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Stability Analysis of an Actin-driven Lamellipodium Model with Pressure

Verfasser

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Abstract

We will consider a new model for actin-driven cell movement by action of a lamellipodium developt in [9], which also takes pressure into account. Numerical simulations indicate the stability of a steady state of the simplified version of this model. This work seeks to verify this behavior by carrying out a linearization around this steady state and imposing rotational symmetry. The resulting system is well posed and by defining a entropy for this system, we will determine the long-time behavior of the solution. This indicates the persistence of the actin-filament network over time for this new model. Furthermore, we will show that another linearization, without the rotational symmetry assumption, possesses smooth solutions for all time.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit einem neuen Model zur Beschreibung der actin-getriebenen Zellbewegung mithilfe von Lamellipodia, entwickelt in [9]. Numerische Simulationen legen die Stabilität eines stationären Punktes einer vereinfachten Version dieses Models nahe. Meine Arbeit zielt nun darauf ab, dieses Verhalten zu bestätigen, in dem ich eine Linearisierung, unter der Annahme von Rotationssymmetrie, in der Nähe des stationären Punktes ausgeführt habe. Das resultierende Problem besitzt dann eine eindeutige Lösung, und durch die Definition einer Entropie für das System können wir auch das Langzeitverhalten der Lösung bestimmen. Diese Ergebnisse legen nun nahe, dass im oben angeführten Model das Netzwerk aus Actinfilamenten auch über längere Zeit erhalten bleibt. Im Kontext einer anderen Linearisierung, die ohne Rotationssymmetrie auskommt, können wir zusätzlich die Existenz von glatten Lösungen für alle Zeiten nachweisen.

Contents

Part I. Introduction

Movement, of whatever sort it may be, is a general key characteristic of living things. So its not surprising that one of the smallest example of them, cells, have many reasons to do so too. Although the field of questions as to why they move is quite interesting, this work treats the questions of how they move. One attempt to mathematically describe the dynamics of cell migration can be found in [9], on which this work is heavily based. This unreleased paper is just one of a structured mathematically approach, that takes place in a series of works (such as [10, 11, 12, 13]). The main goal of this work is to verify, at least mathematically, certain aspects of this model. Therefore, before we proceed to the mainly mathematical work, this introduction is divided into two parts: I will first state some basic facts about cells (especially their movement) and then conclude with the above mentioned mathematical modeling approach in the second part.

1.1 Cell migration: From actin to action

The following descriptions of a cell are mostly based on [2], where more details can be found. The cellular lifeforms can be classified into three principle domains: bacteria, achaean and eukaryotes. The last ones build up all complex lifeforms, such as plants, animals and fungi. In the following, we will only discuss some basic facts about eukaryotes and its "musculoskeletal system", which will help us to understand the modeling assumptions in the second part.

1.1.1 Cell structure

The eukaryote cell differs by the other types of cells by possessing a cell nucleus, which carries most of the genetic material. The kernel is surrounded by cytoplasm, which is filled with other cell organelles like mitochondria and cytosol (mostly water). The cytosol itself is traversed by a network of different type of protein filaments, the so-called cytoskeleton.

Cytoskeleton

Every eukaryote cell is sustained by such a skeleton, which enables it to maintain its shape, move around and organize its cell components. This network gives the cell not only stability, like the skeleton does inside the human body, it has also the ability to change its shape dynamically to fulfill certain tasks, such as moving. The structure consists basically of three types of protein filaments with different assignments:

- Intermediate filaments (structural integrity)
- Microtubulues (intracellular transport, organizing inner cell structure)
- Actin filaments (movement, cell migration and stability)

1.1.2 Lamellipodia

Migration of a eukaryote cell is rather complex, and consists of different forms of movement. In the following, we will attempt to describe one of these forms, the so-called migration by action of a lamellipodium. This kind of movement is used by cells to crawl across substrates and it is believed to be mostly driven by actin filaments, therefore we want to take a closer look at them.

Actin filaments

Actin filaments are rather thin and flexible rodes with a diameter of about 7 nm. These rods, also called F-Actin, are double helix shaped polymers, which are build up by actin molecules that are all directed parallel to the filament. Therefore the actin filaments have plus, so-called barbed, and minus, so-called pointed, ends. The growth of such a rode is the result of polymerization, whereby unattached actin monomers (G-Actin) in the cytosol are carrying ATP (Adenosine triphosphate) and are inserted at the ends of the filament. After entering the filament, the ATP is hydrolyzed to ADP (Adenosine diphosphate), and as a byproduct the stability of the polymer is weakened. This allows proteins like the ADF/cofilin family to remove older (hydrolyzed) monomers at the minus ends or the removal of larger actin parts of the filament by severing proteins like gelsolin. Although the polymerization happens at both ends, it is more likely to take place at the plus ends, where the ATP is not transformed yet, which gives the growth a direction. The removed ADP bound monomers can then again be "recharged" with ATP in the cytosol, which forms the last part of the so called actin cycle. Conjointly, this forms a treadmill-like process, during which a inherent turnover of actin monomers takes place inside the filament. Furthermore, the F-Actin behaves like a poly electrolyte and is negatively charged, which leads in principle to a repulsive force between the rods. Nevertheless, actin filaments are not likely to find themselves isolated, since positively charged polycations can neutralize the negative charge of the filament, and in general, they tend to be part of densely connected durable structures. These structures can fulfill different types of tasks inside the cell and gain their stability through cross-links by actin associated proteins such as filamin. The concentration of actin monomers is in most cells higher along the cell membrane (called cell cortex), which gives this region the ability to form protrusions, such as filopoda and lamellipodia. The former are believed to play a main role in the scout/guide system of the cell, whereby the latter take a major part in the migration of the cell.

Movement by action of a lamellipodium

Although there are many aspects of this movement, the main sequence of events can be described for our purpose in the following model:

The cells are moving by protrusions in the front and retracting in the rear. These protrusions are mainly formed by a complex quasi two-dimensional network of fast growing actin filaments, the lamellipodium. In this network, the barbed ends of the actin filaments are directed to the membrane, and the pointed ends are roughly directed inwards the cell. In order to expand the protrusion, the filaments are extended at the barbed ends by actin polymerization. Furthermore, new filaments can be created at the leading edge by branching of existing filaments. The protein complex Arp2/3 binds at existing filaments and allows it to nucleate a new filament, which results in a dendrite-like growth, or branching. It is also assumed that certain capping proteins can cap the filaments at the branched ends, and therefore block further expansion of the capped filament. As mentioned above, the actin structure forms a dense network by cross-linking and additionally also adhering to the substrate by trans-membrane-proteins (called integrins). Driven by the constant turnover of the actin filaments, the cross-link network, and hence each filament itself, is exposed to bending, twisting and stretching forces. Also the integrins are affected by stretching forces in this process. Therefore the cross-links and integrins are constantly created, destroyed or shifted on the filament by tension forces and (de-)polymerization as the protrusion takes place. As a consequence, the force of polymerization, together with the network of actin rods, drives the membrane forward as the filaments expand. The second important component of this movement is the retraction at the rear. The motor protein myosin has the ability to attach to the pointed end of a filament and slide to the barbed end by ATP hydrolization. If a myosin protein is attached to two filaments, which point in different direction, this creates a contraction force. It is assumed that this actin-myosin interaction plays a major role in the retraction of the cell towards the migration direction.

This kind of description is been used for different kind of cells and it models in many cases the behavior of cell types in natural environments such as tumor cells or leukocytes.

1.2 The mathematical model

In order to set the results of this work in context, I will present the latest model derived in [9] in this section. First I would like to mention the evolution of this model, which captures most of the biological behavior described above. The basic concept of the model was introduced in the paper [13] and was limited by assuming a rotational symmetry of the cell shape. In the incorporated numerical simulations, it was shown that this concept allows the existence of stable steady states, and therefore the successful simulation of the persistence of the tread-milling actin network. Besides the geometrical assumptions, which can not been guaranteed in the dynamical process, this Ansatz was leading to a delay problem and was therefore not local in time. The reason for this was the detailed description of the cross-links and adhesions, which required to know the whole history of the filament dynamic up to the maximal age of these connections.
In a second paper, [11], a limiting process was carried out, which assumes an instantaneous turnover of cross-links and adhesion, in order to get a model local in time. Furthermore, a existence result of solutions local in time was proven by a time step approximation method, which was also used to introduce a numerical scheme for simulations.
In a next step, the geometry was generalized to non-rotational symmetric shapes in [10], whereby also the rotational symmetric case is included. To verify this, a stability analysis for a trivial steady state in the rotational symmetric case was carried out, which shows the same behavior as in the above mentioned first attempt.
The last released paper to this subject, [12], presents a numerical scheme for the general geometrical case. This was used to produce certain simulations, which showed the shape stability of the "cell" influenced by pushing forces. Although this is very promising, the shape holding feature was achieved (in all models) by a artificial membrane model, and was not stabilized by the network dynamic itself.
Finally, we are arriving at the latest model, [9], which is not released yet. This model deals with certain issues which were not included in the previous ones, such as confinement by retraction, branching/capping of filaments and pressure.

1.2.1 Assumptions and structure

Before I state the assumptions, I would like to emphasize the basic model structure, in order to motivate certain aspects of the following. The model is derived by starting at a situation with finite many filaments followed up by a homogenization limit that leads to a continuous model. This is justified by a very high density of filaments within the lamellipodium. The three main features of the model deal with the mechanics of the filament and the dynamics of cross-links and adhesion. Therefore it is built out of three sub-models: two age-structured population models for the links to other rods and substrate and a quasi-static balance of elastic forces determining the position of the filaments. As mentioned above, this leads to a delay model, whereby the past of the dynamic of the filament has to be known up to the maximal age of cross-links and adhesions. In order to get a model local in time, it is assumed that the lifetime of a such connections is short compared to the dynamics of the network. The following assumptions are the foundation of the model, which simplify the movement by only describing the lamellipodium itself:

Geometry

The lamellipodium assumed to be two dimensional and modeled as homogeneous structure which consists out of two families of locally parallel filaments. The filaments are crossing each other at most once transversally, and are polar with so called barbed and pointed ends. This network is assumed to surround the entire cell, which leads to the topology of a ring, whereby the lamellipodium is situated between two curves. The outer curve determines the shape of the cell and the inner one is a merely artificial boundary, which divides the lamellipodium from the rest of the cell. Since this inner boundary will be described by a stochastic process, it is allowed to have two non-identical inner boundaries corresponding to the two families of filaments. We will also assume that all filaments touch the leading edge, which can be interpreted as tethering at the membrane.

Polymerization, branching and capping

The filaments expand at the barbed ends by polymerization with a given, perhaps verifying, speed. In this process, the structural integrity of a filament is weakened, which leads to a severing process and therefore to a removal of monomers or bigger pieces of actin. Removed actin components are assumed to be decomposed immediately. New filaments can be created by branching off an existing filament of the other family near the membrane, influenced by the availability of Arp2/3. Furthermore, certain capping proteins have the ability to block polymerization at the barbed ends, which leads to a fast, in the model instantaneous, removal of this filament.

Filament mechanics and dynamics

The filaments are affected by bending forces and assumed to be inextensible rods. As mentioned above, they can create cross-links at the crossing of two filaments, which build up a stable network. The link-ups are created spontaneously and can also break, which is assumed to be realized in a high order turnover. Furthermore, it is assumed that this process only takes place between two filaments that are not of the same family and there exist at most one cross-link for any pair of filaments. As long as this connection exists, the filaments experience twisting and stretching forces through the cross-links. Another part of the dynamics stems from the linkages by the actin network to the substrate. These adhesions are assumed to be transient, again with a rapid recycling time. We will also assume, that a pulling effect of actin-myosin interaction in the rear of the actin network takes place, which generates the retraction force mentioned above. Finally, a force between all filaments of the same family is claimed, which is modeled as pressure between the rods, due to the negative charging of the filaments.

1.2.2 The model

Since the derivation of the model is not the main topic here, a view details and all computations are left out.

Geometry and polymerization

We start to model this assumptions by representing the filaments by the unknown function $F^{\pm}(\alpha, s, t) \in \mathbb{R}^2$, corresponding to an arc-length parametrization of a filament with index α at time t. The superscript $+$ and $-$ refer to the two filament families, which we will also call clockwise, respectively anti-clockwise. Due to the homogeneous assumption there are infinitely many filaments, we will index them by the first variable α which is a element of $[0, 2\pi)$, owing to the ring topology. Therefore, we also assume that $F^{\pm}$ and all other functions of α are 2π-periodic. This means in particular that we claim the existence of a clockwise and an anti-clockwise filament for every α. The second variable is the negative arc-length parameter s along a filament, measured from the outer boundary. It describes the length of a filament at every time t, whereby s belongs to the interval $[-L^{\pm}(\alpha, t), 0]$ with a maximal length $L^{\pm}(\alpha, t)$, which is modeled as random variable. Hence, we introduce a length distribution function $\eta^{\pm}(\alpha, s, t)$, such that

$$P(-L^{\pm}(\alpha, t) \leq s) = \eta^{\pm}(\alpha, s, t).$$

The function itself takes also severing, branching and capping at the barbed ends into account. Since branching/capping occurs, barbed ends are created and destroyed, and therefore the distribution function is not uniform, i.e.

$\eta^{\pm}(\alpha, 0, t) \neq 1$ is allowed.
As counterpart to this disintegration of filaments, polymerization needs to be described. For fixed t, the parameter s can be interpreted as material coordinate. When we vary t, the polymerization at the barbed ends takes place, and actin molecules travel along the filament towards the pointed ends with non-negative polymerization speed $v^{\pm}(\alpha, t)$. This is modeled by the interpretation of $\sigma^{\pm} = s + \int_0^t v^{\pm}(\alpha, \tau) d\tau$ as Lagrangian variable, which labels monomers along a filament with index α. Hence the path of one actin monomer with label $\sigma^{\pm}$ is given by $F^{\pm}(\alpha, \sigma^{\pm} - \int_0^t v^{\pm}(\alpha, \tau) d\tau, t)$. Since we will later need the orientation of the parametrization of $F^{\pm}$, we will assume that the condition

$$\det(\partial_\alpha F^{\pm}, \partial_s F^{\pm}) > 0 \tag{1.1}$$

is satisfied, which implies a clockwise parametrization by α.
So far, we can observe that the shape of the lamellipodium at time t is represented by the set of all filaments $\mathcal{L}(t) = \mathcal{L}^+(t) \cup \mathcal{L}^-(t)$, with $\mathcal{L}^{\pm}(t) = \{F^{\pm}(\alpha, s, t)(\alpha, s) \in B(t) = [0, 2\pi) \times [-L(\alpha, t), 0]\}$. To ensure the in-extensibility (resp. the arc-length parametrization) of the filaments and our assumption that both families share the leading edge, we formulate the following constraints for our unknown functions:

$$\begin{aligned} &|\partial_s F^{\pm}(\alpha, s, t)| \equiv 1 \qquad \text{for } (\alpha, s) \in B(t),\ t \geq 0 \\ &\{F^+(\alpha, 0, t) : 0 \leq \alpha < 2\pi\} = \{F^-(\alpha, 0, t) : 0 \leq \alpha < 2\pi\}. \end{aligned} \tag{1.2}$$

As last step of the geometrical part, we define the outward unit normal vector along the barbed ends by $\nu(\alpha, t)$, which can be computed through $\partial_\alpha F(\alpha, 0, t)$.

Cross-links and adhesions

Next we have to describe the interactions between filaments. We start by noting that our assumptions do not allow crossings of filaments inside the same family, therefore the map $F^{\pm}(\cdot, \cdot, t) : B(t) \to L^{\pm}(t)$ must be one to one. The set of clockwise and anti-clockwise filament pairs that are crossing at a certain time t is given by

$$\begin{aligned} \mathcal{C}(t) = \{(\alpha^+, \alpha^-) \in [0, 2\pi) \times [0, 2\pi) : \exists s^{\pm}(t, \alpha^+, \alpha^-) \text{ such that} \\ F^+(t, \alpha^+, s^+(t, \alpha^+, \alpha^-)) = F^-(t, \alpha^-, s^-(t, \alpha^+, \alpha^-))\} \end{aligned}$$

Hence, the set of points for potential cross-links is the following:

$$\mathcal{B}^{\pm}(t) = \{(\alpha^{\pm}, s^{\pm}(t, \alpha^+, \alpha^-)) : (\alpha^+, \alpha^-) \in \mathcal{C}(t)\} \subseteq B(t).$$

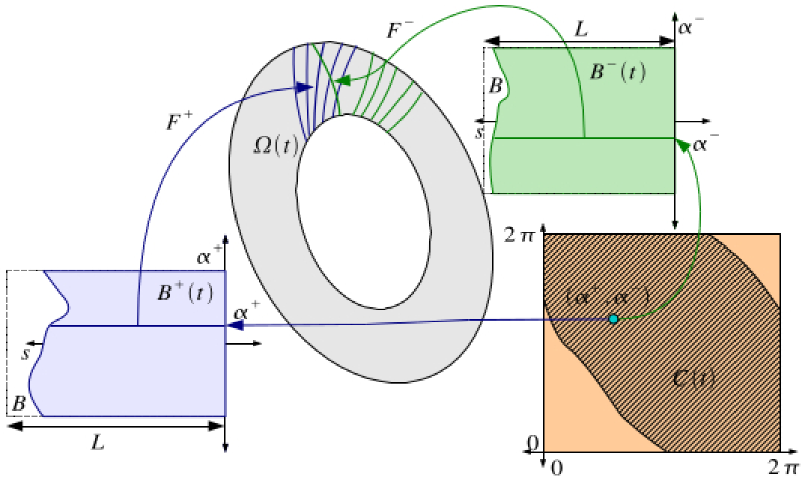

Figure 1.1: Framework of filaments and crosslinks [10]

According to our assumptions, there is a maximum of one cross-link for each pair of filaments. Therefore the transformations that map $\mathcal{C}(t)$ to $\mathcal{B}^{\pm}(t)$,i.e. $(\alpha^+, \alpha^-) \mapsto (\alpha^{\pm}, s^{\pm}(t, \alpha^+, \alpha^-))$, are invertible.

This leads to the existence of a bijective maps $\psi^{\pm} : \mathcal{B}^{\mp}(t) \to \mathcal{B}^{\pm}(t)$, by using the composition of one transformation with the inverse of the other. Furthermore, the pullback by $\psi^{\pm}$ of a clockwise filament gives the anti-clockwise one, corresponding to the domain:

$$F^{\mp} = F^{\pm} \circ \psi^{\pm} \qquad \mathcal{B}^{\mp}(t) \text{ (for fixed t).}$$

To simplify the notation, we will from now on only consider one family of filaments. When both families are present in one term, we will denote the other one with the superscript *.
We are interested in the contribution of the cross-links to the dynamics. So we introduce a measure for the angle-deviation $\varphi - \varphi_0$ of two filaments at the cross-links, with

$$\varphi(t, \alpha, \alpha^*) = \arccos\left[\partial_s F(t, \alpha, s(t, \alpha, \alpha^*) \cdot \partial_s F^*(t, \alpha^*, s^*(t, \alpha, \alpha^*)\right]$$

with $(\alpha, \alpha^*) \in \mathcal{C}(t)$, whereby we compare the angle between them with a equilibrium angle φ_0. Furthermore, we request that $0 \leq \varphi$ and $\varphi_0 \leq \pi$, to exclude degenerate examples for these angles.
The stochastic creating and breaking of cross-links and adhesion is modeled by 2 age-structured population models, which can be solved explicitly after carrying out a limiting process, when the age of such linkages goes to

zero. This corresponds with the assumption that the age of such a connection is short compared to the dynamics of the system. The result is a macroscopic friction effect and resistance to torsion of the continuum at the potential binding sites, by the stiffness parameters μ^S and μ^T. These parameters only act on the set $\mathcal{B}(t)$, since only there the cross-links have to be taken into account. The coefficients itself are then given by $\mu^S[\varphi - \varphi_0]$ and $\mu^T[\varphi - \varphi_0]$, which describes the friction and resistance to torsion with respect to the deviation of the equilibrium angle. Similarly, a macroscopic friction effect for the adhesion is introduced, with a constant friction coefficient μ^A.
At this point, their are no constraints that can control the indefinite expansion of the cell. This can be achieved by introducing pulling forces at the pointed ends of the rods. As described above, this process takes place by actin-myosin bundles that contract in the rear of the lamellipodium. The model for describing this consists of 2 non negative forces: $f_{\tan}(\alpha)$ pulling in the tangential direction and $f_{\text{in}}(\alpha)$ pulling to the center of actin mass. For the first one, the direction is given by $\partial_s F(\alpha, s, t)$. For the second one, we define $V(\alpha)$ as a vector valued normalized quantity directed from the center of mass,

$$C_M := \int_{B(t)} \eta F d(\alpha, s) \left(\int_{B(t)} \eta \; d(\alpha, s) \right)^{-1},$$

to $F(\alpha, -L(\alpha, t))$, the end of the filament with index α. As last step, before we can take at a look the equations of the model, the repulsive forces between filaments of the same family are described by a pressure-like force as $p(\rho) = \Phi'(\rho)\rho^2$. The real valued function $\Phi(\rho)$ is an electrostatic potential and we define

$$\rho := \frac{\eta}{\det(\partial_\alpha F, \partial_s F)}, \tag{1.3}$$

whereby $\rho \; d(\alpha, s)$ for $(\alpha, s) \in \mathcal{L}(t)$ is the filament length in the infinitesimal area element dA of one family of filaments.

The model

Finally we state the model which will be the topic of this work in the next two chapters. As mentioned above, the model is derived by the assumption of a quasi-static force balance, whereby the positions of the filaments, our unknown functions $F(t, \alpha, s)$, are obtained by minimizing an energy functional under the constraints introduced above. To find this minimum, the method of variation is applied, which leads to a weak formulation

$$
\begin{aligned}
0 = &\int_{B(t)} \Bigg(\underbrace{\mu^B \partial_s^2 F \cdot \partial_s^2 \delta F}_{\text{bending}} + \underbrace{\mu^A D_t F \cdot \delta F}_{\text{adhesion}} + \underbrace{\lambda_{\text{inext}} \partial_s F \cdot \partial_s \delta F}_{\text{in-extensibility}} \Bigg) \eta \; d(\alpha, s) \\
&- \int_{B(t)} \underbrace{p(\rho) \left[\det(\partial_\alpha F, \partial_s \delta F) + \det(\partial_\alpha \delta F, \partial_s F) \right]}_{\text{pressure}} \; d(\alpha, s) \\
&+ \int_{C(t)} \Bigg(\underbrace{\mu^S (D_t F - D_t^* F^*) \cdot \delta F}_{\text{cross-link stretching}} \mp \underbrace{\mu^T (\varphi - \varphi_0) \partial_s F^\perp \cdot \partial_s \delta F}_{\text{cross-link twisting}} \Bigg) \eta \eta^* \; d(\alpha, \alpha^*) \\
&+ \underbrace{\int_{[0,2\pi)} \left((f_{\tan} \partial_s F + f_{\text{in}} V) \cdot \delta F \eta \right)\big|_{s=-L} \; d\alpha}_{\text{myosin contraction}} - \underbrace{\int_{[0,2\pi)} \left(\lambda_{\text{tether}} \nu \cdot \delta F \right)\big|_{s=0} \; d\alpha}_{\text{tethering}},
\end{aligned}
$$

with the test function $\delta F : B(t) \mapsto \mathbb{R}^2$, the notation $(x_1, x_2)^\perp = (-x_2, x_1)$ and the material derivative $D_t := \partial_t - v\partial_s$. In order to evaluate F^*, we have to use the above defined transformation ψ^*, which maps $\mathcal{B}(t)$ to $\mathcal{B}^*(t)$.
The first line basically models the linearized deformation of beams, whereby the first term describes the bending, the second term the friction with the substrate, and the third term ensures, with the Lagrange multiplier $\lambda_{\text{inext}}(t, \alpha, s)$, the in-extensibility of the filaments. The second and third line describe the energy contribution of the interactions between one family in the second line, and in the third line the interactions at the cross-links of two different families. In the last line the pulling forces at the pointed ends of the filaments and the second constraint, which states that all filaments touch the leading edge, are included.
Now we want to state the strong formulation, whereby we need to map $\mathcal{C}(t)$ to $\mathcal{B}(t)$. Therefore, we redefine the new stiffness parameters, which also take care of the required Jacobians for the transformation as follows:

$$
\mu^S = \begin{cases} \mu^S \left| \frac{\partial \alpha^*}{\partial s} \right| & \text{in } \mathcal{B}(t), \\ 0 & \text{in } B(t) \backslash \mathcal{B}(t), \end{cases} \quad \mu^T = \begin{cases} \mu^T \left| \frac{\partial \alpha^*}{\partial s} \right| & \text{in } \mathcal{B}(t), \\ 0 & \text{in } B(t) \backslash \mathcal{B}(t). \end{cases}
$$

The Euler-Lagrange equations read then as follows:

$$
\begin{aligned}
0 = &\mu^B \partial_s^2 (\eta \partial_s^2 F) - \partial_s (\eta \lambda_{\text{inext}} \partial_s F) + \mu^A \eta D_t F + \\
&+ \partial_s (p(\rho) \partial_\alpha F^\perp) - \partial_\alpha (p(\rho) \partial_s F^\perp) + \\
&+ \partial_s (\eta \eta^* \mu^T (\varphi - \varphi_0) \partial_s F^\perp) + \eta \eta^* \mu^S (D_t F - D_t^* F^*).
\end{aligned}
$$

The corresponding boundary conditions are given by

$$
\begin{aligned}
-\mu^B \partial_s(\eta \partial_s^2 F) - p(\rho)\partial_\alpha F^\perp + \eta \lambda_{\text{inext}} \partial_s F \mp \eta \eta^* \mu^T (\varphi - \varphi_0) \partial_s F^\perp & \\
= \begin{cases} \eta(f_{\tan}(\alpha)\partial_s F + f_{\text{in}}(\alpha) V(\alpha) & \text{for } s = -L, \\ \lambda_{\text{tether}} \nu & \text{for } s = 0, \end{cases} & \\
\eta \partial_s^2 F = 0 \quad \text{for } s = -L, 0, &
\end{aligned}
$$

where the Lagrange multipliers λ_{inext} and λ_{tether} have to be determined, so that (1.2) holds. The functions F^*, η^* and φ have to be evaluated at $(\alpha^*(\alpha^*, s, t), s^*(\alpha, s, t))$ by using the pullback function, e.g. $F^* \circ \psi^*$.

Part II. Analysis of the simplified pressure model

The main goal of this chapter is to analyze the effect of the newly introduced pressure term. The proposed model in the first chapter is formed by a system of non-linear (quasilinear) partial differential equations, whereby the non-linearity steming from the pressure term possesses also mixed derivatives. Additionally to this difficulty, it is also posed with boundary conditions and the constraints by the Lagrange multipliers. The numerical simulations as presented in [9] indicate the stability of a trivial steady state in a rotational symmetric setting. Under these circumstances, this would imply the persistence of the actin network in this model. Since the analysis of the whole problem is not part of a standard theory, and would therefore exceed the boundaries of this work, I will have to make some assumptions and linearize the equation around the steady state. This will lead us to a situation where, at least for the linearized equation, well-posedness is evident and we can characterize the asymptotical behavior of the solutions. The results for the linearized system indicate that this steady state is stable (in a rotational symmetric way).

2.1 Assumptions

Since the pressure effect takes only place between members of the same family, we will reduce my effort to only one family of filaments. Therefore, also cross-links and adhesions can be neglected, and in addition, we will assume that the length distribution is constant, i.e $\eta = 1$, as well as no polymerization is involved. Furthermore, I will dismiss the bending term and assume that only tangential pulling is present. This leads to the following simplified set of equations,

$$\begin{aligned} 0 &= \mu^A \partial_t F - \partial_s(\lambda_{\text{inext}} \partial_s F) + \partial_s(p(\rho)\partial_\alpha F^\perp) - \partial_\alpha(p(\rho)\partial_s F^\perp), \\ |\partial_s F| &= 1, \end{aligned}$$

whereby the second line corresponds to the Lagrange multiplier λ_{inext}. Since we have dismissed the bending term, this system now presents as a full nonlinear set of equations. The boundary conditions are given by

$$-p(\rho)\partial_\alpha F^\perp + \lambda_{\text{inext}}\partial_s F = \begin{cases} 0 & \text{for } s = 0, \\ f_{\tan}(\alpha)\partial_s F & \text{for } s = -L(\alpha). \end{cases}$$

As next step, we make the assumption that our cell is rotational symmetric. This implies, that all filaments can be described by the rotation of a reference filament $z(s,t) \in \mathbb{R}^2$. We choose a clockwise rotation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

and can deduce that $F(\alpha, s, t) = R(\alpha)z(s,t)$ must hold under this assumption. In the spirit of this geometry, we will transform our equations into polar coordinates, i.e. set $z(s,t) = r(s,t)\exp(i\varphi(s,t))$. Before taking this step, we must choose a pressure term and a pulling force. For the first one, we use the Boltzmann-Poisson model for the mobile carrier density, which leads to $p(\rho) = \mu^P\rho$, with $\mu^P > 0$ constant. The pulling force is assumed to be $f_{\tan} = \mu^{IP}(\pi r^2 - \pi\hat{r}_0^2)_+$, whereby the force gets active, pulls, until a specified equilibrium area with radius r_0 is achieved and the coefficient μ^{IP} determines the inner pulling strength. The by our assumption (1.1) positive density function must be computed from (1.3) and reads as $\rho = \frac{1}{(\partial_s z)z} = \frac{1}{(\partial_s r)r}$. For the sake of brevity, we abbreviate λ_{inert} by λ, and introduce the new notation for time and spatial derivatives by $\partial_s r = r'$ and $\partial_t r = \dot{r}$. After some computations we arrive at the following system

$$\begin{aligned} &\mu^A\dot{r} + \mu^P\left(\frac{1}{r'r}\right)' r - \lambda' r' - \lambda(r'' - (\varphi')r) = 0, \\ &\mu^A r\dot{\varphi} - \lambda'\varphi' r - \lambda(2\varphi' r' + \varphi'' r) = 0, \\ &(r')^2 + (\varphi')^2 r^2 = 1, \end{aligned} \tag{2.1}$$

with boundary conditions:

$$-\frac{\mu^P}{r'} + \lambda r' = \begin{cases} 0, & \text{for } s = 0, \\ \mu^{IP}(\pi r^2 - \pi\hat{r}^2)_+ r', & \text{for } s = -L, \end{cases}$$

$$\lambda r\varphi' = \begin{cases} 0, & \text{for } s = 0, \\ \mu^{IP}(\pi r^2 - \pi\hat{r}^2)_+ \varphi' r, & \text{for } s = -L. \end{cases}$$

2.2 Definitions and framework

This section is devoted to introducing some definitions and results that we will need in the following. We start with some basic definitions and notations and conclude with results of semigroup theory in the second part. In the following, X will be a Banach space over $\{\mathbb{C}, \mathbb{R}\}$ with norm $||\cdot||$ or when we want to specify $||\cdot||_X$. If $A : D(A) \subseteq X \to X$ is a linear operator, we denote its domain by $D(A)$ and by $||A||$ the corresponding operator norm w.r.t. X.

Definition 2.1 ([14]) *Let A be a linear operator, the set*

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ exists as a bounded linear operator}\}$$

is called the resolvent set of A. The set of bounded linear operators $R(\lambda : A) = (\lambda I - A)^{-1} with \lambda \in \rho(A)$, is called resolvent of A. The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called spectrum of A.

Definition 2.2 ([14]) *A linear operator A is called dissipative, if for every $x \in D(A)$, there exists a $x' \in F(x)$, with*

$$F(x) = \left\{x' \in X' : \text{ and } \langle x', x\rangle = ||x||^2 = ||x'||^2\right\},$$

whereby we denote X' as the dual space of X, such that

$$\mathrm{Re}\langle Ax, x'\rangle \leq 0.$$

Definition 2.3 ([3]) *For a open set $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, we define the Sobolev space*

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \,\middle|\, \begin{array}{l} \forall\alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^2(\Omega) \text{ such that} \\ \int_\Omega uD^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega g_\alpha\varphi \quad \forall\varphi \in \mathcal{C}_c^\infty(\Omega) \end{array} \right\}.$$

It can be shown that $H^m(\Omega)$ is a Hilbert space with the following scalar product:

$$\langle u, v\rangle_{H^m} = \sum_{0\leq|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle_{L^2}. \tag{2.2}$$

Theorem 2.4 ([1]) *Let Ω be a domain in $\mathbb{R}^n$ and let Ω^k be the k-dimensional domain obtained by intersecting Ω with a k-dimensional plane in $\mathbb{R}^n$, $1 \leq k \leq n$. Let j and m be non negative integers, and $\partial\Omega$ locally Lipschitz, then for $2m > n > (m-1)2$, such that the following continuously imbedding exists*

$$H^{j+m}(\Omega) \hookrightarrow \mathcal{C}^{j,\lambda}(\overline{\Omega}), \quad 0 < \lambda \leq m - \frac{n}{2},$$

where $\mathcal{C}^{j,\lambda}$ is the Hölder space of j-times differentiable functions with Hölder exponent λ.

Definition 2.5 *Let H_1 and H_2 be two Hilbert spaces, then we define the orthogonal sum $H_1 \oplus H_2$ to be the Hilbert space with $f = (f_1, f_2) \in H_1 \times H_2$ and the inner product*

$$\langle f, g \rangle_{H_1 \oplus H_2} = \langle f_1, g_1 \rangle_{H_1} + \langle f_2, g_2 \rangle_{H_2}.$$

Definition 2.6 ([4]) *The space $\mathcal{C}([0,T], X)$ is defined by all continuous functions $u : [0,T] \to X$ such that*

$$||u||_{\mathcal{C}([0,T],X)} = \max_{0 \le t \le T} ||u(t)|| < \infty.$$

We also define the space $\mathcal{C}^k((0,T), X)$ of all k-times continuous differentiable functions $u : (0,T) \to X$ with the norm

$$||u||_{\mathcal{C}^k((0,T),X)} = \sum_{l=1}^{k} ||u^{(l)}(t)||_{\mathcal{C}([0,T],X)} < \infty.$$

Semigroups

I will now briefly introduce some definitions and Theorems from the field of C_0 semigroups, which will become quite useful to determine the well-posedness and the regularity of our linearized system. The following definitions and Theorems can be found in [14], where proofs and more details are presented rigorously.

Definition 2.7 *Let X be a Banach space. A one parameter family $T(t)$, $0 \le t < \infty$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

- *$T(0) = I$, (I is the identity operator on X),*
- *$T(t+s) = T(t)T(s)$ for every $t, s \ge 0$ (the semigroup property).*

A semigroup of bounded linear operators, $T(t)$, is called strongly continuous (for short C_0 semigroup) if

$$\lim_{t \downarrow 0} ||T(t)x - x||_X = 0 \quad \textit{for every } x \in X.$$

Furthermore, if there exists a linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \textit{ exists} \right\}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = \left. \frac{d^+ T(t)x}{dt} \right|_{t=0} \quad \textit{for } x \in D(A),$$

is called the infinitesimal generator of the semigroup $T(t)$.

Definition 2.8 *A C_0 semigroup is called uniformly bounded, if there exists $M \geq 1$ such that $||T(t)|| \leq M \forall t \geq 0$. If in addition $M = 1$ we call it a C_0 semigroup of contractions.*

Some basic facts about C_0 semigroups:

Theorem 2.9 *Let $T(t)$ be a C_0 semigroup of bounded linear operators on X, then the following holds:*

- *there exist constants $\omega \geq 0$ and $M \geq 1$ such that $||T(t)||_X \leq Me^{\omega t}$ for $0 \leq t < \infty$,*
- *the map $t \to T(t)x$ is continuous, for $x \in X$ and $t \in \mathbb{R}_0^+$,*
- *for $x \in D(A)$ it holds that $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

Lemma 2.10 *Let $T(t)$ be a C_0 semigroup which is differentiable for $t > t_0$ and let A be its infinitesimal generator, then*

- *for $t > nt_0$, $n = 1, 2, \ldots$, $T(t) : X \to D(A^n)$ and $T^{(n)}(t) = A^n T(t)$ is a bounded linear operator.*
- *If $t \mapsto T(t)$ is differentiable for $t > 0$, then this map is infinitely differentiable for $t > 0$.*

In particular, if we can show that A is the infinitesimal generator of a C_0 semigroup of bounded operators, we have solved the abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad t > 0 \\ u(0) &= u_0 \in D(A) \end{aligned} \tag{2.3}$$

with $u(t) = T(t)u_0$. We even get that this solution is unique and $u(t) \in \mathcal{C}([0, \infty), D(A)) \cap \mathcal{C}^1([0, \infty), X)$, as a simple consequence of theorem 2.9.
The main tools to characterize the infinitesimal generators of C_0 semigroups are the following Theorems:

Theorem 2.11 (Lumer-Phillips) *Let A be a linear operator with dense domain $D(A)$ in X.*

- *If A is dissipative and there exists a $\lambda > 0$ such that $R(\lambda I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X, then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.*

Theorem 2.12 (Hille-Yosida) *A linear operator A is the infinitesimal generator of a C_0 semigroup satisfying $||T(t)|| \leq e^{\omega t}$ if and only if*

- *A is closed and $\overline{D(A)} = X$,*
- *The resolvent set $\rho(A)$ of A contains the ray $\{\lambda : \operatorname{Im} \lambda = 0, \lambda > \omega\}$ and for such λ*
$$||R(\lambda : A)|| \leq \frac{1}{\lambda - \omega}.$$

This result can be extended to:

Theorem 2.13 ([6]) *Let A be a infinitesimal generator of a C_0 semigroup such that $||T(t)|| \leq e^{\omega t}$, then we have $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and*

$$||\,\mathrm{R}(\lambda : A)^n|| \leq \frac{1}{(\operatorname{Re} \lambda - \omega)^n}, \quad \forall n \in \mathbb{N},\ \operatorname{Re} \lambda > \omega.$$

Under certain conditions,the asymptotic behavior of a solution of finite dimensional dynamical systems can be determined by the spectrum of the differential operator. Unfortunately, this is not true in the infinitesimal case. One example, where this approach is still possible, are analytical semigroups.

Definition 2.14 *Let $\Delta = \{z : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z), z \in \Delta$ is an analytical semigroup in Δ if the following holds:*

- *$z \to T(z)$ is analytic in Δ,*
- *$T(0) = I$ and $\lim_{z \to 0, z \in \Delta} T(z)x = x$ for every $x \in X$,*
- *$T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.*

The infinitesimal generator is defined similarly to the C_0 semigroup case.

Theorem 2.15 *Let $T(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

1. *$T(t)$ can be extended to an analytical semigroup in a sector $\Delta_\delta = \{z : |\arg z| < \delta\}$ and $||T(z)||$ is uniformly bounded in every closed sub sector $\{z : |\arg z| \leq \delta' < \delta\}$.*
2. *There exists a constant C such that for every $\sigma > 0, \tau \neq 0$*
$$||R(\sigma + i\tau : A)|| \leq \frac{C}{|\tau|}.$$

3. $T(t)$ is differentiable for $t > 0$ and there exists a constant C such that

$$||AT(t)|| \leq \frac{C}{t} \quad \textit{for } t > 0.$$

We can also perturb our infinitesimal generator and still get an analytical semigroup:

Lemma 2.16 *Let A be the infinitesimal generator of an analytical semigroup. If B is a bounded operator then $A + B$ is the infinitesimal generator of an analytical semigroup.*

The asymptotic behavior of a analytical semigroup is determined by the spectrum of the infinitesimal generator A as follows:

Proposition 2.17 ([8]) *Let A be a infinitesimal generator of a analytical semigroup $T(t)$ such that*

$$\sigma = \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda,$$

then for every $\varepsilon > 0$ and $n \in \mathbb{N} \cup \{0\}$ there exists an $M_{n,\varepsilon} > 0$ such that

$$||t^n A^n T(t)|| \leq M_{n,\varepsilon} e^{(\sigma+\varepsilon)t} \quad \forall t > 0.$$

The last definitions are denoted to a special case of Lyapunov functionals, the so-called entropies. These functionals can be used to determine the longtime behavior of solutions, in a more general setting. Since we will only use them in the semigroup case, I have slightly adopted the definitions.

Definition 2.18 ([7]) *Let $(T(t), D(A))$ be a differentiable C_0 semigroup, i.e. $t \mapsto T(t)$ is differentiable, then we call a point $x \in D(A)$ stationary, or steady state, if it fulfills the following equation:*

$$T(t)x = x \quad \textit{for all } t \geq 0.$$

Definition 2.19 ([7]) *Let $H : B \to \mathbb{R}$ be a functional. We call it entropy of a differentiable semigroup $(T(t), D(A))$, for a stationary state x_∞, if the following conditions are satisfied:*

- *Let $x \in D(A)$, then the following Lyapunov property holds*

$$\frac{dH}{dt}[T(t)x] \leq 0 \quad \textit{for all } t > 0,$$

- *H is convex,*

- *There exists a continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(0) = 0$ and*

$$||d(x, x_\infty)|| \leq \Phi(H[T(t)x] - H[x_\infty]) \quad \textit{for all } x \in B$$

The functional $D : B \to \mathbb{R}$, which satisfies

$$\frac{dH}{dt}[T(t)x] \leq -D[T(t)x] \quad t > 0,$$

is called entropy dissipation.

2.3 Linearization

In this section the linearization of (2.1) around the trivial steady state

$$
\begin{aligned}
\varphi_0(s) &= 0,\\
r_0(s) &= s + L + r_I,\\
\lambda_0(s) &= \mu^P \ln \frac{r_I}{s+L+r_I} + C,
\end{aligned}
\tag{2.4}
$$

is carried out. The constants C and r_I are uniquely determined by the boundary conditions as follows:

$$
\begin{aligned}
-\mu^P + C &= \mu^{IP}(\pi r_I^2 - \pi \hat{r}^2),\\
-\mu^P + C &= -\mu^P \left(\ln \frac{r_I}{L+r_I} \right).
\end{aligned}
\tag{2.5}
$$

Since we will only analyze the behavior of the linearized equations and not conclude from this behavior to the whole system, I will not justify the linearization by a functional analytical framework. Instead, we make the Ansatz,

$$
\begin{aligned}
R(s,t) &= r_0(s) + r(s,t)\delta,\\
\Phi(s,t) &= \varphi(s,t)\delta,\\
\Lambda(s,t) &= \lambda_0(s) + \lambda(s,t)\delta,
\end{aligned}
$$

with $\delta > 0$, and plug it into our equations. This will lead us to a formal linearization, which will be sufficient for our purpose.

The main equations

We start with the first equation of (2.1) and compute as follows:

$$
\begin{aligned}
&\mu^A \dot{R} + \mu^P \left(\frac{1}{R'R} \right)' R - \Lambda' R' - \Lambda(R'' - (\Phi')R) =\\
=&\mu^A \dot{r}\delta - \left(\frac{r_0'' + r''\delta}{(r_0' + r'\delta)^2} + \frac{1}{r_0 + r\delta} \right) - \lambda_0' r_0' -\\
&- \left(\lambda' r_0' + \lambda_0' r' \right)\delta - \left(\lambda_0 r_0'' + \lambda_0 r''\delta + \lambda r_0''\delta \right) + O(\delta^2) =\\
=&\mu^A \dot{r}\delta - \mu^P \left(\frac{1}{r_0} + \left(\frac{r''}{r_0'^2} - \frac{r}{r_0^2} \right)\delta \right) - \lambda_0' r_0' -\\
&- \left(\lambda' r_0' + \lambda_0' r' \right)\delta - \left(\lambda_0 r_0'' + \lambda_0 r''\delta + \lambda r_0''\delta \right) + O(\delta^2).
\end{aligned}
$$

Next we can use the fact that the steady state solves the equation and that r_0' is equal to one, which leads, by dividing by δ and letting δ go to zero, to the following linear equation:

$$
\mu^A \dot{r} - \mu^P \left(r'' - \frac{r}{r_0^2} \right) - \lambda' - \lambda_0' r' - \lambda_0 r'' = 0
$$

Similarly we can determine the second equation:

$$\begin{aligned}&\mu^A R\dot{\Phi} - \Lambda'\Phi' R - \Lambda\left(2\Phi' R' + \Phi'' R\right) = \\ &\mu^A r_0\dot{\varphi}\delta - \lambda_0' r_0\varphi'\delta - 2\lambda_0 r_0'\varphi'\delta - \lambda_0 r_0\varphi''\delta + O(\delta^2).\end{aligned}$$

Dividing by δ and take the limit of δ goes to zero, leads to:

$$\mu^A r_0\dot{\varphi} - \lambda_0' r_0\varphi' - 2\lambda_0\varphi' - \lambda_0 r_0\varphi'' = 0.$$

The last equation simplifies to

$$\begin{aligned}(R')^2 + (\Phi')^2 R^2 = 1 &\Leftrightarrow \\ (r_0')^2 + 2r_0' r'\delta + r'\delta^2 = 1 + O(\delta^2) &\Leftrightarrow \\ r' = 0&.\end{aligned} \tag{2.6}$$

Boundary conditions

For the left-hand side of the boundary condition we get

$$-\frac{\mu^P}{R'} + \Lambda R' = -\mu^P\left(\frac{1}{r_0'} - \frac{r'\delta}{(r_0')^2}\right) + \lambda_0 r_0' + r_0'\lambda\delta + \lambda_0 r'\delta + O(\delta^2).$$

The right-hand side is given by

$$\mu^{IP}(\pi R^2 - \pi\hat{r}^2)_+ R' = \mu^{IP}\pi(r_0^2 + 2r_0 r(t)\delta - \hat{r}^2)_+ r_0',$$

where we have used that r' is equal to zero and therefore $r(-L,t) = r(0,t) = r(t)$. For $s = -L$ we observe that $r_0(-L) = r_I$, and can compute for the case $r_I^2 \geq \hat{r}^2$ the following:

$$\begin{aligned}-\mu^P + \lambda_0 + \lambda\delta + \lambda_0 r(t)'\delta = \mu^{IP}\pi(r_I^2 + 2r_I r(t)\delta - \hat{r}^2)_+ + O(\delta^2) &\Leftrightarrow \\ \lambda\delta = \mu^{IP}\pi 2r(t)r_I\delta + O(\delta^2) &\Leftrightarrow \\ \lambda = 2\mu^{IP}\pi r(t) r_I&.\end{aligned}$$

If $r_I^2 < \hat{r}^2$, we remain with $\lambda = 0$ for $s = -L$. For $s = 0$ we compute similarly, and get that $\lambda = 0$ in both cases.
We turn now our attention to the second boundary condition, which leads, for $s = 0$, to

$$\begin{aligned}\Lambda R\Phi' = 0 &\Leftrightarrow \\ \lambda_0 r_I\varphi'\delta = 0 &\Leftrightarrow \\ \varphi' = 0&.\end{aligned}$$

For the other side, we evaluate at $s = -L$ and use the first equations of (2.5) to obtain

$$
\begin{aligned}
\Lambda R\Phi' &= \mu^{IP}(\pi R^2 - \pi\hat{r}^2)_+\Phi' R \Leftrightarrow\\
\lambda_0 r_0\varphi'\delta &= \mu^{IP}(\pi r_I^2 - \pi\hat{r}^2)_+\varphi' r_I\delta + O(\delta^2) \Leftrightarrow\\
C r_I\varphi' &= \mu^{IP}(\pi r_I^2 - \pi\hat{r}^2)_+\varphi' r_I + O(\delta^2) \Leftrightarrow\\
\left(\mu^P r_I + \mu^{IP}(\pi r_I^2 - \pi\hat{r}^2)_+ r_I\right)\varphi' &= \mu^{IP}(\pi r_I^2 - \pi\hat{r}^2)_+\varphi' r_I + O(\delta) \Leftrightarrow\\
\varphi' &= 0.
\end{aligned}
$$

Summary

In summary we have computed the following linearization of (2.1),

$$
\begin{aligned}
&\mu^A\dot{r}(s,t) - \mu^P\left(r''(s,t) - \frac{r(s,t)}{r_0(s)^2}\right) - \lambda'(s,t) - \lambda_0'(s)r'(s,t) - \lambda_0(s)r''(s,t) = 0,\\
&\mu^A r_0(s)\dot{\varphi}(s,t) - \lambda_0'(s)r_0(s)\varphi'(s,t) - 2\lambda_0(s)\varphi'(s,t) - \lambda_0(s)r_0(s)\varphi''(s,t) = 0,\\
&r'(s,t) = 0,
\end{aligned}
$$

where we emphasize also the variables to clarify the dependence of the functions. By using the third equation and r_0' is equal to 1, these equations can be simplified to

$$
\begin{aligned}
\mu^A\dot{r}(t) &= -\frac{\mu^P r(t)}{r_0(s)^2} + \lambda',\\
\dot{\varphi}(s,t) &= \frac{\lambda_0(s)}{\mu^A}\varphi''(s,t) + \frac{1}{r_0(s)\mu^A}\left(\lambda_0'(s)r_0(s) + 2\lambda_0(s)\right)\varphi'(s,t),
\end{aligned}
\tag{2.7}
$$

with boundary conditions for $r_I^2 > \hat{r}^2$

$$
\lambda = \begin{cases} 0, & \text{for } s = 0,\\ 2\mu^{IP}\pi r(t) r_I, & \text{for } s = -L, \end{cases}
\tag{2.8}
$$

$$
\varphi' = 0 \quad s = 0 = -L
\tag{2.9}
$$

and for $r_I^2 \le \hat{r}^2$

$$
\varphi' = 0,\ \lambda = 0,\ \text{for} s = 0 = -L.
\tag{2.10}
$$

2.4 Spectral analysis of the linearized system

Our linearized system reads now as a decoupled one for $\varphi(s,t)$ and $r(t)$, therefore we will discuss the spectral properties of this two equations separately for now. In the last part we conclude with the result for the whole system.

2.4.1 First equation

The first equation is given by

$$\mu^A \dot{r}(t) = -\frac{\mu^P r(t)}{r_0(s)^2} + \lambda'(s,t),$$

with the boundary condition (2.8) or (2.10). Furthermore, we have used (2.6) which implies that r is independent of s. For both boundary conditions we can observe that $\lambda'(s,t)$ is represented as a rational function in s, and therefore λ is smooth with respect to the spatial coordinate.
We will first examine the case $r_0^2 > \hat{r}^2$, hence the boundary condition are given by (2.8). Since $\lambda \in C^\infty$ with respect to s, we can integrate the whole equation in space and use the boundary conditions to obtain

$$\dot{r}(t) = -\frac{1}{L\mu^A}\left(\frac{\mu^P L}{r_I(L+r_I)} + 2\mu^{IP}\pi r_I\right) r(t),$$

Therefore, we can apply Picard-Lindelöf combined with the following growth result:

Theorem 2.20 *Suppose $U = \mathbb{R}\times\mathbb{R}^n$ and for every $T>0$ there are constants $M(T)$, $L(T)$ such that*

$$|f(t,x)| \leq M(T) + L(T)|x|, \quad (t,x) \in [-T,T]\times\mathbb{R}^n. \tag{2.11}$$

Then all solutions of the initial value problem

$$\dot{x}(t) = f(t,x), \quad x(t_0) = x_0, \tag{2.12}$$

with $f \in C(U,\mathbb{R}^n)$ and $(t_0,x_0) \in U$, are defined for all $t \in \mathbb{R}$.

Both results can be found in [16]. Therefore, we have existence and uniqueness of the solution for all $t \in \mathbb{R}$. In our case, the solution can be computed explicitly by using the separation of variables which leads to

$$r(t) = e^{-\frac{1}{L\mu^A}\left(\frac{\mu^P L}{r_I(L+r_I)} + 2\mu^{IP}\pi r_I\right)t} r_{\text{ini}}, \tag{2.13}$$

for a given initial value $r_{\text{ini}} = r(0) \in \mathbb{R}$.
Next we define $K_1 := \frac{1}{L\mu^A}\left(\frac{\mu^P L}{r_I(L+r_I)} + \mu^{IP}\pi 2 r_I\right)$, and derive λ, by using (2.13) and the boundary conditions, directly:

$$\lambda(s,t) = \left(-\mu^A K_1 s - \frac{\mu^P}{r_0(s)} + \frac{\mu^P}{L+r_I}\right) e^{-K_1 t} r_{\text{ini}}. \tag{2.14}$$

In the case of $r_0^2 \leq \hat{r}^2$ we can compute similarly:

$$r(t) = e^{-K_2 t} r_{\text{ini}}, \tag{2.15}$$

$$\lambda(s,t) = \left(-\mu^A K_2 s - \frac{\mu^P}{r_0(s)} + \frac{\mu^P}{L+r_I}\right) e^{-K_2 t} r_{\text{ini}}. \tag{2.16}$$

with $K_2 := \frac{1}{\mu^A}\left(\frac{\mu^P}{r_I(L+r_I)}\right)$.
Hence, in both cases we have shown the existence and uniqueness of r and λ, which both are infinitely differentiable on $[-L, 0]$. Furthermore, they decay exponentially to zero for t goes to infinity.

2.4.2 Second equation

Our next equation is given by the linear partial differential equation

$$\mu^A r_0 \frac{\partial \varphi}{\partial t} = \lambda_0 r_0 \frac{\partial^2 \varphi}{\partial s^2} + \left(\frac{\partial \lambda_0}{\partial s} r_0 + 2\lambda_0\right)\frac{\partial \varphi}{\partial s}, \tag{2.17}$$

with homogeneous Neumann boundary condition, i.e. $\frac{\partial \varphi}{\partial s} = 0$ for $s = -L = 0$. First, we will fit this equation into the general framework by rewriting the equation in divergence form

$$\dot{\varphi} = L\varphi, \qquad \frac{\partial \varphi}{\partial s}(0,t) = \frac{\partial \varphi}{\partial s}(-L,t) = 0, \tag{2.18}$$

for some initial data $\varphi_0(s) = \varphi(s,0), s \in [-L, 0]$, and defining the linear operator

$$L\varphi = \frac{\partial}{\partial s}\left(\frac{\lambda_0}{\mu^A}\frac{\partial \varphi}{\partial s}\right) + \frac{2\lambda_0}{\mu^A r_0}\frac{\partial \varphi}{\partial s}. \tag{2.19}$$

In order to show that this operator is uniformly parabolic, we have to investigate $\lambda_0(s)$. Since the first term of λ_0 is not positive, we have take a look at the second term C from (2.14). This leads to:

$$\lambda_0(s) = \mu^P\left(\ln\left(\frac{r_I}{s+L+r_I}\right) + 1 - \ln\left(\frac{r_I}{L+r_I}\right)\right) = \tag{2.20}$$

$$= \mu^P\left(\ln\left(\frac{r_I+L}{s+L+r_I}\right) + 1\right) \geq \mu^P > 0. \tag{2.21}$$

Therefore, we get the following estimate

$$\frac{\lambda_0}{\mu^A}\xi^2 \geq \frac{\mu^P}{\mu^A}\xi^2 \text{ for all } \xi \in \mathbb{R},$$

and we have shown that our operator is parabolic.
With the aim of analyzing the spectrum of this operator, we will now highlight another perspective of the operator L.

Sturm-Liouville eigenvalue problem

We fix t, and look at our operator L in a ordinary differential equation setting, as follows

$$\mathcal{L}u(s) = p(s)\frac{d^2u}{ds^2}(s) + q(s)\frac{du}{ds}(s), \tag{2.22}$$

where we define: $p(s) = \frac{\lambda_0}{\mu^A}$ and $q(s) = \left(\frac{\lambda_0'}{\mu^A} + \frac{2\lambda_0}{\mu^A r_0}\right)$. By (2.9) we also have homogeneous Neumann boundary conditions,

$$\frac{du}{ds} = 0, \quad \text{for } s = 0 = -L. \tag{2.23}$$

This corresponds to the stationary case of (2.17), i.e. $Lu = 0$. We want to show that this induces a regular Sturm-Liouville problem (for short SL-problem), whereby we use the following definition and the affiliated Theorem from [5].

Definition 2.21 *Let $\mathcal{L}$ be a ordinary differential operator of the form*

$$\mathcal{L} = \frac{d}{dx}\left(e(x)\frac{d}{dx}\right) + r(x).$$

The eigenvalue equation

$$\mathcal{L}u = \lambda\rho(x)u, \quad x \in (a,b),$$

subject to the separated homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = 0, &\quad |\alpha_1| + |\alpha_2| > 0, \\ \beta_1 u(b) + \beta_2 u'(b) = 0, &\quad |\beta_1| + |\beta_2| > 0, \end{aligned}$$

where α_i and β_i are real constants, is called regular SL eigenvalue problem, if e does not vanish on $[a,b]$ and the interval (a,b) is bounded.

Theorem 2.22 *Assuming $e', r, \rho \in C([a,b])$, and $e, \rho > 0$ on $[a,b]$, then the SL eigenvalue problem defined above has a infinite sequence of real eigenvalues λ_n, $n \in \mathbb{N}$. Furthermore, to each eigenvalue λ_n corresponds a single eigenfunction u_n, and the sequence of eigenfunctions $(u_n : n \in \mathbb{N}_0)$ forms an orthogonal base of $L^2_\rho(a,b)$.*

Whereby L^2_ρ is the Hilbert space of $L^2(a,b)$-functions equipped with the following norm:

$$||u||_{L^2_\rho} = \sqrt{\langle \rho u, u\rangle_{L^2}}.$$

This norm is, since ρ is strictly positive and bounded, easily seen to be equivalent with the L^2 norm, so we have:

$$C_{down}||u||^2_{L^2} \leq ||u||^2_{L^2_\rho} \leq C_{up}||u||^2_{L^2}. \tag{2.24}$$

Furthermore, the spectrum $\sigma(L)$ of a SL problem is discrete, and consists only of the eigenvalues of $\mathcal{L}$ (see Teschl, [17] p. 192).
The eigenvalue problem for our operator is given by

$$\mathcal{L}u(s) = \lambda u(s), \quad s \in (-L,0), \tag{2.25}$$

$$u' = 0 \qquad \text{for } s = -L = 0. \tag{2.26}$$

We introduce the new operator

$$\mathcal{L}_\rho := \rho\mathcal{L}u = \frac{d}{ds}\left(e^{\int^s \frac{q(s)}{p(s)}d\tau}\frac{d}{ds}u(s)\right), \tag{2.27}$$

and multiply equation (2.25) with the weight $\rho(s) := \frac{1}{p(s)}e^{\int^s \frac{q(\tau)}{p(\tau)}\,d\tau}$ to obtain

$$\mathcal{L}_\rho u = \lambda\rho u, \tag{2.28}$$

$$u' = 0 \qquad \text{for } s = -L = 0. \tag{2.29}$$

The weight and coefficient functions can be computed explicitly and read as follows:

$$\rho(s) = \mu^A r_0^2(s),$$
$$a(s) := e^{\int^s \frac{q(s)}{p(s)}d\tau} = \lambda_0(s) r_0^2(s),$$

and therefore $a(s), \rho(s) \in \mathcal{C}^\infty(-L,0)$. Equation (2.28) forms a regular SL problem with $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$, since $a(s)$ is by (2.20) strictly positive. Hence Theorem 2.22 provides us with the existence of orthogonal eigenfunctions u_n of $\mathcal{L}_\rho$.
In summary we can proof the following:

Lemma 2.23 *The linear operator $L : D(L) \subset L^2_\rho(-L,0) \to L^2_\rho(-L,0)$ defined by (2.19), with $D(L) = \{u \in H^2_\rho(-L,0) : \frac{\partial u}{\partial s} \in H^1_{\rho,0}(-L,0)\}$, is densely defined and*

$$\sup_{\lambda\in\sigma(L)} \operatorname{Re}\lambda \leq 0. \tag{2.30}$$

Proof:
We begin with the density of $D(L)$. Since the eigenfunction u_n solves the weighted SL problem (2.28), we get the following inclusions by using (2.24),

$$u_n \in \{u \in \mathcal{C}^2(-L,0) : u' = 0 \text{ for } s = -L = 0\} \subseteq D(L) \subseteq L^2_\rho. \tag{2.31}$$

For all $u \in L^2_\rho$ we can therefore find a sequence of eigenfunctions $u_n \in D(L)$, such that

$$u = \sum_{n=1}^{\infty} \langle u, u_n \rangle_\rho u_n, \tag{2.32}$$

which implies $\overline{D(L)} = L^2_\rho$.
In order to proof the rest of the statement, we compute the well known Rayleigh quotient to determine the sign of our eigenvalues. Let λ_n be an eigenvalue with corresponding eigenfunction u_n, then we compute

$$\begin{aligned}\langle \mathcal{L}_\rho u_n, u_n \rangle &= \int_{-L}^{0} \frac{d}{ds}\left(e^{\int^s \frac{q(s)}{p(s)}\, d\tau} \frac{du_n}{ds} \right) u_n \; ds = \\ &= -\int_{-L}^{0} e^{\int^s \frac{q(s)}{p(s)}\, d\tau} \left(\frac{du_n}{ds} \right)^2 \; ds,\end{aligned}$$

where we integrated by parts and used the boundary conditions. Hence, by using the eigenfunction property of u_n,

$$\begin{aligned}\langle \mathcal{L}_\rho u_n, u_n \rangle &= \langle \lambda_n \rho u_n, u_n \rangle \Leftrightarrow \\ -\int_{-L}^{0} e^{\int^s \frac{q(s)}{p(s)}\, d\tau} \left(\frac{du_n}{ds} \right)^2 \; ds &= \lambda_n \int_{-L}^{0} \rho u_n^2 \; ds \Leftrightarrow \\ \lambda_n &= -\frac{\int_{-L}^{0} e^{\int^s \frac{q(s)}{p(s)}\, d\tau} \left(\frac{du_n}{ds} \right)^2 \; ds}{\int_{-L}^{0} \rho u_n^2 \; ds} \le 0, \quad \forall n \in \mathbb{N}.\end{aligned}$$

Since for a SL operator the spectrum only consists eigenvalues, we are finished. □

As a last step, we now put the pieces together and can state the main Theorem of this section. We will therefore formulate the problem as follows: Let $\lambda_i(s,t)$ be the solution of (2.7) with the corresponding boundary conditions, i.e. (2.14) or (2.16). Then the equation for $r(t)$ can be formulated with respect to K_1 and K_2 for $i \in 1,2$, as

$$\dot{r}_i(t) = -K_i r_i(t) \quad i \in 1,2, \tag{2.33}$$

and the system can therefore be stated for $\boldsymbol{u}(s,t) = (r(t), \varphi(s,t))$ as

$$\dot{\boldsymbol{u}} = A\boldsymbol{u} = \begin{pmatrix} -K_i & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} r \\ \varphi \end{pmatrix} \tag{2.34}$$

with boundary and initial conditions

$$\frac{\partial \boldsymbol{u}}{\partial s}(-L,t) = \frac{\partial \boldsymbol{u}}{\partial s}(0,t) = 0, \quad \boldsymbol{u}_0(s,0) = \begin{pmatrix} r_{\mathrm{ini}} \\ \varphi_{\mathrm{ini}} \end{pmatrix}. \tag{2.35}$$

Lemma 2.23 and equation (2.13) now imply the following Theorem:

Theorem 2.24 *The spectrum of the operator A defined in (2.34) with $D(A) = \mathbb{R} \oplus D(L) \subset \mathbb{R} \oplus L^2_\rho = H$, possesses the following spectral bound:*

$$\sigma = \sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda \le 0. \tag{2.36}$$

2.5 Analytical semigroup

As a next step, we will prove that A is the generator of an analytical semigroup. This will lead us to the well-posedness of the problem. Furthermore, the operator L is smoothing initial data, such that after infinitesimal time, we get infinitely differentiability of solutions. We will proof this property first for the operator L, and get the whole result by a perturbation of L.

Lemma 2.25 *The operator L, defined in Lemma 2.23, is the infinitesimal generator of a C_0 semigroup e^{Lt} of contractions.*

Proof:
We will use Theorem 2.11 to proof this result. We already have shown that L is densely defined and linear. To show that L is also dissipative, we use the fact that the dual space of L^2_ρ can be identified with itself. Let $\varphi \in D(L)$, then we compute, by using (2.27) and partial integration,

$$\langle \varphi, L\varphi \rangle_\rho = \int_{-L}^{0} \varphi \frac{\partial}{\partial s}\left(a(s)\frac{\partial \varphi}{\partial s}\right) \text{ ds} = -\int_{-L}^{0} a(s)\left(\frac{\partial \varphi}{\partial s}\right)^2 \text{ ds}.$$

This implies that L is dissipative, so its left to proof the second condition. Therefore, we have to solve the following problem: For all $f \in L^2_\rho$ there exists a $u \in D(L)$ such that

$$\begin{aligned} u - Lu &= f \\ \frac{\partial u}{\partial s} &= 0 \text{ for } s = -L = -0 \end{aligned}$$

holds. Since our operator L is parabolic, hence elliptic in the stationary case, this is a 2nd order elliptic problem with Neumann Boundary conditions. We will therefore use Lax-Milgram with the weak formulation

$$B(u, v) = \langle f, v \rangle_\rho \quad \forall v \in H^1_\rho(-L, 0),$$

where $B(u,v) = \int_{-L}^{0} a(s)\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} \text{ ds} + \langle u, v\rangle_\rho$. The form $B(u,v) : H^1_\rho \to \mathbb{R}$ is coercive, since L is elliptic, and continuous, which can be shown in the usual way. Therefore, Lax Milgram gives us the existence of a function $u \in H^1_\rho$ for all $f \in L^2_\rho$. Furthermore, we have

$$\int_{-L}^{0} a(s)\frac{\partial u}{\partial s}\frac{\partial v}{\partial s} \text{ ds} = \langle f - u, v\rangle_\rho \quad \forall v \in C^1_c(-L, 0),$$

and therefore $a(s)\frac{\partial u}{\partial s} \in H^1_\rho$, since $f - u \in L^2_\rho$, which implies $u \in H^2_\rho$. As a last step, let u be the solution we achieved above, then we integrate by parts as follows

$$\begin{aligned} &B(u, v) - \langle f, v\rangle_\rho = 0 \Leftrightarrow \\ &\langle u - Lu - f, v\rangle_\rho + a(0)\frac{\partial u}{\partial s}(0)v(0) - a(-L)\frac{\partial u}{\partial s}(-L)v(-L) = 0 \end{aligned}$$

and by choosing $v \in H_0^1$ we have that $u - Lu - f = 0$ a.e. So we are left with

$$a(0)\frac{\partial u}{\partial s}(0)v(0) - a(-L)\frac{\partial u}{\partial s}(-L)v(-L) = 0 \quad \forall v \in H_\rho^1(-L,0).$$

This implies that $u \in D(L)$, since $v(s)$ is arbitrary. In summary, we have shown that $R(u - Lu) = L_\rho^2$, which implies the statement of the lemma. $\square$

So far, we have shown that there exists a unique function

$$u \in \mathcal{C}([0,\infty), D(A)) \cap \mathcal{C}^1([0,\infty), L_\rho^2(-L,0))$$

that solves $\dot{u}(t) = Lu(t)$, and also takes the boundary conditions into account. We can also show that this solution is infinitely differentiable in space and time. To achieve this, we will prove that L is in effect the generator of an analytical semigroup.

Lemma 2.26 *The C_0 semigroup e^{Lt} generated by the operator L can be extended to an analytical semigroup and the map $t \mapsto e^{Lt}$ is infinitely differentiable for $t > 0$.*

Proof:
The proof is a simply application of Theorem 2.15, whereby we have to shift our resolvent set, since $0 \notin \rho(L)$. This can be done for every contraction semigroup, since by the previous Lemma, we have that $||e^{Lt}|| \leq 1$. We define a new semigroup by $S(t) = e^{Lt}e^{-\omega}$, $\omega > 0$ and get therefore by Hille-Yosida 2.12 that $S(t)$ is another C_0 semigroup with $(-\omega, \infty) \in \rho(L_S)$, with L_S is the generator of $S(t)$. Another application of Hille-Yosida and Theorem 2.13 implies that for all $\lambda = \sigma + i\tau$, with $\sigma > 0$, $\tau \neq 0$ we set $C = \frac{|\tau|}{\sigma - \omega}$ and get

$$|| \mathrm{R}(\lambda : L_S)|| \leq \frac{1}{\operatorname{Re}\lambda - \omega} = \frac{C}{|\tau|}.$$

Therefore, $S(t)$ can be analytically extended, hence this implies the same for e^{Lt}. The second statement follows from Theorem 2.15 (3) and Lemma 2.10 when we restrict the now analytic semigroup e^{Lz} to $z \in \mathbb{R}^+$ $\square$

Finally we can show

Proposition 2.27 *The parabolic problem defined in (2.18) possesses a unique solution $\varphi(s,t)$, for every initial data $\varphi_{ini}(s) = \varphi(s,0) \in D(L)$ such that*

$$\varphi(s,t) \in C^\infty([-L,0] \times (0,\infty)),$$

the solution depends continuously on the initial data, i.e.

$$||\varphi(s,t)|| \leq ||\varphi_{ini}(s)|| \quad \forall t \leq 0,$$

and

$$\lim_{t\to\infty}\left|\left|\frac{\partial\varphi}{\partial t}(s,t)\right|\right| = 0.$$

Proof:
We begin by using Lemma 2.25 to solve the abstract Cauchy problem as stated in (2.3). Therefore, we have already shown that $\varphi(s,t) = e^{Lt}\varphi_{\text{ini}}(s) \in C([0,\infty), D(L)) \cap C^1([0,\infty), L^2_\rho)$. Lemma 2.10 and 2.26 now imply that $t \mapsto \varphi(t)$ is infinitely differentiable in $D(L)$ for $t > 0$. Furthermore, we also get $\frac{\partial^n\varphi}{\partial t^n} = L^n\varphi \in C^\infty((0,\infty), H^{2n}_\rho)$ for all $n \in \mathbb{N}$. This implies, together with the Sobolev embedding Theorem 2.4 and the second part of Lemma 2.26, the first part of the Theorem. The other two results follow directly by applying (3) of Theorem 2.15 and the fact that we deal with a C_0 semigroup of contractions.

□

Now we can again use the formulation of (2.34), and state:

Theorem 2.28 *The operator A defined in (2.34) with $D(A) = \mathbb{R}\oplus D(L) \subset \mathbb{R}\oplus L^2_\rho = H$ is the infinitesimal generator of a analytic semigroup of contractions. The linearized system of equations defined in (2.34) and (2.35) possesses a unique solution $\boldsymbol{u}(s,t)$, for every initial data $\boldsymbol{u}_0(s) = \boldsymbol{u}(s,0) \in D(A)$ such that*

$$\boldsymbol{u}(s,t) \in C^\infty([0,\infty)) \times C^\infty([-L,0]\times(0,\infty)),$$

the solution depends continuously on the initial data, i.e.

$$||\boldsymbol{u}(s,t)||_H \leq ||\boldsymbol{u}_0(s)||_H \quad \forall t \leq 0,$$

and

$$\lim_{t\to\infty}\left|\left|\frac{\partial\boldsymbol{u}}{\partial t}(s,t)\right|\right|_H = 0.$$

Proof:
Since the sum of an analytical semigroup with a bounded operator is again an analytical semigroup by Lemma 2.16, we begin with the following splitting:

$$A = A_1 + A_2 = \begin{pmatrix} -K_i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}.$$

We already know that L is the generator of a analytical semigroup. Hence, this implies the same for A_2 by using the arguments as in the previous proofs.
Hence, by the continuity of A_1 we get that A is the infinitesimal generator of an analytical semigroup. By using the same techniques as in Proposition

2.27, we conclude that $\boldsymbol{u} \in C^\infty([0,\infty)) \times C^\infty([-L,0]\times(0,\infty))$ is the unique solution of (2.34) with (2.35). The contraction property follows from

$$||\boldsymbol{u}(s,t)||_H^2 = ||r||_2^2 + ||\varphi||_{L^2_\rho}^2 \le e^{-K_i 2t} r_{\text{ini}}^2 + ||\varphi_{\text{ini}}||_{L^2_\rho}^2 \le ||u_0(s)||_H^2,$$

and

$$\begin{aligned} \left|\left|\frac{\partial \boldsymbol{u}}{\partial t}(s,t)\right|\right|_H^2 &= ||(-K_i e^{-K_i t} r_{\text{ini}}, L\varphi_{\text{ini}})^T||_H^2 \le \\ &\le K_i^2 e^{-K_i 2t} r_{\text{ini}}^2 + \frac{C||\varphi_{\text{ini}}||_{L^2_\rho}^2}{t}, \end{aligned}$$

implies the second statement.

$\square$

2.6 An entropy functional for the linearized system

Since we have shown the well-posedness of our problem in the last section, we can now finally analyze the longtime behavior of the solutions. We will again treat the solutions for the two equations separately and combine them later.

Lemma 2.29 *The functional $H_1 : \mathbb{R} \to \mathbb{R}$,*

$$H_1[r] := \frac{1}{2}||r||_2^2 = \frac{1}{2}r^2$$

is a entropy functional.

Proof:

We can compute $\dot{H}_1$ by using the representation of $\dot{r}$ in (2.33):

$$\dot{H}_1[r] = \frac{d}{dt}\frac{1}{2}r^2 = r\dot{r} = -K_i r^2 \leq 0$$

□

We have already seen that solutions of the second equation fulfill $||\varphi(s,t)|| \leq ||\varphi_0(s)||$, since it is a C_0 Semigroup of contractions. First, we observe that we can't expect a decay to zero for $t \to \infty$, since zero is an eigenvalue of L. Nevertheless, we can show that in our case, every solution converges to a constant defined by the mean value of the initial data. In order to proof this, we will need the following result

Lemma 2.30 *Let $\varphi_{ini}(s) = \varphi(s,0) \in D(L)$, then the conservation law*

$$\int_{-L}^{0} \varphi_{ini}(s)\rho(s)\ ds = \int_{-L}^{0} \varphi(s,t)\rho(s)\ ds \quad \forall t \geq 0$$

holds.

Proof:
With (2.27) in mind, we integrate the following equation

$$\frac{\partial \varphi}{\partial t}\rho = \frac{\partial}{\partial s}\left(e^{\int^s \frac{q(s)}{p(s)}}\frac{\partial}{\partial s}\varphi\right),$$

over the interval. This leads to

$$\frac{d}{dt}\int_{-L}^{0}\varphi\rho\ ds = \left(e^{\int^s \frac{q(\tau)}{p(\tau)}d\tau}\varphi_s\right)\Big|_{-L}^{0} = 0.$$

□

Lemma 2.31 *The functional*

$$H_2[\varphi] := \frac{1}{2}||\varphi - \bar{\varphi}||^2_{L^2_\rho} = \frac{1}{2}\int_{-L}^{0} \rho\left(\varphi - \bar{\varphi}\right)^2 \ ds, \tag{2.37}$$

with

$$\bar{\varphi} := \frac{\int_{-L}^{0} \rho\varphi_0 \ ds}{\int_{-L}^{0} \rho \ ds}, \quad \varphi_0(s) \in D(L),$$

defines an entropy for the analytical semigroup $(e^{Lt}, D(L))$.

Proof:
We already have found a weight ρ that turns our operator in an self-adjoint one. So we will now again use the ρ-weighted L^2-Norm as an entropy by defining

$$H_2[\varphi] := \frac{1}{2}||\varphi - \bar{\varphi}||^2_{L^2_\rho} = \frac{1}{2}\int_{-L}^{0} \rho\left(\varphi - \bar{\varphi}\right)^2 \ ds, \tag{2.38}$$

with initial data $\varphi_0(s) = \varphi(s,0)$ and weighted mean value $\bar{\varphi}$. Next, we compute the entropy dissipation of H_2:

$$\begin{aligned}
\frac{d}{dt}H_2[\varphi] = \frac{1}{2}\frac{d}{dt}\int_{-L}^{0} \rho\left(\varphi - \bar{\varphi}\right)^2 \ ds = \\
= \int_{-L}^{0} \varphi_t\varphi\rho \ ds = \int_{-L}^{0} p\varphi_{ss}\rho \ ds + \int_{-L}^{0} q\varphi_s\varphi\rho = \\
= -\int_{-L}^{0} p\varphi_s^2\rho \ ds - \int_{-L}^{0} p_s\varphi_s\varphi\rho \ ds - \int_{-L}^{0} p\varphi_s\varphi\rho_s \ ds + \int_{-L}^{0} q\varphi_s\varphi\rho \ ds
\end{aligned}$$

where we have integrated by parts and used the last line in the proof of Lemma 2.30. The last three terms can be computed explicitly using that

$$p_s\rho + p\rho_s - q\rho = (p\rho)_s - q\rho = \frac{q}{p}\left(e^{\int^s \frac{q}{p}d\tau}\right) - q\rho = 0,$$

which leads to

$$\begin{aligned}
-\int_{-L}^{0} p_s\varphi_s\varphi\rho \ ds - \int_{-L}^{0} p\varphi_s\varphi\rho_s \ ds + \int_{-L}^{0} q\varphi_s\varphi\rho \ ds = \\
= -\int_{-L}^{0} \left(p_s\rho + p\rho_s - q\rho\right)\varphi_s\varphi \ ds = 0,
\end{aligned}$$

and

$$\frac{d}{dt}H_2(\varphi) \leq -\int_{-L}^{0} p\varphi_s^2\rho \ ds \leq 0.$$

This shows the Lyapunov property of H_2.

Now we can estimate the entropy dissipation:

$$\frac{d}{dt}H_2[\varphi] = -D[\varphi] = \int_{-L}^{0} \varphi_t \varphi \rho \; ds = -\int_{-L}^{0} (\varphi_s)^2 \left(e^{\int^s \frac{q(s)}{p(s)}}\right) \; ds \leq$$
$$\leq -C_0 \int_{-L}^{0} (\varphi_s)^2 \rho \; ds.$$

To further estimate the right hand side, we use (2.24) and the Poincare-Friedrich's inequality with $\hat{\varphi} = \frac{1}{L}\int_{-L}^{0} \varphi \; ds$ to get:

$$-C_0||\varphi_s||^2_{L^2_\rho} \leq -C_{down}C_0||\varphi_s||^2_{L^2} \leq \frac{-C_{down}C_0}{C_P}||\varphi - \hat{\varphi}||^2_{L^2} \leq$$
$$\leq \frac{-C_{down}C_0C_{up}}{C_P}||\varphi - \hat{\varphi}||^2_{L^2_\rho} \leq \frac{-C_{down}C_0C_{up}C_1}{C_P}||\varphi - \bar{\varphi}||^2_{L^2_\rho}.$$

This leads again to exponential decay of the entropy in time:

$$H_2[\varphi] = \frac{1}{2}||\varphi - \bar{\varphi}||^2_{L^2_\rho} \leq e^{-2\frac{-C_{down}C_0C_{up}C_1}{C_P}t}||\varphi_{\text{ini}} - \bar{\varphi}||^2_{L^2_\rho}$$

□

As a consequence we can state the following Theorem:

Theorem 2.32 *The solution of (2.34) and (2.35) with initial data* $\boldsymbol{u}_0(s) = (r_{ini}, \varphi_{ini}(s))^T \in D(A)$ *converges, for* t *goes to infinity, to the stationary state* $(0, \bar{\varphi})^T$*, i.e.*

$$||\boldsymbol{u}(s,t) - (0,\bar{\varphi})^T||_H \to 0 \quad \textit{for } t \to \infty.$$

Furthermore, we have found an entropy function $H[\boldsymbol{u}] : D(A) \to \mathbb{R}$ *for this problem, given as*

$$H[\boldsymbol{u}] = H_1[r] + H_2[\varphi],$$

with respect to the analytical semigroup $(e^{At}, D(A))$*.*

Proof:
The previous Lemma implies that $||\varphi||^2_{L^2_\rho} = ||\varphi - \bar{\varphi}||^2_{L^2_\rho}$ decays to zero for $t \to \infty$. Therefore, we can compute

$$||\boldsymbol{u}(s,t) - (0,\bar{\varphi})^T||^2_H = ||r(t)||^2_2 + ||\varphi - \bar{\varphi}||^2_{L^2_\rho},$$

and the result follows from Lemma 2.29. □

Part III. Smooth solutions without boundary conditions

In this last chapter, we will analyse another linearization. As mentioned above, there already exist well-posedness results for the model without pressure term. The following chapter is dedicated to verifying this for a linearized model which incorperates the new pressure term. So the main idea is to show that, at least for this kind of linearization, smooth solutions do exist and therefore one can hope to achieve this also for the whole model.
We will again make a few simplifications with respect to the whole model. First, we will dismiss the boundary conditions and therefore also the myosin pulling force. Instead, we will later impose decay conditions for $|s|$ and $|\alpha|$ tend to infinity. Altough this is an rather unbiological condition, we can imagine this as the situation when we zoom into a small area of the lamellipodium, where boundary effects are negligible. Since the pressure force only acts between members of the same family, we will not take cross-links and adhesions into account. We will also assume that the length distribution is constant equal to one, the polymerisation speed is equal to zero and the pressure is given by $p(\rho) = \mu^P \rho$. This leads us to the following model

$$\begin{aligned}\mu^B \partial_s^4 F + \mu^A D_t F - \partial_s(\lambda_{\text{inext}} \partial_s F) + \mu^P \partial_s(\rho \partial_\alpha F^\perp) - \mu^P \partial_\alpha(\rho \partial_s F^\perp) = 0,\\ |\partial_s F|^2 = 1.\end{aligned}$$

These (quasi-linear) equations still features the rather difficult pressure term. So we will linearize them in a similar manner as before, but since we do not make assumptions to the geometry, we cannot use the approach from the last chapter. Instead, we consider the trivial solution

$$\bar{F}(\alpha, s) = \begin{pmatrix} \bar{F}_1(\alpha, s) \\ \bar{F}_2(\alpha, s) \end{pmatrix} = c\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $c > 0$ constant. This solution corresponds to a situation in which all filaments are lined up next to each other and filaments only differ by a shift in the α-direction. Next, we put this solution into our equation to determine the Lagrangian function, which leads to $\partial_s \bar{\lambda}(\alpha, s) = 0$. In order to simplify the computations, we choose $\bar\lambda(\alpha,s)=\bar\lambda$ to be constant and greater than zero. Hence we make the Ansatz

$$\begin{aligned}\tilde F(\alpha,s,t) &:= \bar F(\alpha,s)+\delta F(\alpha,s,t)=\begin{pmatrix}\bar F_1(\alpha,s)\\ \bar F_2(\alpha,s)\end{pmatrix}+\delta\begin{pmatrix}F_1(\alpha,s,t)\\ F_2(\alpha,s,t)\end{pmatrix},\\ \tilde\lambda(\alpha,s,t) &:= \bar\lambda+\delta\lambda(\alpha,s,t)\end{aligned}$$

and put them into our equation. Since we will only consider local perturbations, will also demand that F and λ decay to zero, if $|s|$ or $|\alpha|$ tend to infinity.

3.1 Linearization

The first two terms can be computed straightforward, so we begin with the third term:

$$\begin{aligned}-\partial_s(\tilde\lambda\partial_s\tilde F) &= -(\delta\partial_s\lambda)(\partial_s\bar F+\delta\partial_s F)-(\bar\lambda+\delta\lambda)(\delta\partial_s^2F)=\\ &= -\begin{pmatrix}\bar\lambda\partial_s^2F_1\\ \partial_s\lambda+\bar\lambda\partial_s^2F_2\end{pmatrix}\delta+O(\delta^2).\end{aligned}$$

Next, we want to linearize the pressure terms, therefore we have to state ρ from (1.3), which, after some computations, reads as

$$\tilde\rho=\frac{1}{\partial_\alpha\bar F_1\partial_s\bar F_2}-\frac{\partial_\alpha F_1\partial_s\bar F_2+\partial_s F_2\partial_\alpha\bar F_1}{(\partial_\alpha\bar F_1\partial_s\bar F_2)^2}\delta+O(\delta^2)=\frac{1}{c}-\frac{\partial_\alpha F_1+c\partial_s F_2}{c^2}\delta+O(\delta^2).$$

The other terms that include ρ are given by

$$\begin{aligned}\partial_s\tilde\rho &= -\frac{\partial_{s,\alpha}F_1+c\partial_s^2F_2}{c^2}\delta+O(\delta^2),\\ \partial_\alpha\tilde\rho &= -\frac{\partial_\alpha^2F_1+c\partial_{s,\alpha}F_2}{c^2}\delta+O(\delta^2).\end{aligned}$$

With these ingredients, we can compute the pressure terms

$$\begin{aligned}\partial_s(\tilde\rho\partial_\alpha\tilde F^\perp) &= \partial_s\tilde\rho\partial_\alpha\tilde F^\perp+\tilde\rho\partial_{s,\alpha}\tilde F^\perp=\\ &= \begin{pmatrix}O(\delta^2)\\ -\frac{\partial_{s,\alpha}F_1+c\partial_s^2F_2}{c}\delta\end{pmatrix}+\frac{1}{c}\begin{pmatrix}-\partial_{s,\alpha}F_2\delta+O(\delta^2)\\ \partial_{s,\alpha}F_1\delta+O(\delta^2)\end{pmatrix}=\\ &= -\frac{1}{c}\begin{pmatrix}\partial_{s,\alpha}F_2\\ c\partial_s^2F_2\end{pmatrix}\delta+O(\delta^2)\end{aligned}$$

and

$$\begin{aligned}\partial_\alpha(\tilde\rho\partial_s\tilde F^\perp) &= \partial_\alpha\tilde\rho\partial_s\tilde F^\perp + \tilde\rho\partial_{\alpha,s}\tilde F^\perp = \\ &= \begin{pmatrix}\frac{\partial_\alpha^2 F_1 + c\partial_{\alpha,s}F_2}{c^2}\delta + O(\delta^2)\\ O(\delta^2)\end{pmatrix} + \frac{1}{c}\begin{pmatrix}-\partial_{\alpha,s}F_2\delta + O(\delta^2)\\ \partial_{\alpha,s}F_1\delta + O(\delta^2)\end{pmatrix} = \\ &= \frac{1}{c^2}\begin{pmatrix}\partial_\alpha^2 F_1\\ c\partial_{\alpha,s}F_1\end{pmatrix}\delta + O(\delta^2).\end{aligned}$$

Now we let δ go to zero and get the following linearization of the first two equations:

$$\begin{aligned}\mu^B\partial_s^4F_1 + \mu^A\partial_tF_1 - \bar\lambda\partial_s^2F_1 - \frac{\mu^P}{c^2}\left(\partial_\alpha^2F_1 + c\partial_{s,\alpha}F_2\right) &= 0,\\ \mu^B\partial_s^4F_2 + \mu^A\partial_tF_2 - \partial_s\lambda - \bar\lambda\partial_s^2F_2 - \frac{\mu^P}{c}\left(c\partial_s^2F_2 + \partial_{\alpha,s}F_1\right) &= 0.\end{aligned} \tag{3.1}$$

Next, we consider the inextensibility condition, which leads to

$$|\partial_s\tilde F|^2 - 1 = (\partial_sF_1)^2\delta^2 + 1 + 2\partial_sF_2\delta + (\partial_sF_2)^2 - 1 = 0.$$

For δ goes to zero, we get $\partial_sF_2 = 0$. Therefore, together with equation (3.1), we have computed

$$\begin{aligned}\mu^B\partial_s^4F_1 + \mu^A\partial_tF_1 - \bar\lambda\partial_s^2F_1 - \frac{\mu^P}{c^2}\partial_\alpha^2F_1 &= 0,\\ \mu^A\partial_tF_2 - \partial_s\lambda - \frac{\mu^P}{c}\partial_{\alpha,s}F_1 &= 0,\\ \partial_sF_2 &= 0.\end{aligned} \tag{3.2}$$

3.2 Existence and long-time behavior

We will now analyse the linearized system (3.2) for $(\alpha, s, t) \in \mathbb{R}^2 \times [0,\infty)$. Furthermore, due to our local perturbation assumptions, we impose the decay conditions

$$\lim_{|s|\to\infty} F(\alpha,s,t) = \lim_{|\alpha|\to\infty} F(\alpha,s,t) = 0 \tag{3.3}$$

and

$$\lim_{|s|\to\infty} \lambda(\alpha,s,t) = \lim_{|\alpha|\to\infty} \lambda(\alpha,s,t) = 0. \tag{3.4}$$

We start by discussing the first equation, whereby we introduce the new abbreviation $F_1(\alpha,s,t) = F_1(x,t)$, $x = (x_1,x_2) = (\alpha,s)$. The equation reads as a linear 4th order equation on the whole space $\mathbb{R}^2$ and therefore we can consider Fourier transformation to solve it. We will need standard

results for Fourier transformation on $\mathbb{R}^2$, which can be found in [17] and [15].
First we define the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^2) \to C_b(\mathbb{R}^2)$ by

$$\mathcal{F}(F(x,t)) = \hat{F}(p,t) = \int_{\mathbb{R}^2} F(x,t)e^{-ip\cdot x}\, dp.$$

We also have the following properties of this operator to our disposal:

- The restriction of $\mathcal{F}$ to $L^2(\mathbb{R}^2)$ is bijective with the inverse

$$\mathcal{F}^{-1}(F(p,t)) = \int_{\mathbb{R}^2} F(p,t)e^{ip\cdot x}\, dx.$$

- The operator interchanges differentiation with multiplication by polynomials as follows

$$(\partial_\alpha F)^\wedge(p) = (ip)^\alpha \hat{F}(p)$$

Furthermore, we will also need the following Lemma, which can be found in [17] as a stated problem.

Lemma 3.1 *Let $X \subseteq \mathbb{R}$, Y be some measure space, and $f : X \times Y \to \mathbb{R}$. Suppose $y \mapsto f(x,y)$ is measuralbe for all x and $x \mapsto f(x,y)$ is differentiable for a.e. y. Then*

$$F(x) = \int_A f(x,y)d\mu(y)$$

is differentiable if there is an integrable function $g(y)$ such that $|\frac{\partial f}{\partial x}| \leq g(y)$. Moreover, $x \mapsto \frac{\partial f}{\partial x}$ is measureable and

$$F'(x) = \int_A \frac{\partial f}{\partial x}(x,y)d\mu(y)$$

in this case.

We now use the property of the Fourier transform, interchanging of differentiation with multiplying by polynoms, and are making the following (for now) formal computations:

$$\begin{aligned}
\mu^B \partial_s^4 F_1 + \mu^A \partial_t F_1 - \bar{\lambda}\partial_s^2 F_1 - \frac{\mu^P}{c^2}\partial_\alpha^2 F_1 = 0 \Leftrightarrow \\
\mu^B p_2^4 \hat{F}_1 + \mu^A \partial_t \hat{F}_1 + \bar{\lambda} p_2^2 \hat{F}_1 + \frac{\mu^P}{c^2} p_1^2 \hat{F}_1 = 0 \Leftrightarrow \\
\hat{F}_1 = e^{P(p_1,p_2)t}\hat{F}_{1,0},
\end{aligned}$$

whereby we define $P(p_1, p_2) = -\frac{(\mu^B p_2^4 + \bar{\lambda} p_2^2 + \frac{\mu^P}{c^2} p_1^2)}{\mu^A}$. The function $\hat{F}_{1,0}(0, p) = \hat{F}_{1,0}(p)$ is given by the Fourier transformation of some initial data $F_1(x, 0) \in L^2(\mathbb{R}^2)$. As next step, we apply the inverse Fourier transformation and get

$$F_1(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{P(p_1,p_2)t} \hat{F}_{1,0}(p) e^{ip\cdot x}\, dp. \tag{3.5}$$

Lemma 3.2 *The function $F_1(x, t)$ defined in (3.5), with $F_{1,0}(x) \in L^2(\mathbb{R}^2)$, has the following properties:*

- $F_1 \in C^\infty(\mathbb{R}^2 \times (0, \infty)) \cap L^2([0, \infty), L^2(\mathbb{R}^2))$,
- $\lim_{t\to\infty} |F_1(x, t)| = 0$,
- $\lim_{t\to\infty} |\partial_t^n F_1(x, t)| = \lim_{t\to\infty} |\partial_{x_i}^n F_1(x, t)| = 0, \forall n \in \mathbb{N}, 1 \le i \le 2,$.

Furthermore,F_1 solves the first equation of (3.2) with the corresponding decay conditions (3.3) and (3.4).

In prepration for this proof, we will need the following simple Proposition:

Proposition 3.3 *The identity*

$$\int_{\mathbb{R}} x^{2n} e^{-x^2}\, dx = \frac{(2n-1)!\sqrt{\pi}}{2^{2n-1}(n-1)!}$$

holds for all $n \in \mathbb{N}$.

Proof:
We will show this by induction, so let $n = 1$, and we compute

$$\int_{\mathbb{R}} x(xe^{-x^2})\, dx = x^2 e^{-x^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{\mathbb{R}} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

The induction step for $n + 1$ is now carried out as follows:

$$\begin{aligned}\int_{\mathbb{R}} x^{2n+1}(xe^{-x^2})\, dx &= x^{2n+2}e^{-x^2}\Big|_{-\infty}^{\infty} + \frac{2n+1}{2}\int_{\mathbb{R}} x^{2n}e^{-x^2}\, dx = \\ &= \frac{2n+1}{2}\frac{(2n-1)!\sqrt{\pi}}{2^{2n-1}(n-1)!} = \frac{(2(n+1)-1)!\sqrt{\pi}}{2^{2(n+1)-1}((n+1)-1)!}.\end{aligned}$$

□

Now we can continue with the proof of Lemma 3.2.
Proof:
We begin by verifying that $\hat{F}_1$ is in $L^2([0, \infty), L^2(\mathbb{R}^2))$. This follows immediatly from $||e^{P(p_1,p_2)t}||_\infty \le 1$ for all $t > 0$ and the L^2 assumption for the

initial data. Hence, $F_1 \in L^2([0,\infty), L^2(\mathbb{R}^2))$, which implies that our solution is well defined. To show the decay statement, we use the transformation $\varphi(p_1,p_2) = \left(\frac{c\sqrt{\mu^A}\tau_2}{\sqrt{2\mu^P t}}, \frac{\sqrt{\mu^A}\tau_1}{\sqrt{2\bar\lambda t}})\right)$, and compute

$$\begin{aligned} ||e^{P(p_1,p_2)t}||_{L^2}^2 &\leq ||e^{-2\frac{\mu^B p_2^4}{\mu^A}}||_\infty \int_{\mathbb{R}^2} e^{-2\frac{(\bar\lambda p_2^2+\frac{\mu^P}{c^2}p_1^2)t}{\mu^A}}\ dp = \\ &= \frac{\mu^A C}{2\sqrt{\bar\lambda \mu^P}t} \int_{\mathbb{R}^2} e^{-(\tau_1^2+\tau_2^2)}\ dp = \frac{\pi\mu^A C}{2\sqrt{\bar\lambda\mu^P}t}, \end{aligned}$$

for some constant $C > 0$. Together with the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |F_1(x,t)|^2 &\leq \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{P(p_1,p_2)t}|\hat F_{1,0}|\ dp\right)^2 \leq \\ &\leq C||e^{P(p_1,p_2)t}||_{L^2}^2||\hat F_{1,0}||_{L^2}^2 \leq C\frac{\pi\mu^A c}{2\sqrt{\bar\lambda\mu^P}t}||\hat F_{1,0}||_{L^2}^2, \end{aligned}$$

which implies the second statement. In order to proof the first statement, we will use Lemma 3.1, therefore we will show that

$$\left|\frac{\partial^n}{\partial x_i^n} e^{P(p_1,p_2)t}\hat F_{1,0}(p,0)e^{ip\cdot x}\right|$$

is integrable for $i \in \{1,2\}$ and for all $n \in \mathbb{R}$. So we compute, by using Hölders inequality and Proposition 3.3,

$$\begin{aligned} &\left|\left|\frac{\partial^n}{\partial x_i^n} e^{P(p_1,p_2)t}\hat F_{1,0}(p)e^{ip\cdot x}\right|\right|_{L^1}^2 = ||p_i^n e^{P(p_1,p_2)t}\hat F_{1,0}(p)||_{L^1}^2 \leq \\ &\leq ||\hat F_{1,0}(p)||_{L^2}^2||p_i^n e^{P(p_1,p_2)t}||_{L^2}^2 \leq C\int_{\mathbb{R}^2} p_i^{2n} e^{-2\frac{(\bar\lambda p_2^2+\frac{\mu^P}{c^2}p_1^2)t}{\mu^A}}\ dp = \\ &= C\int_{\mathbb{R}} e^{-C_j(t)p_j^2}\int_{\mathbb{R}} p_i^{2n}e^{-C_i(t)p_i^2}\ dp_i\ dp_j = \\ &= C\int_{\mathbb{R}} e^{-C_j(t)p_j^2}\frac{1}{C_i(t)^{\frac{2n+1}{2}}}\int_{\mathbb{R}}\tau^{2n}e^{-\tau^2}\ d\tau\ dp_j = \\ &= \frac{C}{C_i(t)^{\frac{2n+1}{2}}}\frac{(2n-1)!\sqrt{\pi}}{2^{2n-1}(n-1)!}\int_{\mathbb{R}} e^{-C_j(t)p_j^2}\ dp_j = O\left(\frac{1}{t^n}\right), \end{aligned} \tag{3.6}$$

with some constant $C > 0$, the two t depending coefficicents $C_1(t) = 2\frac{\mu^P}{\mu^A c^2}t$, $C_2(t) = \frac{2\bar\lambda t}{\mu^A}$, $j \neq i$ and $j \in \{1,2\}$. We are left to show the differentiabilty in time, hence

$$\left|\frac{\partial^n}{\partial t^n} e^{P(p_1,p_2)t}\hat F_{1,0}(0,p)e^{ip\cdot x}\right|$$

is integrable for all $n \in \mathbb{R}$. Therefore, will again use the same techniques as for the space variables and compute

$$
\begin{aligned}
&\left|\left|\frac{\partial^n}{\partial t^n} e^{P(p_1,p_2)t}\hat{F}_{1,0}(p)e^{ip\cdot x}\right|\right|_{L^1}^2 = ||P^n(p_1,p_2)e^{P(p_1,p_2)t}\hat{F}_{1,0}(p)||_{L^1}^2 \leq \\
&\leq ||\hat{F}_{1,0}(p)||_{L^2}^2 ||P^n(p_1,p_2)e^{P(p_1,p_2)t}||_{L^2}^2 \leq \\
&\leq C\int_{\mathbb{R}^2}\left(\frac{\mu^B p_2^4 + \bar{\lambda}p_2^2 + \frac{\mu^P}{c^2}p_1^2}{\mu^A}\right)^{2n} e^{-2\frac{(\bar{\lambda}p_2^2+\frac{\mu^P}{c^2}p_1^2)t}{\mu^A}}\, dp.
\end{aligned}
$$

We can now expand the term $\left(\frac{\mu^B p_2^4+\bar{\lambda}p_2^2+\frac{\mu^P}{c^2}p_1^2}{\mu^A}\right)^{2n}$. Every term in the sum has the following generic form:

$$
C\int_{\mathbb{R}^2} p_1^{2k}p_2^{2l}e^{-\frac{(C_1(t)p_1^2+C_2(t)p_2^2)}{\mu^A}}\, dp,
$$

for some $1 \leq k, l \leq 16n$. Therefore we can again apply Proposition 3.3 as follows:

(3.7)
$$
\begin{aligned}
&C\int_{\mathbb{R}^2} p_1^{2k}p_2^{2l}e^{-(C_1(t)p_1^2+C_2(t)p_2^2)}\, dp = \\
&= C\frac{(2k-1)!\sqrt{\pi}}{2^{2k-1}(k-1)!C_1^{\frac{2k+1}{2}}(t)}\frac{(2l-1)!\sqrt{\pi}}{2^{2l-1}(l-1)!C_2^{\frac{2l+1}{2}}(t)} = O\left(\frac{1}{t^{\frac{1}{2}(2k+1)(2l+1)}}\right).
\end{aligned}
$$

With the estimates from above, we can apply Lemma 3.1 for all (mixed) derivatives of F_1, hence the first statement follows. Furthermore, the estimates in the equations (3.6) and (3.7) imply the decay properties for F_1. This also verifies our formal computation, which implies that F_1 is a unique solution of the equation. □

Next, now that we have established the existence of a solution of the first equation, we will consider the second equation of (3.2),

$$
\partial_t F_2 = \frac{1}{\mu^A}\partial_s(\lambda + \frac{\mu^P}{c}\partial_\alpha F_1).
$$

The decay conditions imply that F_2 must be independet of t, since otherwise the structure of the equation would lead to a unbounded linear growth in s for λ. Hence, we compute

$$
\begin{aligned}
0 &= \frac{1}{\mu^A}\partial_s(\lambda + \frac{\mu^P}{c}\partial_\alpha F_1), \\
\lambda &= -\frac{\mu^P}{c\mu^A}\partial_\alpha F_1 + \lambda_0(t,\alpha),
\end{aligned}
$$

for some abbritrary function $\lambda_0(t,\alpha)$ with $\lim_{|\alpha|\to\infty}\lambda_0(t,\alpha)=0$. One can interpret this set of solutions as shifts in the s direction, which, after the F_1 component has vanished, only translate the filament lengthwise for all time. Such perturbations are therefore neither unique nor will they decay to zero on the whole space. Hence, the only solution which fulfills the decay conditions is F_2 and λ_0 equal to zero.

Theorem 3.4 *There exists a unique solution of the linearized system (3.2) for initial conditions $F(0,\alpha,s)=(F_{1,0}(\alpha,s),0)$, with $F_{1,0}\in L^2(\mathbb{R}^2)$ and in respect to the decay conditions (3.3) and (3.4). The solution is given by*

$$F(\alpha,s,t)=(F_1(\alpha,s,t),0)=\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{P(p_1,p_2)t}\hat{F}_{1,0}(p)e^{ip\cdot x}\ dp,0\right)$$

and

$$\lambda(\alpha,s,t)=-\frac{\mu^P}{c\mu^A}\partial_\alpha F_1.$$

Furthermore, the solution has the following properties:

- $F_1\in C^\infty((0,\infty)\times\mathbb{R}^2)\cap L^2([0,t),L^2(\mathbb{R}^2))$,
- $\lambda\in C^\infty((0,\infty)$,
- $\lim_{t\to\infty}|F(\alpha,s,t)|=0$, $\lim_{t\to\infty}|\lambda(\alpha,s,t)|=0$.

Proof:
The previous considerations imply that $(F_1,0)$ is a solution of the system with the first two properties. Hence, only the last property and the decay contitions for λ are left to proof. We use Hlders inequality and (3.6) to estimate

$$|\lambda(\alpha,s,t)|^2=|\partial_\alpha F_1(\alpha,s,t)|^2\leq||\hat{F}_{1,0}(p)||^2_{L^2}||p_1e^{P(p_1,p_2)t}||^2_{L^2}\leq O\left(\frac{1}{t}\right),$$

which implies the last statement for λ. The decay conditions follow, since λ is the Fourier transformation of the function $p_1e^{P(p_1,p_2)t}\hat{F}_{1,0}(p)$. Hence, we can again use the estimate (3.6) of the previous Lemma, and conclude that this function is in $L^1(\mathbb{R}^2)$ and therefore Riemann-Lebesgues Lemma implies that $\lambda\in C_0(\mathbb{R}^2)$. □

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