subject, we follow [4].

2.5.12. *Definition.* We define an *n-dimensional* (*combinatorial*) *complex* by a recursion (on dimension). The definition of an *open cell* is defined by a simultaneous recursion. If K_1 and K_2 are complexes, then a continuous map $K_1 \to K_2$ is said to be *combinatorial* if its restriction to each open cell of K_1 is a *homeomorphism* onto an open cell of K_2.

A *complex of dimension* 0 is simply a discrete set, the *vertices*; each point is an *open cell.* Having defined $(n-1)$-dimensional complexes and their open cells, one constructs n-dimensional complexes as follows:

Take the disjoint union of an $(n-1)$-dimensional complex $K^{(n-1)}$ and a family $(e_\lambda \mid \lambda \in \Lambda)$ of copies of *closed* n-dimensional discs (i.e. closed balls in $\mathbb{R}^n$). It is no restriction to consider only *unit* discs. Suppose that for each $\lambda \in \Lambda$ a *homeomorphism* is given from ∂e_λ (a sphere) to an $(n-1)$-dimensional complex S_λ and that a *combinatorial* map $S_\lambda \to K^{(n-1)}$ is also given. Let $\phi_\lambda \colon \partial e_\lambda \to K^{(n-1)}$ be the composition of these maps, the *attaching map.* Define K to be the quotient of $K^{(n-1)} \sqcup \bigsqcup_{\lambda\in\Lambda} e_\lambda$ by the equivalence relation generated by $t \sim \phi_\lambda(t)$ for all $\lambda \in \Lambda$ and $t \in \partial e_\lambda$. Then K, with the quotient topology, is an *n-dimensional* (*combinatorial*) *complex* whose *open cells* are the (images of) open cells in $K^{(n-1)}$ and the interiors of the e_λ.

As it was mentioned already we call 0-cells *vertices* and additionally 1-cells are called *edges* and 2-cells are called *faces.*

2.5.13. *Remark.* A 1-complex can be interpreted as a graph (compare with Remark 2.3.4). Indeed, the vertices of the graph, say V, are the vertices (0-cells) of the complex. A closed 1-dimensional unit disc e_λ is a copy of the interval $[0,1]$ and $\partial e_\lambda = \{0,1\}$. The attaching map $\phi_\lambda \colon \{0,1\} \to V$ sends first $\{0,1\}$ to a (homeomorphic) 0-dimensional complex S_λ (a graph with two vertices and no edges) and then to a single vertex (which creates a loop) or to two distinct vertices (which creates an "ordinary" edge). Note that multiple edges are possible. Moreover, the open 1-cells are the edges *without* their incident vertices.

2.5.14. *Remark.* We are interested primarily in 2-dimensional complexes. By the above remark we start with a metric graph X. A closed 2-dimensional unit disc is denoted by D^2 (the set of vectors in $\mathbb{R}^2$ of euclidean length at most 1) and $\partial D^2 = S^1$ (the set of vectors of euclidean length $= 1$). The attaching maps $\phi_\lambda \to X$ are cycles (combinatorial loops, edge loops, see [4]) in X. In general, we have only cycles and *not* circuits. Hence we can view a 2-dimensional complex as a metric graph were some cycles are "filled up".

2.5.15. *Definition.* Let D^2 and S^1 denote as above the unit disc and the unit sphere respectively.

A *triangulation of* D^2 is a homeomorphism $P\colon D^2 \to K$, where K is a 2-complex in which every 2-cell is a 3-gon (i.e. a triangle).

We endow D^2 with the induced cell structure and refer to the preimages under P of *0-cells, 1-cells* and *2-cells* as, resprectively, the *vertices, edges* and *faces of* P.

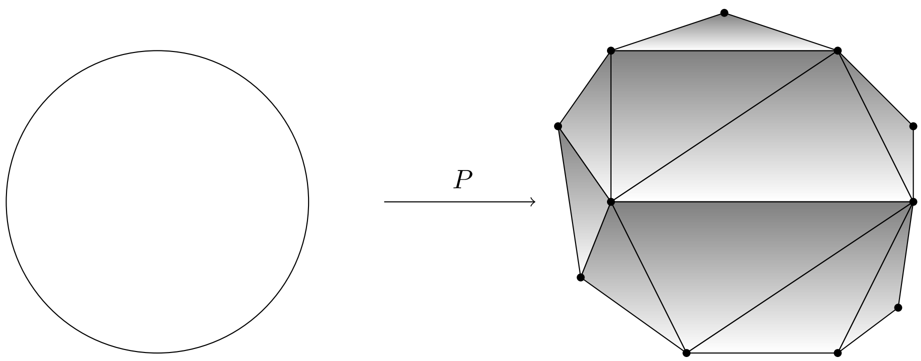

FIGURE 4. Triangulation of D^2

.

Let X be a metric space. Note that we can view a loop $c\colon [0,1] \to X$ as a map $c : S^1 \to X$, which we assume to be a rectifiable.

An ε-*filling* (P, Φ) of c consists of a triangulation P of D^2 and a (not necessarily continuous) map $\Phi\colon D^2 \to X$ such that $\Phi|_{S^1} = c$ and the image under Φ of each face of P is a set of diameter at most ε in X.

We write $|\Phi|$ to denote the *number of faces* of P and refer to this as the *area of the filling.* The ε-*area* of c is defined to be:

$$\mathrm{Area}_\varepsilon(c) := \min\{|\Phi| \mid \Phi \text{ is an } \varepsilon\text{-filling of } c\}.$$

If there is no ε-filling we set $\mathrm{Area}_\varepsilon(c) := \infty$.

A function $f\colon [0,\infty) \to [0,\infty)$ is called a *coarse isoperimetric bound* on X if there exists $\varepsilon > 0$ such that *every* rectifiable loop c in X has an ε-filling and $\mathrm{Area}_\varepsilon(c) \leq f(l(c))$.

If f is linear then wen say that X satisfies a (*coarse*) *linear isomerimetric inequality.* The key point is that this property is equivalent with hyperbolicity.

2.5.16. *Theorem.* (see Proposition 2.7 and cf. Theorem 2.9 in [4], p.417,419) Let X be a geodesic metric space.

Then X is hyperbolic if and only if X satisfies a linear isoperimetric inequality.

We now take a closer look on how one can "view" such fillings in metric graphs.

2.5.17. *Remark.* Since ε-fillings need *not* to be continuous we we can argue as follows:

Suppose we have specified a *filling-map* Φ only on the *vertices* of a triangulation of the disc. Then one can extend Φ across edges and triangles in the interior of the disc by simply sending them to the image of a vertex in their *boundary.* In particular, given a loop, if one wishes to exhibit the existence of an ε-filling of a given area then one need only specify an appropriate triangulation of the disc and specify a map on the vertices of the triangulation, ensuring that the *vertex set* of each triangle is sent to a set of diameter at most ε.

2.5.18. *Remark.* Let X be a metric graph in which all edges have length one. Given a loop $c\colon S^1 \to X$, we proceed along c from a vertex v_1 in the image and we record the vertices that c visits $v_1, v_2, \ldots, v_n = v_1$, where we record v_i only if it is distinct from v_{i-1} (if c does not hit any vertex the situation is trivial). We set $c(t_i) = v_i$.

Let c' be the cycle (combinatorial loop) in X obtained by concatenating the edges $[v_{i-1}, v_i]$. If $\varepsilon \geq 2$ then in order to construct an ε-filling of c we start by inscribing and n-gon in S^1 with vertices t_i. Then we map this n-gon to X by c' in the obvious way. Fill the n-gon with an ε-filling of c' and extend the given triangulation of the n-gon to a triangulation of the disc by introducing extra edges and vertices (if necessary) in the sectors bounded by the arcs and chords $[t_{i-1}, t_i]$: the image of the boundary of each of these sectors has diameter less than 2, so it suffices (see Remark 2.5.17 above) to introduce a new vertex on the arc joining t_i to t_{i+1}, say s_i, and to introduce an edge joining this vertex to each of the vertices of the filling of c' that lie on $[t_{i-1}, t_i]$.

This construction bounds $\mathrm{Area}_\varepsilon(c)$ by a linear function of $\mathrm{Area}_\varepsilon(c')$ and $l(c)$.

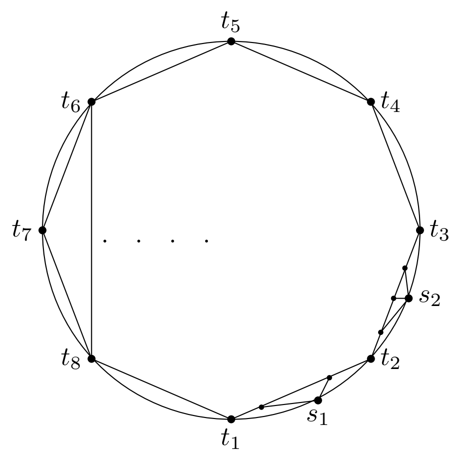

FIGURE 5. Triangulation of discs in metric graphs I

.

2.5.19. *Remark.* Let X be a metric graph with unit edge lengths and c be a cycle in X of length n, i.e. c traverses n edges. The figure below should bring the following inequality out,

$$\mathrm{Area}_n(c) \leq n,$$

since the *vertex set* of every triangle has diameter at most n (see Remark 2.5.17 above).

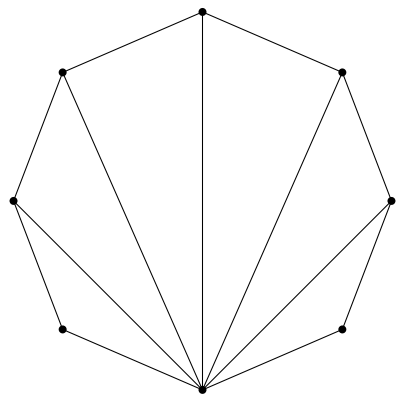

FIGURE 6. Triangulation of discs in metric graphs II

.

We next state a well-known result that hyperbolic groups can be characterized by particular group presentations. For this we review the so called *Word Problem*, which was first formulated and studied by Dehn in [8]:

An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.

The following algorithm given by Dehn is perhaps the most direct approach that one can hope for to solve the word problem, using directly the information given by a finite presentation:

Let S be a finite set of generators for a group G. Suppose we have a *finite* list of words $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$ such that the following holds:

(1) $u_i = v_i$ in G,
(2) the length of these words satisfy $|v_i| < |u_i|$ for every $1 \leq i \leq n$, and
(3) *each* word w, which represents the identity in G, has at least one u_i as a subword for some i.

It is clear that condition (3) is an essential ingredient of the algorithm.

Indeed, let w be a (freely reduced) word in S. First one looks for a subword of the form u_i. If there is no such subword by condition (3) we know that w cannot represent the identity in G. Otherwise we interchange the present u_i by the corresponding v_i obtaining a *strictly shorter* word w' by condition (2) which represents the same element as w by condition (1). We obtain that after at most $|w|$ steps one will have either reduced to the empty word, i.e $w = 1$ in G, or else verified that w does not represent the identity in G.

2.5.20. *Definition.* A finite presentation $\langle S|R\rangle$ of a group G is called a *Dehn presentation* if $R = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\}$, where the words u_i, v_i satisfy the conditions (1), (2), (3) as formulated above.

We will prove the following theorem in full detail since the main aim of this thesis is to generalize this theorem to so-called *relatively hyperbolic groups* and we want to "transport" this proof rigorously to the relative case (see next section).

Prior to this we introduce one last notion: For a metric graph X and a cycle c a *standard* ε-filling of c is an ε-filling given by a triangulation of the disc such that all of the vertices of the *boundary circle* are points of concatenation of c and each edge of the triangulation is either mapped to a concatenation of edges in X or else is sent to a vertex in X by a constant map.

2.5.21. *Theorem.* (see Theorem 2.6 in [4], p.450) A (finitely generated) group G is hyperbolic if and only if it admits a (finite) Dehn presentation.

Proof. First suppose that $\langle S|R\rangle$ is a finite Dehn presentation for G and let ρ be the length of the longest word in R. We will show that the Cayley graph $\Gamma := \Gamma(G,S)$ admits a linear isoperimetric inequality. By Remark 2.5.18, it suffices to consider cycles, i.e. let c be a cycle in Γ. This is labelled by a word w in $S^{\pm}$ which represents the identity in G. Condition (3) on the set R yields a subword u_i of w which constitutes a subpath of c which is *not* geodesic, since there is a shorter path with the same endpoints that is labelled by v_i (by Condition (2)). Let c' be the cycle obtained by replacing the subpath of c labelled u_i by the path labelled v_i. Given a standard ρ-filling of c' one can obtain a standard ρ-filling of c as follows: We add an extra *polyhedral face* to the filling of c' and adding extra edges to dive this face into $\leq \rho$ triangles, see the figure below and Remark 2.5.19. By induction on $n = l(c)$ we suppose that there is a standard ρ-filling of c' whose ρ-area is $\leq \rho l(c')$ (the first steps of the induction hold trivially). We obtain that $\mathrm{Area}_\rho(c) \leq \rho l(c)$, thus Γ admits a linear isoperimetric inequality and thus by Theorem 2.5.16 Γ is hyperbolic, i.e. G is a hyperbolic group.

Suppose conversely that G is hyperbolic, i.e. Γ is hyperbolic (as a metric graph). We set $D := [8\delta] + 1$ (clearly it would suffice to fix an integer $> 8\delta$). By Lemma 2.5.10 every cycle in Γ contains a subpath p of length at most D that has its endpoints at vertices and is *not* a geodesic. At the same time every cycle corresponds to a word w in S which represents the identity and the subpath p to a subword, say u, that is equal in G to a *strictly shorter* word v (the label of a geodesic connecting the endpoints of p). Thus, we define R as the set of words uv^{-1}, where u varies over *all* words of length at most D in $S^{\pm}$ and v is a word of minimal length that is equal to u in G. More formally, if $\pi\colon F(S) \to G$ is the canonical epimorphism (without loss we assume all words to be freely reduced) we set

$$R := \{uv^{-1} \mid u, v \text{ are words in } F(S), |u| \leq D$$

$$|u| > |\pi(u)| \text{ and } |v| < |u| \text{ with } \pi(u) = \pi(v)\}.$$

We denoted $|\pi(u)| := d_X(e_G, \pi(u))$. We have to show that $\langle\langle R\rangle\rangle = \ker(\pi)$. By construction it holds that $R \subseteq \ker(\pi)$, since $\pi(u) = \pi(v)$, thus $\langle\langle R\rangle\rangle \subseteq \ker(\pi)$. Suppose conversely that $w \in \ker(\pi)$. We prove the statement by induction on $|w|$. If $|w| = 0$ then w is the empty word and hence trivially $w \in \langle\langle R\rangle\rangle$. Hence let $|w| > 0$. Since w induces a cycle in Γ we can write $w = w'uw''$ (as described above). Let v be as above then

$$e_G = \pi(w) = \pi(w'uw'') = \pi(w'vw''),$$

where $w'vw''$ is *strictly shorter* than w. By induction hypothesis we have $w'vw'' \in \langle\langle R\rangle\rangle$. By construction of R we have $uv^{-1} \in R$ and thus conjugating by the element w' implies

$$\begin{aligned}\langle\langle R\rangle\rangle \ni w'uv^{-1}(w')^{-1} &= w'u(w''(w'')^{-1})v^{-1}(w')^{-1} = \\ &= (w'uw'')\underbrace{(w'vw'')^{-1}}_{\in\langle\langle R\rangle\rangle}.\end{aligned}$$

Hence $w = w'uw'' \in \langle\langle R\rangle\rangle$, establishing the claim and thus $\langle S \mid R\rangle$ is indeed a (finite) presentation for G and by the above arguments a Dehn presentation. $\square$

2.5.22. *Remark.* Since we have a *finite* generating set S, in other words a *finite* alphabet, the set R is indeed *finite.* Moreover, most of the arguments in the above proof rely on theorems holding generally in hyperbolic spaces and hyperbolic metric graphs respectively. This is the most crucial observation of this subsection.

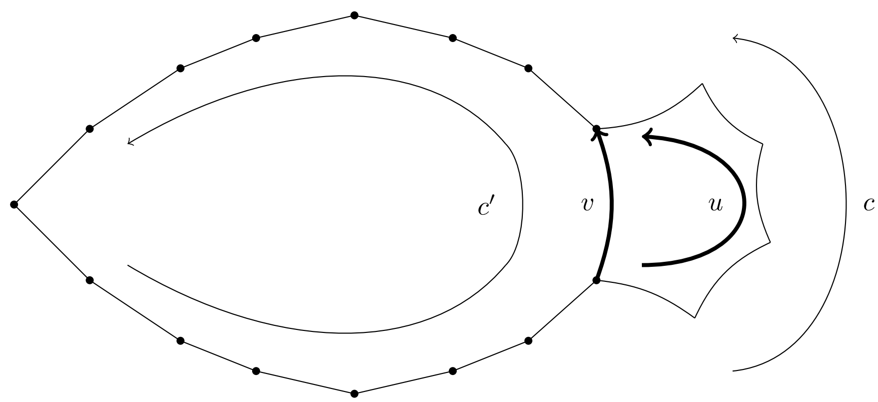

FIGURE 7. Applying a relation in Dehn's algorithm
(cf. Fig. Γ.3 in [4], p. 451)

3. Relative pairs

3.1. **Relative presentations.** We now lay the foundations to enter the main topic of this thesis, so-called *relatively hyperbolic groups.*

3.1.1. *Definition.* Let G be a group and let $\mathbb{P} = \{P_i\}_{i \in I}$ be a *non-empty collection of subgroups* of G. We call $(G, \mathbb{P})$ a *relative pair* and the subgroups $P_i, i \in I$, are called *peripheral subgroups.* We call $X \subseteq G$ a *relative generating set* or $\mathbb{P}$-*generating set* if

$$G \text{ is generated by } X \cup (\bigcup_{i \in I} P_i).$$

We say that G is *finitely* $\mathbb{P}$-generated if it admits a *finite* $\mathbb{P}$-generating set. We start with an relative analogue of presentations (see [16]).

Let G, X and $\mathbb{P}$ be as above. Consider the *disjoint union*

$$\mathcal{P} := \bigsqcup_{i \in I} \tilde{P}_i \backslash \{1\},$$

where $\tilde{P}_i$ is an isomorphic copy of P_i for every $i \in I$. Define

$$F = F(\mathbb{P}, X) := (*_{i \in I} \tilde{P}_i) * F(X)$$

where $F(X)$ denotes the free group with basis X. For every $i \in I$ define S_i as the set of words in the alphabet $\tilde{P}_i \backslash \{1\}$ that represent the identity in F. In other words S_i is the set of relations which hold in $P_i \cong \tilde{P}_i$. Moreover, consider the (again disjoint) union

$$\mathcal{S} := \bigsqcup_{i \in I} S_i.$$

Clearly F has the (ordinary) presentation

$$\langle X \sqcup \mathcal{P} | \mathcal{S} \rangle.$$

Consider the canonical epimorphism

$$\pi \colon F \to G$$

and its kernel N, i.e. $G \cong F/N$. Let $R \subseteq (X \cup \mathcal{P})^*$ and suppose $\langle\langle R \rangle\rangle = N$. Then we say that G has the *relative* ($\mathbb{P}$-)*presentation*

$$\langle X, \mathcal{P} \mid s = 1, s \in \mathcal{S}, r = 1, r \in R \rangle.$$

For brevity we use the following reduced record for the relative presentation

$$\langle X, P_i, i \in I \mid R \rangle.$$

The relative presentation is called *finite* if both the sets R and X are *finite*. In this case we say that G is *finitely* ($\mathbb{P}$-)*presented.*

3.1.2. *Remark.* We see that the group $F = F(\mathbb{P}, X)$ above plays the role of a free group $F(X)$ in the relative case. In this sense we fix a length assigned to the elements of F.

By definition of a free product we know that elements of F are equivalence classes of words in $(F(X) \cup \bigcup_{i\in I} \tilde{P}_i)^*$. We also know that each equivalence class contains a *unique* reduced word. Thus from now on when we talk about an element, say $[w]$, in F we mean *the* reduced word representing this equivalence class and simply write $w \in F$.

By definition a word $v = x_1 x_2 \dots x_n$ with $x_i \in X$ is a *one letter* word in F, but a word with n letters in $F(X)$. In this sense we will set the length of such a word v to be n also in F, denoting it by $|v|$. We will transfer this *length* onto elements of F in the obvious way.

For example the element

$$w = x_1 x_2 x_3 p q x_2 x_5 q^2 p,$$

where $x_i \in X$, $p \in \tilde{P} \backslash \{1\}$, $q \in \tilde{Q} \backslash \{1\}$ with $\tilde{P} \neq \tilde{Q}$, would have length $|w| = 3+1+1+2+1+1 = 9$ (if q is *not* an element of order 2 in $\tilde{Q}$). We will also refer to this as the *F-length* of $w \in F$ (this is also called the *syllable length* of the free product w.r.t the generating sets X and $\mathcal{P}$). Similarly we have the following

3.1.3. *Definition.* Let G, X and $\mathbb{P}$ be as above. To each element $g \in G$ we assign its *relative* ($\mathbb{P}$-)*length* $|g|_{X\cup\mathcal{P}}$, that is the length of a *shortest word* from $(X \cup \mathcal{P})^*$ representing g in G. This is nothing else than the distance

$$d_{X\cup\mathcal{P}}(e_G, g)$$

in the Cayley graph of G with respect to the generating set $X \cup \bigcup_i P_i$.

3.1.4. *Remark.* In general we now leave the "universe" of finitely generated groups and hence of locally finite (Cayley) graphs. Many standard arguments fail, e.g. that Cayley graphs with respect to different generating sets are quasi-isometric. Nonetheless, if we only change finite $\mathbb{P}$-generating sets (ore more generally bounded sets), while we are fixing $\mathbb{P}$, we do *not* leave the quasi-isometric class.

3.1.5. *Definition.* Let G, X and $\mathbb{P}$ be as above. We set $\Gamma(G, X \cup \mathcal{P})$ to denote the *Cayley graph* of G with respect to the set $X \cup \bigcup P_i$.

Let $Y \subseteq G$. We call Y a *bounded X-set* if Y is a bounded (vertex) set in $\Gamma(G, X \cup \mathcal{P})$. In particular this means that *all* elements of Y can be written with at most c letters with c some constant. If Y is *finite* then it is trivially a bounded X-set.

3.1.6. *Lemma.* (see Proposition 2.8 in [16]) Let $(G, \mathbb{P})$ be a relative pair and let X and Y be two $\mathbb{P}$-generating sets of G.

If X is a bounded Y-set and Y is a bounded X-set then

$$\Gamma(G, X \cup \mathcal{P}) \sim_{QI} \Gamma(G, Y \cup \mathcal{P})$$

Proof. This proof is just mimicking the "ordinary" one. It is clear, that it suffices to construct a quasi-isometry on the vertices of the graphs in question (because all edges have length equal 1). We denote by d_1 and d_2 the word metric w.r.t $X \cup \mathcal{P}$ and $Y \cup \mathcal{P}$ respectively.

By assumption there exists a constant M, such that we can fix for each $y \in Y$ a word $v_y \in (X \cup \mathcal{P})^*$ that represents y and has length $\leq M$, i.e

$$|v_y| \leq M \text{ for all } y \in Y$$

where $|.|$ denotes the length of the word.

Let now $g \in G$ and $w \in (Y \cup \mathcal{P})^*$ be a *shortest* word representing g. Replace every appearance of a letter $y \in Y$ in w by v_y. Hence we obtain a word $u \in (X \cup \mathcal{P})^*$. This implies

$$d_1(e, g) \leq |u| \leq M|w| = Md_2(e, g)$$

Thus for $g, h \in G$ we have

$$d_1(g, h) = d_1(e, g^{-1}h) \leq Md_2(e, g^{-1}h) = Md_2(g, h)$$

Repeating the above argument with X and Y interchanged and a constant N such that

$$|v_x| \leq N \text{ for all } x \in X,$$

we analogously get

$$d_2(g, h) \leq Nd_1(g, h)$$

and thus with $\lambda := \max\{M, N\}$ we conclude, that the *identity* is a $(\lambda, 0)$-quasi isometry (in particular bi-lipschitz). (The identity is bijective hence surjective, thus we don't need a quasi-dense constant)

□

3.2. **The coned-off Cayley graph.** We now introduce the *coned-off Cayley graph* of a relative pair $(G, \mathbb{P})$ which will play the role of the Cayley graph in the relative case.

Let X be a $\mathbb{P}$-generating set for G and denote $\Gamma := \Gamma(G, X)$. Note that this graph is in general *not* connected (since X is not necessarily a generating set for G).

We define the *coned-off Cayley graph* $\hat{\Gamma}(G, \mathbb{P}, X)$ with the following construction out of Γ:

(1) For *each* left coset gP with $g \in G$ and $P \in \mathbb{P}$ add a vertex v_{gP} to Γ.
(2) Add an edge of length $\frac{1}{2}$ from v_{gP} to *each* element of gP.

We view $\Gamma(G, X)$ as a subgraph of $\hat{\Gamma}(G, \mathbb{P}, X)$. Indeed, we will refer to the vertices of Γ as *first floor vertices* and refer to the vertices v_{gP} for some $g \in G$ and $P \in \mathbb{P}$ as *second floor vertices.*

3.2.1. *Example.* Consider $G := \mathbb{Z}^2$, $a := (1,0)$, $b := (0,1)$ and $\mathbb{P} := \{\langle a \rangle\}$. Then for $X := \{b, b^{-1}\}$ the following figure indicates the structure of $\hat{\Gamma}(G, \mathbb{P}, X)$, where we abbreviate $\langle a \rangle =: P$.

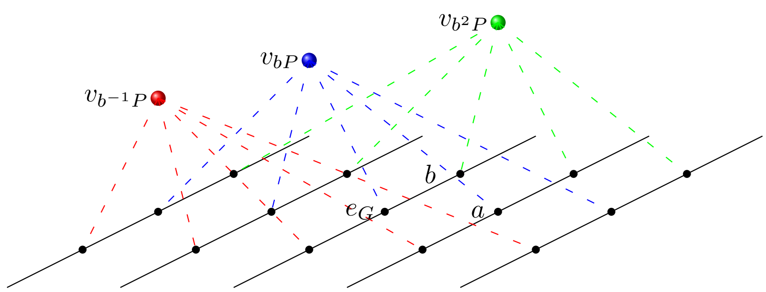

FIGURE 8. The shape of $\hat{\Gamma}(\mathbb{Z}^2, \mathbb{Z})$

3.2.2. *Remark.* It may look a bit artificial that the edges from the first floor to the second floor have length $\frac{1}{2}$, but we will see that this is indeed quite useful. Being more precise, it is an important feature of the ordinary Cayley graph that walks and words in the generators are in one-to-one correspondence. We will see that this is also true for the coned-off Cayley graph. Recall that we set X to be a *symmetrized* $\mathbb{P}$-generating set for the group G. Moreover, we set the $\mathbb{P}$-*length* of a (combinatorial) walk of $\Gamma(G, \mathbb{P}, X)$ to be the sum of the length of the edges traversed by this walk.

For this we fix the following convention. If we walk from a first floor vertex to a first floor vertex along an edge we record the corresponding label (*letter*) of this edge, just as in the usual way.

Consider for *distinct* $g_1, g_2 \in G$ the walk (g_1, v_{hP}, g_2) for some left coset of $P \in \mathbb{P}$ (in the literature this is called *coset penetration*). Note that this walk has $\mathbb{P}$-length 1. Moreover, by construction of $\hat{\Gamma}(G, \mathbb{P}, X)$ we have that $g_1, g_2 \in hP$, i.e $g_1 = hp_1$ and $g_2 = hp_2$ for some *distinct* $p_1, p_2 \in P$. We record for such a walk the *letter* $p_3 := p_1^{-1}p_2 \in P$. By the following computation

$$p_1^{-1}p_2 = p_1^{-1}h^{-1}hp_2 = (hp_1)^{-1}hp_2 = g_1^{-1}g_2,$$

we see that g_1 and g_2 are equal if and only if p_1 and p_2 are, i.e. we would record the identity $e = p_3$ if and only if the above walk is backtracking.

The above mentioned describes how we get a word in $(X \cup \mathcal{P})^*$ from a walk starting and ending at a first floor vertex. Moreover, the F-length of the obtained word coincides with the $\mathbb{P}$-length of the walk, if we delete backtrackings of the form (g, v_{hP}, g). From now on we will always assume that our walks are without backtrackings of the above form and call such walks, *proper walks.*

Conversely, given a (reduced) word in $(X \cup \mathcal{P})^*$. If the word consists only of letters of X we get a walk in $\Gamma(G, X)$ (considered as a subgraph in $\hat{\Gamma}(G, \mathbb{P}, X)$). We describe the procedure by a concrete example. Consider the word $w = x_1x_2x_3pqx_4x_5rx_6$, where $x_i \in X$, $p \in P\backslash\{1\}$, $q \in Q\backslash\{1\}$, $r \in R\backslash\{1\}$ for *distinct* $P, Q, R \in \mathbb{P}$ (we omitted the "tilde" notation for notational convenience). We start the walk at some first floor vertex x.

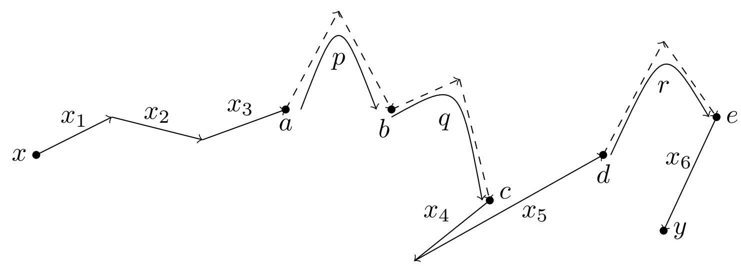

FIGURE 9. A "coned-off" walk in a coned-off Cayley graph.

We start to walk along the letters x_1, x_2 and x_3 as usual and stop at a first floor vertex, say a. Now we want to read the letter $p \in P\backslash\{1\}$. We know that $a \in \bar{a}P$ for some *unique* left coset of P, i.e $a = \bar{a}p_1$. Thus reading the letter p is viewed as walking from $a = \bar{a}p_1$ to $\bar{a}p_1p =: b \in \bar{a}P$. Proceeding in this manner, we walk from $b = \bar{b}q_1 \in \bar{b}Q$ to $\bar{b}q_1q =: c \in \bar{b}Q$. We now read the letters x_4 and x_5 and stop at the element $cx_4x_5 =: d = \bar{d}r_1 \in \bar{d}R$. We now read the letter r and stop at the element $\bar{d}r_1r =: e \in \bar{d}R$. Finally with the last letter x_6 we end at the element $ex_6 =: y$. A trivial computation shows that

$$xw = \underbrace{xx_1x_2x_3}_{=a}pqx_4x_5rx_6 = \underbrace{ap}_{=b}qx_4x_5rx_6 = \underbrace{bq}_{=c}x_4x_5rx_6 =$$
$$= \underbrace{cx_4x_5}_{=d}rx_6 = \underbrace{dr}_{=e}x_6 = ex_6 = y,$$

hence the word w represents the element $x^{-1}y$. As before the F-length of the word w concides with the $\mathbb{P}$-length of the obtained walk, which is without back-tracking described as above, i.e it is a proper walk.

Finally, we note that different proper walks between two first floor vertices induce different words and vice versa. We summarize the above observations in the following

3.2.3. *Proposition.* Let $(G, \mathbb{P})$ be a relative pair with X a relative $\mathbb{P}$-generating set for G. Then there is a one-to-one correspondence between proper walks connecting two first floor vertices, say x, y, in $\hat{\Gamma}(G, \mathbb{P}, X)$ of $\mathbb{P}$-length k and words in $(X \cup \mathcal{P})^*$ representing $x^{-1}y$ of F-length k.

The following is evident from the above considerations.

3.2.4. *Lemma.* Let $(G, \mathbb{P})$ be a relative pair and let $X \subseteq G$. Then $\hat{\Gamma}(G, \mathbb{P}, X)$ is connected if and only if X is a $\mathbb{P}$-generating set.

3.2.5. *Lemma.* Let $(G, \mathbb{P})$ be a relative pair and let X and Y be two $\mathbb{P}$-generating sets for G. If X is a bounded Y-set and Y is a bounded X-set then

$$\hat{\Gamma}(G, \mathbb{P}, X) \sim_{QI} \hat{\Gamma}(G, \mathbb{P}, Y).$$

Proof. This proof works similar to the one of Lemma 3.1.6. It is again sufficient to construct a quasi-isometry on the vertices only. Analogously we choose a constant M and fix a word v_y in $(X \cup \mathcal{P})^*$ representing $y \in Y$, such that

$$|v_y| \leq M \text{ for all } y \in Y.$$

Similarly we choose a constant N, such that

$$|v_x| \leq N \text{ for all } x \in X,$$

for words v_x in $(Y \cup \mathcal{P})^*$ representing $x \in X$. We abbreviate $\hat{\Gamma}(Y) := \hat{\Gamma}(G, \mathbb{P}, Y)$ and $\hat{\Gamma}(X) := \hat{\Gamma}(G, \mathbb{P}, X)$. Moreover, we denote as d_1 and d_2 the metrics of $\hat{\Gamma}(Y)$ and $\hat{\Gamma}(X)$ respectively.

Let x and y be two first floor vertices of $\hat{\Gamma}(Y)$ and $d_1(x, y) = k \in \mathbb{N}$. By Proposition 3.2.3 we obtain a word w of *minimal* length k which represents $x^{-1}y$ with letters in $(Y \cup \mathcal{P})$. Replacing every appearance of $y \in Y$ by v_y gives a word $\tilde{w}$ in $(X \cup \mathcal{P})$ representing $x^{-1}y$ of length $\leq Mk$. Since $\tilde{w}$ is *some* word representing $x^{-1}y$ we obtain

$$d_2(x, y) \leq Mk = Md_1(x, y).$$

Analogously interchanging X and Y we obtain

$$d_1(x, y) \leq Nd_2(x, y).$$

We set $\lambda := \max\{M, N\}$ and note that $\lambda \geq 1$.

Let now v_{gP} and v_{hQ} be arbitrary second floor vertices of $\hat{\Gamma}(X)$ and $\hat{\Gamma}(Y)$. The distance of these two points can be described as follows:

$$d_1(v_{gP}, v_{hQ}) = \frac{1}{2} + d_1(gp, hq) + \frac{1}{2}$$

and

$$d_2(v_{gP}, v_{hQ}) = \frac{1}{2} + d_1(g\tilde{p}, h\tilde{q}) + \frac{1}{2},$$

for some $p, \tilde{p} \in P$ and $q, \tilde{q} \in Q$. A simple computation shows

$$\begin{aligned} d_1(gp, hq) &\leq d_1(gp, g\tilde{p}) + d_1(g\tilde{p}, h\tilde{q}) + d_1(h\tilde{p}, hq) \leq \\ &\leq 1 + d_1(g\tilde{p}, h\tilde{q}) + 1 \leq \lambda d_2(g\tilde{p}, h\tilde{q}) + 2, \end{aligned}$$

since gp and $g\tilde{p}$ as well as hp and $h\tilde{p}$ are elements of the *same* left coset. Thus

$$\begin{aligned} d_1(v_{gP}, v_{hQ}) &\leq \lambda d_2(g\tilde{p}, h\tilde{q}) + 1 + 2 \leq \\ &\leq \lambda(d_2(g\tilde{p}, h\tilde{q}) + 1) + 2 = \lambda d_2(v_{gP}, v_{hQ}) + 2. \end{aligned}$$

The same computation gives

$$d_2(v_{gP}, v_{hQ}) \leq \lambda d_1(v_{gP}, v_{hQ}) + 2.$$

The same arguments show that for some first floor vertex v and second floor vertex v_{gP} it holds that

$$d_1(v, v_{gP}) \leq \lambda d_2(v, v_{gP}) \text{ and } d_2(v, v_{gP}) \leq \lambda d_1(v, v_{gP}).$$

In summary, we obtain that the identity on the vertices gives a $(\lambda, 2)$-quasi-isometry from $\hat{\Gamma}(X)$ to $\hat{\Gamma}(Y)$ (again, since we have the same vertex set, we don't need a quasi-dense constant). $\square$

3.2.6. *Lemma.* Let $(G, \mathbb{P})$ be a relative pair and X a $\mathbb{P}$-generating set for G. Then

$$\hat{\Gamma}(G, \mathbb{P}, X) \sim_{QI} \Gamma(G, X \cup \mathcal{P}).$$

Proof. It suffices to construct a quasi-isometry on the corresponging vertex sets. Let ϕ be the inclusion of the vertex set of $\Gamma(G, X \cup \mathcal{P})$ into the vertex set of $\hat{\Gamma}(G, \mathbb{P}, X)$. Clearly, $\frac{1}{2}$ serves as a quasi-dense constant. But on the first floor vertices the distances in both graphs coincide. Hence we are done. $\square$

3.3. **Ends of relative pairs.** There is a vast literature on ends of (topological) spaces, graphs, and groups respectively. Usually they are studied in locally compact (Hausdorff) spaces, locally finite graphs, and finitely generated groups respectively. Roughly speaking, ends are "possible directions" to go to infinity. For example considering the real line $\mathbb{R}$, one has two ends. We briefly recall the required notions. We follow [12].

3.3.1. *Definition.* Let X be a graph with vertex set V. A *ray* is an infinite sequence $(v_1, v_2, \dots)$ of *distinct* vertices $v_i \in V$. For a subset $C \subseteq V$ the *vertex boundary* of C is defined as the set of vertices in C^c adjacent with a vertex in C and will be denoted by NC. The *edge boundary* δC of C is defined as the set of edges which are incident with a vertex of C and a vertex of C^c.

We call a non-empty set of vertices C a *vertex* or *edge cut* if NC or δC are finite respectively. Note that every *edge cut* is also a *vertex cut.* The converse is true if X is locally finite. In general the converse is false.

We say that a ray R *lies* in a set $C \subseteq V$, if all but finitely many vertices of R are contained in C. A set of vertices C *separates* two rays if one of them lies in C and the other lies in C^c.

Two rays are called *vertex equivalent* or *edge equivalent* if they cannot be separated by a vertex or edge cut respectively. It is an easy exercise to show that these relations are indeed *equivalence relations.* Their equivalence classes are called *vertex* or *edge ends* of X respectively.

The keypoint to make this (graph theoretic) notion interesting for finitely generated groups is the following proposition. For a detailed proof see e.g. Theorem 3.2.8 in [17].

3.3.2. *Proposition.* (see Lemma 21.4 in [18]) Let X and Y be quasi-isometric, connected locally finite graphs. Then the quasi-isometry induces a homeomorphism from the vertex (edge) ends of X onto the vertex (edge) ends of Y.

We know that Cayley graphs with respect to different *finite* generating sets are quasi-isometric and hence we can define the *number of ends* of a group as the number of ends of *some* Cayley graph of this group with respect to a finite generating set.

3.3.3. *Remark.* The above proposition is in general *not* true for non-locally finite graphs. Indeed, consider $K = K_{\mathbb{N}}$ the complete graph on infinitely many vertices, i.e. each pair of distinct vertices is adjacent. This graph has one vertex end. Consider two copies of K which are glued together at one vertex. The resulting graph Y has two vertex ends. But both graphs are bounded and hence are quasi-isometric.

A famous theorem for finitely generated groups is the following *Freudenthal-Hopf-Theorem.*

3.3.4. *Theorem.* (see Theorem 11.27 in [14]) Let G be a finitely generated group. Then G has at most 2 or infinitely many ends.

The proof of the above theorem only depends on the fact that the Cayley graph of a group is transitive. Therefore, the proof also works for almost transitive graphs.

In general the coned-off Cayley graph of a relative pair $(G, \mathbb{P})$ is non-locally finite. Hence we cannot use the above proposition. Fortunately there is one more notion of ends, which should be regarded as the "right" notion of ends in non-locally finite graphs. In locally finite graphs all notions are equivalent. Again we follow [12].

3.3.5. *Definition.* A *metric cut* is a set of vertices with a vertex boundary of *finite diameter.* A ray whose infinite subsequences all have infinite diameter is called *metric ray.* Two metric rays are *metrically equivalent* if they cannot be separated by a metric cut.

Again it is an easy exercise to show that this relation defines an equivalence relation on the set of metric rays. We call its equivalence classes *metric ends.*

One can show that the set of metric cuts is closed under finite intersection, i.e. it forms a basis for a topology, the so-called (*metric*) *end topology.*

3.3.6. *Theorem.* (see Theorem 6 in [12]) Let X and Y be connected quasi-isometric graphs. Then the quasi-isometry induces a homeomorphism from the metric ends of X onto the metric ends of Y.

The above theorem allows to introduce ends for relative pairs.

3.3.7. *Definition.* Let $(G, \mathbb{P})$ be a relative pair which is finitely $\mathbb{P}$-generated. Then the *ends* of $(G, \mathbb{P})$ are the metric ends of $\hat{\Gamma}(G, \mathbb{P}, X)$ for some finite $\mathbb{P}$-generating set X of G.

3.3.8. *Definition.* A group G acts *metrically almost transitive* on a graph X if there is an integer r, such that

$$\bigcup_{g \in G} gB(x, r) = VX,$$

for any vertex x.

The smallest integer r with this property is called the *covering radius* (of the action of G on X). A graph is called *metrically almost transitive* if its group of automorphisms acts metrically almost transitively.

3.3.9. *Theorem.* (see Corollary 3.15 in [12]) Let X be a metrically almost transitive graph with infinite diameter. Then X has 1,2 or infinitely many metric ends.

Wel now prove that for a relative pair $(G,\mathbb{P})$ the coned-off Cayley graph is metrically almost transitive.

Indeed, let X be a finite $\mathbb{P}$-generating set for G. For notational convenience we write v_h to denote the vertex which corresponds to the element $h \in G$. We first define an action of G onto $\hat{\Gamma}(G,\mathbb{P},X)$ as follows:

Let $g \in G$ and consider the first floor vertex v_h. Then we set $g \cdot v_h := v_{gh}$, i.e. the action is simply left multiplication. Similarly for a second floor vertex v_{hP} we set $g \cdot v_{hP} := v_{ghP}$. We note that this action sends first floor vertices to first floor vertices and second floor vertices to second floor vertices. We take a closer look onto the (edge and vertex) orbits and all stabilizers of this action:

Vertex orbits: Let v_g and v_h be two first floor vertices. With the action of the element $a := hg^{-1}$ we see that $a \cdot v_g = v_h$, i.e. we can reach *each* first floor vertex but clearly not more.

Let v_{gP} for some $P \in \mathbb{P}$. As above we see that $a \cdot v_{gP} = v_{hP}$, but we can clearly not reach any other second floor vertex corresponding to a different $Q \in \mathbb{P}$. We conclude that there is a vertex orbit for each $P \in \mathbb{P}$. In particular we see that the coned-off Cayley graph is almost transitive if and only if $\mathbb{P}$ is a *finite* collection of subgroups of G. Nonetheless, it is *always* metrically-almost transitive. Indeed, take a first floor vertex v_g. Since g is contained in some left coset of any $P \in \mathbb{P}$, we see that a ball of radius $\frac{1}{2}$ contains a representative of each orbit. For a second floor vertex we need a ball of radius 1. In particular the coned-off Cayley graph is metrically almost transitive, indepent of the finiteness of $\mathbb{P}$ and also of X.

Edge orbits: First we need to fix how our action onto the vertices induces an action onto the edges. Let e be an edge between two first floor vertices v_g and v_{gx} with label $x \in X$. Then an element $h \in G$ sends e to the edge between v_{hg} and v_{hgx} (which are connected by an edge by construction of the Cayley graph) with the label x. Let f be the edge between v_g and v_{hP}, i.e. $g \in hP$ for some $P \in \mathbb{P}$. Since $g = hp$ for some $p \in \mathbb{P}$, we see that for $a \in G$ it holds that $ag = ahp \in ahP$, i.e. there is an edge between $a \cdot v_g$ and $a \cdot v_{hP}$. We set $a \cdot f$ to be this edge.

The above description of the action onto the edges makes it evident that edges connecting two first floor vertices are in the same edge orbit if and only if they have the same label $x \in X$. Thus, there finitely many edge orbits on the first floor if and only if X is a finite $\mathbb{P}$-generating set.

Let now f be the edge between v_{gp} and v_{gP} for some $P \in \mathbb{P}$ and consider $v_{g\tilde{p}}$ for some arbitrary $g\tilde{p} \in gP$. Set $a := g\tilde{p}p^{-1}g^{-1} \in gPg^{-1}$, then $a \cdot v_{gp} = v_{g\tilde{p}}$ and $a \cdot v_{gP} = v_{gP}$. Thus, we see that the edge orbit of f *contains* all edges incident with v_{gP}. Moreover, let hP be an arbitrary left coset of P and consider v_{hP} and v_{hp} which are connected by an edge, say $\tilde{f}$. If we set $b := hg^{-1}$, we see that $b \cdot v_{gp} = v_{hp}$ and $b \cdot v_{gP} = v_{hP}$, i.e. b sends f to the edge $\tilde{f}$. We conclude that the edge orbit of f *coincides* with all edges incident with v_{hP} for *any* left coset hP of P. Thus, there are finitely many edges orbits of this type if and only if $\mathbb{P}$ is finite.

Thus, there are finitely many edge orbits if and only if X is a finite $\mathbb{P}$-generating set and $\mathbb{P}$ is a finite collection of subgroups.

Vertex Stabilizers: Consider $H := \mathrm{Stab}(v_{gP})$ for some left coset of $P \in \mathbb{P}$. If $h \in H$ then $h \cdot v_{gP} = v_{gP}$, i.e. $hgP = gP$. This implies that $g^{-1}hg \in P$ which is equivalent with $h \in gPg^{-1}$. Conversely, let $h \in gPg^{-1}$, i.e. $h = gpg^{-1}$ for some $p \in P$. Thus, $h \cdot v_{gP} = v_{gpg^{-1}gP} = v_{gP}$, i.e. $h \in \mathrm{Stab}(v_{gP})$. In particular, we see that H is *conjugate* to P.

Consider $\mathrm{Stab}(v_g)$ for some first floor vertex v_g. An element $h \in G$ which fixes v_g satisfies the equation $hg = g$, thus $h = e_G$. In particular, all vertex stabilizers of first floor vertices are trivial.

Edge Stabilizers: Since an element which fixes an edge must (by construction) fix also the vertices incident with this edge, the above considerations imply that all edge stabilizers are trivial, since each edge is incident with at least one first floor vertex.

We summarize the above considerations in the following

3.3.10. *Proposition.* Let $(G, \mathbb{P})$ be a relative pair and X a $\mathbb{P}$-generating set for G. For $\hat{\Gamma} := \hat{\Gamma}(G, \mathbb{P}, X)$ we have:

(1) $\hat{\Gamma}$ is metrically almost transitive.
(2) $\hat{\Gamma}$ is almost transitive if and only if $\mathbb{P}$ is finite.
(3) $\hat{\Gamma}$ admits finitely many edge orbits if and only if X and $\mathbb{P}$ are finite.
(4) $\mathbb{P}$ contains a set of representatives of conjugacy classes of non-trivial vertex stabilizers of $\hat{\Gamma}$.
(5) All edge stabilizers of $\hat{\Gamma}$ are trivial.

Regarding item (4) we have shown that each non-trivial vertex stabilizer is conjugate to some $P \in \mathbb{P}$, but since we have no control over particular $P, Q \in \mathbb{P}$ it could be that $P = gQg^{-1}$ for some $g \in G$. Item (1) of the above Proposition and Theorem 3.3.9 directly imply the following

3.3.11. *Theorem.* Let $(G, \mathbb{P})$ be a relative pair which is finitely $\mathbb{P}$-generated. Then it has at most 2 or infinitely many ends.

One trivial but important property of locally finite graphs is that bounded sets are finite, or in other words, infinite sets are necessarily unbounded. For non-locally finite graphs this is (of course) false in general. For metrically almost transitive graphs there is at least only some 'finite condition' to check that sets are unbounded.

3.3.12. *Lemma.* (see Corollary 3.7 in [12]) Let X be a connected metrically almost transitive graph with covering radius ρ. If there is a bounded and connected set T and components C_1 and C_2 of $VX \backslash T$ which contain vertices x_1 and x_2, respectively, such that

$$\min\{d(x_1, T), d(x_2, T)\} > \mathrm{diam} T + \rho,$$

then C_1 and C_2 are both unbounded.

Note that C_1 and C_2 in the above lemma are metric cuts. Moreover, one can prove that unbounded metric cuts in metrically almost transitive graphs necessarily contain metric rays (see Corollary 3.14 in [12]).

It is well-known that the number of ends of a graph on which a group acts (metrically) almost transitive corresponds to the (non-) existence of torsion elements of the group in question.

3.3.13. *Theorem.* (cf. Theorem 3.8 in [12]) Let G be a torsion group which acts metrically almost transitively on a connected graph X. Then X has at most one metric end.

Immediate is the following

3.3.14. *Theorem.* (cf. Theorem 3.5 and Corollary 3.6 in [12]) Let G act almost transitively on a connected graph with more than one metric end. Then G contains an element of infinite order.

Applying this theorem to relative pairs we obtain

3.3.15. *Corollary.* Let $(G, \mathbb{P})$ be relative pair with more than one end. Then G contains an element of infinite order.

4. Relatively hyperbolic groups

We now introduce the relative analogue of hyperbolic groups for relative pairs, that is *relative hyperbolicity.*

4.1. Osin's relative hyperbolicity.

The approach of Osin in [16] seems to be the most general one. For this let $(G, \mathbb{P})$ be a relative group which is given by a relative presentation

$$\langle X, P_i, i \in I \mid r = 1, r \in R \rangle.$$

For a word $w \in (X \cup \mathcal{P})^*$, which represents the identity in G, there exists an expression

$$w = \prod_{i=1}^{k} f_i^{-1} r_i f_i,$$

where equality holds as a word in F, $r_i \in R$ and $f_i \in F$ for any i.

4.1.1. *Definition.* We say that a function $f\colon \mathbb{N} \to \mathbb{N}$ is a *relative isoperimetric function* of the presentation for $(G, \mathbb{P})$ as above if for any $n \in \mathbb{N}$ and any word $w \in (X \cup \mathcal{P})^*$ of length $\leq n$ representing the identity in G, one can write w as a product as above with $k \leq f(n)$.

The *smallest* relative isoperimetric function is called the *relative Dehn function* (of the presentation) of $(G, \mathbb{P})$.

4.1.2. *Remark.* (cf. Example 1.3 in [16]) We note that it can happen that there exists no relative isoperimetric function, since the number of words of bounded relative length can be infinite. Indeed, consider the group

$$G = \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

as in Example 3.2.1 and the cyclic subgroup P generated by a. Then $X := \{b\}$ is a finite relative generating set for G. Consider the word $w_n = [a^n, b]$. Then w_n has length 4 as a word over $X \cup P$ for every n (see the figure in Example 3.2.1). However, the minimal number of factors in the expression as above corresponding to w_n growths linearly as $n \to \infty$.

In such a case as above, we say that the relative Dehn function is *not well-defined.*

4.1.3. *Remark.* Strictly speaking, the relative Dehn function depends on the sets X and R. But if both sets are finite, i.e. the relative pair $(G, \mathbb{P})$ is finitely $\mathbb{P}$-presented it can be shown that it is independent up to some appropriate equivalence relation on functions $\mathbb{N} \to \mathbb{N}$. We refer to [16], especially Definition 2.33 and Theorem 2.34.

4.1.4. *Definition.* Let $(G, \mathbb{P})$ be a relative pair. We say that G is *hyperbolic relative to* $\mathbb{P}$ if G is finitely $\mathbb{P}$-presented and the relative Dehn function of $(G, \mathbb{P})$ is *linear.*

4.1.5. *Theorem.* (see Theorem 1.7 in [16]) Suppose that $(G, \mathbb{P})$ is finitely $\mathbb{P}$-presented and the relative Dehn function of $(G, \mathbb{P})$ is well-defined. Then the following are equivalent.

(1) G is hyperbolic relative to $\mathbb{P}$.
(2) The graph $\Gamma(G, X \cup \mathcal{P})$ is a hyperbolic space.

With the above theorem we can easily transfer the proof of Theorem 2.5.21 to the relative case. First we introduce a relative analogue of a Dehn presentation.

4.1.6. *Definition.* Let $(G, \mathbb{P})$ be a relative pair and X a $\mathbb{P}$-generating set of G. Let R be a set of words in F of the following form:

(1) $R := \{uv^{-1} \mid u, v \text{ words in F}\}$.
(2) There is a universal bound on the F-lengths of all words of R.
(3) For every $uv^{-1} \in R$ it holds that $|v| < |u|$.
(4) Every word w which represents the identity in G contains some u as a subword.

We call property (4) the *algorithmic condition* on the set R. Property (2) means that there is a constant c such that all words $uv^{-1} \in R$ can be written with at most c letters in $(X \cup \mathcal{P})^*$.

We call the set R as above a *relative Dehn set* with respect to X. We say that $(G, \mathbb{P})$ admits a *relative Dehn presentation with respect to* X if

$$\langle X, P_i, i \in I \mid r = 1, r \in R \rangle$$

is a relative $\mathbb{P}$-presentation of G and R is a relative Dehn set with respect to X. We call the relative Dehn presentation *finite* if X and R are finite. Our generalization of Theorem 2.5.21 is the following.

4.1.7. *Theorem.* Let $(G, \mathbb{P})$ be a relative pair and X a $\mathbb{P}$-generating set for G. Then the following are equivalent:

(1) $\hat{\Gamma}(G, \mathbb{P}, X)$ is hyperbolic.
(2) $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to X.

Proof. Suppose that $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to X. Let R be the corresponding relative Dehn set. We abbreviate $\hat{\Gamma} = \hat{\Gamma}(G, \mathbb{P}, X)$. We argue as in the proof of Theorem 2.5.21. First we observe that it suffices to consider cycles (see Remark 2.5.18). Although in Remark 2.5.18 one only considers graphs of unit edge lengths it is clearly no restriction to consider a graph with unit edge lengths and additionally with $\frac{1}{2}$-edge lengths. Moreover, we can assume that our cycle is *proper*, because shortening a general cycle to a proper cycle is clearly a linear process with respect to a given ε-filling. Hence let c be a proper cycle in $\hat{\Gamma}$ of length $k \in \mathbb{N}$. By Proposition 3.2.3 we obtain a word w in $(X \cup \mathcal{P})^*$ of length k which represents the identity in G. By the algorithmic condition on the set R we have $uv^{-1} \in R$ such that u is a subword of w. Note that by condition (2) of R all words uv^{-1} are of length *at most* ρ. Hence we can argue word by word as in Theorem 2.5.21 to obtain that $\hat{\Gamma}$ admits a linear isoperimetric bound and hence is hyperbolic.

Suppose conversely that $\hat{\Gamma}$ is hyperbolic with hyperbolic constant $\delta \geq 0$. Consider a word w representing the identity in G of length $k \geq D$ where $D = [8\delta] + 1$. As in the proof of Proposition 3.2.3 we obtain a *proper* cycle c in $\hat{\Gamma}$ of the same length. We can apply Lemma 2.5.10 (note that although this lemma requires metric graphs with *integer* edge lengths, the proof in [4] still works for our $\hat{\Gamma}$) to obtain a subpath c_u, which is *not* geodesic. Since c is proper c_u is proper too. If both endvertices of c_u are on the first floor it constitutes a word of the *same length*, say u. A geodesic segment with the same endpoints as c_u, say c_v, is also proper (since it is geodesic) hence also constitutes in a word, say v, of the same length (as c_v). We have to distinguish two more cases, where we again denote by c_v the geodesic with the same endpoints as c_u:

Case 1: The starting vertex of c_u is on the first floor, say v_g, and the ending vertex is on the second floor, say v_{hP}. Let v_{hp_1} and v_{hp_2} be the predecessor vertex of v_{hP} in c_u and c_v respectively. Moreover, let v_{hp_3} be the successor vertex of v_{hP} on the cycle c. Let $c_{\tilde{v}}$ be the subpath of c_v from v_g to v_{hp_2} and similarly $c_{\tilde{u}}$ be the subpath of c_u from v_g to v_{hp_1}.

If $v_{hp_3} \neq v_{hp_2}$, then the concatenation of $c_{\tilde{u}}$ and $\{v_{hp_1}, v_{hP}, v_{hp_3}\}$ is a subpath of c (hence proper) which is strictly longer than the concatenation of $c_{\tilde{v}}$ and $\{v_{hp_2}, v_{hP}, v_{hp_3}\}$. Since both paths have the same endpoints on the first floor they constitute in words of the same respective length which represent the *same* group element of G.

If $v_{hp_3} = v_{hp_2}$, then it suffices to consider $c_{\tilde{v}}$ to obtain a strictly shorter path connecting two first floor vertices on c.

Case 2: Both vertices of c_u are on the second floor. This case is treaten with an analogous argument as in Case 1.

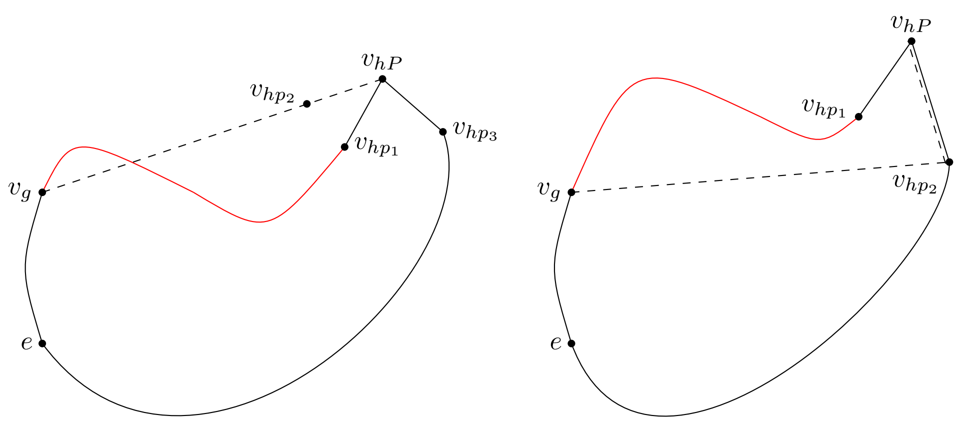

FIGURE 10. Shortening a cycle
in a hyperbolic coned-off Cayley graph.

Thus we can define the following set

$$R := \{uv^{-1} \mid u, v \text{ are words in } F, |u| \leq D+1$$

$$|u| > |\pi(u)| \text{ and } |v| < |u| \text{ with } \pi(u) = \pi(v)\},$$

where $\pi\colon F \to G$ is the canonical epimorphism, $|.|$ denotes the F-length of a word and $|\pi(u)|$ is the length of a geodesic path in $\hat{\Gamma}(G, \mathbb{P}, X)$ from the identity to the vertex $v_{\pi(u)}$. We can copy the arguments of Theorem 2.5.21 word by word to obtain that the set R is indeed a relative Dehn set and that $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to X. Note that the set R is in general *not* finite. □

The above theorem and Lemma 3.2.5 directly imply the following

4.1.8. *Corollary.* Let $(G, \mathbb{P})$ be a relative pair and let X, Y be two $\mathbb{P}$-generating sets for G. If X is a bounded Y-set and Y is a bounded X-set then $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to X if and only if $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to Y.

4.1.9. *Definition.* Let $(G, \mathbb{P})$ be a relative pair. We say that $(G, \mathbb{P})$ admits a *relative Dehn presentation* if $(G, \mathbb{P})$ admits a relative Dehn presentation with respect to *some* finite $\mathbb{P}$-generating set for G.

The above considerations together with Lemma 3.2.6 directly imply the following

4.1.10. *Corollary.* Suppose that $(G, \mathbb{P})$ is finitely $\mathbb{P}$-presented and the relative Dehn function of $(G, \mathbb{P})$ is well-defined. Then the following are equivalent:

(1) G is hyperbolic relative to $\mathbb{P}$.
(2) $(G, \mathbb{P})$ admits a relative Dehn presentation.

4.2. **Finite relative Dehn presentations.** In the above proof of Theorem 4.1.10 it is not possible to prove that the relative Dehn set is *finite*. Moreover, if one proves that a finite relative Dehn presentation implies that the relative Dehn function is linear (and in particular well-defined) than one would have another definition of relative hyperbolicity slightly different from Osin's one.

Nonetheless, there are more (quite different) definitions of relative hyperbolicity. In [11] Hruska analyzed six different definitions and showed that they are equivalent (one of them being Osin's one). But for this, one needs to make the following assumptions:

(1) The group G is at most *countable*.
(2) The set $\mathbb{P}$ is a *finite* collection of *infinite* groups.

We will call such a relative pair $(G, \mathbb{P})$ *of type A*.

We will mainly work with two other definitions, one by Bowditch (see [3]) and one by Farb (see [9]). For this we need to introduce some notions.

4.2.1. *Definition.* A graph X is *fine* if each edge of X is contained in only finitely many circuits of length n for each n. Note that we obtain an equivalent notion if we consider cycles instead of circuits (see Proposition 2.1 in [3]).

4.2.2. *Example.* Locally finite graphs are trivially fine. Also trees (not necessarily locally finite) are obviously fine. However the complete graph $K_{\mathbb{N}}$ and the graph $\hat{\Gamma}(\mathbb{Z}^2, \mathbb{Z}, X)$ of Example 3.2.1 are not fine (cf. Remark 4.1.2). One could regard fine graphs as graphs which are somewhere between being locally finite and non-locally finite.

4.2.3. *Definition.* Let $(G, \mathbb{P})$ be a relative pair of type A. Suppose G acts on a hyperbolic graph Z with finite edge stabilizers and finitely many orbits of edges. If Z is fine and $\mathbb{P}$ is a set of representatives of the conjugacy classes of infinite vertex stabilizers then G is called B-hyperbolic relative to $\mathbb{P}$ (B stands for Bowditch).

4.2.4. *Definition.* Let $(G, \mathbb{P})$ be a relative pair, X a $\mathbb{P}$-generating set for G and $P \in \mathbb{P}$. Given an oriented path γ in $\hat{\Gamma}(G, \mathbb{P}, X)$, we say that γ *penetrates* the coset gP if γ passes through the second floor vertex v_{gP}. A vertex v of γ immediately preceding the vertex v_{gP} is an *entering vertex* of γ in the coset gP. *Exiting vertices* are defined similarly. A path γ is said to be *without backtracking* if for every coset gP which γ penetrates, γ never returns to gP after leaving gP.

The triple $(G, \mathbb{P}, X)$ satisfies *Bounded Coset Penetration* (abbreviated by *BCP*) if for each $\lambda \geq 1$, there is a constant $a_\lambda > 0$ such that if γ and γ' are $(\lambda, 0)$-quasigeodesics without backtracking in $\hat{\Gamma}(G, \mathbb{P}, X)$ with initial endpoints $\gamma_- = \gamma'_-$ and terminal endpoints γ_+ and γ'_+ such that $d_X(\gamma_+, \gamma'_+) \leq 1$, then the following two conditions hold:

(1) If γ penetrates a coset gP but γ' does not penetrate gP, then the entering vertex and exiting vertex of γ in gP are at an X-distance at most a_λ from each other.
(2) If γ and γ' both penetrate a coset gP, then the entering vertices of γ and γ' in gP are at an X-distance at most a_λ from each other. Similarly, the exciting vertices are at an X-distance at most a_λ from each other.

4.2.5. *Remark.* In [7] it is proven that property (1) already implies Property (2). Although our Definition (from [11]) is slightly different from the one in [7] (which is the original one from [9]) we can adapt this proof almost verbatim. For this (and later use) we state the following

4.2.6. *Lemma.* Let X be a graph. Let p be an edge-path (or combinatorial path) which is an (c, b)-quasi-geodesic in X which ends at a vertex, say v. Suppose we extend p along an edge, say e, of length λ incident with v and a vertex $\tilde{v}$ which is not contained in p. Then the so obtained path $\tilde{p}$ is a $(\tilde{c}, b)$-quasi-geodesic with $\tilde{c} \geq c$. In particular the constant b has not changed.

Proof. We first note that by definition $c \geq 1$ and that it is enough to consider the respective inequalities for the vertices only. Therefore, let $\{v_1, v_2, \ldots, v_n\}$ be the vertices traversed by the path p (where $v_n = v$). Since p is already a quasi-geodesic we only have to compare pairs of vertices of the form $(v_i, \tilde{v})$. Therefore, we set

$$m := \min_i\{d_X(v_i, \tilde{v})\} \quad \text{and} \quad M := \max_i\{d_X(v_i, v)\}.$$

We directly obtain the following inequality

$$d_X(v_i, \tilde{v}) \leq d_X(v_i, v) + d_X(v, \tilde{v}) \leq c d_p(v_i, v) + b + \lambda \leq c(\underbrace{d_p(v_i, v) + \lambda}_{=d_{\tilde{p}}(v_i, \tilde{v})}) + b,$$

where we set d_p (and $d_{\tilde{p}}$) the distance in the codomain (an intervall intersected with $\mathbb{Z}$) of p (and of $\tilde{p}$). The first inequality is just the triangle inequality in X, the second a (c,b)-inequality for p and the fact that $d_X(v,\tilde{v}) \leq \lambda$, and the last inequality is just $\lambda \leq c\lambda$. Note that the equality $d_p(v_i,v)+\lambda = d_{\tilde{p}}(v_i,\tilde{v})$ holds since $\tilde{v}$ is not contained in p. All in all, this gives the second inequality in the definition of quasi-isometric embeddings. For the first inequality we have to bound $d_X(v_i,\tilde{v})$ from below. More precisely we have to show the following inequality

$$\frac{1}{\tilde{c}} d_{\tilde{p}}(v_i,\tilde{v}) - b \leq d_X(v_i,\tilde{v}) \text{ for some } \tilde{c},$$

which is equivalent to

$$\frac{d_{\tilde{p}}(v_i,\tilde{v})}{d_X(v_i,\tilde{v})+b} \leq \tilde{c}.$$

Firstly we have $0 \neq m \leq d_X(v_i,\tilde{v})$ hence

$$\frac{1}{d_X(v_i,\tilde{v})+b} \leq \frac{1}{m}.$$

Moreover by the (c,b)-inequalities for p and $d_X(v_i,v) \leq M$ and the fact that p and $\tilde{p}$ coincide on the vertices $v_1, \dots v_n = v$ we obtain

$$d_{\tilde{p}}(v_i,\tilde{v}) \leq d_p(v_i,v) + d_{\tilde{p}}(v,\tilde{v}) \leq c(M+b)+\lambda.$$

Thus, if we set $\tilde{c} := \max\{\frac{1}{m}(c(M+b)+\lambda), c\}$ we are done.

□

4.2.7. *Lemma.* (see [7], p.80) In the definition of BCP Property (1) implies Property (2).

Proof. Suppose γ_1 and γ_2 are two $(\lambda,0)$-quasigeodesics such as in the definition of BCP which both penetrate a coset gP. Let γ_1' be the subpath of γ_1 from its starting point to its entering vertex in gP. Let γ_2' be the subpath of γ_2 from its starting point to the vertex v_{gP}. Let γ_2'' be the path obtained from γ_2' by adding one edge to reach the ending vertex of γ_1'. Clearly γ_1' is still a $(\lambda,0)$-quasigeodesic where γ_2'' is a $(\tilde{\lambda},0)$-quasigeodesic (with a different $\tilde{\lambda}$). In particular we can apply Property (1) to the paths γ_1' and γ_2'' which exactly gives Property (2) for γ_1 and γ_2 (with different constants). □

Regardless of the last lemma we state the following

4.2.8. *Definition.* Let $(G,\mathbb{P})$ be a relative pair of type A. G is called F-hyperbolic relative to $\mathbb{P}$ (F stands for Farb) if $\hat{\Gamma}(G,\mathbb{P},X)$ is hyperbolic and $(G,\mathbb{P},X)$ satisfies BCP for some (every) finite $\mathbb{P}$-generating set X of G.

Note that it is not obvious that BCP is independent of the finite $\mathbb{P}$-generating set. This will follow from Proposition 4.2.13 below.

But before that let us take a closer look at those three different definitions of relative hyperbolicity. More precisely, how are the elements of $\mathbb{P}$ (subgroups of G relative to which we want hyperbolicity) related to *each other*?

Roughly speaking all three definitions can be split up into two parts. A "hyperbolic part" which describes "something being hyperbolic" and some "additional property".

Indeed, consider a relative pair $(G, \mathbb{P})$ and firstly Osin's definition: The Dehn function must be linear which describes a *hyperbolic condition* (cf. linear isoperimetric inequality for metric spaces and Theorem 2.5.16) and $(G, \mathbb{P})$ must be finitely $\mathbb{P}$-presented which describes an *additional property*.

Consider Bowditch's definition: The group G acts on a hyperbolic graph Z which describes a *hyperbolic condition* and the graph Z is fine which describes an *additional property*.

Consider Farb's definition: The graph $\hat{\Gamma}(G, \mathbb{P}, X)$ is hyperbolic which describes a *hyperbolic condition* and $(G, \mathbb{P}, X)$ satisfies *BCP* which describes an *additional property*.

In the theorems and their proofs above (and below) we saw (and will see) that the respective *hyperbolic conditions* are (of course) directly related to each other. But what about the *additional properties*?

4.2.9. *Definition.* A subgroup P in a group G is called *malnormal* if for any $g \in G \backslash P$ it holds that $P \cap gPg^{-1} = \{1\}$.

A subgroup P in G is called *almost malnormal* if for any $g \in G \backslash P$ it holds that $P \cap gPg^{-1}$ is *finite*.

A collection $\mathbb{P} = \{P_i\}_{i \in I}$ of subgroups in G is called *almost malnormal* if for all $g \in G$ it holds that if $gP_ig^{-1} \cap P_j$ is infinite then $i = j$ and $g \in P_i$.

In [9] Farb considers the case when G is torsion free and indicates that one easily proves that *BCP* implies that the subgroup P (for simplicity he works with one peripheral subgroup, where he writes that the arguments are easily translated to the general case for finitely many peripheral subgroups) is malnormal (see p.811 in [9]). Note that our definition of relative hyperbolicity in the sense of Farb (as in [11]) is slightly different from the orginial one of [9]. For simplicity we will also restrict ourselves to the case of one peripheral subgroup. The basic idea of the following proof is due to Osin.

4.2.10. *Proposition.* Let G be a group, $P \leq G$ a subgroup and X a finite $\{P\}$-generating set which has empty intersection with P. If $(G, \{P\}, X)$ satisfies *BCP* then P is almost malnormal.

Proof. We argue by contradiction. Hence, suppose there exists $g \in G \backslash P$ such that the intersection $P \cap gPg^{-1}$ is infinite. We set $|g| = a$, i.e. a is the length of a minimal word in $X \cup P$ which represents g. We can assume that g does *not* start with a letter of P. Indeed if $g = \tilde{p}\tilde{g}$ then, since $g \notin P$, also $\tilde{g} \notin P$. Moreover, it is a trivial computation to show that in this case also $P \cap \tilde{g}P\tilde{g}^{-1}$ is infinite. Similarly, we can assume that g does not end with a letter of P. Now consider such a $p \in P \cap gPg^{-1}$, i.e. $p = g\tilde{p}g^{-1}$ for some $\tilde{p} \in P$ and the geodesic quadrangle with endpoints $1, g, gp$ and p. Note that $p \in gPg^{-1}$ implies that $pg \in gP$, thus the picture is as follows.

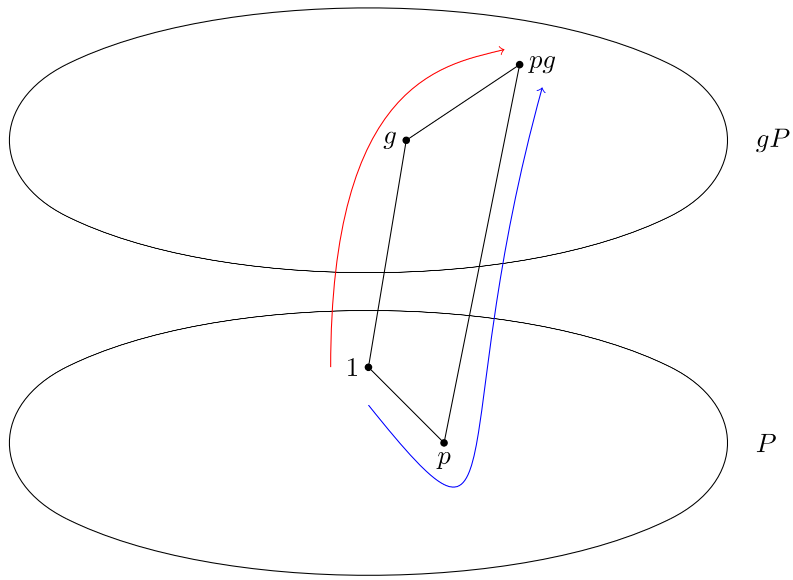

FIGURE 11. A geodesic quadrangle

Consider the geodesic side $[g, pg]$ which is "inside gP". Since both elements g, pg are in gP they can certainly be connected by a geodesic of the form $\{g, v_{gP}, pg\}$ of length $\frac{1}{2} + \frac{1}{2} = 1$ with the second floor vertex v_{gP}. It is also possible that they are connected by a first floor edge (of length 1) which would also be a geodesic. But this corresponds to an edge with a label $x \in X$ which is finite by assumption. Hence we can assume that for infinitely many such p we *must* be in the first case. We have an analogous situation for the geodesic side $[1, p]$ which is "inside P". We will prove that the concatenations $[1, p] \circ [p, pg]$ and $[1, g] \circ [g, pg]$ are apropriate quasi-geodesics such as in the definition of BCP. For the geodesics $[1, g]$ and $[p, pg]$ we choose those segments corresponding to the geodesic word of g from above (of length a).

We start with showing that they are *without backtracking*. Indeed consider $[1, g] \circ [g, pg]$. There cannot be any backtracking on $[1, g]$ since otherwise it wouldn't be a geodesic. The geodesic $[g, pg]$ penetrates the coset gP. Hence suppose that $[1, g]$ also penetrates gP. It is easily seen that this would imply that g ends with a letter of P, which is forbidden. The situation for $[1, p] \circ [p, pg]$ is treaten analogously (where we need that g cannot *start* with a letter of P). In summary, we have that both paths are without backtracking.

Let us prove that these paths are quasi-geodesics. We start with $[1, p] \circ [p, pg]$. Since $[1, p]$ is a geodesic it is thus a $(1, 0)$-quasi-geodesic. By the same arguments as above, v_{gP} cannot be contained in $[1, g]$. Hence, we can apply Lemma 4.2.6 and obtain a $(\lambda_1, 0)$-quasi-geodesic. We can clearly apply it a second time to obtain that $[1, p] \circ [p, pg]$ is a $(\lambda_2, 0)$-quasi-geodesic (indeed one can show by a straightforward computation that it is a $(a + 1, 0)$-quasi geodesic). The situation for the other path

$[1,g] \circ [g,pg]$ is treaten analogously. Thus, we have two $(\lambda_p, 0)$-quasi-geodesics in $\hat{\Gamma}(G, \{P\}, X)$ without backtracking (indeed they are both $(a+1, 0)$-quasi-geodesics).

Now the above holds for any non-trivial $p \in P \cap gPg^{-1}$. Thus we have infinitely many $(\lambda, 0)$-quasi geodesics (we can take the same λ for all paths since looking at the proof of Lemma 4.2.6 and the fact that $P, gP \subseteq B_{\hat{\Gamma}}(1, a+1)$ we see that all constants involved are bounded from above), say $c_i := [1,g] \circ [g, p_i g]$ and $d_i := [1, p_i] \circ [p_i, p_i g]$ for $i \in I$, where each pair (c_i, d_i) has the same endpoints (namely 1 and $p_i g$). Any c_i penetrates the coset gP. Thus, if we take some d_j and suppose it does *not* penetrate gP we obtain, by Property (1) of *BCP*, that for all $i \in I$ the entering vertices of c_i, i.e. g, and all exciting vertices of c_i, i.e. gp_i, are at an X-distance at most a_λ (some constant depending on λ only) from each other. But g is fixed and X is finite hence this can only occur at most finitely many times. Hence for infinitely $j \in I$ we have that d_j penetrates the coset gP. But similar to above we would obtain that g ends with a letter of P. □

4.2.11. *Corollary.* (see p.811 in [9]) Let G be a torsion free group, $P \leq G$ a subgroup and X a finite $\{P\}$-generating set which has empty intersection with P. If $(G, \{P\}, X)$ satisfies *BCP* then P is malnormal

Proof. Suppose there exists $g \in G \backslash P$ such that $P \cap gPg^{-1}$ is non-trivial. By the above Proposition it can be at most finite. But if $|P \cap gPg^{-1}| \geq 2$ this would give us a torsion element (elements of finite groups have finite order). □

We turn to Bowditch's definition. As one may assume it is the *fineness* of the graph (plus some extra properties), which implies that the peripheral subgroups form a almost malnormal collection. Indeed, Dahmani proved that *BCP* implies that the coned-off Cayley graph is fine (see below). The idea of the following proof is due to Osin.

4.2.12. *Proposition.* Let $(G, \mathbb{P})$ be a relative pair. Suppose that G acts on a (connected) *fine* graph X with finite edge stabilizers and $\mathbb{P}$ is a set of representatives of conjugacy classes of infinite vertex stabilizers. Then the collection $\mathbb{P}$ is almost malnormal.

Proof. We argue by contradiction. Hence, suppose we have distinct $P_1, P_2 \in \mathbb{P}$ and $g \in G$ such that $P_1 \cap gP_2g^{-1}$ is infinite. In particular, this implies that P_1 and P_2 are infinite. Let v_1 and v_2 be vertices which are stabilized by P_1 and gP_2g^{-1} respectively. We can choose such a pair of vertices with minimal distance, that is for a geodesic segment c connecting v_1 and v_2 in X no interior vertex (i.e. vertices except v_1 and v_2) is stabilized by P_1 or gP_2g^{-1}. By construction, for any $p \in P_1 \cap gP_2g^{-1}$ the path $p \cdot c$ and c have the same endpoints, namely v_1 and v_2. But by assumption the action is with finite edge stabilizers, hence although it can happen that $p \cdot c$ and c have interior points in common, this can only be the case for at most *finitely* many such p. But we assumed that $P_1 \cap gP_2g^{-1}$ is infinite, i.e for infinitely many i we have that c and $p_i \cdot c$ have no interior vertex in common. Hence, their concatenation is a curcuit of length $2|c|$. In particular every edge of c is contained in infinitely many circuits of length $2|c|$, a contradiction. □

4.2.13. *Proposition.* (cf Theorem 5.1 in [11]) Let $(G, \mathbb{P})$ be a relative pair of type A. Then G is B-hyperbolic relative to $\mathbb{P}$ if and only if it is F-hyperbolic relative to $\mathbb{P}$ if and only if it is hyperbolic relative to $\mathbb{P}$ (in the sense of Osin).

Proof. For simplicity we will only sketch the proof of the direct equivalence of relative F-hyperbolicity and relative B-hyperbolicity. Suppose that G acts on a fine, hyperbolic graph Z with finite edge stabilizers and finitely many orbits of edges. It can be shown by standard arguments that G is finitely $\mathbb{P}$-generated. Let X be an arbitrary finite $\mathbb{P}$-generating set for G. By Proposition 3.3.10 we know that $\hat{\Gamma} = \hat{\Gamma}(G, \mathbb{P}, X)$ has trivial edge stabilizers and finitely many orbits of edges. An argument in [7] (Lemma A.4) produces a fine graph Z' with finite edge stabilizers and finitely many orbits of edges such that Z and $\hat{\Gamma}$ both embed in an apropriate way (simplicially and equivariantly) into Z'. Since any subgraph of a fine graph is fine we obtain that $\hat{\Gamma}$ is fine. Furthermore a connected equivariant subgraph of Z' is quasi-isometric to Z'. Since Z is hyperbolic it follows that $\hat{\Gamma}$ is hyperbolic. By an argument in [7] (Lemma A.5) the fineness of the connected $\hat{\Gamma}$ implies that $(G, \mathbb{P}, X)$ has BCP.

Suppose conversely that G is F-hyperbolic relative to $\mathbb{P}$ and let X be some finite $\mathbb{P}$-generating set. In [7] (Proposition A.1) it is proven that the property BCP implies that $\hat{\Gamma}$ is fine. $\square$

Thus when we speak of a *relatively hyperbolic* group G (relative to some $\mathbb{P}$) we mean one of the above equivalent definitions. We will prove that the *fineness* of $\hat{\Gamma}$ is equivalent to the *finiteness* of the corresponding relative Dehn set.

4.2.14. *Lemma.* Let $(G, \mathbb{P})$ be a relative pair of type A and G be hyperbolic relative to $\mathbb{P}$. Then the obtained relative Dehn set R is *finite*.

Proof. Let $\mathbb{P} = \{P_1, \ldots, P_m\}$ and X be a finite $\mathbb{P}$-generating set. The hyperbolic graph $\hat{\Gamma} = \hat{\Gamma}(G, \mathbb{P}, X)$ has property BCP. As noted in the above proof this implies that $\hat{\Gamma}$ is *fine*. Let R be a as in the proof of Theorem 4.1.7, hence $(D = [8\delta] + 1)$

$$R := \{uv^{-1} \mid u, v \text{ are words in } F, |u| \leq D + 1$$
$$|u| > |\pi(u)| \text{ and } |v| < |u| \text{ with } \pi(u) = \pi(v)\}.$$

We have to prove that R is a *finite* set.

Each uv^{-1} yealds a cycle in $\hat{\Gamma}$ up to length $2D$, starting at the identity $e_G \in G$ (on the first floor), which has finite degree (because we have a *finite* collection of subgroups $\mathbb{P}$). In particular, each such cycle must 'start' with an edge incident with e_G. Let $X := \{x_1, x_2, \ldots, x_n\}$, then for each x_i there is exactly one edge incident with e_G having x_i as a label, say e_i. Thus, every word which starts with x_i gives rise to a cycle containing the edge e_i. Since by assumption $\hat{\Gamma}$ is fine and we have a bound on the length of all these cycles, these are only *finitely* many.

What if a word starts with $p \in P_i$ for some i? The obtained cycle *must* walk along the edge $\{e_G, v_{P_i}\}$ independent of the element $p \in P_i$. Thus, again by assumption there can only be finitely many such cycles. Since by assumption we have a *finite* collection of subgroups $\mathbb{P} = \{P_1, P_2, \ldots, P_m\}$ and by Proposition 3.2.3 different words give different cycles and vice versa the set R is indeed *finite*. $\square$

We note that it is crucial in the above proof that we have a *finite* collection $\mathbb{P}$ of subgroups.

4.2.15. *Lemma.* Let $(G, \mathbb{P})$ be a relative pair of type A which admits a *finite* relative Dehn presentation, then $\hat{\Gamma}(G, X, \mathbb{P})$ is *fine* (and hyperbolic).

Proof. We prove the assertion by induction on the (combinatorial) length of circuits. First we note that every vertex of the first floor has finite degree, because firstly, there is one edge for each letter of the *finite* $\mathbb{P}$-generating set X. Secondly, the vertex, say $g \in G$, is contained in exactly one coset of each P_i. Hence one edge for each i. Thus it has finite degree (finitely many neighbours).

Induction start $n = 3$: Let $e = \{v, w\}$ be an edge. We have two cases:

Both v, w are on the first floor, i.e. are elements of G. A circuit of length 3 containing e is of the form $\{v, w, z\}$ for some vertex z (circuit considered as a vertex set). But w and v both have finite degree, hence there are only finitely many posibilities for z and thus only finitely many circuits of length 3 containing e.

At least one vertex of e is some second floor vertex, say $v = v_{gP_i}$ for some i and some $g \in G$. Thus a circuit of length 3 must contain an edge $\{z, v_{gP_i}\}$ of length $\frac{1}{2}$, i.e. $z \in gP_i$. But there are at most finitely many edges from w to z (one for each $x \in X$).

Induction step: Suppose that for every edge the set of circuits containing this edge of length at most $n-1$ is finite. By assumption R (relative Dehn set as in the proof of Theorem 4.1.7) is finite, i.e. we have some $u_1, \ldots, u_k$ and corresponding *shorter* $v_1, \ldots, v_k$. Let e be an edge and suppose that there are infinitely many circuits of length n containing e. Each such circuit correspods to a word representing $e_G \in G$. Thus by assumption it contains some u_i as a subword and hence as a subpath. Let T_i be the set of circuits which contain u_i (these sets are not necessarily disjoint). Thus at least one of these sets is infinite. Without loss of generality we assume that T_1 is infinite. We have to distinguish two cases:

First consider those circuits of T_1 such that e is *not* contained in the subpath u_1 (more precisely at most one vertex of e is contained). In other words, when we interchange u_1 with v_1 we obtain a shorter circuit which still contains e. We reduce each such circuit to a necessarily shorter circuit still containing e. By induction there are only finitely many such circuits thus there must be some circuit $\bar{c}$ (of length strictly smaller than n) to which infinitely many circuits c of length n reduce. But we can reverse the above procedure in an appropriate way. More precisely, we can take an occurance of v_1 in $\bar{c}$ and interchange it with u_1. It is (trivial but) important to note that this interchanging is *unique*, by Proposition 3.2.3. If one starts to read a word beginning at some vertex, the obtained path is *unique.* In $\hat{\Gamma}$ it holds that for each vertex and each word there is *at most one* path starting at this vertex and the given word as label. Turning back to our cycle $\bar{c}$, we see that for each occurance of v_1 in $\bar{c}$ we can obtain one circuit c of length n which reduces to $\bar{c}$. But there are only finitely many possibilities of occurances of v_1 in $\bar{c}$, hence we have only finitely many circuits in T_1 of the above type.

Secondly, consider those circuits of T_1 such that e is contained in the subpath u_1. If interchanging u_1 with v_1 does not delete e, we are in the above case. Otherwise we again divide these circuits into finitely many sets S_j, such that e is the $j-th$ edge of u_1. Consider S_1, i.e circuits such that e is the first edge of u_1. Interchanging

u_1 with v_1 we obtain a circuit which is shorter but does *not* contain e. If we consider the first edge of v_1, say $\tilde{e}$, than the induction hypothesis for $\tilde{e}$ implies that S_1 is finite. Clearly the same argument works for every S_j, hence T_1 can *not* be infinite.

All in all, we have proven that there are only finitely many circuits containing e, hence the assertion is proven. □

Combining the above considerations we directly obtain the following

4.2.16. *Theorem.* Let $(G, \mathbb{P})$ be a relative pair of type A. Then the following are equivalent.

(1) G is hyperbolic relative to $\mathbb{P}$.
(2) $(G, \mathbb{P})$ admits a *finite* relative Dehn presentation.

Clearly, one now tries to obtain results on the word problem. Of course, one has to assume that the word problem is solvable for the peripheral subgroups. One could apply Dehn's algorithm in the usual way. Nonetheless, this has already been proved (differently) in [9] (see Theorem 4.14) and slightly more general in [16] (see Theorem 5.1). We have a similar situation for the conjugacy problem (see [5]).

4.3. **Examples and some properties.** For simplicity we will often restrict ourselves to the case where $\mathbb{P} = \{H\}$ and write (G, H) rather than $(G, \{H\})$ for such a relative pair. We start with non-hyperbolic relative groups. Moreover, most groups we encounter will be finitely generated.

4.3.1. *Free abelian groups.* Consider the free abelian group of rank 2, $\mathbb{Z}^2$, given by the (ordinary) presentation $\langle a, b \mid [a, b] = 1 \rangle$ and $H := \langle a \rangle \cong \mathbb{Z}$ as well as $X := \{b, b^{-1}\}$. The group $\mathbb{Z}^2$ is not hyperbolic since one easily shows that the inclusion $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ is a quasi-isometry (more generally one can do this for $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$). Moreover, the hyperbolic geometry of a hyperbolic group inhibits the group to even contain $\mathbb{Z}^2$ (see Corollary 3.10 in [4]).

Despite the above mentioned, it is easily shown that $\hat{\Gamma} := \hat{\Gamma}(\mathbb{Z}^2, H, X)$ is δ-hyperbolic with $\delta = 1$. Nonetheless, $\hat{\Gamma}$ is *not* fine hence $\mathbb{Z}^2$ is *not* hyperbolic relative to H but so-called *weakly hyperbolic*, i.e. the coned-off Cayley graph is hyperbolic but we don't have *BCP* (or equivalently fineness). Regardless of that, one could write down a (non-finite) relative Dehn presentation.

With little effort one shows that we have a similiar situation for *any* cyclic subgroup $H \leq \mathbb{Z}^2$. More precisely, in these cases $\hat{\Gamma}(\mathbb{Z}^2, H, X)$ (with a corresponding finite relative generating set X) is hyperbolic but not fine. It is a fact, that any subgroup of $\mathbb{Z}^2$ is either isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$. In the latter case it has finite index and thus the corresponding coned-off Cayley graph is bounded, hence hyperbolic. It is clear, that in this case the graph is not fine.

More generally, any subgroup of $\mathbb{Z}^n$ is a free abelian group of rank $k \leq n$. Moreover, a coned-off Cayley graph of $(\mathbb{Z}^n, \mathbb{Z}^k)$ is quasi-isometric to an (ordinary) Cayley graph of $\mathbb{Z}^{(n-k)}$. Thus, $\mathbb{Z}^n$ is weakly hyperbolic relative to $\mathbb{Z}^k$ if and only if $k = n - 1$, but never relatively hyperbolic (since the corresponding graphs are not fine).

We already mentioned that a hyperbolic group cannot contain $\mathbb{Z}^2$. One can say even more. Indeed, a hyperbolic group cannot contain a subgroup isomorphic to a *Baumslag-Solitar group*, i.e. a group given by the presentation

$$\langle a, b \mid ba^m b^{-1} = a^n \rangle =: \mathrm{BS}(m, n),$$

for some integers m, n. Note that $\mathbb{Z}^2 = \mathrm{BS}(1,1)$. So what about the analogue for hyperbolic relative groups? Consider a relative pair $(G, \mathbb{P})$ of type A. Since we have no control over the groups of $\mathbb{P}$ the exact same statement cannot be true. Clearly, we have to adapt the statement to the right setting.

4.3.2. *Lemma.* (see Corollary 4.22 in [16]) Let $(G, \mathbb{P})$ be of type A and G be hyperbolic hyperbolic relative to $\mathbb{P}$. Assume additionally that G is finitely generated and let B be a subgroup of G. If B is isomorphic to a Baumslag-Solitar group, then B is conjugate to a subgroup of P_i for some $i \in I$.

Another well-known fact about hyperbolic groups states that every infinite hyperbolic group contains an element of infinite order (see Proposition 2.22 in [4]). In particular, this shows that finitely generated infinite torsion groups cannot be hyperbolic. We have a similar result in the relative case, where we call an element $g \in G$ *hyperbolic* if it is *not* conjugated to an element of any peripheral subgroup.

4.3.3. *Lemma.* (see Corollary 4.5 in [15]) Let G be hyperbolic relative to $\mathbb{P}$. Assume additionally that G is infinite and all peripheral subgroups are proper. Then G contains a hyperbolic element of infinite order.

In particular, this shows that a (not necessarily finitely generated) infinite torsion group cannot be relatively hyperbolic.

Basically it seems that it is hard to prove that a group is *not* relatively hyperbolic since one would have to check for *any* collection of peripheral subgroups. In [1] a simple combinatorial criterion is described which enables one to decide if the group in question is non-relatively hyperbolic.

4.3.4. *What are the right peripheral subgroups?* We have already seen that $\mathbb{Z}^n$ is *not* relatively hyperbolic (relative to some subgroup) by considering the coned-off Cayley graph only. Although in this case the coned-off Cayley graph was hyperbolic we didn't have *BCP* or equivalently fineness. We know already at least a necessary condition for the peripheral subgroups, such that these additional properties (*BCP*, fineness) hold, that is (almost) malnormality.

First we note that in an arbitrary group G the trivial subgroup $\{e_G\}$ and G itself are clearly (almost) malnormal. Similarly, these are trivially normal subgroups. Recall that a subgroup $N \leq G$ is *normal* if $gN = Ng$ for all $g \in G$. This is equivalent with $N = gNg^{-1}$ for all $g \in G$. Hence, the properties being normal and being malnormal are somehow "inverse" to each other. More precisely, if a subgroup $N \leq G$ is malnormal *and* normal then $N = \{e_G\}$ or $N = G$. As one may assume we want *infinite* peripheral subgroups. In this sense if an *infinite* subgroup $N \leq G$ is normal *and* almost malnormal then $N = G$. Since we don't really want G to be part of the peripheral structure of G we have to *avoid* normal (infinite) subgroups. But there are groups where we can't do that, that is *abelian* groups.

Considering a finitely generated abelian group G it is well-known, that $G \cong \mathbb{Z}^n \times F$, where n is some non-negative integer and F some *finite* group. Moreover, every subgroup of G is finitely generated (which is clearly a false statement in non-abelian groups). Many authors consider finitely generated groups and infinite finitely generated (peripheral or sometimes called parabolic) subgroups. Indeed, the above mentioned gives one way to understand the following quote of [6], p. 2:

"*However, in my opinion, the most natural examples of relatively hyperbolic groups have abelian or virtually abelian parabolic subgroups.*"

4.3.5. *Free products and quasiconvex subgroups.* As already noted, several authors started to investigate *finitely generated* relatively hyperbolic groups. It is quite plausible to try generalizing theorems of hyperbolic groups to the relative case. In this matter the study of *quasiconvex subgroups* plays a central role, i.e. subgroups which inherit the hyperbolic geometry. Recall the following definition:

A subspace C of a geodesic metric space X is said to be *quasiconvex* if there exists a constant $k > 0$ such that for all $x, y \in C$ each geodesic joining x to y is contained in the k-neighbourhood of C.

One calls a subgroup $H \leq G$ *quasiconvex* if H is quasiconvex in the Cayley graph of G (with respect to some finite generating set of G). It is now straightforward to prove that H is necessarily finitely generated and that $H \hookrightarrow G$ is a quasi-isometric embedding (see Lemma 3.5 in [4]) and moreover, quasiconvex subgroups of hyperbolic groups are hyperbolic themselves (see Proposition 3.7 in [4]). A second important property is that the intersection of two quasiconvex subgroups is again quasiconvex (see Proposition 3.9 in [4]).

Indeed, these two fundamental properties of quasiconvex subgroups of a hyperbolic group lead Hruska in [11] to study non-finitely generated relatively hyperbolic groups. It seems a necessity for a satisfying theory of *relatively quasiconvex subgroups*, since for example the intersection of two relatively quasiconvex subgroups (in the sense of Hruska, see section 6 in [11]) is often non-finitely generated.

A somehow basic example of a non-finitely generated relatively hyperbolic group is a free product. Indeed consider $G = A * B$ where A and B are non-finitely generated. Then G is hyperbolic relative to $\mathbb{P} = \{A, B\}$. Clearly we have a relative presentation with an empty set of relations, which in turn is trivially a relative Dehn presentation. In this case the coned-off Cayley graph has *no* first floor edges.

What if we only consider *one* factor, i.e. $\mathbb{P} = \{A\}$? Do we still have a relatively hyperbolic group? The answer is no, in general, since we have no control over the second free factor B. For example consider $A := \mathbb{Z}$ and $B := \mathbb{Z}^2$. Then first of all $G = A * B$ is not hyperbolic since it contains $\mathbb{Z}^2$ as a subgroup. But G is also *not* relatively hyperbolic with respect to $\mathbb{P} = \{A\}$. Indeed, let $X = \{a, b\}$ be a generating set for B such as in Example 3.2.1. Then, by definition of relative presentations, we have

$$G = A * B \cong (A * F(X))/\langle\langle R \rangle\rangle.$$

Since there are no relations between elements of A and B (by definition of a free product) the words in R only consist of words in X. Hence, these define relations which hold in B only. If R would be a relative Dehn set for G it also would be a Dehn set for B, but then $B = \mathbb{Z}^2$ would be hyperbolic, which is a contradiction.

4.3.6. *Hyperbolic groups and malnormality.* Clearly, one attempt of relatively hyperbolic groups is to generalize hyperbolic groups. Indeed, if G is a hyperbolic group than it is hyperbolic relative to the trivial subgroup. Obviously a group is hyperbolic if and only if it is hyperbolic relative to the trivial subgroup. However, there are other, maybe more subtle, hyperbolic peripheral structures in a hyperbolic group. Actually, we know already some "special" subgroups of hyperbolic groups, namely quasi-convex subgroups. Indeed, Bowditch proved in [3] Theorem 7.11, that a hyperbolic group is hyperbolic relative to an almost malnormal collection of quasi-convex subgroups.

References

[1] Anderson J. W., Aramayona J., Shackleton K. J., *An obstruction to the strong relative hyperbolicity of a group.* Journal of Group Theory, 10(6) (2007), 749-756.

[2] Bogopolski O., *Introduction to group theory.* EMS Textbooks in Mathematics, Zürich, 2008.

[3] Bowditch B. H., *Relatively hyperbolic groups.* Internat. J. Algebra and Computation, 22 (2012), 1250016, 66pp.

[4] Bridson M. R., Haeflinger A., *Metric spaces of non-positive curvature.* Springer Verlag, Berlin, 1999.

[5] Bumagin I., *The conjugacy problem for relatively hyperbolic groups.* Algebraic and Geometric Topology, 4 (2004), 1013-1040.

[6] Dahmani F., *Existential questions in (relatively) hyperbolic groups.* Israel J. Math., 173 (2009), 91-124.

[7] Dahmani F., *Les groupes relativement hyperboliques et leurs bonds.* PhD thesis, Univ. Louis Pasteur, Strasbourg, 2003.

[8] Dehn M., *Über unendliche diskontinuierliche Gruppen.* Math. Ann., 71 (1912), 116-144.

[9] Farb, B., *Relatively hyperbolic groups.* Geom. Funct. Anal., 8(5) (1998), 810-840.

[10] Harpe P. de la, *Topics in geometric group theory.* University of Chicago Press, Chicago, 2000.

[11] Hruska G. C., *Relative hyperbolicity and relative quasiconvexity for countable groups.* Algebr. Geom. Topol., 10 (2010), 1807-1856.

[12] Krön B., *End compactifications in non-locally-finite graphs.* Math. Proc. Cambridge Philos. Soc., 131 (2001), 427-443.

[13] Lyndon R.C, Schupp P.E., *Combinatorial group theory.* Springer Verlag, Berlin, 1977.

[14] Meier J., *Groups, Graphs and Trees. An Introduction to the geometry of infinite groups.* Cambridge University Press, 2008.

[15] Osin D. V., *Elementary subgroups of relatively hyperbolic groups and bounded generation*, Internat. J. Algebra Comput., 16 (2006), no.1, 99-118.

[16] Osin D. V., *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties and algorithmic problems.* Memoirs Amer. Math. Soc., 179 (2006), no. 843, vi+100 pp.

[17] Urbina M., *Ends and spectra of graphs.* Diploma thesis, Vienna, 2014.

[18] Woess W., *Random walks on infinite graphs and groups.* Cambridge University Press, 2000.