DIPLOMARBEIT

Small group learning in mathematics

A connection between Berry and Sahlberg's typology of tasks and Hußmann's intentional problems in school mathematics

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Synopsis

0 Note of thanks, Abstract, Zusammenfassung

0.1 Note of thanks

Overall, I would like to thank my parents Hermine and Gerhard Jodl for always believing in me and for giving me the feeling of being the most important part of their lives. When looking back on my childhood, I can remember them telling me that I can succeed in everything. I achieved all my goals because of their love and support.

I would like to thank Dr. Stefan Götz, who accompanied me during writing this paper with patience and professional guidance. Right from the beginning, he believed in my abilities as he didn't restrict my ideas. Even in other mathematical courses, he reaches students by making clear statements and evoking lively discussions.

I would like to thank Dr. John Monaghan, who teaches at the school of education at the University of Leeds in Great Britain. During my exchange stay in Leeds for half a year, I took one of his courses called 'Mathematical Education'. Luckily, Berry and Sahlberg's paper called 'Mathematical tasks in small group learning', which was presented in Monaghan's course, covers important topics of this paper, which makes it an important piece of literature I refer to.

0.2 Abstract

At the beginning of my 'Diplomarbeit' called 'Small group learning in mathematics. A connection between Berry and Sahlberg's typology of tasks and Hußmann's intentional problems in school mathematics', I combine small group learning in a mathematics lesson with other pedagogical ideas of learning, which were introduced by Hußmann, Peirce, Buchberger, Gardiner, Alrø, Skovsmose, Berry and Sahlberg as well as antique philosophers like Platon, Aristoteles and finally Kant. The extended 'learning arrangement' (page 7-9), which was designed by Stephan Hußmann originally, illustrates the adaptability of my 'Diplomarbeit'.

Berry and Sahlberg belong to those English educators, who reserach the practical use of tasks for small group learning in mathematics lessons in numerous studies. In chapter 5.1.10, I refer to one of these studies.
Overall, the fifth chapter deals with Berry and Sahlberg`s typology of tasks in school mathematics, which leads to a challenging and exciting class of tasks, called 'intentional problems' (chapter 5.2). In contrast to traditional mathematical tasks, intentional problems are more open as some of them don't contain any questions and all of them stimulate research in various ways. In chapter 5.2 I prepare the integral calculus for school mathematics.

In the end, I discuss several advantages and disadvantages of small group learning in mathematics. By putting my focus on the advantages, I clearly approve small group learning in mathematics.

0.3 Zusammenfassung

Zu Beginn meiner Diplomarbeit mit dem Titel „Small group learning in mathematics. A connection between Berry and Sahlberg's typology of tasks and Hußmann's intentional problems in school mathematics“ verbinde ich Kleingruppenlernen als Unterrichtsgestaltung mit anderen pädagogischen Auffassungen des Lernens, die von Hußmann, Peirce, Buchberger, Gardiner, Alrø, Skovsmose, Berry und Sahlberg sowie den antiken Philosophen Platon, Aristoteles und – in der Neuzeit – Kant eingeführt wurden. Dadurch werden die Konzepte einiger Pädagogen erweitert oder geändert. Meine Interpretation des „learning arrangement“, welches ursprünglich von Stephan Hußmann entworfen wurde, illustriert die Adaptabilität meiner Diplomarbeit.

Da Berry und Sahlberg zu jenen englischsprachigen Pädagog/innen zählen, die in von mir referierten Studien die praktischen Anwendungen von Aufgaben für das Lernen in Kleingruppen im Mathematikunterricht untersuchen, beschäftige ich mich im fünften Kapitel mit deren „typology of tasks in school mathematics“, die zu einem anspruchsvollen und spannenden Aufgabentypus namens „intentionale Probleme“ (Kapitel 5.2) führt. Im Vergleich zu konventionellen mathematischen Aufgaben besitzen intentionale Probleme weitaus mehr Offenheit, was sich unter anderem darin zeigt, dass sie teilweise keine Frage stellen und die dargestellten (Problem-)Situationen zahlreiche Untersuchungen auf unterschiedlichen Wegen anregen. Im Zuge der Darstellung von intentionalen Problemen bereite ich die Integralrechnung für den Unterricht an höheren Schulen auf.

Das Ende meiner Diplomarbeit bildet eine Diskussion über die Vor- und Nachteile von Kleingruppen im Mathematikunterricht. Durch die stärkere Gewichtung der Vorteile, die das Lernen in Kleingruppen mit sich bringt, beziehe ich eine klare Position in dieser Diskussion, die bereits seit mehreren Jahren vonstatten geht.

1 Introduction

I hear and I forget,
I see and I remember,
I act and I understand.
[Konfuzius, about 500 B.C. – out of Hu, 2003, 10]

On average, a person keeps 90% of all the things he/she does him/herself, 70% of what he/she says him/herself and 50% of what he/she sees, reads and hears. But he/she can only remember 30% of what he/she sees only, 20% of what he/she hears only and solely 10% of what he/she reads only [Hu, 2003, 10].
This means that no matter whether I see or touch something, I have to act on my own to ensure that certain information turns into a component of my knowledge. Accordingly, teachers are forced to address many senses at the same time. It is important that pupils get opportunities to be in action in the classroom.
Similar to Bauer and Hußmann, Friere points out the importance of cooperation between 'hand and head' by saying that *hand and head have to go together. Acting without reflecting would end up in pure activism and reflection without action would result in verbalism* [Fr, 1972, 75f], [compare to Hu, 2003, Bau, 1997].

Cooperative learning, like small group learning (SGL), is one possibility to ensure action in a mathematics lesson. Hußmann understands cooperative learning as talking about the group's target and how to reach it and arousing coordination amongst all group members [Hu, 2003, 18].

SGL is based on pedagogic ideas presented over a decade ago. Still, teachers and educators are sceptical. Because of the common view of mathematics as a body of knowledge, i.e. number, algebra, analysis, stochastics and geometry, school leavers have learned some content, faced with traditional teacher-centred learning methods, that can be carried out more efficiently by information communication technology. Unfortunately, the role of all group learning seems to be seen by most teachers as to do mathematics in problem solving situations and to introduce social skills and discussion, but not as a way to learn new mathematical knowledge and skills [Ber I, 2002, 1-3].

These facts lead to a fundamental question: Is it better to establish new mathematical concepts in the first place together with pupils and let them practice new terms, definitions and corresponding algorithms afterwards or does it show more effect if students try to find new terms, definitons and algorithms autonomously in the first place and discuss their results together with a teacher afterwards? I discuss both views. Not only one way hits the spot, but both include certain advantages. First, the question leads to the white box/black box principle, which has been introduced by Bruno Buchberger in 1990. In this case, all students of a classroom achieve 'useful' knowledge right away when a teacher introduces them to a new theme (= white box). Afterwards they get confronted with eventualities to practice these new skills. Therefore, they often don't need to enlarge the knowledge the teacher gave them beforehand (= black box). I see the first three categories of my typology of tasks in school mathematics (namely 'drilling basic skills' (A1), 'applying a formula or algorithm' (A2) and 'measuring and collecting data' (A3)) similar to the black box in the white box/black box principle (compare to 5.1 A proposal on typology of tasks in school mathematics). As a result, SGL can be used in an old fashioned way to practice. In contrast, the black box/white box principle (also introduced by Bruno Buchberger in 1990) turns the situation upside down and uses SGL as a way to build up new (mathematical) knowledge with colleagues (= black box). In this case, pupils get the chance to come up with their own ideas while no teacher limits them or corrects them as soon as they are about to take a wrong path. Therefore, pupils listen to their intuition. The first part is called black box because the students' ideas aren't evaluated until every group/student is finished or the time for non-guided work is over. Only then, a teacher explains the common way of solving the discussed problem and shows accepted algorithms and terms according to the new theme (= white box). This part is called white box because a teacher enlightens the students' brains and he/she ensures that students don't learn any wrong concepts [compare to Heu, 1996, 158 – 179].

Berry and Sahlberg's study (see 5.1.10 A study design by Berry and Sahlberg) draws a vivid image of the way teachers understand mathematics. Mathematics teachers seem to believe that *[...] small group work is for collecting data and doing mathematics rather than learning new mathematical concepts and skills* [Ber I, 2002, 28]. Those teachers, who share this view, won't give pupils opportunities to experience the black box/white box principle, which seems to be similar to SGL. Unfortunately, the teachers' view of mathematics *seems to be 'learn mathematics alone and do mathematics together'* [Ber I, 2002, 31].

An increasing emphasis on real problem solving, investigation, projects and other forms of applying mathematical knowledge and skills in everyday life situations is changing the nature of mathematics in school. There has been a growing demand from both professional and business people and education policy-makers to stimulate active learning, promote effective teaching and encourage appropriate assessment methods to be utilised in the teaching and learning of mathematics [Bla, 1996, chapter 3 and chapter 4]. Constructivist education has been interpreted to mean more participatory learning methods, such as working on real life problems, communication and cooperation although its practical implications vary considerably [compare to Fo, 2001]. Hence, many school improvement experts and teachers, who are looking for better quality teaching and learning, have turned their attention to SGL methods that have been developed since the late 1960s [Ber I, 2002, 2].

It is essential to note a topical development, described by Gardiner in 'Back to the future'. He points out that pupils interpret all mathematical expressions – whether *'$\pi/2 + 2 - 2x(\pi/4)$', '1/3' or 'cos($\pi/2$)'* – as algorithms to be evaluated. They don't perceive them as algebraic expressions to be simplified or comprehended. As a result, students don't understand the idea behind such expressions. Unfortunately, teachers feel that students have lost the liberating power of algebra, which is to gain an insight into why things are true. In chapter 8.1 Advantages of SGL, this power is called 'achieving fluency from the insight'. I understand it that SGL is one possibility to slow this development, which is pointed out by Gardiner, down. SGL shows pupils how to make sense of mathematics *by making their own tentative conjectures and constructions and linking them with prior schemata* [Jon II, 1999, 18]. During SGL, pupils have to express themselves and their mathematical thoughts more often compared to working alone or listening to a teacher [compare to Ga, 1995, 527f].

However, there are several different models of cooperative learning that vary by their epistemological orientation and practical implication to the role of the teacher and the learner. Particularly in mathematics, cooperative learning can be used in conjunction with practising skills, doing investigations, collecting data, discussing concepts and principles, or solving mathematical problems [compare to Da, 1991].

2 Learning arrangement by Hußmann

There is a difference between individuals constructing their world by making sense of it, and being constructed by their world through participating in it [compare to Jon II, 1999, 2f].

Hußmann states that a learning arrangement combines various aspects of teaching, which are formed by a teacher and have an impact on the way students are learning. Figure 1 shows a vivid image of such an arrangement as a net of factors influencing each other. Intentional problems can be seen as the starting point for learning, self-navigated learning as the way from theory to students and the moderation of learning processes as the teacher's impact.
These aspects are characterized by the students' opinion of mathematics, the school lesson and learning in general. The teacher's opinion flows into the part of moderating learning processes.
Each learning arrangement, which includes intentional problems (see 5.2 Intentional problems of the integral calculus by Hußmann), calls for the reduction of routine calculation so that students develop mathematical concepts as independent as possible. At this point I can see a clear difference to traditional teaching methods in mathematics [Hu, 2003, 8].

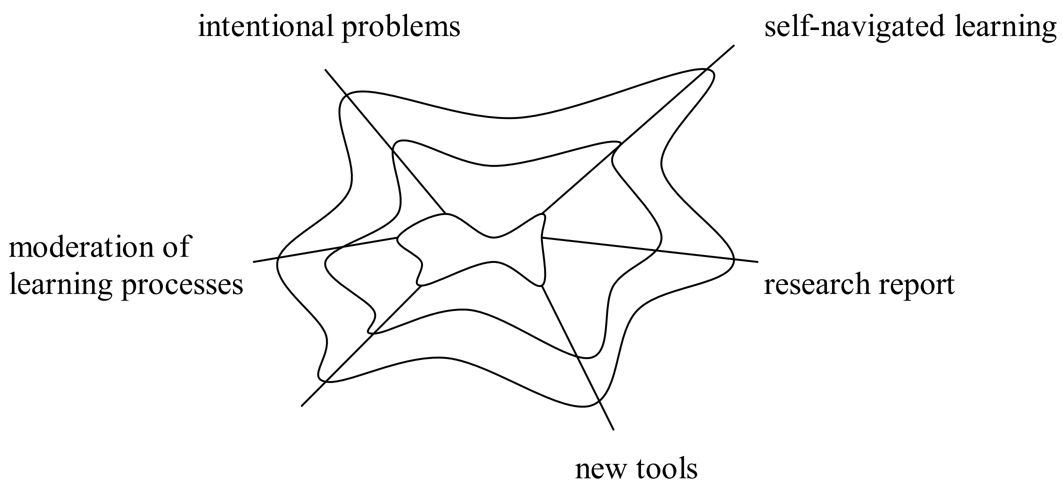


Figure 1: The learning arrangement [Hu, 2003, 8]

The net shown above can be extended as illustrated in Figure 2, which emphasises on learning arrangements for SGL. In Figure 2, I add self-confidence, stamina and self-examination as crucial aspects of SGL. In addition, I replace self-navigated learning by group-navigated learning, which takes place during SGL more often than during traditional teaching methods.

The necessarity of self-examination is one major advantage of SGL. The way I see it, members of a small group have to control each step of their working process to continue their argument. Hence, they have to discuss the problem from various perspectives, which enlarges their knowledge about a given topic [compare to Hu, 2003, 8].

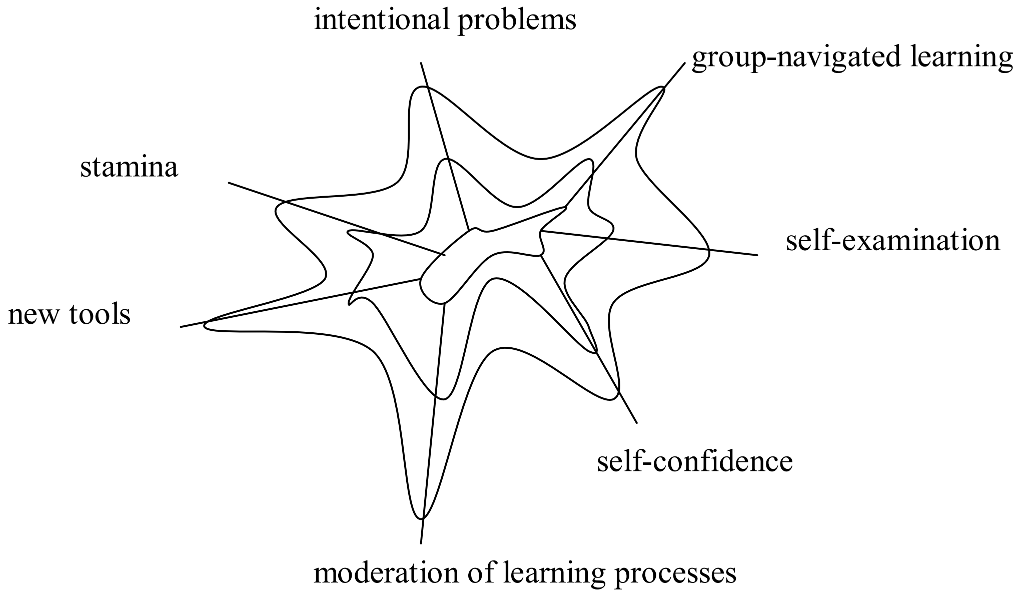


Figure 2: A proposal on a learning arrangement for SGL [compare to Hu, 2003, 8]

I believe that Hußmann uses the image of a net to show how a human brain works, which is linking ‘old’ knowledge to work with ‘new’ information. Personally, I see mathematics as a (school) subject, which builds up one’s knowledge by refering to available knowledge. Deducing the area formula of a triangle by remembering the area formulae of certain quadrilaterals is one example. Learning is a mental effort. Instead of watching the process of learning the moment it takes place, one can watch the changes of a student’s personality [Hu, 2003, 13].

Hußmann understands learning as the passing of information. He states that teachers are endangered to turn their perspective to the only criterion. Far too soon we tell students our own perspective without being asked for it. Moreover, we prefer to help students to choose a path instead of giving them enough time to make decisions autonomously. As a result, students tend to rely on a teacher who tells them when they are about to take a wrong path, which goes against the idea of SGL, namely making consumptions independently. In order to avoid these situations, both students and teachers need an appropriate image of human beings, which is seeing each other as autonomous, sensing, acting and independently thinking people. This attitude is the centre of successful open learning arrangements. Self-navigation/Group-navigation means having an impact on the decision process when it is about to decide what, how, when and why learning takes place. In real life, learning is guided both by the person himself/herself and the influences around him/her. As soon as the amount of self-navigation/group-navigation is high, we talk about 'autonomous learning' [compare to Hu, 2003, 14/15].

Figure 3 describes the three stages of self-navigated learning. In order to show its relation to SGL, I highlight the places where SGL can take place. I use scaffolder as a synonymon for teacher as they have a similar position in a mathematics lesson [compare to Hu, 2003, 16].

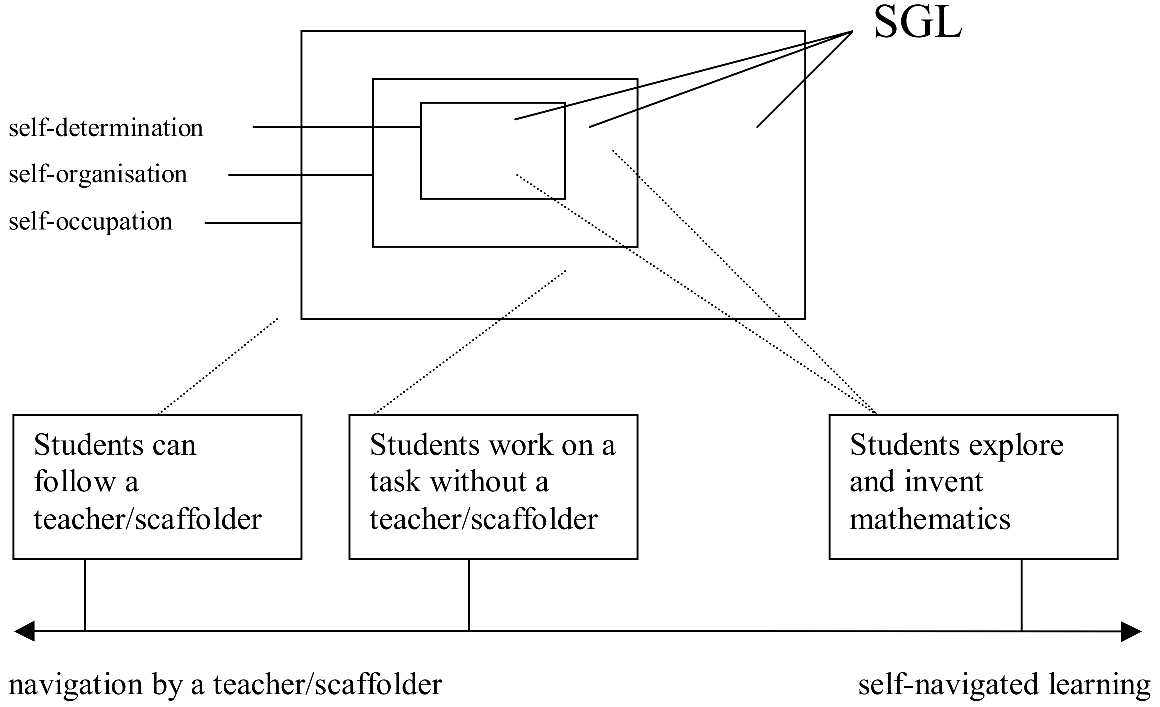


Figure 3: The three stages of self-navigated learning [compare to Hu, 2003, 16]

In conclusion, the amount of self- and group-navigated learning is dependent on the amount of freedom in action and thinking [Hu, 2003, 14-15].

3 Development of mathematical thinking

Thinking mathematically is not [...] *an end in itself. Rather it is a process through which we make sense of our world* [Mas, 1982, 178].

In 'Theaitetos', Platon gives three conditions for knowledge:
An individual x knows that p if and only if

1. *x believes that p,*
2. *p is true and*
3. *x can reason p* [Ho, 2005, 15].

Effective SGL needs to go through these three steps to ensure that the group members' stamina lasts till the end. Personally, I put the emphasis on the third step as it calls for the biggest amount of networking. Pupils aim to generalize their knowledge to use it as often as possible. During SGL, students get confronted with various ideas how to generalize their results, which helps them to decide for the right path. Such generalization is the basis for any further development of one's cognizance [Ho, 2005, 8].

SGL accompanies with the constructivist viewpoint, in which the learner is considered as an active purveyor of meaning. For making a clear statement, it is important that I give a definition of 'mathematical thinking'. *Mathematical thinking is not thinking about the subject matter of mathematics but a style of thinking that is a function of particular operations, processes and dynamics recognisably mathematical* [Bu, 1984, 35]. It is necessary to state that learning to think mathematically is more than just learning to use mathematical techniques. Consequently, pupils seem to achieve a special kind of knowledge compared to other subjects, i.e. languages if they are taught traditionally. The support by small group members is strongly needed and desired. To sum up, learning to think mathematically includes developing a mathematical point of view, namely valuing the processes of mathematisation and abstraction and having the predilection to apply them, and competence in using mathematical tools and those which serve the goal of understanding. Both developments are combined in the term 'mathematical sense making' [Schoe, 1994, 60].

Refering to Kant, Hoffmann describes a major pillar of mathematical thinking, namely 'a term's construction throughout visualization' (equal to 'Konstruktion des Begriffs in der Anschauung'). It means that drawing a diagram, which is corresponding to a certain mathematical term, helps students to understand this term. By creating an appropriate visualization of a mathematical term, students contemplate essential aspects of this term, which would have kept hidden without the act of drawing a diagram. For example, Figure 4 explains the 'Winkelsummensatz' better than any equation can do. In this case, finding the 'right' diagram calls for the biggest effort [compare to Ho, 2005, 102f].

Winkelsummensatz

For each triangle $\angle\alpha,\beta,\chi$ *(let α, β, χ be the triangle's angles) is* $\alpha+\beta+\chi=180°$ [compare to Ho, 2005, 103].

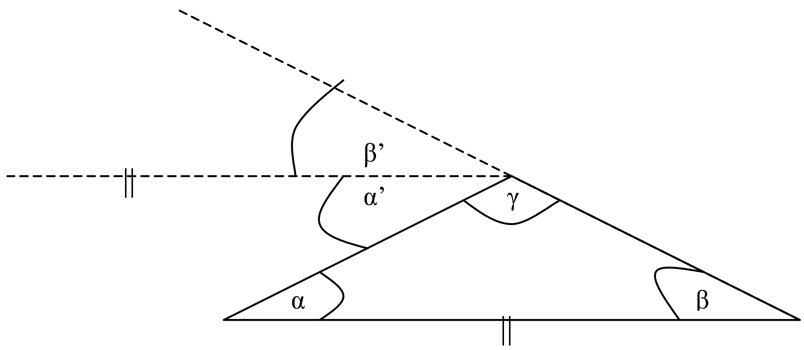


Figure 4: Kant's proof of the 'Winkelsummensatz' [Ho, 2005, 103]

Many authors emphasize the importance of students' participation in reflective classroom discourse (see [Cob, 1997], [Ber I, 2002], [Jon II, 1999]). While SGL, pupils tend to build up a close relation to mathematical concepts. As a consequence, they fulfill Confrey's requirement to believe in what they are doing. He continues by saying that *ironically, in most formal knowledge, students distinguish between believing and knowing [...] To a constructivist, knowledge without belief is contradictory. Thus I wish to assert, that personal authority is the backbone of the process of construction* [Con, 1990, 111].

Learning should mean learning for citizenship because learning for citizenship includes competencies which are essential to participate in the democratic life and to develope maturity [Al, 2004, 134]. Skovsmose and Valero [Sk, 2001] list three possible relationships between mathematics education and democracy. Selected literature emphasises that mathematics education maintains an intrinsic resonance with democratic ideals because of the very nature of mathematics. Moreover, its logical structure makes clear argumentation inescapable where dogmatism has no role to play (except for axiomatic rules). Because mathematics is a topic where only sound arguments are accepted, mathematical thinking, especially the type of thinking which emerges from SGL, becomes an excellent preparation for reasoning and dialogue, which ought to characterise a democracy. The way I see it, pupils, who develop mathematical knowledge in small groups, enlarge their authority widely [compare to Al, 2004, 4, 134].

In general, various authors argue that knowledge develops in different ways. For example, Plato and Aristoteles understand learning as remembering information. They say that a newborn holds all information without being aware of it, whereas John Locke calls a newborn's mind 'tabula rasa'. Plato draws a stunning picture to explain the term 'episteme', which means 'knowledge' or 'cognizance'. His understanding of a newborn's brain justifies his seperation between a *true opinion* [Pla, 1981, 187f] and a *wrong opinion* [McD, 1973, 212]. In Theaitetos, he claims that knowledge is the same as a *true opinion*. This definition shows the importance of SGL to give pupils space to discuss divergent opinions so that they may find a *true opinion*. To understand the meaning of a *wrong opinion*, Plato pictures a wax block holding all information one has in his/her soul. A wrong opinion is a mismatching of perceptions with imprints.

Moreover, Plato differs *recognizing a particular circumstance (1)* from *thinking without cognition (2).* He refers to recognizing information (1) as an act of cognition in which an appropriate engraving of the wax block has to be activated. According to thinking without cognition (2), Plato gives the image of persistent knowledge flying around in one's brain like a dove. Plato distinguishes between possessing pieces of knowledge and holding them [McD, 1973, 212]. In 'Erkenntnisentwicklung', Hoffmann deals with this image by giving the following example. He refers to Plato by stating that 7 + 5 can be solved without cognition. He carries on by explaining his example and uses 'catching and locking doves up' as synonym for learning. I disagree with Hoffmann because 'catching the right dove' certainly calls for cognition. Still, this constructive approach to the structure of knowledge fascinates me. Similar to Plato, Aristoteles claims that understanding arises from bridging the spread between available and new knowledge. *In one way we know, in another way we don't* [Ho, 2005, 73]. SGL concentrates on the first part of Hoffmann's sentence, i.e. '*in one way we know'*. During cooperative learning, at least one student may say 'in one way I know'. Understanding Plato's idea of (building up) knowledge helps a teacher to (re)act at his/her best [compare to Ho, 2005, 69 – 73].

Kant divides two sources for the mind, which are starting points for developing cognizance, namely outlooks (1), which train one's decoding skills, and the spontaneous mind (2). *Ohne Sinnlichkeit würde uns kein Gegenstand gegeben und ohne Verstand keiner gedacht werden. Gedanken ohne Inhalt sind leer, Anschauungen ohne Begriffe sind blind* [Kan, 1998, 74f]. SGL is acting on instinct, which is how our mind works naturally [compare to Ho, 2005, 7].

4 Activity within small groups

Whatever a child achieves together with colleagues today, it will accomplish autonomously tomorrow [Vyg IV, 1991, 240].

Berry and Sahlberg claim that small groups should be made up of two unto six pupils. I disagree with the number of participants in a certain way. If six pupils were working together on one problem, it would be rather difficult and time-consuming to bring their ideas and concepts together. Personally, I would bring two to four pupils together in one small group and assign specific tasks to these groups that encourage cooperation in their learning. It may be interesting to let groups of four students split up into groups of two to subdivide a problem and intensify the need for communication to merge the ideas of the subgroups. If each member or each subgroup is challenged to solve the task using a different method and then a group discussion follows about the route to the solution then we will have the possibility of a rich dialogue and equal exchange of ideas about the appropriateness of different approaches to mathematics problems. Only intentional problems (see 5.2 Intentional problems of the integral calculus by Hußmann) seem to be appropriate for the number of group members suggested by Berry and Sahlberg [compare to Ber I, 2002, 2].

4.1 Milieus of learning

School mathematics tradition is characterised by certain ways of organising the classroom. Alrø and Skovsmose divide a traditional mathematics lesson into two parts. *First, the teacher presents some mathematical ideas and techniques. This presentation is normally closely related to the presentation in the given textbook. Secondly, the students work with selected exercises. These exercises can be solved by using the just presented techniques. The solutions are checked by the teacher. An essential part of the students' homework is to solve exercises from the textbook.* [The second part is related to] *one particular aspect of the school mathematics tradition, the exercise paradigm* [...] *This paradigm has a deep influence on mathematics education, concerning the organisation of the individual lessons, the patterns of communication between teacher and students, as well as the social role that mathematics may play in society, for instance operating as a gatekeeper (the mathematical exercises fit nicely into processes of exams and tests). Normally, exercises in mathematics are formulated by an* *authority from outside the classroom. It is neither the teacher nor the students who have formulated the exercises. They are set by an author of a textbook. This means that the justification of the relevance of the exercises is not part of the mathematics lesson itself. Most often, the mathematical texts and exercises represent a 'given' for the classroom practice, including the classroom communication* [...] *To put it more generally, the exercise paradigm can be contrasted by investigative approaches. We* [Alrø and Skovsmose] *see the activities of solving exercises as being much more restrictive for the students than being involved in investigations. We want to elaborate on learning as action and not as a forced activity* [Al, 2004, 45f].

Most mathematical exercises are dealing with the semi-reality. Unfortunately, they often contain implicit agreements, which simplify mathematical exercises. Alrø and Skovsmose explain this theory by stating that *the semi-reality is fully described by the text of the exercise. No other information concerning the semi-reality is relevant in order to solve the exercise, and accordingly not relevant at all. The whole purpose of presenting the exercise is to solve the exercise* [Al, 2004, 47]. Then pupils are not encouraged to deal with exercises, which are presented in the classroom, long enough to build up a connection to (mathematical) fields, which are not mentioned in the first place. In contrast, intentional problems (see 5.2 Intentional problems of the integral calculus by Hußmann) enforce combined thinking, which makes them suitable for SGL. Exercises like the ones described by Alrø and Skovsmose don't offer many opportunities for equal exchange (see 6 Equal exchange model by Cohen) and cooperation. From my point of view, many mathematical exercises out of school books seem narrow-minded and straight forward without kepping in mind that the human brain and reality itself is build up like a network in contrast to a one way street. As stated above, the exercise paradigm can be contrasted by investigative approaches, which elaborate on learning as action and not as a forced activity. 'Openess from the start' [Al, 2004, 46] can be found in Hußmann's literature as well [Hu, 2003, part 2 'The Intentional Problems'].

Alrø and Skovsmose combine the paradigm of exercise and landscapes of investigation and entitle this combination as 'milieus of learning' (see Table 1). I state that all milieus of learning can be suitable for SGL if they possess an open nature. Still, landscapes of investigation are more appropriate for SGL than exercises out of categories (1) or (2) [compare to Hu, 2003, chapter 2]. Landscapes of investigation are substituted by some kind of scene setting that introduces the landscape. Students get encouraged to formulate questions and plan different routes of investigation. They can become part of an inquiry process. In a landscape of investigation, the teacher's 'How can you explain your statement?' often turns into the students' 'Yes, how can I explain my statement?' eventually. Personally, I embrace the authors' idea of varying the landscapes of investigation in their references to various aspects. Exercises with references to pure mathematics can be vitalized by questions including the key-word 'discuss'. In particular, Table 1 emphasises that different forms of references provide different milieus of learning [compare to Al, 2004, 46 - 51].

	Paradigm of exercise	Landscapes of investigation
References to pure mathematics	(1)	(3)
References to a semi-reality	(2)	(4)

Table 1: Milieus of learning [compare to Al, 2004, 50]

When one compares Table 1 with Alrø and Skovsmose's suggestion on page 50 in 'Dialogue and Learning in Mathematics Education. Intention, Reflection, Critique' [Al, 2004], he/she finds that I left out the unit about real references in Table 1. In performing this operation, I want to illustrate my understanding of mathematical exercises, which are suitable for school mathematics.

As I will mention later on in chapter 5.1.8 'Proving mathematical ideas, concepts or statements (F)', school mathematics always deals with models of the reality (= *references to a semi-reality* [Al, 2004, 50]). From my point of view, teachers need to simplify the reality in school mathematics to reach their goal of enlarging the students' knowledge without confusing them too much. Consecutively, I don't understand *real references* [Al, 2004, 50] as part of school mathematics.

Moreover, I numbered the categories in a different way to emphasize the difference between the paradigm of exercise, which does not lead to rich dialogue and equal exchange of ideas in a small group, and landscapes of investigation, which evoke active team work [compare to Al, 2004, 50].

4.1.1 Examples for small group learning (SGL)

(1) Paradigm of exercises. References to pure mathematics

Solve the equations: $3x + 2 = 7$

$2x - 3 = 15$

$x + 4 = 5x - 9$ [Ber I, 2002, 11]

(2) Paradigm of exercises. References to a semi-reality

The following equations are given:

$3P + 2 = 72\ 002$

$P + 3\ 000 = 5P - 9\ 000$

$2P - 3 = 15\ 491$

$3\ 000 + 100T = 100\ 000 - P$

$10P + 30 = 50 + 5T$

(a) Which equations can you draw without reformulating the given expression and without any CAS? Draw them!
(b) Solve each equation! Draw those which you haven't pictured in (a)!
(c) Describe a realistic situation in which each equation can be used! Describe the meaning of each equation (let T stand for time in months and P for price in €)! What does it mean if T does not occur in some equations [compare to Ber I, 2002, 11]?

The first, second and third equation can be solved in a geometric way by letting each side of the equal sign be a linear/constant function and afterwards intersecting them. Pupils have to use different distances for the axes which may challenge them more than drawing these three equations ***after*** *reformulating them.*

(3) Landscapes of investigation. References to pure mathematics

Bracelets: Choose a number between 1 and 18. Apply the following rule: multiply the number of units by 2 and add the number of tens. This gives a new number. What happens if you apply the rule to your new number? Investigate what happens if you continue to apply the rule [Ber I, 2002, 11]!

(4) Landscapes of investigation. References to a semi-reality

(The way I see it, Hußmann's intentional problems appear as synonym for this category.)

Riding a bicycle: Tom says that he can constantly raise his speed to 60 km/h. He wants to prove his hypothesis and starts bicycling right away. He needs 15 minutes for a way of 8 km. Afterwards he tells his friends that he exceeded his own speed limit of 60 km/h.

The following table shows his way dependent on time. The function *S* allows you to calculate the value of kilometers Tom already managed, at each time. Figure 5 visualizes Tom's way.

$$S(t)=\begin{cases} \dfrac{(t+3)^2-9}{36} & 0\le t\le 14 \\ \left(t-\dfrac{\sqrt{120}}{18}-14\right)^3+\dfrac{17\cdot\sqrt{102}}{972}+\dfrac{70}{9} & 14<t\le\dfrac{\sqrt{102}}{18}+14 \\ \dfrac{17\cdot\sqrt{102}}{972}+\dfrac{70}{9} & \dfrac{\sqrt{102}}{18}+14<t \end{cases}$$

[Hu, 2003, 90]

time (min)	way (km)
1	0.194
2	0.444
5	1.53
8	3.11
10	4.44
13	6.86
16	7.95

Table 2: Tom's way [Hu, 2003, 90]

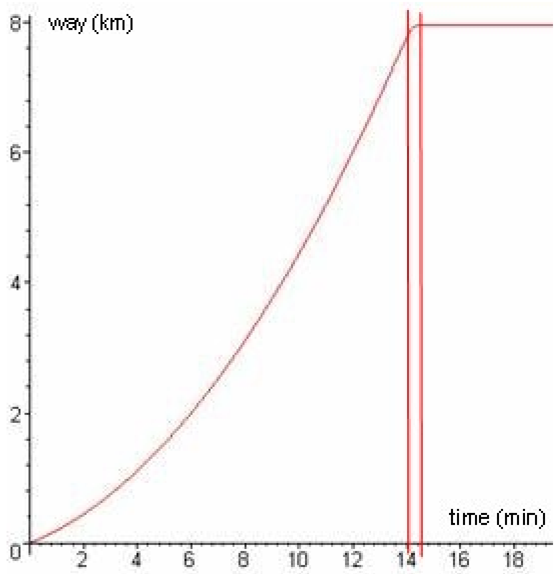


Figure 5: visualization [Hu, 2003, 90]

Mathematical irritation by simplifying real world problems for school mathematics is illustrated by this example.

Function *S* is described by three different terms although *t* runs only from 14 to – approximately – 14.56 in the second case. Studying Figure 5 leads to an explanation. The second term ensures a smooth crossover betwen the first and the third term (visualized by red verticals in Figure 5). In school mathematics, we tell students that the first and second term achieve almost the same result for *t* = 14 which makes them 'good enough'.

4.1.2 The Students' Inquiry Cooperation Model (SIC-model)

The Inquiry Cooperation Model (IC-model) explains the processes between (small) group members when they are working on exercises out of categories (3) and (4) of table 1. Alrø and Skovsmose present the IC-model as a system between a teacher and students. For my own purpose, I adapted it to (equal) exchange between students themselves. Therefore, I named it 'Students' Inquiry Cooperation Model' (SIC-model), while letting the processes between all parties stay the same. The authors' ideas can easily be transmitted to students only instead of a teacher and students. Moreover, I can see all processes taking place during SGL as well. Consequently, I enlarged the model to an appropriate number of small group participants [compare to Al, 2004, 63].

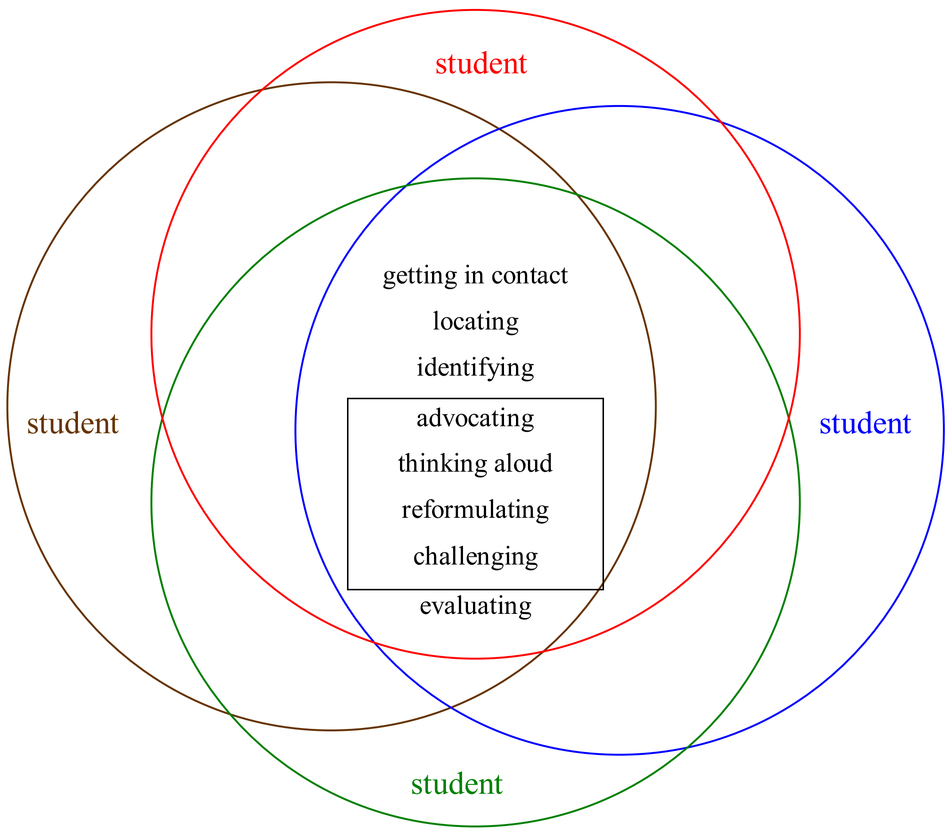


Figure 6: Students' Inquiry Cooperation Model (SIC-model) [compare to Al, 2004, 63]

As stated above, my SIC-model consists of the same activities as the IC-model suggested by Alrø and Skovsmose, namely getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging and evaluating.

First, ***getting in contact*** often takes place faster in a small group than together with a teacher because ‘children speak the same language’. In particular, they are likely to think the same way which explains my phrase ‘children speak the same language’. Tag questions, which end with ‘right?’, often help pupils to get in contact with each other. Lakoff introduces the term ‘tag question’ in order to describe women’s use of language when they avoid asserting their opinions. From my point of view, tag questions have functions as invitation and entering a dialogue, which explains why they are welcome during SGL [compare to La, 1975].
Another common sort of questions during SGL are hypothetical questions which result in ‘I suppose?’. *Hypothetical questions can be understood as a wondering attitude, as an openness and willingness to examine possibilities, instead of just rejecting or leaving the subject* [Al, 2004, 73].
After getting in contact, various perspectives and ways of approaching the task or problem can be defined. *The process of* ***locating*** *is a process of examining possibilities and trying things out. Thus, hypothetical questions like ‘what if’ questions can also be understood in terms of openness and willingness to locate new possibilities. Locating means zooming-in and dwelling on a subject before rejecting it, for instance by questioning and examining an algorithm though it might turn out to be useless* [Al, 2004, 102].
Only when students of a small group are able to express their ideas, they can be identified in mathematical terms. The process of ***identifying*** will lead to further inquiry. What-if questions can be related to the process of locating, while, in many cases, why-questions can be related to the process of identifying. Why-questions lead to justification, too. A proof is the most common and most professional form of justification. In contrast to Alrø and Skovsmose, I see a constantly changing sequence of why-questions and what-if questions. Some pupils ask why-questions before what-if questions, other students do it the other way around [compare to Al, 2004, 104f].
Advocating *can contribute to establishing a shared understanding of a perspective. Advocating means stating what you think and at the same time being willing to examine your understandings and preunderstandings. In that sense advocating involves stating or arguing with the purpose of mutual inquiry of a topic or a perspective in order to clarify what this perspective might involve. This is opposed to claiming, which means trying to convince the other that you are right without being involved in a process of justification. Advocating opens inquiry [...] thus, another important implication of advocating is to keep the focus and dwell on a statement or suggestion and to examine it before it is accepted or rejected* [Al, 2004, 106]. Trying lines of argumentation out in order to illuminate a perspective presupposes

advocating. *Advocating means putting forward ideas or points of view not as the absolute truth, but as something that can be examined. An examination may lead to a reconsideration of perspectives or to further inquiry. Advocating means proposing arguments for certain positions, but it does not mean sticking to a particular position at any cost* [Al, 2004, 63]. I see advocating in relation with 'Proving mathematical ideas, concepts or statements (F)' (see 5.1 A proposal on typology of tasks in school mathematics). In terms of mathematics, advocating means trying out suggestions to prove a statement. Therefore, proving becomes interpreted as a result of a collective process of inquiry. It is to say that pupils enjoy the non-competitive atmosphere during SGL. As a result, they compare their notes and talk about their ideas. This explains the need for advocating [compare to Al, 2004, 63, 106, 120].

Advocating can be seen as a form of ***thinking aloud*** because thinking aloud means expressing one's thoughts, ideas and feelings during the process of inquiry. Perspectives become visible and accessible to collective inquiry by formulating what is inside one's mind. Especially during SGL, hypothetical questions appear as thinking aloud, and serve as an invitation for further inquiry [compare to Al, 2004, 63, 73].

Although ***reformulating*** stands for repeating what has just been said only in a slightly different word or tone of voice, it leads to important (group) processes. It can mean paraphrasing, which focuses the group members' attention to key terms and ideas. Paraphrasing shows that one participant has heard what someone else said and serves as an invitation for further reflection. *In that way the participants can confirm a mutual understanding or conversely, they can become aware of differences that need to be clarified* [Al, 2004, 108]. 'Paraphrase plus' stands for adding ones own contribution to enlighten the paraphrase in order to promote further cooperative examination and has been suggested by Stewart and Logan [Stew, 1999[7], 229]. Luckily, SGL gives space for pupils to add contributions to their colleague's ideas. *A reformulation can be initiated by check-questions through which one can check up with other group members whether they understood each other in the right way, or one can check whether his/her ideas are really the case. Check-questions can serve as a tool for clarification that is important in any process of advocating as well as in an inquiry process in general. Closely related to the term of reformulating and of thinking aloud is the completing of utterances* [Al, 2004, 108].

Challenging often pushes into a new direction. It can happen through hypothetical questions starting with a 'what-if'. What-if questions can be associated with locating. Locating an alternative perspective can mean a strong challenge to all group members. In my interpretation of the SIC-model, advocating, thinking aloud, reformulating and challenging occur almost at the same time. Therefore, I framed them in Figure 6 [compare to Al, 2004, 64, 102].

Although I adopted Alrø and Skovsmose's model to SGL, I didn't exclude ***evaluating*** as the final process between all parties. Even students can evaluate themselves without paying attention to a teacher as a given. Correction of mistakes, negative critique, constructive critique, advice, unconditional support, praise or new examination are examples for several ways of evaluation in a small group. In general, an evaluation can be made by others and by oneself. Questions like 'Did we see the same problem?', 'Did we look at the problem from the same point of view?' or 'Did we try to solve the task the same way?' can be seen as the starting point for an evaluation [compare to Al, 2004, 64].

Alrø and Skovsmose summarise the significant features of the IC-model's stages by saying that *getting in contact involves inquiring questions, paying attention, tag questions, mutual confirmation, support and humour. Locating has been specified with the clues of inquiring, wondering, widening and clarifying questions, zooming in, check-questions, examining possibilities and hypothetical questions. Identifying involves questions of explanation and justification and crystallising mathematical ideas. Advocating is crucial to the particular trying out of possible justifications, and it is closely related to arguing and considering. Thinking aloud often occurs as hypothetical questions and expression of thoughts and feelings. Reformulating can occur as paraphrasing, completing of utterances and staying in contact. Challenging can be made through hypothetical questions, examining new possibilities, clarifying perspectives, and it can be a turning point of investigation. Evaluating implies constructive feedback, support and critique* [Al, 2004, 110].

Students have to activate or even learn several abilities and attitudes to ensure the successful completion of all stages of my SIC-model. Berry and Sahlberg describe two types of problem solving, which take place during SGL. *On the one hand they* [pupils] *are attempting to solve a mathematical problem and on the other they are using their social skills of working interactively and productively together* [Ber I, 2002, 8].

The following list of selected human features combines 'new' abilities for SGL, which normally don't occur during traditional mathematics lessons, and 'old' abilities, which have to be available during SGL as well as during traditional learning. I italicized all 'new' abilities [compare to Hu, 2003, Al, 2004, Ber I, 2002, Ber II, 2003].

- *It seems likely that self regulation is integral to active learning* [Jon II, 1999, 7].
- Social skills, namely *persuading*, listening, *encouraging.*
- *Staying power.*
- *Self-confidence and confidence in the remaining group members.*
- *Ability to discretly search for helpful materials.*
- Ability to express one's own thoughts and ideas in detail.
- *Ability to prove and explain single steps of one's own solution.*
- *Ability to describe one's own thoughts and ideas in numerous ways to reach all group members.*
- *Being creative, especially as far as task solutions are concerned.*
- Ability to remember and use knowledge one has already achieved to develop new knowledge.

[Compare to Hu, 2003, Al, 2004, Ber I, 2002, Ber II, 2003]

4.2 Dialogue and learning

The qualities of communication in the classroom influence the qualities of learning mathematics [Al, 2004, 1].

The act of verbalising is associated with bringing the subconscious into the conscious and the consequent development of reflective awareness [Jon II, 1999, 8].

The main condition of communication is active listening. *It is called 'active' because the listener has a very definite responsibility. He does not passively absorb the words which are spoken to him. He actively tries to grasp the facts and the feelings in what he hears, and he tries, by his listening, to help the speaker work out his own problems* [Rog, 1969, 481]. Rogers and Rarson mean asking questions, giving non-verbal support, while finding out what the other is getting at, and students, who are getting in contact, as essential parts of active listening. Davis uses the term ‘hermeneutic listening’ for a similar activity, which includes participation in the *unfolding of possibilities through collective action* [Dav, 1996, 53].

Stewart and Logan present three different types of listening:
reflective-analytical listening (1), emphatic listening (2) and dialogic listening (3). The first pole is reflective, analytical listening where you reflect and try to understand your associate’s statements in relation to your own conceptions. The second pole is emphatic listening where you are about to understand your colleagues and their way of seeing things. Most important for SGL, dialogic listening means being more interested in creating a meaning than in being right. To sum it up, reflective-analytical listening is related to oneself, emphatic listening is related to group members and dialogic listening focuses on both parties in a mutual process [compare to Stew, 1999[7]].

Although learning takes place in the social context of interpersonal relationships, it is personal as well. At this point, SGL covers both aspects of learning. In a small group, pupils are able to take themselves back of a discussion at certain times without getting reproved for not speaking their thoughts out loud. Moreover, pupils who are working in a small group decide by themselves when they are about to enter the common discussion again. Overall, getting in contact with colleagues is highly important as the context in which people communicate affects what is learned by all parties.
As soon as we look back in history, we realize the relevance of communicating ideas. For example, Plato presented his ideas as dialogues, Galileo Galilei wrote ‘The Dialogue Concerning the two Chief World Systems’ in 1632 and Imre Lakatos invested the logic of mathematical discovery in the form of a dialogue taking place in an imaginary classroom [compare to Al, 2004, 2].
In philosophy, dialogue refers to the presentation and confrontation of different points of view, with the aim of identifying a conclusion that all parties can accept. ‘Menon’ seems to be the most familiar dialogue in philosophy, partly because it was written by Plato to describe a certain style of teaching. He asks the protagonist Menon appropriate questions to make him realize how to double the area of a square. This way, he wants to show that knowledge resembles memory while he does not exactly pass information to Menon. The dialogue shows that knowledge cannot be given from one person to another. The addressee has to come up with essential ideas himself while answering adequate questions. The same process appears during SGL. *Plato is in accordance with modern educational theory as, for example, radical*

constructivism […] which also claims that knowledge cannot be transferred from one person to another. According to Plato, and this is different from radical constructivism, knowledge already exists within the person, but knowledge is 'slumbering'. Plato claims that all kinds of knowledge exist beforehand within the person, but that the person has forgotten what he or she somehow already knows. The right method of teaching is, consequently, to bring the person to remember what he or she already knows [Al, 2004, 114]. This quality of learning verifies the need for SGL. When pupils work without continuous help from a teacher, they tend towards not knowing an entire answer for quite a long time. Instead, they try to get as close to an answer as feasible. Hence, they question their ideas continuously to proceed as much as possible. Listening to pupils of a small group reminds me of the Socratic dialogue. When 'to learn' means 'to remember', nothing seems more natural than making sure that tasks offer enough opportunities for pupils to ask themselves many questions. Such explanations can be used as inspiration for further specification. *Dialogues that are truly dialogic interactions* […] *are 'exploratory, tentative and invitational'* [Li, 1999, 243].

4.3 Funnelling, focusing and scaffolding

Here is a famous word which is absolutely correct:
Anyone who teaches colleagues, teaches himself;
not only because iteration stabilizes knowledge, but also because he deeply invades matters
[Com, 1961, 171f].

The inquirer expresses an orientation toward partner and topic that is uncertain and invitational. It is this stance of uncertainty and invitation towards the other that guides the inquirer in the news land and in his or her seeking for help and assistance [Al, 2004, 119].

Jones and Tanner explain two *different forms of interaction which might be described as scaffolding to support pupils' learning: funnelling and focusing* [Jon II, 1999, 5]. Different from Berry and Sahlberg, I understand funnelling and focusing as preliminary stage of scaffolding. Each SGL calls for funnelling and focusing. In **funnelling** it is the person with most knowledge, *who selects the thinking strategies and controls the decision processes to lead the discourse to a predetermined solution*. [In **focusing** the attention is drawn to] *critical features of the problem which might not yet be understood. The pupil is then expected to resolve perturbations which have thus been created. [...] Pupils as well as the* [scaffolder] *are* *expected to question assumptions and make conjectures and the responsibility for identifying strategies and making decisions is devolved to the pupils. The achievement of higher levels of understanding and thoughtfulness in mathematics are claimed for such discourse-based models of teaching* [Jon II, 1999, 5].

I start by claiming that scaffolding takes place during every kind of SGL. Therefore, I have to explain the meaning of scaffolding. Bruner calls a competent peer (or adult), who acts *as a vicarious form of consciousness until such time as the learner is able to master his own action through his own consciousness and control* [Bru, 1985, 24f], a scaffolder. As Askew and his colleagues state, *scaffolding in the classroom must be* [...] *dynamic and ways must be found for the teacher and pupils to work jointly on activities* [As, 1995, 216].

Alrø and Skovsmose list pivotal acts of inquiry which can be seen as characteristics of scaffolding, namely explaining, elaborating, suggesting, supporting and considering consequences. *They are identified as inquiry acts because they are attempts to go beyond, and help others to go beyond their present thinking* [Al, 2004, 119].

My SIC-model (see 4.1.2 The Students' Inquiry Cooperation Model (SIC-model)) can be seen as scaffolding that takes place in the zone of proximal development (see 7 Zone of Proximal Development by Vygotsky), [compare to Al, 2004, Vyg I – Vyg IV, 1962, 1978, 1981, 1991].

4.3.1 Mathematical Thinking Skills Project by Jones and Tanner

In 'Scaffolding Metacognition: reflective discourse and the development of mathematical thinking' [Jon II, 1999], Jones and Tanner report on their Mathematical Thinking Skills Project, which they initiated to develop and evaluate certain thinking skills to accelerate students' cognitive development in mathematics. *The course was based on the results of the earlier Practical Applications of Mathematics Project* [Jon I, 1994], *which had identified the metacognitive skills of planning, monitoring and evaluating as necessary for practical problem solving and modelling and had reported classroom practices which appeared to be successful in facilitating the development of such skills. These practices included strategies to scaffold pupils' thinking and to encourage reflective discourse. The Mathematical Thinking Skills Project aimed to evaluate the effect of a course based on such strategies in a quasi-experiment [...] We* [Jones and Tanner] *wished to explore the extent to which metacognitive* *knowledge and skills could be taught, the teaching approaches which were most effective and whether a focus on metacognition would lead to improved mathematical performance in other areas* [Jon II, 1999, 1].

In the course of the Mathematical Thinking Skills Project, Jones and Tanner distinguished between four different teaching styles, namely *taskers* [1], *rigid scaffolders* [2], *dynamic scaffolders* [3] *and reflective scaffolders* [4] [Jon II, 1999, 14].

Jones' and Tanner's study shows the difficulties SGL can cause for a teacher and students not used to such learning styles. Jones and Tanner report that initially the *taskers* [1] represented a constructivist teaching style by thinking that they shouldn't interfere. They gave the impression that their students either have to sink or to swim. *In the later stages the teachers* [or scaffolders] *provided more structure especially for the planning stage, but the attention of the pupils was not drawn to the underpinning strategies and many pupils remained focused on the superficial objectives of the task rather than the intended learning objectives* [Jon II, 1999, 14].

The rigid scaffolders [2] *were far more directive in their approach to planning* [Jon II, 1999, 14]. Still, they concentrated on their own previously identified plan rather than helping group members to develop their own plan. They forced pupils to follow their plan *as if it were a recipe* [Jon II, 1999, 14]. Although organisational prompts, i.e. *questions designed to help pupils to develop a framework to organise their thoughts* [Jon II, 1999, 14], were used, rigid scaffolders used these prompts to constrain pupils' thinking and to funnel them down to a pre-determined path. These two groups of scaffolding, namely *taskers* [1] and *rigid scaffolders* [2], were the least successful ones in the Mathematical Thinking Skills Project. In contrast, the *dynamic scaffolders* [3] and the *reflective scaffolders* [4] were more successful [compare to Jon II, 1999].

The dynamic scaffolders [3] *made full use of the social structure of Start-stop-go* [Jon II, 1999, 14]. *One key strategy was referred to as Start-stop-go in which pupils were asked to read the problem, think in silence for a few minutes, and then discuss possible plans in small groups before the teacher led a brainstorming session which focused attention on key features. The intention was to constrain pupils to act as experts rather than novices by slowing down impulsive behaviour and encouraging the examination of several problem formulations. After the class had started work they were stopped at intervals for groups to report on progress; this was intended to encourage monitoring by pupils of their progress* *against their plans, and also against the progress of others* [Jon II, 1999, 9]. Therefore pupils were allowed being significant autonomy, especially in the early stages of planning.

Their scaffolding was dynamic in character and was based on participation in a discourse in which differences in perspective were welcomed and encouraged. [Still, the students' participation was limited by the scaffolders' desire] *to negotiate a plan to a pre-determined template.* [We call it] *a legitimate peripheral participation.* [In other words, the scaffolders took overall responsibility for the design. Both procedural knowledge, like strategies and processes, and conceptual knowledge were forced by dynamic scaffolders.] *During the planning and monitoring sessions, articulation and objectification of explanation was encouraged, i.e. the explanation itself became the object of the discourse. This was the only form of evaluation or reflection used by the dynamic scaffolders, however, and is characterised as 'reflection in action' as opposed to 'reflection on action'* [Jon II, 1999, 14f]. In the end, dynamic scaffolders *were very effective in teaching for near transfer,* [but] *they failed to achieve far transfer* [Jon II, 1999, 17].

Finally, the *reflective scaffolders* [4] were most successful. They used the social structure of Start-stop-go and *granted their pupils more autonomy* [by] *encouraging several approaches to the problems rather than constraining the discourse to produce a class plan. Pupils thus had to evaluate their own plans in comparison with the other plans in the posing, planning and monitoring phases of the lessons. The participation framework required pupils to draw on the help of the expert but to take a* ***greater responsibility*** *for an end product of their own design [...] The characteristic feature of the reflective scaffolders, however, was their* ***focus on evaluation and reflection*** [Jon II, 1999, 16].

5 Tasks

In England, public examinations often provide clear instructions as what to do. This development ignores the intention of SGL, which is developing stamina, stroving for independence, networking and reasoning logically and deductively. For example, where a solution cannot be directly extracted from the formula book, the method is often generally given in the question, like ‘Evaluate the integral [...] by using the substitution [...]’. Students are thus mislead as they think that mathematical problems can be solved by applying a recipe shown by a teacher and mathematics only involves carrying out one-step routine procedures. There is no doubt that such procedures are inappropriate for SGL [compare to Al, 2004, 45f].

Alrø’s implementations lead to an essential question: Which type of tasks is appropriate for SGL? Therefore, I make an attempt to summarise all tasks in school mathematics and to test them on their suitability for SGL (see 5.1 A proposal on typology of tasks in school mathematics), [compare to Ber I, 2002, 11].

In general, the role of tasks in teaching and learning mathematics has major influences on the nature and view of mathematics held by the teacher and learner. For example, pupils often have a narrow view of how to solve mathematical problems as a teacher only offers one solution because of her/his time management. Instead of realising that various ways can lead to the solution, students tend to give a task up as soon as the ‘traditional’ way of solving the problem doesn’t work. The most effective way to avoid such a situation is to encourage lively discussions among all small group members. Widening a student’s view of mathematical problem solving and initializing a discussion can be trained in traditional mathematics lessons as well as during SGL. From my point of view, many tasks presented in mathematics lessons should contain the request ‘Try to find more than one solution!’. Discussing various solutions instead of writing them down, which can claim too much time during a school lesson, is the first step [compare to Ber I, 2002, 5f].
As Berry and Sahlberg state, a small group task should be designed in such way that requires team effort. Experienced scholars like Sahlberg and Berry [Ber I, 2002], [Ber II, 2003], Sharan [Sh, 1999[2]] and Cohen [Co, 1994[2]] say that in most of the situations, when groups do not seem to work well, the reason is an inappropriate learning task, that is based on individualistic modes of teaching and learning. This is especially true in school mathematics. Most of the tasks and exercises that are available in textbooks and additional materials are

designed for individuals, not for teams. They contain procedural questions, i.e. questions that can be answered by a simplistic rehearsal of a rule, method or formula. ‘Calculate the mean of a set of numbers’ is one example for procedural questions. This implies that these tasks do not necessarily challenge a small group to work together towards a common goal. In contrast, conceptual questions require the use of (unusual) ideas and rules or methods committed to memory. These questions are suitable for small group learning, for example ‘A curve has the equation $x^3 - xy + y^3 = c$ where c is a constant. Prove that if $27c < -1$ then the curve cannot have a tangent parallel to the x-axis’. The fact that this exercise turns the way pupils are introduced to the differential calculation upside down makes it suitable for SGL. Not all tasks are suitable for cooperative learning purposes. According to Berry and Sahlberg, solving simple equations does not require a group to complete the assignment successfully. I agree that the methods used to solve a simple equation don’t require creative thinking, but understanding why certain transformations of the equation are legitimate calls for the ability of abstract thinking. Pupils have to understand that (and why) transforming each side of an equation the same way is permitted. Such abstract thinking can be discussed in a small group. Similar to Berry and Sahlberg, I see the preparation of a plan or the investigation of the properties of numbers as fruitful opportunities for real cooperation to flourish in most, if not in all small groups [compare to Ber I, 2002].

Sharan and Sharan invent three key conditions that a task appropriate for cooperative learning sessons has to fulfill, namely stimulating and promoting cooperation (1), activating all pupils to participate (2) and leading to productive learning (3). In other words, all members of the group have reasons to participate, the task provides all group members opportunities to talk and group members need to make choices and decisions [Ber I, 2002, 6f].

First, the most important guideline in designing an interactive learning task is to encourage every group member to participate. It is obvious that any good learning task should be interesting to students and connected to their previous knowledge and skills. Second, studies of experienced scholars like Cohen [Co, 1994[2]] show that the amount of productive talk positively correlates with the quality of learning, i.e. the more pupils talk and communicate about what they are studying the more they learn about it. The task should pose a problem or question that invites more than one solution or answer. That is why open questions like intentional problems [Hu, 2003, part 2] or complex problems are particularly suitable for cooperative learning purposes. A task that promotes interactive talk is one that engages

students in sharing their ideas, exchanging information, making choices and decisions in terms of procedures, working methods and even optimal solutions to the problem.
Third, as soon as students encounter a situation in which there is no one way forward they need to plan and decide what is the best think to do. Making choices and decisions helps pupils relate what they study. Furthermore, the task should provide somethink to do or say for all, including those who might not be able to complete the entire task. Tasks for SGL truly challenge the group to think what they should do in order to complete the task. I claim that the following statement about features that make tasks suitable for SGL is the most important one: *Everybody knows something about the task in hand, but nobody knows everything about it* [Ber I, 2002, 7].

To summarise the needed qualities of tasks for SGL, Berry and Sahlberg state that such tasks have to encourage as much talk as possible [Ber I, 2002, 32f]. Like many other educators, they realise that such tasks don't dominate school textbooks. Asked for the reasons, they list two major ones. First, textbook writers seem to be pandering to the needs of teachers, parents and school authorities that pupils need to be prepared for tests. *Problem solving, investigational and mathematical thinking skills are difficult, if not impossible, to develop and assess in a regime of timed tests* [Ber I, 2002, 32]. Second, it's not enough to change the range of tasks to fit SGL in mathematics, because the use of tasks appropriate for SGL is different to traditional task use. *For example, [...] mathematical modelling tasks (such as 'Length of a toilet roll paper') could be included as an exercise in the appropriate section of a mathematics textbook, but its use as a closed task with each pupil working alone does not encourage the rich exchange of ideas between pupils that is possible when used as a group task. Simply including many open ended tasks in a textbook is unlikely to change classroom practice. It is essential to develop resources, such as teacher guides, that lead teachers through using such tasks in ways that encourage all the good ingredients of small group learning* [Ber I, 2002, 33].

5.1 A proposal on typology of tasks in school mathematics

Berry and Sahlberg propose a tentative typology of group tasks in school mathematics. It is essential to note that I added my own ideas to their table and italicized these parts. The motivation for my changes is registered at the end of each category. Moreover, I alphabetized all categories instead of numbering them consecutively. Therefore, I think of different categories as belonging to each other than Berry and Sahlberg. For instance, I disagree with Berry and Sahlberg who claim that 'real problem solving' (equal to category (B) according to my typology) belongs to 'applying a formula or algorithm' (A2) and 'measuring and collecting data' (A3). From my point of view, category (B) is far too extensive to relate it to (A2) and (A3) because those categories require rather basic skills. In addition, I put the categories in ascending order according to the mathematical and social skills, which are called for. Hence, I put mathematical investigations before mathematical modelling. I attached the subcategory 'suitability for SGL' on my own to emphasize the conductive relation between mathematical tasks and small group learning [compare to Ber I, 2002, 11].

All mentioned tasks of the following list are closed in terms of the description of the problem because they include questions. An educated person may ask himself/herself how to solve a mathematical problem which doesn't mention any questions. Hußmann [Hu, 2003, part 2] gives examples for the idea of open mathematical tasks. He asks the authors of mathematical tasks not to mention any questions but to let the students think of their own questions while reading the problem. These tasks are called 'intentional problems' (see 5.2 Intentional problems of the integral calculus by Hußmann), [compare to Hu, 2003, part 2].

5.1.1 Drilling basic skills (A1)

Description: examples and exercises that are traditionally found in mathematics textbooks and other classroom resources. They usually provide students with opportunities to practice the procedures, rules and skills of traditional mathematics curricula. *Students don't have to come up with own ideas as they achieve the goal by using the methods shown by a teacher.*

Nature of task: closed in terms of method and outcomes

Suitability for SGL: Suitable only if the category is new to children. In this case the task has to contain several subtasks to split them up between the members of a group. Only if pupils have to solve the task in an unusual way or they can deal with it in different manners, they will benefit from working on it in a small group. For example, such a task should contain several equations where some can only be solved graphically and others can be solved both graphically and numerically (see 4.1.1 Examples for small group learning (SGL), (2) Paradigm of exercises. References to a semi-reality). Otherwise the task would be too easy and would take too little time to cause a discussion or fruitful interaction.

Motivation for addendum: By noting the fact that pupils don't have to come up with own ideas while solving tasks of category A1, I highlight how constitutional these tasks are. Still, they are necessary to give students the opportunity to gain routine in working for instance with (in)equations [compare to Ber I, 2002, 11].

5.1.2 Applying a formula or algorithm (A2)

Description: see A1

Nature of task: typically closed in terms of outcomes and also the methodology

Suitability for SGL: see A1 [Ber I, 2002, 11]

5.1.3 Measuring and collecting data (A3)

Description: an aggregation of A1 and A2, activities and tasks which are normally well-defined in terms of what to do. *The more methods and instruments for measuring and collecting data are available the easier these tasks get.*

Nature of task: some openness in terms of methodology but rather closed in terms of outcomes

Suitability for SGL: discussion will only occur if pupils cannot remember which methods and algorithms to use [compare to Ber I, 2002, 11]

5.1.4 Archetyps of real world problems (B)

Description: tasks that provide opportunities for students to use their procedures, rules and skills to solve world problems of a traditional or familiar nature

Nature of task: Real problems are those encountered in everyday life. They may or may not involve mathematical models. Openness of these tasks may vary from rather closed to wide open.

Suitability for SGL: Pupils need a reliable understanding of how to use familiar mathematical concepts in new and unusual situations. Therefore, they may have to look at the problem in various ways and some of them may even vary from those they have already learned.

Motivation for change: I named category (B) ‘archetyps of real world problems’. Tasks out of the real world would be far too complicated to address in a classroom. As a result, the tasks used at school are simplified versions of real world problems and may not even be presented to employees in the workplace nowadays because computers are faster in solving them. Keeping this fact in mind, teachers have to know that it may be difficult to give reasons why students should solve such problems anyway. Nevertheless, omiting these tasks, which is a possible alternative, doesn’t turn pupil’s attention to a rudimentary mathematical skill namely modelling. Consequently, students may not be able to reduce real world problems to appropriate models which can be dealt with. These considerations lead to my final statement about real world problems: Although the name should be changed into ‘archetyps of real world problems’ to make sure that pupils don’t get a wrong impression, teachers must not omit these tasks [compare to Ber I, 2002, 11].

5.1.5 Mathematical Investigations (C)

Description: tasks that often require students to explore their own conjectures in order to meet some criteria. Mathematical investigations are pure mathematical puzzle type problems that involve the exploration of mathematical problems which do not necessarily involve real applications. Investigations provide opportunities for pupils to express and explore ideas. They encourage them to follow their own lines of enquiry.

Nature of task: Basic investigations are often closed in terms of outcomes but open to various methods. Extended investigations are typically open tasks.

Suitability for SGL: The openness of these tasks often leads to lively discussions about various methods. Moreover, the challenge of these tasks appears when pupils have to write their intuitional ideas down in a correct mathematical way. Such challenges give several opportunities for interaction [compare to Ber I, 2002, 11].

5.1.6 Mathematical Modelling (D)

Description: an aggregation of B and C, tasks that often require students to develop their own models in order to meet some criteria. They often provide good opportunities for students to develop problem solving skills that are useful in showing the role of mathematicians in the workplace.

Nature of task: Modelling tasks are typically real problems that require mathematical principles and formulas. These tasks are open in terms of procedures and outcomes.

Suitability for SGL: Because of the tasks' relation to the 'real world' without entering the level of abstraction right away, they arouse the students' interest and almost guarantee rich exchange in a small group [compare to Ber I, 2002, 11].

5.1.7 Designing projects and studies in mathematics (E)

Description: mathematical tasks providing students with opportunities to define their own framework for the problem which could be of a mathematical modelling or pure mathematical investigation type of activity.

Nature of task: Projects and studies are an important category of open mathematical tasks. The openness includes the setting of questions and selection of methods. *Still, projects give you a quite detailled instruction of what you should come up with in the end.*

Suitability for SGL: Those tasks are particularly suitable for SGL because enough work can be found for everyone [compare to Ber I, 2002, 11].

5.1.8 Proving mathematical ideas, concepts or statements (F)

I adjoin this final category named 'proving mathematical ideas, concepts or statements', which is most suitable for SGL and is not mentioned by Berry and Sahlberg.

Description: tasks that force pupils to combine familiar knowledge from different areas to verify a statement. They don't give the reader a hint of what to do apart from which mathematical statement to prove. They unify not only organizational thinking, but all mathematical skills which should be learned in high school, namely calculating, which is the basic mathematical skill, making as few mistakes as possible because they have the biggest and worst bearing on proofs (every single step is related to the next one), motivating one's ideas, recalling certain aspects of one's knowledge, combining knowledge from different areas, being creative, showing staying power and focusing.

Nature of task: most possibilities in certain cases as far as method is concerned.

Suitability for SGL: some mathematical proofs take years for only one person to solve. Even those proofs, which are suitable for school mathematics, often require pupils to recall information they got from their mathematics teacher over many years. The processes of a small group force students not only to prove that one idea is right, but also to control why another one may be wrong or also leads to the goal.

Motivation for addendum: I refer to (F) as a category, which combines most mathematical and social skills. Dealing with mathematical proofs allows pupils to immerse themselfes in working step by step and also maintaining the outline no matter how interminable a task may be [compare to Ber I, 2002, 11].

5.1.9 Examples for SGL

My assumption indicates that a task's structure and its interestingness, for which reason it should arouse many questions in pupil's minds, is far more important than the category it belongs to. In the course of my inquiries, I choose one example for each category which is suitable for SGL. Most of the following examples, namely those for A1, A2, C, D and E, are mentioned by Berry and Sahlberg. As long as I don't come up with an example myself, I state the reference at the end of a task [compare to Ber I, 2002, 11-13].

Drilling basic skills (A1)

See 4.1.1 Examples for small group learning (SGL), (2) Paradigm of exercises. References to a semi-reality

Need for SGL

Pupils have to recall transformation rules to receive the price either dependent on the time or not. If they aren't able to remember those rules, other members of a small group can be helpful. In addition, they have to interpret their results by thinking of a realistic situation to find a meaning for all those numbers and variables. Interpretation is always open in terms of outcomes. Hence, it allows rich exchange in small groups. Consequently, I used T for time and P for price instead of x and y as they would be pretty meaningless in such an example.
I would expect students to give explanations similar to the following ones:

- The first, second and third equation tell you the price P of a car or something else. As you can see, time doesn't play a role which means that the price doesn't change over time.
- In the fourth case you already bought a car some time ago. Imagine you want to sell it by putting an advertisement into your local newspaper. The money you will receive for your car (P) will decrease with time because the possibility for damages increases with time. Therefore, T stands for months, which passed since the date you bought the car.
- The fifth equation cannot describe the value of your own car as the price increases with time. It may demonstrate the value of an antique piece of furniture, e.g. an old chair [compare to Ber I, 2002, 11].

Applying a formula or algorithm (A2)

Sara's shoes: The initial price was €48.50 and there was a 30% sale in that store. They gave an additional 20% discount on shoes. How much money did Sara need for her shoes?
Before you start calculating the final price of the shoes, ***discuss*** in your group whether you think that they are cheaper the way described above than after a 50% sale right away. Check your assumption by calculating the price after both kinds of sale [compare to Ber I, 2002, 11]!

Need for SGL

The discussion clearly calls for small groups to confront pupils with opposite opinions. While talking about their assumptions they should receive an intutional feeling for percentage calculation without actually calculating right away. Finally, they may be surprised by their results [compare to Ber I, 2002, 11].

Measuring and collecting data (A3)

The table and the diagram below show the tax values of an engine in the year it was bought and several years afterwards. They carry out a constant fiscal disprofit.

(1) Study the table and the diagram to verbalize an algorithm to calculate the value of the engine at different times!

(2) Let the variable *t* stand for the time after buying the engine in years. It can be replaced by one of the numbers 0, 1, 2, 3, 4, 5 or 6. What's the value of the engine at data *t*?

(3) Using the data, estimate when the value of the engine is 6 000 € [compare to Bue I, 2000^2, 51]!

years after buying the engine	value in €
0	60 000
1	54 000
2	48 000
3	42 000
4	36 000
5	30 000
6	24 000

Table 2: The engine's value [Bue I, 2000^2, 51]

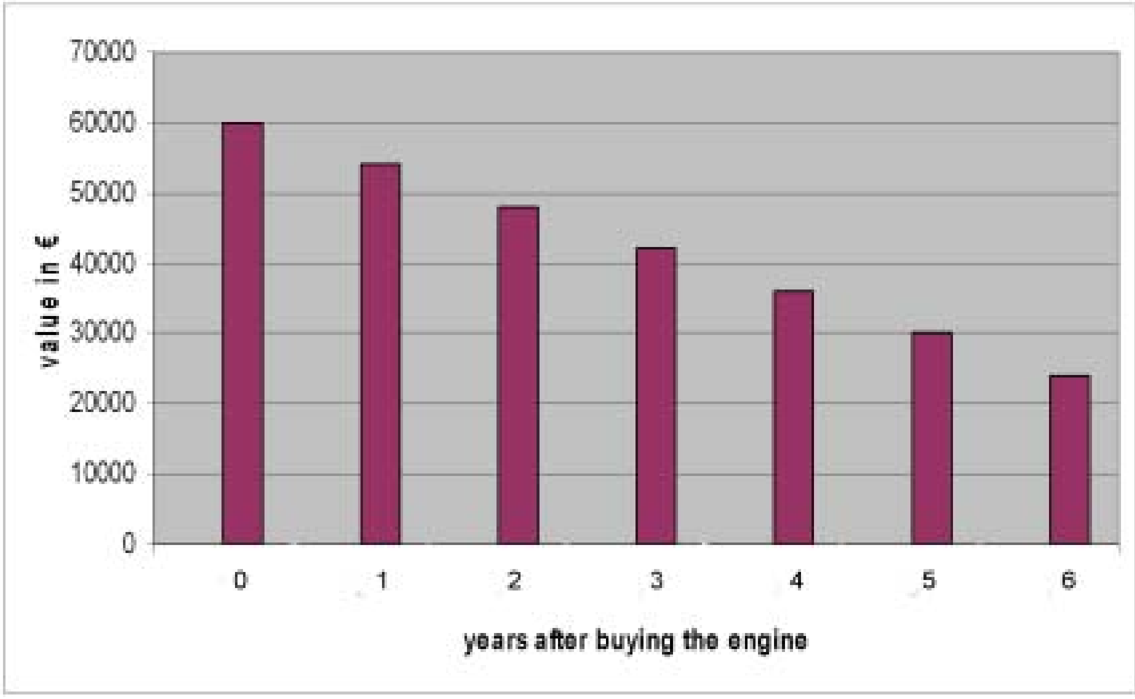


Figure 7: visualization of the engine's decreasing value [Bue I, 2000[2], 51]

Need for SGL

The task's layout may entice small group members to divide it up amongst all participants. As soon as they come together after everyone finished his/her individual work, they will realize that those who worked on the third statement had to answer the first and second question in their mind as well. Accordingly, students should get a feeling for whether it makes sense to divide a task up.

Archetypes of real world problems (B)

Zwei Sportvereine A und B haben je eine Faustball- und eine Volleyballmannschaft. Man weiß aus den bisherigen Spielen, dass die Faustballmannschaft von A gegen jene von B mit der Wahrscheinlichkeit 0,32 gewinnt, während für die Volleyballmannschaften die entsprechende Wahrscheinlichkeit 0,58 beträgt. An einem Tag spielen die Mansnchaften der beiden Vereine gegeneinander.

- *a) Stelle alle möglichen Ausfälle durch ein Baumdiagramm dar!*
- *b) Verein A würde gerne wenigstens eines der beiden Spiele gewinnen. Wie groß ist die Wahrscheinlichkeit dafür?*
- *c) Berechne die Wahrscheinlichkeit, dass ein und derselbe Verein beide Spiele gewinnt* [Bue II, 2000[3], 285]!

Need for SGL

Similar to 'Measuring and collecting data (A3)', this example may be divided up amongst all group members. In constrast to (A3), each subtask can be answered autonomously. Still, a teacher is responsible to ensure that all group members report each other about their results.

Mathematical Investigations (C)

See 4.1.1 Examples for small group learning (SGL), (3) Landscapes of investigation. References to pure mathematics

Need for SGL

Because the task does not narrow the algorithm down to one specific number, small group members can try it out many times until everyone is convinced they found an algorithm for all integers between 1 and 18. Moreover, students get confronted with various ways of solving such a task as some pupils will be able to explain the described phenomenon right away whereas other ones need to write it down to understand what is happening.

Eventually, the members of a small group should describe the algorithm similar to 'I choose an integer $x \in \{1, 2, 3, \ldots 17, 18\}$, multiply the units of x by 2 and add the number of tens of x. As a result, I receive all integers out of $\{1, 2, 3, \ldots 17, 18\}$ in a random order.'.

Students, who are used to working on intentional problems (see 5.2 Intentional problems of the integral calculus by Hußmann), are likely to extend the given task by asking themselves whether the arrangement of integers, which results from the algorithm, is random indeed. They may try to find an algorithm for the arrangement of the resulting integers [compare to Ber I, 2002, 11].

Mathematical Modelling (D)

Take a rope and measure the distance between its ends. Tie an overhand knot in the rope and measure the distance between its ends now. The rope has shortened. Find a relationship between the length of the rope and the diameter of the rope [Ber I, 2002, 13].

Need for SGL

Difficulties might occur when working alone on this task:

- What does a knot look like geometrically?
- Do the elastic properties of the rope make any differences?
- How tight should I make the knot?

Although the last two questions aren't mathematical ones, they may unsettle students much more when working alone than when talking through the problem with colleagues.

Overall, it is always helfpul to work together with colleagues as soon as measuring objects is required [compare to Ber I, 2002, 13].

Designing projects and studies in mathematics (E)

(a) Planning a party

It's your birthday. Plan a party for your friends. Make sure that you arrange for enough food and drink. Find a realistic amount of money you are going to receive from your parents for your party. Be careful to include everything in your estimate of the cost. Decide how you are going to report your work. You may do a poster or a written report [compare to Ber I, 2002, 11].

(b) Travel agency

A Danish family would like to rent a summer cottage or camp in Denmark. They would also be interested in going abroad. They think they prefer going by bus, train or airplane. Would you please help the family find out what the possibilities are? What would they need to know before leaving? What would they need to do?

Additional information

The teacher should emphasise that it could be helpful to use the first ten minutes inventing a family and deciding upon family members, names, age, residence, interests and occupation. Although the teacher doesn't introduce specific exercises, he/she has to prepar different possible problems with mathematical content: travel charges and the estimation of costs, which could be part of a budget planning, maybe using a spreadsheet; distances (scales) and the reading of maps, which would be a natural part of the overall planning of the trip as well as of the choice of the particular routes to be taken; transport time and speed, which have implications for where to book places overnight during the trip; weather conditions (statistics), which may be relevant for a decision about the destination; currency and exchange rates come almost as a given. However, the groups decide upon their own tasks and do their own problem posing/solving. The teacher has to make at least magazines, statistical outlines, atlas, newspapers and material from travel agencies available for his/her students. Independent from the time of year, planning holidays occupies some of the students and their families at the present time. Consequently, holiday planning easily becomes both relevant and realistic. There is no restriction in the possibilities of real-life references to the proposed classroom activities [Al, 2004, 138f].

Need for SGL

Time management is a major argument for using these tasks for SGL. In addition, designing projects in a group gives pupils a realistic idea of what is expected at many working places in real life. It's not just about mathematics, but also about organising one's work as the task only gives you a vague idea of what to do, but not of how to do it. Compared to Berry and Sahlberg's task for category (E), I added 'Find a realistic amount of money you are going to receive from your parents for your party'. The extention's purpose is to show members of a small group how different their ideas of 'enough money for a party' may be [compare to Ber I, 2002, 11].

Proving mathematical ideas, concepts or statements (F)

See 5 Tasks, page 30, first paragraph.

Need for SGL

I listed several reasons for working on a mathematical proof in a small group when I described category (F) of my typology of tasks in school mathematics (see 5.1 A proposal on typology of tasks in school mathematics), [compare to Ber I, 2002, 11].

5.1.10 A study design by Berry and Sahlberg

To confront pupils and teachers with the typology of mathematical school tasks, Berry and Sahlberg designed a study both for England and Finland. Their initial hypothesis consisted of the following aspects. First, they wanted to prove that *if small groups are used in conjunction with the careful redesigning of mathematical tasks, then pupils have more opportunities and reasons to work and learn together.* [Second, they] *have established a position that small groups are not used according to the basic elements that promote higher quality interaction in mathematics lessons*. [They wanted to gain] *a better understanding of what are the conditions that make small group learning successful in mathematics and what teachers and pupils think about various types of mathematical tasks that are regularly used in [...] inservice training courses.* [In the following study, they] *wanted to know what kind of tasks teachers use in small group learning situations in primary (years five and six) and secondary school (years eight to ten) mathematics teaching in Finland and England.* [The purpose of Berry and Sahlberg's study is to] *draw more educators' and researchers' attention to the importance of task design and reformulation in small group learning in school mathematics* [Ber I, 2002, 15].

[Berry and Sahlberg started their work by asking the following questions]

1. *Do teachers classify mathematics tasks according to the different categories identified in the typology?*
2. *How do pupils classify mathematics tasks in terms of using group learning to solve them?* [Ber I, 2002, 15].

Both questions will be answered in a proper way later on by displaying appropriate diagrams.
Berry and Sahlberg argue for the need of their study by saying that *good learning will only succeed if appropriate tasks are used that encourage conceptual understanding, mathematical thinking as well as skill development.* [They prepared their study by developing] *a series of questionnaires and interviews to collect both quantitative and qualitative data.* [They] *wanted to find out about the use of small group learning in a 'naturalistic' way* [... They] *formed a project team consisting of the two authors (the lead researchers) and two research assistants who conducted the interviews with the teachers using a carefully prepared and piloted interview proforma* [...] *The questionnaire had four parts: (a) background data, (b) teaching styles in mathematics, (c) use of small group learning in mathematics, and (d) a selection of mathematical tasks for various purposes. The sample consisted of two kinds of teachers: those*

who teach mathematics in primary school (year 5/6), and those who teach mathematics in secondary school (years 9/10). [Berry and Sahlberg] *selected three municipalities, two in Finland and one in England and sent the questionnaires in March 2001 to the school principals in all schools. A total of 77 teachers responded, of which 37 were in England and 40 were in Finland. About 23% had more than 15 years of teaching experience, whilst 17% had less than four years teaching career. Furthermore, 61% of the sample represents subject teachers, the others are class teachers except one who is not qualified to teach. Out of the total sample 58% teachers were women* [Ber I, 2002, 16f].

The following tasks were used for this study.

(A1) Practice the rule

See 4.1.1 Examples for small group learning (SGL), (1) paradigm of exercises. References to pure mathematics

(A2) Sarah's shoes

See 5.1.9 Examples for SGL

(A3) Books of stamps

In 1985 a book of stamps cost £1. First class stamps cost 17p and second class stamps cost 13p. How many different ways could the book have been made up? Which do you think was the most useful?

What are prices of stamps and a book of stamps today? How many different ways could the book have been made up [Ber II, 2003]?

(B) The drink can

A simple model of a drink can is a cylinder of radius r with circular ends. If its volume is 330 ml, find an expression for h, the height of the can, and hence an expression for A, the total surface area of the can. Find the dimensions of the can that has the smallest surface area for this volume. Comment on your answer [Ber I, 2002, 11].

(C) Bracelets

See 4.1.1 Examples for small group learning (SGL), (3) landscapes of investigation. References to pure mathematics

(D1) Length of a toilet roll

You have a roll of kitchen paper and one sheet of identical paper. Estimate the number of sheets that your kitchen roll has in total [Ber I, 2002, 11].

(D2) **Ropes**: see 5.1.9 Examples for SGL

(E) **Planning a party**: see 5.1.9 Examples for SGL

5.1.10.1 **Teachers' ranking of best and worst task for SGL**

Type of task	Best task for SGL		Worst task for SGL	
	Primary (%)	Secondary (%)	Primary (%)	Secondary (%)
Mathematics problem (A1)	0	0	69	47.9
[Arechetyps of] real [world] problems (B)	48.3	31.2	17.2	8.3
Mathematical investigation (C)	48.3	66.7	0	4.2
Mathematical modelling (D1)	3.4	2.1	13.8	39.6

Table 4: Teachers' ranking of best and worst task for SGL [Ber I, 2002, 19]

Mathematics problem task was a set of linear equations to be drawn in the frame of reference. The second, a real problem solving task was a direct application of real data to equations. The third task [was a mathematical investigation (C) on the properties of numbers]. *Finally, the fourth task was a mathematical* [modelling activity (D), which required finding out how many sheets of paper are in a kitchen roll]. *According to teachers, the most appropriate tasks for small group learning purposes are mathematical investigations* [C] *and real problems* [B]. *Two thirds of secondary school mathematics teachers and about half of primary school teachers claimed that mathematical investigations are most suitable for small group learning. One third of secondary school mathematics teachers and half of primary teachers chose* [archetyps of] *real* [world] *problems as most appropriate for cooperative mathematics. Furthermore, most teachers suggested that routine mathematics problems are not as suitable for small group learning as the problems in the other categories.* [Sadly,] *about 40% of secondary school teachers thought that the mathematical modelling task is almost as bad a choice as the routine mathematics problems for using as an interactive, rich equal exchange task* [Ber I, 2002, 19f].

Berry and Sahlberg summarise the teachers' reasons for a task being suitable for SGL:

- *There is a need for discussion and listening to others*
- *Open ended with many routes through the problem*
- *There is a need to find a rule and explore 'what if...?' questions*
- *Everyone can take part in the problem*
- *There are practical, hands-on activities*
- *The task is interesting, fun or there is some novelty*
- *The task is difficult to do alone* [Ber I, 2002, 27].

5.1.10.2 **Pupils' ranking of best and worst task for SGL**

The following diagrams illustrate that pupils in Finland and in England (N = 999) show some difference to the teachers, who are representing their nation. *In the questionaire there were* [seven] *tasks* [out of each category of 5.1 A proposal on typology of tasks in school mathematics] *given in random order and the pupils were asked to select their first choice task for individual learning and their first choice task for small group learning* [Ber I, 2002, 22].

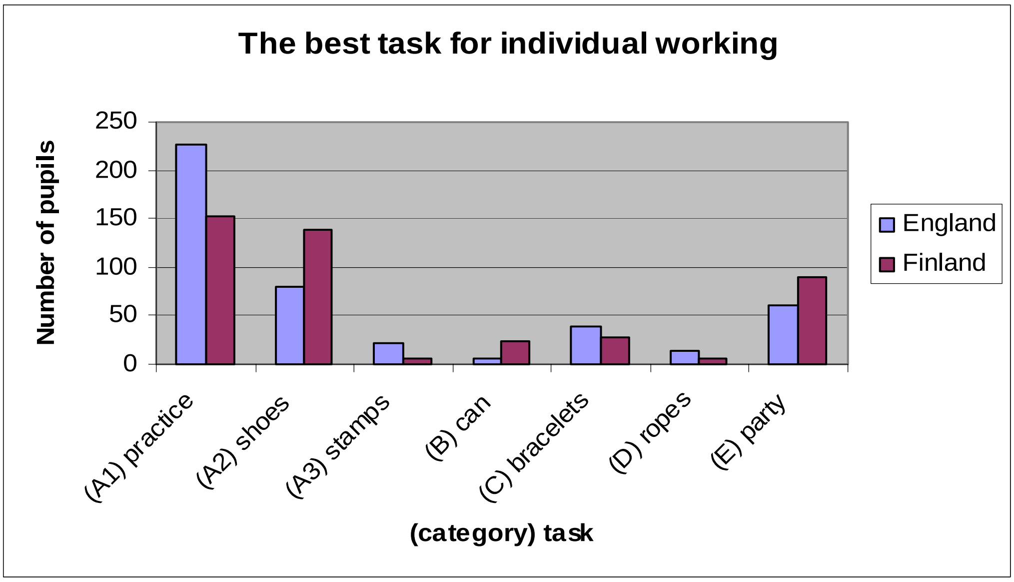


Figure 8: The best task for individual working according to the pupils (N = 999) [Ber I, 2002, 23]

Figure 8 shows that 'Practice the rule' (Finland 30% and England 46.5%) and 'Sarah's shoes' (Finland 27.6% and England 16.7%) are the most common choices for working individually. *These two tasks require little discussion between pupils and they* [the pupils] *clearly feel that tasks of this type are best done alone* [Ber I, 2002, 23].

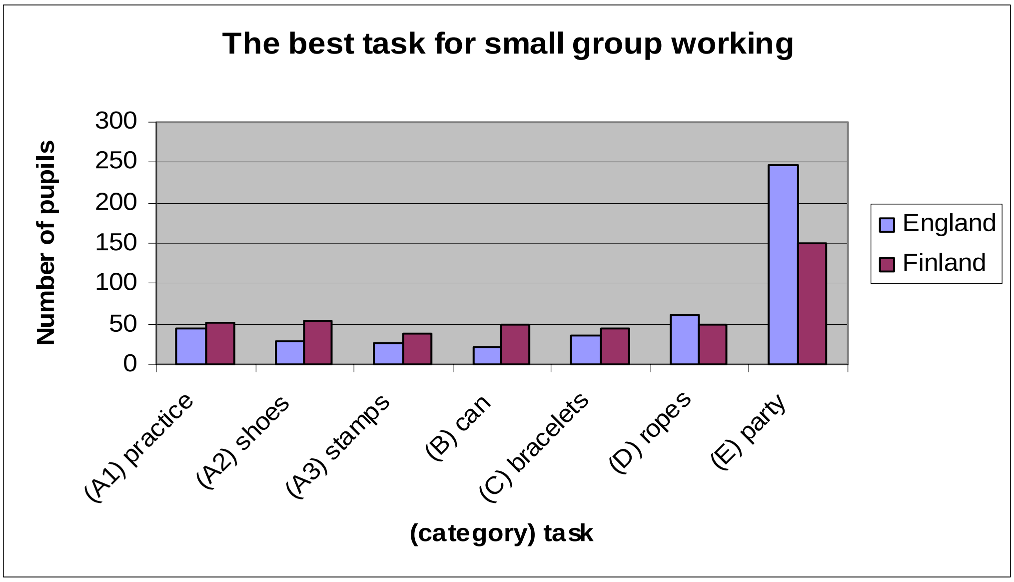


Figure 9: The best task for small group working according to pupils (N = 999) [Ber I, 2002, 23]

Figure 9 illustrates that *for each group of pupils* [category (E)] *task 'Planning a party' (Finland 27.4% and England 50.2%) is the most common choice for working in small groups. However, in this case there is quite a contrast between the two groups of pupils. For the English pupils the choice is clear with about a half of the pupils opting for the* [category (E)] *type of tasks such as 'Planning a party' task and* [category (D)] *tasks like 'Ropes' being next at 12.4%. For the Finish pupils the decision is not so clear cut. There is a fairly even distribution of the other tasks* [Ber I, 2002, 23].

The most common reasons for the ranking of tasks as good for small group working were the need for discussion and the task should be open-ended with many different routes through the problem [Ber I, 2002, 26]. *They* [the pupils] *seem to have beliefs that good tasks for group work need to able to be broken down into smaller tasks for different people to do. They also comment that the notion of breaking down a task should actually make it easier to do* [Ber I, 2002, 28].

5.1.10.3 Teachers' view of SGL tasks in mathematics

Teachers usually use textbooks as the main resource in planning their lessons. As Table 4 points out, *approximately 55% of the teachers in this study responded that it is difficult to find appropriate mathematical tasks for small group learning in mathematics lessons. Two thirds of teachers recognised that small group learning tasks have some specific characteristics compared to individual tasks. Furthermore, nearly half of them are using problems and exercises from their textbook as small group learning tasks. This implies that teachers need* *resources from elsewhere when they are planning to conduct cooperative learning lessons in mathematics* [Ber I, 2002, 25].

	It is difficult to find appropriate mathematical tasks for SGL		Small group learning tasks have specific characteristics		I normally use textbook tasks in SGL	
	N	%	N	%	N	%
Totally agree	4	5.2	6	7.8	5	6.5
Agree	19	24.7	28	36.3	22	28.5
Almost agree	20	26.0	18	23.4	10	13.0
Almost disagree	7	9.0	12	15.6	6	7.8
Disagree	17	22.1	12	15.6	19	24.7
Totally disagree	10	13.0	1	1.3	15	19.5

Table 5: Teachers' view of small group learning tasks in mathematics (N = 77) [Ber I, 2002, 25]

Up to one third of teachers teaching mathematics think that small group learning tasks do not have any specific features compared to any other mathematics tasks. This may explain partially the negative experiences that many teachers have of small group learning in mathematics. Closed routine mathematics problems or linear real problems do not promote equal exchange of ideas, positive social interdependency or mutual communication that are essential elements of cooperative learning [Ber I, 2002, 25].

5.2 *Intentional problems of the integral calculus by Hußmann*

5.2.1 Introduction

Pupils are sitting widespreadly in the classroom while making a lot of noise. Small groups of four pupils are working on the same problem, but each group discusses a different question related to the problem. No teacher tells them to be quiet or to follow her/his trains of thoughts [Hu, 2003, 5].

Personally, I call this situation ‘school as a place for Muße’. In my opinion, the german term ‘Muße’, which comes from the greak vocable ‘scholè’ and leads to ‘school’, is much more appropriate then ‘leisure’ as one may call it. ‘Leisure’ often implies laziness. Bartels tells us that ‘Muße’ originally meant the opposite of laziness. It represented tenses in which life is characterized by meaningful completion. On the other hand, ‘Unmuße’ is linked directly to labor necessary for living [Bar, 1993, 57f].

One way to reach meaningful completion in school is giving pupils space to develope their own trains of thoughts. Intentional problems urge people to come up with their own considerations as they contain only a few questions formulated mathematically or even none at all.

Fundamentally, Hußmann doesn’t understand mathematics as a finished product, but as developing knowledge. He wants students to take responsibility for such development and to force it [Hu, 2003, 23].

Intentional problems contain several intentions. First, they represent the society’s perspective by preparing students for their examinations. Second, intentional problems open the way to enclosed mathematical topics, e.g. the integral calculus and statistics. Therefore, they include essential terms and the basic understanding of a domain. Moreover, they evoke the guiding ideas of mathematics and the traditional way of mathematical thinking. In addition, intentional problems cannot be seperated from the teacher’s interests, leaning and pre-experience. These aspects are reflected in the topical planning. Hußmann gives the example of a mathematics teacher who teaches physics, too. Such a teacher is likely to select intentional problems, which describe the connection between speed and way or force and work, to guide through the integral calculus. In my own case, I combine mathematics and psychology or mathematics and philosophy in the course of intentional problems. The first combination of school subjects allows the construction of intentional problems about statistics and the second combination can look back to the achievements of prior educators in both mathematics and philosophy, e.g. Rene Descartes [compare to Hu, 2003, 23].

Intentional problems try to be as closest to the real world as possible. The fact that they should be real can mean that they are real for the students in means of being interesting for them. As stated in my typology of school tasks (see 5.1 A proposal on typology of tasks in school mathematics), mathematical tasks are archetyps of real-world problems when they get simplified, classified and enriched by the process of modelling [Hu, 2003, 24].

The definition of mathematical literacy focuses on developing mathematical terms instead of routine calculation which explains the need for intentional problems in SGL.
Mathematische Grundbildung ist die Fähigkeit einer Person, die Rolle zu erkennen und zu verstehen, die Mathematik in der Welt spielt, fundierte mathematische Urteile abzugeben und sich auf eine Weise mit der Mathematik zu befassen, die den Anforderungen des gegenwärtigen und künftigen Lebens dieser Person als konstruktivem, engagiertem und reflektierendem Bürger entspricht [Hu, 2003, 24].
This definition states the importance of the following competences:

- modelling in a mathematical way
- using mathematical visualizations
- using symbolic, formal and technical elements out of mathematics practically and reflecting them
- arguing and communicating mathematically
- identifying and solving problems

  Identifying a problem is a main characteristic of intentional problems as they often don't contain a question to tell students what to do [Hu, 2003, 24].

To put it generally, intentional problems are open problem situations, which offer an access to a (mathematical) topic. Students develop essential terms and procedures for solving the problem on their own. Each problem's design forces them to discover/invent constitutional terms of one subject area. While working on intentional problems, pupils enlarge their cumulative knowledge by activating facts from various domains. They can develop facts

along their individual ideas and interests because they navigate their working style autonomously.
When used in school, intentional problems should be connected so that they can complete each other, which is one of their strenghts. Students can only build up mathematical terms individually if they get confronted with many problem situations concerning the same topic [compare to Hu, 2003, 6 – 9].

The following statement of Isaacs motivates tasks which don't consist of questions as, for example, intentional problems: *'Inquiry' comes from the Latin 'inquaerere', to seek within […] The word decision, from the Latin 'decidere', literally means to 'murder alternatives'. It is best to approach dialogue with no result in mind, but with the intention of developing deeper inquiry, wherever it leads you* [Is I, 1994, 375].

As far as I understand Hußmann's idea of intentional problems, they can be taken out of four categories of my typology of school tasks in mathematics, namely 'archetyps of real world problems (B)', 'mathematical investigations (C)', 'mathematical modelling (D)' and 'designing projects and studies in mathematics (E)' (see 5.1 A proposal on typology of tasks in school mathematics).

In England and Finland, teachers state that *statistics including data handling and designing questionnaires* is the area of the mathematics curriculum they might use for SGL as their first choice [Ber I, 2002, 27]. Hußmann seems to be aware of the appropriateness of such topics as far as SGL is concerned. In 'Mathematik entdecken und erforschen' the ninth chapter called 'Vektorrechnung' includes statistics [Hu, 2003, chapter 9].

5.2.2 The integral calculus

Traditionally, teachers represent the integral calculus after working on the differential calculus instead of dealing with both topics at the same time. Hußmann prefers a parallel treatment of these topics to ensure networking [Hu, 2003, 71].

Reichel explains the image behind integral calculation by saying
Man geht aus von einem Vektorraum V(K) "einfacher" Funktionen und versucht, diesen Funktionen f [aus] *V(K) eine Zahl I(f)* [aus] *K mit I(λf) = λ I(f), λ* [aus] *K, und I(f + g) = I(f) + I(g) zuzuordnen. Man konstruiert also ein lineares Funktional I auf V(K), welches man als das Integral von f bezeichnet; zusätzlich verlangt man I(f)≥0, wenn stets f(x)≥0; der Körper K steht dabei in den allermeisten Fällen für* [die Menge der reellen Zahlen oder die Menge der komplexen Zahlen]. *Danach versucht man, zumeist durch gewisse „Grenzwertbildungen", dieses Funktional auf größere Klassen (i.e. Vektorräume über* [dem Körper] *K) von Funktionen auszudehnen:* $V(K) \subset V_1(K) \subset V_2(K) \subset ...$ *Das Lebesque-Integral reeller Funktionen liefert den Prototyp für alle derartigen Integrationstheorien* [Re, 1974, 168].

Looking back in history, the Lebesque-Integral is the most dominant approach.
Die Vorherrschaft des L-Integrals hat aber auch tiefere Wurzeln: eine der Grundaufgaben der Integralrechnung ist nicht zuletzt die Aufgabe, ein „kontinuierliches Analogon" zur arithmetischen Mittelwertbildung bereitzustellen. Das sollte in der Schule u.a. als Motivation besonders hervorgehoben werden, z.B. an Hand physikalischer Problemstellungen: $m = \frac{1}{b-a}\int_a^b f(x)dx$ *als kontinuierliches Analogon zu* $m = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$ [Re, 1974, 168].

Hußmann lists four aspects and suggests discussing them at school:
the area-aspect (a-aspect), the primitive-aspect (p-aspect), the mean-aspect (m-aspect), the approximation-aspect (ap-aspect) [Hu, 2003, 73f].

I illustrate the a-aspect, the m-aspect and the ap-aspect for $f, f(x) = ax^3 + bx^2 + cx + d$ with *a, b, c, d* being elements of the set of real numbers; $[x_1, x_2]$ is a subset of the real numbers.

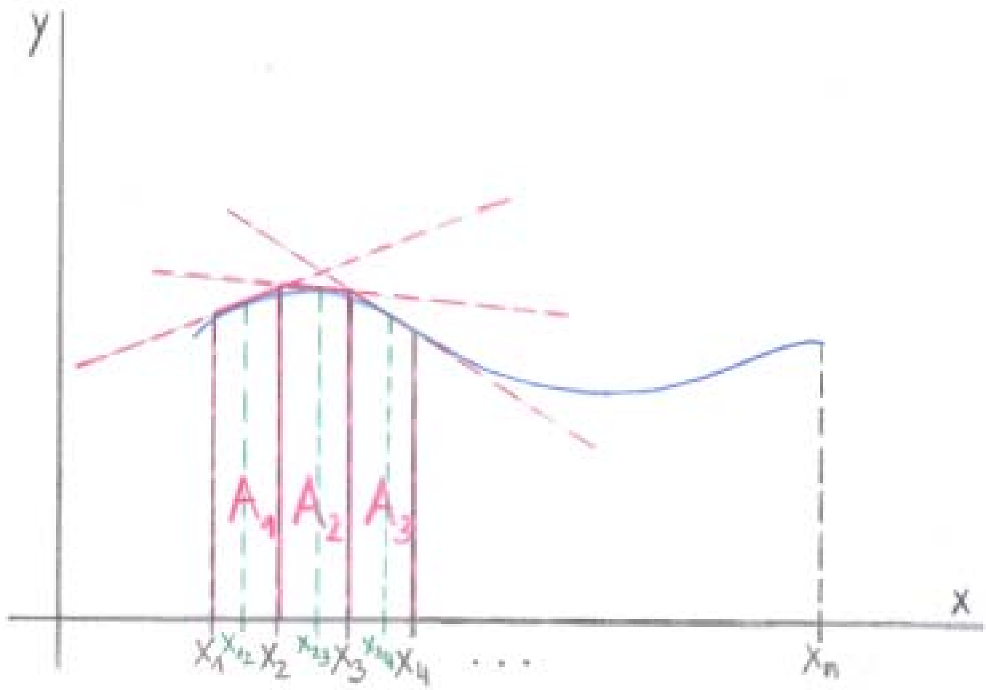


Figure 10: the a-aspect [compare to Re, 1974, 170]

I use the trapezoid rule to illustrate the a-aspect. Therefore, I draw the tangent on *f* in a point between x_1 and x_2 (named 'x_{12}') to replace *f* in $[x_1; x_2]$ (see red dashed lines). The red lines signify the area under those tangents (called 'A_1', 'A_2' and 'A_3').

Continuing this process until I use the tangent on *f* in a point between x_{n-1} and x_n leads to all trapezoids needed for calculating approximately the area under *f* when using the trapezoid rule.

As a result, we reiceive 'almost' the exact area under *f* (more exactly expressed: between the graph of *f* and the *x*-axis). As teachers, we are satisfied with 'almost the exact area under *f*' as long as pupils can recall my explanations above.

To illustrate the p-aspect, I visualize the *Mittelwertsatz der Integralrechnung* (MWS d. IR) for function z with $z(x) = x^3$ beforehand.
To calculate the area under z in $[x_1; x_2]$, I use a rectangle to approximate this area. (This is not the MWS d. IR, but it demonstrates the basic idea of this theorem.)

(a)

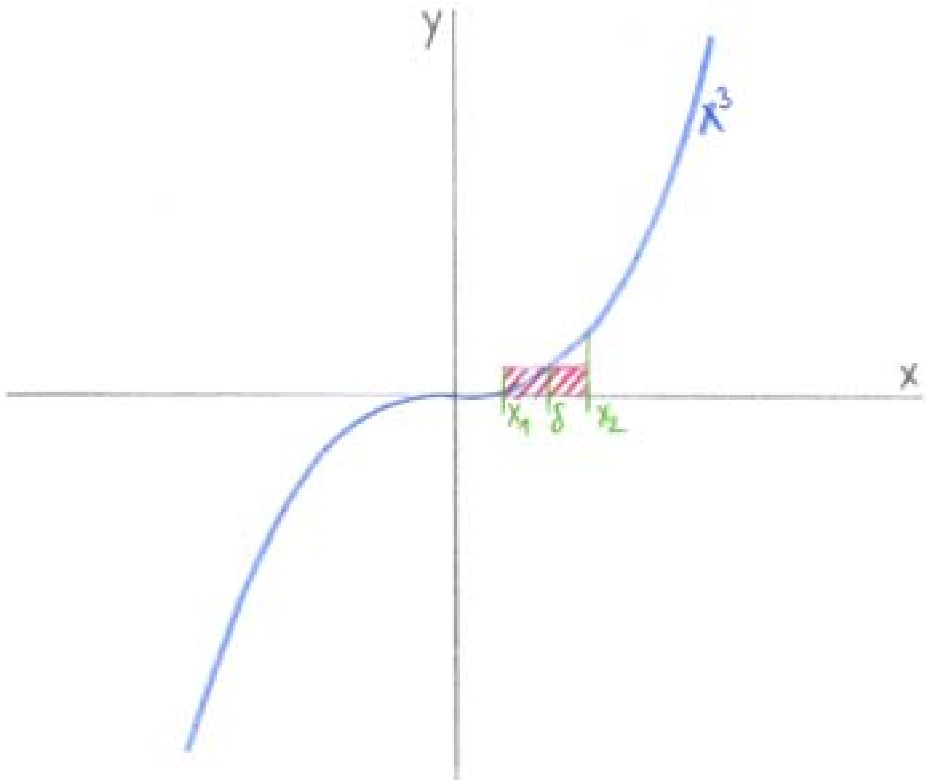


Figure 11: *Mittelwertsatz der Integralrechnung*

The MWS d. IR says that there exists a $\delta \in [x_1, x_2]$ so that the area of the red rectangle (see Figure 11) is equal to $(x_2 - x_1) \cdot f(\delta)$ [compare to *Mittelwertsatz der Integralrechnung: Ist f:[a, b]→ [Menge der reellen Zahlen] stetig auf ganz [a,b] mit a, b aus der Menge der reellen Zahlen, dann existiert ein c aus [a,b] mit* $\int_a^b f(x)dx = f(c) \cdot (b-a)$*;* Mi, 2004/05].

(b)

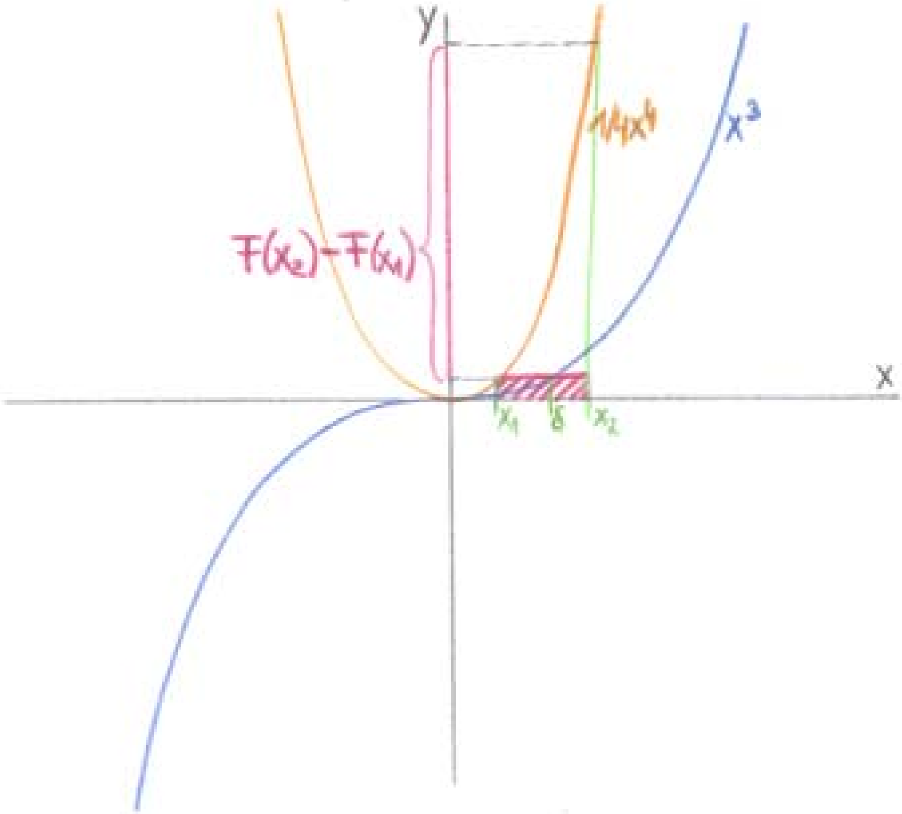


Figure 12: the p-aspect

Figure 12 shows $F(x_2) - F(x_1)$ and $f(\delta) \cdot (x_2 - x_1)$. 'There are almost as many dots in the red rectangle as on the red line' is an outstanding illustration for this coherence. At this point we use the p-aspect. It means that integration and derivation are inverse procedures: $F'(x) = f(x)$, we say: F is the antiderivative of f.

In case, time doesn't allow explaining the p-aspect (in such a detailed way); I isolate the m-aspect in Figure 13 while letting the p-aspect stay out of focus.

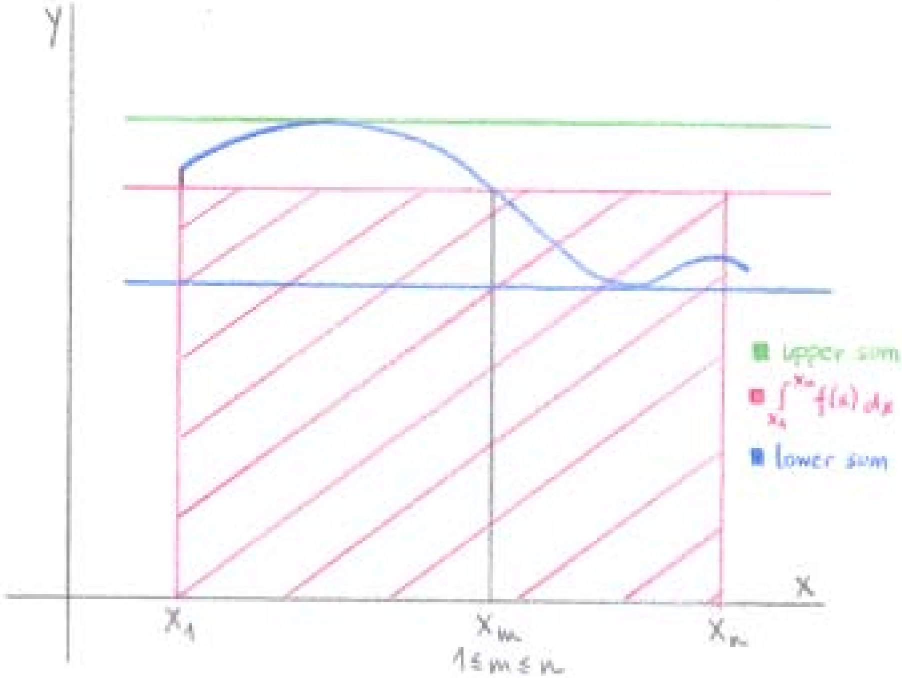


Figure 13: the m-aspect [compare to Mi, 2004/05]

In this case, pupils are likely to be satisfied by watching a teacher drawing and meanwhile explaining Figure 13. Basically, saying 'You can see that the area under *the green line* (see Figure 13) is too big and the area under *the blue line* (see Figure 13) is too small. So the shaded area is much more likely to give us the 'real' area under f ' satisfies most pupils without confusing those students who don't get along with abstact (mathematical) thinking.

We can interpret $f(x_m)$ as the mean value of f in the interval $[x_1,x_n]$, because as we mentioned is $f(x_m)\cdot(x_n - x_1) = \int_{x_1}^{x_n} f(x)dx = F(x_n) - F(x_1)$ and therefore the equation

$$f(x_m) = \frac{F(x_n) - F(x_1)}{x_n - x_1} = \frac{\int_{x_1}^{x_n} f(x)dx}{x_n - x_1}$$ holds.

Compare with the arithmetic mean of the data $x_1, \ldots, x_n,$ $\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$, we see that we have found the continuous analogon to this discrete value.

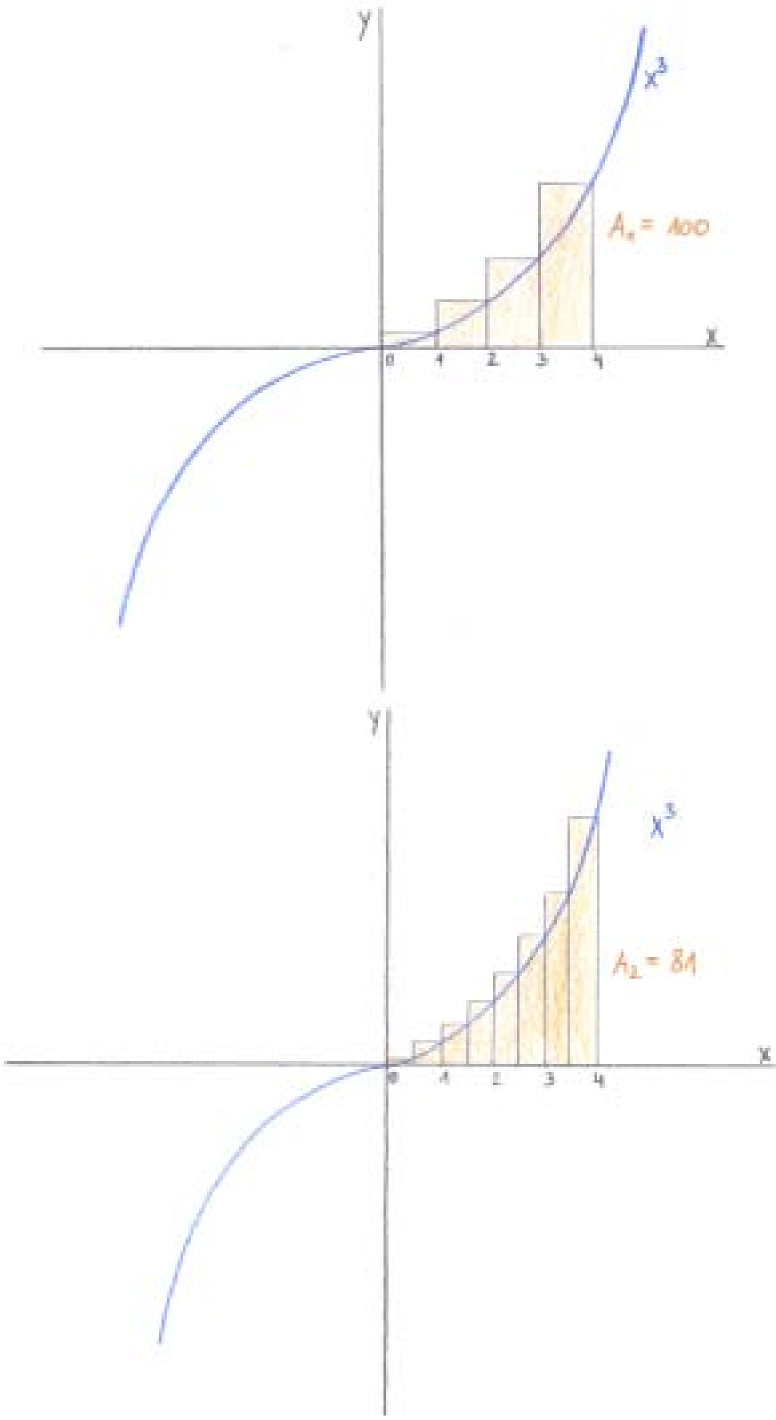


Figure 14: the ap-aspect [compare to Mal, 2006, 3]

In this case, I use upper sums to approximate the area under *f*: $y = x^3$ in $[x_1; x_n]$ and name the sum of the receiving rectangles 'A_1' and 'A_2', respectively. As one can see, the sum of the rectangles gets smaller and – more important – closer to the area between the graph of *f* and the *x*-axis each time I refine the partition for [0; 4], i.e. it approximates $\int_0^4 x^3 dx$.

It is to say that these four aspects can be presented in a different order. Moreover, a teacher can concentrate on one/several of them because of his/her time managment. Then, the drawings may change [dependent on the aspects' order].

But this is not the end of the line, a critical reflection of what we have gained dealing with the integral conception is necessary. It depends not on the number or the choice of aspects which are discussed in the mathematics lessons before.

As Karcher says
Nun folgt der in Bezug auf die Bildungsaufgabe der Höheren Schulen vielleicht wichtigste Teil, der jedenfalls angerissen werden sollte, wie wenig Zeit auch zur Verfügung steht. Es handelt sich dabei um eine Kritik des Gelernten und damit im Zusammenhang eine Kritik der Begriffe „Flächeninhalt" und „Wahrscheinlichkeit" (letzteres nur in heuristischer Form) [Kar, 1973, 179].
I suggest confronting pupils with the real function g defined as

$g(x) = \sin\frac{1}{x}$ for $x \neq 0$, $g(x) = 0$ for $x = 0$.

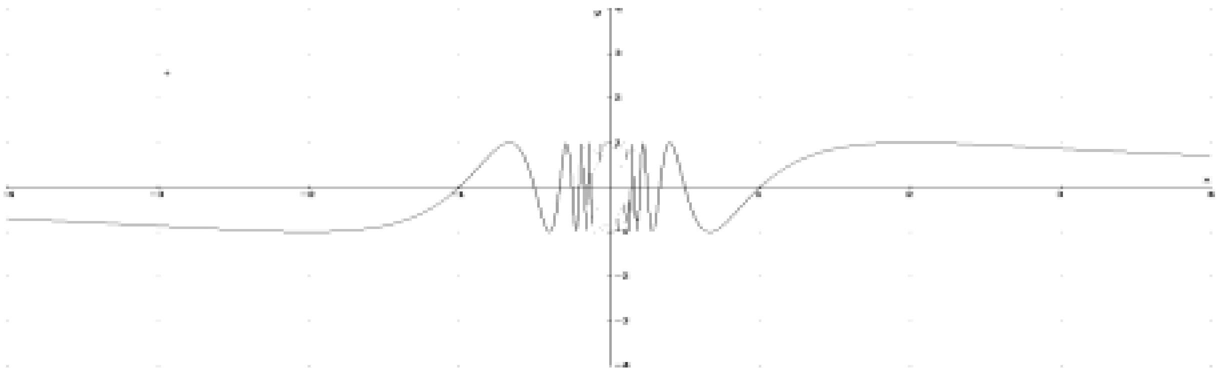


Figure 15: Visualization of function g

First, I prove that $\lim_{x \to 0} g(x)$ doesn't exist to show the function's discontinuity at the point 0.

Take three series (x_n), (y_n) and (z_n) with $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} > 0$ for all natural numbers,

$y_n = \frac{1}{-\frac{\pi}{2} + 2n\pi} > 0$ for all natural numbers and $z_n = \frac{1}{2n\pi} > 0$ for all natural numbers.

It follows that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{\frac{\pi}{2}+2n\pi} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{\pi}{2n}+2\pi} = 0$,

$\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \frac{1}{-\frac{\pi}{2}+2n\pi} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{-\frac{\pi}{2n}+2\pi} = 0$ and $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} \frac{1}{2n\pi} = 0$.

But $\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty} \sin\frac{1}{x_n} = \lim\limits_{n\to\infty} \sin\left(\frac{\pi}{2}+2n\pi\right) = 1$,

$\lim\limits_{n\to\infty} g(y_n) = \lim\limits_{n\to\infty} \sin\frac{1}{y_n} = \lim\limits_{n\to\infty} \sin\left(-\frac{\pi}{2}+2n\pi\right) = -1$ and

$\lim\limits_{n\to\infty} g(z_n) = \lim\limits_{n\to\infty} \sin\frac{1}{z_n} = \lim\limits_{n\to\infty} \sin(2n\pi) = 0$; qed. [Compare to Mi, 2004/05].

Second, the function g is R-integrable, i.e. one can make the difference between upper sum O and lower sum U arbitrary small, even the considered interval includes 0. Take an interval $I = [-\delta; \delta]$ for $\varepsilon > 0$ with $0 < \delta < \varepsilon/4$. Then follows that (thinking of the series x_n and y_n before) $O_I = 2 \cdot 2 \cdot \delta = 4\delta$ and $U_I = 0 \cdot 2\delta = 0$ (look at series z_n before).
Consequently, $O_I - U_I = 4\delta < \varepsilon$ for $\delta < \varepsilon/4$. Moreover, g is continuous in $[a, b]$ with $0 \notin [a, b]$ and therefore R-integrable in $[a, b]$.
BUT: g cannot be uniformly approximated by step functions, i.e. g is not integrable in this sense because the function's values vary 'too much' on $[-\delta; \delta]$.
So we 'loose' some functions if we take the integration concept which means uniform approximation by step functions as basis compared to the set of R-integrable functions.

All aspects mentioned above can be reunited with the main sentence of the integral calculus to the basic image about the integral: accumulation. *Accumulation means to sum up the products* [Hu, 2003, 74].

By understanding the process of accumulation, students can interpret negative values of the area between the function and the x-axis. For example, a function f_H describes the rate of change of water flowing into several households in a city. Negative values of f_H stand for 'the flowing away of water'.
The overall area [between the function and the x-axis] *is called overall effect. The overall effect is the target, accumulation is the way to reach it* [Hu, 2003, 74].

(a) When starting with accumulation and the overall effect, students reach the integral function $I_a(x) := \int_a^x f(t)dt$ by working on the effect of the rate of change f in $[a, x]$. Comparing f and I_a leads to the p-aspect and to the main sentence of the integral calculus.

(b) When starting with the p-aspect, they are likely to reach the idea of accumulation and overall effect.

Both ways lead to an essential idea of the integral calculus: The value of the integral $\int_a^b f$ of a function f in $[a, b]$ is the result of a limit of an accumulation [compare to Hu, 2003, 75].

5.2.3 Example 1: Water Consumption

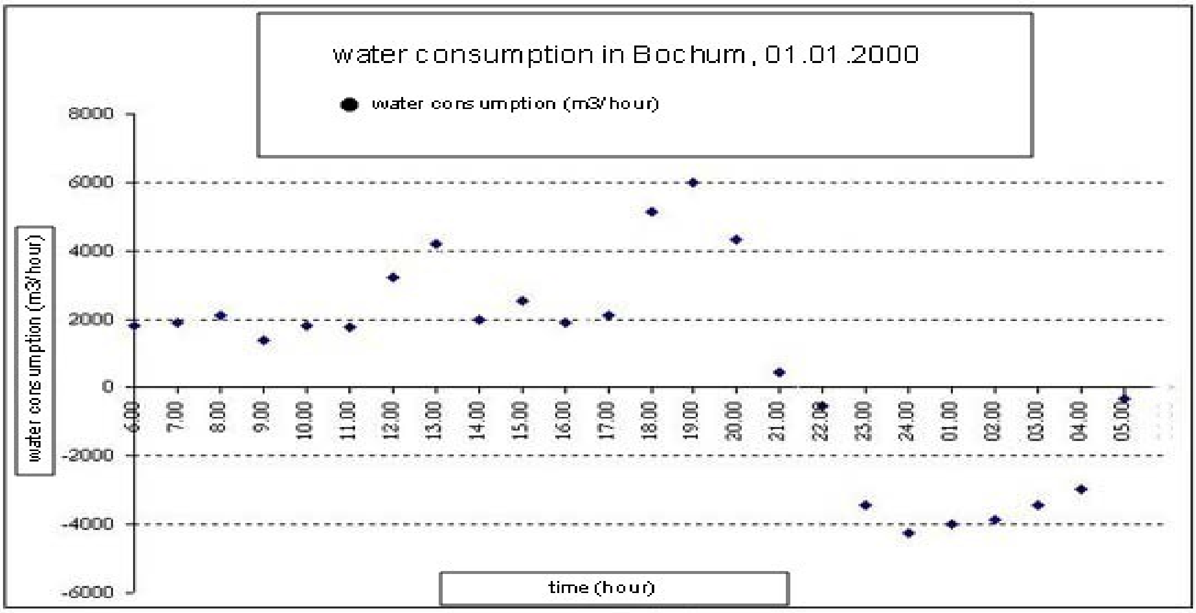


Figure 16: Water consumption in Bochum - Visualisation [Hu, 2003, 68]

time (hour)	water consumption (m^3/hour)
6.00	1823
7.00	1900
8.00	2100
9.00	1400
10.00	1800
11.00	1765
12.00	3236
13.00	4219
14.00	1989
15.00	2549
16.00	1888
17.00	2119
18.00	5129
19.00	5988
20.00	4322
21.00	430
22.00	-543
23.00	-3450
24.00	-4229
1.00	-3998
2.00	-3887
3.00	-3456
4.00	-2987
5.00	-327

Table 6: Water consumption in Bochum – time/amount relation [Hu, 2003, 68]

The water supply companies of Stiepel and Essen-Horst supply the biggest part of Bochum with drinking water with exception of Werne and Langendreer. The water supply company of Witten is responsible for those two cities. The water supply company Stiepel was built in 1910. It includes for example a hydroelectric installation and a pump station.

To guarantee the supply of the population, the water supply company aspires optimum efficiency. The starting point therefore is the current consumption of the population, which is measured by sensitive antennae. The figure and the table show the consumption of one day. The values stand for the current consumption. The water supply companies of Bochum use figures like the one above to forecast the following days' optimum water exploitation [Hu, 2003, 68].

This example, which is less complex than following example 2 or following example 3, is an example for ‘archetyps of real world problems (B)’ because of its connection to the city of Bochum. In every school, a city closely located can be found to reach the students. Obviously, only authentic amounts of water consumption make this example an archetype of real world problems [Hu, 2003, 77].

In example 1, students have to face the difficulty of not finding any questions to work on. Personally, I am convinced that students will work on the following questions:

1. How can I explain a negative amount of water consumption?

   In these periods there comes more water in the tanks than it is taken from them.

2. How big is the water consumption for several hours/a day?

This question leads to the integral calculus. First, the given amounts don’t allow the finding of only one function out of those, which are introduced in school mathematics, to describe the water consumption of Bochum on the first of January, 2000. Accordingly, students have to approximate the amounts of water consumption by using step functions or partly defined linear functions. Some students may use rational functions of a high order, which makes this example’s complexity increase [compare to Hu, 2003, 77].

In case pupils don’t come up with ‘How big is the water consumption for several hours/a day?’ themselves, I suggest the following task:

Calculate the water consumption in Bochum for at least 6 hours on the 1^{st} of January in 2000 by (a) calculating the arithmetic mean

(b) using integral calculation.

Divide (a) and (b) up amongst your group members and come together afterwards to discuss your results!

Ad (b): For example, a linear function g with $g(x)$ = 2000, x standing for the water consumption (m^3/h), is predestinated to calculate the water consumption in Bochum from 6 am to 11 am. To calculate the water consumption from midday to 5 am, quadratic functions (piecewise defined) are likely to be chosen by pupils.

Water consumption from 5 pm to 9 pm:

Use function f with $f(-2) = 2000$, $f(0) = 6000$, $f(2) = 400$ and $y = a + bx + cx^2 = f(x)$ *to approximate the values for x in [5 pm; 9 pm]. So we get f:* $y = 6000 - 400x - 1200x^2$ and therefore $\int_{-2}^{2} f(x)dx = 17600$.

Compare this result with the discret analogon 2119 + 5129 + 5988 + 4322 + 430 = 17988!
Between 5 pm and 9 pm, Bochum consumes – approximately – 18000 litre water.

Hußmann lists several other questions and entitles them ‘Kernideen’:

1. What happens in Bochum’s households at one o’clock?
2. What is the mean use of the water supply company?
3. Does the population of Bochum consume more water than they become?
4. What does the amount of water consumption at 7 o’clock stand for: the water consumption at 7 o’clock or the value between 6 and 7 o’clock?
5. How does the diagram of the water consumption of one day help to predict the water consumption of the following days or even a whole week?
6. Why do predictions make sense? [Hu, 2003, 77]

Possible solutions:

1. The curve reaches a local maximum.
2. Mean use of the water supply company:
    a. at day (6 am – midnight):
    $$\frac{1823 + 1900 + 2100 + 1400 + 1800 + 1765 + \ldots + (-3450) + (-4229)}{19} \cong 1812.368421$$
    b. at night (1 am – 6 am)
    $$\frac{(-3998) + (-3887) + (-3456) + (-2987) + (-327)}{5} = (-)2931$$
3. Yes, because the arithmetic mean
    $\frac{1823 + 1900 + 2100 + \ldots + (-2987) + (-327)}{24} \cong 824.1\dot{6}$ is greater than zero.
4. The water consumption between 6 and 7 o’clock.
5. The curve for the water consumption in a certain city is similar each day a year.
6. In this case: to save water and therefore protect nature [compare to Hu, 2003, 78f].

5.2.4 Example 2: Tachometer

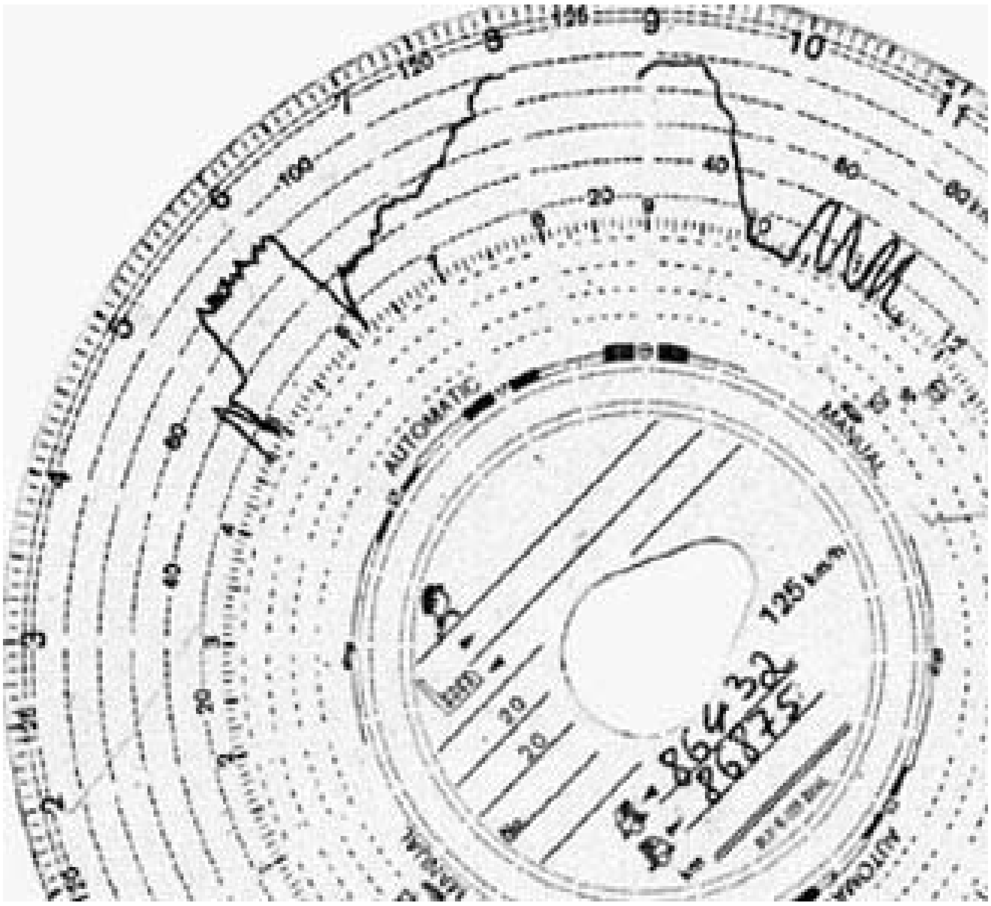


Figure 17: tachometer [Hu, 2003, 70]

Several men and women work at a transport company in Bochum. They have to stock supermarkets in Germany. One day, Miss Grat, who works at the transport company, drives from Muniche to Bochum. Suddenly the police stops her. In general, the inspections of the police ensure the safety of the transport machines on the road. While inspecting the tachograph, the police discovers a relatively long time where no speed has been recorded. A tachograph consists of velocities which are recorded in a time-speed-diagramm during the whole journey of the vehicle. On enquiry, Miss Grat tells the police that she took a break at that time [Hu, 2003, 70].

To solve this problem, students have to understand the relation between speed and way. In this case, the given situation easily motivates students to search for related information, e.g. the legaly approved maximum speed for trucks. Moreover, this example is constructed like the description of a detective case and turns mathematics into a discipline where tricky problems need to be solved [Hu, 2003, 80].

First, students, who work on this task in a small group, try to answer the question 'Did Miss Grat take a break between 8 and 9 o'clock?' by taking a close look at the diagram. They should find out that Miss Grat lied because if Miss Grat took a break between 8 and 9 o'clock, the speed curve would fall down to 0 km/h both at 8 o'clock and at 9 o'clock. This insight leads to the question, why no speed has been recorded between 8 and 9 o'clock. One may assume that Miss Grat crossed the legally approved maximum speed for trucks and therefore didn't use the tachograph at this time to avoid a fine.

Moreover, students should try to calculate the amount of kilometers of Miss Grat's tour. Therefore, an approximation with step functions is inescapable.

In addition, this example allows mathematical inspections, which aren't linked to the integral calculus. For example, students may find the maximum speed. Furthermore, a teacher can test his/her students' understanding of the idea behind a two-dimensional coordinate system. Only students, who understand the meaning of a two-dimensional coordinate system, are able to insert the given data of the tachometer into a two-dimensional coordinate system [compare to Hu, 2003, 80-83].

Students calculate the amount of kilometers of Miss Grat's tour by finding a useful fragmentation of the time axis. Therefore, it is best to read off the mean speed on an obvious fragment and multiply it by the length of the time intervall. For example, the mean speed between 5 and 6 o'clock can be seen as 80 km/h. When the speed values are changing very often although not much time passed, it helps to use a more accurate fragmentation of the time axis. Such an approach contains two advantages: First, the ap-aspect becomes accented. Second, students realise that an equidistant fragmentation, which can be used for example 1, can be worse than an inequidistant fragmentation, which should be used for this example [Hu, 2003, 81].

Hußmann illustrates the tachometer's solving process in Table 7.

Kernidee	**Symbol/archetype**
Did Miss Grat lie?	
How long is the way of Miss Grat's tour?	The way is the area under the graph.
How do I get the area under the graph?	Use rectangles, triangles or trapezoids. Think of accumulation and calculate the product sums.
How do I calculate the product sums as accurate as possible?	Breakdown into many small parts
How do I calculate the area (= the product sums) for any graph?	Calculate the limit of the product sums.
...	...
Pro Miss Grat Contra Miss Grat Way of Miss Grat's tour	Calculate the lower sum. Calculate the upper sum. Limit of lower and upper sums

Table 7: Kernideen [Hu, 2003, 38]

5.2.5 Example 3: Growth of Boys and Girls

The curves in the diagram illustrate the growth rate of a group of boys and girls until their eighteenth year. The curves approximate the values of a middle European study for a group of people. The basic data for the curves are mean values of the measured growth values of each group. To ensure a smooth reading of the curves, you cannot see those values.

What information can be calculated when using the given data? Calculate the intervalls in which boys are taller/shorter than girls!

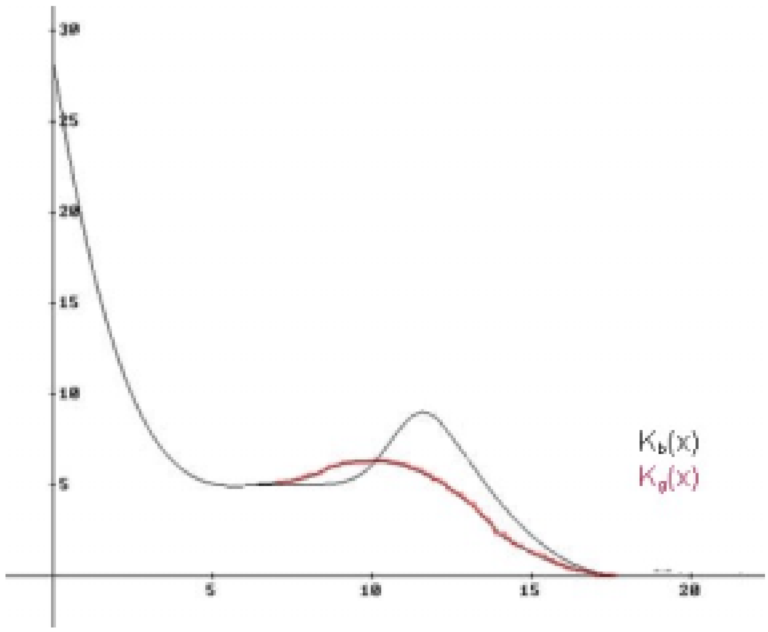


Figure 18: Growth of boys and girls [Hu, 2003, 69]

$$K_b(x) := \begin{cases} -0.0973 \cdot x^3 + 1.8312 \cdot x^2 - 11.407 \cdot x + 28.4909 & 0 < x \le 6 \\ 4 \cdot e^{\frac{-(x-11.6)^2}{2}} + 5 & 6 < x \le 13 \\ 0.275762 \cdot x^2 - 9.84747 \cdot x + 87.9133 & 13 < x \le 18 \end{cases}$$

$$K_g(x) := \begin{cases} -0.0973 \cdot x^3 + 1.8312 \cdot x^2 - 11.407 \cdot x + 28.4909 & 0 < x \le 6 \\ 0.00017733 \cdot e^x + 4.928 & 6 < x \le 8.5 \\ -0.242877 \cdot x^2 + 4.84906 \cdot x - 17.7548 & 8.5 < x \le 14 \\ \frac{23}{160} \cdot (x-18)^2 & 14 < x \le 18 \end{cases}$$

[Compare to Hu, 2003, 69]

The example ‘Growth of Boys and Girls’ is the most difficult, the most complex and the most comprehensive one above the three mentioned. This example can be solved either by using the terms of the functions or by using the graphs of the functions only. Subsequently, the problem can be explained without mentioning the terms of the functions. Still, teachers need to make these terms available for their students.

In ‘Growth of Boys and Girls’, the functions were created by using the data of one of Wacker’s research studies in the seventies. In this study, Wackler shows that you cannot find any laws for the growth of people to make long-term assumptions [Wa, 1986]. Therefore, the interpolation does not approximate the data in a perfect way, but it illustrates the data with certain functions to allow the work on this problem situation.

Different to the first and the second example, the third example consists of a question to make sure that students work on this pivotal part of the problem situation. Still, many other questions can be found to work on [Hu, 2003, 83f].

I put myself in the role of a pupil and solved this example. When I worked on the first question (What information can be calculated when using the given data?), I came up with the following tasks:

1. List the age intervalls, in which boys grow faster than girls/girls grow faster than boys on average.
2. Calculate the age of maximum growth [lokales Maximum] for both sexes.
3. Calculate the age of minimum growth [lokales Minimum] for both sexes.
4. Calculate the average growth rate for both sexes.

Students, who are used to open problem situations, may come up with more questions:

(5) When I calculate the growth of both sexes for age 0, I get 28.4904. Why?

(6) Can I draw each partly defined function on $[-\infty, \infty]$ approximately without using any CAS?

Solutions:

1. Boys grow faster than girls in [10; 18], girls grow faster than boys in [6; 10], equal growth rate in [0; 6] [compare to Hu, 2003, 73f].
2. $x = 11,6$ is the local maximum for boys, $x = 9.98$ is the local maximum for girls.
3. Discuss useful parts of $K_b(x)$ and $K_g(x)$ by refering to Figure 18 [compare to 2.]!

4.

a. boys

Arithmetic mean in cm/year		*Integral in cm/year*
9.2	$1 \le x \le 6$	11
6.4	$7 \le x \le 13$	6.3
1.5	$14 \le x \le 18$	2.1

b. girls

Arithmetic mean in cm/year		*Integral in cm/year*
9.2	$1 \le x \le 6$	11
5.6	$7 \le x \le 9$	5.2
5.0	$10 \le x \le 14$	5.4
0.5	$15 \le x \le 18$	0.8

(1) 28.4904 stands for the average body height of a newborn.

(2) $K_b(x)/K_g(x)$

i. $K_b(x)$

Polynom function of 3rd order for x = 1, 2, 3, 4, 5, 6

Exponential function [combined with a polynom function of 2nd order] for x = 7, 8, 9, 10, 11, 12, 13

Polynom function of 2nd order for x = 14, 15, 16, 17, 18

ii. $K_g(x)$

Polynom function of 3rd order for x = 1, 2, 3, 4, 5, 6

Exponential function for x = 7, 8, 9

Polynom function of 2nd order for x = 10, 11, 12, 13, 14

Polynom function of 2nd order for x = 15, 16, 17, 18

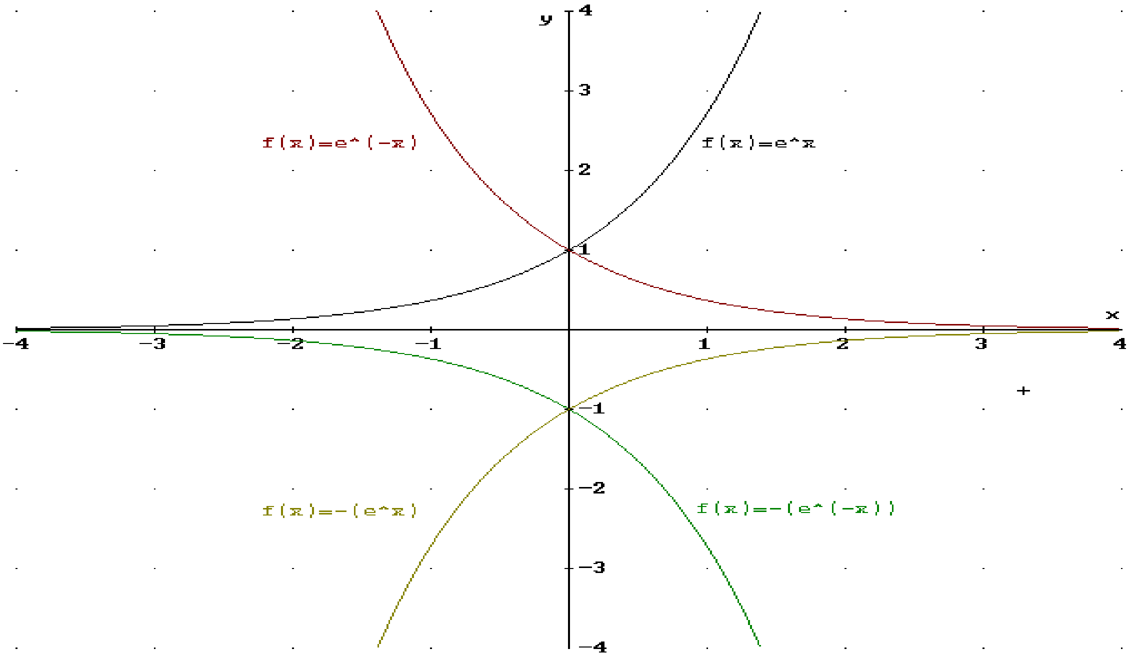
[compare to Hu, 2003, 83f]

Excursion: As teacher, I expect pupils to understand and recall my summary of functions, which are dealt with at school [compare to solution (2)].

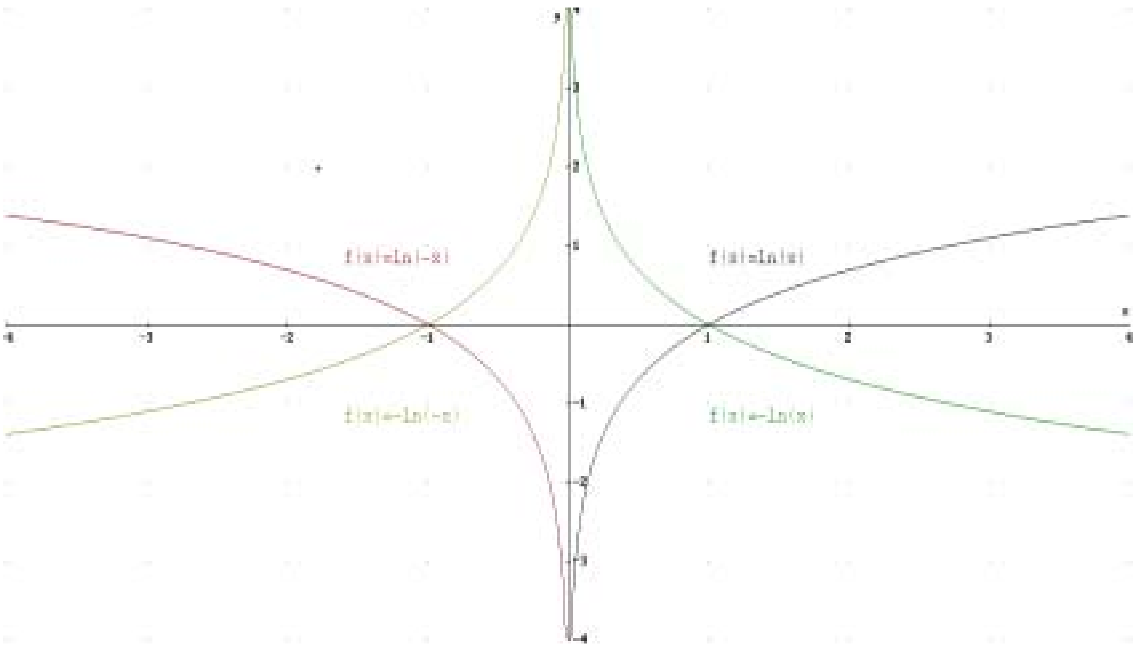
§1 Funktionen – Gleichungen und Graphen

(1) **Exponentialfunktion:** $f(x) = e^{x}$

e wird Grundzahl/Basis genannt
x wird Hochzahl/Exponent genannt
e … die Euler'sche Zahl [$\cong 2.72$]

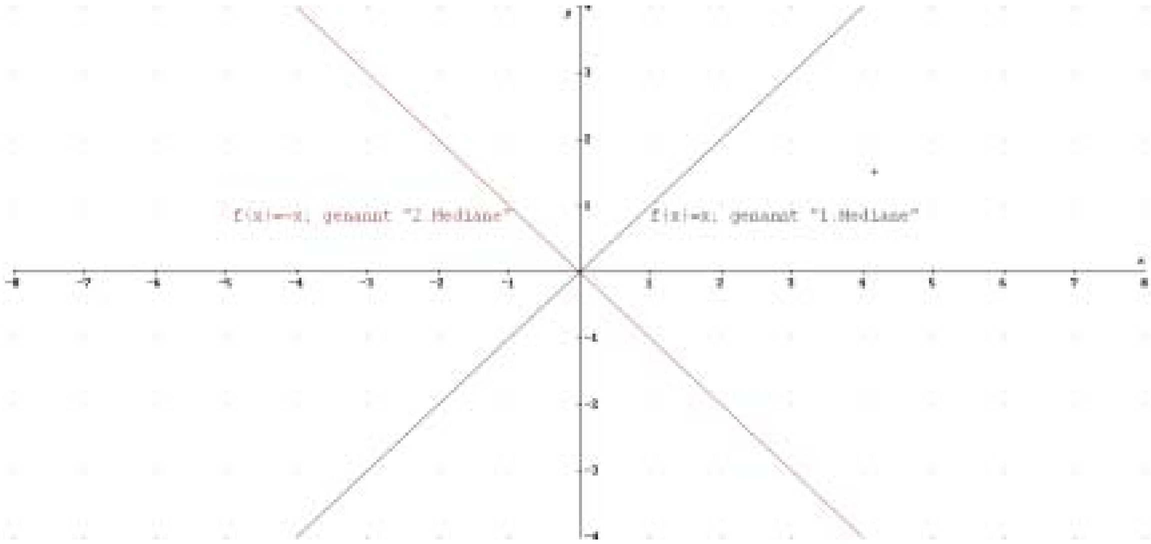


(2) **Logarithmusfunktion:** $f(x) = \ln x$



(3) **Gerade:** $f(x) = k \cdot x + d$

k … Steigung
d … Abstand zum Koordinatenursprung entlang der y-Achse



(4) Polynomfunktion/Potenzfunktion

a. $f(x) = x^n$

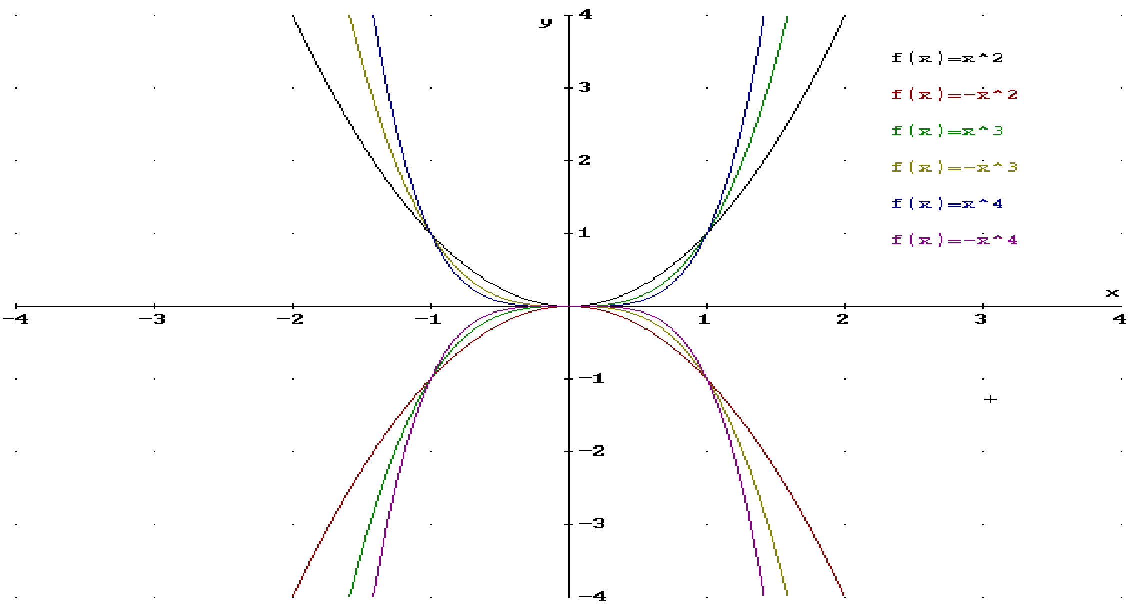
n ungerade: Graph punktsymm. bzgl. des Koordinatenursprungs
n gerade: Graph symm. bzgl. der y-Achse

b. $f(x) = m \cdot (x+s)^k + q$

$m > 1$: je größer m, desto steiler verläuft der Graph
$0 < m < 1$: je näher m bei 0, desto flacher verläuft der Graph

s … wenn < 0, dann Verschiebung entlang der x-Achse nach rechts
s … wenn > 0, dann Verschiebung entlang der x-Achse nach links

q … Verschiebung entlang der y-Achse

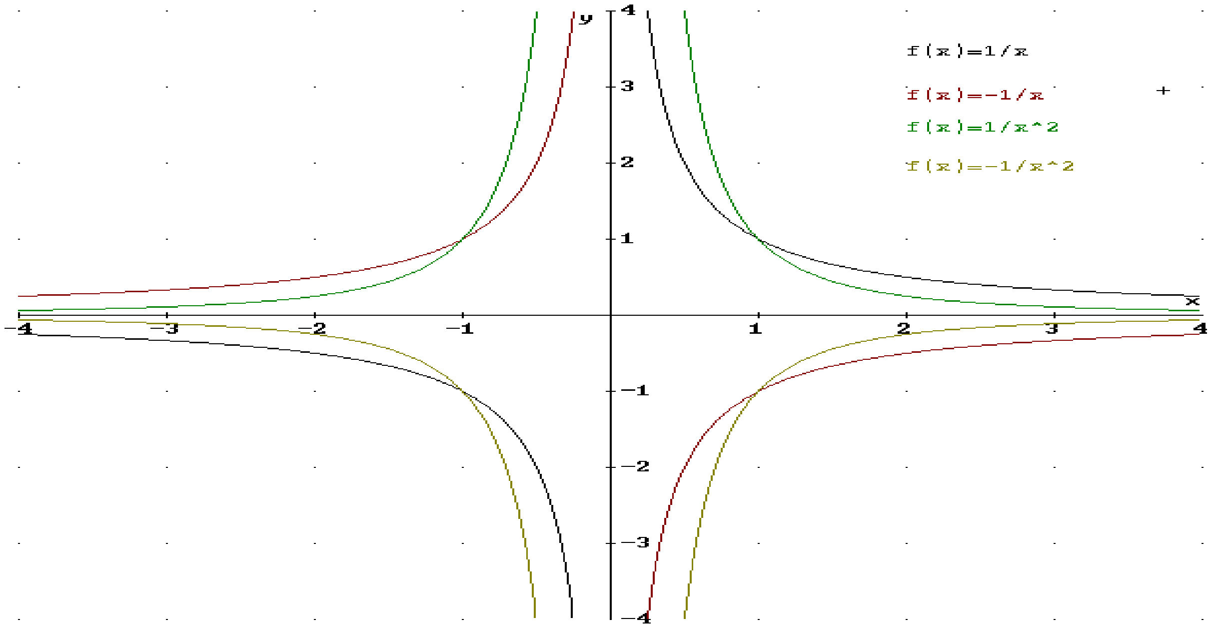


(5) Reziproke Funktion [Variable befindet sich im Nenner]

$$f(x) = \frac{c}{x^m}$$

c … je größer c, desto weiter entfernt von den Koordinatenachsen liegt der Graph der Funktion f

m … je größer m, desto rascher konvergiert der Graph von f gegen 0 für x gegen $\pm\infty$ bzw. gegen $\pm\infty$ für x gegen 0



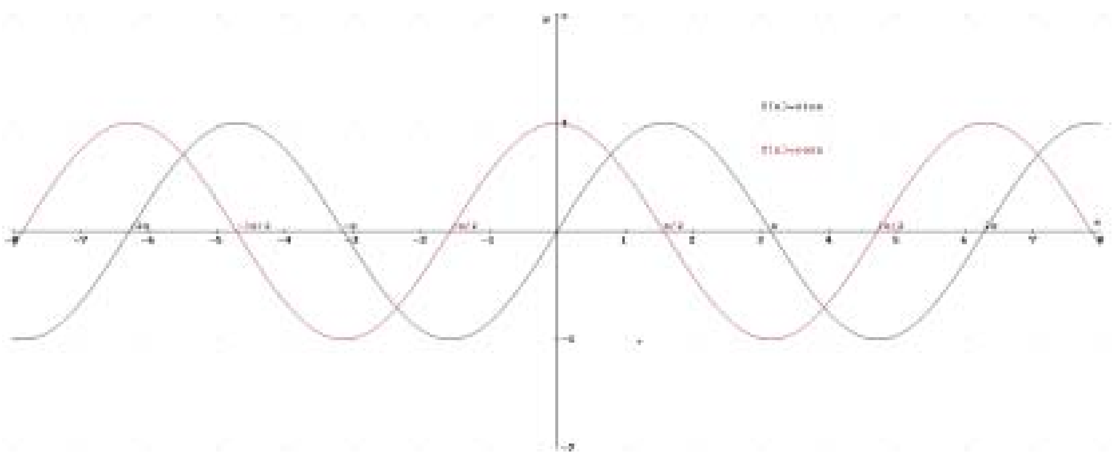
(6) Winkelfunktion

a. **Sinusfunktion:** $f(x) = k \cdot \sin(l \cdot x + j) + m$

k … Graph wird höher/niedriger
l … Graph wird enger/weiter
m … Verschiebung entlang der y-Achse
j … Verschiebung entlang der x-Achse

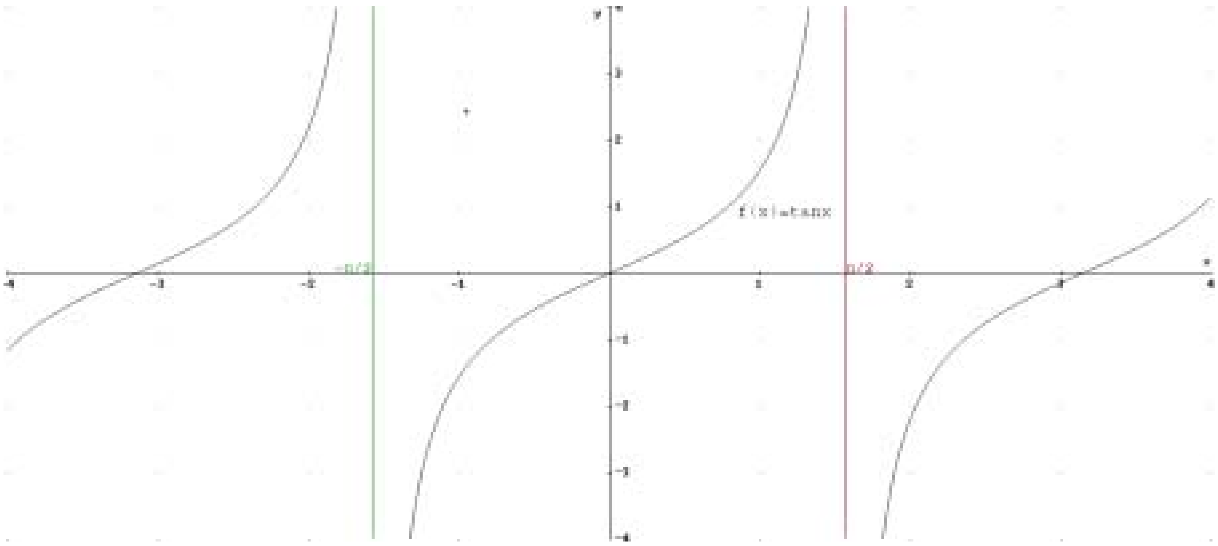
b. **Kosinusfunktion:** $f(x) = k \cdot \cos(l \cdot x + j) + m$

Parameter siehe (6)a. Sinusfunktion



c. **Tangensfunktion:** $f(x) = k \cdot \tan(l \cdot x + j) + m$ k … Graph wird steiler/flacher

Restliche Parameter siehe (6)a. Sinusfunktion



§2 Zuordnungen/Relationen – Gleichungen und Graphen

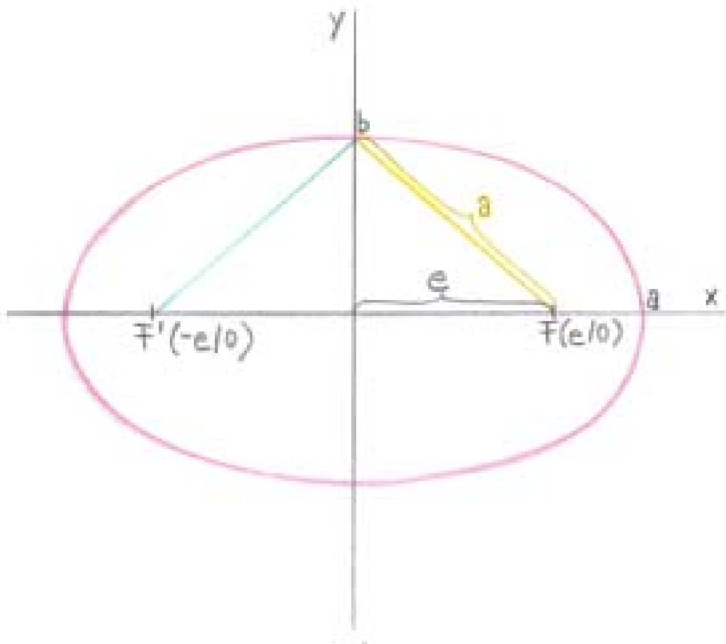
(1) Ellipse

$b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2$, wobei $a^2 = e^2 + b^2$ [e … siehe Skizze]

gleichseitige Ellipse ($a = b$):

$a^2 \cdot x^2 + a^2 \cdot y^2 = a^4 \,/ : a^2 > 0$

$x^2 + y^2 = a^2$

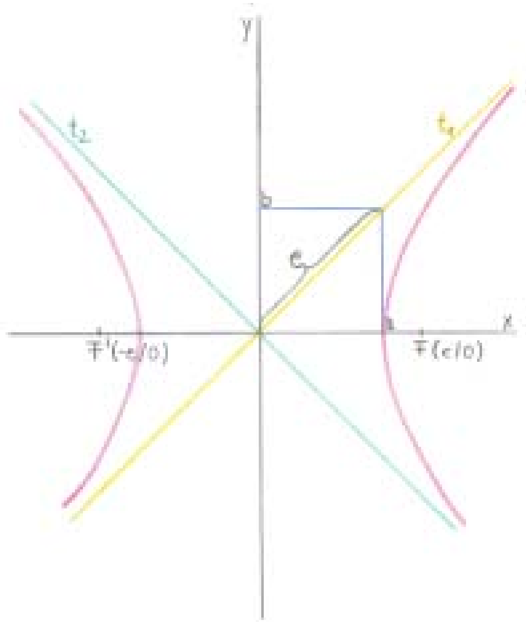


1. Hauptlage
F, F' … Brennpunkte

(2) Hyperbel

$b^2 \cdot x^2 - a^2 \cdot y^2 = a^2 \cdot b^2$, wobei $a^2 = e^2 - b^2$ [e … siehe Skizze]

gleichseitige Hyperbel ($a = b$):
$x^2 - y^2 = a^2$

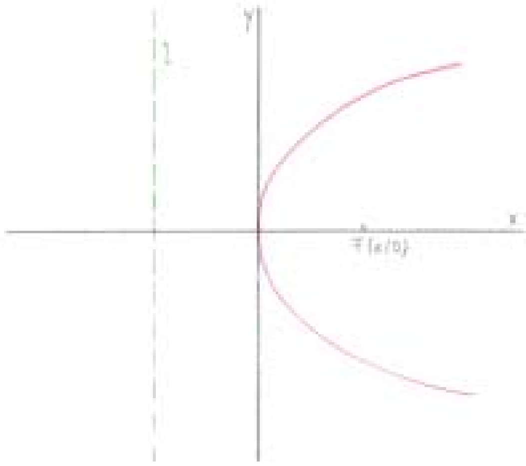


1. Hauptlage

t_1, t_2 … Asymptoten an f mit $t_1(x) = \frac{b}{a} \cdot x$, $t_2(x) = -\frac{b}{a} \cdot x$

(3) Parabel

$y^2 = 2 \cdot p \cdot x$ $\quad$ e [Brennweite] $= p/2$
F [Brennpunkt] $(e/0)$
l … Leitgerade

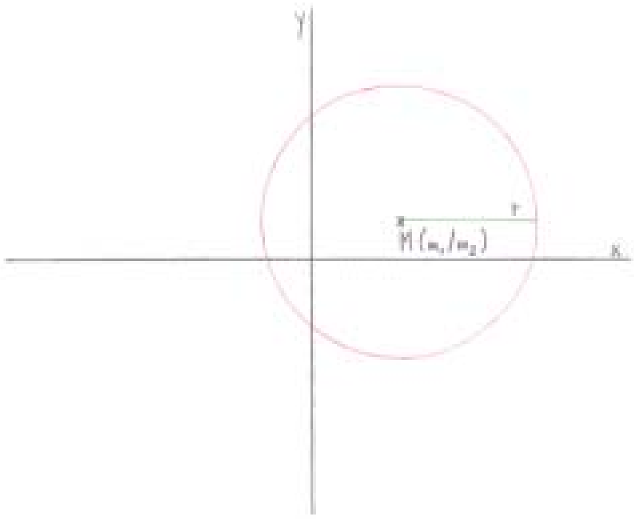


1. Hauptlage

(4) Kreis

$(x - m_1)^2 + (y - m_2)^2 = r^2$

m_1 … x-Koordinate/erste Koordinate des Mittelpunkts
m_2 … y-Koordinate/zweite Koordinate des Mittelpunkts
r … Radius



End of excursion

‘Calculate the intervalls in which boys are taller/shorter than girls’ leads to the integral calculus. Because several students will remember that growth is similar to the rate of change of body height, which can be calculated by using the differentiation, they are likely to search for the antiderivatives of the given functions. The antiderivatives of most given functions can be calculated manually or by using computer algebra systems. Students have to face the difficulty of not finding an explicit antiderivative of the second boy function, which calls for an approximation by step functions. At this point, a teacher/scaffolder should initiate a discussion about the function f(x) = e^{-x^2} [Hu, 2003, 87].

To sum up the research on the mentioned examples of the integral calculus, I introduce the term *difference of constitution* [Hu, 2003, 29]. ‘Water Consumption’, ‘Tachometer’ and ‘Growth of Boys and Girls’ lead to various aspects of the integral calculus: ‘Water Consumption’ calls for an approximation with step functions and an equidistant fragmentation, ‘Tachometer’ focuses on the a-aspect and inequidistant fragmentation and ‘Growth of Boys and Girls’ leads to the p-aspect, the m-aspect, the ap-aspect and the rules of integration. Despite their different constitutions, all three examples consist of vivid images of accumulation and overall effect [compare to Hu, 2003].

6 Equal exchange model by Cohen

Cohen distinguishes between two types of tasks using the notion of exchange models to describe the quality of exchange that takes place when students work in small groups. He says that students either have a limited exchange model of working together (first type) or equal exchange (second type) because of the task's suitability for group work. Solving (quadratic) equations and calculating the area of a triangle are examples for the first type, intentional problems for the second type. As it is also illustrated by Berry and Sahlberg, most group members have a greater benefit when working on equal exchange tasks in mathematics [Ber I, 2002, 12-15].

Berry and Sahlberg explain both types of tasks by saying that first *each group member may take it in turns to solve a problem and the other students watch to check that the method and solution are correct and that they understand what is happening. They may each solve the same problem and talk through the solution.* [They call the exchange of] *information on how to proceed and information on content* the *major reason for students to interact* [Ber I, 2002, 12].

[Second,] *equal exchange requires a true group task that demands the use of multiple abilities where no one student could easily manage the task alone* [Ber I, 2002, 13].

Example for equal exchange: see 5.1.9 Examples for SGL, category (D)

I took Table 8 out of [Ber I, 2002] and adopted it to my paper by adding the last unit about 'Proving mathematical ideas, concepts or statements (F)' and changing several statements. All my changes are italized to emphasize my own understanding of the mathematical tasks' nature compared to Berry and Sahlberg's understanding. Table 8 combines all categories of my typology of school tasks (see 5.1 A proposal on typology of tasks in school mathematics) with Cohen's exchange model. Furthermore, it recalls each category's nature for a better understanding [compare to Ber I, 2002, 13].

Category	**Type of exchange model**	**Nature of task**
Drilling basic skills (A1)	Very limited in each way	Closed in terms of method and outcomes
Applying a formula or algorithm (A2)	(Very) limited in how to proceed or check results	Typically closed in terms of outcomes and also the methodology
Measuring and collecting data (A3)	Equal exchange is possible	Some openness in terms of methodology but rather closed in terms of outcomes
Archetypes of real world problems (B)	Equal exchange	Encountered in everyday life. *Should envolve mathematical models.* Openness may vary from rather closed to open.
Mathematical investigations (C)	Rich equal exchange is possible	Basic investigations are often closed in terms of outcomes but open to various methods. Extended investigations are typically open tasks.
Mathematical modelling (D)	Rich equal exchange is possible	Modelling tasks (are typically real problems that) require mathematical principles and formulas in order to be solved. These tasks are open in terms of procedures and outcomes.
Designing projects and studies in mathematics (E)	Rich equal exchange	Very open in all terms. The openness includes the setting of questions and selection of methods.
Proving mathematical ideas, concepts or statements (F)	*Very rich equal exchange*	*Most open mathematical tasks which are suitable for school after careful preparation.*

Table 8: Cohen's exchange model combined with my typology of school tasks [compare to Ber I, 2002, 13]

A task works well for equal exchange in a small group if it has the following features:

- *The problem has no unique correct answer*
- *Start with a problem or situation that is real to the student*
- *The problem is intrinsically interesting and rewarding*
- *The problem has a variety of approaches and uses a range of mathematical ideas*
- *It can last for a few lessons to many weeks developing 'staying power'*
- *The problem is open ended and contains many questions*
- *The activity ends with an outcome in the material world*

In contrast, a task does not work well for the equal exchange model if it has the following features:

- *The problem has a unique correct answer*
- *The problem can be solved more quickly and efficiently by one person than by a group*
- *The problem involves simple memorisation or learning routine rules or algorithms* [Ber I, 2002, 14].

7 Zone of Proximal Development by Vygotsky

During studying Vygotsky's literature, I found several statements which can be seen as a proof for the necessarity of SGL at school. Vygotsky claims that *consciuosness and control appear only at a late stage in the development of a function, after it has been used and practiced unconsciously and spontaneously* [Vyg I, 1962, 90]. SGL is famous for working spontaneously, e.g. on problems, which are linked to the given task indirectly. *This suggests that unconscious self regulation should precede conscious self regulation, presumably appearing first on the social level between people (interpsychological) and then inside the child (intrapsychological) in an unconscious form* [Vyg II, 1978, 57].

SGL helps all group members to proceed from their actual developmental level to the level of potential development even without impact from people outside the group. *Good problem solving calls for using efficiently what you know* [Jon II, 1999, 7]. Pupils reach a high potential development while working with colleagues, which takes away their insecurity among a teacher. During SGL, group members with most knowledge and mathematical understanding offer the help needed to reach a higher potential development (compare to 4.3 Funnelling, focusing and scaffolding). As one can clearly see, SGL helps students bridging the so-called 'Zone of Proximal Development' (ZPD), which is

the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers [Vyg II, 1978, 86].

Personally, I refer to the Zone of Proximal Development as 'die der momentanen geistigen Leistungsfähigkeit nahe gelegene Erkenntnisentwicklung'. The school's intention is to pass this stage of understanding for a higher one [compare to Vyg I – Vyg IV, 1962, 1978, 1981, 1991].

Berry and Sahlberg refer Jones and Tanner, who use the phrase 'the learning zone' as synonym for ZPD. Table 9 concentrates on Berry and Sahlberg's categorisation of the effects of activities which are out of range, within the learning zone or at the current level of pupils' knowledge and understanding [compare to Ber I, 2002, 7/8].

Task's place	Out of range	Within the learning zone/ZPD	Current level
Description	Activities are too hard, the work goes over the students' heads	Activities can only be completed with help/guidance	Activities ask for familiar and secure knowledge
Students' and teacher's/*scaffolder's* behaviour	Off-task behaviour, poor discipline, teacher/*scaffolder* needed all the time	Working noise, teacher/*scaffolder* kept busy	Silent working, teacher/*scaffolder* not needed
Learning effect	No learning	Maximum learning	No *mathematical* learning or very little *mathematical* learning, *consolidation of current knowledge*
Suitable categories out of my typology of tasks in school mathematics	---	*B, C, D, E, F*	*A1, A2, A3*

Table 9: The effects of tasks either within the ZPD or out of range [compare to Ber I, 2002, 8]

I adopted Table 9 to SGL and combined it with my typology of tasks in school mathematics. All my table's changes compared to Berry and Sahlberg's suggestion are italiziced. First, I put 'scaffolder' (see 4.3 Funnelling, focusing and scaffolding) next to 'teacher'. Although small groups may need the guidance of a teacher in certain situations, a scaffolder can take over the teacher's place. As Berry and Sahlberg quote, *in the context of quality interaction between pupils working in collaborative small groups, the role of the busy teacher also includes the pupils, who are giving explanations to their peers* [Ber I, 2002, 8].

Second, I stressed the importance of dealing with activites placed at the current level from time to time to consolidate current knowledge. Therefore, I added 'consolidation of current knowledge' to 'no mathematical learning or very little mathematical learning' in the column about the learning effect at one's current level. I quote 'mathematical learning' instead of 'learning' in this column. Tasks addressing the students' current level lead to no mathematical learning/very little mathematical learning indead, but learning (e.g. learning social skills) always takes place during SGL.

Berry and Sahlberg point out the most important feature of small group tasks by saying that *productive interactive tasks that lead to good learning of new knowledge and skills need to fit into the learning zone* [Ber I, 2002, 8].

8 Conclusion and discussion

8.1 Advantages of SGL

Small group arrangements enrich the traditionally one-way communication structure of classrooms towards multi-lateral interactive learning environments in which new abilities and attitudes are necessary in order to create productive settings for learning and understanding. Those new abilities and attitudes consist of social skills, self-confidence, taking comfort in dealing with new mathematical problems, independency, the ability to search for helpful material discretly and to express one's own thoughts and ideas. In addition, pupils develop staying power because the work of a small group can last for a few days or even a week. I will give further explanations in the following paragraphs.

In mathematics, the '='-symbol conveys a moral message. It not only declares that the left-hand side is exactly equal to the right-hand side, but also imposes on the person who writes it the responsibility of being able to explain how one can transform one side strictly according to the rules to obtain the other side. In contrast, the '='-button on the calculator is a completely different animal. Its nearest equivalent in everyday language is the magician's abracadabra. In the century of computer algebra systems (CAS), students are likely to rely on CAS such as calculators without proving the results they get. Gardiner states that he has seen a marked increase in the 'infantile' assumption that it is the job of the teacher to lean over backwards to try to make sense of whatever the student happens to utter verbally or in writing [Ga, 1995, 528]. Serendipitously, pupils who work in small groups seem to be aware of their own responsibility to be capable of proving and explaining single steps of their solution to other group members. Gardiner's statements are directly linked to Alrø and Skovsmose's comments about inquiry: *Entering an inquiry means taking control of the activity in terms of ownership. The inquiry participants own their activity and they are responsible for the way it develops and what they can learn from it* [Al, 2004, 119]. To sum up Gardiner's idea, communication is directly linked to concepts like 'empowerment' and 'emancipation' [compare to Ga, 1995].

Berry and Sahlberg compliment a couple of categories of positive effects of SGL. First, they believe that cooperative learning, if conducted appropriately and carefully, brings about equal academic achievement among all students compared to more traditional methods of teaching. It is apparent that this fact can be explained by the ‘developing levels’ of Vygotsky. He claims that learning is a movement from a lower developing level to a higher one. Learning takes place in a small group because the difference between various levels leads to confusion on the side of the individuum at the lower developing level. An increasing balance between group members results. In other words, people from lower developing levels automatically scale to a higher level. Members of higher levels take advantage of this development, too, as they consolidate their knowledge by explaining mathematical concepts to their colleagues. The best learning effect results of group members who are located at different developing levels. Notwithstanding this, the levels shouldn’t be too far away from each other [compare to Hu, 2003, 18, Vyg I – Vyg IV, 1962, 1978, 1981, 1991].

Second, small group members come up with many constructive activities. However, it is paramount to note that in many cooperative learning situations students need to be trained and guided to perform and think according to the principles of learning together. In particular, the promotion of helping behaviour during cooperative learning (see 4.3 Funnelling, focusing and scaffolding) has appeared to be one precondition for successful implementation of SGL methods.

Third, SGL has a positive effect on pupil’s meta-cognitive development, often closely-related to problem-solving processes [Ber I, 2002, 4f].

The third positive effect listed by Berry and Sahlberg is directly related to Hußmann’s methodical skills, which are part of his arrangement of all skills necessary for effective SGL. Examples for methodical skills are

- structuring the working process (planing, developing, reflecting, presenting),
- using tools,
- methods of problem solving,
- strategies of argumentation and communication techniques,
- learning strategies (metacognition),
- motivating oneself and other group members,
- being able to evaluate one’s own abilities,
- record ideas and solutions in a written way.

He mentions two more categories of skills, namely professional skills and personal skills.

According to his theory, professional demands should be covered by the pupils themselves. A teacher can point out the steps of their colleagues or create subtasks to make sure that pupils conquer their mental barriers.

It is essential to note that Hußmann divides personal skills into those which concern one and those which mean an appropriate association with colleagues. First, he points out the necessarity of presenting oneself in an advisable way and holding a fitting self-perception. Second, pupils need to possess a moderate idea of man so that they treat feelings and adjustments of every person respectfully [Hu, 2003, 58f].

In addition, he identifies the fact that it's not the smooth representation of terms and relations that leads to appreciation, but coming across barriers and resolving them enforces the learning process. Such irritation is especially important for SGL as it arouses the student's interest much more than the simple representation of a mathematical task. Then, students experience their own position as unsecure and try to clear such imbalance. Piaget calls it accommodation and assimilation. Assimilation means adapting the environment to your own structures. If it's impossible to do so, you will adapt your structures to the environment, which is called accomodation. Assimilation helps to stabilize your knowledge, whereas accommodation produces new knowledge. Learning arrangements should include both accommodation and assimilation [Hu, 2003, 13/14].

Freire's [Fr, 1972, 75f] understanding of cooperation goes along with the personal skills mentioned by Hußmann. According to Freire, taking part in a discussion presupposes some kind of humility. He says that *you cannot enter a dialogic relationship being self-sufficient. The participants have to believe in each other and to be open-minded towards each other in order to create an equal and faithful relationship* [Fr, 1972, 80f].

Pauli stresses another advantage of cooperative learning which improves the social structure of a classroom. By SGL, minorities in the social landscape of a classroom get involved much more than by a teacher directly addressing students, which may be too daunting for those minorities [Pa, 1998, 7].

Moreover, person-centered modes like SGL prepare students for democracy, whereas traditional modes socialise students to obey power and control. In the person-centred mode, the environment is trustful and responsibility for learning processes is shared. Rogers mentions that *the facilitator provides learning resources* [and] *the students develop their program of learning alone and in co-operation with others*. [Pupils get opportunities to learn how to learn and to guarantee a continuing process of learning by establishing self-discipline and self-evaluation.]; [Ro, 1994[3], 214].

Berry and Sahlberg mention that cooperative learning is promising a more positive impact on one's level of confidence. In my opinion, this can be explained by the fact that while working together in small groups pupils aren't afraid of asking for explanations for mathematical topics they don't understand. Colleagues, in contrast to teachers, don't blame them for not concentrating on the topic or being unable to follow [compare to Ber I, 2002, 4].

In compliance with Pauli, I hold SGL in high regard for its non-competitive learning environment. During SGL, students don't feel pressure to stand up against their colleagues. Notwithstanding, small group members get confronted with controversial opinions many times. Those disputes should evoke an epistemological interest, which is characterised by the urge for looking up new information and allows students to dissolve any problem constructively [Pa, 1998, 27].

Stebler explains the current concept of mathematics as a science. The 'science of number and space' changed to a 'science of patterns'. For example, it is more important that students can prove whether a function is increasing than to calculate the amount of increase right away. Personally, I am convinced that SGL boosts the current concept of mathematics as a science. Small groups talk about a task much more than pupils who are working alone or with a teacher in a traditional way so they are more likely to listen to their intuition and concentrate on qualitative aspects before determining quantiative aspects [Ste, 1999, 8].

Gardiner draws an outstanding picture of the most important feature of learning. *Some things in life can only be learned and understood by achieving fluency from the inside. If you wish to learn swimming you have to go into the water* [Ga, 1995, 527]. In a small group, students have to find appropriate ways of dealing with difficulties independently by getting an insight. Without any doubt, it is an advantage that those approaches, which develop during SGL, fit nicely with the students' ways of thinking.
During SGL, pupils learn to ask genuine questions, *ones that seek to understand the rules that govern why people do what they do as much as to challenge what they do […] balancing advocacy and inquiry means stating clearly and confidently what one thinks and why one thinks it, while at the same time being open to being wrong* [Is II, 1999, 188].

SGL is one way to highlight ‘significant learning’: *By significant learning I mean learning which is more than accumulation of facts. It is learning which makes a difference – in the individual’s behaviour, in the course of action he chooses in the future, in his attitudes and in his personality. It is pervasive learning which is not just an accretion of knowledge, but which interpenetrates with every portion of his existence* [Ro, 1994^3, 280].
Significant learning seems to correspond with Colaizzi’s ‘genuine learning’. *Genuine learning means having integrated the subject matter in ones person (body and mind). Significant or genuine learning can never be forced upon anybody. This kind of learning always occurs in relation to others, and it changes the learner in a radical way that cannot be predicted. In that sense learning is also connected with taking risks* [Al, 2004, 133].

8.2 Disadvantages of SGL

Although SGL trains students to express themselves correctly and comprehensibly, it leads to a disadvantage mentioned by Gardiner. The basic language of mathematics, which includes the algebra of expressions or forms, is one of its central pillars. Both the objects of this language and the methods, which are used by mathematics to transform and analyse these objects, are in principle absolutely exact. Mathematical teachers aspire such exactness. Consequently, they are afraid that students don’t use the mathematical language and its terms correctly [Ga, 1995, 527].

Rogers mentions a problematic aspect of cooperative learning: *I have slowly come to realize that it is in its politics that a person-centred approach to learning is most threatening. The teacher or administrator who considers using such an approach* [SGL, for instance] *must face up to the fearful aspects of sharing of power and control. Who knows whether students or teachers can be trusted, whether a process can be trusted? One can only take the risk, and risk is frightening* [Ro, 1994^3, 214].

Another disadvantage of small group activities is how much time they strain. This one may be the argument for not using cooperative learning in mathematics mentioned most of all.
To list pivotal disadvantages according to small groups in mathematics is inescapable to counteract too much enthusiasm about such kind of learning [compare to Ber I, 2002].

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10 Attachments

10.1 Curriculum vitae

Personal data

Name	Marlene Maria Jodl
Address	Kapuzinerstraße 1, 4020 Linz, Austria
E-mail	marlene.jodl@gmx.net
Place of birth	Linz
Date of birth	7th of July in 1986
Nationality	Austrian

Education

1992 – 1996	primary school at ‘Kreuzschwestern Linz’
1996 – 2004	secondary school with an emphasis on economy and social studies at ‘Kreuzschwestern Linz’
Oct 2004 – Feb 2009	teacher-training studies in mathematics, psychology and philosophy at the University of Vienna, Austria
Sept 2006 – Jan 2007	exchange studies in mathematics at the University of Leeds, Great Britain

Traineeship

08/07/ – 02/08/ 2002	office assistent at ‘Seniorenzentrum Franckviertel’ in Linz
Aug/Sept 05, Aug 06	mathematics teacher at ‘Institut für Lernhilfe’ (IFL) in Linz
Sept 2006 – Jan 2007	University of Leeds primary tutoring volunteering scheme at Blenheim Primary School (Great Britain)
Nov 2007 – Aug 2008	mathematics teacher at ‘Landstraßer Gymnasium’ in Vienna

Voluntary work

2002 – 2004	assistent at church champs in Linz

Other qualifications

EDV	European computer driving license (18/02/2003)
Communication	Schul-Mediatorin (2000)

For written confirmation and verification of my qualifications/my personal data contact me.