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“Stability of 1-dimensional A-branes”

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Fabian Haiden

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1 Introduction

The study of stability in triangulated categories was initiated by Bridgeland [4] based on a proposal of M.R. Douglas. The majority of the activity since then has focused on categories of algebro-geometric origin, see e.g. [2, 5, 6, 16, 21]. For symplectic manifolds, on the other hand, conjectural ideas relating special Lagrangians and mean curvature flow to algebraic notions of stability have been around for some time, see Thomas [30], Thomas–Yau [31], and more recently Joyce [15], motivating the study of stability conditions on Fukaya-type categories. Significant work in this direction is due to Bridgeland–Smith [8] and Smith [28], and we will explain the relation to their results below. In the present paper we verify Bridgeland’s axioms in the lowest dimensional case, that of surfaces. Let us explain what we mean by that.

Consider a compact Riemann surface $\overline{C}$ with quadratic differential Ω which is holomorphic and non-vanishing on the complement $C = \overline{C} \setminus S$ of a finite set S . Near points of S we require Ω to be meromorphic or else of the form

$$e^{f(z)}g(z)dz^2 \quad (1.1)$$

with f, g meromorphic and f having a pole at 0. There is a natural flat metric, $|\Omega|$, on C , and the metric completion, F , of C adds a finite number of conical-type singularities.

Associated with (C, Ω) is a Fukaya-type A_∞ -category $\mathcal{A}$ which depends only on topological data: The genus of C , the number and types of singularities of Ω , and the monodromy of Ω . An elementary definition of this category will be given in the main text below. Very roughly, its objects are curves in C . In particular, all finite-length geodesics in F give objects in $\mathcal{A}$. The main result of this work is the following (Theorem 5.2 in the main text).

Theorem. *Let (C, Ω) be a Riemann surface with quadratic differential as above, so that Ω has no higher order poles, $\mathcal{A}$ the associated Fukaya A_∞ -category. Then the collection of all objects in the triangulated category $H^0(\mathcal{A})$ corresponding to smooth geodesics with local system satisfy the axioms of a stability condition.*

This is a consequence of a striking interplay between the geometry of flat surfaces, the properties of Fukaya categories, and the definition of a stability condition, summarized in the following table.

Flat geometry	Stability condition
finite smooth geodesic	stable object
broken geodesic	Harder–Narasimhan tower
angle	phase
period map $H_1 \rightarrow \mathbb{C}$	central charge $K_0 \rightarrow \mathbb{C}$

These correspondences played a guiding role in [9].

In the process of proving our main theorem we also establish a number of results on Fukaya categories of (non-closed) surfaces which have, to our knowledge, not appeared in this generality, including:

- Classification of objects (Theorem 3.3),
- Description by quivers with relations (Subsection 3.2),
- Regularity properties (Corollary 3.1),
- Computation of K_0 (Theorem 5.1).

Quadratic differential are also found to give stability conditions on triangulated categories in the work of Bridgeland–Smith [8], who furthermore identify the relevant moduli spaces. However, there is essentially no overlap between [8] and the present work for the following reasons. First, the categories considered in [8] are CY3, and indeed turn out to be subcategories of Fukaya categories of

quasi-projective 3-folds [28]. The CY3 property is also required in the definition of “categorical” Donaldson–Thomas invariants [18]. In contrast, the Fukaya category of a surface is never CY3. Second, the quadratic differentials in [8] are required to have simple zeros and at least one pole, which is of higher order at a generic point in the moduli space. The case of higher order poles is somewhat degenerate from our perspective, as one does not get a stability condition on the full Fukaya category, but only on a subcategory. More recently, an extension of the results of Bridgeland–Smith to certain polynomial quadratic differentials on $\mathbb{C}$ was carried out in [14, 7].

Let us describe the structure of the paper. The purpose of Section 2 is to give an elementary definition of the Fukaya category of a surface with boundary. We do this by associating an explicitly defined A_∞ -category $\mathcal{F}_A$ to a suitable collection A of arcs, and then showing that the category of twisted complexes $\mathcal{F} = \text{Tw}(\mathcal{F}_A)$ is independent of the choice of arcs. In Section 3 we solve a tame classification problem which allows us to prove *geometricity* — that objects in $\mathcal{F}$ come from curves with local system. While also of independent interest, this result is essential in our strategy of verifying the axioms of a stability condition. In Section 4 we recall some facts about quadratic differentials and flat surfaces. What is new here is the analysis of the exponential-type singularities (1.1). Finally, in Section 5 we are ready to state and prove the main result, after recalling Bridgeland’s definition and a computation of K_0 .

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2 A_∞ -structures from arc systems on surfaces

The main goal of this section is to give an elementary definition of the partially wrapped Fukaya category of a surface. We start by reviewing the notion of *grading*, adapted to the case of surfaces in Subsection 2.1. In the second subsection we define a class of surfaces with corners and marked boundary and explain how they can be encoded combinatorially through ribbon graphs with grading. In Subsection 2.3 we construct a minimal A_∞ -category $\mathcal{F}_A(S, M, \Omega)$ depending on a collection of arcs A . Twisted complexes over A_∞ -categories are reviewed in Subsection 2.4. The considerations in Subsection 2.5 then allow us to define an A_∞ -category $\mathcal{F}(S, M, \Omega)$ which is independent of the choice of arcs. The last subsection discusses the relation between localization and removing boundary arcs.

2.1 Grading of surfaces and curves

A generalization of the discussion in this subsection to symplectic manifolds with compatible almost complex structure and Lagrangian submanifolds is due independently to Kontsevich and Seidel [17, 26].

For a real vector bundle V we denote by $\mathbb{P}(V)$ its projectivization. A **grading** of a smooth oriented surface S is a section Ω of the projectivized tangent bundle, i.e.

$$\Omega \in \Gamma(S, \mathbb{P}(TS)).$$

The pair (S, Ω) is a **graded surface**. A morphism of graded surfaces $(S_1, \Omega_1) \rightarrow (S_2, \Omega_2)$ is a pair $(f, \tilde{f})$ where

- $f : S_1 \rightarrow S_2$ is an orientation preserving local diffeomorphism,
- $\tilde{f} \in \Pi_1(\Gamma(S_1, \mathbb{P}(TS_1)), f^*\Omega_2, \Omega_1)$.

Here we use the notation $\Pi_1(X, x, y)$ for the set of homotopy classes of paths from x to y in a space X . Composition is given by

$$(f, \tilde{f}) \circ (g, \tilde{g}) = (f \circ g, (g^*\tilde{f}) \cdot \tilde{g}) \quad (2.1)$$

where $\alpha \cdot \beta$ denotes concatenation of paths. Every graded surface S has a **shift** automorphism given by the pair (id_S, σ) where σ restricts, for every $x \in S$, to the generator of $\pi_1(\mathbb{P}(T_x S))$ given by the orientation of S (i.e. the path which rotates a line counterclockwise by an angle of π). Anticipating the connection with triangulated categories, we write this automorphism as $[1]$ and its powers as $[n]$.

Next, we consider immersed curves in graded surfaces. A **graded curve** in a graded surface (S, Ω) is a triple $(I, c, \tilde{c})$ where

- I is a 1-manifold, possibly disconnected and with boundary,
- $c : I \rightarrow S$ is an immersion,
- $\tilde{c} \in \Pi_1(\Gamma(I, c^*\mathbb{P}(TS)), c^*\Omega, \dot{c})$.

In other words, $\tilde{c}$ is given by a homotopy class of paths in $\mathbb{P}(T_{c(t)}S)$ from the subspace given by the grading to the tangent space of the curve, varying continuously with $t \in I$. The pushforward of a graded curve as above by a graded morphism $(f, \tilde{f})$ is given by $(I, f \circ c, (c^*\tilde{f}) \cdot \tilde{c})$.

A point of transverse intersection of a pair $(I_1, c_1, \tilde{c}_1), (I_2, c_2, \tilde{c}_2)$ of graded curves determines an integer as follows. Suppose $t_i \in I_i$ with

$$c_1(t_1) = c_2(t_2) = p \in S, \quad \dot{c}_1(t_1) \neq \dot{c}_2(t_2) \in \mathbb{P}(T_p S). \quad (2.2)$$

We have homotopy classes of paths in $\mathbb{P}(T_p S)$

1. $\tilde{c}_1(t_1)$ from $\Omega(p)$ to $\dot{c}_1(t_1)$,
2. $\tilde{c}_2(t_2)$ from $\Omega(p)$ to $\dot{c}_2(t_2)$,

3. κ from $\dot{c}_1(t_1)$ to $\dot{c}_2(t_2)$ given by counterclockwise rotation in $T_p S$ by an angle $< \pi$.

Define the **intersection index** of c_1, c_2 at p

$$i_p(c_1, c_2) = \tilde{c}_1(t_1) \cdot \kappa \cdot \tilde{c}_2(t_2)^{-1} \in \pi_1(\mathbb{P}(T_p S)) \cong \mathbb{Z}. \quad (2.3)$$

More precisely, this notation is correct only if c_1, c_2 pass through p exactly once, in particular if c_i are in general position, otherwise the index may depend on the t_i as well. We note that

$$i_p(c_1, c_2) + i_p(c_2, c_1) = 1 \quad (2.4)$$

and

$$i_p(c_1[m], c_2[n]) = i_p(c_1, c_2) + m - n. \quad (2.5)$$

Another observation which will be used repeatedly, is that if graded curves X and Y intersect in p with $i_p(X, Y) = 1$, then we may perform a kind of smoothing near p which produces again a grade curve (see Figure 1).

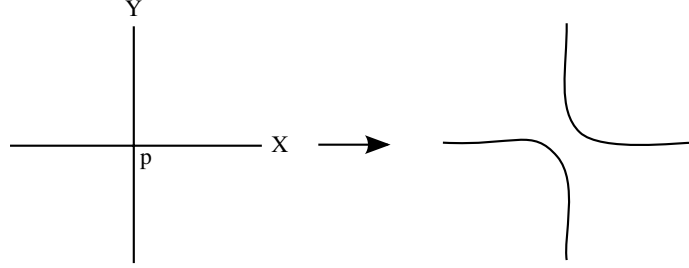


Figure 1: Smoothing an intersection with $i_p(X, Y) = 1$.

2.2 Surfaces with marked boundary

A **marked surface** is an oriented surface with corners together with a closed subset $M \subset \partial S$ of the boundary (union of 0- and 1-strata) such that the point-set topological boundary of M in ∂S is the set of corners of S , and each component of ∂S contains a point of M . In particular, if S is compact then each component of ∂S is either a circle which belongs entirely to M or a sequence of intervals which alternately belong to M and its complement.

A **boundary arc** is the closure of a component of $\partial S \setminus M$. An **internal arc** in a marked surface S is an embedded interval intersecting ∂S precisely in the endpoints, which should lie in the interior of M , and also not isotopic to a boundary arc or a path in M . An **arc** is one of the above. An **arc system** in (S, M) is a collection of pairwise disjoint and non-isotopic arcs. The arc system is **full** if it includes all the boundary arcs and cuts the surface into contractible, relatively compact components (polygons).

A map of marked surfaces $f : (S_1, M_1) \rightarrow (S_2, M_2)$ is an orientation preserving immersion with $f(M_1) \subset M_2$ mapping boundary arcs of S_1 to disjoint non-isotopic arcs in S_2 . The condition insures that a full arc system in S_2 which includes all the images of boundary arcs under f can be lifted to a full arc system on S_1 . It is however not closed under composition in general.

Combinatorial description: graded ribbon graphs

Let (S, M, Ω) be a graded marked surface and A a full system of graded arcs on it. Such a collection of arcs is dual to a graph on S having an edge for each element of A and an n -valent vertex for each $2n$ -gon cut out by A (see Figure 2). By definition, a *half-edge* is an edge with orientation pointing towards a vertex of

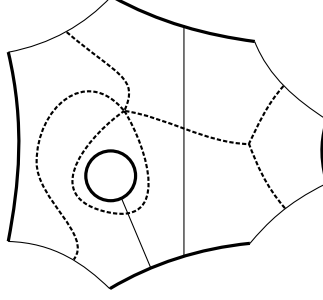


Figure 2: Full system of arcs in a marked surface and dual ribbon graph (dashed).

the graph (but not a boundary arc). Since the graph is embedded in an oriented surface, there is a clockwise cyclic order on the set of half-edges pointing towards a given vertex (i.e. the graph has the structure of a *ribbon graph*). Moreover, for any half-edge h the grading determines an integer $d(h)$ as follows. Let $\sigma(h)$ be the half edge following h in the cyclic order. Then $h, \sigma(h)$ correspond to a pair of arcs α, β in A . Let $c : [0, 1] \rightarrow M$ be the embedded curve which starts on α , ends on β , follows the boundary with S to the right, and bounds the polygon corresponding to the vertex that $h, \sigma(h)$ point to. Also choose an arbitrary grading on c . Then

$$d(h) = i_{c(0)}(\alpha, c) - i_{c(1)}(\beta, c) \quad (2.6)$$

is independent of the choice of grading on c .

The combinatorial data determined by a full system of graded arcs has the structure of a **graded ribbon graph**, which is given by

- X — the set of *half-edges*
- $X_{\text{int}} \subset X$ — the subset of *internal edges*.
- $\tau : X_{\text{int}} \rightarrow X_{\text{int}}$ — a fixed-point-free involution.
- $\sigma : X \rightarrow X$ — a bijection with finite orbits
- $d : X \rightarrow \mathbb{Z}$ — the *grading*, satisfying for any orbit $v \in X/\sigma$:

$$\sum_{e \in v} d(e) = |v| - 2 \quad (2.7)$$

The complement $X_{\text{ext}} := X \setminus X_{\text{int}}$ is the set of *external edges*. The set of *edges* is $X_{\text{ext}} \sqcup X_{\text{int}}/\tau$. The set of *vertices* is X/σ . The *valency* of a vertex $v \in X/\sigma$ is $|v|$, the size of the orbit. One generally assumes that all vertices have valency ≥ 3 .

Conversely, a marked graded surface with full system of arcs may be constructed from any graded ribbon graph. In particular, the surface may be recovered from an associated graded ribbon graph, up to graded diffeomorphism. This is a convenient way to specify compact graded marked surface by a finite amount of data.

A morphism of graded ribbon graphs

$$(X, X_{\text{int}}, \sigma_X, \tau_X, d_X) \rightarrow (Y, Y_{\text{int}}, \sigma_Y, \tau_Y, d_Y) \quad (2.8)$$

is a function $f : X \rightarrow Y$ compatible with all the structure, so

$$f(X_{\text{int}}) \subset Y_{\text{int}}, \quad f \circ \sigma_X = \sigma_Y \circ f, \quad f \circ \tau_X = \tau_Y \circ f, \quad d_X = d_Y \circ f \quad (2.9)$$

Note that we do *not* require that $f(X_{\text{ext}}) \subset Y_{\text{ext}}$ in general, and call f a **cov-ering** if it has this property. It follows from (2.7) that f preserves valency.

2.3 Minimal A_∞ -category of an arc system

Fix from now on a field of scalars $\mathbb{K}$. In this subsection we assign a strictly unital, minimal A_∞ -category over $\mathbb{K}$ to any system of graded arcs in a graded marked surface. Let us first recall the relevant notions. For more on the A_∞ language, see e.g. [27, 19].

An A_∞ -**category**, $\mathcal{A}$, consists of a set $\text{Ob}(\mathcal{A})$ of objects, for each pair $X, Y \in \text{Ob}(\mathcal{A})$ a $\mathbb{Z}$ -graded vector space $\text{Hom}(X, Y)$, and structure maps

$$\mu^n : \text{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \text{Hom}(X_0, X_1) \rightarrow \text{Hom}(X_0, X_n)[2 - n]$$

for each $n \geq 1$, satisfying the A_∞ -relations

$$\sum_{i+j+k=n} (-1)^{\|a_k\| + \cdots + \|a_1\|} \mu^{i+1+k}(a_n, \dots, a_{n-i+1}, \mu^j(a_{n-i}, \dots, a_{k+1}), a_k, \dots, a_1) = 0 \quad (2.10)$$

where $\|a\| = |a| - 1$ is the reduced degree.

An A_∞ -category $\mathcal{A}$ is **strictly unital** if for every object X there is a $1_X \in \text{Hom}^0(X, X)$ such that

$$\mu^1(1_X) = 0 \quad (2.11)$$

$$\mu^2(a, 1_X) = (-1)^{|a|} \mu^2(1_Y, a) = a, \quad a \in \text{Hom}(X, Y) \quad (2.12)$$

$$\mu^k(\dots, 1_X, \dots) = 0 \text{ for } k \geq 3 \quad (2.13)$$

Strictly unital A_∞ -categories with $\mu^k = 0$ for $k \geq 3$ correspond to small dg-categories with

$$da = (-1)^{|a|} \mu^1(a), \quad ab = (-1)^{|b|} \mu^2(a, b). \quad (2.14)$$

Indeed, in this case the first three A_∞ -relations correspond to $d^2 = 0$, the Leibniz rule, and associativity of the product.

Suppose now that (S, M, Ω) is a graded marked surface with system of graded arcs A . We define a strictly unital A_∞ -category $\mathcal{F}_A(S, M, \Omega)$ with $\mu^1 = 0$ (c.f. [3] for a similar approach in the context of dimers).

- **Objects:** The set of arc in A .
- **Morphisms:** A **boundary path** is a non-constant path in M which follows the reverse orientation of the boundary (i.e. the surface lies to the right). Given arcs X and Y , a basis of morphisms from X to Y is given by boundary paths, up to reparameterization, starting at an endpoint of X and ending at an endpoint of Y , as well as the identity morphism if $X = Y$. The degree of a boundary path a from p to q joining arcs X and Y is by definition

$$|a| = i_p(X, a) - i_q(Y, a) \quad (2.15)$$

for arbitrary grading of a .

- **Composition:** Let a, b be boundary paths defining morphisms from X to Y and Y to Z respectively. If a and b are composable then $(-1)^{|a|}\mu^2(b, a) = a \cdot b$, otherwise $\mu^2(b, a) = 0$.
- **Higher operations:** Consider a marked surface (S', M') which is topologically a closed disk with M' having $n \geq 3$ components. Let $a_1, \dots, a_n$ be the distinct boundary paths (components of M') ending in boundary arcs in clockwise order. Given a map $f : (S', M') \rightarrow (S, M)$ sending all boundary arcs of (S', M') to arcs in A we get a sequence $f \circ a_1, \dots, f \circ a_n$ of boundary paths in (S, M) , and call any such sequence a **disk sequence**.

We define higher A_∞ -operations so that if $a_1, \dots, a_n$ is a disk sequence then

$$\mu^n(a_n, \dots, a_1 b) = (-1)^{|b|} b \quad (2.16)$$

for basis morphisms b with $a_1 b \neq 0$, and

$$\mu^n(b a_n, \dots, a_1) = b \quad (2.17)$$

for paths b with $b a_n \neq 0$, and μ^n vanishes on all sequences of paths not of the above forms. The lemma below ensures that this is really well-defined.

Lemma 2.1. *For a sequence of composable basis morphisms $a_n, \dots, a_1$ there is at most one factorization $a_1 = a'_1 b$ with $a'_1, a_2, \dots, a_n$ a disk sequence. If such a factorization exists with b not an identity, then there is no factorization $a_n = c a'_n$ with $a_1, \dots, a_{n-1}, a'_n$ a disk sequence and c not an identity. The dual statement also holds.*

Proof. To see the first statement note that if we have such a factorization and α_i is the arc on which a_i ends, then the concatenation $a'_1 \cdot \alpha_1 \cdots a_n \cdot \alpha_n$ is a null-homotopic loop. Hence, b must be homotopic (relative endpoints) to the concatenation $a_1 \cdot \alpha_1 \cdots a_n \cdot \alpha_n$ and is thus uniquely determined as a morphism.

For the second statement, assume such a factorization exists. Then $c = b$ by the same argument as before, but b and a_n end at a different endpoints of α_n , contradicting $a_n = c a'_n$. $\square$

Proposition 2.1. *With the structure defined above, $\mathcal{F}_A(S, M, \Omega)$ is a strictly unital A_∞ -category.*

Proof. Let $a_n, \dots, a_1$ be a composable sequence of morphisms corresponding boundary paths. The claim is that the A_∞ -equation (2.10) holds. For $n = 1, 2, 3$ there is nothing to prove, so we may assume $n \geq 4$. The following types of non-zero terms cancel pairwise, ignoring signs for the moment:

$$\begin{aligned}
& \mu^2(\mu^i(a_n, \dots, a_1 b), c) \\
& \mu^i(a_n, \dots, a_2, \mu^2(a_1 b, c)) \\
\\
& \mu^2(a, \mu^i(bc_n, \dots, c_1)) \\
& \mu^i(\mu^2(a, bc_n), c_{n-1}, \dots, c_1) \\
\\
& \mu^i(a_i, \dots, a_{j+1}, \mu^k(a_j b_k, \dots, b_1), a_{j-1}, \dots, a_1 c) \\
& \mu^{i+k-2}(a_i, \dots, a_j b_k, \dots, \mu^2(b_1, a_{j-1}), a_{j-2}, \dots, a_1 c) \\
\\
& \mu^i(a_i, \dots, a_{j+1}, \mu^k(b_k, \dots, b_1 a_j), a_{j-1}, \dots, a_1 c) \\
& \mu^{i+k-2}(a_i, \dots, a_{j+2}, \mu^2(a_{j+1}, b_k), \dots, b_1 a_j, a_{j-1}, \dots, a_1 c)
\end{aligned}$$

as well as variants of the previous two with c on the other side, and

$$\begin{aligned}
& \mu^i(a_i, \dots, a_2, \mu^j(a_1 b c_j, c_{j-1}, \dots, c_1)) \\
& \mu^j(\mu^i(a_i, \dots, a_2, a_1 b c_j), c_{j-1}, \dots, c_1)
\end{aligned}$$

All non-zero terms which can appear must indeed belong to one of the above pairs. To check that the signs are in fact opposite, one uses that

$$\sum \|a_i\| = -2 \tag{2.18}$$

for any disk sequence $a_1, \dots, a_n$. $\square$

There are two special cases for which the above description needs to be somewhat amended, as disks also contribute to μ^2 . The first is that of a square, in which the two boundary arcs are isotopic. They represent the same object X , which has $\text{End}(X) = \mathbb{K}$. The second case is when S is a cylinder and $M = \partial S$. Up to isotopy there is again only one arc, and the corresponding object X has $\text{End}(X) = \mathbb{K}[z^{\pm 1}]$, with the degree of z depending on Ω .

Next, consider a map of graded marked surfaces $f : (S_1, M_1, \Omega_1) \rightarrow (S_2, M_2, \Omega_2)$ and arc systems A_i in S_i such that f maps graded arcs in A_1 to graded arcs in A_2 . Then f induces a strict A_∞ -functor f_* from $\mathcal{F}_{A_1}(S_1, M_1, \Omega_1)$ to $\mathcal{F}_{A_2}(S_2, M_2, \Omega_2)$. This just follows from the following fact: Suppose we have an immersed disk D in S_2 whose boundary consists of arcs and boundary curves, then if ∂D lifts to S_1 , so does D .

2.4 Twisted complexes

We briefly recall the construction of the category of twisted complexes, $\text{Tw}\mathcal{A}$, over an A_∞ -category $\mathcal{A}$. Twisted complexes can be thought of as formal vector bundles with flat connection.

The first step is to form $\text{add}\mathbb{Z}\mathcal{A}$ whose objects are formal sums of tensor products of the form

$$V = \bigoplus_{X \in \text{Ob}\mathcal{A}} V_X \otimes X \quad (2.19)$$

with V_X finite-dimensional graded vector spaces, zero for all but finitely many X . Morphism spaces are defined by

$$\text{Hom}(V \otimes X, W \otimes Y) = \text{Hom}(V, W) \otimes \text{Hom}(X, Y) \quad (2.20)$$

and additivity. When extending the μ^k a Koszul sign gets introduced:

$$\mu^k(\phi_k \otimes a_k, \dots, \phi_1 \otimes a_1) = (-1)^{\sum_{i < j} |\phi_i| |a_j|} \phi_k \cdots \phi_1 \otimes \mu^k(a_k, \dots, a_1) \quad (2.21)$$

Note that $\text{add}\mathbb{Z}\mathcal{A}$ has a natural shift functor and formal finite direct sums. Strictly speaking, $\text{Ob}(\text{add}\mathbb{Z}\mathcal{A})$ is not a set, but since the category of finite-dimensional vector spaces is essentially small, this is a non-issue.

An object in $\text{Tw}\mathcal{A}$ is given by a pair (V, δ) with $V \in \text{add}\mathbb{Z}\mathcal{A}$ and $\delta \in \text{Hom}^1(V, V)$. The first condition is that there is a direct sum decomposition of V (i.e. of $\bigoplus V_X$ as an $\text{Ob}\mathcal{A} \times \mathbb{Z}$ -graded vector space) so that δ is strictly upper triangular. This ensures that $\mu^k(\delta, \dots, \delta) = 0$ for k big, so that the second condition, which is

$$\sum_{k \geq 1} \mu^k(\delta, \dots, \delta) = 0 \quad (2.22)$$

makes sense. Morphism spaces are just

$$\text{Hom}((V, \delta), (W, \epsilon)) = \text{Hom}(V, W) \quad (2.23)$$

and the structure maps are

$$\mu^k(a_k, \dots, a_1) = \sum_{n_0, \dots, n_k \geq 0} \mu^{k+n_0+\dots+n_k}(\underbrace{\delta_k, \dots, \delta_k}_{n_k \text{ times}}, a_k, \dots, a_1, \underbrace{\delta_0, \dots, \delta_0}_{n_0 \text{ times}}) \quad (2.24)$$

where $a_i \in \text{Hom}((V_{i-1}, \delta_{i-1}), (V_i, \delta_i))$. Again, this sum is actually finite by our requirement on the δ_i .

The main fact we need about $\text{Tw}\mathcal{A}$ is that its homotopy category, $H^0(\text{Tw}\mathcal{A})$, is triangulated. When we write $K_0(\text{Tw}\mathcal{A})$, we always mean the K_0 -group of this triangulated category.

2.5 Morita invariance

Consider the simplest case when (S, M, Ω) is a graded marked surface which is topologically a disk and M has $n \geq 3$ components, and A is the arc system containing exactly the boundary arcs, with some arbitrary grading. The A_∞ -category $\mathcal{F}_A(S, M, \Omega)$ has objects E_k , $k \in \mathbb{Z}/n$, and morphisms $a_k : E_k \rightarrow E_{k+1}$ of some degrees $|a_k| \in \mathbb{Z}$ with $\sum |a_k| = n - 2$ which, together with the identity morphisms, form a basis of all morphisms. The only non-zero A_∞ -terms come from strict unitality and

$$\mu^n(a_{k+n-1}, \dots, a_k) = 1_{E_k}. \quad (2.25)$$

See also [22] for a discussion of these categories.

We make a simple observation which will be essential in what follows. Namely, that the twisted complex

$$E_1 \xrightarrow{a_1} E_2[\|a_1\|] \longrightarrow \dots \xrightarrow{a_{n-2}} E_{n-1}[\|a_1\| + \dots + \|a_{n-2}\|]$$

is isomorphic to $E_n[-|a_n|]$, the inverse isomorphisms being given by a_{n-1} and a_n . In other words, $n-1$ of the boundary arcs already generate all of $\text{Tw}(\mathcal{F}_A(S, M, \Omega))$. By functoriality, the same holds for the images of these arcs under some map of graded marked surfaces $(S, M, \Omega) \rightarrow (S', M', \Omega')$ as long as no other arc has the same image as E_n .

What we take away from the previous paragraph is that when A and B are full arc systems in a general graded marked surface (S, M, Ω) with A included in B , then the functor $\mathcal{F}_A(S, M, \Omega) \rightarrow \mathcal{F}_B(S, M, \Omega)$ induces a quasi-equivalence of A_∞ -categories

$$\text{Tw}(\mathcal{F}_A(S, M, \Omega)) \rightarrow \text{Tw}(\mathcal{F}_B(S, M, \Omega)). \quad (2.26)$$

Furthermore, it is clear that we also get an equivalence as above (in fact an isomorphism) in the case when the two arc systems differ just by grading. Consider the set $\mathcal{A}$ of full arc systems up to isotopy on (S, M) , partially ordered by inclusion. If we view $\mathcal{A}$ as a category in the usual way, then the arguments above show that we get a functor $A \mapsto \text{Tw}\mathcal{F}_A(S, M, \Omega)$ from $\mathcal{A}$ to the category of strictly unital A_∞ categories and quasi-equivalences. From contractability of the classifying space $|\mathcal{A}|$ of full arc systems or ribbon graphs (see [12, 13]), it now follows that the various categories $\text{Tw}\mathcal{F}_A(S, M, \Omega)$ are canonically quasi-isomorphic, and we will denote any one of them by $\mathcal{F}(S, M, \Omega)$.

Remark. A $\mathbb{Z}/2$ -graded version of the category $\mathcal{F}(S, M, \Omega)$ was defined by Dyckerhoff–Kapranov [10] using dg-categories and homotopical algebra methods.

2.6 Localization

We begin by recalling a version Drinfeld’s construction for strictly unital A_∞ -categories, studied in detail in [20]. Let $\mathcal{A}$ be a strictly unital A_∞ -category over a field $\mathbb{K}$ and $E \in \text{Ob}(\mathcal{A})$. It will be convenient to use the notation $\mathcal{A}(X, Y)$ for $\text{Hom}(X, Y)$ in this subsection. We will define the quotient category $\mathcal{A}/E = \mathcal{B}$ of $\mathcal{A}$ by E , which will again be a strictly unital A_∞ -category. Informally, $\mathcal{B}$ is obtained by freely adjoining a morphism $\varepsilon \in \mathcal{B}^{-1}(E, E)$ with $\mu^1(\varepsilon) = 1_E$. Set

$$\begin{aligned} \mathcal{B}(X, Y) = & \mathcal{A}(X, Y) \oplus \\ & \oplus \left(\mathcal{A}(E, Y) \otimes \mathbb{K}[1] \otimes \left(\bigoplus_{n \geq 0} (\mathcal{A}(E, E) \otimes \mathbb{K}[1])^{\otimes n} \right) \otimes \mathcal{A}(X, E) \right) \end{aligned} \quad (2.27)$$

as a graded vector space. Write generators of the summand

$$\mathcal{A}(E, Y) \otimes \mathbb{K}[1] \otimes (\mathcal{A}(E, E) \otimes \mathbb{K}[1])^{\otimes n-2} \otimes \mathcal{A}(X, E)$$

as

$$a_n \cdot \dots \cdot a_1$$

with $a_n \in \mathcal{A}(E, Y)$, $a_1 \in \mathcal{A}(X, E)$, and $a_i \in \mathcal{A}(E, E)$ for $2 \leq i \leq n-1$, then

$$|a_n \cdots \cdots a_1| = |a_n| + \dots + |a_1| - n + 1 \quad (2.28)$$

$$\|a_n \cdots \cdots a_1\| = \|a_n\| + \dots + \|a_1\|. \quad (2.29)$$

Structure maps are given by

$$\begin{aligned} \mu^r(a_{n_r} \cdots \cdots a_{n_{r-1}+1}, \dots, a_{n_1} \cdots \cdots a_1) := \\ \sum_{\substack{i+j+k=n_r \\ j+k > n_{r-1} \\ k < n_1}} (-1)^{\|a_k\| + \dots + \|a_1\|} a_{n_r} \cdots \cdots \mu^j(a_{j+k}, \dots, a_{k+1}) \cdot a_k \cdots \cdots a_1 \end{aligned} \quad (2.30)$$

where $0 = n_0 < n_1 < \dots < n_r$, $r \geq 1$. All this generalizes in a straightforward manner to quotients by full subcategories $\mathcal{E} \subset \mathcal{A}$.

We return to the setting of surfaces with marked boundary where we consider the following modification. If (S, M) is a marked surface and E a boundary arc, we get a new pair (S', M') by adding E to the marked points and smoothing the two corners on which E ends. We do not quite get a morphism $(S, M) \rightarrow (S', M')$, since E would need to map to an arc which isotopic to a path in M' . For this reason we allow such arcs, which we call **null**, in this subsection. The definition of the A_∞ -category of a system of arcs works as before with the following small change: The category is no longer minimal, and the definition of μ^1 is modeled on the one for the higher μ^i . This means that the boundary path which is isotopic to a null arc E is a disk-sequence of length 1 and has differential the identity morphisms of that arc.

Proposition 2.2. *Let A be an arc system on (S, M, Ω) including the boundary arc E . By the modification as above we get (S', M', Ω') with arc system A' containing a null arc. Then there is a natural equivalence of A_∞ -categories*

$$\mathcal{F}_A(S, M, \Omega)/E \cong \mathcal{F}_{A'}(S', M', \Omega') \quad (2.31)$$

under $\mathcal{F}_A(S, M, \Omega)$.

Proof. Comparing definitions, one notices an evident strict A_∞ -functor

$$G : \mathcal{F}_{A'}(S', M', \Omega') \rightarrow \mathcal{F}_A(S, M, \Omega)/E \quad (2.32)$$

compatible with the functors from $\mathcal{F}_A(S, M, \Omega)$. It is obtained by factoring a boundary path in M' into its pieces which are alternately contained in M and E . The functor G is not an isomorphism of A_∞ -categories, but becomes one once we drop E from both the target and the source category. By general properties, $E \in H^*(\mathcal{F}_A(S, M, \Omega)/E)$ is a zero-object, and it is easily seen that the same is true in $H^*(\mathcal{F}_{A'}(S', M', \Omega'))$, implying that G is a quasi-equivalence. $\square$

3 Tameness and geometricity

The main result of this section is that equivalence classes of objects in $\mathcal{F}(S, M, \Omega)$ correspond to certain graded curves with local system in S . We refer to this

phenomenon as *geometricity*, and it implies that the classification of objects in $\mathcal{F}(S, M, \Omega)$ is a tame problem in representation theory.

The purpose of the first subsection is to show that certain *admissible* curves correspond to objects in $\mathcal{F}(S, M, \Omega)$. In Subsection 3.2 we discuss a class of quivers with relations which describe Fukaya categories of surfaces in many cases. The next subsection, which forms the heart of the proof, is concerned with linear representations of combinatorial structures called *nets*. We study twisted complexes over graded linear categories satisfying a nilpotence property in Subsection 3.4. The results of these subsections are applied in the final one to prove the classification.

3.1 Twisted complexes from curves

Fix a graded marked surface (S, M, Ω) and a ground field $\mathbb{K}$. An immersed curve c in S is **unobstructed** if it does not bound an immersed teardrop, which is a map from the closed disk D to the surface which takes ∂D to c and which is a smooth immersion at every point except one point of ∂D (see Figure 3). An **admissible curve** is an unobstructed graded curve c such that one of the

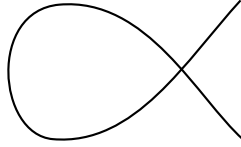


Figure 3: Curve bounding a teardrop.

following holds:

1. The domain of c is S^1 , the image of c is disjoint from ∂S , and c represents a primitive class in $\pi_1(S)$.
2. The domain of c is $[0, 1]$, c intersects ∂S transversely in M and only in the endpoints, and c is not homotopic relative endpoints to a path in M .

Suppose that S is compact so that we have a category $\mathcal{F}(S, M, \Omega)$, well defined up to canonical equivalence. The purpose of this subsection is to show that an admissible curve together with a local system of finite dimensional $\mathbb{K}$ -vector spaces (on its domain) gives an equivalence class of objects in $\mathcal{F}(S, M, \Omega)$.

First, in the case when S is topologically a disk any admissible curve c is a graded arc. Thus we can find a full arc system A which includes c , so that c is an object of $\mathcal{F}_A(S, M, \Omega)$ by definition. The isomorphism class of that object is clearly well-defined in $\mathcal{F}(S, M, \Omega)$ independently of the arc system.

Returning to the case of general S , we will first deal with admissible curves c which have domain $[0, 1]$. For any such c we can find a (S', M', Ω') which is of disk-type and with a map f to (S, M, Ω) so that c is the image of an admissible curve $\tilde{c}$ under f . To see this, consider the universal cover $\tilde{S}$ of S and lift a full arc system A on S to $\tilde{A}$ on $\tilde{S}$. Lift c to $\tilde{c}$ on $\tilde{S}$ and take as S' a closed disk which is cut out by arcs in $\tilde{A}$ and which contains $\tilde{c}$. Now, $\tilde{c}$ gives an equivalence class of objects in $\mathcal{F}(S', M', \Omega')$, and the image under the functor

$\mathcal{F}(S', M', \Omega') \rightarrow \mathcal{F}(S, M, \Omega)$ is independent of the choice of (S', M', Ω') . This follows from the fact that if we have S', S'' as above, then the maps $S' \rightarrow S, S'' \rightarrow S$ both factor through a third $S''' \rightarrow S$ as can be seen by looking at the universal cover again.

Suppose now instead that c is an admissible curve with domain S^1 and local system V of finite-dimensional vector spaces on it. We will follow the same strategy as before and assume first that (S, M) is of annular type, i.e. topologically a compact annulus with corners on each boundary component. Choose a cyclic sequence of disjoint non-isotopic arcs $X_i, i \in \mathbb{Z}/n$ so that at least one connects the two components of ∂S and such that every component of M contains either exactly two endpoints of the arcs, belonging X_i, X_{i+1} for some i , or none of the endpoints (see Figure 4). Thus we get a sequence a_i ,

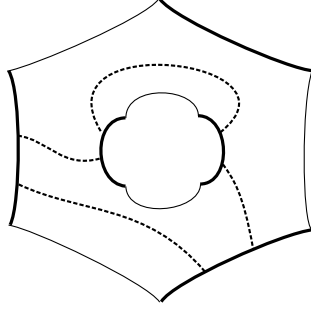


Figure 4: Example of allowed sequence of arcs (dashed).

$i \in \mathbb{Z}/n$ of distinct boundary paths so that a_i connects endpoints of X_i, X_{i+1} . If we follow $X_0, a_0, X_1, a_1, \dots$ we get a path which after suitable smoothing near the intersection points becomes a simple closed loop isotopic to c . It is possible to choose grading and a local systems on X_i, a_i , so that the smoothed path is isotopic to c as a graded curve with local system. As a result, each a_i will be morphism of degree 1 either from X_i to X_{i+1} or in the other direction. Further we get a vector space V_i of sections of over X_i and parallel transport $T_i : V_i \rightarrow V_{i+1}$ along a_i . Consider the twisted complex

$$\left(\bigoplus V_i \otimes X_i, \sum T_i^{\pm 1} \otimes a_i \right) \quad (3.1)$$

where the signs are determined by the direction of the morphisms a_i . This is the object of $\mathcal{F}(S, M, \Omega)$ we assign to c .

Lemma 3.1. *The equivalence class of the twisted complex constructed depends only on the isotopy class of the graded curve c with local system.*

Proof. We claim first that we can replace $X_1, \dots, X_{n-1}$ be a single arc Y with the pair X_0, Y giving an isomorphic twisted complex. Indeed, cutting S along X_0 we get a surface of disk type (S', M', Ω') in which the sequence of arcs $X_1, \dots, X_{n-1}$ concatenates to a single graded arc Y . The corresponding isomorphism of twisted complexes formed from $X_1, \dots, X_{n-1}$ and Y respectively was established in Subsection 2.5, and the claimed isomorphism of twisted complexes formed from $X_0, \dots, X_{n-1}$ and X_0, Y follows.

We have reduced the problem to the case of two arcs. Any two pairs of arcs satisfying our requirements are related by Dehn twists (automorphisms) of S . It is clear that the isotopy class of c is invariant under Dehn twists. To finish the proof we need to show that the twisted complex associated with c is invariant under the induced autoequivalence, up to isomorphism. This can be checked by direct computation, or by using the equivalence

$$H^0(\mathrm{Tw}(\mathcal{F}_{X_0,Y}(S, M, \Omega))) = D^b(\mathbb{P}^1) \quad (3.2)$$

under which X_0, Y correspond to $O, O(1)[-1]$, the Dehn-twist to $\otimes O(1)$, and the twisted complex assigned to c to a torsion sheaf. $\square$

We have excluded above the case when one or both boundary components are entirely contained in M . These cases can be handled fairly easily directly, or alternatively by the localization construction of the previous section. The case of general (S, M, Ω) is handled by finding maps from surfaces of annular type. Here the argument uses the annular covering associated with c instead of the universal covering.

3.2 Graded quivers with quadratic monomial relations

To set up some notation, a **graded quiver Q with quadratic monomial relations** is given by

- Q_0 – set of vertices
- Q_1 – set of arrows
- $\partial_0, \partial_1 : Q_1 \rightarrow Q_0$ – source and target maps
- $|\cdot| : Q_1 \rightarrow \mathbb{Z}$ – grading of arrows
- $R \subset Q_1 \times_{Q_0} Q_1$ – quadratic monomial relations, i.e. a collection of pairs of composable arrows

Given a ground field $\mathbb{K}$ we may form the path category $\mathbb{K}Q = \mathbb{K}Q/R$, which is a graded linear category with set of objects Q_0 and basis of morphisms given by paths of arrows not containing any of the relations.

Let (S, M, Ω) be a graded marked surface, compact for simplicity. Note first that if A is any arc system on S , then the graded category obtained from $\mathcal{F}_A(S, M, \Omega)$ by forgetting the higher μ^k is of the form $\mathbb{K}Q$ for some graded quiver with quadratic monomial relations. Namely, take as arrows the boundary paths which start and end at arcs of A but do not cross any other endpoints, and quadratic relations coming from composable arrows which do not correspond to composable paths. When do the higher A_∞ operations of $\mathcal{F}_A(S, M, \Omega)$ actually vanish? This will be the case if the system of arcs A has the following property: Any disk cut out by A is bounded by a boundary arc of S which does not belong to A . We call such systems of arcs **formal**. Thus, under this condition, the category $\mathcal{F}_A(S, M, \Omega)$ is of the form $\mathbb{K}Q$ for some graded quiver with quadratic monomial relations Q . Call a formal system of arcs **full** if it cuts S into disks each of which has exactly a single boundary arc (of itself) not belonging to A . For a full formal system of arcs A the category $\mathrm{Tw}(\mathcal{F}_A(S, M, \Omega))$ is quasi-equivalent to $\mathcal{F}(S, M, \Omega)$.

Lemma 3.2. *Let (S, M, Ω) be a graded marked surface with S compact and connected. If $\partial S \neq M$, i.e. S has boundary arcs, then it has a full formal system of arcs.*

Proof. This follows from the fact that we can find a system of arcs on (S, M) which cuts S into a single disk, the boundary of which must include all boundary arcs of S . This in turn is seen from the dual ribbon graph. If it has more than one vertex we can contract some edges to reduce the number of vertices to one eventually. $\square$

We say a graded quiver with relations is of **type F1** if it arises as above from a full formal system of arcs. It is not difficult to see that a graded quiver with quadratic monomial relations is of type F1 if and only if

1. There are no cycles $a_1, \dots, a_n$, $n \geq 1$, with $a_i a_{i+1} = 0$ for all $i \in \mathbb{Z}/n$.
2. Each vertex has at most two incoming and outgoing arrows.
3. Let $a, b \neq c$ be arrows. If ab, ac are defined, then $ab = 0$ or $ac = 0$, but not both. If ba, ca are defined, then $ba = 0$ or $ca = 0$, but not both.

Resolution of the diagonal

Let Q be a graded quiver with quadratic monomial relations. In this subsection it will be convenient to let $A = \mathbb{K}Q/R$ denote the path algebra instead of the path category. Denote the constant path at the vertex i by e_i . There is a splitting of A

$$A = \bigoplus_{i \in Q_0} A e_i, \quad A = \bigoplus_{i \in Q_0} e_i A \quad (3.3)$$

as a left (resp. right) module over A .

Define $A^{\text{op}} \otimes A$ -modules

$$M_n = \bigoplus_{\substack{\alpha_1, \dots, \alpha_n \in Q_1 \\ (\alpha_i, \alpha_{i+1}) \in R}} A e_{\partial_0 \alpha_1} \otimes e_{\partial_1 \alpha_n} A, \quad n \geq 1 \quad (3.4)$$

$$M_0 = \bigoplus_{i \in Q_0} A e_i \otimes e_i A \quad (3.5)$$

connected by maps $f_n : M_n \rightarrow M_{n-1}$ such that for $a \otimes b$ in the $(\alpha_1, \dots, \alpha_n)$ -component of M_n

$$f_n(a \otimes b) = a \alpha_1 \otimes b + (-1)^n a \otimes \alpha_n b \quad (3.6)$$

and $f_0 : M_0 \rightarrow A$, $f_0(a \otimes b) = ab$.

Proposition 3.1. *The sequence of bimodules*

$$\dots \longrightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} A \longrightarrow 0 \quad (3.7)$$

is exact.

Proof. For $a \otimes b \in M_n$ in the $(\alpha_1, \dots, \alpha_n)$ -component we have

$$\begin{aligned} f_{n-1}(f_n(a \otimes b)) &= \\ a\alpha_1\alpha_2 \otimes b + (-1)^{n-1}a\alpha_1 \otimes \alpha_n b + (-1)^n a\alpha_1 \otimes \alpha_n b - a \otimes \alpha_{n-1}\alpha_n b \\ &= 0 \end{aligned}$$

hence $f_{n-1} \circ f_n = 0$.

It remains to show that $\text{Ker}(f_n) \subset \text{Im}(f_{n+1})$. An element $x \in M_n$ is uniquely written as

$$x = \sum_{\substack{\alpha_1, \dots, \alpha_n \in Q_1 \\ (\alpha_i, \alpha_{i+1}) \in R}} \sum_{\substack{\alpha \text{ path, } \neq 0 \text{ in } A \\ \partial_1 \alpha = \partial_0 \alpha_1}} \alpha \otimes b_{\alpha_1, \dots, \alpha_n, \alpha} \quad (3.8)$$

with $b_{\alpha_1, \dots, \alpha_n, \alpha}$ almost all zero. For $x \neq 0$ define

$$l(x) = \max \{l(\alpha) \mid b_{\alpha_1, \dots, \alpha_n, \alpha} \neq 0\} \quad (3.9)$$

where $l(\alpha)$ is the length (i.e. number of arrows) of the path α . Observe that by definition of f_n , for every $x \neq 0$ there is a $y \in M_n$ with

$$l(y) = 0, \quad x - y \in \text{Im}(f_{n+1}). \quad (3.10)$$

Suppose now $x \in M_n$ with $l(x) = 0$, so we can write

$$x = \sum_{\substack{\alpha_1, \dots, \alpha_n \\ (\alpha_i, \alpha_{i+1}) \in R}} e_{\partial_0 \alpha_1} \otimes b_{\alpha_1, \dots, \alpha_n}. \quad (3.11)$$

The projection of $f_n(x)$ to the $\alpha_1, \dots, \alpha_{n-1}$ -component is

$$\sum_{\substack{\alpha \in Q_1 \\ (\alpha, \alpha_1) \in R}} \alpha \otimes b_{\alpha, \alpha_1, \dots, \alpha_{n-1}} + (-1)^n \sum_{\substack{\alpha \in Q_1 \\ (\alpha_{n-1}, \alpha) \in R}} e_{\partial_0 \alpha_1} \otimes \alpha b_{\alpha_1, \dots, \alpha_{n-1}, \alpha}. \quad (3.12)$$

Therefore, $f_n(x) = 0$ implies $b_{\alpha_1, \dots, \alpha_n} = 0$ for all $\alpha_1, \dots, \alpha_n \in Q_1$, $(\alpha_i, \alpha_{i+1}) \in R$, so $x = 0$, which completes the proof. $\square$

Proposition 3.2. *Let Q be a graded quiver with quadratic monomial relations, $A = \mathbb{K}Q/R$ its path algebra.*

1. *Suppose there are no cyclic paths $\alpha_1 \cdots \alpha_n$ in Q with $(\alpha_i, \alpha_{i+1}) \in R$, $i \in \mathbb{Z}/n$, then A is homologically smooth.*
2. *Suppose there are no cyclic paths $\alpha_1 \cdots \alpha_n$ in Q with $(\alpha_i, \alpha_{i+1}) \notin R$, $i \in \mathbb{Z}/n$, then A is proper.*

Proof. 1. Under the stated condition on cyclic paths, we see that $M_n = 0$ for $n \gg 0$, hence A is perfect as an $A^{\text{op}} \otimes A$ -module.

2. The condition implies that only finitely many paths are non-zero in A , so A has finite rank over $\mathbb{K}$. $\square$

We apply these results to graded linear models of $\mathcal{F}(S, M, \Omega)$. In this case, the first condition of Proposition 3.2 is always satisfied, while the second is satisfied if S has no boundary components without corners.

Corollary 3.1. *Let (S, M, Ω) be a graded marked surface without boundary components diffeomorphic to S^1 , then $\mathcal{F}(S, M, \Omega)$ is homologically smooth and proper.*

3.3 Representations of nets

In this subsection we classify representations of certain combinatorial structures, called *nets*, based on ideas in the work of Nazarova–Roiter [23] on a class of tame linear algebra problems. We will later use this result to derive the classification of objects in Fukaya categories of surfaces with boundary.

A **net** is a quadruple $\mathcal{X} = (A, \alpha, B, \beta)$ where

- A is a finite set,
- α is a fixed-point-free involution on A ,
- B is a finite set with partition $B = \bigsqcup_{i \in A} B_i$ into totally ordered sets,
- β is a fixed-point-free involution on $\text{Dom}(\beta) \subset B$.

A morphism of nets $f : (A, \alpha, B, \beta) \rightarrow (A', \alpha', B', \beta')$ is given by maps $f_1 : A \rightarrow A'$, $f_2 : B \rightarrow B'$ such that

$$f_1 \circ \alpha = \alpha' \circ f_1 \quad (3.13)$$

$$f_2(\text{Dom} \beta) \subset \text{Dom} \beta' \quad (3.14)$$

$$f_2(B \setminus \text{Dom} \beta) \subset B' \setminus \text{Dom} \beta' \quad (3.15)$$

$$f_2|_{\text{Dom} \beta} \circ \beta = \beta' \circ f_2|_{\text{Dom} \beta} \quad (3.16)$$

$$f_2(B_i) \subset B'_{f_1(i)} \text{ and } f_2|_{B_i} \text{ is increasing} \quad (3.17)$$

The **height** of a net $\mathcal{X} = (A, \alpha, B, \beta)$ is defined as

$$h(\mathcal{X}) = \max_{i \in A} |B_i|. \quad (3.18)$$

Nets of height 1 are disjoint unions of the following types of nets:

$$\bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \dots \xleftarrow{\alpha} \bullet \quad (3.19)$$

and

$$\bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \dots \xleftarrow{\alpha} \bullet \quad (3.20)$$

β

where we have indicated elements of $A = B$ by dots and the action of α, β by arrows. In both cases $|A| \geq 2$.

A **representation of a net** (A, α, B, β) is given by

- a finite-dimensional vector space V with a direct sum decomposition

$$V = \bigoplus_{i \in A/\alpha} V_i,$$

- for each $i \in A$ an increasing exhaustive filtration $\{F_j\}_{j \in B_i}$ on V_i (i.e. $j \leq k \implies F_j \subset F_k$, $F_{\max B_i} = V_i$),

- isomorphisms

$$\phi_i : \text{gr}_i V \rightarrow \text{gr}_{\beta(i)} V, \quad i \in \text{Dom}(\beta)$$

with $\phi_{\beta(i)} = \phi_i^{-1}$. Here $\text{gr}_i V = F_i/F_{<i}$ is the associated graded.

Representations of a net form a linear category with morphisms $(V, F_i, \phi_i) \rightarrow (W, G_i, \psi_i)$ being linear maps $f : V \rightarrow W$ such that $f(F_i) \subset G_i$ and

$$\begin{array}{ccc} \text{gr}_i V & \xrightarrow{\text{gr}_i f} & \text{gr}_i W \\ \downarrow \phi_i & & \downarrow \psi_i \\ \text{gr}_{\beta(i)} V & \xrightarrow{\text{gr}_{\beta(i)} f} & \text{gr}_{\beta(i)} W \end{array}$$

commutes for all $i \in \text{Dom}(\beta)$. We note that the category of representations of a net as in (3.19) is the category of finite dimensional vector spaces, while for (3.20) we get the category of finite dimensional representations of $\mathbb{Z}$ (up to equivalence).

Given a morphism of nets $f = (f_1, f_2) : \mathcal{X} = (A, \alpha, B, \beta) \rightarrow \mathcal{X}' = (A', \alpha', B', \beta')$ with f_2 injective on each B_i , there is an induced **pushforward** functor f_* on the corresponding categories of representations. If $V = (V, F_i, \phi_i)$ is a representation of $\mathcal{X}$, then we produce a representation $f_* V = W = (W, G_i, \psi_i)$ of $\mathcal{X}'$ with $W = V$ as vector spaces and

$$W_j = \bigoplus_{i \in A/\alpha, f_1(i)=j} V_i, \quad j \in A'/\alpha' \quad (3.21)$$

$$G_l = \bigoplus_{i \in A, l \in B'_{f_1(i)}} F_k, \quad \text{where } k = \max\{r \in B_i \mid f_2(r) \leq l\} \quad (3.22)$$

for which we have

$$\text{gr}_j W = \bigoplus_{i \in B, f_2(i)=j} \text{gr}_i V \quad (3.23)$$

so that we can define

$$\psi_j = \bigoplus_{i \in B, f_2(i)=j} \phi_i. \quad (3.24)$$

If f_2 fails to be injective on some B_i 's, then one can still define $f_* V$ given additional choices on V (i.e. in a non-functorial way). Namely, for each $i \in A$ and $j \in B'_{f_1(i)} \cap \text{Dom}(\beta')$ with

$$(f_2|_{B_i})^{-1}(j) \cong \{1, \dots, n\}, \quad n \geq 2 \quad (3.25)$$

choose a splitting of the inclusions $F_1 V \subset \dots \subset F_n V$. This gives us an isomorphism

$$\text{gr}_1 V \oplus \dots \oplus \text{gr}_n V \cong F_n V / F_{<1} V = \text{gr}_j W \quad (3.26)$$

allowing us to define ψ_j as direct sums of $\psi_1 \oplus \dots \oplus \psi_n$ for various i . Different choices of splittings give isomorphic $f_* V$.

Theorem 3.1. *Let $\mathcal{X}$ be a net, V a representation of $\mathcal{X}$. Then there is a net $\mathcal{X}'$ of height 1, a morphism $f : \mathcal{X}' \rightarrow \mathcal{X}$, and representation V' of $\mathcal{X}'$ such that $f_* V' = V$.*

As a preliminary to the proof of the theorem, we discuss diagrams

$$X \begin{array}{c} \xrightarrow{A} \\ \xleftarrow{B} \end{array} Y$$

of parallel surjective linear maps between finite dimensional vector spaces. First, we have two increasing filtrations on X :

$$A^{-1}0 \subset A^{-1}BA^{-1}0 \subset A^{-1}BA^{-1}BA^{-1}0 \subset \dots \subset \underbrace{A^{-1}(BA^{-1})^{k-1}0}_{=:F_k} \subset \dots \quad (3.27)$$

$$B^{-1}0 \subset B^{-1}AB^{-1}0 \subset B^{-1}AB^{-1}AB^{-1}0 \subset \dots \subset \underbrace{B^{-1}(AB^{-1})^{k-1}0}_{=:G_k} \subset \dots \quad (3.28)$$

They necessarily stabilize with $F_\infty = F_N$, $G_\infty = G_N$ for N big. In general, these filtrations are not exhaustive, but

$$F_\infty = G_\infty. \quad (3.29)$$

Furthermore, there are inverse isomorphisms

$$M_{i,j} \begin{array}{c} \xleftarrow{B^{-1}A} \\ \xrightarrow{A^{-1}B} \end{array} M_{i-1,j+1}$$

for $i \geq 0$, $j \geq 1$, where

$$M_{i,j} = \frac{F_i \cap G_j}{(F_{i-1} \cap G_j) + (F_i \cap G_{j-1})} \quad (3.30)$$

for $i, j \geq 1$. In particular,

- $\text{Ker}(B)$ has an exhaustive filtration $F \cap \text{Ker}(B)$ with associated graded pieces $M_{i,1}$,
- $\text{Ker}(A)$ has an exhaustive filtration $G \cap \text{Ker}(A)$ with associated graded pieces $M_{1,j}$,
- there are isomorphisms $M_{i,1} \longrightarrow M_{1,i}$.

We proceed with the proof of the theorem.

Proof. Let $(V, \{F_i\}, \phi_i)$ be a representation of a net $\mathcal{X} = (A, \alpha, B, \beta)$. The proof is by induction over

$$\sum_{|B_i| \geq 2} \dim V_i \quad (3.31)$$

for all nets simultaneously. If $h(\mathcal{X}) = 1$ we are done, so let us assume that $h(\mathcal{X}) \geq 2$. We can also assume, by passing to subsets of A and the B_i , that all associated graded $\text{gr}_i V$ are non-zero and all B_i are non-empty.

Let $r \in A$ with $|B_r| = h(\mathcal{X})$. Let $n = \max B_r$ and find the unique $k \in B_{\alpha(r)}$ such that

$$F_{<k} + F_{<n} \subsetneq V_r, \quad F_k + F_{<n} = V_r. \quad (3.32)$$

Set $X_1 = F_{<k} + F_{<n}$, $X_2 = F_k \cap (F_{<k} + F_{<n})$ and consider the refinements of the two filtrations of V_r :

$$\dots \subset F_{<n} \subset X_1 \subset F_n = V_r, \quad \dots \subset F_{<k} \subset X_2 \subset F_k \subset \dots \quad (3.33)$$

Note that the identity map induces an isomorphism of associated graded

$$F_n/X_1 \longrightarrow F_k/X_2 \quad (3.34)$$

which follows from the general isomorphism theorem $S/(S \cap T) \cong (S + T)/T$.

Case $\beta(k) \neq n$.

- If $n \in \text{Dom}(\beta)$, $\beta(n) \in B_p$, can refine the filtration on V_p to

$$\dots \subset F_{<\beta(n)} \subset X_3 \subset F_{\beta(n)} \subset \dots$$

where X_3 is the preimage of $\phi_n(X_1/F_{<n})$ under the quotient map $F_{\beta(n)} \twoheadrightarrow \text{gr}_{\beta(n)}X$.

- Similarly, if $k \in \text{Dom}(\beta)$, $\beta(k) \in B_p$, can refine the filtration on V_p to

$$\dots \subset F_{<\beta(k)} \subset X_4 \subset F_{\beta(k)} \subset \dots$$

where X_4 is the preimage of $\phi_k(X_2/F_{<k})$ under the quotient map $F_{\beta(k)} \twoheadrightarrow \text{gr}_{\beta(k)}X$.

We construct a modified net $\mathcal{X}' = (A', \alpha', B', \beta')$ with

$$A' = A \sqcup \{1, 2\}, \quad \alpha'(1) = 2, \quad \alpha'|_A = \alpha \quad (3.35)$$

and $B'_1 = \{1\}$, $B'_2 = \{2\}$. Moreover, if $n \in \text{Dom}\beta$ we insert an element “3” into B just before $\beta(n)$ with $\beta'(1) = 3$, and similarly if $k \in \text{Dom}\beta$ we insert an element “4” before $\beta(k)$ with $\beta'(2) = 4$. Note that the new elements of B correspond to the additional pieces of the filtrations described above. There is a morphism of nets $f : \mathcal{X}' \rightarrow \mathcal{X}$ with $1 \mapsto n$, $2 \mapsto k$, $3 \mapsto \beta(n)$, $4 \mapsto \beta(k)$.

V determines a representation $V' = (V', F_i, \psi_i)$ of $\mathcal{X}'$ with

$$V'_1 = V'_2 = F_n/X_1 \cong F_k/X_2, \quad V'_r = V'_{\alpha(r)} = X_1 \quad (3.36)$$

where we suppress the canonical isomorphism $F_n/X_1 \cong F_k/X_2$ from now on. The filtrations on V'_r are obtained from those on V_r by restriction, and the filtrations containing F_3, F_4 by refinement. The isomorphisms $\psi_n, \psi_k, \psi_1, \psi_2$ and their inverses are induced by ϕ_n, ϕ_k and their inverses respectively. We claim that f_*V' is isomorphic to V . Choose complements

$$X_1/F_{<n} \oplus Y_1 = \text{gr}_n V \quad (3.37)$$

$$X_2/F_{<k} \oplus Y_2 = \text{gr}_k V. \quad (3.38)$$

If $n \in \text{Dom}\beta$, then the choice of Y_1 is equivalent to a choice of complement of $X_3/F_{<\beta(n)} \subset \text{gr}_{\beta(n)}V$, and similarly for Y_2 . We use these complements in the definition of f_*V' .

We claim that we can find a $U \subset F_k$ such that

$$F_n \rightarrow \text{gr}_n V \text{ induces an isomorphism } U \rightarrow Y_1 \quad (3.39)$$

$$F_k \rightarrow \text{gr}_k V \text{ induces an isomorphism } U \rightarrow Y_2 \quad (3.40)$$

Namely, choose a $U \subset F_k$ with

$$X_2 \oplus U = F_k \quad (3.41)$$

such that (3.40) holds. We also have a direct sum $X_1 \oplus U = V_r$, but (3.39) may fail. However, we can ensure (3.39) after shearing U in the directions of $F_{<k}$, which does not affect (3.40).

The choice of U gives an identification

$$V_r = X_1 \oplus U = X_1 \oplus (F_n/X_1) = V'_r \oplus V'_1 \quad (3.42)$$

which we use to define the isomorphism $V \rightarrow f_* V'$. Compatibility with filtrations follows from $U \subset F_k$, and compatibility with ϕ_n, ϕ_k since they are block-diagonal with respect to the chosen splittings of associated graded.

Case $\beta(k) = n$. We have a diagram of parallel surjections:

$$\begin{array}{ccc} \text{gr}_n V & \xrightarrow{p_1} & F_n/X_1 \\ \cong \downarrow \phi_n & & \downarrow \cong \\ \text{gr}_k V & \xrightarrow{p_2} & F_k/X_2 \end{array}$$

By the discussion preceding the proof we get first of all a pair of non-exhaustive filtrations

$$0 = G_0 \subset G_1 \subset \dots \subset G_m, \quad 0 = H_0 \subset H_1 \subset \dots \subset H_m = G_m \quad (3.43)$$

of $\text{gr}_n V$, where $G_1 = \text{Ker}(p_1)$ and $H_1 = \phi_n^{-1}(\text{Ker}(p_2))$. For the associated double graded M_{ij} , $1 \leq i, j$, $i + j \leq m + 1$, we have isomorphisms $M_{ij} \rightarrow M_{i+1, j-1}$ induced by ϕ_n . Further, we have restricted filtrations on $\text{Ker}(p_1), \text{Ker}(p_2)$ lifting to refinements of the filtrations on V_r :

$$\dots \subset F_{<n} = X_{1,0} \subset X_{1,1} \subset \dots \subset X_{1,m} = X_1 \subset F_n \quad (3.44)$$

$$\dots \subset F_{<k} = X_{2,0} \subset X_{2,1} \subset \dots \subset X_{2,m} = X_2 \subset F_k \subset \dots \quad (3.45)$$

with

$$X_{1,i}/X_{1,i-1} \cong M_{1,i}, \quad X_{2,i}/X_{2,i-1} \cong M_{i,1} \quad (3.46)$$

We construct a modified net $\mathcal{X}' = (A', \alpha', B', \beta')$ with

$$A' = A \sqcup \{1, 2\} \sqcup \{(l, i, j) \mid l \in \{1, 2\}, i \geq 2, j \geq 2, i + j \leq m + 1\} \quad (3.47)$$

$$\alpha'(1) = 2, \quad \alpha'(1, i, j) = (2, i, j), \quad \alpha'|_A = \alpha \quad (3.48)$$

$$B'_1 = \{1\}, \quad B'_2 = \{2\}, \quad B'_{l,i,j} = \{l, i, j\} \quad (3.49)$$

$$B'_r = (B_r \setminus \{n\}) \sqcup \{(1, 1, 1) < (1, 1, 2) < \dots < (1, 1, m)\} \quad (3.50)$$

$$B'_{\alpha(r)} = (B_{\alpha(r)} \setminus \{k\}) \sqcup \{(2, 1, 1) < (2, 2, 1) < \dots < (2, m, 1)\} \quad (3.51)$$

$$\beta'(1) = 2, \quad \beta'(1, i, j + 1) = (2, i + 1, j), \quad i \geq 1, j \geq 1, i + j \leq m \quad (3.52)$$

where in B'_r (resp. $B'_{\alpha(r)}$) the new elements replace n (resp. k) in the total order. There is a morphism of nets $f : \mathcal{X}' \rightarrow \mathcal{X}$ with $1, (1, i, j) \mapsto n$, and $2, (2, i, j) \mapsto k$.

Let G be the preimage of $G_m = H_m$ under the projection $F_n \rightarrow \text{gr}_n V$. V determines a representation $V' = (V', F_i, \psi_i)$ of $\mathcal{X}'$ with

$$V'_r = X_1, \quad V'_1 = F_n/G, \quad V'_{1,i,j} = M_{i,j}, \quad (3.53)$$

the restrictions of the filtrations (3.44) to X_1 , and $\psi_1, \psi_{1,i,j}$ induced by ϕ_n . We claim that $f_* V'$ is isomorphic to V . Choose a complement Y to $G \subset F_n$ with $Y \subset F_k$. Further, let $Y_{1,1}, \dots, Y_{1,m} \subset \text{gr}_n V$ with

$$X_{1,j}/F_{<n} = Y_{1,1} \oplus \dots \oplus Y_{1,j} \quad (3.54)$$

and define $Y_{i+1,j} = \phi_n(Y_{i,j+1})$. We get $Y_{ij} \cong M_{ij}$ and

$$G/X_1 = \bigoplus_{\substack{i,j \geq 2 \\ i+j < m+1}} Y_{ij}. \quad (3.55)$$

Also we can use the $X_{1,j}$ and $X_{i,1}$ in the definition of $f_* V'$. Choose a complement Z to $X_1 \subset G$ with $Z \subset F_k$, allowing us to lift $Y_{i,j}$ to $Z_{i,j} \subset F_k$. Combining the various splittings we obtain an isomorphism

$$X_1 \oplus (F_n/G) \oplus \bigoplus_{\substack{i,j \geq 2 \\ i+j < m+1}} M_{ij} \rightarrow V_r \quad (3.56)$$

which we use to identify $f_* V'$ with V . $\square$

Call a net as in (3.20) a *cycle*. Note that for any cycle and any $n \geq 2$ there is a morphism from another cycle which is $n : 1$, i.e. an n -fold “covering”. These, and isomorphisms, are the only morphisms between connected nets of height 1. We use this to formulate a strengthening of the previous theorem.

Theorem 3.2. *Let $\mathcal{X}$ be a net, V an indecomposable representation of $\mathcal{X}$. Then there exists a connected net $\mathcal{X}'$ of height 1, a morphism $f : \mathcal{X}' \rightarrow \mathcal{X}$ which does not factor through a covering, and an indecomposable representation V' of $\mathcal{X}'$ such that $f_* V' = V$. Moreover, for any triple $\mathcal{X}'', f'', V''$ with these properties there is an isomorphism of nets $g : \mathcal{X}' \rightarrow \mathcal{X}''$ over $\mathcal{X}$ such that $g_* V' \cong V''$.*

Proof. Existence follows from Theorem 3.1, which gives us $\mathcal{X}'$, f , V' , with $f_* V' = V$. Since V is indecomposable, V' must be as well, and can only be supported on a single component of $\mathcal{X}'$ so we can take $\mathcal{X}'$ to be connected. If f factors through some covering, we just push V' forward along it.

For uniqueness, consider the inductive procedure in the proof of Theorem 3.1. After choosing a total order on A , the output $\mathcal{X}', f, V'$ depends, up to isomorphism, only on the isomorphism class of V . Moreover, if V is of the form $f_* V'$ for some connected net of height 1, $f : \mathcal{X}' \rightarrow \mathcal{X}$ not factoring through a covering, and V' indecomposable, then the output of the procedure applied to V is isomorphic to the same $(\mathcal{X}', f, V')$. $\square$

In applications of this theorem below it will be natural to consider nets which are not finite (only individual B_i are). Still, all our results extend to this case, as finite dimensional representations are supported on a finite subnet.

3.4 Minimal twisted complexes

Let $\mathcal{A}$ be a graded linear category which is **augmented** in the sense that there are splittings

$$\mathcal{A}(X, Y) = \mathcal{A}_e(X, Y) \oplus \mathcal{A}_r(X, Y) \quad (3.57)$$

such that $\mathcal{A}_e(X, Y)$ is $\mathbb{K}1_X$ for $X = Y$ and zero otherwise, and $\mathcal{A}_r(X, Y)$ are closed under composition. We view $\mathcal{A}_r$ as a non-unital category with the same objects as $\mathcal{A}$. Assume for the rest of the subsection that $\mathcal{A}_r$ is nilpotent in the sense that there exists an integer $N > 0$ such that any composition of N morphisms vanishes. Recall that a functor F is said to *reflect isomorphisms* if f is invertible whenever $F(f)$ is.

Lemma 3.3. *The functor $T : \text{add}\mathbb{Z}(\mathcal{A}) \rightarrow \text{add}\mathbb{Z}(\mathcal{A}_e)$ induced by the augmentation reflects isomorphisms.*

Proof. Let $M, N \in \text{add}\mathbb{Z}(\mathcal{A})$ and $\phi : M \rightarrow N$ such that $T(\phi) = \bar{\phi}$ is an isomorphism. Thus we have graded vector spaces $M(X), N(X)$ for each $X \in \text{Ob}(\mathcal{A})$, and components of ϕ

$$\begin{aligned} \phi_X &\in \text{Hom}(M(X), N(X)) \oplus (\text{Hom}(M(X), N(X)) \otimes \mathcal{A}_r(X, X)) \\ \phi_{X,Y} &\in \text{Hom}(M(X), N(Y)) \otimes \mathcal{A}_r(X, Y), \quad X \neq Y \end{aligned}$$

Let $\bar{\phi}_X$ be the component of ϕ_X in $\text{Hom}(M(X), N(X))$. By assumption, the $\bar{\phi}_X$ are isomorphisms. Composing ϕ with the morphism $\bar{\phi}^{-1}$ with components $\bar{\phi}_X^{-1}$, we may assume that $M(X) = N(X)$ and all $\bar{\phi}_X$ are identity morphisms. Thus,

$$\phi = 1 - \epsilon \in \text{End}^0(M), \quad \epsilon^N = 0 \quad (3.58)$$

which clearly has an inverse. $\square$

A two-sided twisted complex over $\mathcal{A}$ is given by a pair (M, δ) with $M \in \text{Ob}(\text{add}\mathbb{Z}\mathcal{A})$ and $\delta \in \text{End}^1(M)$ with $\delta^2 = 0$. We say that (M, δ) is **minimal** if the image $\bar{\delta}$ of δ under the functor $\text{add}\mathbb{Z}\mathcal{A} \rightarrow \text{add}\mathbb{Z}\mathcal{A}_e$ vanishes, i.e. δ has components in (the degree 1 part of)

$$\text{Hom}(M(X), M(Y)) \otimes \mathcal{A}_r(X, Y). \quad (3.59)$$

Proposition 3.3. *1. Every twisted complex $A \in \text{Tw}\mathcal{A}$ is isomorphic to a direct sum $A = A_m \oplus A_c$ with A_m minimal and A_c contractible.*

2. Any homotopy equivalence between minimal twisted complexes is an isomorphism.

Proof. 1. Let (M, δ) be a twisted complex over $\mathcal{A}$, $(M, \bar{\delta})$ its image in $\text{Tw}(\mathcal{A}_e)$. By semisimplicity of $\mathcal{A}_e$, $(M, \bar{\delta})$ is isomorphic to a direct sum $B_c \oplus B_m$ where B_c is contractible and B_m has trivial differential. Therefore, (M, δ) is isomorphic to a twisted complex of the form

$$\left(K \oplus K[-1] \oplus L, \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ 1_K + \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \right)$$

with $\bar{\delta}_{ij} = 0$. Using elementary row and column operations (automorphisms of $K \oplus K[-1] \oplus L$) one reduces the matrix to the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1_K & 0 & 0 \\ 0 & 0 & \delta_L \end{pmatrix}$$

with $\bar{\delta}_L = 0$, thus providing the direct sum decomposition.

2. Suppose (M, δ^M) and (N, δ^N) are minimal twisted complexes, $f : M \rightarrow N$ and $g : N \rightarrow M$ closed morphisms of degree 0, $G : M \rightarrow M$ and $H : N \rightarrow N$ morphisms of degree -1 such that

$$1_M - gf = \delta^M G + G\delta^M, \quad 1_N - fg = \delta^N H + H\delta^N. \quad (3.60)$$

Using minimality, $\overline{gf} = 1_M$ and $\overline{fg} = 1_N$, i.e. $\overline{f}, \overline{g}$ are inverse isomorphisms. Hence, the claim follows from the previous lemma. $\square$

3.5 Classification of objects

We are now ready to state and prove the main result of this section.

Theorem 3.3. *Let (S, M, Ω) be a compact graded marked surface, then the construction of Subsection 3.1 sets up a bijection between isomorphism classes of indecomposable objects in $H^0(\mathcal{F}(S, M, \Omega))$ and isotopy classes of admissible curves with indecomposable local system.*

Proof. Let Q be a graded quiver with quadratic monomial relations of type F1, i.e. associated with some systems of arcs A on a compact graded marked surface (S, M, Ω) . The graded linear category $\mathbb{K}Q$ has a natural augmentation with $(\mathbb{K}Q)_r$ generated by non-identity paths. Assume that sufficiently long paths are zero in $\mathbb{K}Q$, equivalently that no components of M are diffeomorphic to S^1 , so that $(\mathbb{K}Q)_r$ is nilpotent and we can apply the results of the previous subsection.

Consider first the case when Q is of the form

$$\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet \quad (3.61)$$

without relations and arbitrary grading. Let us number the vertices from left to right by $\{1, \dots, n\}$ and let α_i denote the arrow from $i+1$ to i . A minimal twisted complex over $\mathbb{K}Q$ is then given by finite-dimensional graded vector spaces $V_1, \dots, V_n$, and for $1 \leq i < j \leq n$ a linear map

$$\delta_{ij} : V_j \rightarrow V_i, \quad \deg(\delta_{ij}) = \deg(\alpha_i) + \dots + \deg(\alpha_{j-1}) - 1 \quad (3.62)$$

such that the matrix $\delta = (\delta_{ij})$ has square zero.

Consider the direct sum

$$V = \bigoplus_{i=1}^n V_i. \quad (3.63)$$

which has the filtration

$$V_1 \subset V_1 \oplus V_2 \subset \dots \subset V_1 \oplus \dots \oplus V_n. \quad (3.64)$$

The endomorphism δ gives a three-step filtration

$$\text{Im}\delta \subset \text{Ker}\delta \subset V \quad (3.65)$$

and an isomorphism of associated graded $V/\text{Ker}\delta \cong \text{Im}\delta$. We find that we have a representation of the net

$$(\{1, 2\} \times \mathbb{Z}, (12) \times \text{id}, (\{1, \dots, n\} \sqcup \{1, 2, 3\}) \times \mathbb{Z}, (13) \times \text{id}) \quad (3.66)$$

with total order so that $(i, k + d(\alpha_i) - 1) < (i + 1, k)$ on $\{1, \dots, n\} \times \mathbb{Z}$ and $(1, n) < (2, n) < (3, n)$ on $\{1, 2, 3\} \times \mathbb{Z}$. In fact, we get an embedding of the category of minimal twisted complexes into the category of representations of the above net as a full subcategory. We do not get an equivalence of categories, since we only consider δ which decrease the filtration (3.64).

Turn now to the case of general Q . Let D be the set of maximal non-zero paths in $\mathbb{K}Q$. Then for each element of D there is a sub-quiver of Q which is of the simple form above, and Q is obtained from their disjoint union by identifying some pairs of vertices. The corresponding net is now given by

$$A = D \times \{1, 2\} \times \mathbb{Z} \quad (3.67)$$

$$\alpha = \text{id} \times (12) \times \text{id} \quad (3.68)$$

$$B = (\{(v, d) \in Q_0 \times D \mid v \text{ on } d\} \sqcup (D \times \{1, 2, 3\})) \times \mathbb{Z} =: B_g \sqcup B_a \quad (3.69)$$

with partial order defined so that for an arrow α in $d \in D$ we have $(\partial_1(\alpha), n + d(\alpha) - 1) < (\partial_0(\alpha), n)$ and $(d, 1, n) < (d, 2, n) < (d, 3, n)$. Further,

$$\beta = (\tau \sqcup (\text{id} \times (13))) \times \text{id} \quad (3.70)$$

where τ sends a pair $(v, d_1) \in Q_0 \times D$ with v on $d_1, d_2 \in D$, $d_1 \neq d_2$ to (v, d_2) . There is a fully-faithful functor from the category of minimal twisted complexes over Q to the category of representations of (A, α, B, β) . Theorem 3.2 gives us a classification of indecomposable minimal twisted complexes over $\mathbb{K}Q$, equivalently isomorphism classes in $H^0(\mathcal{F}(S, M, \Omega))$ by Proposition 3.3. Following the definitions, one identifies isotopy classes of admissible curves with local system with morphisms from nets of height 1 with indecomposable representation. Of course, nets of the type (3.19) (resp. (3.20)) correspond to admissible curves with domain $[0, 1]$ (resp. S^1).

It remains to deal with the case when M has components diffeomorphic to S^1 . Let (S', M', Ω') be the surface obtained by adding two corners and a boundary arc on each such component of M . Then $H^0(\mathcal{F}(S, M, \Omega))$ is a localization of the triangulated category $H^0(\mathcal{F}(S', M', \Omega'))$, and the localization functor is essentially surjective. Moreover, if the images of $E, F \in H^0(\mathcal{F}(S, M, \Omega))$ corresponding to admissible curves become isomorphic under localization, then E and F differ by extensions by boundary arcs, so map to isotopic curves in (S, M, Ω) . $\square$

4 Flat metrics on surfaces

The focus of this section are quadratic differentials on Riemann surfaces and the associated flat metrics. In Subsection 4.1 we define a class of quadratic differentials on open Riemann surfaces with at worst exponential singularities at infinity. The local behavior of the associated flat metric is reviewed in Subsection 4.2. In the following subsection we discuss some facts about geodesics on flat surfaces. The last subsection is concerned with the period map defined on cycles with certain twisted coefficients.

4.1 Riemann surfaces with Calabi–Yau structure

A **CY-structure** on a Riemann surface X is given by a holomorphic section

$$\Omega \in \Gamma(X, (\Omega_X^{1,0})^{\otimes 2}) \quad (4.1)$$

which is everywhere non-vanishing. Sections of $(\Omega_X^{1,0})^{\otimes 2}$ are called *quadratic differentials*.

A CY-structure determines a flat Riemannian metric $|\Omega|$ and a **horizontal foliation** of vectors $v \in T_p X$ with $\Omega(v, v) \geq 0$. The charts which give local isometries with the Euclidean plane are the indefinite integrals

$$f : X \supset U \rightarrow \mathbb{C}, \quad f(z) = \int_p^z \sqrt{\Omega} \quad (4.2)$$

where U is simply connected, $p \in U$. Conversely, a topological surface with flat Riemannian metric with monodromy contained in $\{1, -1\}$ together with a choice of line in the tangent space at some point in each connected component determines a complex structure and a non-vanishing quadratic differential on X .

Since a compact surface can only have a CY-structure if it is the torus, we will mostly consider open surfaces. In this case it is necessary to impose some conditions on the behavior at infinity. First, let us call a transcendental singularity of a quadratic differential of the form

$$e^{f(z)} g(z) dz^2, \quad f, g \text{ meromorphic} \quad (4.3)$$

in some local coordinate z , an **exponential-type singularity**. We say that a Riemann surface X with CY-structure Ω is **tame at infinity** if there is a compact Riemann surface $\bar{X}$ with meromorphic quadratic differential $\bar{\Omega}$, defined away from a finite set of points where it has exponential-type singularities, and a finite set $S \subset X$ so that there is an isomorphism

$$(X, \Omega) \cong (\bar{X} \setminus S, \bar{\Omega}|_{\bar{X} \setminus S}). \quad (4.4)$$

Note that S necessarily includes the zeros, poles, and exponential-type singularities of $\bar{\Omega}$, but we allow it to contain additional points as well.

Grading on surfaces with CY-structure

Let S be a Riemann surface with non-vanishing holomorphic quadratic differential $\Omega \in \Gamma(S, (T^*S)^{\otimes 2})$, i.e. a CY-structure. Recall that the holomorphic and real tangent bundles may be identified, and we associate to Ω the unique section of $\mathbb{P}(TS)$ on which Ω is real and positive (i.e. the horizontal foliation). Moreover, we have an identification of $\mathbb{P}(TS)$ with the bundle with constant fiber $\mathbb{C}^*/\mathbb{R}_{>0} \cong U(1)$ under which the section simply corresponds to $1 \in U(1)$.

For an immersed curve $\alpha : I \rightarrow S$ a grading is defined by a function $\phi : I \rightarrow \mathbb{R}$ such that

$$\Omega(\dot{\alpha}(t), \dot{\alpha}(t)) \in \mathbb{R}_{>0} e^{2\pi i \phi(t)}. \quad (4.5)$$

The corresponding path $\alpha^* \Omega \rightsquigarrow \dot{\alpha}$ is given by

$$s \mapsto e^{2\pi i \phi(t)s}, \quad s \in [0, 1] \quad (4.6)$$

under the identification as above. Note that if α is geodesic if and only if ϕ is locally constant.

Suppose we have curves α_1, α_2 with grading given by real-valued functions ϕ_1, ϕ_2 intersecting transversely in $p \in S$. Then the intersection index is simply

$$i_p(\alpha_1, \alpha_2) = \lceil \phi_1(p) - \phi_2(p) \rceil. \quad (4.7)$$

4.2 Local flat geometry

The flat metric on a complex surfaces with CY-structure is in general not complete, and its metric completion not a smooth Riemannian manifold. In the case of CY-structures tame at infinity the completion adds only a finite number of points, which are all singularities of conical type, as we will explain in this subsection. Lets us consider the various special points of a quadratic differential φ .

- *Zero of order n :* We can find a local coordinate with $\varphi = z^n dz^2$. Near the zero, the metric case a conical point with cone angle $(n + 2)\pi$.
- *First order pole:* We can find a local coordinate with $\varphi = \frac{dz^2}{z}$. Near the pole, the metric case a conical point with cone angle π .
- *Second order pole:* We can find a local coordinate with $\varphi = \frac{adz^2}{z^2}$ for some $a \in \mathbb{C}^*$. A neighborhood is isometric to a semi-infinite cylinder, in particular contains infinite area.
- *Pole of odd order $n \geq 3$:* We can find a local coordinate with $\varphi = \frac{dz^2}{z^n}$. The metric is smooth with infinite area.
- *Pole of even order $n \geq 4$:* We can find a local coordinate with

$$\varphi = \left(\frac{1}{z^n} + \frac{a}{z^2} \right) dz^2$$

for some $a \in \mathbb{C}$. The metric is smooth with infinite area.

- *Exponential-type singularity:* We can find a local coordinate with

$$\varphi = e^{z^{-n}} f(z) dz^2$$

for some $n \geq 1$ and meromorphic f . The completed metric has n infinite-angle singularities, and infinite area.

The claims are easily verified in all cases except the last (transcendental) one, c.f. [29]. The case of exponential-type singularities will be analyzed in detail in the following.

Exponential-type singularities

Lemma 4.1. *Let $\alpha > 1$, $\lambda > 0$, then there is a $C = C(\lambda)$ such that*

$$\int_0^\infty (\alpha + t)^\lambda e^{-t} dt \leq C e^{\alpha/2}.$$

Proof. By change of variables $s = \alpha + t$,

$$\int_0^\infty (\alpha + t)^\lambda e^{-t} dt = e^\alpha \int_\alpha^\infty s^\lambda e^{-s} ds \quad (4.8)$$

Find $C > 0$ such that $s^\lambda e^{-s} \leq C/2 e^{-s/2}$ for $s \geq 0$, then

$$e^\alpha \int_\alpha^\infty s^\lambda e^{-s} ds \leq \frac{C e^\alpha}{2} \int_\alpha^\infty e^{-s/2} ds \quad (4.9)$$

$$= C e^{\alpha/2} \quad (4.10)$$

□

Lemma 4.2. *Let $\rho \geq 1$, $\pi/2 < \varepsilon < \pi$, $A = \{r e^{i\phi} \mid r \geq \rho, |\phi| \leq \varepsilon\} \subset \mathbb{C}$, $h(z)$ a smooth function on A such that for some $C_1, C_2 > 0, \lambda \in \mathbb{R}$*

$$C_1 |z|^\lambda \leq h(z) \leq C_2 |z|^\lambda \quad (4.11)$$

Then the completion of A with respect to the metric

$$(e^{-\operatorname{Re}(z)} h(z))^2 |dz|^2 \quad (4.12)$$

has a single additional point.

Proof. 1. Define $\eta(z) = (\max\{1, |\operatorname{Im}(z)|\})^\lambda e^{-\operatorname{Re}(z)}$, then for sufficiently small $\varepsilon > 0$, the boundary of $\{\eta \geq \varepsilon\}$ in A is given by the curve

$$\operatorname{Re}(z) = \lambda \log(\max\{1, |\operatorname{Im}(z)|\}) - \log(\varepsilon). \quad (4.13)$$

Moreover, for distinct values of ε , these curves have positive distance with respect to the Euclidean metric.

2. We claim that $e^{-\operatorname{Re}(z)} h(z)$ is bounded below on $\{\eta \geq \varepsilon\}$ for any $\varepsilon > 0$. By assumption,

$$e^{-\operatorname{Re}(z)} h(z) \geq C_1 |z|^\lambda e^{-\operatorname{Re}(z)}. \quad (4.14)$$

Case $\lambda \geq 0$: For $|\operatorname{Im}(z)| \geq 1$ we have

$$C_1|z|^\lambda e^{-\operatorname{Re}(z)} \geq C_1|\operatorname{Im}(z)|^\lambda e^{-\operatorname{Re}(z)} \quad (4.15)$$

$$= C_1\eta(z) \quad (4.16)$$

$$\geq C_1\varepsilon \quad (4.17)$$

Case $\lambda < 0$:

$$C_1|z|^\lambda e^{-\operatorname{Re}(z)} \geq C_1 \left(|\operatorname{Re}(z)|e^{-\operatorname{Re}(z)/\lambda} + |\operatorname{Im}(z)|e^{-\operatorname{Re}(z)/\lambda} \right)^\lambda \quad (4.18)$$

The first term is bounded above, as $\operatorname{Re}(z)$ is bounded above by $-\log(\varepsilon)$, the second term is bounded above by $\varepsilon^{1/\lambda}$.

3. We claim that for every $\varepsilon > 0$, the set $\{\eta \geq \varepsilon\}$ is complete with respect to the metric $g = (e^{-\operatorname{Re}(z)}h(z))^2|dz|^2$, thus any Cauchy sequence $z_j \in A$ without limit must satisfy $\eta(z_j) \rightarrow 0$. Namely, we have $g \geq Cg_{\text{eucl}}$ on $\{\eta \geq \varepsilon/2\}$ be the previous step, and the boundary curves $\{\eta = \varepsilon\}$, $\{\eta = \varepsilon/2\}$ have some positive distance δ , so

$$d(z_1, z_2) \geq C \min\{|z_1 - z_2|, 2\delta\} \quad (4.19)$$

for $z_1, z_2 \in \{\eta \geq \varepsilon\}$. Hence, any sequence which is Cauchy with respect to g is Cauchy for the standard metric.

4. We claim that $\operatorname{diam}\{\eta \leq \varepsilon\} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Assume $1 > \varepsilon > 0$ is sufficiently small so that $\{\eta \leq \varepsilon\}$ contains no z with $\operatorname{Re}(z) \leq 0$, $|\operatorname{Im}(z)| \leq \rho$, then for $z \in \{\eta \leq \varepsilon\}$ the path $\alpha(t) = z + t$, $t \in [0, \infty)$ is contained in $\{\eta \leq \varepsilon\}$. We compute the length of α ,

$$l(\alpha) = \int_0^\infty e^{-\operatorname{Re}(z)-t} h(z+t) dt \quad (4.20)$$

$$\leq C_2 \int_0^\infty e^{-\operatorname{Re}(z)-t} |z+t|^\lambda dt =: C_2 I \quad (4.21)$$

Case $\lambda > 0$:

$$I \leq \int_0^\infty \left(|\operatorname{Re}(z) + t|e^{-(\operatorname{Re}(z)+t)/\lambda} + |\operatorname{Im}(z)|e^{-(\operatorname{Re}(z)+t)/\lambda} \right)^\lambda dt \quad (4.22)$$

As $\eta(z) \leq \varepsilon$, $\operatorname{Re}(z) \geq -\log(\varepsilon) > 0$, the first summand is bounded above by $(-\log(\varepsilon) + t)\varepsilon^{1/\lambda}e^{-t/\lambda}$, and the second by $\varepsilon^{1/\lambda}e^{-t/\lambda}$, so

$$I \leq \varepsilon \int_0^\infty (1 - \log(\varepsilon) + t)^\lambda e^{-t} dt \quad (4.23)$$

Together with the previous lemma we get $I \leq 2C\varepsilon e^{(1-\log(\varepsilon))/2} = 2C\sqrt{e\varepsilon}$ which goes to 0 for $\varepsilon \rightarrow 0$.

Case $\lambda \leq 0$:

$$I \leq \int_0^\infty e^{-\operatorname{Re}(z)-t} (\max\{1, |\operatorname{Im}(z)|\})^\lambda dt \quad (4.24)$$

$$= \eta(z) \leq \varepsilon \quad (4.25)$$

To prove the claim, it remains to show that for different values of z_1, z_2 , the corresponding horizontal curves starting at z_1, z_2 have vanishing distance. Let $\beta(t) = x + it$, $t \in [y_1, y_2]$, $x > \rho$, then

$$l(\beta) = \int_{y_1}^{y_2} e^{-x} h(x + it) dt \quad (4.26)$$

$$\leq C_2 e^{-x} \int_{y_1}^{y_2} |x + it|^\lambda dt \xrightarrow{x \rightarrow \infty} 0 \quad (4.27)$$

Which completes the proof of the lemma. $\square$

Theorem 4.1. *Let*

$$\varphi = e^{z^{-n}} z^m g(z) dz^2 \quad (4.28)$$

with $n \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}$, g holomorphic, non-vanishing on a neighborhood U of $0 \in \mathbb{C}$. Let

$$D = \{0 < |z| \leq r\} \subset U \quad (4.29)$$

then the completion of D with respect to $|\phi|$ has n additional points, which are ∞ -angle singularities.

Proof. After a change of coordinates $w = e^{\pi i/n}/z$, φ is of the form

$$\varphi = e^{-w^n} h(w) dw^2 \quad (4.30)$$

with h meromorphic in a neighborhood of ∞ . Let $B = D^{-1}$ and fix $\varepsilon \in (\frac{1}{2}, 1)$. Define sectors

$$V_n = \left\{ w \in B \mid \left| \arg(w) - \frac{2\pi k}{n} \right| < \frac{\pi \varepsilon}{n} \right\} \quad (4.31)$$

and their complement

$$A = B \setminus \bigcup_{k=0}^{n-1} V_n. \quad (4.32)$$

Then for $z = re^{i\phi} \in A$ we find $\cos(n\phi) \leq \cos(\pi\varepsilon) < 0$, hence

$$|\varphi| = e^{-r^n \cos(n\phi)} |h dz^2| \geq C |dz|^2 \quad (4.33)$$

for some $C > 0$. Since the sets V_k have positive distance with respect to the standard metric, this shows that every Cauchy sequence in B without limit is eventually contained in one of the V_k .

On V_k we perform a change of coordinates $u = w^n$, so

$$\varphi = e^{-2u} f(u) du^2 \quad (4.34)$$

and there are $C_1, C_2 > 0$ with

$$C_1 |u|^\lambda \leq \sqrt{|f(u)|} \leq C_2 |u|^\lambda, \quad \lambda = \frac{m-n+1}{2n} \quad (4.35)$$

where $m = \text{ord}_\infty h$. Applying the previous lemma, the first part of the theorem follows.

By the first part of the proof,

$$\bar{B} =: B \cup \{a_1, \dots, a_n\} \quad (4.36)$$

is a complete metric space. Let $a = a_k$, then we must show that a is an ∞ -angle singularity of $\bar{B}$.

As V_k is simply connected, we may choose a holomorphic $f : V_k \rightarrow \mathbb{C}$ with $(df)^2 = \varphi$. Choose $r > 0$ such that $D_{2r}^*(0) \subset V_k$. Then for any path α in D_{2r}^* with endpoint $z_1, z_2 \in D_r^*(0)$ we compute

$$|f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} \sqrt{\varphi} \right| \leq \int_{\alpha} \sqrt{|\varphi|} \quad (4.37)$$

hence

$$|f(z_2) - f(z_1)| \leq d(z_1, z_2) \quad (4.38)$$

so that f extends to a continuous map on $V_k \cup \{a\}$.

Without loss of generality, $f(a) = 0$. We claim that f restricts to a covering $D_r^*(a) \rightarrow D_r^*(0)$. By (4.38), the image of $D_r^*(a)$ under f is contained in $D_r(0)$. The radius of injectivity at $z \in D_r^*(a)$ is $r_1 = d(z, a)$. It cannot be larger, as the singular point a would have to be contained in any disk of radius $> r_1$ centered at z , and it cannot be smaller by completeness of $\bar{B}$, and since any geodesic may be extended until it hits a or leaves $D_{2r}(a)$. We conclude that $D_{r_1}(z)$ is mapped isometrically to $D_{r_1}(f(z))$ by f , and thus $r_1 = |f(z)|$. This shows that f restricts to a covering $D_r^*(a) \rightarrow D_r^*(0)$.

From the proof of the lemma, it follows that a admits a fundamental system of neighborhoods W with the property that $W \setminus \{a\}$ is simply-connected. Thus $D_r^*(a)$ must be simply connected, and $f : D_r^*(a) \rightarrow D_r^*(0)$ is a universal covering. $\square$

4.3 Geodesics

A **geodesic** in a metric space is a curve which is locally length minimizing. In the case of a complete flat surface with conical singularities

$$F = F_{\text{sm}} \cup F_{\text{sg}}$$

where F_{sm} is the set of smooth points and F_{sg} is the set of singular points, all geodesics are piece-wise smooth. In this section we describe their local behavior as well as the maximal geodesics of finite length.

Local behavior

First, near smooth points $p \in F_{\text{sm}}$, F admits a Euclidean chart $\phi : U \rightarrow \mathbb{R}^2$, and geodesics are the preimages of straight lines in $\mathbb{R}^2$.

Let us consider the case of a conical singularity $p \in F_{\text{sg}}$ with finite cone angle ρ . Any ray starting at the conical point is a geodesic. Moreover, the concatenation of two such rays will be a geodesic iff the angles between them, measured on both sides, are $\geq \pi$. In particular, if $\rho < 2\pi$ no geodesic can pass through the conical point, and if $\rho > 2\pi$ there are many ways of extending a geodesic past the conical point. The case of an infinite-angle singularity is similar, except that there is only one (arbitrarily large) angle between any two rays.

By an **unbroken geodesic** (or **smooth geodesic**) we understand a geodesic on F such that whenever it passes through a conical point the incoming and outgoing rays form an angle equal to π . This is equivalent to saying that we can locally approximate by geodesics in F_{sm} . A **broken geodesic** is by definition a geodesic which is not smooth, so it passes through a conical point such that the incoming and outgoing rays form only angles $> \pi$.

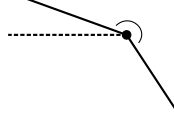


Figure 5: A broken geodesic passing through a conical point. We think of the surface as covering the plane with a branch cut along the dashed line.

Finite geodesics

Consider a maximal geodesic c in F_{sm} of finite length, there are two possibilities:

1. **Saddle connections:** c converges in both directions to some points $p, q \in F_{\text{sg}}$, possibly $p = q$.
2. **Closed geodesics:** c is a map from S^1 . Let us exclude multiple covers. The geodesics in the isotopy class of c foliate a cylinder, possibly of infinite height or bounded by saddle connections.

4.4 Homology and period map

Let (S, Ω) be a graded surface. For each $x \in S$ there is a $\mathbb{Z}/2$ -torsor of orientations of $\Omega(x)$, together giving a local system $\pm\sqrt{\Omega}$ of $\mathbb{Z}/2$ -torsors on S . There is a corresponding local system of abelian groups $\mathbb{Z}\sqrt{\Omega}$, which is the natural choice of coefficients when assigning homology classes to graded curves.

For a graded marked surface (S, M, Ω) consider homology with twisted coefficients

$$\Gamma = H_1(S, M; \mathbb{Z}\sqrt{\Omega}). \quad (4.39)$$

Γ is easily computed in the case when S is compact and connected by finding an explicit CW-decomposition. Let $g = g(S)$ be the genus of S , $m = |\pi_0(M)|$ the number of components of M , and n the number of components of ∂S which are not entirely contained in M . If $n > 0$, i.e. $\partial S \neq M$, then

$$\Gamma = \mathbb{Z}^{2g-2+m+n}. \quad (4.40)$$

If $\partial S = M$, i.e. all components of M are diffeomorphic to S^1 , then Γ depends on Ω . If $\pm\sqrt{\Omega}$ is a trivial local system, i.e. Ω has a global square root, then

$$\Gamma = \mathbb{Z}^{2g-2+m} \quad (4.41)$$

otherwise

$$\Gamma = \mathbb{Z}^{2g-1+m} \times \mathbb{Z}/2. \quad (4.42)$$

Given a surface with CY-structure (C, Ω) which is tame at infinity, we associate to it a surface with marked boundary (S, M) embedded in C so that $C \setminus S$ is a disjoint union of annuli around each singularity of Ω , and so that the component of ∂S around a meromorphic singularity of Ω is diffeomorphic to S^1 and the component of ∂S around an exponential type singularity of Ω has corners and a bijection of the components of M on it with the infinite angle singularities.

In terms of the flat surface $F = F_{\text{sm}} \cup F_{\text{sg}}$ determined by (C, Ω) we can write

$$\Gamma = H_1(F, F_{\text{sg}}; \mathbb{Z}\sqrt{\Omega}). \quad (4.43)$$

Suppose α is a closed singular 1-chain with values in $\mathbb{Z}\sqrt{\Omega}$ which is geodesic near F_{sg} . Then the integral

$$\int_{\alpha} \sqrt{\Omega} \quad (4.44)$$

is a well defined complex number. Since $\sqrt{\Omega}$ is closed, it depends only on the class of α in Γ and so we get a map

$$Z : \Gamma \rightarrow \mathbb{C} \quad (4.45)$$

called the **period map**.

5 Stability conditions determined by flat metrics

The connection between flat geometry and stability conditions is the subject of this final section. We begin by reviewing Bridgeland's axioms in Subsection 5.1. In Subsection 5.2 we then compute K_0 of Fukaya categories of open surfaces in terms of singular homology. This allows us to state our main result in Subsection 5.3, the proof of which occupies the remainder of this section.

5.1 Bridgeland's axioms for stability

Fix a triangulated category $\mathcal{C}$ and a homomorphism $\text{cl} : K_0(\mathcal{C}) \rightarrow \Gamma$ to a finitely generated abelian group Γ . A **stability condition** (c.f. [4], [18]) on $\mathcal{C}$ is given by

- for each $\phi \in \mathbb{R}$ a full additive subcategory $\mathcal{C}^{\phi} \subset \mathcal{C}$ of **semistable objects of phase ϕ** , and
- an additive map $Z : \Gamma \rightarrow \mathbb{C}$, the **central charge**.

This data has to satisfy the following axioms.

1. $\mathcal{C}^{\phi}[1] = \mathcal{C}^{\phi+1}$
2. If $E \in \mathcal{C}^{\phi_1}$, $F \in \mathcal{C}^{\phi_2}$, $\phi_1 > \phi_2$, then $\text{Hom}(E, F) = 0$.

3. Every $E \in \mathcal{C}$ has a **Harder–Narasimhan** decomposition: A tower of triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & & \cdots & & E_{n-1} \longrightarrow E_n = E \\
 & \searrow \text{dotted} & \downarrow & & & & \searrow \text{dotted} \downarrow \\
 & & A_1 & & & & A_n
 \end{array}$$

with $0 \neq A_i \in \mathcal{C}^{\phi_i}$ and $\phi_1 > \phi_2 > \dots > \phi_n$.

4. If $0 \neq E \in \mathcal{C}^\phi$ then $Z(E) := Z(\text{cl}([E])) \in \mathbb{R}_{>0} e^{\pi i \phi}$.
5. The **support property**: For some norm $\|\cdot\|$ on $\Gamma \otimes \mathbb{R}$ and $C > 0$ we have an estimate

$$\|\text{cl}(E)\| \leq C|Z(E)| \quad (5.1)$$

for $E \in \mathcal{C}^\phi$.

The main result of [4] (see also [18]) is that the set $\text{Stab}(\mathcal{C}, \Gamma)$ of all stability conditions on a triangulated category $\mathcal{C}$ has the structure of a complex manifold of dimension $\text{rk}(\Gamma)$ (if it is non-empty), so that the projection

$$\text{Stab}(\mathcal{C}, \Gamma) \rightarrow \text{Hom}(\Gamma, \mathbb{C})$$

becomes a complex analytic local homeomorphism.

5.2 K_0 and singular homology

In order to define the central charge as the period map, we need to relate K_0 of the Fukaya category to singular homology. The precise statement is the following.

Theorem 5.1. *Let (S, M) be a compact graded marked surface, then there is a natural isomorphism of abelian groups*

$$K_0(\mathcal{F}(S, M, \Omega)) \cong H_1(S, M; \mathbb{Z}\sqrt{\Omega}).$$

Proof. Choose a full system A of graded arcs on (S, M) . Each arc defines a class in $H = H_1(S, M; \mathbb{Z}\sqrt{\Omega})$. From cellular homology it follows that H is the group generated by these arcs and with relations of the form $\pm X_1 \pm \dots \pm X_n = 0$ where $X_1, \dots, X_n$ are the arcs bounding a disk cut out by A . Correspondingly, it is also clear that the arcs in A generate

$$K := K_0(\mathcal{F}(S, M, \Omega)) = K_0(\text{Tw}(\mathcal{F}_A(S, M, \Omega))) \quad (5.2)$$

and satisfy the same relations, which follows from the observation in Subsection 2.5. By construction we get a surjective homomorphism $H \rightarrow K$ which is independent of the choice of arcs and natural with respect to maps of graded marked surfaces. It remains to show that no additional relations are needed to present K .

Suppose first that M has no components diffeomorphic to S^1 . Choosing a formal collection A of arcs, we can present $\mathcal{F}(S, M, \Omega)$ as the category $\text{Tw}(\mathbb{K}Q)$

of twisted complex over the path category of a graded quiver Q with quadratic monomial relations. Note that H is freely generated by A , so we need to verify that K is freely generated by the vertices Q_0 of Q . One way to see this is by using the explicit resolution of the diagonal of $\mathbb{K}Q$, showing that, as a bimodule, $\mathbb{K}Q$ is a repeated cone of Yoneda bimodules. As a consequence, $\text{Tw}(\mathbb{K}Q) \cong \text{Perf}(\mathbb{K}Q)$, and an inverse of the map

$$\mathbb{Z}^{Q_0} \rightarrow K_0(\text{Perf}(\mathbb{K}Q)) \quad (5.3)$$

sending a vertex to the corresponding simple module, is induced by the dimension vector of a module.

For a general (S, M, Ω) there is a (S', M', Ω') obtained by adding two corners to each component of ∂S diffeomorphic to S^1 . Let N denote set of boundary arcs of (S', M') created in this process, and $\mathcal{N}$ the corresponding full triangulated subcategory of $H^0(\mathcal{F}(S', M', \Omega'))$ generated by N . Each arc in N has endomorphism algebra of the form $\mathbb{K}[x]/x^2$ with $|x| \in \mathbb{Z}$, so $\mathcal{N}$ is a product of categories of the form $H^0(\text{Tw}(\mathbb{K}[x]/x^2))$. We get a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z}^N & \longrightarrow & H_1(S', M', \mathbb{Z}\sqrt{\Omega'}) & \longrightarrow & H_1(S, M, \mathbb{Z}\sqrt{\Omega}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K_0(\mathcal{N}) & \longrightarrow & K_0(\mathcal{F}(S', M', \Omega')) & \longrightarrow & K_0(\mathcal{F}(S, M, \Omega)) & \longrightarrow & 0 \end{array} \quad (5.4)$$

where we use Proposition 2.2 to get the bottom row. We claim that the rows are exact. For the top row this is just the exact sequence of a triple. For the bottom row this uses the fact that $\mathcal{N}$ is idempotent complete, thus a thick subcategory so that Proposition 3.1 in [11] can be applied. Since the left and middle horizontal maps are isomorphisms, so is the right one. $\square$

5.3 Statement of main theorem

We now have all the ingredients needed for the precise formulation of our main result.

Theorem 5.2. *Let (C, Ω) be a Riemann surface with CY-structure (non-compact, tame at infinity without higher order poles), (S, M) the corresponding surface with marked boundary. Define subcategories $\mathcal{C}^\phi \subset H^0(\mathcal{F}(S, M, \Omega))$ so that isomorphism classes of indecomposable objects correspond to unbroken graded geodesics of phase ϕ with indecomposable local system, and let*

$$Z : K_0(\mathcal{F}(S, M, \Omega)) \cong H_1(S, M; \mathbb{Z}\sqrt{\Omega}) \rightarrow \mathbb{C}$$

be the period map. Then this data satisfies the axioms of a stability condition.

Remark. When C is compact, then necessarily $g(C) = 1$. In this case one can use homological mirror symmetry for elliptic curves [25] and Atiyah's classification [1] to derive a version of Theorem 5.2, where K_0 is replaced by the numerical K-group $K_{\text{num}} \cong \mathbb{Z}^2$.

Remark. There are possibly two ways to extend the result to the case of higher order poles. Either one restricts to a full subcategory of $\mathcal{F}(S, M, \Omega)$ with objects corresponding to admissible curves which avoid the higher order poles, or one defines a type of limiting stability where Z is allowed to become infinite.

Remark. Let us emphasize that we put no restrictions on the ground field $\mathbb{K}$ of $\mathcal{F}(S, M, \Omega)$. In fact this category is defined over any ring, but for example $\text{Perf}(\mathbb{Z})$ does not admit stability conditions.

5.4 Support property

Let (C, Ω) be an open Riemann surface with CY-structure coming from a quadratic differential without higher order poles, and let (S, M) be the corresponding marked surface. We have as before

$$\Gamma = H_1(S, M; \mathbb{Z}\sqrt{\Omega}), \quad \Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R} = H_1(S, M; \mathbb{R}\sqrt{\Omega}) \quad (5.5)$$

Proposition 5.1. *There is a norm $\|\cdot\|$ on $\Gamma_{\mathbb{R}}$, $C > 0$ such that for $\gamma \in \Gamma$ the class of a finite geodesic on (C, Ω) there is an estimate*

$$\|\gamma\| \leq C|Z(\gamma)| \quad (5.6)$$

where $Z : \Gamma \rightarrow \mathbb{C}$ is the period map.

Proof. We do first the case without exponential type singularities, so that $M = \partial S$. The locally constant sheaf $\mathbb{R}\sqrt{\Omega}$ is given by flat sections of a flat line bundle with metric for which we use the same notation. By Poincaré duality

$$\Gamma_{\mathbb{R}}^* = H_1(S, \partial S; \mathbb{R}\sqrt{\Omega})^* \quad (5.7)$$

$$= H^1(S, \partial S; \mathbb{R}\sqrt{\Omega}) \quad (5.8)$$

$$= H_{dR}^1(S; \mathbb{R}\sqrt{\Omega})^* \quad (5.9)$$

$$= H_c^1(S; \mathbb{R}\sqrt{\Omega}). \quad (5.10)$$

Choose compactly supported forms $\omega_1, \dots, \omega_n$ representing a basis of $H_c^1(S; \mathbb{R}\sqrt{\Omega})$. Define a norm on $\Gamma_{\mathbb{R}}$ by

$$\|\gamma\| = \sum_i |[\omega_i](\gamma)| \quad (5.11)$$

and set

$$C = \sum_i \|\omega_i\|_{\infty} \quad (5.12)$$

where the norm is taken with respect to the flat metric. Let α be a finite geodesic on (C, Ω) with class $\gamma \in \Gamma$, then

$$l(\alpha) = |Z(\gamma)| \quad (5.13)$$

where $l(\alpha)$ is the length. Thus

$$\|\gamma\| = \sum_i |[\omega_i](\gamma)| \quad (5.14)$$

$$= \sum_i \left| \int_{\alpha} \omega_i \right| \quad (5.15)$$

$$= \sum_i l(\alpha) \|\omega_i\|_{\infty} \quad (5.16)$$

$$= C |Z(\gamma)|. \quad (5.17)$$

In the case of exponential type singularities some modifications are needed. First, we consider forms which are supported on the complement of a neighborhood of the singularities of the flat metric. Second, we use the fact that all finite geodesics are contained in a compact subset K of the completed flat surface, see Proposition 5.2 below, and take $\|\omega_i|_K\|_{\infty}$ in (5.12). $\square$

Proposition 5.2. *Let (C, Ω) be a Riemann surface with CY-structure tame at infinity, F the metric completion of C . Then there exists a compact $K \subset F$ such that all finite geodesics on F are contained in K .*

Proof. Choose K so that its boundary is a union of (possibly broken) geodesics connecting consecutive ∞ -angle singularities corresponding to the same exponential singularity of Ω . Each component of $F \setminus K$ is simply connected and has a boundary (in F) which is a sequence of saddle connections meeting in angles $\geq \pi$. Thus, no geodesic can pass through ∂K twice, so any finite geodesic must be contained in K . $\square$

5.5 A_n (disk type)

In this subsection we consider the case when $C = \mathbb{C}$, $\overline{C} = \mathbb{CP}^1$ and Ω has a single exponential-type singularity at ∞ . It is not difficult to see that in fact

$$\Omega(z) = e^{P(z)} dz^2, \quad z \in \mathbb{C} \quad (5.18)$$

with P a polynomial of some degree $n+1$. The corresponding marked surface (S, M) is of type A_n , i.e. topologically a disk with M having $n+1$ components. We have as before a candidate for a stability condition on $\mathcal{F}(S, M, \Omega)$.

We first check that $\text{Hom}(A, B) = 0$ for $\phi(A) > \phi(B)$. Note that it suffices to do this for A, B stable, so that they correspond to graded arcs. We consider the various cases.

1. If A and B differ only by a grading shift, the claim follows from $\text{Ext}^{<0}(A, A) = 0$.
2. If A and B are disjoint, then $\text{Hom}(A, B) = 0$.
3. If A and B intersect in a smooth point p of the surface, then $\text{Ext}(A, B)$ is concentrated in degree

$$i_p(A, B) = \lceil \phi(A) - \phi(B) \rceil \quad (5.19)$$

so $\phi(A) > \phi(B)$ implies $\text{Ext}^0(A, B) = 0$.

4. If A and B meet in an ∞ -angle singularity p in an angle $\phi > 0$, then

$$i_p(A, B) = \phi(A) - \phi(B) + \frac{\phi}{\pi} \quad (5.20)$$

and $\text{Ext}(A, B)$ is either concentrated in degree $i_p(A, B)$ or zero, depending on the order in which they meet p .

Next we check the HN-property. It suffices to do this for indecomposable objects X . By the classification, X corresponds to a graded arc α connecting two ∞ -angle singularities. A geodesic representative of α is a concatenation of graded saddle connections $\alpha_1, \dots, \alpha_k$ corresponding to objects $X_1, \dots, X_n$ in $\mathcal{F}(S, M, \Omega)$. Let

$$\phi_1 > \dots > \phi_l \quad (5.21)$$

be the distinct phases of the α_i (generically, $l = k$). For $1 \leq i \leq l$ there is a semistable G_i given by a twisted complex

$$G_i = \left(\bigoplus_{\phi(X_j) = \phi_i} X_j, \delta_i \right) \quad (5.22)$$

where δ_i has non-zero coefficients corresponding to the singular points where the geodesic representative of α passes through without changing phase equal to ϕ_i . Also,

$$X = \left(\bigoplus_j X_j, \delta \right) \quad (5.23)$$

where δ has non-zero coefficients corresponding to all the singular points the geodesic representative passes through (i.e. where α_i meets α_{i+1}). Note that this really gives morphisms of degree 1 since the concatenation of the saddle connections smooths to a graded curve. To show that the twisted complex representation of X in terms of the G_i is a HN-tower, we need to check that components of δ increase phase. For this we use that fact that the concatenation of the α_i is a geodesic, and so α_i, α_{i+1} necessarily meet at a singular point in an angle $\phi \geq \pi$, thus $\phi > \pi$ if the phase jumps. The corresponding morphism, without loss of generality from X_i to X_{i+1} , has degree 1, and so

$$1 = \phi(X_i) - \phi(X_{i+1}) + \frac{\phi}{\pi} \quad (5.24)$$

which implies $\phi(X_i) < \phi(X_{i+1})$.

We can bootstrap the above analysis to the case of general surfaces, as long as only admissible curves with domain $[0, 1]$ are involved. The first claim is that $\text{Hom}(A, B) = 0$ for a pair of graded saddle connections A, B with $\phi(A) > \phi(B)$. To see this, pass to the universal cover $\tilde{C}$ of (C, Ω) , which is a flat surface with only ∞ -angle singularities. Lift A, B to curves $\tilde{A}, \tilde{B}$ on $\tilde{C}$, graded with $\phi(\tilde{A}) = \phi(A)$, $\phi(\tilde{B}) = \phi(B)$. We have

$$\text{Hom}(A, B) = \bigoplus_{g \in \pi_1(C)} \text{Hom}(\tilde{A}, g\tilde{B}) \quad (5.25)$$

and hence it suffices to prove the claim for $\tilde{C}$. But any two saddle connections are objects in some category coming from a surface of disk-type, and so our previous arguments can be applied.

Second, we claim that the HN property holds for admissible curves c with domain $[0, 1]$. To see this, lift c to a graded curve $\tilde{c}$ in the universal cover. We work in the category generated by all the saddle connections in a geodesic representative of $\tilde{c}$. It corresponds to some surface of disk-type in the universal cover. We get a HN-tower for $\tilde{c}$ in this category and push it forward to $\mathcal{F}(S, M, \Omega)$ to get a tower for c in that category.

Moduli spaces

When (S, M, Ω) is of disk-type the category $\mathcal{F}(S, M, \Omega)$ has only finitely many indecomposables up to shift, allowing an easy classification of t-structures. The following is not used in the proof of Theorem 5.2.

Theorem 5.3. *Let (S, M, Ω) be a graded marked surface of disk-type, $n + 1 = |M| \geq 3$, $\mathcal{T} = H^0(\mathcal{F}(S, M, \Omega))$, then*

$$\text{Stab}(\mathcal{T})/\text{Aut}(\mathcal{T}) = \left\{ e^{P(z)} dz^2 \mid \deg P = n + 1 \right\} / \text{Aut}(\mathbb{C}) \quad (5.26)$$

$$= \left\{ e^{z^{n+1} + a_{n-1}z^{n-1} + \dots + a_0} dz^2 \right\} / \mathbb{Z}/(n+1)\mathbb{Z} \quad (5.27)$$

as complex orbifolds.

The case $n = 2$ of this theorem is contained in [7]. Our proof uses a classification result of R. Nevanlinna [24].

Proof. The category $\mathcal{T}$ has, up to shift, only finitely many indecomposable objects. This implies that any t-structure on $\mathcal{T}$ is *Artinian*, i.e. objects in its heart are all of finite length. Moreover, as $K_0(\mathcal{T}) \cong \mathbb{Z}^n$, the heart of any bounded t-structure on $\mathcal{T}$ has exactly n simple objects up to isomorphism. In particular, any central charge $Z : K_0(\mathcal{T}) \rightarrow \mathbb{C}$ sending these simple objects to

$$H_- = \{ r e^{i\phi} \mid r > 0, \phi \in (0, \pi] \} \quad (5.28)$$

defines a stability condition on $\mathcal{T}$ and all stability conditions are obtained in this way.

Let us classify t-structures on $\mathcal{T}$. Suppose $\mathcal{A} \subset \mathcal{T}$ is the heart of a bounded t-structure with simple objects $S_1, \dots, S_n$. The S_i correspond to graded arcs in (S, M, Ω) for which we use the same symbols. By general properties of t-structures,

$$\text{Ext}^{\leq 0}(S_i, S_j) = 0, \quad i \neq j \quad (5.29)$$

which implies that $S_1, \dots, S_n$ form an arc system A . It is easily verified that A is formal, and conversely that any full formal system of graded arcs satisfying (5.29) gives precisely the simple objects in a bounded t-structure on $\mathcal{T}$.

We have already seen that a quadratic differential $\Omega = e^{P(z)} dz^2$ gives a stability condition on $\mathcal{T}$. What are the simple objects in the associated t-structure, $\mathcal{A}$? To answer this, consider the horizontal foliation of Ω . There

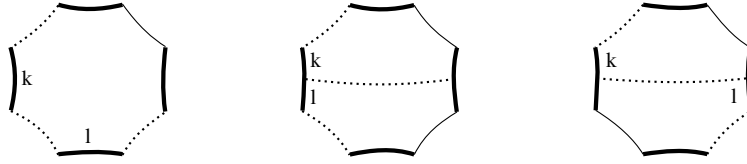


Figure 6: Types of full formal systems of graded arcs (dotted lines) in the case $n = 3$. The positive integers k, l indicate degrees of morphisms.

are three types of leaves: Those which do not converge in F in either direction (*infinite leaves*), those which converge to a point in F_{sg} in one direction, and those which converge to a point in F_{sg} in both directions, i.e. saddle connections. The latter two types are called *convergent*. From the local structure of the flat metric we see that there is a $\mathbb{Z}$ -torsor of convergent leaves starting at any ∞ -angle singularity, cutting F into strips. The saddle connections which do not cross a convergent leaf correspond to the simple objects in $\mathcal{A}$.

Assume now that no central charges lie on $\mathbb{R}$, which can always be achieved after rotating phases by a small amount. The flat surface can then be described as being glued along convergent trajectories from rectangular pieces of the form $\mathbb{R} \times [0, h]$ for each simple object S_i , where $h = \text{Im}Z(S_i)$, and an infinite number of rectangular pieces of the form $\mathbb{R} \times [0, \infty)$. This description makes it clear how to construct a simply connected flat surface with $n+1$ infinity-angle singularities from any stability condition on $\mathcal{T}$. However, it is not obvious that such a flat surface comes from a pair $(\mathbb{C}, e^{P(z)}dz^2)$. This follows from results of Nevanlinna in [24].

To complete the proof one must show that $\text{Aut}(\mathcal{T})$ coincides with graded automorphisms of (S, M) up to isotopy. This follows from the classifications of indecomposables in $\mathcal{T}$ and that any autoequivalence of $\mathcal{T}$ which acts trivially on objects is naturally isomorphic to the identity. $\square$

In order to study the wall and chamber structure on $\text{Stab}(\mathcal{T})/\text{Aut}(\mathcal{T})$ one can consider the convex hull K of the $(n+1)$ singularities. It is a union of 1-dimensional pieces which are straight lines and 2-dimensional pieces which are polygons, possibly non-convex and non-planar. Figure 7 gives some examples for $n = 3$.

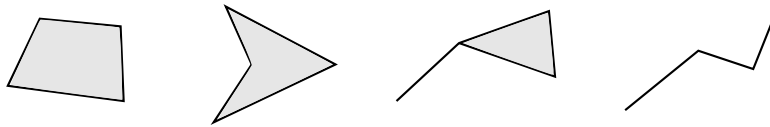


Figure 7: Possible topologies of K when $n = 4$ for a generic stability condition. Note that the number of stable objects up to shift is 6, 5, 4, 3 respectively.

5.6 $A_{m,n}$ (annular type)

We turn to the case $C = \mathbb{C}^*$, with Ω having exponential singularities at 0 and ∞ , and so that the unit circle $S^1 \subset \mathbb{C}^*$ is gradable. Then Ω is of the form

$$\Omega = e^{P(z)} \frac{dz^2}{z^2} \quad (5.30)$$

with P a Laurent polynomial with poles of orders m and n at 0 and ∞ respectively. The associated marked surface (S, M) is an topologically an annulus with m and n marked boundary intervals on the two components of ∂S respectively.

First we check that $\text{Hom}(A, B) = 0$ for stable A, B with $\phi(A) > \phi(B)$. It remains to consider the cases when either A or B , or both, are closed geodesics with simple local system. We omit computations of $\text{Ext}^*(A, B)$ for specific twisted complexes A, B , which can be done directly or by relating $\mathcal{F}(S, M, \Omega)$ to Floer-theoretic or B-model definitions.

1. If A and B both correspond to loops with simple local system, then they either differ by a shift or else are orthogonal, i.e. $\text{Ext}^*(A, B) = \text{Ext}^*(B, A) = 0$. In the first case it suffices to note that $\text{Ext}^{<0}(A, A) = 0$ since A lies in an abelian category.
2. If one of A or B is a geodesic loop and the other a saddle connection tending in both directions to the same exponential singularity of Ω , then A and B are orthogonal.
3. If one of A or B is a geodesic loop and the other a saddle connections tending towards both singularities of Ω , then A and B intersect in a unique point p and $\text{Ext}^*(A, B)$ is concentrated in degree $i_p(A, B) = \lceil \phi(A) - \phi(B) \rceil$.

By passing to the annular covering we see, as in the case of two saddle connections, that the statement is true for general (C, Ω) when at least one of A or B is a saddle connection.

There is, up to isotopy and grading, a unique admissible curve c with domain S^1 on (S, M, Ω) . We need to check that HN-towers exist for the corresponding indecomposable objects X . If (X, C) has a geodesic loop (necessarily isotopic to c), then X is semistable, so we are done. If not, then c has a geodesic representative which is a concatenation of saddle connections. The corresponding collection of arcs satisfies the properties we required when assigning a twisted complex to c . The same argument as in the A_n case shows that this twisted complex gives a HN-tower for X . Again, passing to the annular covering, we find that HN-towers exist for objects corresponding to admissible curves with local system and domain S^1 on general (C, Ω) . Since all isomorphism classes of indecomposable objects in $H^0(\mathcal{F}(S, M, \Omega))$ come from admissible curves, this completes the proof of the HN-property.

5.7 The general case

Let (C, Ω) be a general Riemann surface with CY-structure, (S, M) the associated marked surface. To finish the proof of Theorem 5.2 we need to show that for two objects $A, B \in H^0(\mathcal{F}(S, M, \Omega))$ corresponding to closed geodesics α, β with local systems and $\phi(A) > \phi(B)$ we have $\text{Hom}(A, B) = 0$. We may also assume that α and β are non-isotopic as simple closed curves.

Let Z denote the flat cylinder foliated by closed geodesics isotopic to α . Since Ω is assumed to not have any second-order poles, Z has finite height with singular points on each boundary component, see Figure 8. The other loop β

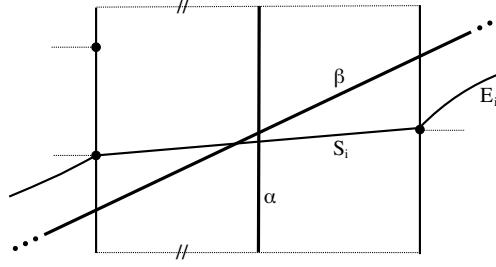


Figure 8: The cylinder Z , cut along a dotted line, closed geodesics α, β , saddle connections S_i , arcs E_i .

can cross Z any number of times, but does so always with the same slope. The we can represent B as an extension of graded saddle connections with local systems S_i , which are contained in Z and connect the two components of its boundary, and arcs E_i which are disjoint from the interior of Z . Moreover, these saddle connections can be chosen so that $\phi(S_i) < \phi(B)$. Namely, there is an isotopy moving β to the concatenation of the S_i and E_i , and it should move any point of intersection with ∂Z so that Z lies to the right. From what we have shown in the annular case, it follows that $\text{Hom}(A, S_i) = 0$ for all i . Further, A is orthogonal to any E_i , so $\text{Hom}(A, B) = 0$ follows. This completes the proof of Theorem 5.2.

Zusammenfassung auf Deutsch

Die Untersuchung von Stabilität in triangulierten Kategorien wurde, basierend auf Ideen des Physikers M.R. Douglas, von Bridgeland eingeleitet [4]. Seitdem beschäftigt sich die Mehrheit der Arbeiten in diesem Gebiet mit Kategorien algebraisch-geometrischen Ursprungs, siehe z.B. [2, 5, 6, 16, 21]. Andererseits wird im Zusammenhang mit symplektischen Mannigfaltigkeiten schon länger eine Verbindung zwischen speziellen lagrangeschen Untermannigfaltigkeiten und dem Fluss entlang mittlerer Krümmung mit algebraischen Konzepten von Stabilität vermutet, siehe Thomas [30], Thomas–Yau [31], aktualisiert in Joyce [15]. Dies motiviert die Untersuchung von Stabilitätsbedingungen auf Fukayakategorien. In dieser Arbeit bestätigen wir die Axiome von Bridgeland in der niedrigst möglichen Dimension, der von Flächen.

Im Detail bedeutet dies Folgendes. Gegeben sei eine riemannsche Fläche $\overline{C}$ mit quadratischem Differential Ω welches auf dem Komplement $C = \overline{C} \setminus S$ einer endlichen Menge S holomorph und nirgends verschwindend ist. Weiters nehmen wir an, dass sich Ω in den Punkten S entweder meromorph fortsetzen lässt, oder eine transzendente Singularität der Form

$$e^{f(z)}g(z)dz^2$$

besitzt, wobei f und g meromorph sind und 0 eine Polstelle von f ist. Der Absolutwert des Differentials, $|\Omega|$, definiert eine flache Metrik auf C . Die metrische Vervollständigung F von C besitzt zusätzlich eine endliche Anzahl von konischen Punkten.

Das Paar (C, Ω) bestimmt eine A_∞ -Kategorie $\mathcal{A}$ vom Fukaya-Typ welche nur von topologischen Eigenschaften abhängt: dem Geschlecht von C , der Anzahl und Arten von Singularitäten von Ω , und der Monodromie von Ω . Grob gesprochen sind die Objekte von $\mathcal{A}$ Kurven in C . Insbesondere lassen sich Geodäten in F mit endlicher Länge als Objekte von $\mathcal{A}$ verstehen. Das Hauptresultat dieser Arbeit ist das folgende Theorem.

Theorem. *Sei (C, Ω) wie oben ein quadratisches Differential ohne Polstellen höherer Ordnung, und $\mathcal{A}$ die assoziierte Fukayakategorie. Dann bestimmt die Menge der Objekte in der triangulierten Kategorie $H^0(\mathcal{A})$ welche zu ungebrochenen Geodäten gehören eine Stabilitätsbedingung auf dieser Kategorie.*

Die Arbeit ist inhaltlich folgendermaßen aufgebaut. Ziel des ersten Abschnittes ist, es eine elementare Definition der Fukayakategorie einer Fläche mit Rand zu geben. Dafür wird jedem geeigneten System von Schnitten A eine explizit gegebene A_∞ -Kategorie $\mathcal{F}_A$ zugeordnet und anschließend gezeigt, dass die assoziierte Kategorie von Komplexen $\mathcal{F} = \text{Tw}(\mathcal{F}_A)$ unabhängig von der Wahl von A ist. Im Abschnitt 3 lösen wir ein zahmes Klassifikationsproblem welches die *Geometrizität* von Objekten in $\mathcal{F}$ impliziert. Dieses Resultat ist wesentlich für den Beweis des Haupttheorems, ist aber auch unabhängig davon von Interesse in der Darstellungstheorie. Im Abschnitt 4 fassen wir einige Eigenschaften von quadratischen Differentialen und Flächen mit konischen Punkten zusammen, wobei besonders die Singularitäten vom exponentiellen Typ im Detail behandelt werden. Schlussendlich erfolgt im Abschnitt 5 eine Berechnung der relevanten K_0 -Gruppe sowie der Beweis des Haupttheorems.

Lebenslauf

- Matura im Juni 2004 am Gymnasium Klosterneuburg
- Diplomstudium Mathematik an der Universität Wien von WS 2004 bis SS 2010. Betreuer: Stefan Haller, Diplomarbeit: “Refined Combinatorial Torsion”.
- Zivildienst von Februar 2005 bis Januar 2006
- Doktoratsstudium der Naturwissenschaften an der Universität Wien seit WS 2010. Betreuer: Ludmil Katzarkov und Maxim Kontsevich, Dissertation: “Stability of 1-dimensional A-branes”.

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