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Existence of a Shadow Price in Discrete Time

Verfasser

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To my alter ego, my alternative friends and the alternative life I would had in a parallel universe where I managed to become a much better mathematician than in this universe

Abstract

Portfolio optimization problems, with respect to a (concave) utility function, are a quite new research field in financial mathematics, especially when someone considers transaction costs to be present. Whereas there are rigorous proofs and examples in the case without transaction costs, in market models with frictions many problems can occur. Usually someone wants to find a so called shadow price in the associated frictionless market, so that many problems can be solved again. This thesis gives a small outline of the existence of a shadow price in a model with discrete time. Subject matter will be the case with a finite and an infinite probability space. It will be shown that in case of the discrete finite probability space a shadow price always exists, whereas in the other case it usually fails to exists. Finally it will be derived, which assumptions are necessary to guarantee the existence in a model with an infinite probability space.

Zusammenfassung

Ein großes Forschungsgebiet der Finanzmathematik ist die Nutzenoptimierung eines Portfolios in einem Marktmodell, unter Berücksichtigung einer gegebenen (konkaven) Nutzenfunktion des Investors. Das Vorhandensein der Transaktionskosten ist eine sehr neue Annahme die häufig zu Problemen bei der Lösbarkeit der Fragestellung der Nutzenoptimierung führt. Viele Fragen können jedoch beantwortet werden, wenn ein so genannter shadow price im assoziierten Markt ohne Transaktionskosten existiert. Die vorliegende Arbeit befasst sich mit der Existenz eines solchen shadow price in einem Modell mit diskreter Zeit. Betrachtet werden die beiden Fälle mit endlichem Wahrscheinlichkeitsraum und unendlichem Wahrscheinlichkeitsraum. Es wird gezeigt, dass im Fall eines endlichen Wahrscheinlichkeitsraumes ein shadow price immer existiert, wogegen im anderen Fall üblicherweise keiner existiert. Abschließend wird noch gezeigt, unter welchen Bedingungen eine Existenz im Falle eines unendlichen Wahrscheinlichkeitsraumes gewährleistet werden kann.

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Contents

Chapter 1

Introduction

1.1 Problem

One of the most discussed and researched problems in financial mathematics is the portfolio choice question under transaction costs. The main question here is: how would a risk affine investor choose his position in either a risk free account (bond, or something similar) and/or in risky assets (e.g. stock, derivatives and so on) under proportional transaction costs. The risk aversion is usually modelled as a function that has to be concave. This seems rather logical, because it is (usually) better for an investor to higher his position in an asset where he holds not so many shares than in an asset where he already has much shares.

There are two ways how we can solve this utility optimization problem: either by using stochastic control theory or by using the methods of a shadow price (which in most cases leads to methods of dual problems and convex analysis). The second approached is based upon the paper of Jouini [1] and is the one that will be dealt with in this thesis. A shadow price is a (fictitious) stock price, that is located in the associated market without transaction costs, and that gives the opportunity to solve the optimization problem in the much easier frictionless case. The next question someone might ask is whether such a shadow price exists or not, and under which conditions. In other words, if there is a connection between the market without transaction costs and the market with transaction costs via the shadow price, which should we use for the optimization problem?

This master thesis will give an overview over some results of this questions in discrete time with finite and infinite probability spaces. There are also many interesting results about the same question in continuous time, but this would be beyond the scope of this thesis.

In the second part of chapter 1 we will have a look at the exact problem we are dealing with, and give the parameters of our setting.

In chapter 2 we will discuss the setting with finite probability spaces, which is quite convenient and leads to a general result. Here I will follow the paper of Kallsen and Muhle-Karbe [2].

In the main section, Chapter 3, we will have a look at the infinite probability space (again in discrete time), and show that even in these still simple surroundings the shadow price fails to exist in general. Here I will follow the paper of Czichowsky, Muhle-Karbe and Schachermayer [3] and will also give a rather easily understandable counter example for the existence of a shadow price. On the other hand we will also have a look on how we can change the conditions, so that we obtain a solution in this setting.

1.2 Market Model

In this section we will define and summarize a few things that we are using all over the whole thesis. In some cases I will not be 100 % explicit. For example if a process is adapted to a finite or an infinite probability space, because we will deal with both cases in further sections. In such cases the exact definitions will be given in the chapters later on. The idea behind this is that in many papers which deal with this matter someone might usually come across the phrase ”we use the usual setting” and so I wanted to write these basics down in more detail.

First of all we define our **market model** S which consists of two assets:

- the risk free asset, which we could call **bond** when thinking economical, that just models our bank account. We will denote this process with $S^0 = (S^0_t)_{t=0}^T$. We will choose S^0 as a numéraire, so we will normalize our process in each time step with respect to S^0, that means we can assume that $S^0 \equiv 1$ at all time.
- the second asset is the risky asset, which could model for example a **stock price**. This process $S^1 = (S^1_t)_{t=0}^T$ is adapted to a probability space that can be finite or infinite. We assume this process to be a $\mathbb{R}_+$-valued process, again economically a negative value for a stock price won't make much sense and since we deal with discounted values, positions in stocks won't just take values in $\mathbb{N}$. We will have a more explicit look at this asset in each section.

Just a small explanation of notation, when we talk about the market model S we assume a combination of both processes, bond and stock. The lower index (in the market model or at the single assets), tell us the time when we look at the price, whereas when an upper index is present, we only deal with one particular asset (the bond if it's 0 and the stock when it's 1).

Next we will define what is meant by transaction costs. For a long time it was assumed that there are no transaction costs in a market model, which is quite a strong restriction, because "in reality" we usually have to pay transaction costs.

Mathematically we define **transaction costs** by a factor $\lambda \geq 0$ (whereas the case $\lambda = 0$ just refers to the frictionless case, and the case $\lambda > 1$ makes economically not much sense as clarified below), such that together with the stock price S^1 we can define the **bid/ask-spread** as the $\mathbb{R}^2_+$ process $(\underline{S}_t, \overline{S}_t) = ((1-\lambda)S^1_t, S^1_t)_{t=0}^T$. We can interpret this in the following way: at time t we can buy stocks at the higher ask-price $\overline{S}_t$ and when we sell stocks we only receive the lower bid-price $\underline{S}_t = (1-\lambda)S_t$. We usually denote the bid/ask with $[\underline{S}_t, \overline{S}_t]$ because we will rather use it as a closed interval from $\underline{S}$ to $\overline{S}$, than as a 2 dimensional process.

Notify that we do not assume that there are transaction costs for "buying or selling" positions in the bond S^0. So there are no fees for money transfers from or to the bond. For the exact definition why this is a valid assumption see e.g. the paper of Davis and Norman [4]. This makes sense in most of the cases, a counter example would be if we deal with a foreign currency market, but this is not part of this work. Results for that case can be found in the work of Kabanov [5].

Since we now have defined how the stock price evolves, we need a model, that explains how much of our wealth we hold in stock and how much is located in the bond. This is done via a **trading strategy**, which is a predictable, $\mathbb{R}^2$-valued process $\varphi_t = (\varphi^0_t, \varphi^1_t)_{t=0}^{T+1}$. Since we have to decide before the stock price evolves at each time, this process is $\mathcal{F}_{t-1}$-measurable (which we already denoted by predictable). We denote the set of all predictable trading strategies by $\mathcal{P}$. A trading strategy can be interpreted in the following way: the first part, with upper index 0, gives the numbers of shares held between time $t-1$ and t (after we buy or sell at time $t-1$), the second part, with upper index 1, therefore gives the positions in stocks. It is important to observe that we have to use different indices of time when we put S and φ together, see e.g. the stochastic integral in the next paragraph.

Through this whole thesis we will use the notation, that a trading strategy without an upper index refers to the 2-dimensional process, and with an index always means just one part of the process. Furthermore we will denote the trading strategy for the frictionless market with ψ, whereas an optimal strategy is denoted by $\hat{\varphi}$ and $\hat{\psi}$ respectively.

We also assume that at time T we liquidate our stocks. This means selling all of them at bid price $\underline{S}_T = (1-\lambda)S_T$ or at the shadow price $\tilde{S}_T$ when we deal with the associated frictionless market. As long as we talk about the frictionless case we can also define the **stochastic integral**, which gives an information about the total wealth at a specific time

$$(\varphi \cdot S)_s = \sum_{t=1}^{s} (\varphi_{t-1}(S_t - S_{t-1})$$

We call a trading strategy **self financing** if we don't add (or take out) money from the market during the whole process. In other words all money we need to buy stocks is taken from the bond S^0 and all money we get from selling stocks is transferred to it. We will need this assumption for example to forbid situations where the investor could avoid bankruptcy, when his total wealth went negative but he could rebalance it by adding money from outside. We will give the mathematically accurate definition of self-financing in each section.

An often used concept (and even more often a condition of the used theorem later on) is the so called **No-Arbitrage condition**. An Arbitrage is the possibility to generate money "out of nothing", so it is a chance to obtain a riskless profit. It is important that, when we are talking about the case where transaction costs are present, that the possibility to gain an arbitrage depends (logically) on the choice of λ.

Definition 1.2.1 (*No-Arbitrage*)

The process $S = (S_t)_{t=0}^T$ admits for arbitrages under transaction costs $0 \leq \lambda < 1$ if there is a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)$, such that

$$(\varphi_{T+1}^0, \varphi_{T+1}^1) \geq (0,0) \quad \mathbb{P}\text{-almost surely}$$

and

$$\mathbb{P}((\varphi_{T+1}^0, \varphi_{T+1}^1) \geq (0,0)) > 0$$

The process satisfies the No-Arbitrage condition (NA^λ) if it does not allow for an arbitrage under transaction costs λ.

1.3 Utility function and Shadow Price

A **utility function** is a function $U : (0, \infty) \mapsto \mathbb{R}$ (we could also define the function on $\mathbb{R} \mapsto \mathbb{R}$, but in our setting defining it on the positive real numbers is enough) that is

- increasing on $(0, \infty)$
- continuous on $(0, \infty)$
- strictly concave on the interior of $(0, \infty)$
- satisfies the **Inada conditions**

$$U'(x) := \lim_{x \to 0} U'(x) = \infty$$
$$U'(x) := \lim_{x \to \infty} U'(x) = 0$$

There are many examples for utility functions, here are some examples that are commonly used:

- $U(x) = \log(x)$
- $U(x) = \frac{x^\gamma}{\gamma}$ for a $\gamma \in (0, 1)$ or $\gamma \in (-\infty, 0)$. Often used is for example $\frac{x^{1/2}}{1/2}$
- $U(x) = -e^{-x}$

The main goal of a risk affine investor is to maximize his utility function, or strictly speaking to maximize the expectation of the utility function. In mathematical terms that can be simply stated as

$$E[U(g)] \to \max$$

Where g runs through the set of all non-negative positions of cash, that can be achieved via a self financing trading strategy φ_t from a defined initial endowment. This set will later on be defined as the set of all admissible trading strategies.

Once again we will have a closer look at the exact utility maximization problem in each chapter.

And at last we define the **shadow price** itself. First we will make the idea clear and then I will give a proper definition.

The idea behind a shadow price is, that we seek for a (fictitious) frictionless market where the stock price takes values within the Bid/Ask-Spread $[\underline{S}_t, \overline{S}_t]_{t=0}^T$ of our original market. We also assume that the utility optimization problem in this frictionless case leads to the same result as in the original market. So to be more exact the maximal expected utility in both markets coincide. And last but not least, we have to claim that we only buy or sell stocks in the frictionless case, whenever the shadow price coincide with one of its borders, so the bid- or the ask-price of the original market. We will call such a process $\tilde{S}^1 = (\tilde{S}_t^1)_{t=0}^T$.

Definition 1.3.1 (*Shadow Price*)

Let $S = (S_t)_{t=0}^T$ be a process that models a stock price and $0 \leq \lambda < 1$ the associated transaction cost-factor under which the No-Arbitrage condition (NA^λ) is satisfied. Let $U : (0, \infty) \to \mathbb{R}$ be the utility function, satisfying the Inada conditions as above and $(x, 0) \in \mathbb{R}^2$ the initial endowment, meaning $\varphi_0 = (\varphi_0^0, \varphi_0^1) = (x, 0)$.
Let now $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ be an adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ that takes values within the bid/ask spread $[\underline{S}_t, \overline{S}_t] = ([(1-\lambda)S_t, S_t])_{t=0}^T$.
$\tilde{S}$ is called a shadow price process for S if there is an optimizer $(\hat{\psi}_t)_{t=0}^T$ for the frictionless market $\tilde{S}$, meaning

$$E[U(x + (\hat{\psi} \cdot \tilde{S})_T)] = \max\{E[U(x + (\psi \cdot \tilde{S})_T)] : \psi \in \mathcal{P}\}$$

(where $\mathcal{P}$ is the space of all predictable $\mathbb{R}$-valued trading strategies) such that

$$\{\Delta \hat{\psi}_t > 0\} \subseteq \{\tilde{S}_{t-1} = S_{t-1}\} \quad t = 1, \dots, T$$

and

$$\{\Delta \hat{\psi}_t < 0\} \subseteq \{\tilde{S}_{t-1} = (1-\lambda)S_{t-1}\} \quad t = 1, \dots, T$$

Whenever such a shadow process exists we can transfer the information from a trading strategy, that optimizes the above mentioned frictionless utility problem, to the market with transaction costs. So the existence of a shadow price makes life much easier. The connection of the shadow price and the utility optimization problem is given by the following Theorem:

Theorem 1.3.1

Let $\tilde{S}_t$ be an shadow price like in Definition 1.3.1., as well as $\hat{\psi}$ be the trading strategy that maximizes

$$E[U(x + (\psi \cdot \tilde{S})_T)] \to \max$$

Furthermore be $U : (0, \infty) \mapsto \mathbb{R}$ a utility function that satisfies the above conditions, $(x, 0)$ the initial endowment and λ the transaction cost with $0 \leq \lambda < 1$.

Then the optimal trading strategy $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t=1}^T$ *for the market* S *under transaction cost can be obtained via the following identification*

$$\hat{\varphi}_t^1 = \hat{\psi}_t^1 \qquad t = 1, \ldots, T$$

$$\hat{\varphi}_t^0 = x + (\hat{\psi}_t \cdot \tilde{S})_t - \hat{\psi}_t^1 \tilde{S}_t \qquad t = 1, \ldots, T$$

We will only have a look at a rather economical proof, which may not be 100% accurate in a mathematical sense, but shows pretty well how the idea behind this works. Besides that, we will have a look at the connection between optimal strategy and shadow price through the next chapters as well.

Proof: The proof is reduced to a rather simple but really useful observation: trading in the frictionless market with process $\tilde{S}$ will always lead to a better or equal result, when it comes to utility maximization, than trading under transaction costs. In a much more mathematical way this can be explained via the following inclusion

$$\mathcal{C}^\lambda \subseteq \mathcal{C}^{\tilde{S}}$$

Where $\mathcal{C}^\lambda$ is the set of claims, that can be attained from initial endowment $(0, 0)$ and $\mathcal{C}^{\tilde{S}}$ denotes the cone of random variables φ_T^0 that can be replicated in a frictionless financial $\tilde{S}$ with no initial endowment. This inclusion can be derived from the Fundamental Theorem of Asset Pricing, in the version with transaction costs, the idea goes originally back to [1].
So we therefore can transform the frictionless trading strategy $\hat{\psi}$ into a trading strategy $\hat{\varphi}$ for S. ■

Further information about this theorem can be found in the lecture note of Schachermayer [6] (when this thesis was created this paper was not yet published). This theorem will help us find an optimal strategy in a market under transaction costs, because as soon as we have find a shadow price, we can easily find an optimizer for the frictionless market (see for example [7]), where the shadow price is settled in. And with this strategy we can calculate the strategy for the market with frictions.

The problem that we are facing now is that a shadow price usually only exists in rather simple models. This we will show in the upcoming chapters of this thesis.

Chapter 2

Existence in Finite Discrete Time and Discrete Finite Probability Space

2.1 Model

In this section we will follow the paper of Kallsen and Muhle-Karbe [2] and just adapt it so the connection between this and the next chapter will be easier to see. We will now have a look at the finite discrete time case, with finite probability space and give a proof that in this simple and nice setting shadow prices always exist (except for the trivial case where there is no possible solution for the optimization problem).

We will use the following definitions from the previous section:

- the Market model S, that is a bond process S_t^0 and a stock process S_t^1 (the theorem presented in these chapter can also be done for d risky assets, see [2])
- the trading strategy process $\varphi_t = (\varphi_t^0, \varphi_t^1)_{t=0}^{T+1}$, which is predictable, and therefore $\mathcal{F}_{t-1}$-measurable
- the bid/ask-spread $[\underline{S}, \overline{S}]$ with $\underline{S}_{t=0}^T = (1-\lambda)S_{t=0}^T$ and $\overline{S}_{t=0}^T = S_{t=0}^T$
- and the definition of the shadow price process $\tilde{S}$

We are working in the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where both $\Omega = \{\omega_1, \ldots, \omega_K\}$ and $\{0, 1, \ldots, T\}$ are **finite**. We also assume (to make notations simpler) that $\mathbb{P}(\{\omega_k\}) > 0$ for all $k \in \{1, \ldots, K\}$ and that $\mathcal{F}$ is the filtration generated by Ω, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

We also assume the presence of a **consumption process** c_t, this is a (discounted) adapted, $\mathbb{R}$-valued process that represents the consumed amount at time $t = 0, \ldots, T$. The pair $(\varphi_t, c_t) = ((\varphi_t^0, \varphi_t^1), c_t)$ is called a **portfolio/consumption pair**.

Definition 2.1.1 (*self-financing strategy*)

A portfolio/consumption pair $(\varphi, c_t) = ((\varphi_t^0, \varphi_t^1), c_t)$ is called self-financing if for all t

$$\Delta\varphi_{t+1}^0 = \underline{S}_t \Delta\varphi_{t+1}^{\downarrow} - \overline{S}_t \Delta\varphi_{t+1}^{\uparrow} - c_t$$

where the cumulated purchases are defined as

$$\Delta\varphi_{t+1}^{\downarrow} := \varphi_{t+1}^- - \varphi_t^- := (\max(-\varphi_{t+1}^0, 0), \max(-\varphi_{t+1}^1, 0)) - (\max(-\varphi_t^0, 0), \max(-\varphi_t^1, 0))$$

and

$$\Delta\varphi_{t+1}^{\uparrow} := \varphi_{t+1}^+ - \varphi_t^+ := (\max(\varphi_{t+1}^0, 0), \max(\varphi_{t+1}^1, 0)) - (\max(\varphi_t^0, 0), \max(\varphi_t^1, 0))$$

For a better comparability we will just use a very special case, where $c_1 = \ldots = c_{T-1} = 0$ (the proofs we will use carry on with small changes, to the case where this assumption isn't present). We also need an additional concept for our main result that we mentioned in the introducing chapter, but that has to be defined as well

Definition 2.1.2 (*admissible and optimal*)

A self-financing portfolio/consumption pair (φ_t, c_t) is called ***admissible*** *if $\varphi_{-1} = (\varphi_0^0, \varphi_0^1) = (x, 0)$ and $(\varphi_{T+1}^0, \varphi_{T+1}^1) = (0, 0)$. We will denote the set of all admissible strategies starting from initial endowment $(x, 0)$ with $\mathcal{A}(x)$.*

An admissible portfolio/consumption pair (φ_t, c_t) is called ***optimal*** *if it maximizes*

$$E\left[U_T(c_T)\right] \to \max$$

The function is maximized over all admissible portfolio/consumption pairs $((\varphi_t^0, \varphi_t^1), c_t) \in \mathcal{A}(x)$. We will denote the optimal portfolio/consumption with $(\hat{\varphi}, \hat{c}_t) = ((\hat{\varphi}_t^0, \hat{\varphi}_t^1), \hat{c}_t)$

For $U : \Omega \times \{0, \ldots, T\} \times \mathbb{R} \mapsto [-\infty, \infty)$ the utility function, we have:

- *$(\omega, t) \mapsto U_t(\omega, x)$ is predictable for any $x \in \mathbb{R}$ and for any $(\omega, t) \in \Omega \times \{0, \ldots, T\}$*
- *$x \mapsto U_t(\omega, x)$ is a proper, upper-semicontinuous, concave function, which is strictly increasing on $\{x \in \mathbb{R} : U_t(\omega, x) > -\infty\}$*

2.2 Existence Theorem

We will now derive the main theorem for the existence of a shadow price in discrete time and a finite discrete probability space.

Theorem 2.2.1 (*Existence in discrete time*)

Let $(\hat{\varphi}_t, c_t) = ((\hat{\varphi}^0_t, \hat{\varphi}^1_t), \hat{c}_t)$ be an optimal portfolio/consumption pair for a market with bid process $\underline{S}_t$ and ask process $\overline{S}_t$. If $E[U_T(c_T)] > -\infty$, then a shadow price $\tilde{S}$ for the frictionless market exists.

Proof: Due to the fact, that U_t is strictly increasing (by definition of a utility function), we can assume the following property: even if we buy and sell at the same time, it won't change the maximal expected utility. To be more precise:

Since U_t is strictly increasing, maximizing

$$E\left[U_T(c_T)\right] \rightarrow \max$$

over all admissible portfolio/consumption pairs $((\varphi^0, \varphi^1), c_t) \in \mathcal{A}(x)$ will lead to the same result (meaning maximal expected utility) as maximizing over all strategies where we split up the "buying stocks" part and the "selling stocks" part of φ^1. In other words we track down separately how many shares we sell at time t, called $\varphi^{1,\uparrow}$ and how much we buy $\varphi^{1,\downarrow}$. So we will therefore maximize over $((\varphi^0, \varphi^{1,\uparrow}, \varphi^{1,\downarrow}), c_t)$.

Here the process that should model the holdings in bond $(\varphi^0)_t$ is an $\mathbb{R}$-valued process, which is predictable, with $\varphi^0_0 = x$ (starting with initial endowment x) and also $\varphi^0_{T+1} = 0$.

On the other hand both processes $(\varphi^{1,\downarrow})_t$ and $(\varphi^{1,\uparrow})_t$ for $t = 0, \ldots, T+1$ are increasing by definition, since they track down how many positions have been bought or sold so far and are also $\mathbb{R}$-valued predictable processes, that furthermore satisfy

$$\varphi^{1,\uparrow}_0 = 0 \qquad \varphi^{1,\downarrow}_0 = 0 \qquad \varphi^{1,\uparrow}_{T+1} - \varphi^{1,\downarrow}_{T+1} = 0$$

starting at 0 and fulfil at the end, that the all over bought and sold stocks are equivalent.

Then finally let c_t be a consumption process such that

$$\Delta\varphi^0_{t+1} = \underline{S}_t \Delta\varphi^{1,\downarrow}_{t+1} - \overline{S}_t \Delta\varphi^{1,\uparrow}_{t+1} - c_t$$

holds true for all $t = 0, \ldots, T$. Remember that we assume that $c_1 = \ldots = c_{T-1} = 0$.

We can therefore set the strategies that represent how much was bought (or sold) until time t as:

$$\varphi_t^{1,\uparrow} := \sum_t \Delta\varphi_t^{1,\uparrow} = \sum_t (\max(\varphi_{t+1}^0, 0), \max(\varphi_{t+1}^1, 0)) - (\max(\varphi_t^0, 0), \max(\varphi_t^1, 0))$$

$$\varphi_t^{1,\downarrow} := \sum_t \Delta\varphi_t^{1,\downarrow} = \sum_t \left(\max(-\varphi_{t+1}^0, 0), \max(-\varphi_{t+1}^1, 0)) - (\max(-\varphi_t^0, 0), \max(-\varphi_t^1, 0)\right)$$

that completes the connection between the processes and so if $(\hat{\varphi}_t, \hat{c}_t)$ is an optimal portfolio consumption pair, this split up strategy $((\hat{\varphi}^0, \hat{\varphi}^{1,\uparrow}, \hat{\varphi}^{1,\downarrow}), c_t)$ is optimal as well.

Now let $F_t^1, \ldots, F_t^{m_t}$ be the partition of Ω that generates the filtration $\mathcal{F}_t$ for $t \in \{0, \ldots, T\}$. We observe once again that the processes $(\varphi^0, \varphi^{1,\uparrow}, \varphi^{1,\downarrow})$ satisfy the following conditions:

- $(\varphi^0)_t$ is an $\mathbb{R}$-valued process, which is predictable
- both processes $(\varphi^{1,\downarrow})_t$ and $(\varphi^{1,\uparrow})_t$ are increasing, and $\mathbb{R}$-valued predictable processes as well
- the self-financing property is satisfied

Now since a mapping is $\mathcal{F}_t$-measurable if and only if it is constant on the sets F_t^j for $j = 1, \ldots, m_t$ of the partition, we can identify $((\varphi^{1,\uparrow}, \varphi^{1,\downarrow}), c_t)$ with

$$\mathbb{R}_+^{2n} \times \mathbb{R}^n := (\mathbb{R}_+^1 \times \ldots \times \mathbb{R}_+^{m_T}) \times (\mathbb{R}_+^1 \times \ldots \times \mathbb{R}_+^{m_T}) \times (\mathbb{R}^1 \times \ldots \times \mathbb{R}^{m_T})$$

So the other way round: it holds also true that we can identify $(\Delta\varphi^{1,\uparrow}, \Delta\varphi^{1,\downarrow}, c)$ as

$$(\Delta\varphi_t^{1,\uparrow}, \Delta\varphi_t^{1,\downarrow}, c_t) := (\Delta\varphi_1^{1,\uparrow,1}, \ldots, \Delta\varphi_{T+1}^{1,\uparrow,m_T}, \Delta\varphi_1^{1,\downarrow,1}, \ldots, \Delta\varphi_{T+1}^{1,\downarrow,m_T}, c_0^1, \ldots, c_T^{m_T})$$

where we shortened notation $\Delta\varphi_t^{1,\uparrow,j} := \Delta\varphi_t^{1,\uparrow,j}(\omega)$ with $t = 1, \ldots, T$, $j = 1, \ldots, m_T$ and $\omega \in F_t^j$. Same goes for $\Delta\varphi_t^{1,\downarrow,j}, c_t, \underline{S}_t, \overline{S}_t$.

With this identification, we can reformulate our problems in terms of dynamic programming, with the following objective function

$$f((\Delta\varphi_t^{1,\uparrow}, \Delta\varphi_t^{1,\downarrow}), c_t) := -E\left[U_T(c_T)\right]$$

which is defined on $\mathbb{R}_+^{2n} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and the constraints for $j = 1, \ldots, m_T$

$$h_0^j((\Delta\varphi_t^{1,\uparrow},\Delta\varphi_t^{1,\downarrow}),c_t) := x + \sum_{t=1}^{T}\left(\underline{S}_{t-1}^j\Delta\varphi_t^{1,\downarrow,j} - \overline{S}_{t-1}^j\Delta\varphi_t^{1,\uparrow,j}\right) - \sum_{t=0}^{T} c_t$$

$$h^j)(\Delta\varphi_t^{1,\uparrow},\Delta\varphi_t^{1,\downarrow}),c_t) := \sum_{t=1}^{T+1}\left(\Delta\varphi_t^{1,\uparrow,j} - \Delta\varphi_t^{1,\downarrow,j}\right)$$

where both are defined on $\mathbb{R}_+^{2n}\times\mathbb{R}^n\to\mathbb{R}$

We now can reformulate our problem of finding an optimal trading strategy $((\Delta\hat{\varphi}^{1,\uparrow},\Delta\hat{\varphi}^{1,\downarrow}),c)$ by: minimizing the function f over $\mathbb{R}_+^{2n}\times\mathbb{R}$, so

$$-E\left(U_T(c_T)\right)\to\min$$

with the constraints

$$h_0^j((\Delta\varphi_t^{1,\uparrow},\Delta\varphi_t^{1,\downarrow}),c_t) = 0 \quad \forall j=1,\dots,m_T, t=0,\dots,T$$

$$h^j((\Delta\varphi_t^{1,\uparrow},\Delta\varphi_t^{1,\downarrow}),c_t) = 0 \quad \forall j=1,\dots,m_T, t=0,\dots,T$$

All of these function are convex functions on $\mathbb{R}_+^{2n}$. So we will use methods from convex analysis, for which we need some additional constraints $g_t^{\uparrow,j}$ and $g_t^{\downarrow,j}$, both convex mappings from $\mathbb{R}^{2n}\to\mathbb{R}$

$$g_t^{\uparrow,j}((\Delta\varphi_t^{\uparrow},\Delta\varphi_t^{\downarrow}),c_t) := -\Delta\varphi_{t+1}^{\uparrow,j} \quad \text{for } \ t=0,\dots,T \ \text{ and } \ j=1,\dots,m_t$$

$$g_t^{\downarrow,j}((\Delta\varphi_t^{\uparrow},\Delta\varphi_t^{\downarrow}),c_t) := -\Delta\varphi_{t+1}^{\downarrow,j} \quad \text{for } \ t=0,\dots,T \ \text{ and } \ j=1,\dots,m_t$$

and therefore minimizing f under the above constraints, is equivalent to: the optimizer $((\Delta\hat{\varphi}_t^{\uparrow},\Delta\hat{\varphi}_t^{\downarrow}),c_t)$ minimizes f with $h_0^j=0$, $h^j=0$ (for $j=1,\dots,m_T$) and $g_t^{\downarrow,j},g_t^{\uparrow,j}\le 0$ (for $t=0,\dots,T$ and $j=1,\dots,m_t$).

We now need to get some information about this convex optimization problems, so for the next step we need two theorems from [8]

Theorem 2.2.2 (*Theorem 28.2. in [8]*)

Let (P) be an ordinary convex program, and let I be the set of indices $i\neq 0$ such that f_i is not affine. Assume that the optimal value in (P) is not $-\infty$ and that (P) has at least one feasible solution in the relative interior of C which satisfies (with strict inequality) all the inequality constraints for $i\in I$. Then a Kuhn-Tucker vector exits for (P).

Proof and further definitions for this and the next theorem can be found in the book of Rockafellar, chapter 28 [8]. For our purpose it's enough to know that the requirement for this theorem is fulfilled with our above defined program. We also need

Theorem 2.2.3 (*Theorem 28.3. in [8]*)

Let (P) be an ordinary convex program and let $\overline{u}^* \in \mathbb{R}^m$ and $\overline{x} \in \mathbb{R}^n$ be vectors. Then $\overline{u}^*$ is a Kuhn-Tucker vector for (P) and $\overline{x}$ is an optimal solution for (P) if and only if $(\overline{u}^*, \overline{x})$ is a saddle point of the Lagrangian L of (P). Moreover, this condition holds if and only if $\overline{x}$ and the components λ_i of $\overline{u}^*$ satisfy

1. $\lambda_i \geq 0, f_i(\overline{x}) \leq 0$ *and* $\lambda_i f_i(\overline{x}) = 0$ *for* $i = 1, \ldots, r$
2. $f_i(\overline{x}) = 0$ *for* $i = r+1, \ldots, m$
3. $0 \in [\partial f_0(\overline{x}) + \lambda_1 \partial f_1(\overline{x}) + \ldots + \lambda_m \partial f_m(\overline{x})]$

Using this two theorems in our setting we get: $((\Delta\hat{\varphi}_t^{\uparrow}, \Delta\hat{\varphi}_t^{\downarrow}), \hat{c}_t)$ is optimal if and only if there exists a set of Lagrange multipliers. These are real numbers ν^j, μ^j (for $j = 1, \ldots, m_T$) and $\lambda_t^{\uparrow,j}, \lambda_t^{\downarrow,j}$ (for $t = 0, \ldots, T$ and $j = 1, \ldots, m_t$) such that the following points hold true:

1. For $t = 0, \ldots, T$ and $j = 1, \ldots, m_t$, the following equations are valid
   - $\lambda_t^{\uparrow,j}, \lambda_t^{\downarrow,j} \geq 0$
   - $g_t^{1,\uparrow,j}((\Delta\hat{\varphi}_t{}^{\uparrow}, \Delta\hat{\varphi}_t{}^{\downarrow}), \hat{c}_t) \leq 0$
   - $g_t^{1,\downarrow,j}((\Delta\hat{\varphi}_t{}^{\uparrow}, \Delta\hat{\varphi}_t{}^{\downarrow}), \hat{c}_t) \leq 0$
   - $\lambda_t^{\uparrow,j} g_t^{1,\uparrow,j}((\Delta\hat{\varphi}_t{}^{\uparrow}, \Delta\hat{\varphi}_t{}^{\downarrow}), \hat{c}_t) = 0$
   - $\lambda_t^{\downarrow,j} g_t^{1,\downarrow,j}((\Delta\hat{\varphi}_t{}^{\uparrow}, \Delta\hat{\varphi}_t{}^{\downarrow}), \hat{c}_t) = 0$
2. $h_0^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) = 0$ and $h^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) = 0$ for $j = 1, \ldots, m_T$
3. Let ∂ denote the subdifferential of a convex mapping, then 0 is located in:

$$0 \in \partial f((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{j=1}^{m_T} \nu^j \partial h_0^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{j=1}^{m_T} \mu^j \partial h^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) +$$

$$+ \sum_{t=0}^{T} \sum_{j=1}^{m_t} \lambda_t^{\uparrow,j} \partial g_t^{\uparrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{t=0}^{T} \sum_{j=1}^{m_t} \lambda_t^{\downarrow,j} \partial g_t^{\downarrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t)$$

By the definition of the functions we did above, the first two points are already fulfilled, so we just need to show point 3. For that we need a Theorem from [9]

Theorem 2.2.4 (*Theorem 10.5 in [9]*)

Let $f(x) = f_1(x_1) + \ldots + f_m(x_m)$ for some lower semicontinuous functions $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$, where $x \in \mathbb{R}^n$ is expressed as $(x_1, \ldots, x_m)$ with $x \in \mathbb{R}^{n_i}$, than at any $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_m)$ with $f(\overline{x}) < \infty$ and $df_i(\overline{x}_i)(0) = 0$ we have

$$\partial f(\overline{x}) = \partial f_1(\overline{x}_1) \times \ldots \times \partial f_m(\overline{x}_m)$$

So we can now split up point 3 from above into many smaller (but similar) statements, by replacing the subdifferentials with partial subdifferentials, referring to $\Delta\hat{\varphi}_1^{1,\uparrow,1},\dots,\Delta\hat{\varphi}_{T+1}^{1,\uparrow,m_T}$, $\Delta\hat{\varphi}_1^{1,\downarrow,1},\dots,\Delta\hat{\varphi}_{T+1}^{1,\downarrow,m_T}$ and $c_t^1,\dots,c_t^{m_T}$. With that we get for c_T^j (with $j=1,\dots,m_T$)

$$0 \in \partial_{c_T^j} f((\Delta\hat{\varphi}_t^{1,\uparrow},\Delta\hat{\varphi}_t^{1,\downarrow}),c_t) - \nu^j$$

here $\partial_{c_T^j}$ denotes the partial subdifferential of a convex function relative to the vector c_T^j. Because we defined f as $-E[U_T(c_T)]$, it is strictly decreasing in c_T^j and therefore must $\nu^j < 0$ for $j=1,\dots,m_T$ such that the above equation holds true.

Again we will need a Theorem from the Book of Rockafellar [8]

Theorem 2.2.5 (*Theorem 25.1 from [8]*)

Let f be a convex function, and let x be a point where $f(x)<\infty$. If f is differentiable at x then $\nabla f(x)$ is the unique subgradient of f at x, so that in particular

$$f(z) \geq f(x) + \langle \nabla f(x), z-x\rangle \qquad \forall z \in dom(f)$$

Conversely, if f has a unique subgradient at x, then f is differentiable at x

So since $g_t^{\uparrow,j}, g_t^{\downarrow,j}$ (for $t=0,\dots,T$ and $j=1,\dots,m_t$) and h_0^j, h^j (for $j=1,\dots,m_T$) are all differentiable, their partial subdifferential are equal to their partial derivatives.

If we take point 3 from above and take the partial derivatives with respect to $\Delta\hat{\varphi}_{t+1}^{\uparrow,j}$ and $\Delta\hat{\varphi}_{t+1}^{\downarrow,j}$ (again for $t=0,\dots,T$ and $j=(1,\dots,m_t)$), we can follow that

$$0 = \sum_{k:\omega_k\in F_t^j} \mu^j - \left(\sum_{k:\omega_k\in F_t^j} \nu^k\right) \overline{S}_t^j - \lambda_t^{\uparrow,j} =$$

$$= \sum_{k:\omega_k\in F_t^j} \mu^j - \left(\sum_{k:\omega_k\in F_t^j} \nu^k\right)\left(1+\frac{\lambda_t^{\uparrow,j}}{\overline{S}_t^j\sum_{k:\omega_k\in F_t^j}\nu^k}\right)\overline{S}_t^j$$

and as well (with the same calculation):

$$0 = \sum_{k:\omega_k\in F_t^j} \mu^j - \left(\sum_{k:\omega_k\in F_t^j} \nu^k\right)\left(1-\frac{\lambda_t^{\downarrow,j}}{\underline{S}_t^j\sum_{k:\omega_k\in F_t^j}\nu^k}\right)\underline{S}_t^j$$

if we combine these 2 equations we get, for $t=0,\dots,T$ and $j=1,\dots,m_t$

$$\tilde{S}_t^j := \left(1+\frac{\lambda_t^{\uparrow,j}}{\overline{S}_t^j\sum_{k:\omega_k\in F_t^j}\nu^k}\right)\overline{S}_t^j = \left(1-\frac{\lambda_t^{\downarrow,j}}{\underline{S}_t^j\sum_{k:\omega_k\in F_t^j}\nu^k}\right)\underline{S}_t^j$$

By definition this process $\tilde{S}$ is constant on F_t^j, and furthermore it is an adapted process. Because $\nu_k < 0$ for $k = 1, \ldots, m_T$ and by statement 1 from above we can deduce that $\underline{S} \leq \tilde{S} \leq \overline{S}$. Another thing we get from statement 1 is that the process $\tilde{S}$ satisfies the boundary conditions we demand from a shadow price

$$\tilde{S} = \overline{S} \text{ on } \left\{\Delta\hat{\varphi}^{\uparrow} < 0\right\} \quad \text{and} \quad \tilde{S} = \underline{S} \text{ on } \left\{\Delta\hat{\varphi}^{\downarrow} < 0\right\}$$

We now define Lagrange multiplier for our shadow price process:

- $\tilde{\mu}^j := \mu^j$ (for $j = 1, \ldots, m_T$)
- $\tilde{\nu}^j := \nu^j$ (also for $j = 1, \ldots, m_T$)
- since we are dealing with the minimized process $\tilde{\lambda}^{\uparrow,j}, \tilde{\lambda}^{\downarrow,j} := 0$ (for $t = 0, \ldots, T$ and $j = 0, \ldots, m_t$)

We now have to redefine our functions $\tilde{f}, \tilde{h}_0^j, \tilde{h}^j, \tilde{g}_t^{\uparrow,j}$ and $\tilde{g}_t^{\downarrow,j}$ for the dynamic programming for the shadow price by setting $\overline{S} = \underline{S} = \tilde{S}$

$$f((\Delta\varphi_t^{1,\uparrow}, \Delta\varphi_t^{1,\downarrow}), c_t) := -E\left[U_T(c_T)\right]$$

$$\tilde{h}_0^j((\Delta\varphi_t^{1,\uparrow}, \Delta\varphi_t^{1,\downarrow}), c_t) := x + \sum_{t=1}^{T} \left(\tilde{S}_{t-1}^j \Delta\varphi_t^{1,\downarrow,j} - \tilde{S}_{t-1}^j \Delta\varphi_t^{1,\uparrow,j}\right) - c_T$$

$$\tilde{h}^j((\Delta\varphi_t^{1,\uparrow}, \Delta\varphi_t^{1,\downarrow}), c_t) := \sum_{t=1}^{T+1} \left(\Delta\varphi_t^{1,\uparrow,j} - \Delta\varphi_t^{1,\downarrow,j}\right)$$

$$\tilde{g}_t^{\uparrow,j}((\Delta\varphi_t^{\uparrow}, \Delta\varphi_t^{\downarrow}), c_t) := -\Delta\varphi_{t+1}^{\uparrow,j}$$

$$\tilde{g}_t^{\downarrow,j}((\Delta\varphi_t^{\uparrow}, \Delta\varphi_t^{\downarrow}), c_t) := -\Delta\varphi_{t+1}^{\downarrow,j}$$

If we now merge this all together we get the following results for the three statements from before

1. For $t = 0, \ldots, T$ and $j = 1, \ldots, m_t$, the following equations are valid
   - $\tilde{\lambda}_t^{\uparrow,j}, \tilde{\lambda}_t^{\downarrow,j} \geq 0$
   - $\tilde{g}_t^{\uparrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) \leq 0$
   - $\tilde{g}_t^{\downarrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) \leq 0$
   - $\tilde{\lambda}_t^{\uparrow,j} \tilde{g}_t^{\uparrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) = 0$
   - $\tilde{\lambda}_t^{\downarrow,j} \tilde{g}_t^{\downarrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) = 0$
2. $\tilde{h}_0^j(\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}, \hat{c}_t) = 0$ and $\tilde{h}^j(\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}, \hat{c}_t) = 0$ for $j = 1, \ldots, m_T$

3. With ∂ denote the subdifferential of a convex mapping, 0 is located in:

$$0 \in \partial\tilde{f}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{j=1}^{m_T} \tilde{\nu}^j \partial\tilde{h}0^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{j=1}^{m_T} \tilde{\mu}^j \partial\tilde{h}^j((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) +$$

$$+ \sum_{t=0}^{T}\sum_{j=1}^{m_T} \tilde{\lambda}_t^{\uparrow,j} \partial\tilde{g}_t^{\uparrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t) + \sum_{t=0}^{T}\sum_{j=1}^{m_T} \tilde{\lambda}_t^{\downarrow,j} \partial\tilde{g}_t^{\downarrow,j}((\Delta\hat{\varphi}_t{}^{1,\uparrow}, \Delta\hat{\varphi}_t{}^{1,\downarrow}), \hat{c}_t)$$

And with that we can again follow from the above theorem 2.2.2. that statement 1 and 2 therefore translate into: $(\hat{\varphi}, \hat{c}_t)$ is an optimal trading strategy for the market with bid/ask spread $[\underline{S}, \overline{S}]$ and also for the market with "spread" $[\tilde{S}, \tilde{S}]$, so nothing else but the frictionless case. If we take this $\tilde{S}$ for our shadow price, we are finished. ■

Chapter 3

Existence in Finite Discrete Time

3.1 Model

In this section we will derive our main approach, the qualitative information whether or not a shadow price exists in a model with discrete time, but in contrary to the last chapter, with an infinite probability space. Following the paper from Czichowsky, Muhle-Karbe und Schachermayer [3], we will first have a look at a rather simple counter-example and then give a proof for the general case.

Again we will use the following definitions from the first section:

- Market model, that is a bond process S_t^0 and a stock process S_t^1
- trading strategy $\varphi_{t=0}^{T+1} = (\varphi_t^0, \varphi_t^1)$, for the market with transaction costs (predictable, $\mathbb{R}^2$ valued process), when we deal with the frictionless case we will denote the strategy with $\psi_{t=0}^{T+1} = (\psi_t^0, \psi_t^1)$.
- the bid/ask-spread $[\underline{S}_t, \overline{S}]$ with bid-price process, $\underline{S}_{t=0}^T = (1-\lambda) S_{t=0}^T$ and the ask-price process $\overline{S}_{t=0}^T = S_{t=0}^T$
- and the shadow price process $\tilde{S}$

Our price process is again on one hand strictly positive and on the other adapted to the probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t=0}^T, \mathbb{P})$ where we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial and $T \in \mathbb{N}$ is a fixed time horizon.

Since we are now working without a consumption process we have to adapt the definition of self-financing:

Definition 3.1.1 (*self-financing strategy*)

We call a strategy φ_t self-financing, if the following inequality holds true

$$\Delta\varphi^0_{t+1} \leq -(\Delta\varphi^1_{t+1})^+\overline{S}_t + (\Delta\varphi^1_{t+1})^-\underline{S}_t$$

where $(\Delta\varphi^1_{t+1})^+ := (\max(\varphi^1_{t+1}, 0) - (\max(\varphi^1_t, 0))$ *and* $(\Delta\varphi^1_{t+1})^- := (\max(-\varphi^1_{t+1}, 0) - (\max(-\varphi^1_t, 0))$. *We will denote the set of all self financing strategies as* $\mathcal{A}$.

We also assume that a self-financing strategy starts from initial endowment $(x, 0)$ and that we liquidate our holdings in stocks at the terminal time T. Equally we can say that we start at $\varphi_0 = (\varphi^0_0, \varphi^1_0) = (x, 0)$ and $\varphi^0_{T+1} \geq 0$ and $\varphi^1_{T+1} = 0$, we will denote the set of all self-financing strategies that start at initial endowment x by $\mathcal{A}(x)$. If we are working in the frictionless market with shadow price $\tilde{S}$ we will denote the set with $\mathcal{A}(x, \tilde{S})$.

Our main goal is again, to maximize the expected utility, where we model the utility via the utility function $U : (0, \infty) \to \mathbb{R}$, that satisfies the Inada conditions, mentioned in the first chapter. So the aim is

$$E[U(g)] \to \max$$

where g runs through $\mathcal{C}(x) = \left\{ \varphi^0_{T+1} \in L^0_x(P) \middle| \exists(\varphi^0, \varphi^1) \in \mathcal{A}(x) \right\}$. These are the non-negative cash positions, where a self-financing trading strategy from initial endowment x exists. Again when working in the frictionless market with shadow price $\tilde{S}$ we will denote the set with $\mathcal{C}(x, \tilde{S})$.

Definition 3.1.2 (*shadow price*)

An adapted process $(\tilde{S})^T_{t=0}$ *is called a shadow price if it takes values only within the bid/ask spread, so* $\underline{S} \leq \tilde{S} \leq \overline{S}$ *and if there is a solution* $(\hat{\psi}^0, \hat{\psi}^1)$ *to the associated frictionless markets utility maximization problem*

$$E[U(\psi^0_{T+1})] = E[U(x + (\psi^1_{T+1} \cdot \tilde{S}_T)) \to \max \quad \text{with} \quad (\psi^0, \psi^1) \in \mathcal{A}(x, \tilde{S})$$

where $\mathcal{A}(x, \tilde{S})$ *is the set of all self-financing trading strategies for the frictionless market starting from initial endowment* $(x, 0)$, *that only trades when its value reaches either the bid or the ask price*

$$\left\{\Delta\psi^1_{t+1} > 0\right\} \subseteq \left\{\tilde{S}_t = \overline{S}_t\right\} \quad \text{and} \quad \left\{\Delta\psi^1_{t+1} < 0\right\} \subseteq \left\{\tilde{S}_t = \underline{S}_t\right\}$$

To make the idea behind this definition clear we will again give a more or less economic interpretation. Since the shadow price $\tilde{S}$ always lies within the bid ask spread $\underline{S} \leq \tilde{S} \leq \overline{S}$, trading at the frictionless case is always better (or at least exactly as favourable as in the case with transaction costs). Because when we want to buy at the shadow price $\tilde{S}$ this is always cheaper than buying at $\overline{S} \geq \tilde{S}$ and on the other side selling the asset at price $\tilde{S}$ is again better than at the bid price $\underline{S} \leq \tilde{S}$.

And this implies first of all, that as long as our shadow price takes values between $\underline{S}$ and $\overline{S}$ we don't have to buy or sell anything, because this will always lead to a situation where the utility is not maximal any more, because we would have bought (or sold) at a higher/lower price as we could have achieved. So as long as we just sell or buy when the shadow price hits the bid/ask spread boarder, the optimizer for $\tilde{S}$ coincides with the one for the market with transaction costs.

On the other hand, our utility maximization problem in the market with transaction costs can always be dominated by a maximization problem in the frictionless case (so with shadow price $\tilde{S}$). So in other words:

$$u := \sup_{\varphi_{T+1} \in \mathcal{C}(x)} E[U(\varphi^0_{T+1})] \leq \inf_{\tilde{S} \in [\underline{S}, \overline{S}]} \sup_{\psi_{T+1} \in \mathcal{C}(x, \tilde{S})} E[U(\psi^0_{T+1})]$$

$$\inf_{\tilde{S} \in [\underline{S}, \overline{S}]} u(x, \tilde{S}) := \inf_{\tilde{S} \in [\underline{S}, \overline{S}]} \sup_{\varphi_{T+1} \in \mathcal{C}(x, \tilde{S})} E[U(\psi^0_{T+1})]$$

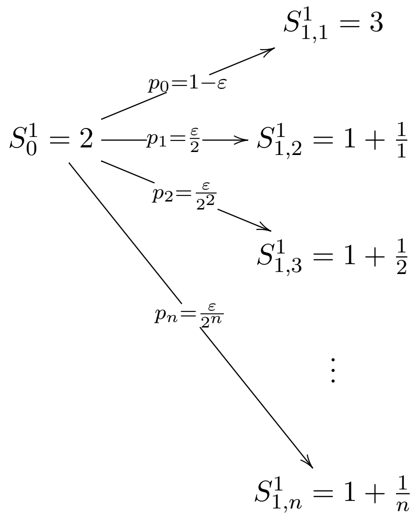
3.2 Counter- Example

We will now have a look a at really simple example, and even here a shadow price fails to exist. We will first have a look at it from a rather "non-mathematic" view and then give the exact explanation. In the next subsection we will then prove the general result.

This example was originally published by Kramkov and Schachermayer [10]. In the original example the approach was a bit less economic and also different in some ways but led to the same result (or strictly speaking to an even stronger result): actually under less conditions (even one time period and just a countable probability set is already enough) someone can construct a counter example to the existence of a shadow price. As we will see in a moment the major problem will be, that in the present setting a process that would fulfill the other properties of a shadow price is usually no martingale (not even a local martingale, but a supermartingale).

Let our model consist of one stock and a bank account. Our investor uses a simple utility function, namely $U(x) := \log(x)$. Our model consists of two time periods (and the starting time 0), so $t = \{0, 1, 2\}$. Also our investor will start with initial endowment

$\varphi_0 = (1, 0)$ and the stock price starts at $S_0 = 2$. In the first step our stock price S_0 evolves in the following way:



Here $S_{1,j}$ denotes the value of the stock at time $t = 1$ and the case $\omega_j \in \mathcal{F}_1$ turned out to be true. As we can see here, there are countably many results how our stock price can evolve within the range of [1, 3]. We will first take just a look at this first time period, and since the investor has to choose his position in stock or bond at this moment we have no problems doing it the same way.

Furthermore if ε is chosen small enough, it is pretty sure that the stock will rise in value rather than fall. And the smaller the stock price would get the more unlikely is this event to happen. It is important to recognize, that the probability converges "faster" to 0, than the stock price S converges to 1.

In this setting even a risk averse investor would buy as much of the stock as possible. Since the trading strategy should be self-financing (and we do not allow bankruptcy), the investor is only limited by his initial endowment $x = 1$. Meaning that the amount of holdings in Stock $\Delta\varphi^1_1$ at time $t = 1$ must satisfy

$$\Delta\varphi^1_1 \overline{S}^1_0 - \Delta\varphi^1_1 \underline{S}^1_1 \leq 1$$

for all possible values of $S^1_{1,j}$. Since we could (and should) choose $\Delta\varphi^1_1$ in a way, so that the left hand side gets as close to 1 as possible (because we assume the stock to rise in value and holding as much positions in stocks will optimize our utility) and the upper bound for $\Delta\varphi^1_1$ is $\frac{1}{1+\lambda}$. If we now let ε get really small (we will make this more precise later on), so that the possibility of S^1_1 falling nearly vanishes, then this value is not just the limit but indeed the actual value for $\Delta\varphi^1_1$ which we should choose.

Now we will look at the second time period, here the stock price from the first period evolves in the following way,

$$\begin{tikzcd}[column sep=small] & S_2^1 = \frac{3}{1-\lambda} \\ S_{1,n}^1 = 1 + \frac{1}{n} \arrow[ur, "p_n = 1-\varepsilon_n"] \arrow[dr, "1-p_n=\varepsilon_n"'] & \\ & S_2^1 = 1 + \frac{1}{n+1} \end{tikzcd}$$

with one exception, namely if the case $S_{1,1}$ happened at time $t = 1$ then we have

$$\begin{tikzcd}[column sep=small] & S_2^1 = \frac{4}{1-\lambda} \\ S_{1,1}^1 = 3 \arrow[ur, "p_1 = 1-\varepsilon_1"] \arrow[dr, "1-p_1=\varepsilon_1"'] & \\ & S_2^1 = 2 \end{tikzcd}$$

Again we will choose all the ε_j for $j = 1, \dots, n$ small.

So our investor is facing a quite similar situation at time $t = 2$, as before: the stock prices (independent of which particular price it has at time $t = 1$) is more likely to rise than to fall. More precisely with probably the highest possibility (depending on the ε and ε_j) the stock will rise up to $\frac{4}{1-\lambda}$, if the stock rises at both times, or at least rises with high possibility at time $t = 1$ (at all event should $1 - \varepsilon_j \gg \varepsilon_j$ be valid). So if possible the investor will again raise his position in stocks (or at least won't sell them).

Summing up: our investor always will hold a positive amount of stocks that are near (or with the ε chosen small enough even equivalent) the amount that is affordable. After the two time periods he is going to liquidate everything.

But now let us take a look at a possible shadow price $\tilde{S}$. Since the shadow price must always be between the bid- and the ask-price, so $\underline{S} \leq \tilde{S} \leq \overline{S}$, but on the other hand has to coincide with $\underline{S}$ whenever we buy or with $\overline{S}$ whenever we sell stocks, it must look like this:

$$\tilde{S}_0 = \overline{S}_0^1$$

$$\tilde{S}_1 = \overline{S}_1^1$$

$$\tilde{S}_2 = \underline{S}_2^1 = (1 - \lambda) S_2^1$$

This is, because at time $t = 0$ as well as at $t = 1$, the investor is buying stocks (to increase his utility due to the always high probably raising stock price), the shadow price has to coincide with the ask price, and at time $t = 2$ we sell everything so we have to take the bid-price for $\tilde{S}$.

The problem is now, that this process $\tilde{S}$ does not lead to an optimal trading strategy in the frictionless market. This is caused by the transaction costs: since for an investor in the frictionless market it is possible to invest up to his total initial endowment (so up to $\Delta\varphi_1^1 = 1$) where an investor in the frictional market is only able to invest up to $\Delta\varphi_1^1 = \frac{1}{1+\lambda} < 1$. And as long as we choose the probabilities for a downward move in the stock price small enough, this strategy (where $\Delta\varphi$ is strictly larger than the possible strategy in the market with transaction costs) does not lead to an optimal strategy. And since this process $\tilde{S}$ was the only possible candidate for a shadow price, there is no shadow price, even for a model this simple. But now let's have a closer (more mathematical) look at this example.

3.3 Proof of the Counter-Example

The results we got from our example can be summarized in the following theorem, we will already have a short look at the dual problem to our problem here, which we will define this exactly in section 3.4.

Theorem 3.3.1 (*Non-Exsitence of shadow price in 2-period model*)

Let $\lambda \in (0, 1)$ be (fixed) transaction costs. Then there is an arbitrage-free bounded process $(S)_{t=\{0,1,2\}}$ based on a countable probability space $\Omega = \{\omega_{n,1}, \omega_{n,2}\}_{n=0}^{\infty}$ with the following properties:

1. *For the utility maximization problem with transaction costs λ with utility function $U(x) : x \mapsto \log(x)$ and initial endowment $\varphi_0 = (1, 0)$, there is a unique solution $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t=\{0,1,2,3\}}$ for*

$$E[U(\varphi_3^0)] \to \max \quad \textit{with} \quad (\varphi^0, \varphi^1) \in \mathcal{A}(1)$$

where $\mathcal{A}(x)$ is the set of all self-financing trading strategies starting from initial endowment x. Let V be the Legendre transformation to $-U(x)$, so $V(x) := \sup_{x>0}(U(x) - xy)$, in our case this is $V(y) = -\log(y) - 1$ Then we got for the dual problem

$$E[V(Y_2^0)] = E[-\log(Y_2^0) - 1] \to \min \quad \textit{with} \quad (Y^0, Y^1) \in \mathcal{B}(1)$$

for $\hat{y}(x) = 1$ *there is also a unique solution* $\hat{Y}_t = (Y_t^0, Y_t^1)_{t=\{0,1,2\}}$. *Here* $\mathcal{B}(1)$ *is defined as*

$$\mathcal{B}(y) = \{(Y^0, Y^1) \geq 0 \big| Y_0^0 = y, \frac{Y^1}{Y^0} \in [\underline{S}, \overline{S}] \text{ and } Y^0(\varphi^0 + \varphi^1 \frac{Y^1}{Y^0}) = (Y^0\varphi^0 + Y^1\varphi^1)$$

$$\text{is a non-negative supermaritngale for all } (\varphi^0, \varphi^1) \in \mathcal{A}(1)\}$$

2. *There is a (unique) candidate for a shadow price* $(\tilde{S})_{t=\{0,1,2\}} = (\hat{Y}_t^1/\hat{Y}_t^2)_{t=\{0,1,2\}}$ *in the corresponding frictionless (and arbitrage free) market, with* $\underline{S} \leq \tilde{S} \leq \overline{S}$.

3. *For the utility optimization problem*

$$E[\log(\psi_3^0)] = E[\log(1 + \psi_3^1 \cdot \hat{S}_2)] \to \max \quad \text{with} \quad (\psi^0, \psi^1) \in \mathcal{A}(1, \tilde{S})$$

the shadow price (candidate) $\tilde{S}$ *from point 2 leads to a higher solution than the solution for the market with transaction.*

So there exists no shadow price for this model.

To be a bit more precise we will now give some formal definitions of the stock price process S and the used constants. So first of all the probabilities of the up- or downward movements of the stock price are defined with:

$$\mathbb{P}(\{\omega_{0,1}\}) = 1 - \varepsilon$$

$$\mathbb{P}(\{\omega_{0,1}, \omega_{0,2}\}) = (1 - \varepsilon)p_{1,1} := (1 - \varepsilon)(1 - \varepsilon_1)$$

$$p_n := \mathbb{P}(\{\omega_{n,1}\}) = \varepsilon 2^{-n}$$

$$\mathbb{P}(\{\omega_{n,1}, \omega_{n,2}\}) = p_n(1 - \varepsilon_n)$$

here the first index of ω tells us on which "step" of the ladder we are at the moment, and the second describes the time. As we already mentioned $\varepsilon, \varepsilon_{i,1}$ should be "sufficiently small" for now we will take $\varepsilon \in (0, \frac{1}{3})$ and set the other constants as:

$$q_0 \in \left(0, \frac{1 - \lambda}{1 + 3\lambda + 2\lambda^2}\right)$$

$$q_1 \in \left(0, \frac{1 - \lambda}{1 + 4\lambda + 3\lambda^2}\right)$$

and

$$\varepsilon_1 = 1 - \frac{(1+2\lambda)(3+q_0+\lambda+q_0\lambda)}{2(1+\lambda)(2+\lambda)}$$

$$\varepsilon_n = 1 - \frac{(1+n(2+n)\lambda)((2n-1)q_1(1+\lambda)+n^2(2+\lambda))}{n^2(1+n\lambda)(1+2\lambda+n(2+\lambda))}$$

These rather technical looking bonds will became clearer later on.

To solve this maximization problem for our optimal strategy we will use methods from dynamic programming in discrete time.

First of all we reformulate our problem in therms of dynamic programming, so we have a 2-dimensional objective function

$$\mathcal{U}(x,y) := \begin{cases} \log(x), & \text{if } y \geq 0, x > 0 \\ -\infty, & \text{else} \end{cases}$$

and our optimization problem translates into

$$E[\mathcal{U}(\varphi_3^0, \varphi_3^1)] \to \max \quad \text{with} \quad \varphi \in \mathcal{A}(1)$$

and our value function is given by

$$\mathcal{U}_t(x,y) = \operatorname*{ess\,sup}_{(\tilde{\varphi}^0,\tilde{\varphi}^1)\in\mathcal{A}(1)} E[\mathcal{U}_t(x+(\tilde{\varphi}_3^0-\tilde{\varphi}_t^0), y+(\tilde{\varphi}_3^1-\tilde{\varphi}_t^1))\big|\mathcal{F}_t]$$

where $(\tilde{\varphi}^0, \tilde{\varphi}^1)$ is taken from the set of self-financing trading strategies starting from initial endowment 1, $(x,y) \in \mathbb{R}^2$ and $t = \{0,1,2\}$. Furthermore the essential supremum for a function $f : X \to Y$ is defined as:

$$\operatorname*{ess\,sup}_{x\in X}(f(x)) = \inf\{\alpha \in [0,\infty] : |f| \leq \alpha \quad a.s.\}$$

If there is no existing $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}(1)$, such that in the above defined value function $(x+(\tilde{\varphi}_3^0-\tilde{\varphi}_t^0), y+(\tilde{\varphi}_3^1-\tilde{\varphi}_t^1)) \in dom(\mathcal{U})$ we then set $\mathcal{U}_t = -\infty$.

To have a better look at this matter we will define the **liquidation value**, that is

$$l_t(x,y) := x + y^+\underline{S}_t - y^-\overline{S}_t$$

The liquidation value holds the information about how much our position in $(x,y) \in \mathbb{R}^2$ is worth at time t. So if it is positive we are still liquid, and there is still money left in the bank account after buying or selling all the planed stocks. If it gets negative we will suffer of bankruptcy. It is a simple observation, that as long as there exists a self-financing trading strategy our liquidation value is positive and vice versa, so

$$(x + (\tilde{\varphi}_3^0 - \tilde{\varphi}_t^0), y + (\tilde{\varphi}_3^1 - \tilde{\varphi}_t^1)) \in dom(\mathcal{U}) \leftrightarrow l_t(x,y) > 0$$

in other words: as long as we stay within a positive liquidation value (which is anyway a condition in our model), we have a defined value function $\mathcal{U}$. We can therefore follow that our dynamic programming property translates into:

$$\mathcal{U}_t(x,y) = \operatorname*{ess\,sup}_{l_t(-\Delta\varphi_{t+1})\geq 0} E[\mathcal{U}_t(x + \Delta\tilde{\varphi}_{t+1}^0, y + \Delta\tilde{\varphi}_{t+1}^1))\big|\mathcal{F}_t]$$

We can now start to compute the solution $\hat{\varphi}$ out of the last equation recursively by the following Lemma:

Lemma 3.3.1

The solution $(\hat{\varphi}^0, \hat{\varphi}^1)$ for the problem

$$E[U(\varphi_3^0)] \to \max \quad \textit{with} \quad \varphi \in \mathcal{A}(1)$$

is given by the starting position $(\hat{\varphi}_0^0, \hat{\varphi}_0^1) = (1,0)$ and

$$\Delta\hat{\varphi}_1^1 = \frac{1}{1+\lambda} \qquad \Delta\hat{\varphi}_2^1(\omega_{1,1}) = \Delta\hat{\varphi}_2^1(\omega_{1,2}) = q_0$$

$$\Delta\hat{\varphi}_2^1(\omega_{n,1}) = \Delta\hat{\varphi}_2^1(\omega_{n,2}) = \frac{q_1}{n} \quad \textit{for} \quad n \in \mathbb{N}, \qquad \Delta\hat{\varphi}_3^1 = -\hat{\varphi}_2^1$$

as well as $\hat{\varphi}^0$ is defined by

$$\Delta\hat{\varphi}_{t+1}^0 = -(\Delta\hat{\varphi}_{t+1}^1)^+\overline{S}_t + (\Delta\hat{\varphi}_{t+1}^1)^-\underline{S}_t$$

Furthermore the Superdifferential $\partial\mathcal{U}(\hat{\varphi})$ of the value function along the optimal strategy $\hat{H}$ is given by

$$\partial\mathcal{U}_0(\hat{\varphi}_0) = \left(\frac{1}{\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0}, \frac{S_0}{\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0}\right) \qquad \partial\mathcal{U}_1(\hat{\varphi}_1) = \left(\frac{1}{\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1}, \frac{S_1}{\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1}\right)$$

$$\partial\mathcal{U}_2(\hat{\varphi}_2) = \left(\frac{1}{\hat{\varphi}_2^0 + \hat{\varphi}_2^1 \underline{S}_2}, \frac{(1-\lambda)S_2}{\hat{\varphi}_2^0 + \hat{\varphi}_2^1 \underline{S}_2}\right)$$

In the first section of this lemma we will proof exactly what we already observed in the last section at the counter example: independently of the event that will happen at time $t = 1$ the investor will buy as much of the stock positions as he can (in mathematical terms $\Delta\hat{\varphi}_1^1 = \frac{1}{1+\lambda}$). In the second time period he will still buy, depending on how the stock has evolved (given in terms of the constants q_0 and q_1). In the last time period he has to sell everything ($\Delta\hat{\varphi}_3^1 = -\hat{\varphi}_2^1$). We will now prove this

Proof: Because the investor has to liquidate the stock after time $t = 2$ and we are using $U(x) : x \mapsto \log(x)$ as utility function, we get for the dynamic programming property

$$\mathcal{U}_2(\varphi_2) = \mathcal{U}_2(\varphi_2^0, \varphi_2^1) = \log((\varphi_2^0 + (\varphi_2^1)^+\underline{S}_2 - (\varphi_2^1)^-\overline{S}_2))$$

for all $\varphi \in \mathcal{A}(x)$. We now define another function f_n, that will be used as a constraint

$$f_n(\Delta\tilde{\varphi}_2^1(\omega_{n,i}); \varphi_1) := E\left[\log(\varphi_1^0 + \varphi_1^1\overline{S}_1 + (\varphi_1^1 + \Delta\tilde{\varphi}_2^1)(\underline{S}_2 - \overline{S}_1))\middle|\mathcal{F}_1\right](\omega_{n,i})$$

for $i = 1, 2$ and $n \in \mathbb{N}_0$. At time $t = 1$ this function coincides with the expectation of our value function, so

$$E[\mathcal{U}_2(\varphi_1^0 + \Delta\tilde{\varphi}_2^0, \varphi_1^1 + \Delta\tilde{\varphi}_2^1)\big|\mathcal{F}_t)](\omega_{n,i}) = f_n(\Delta\tilde{\varphi}_2^1(\omega_{n,i}); \varphi_1)$$

as long as $\Delta\tilde{\varphi}_2^1 \geq 0$ and $\varphi_1^1 + \Delta\tilde{\varphi}_2^1 \geq 0$ is satisfied.

Therefore we can replace our original optimization problem with an optimization problem for f_n as long as it is satisfied that the liquidation function is positive, e.g.

$$f_n(\Delta\tilde{\varphi}_2^1(\omega_{n,i}), \varphi_1^0, \varphi_1^1) \to \max \quad \text{with} \quad \Delta\tilde{\varphi}_2^1(\omega_{n,i}) \in \mathbb{R}$$

again for $i = 1, 2$ and $n \in \mathbb{N}_0$. Because f_n is differentiable with respect to $\Delta\tilde{\varphi}_2^1(\omega_{n,i})$, we can determine the maximum of the (new) optimization problem, by simply setting the first derivative of f_n equal to zero, $f_n'(\Delta\tilde{\varphi}_2^1(\omega_{n,i}); \varphi_1) = 0$. From this equation we get the following solution, that defines the optimizer for $\Delta\hat{\varphi}_2^1$ recursively by

$$\Delta\hat{\varphi}_2^1(\varphi_1, 0) := (\varphi_1^0 + \varphi_1^1 S_1(\omega_{0,i}))\frac{1+\lambda}{\lambda+2}\left(\frac{1}{1+\lambda} + q_0\right) - \varphi_1^1$$

$$\Delta\hat{\varphi}_2^1(\varphi_1, n) := (\varphi_1^0 + \varphi_1^1 S_1(\omega_{n,i}))\frac{1+\lambda}{\lambda+\frac{1}{n}}\left(\frac{1}{1+\lambda} + \frac{q_1}{n}\right) - \varphi_1^1 \quad n \in \mathbb{N}$$

and this is a maximizer for f_n due to our choice of ε_n. Furthermore since

$$(\varphi_1^0 + \varphi_1^1 S_1(\omega_{0,i})) \geq (\varphi_1^0 + \varphi_1^1 S_1)(\omega_{n,i}) \geq \frac{\lambda + \frac{1}{n}}{1+\lambda}$$

for $i = 1, 2$, all $n \in \mathbb{N}_0$ and $\varphi \in \mathcal{A}(1)$ with $\varphi_1^1 > 0$, we also get that

$$\Delta\hat{\varphi}_2^1(\varphi_1, n) \geq 0 \quad \text{and also} \quad \varphi_1^1 + \Delta\hat{\varphi}_2^1(\varphi_1, n) \geq 0$$

for all $n \in \mathbb{N}_0$ and $\varphi \in \mathcal{A}(1)$ again with $\varphi_1^1 > 0$. So it is true for every $\varphi \in \mathcal{A}(1)$ because for $\varphi_1^1 \leq 0$ it is always true. Since it is valid for all necessary $\varphi \in \mathcal{A}(1)$ we now take the equation for $\Delta\hat{\varphi}_2^1(\varphi_1)$ into the dynamic programming property

$$\mathcal{U}_t(x, y) = \operatorname*{ess\,sup}_{l_t(-\Delta\varphi_{t+1}) \geq 0} E[\mathcal{U}_{t+1}(x + \Delta\tilde{\varphi}_{t+1}^0, y + \Delta\tilde{\varphi}_{t+1}^1) \big| \mathcal{F}_t)]$$

we get

$$\mathcal{U}_1(\varphi_1) = \log\left(\varphi_1^0 + \varphi_1^1 S_1\right) + \underbrace{E\left[\log\left(1 + \left(\frac{1}{1+\lambda} + \frac{q_1}{n}\right)(\underline{S}_2 - S_1)\right) |\mathcal{F}_1\right]}_{C_1}$$

We can now do a similar observation as we did just now for $t = 1$ for $t = 0$. We start with the definition of the counterpart to function f

$$h(\Delta\varphi_1^1; \varphi_0) := E[\log(\varphi_0^0 + \varphi_0^1 S_0 + (\varphi_0^1 + \Delta\varphi_1^1)(S_1 - S_0)) + C_1]$$

and again see the connection with the value function at time $T = 0$

$$E[\mathcal{U}_1(\varphi_0^0 + \Delta\tilde{\varphi}_1^0, \varphi_0^1 + \Delta\tilde{\varphi}_1^1] = h(\Delta\tilde{\varphi}_1^1; \varphi_0)$$

also with $\Delta\tilde{\varphi}_1^1 \geq 0$ and $\varphi_0^1 + \Delta\tilde{\varphi}_1^1 \geq 0$ has to be fulfilled.

It's now possible to follow the same procedure as we did before for time $t = 1$, so first we observe, that we can replace the original optimization with an optimization for function h, so by

$$h(\Delta\tilde{\varphi}_1^1, \varphi_0^0, \varphi_0^1) \to \max \quad \text{with} \quad \Delta\tilde{\varphi}_1^1 \in \left(-\frac{1}{1+\lambda}, \frac{1}{1+\lambda}\right]$$

where we had to change the borders for $\Delta\tilde{\varphi}_1^1$ according to our solvency constraint. Again we have to make sure, that the maximizer of the above function is in the domain, so that both function coincide, and exactly as before this is the case when we take strategies that fulfil the liquidation property. As the function f before h is also differentiable with respect to $\Delta\tilde{\varphi}_1^1$ and the first derivative is

$$\frac{\partial h}{\partial \Delta\tilde{\varphi}_1^1}(\Delta\tilde{\varphi}_1^1, \varphi_0) = (1 - \varepsilon)\frac{2}{\varphi_0^0 + \varphi_0^1 + \Delta\tilde{\varphi}_1^1} + \varepsilon\sum_{n=1}^{\infty} 2^{-n}\frac{2(\frac{1}{n} - 1)}{\varphi_0^0 + \varphi_0^1 + \Delta\tilde{\varphi}_1^1(\frac{1}{n} - 1)}$$

When we now insert our starting conditions, make $\varphi_0^0 = 1$ and $\varphi_1^0 = 0$ as well as the results we obtain from above $\Delta\tilde{\varphi}_1^1 = \frac{1}{1+\lambda}$ into the derivative of h we get

$$\begin{aligned}
\frac{\partial h}{\partial \Delta\tilde{\varphi}_1^1}\left(\frac{1}{1+\lambda};(1,0)\right) &= (1-\varepsilon)\frac{2}{1+\frac{1}{1+\lambda}} + \varepsilon\sum_{n=1}^{\infty} 2^{-n}\frac{2(\frac{1}{n}-1)}{1+\frac{1}{1+\lambda}(\frac{1}{n}-1)} \\
&\geq (1+\lambda)\left((1-\varepsilon)\frac{1}{2} + \varepsilon\sum_{n=1}^{\infty} 2^{-n}(1-n)\right) \\
&\geq (1+\lambda)\left((1-\varepsilon)\frac{1}{2} + \varepsilon\left(1 - \frac{\frac{1}{2}}{(\frac{1}{2}-1)^2}\right)\right) \\
&= (1+\lambda)\left(\frac{1}{2} - \varepsilon\frac{3}{2}\right) > 0
\end{aligned}$$

So as long as we take $\varepsilon \in (0, \frac{1}{3})$ the function h with starting value $(1,0)$ is concave. This function therefore must obtain it's maximum (over the range $\Delta\tilde{\varphi}_1^1 \in \left(-\frac{1}{1+\lambda}, \frac{1}{1+\lambda}\right]$) at

$$\Delta\tilde{\varphi}_1^1 = \frac{1}{1+\lambda}$$

When we now insert this first result in the original optimization problem we get

$$\mathcal{U}_0(\varphi_0) = \mathcal{U}_0(\varphi_0^0, \varphi_0^1) = \log(\varphi_0^0 + \varphi_0^1 S_0) + \underbrace{E\left[\log(1 + \frac{1}{1+\lambda}(S_1 - S_0)) + C_1\right]}_{C_0}$$

for all $\varphi \in \mathcal{A}(1)$ with the C_1 as defined before.

We still have to check whether the above mentioned strategy is optimal, for that we obverse

$$E[\mathcal{U}(\hat{\varphi}_3)] = \mathcal{U}_0(1,0) = \sup_{\varphi\in\mathcal{A}(1)} E[\mathcal{U}(\varphi_3)]$$

with the considerations we made in the last paragraph, this implies already that $\hat{\varphi}$ is a solution to the original optimization problem.

Because of $\hat{\varphi}_2^1 > 0$ we can derive the first superdifferential from

$$\mathcal{U}_2(\varphi_2) = \log(\varphi_2^0 + (\varphi_2^1)^+\underline{S}_2 - (\varphi_2^1)^-\overline{S}_2)$$

$$\Rightarrow \partial\mathcal{U}_2(\hat{\varphi}_2) = \left\{\frac{(1, \underline{S}_2)}{\hat{\varphi}_2^0 + \hat{\varphi}_2^1(1-\lambda)S_2}\right\}$$

To get the second superdifferential $\partial\mathcal{U}_1(\hat{\varphi}_1)$ we remember that our recursively defined equation for $\Delta\hat{\varphi}_2^1$ are continuous with respect to φ and that the following two conditions also hold true for all $n \in \mathbb{N}$

$$\Delta\hat{\varphi}_2^1(\varphi_1^0, \varphi_1^1, n) > 0 \quad \text{and} \quad \varphi_1^1 + \Delta\hat{\varphi}_2^1(\varphi_1^0, \varphi_1^1, n) > 0$$

We can then have a look at the formula we derived for $\mathcal{U}_1$. This formula also holds true for some sufficiently small open neighbourhood around $(\hat{\varphi}_1)$, so we just have to differentiate and get

$$\mathcal{U}_1(\hat{\varphi}_1) = \log\left(\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1\right) + C_1 \Rightarrow \partial\mathcal{U}_1(\hat{\varphi}_1) = \left\{\frac{(1, S_1)}{\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1}\right\}$$

And for the last superdifferential we again just look at a (sufficiently small) open Neighbourhood around the starting condition $(0, 1)$, there φ is solvent (so the strategy is self-financing) at time $t = 1$ as long as $\Delta\varphi_1^1 \in \left(-\frac{\varphi_0^0+\varphi_0^1 S_0}{1+\lambda}, \frac{\varphi_0^0+\varphi_0^1 S_0}{1+\lambda}\right]$, so our last formula for $\mathcal{U}_0$ also holds true.

And again we get the superdifferental by differentiation.

$$\mathcal{U}_0(\hat{\varphi}_0) = \log(\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0) + C_0 \Rightarrow \partial\mathcal{U}_0(\hat{\varphi}_0^0, \hat{\varphi}_0^1) = \left\{\frac{(1, S_0)}{\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0}\right\}$$

And with that our proof is finished. ■

So we now see that the strategy we described in a formal way in the last section is indeed a optimal strategy for our primal optimization problem. We will now derive that the optimizer for the dual problem also exists but fails to be a martingale.

Lemma 3.3.2

For our dual problem

$$E[V(Y_2^0)] = E[-\log(Y_2^0) - 1] \to \min \quad \textit{with} \quad (Y^0, Y^1) \in \mathcal{B}(1)$$

where $\mathcal{B}(1)$ is defined as in Theorem 3.2.1, there exists a solution $\hat{Y}_t = (\hat{Y}_t^0, \hat{Y}_t^1)$ which is given by

$$\hat{Y}_0 = \left(\frac{1}{\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0}, \frac{S_0}{\hat{\varphi}_0^0 + \hat{\varphi}_0^1 S_0}\right) \qquad \hat{Y}_1 = \left(\frac{1}{\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1}, \frac{S_1}{\hat{\varphi}_1^0 + \hat{\varphi}_1^1 S_1}\right)$$

$$\hat{Y}_2 = \left(\frac{1}{\hat{\varphi}_2^0 + \hat{\varphi}_2^1 \underline{S}_2}, \frac{(1-\lambda) S_2}{\hat{\varphi}_2^0 + \hat{\varphi}_2^1 \underline{S}_2}\right)$$

Both parts of the solution $\hat{Y}^0$ and $\hat{Y}^1$ are supermartingales, but they are strict, so they will not be martingales.
But still the process $(\tilde{S}_t)_{t=\{0,1,2\}} := (\hat{Y}_t^1/\hat{Y}_t^0)_{t=\{0,1,2\}}$ ranges within the bid/ask-spread $[\underline{S}, \overline{S}]$ and the corresponding frictionless market satisfies (NA^λ).

Proof: By proposition 3.5.4. (see section 3.5. later on), we have that: the $\mathcal{F}_t$-conditional value function $\mathcal{U}_t$, that is for every $(\psi^0, \psi^1) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2)$

$$\mathcal{U}_t(\psi^0, \psi^1) := \operatorname*{ess\,sup}_{(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}} E(\mathcal{U}(\psi^0 + (\tilde{\varphi}^0_{T+1} - \tilde{\varphi}^0_t), \psi^1 + (\tilde{\varphi}^1_{T+1} - \tilde{\varphi}^1_t)) \big| \mathcal{F}_t)$$

has the following property for a dual minimizer $\hat{Y} \in \mathcal{B}(\hat{y}(x)$

$$\mathcal{U}_t(\psi^0, \psi^1) \leq \mathcal{U}_t(\hat{\varphi}^0_t, \hat{\varphi}^1_t) + (\psi^0 - \hat{\varphi}^0_t)\hat{Y}^0_t + (\psi^1 - \hat{\varphi}^1_t)\hat{Y}^1_t$$

almost surely for all $(\psi^0_t, \psi^1_t) \in L^0(\Omega, \mathcal{F}_t, P, \mathbb{R})$ and all $t = 0, \ldots, T$. In other words the solution $\hat{Y}$ of the dual problem

$$E[V(\hat{Y}^0_2)] = E[-\log(\hat{Y}^0_2) - 1] \to \min$$

is valued in the $\mathcal{F}_t$-conditional superdifferential of the $\mathcal{F}_t$-conditional value function (which will be defined later on, but we don't really need the exact definition, as we will see in a minute) along the path of the optimal strategy $\hat{\varphi}$. Since we are working in a Markovian setting this $\mathcal{F}_t$-conditional superdifferential is equal to the normal superdifferential value function. By the last Lemma this is a single-valued, unique process $\hat{Y}$ given by

$$\hat{Y}_0 = \left(\frac{1}{\hat{\varphi}^0_0 + \hat{\varphi}^1_0 S_0}, \frac{S_0}{\hat{\varphi}^0_0 + \hat{\varphi}^1_0 S_0} \right) \qquad \hat{Y}_1 = \left(\frac{1}{\hat{\varphi}^0_1 + \hat{\varphi}^1_1 S_1}, \frac{S_1}{\hat{\varphi}^0_1 + \hat{\varphi}^1_1 S_1} \right)$$

$$\hat{Y}_2 = \left(\frac{1}{\hat{\varphi}^0_2 + \hat{\varphi}^1_2 \underline{S}_2}, \frac{(1-\lambda) S_2}{\hat{\varphi}^0_2 + \hat{\varphi}^1_2 \underline{S}_2} \right)$$

which is the solution to the dual problem.

To proof the second part of the lemma, so that $\hat{Y}^0$ and $\hat{Y}^1$ are supermartinagales (but fail to be martingales) we first observe that

$$E[\hat{Y}^0_1 (S_1 - S_0) \big| \mathcal{F}_0] = h' \left(\frac{1}{1+\lambda}; (1,0) \right)$$

where the function h is defined as in the last proof.

$$h(\Delta\varphi^1_1; \varphi_0) = E[\log(\varphi^0_0 + \varphi^1_0 S_0 + (\varphi^1_0 + \Delta\varphi^1_1)(S_1 - S_0)) + C_1]$$

Furthermore we got from the definition of h that $h'\left(\frac{1}{1+\lambda}; (1,0)\right) \in \left(0, \frac{1+\lambda}{2}\right)$ and therefore we get

$$E\left[\frac{\hat{Y}^0_1}{\hat{Y}^0_0} \bigg| \mathcal{F}_0 \right] = 1 - \hat{\varphi}^1_0 h' \left(\frac{1}{1+\lambda}; (1,0) \right) = 1 - \hat{\varphi}^1_0 E[\hat{Y}^0_1 (S_1 - S_0) \big| \mathcal{F}_0] < 1$$

So further on we can also see that

$$E[\hat{Y}_1^1 - \hat{Y}_0^1 | \mathcal{F}_0] = E[\hat{Y}_1^0(S_1 - S_0) + (\hat{Y}_1^0 - \hat{Y}_0^0)S_0 | \mathcal{F}_0] =$$

$$= E[\hat{Y}_1^0(S_1 - S_0) | \mathcal{F}_0] < 0$$

and so $\hat{Y}^0$ are $\hat{Y}^1$ both supermartingales but (because the last inequality is strict), no martingales.

To see that $(\hat{S}_t)_{t=\{0,1,2\}} = (\hat{Y}_t^1/\hat{Y}_t^0)_{t=\{0,1,2\}}$ has the properties we claimed in the lemma we just observed that

- $(\tilde{S}_t)_{t=\{0,1,2\}}$ can go up or down with positive probability at time $t = 0,1,2$ and there for it does not admit for arbitrage.
- $(\tilde{S}_t)_{t=\{0,1,2\}}$ takes values in $(\underline{S}, \overline{S})$, by definition. ■

With the next and last lemma in this section we are going to define the solution for the frictionless optimization problem with log-utility and the above defined price process $(\tilde{S}_t)_{t=\{0,1,2\}} := (\hat{Y}_t^1/\hat{Y}_t^0)_{t=\{0,1,2\}}$. We will therefore also complete the proof of the theorem at the beginning of this section, so that the solution for the frictionless market and the market with transaction costs does not coincide.

Lemma 3.3.3

For the frictionless log-utility optimization problem

$$E[\log(\psi_3^0)] = E[\log(1 + \psi_3^1 \cdot \hat{S}_2)] \to \max \quad \textit{with} \quad \psi \in \mathcal{A}(1, \hat{S})$$

the solution $(\hat{\psi}^0, \hat{\psi}^1)$ *is given by*

$$(\hat{\psi}_0^0, \hat{\psi}_0^1) = (1, 0) \qquad \Delta\hat{\psi}_1^1 = 1$$

$$\Delta\hat{\psi}_2^1(\omega_{1,1}) = \Delta\hat{\psi}_2^1(\omega_{1,2}) = 2\frac{1+\lambda}{\lambda+2}\left(\frac{1}{1+\lambda} + q_0\right) - 1$$

$$\Delta\hat{\psi}_2^1(\omega_{n,1}) = \Delta\hat{\psi}_2^1(\omega_{n,2}) = \frac{1}{n}\frac{1+\lambda}{\lambda+\frac{1}{n}}\left(\frac{1}{1+\lambda} + \frac{q_1}{n}\right) - 1$$

$$\Delta\hat{\psi}_3^1 = -\hat{\psi}_2^1$$

where q_0 *and* q_1 *are defined as at the beginning of the chapter and* $n \in \mathbb{N}$*. The positions of the bond* $\hat{\psi}^0_{t=\{1,2,3\}}$ *are again determined by the self-financing property*

$$\Delta\hat{\psi}_{t+1}^0 = -(\Delta\hat{\psi}_{t+1}^1)^+ \overline{S}_t + (\Delta\hat{\psi}_{t+1}^1)^+ \underline{S}_t$$

Then the maximal expected utility in the frictionless case is strictly bigger than the expected utility in the original market

$$\sup_{(\varphi^0,\varphi^1)\in\mathcal{A}(1,\tilde{S})} E[\log(\varphi_3^0)] > \sup_{(\varphi^0,\varphi^1)\in\mathcal{A}(1)} E[\log(\varphi_3^0)]$$

and furthermore the dual optimizer for the frictionless market $\hat{Y} = \frac{1}{\hat{\psi}^0+\hat{\psi}^1\tilde{S}}$ *does not coincide with the optimizer of the market with transaction costs, and so no shadow price exists.*

Proof: We first have to solve the primal problem, this can be done in a similar way as in Lemma 3.3.1. recursively by dynamic programming. We start by solving the following equation for time $t = 1$

$$E[\log(1+\varphi_1^1\Delta\tilde{S}_1+\varphi_2^1\Delta\tilde{S}_2)\big|\mathcal{F}_1](\omega_{n,i}) = f_n(\varphi_2^1;(1+\varphi_1^1\Delta\tilde{S}_1,0)) \to \max$$

where $\varphi_2^1 \in \mathbb{R}$. By the recursive solution of this equation (see Lemma 3.3.1.) we get the following solution by the same calculation as above

$$\hat{\psi}_2^1(\omega_{0,1}) = \hat{\psi}_2^1(\omega_{0,2}) = (1+\varphi_1^1\Delta\tilde{S}_1)\frac{1+\lambda}{\lambda+2}\left(\frac{1}{1+\lambda}+q_0\right)$$

$$\hat{\psi}_2^1(\omega_{n,1}) = \hat{\psi}_2^1(\omega_{n,2}) = (1+\varphi_1^1\Delta\tilde{S}_1)\frac{1+\lambda}{\lambda+\frac{1}{n}}\left(\frac{1}{1+\lambda}+\frac{q_1}{n}\right)$$

again where $q_0 \in \left(0, \frac{1-\lambda}{1+3\lambda+2\lambda^2}\right)$, $q_1 \in \left(0, \frac{1-\lambda}{1+4\lambda+3\lambda^2}\right)$ and $n \in \mathbb{N}$.

The agent in the frictionless market is able to buy (strictly) more stocks than the agent in the other market, because he does not have to pay transaction costs. So his position after time $t = 1$ in stocks lies within the range $\varphi_1^1 = (-1, 1]$. Therefore we get the optimal (frictionless) strategy by solving the following equation:

$$E[\log(1+\varphi_1^1\Delta\tilde{S}_1)+C_1] = h(\varphi_1^1;(1,0)) \to \max \quad \text{with} \quad \varphi_1^1 \in (-1,1]$$

Following the same calculations we did as in the proof for Lemma 3.3.1., we get that the first derivative of h with our starting values is given by

$$h'(1;(1,0)) = \left(\frac{1}{2}-\varepsilon\frac{3}{2}\right) > 0$$

so the function h is concave and attains its maximum (over the interval $(-1,1]$) at $\hat{\psi}_1 = 1$. And so we can follow that

$$\sup_{\varphi\in\mathcal{A}(1,\tilde{S})} E[\log(1+\varphi^1\cdot\hat{S}_2)] = h(\hat{\psi}_1;(1,0)) > h(\Delta\hat{\varphi}_0^1;(1,0)) = \sup_{\varphi\in\mathcal{A}(1)} E[\mathcal{U}(\varphi_3)]$$

where on the far left side we can get rid of the supremum when we take the optimizer:

$$\sup_{\varphi \in \mathcal{A}(1,\hat{S})} E[\log(1+\varphi^1 \cdot \hat{S}_2)] = E[\log(1+\hat{\psi}^1 \cdot \tilde{S}_2)]$$

And when we take a closer look on the values inside the inner braces we can observe (by our definition of the process):

$$(1+\hat{\psi}^1 \cdot \tilde{S}_1)(\omega_{0,1}) = (1+\hat{\psi}^1 \cdot \hat{S}_1)(\omega_{0,2}) = 2$$

$$(1+\hat{\psi}^1 \cdot \tilde{S}_1)(\omega_{n,1}) = (1+\hat{\psi}^1 \cdot \hat{S}_1)(\omega_{n,2}) = \frac{1}{n}$$

for $n \in \mathbb{N}$. When we now plug this into

$$E[\log(\varphi_3^0)] = E[\log(1+\varphi^1 \cdot \tilde{S}_2)] \to \max \quad \text{with} \quad \varphi \in \mathcal{A}(1,\hat{S})$$

we get exactly the solution for $(\hat{\psi}^0, \hat{\psi}^1)$ as it was given in the lemma.

To check the last property of the lemma we first see that our dual optimizer for the frictionless case is given by:

$$\hat{Y} = \frac{1}{1+\hat{\psi}^1 \cdot S}$$

so therefore $\hat{Y}_2 \frac{1}{1+\hat{\psi}^1 \cdot S_2}$ and by [10] Theorem 2.2, we observe that the product

$$\left(1+\hat{\psi}^1 \cdot S\right)\hat{Y}$$

is also a martingale. And so the proof is complete. ■

Now we can observe that the expected utility for the frictionless case it strictly bigger than the utility for the original market with transaction costs. Since the process $(\tilde{S}_t)_{t=\{0,1,2\}} = (\hat{Y}_t^1/\hat{Y}_t^0)_{t=\{0,1,2\}}$ was the only candidate for a shadow price but the expected utility doesn't match, there is no way a shadow price may exist. In the next section we will have a general look at how the problem can still be solved in some cases and what approaches are still possible to be made.

3.4 Duality

In this section we will formulate and prove our main result, namely what type of process we can find in the case of finite discrete time, but with an infinite probability space. For that we need some help from the field of convex analysis, namely the theory of dual functions. So our aim will be, that we search a conjugated function to our original maximization problem, that then turns into a minimization problem of a convex function. Some detailed information about dual functions can be found in [8] mostly chapter 26-28.

As before we will stay close to the paper of Czichowsky, Muhle-Karbe und Schachermayer [3] whereas the problem is solved via some more abstract work in [10]. First we will give some proper definition of a concept we already have used a bit in the last section

Definition 3.4.1 (*conjugated functions*)

For a function $U : (0, \infty) \mapsto \mathbb{R}$ we define the ***conjugated function*** *V by*

$$V(y) := \sup_{x>0}(U(x) - xy) \quad \textit{for} \quad \forall y > 0$$

this is the Legendre transformation of $-U(-x)$

The first question we have to deal with is: what would be a sufficient domain for the dual function, which we need to fins the candidate for a shadow price process $\tilde{S}_t$ in the frictionless market. As a result of the work of Kramkov and Schachermayer [10], see chapter 2, a good candidate for such a domain would be

$$\mathcal{Y}(y, \tilde{S}) = \{Y = (Y)_{t=0}^T \geq 0 \big| Y_0 = y \text{ and } Y(\varphi^0 + \varphi^1\tilde{S}) = (Y_t(\varphi^0 + \varphi^1\tilde{S}))_{t=0}^T$$

$$\text{is a non-negative supermaritngale for all } (\varphi^0, \varphi^1) \in \mathcal{A}(1, \tilde{S})\}$$

This is the set of all non-negative semimartingales, with $Y_0 = y$ and such that for any admissible strategy $(\varphi^0 + \varphi^1\tilde{S})$ the product becomes a supermartinagle. Within this set all admissible wealth processes (referring to the frictionless price process $\tilde{S}$) are transformed into supermartingales. Since we want to bring in terminal values in our model, we have to define some equivalent set to $\mathcal{A}$ and $\mathcal{C}$ from the previous sections. We start with the counterpart to $\mathcal{C}$

$$\mathcal{D}(y, \tilde{S}) = \Big\{h \in L^0_+(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2) \big| \exists Y \in \mathcal{Y}(y, \tilde{S}) \text{ with } h \leq Y_T\Big\}$$

this we have for $y > 0$. In other words this is the set of random variables dominated by a dual element. With that we can reformulate our original maximization problem

$$E[U(g)] \to \max \quad g \in \mathcal{A}(x)$$

into

$$E[V(Y_T)] \to \min \quad \text{for} \quad Y \in \mathcal{Y}(y, \tilde{S})$$

or in the notation of the above set $\mathcal{D}$ equivalently

$$E[V(h)] \to \min \quad \text{for} \quad h \in \mathcal{D}(y, \tilde{S})$$

The question that arises now is: which random variables should be used in the conjugated set, because any frictionless shadow price process $\tilde{S}$ can normally be used to model the position in stocks, as long as this process ranges within the bid/ask-spread $\underline{S}, \overline{S}$. So the dual set therefore would be the set of the consistent price systems, which we will define now:

Definition 3.4.2 (*consistent price system*)

*Be $S_{t=0}^T$ a price process. This process has a **consistent price system** under transaction costs λ, also called (CPS^λ) when there is a shadow price process $\tilde{S}$ with values in the bid/ask-spread $\underline{S} \leq \tilde{S} \leq \overline{S}$ that is adapted to $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^T, \mathbb{P})$. And in addition there is a probability measure $\mathbb{Q}$ on $\mathcal{F}$ that is equivalent to $\mathbb{P}$ under which $\tilde{S}$ is a (local) martingale.*
We will denote the set of all λ-consistent price systems by $\mathcal{Z}$.

Since we work in a setting with discrete time, local martingales becomes general martingales.

Unfortunately with this definition, this set is too small to contain the dual minimizer (since we already observe that it will be a supermartingale), we have to expand the set $\mathcal{A}$ a little and use the following definition:

$$\mathcal{B}(y) := \{(Y^0, Y^1) \geq 0 \big| Y_0^0 = y, \frac{Y^1}{Y^0} \in [\underline{S}, \overline{S}] \text{ and } Y^0(\varphi^0 + \varphi^1 \frac{Y^1}{Y^0}) = (Y^0\varphi^0 + Y^1\varphi^1)$$
$$\text{is a non-negative supermartingale for all } (\varphi^0, \varphi^1) \in \mathcal{A}(1)\}$$

and with that the above set $\mathcal{D}$ transforms into

$$\mathcal{D}(y, \tilde{S}) = \big\{h \in L_+^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2) \big| \exists (Y^0, Y^1) \in \mathcal{B}(y) \text{ with } h \leq Y_T^0\big\} \quad \text{for} \quad y > 0$$

When we now take a closer look at the last definition of $\mathcal{B}(y)$: here the position is not modelled by a shadow price process $\tilde{S}$ any more but by the process $\frac{Y^1}{Y^0}$, which is also (by definition) frictionless. As we mentioned before Y^0 isn't a martingale any more but only a supermartingale. Therefore when we multiply our wealth process (which is also frictionless) $\varphi^0 + \varphi^1 \frac{Y^1}{Y^0}$ with Y^0 we turn it into a supermartingale, that is on the one hand still generated by the trading strategy (φ^0, φ^1) (and also admissible in the original market) but on the other hand fails to be a martingale. Further on this set is a (strict) subset of the original set of all admissible strategies in the frictionless market, where the price process was given by $\frac{Y^1}{Y^0}$. This implies that Y^0 does not mandatory need to be an admissible dual variable for the frictionless market. And exactly that is the problem with this model: as long as (Y^0, Y^1) is only a supermartigale we won't get such nice results as in chapter 2. But when we assume as an additional constraint, that this optimizer $\frac{Y^1}{Y^0}$ has to be a martingale, we will get the same nice qualitative result as in chapter 2, so that a shadow price will exist.

It is possible to rewrite the whole above definitions with the notation of (CPS^{λ}), seen in the original paper. We will now give the main result of this section, for that we will also need the asymptotic elasticity:

Definition 3.4.3 (*asymptotic elasticity, AE*)

The asymptotic elasticity of a utility function U is given by

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}$$

Just a small note: we will deal with the general case where we work with constraint for the asymptotic elasticity. Otherwise many "usually" used utility functions, like e.g. $U(x) = \log(x)$ always has $AE(U) < 1$.

Theorem 3.4.1 (*Duality*)

Let S be a price process with transaction costs $\lambda \in (0,1)$, that satisfies $(CPS^{\lambda'})$ for some $\lambda' \in [0,1)$. Let the utility function $U : (0,\infty) \to \mathbb{R}$ have asymptotic elasticity smaller 1, $AE(U) < 1$, and the expected utility be finite, so

$$u(x) = \sup_{g\in\mathcal{C}(x)} E[U(g)] < \infty \quad \textit{for some} \quad x \in (0,\infty)$$

Then the following results are valid:

1. *The primal value function $u(x)$ above and the dual value function*

$$v(y) := \inf_{h\in\mathcal{D}(y)} E[V(h)]$$

are conjugated and continuous differentiable on $(0,\infty)$*, in other words*

$$u(x) = \inf_{y>0}(v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x>0}(u(x) - xy)$$

u *and* $-v$ *are both strictly concave, strictly increasing and satisfy the Inada conditions*

$$u'(x) := \lim_{x\to 0} u'(x) = \infty \qquad -v'(y) := \lim_{y\to 0} -v'(y) = \infty$$

$$u'(x) := \lim_{x\to \infty} u'(x) = 0 \qquad -v'(y) := \lim_{y\to \infty} -v'(y) = 0$$

2. *For all* $x, y > 0$ *the solution* $\hat{g}(x)$*, for the primal problem, and* $\hat{h}(y)$*, for the dual problem:*

$$E[U(g(x))] \to \max \qquad \text{with} \qquad g(x) \in \mathcal{C}(x)$$

$$E[V(h(y))] \to \min \qquad \text{with} \qquad h(y) \in \mathcal{D}(y)$$

exists and are unique. Furthermore there are (not necessarily unique) $(\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in \mathcal{A}(x)$ *such that*

$$\hat{\varphi}^0_{T+1}(x) = \hat{g}(x)$$

and $(\hat{Y}^0(y), \hat{Y}^1(y)) \in \mathcal{B}(y)$ *such that*

$$\hat{Y}^0_T(y) = \hat{h}(y)$$

But, even if the process $\hat{\varphi}_t$ *and* $\hat{Y}_t$ *do not need to be unique, the optimizer* $\hat{\varphi}^0_{T+1}$ *and* $\hat{Y}^0_{T+1}$ *are unique.*

3. *For all* $x > 0$*, let* $\hat{y}(x) = u'(x) > 0$ *which is the unique solution to*

$$v(y) + xy \to \min \qquad \text{for} \quad y > 0$$

then $\hat{g}(x)$ *and* $\hat{h}(\hat{y}(x))$ *are given by*

$$\hat{g}(x) = (U')^{-1}(\hat{h}(\hat{y}(x))) \quad \text{and} \quad \hat{h}(\hat{y}(x)) = U'(\hat{g}(x))$$

and we have $E[\hat{g}(x)\hat{h}(\hat{y}(x))] = x\hat{y}(x)$*. We also get that the following process*

$$\hat{Y}^0(\hat{y}(x))\hat{\varphi}^0(x) + \hat{Y}^1(\hat{y}(x))\hat{\varphi}^1(x) = (\hat{Y}^0_t(\hat{y}(x))\hat{\varphi}^0_t(x) + \hat{Y}^1_t(\hat{y}(x))\hat{\varphi}^1_t(x))_{t=0}^T$$

is a martingale for all $(\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in \mathcal{A}(x)$ *and* $(\hat{Y}^0(y), \hat{Y}^1(y)) \in \mathcal{B}(\hat{y}(x))$ *satisfying*

$$\hat{\varphi}^0_{T+1}(x) = \hat{g}(x) \quad \text{and} \quad \hat{Y}^0_T(\hat{y}(x)) = \hat{h}(\hat{y}(x))$$

4. $\hat{Y}^0(\hat{y}(x))\hat{\varphi}^0(x) + \hat{Y}^1(\hat{y}(x))\hat{\varphi}^1(x) = \hat{Y}^0(\hat{y}(x))\left(x + \hat{\varphi}(x) \cdot \frac{\hat{Y}^1}{\hat{Y}^0}\right)$ *and therefore for all* $t = 0, \dots, T$

$$\{\Delta\hat{\varphi}^1_{t+1} > 0\} \subseteq \left\{\frac{\hat{Y}^1_t}{\hat{Y}^0_t} = \overline{S}_t\right\} \quad \text{and} \quad \{\Delta\hat{\varphi}^1_{t+1} < 0\} \subseteq \left\{\frac{\hat{Y}^1_t}{\hat{Y}^0_t} = \underline{S}_t\right\}$$

5. *For Martingales* $(Z^0, Z^1) \in \mathcal{Z}$ *such that* $\frac{Z^1}{Z^0} \in [\underline{S}, \overline{S}]$ *we have*

$$v(y) = \inf_{(Z^0,Z^1)\in\mathcal{Z}} E(V(yZ^0_T))$$

For the proof of this theorem we need two additional definitions, two theorems and a proposition, that are all from the paper of Kramkov and Schachermayer [10]. All of the proofs can also be found in that paper, since it would be a little too much for the extent of this master-thesis, to write them all down.

Definition 3.4.4 (*solid*)

A set $\mathcal{G} \subseteq L^0_+(P)$ *is called* ***solid****, if for a function* $g \in \mathcal{G}$ *we have* $0 \leq f \leq g$ *then this implies that* $f \in \mathcal{G}$.

Definition 3.4.5

Let the sets $\mathcal{C}$ *and* $\mathcal{D}$ *be defined as above, then we have:*

$$x\mathcal{C} := \mathcal{C}(x) = x\mathcal{C}(1) = \{xg : g \in \mathcal{C}\} \quad \text{for} \quad x > 0$$

$$y\mathcal{D} := \mathcal{D}(y) = y\mathcal{D}(1) = \{yh : h \in \mathcal{D}\} \quad \text{for} \quad y > 0$$

Proposition 3.4.1 (*Proposition 3.1. from [10]*)

Let S *be an* $\mathbb{R}^d$ *valued semi-martingale, where the set of equivalent martingale measures* $\mathcal{M}^e$ *is not empty, and let the set* $\mathcal{C}$ *and* $\mathcal{D}$ *be defined as above, then we have*

1. $\mathcal{C}, \mathcal{D} \subset L^0_+(\Omega, \mathcal{F}, P)$*, both are convex, solid and and closed in the topology of convergence in measure.*

2.
$$g \in \mathcal{C} \Leftrightarrow E[gh] \leq 1 \quad \text{for all} \quad h \in \mathcal{D}$$
$$h \in \mathcal{D} \Leftrightarrow E[gh] \leq 1 \quad \text{for all} \quad g \in \mathcal{C}$$

3. *The constant function* $\mathbb{1}$ *is in* $\mathcal{C}$.

Theorem 3.4.2 (*Theorem 3.1. from [10]*)

Let $\mathcal{C}$ and $\mathcal{D}$ again be defined as above and the assertions of Proposition 3.4.1. hold true. Let also $U(x)$ be a utility function that satisfies the Inada conditions and

$$U(x) < \infty \quad \text{for some} \;\; x > 0$$

hold true, then we have

1. $u(x) < \infty$ *for all* $x > 0$ *and there exists a* y_0 *such that* $v(y) < \infty$ *for all* $y > y_0$. *Furthermore the value functions* u *and* v *are conjugated, meaning*

$$u(x) = \inf_{y>0}(v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x>0}(u(x) - xy)$$

   u *is continuously differentiable on* $(0, \infty)$ *and* v *is strictly convex on the set* $\{y < \infty\}$. *Also* u' *and* v' *satisfy*

$$u'(0) := \lim_{x\to 0} u'(x) = \infty \quad v'(\infty) := \lim_{y\to\infty} v'(x) = 0$$

2. *If* $v(y) < \infty$ *then the optimizer* $\hat{h}(y) \in \mathcal{D}$ *to*

$$v(y) := \inf_{h\in\mathcal{D}(y)} E(V(h))$$

   exists and is unique.

Theorem 3.4.3 (*Theorem 3.2. from [10]*)

In addition to all prerequisites of Theorem 3.4.2. we also assume that

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$$

then we get the additional results:

1. $v(y) < \infty$ *for all* $y > 0$ *both value functions* u *and* v *are continuously differentiable on* $(0, \infty)$, *the derivatives* u' *and* $-v'$ *are strictly decreasing and satisfy*

$$u'(\infty) := \lim_{x\to\infty} u'(x) = \infty \quad -v'(0) := \lim_{y\to 0} v'(x) = \infty$$

   and the asymptotic elasticity of u *is smaller or equal to them of* U, *so*

$$AE(u)_+ \leq AE(U)_+ \leq 1$$

2. *The optimizer $\hat{g}(x) \in \mathcal{C}(x)$ to*

$$u(x) = \sup_{g \in \mathcal{D}(x)} E(U(g))$$

exists and is unique. If also $\hat{h}(y) \in \mathcal{D}(y)$ is the optimal solution to the conjugated problem, then we have the dual relation

$$\hat{g}(x) = \hat{h}(y)^{-1}$$

$$E[\hat{g}(x)\hat{h}(y)] = xy$$

3. *We have the following relations:*

$$u(x)' = E\left(\frac{\hat{g}(x)U'(\hat{g}(x))}{x}\right) \quad \text{and} \quad v(y)' = E\left(\frac{\hat{h}(y)V'(\hat{h}(y))}{y}\right)$$

For the proof of our main result Theorem 3.4.1., we now sum up the results of the above theorems and corollaries and give a proof of the connections that are not trivial or in the paper [10].

Lemma 3.4.1

Let the process S satisfy $(CPS^{\lambda'})$ for some $\lambda' \in (0, \lambda)$, then we have

1. *The sets $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are convex, solid and closed in the topology of convergence in measure.*

2. *$g \in \mathcal{C} \Leftrightarrow E[gh] \leq 1$ for all $h \in \mathcal{D}$ and $h \in \mathcal{D} \Leftrightarrow E[gh] \leq 1$ for all $g \in \mathcal{C}$.*

3. *$\mathcal{C}$ is bounded and a subset of $L^0_+(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2)$, and for the constant function we have $\mathbb{1} \in \mathcal{C}$.*

4. *The set $D := \{Z^0_T \big| (Z^0, Z^1) \in \mathcal{Z}\}$ is closed under countable convex combinations.*

Proof: 1.

By the definition of the sets above, $\mathcal{C}$ and $\mathcal{D}$, are both solid and convex. Because we assumed that S satisfies $(CPS^{\lambda'})$ this implies also that a strictly consistent price system exists. A strictly consistent price system is a process Z_t that is a martingale under $\mathbb{P}$ and for all $t = 0, \ldots, T$ we have a.s. that Z_t lies in the relative interior of $\hat{K}^*_t(\omega)$. Here $\hat{K}^*_t(\omega)$ is the polar of the cone $-\hat{K}_t$ and this is furthermore the solvency cone. For further definitions and explanations see [11]. We now take a look at the set

$\hat{A}$, which is the cone formed by all self financing trading strategies, and in our setting this is

$$\hat{A}_T := \{(\varphi_T^0, \varphi_T^1) \big| (\varphi^0, \varphi^1) \in \mathcal{A}\}$$

From that we can deduce that

$$\mathcal{C} = ((1,0) + \hat{A}_T) \cap L_+^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2) \times \{0\}$$

is closed under convergence in measure. That follows from:

S allows a strictly consistent price system $\Leftrightarrow$ (by Theorem 1.7 in [11]) the bid ask process $(\underline{S}_t, \overline{S}_t)_{t=0}^T$ satisfies the robust no Arbitrage condition (NA^r) $\Leftrightarrow$ (by Theorem 2.1 in [11]) the convex cone $\hat{A}_T$ is closed in $L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R})$ (here we identified $L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}_+ \times 0)$ with $L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R})$).

To see that $\mathcal{D}$ is also closed we will follow a proof of Lemma 4.1 in [10], we take a sequence of functions $(h)^n \in \mathcal{D}$ with $n \in \mathbb{N}$ that converges in measure to some function h. From the definitions of our sets therefore exists a sequence $(Y^{n,0}, Y^{n,1}) \in \mathcal{B}(1)$, $n \in \mathbb{N}$ and with

$$Y_T^{0,n} \geq h^n$$

for all $n \in \mathbb{N}$. Next we need the following Lemma (see [12], Lemma 5.2.1): Let $(X)^n$ with $n \in \mathbb{N}$ be a sequence of supermartingales which are uniformly bounded from below and $X_0^n = 0$ for all $n \geq 1$. In addition let T be a dense countable subset of $\mathbb{R}^+$. Then there is a sequence $Y^n \in conv(X^n, X^{n+1}, \ldots)$ (the minimal convex set containing $(X^n, X^{n+1}, \ldots)$ for $n \geq 1$ and a supermartingale Y such that $Y_0 \leq 0$ and Y^n with $n \geq 1$ is Fatou convergent (so uniformly bounded from below and convergence a.s.).

So since $Y^{n,0}$ and $Y^{n,1}$ are (non-negative) supermartingales there exists a sequence $(\tilde{Y}^{0,n}, \tilde{Y}^{1,n})$ and supermartingales $\tilde{Y}^0$ and $\tilde{Y}^1$ such that

$$(\tilde{Y}^{n,0}) \rightarrow^{a.s.} \tilde{Y}^0 \quad \text{a. s.}$$

$$(\tilde{Y}^{n,1}) \rightarrow^{a.s.} \tilde{Y}^1 \quad \text{a. s.}$$

for all $t = 0, \ldots, T$. This is already enough to imply that $(\tilde{Y}^{0,n}, \tilde{Y}^{1,n})$ is a process in $\mathbb{R}_+^2$, so positive, with $\tilde{Y}_0^0 = 1$. Further on

$$\tilde{Y}^0 \varphi^0 + \tilde{Y}^1 \varphi^1$$

is a non-negative supermartingale (for all $(\varphi^0, \varphi^1) \in \mathcal{A}(1)$ and $\tilde{Y}_T^0 \geq h$). And so at last we have that $(\tilde{Y}^0, \tilde{Y}^1) \in \mathcal{B}(1)$ and therefore $h \in \mathcal{D}$ which shows that $\mathcal{D}$ is closed.

2.

It is easy to see that $D \subset \mathcal{D}$. Now we need an equivalence from the superreplication theorem under transaction costs:

There is a self-financing trading strategy φ_t and a $v \in \mathbb{R}$ such that $\varphi_t \leq v + \varphi_T \Leftrightarrow$ for a process Z_t that satisfies $(CPS^{\lambda'})$ we have $E[(\varphi, Z_T)] \leq (v, Z_0)$ for a $v \in \mathbb{R}^d$ (see also [11], Theorem 4.1.) from there we get the first assertion.

The second assertion follows directly from Proposition 3.4.1.

3.

Again due to our definition of $\mathcal{C}$, the constant function $\mathbb{1}$ lies in the set $\mathcal{C}$. By the definition of set $\mathcal{D}$ we get that it must contain a strictly positiv element, and this implies the $L^0_+(P)$-boundedness.

4.

By martingale convergence it follows already that the set $D := \{Z^0_T | (Z^0, Z^1) \in \mathcal{Z}\}$ is closed in the norm of $L^1(P)$, and therefore it is also closed under countable convex combinations. ■

The last part of Theorem 3.4.1. we need to show, is Part 4. This we do now in this separate proof.

Proof: By the above proof of part 1 of Lemma 3.3.1. and also by part 5 of Theorem 3.4.1. we have a sequence of $(Z^{0,n}, Z^{1,n}) \in \mathcal{Z}$ with

$$\hat{y}(x)Z^{0,n}_t \to \hat{Y}^0_t(\hat{y}(x)) \quad \text{almost surely}$$

$$\hat{y}(x)Z^{1,n}_t \to \hat{Y}^1_t(\hat{y}(x)) \quad \text{almost surely}$$

for all $t = 0, \ldots, T$. By definition of $\mathcal{Z}$ the fraction of these sequences lies within the bid/ask-spread $[\underline{S}, \overline{S}]$,

$$\frac{Z^{0,n}}{Z^{1,n}} \in [\underline{S}, \overline{S}]$$

and each strategy $(\varphi^0, \varphi^1) \in \mathcal{A}(x)$ is also a self financing strategy in the frictionless market with price process $\frac{Z^{1,n}}{Z^{0,n}}$. Also we have by definition of the sequences that

$$Z^{0,n}\left(x + \varphi^1 \cdot \frac{Z^{1,n}}{Z^{0,n}}\right)$$

is a local martingale that is bounded from below, and therefore also a supermartingale. From that we get the following result

$$
\begin{aligned}
E[Z_T^{0,n}\hat{\varphi}_{T+1}^0(x)] &= E\left[Z_T^{0,n}\left(\sum_{k=1}^{T+1}\left(\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)\right)+x+\hat{\varphi}^1(x)\cdot\frac{Z_T^{1,n}}{Z_T^{0,n}}\right)\right]\\
&\leq E\left[Z_T^{0,n}\left(\sum_{k=1}^{T+1}\left(\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)\right)\right)\right]+x
\end{aligned}
$$

And by applying Fatous Lemma two times we get

$$
\begin{aligned}
x\hat{y}(x) = E[\hat{Y}_T^0(\hat{y}(x))\hat{\varphi}_{T+1}^0(x)] &\leq \liminf_{n\to\infty} E[\hat{y}(x)Z_T^{0,n}\hat{\varphi}_{T+1}^0(x)]\\
&= \liminf_{n\to\infty} E\left[\hat{y}(x)Z_T^{0,n}\left(\sum_{k=1}^{T+1}\left(\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)\right)\right)\right]+x\hat{y}(x)\\
&\leq \limsup_{n\to\infty} E\left[\hat{y}(x)Z_T^{0,n}\left(\sum_{k=1}^{T+1}\left(\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)\right)\right)\right]+x\hat{y}(x)\\
&\leq E\left[\hat{Y}_T^0(\hat{y}(x))\left(\sum_{k=1}^{T+1}\left(\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)\right)\right)\right]+x\hat{y}(x)
\end{aligned}
$$

And so we get for $k=1,\ldots,T+1$ that

$$\Delta\hat{\varphi}_k^0(x)+\frac{Z_{k-1}^{1,n}}{Z_{k-1}^{0,n}}\Delta\hat{\varphi}_k^1(x)=0$$

and this implies the last part of the Theorem 3.4.1. ■

3.5 Existence under stricter conditions

In this last section we will have a small look at what we can do, to get better results from the main theorem of the last section for the utility optimization problem. As shown in the last section a shadow price usually fails to exist. Key problem here is that the dual minimizer $\frac{\hat{Y}^1}{\hat{Y}^0}$ must not in any case be a martingale, which would be necessary for a shadow price. Otherwise if we assume some stronger restrictions (namely this martingale property) we can derive better results:

Proposition 3.5.1

If there is a minimizer $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}(\hat{y}(x))$ *of the dual problem*

$$E[V(Y_T)] \to \min \quad \textit{for} \quad Y \in \mathcal{Y}(y, \tilde{S})$$

and it is a martingale, then $\hat{S} := \frac{\hat{Y}^1}{\hat{Y}^0}$ *is a shadow price.*

Proof: Let us take a martingale $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}(\hat{y}(x))$ If we now have a look at the following process

$$\hat{Y}^0 \varphi^0 + \hat{Y}^1 \varphi^1 = \hat{Y}^0 \left(x + \varphi^1 \cdot \frac{\hat{Y}^1}{\hat{Y}^0} \right)$$

we can see that this is a non-negative (local) martingale and (as long as we forbid short selling e.g. no negative values for φ^1) also a supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}(x, \frac{\hat{Y}^1}{\hat{Y}^0})$. And so we can follow that $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x), \frac{\hat{Y}^1}{\hat{Y}^0})$. By our definition of the duality we get that

$$\hat{\varphi}^0_{T+1} = (U')^{-1}(\hat{Y}^0_T) \Leftrightarrow \hat{Y}^0_T = U'(\hat{\varphi}^0_{T+1})$$

and therefore the above supermartingale $\hat{Y}^0 \left(x + \varphi^1 \cdot \frac{\hat{Y}^1}{\hat{Y}^0} \right)$ is by Theorem 3.4.1. also a martingale.

Again because of our definition of the dual functions (and by Theorem 2.2. from [10]) we got that $(\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{A}(x, \frac{\hat{Y}^1}{\hat{Y}^0})$ is a solution for the primal problem

$$E[U(\varphi^0_{T+1})] \to \max \quad \text{with} \quad (\varphi^0, \varphi^1) \in \mathcal{A}(1)$$

and $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x), \frac{\hat{Y}^1}{\hat{Y}^0})$ is a solution for the dual problem for $\frac{\hat{Y}^1}{\hat{Y}^0}$

$$E[V(Y_T)] \to \min \quad \text{for} \quad Y \in \mathcal{Y}(y, \tilde{S})$$

if $\hat{y}(x, \frac{\hat{Y}^1}{\hat{Y}^0}) = \hat{y}(x)$, what we will check now:

By definition of the value functions u and v and Theorem 3.4.2. we have that

$$u(x) = v(\hat{y}(x)) + x\hat{y}(x)$$

and by

$$\sup_{\varphi_{T+1}\in\mathcal{C}(x)} E[U(\varphi^0_{T+1})] \leq \inf_{\tilde{S}\in[\underline{S},\overline{S}]} \sup_{\varphi_{T+1}\in\mathcal{C}(x,\tilde{S})} E[U(\varphi^0_{T+1})]$$

we got that

$$u(x) \leq u\left(x, \frac{\hat{Y}^1}{\hat{Y}^0}\right) \leq v\left(\hat{y}(x), \frac{\hat{Y}^1}{\hat{Y}^0}\right) + x\hat{y}(x)$$

And finally since $v(\hat{y}(x)) = E[V(\hat{Y}^0_T)]$, $E[\hat{\varphi}^0_{T+1}\hat{Y}^0_T] = x\hat{y}(x)$ and $\hat{Y}^0 \in \mathcal{Y}(\hat{y}(x), \frac{\hat{Y}^1}{\hat{Y}^0})$ we got that $\hat{y}(x, \frac{\hat{Y}^1}{\hat{Y}^0}) = \hat{y}(x)$, so the above assumption is correct and the proof is finished. ■

We now just showed, that under the assumption of the martingale property a shadow price is again able to be derived, we will now show that it is also true that if a shadow price exists it has to coincide with the strategy derived from the dual problem.

Proposition 3.5.2

If there exists a shadow price $\hat{S}$, then it is given by $\hat{S} = \frac{\hat{Y}^1}{\hat{Y}^0}$ for a minimizer $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}(\hat{y}(x))$ of

$$E[V(Y_T)] \to \min \quad \textit{for} \quad Y \in \mathcal{Y}(y, \tilde{S})$$

Proof: For our frictionless optimization problem

$$E[U(\varphi^0_{T+1})] = E[U(x + \varphi^1 \cdot \tilde{S}_T)] \to \max$$

there exists a solution $(\varphi^0, \varphi^1) \in \mathcal{A}(x, \tilde{S})$ because of the definition of a shadow price. We now need to check that the process $(\tilde{S})^T_{t=0}$ is arbitrage free. For that we assume that there is a arbitrage possibility. Because we are working in discrete time this is equivalent to the existence of an arbitrage opportunity (ξ^0, ξ^1) at one particular time period. If we now consider this in our solution $(\varphi^0, \varphi^1) \in \mathcal{A}(x, \hat{S})$ for the above optimization problem, this strategy would lead to a strictly higher expected utility:

$$E[U(\varphi^0_{T+1})] < E[U((\varphi^0 + \xi^0)_{T+1})]$$

(for detailed information how to add the arbitrage opportunity see proof of Theorem 3.3 in [13]). But then would $(\varphi^0 + \xi^0, \varphi^1 + \xi^1) \in \mathcal{A}(x, \tilde{S})$ and this would be a contradiction to the assumption that (φ^0, φ^1) is a solution to $E[U(\varphi^0_{T+1})] = E[U(x+\varphi^1\cdot\tilde{S}_T)] \to \max$, and so $(\tilde{S})^T_{t=0}$ is arbitrage free. By the fundamental theorem of asset pricing we get that there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ for $\tilde{S}$ under which $\tilde{S}$ is still a martingale. And so we get that $\mathcal{Y}(y, \hat{Y}) \neq \emptyset$ for all $y > 0$. Because it is a shadow price $\hat{S}$ it is valued within the bid/ask spread $[\underline{S}, \overline{S}]$ any solution $(\varphi^0, \varphi^1) \in \mathcal{A}(x)$ is also self-financing in the market without frictions, and so $\mathcal{A}(x) \subseteq \mathcal{A}(x, \tilde{S})$. And with that information we get for the dual function that $(Y^0, Y^1) := (Y, Y\tilde{S}) \in \mathcal{B}(y)$ for all $Y \in \mathcal{Y}(y, \tilde{S})$ and so we get a similar result for the inequality referring to the value function:

$$u = \sup_{\varphi_{T+1} \in \mathcal{C}(x)} E[U(\varphi^0_{T+1})] \leq \inf_{\tilde{S} \in [\underline{S}, \overline{S}]} \sup_{\varphi_{T+1} \in \mathcal{C}(x, \tilde{S})} E[U(\varphi^0_{T+1})]$$

that is

$$v(y) = \inf_{(Y^0, Y^1) \in \mathcal{B}(y)} E[V(Y^0_T)] \leq \inf_{Y \in \mathcal{Y}(y, \tilde{S})} E[V(Y_T)] = v(y; \tilde{S})$$

If we observe the following chain of inequalities

$$u(x) = v(\hat{y}(x)) + x\hat{y}(x) \leq v(\hat{y}(x, \tilde{S})) + x\hat{y}(x, \tilde{S}) \leq v(\hat{y}(x, \tilde{S}), \tilde{S}) + x\hat{y}(x, \tilde{S}) = u(x, \tilde{S}) = u(x)$$

We get $\hat{y}(x) = \hat{y}(x, \tilde{S})$ and further more $(\hat{Y}^0, \hat{Y}^1) = (\hat{Y}, \hat{Y}\tilde{S}) \in \mathcal{B}(\hat{y}(x))$ is a solution to

$$E[V(h)] \to \min$$

where $\hat{Y} \in \mathcal{Y}(\hat{y}(\hat{x}, \hat{S}))$ is a solution to

$$E[V(Y_T)] \to \min$$

which completes the proof. ■

With that we now take again a look at our example from section 3.2. If we would take the optimizer from Theorem 3.4.2., we would face the problem that the solution is not a shadow price, at least not in the global sense. But it is still possible to get a local solution for each period $(t, t+1)$. For that we first of all rewrite our initial optimization problem

$$E[U(g)] \to \max \quad \text{for} \quad g \in \mathcal{C}(x)$$

into the notation of dynamic programming (see Theorem 3.1.)

$$E[\mathcal{U}(\varphi^0_{T+1}, \varphi^1_{T+1})] \to \max \quad \text{for} \quad (\varphi^0, \varphi^1) \in \mathcal{A}(x)$$

Here $\mathcal{U}$ is a 2-dimensional optimization function with

$$\mathcal{U}(x, y) := \begin{cases} U(x), & \text{if } y \geq 0, x > 0 \\ -\infty, & \text{else} \end{cases}$$

We now need to define a $\mathcal{F}_t$-**conditional value function** for all $t = 0, \dots, T$ and every $(\psi^0, \psi^1) \in L(\Omega, \mathcal{F}_t, P, \mathbb{R}^2)$ that is

$$\mathcal{U}_t(\varphi^0, \varphi^1) := \operatorname*{ess\,sup}_{(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}} E(\mathcal{U}(\psi^0 + (\tilde{\varphi}^0_{T+1} - \tilde{\varphi}^0_t), \psi^1 + (\tilde{\varphi}^1_{T+1} - \tilde{\varphi}^1_t)) \big| \mathcal{F}_t) =$$

where the right hand side of the equation is defined as the difference between the positive and the negative part of the conditional value function. Since parts of this function can obtain values ∞ and $-\infty$ we are going to set $+\infty - \infty = -\infty$

With that definition, we first of all get, that all dual minimizers lie in the $\mathcal{F}_t$-conditional superdifferential (along the optimal strategy $(\hat{\varphi}^0, \hat{\varphi}^1)$) of this $\mathcal{F}_t$-conditional concave function in the following sense

Proposition 3.5.3

For any dual minimizer $(\hat{Y}^0, \hat{Y}^1 \in \mathcal{B}(\hat{y}(x))$ we have for the $\mathcal{F}_t$-conditional value function

$$\mathcal{U}_t(\psi^0, \psi^1) \leq \mathcal{U}_t(\hat{\varphi}^0_t, \hat{\varphi}^1_t) + (\psi^0 - \hat{\varphi}^0_t)\hat{Y}^0_t + (\psi^1 - \hat{\varphi}^1_t)\hat{Y}^1_t$$

almost surely for all $(\psi^0, \psi^1) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2)$ and all $t = 0, \dots, T$.

Proof: Similar to the primal case where we defined the objective function $\mathcal{U}$, we now define the 2-dimensional objective function for V as

$$\mathcal{V}(x, y) := \begin{cases} V(x), & \text{if } y \geq 0, x > 0 \\ \infty, & \text{else} \end{cases}$$

We can reformulate our dual problem as

$$E[\mathcal{V}(Y^0_T, Y^1_T)] \to \max \quad \text{with} \quad (Y^0, Y^1) \in \mathcal{B}(y)$$

so we now have to rewrite this optimization problem in terms of dynamic programming, therefore we set for $(Y^0, Y^1) \in \mathcal{B}(y)$

$$\mathcal{B}(Y^0, Y^1, t) = \left\{ (\tilde{Y}^0, \tilde{Y}^1) \in \mathcal{B}(y) \big| (\tilde{Y}^0_s, \tilde{Y}^1_s) = (Y^0_s, Y^1_s) \text{ for } s = 0, \dots, t \right\}$$

and again similar to the primal case we have

$$\mathcal{V}_t(Y^0, Y^1) := \operatorname*{ess\,inf}_{(\tilde{Y}^0, \tilde{Y}^1) \in \mathcal{B}((Y^0, Y^1); t)} E[\mathcal{V}(\tilde{Y}^0_T, \tilde{Y}^1_T) \big| \mathcal{F}_t]$$

When we apply the martingale optimality principle (from martingale property follows solution to corresponding dynamic programming problem) we get

$$\mathcal{U}_t(\hat{\varphi}^0_t, \hat{\varphi}^1_t) = E[\mathcal{U}(\hat{\varphi}^0_{T+1}, \hat{\varphi}^1_{T+1})\big|\mathcal{F}_t] \Leftrightarrow$$

$$(\hat{\varphi}^0, \hat{\varphi}^1) \text{ is a solution to } E[\mathcal{U}(\varphi^0_{T+1}, \varphi^1_{T+1}] \to \max$$

$$\mathcal{V}_t(\hat{Y}^0, \hat{Y}^1) = E[\mathcal{V}(\hat{Y}^0_T, \hat{Y}^1_T)\big|\mathcal{F}_t] \Leftrightarrow$$

$$(\hat{Y}^0, \hat{Y}^1) \text{ is a solution to } E[\mathcal{V}(Y^0_T, Y^1_T)] \to \max$$

We now can use Fenchels' inequality (for $x \in X$ and $y \in X^*$ we have $\langle y, x\rangle \leq f(x) + f^+(y)$) to get for all $(\psi^0, \psi^1) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2)$

$$\mathcal{U}(\psi^0 + (\tilde{\varphi}^0_{T+1} - \tilde{\varphi}^0_t), \psi^1 + (\tilde{\varphi}^1_{T+1} - \tilde{\varphi}^1_t)) \leq \mathcal{V}(\hat{Y}^0_T, \hat{Y}^1_T) + (\psi^0 + (\tilde{\varphi}^0_{T+1} - \tilde{\varphi}^0_t))\hat{Y}^0_T + (\psi^1 + (\tilde{\varphi}^1_{T+1} - \tilde{\varphi}^1_t))\hat{Y}^1_T$$

for all $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}(x)$. We also get from part 3 of Theorem 3.4.1.

$$U(\hat{\varphi}^0_{T+1}) = \mathcal{U}(\hat{\varphi}^0_{T+1}, \hat{\varphi}^1_{T+1}) = \mathcal{V}(\hat{Y}^0_T, \hat{Y}^1_T) + \hat{\varphi}^0_{T+1}\hat{Y}^0_T + \hat{\varphi}^1_{T+1}\hat{Y}^1_T$$

Taking conditional expectation as well as optimizing the equations gives us

$$\mathcal{U}_t(\psi^0, \psi^1) \leq \mathcal{V}_t(\hat{Y}^0, \hat{Y}^1) + \psi^0_t\hat{Y}^0_t + \psi^1_t\hat{Y}^1_t$$

$$\mathcal{U}_t(\hat{\varphi}^0, \hat{\varphi}^1) = \mathcal{V}_t(\hat{Y}^0, \hat{Y}^1) + \hat{\varphi}^0_t\hat{Y}^0_t + \hat{\varphi}^1_t\hat{Y}^1_t$$

When we look at the first inequality (and cancel out the case $\mathcal{U}_t = -\infty$, where the inequality is trivially satisfied) we can see that there exists a strategy $(\varphi^0, \varphi^1) \in \mathcal{A}(\tilde{x})$ for some $\tilde{x} > 0$ so that $(\varphi^0_{T+1}, \varphi^1_{T+1}) = (\psi^0 + (\tilde{\varphi}^0_{T+1} - \tilde{\varphi}^0_t), \psi^1 + (\tilde{\varphi}^1_{T+1} - \tilde{\varphi}^1_t))$ and further more the process $(\hat{Y}^0_t\varphi^0_{t+1} + \hat{Y}^1_t\varphi^1_{t+1})^T_{t=0}$ is a supermartingale with

$$E[\hat{Y}^0_T\varphi^0_{T+1} + \hat{Y}^1_T\varphi^1_{T+1}\big|\mathcal{F}_t] \leq \hat{Y}^0_t\psi^0 + \hat{Y}^1_t\psi^1$$

and finally by combining the above equation and inequality we get our result:

$$\mathcal{U}_t(\psi^0, \psi^1) \leq \mathcal{U}_t(\hat{\varphi}^0_t, \hat{\varphi}^1_t) + (\psi^0 - \hat{\varphi}^0_t)\hat{Y}^0_t + (\psi^1 - \hat{\varphi}^1_t)\hat{Y}^1_t$$

for all $(\psi^0, \psi^1) \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^2)$ almost surely. ■

Again we will have a look at our example: so let the investor follow the optimal strategy $\hat{\varphi}$ until time $t-1$. Then at time t he has the possibility to change his position between stock and bond, but at the frictionless price $\frac{\hat{Y}^1}{\hat{Y}^0}$. Say he changes the position by a value

ν. So the positions ψ_t^0 and ψ_t^1 changed in the following way

$$\psi_t^1 = \hat{\varphi}_t^1 + \nu \quad \text{and} \quad \psi_t^0 = \hat{\varphi}_t^0 - \nu \frac{\hat{Y}^1}{\hat{Y}^0}$$

By the inequality of the above Proposition 3.5.2, this doesn't change the (indirect) utility of the trader

$$\mathcal{U}_t(\psi_t^0, \psi_t^1) \leq \mathcal{U}_t(\hat{\varphi}_t^0, \hat{\varphi}_t^1)$$

But this means, that if we allow the trader to change position at one time t, even if he is allowed to trade at the more favourable frictionless price $\frac{\hat{Y}^1}{\hat{Y}^0}$, this is not enough to force him to give up the strategy in the market with transaction costs (because as above mentioned, it will not change the utility or at least it will not raise the utility above the utility from the frictional optimal trading strategy).

However the problem that we have to deal with now is, that the interpretation (in contrast to the original definition of a shadow price, see Definition 1.3.1.) just focuses on one particular time spot t saying, this result is just "local". Contrary to the above corollary where both sides of the inequality were said to be valid at all time.

And now one last time looking at our example from above section 3.2.: if the investor would trade at time $t = 1$ beyond the optimal strategy from the market with transaction costs, this would lead to bankruptcy when he must liquidate (at the bid/ask-price) at the end of trading period $t = 2$ (explanation see section 3.2.). Otherwise if he would consider the frictionless market price $\frac{\hat{Y}^1}{\hat{Y}^0}$ (which is derived from the dual minimization problem) at all trading periods, this problem will not occur.

Appendix A

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Appendix B

Curriculum Vitae

Persönliche Daten

Name	Stefan BACHNER, BSc.
Wohnort	Wien

Ausbildung

Master-Studium: Mathemtik	10/2010 - 07/2015
	Universität Wien
Studiumsschwerpunkt	Stochastik und dynamische Prozesse
Master-Arbeit	Existence of a Shadow Price in Discrete Time
Betreuer	Prof. Walter Schachermayer
Bachelor-Studium: Mathemtik	10/2007 - 10/2010
	Universität Wien
Studiumsschwerpunkt	Vorbereitung auf wissenschaftliche Arbeit
HTL-Ausbildung	09/2001 - 06/2006
	HTBLVA für chemische Industrie, Wien 1170
Schwerpunkt	Biochemie, Biotechnologie und Gentechnik