DISSERTATION

Titel der Dissertation

„On characteristic Cauchy problems in general relativity"

Verfasser

Dipl.-Phys. Tim-Torben Paetz

angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2014

Studienkennzahl lt. Studienblatt:	A 791 411
Dissertationsgebiet lt. Studienblatt:	Physik
Betreuerin / Betreuer:	Univ.-Prof. Piotr T. Chruściel

Acknowledgments

First and foremost I would like to express my deep gratitude to my advisor, Prof. Piotr T. Chruściel. He gave me the opportunity to work on an extremely exciting and fascinating topical branch of general relativity. His dedication to science has been as motivating as inspiring for me. I am grateful for his continuous support and for being available whenever questions arose, but also for giving me a lot of freedom to focus attention on what I was interested in.

I should like to thank Prof. Helmut Friedrich for clarifying a couple of issues I had to face during this thesis and also for a joint project with him and Prof. Chruściel. It was a pleasure for me to witness both of them working.

The first lecture which I took on general relativity was given by Prof. Stefan Hollands during my diploma studies at the University of Göttingen. I am thankful to him for sparking my fascination for this beautiful theory.

Furthermore, many thanks to Carla Cederbaum, Beate Gula, Michael Hörzinger and Christa Ölz for proofreading and various useful comments.

I would also like to thank the members and former members of the Vienna gravity group, who made me always feel comfortable at the institute. In particular, I should mention
– Walter Simon for a couple of get-togethers outside of science, but also for a joint project within science which led to a paper which unfortunately could not be placed in this thesis;
– Patrick Arnold for being a great aid to understand the Austrians, in the truest sense of the word (I know it is still a long way to go), which helped me a lot to get along and acclimatize in Vienna;
– all the various people who had to share an office with me during the last 3.5 years.

Beyond my doctoral studies, improvisational theater became a very important part of my life. I am beholden to all the people who I met there on one or the other session, workshop etc. My special thanks go to the "inner core" of this group which meanwhile has been playing and clinging together for several semesters: Andi, Bernadette, Claudia, Clemens, Fritz, Lukas and Verena.

Finally, I am deeply thankful to my parents for their constant support and for giving me the opportunity to pursue and realize my aims and wishes.

This work was supported in part by the Austrian Science Fund (FWF) P 24170-N16.

"Study and in general the pursuit of truth and beauty is a sphere of activity in which we are permitted to remain children all our lives."
(Albert Einstein)

Contents

CHAPTER 1

Introduction

"Scarcely anyone who fully understands this theory can escape from its magic."
ALBERT EINSTEIN (1879–1955)

1.1 General relativity and Einstein's field equations

Einstein's theory of general relativity [43] (compare e.g. [13, 28, 62, 86]) provides a well-accepted geometric description of *gravity*, one of the four fundamental interactions in physics, beside the *strong* and *weak nuclear interactions* and *electromagnetism*. Albeit being by far the weakest of these interactions, it is the dominant one on large scales: Both nuclear interactions are short-ranged, and macroscopic objects tend to be electrically neutral, so that repulsion of like charges and attraction of opposite charges is nearly balanced, whereas gravity appears to be always attractive.

General relativity is based on merely a few simple principles and, due to its geometric elegance, often regarded as one of the most beautiful theories in the natural sciences. The most crucial input into Einstein's theory of space, time and gravity is the observation that all particles are affected by gravity in exactly the same manner. This universality, the *equivalence principle*, allows for a treatment of gravity as a purely geometric effect rather than a force, as which it is described in Newtonian mechanics. Properties of the gravitational field can be ascribed to the structure of space-time itself, namely its curvature: space-time is not flat but curved. Test bodies, only subject to gravity, still move along paths that are "as straight as possible", as it has already been postulated in Newtonian mechanics. In the relativistic setting, though, this means that their world lines are *geodesics*, a generalization of "straight lines" to curved geometry. A consequence is that the earth does not orbit around the sun because the sun generates a gravitational force field attracting the earth, but because its mass curves space-time in such a way that the new "straight lines" are circles, ellipsoids etc. around the sun. The notion of an absolute gravitational force is generally meaningless in general relativity. Nonetheless, it can be defined in certain limits, whence, in such a limit, the motion of an object in a curved geometry looks like its motion in a flat background with a force acting on it, as one is used to from Newtonian mechanics.

At the heart of general relativity lie *Einstein's field equations* which read, in units where $c = G = 1$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} \;=\; 8\pi T_{\mu\nu} \;. \tag{1.1}$$

Here, $g = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the *metric tensor*, $R_{\mu\nu}$ the *Ricci tensor*, $R = g^{\mu\nu}R_{\mu\nu}$ the *curvature scalar*, λ the *cosmological constant*, and $T_{\mu\nu}$ the *stress-energy-momentum tensor*. Einstein's equations describe in which way the presence of matter curves space-time, and how the curvature of space-time in turn influences the motion of the matter. They form a complicated system of non-linear partial differential equations (PDEs) for the gravitational field, which is represented in general relativity as a pseudo-Riemannian metric g of Lorentzian signature $(-,+,\dots,+)$ on the underlying topological space-time manifold $\mathscr{M}$. The non-linearity of the Einstein equations expresses a distinctive feature of the gravitational field, namely that it is self-interacting.

To restrict attention to a physically reasonable class of energy-momentum tensors, $T_{\mu\nu}$ is required to satisfy certain *energy conditions*. In vacuum, characterized by the absence of matter, that is by a vanishing energy-momentum tensor, the Einstein equations reduce to

$$R_{\mu\nu} \;=\; \frac{2}{n-1}\lambda g_{\mu\nu} \;, \tag{1.2}$$

where n denotes the dimension of space. Even the vacuum case turns out to be fairly complicated to analyze, and already comes along with a huge variety of solutions describing gravitational waves, black holes, singularities, etc.

Throughout we assume $n \geq 3$, though some of the results might remain equally valid for $n = 2$.

1.2 Outline

On the way to gaining a complete understanding of (the implications of) Einstein's general relativity, a thorough analysis of properties and peculiarities of solutions to his field equations is essential. The only known way to systematically construct general solutions thereof without any simplifying symmetry assumptions is in terms of initial (boundary) value problems, or *Cauchy problems*, where existence and uniqueness of solutions is deduced from appropriately specified initial data and initial surfaces. The two "standard" Cauchy problems in general relativity are the space-like and the characteristic Cauchy problem, where data are prescribed on space-like and null hypersurfaces, respectively. Here, local-in-(retarded-)time well-posedness results are available [16, 19, 27, 44, 67, 76]. These results as well as the *wave-map gauge condition* which we shall impose throughout this work are recalled in Chapter 2.

This thesis is based upon the papers [69–72] and, in collaboration with P.T. Chruściel, [34–38], all of which are attached as Chapters 7-15. Its main focus lies on the construction of solutions to Einstein's field equations via (characteristic) initial value problems such that the emerging space-times exhibit specific properties:

- Instead of "ordinary" initial value problems one may consider "asymptotic" initial value problems where (parts of) the data are prescribed "at infinity". This is extremely useful for the construction of space-times, solving Einstein's vacuum field equations, which have some global properties in the sense that they have a certain "asymptotically flat or de Sitter-like"-structure at infinity. In particular, asymptotic initial value problems provide a mean to construct solutions which are "large" in the sense that they have an infinite extension, at least in certain directions. Well-posedness results are available in

the space-like case [52], i.e. for a space-like hypersurface which represents "null infinity" (which requires $\lambda > 0$), and for two transversally intersecting null hypersurfaces, one of which representing "null infinity" [63] (this requires $\lambda = 0$).
One main result of this thesis is to establish a well-posedness result where data are prescribed on the light-cone emanating from a point which represents "past time-like infinity" [36], attached as Chapter 9. It relies on previous work by Friedrich [57], Chruściel [27], and a novel system of wave equations derived in [69], attached as Chapter 8, which provides a substitute to Friedrich's conformal field equations.

- In [38, 72], attached as Chapters 10 and 11, we provide the basis to establish another result for the construction of asymptotically flat vacuum space-times starting from an ordinary characteristic Cauchy problem. We extract necessary and sufficient conditions which ensure that, on the initial surface, a smooth extension of the space-time is possible "across infinity". This is achieved by using a novel gauge scheme to integrate the characteristic constraint equations which we develop in [34], attached as Chapter 7, compare [72], attached as Chapter 11. It will be proved elsewhere that such initial data, indeed, lead to asymptotically flat space-times.

- Another important class of space-times consists of those which possess non-trivial isometry groups. These are generated by Killing vector fields. One way of systematically constructing vacuum space-times with Killing vector fields is to do so in terms of a space-like Cauchy problem for appropriately chosen initial data. This is described in [8]. In [35], attached as Chapter 13, we follow the same procedure for the characteristic Cauchy problem. In [70, 71], attached as Chapters 14 and 15, we extend these results to the asymptotic space-like and characteristic Cauchy problems.

- Finally, an important quantity characterizing asymptotically flat space-times is its mass, such as its ADM and Bondi mass [4, 9, 78]. The Bondi mass provides a way to measure the amount of energy a system is loosing by virtue of radiation. In [37], attached as Chapter 12, we give an elementary derivation of a formula which expresses the Bondi mass of a globally smooth light-cone in terms of the initial data given there, and which, in addition, is manifestly positive. We thus obtain a simple and direct proof for the positivity of the Bondi mass in this setting. Moreover, such a formula might be useful to derive a priori estimates for e.g. the stability analysis of black holes.
Here, we present a formula somewhat more general than that in [37], allowing more flexible gauges, which is why some of the calculations are included in Chapter 5.

The definition of asymptotically flat and asymptotically de Sitter space-times in terms of Penrose's method of conformal rescaling is recalled in Chapter 3. There, we also introduce Friedrich's conformal field equations, which replace Einstein's vacuum field equations in a conformally rescaled space-time, and give an overview over the relevant gauge degrees of freedom. In Chapter 4, we discuss the various types of asymptotic initial value problems and present our system of conformal wave equations. Chapter 5 contains an introduction to the notion of mass (or rather energy) in general relativity and we describe our approach to determine the Bondi mass of a globally smooth light-cone. In Chapter 6, we explain the construction of vacuum space-times with Killing vector fields via Killing initial data sets for the various ordinary and asymptotic Cauchy problems discussed afore. We conclude with a discussion of our results in Chapter 16, where we also raise some open questions related to the topics addressed in this thesis. In Appendix A, we collect a couple of mathematical well-posedness results on wave equations which play a significant role in this work.

Light-cones and two transversally intersecting null hypersurfaces are characteristic initial surfaces of particular interest. Here, we shall focus on light-cones. Two intersecting null hypersurfaces can be treated similarly, although their analysis requires a discussion of the free data on the intersection manifold which we want to avoid here for reasons of simplicity. The reader will be referred to the references given in the text, where both cases are treated in detail.

Besides providing an overview of the current state of research and giving a brief description of the main results obtained in the attached papers, Chapters 7-15, we also want to stress analogies between the space-like and the characteristic case, which is why we shall treat them simultaneously whenever feasible. We shall analyze and compare their peculiarities and difficulties whenever things are differing. Since we are particularly interested in summarizing and relating our results to each other and providing some physical motivation for their underlying questions, we will ignore the technical details here. They are taken care of in detail in the attached papers.

If not explicitly stated otherwise, all manifolds, fields, expansion coefficients etc. are assumed to be smooth.

1.3 Overview of the attached papers

In Chapters 7-15 we have attached all the papers on which this thesis is based. Here we give an overview over all the manuscripts, where they are published or submitted etc., cf. page 6 for a statement on the co-authored papers:

1. "The many ways of the characteristic Cauchy problem", with P. T. Chruściel, published in Classical and Quantum Gravity **29** (2012) 145006, arXiv:1203.4534 [gr-qc].

   Abstract. *We review various aspects of the characteristic initial value problem for the Einstein equations, presenting new approaches to some of the issues arising.*

2. "Conformally covariant systems of wave equations and their equivalence to Einstein's field equations", accepted for publication in Annales Henri Poincaré (28.04.2014), arXiv:1306.6204 [gr-qc].

   Abstract. *We derive, in* $3+1$ *spacetime dimensions, two alternative systems of quasi-linear wave equations, based on Friedrich's conformal field equations. We analyse their equivalence to Einstein's vacuum field equations when appropriate constraint equations are satisfied by the initial data. As an application, the characteristic initial value problem for the Einstein equations with data on past null infinity is reduced to a characteristic initial value problem for wave equations with data on an ordinary light-cone.*

3. "Solutions of the vacuum Einstein equations with initial data on past null infinity", with P. T. Chruściel, published in Classical and Quantum Gravity **30** (2013) 235037, arXiv:1307.0321 [gr-qc].

   Abstract. *We prove existence of vacuum space-times with freely prescribable cone-smooth initial data on past null infinity.*

4. "Characteristic initial data and smoothness of Scri. I. Framework and results", with P. T. Chruściel, submitted to Annales Henri Poincaré (17.03.2014), arXiv:1403.3558 [gr-qc].

Abstract. *We analyze the Cauchy problem for the vacuum Einstein equations with data on a complete light-cone in an asymptotically Minkowskian space-time. We provide conditions on the free initial data which guarantee existence of global solutions of the characteristic constraint equations. We present necessary-and-sufficient conditions on characteristic initial data in $3+1$ dimensions to have no logarithmic terms in an asymptotic expansion at null infinity.*

5. "Characteristic initial data and smoothness of Scri. II. Asymptotic expansions and construction of conformally smooth data sets", submitted to the Journal of Mathematical Physics (31.03.2014), arXiv:1403.3560 [gr-qc].

   Abstract. *We derive necessary-and-sufficient conditions on characteristic initial data for Friedrich's conformal field equations in $3+1$ dimensions to have no logarithmic terms in an asymptotic expansion at null infinity.*

6. "The mass of light-cones" , with P. T. Chruściel, published as a Fast Track Communication in Classical and Quantum Gravity **31** (2014), 102001, arXiv:1401.3789 [gr-qc].

   Abstract. *We give an elementary proof of positivity of the Trautman-Bondi mass of light-cones with complete generators.*

7. "KIDs like cones", with P. T. Chruściel, published in Classical and Quantum Gravity **30** (2013) 235036, arXiv:1305.7468 [gr-qc].

   Abstract. *We analyze vacuum Killing Initial Data on characteristic Cauchy surfaces. A general theorem on existence of Killing vectors in the domain of dependence is proved, and some special cases are analyzed in detail, including the case of bifurcate Killing horizons.*

8. "KIDs prefer special cones", published in Classical and Quantum Gravity **31** (2014) 085007, arXiv:1311.3692 [gr-qc].

   Abstract. *As complement to Class. Quantum Grav. 30 (2013) 235036 we analyze Killing initial data on characteristic Cauchy surfaces in conformally rescaled vacuum spacetimes satisfying Friedrich's conformal field equations. As an application, we derive and discuss the KID equations on a light-cone with vertex at past timelike infinity.*

9. "Killing Initial Data on space-like conformal boundaries", submitted to the Journal of Geometry and Physics (11.03.2014), arXiv:1403.2682 [gr-qc].

   Abstract. *We analyze Killing Initial Data on Cauchy surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. As an application, we derive the KID equations on a spacelike $\mathscr{I}^-$.*

Univ. Prof. Dr. Piotr T. Chruściel
Gravitational Physics, Head
Boltzmanngasse 5
A 1090 Wien, Austria

May 28, 2014

PhD Committee
Faculty of Physics
University of Vienna

Dear Colleagues,

Tim Paetz's Thesis consists of an extensive introduction, reviewing the field and summarizing the work done, written by him, and several papers which were written either by himself, or with me. Hereby I declare that our joint papers have been written in an evident scientific collaboration, with many key and substantial contributions from Tim. In each case it is impossible to say precisely which part of the paper is due to him and which to me, as the papers arose as a result of many conversations concerning the problems at hand. I would estimate that on average across all *joint* papers, about half of the ideas are due to me and half to Tim. This concerns both key and technical ideas. While the papers written by Tim as sole author are clearly related to the remaining ones, they are entirely his own work. None of the papers would have been conceived, started, and completed without his contributions. He has moreover done most of the difficult calculations arising.

Sincerely yours,

Piotr T. Chruściel
Professor of Gravitational Physics

CHAPTER 2

Initial value problems in general relativity

"Prediction is very difficult, especially about the future."
NIELS H. D. BOHR (1885–1962)

2.1 Initial value problems

A basic issue in theoretical physics is the so-called *Cauchy problem* or *initial value problem.* One considers a freely evolving physical system at a certain instant of time and asks, if the specification of certain quantities guarantees the existence and uniqueness of a solution of the corresponding dynamical system which describes the evolution. The Cauchy problem is of conceptual importance since it concerns the predictability of the underlying theory. Practically, the state of a system can only be measured with finite accuracy, so the system should not be strongly affected by small changes of the initial conditions. One therefore asks for a continuous dependence of the solution on the prescribed data ("*Cauchy stability*"). If all these requirements are satisfied the Cauchy problem is *well-posed.* In classical mechanics, for instance, where the dynamical system is described by a system of ordinary differential equations (ODEs) the Cauchy problem is satisfactorily solved by the Picard-Lindelöf theorem (cf. e.g. [3]), a well-posedness result for ODEs.

The Cauchy problem also arises in general relativity, where of course the concept of an absolute time is missing. However, one natural possibility there is to regard a space-like n-dimensional hypersurface as representing, roughly speaking, an instant of time, this leads to the *space-like Cauchy problem in general relativity.* Due to the Lorentzian structure of the space-time manifold there are other types of "initial surfaces" for which a Cauchy problem can be formulated:

- A *characteristic Cauchy problem* where data are given on (piecewise) null hypersurfaces, such as a light-cone or two null hypersurfaces intersecting transversally.
- Later on we shall also consider *asymptotic Cauchy problems* where these surfaces are (partly) located "at infinity".
- Another possibility is to prescribe (for $\lambda = 0$) data on an asymptotically hyperbolic space-like hypersurface which intersects "null infinity" in a space-like sphere (*"hyper-*

boloidal Cauchy problem") [1, 2, 49, 53]. In fact, this is another space-like Cauchy problem, though in the "classical" cases the initial manifold will be chosen to be either compact for cosmological models or asymptotically Euclidean to describe isolated bodies.

- Also *initial boundary value problems* [59] play a significant role, for instance in numerical relativity or when studying anti-de-Sitter-like space-times. Here, part of the data need to be specified on a time-like boundary, cf. e.g. [80] for an overview.
- Various mixtures of all these kind of Cauchy problems.

Since a space-like hypersurface represents the gravitational system at "an instant of time", the space-like Cauchy problem is closer related to the "classical" Cauchy problem than the characteristic one. Most of the results obtained in the course of this thesis are related to the characteristic Cauchy problem, let us therefore provide some motivation why it is interesting and relevant from a physical point of view, as well (cf. [76]).

First of all, a light-cone appears in a very natural way as initial (or rather as "final") surface. All the observations which are made of our universe lie on a family of past light-cones from the time when people have started observing the sky until now. This covers a rather modest-in-size set as compared to the observational scales involved. For many observational purposes this set is so thin that it can be considered to be a single past light-cone. The characteristic Cauchy problem with data on a past light-cone thus reflects the issue to what extent observational data determine a unique evolution of our universe backwards in time.

Furthermore, null hypersurfaces appear naturally in form of e.g. horizons, and one would like to have the possibility of prescribing data on these distinguished surfaces. Finally, gravitational radiation far away from its source is most conveniently analyzed "at null infinity": Whereas the (asymptotically flat) space-like Cauchy problem might be well-suited to tackle issues "at spatial infinity", the characteristic one is natural for questions related to null infinity, which, by means of an appropriate conformal rescaling, can be represented (for $\lambda = 0$) by null hypersurfaces (cf. Chapter 3). The characteristic Cauchy problem therefore appears naturally when dealing with the "asymptotic Cauchy problems" mentioned above and discussed in Chapter 4.

While, for the reasons just explained, it is natural to think of Cauchy problems with all these different types of initial surfaces, it is far away from being evident that such a Cauchy problem is well-posed in general relativity. Indeed, the Einstein equations do not have a structure to which any of the known mathematical well-posedness results for PDEs directly applies. Now, it is a feature of general relativity, expressed by the tensorial nature of Einstein's equations, that they are coordinate-independent, leaving a lot of gauge freedom hidden in the equations. Due to this gauge freedom one actually cannot expect them to admit unique solutions. What one can expect at most is uniqueness up to gauge-, i.e. diffeomorphism-invariance. In fact, diffeomorphic space-times are physically indistinguishable.

It was the monumental discovery of Choquet-Bruhat [44] that imposing a *harmonic gauge condition* on the coordinates, the disturbing gauge degrees of freedom are removed and the Einstein equations do split into a system of hyperbolic wave equations, the evolution equations, for which well-posedness results *are* available, and a system of constraint equations, propagated by the evolution equations. It is the hyperbolic character of the evolution equations which ensures a causal propagation of the gravitational field.

In the case of a space-like Cauchy problem, the constraint equations can be written as an elliptic PDE-system. It is one of the attractive features of the characteristic Cauchy problem that the constraints can be read as an ODE-system, which is much easier to deal with, whence it comes along with several advantages from a technical point of view to be discussed below.

2.2 Hyperbolic reduction of Einstein's field equations

To realize the above mentioned splitting of the Einstein equations into constraint and evolution equations we impose an appropriate gauge condition. We shall work in a *($\hat{g}$-generalized) wave-map gauge* [16, 50, 54],

$$H^\lambda = 0 , \tag{2.1}$$

which is characterized by the vanishing of the *(generalized) wave-gauge vector*

$$H^\lambda := \Gamma^\lambda - V^\lambda , \quad \text{where} \quad V^\lambda := \hat{\Gamma}^\lambda + W^\lambda , \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} , \quad \hat{\Gamma}^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta} . \tag{2.2}$$

We use the hat-symbol "ˆ" to denote objects associated with some *target metric* $\hat{g}$. The vector field $W^\lambda = W^\lambda(x^\mu, g_{\mu\nu})$ can be freely specified or chosen to fulfill ad hoc equations. It is allowed to depend upon the coordinates and $g_{\mu\nu}$, but not upon derivatives thereof, and reflects the freedom to choose coordinates off the initial surface. Indeed, by an appropriate choice of coordinates it can locally be given any preassigned form, and conversely W^λ determines coordinates via a system of wave equations and suitable data on the initial surface [54], as which we will take coordinates adapted to the geometry. When solving an initial value problem it is the other way around. The coordinates are given and the metric tensor is constructed in such a way that the coordinates become *generalized wave coordinates* which satisfy (2.1).

The wave-map gauge is a generalization of Choquet-Bruhat's classical harmonic gauge. As will be discussed later on it is the addition of the *gauge source functions W^λ*, introduced by Friedrich [50], which, in the characteristic case, bring in a flexibility crucial to solve a couple of problems we shall be dealing with.

A *hyperbolic reduction* of the Einstein equations is realized when the Ricci tensor is replaced by the *reduced Ricci tensor in wave-map gauge*, namely

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma . \tag{2.3}$$

The so-obtained *reduced Einstein equations*

$$R^{(H)}_{\mu\nu} - \frac{1}{2}R^{(H)}g_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi T_{\mu\nu} , \tag{2.4}$$

form, in vacuum or for suitable matter models, a system of quasi-linear wave equations for which well-posedness results are available, cf. Appendix A (in the non-vacuum case equations for the matter fields need to be added).

However, once (2.4) has been solved, one needs to make sure that the so-obtained solution is consistent with the gauge condition, i.e. it should satisfy $H^\lambda = 0$. Only in that case one would end up with a solution of the original equations (1.1). Assuming that the stress-energy tensor satisfies the conservation law $\nabla_\nu T_\mu{}^\nu = 0$, it follows from (2.4) and the second Bianchi identity that the components of the wave-gauge vector fulfill a linear, homogeneous system of wave equations,

$$\nabla^\nu\hat{\nabla}_\nu H^\mu + 2g^{\mu\nu}\nabla_{[\sigma}\hat{\nabla}_{\nu]}H^\sigma = g^{\alpha\beta}\partial_\alpha\partial_\beta H^\mu + f_\alpha{}^\mu H^\alpha + f_\alpha{}^{\beta\mu}\partial_\beta H^\alpha = 0 , \tag{2.5}$$

for appropriate fields $f_\alpha{}^\mu$ and $f_\alpha{}^{\beta\mu}$. Due to standard uniqueness results for such equations [45, 84] the wave-gauge vector will vanish if and only the initial data for (2.5) vanish, which we denote by $[H^\lambda]$ (as a matter of course, which data need to be specified depends on the type of the initial surface). Equation (2.5) thus reflects the key fact that the gauge condition (2.1) and the constraint equations derived below are propagated by the evolutionary part of the Einstein equations, whence it merely needs to be ensured that they are initially satisfied.

We conclude that we can replace the Einstein equations by the reduced Einstein equations, which are much easier to handle mathematically, as long as we make sure that $[H^\lambda] = 0$. Note that $[H^\lambda]$ contains metric components (or rather transverse derivatives thereof) which are *not* part of the initial data for (2.4), so it is a non-trivial issue to make sure that $[H^\lambda]$ vanishes. A priori, it is not even clear whether this condition can be satisfied at all. It was the great discovery of Choquet-Bruhat [44] that the equations have a structure due to which it is possible. The geometric Cauchy problem for Einstein's field equations can be reduced to a Cauchy problem for a hyperbolic system, together with a system of constraint equations for the initial data.

In the next sections we shall review the space-like Cauchy problem for Einstein's field equations [7, 21, 25, 60] whose solution relies on the classical papers by Choquet-Bruhat [44] and Choquet-Bruhat & Geroch [19], and the characteristic one as studied most notably by Rendall [76] and Choquet-Bruhat, Chruściel & Martín-García [16]. Since the predominant part of this thesis deals with the characteristic case and only a minor part with the space-like case, our treatment of the characteristic Cauchy problem will be somewhat more detailed.

We shall write down most of the equations for a non-vanishing energy-momentum tensor. However, for some of the arguments below to work, one has to make certain assumptions on the matter models used. Since we are particularly interested in the vacuum case we have not studied these problems.

2.3 Space-like Cauchy problem

Consider a space-like hypersurface $\Sigma \subset \mathscr{M}$ in some Lorentzian manifold $(\mathscr{M}, g)$[1] and introduce *adapted coordinates* $(x^0 = t, x^i)$, $i = 1, \dots n+1$, such that $\Sigma = \{t = 0\}$ (on the portion covered by the coordinate system). The data for the reduced Einstein equations (2.4) are $g_{\mu\nu}|_\Sigma$ and $\partial_t g_{\mu\nu}|_\Sigma$, the data for (2.5) are $H^\lambda|_\Sigma$ and $\partial_t H^\lambda|_\Sigma$. We denote by n^μ a future directed unit time-like vector field, normal to the initial surface Σ, and consider the contraction of the reduced Einstein equations with n^μ,

$$(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} - 8\pi T_{\mu\nu}) n^\nu - g_{\sigma(\mu} n^\nu \hat{\nabla}_{\nu)} H^\sigma + \frac{1}{2} n_\mu \hat{\nabla}_\sigma H^\sigma \; = \; 0 \, . \tag{2.6}$$

An analysis of this equation shows that it *implies* the vanishing of $\partial_t H^\lambda|_\Sigma$, supposing that the equations

$$(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} - 8\pi T_{\mu\nu}) n^\nu|_\Sigma = 0 \quad \text{and} \quad H^\lambda|_\Sigma \; = \; 0 \tag{2.7}$$

hold. One checks that, in vacuum or for appropriate matter models, both equations in (2.7), which are clearly necessary to obtain a solution to the full Einstein equations in wave-map gauge, just depend on $g_{\mu\nu}|_\Sigma$ and $\partial_t g_{\mu\nu}|_\Sigma$, i.e. the initial data.

More specifically, let us denote by

$$h_{ij} := g_{ij}|_\Sigma \tag{2.8}$$

the *induced metric* or *first fundamental form*, and by

$$K_{ij} := \frac{1}{2} \mathscr{L}_n g_{ij}|_\Sigma = \frac{1}{2} \sqrt{|g^{tt}|} (\partial_t g_{ij} - 2 \mathscr{D}_{(i} g_{j)t})|_\Sigma \tag{2.9}$$

[1] At this stage, we assume that a Lorentz manifold, solution to Einstein's field equations, has been given, later on it will be constructed from Σ and the data given there.

the *extrinsic curvature* or *second fundamental form*, where $\mathscr{L}$ is the Lie derivative. We denote by $\mathscr{D}$, $R[h]$ and $\Gamma[h]^k_{ij}$ covariant derivative, curvature scalar and connection coefficients associated with the Riemannian metric $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$. One then shows that (2.7) is equivalent to

$$
\begin{aligned}
(R[h] - 2\lambda - |K|^2 + K^2 - 16\pi T_{\mu\nu}n^\mu n^\nu)|_\Sigma &= 0\,, &(2.10)\\
(\mathscr{D}_j K_i{}^j - \mathscr{D}_i K - 8\pi h_{ij}g^{j\nu}T_{\mu\nu}n^\mu)|_\Sigma &= 0\,, &(2.11)\\
(\partial_t g^{tt} - 2\sqrt{|g^{tt}|}K - g^{tt}g^{ti}\mathscr{D}_i g_{tt} - 2g^{ti}g^{tj}\mathscr{D}_i g_{tj} + 2V^t)|_\Sigma &= 0\,, &(2.12)\\
(g^{tt}\partial_t g_{tk} + 2|g^{tt}|^{-\frac{1}{2}}g^{ti}K_{ik} - \frac{1}{2}g^{tt}\mathscr{D}_k g_{tt} + 2g^{ti}\mathscr{D}_i g_{tk} + h_{kl}g^{ij}\Gamma[h]^l_{ij} - V_k)|_\Sigma &= 0\,. &(2.13)
\end{aligned}
$$

Here we have set $K := h^{ij}K_{ij}$ and $|K|^2 := h^{ik}h^{jl}K_{ij}K_{kl}$.

The wave-map gauge condition $H^\lambda = 0$ for the arbitrarily prescribed vector field W^λ does not fully exploit the gauge freedom to choose coordinates. It is well-known (cf. e.g. [25]) that there remains the freedom to prescribe

$$g^{tt}|_\Sigma < 0 \quad \text{and} \quad g^{ti}|_\Sigma\,, \tag{2.14}$$

or, equivalently,

$$g_{tt}|_\Sigma \quad \text{and} \quad g_{ti}|_\Sigma \quad \text{such that} \quad (g_{tt} - h^{ij}g_{ti}g_{tj})|_\Sigma < 0\,. \tag{2.15}$$

Taking this into account when analyzing the constraint equations (2.10)-(2.13), we observe that, in vacuum say, (2.10)-(2.11) provide constraints on (h_{ij}, K_{ij}) (observe that $\partial_t g_{ij}|_\Sigma$ can be computed from K_{ij}). Once we have found a solution, $\partial_t g_{t\mu}|_\Sigma$ can be algebraically determined from (2.12)-(2.13).[2] In fact, due to the freedom to choose the gauge source functions W^λ, the values of $\partial_t g_{t\mu}|_\Sigma$ are just a matter of gauge and can be prescribed arbitrarily. The equations (2.12)-(2.13) then need to be read as algebraic equations for $W^t|_\Sigma$ and $W^i|_\Sigma$. Contrary to the characteristic case treated below, the additional freedom arising from the gauge source functions cannot be used to simplify the constraint equations significantly (nonetheless, they play an important role mathematical and numerical relativity, cf. e.g. [75, 77]). The proper, non-trivial constraints are given by the *Hamiltonian constraint* (2.10) and the *momentum constraint* (2.11) for the *geometric* data (h_{ij}, K_{ij}) on Σ, comprising its first and second fundamental form. In vacuum, these constraints become

$$
\begin{aligned}
R[h] - 2\lambda - |K|^2 + K^2 &= 0\,, &(2.16)\\
\mathscr{D}_j K_i{}^j - \mathscr{D}_i K &= 0\,, &(2.17)
\end{aligned}
$$

Now, given Σ as a manifold on its own (cf. below) there are various ways of constructing the same (up to isometries) space-time which does not arise from a freedom in the choice of coordinates, but from a freedom to embed Σ into the prospective space-time manifold. This is sometimes regarded as another gauge freedom, captured by the freedom to prescribe the mean curvature τ of Σ, though one has to be careful since e.g. a *constant mean curvature (CMC)-gauge*, where $\tau = \text{const}$, *is* a geometric restriction [31].

As an illustration, and for later reference we note that working in a gauge where

$$g_{tt}|_\Sigma = -1\,, \quad g_{ti}|_\Sigma = 0\,, \quad \partial_t g_{t\mu}|_\Sigma = 0 \tag{2.18}$$

and using a Minkowski target $\hat{g} = \eta = -dt^2 + \sum_i (dx^i)^2$, we need to take

$$
\begin{aligned}
W^t|_\Sigma &= K\,, &(2.19)\\
W^k|_\Sigma &= h^{ij}\Gamma[h]^k_{ij}\,. &(2.20)
\end{aligned}
$$

[2] Some care is needed when the W^λ's are allowed to depend on the metric.

There have been many successful approaches to construct solutions of (2.10)-(2.11) or (2.16)-(2.17) (cf. e.g. [7, 21, 25, 31] and the references given therein for an overview over the various methods). Ideally, one would like to extract certain components of (h_{ij}, K_{ij}) as *reduced "seed" data* which can be prescribed completely freely, while the remaining components are determined by the constraint equations. In general, though, there is no known method to do this.

One common approach is to transform the constraint equations via the *Choquet-Bruhat-Lichnerowicz-York conformal method* [13, 21, 25] into a more convenient PDE-system. A standard choice is to regard the conformal class $[h_{ij}]$ of the induced metric h_{ij}, a symmetric, trace-free tensor $\tilde{L}_{ij}$ and, depending on the viewpoint, the mean curvature τ as the non-gauge data. Denote by $\tilde{h}_{ij}$ a representative of $[h_{ij}]$. The vacuum constraints (2.16)-(2.17) expressed in terms of $\tilde{L}_{ij}$ and a conformal factor $\phi > 0$ relating $\tilde{h}$ and h, then read

$$\tilde{\mathscr{D}}_j \tilde{L}_i{}^j - \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\tilde{\mathscr{D}}_i\tau \quad = \quad 0\,, \tag{2.21}$$

$$4\frac{n-1}{n-2}\Delta_{\tilde{h}}\phi - \tilde{R}[\tilde{h}]\phi + |\tilde{L}|^2\phi^{-\frac{3n-2}{n-2}} - \Big(\frac{n-1}{n}\tau^2 - 2\lambda\Big)\phi^{\frac{n+2}{n-2}} \quad = \quad 0\,, \tag{2.22}$$

where $\tilde{\mathscr{D}}$ denotes the Levi-Civita connection of $\tilde{h}$. Any solution $(\phi > 0, \tilde{L}_{ij})$ leads via

$$h_{ij} \;=\; \phi^{\frac{4}{n-2}}\tilde{h}_{ij} \quad \text{and} \quad K_{ij} \;=\; \phi^{-2\frac{n+2}{n-2}}\tilde{L}_{ij} + \frac{\tau}{n}h_{ij} \tag{2.23}$$

to a solution of the vacuum constraints. Writing

$$\tilde{L}_{ij} = \tilde{B}_{ij} + 2(\tilde{\mathscr{D}}_{(i}Y_{j)})\breve{}\,, \tag{2.24}$$

where "$\breve{}$" denotes the $\tilde{h}$-trace-free part of the corresponding 2-tensor, and where the symmetric, trace-free "seed" tensor field $\tilde{B}_{ij}$ rather than $\tilde{L}_{ij}$ is regarded as given, the constraints (2.21)-(2.22) become a semi-linear elliptic PDE-system for the conformal factor ϕ and the vector field Y.

A characteristic property in a CMC-gauge is that the equations for ϕ and $\tilde{L}_{ij}$ decouple, with (2.21) reducing to $\operatorname{div}\tilde{L} = 0$. In that situation $[h_{ij}]$ and a symmetric trace- and divergence-free tensor (*TT-tensor*) $\tilde{L}_{ij}$ provide the free reduced non-gauge data. The splitting (2.24) then provides a method due to York to construct TT-tensors from trace-free tensors via a linear elliptic PDE-system. It then remains to solve *Lichnerowicz equation* (2.22) (with $\phi > 0$), a semi-linear elliptic scalar equation.

While the CMC-case is quite well-understood in many situations of physical interest such as on compact, asymptotically Euclidean and asymptotically hyperbolic hypersurfaces, it is particularly the non-CMC-case where the existence of a solution to the constraint equations can generally not be guaranteed. This will be completely different in the characteristic case, where it is possible to provide an exhaustive class of freely prescribable initial data.

To conclude, the wave-map gauge condition splits the Einstein equations into a system of hyperbolic evolution equations (2.4), the reduced Einstein equations, and an (elliptic) system of constraint equations (2.10)-(2.13), which needs to be satisfied on the initial manifold. Taking well-known well-posedness results for quasi-linear wave equations into account (cf. Appendix A) it follows that the constraint equations (2.10)-(2.11) are necessary and sufficient for the existence of an, in this gauge, locally unique globally hyperbolic development which solves the space-like Cauchy problem for the vacuum field equations. One thereupon shows that the solution is locally unique up to isometries whatever coordinates have been selected.

Strictly speaking the procedure described above to establish well-posedness of the space-like Cauchy problem works for data given on some coordinate patch of the initial surface $\Sigma \subset \mathscr{M}$. To construct a solution global in space (cf. e.g. [25]) one needs to patch together the solutions obtained in each neighborhood to a globally hyperbolic development with Cauchy surface Σ. This is possible owing to the fact that all solutions with the same initial data are locally isometric. The solution is unique (up to isometries) in a neighborhood of Σ.

So far we have assumed that $\Sigma \subset \mathscr{M}$, though one would rather like to consider Σ as an n-manifold on its own. Given data (Σ, h_{ij}, K_{ij}), with h_{ij} a Riemannian metric and K_{ij} a symmetric tensor of valence two on Σ, solution to the vacuum constraint equations (2.16)-(2.17), one shows that Σ can be equipped with an embedding ι into a new manifold $\mathscr{M}$ with topology $\Sigma \times \mathbb{R}$ endowed with a Lorentzian metric g, such that $\iota(\Sigma)$ is a space-like Cauchy surface in $(\mathscr{M}, g)$, the pull-backs of the induced metric and the second fundamental form on $\iota(\Sigma)$ to Σ coincide with h_{ij} and K_{ij}, respectively, and the metric g solves the vacuum Einstein equations and is locally unique up to isometries.

Consider the triple (Σ, h_{ij}, K_{ij}) for say $n = 3$. Taking the number of constraint equations, the gauge character of the mean curvature and the fact that diffeomorphic initial data sets determine physically equivalent space-times into account, one finds that (Σ, h_{ij}, K_{ij}) conceals 4 free "non-gauge" functions, which means that the gravitational field has 2 degrees of freedom per point in space [86]. This number will be retrieved in all the other Cauchy problems we shall discuss here.

2.3.1 Maximal globally hyperbolic developments

The space-time $(\mathscr{M}, g)$ constructed this way is not unique. For instance, any proper subset of $\mathscr{M} \supset \iota(\Sigma)$ gives another, non-isometric solution to the Cauchy problem, which, though, can be isometrically embedded into the first one, keeping the Cauchy surface fixed. This defines a partial order. Any totally ordered subset has an upper bound which is obtained by taking the union of all solutions in this subset and identifying points via isometric embedding. One therefore may employ Zorn's lemma to conclude that a maximal globally hyperbolic development $(\mathscr{M}_{\max}, g_{\max})$ exists. This can be shown [19] to be unique in the sense that every other globally hyperbolic solution to the Cauchy problem can be isometrically embedded into $(\mathscr{M}_{\max}, g_{\max})$.

THEOREM 2.3.1 *A triple (Σ, h_{ij}, K_{ij}) defines a unique (up to isometries) maximal globally hyperbolic development of the vacuum Einstein equations, which depends in a continuous manner on the initial data,*[3] *if and only if the constraint equations (2.16)-(2.17) are fulfilled.*[4]

2.4 Characteristic Cauchy problem

Let us consider a (future) light-cone $C_O \subset \mathscr{M}$ with vertex $O \in \mathscr{M}$ in a space-time $(\mathscr{M}, g)$ which satisfies Einstein's field equations. As compared to two transversally intersecting null-hypersurfaces, treated e.g. in [34, 76], some additional technical difficulties arise due to the non-smoothness of C_O at its vertex, which most comprehensively have been solved by Chruściel [27], compare [17, 18]

[3]In this work we focus on existence and uniqueness results, so here and elsewhere we will not go into detail by e.g. specifying the topology w.r.t. which continuity holds.

[4]Similar results can be derived e.g. for scalar, Maxwell, Yang-Mills and Vlasov fields, supposing that additional data for the matter fields are given [13].

We introduce *adapted null coordinates* $(x^0 = u, x^1 = r, x^A)$, $A = 2, \dots n+1$, singular at the tip of the cone, such that $C_O \setminus \{O\} = \{u = 0\}$, $r > 0$ parameterizes the null geodesics issued from O and generating the cone such that O is approached as r goes to zero, while the x^A's are local coordinates on the level sets $\Sigma_r = \{u = 0, r = \text{const.}\} \cong S^{n-1}$ (cf. [16]). In these coordinates the trace of the metric on the cone adopts the form (the symbols x^0 and u will be used interchangeably)

$$\overline{g} = \overline{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \overline{g}_{00}(\mathrm{d}u)^2 + 2\nu_0 \mathrm{d}u\mathrm{d}r + 2\nu_A \mathrm{d}u\mathrm{d}x^A + \check{g}\ , \tag{2.25}$$

where we have set

$$\nu_0 := \overline{g}_{0r}\ , \quad \nu_A := \overline{g}_{0A}\ , \quad \check{g} := \overline{g}_{AB}\mathrm{d}x^A \mathrm{d}x^B\ . \tag{2.26}$$

Here and henceforth we use an overbar to denote the restriction of a space-time object to C_O, or, more general, to the respective initial surface (we shall use this notation for space-like initial surfaces, as well).

The *induced metric* is given by the degenerate quadric form $\check{g}$, which induces on each slice Σ_r an r-dependent Riemannian metric $\check{g}_{\Sigma_r}$. For the convenience of the reader, let us also note that the restriction to the cone of the inverse metric takes the form

$$\overline{g}^{\#} = 2\nu^0 \partial_u \partial_r + \overline{g}^{rr}\partial_r\partial_r + 2\overline{g}^{rA}\partial_r\partial_A + \overline{g}^{AB}\partial_A\partial_B\ , \tag{2.27}$$

where $\overline{g}^{AB}$ is the inverse of $\overline{g}_{AB}$, and where

$$\nu^0 = \overline{g}^{0r} = (\nu_0)^{-1}\ ,\ \ \overline{g}^{rA} = -\nu^0\nu^A = -\nu^0\overline{g}^{AB}\nu_B\ ,\ \ \overline{g}^{rr} = (\nu^0)^2(\nu^A\nu_A - \overline{g}_{00})\ . \tag{2.28}$$

In the characteristic case the data for the reduced Einstein equations (2.4) are $\overline{g}_{\mu\nu}$, the data for (2.5) are $\overline{H}^\lambda$ (cf. Appendix A).

Recall that in the space-like case certain components of the trace of the full Einstein equations to the initial surface involve only metric components and transverse derivatives thereof which are part of the initial data for the evolution equations. This is the reason why one is straightforwardly led to the geometric constraint equations (2.10)-(2.11). In the characteristic case this is not true anymore, whence things are somewhat more involved. The Einstein equations have to be combined with the wave-gauge vector to eliminate transverse derivatives of the metric which do not belong to the initial data for the evolution equations. However, it is this peculiarity which allows for a simplification of the integration scheme for the characteristic constraint equations by the use of non-vanishing gauge source functions.

Denote by ℓ the vector ∂_r. The *null second fundamental form of* C_O is defined as

$$\chi_{ij} := \frac{1}{2}\mathscr{L}_\ell \overline{g}_{ij}\ . \tag{2.29}$$

Since the normal vector ℓ is also tangent, χ shows very different properties in comparison with the space-like case. Most notably, it is intrinsically defined and does not depend on transverse derivatives of the metric. In adapted null coordinates we have

$$\chi_{AB} = \frac{1}{2}\partial_r\overline{g}_{AB}\ , \quad \chi_{1i} = 0\ . \tag{2.30}$$

The trace of the null second fundamental form

$$\tau = \overline{g}^{AB}\chi_{AB} \tag{2.31}$$

is called the *expansion* or *divergence of* C_O, while its trace-free part

$$\sigma_A{}^B = (\overline{g}^{BC}\chi_{AC})\breve{} = \check{\chi}_A{}^B = \chi_A{}^B - \frac{1}{n-1}\delta_A{}^B\tau = \frac{1}{2}[\overline{g}^{BC}](\partial_r[\overline{g}_{AB}])\breve{} \tag{2.32}$$

denotes the *shear of* C_O. It merely depends upon the conformal class $[\breve{g}]$ of $\breve{g}$. Recall that we decorate a 2-tensor with the symbol "$\breve{}$" whenever its trace-free part is meant.

It turns out [16] that, in vacuum or for appropriate matter models, the equations

$$
\begin{aligned}
(\overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}\overline{g}_{\mu\nu} + \lambda\overline{g}_{\mu\nu} - 8\pi\overline{T}_{\mu\nu})\ell^\mu\ell^\nu &= -\frac{1}{2}\tau\overline{H}_r \;, && (2.33)\\
(\overline{R}_{A\nu} - \frac{1}{2}\overline{R}\overline{g}_{A\nu} + \lambda\overline{g}_{A\nu} - 8\pi\overline{T}_{A\nu})\ell^\nu &= -\frac{1}{2}(\partial_r + \tau)\overline{H}_A + \frac{1}{2}\partial_A\overline{H}_r \;, && (2.34)\\
2\nu^0(\overline{R}_{0\nu} - \frac{1}{2}\overline{R}\overline{g}_{0\nu} + \lambda\overline{g}_{0\nu} - 8\pi\overline{T}_{0\nu})\ell^\nu &= (2\nu^0\partial_r\nu^A - \overline{g}^{rA}\partial_r + \overline{g}^{rA}\overline{V}_r)\overline{H}_A \\
&\quad -(\check{\nabla}_A - \xi_A)\overline{H}^A - (2\nu^0\partial_r + \tau\nu^0 - \overline{V}^0)\overline{H}_0 - \Big(\overline{g}^{rr}(\partial_r + \nu^0\partial_r\nu_0) \\
&\quad +2(\nu^0)^2\nu_A\partial_r\nu^A + \nu^0\overline{g}^{Ai}\partial_A\overline{g}_{0i} - \nu^0\overline{V}_0 - \overline{g}^{rr}\overline{V}_r\Big)\overline{H}_r + \frac{1}{2}|\overline{H}|^2 \;, && (2.35)
\end{aligned}
$$

with $H_\mu := g_{\mu\nu}H^\nu$, $|H|^2 := H^\mu H_\mu$, and with ξ_A being defined in equation (2.38) below,[5] do not involve any transverse derivative of the metric tensor. It is clear, that they necessarily need to be satisfied by any solution of the Einstein equations in wave-map gauge.

The system (2.33)-(2.35) is equivalent to the so-called *Einstein wave-map gauge constraints* derived in [16] (here we have added a cosmological constant λ),

$$
\begin{aligned}
(\partial_r - \kappa)\tau + \frac{1}{n-1}\tau^2 &= -|\sigma|^2 - 8\pi\overline{T}_{rr} \;, && (2.36)\\
(\partial_r + \frac{1}{2}\tau + \kappa)\nu^0 &= -\frac{1}{2}\overline{V}^0 \;, && (2.37)\\
(\partial_r + \tau)\xi_A &= 2\check{\nabla}_B\sigma_A{}^B - 2\frac{n-2}{n-1}\partial_A\tau - 2\partial_A\kappa - 16\pi\overline{T}_{rA} \;, && (2.38)\\
(\partial_r - \frac{1}{2}\nu_0\overline{V}^0)\nu^A &= \frac{1}{2}\nu_0(\overline{V}^A - \xi^A - \overline{g}^{BC}\check{\Gamma}^A_{BC}) \;, && (2.39)\\
(\partial_r + \tau + \kappa)\zeta &= \frac{1}{2}|\xi|^2 - \check{\nabla}_A\xi^A - \check{R} + 8\pi(\overline{g}^{AB}\overline{T}_{AB} - \overline{T}) + 2\lambda \;, && (2.40)\\
(\partial_r + \frac{1}{2}\tau + \kappa)\overline{g}^{rr} &= \frac{1}{2}\zeta - \overline{V}^r \;, && (2.41)
\end{aligned}
$$

with

$$
|\sigma|^2 := \sigma_A{}^B\sigma_B{}^A \;, \quad |\xi|^2 := \overline{g}^{AB}\xi_A\xi_B \;, \quad \xi^A := \overline{g}^{AB}\xi_B \;, \quad \overline{T} := \overline{g}^{\mu\nu}\overline{T}_{\mu\nu} \;. \tag{2.42}
$$

We use the check symbol "$\check{}$" to denote objects associated with the one-parameter family of Riemannian metrics $r \mapsto \overline{g}_{AB}(r, x^C)\mathrm{d}x^A\mathrm{d}x^B$ on S^{n-1}. The relevant boundary conditions to integrate (2.36)-(2.41) follow from regularity conditions at the tip of the cone [16, Section 4.5].

The function κ appearing in (2.36)-(2.41) provides another gauge function. It reflects the freedom to choose the r-coordinate which parameterizes the null geodesics generating the cone. Once a solution of the reduced Einstein equations in wave-map gauge has been constructed, κ and the "auxiliary" fields ξ_A and ζ, which have been introduced to transform (2.33)-(2.35) into first-order equations, turn out to be related to certain connection coefficients,

$$
\kappa = \overline{\Gamma}^r_{rr} \;, \quad \xi_A = -2\overline{\Gamma}^r_{rA} \;, \quad \zeta = 2\overline{g}^{AB}\overline{\Gamma}^r_{AB} + \tau\overline{g}^{rr} \;. \tag{2.43}
$$

[5] The reader is warned that in [16] a different convention has been used how the indices of the wave-gauge vector are lowered.

In fact, the function ζ has a physical meaning beyond its relation to a connection coefficient: Consider two transversally intersecting null hypersurfaces $\mathscr{N}_1 = \{u = 0\}$ and $\mathscr{N}_2 = \{r = 0\}$ in adapted null coordinates. Then, on the intersection manifold S, $\zeta_{N_1} = -2\overline{g}^{ur}\tau_{N_2}$, i.e. the function ζ associated to N_1 is closely related to the expansion τ of N_2.

It has been shown in [16] that any solution of the reduced Einstein equations with initial data $\overline{g}_{\mu\nu}$ fulfilling the wave-map gauge constraints *implies* a system of transport equations for $\overline{H}^\lambda$ along the null geodesic generators of C_O. Assuming that the solution is regular at the tip of the cone, by which we mean that it can be smoothly extended through O, one shows that $\overline{H}^\lambda = 0$ is the only solution. Consequently, the Einstein equations split, again, into a system of hyperbolic evolution equations (2.4), the reduced Einstein equations, and a system of constraint equations (2.36)-(2.41) which is propagated by the Einstein equations.

A few comments how these equations can be solved are in order: In contrast to the space-like case one can easily extract components from the data $\overline{g}_{\mu\nu}$ for the evolution system which are freely prescribable. The "standard way" to achieve that, which goes back to Rendall [76], compare [79], is to regard the conformal class of $\check{g}$, denoted by $[\gamma]$, supplemented by a choice of κ and the gauge source functions W^λ, as the unconstrained "reduced" data (Rendall's choice was $\kappa = 0 = W^\lambda$). We denote the conformal factor relating $\check{g}$ and a given representative of $[\gamma]$, $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$, by Ω,

$$\check{g} = \Omega^2\gamma\,. \tag{2.44}$$

Let us now define the subsidiary function

$$\varphi := \Omega\Big(\frac{\det\gamma}{\det s}\Big)^{1/(2n-2)} = \Big(\frac{\det\check{g}_{\Sigma_r}}{\det s}\Big)^{1/(2n-2)}\,, \tag{2.45}$$

where $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$ denotes the unit round metric on S^{n-1}. Then the relation

$$\tau = (n-1)\partial_r \log\varphi \tag{2.46}$$

holds, which transforms the Raychaudhuri equation (2.36) into a linear, second-order equation,

$$\Big(\partial^2_{rr} - \kappa\partial_r + \frac{|\sigma|^2 + 8\pi\overline{T}_{rr}}{n-1}\Big)\varphi = 0\,. \tag{2.47}$$

The system (2.47), (2.37)-(2.41) can be read as a hierarchical *linear* ODE-system along each null geodesic generator of the cone (as usual, in vacuum or for appropriate matter models).[6] It can be integrated step-by-step to determine successively from the data $[\gamma]$ all the components of the space-time metric $\overline{g}$ on C_O, which in turn provide the data for the reduced Einstein equations. Some care is needed since the metric $\overline{g}$ degenerates when φ or ν^0 have zeros, thus leading to coordinate or space-time singularities, so a solution to the constraints may only exists in some neighborhood of the tip of the cone. Sufficient conditions which ensure the existence of a global solution on C_O are derived in [38, Section 2.4], attached as Chapter 10, compare [16].

Apart from the fact that in the characteristic case the constraint equations are just ODEs for the unknown metric coefficients rather than elliptic PDEs, which, even better, can be transformed into linear ODEs, the gauge source functions W^λ can be utilized to transform some of them into algebraic equations, cf. the next section. Since the constraints are thus much more convenient to solve, the characteristic Cauchy problem plays a significant role in

[6] If the gauge source functions are allowed to depend on the metric, certain restrictions are necessary to preserve the hierarchical ODE-structure.

numerical relativity [87]. Note, however, that the equations (2.36), (2.38) and (2.40) are to a large extent gauge-independent: They merely dependent on the gauge function κ, which can be employed to simplify the construction of solutions of the Raychaudhuri equation (cf. below), whereas there seems to be no way to simplify the equations for ξ_A and ζ.

The well-posedness result for given data $[\gamma]$ and gauge functions κ and W^λ, which has been established in [16, 27], relies on Dossa's well-posedness result for quasi-linear wave equations [41], cf. Appendix A, and guarantees existence of a solution only in some neighborhood to the future of the vertex of the cone (there is one subtlety which will be addressed in Section 2.4.1). The result has been improved by Luk [67] who showed that well-posedness holds in fact in a neighborhood to the future of the whole cone, assuming that the constraint equations can be integrated that far, i.e. that no conjugate points appear. While Dossa's result and also the corresponding result of Rendall [76] for two transversally intersecting null hypersurfaces, are valid for large classes of quasi-linear wave equations, Luk makes use of the particular structure of Einstein's vacuum field equations.

2.4.1 The many ways of the characteristic Cauchy problem

As has already been indicated, the characteristic Cauchy problem comes along with a huge flexibility in prescribing data. There are many ways other than Rendall's one of constructing solutions of Einstein's wave-map gauge constraints. A discussion of this point is the main object of [34], compare also [72]. The paper [34] is attached as Chapter 7, [72] as Chapter 11.

Our motivation to take non-vanishing gauge source functions and appropriate variations of the gauge scheme into account has been two-fold:

(i) While Rendall's approach works well in vacuum and also for a certain class of matter models such as scalar, Maxwell and Yang-Mills fields, non-vanishing gauge source functions allow one to handle a larger class of matter models such as e.g. Vlasov matter.

(ii) As the no-go result in [38, Section 3] reveals, Rendall's approach with $\kappa = 0 = W^\lambda$, or rather any $(\kappa = 0, \overline{W}^0 = 0)$-gauge, inevitably produces logarithmic terms at infinity whenever non-flat data are prescribed. The need to get rid of these log terms when constructing solutions which are smooth at null infinity (cf. Section 4.4), led us to the search of alternative, more flexible schemes to construct solutions of the wave-map gauge constraints. Thereby it turned out that most, but not all, logarithmic terms are gauge-dependent and eliminable by means of suitable coordinate transformations.

In [34] we recall, develop and discuss several methods of choosing a gauge and prescribing data to construct solutions to Einstein's wave-map gauge constraint equations. Let us summarize what has been done there.

- Rendall's scheme with the classical gauge choices $\kappa = 0 = W^\lambda$, or rather its generalization where the conformal class $[\gamma]$ of $\breve{g}$ together with arbitrary gauge functions κ and W^λ are prescribed and which brings in more flexibility, has the advantage that one has a clear-cut separation between physical and gauge degrees of freedom. Moreover, the initial data $[\gamma]$ do not need to satisfy any constraint equation.

- The induced metric, i.e. the degenerate quadric form $\breve{g}$ is a physically and geometrically more natural object than a family of conformal metrics $[\gamma]$. From this point of view one should regard $\breve{g}$ together with the gauge functions κ and W^λ as initial data as in [14] or [34, Section 5], where the pair $(\breve{g}, \kappa)$ then needs to satisfy the Raychaudhuri equation (2.36), which in this setting plays the role of a constraint equation.

Assume that $\breve{g}$ has been given. In the regions where τ has no zeros, as it needs to be the case near the vertex of the cone,[7] κ can be algebraically computed from the Raychaudhuri equation (2.36), so that, again, there are no constraints if $\breve{g}$ together with W^λ are regarded as the free data.[8]
In this scheme physical and gauge degrees of freedom are somewhat mixed.

- Another approach is to prescribe $\overline{g}_{0\mu}$ rather than $\overline{W}^\lambda$ and regard (2.37), (2.39) and (2.41) as *algebraic* equations for $\overline{W}^\lambda$. This saves a couple of integrations. As before, one may prescribe $\breve{g}$ rather than its conformal class $[\gamma]$. This way it becomes possible to prescribe the full space-time metric $\overline{g}$ of the space-time-to-be-constructed restricted to C_O [34, Section 3], together with a choice of κ, where, again, the Raychaudhuri equation provides a constraint on $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$ and κ.
At least sufficiently close to the tip, where the positivity of τ is necessary, κ is algebraically determined by the Raychaudhuri equation, cf. footnote 7. In those regions, no constraints need to be imposed on the initial data $\overline{g}$ for the evolution equations, at the expense, though, that physical and gauge degrees of freedom are completely mixed. In other words, many different initial data $\overline{g}$ will evolve into geometrically the same space-time, whereas in Rendall's scheme only minor gauge degrees of freedom remain hidden in the data (e.g., a choice of κ does not determine the r-coordinate uniquely).

- A modification of this scheme which takes care of this separation, discussed in [72, Section 4], is to prescribe $[\gamma]$ as the physical data and, instead of κ and $\overline{W}^\lambda$, regard φ and $\overline{g}_{0\mu}$ as gauge functions. Proceeding this way, the separation between physical and gauge degrees of freedom is sustained, while some of the metric components can be arranged to take convenient values.

- Instead of (some components of) $\overline{g}$, one may alternatively prescribe the shear σ_{AB} of C_O [22], or certain components of the Weyl tensor, $\overline{C}_{rArB}$ [46]. However, one has to make sure that, once $\breve{g}$ has been computed, the relation $\overline{g}^{AB}\sigma_{AB} = 0$ or $\overline{g}^{AB}\overline{C}_{rArB} = 0$, respectively, holds. The most convenient way to arrange this is to introduce a frame formalism and prescribe the data w.r.t. the frame field [34, Section 5].

There is one subtlety which we have ignored so far: In order to apply Dossa's well-posedness result [41] for the reduced Einstein equations with data on a light-cone, cf. Appendix A, one needs to make sure that $\overline{g}$ is the restriction to the cone of some smooth space-time metric. Clearly, this is necessary for a smooth solution of the Einstein equations to exist. Nevertheless, it is a non-trivial issue near the tip of a cone. A similar issue arises for the gauge source functions $\overline{W}^\lambda$ which, if algebraically computed from the constraint equations, need to admit an extension to a smooth space-time vector field W^λ as needed to define the wave-gauge vector H^λ properly.

In [27] (cf. [17, 18]) it has been analyzed in detail how such admissible data are constructed for the various schemes. When $\breve{g}$ is prescribed, it is shown that Dossa's theorem is applicable if $\breve{g}$ is the trace on the cone of a smooth space-time metric. In Rendall's scheme, one needs to make sure that γ is the trace on the cone of a smooth space-time metric to end up with a solution $\overline{g}$ of Einstein's vacuum wave-map gauge constraints to which Dossa's theorem is applicable. If the whole metric field $\overline{g}$ is prescribed, and it is the restriction to the cone of a

[7] If the data $\breve{g}$ (or $\overline{g}$, cf. below) are taken to be the restriction of some smooth metric in space-time, as it is necessary for the existence of a solution anyway, one will automatically have $\tau > 0$ near the tip.

[8] Note that the r-coordinate w.r.t. which $\breve{g}$ is given is then implicitly defined. Its actual meaning in the emerging space-time, like e.g. its deviation from an affine parameter, is only known once the Raychaudhuri equation has been solved for κ.

smooth space-time metric, the vector field $\overline{W}^\lambda$, computed from the constraint equations, can be extended to a smooth space-time field and well-posedness follows again.

Similar to the space-like case one may regard the initial surface $C_O \setminus \{O\} \cong \mathbb{R}^n \setminus \{0\}$ with its vertex removed as a (then smooth) manifold on its own [37], or, as it is more convenient in view of the results in [27], as a light-cone in some space-time $(\mathscr{M}^{(0)}, g^{(0)})$. As data one then takes, depending on the scheme, the trace of $g^{(0)}$ on C_O, the induced degenerate quadric form $\breve{g}^{(0)}$, its conformal class $[\breve{g}^{(0)}]$, etc. The results described above then guarantee that the prospective vacuum space-time has a cone-like past boundary which can be identified with the embedded null cone C_O.

THEOREM 2.4.1 *Consider a light-cone C_O in some space-time $(\mathscr{M}^{(0)}, g^{(0)})$ near its vertex where $\tau^{(0)} > 0$. The tuples*

(i) $(\gamma = [g^{(0)}_{AB}]\mathrm{d}x^A \mathrm{d}x^B|_{C_O}, \kappa = 0)$,

(ii) $\breve{g} = g^{(0)}_{AB}\mathrm{d}x^A \mathrm{d}x^B|_{C_O}$,

(iii) $g^{(0)}|_{C_O}$,

depending on the scheme, define locally an (up to isometries) unique globally hyperbolic vacuum space-time which is bounded in the past by the embedded null cone C_O and which induces the prescribed data there.[9] *The solution depends continuously on the initial data.*

Here and henceforth, *local uniqueness (up to isometries)* in connection with a characteristic Cauchy problem is to be understood as follows: For given gauge functions the solution is unique. Given two different sets of gauge functions, there exists a common neighborhood where the solutions coincide (up to isometries). Nonetheless, also in the characteristic case it should be viable to establish uniqueness in the sense that there exists a unique maximal globally hyperbolic development by an appropriate adaption of the Choquet-Bruhat-Geroch-method.

[9]Note that we have restricted attention to the region where $\tau^{(0)} > 0$, an inequality which does not need to hold away from a neighborhood of the tip. In those regions where $\tau^{(0)} = 0$, (ii) and (iii) need to be supplemented by a choice of κ, and the pair $(\breve{g} = g^{(0)}_{AB}\mathrm{d}x^A \mathrm{d}x^B|_{C_O}, \kappa)$ needs to satisfy the Raychaudhuri equation.

CHAPTER 3

Asymptotically flat and asymptotically de Sitter space-times

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite [...]."
GEORG F. L. P. CANTOR (1845–1918)

3.1 Introduction

In the previous section we recalled local in (retarded) time well-posedness results for the space-like and the characteristic Cauchy problem. However, even if one invokes Zorn's lemma to establish the existence of unique maximal globally hyperbolic developments, this does not say anything about how large the emerging space-time will actually be. Moreover, one does not know anything about local and, even more, global properties of the so-obtained space-times.

There have been intensive efforts to construct space-times with controlled global properties. A decisive milestone was Friedrich's proof of the non-linear stability of the de Sitter space-time [53]. He further showed (cf. also [1, 2]) that for sufficiently small hyperboloidal data the space-like initial value problem yields a space-time which has globally (to the future of the initial surface) the same asymptotic structure as that part of Minkowski space-time which lies to the future of a hyperboloid. Another milestone was Christodoulou's and Klainerman's proof of the non-linear stability of the Minkowski space-time [23]. Christodoulou [22] used similar methods to construct space-times from characteristic surfaces which contain *trapped surfaces* (which provide a quasi-local approach to black holes) by using a regular *short-pulse-ansatz* for the initial data. The class of characteristic data leading to the formation of trapped surfaces has later been enlarged by Klainerman and Rodnianski [64].

In what follows we are interested in the construction of vacuum space-times with controlled asymptotic behavior. More precisely, we would like to analyze the existence of space-times which extend "arbitrarily far", at least in certain directions, and where the gravitational field shows a specific "asymptotically flat" or "asymptotically de Sitter" structure at infinity. In order to avoid dealing with asymptotic limiting processes, the space-time will be conformally rescaled so that infinity is represented by a set of regular points, an idea due to Penrose [73]. By prescribing data on these regular sets, it becomes feasible to construct space-times with such an asymptotic structure via local existence results.

3.2 Penrose's conformal technique

Due to the absence of non-dynamical background fields w.r.t. which the asymptotic fall-off rates of the curvature can be measured, it is a delicate issue to define a notion of *asymptotic flatness* or *asymptotic de Sitterness* in general relativity. On the other hand, it is important to have such a notion to be able to describe e.g. purely radiative space-times or isolated gravitational systems. Penrose provided a very elegant geometric foundation to tackle this issue [73, 74], cf. [61] for an overview.

Consider an $n+1$-dimensional space-time $(\tilde{\mathscr{M}}, \tilde{g})$, the *physical space-time*. The basic idea is to conformally rescale the metric $\tilde{g}$ and add a boundary, so that infinity is represented as something finite. For this, one assumes that (a part of) $(\tilde{\mathscr{M}}, \tilde{g})$ can be conformally embedded into an *unphysical space-time* $(\mathscr{M}, g)$,

$$\tilde{g} \overset{\phi}{\mapsto} g := \Theta^2 \tilde{g} \;, \quad \tilde{\mathscr{M}} \overset{\phi}{\hookrightarrow} \mathscr{M} \;, \quad \Theta|_{\phi(\tilde{\mathscr{M}})} > 0 \;,$$

with $\phi(\tilde{\mathscr{M}}) \subset \mathscr{M}$ being relatively compact and with $\Theta : \mathscr{M} \to \mathbb{R}$ being a smooth function.

Let us add a comment concerning notation. Up to now we have denoted the physical space-time by $(\mathscr{M}, g)$. However, when dealing with problems where the unphysical space-time is involved, it will be more convenient to denote the physical space-time by $(\tilde{\mathscr{M}}, \tilde{g})$ and reserve the "non-tilde" objects for the unphysical space-time (the only exception will be Section 4.4, which is stressed there again).

The part of $\partial\phi(\tilde{\mathscr{M}})$ where the conformal factor Θ relating the physical and the unphysical space-time vanishes corresponds to "infinity" of the original physical space-time. Indeed, the affine parameter along $\tilde{g}$-geodesics diverges when approaching this part of the conformal boundary. The subset $\{\Theta = 0 \;, \; \mathrm{d}\Theta \neq 0\} \subset \partial\phi(\tilde{\mathscr{M}})$ is called *Scri*, denoted by $\mathscr{I}$. Since null geodesics in $(\mathscr{M}, g)$ acquire start- or end-points on $\mathscr{I}$, it is regarded as a representation of null infinity. One further distinguishes two components of $\mathscr{I}$, past and future null infinity $\mathscr{I}^-$ and $\mathscr{I}^+$, which are generated by the past and future endpoints of null geodesics in $\mathscr{M}$, respectively.

Penrose proposed to characterize space-times with a "(null) asymptotically flat" or "asymptotically de Sitter" structure (in certain null directions) by requiring that the unphysical metric g extends smoothly across (a part of) $\mathscr{I}$.[1] The idea is that only gravitational fields with an appropriate "asymptotically flat or de Sitter-like" fall-off behavior for $\lambda = 0$ and $\lambda > 0$, respectively, admit such a smooth extension through conformal infinity.

Supposing that a regular $\mathscr{I}$ exists, Einstein's field equations with vanishing cosmological constant imply (in vacuum or for appropriate matter models) that $\mathscr{I}$ is a null hypersurface, while for positive cosmological constant it is a space-like hypersurface in $(\mathscr{M}, g)$. In vacuum this follows directly from equation (3.8) below.

We have avoided a too stringent definition of an asymptotically flat or de Sitter space-time e.g. by requiring $\mathscr{I}$ to be of a certain topology, or by requiring completeness in the sense that every null geodesic in $(\mathscr{M}, g)$ has two distinct points on $\mathscr{I}$ (this leads to the notion of *asymptotically simple* space-times which e.g. exclude black holes [62]). The reason for that being that we will be interested in local problems near $\mathscr{I}$ and in the construction of space-times with "a piece of a smooth $\mathscr{I}$", so that it is not necessary to impose any global restrictions. Our definition rather corresponds to so-called *weakly asymptotically simple space-times*, cf. [62]. Note that Penrose's construction leaves some ambiguity in that there might exist non-equivalent conformal completions; this question is addressed in [24].

[1] Originally, Penrose required the metric to be C^3.

Given, in $3+1$ dimensions, a space-time which admits a smooth $\mathscr{I}$ with topology $\mathbb{R}\times S^2$, it is possible to construct coordinates in which the physical metric becomes manifestly flat at an appropriate fall-off rate as one approaches null infinity. In fact, these coordinates can be taken to be *Bondi coordinates* [83], which will be discussed in Section 5.2, whereby the classical coordinate approach by Bondi *et al.* [9] and Sachs [78] is recovered, who analyzed the radiative structure of the gravitational field at large distances from its source.[2] Asymptotically de Sitter space-times have a conformal structure similar to the one of the de Sitter space-time, though the metric does not necessarily need to approach the de Sitter metric at infinity.

The prototypes where Penrose's construction is possible are of course the Minkowski space-time for $\lambda = 0$ and the de Sitter space-time for $\lambda > 0$. In the latter case, conformal infinity consists of two space-like hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$. For Minkowski space-time the asymptotic structure is somewhat richer [74]: The null hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$ form future and past light-cones with vertex i^- and i^+ representing *future* and *past time-like infinity*, respectively. The intersection of the two light-cones is considered as one point, i^0, representing *space-like infinity*. In other words, $\mathscr{I}^-$ and $\mathscr{I}^+$ form "closed up" cones with future and past vertex i^0, respectively. At $i^\pm$ and i^0 the one-form $\mathrm{d}\Theta$ vanishes. These points do *not* belong to $\mathscr{I}$. While in the Minkowski case the metric is also smooth at $i^\pm$ and i^0, this is generally not expected from space-times which one would still regard as "asymptotically flat" (such as e.g. Schwarzschild space-time). In particular, a smoothness requirement at i^0, where the space-like geodesics meet, is too restrictive (the unphysical metric fails to be C^1 whenever the total mass of the space-time is non-zero [5]).

A class of particular interest in this context are *isolated gravitational systems* (for $\lambda = 0$), which one stipulates to be asymptotically flat at both spatial and null infinity. Since isolated bodies may remain present at arbitrary early or late times, one does not require $i^\pm$ to be regular points. A precise definition is given in [6, 61].

Another class of important gravitating systems is comprised by so-called *purely radiative space-times* [51] (for $\lambda = 0$), i.e. space-times which are generated solely by gravitational radiation coming in from past null infinity and interacting with itself. Information coming in from past time-like infinity is excluded. In this case, one does not only stipulate the space-time to be smooth at null infinity but also that it admits a smooth extension across i^-, and that $\mathscr{I}^-$ forms a regular (at least near i^-) future light-cone whose vertex is given by i^-. At sufficiently early times, the space-time is required to possess a conformal structure similar to the Minkowskian one. If there is only gravitational radiation interacting with itself and dispersing completely to infinity again, an analogous behavior is required from the gravitational field at i^+. If, in addition, certain global conditions are satisfied, the corresponding space-time will be called *purely radiative* [51]. For our purposes purely radiative space-times provide a motivation why it is physically relevant to regard i^- as a regular point.

Due to Penrose's geometric construction, the asymptotic behavior of the gravitational field can be analyzed in terms of a local problem in some neighborhood of an ordinary hypersurface boundary $\mathscr{I}$ (possibly supplemented by the points $i^\pm$ and i^0), to which all the local techniques of differential geometry are applicable. Now, it is a crucial issue to understand the interplay between Penrose's geometric concept of asymptotically flat and de Sitter space-times on the one hand and Einstein's field equations on the other hand, and whether all relevant physical systems which one would regard as "asymptotically flat or de Sitter" are compatible with the notion of a regular conformal infinity. The Bondi-Sachs metrics [9, 78] as well as certain other classes of metrics (cf. [74] for the relevant

[2] To prove the existence of Bondi coordinates it suffices to have a polyhomogeneous $\mathscr{I}$ [33].

literature) which should be expected to be "asymptotically flat or de Sitter", do admit smooth extensions à la Penrose at null infinity. This indicates that Penrose's approach includes a reasonable class of space-times, even though a more careful analysis is needed to obtain a deeper understanding of his construction (compare with [1, 2, 33] where it is argued that a polyhomogeneous $\mathscr{I}$ might provide a more suitable setting to model "generic" asymptotically flat space-times).

A systematic approach to establish existence of large classes of such space-times is via *asymptotic initial value problems*, where data are prescribed in the unphysical space-time on initial surfaces which comprise (a part of) $\mathscr{I}^-$ (cf. Chapter 4). Once a (local) solution to this problem has been constructed, the physical space-time is of infinite extent, at least in certain null directions, and possesses there an asymptotically flat or de Sitter structure. Strictly speaking, the data are prescribed on an appropriate initial manifold which *a posteriori* becomes $\mathscr{I}^-$, once the initial value problem has been solved and a space-time has been constructed (with $\Theta = 0$ and $\mathrm{d}\Theta \neq 0$ on the initial surface). However, we will be a bit sloppy with this terminology.

While these kinds of initial value problems might be useful for the construction of asymptotically flat or de Sitter space-times, they do not give any insight how "generic" these space-times are. By prescribing data on $\mathscr{I}^-$ and constructing smooth solutions out of them, a substantial part of the desired structure at infinity is built-in from the outset. We will therefore also consider ordinary null hypersurfaces which intersect $\mathscr{I}$ and which may be used to construct such space-times from a standard characteristic initial value problem (cf. Section 4.4). For the hyperboloidal Cauchy problem such a result is already available [49] for large classes of "non-generic" hyperboloidal data [1, 2]. Moreover, Corvino's gluing technique [39] provides a tool to construct null asymptotically flat vacuum space-times from asymptotically Euclidean space-like initial surfaces, cf. [29] and the references given therein.

There is, however, a problem: To formulate and, even more, to solve such asymptotic Cauchy problems, one needs well-behaved equations. The Einstein equations, say in vacuum, where they reduce to (1.2), expressed in terms of the conformally rescaled metric $g = \Theta^2\tilde{g}$, with the conformal factor Θ being some given smooth function, adopt the form

$$R_{\mu\nu}[g] + (n-1)\Theta^{-1}\nabla_\mu\nabla_\nu\Theta + (\Theta^{-1}\Box_g\Theta - n\Theta^{-2}\nabla^\sigma\Theta\nabla_\sigma\Theta)g_{\mu\nu} = \lambda\Theta^{-2}g_{\mu\nu} \;, \qquad (3.1)$$

where we have set $\Box_g := g^{\mu\nu}\nabla_\mu\nabla_\nu$. When treated as equations for the unphysical metric g, the vacuum equations become (formally) singular at conformal infinity, where Θ vanishes.

3.3 Friedrich's conformal field equations (CFE)

Due to its singular behavior at $\mathscr{I}$ the system (3.1) does not provide a convenient evolution system to study the gravitational field near conformal infinity. Fortunately, Friedrich [46, 47, 56] was able to find a representation of Einstein's vacuum field equations, the *conformal field equations (CFE)*, which *do* remain regular even if Θ vanishes, and which are equivalent to Einstein's equations

$$\tilde{R}_{\mu\nu}[\tilde{g}] \;=\; \lambda\tilde{g}_{\mu\nu} \;, \quad \tilde{g}_{\mu\nu} \;=\; \Theta^{-2}g_{\mu\nu} \;,$$

wherever Θ is non-vanishing. It is crucial for their derivation that Einstein's vacuum equations exhibit a conformally invariant substructure, and it is this feature which matches with Penrose's geometric description of the asymptotic structure of the gravitational field [52].

The curvature of a space-time is measured by the Riemann curvature tensor $R_{\mu\nu\sigma}{}^\rho$, which can be decomposed into the trace-free *conformal Weyl tensor* $W_{\mu\nu\sigma}{}^\rho$ and a second term which involves the *Schouten tensor* $L_{\mu\nu}$,

$$R_{\mu\nu\sigma}{}^{\rho} = W_{\mu\nu\sigma}{}^{\rho} + 2(g_{\sigma[\mu}L_{\nu]}{}^{\rho} - \delta_{[\mu}{}^{\rho}L_{\nu]\sigma}) \ .$$

The Schouten tensor $L_{\mu\nu}$ is in one-to-one correspondence with the Ricci tensor $R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}$,

$$L_{\mu\nu} := \frac{1}{n-1}R_{\mu\nu} - \frac{1}{2n(n-1)}Rg_{\mu\nu} \ .$$

The Weyl tensor $W_{\mu\nu\sigma}{}^{\rho}$ is invariant under conformal transformations. It is often considered to represent the radiation part of the gravitational field.

Let us further introduce the *rescaled Weyl tensor*

$$d_{\mu\nu\sigma}{}^{\rho} := \Theta^{2-n}W_{\mu\nu\sigma}{}^{\rho} \ , \tag{3.2}$$

and the scalar function

$$s := \frac{1}{n+1}\Box_g\Theta + \frac{1}{2n(n+1)}R\Theta \ .$$

There are different versions of the CFE, depending on which fields are treated as the unknowns. The version of the CFE which we shall primarily pay attention to are the *metric conformal field equations (MCFE)* [56],

$$R_{\mu\nu\sigma}{}^{\kappa}[g] = \Theta^{n-2}d_{\mu\nu\sigma}{}^{\kappa} + 2\left(g_{\sigma[\mu}L_{\nu]}{}^{\kappa} - \delta_{[\mu}{}^{\kappa}L_{\nu]\sigma}\right) \ , \tag{3.3}$$
$$\nabla_\rho d_{\mu\nu\sigma}{}^{\rho} = 0 \ , \tag{3.4}$$
$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \Theta^{n-3}\nabla_\rho\Theta\, d_{\nu\mu\sigma}{}^{\rho} \ , \tag{3.5}$$
$$\nabla_\mu\nabla_\nu\Theta = -\Theta L_{\mu\nu} + sg_{\mu\nu} \ , \tag{3.6}$$
$$\nabla_\mu s = -L_{\mu\nu}\nabla^\nu\Theta \ , \tag{3.7}$$
$$2\Theta s - \nabla_\mu\Theta\nabla^\mu\Theta = \lambda/n \ , \tag{3.8}$$

where the unknowns are

$$g_{\mu\nu} \ , \quad d_{\mu\nu\sigma}{}^{\rho} \ , \quad L_{\mu\nu} \ , \quad \Theta \ , \quad s \ ,$$

which are regarded as independent.

Consider now any frame field $e_i = e^{\mu}{}_i\partial_\mu$ for which the $g(e_i, e_j) \equiv g_{ij}$'s are constants. The *general conformal field equations (GCFE)* [56] for the unknowns

$$e^{\mu}{}_k \ , \quad \Gamma_i{}^k{}_j \ , \quad d_{ijk}{}^l \ , \quad L_{ij} \ , \quad \Theta \ , \quad s$$

read (Latin indices are used here to denote frame-components)

$$[e_p, e_q] = (\Gamma_p{}^l{}_q - \Gamma_q{}^l{}_p)e_l \ , \tag{3.9}$$
$$e_{[p}(\Gamma_{q]}{}^i{}_j) - \Gamma_k{}^i{}_j\Gamma_{[p}{}^k{}_{q]} + \Gamma_{[p}{}^i{}_{|k|}\Gamma_{q]}{}^k{}_j = \delta_{[p}{}^iL_{q]j} - g_{j[p}L_{q]}{}^i - \frac{1}{2}\Theta^{n-2}d_{pqj}{}^i \ , \tag{3.10}$$
$$\nabla_i d_{pqj}{}^i = 0 \ , \tag{3.11}$$
$$\nabla_i L_{jk} - \nabla_j L_{ik} = \Theta^{n-3}\nabla_l\Theta d_{jik}{}^l \ , \tag{3.12}$$
$$\nabla_i\nabla_j\Theta = -\Theta L_{ij} + sg_{ij} \ , \tag{3.13}$$
$$\nabla_i s = -L_{ij}\nabla^j\Theta \ , \tag{3.14}$$
$$2\Theta s - \nabla_j\Theta\nabla^j\Theta = \lambda/n \ , \tag{3.15}$$

where the $\Gamma_i{}^j{}_k$'s denote the Levi-Civita connection coefficients in the frame e_k.

One easily shows that (3.8) and (3.15) are consequences of (3.6)-(3.7) and (3.13)-(3.14), respectively, if they are arranged to hold at just one point. This can be ensured e.g. by an appropriate choice of the initial data.

The CFE constitute a complicated and highly overdetermined system of PDEs. Like the Einstein equations, they can be split into constraint and evolution equations. To this end, it is convenient to impose geometric gauge conditions on the coordinates and the frame rather than the classical harmonic gauge condition. A specific property in the $3+1$-dimensional case is that, irrespective of the sign of Θ, the propagational part of the CFE, the *reduced CFE*, provide a first-order quasi-linear symmetric hyperbolic system [46, 50, 54]. This is related to the fact that in $3+1$ dimensions the contracted second Bianchi identity is equivalent to the non-contracted one [56]. In higher dimensions this is no longer true, which is why the CFE seem to provide a nice evolution system solely in $3+1$ dimensions. Equipped with some convenient mathematical properties they provide an indispensable tool to study those solutions of the Einstein equations which admit an asymptotically flat or de Sitter structure at conformal infinity à la Penrose.

In the remainder of this work we shall always assume $3+1$ space-time dimensions whenever the unphysical space-time and the CFE are involved.

3.4 Gauge freedom inherent to CFE

There is considerable gauge freedom contained in the CFE, which arises from the freedom to choose the conformal factor and the usual freedom to choose coordinates. Let us focus on the MCFE. As described in Section 2, one may use coordinates adapted to the initial surface and impose a generalized wave-map gauge condition $H^\lambda = 0$ with arbitrarily prescribed gauge source functions W^λ and κ to exploit the latter freedom.

Instead of the conformal factor, which is treated as an unknown in the MCFE, one needs to identify another function which captures its gauge freedom. The standard choice is the unphysical curvature scalar R (cf. e.g. [56, 69]). Indeed, given a solution $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ of the MCFE, one may construct another one, which corresponds to the same physical solution, by replacing Θ by $\phi\Theta$, with ϕ some positive function, and by transforming the other fields according to their usual behavior under conformal rescalings of the metric. This expresses the *conformal covariance* of the CFE. Choosing ϕ such that it satisfies the wave equation

$$6\Box_g \phi - R\phi + R^*\phi^3 = 0 , \tag{3.16}$$

where R is the original curvature scalar and R^* a prescribed function, the new curvature scalar will coincide with R^*. Since well-posedness results for wave equations guarantee the existence of a positive solution to (3.16) (cf. Appendix A), at least locally, it is possible to regard R as a *conformal gauge source function.*

In addition, there remains the freedom to prescribe appropriate data $[\phi]$ on the initial surface. Depending on the type of the initial surface, this leads to some further gauge freedom:

- If the initial surface is $\mathscr{I}^-$, for $\lambda > 0$, one may prescribe $s|_{\mathscr{I}^-}$ and only the conformal class of the induced metric on $\mathscr{I}^-$ matters geometrically (cf. e.g. [71, Section 2]).
- If the initial surface is, for $\lambda = 0$, the null cone $C_{i^-} = \mathscr{I}^- \cup \{i^-\}$, one may prescribe $s|_{C_{i^-}}$ with $s(i^-) > 0$ (cf. e.g. [69, Sections 2 and 4], compare [57, Section 4] for an alternative choice).

CHAPTER 4

Asymptotic initial value problems in general relativity

"Infinity is the end."
Dejan Stojanović (*1959)

4.1 Asymptotic initial value problems

Having an appropriate system of equations at our disposal, we are now in a position to discuss various types of *asymptotic Cauchy problems* in the unphysical space-time, where data are prescribed on surfaces which intersect or form a part of $\mathscr{I}$, some of which will be analyzed in greater detail in the course of this chapter. Recall that we restrict attention to $3+1$ space-time dimensions.

1. *A space-like Cauchy problem for $\lambda > 0$ with data on $\mathscr{I}^-$*: Well-posedness has been established by Friedrich [52], who took advantage of the fact that the CFE imply a symmetric hyperbolic system of evolution equations for which standard well-posedness results are available. We provide an alternative proof in [71], attached as Chapter 8, which is based on the wave equations (4.5)-(4.9) below. The unconstrained data on $\mathscr{I}^-$ are, in the adapted coordinates introduced in Section 2.3, the conformal class $[h]$ of the induced metric $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and a symmetric trace- and divergence-free tensor field $D = D_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The constraint equations, from which all the other data relevant for the evolution equations are computed, will be given in Section 4.2.2 below. In the final vacuum space-time one will have $d_{titj}|_{\mathscr{I}^-} = D_{ij}$, whence D is interpreted as the *incoming radiation field*.

2. *A characteristic Cauchy problem with data on the light-cone $C_{i^-} = \mathscr{I}^- \cup \{i^-\}$ for $\lambda = 0$*: More precisely, one prescribes data on a Minkowskian cone C_O, so that the emerging vacuum space-time admits a smooth conformal completion where the C_{i^-}-cone is represented by C_O. This will be discussed in Section 4.3.

3. *A characteristic Cauchy problem for $\lambda = 0$ with data on an incoming null hypersurface and a part of $\mathscr{I}^-$*: Well-posedness near the intersection manifold has been shown by

Friedrich [48] in the analytic case. His result has been extended by Kánnár [63] to the smooth case, exploiting the fact that the CFE imply a symmetric hyperbolic system to which Rendall's well-posedness result [76] applies. As free data one may prescribe, in an adapted coordinate frame, the conformal class of $\breve{g}$ on the ordinary null surface and the tensor field $\overline{d}_{rArB}$ on $\mathscr{I}^-$, supplemented by certain data on the intersection manifold [63].

4. *An ordinary characteristic initial value problem with data on a light-cone (or on two transversally intersecting null hypersurfaces)*: As an extension of Luk's result [67] (for $\lambda = 0$), one would like to establish the existence of a smooth solution in some region to the future of the initial surface which admits a "piece of a smooth $\mathscr{I}^+$". This is expected to be possible for suitably specified data, namely those for which the solution to Einstein's wave-map gauge constraints is smooth at conformal infinity. Smoothness of the relevant fields where the initial surface intersects $\mathscr{I}^+$ will be discussed in Section 4.4. As indicated above, an analysis of this issue will provide some insights how generic the asymptotically flat space-times à la Penrose are.
   Since we prescribe smooth data on the cone up-to-and-including conformal infinity and since we demand the emerging space-time to admit a smooth $\mathscr{I}$, we regard this as an asymptotic Cauchy problem as well.

5. *A hyperboloidal initial value problem for* $\lambda = 0$: "Hyperboloidal data" with an appropriate fall-off behavior are prescribed on a space-like hypersurface which intersects $\mathscr{I}^+$ in a smooth spherical cross-section. An advantage in comparison with the asymptotically flat space-like case is that difficulties at i^0 are avoided. Friedrich [49, 53] showed that in this case the emerging space-time does contain a piece of a smooth $\mathscr{I}^+$, while for sufficiently small data one can even predict the existence of a space-time which admits a regular C_{i^+}-cone.

6. Other initial surfaces of interest could e.g. be a light-cone with vertex at $\mathscr{I}^-$ for $\lambda \geq 0$, or two transversally intersecting null hypersurfaces where the intersection manifold belongs to $\mathscr{I}^-$ for $\lambda \geq 0$.

These kinds of initial value problems permit the construction of space-times compatible with a Penrose-type conformal completion at infinity in a systematic manner. Although most of the results are local in (retarded) time in the unphysical space-time, one ends up with solutions of the vacuum Einstein equations which exist all the way to null infinity when going back to the physical space-time (at least in certain null directions)

As described above, Penrose proposed to distinguish asymptotically flat or de Sitter space-times by requiring the unphysical metric g to be smooth at $\mathscr{I}$. However, when working with the CFE to construct smooth solutions of the vacuum equations via asymptotic initial value problems, we need a metric and a rescaled Weyl tensor which are smooth at $\mathscr{I}$ (and, if necessary, even at i^-). Since the conformal Weyl tensor of g vanishes on $\mathscr{I}$ [74] and by definition $\mathrm{d}\Theta|_{\mathscr{I}} \neq 0$, the rescaled Weyl tensor will be regular at $\mathscr{I}$. If we consider data on the C_{i^-}-cone, though, the same conclusion cannot be drawn at i^- where $\mathrm{d}\Theta = 0$. We will thus confine attention to the class of solutions where both $g_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ admit smooth extensions across i^- when discussing the corresponding well-posedness result (cf., however, the comments at the end of Section 4.2.1).

4.2 A system of conformal wave equations

4.2.1 Derivation of the conformal wave equations (CWE)

On the way towards the construction of purely radiative space-times a first step is to derive a well-posedness result for the CFE with data given on the C_{i^-}-cone (at least locally near i^-), which of course provides an interesting result by itself. Since well-posedness results are available for quasi-linear wave equations on a light-cone (cf. Apppendix A), but not for symmetric hyperbolic systems, we will derive a system of wave equations which substitutes Friedrich's reduced CFE and which is equivalent to the CFE when supplemented by certain constraint equations on the initial surface. This has been accomplished in [69], attached as Chapter 8, compare [71], attached as Chapter 15, where a space-like $\mathscr{I}^-$ is treated.

Our starting point are the MCFE (3.3)-(3.8). Let us, for the time being, assume that the metric tensor is given. Then wave equations for $L_{\mu\nu}$ and s are straightforwardly derived by taking the divergence of (3.5) and (3.7), respectively, employing the second Bianchi identity as well as (3.3) and (3.6). A wave equation for Θ is simply obtained by taking the trace of (3.6). We further need a wave equation for $d_{\mu\nu\sigma}{}^{\rho}$. Taking into account that, in 4 dimensions, (3.4) is equivalent to $\nabla_{[\lambda}d_{\mu\nu]\sigma}{}^{\rho}=0$, this is achieved by taking the divergence of the latter equation and invoking (3.4). Proceeding this way, we find the system [69, Section 3]

$$\Box_g L_{\mu\nu} = 4L_{\mu\kappa}L_{\nu}{}^{\kappa} - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^{\rho}L_{\rho}{}^{\sigma} + \frac{1}{6}\nabla_\mu\nabla_\nu R\,, \tag{4.1}$$

$$\Box_g s = \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \frac{1}{6}sR\,, \tag{4.2}$$

$$\Box_g\Theta = 4s - \frac{1}{6}\Theta R\,, \tag{4.3}$$

$$\Box_g d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^{\alpha}d_{\sigma\rho\alpha}{}^{\kappa} - 4\Theta d_{\sigma\kappa[\mu}{}^{\alpha}d_{\nu]\alpha\rho}{}^{\kappa} + \frac{1}{2}Rd_{\mu\nu\sigma\rho}\,. \tag{4.4}$$

If the metric field is regarded as an unknown, we need to derive a wave equation for it as well. Moreover, when acting on tensor fields of non-zero valence, the principal part of $\Box_g$ will not be a d'Alembert operator anymore. To resolve the second issue we introduce a *reduced wave operator* $\Box_g^{(H)}$, which we define via its action on covector fields $v=v_\mu \mathrm{d}x^\mu$,

$$\Box_g^{(H)}v_\lambda := \Box_g v_\lambda - g_{\sigma[\lambda}(\hat\nabla_{\mu]}H^\sigma)v^\mu + (2L_{\mu\lambda} - R^{(H)}_{\mu\lambda} + \frac{1}{6}Rg_{\mu\lambda})v^\mu\,,$$

similarly for higher valence tensor fields. This, indeed, defines a wave-operator with principal part $g^{\mu\nu}\partial_\mu\partial_\nu$ which, in wave-map gauge and if (3.3) holds, coincides with the action of $\Box_g$. Finally, taking the trace of (3.3) and replacing the Ricci tensor by the reduced Ricci tensor yields a wave equation for the metric tensor.

Altogether, we end up with the following system of *conformal wave equations (CWE)*,

$$\Box_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa}L_{\nu}{}^{\kappa} - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^{\rho}L_{\rho}{}^{\sigma} + \frac{1}{6}\nabla_\mu\nabla_\nu R\,, \tag{4.5}$$

$$\Box_g s = \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \frac{1}{6}sR\,, \tag{4.6}$$

$$\Box_g\Theta = 4s - \frac{1}{6}\Theta R\,, \tag{4.7}$$

$$\Box_g^{(H)} d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^{\alpha}d_{\sigma\rho\alpha}{}^{\kappa} - 4\Theta d_{\sigma\kappa[\mu}{}^{\alpha}d_{\nu]\alpha\rho}{}^{\kappa} + \frac{1}{2}Rd_{\mu\nu\sigma\rho}\,, \tag{4.8}$$

$$R^{(H)}_{\mu\nu}[g] = 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}\,, \tag{4.9}$$

for the unknowns
$$g_{\mu\nu}\,, \quad d_{\mu\nu\sigma}{}^{\rho}\,, \quad L_{\mu\nu}\,, \quad \Theta\,, \quad s\,.$$
Recall that the curvature scalar is regarded as a conformal gauge source function, so the second-order derivative appearing in (4.5) does not disturb.

In [69, Section 6] we derive an alternative system of wave equations which treats the Weyl and the Cotton tensor rather than the rescaled Weyl tensor as unknowns. It has the property that the equations for the conformal factor Θ and the function s decouple from the remaining ones, and form a linear system of wave equation once the latter ones have been solved. Moreover, the alternative system might be suited to establish the existence of solutions of the MCFE where the metric but not the rescaled Weyl tensor is regular at past time-like infinity (cf. the discussion in [69, Section 7.1]). This way, the assumptions formulated at the end of Section 4.1 might be weakened. An alternative system of wave equations based on equations by Choquet-Bruhat and Novello [20] has been used by Dossa [42], though the equivalence issue is ignored there.

Similar to Friedrich's reduced CFE, the CWE form a subset of the CFE which, in a convenient gauge, form a mathematically nice evolution system. By their derivation, any solution of the MCFE in wave-map gauge will be a solution of the CWE. Conversely, a solution of the CWE will generally not be a solution of the MCFE. Since, roughly speaking, the CWE follow from the MCFE by differentiation, one essentially needs to make sure that the MCFE are initially satisfied, including their transverse derivatives in the space-like case (cf. [69, Theorem 3.7] and [71, Theorem A.1]). One should expect this to be the case whenever an appropriate set of constraint equations induced by the MCFE on the initial surface is fulfilled by the data given there. Indeed, this has been rigorously proved in the case where the initial surface is $\mathscr{I}^-$ for $\lambda > 0$ [71, Appendix A] and in the case where it is the C_{i^-}-cone with $\lambda = 0$ [69, Sections 3 and 5].

4.2.2 Equivalence of CWE & MCFE with data on a space-like $\mathscr{I}^-$

First of all, one needs to derive the constraint equations for the fields $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$, $d_{\mu\nu\sigma\rho}$ as well as their transverse derivatives, induced by the MCFE on a space-like $\mathscr{I}^-$ (i.e. we assume $\lambda > 0$). This has been done in [71, Section 2] (compare [52]) in adapted coordinates $(x^0 = t, x^i)$, where $\mathscr{I}^- = \{t = 0\}$ (at least locally), and by imposing the gauge conditions

$$R = 0\,, \quad \overline{s} = 0\,, \quad \overline{g}_{tt} = -1\,, \quad \overline{g}_{ti} = 0\,, \quad W^{\sigma} = 0\,, \quad \hat{g}_{\mu\nu} = \overline{g}_{\mu\nu}\,. \tag{4.10}$$

Recall that the relevant data are the conformal class $[h]$ of the induced metric on $\mathscr{I}^-$ and the symmetric 2-tensor $D_{ij} = d_{titj}|_{\mathscr{I}^-}$. Then, the following set of constraint equations is enforced by the MCFE on $\mathscr{I}^-$ on the fields $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$, $d_{\mu\nu\sigma\rho}$ (apart from $\overline{\Theta} = \overline{s} = 0$), which are further required to satisfy their usual algebraic symmetry properties,

$$h^{ij} D_{ij} = 0\,, \quad \mathscr{D}^j D_{ij} = 0\,, \tag{4.11}$$
$$\overline{g}_{tt} = -1\,, \quad \overline{g}_{ti} = 0\,, \quad \overline{g}_{ij} = h_{ij}\,, \quad \overline{\partial_t g_{\mu\nu}} = 0\,, \tag{4.12}$$
$$\overline{\partial_t \Theta} = \sqrt{\tfrac{\lambda}{3}}\,, \quad \overline{\partial_t s} = \sqrt{\tfrac{\lambda}{48}}\,\tilde{R}\,, \tag{4.13}$$
$$\overline{L}_{ij} = \tilde{L}_{ij}\,, \quad \overline{L}_{ti} = 0\,, \quad \overline{L}_{tt} = \tfrac{1}{4}\tilde{R}\,, \tag{4.14}$$
$$\overline{\partial_t L_{ij}} = -\sqrt{\tfrac{\lambda}{3}}\, D_{ij}\,, \quad \overline{\partial_t L_{ti}} = \tfrac{1}{4}\mathscr{D}_i \tilde{R}\,, \quad \overline{\partial_t L_{tt}} = 0\,, \tag{4.15}$$
$$\overline{d}_{titj} = D_{ij}\,, \quad \overline{d}_{tijk} = \sqrt{\tfrac{3}{\lambda}}\tilde{C}_{ijk}\,, \tag{4.16}$$
$$\overline{\partial_t d_{titj}} = \sqrt{\tfrac{3}{\lambda}}\tilde{B}_{ij}\,, \quad \overline{\partial_t d_{tijk}} = 2\mathscr{D}_{[j} D_{k]i}\,, \tag{4.17}$$

where $\tilde{L}_{ij}$, $\tilde{C}_{ijk}$ and $\tilde{B}_{ij}$ are the Schouten, Cotton and Bach tensor of the induced metric h_{ij}. We observe that given a representative h_{ij} of $[h_{ij}]$ and D_{ij}, the equations (4.12)-(4.17) are trivial to solve. The remaining constraints are (4.11), which correspond to the scalar and the vector vacuum constraint (2.16) and (2.17) in the ordinary space-like case. However, in the asymptotic case they "degenerate" to much simpler equations in that the tensor D_{ij} just needs to be trace- and divergence-free w.r.t. the induced metric $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ (a systematic construction of TT-tensors via York's decomposition method was recalled in Section 2.3). We observe that there is the same number of freely prescribable components as in the ordinary space-like say CMC-case (that the "gauge freedom" to prescribe the mean curvature has no counterpart appears to be reasonable due to the very special nature of the hypersurface $\mathscr{I}^-$).

The following theorem is proved in [71]:

Theorem 4.2.1 *Suppose we have been given a Riemannian metric h_{ij} and a symmetric tensor field D_{ij} on a space-like $\mathscr{I}^-$ (i.e. $\lambda > 0$). A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \Theta, s)$ of the CWE (4.5)-(4.9) to the future of $\mathscr{I}^-$ which induces these data on $\mathscr{I}^-$ is a solution of the MCFE (3.3)-(3.8) in the $(R = 0, \overline{s} = 0, \overline{g}_{tt} = -1, \overline{g}_{ti} = 0, W^\lambda = 0, \hat{g}_{\mu\nu} = \overline{g}_{\mu\nu})$-wave-map gauge if and only if the fields $(\overline{g}_{\mu\nu}, \overline{\partial_t g_{\mu\nu}}, \overline{L}_{\mu\nu}, \overline{\partial_t L_{\mu\nu}}, \overline{d}_{\mu\nu\sigma}{}^\rho, \overline{\partial_t d_{\mu\nu\sigma}{}^\rho}, \overline{\partial_t \Theta}, \overline{\partial_t s})$ have their usual algebraic symmetry properties and fulfill the constraint equations (4.11)-(4.17).*

Applying standard well-posedness results for wave equations with data on a space-like hypersurface (cf. Appendix A), one immediately recovers Friedrich's well-posedness result [52] for the asymptotic Cauchy problem on a space-like $\mathscr{I}^-$ for $\lambda > 0$ (a system of wave equations might be advantageous in certain situations from a numerical point of view [65]):

Theorem 4.2.2 *The tuple (Σ, h_{ij}, D_{ij}), with h_{ij} a Riemannian metric and D_{ij} a symmetric 2-tensor, defines, for $\lambda > 0$, an (up to isometries) unique maximal globally hyperbolic development (in the unphysical space-time) of the vacuum field equations where $\iota(\Sigma)$ represents $\mathscr{I}^-$ (i.e. $\overline{\Theta} = 0$ and $\overline{\mathrm{d}\Theta} \neq 0$) with $\iota^* g_{ij}|_\Sigma = h_{ij}$ and $\iota^* d_{titj}|_\Sigma = D_{ij}$, if and only if D_{ij} is trace- and divergence-free. The solution depends continuously on the initial data.*

4.2.3 Equivalence of CWE & MCFE with data on C_{i^-}

As it is the main object of [69], attached as Chapter 8, the same analysis can be carried out on the C_{i^-}-cone. We first give the constraint equations for the fields $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma\rho}$ induced by the MCFE on C_{i^-} [69, Section 4] (we now need to assume $\lambda = 0$). In adapted null coordinates, imposing a wave-map gauge condition with

$$R = 0\,, \quad \overline{s} = -2\,, \quad \kappa = 0\,, \quad W^\lambda = 0\,, \quad \hat{g} = \eta\,, \tag{4.18}$$

(here η denotes the Minkowski metric in adapted coordinates), and assuming regularity of the fields at the vertex of the cone, cf. [16, Section 4.5], we discover that, beside their usual algebraic symmetries, $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma\rho}$ need to satisfy the following set of constraint equations (cf. [47, 51, 63] where they are given in a spin frame):

$$\overline{g}_{\mu\nu} = \eta_{\mu\nu}\,, \tag{4.19}$$

$$\overline{L}_{r\mu} = 0\,, \quad \overline{L}_{0A} = \frac{1}{2}\check{\nabla}^B\lambda_{AB}\,, \quad \overline{g}^{AB}\overline{L}_{AB} = 0\,, \quad \breve{\overline{L}}_{AB} = \omega_{AB}\,, \tag{4.20}$$

$$\overline{d}_{rArB} = -\frac{1}{2}\partial_r(r^{-1}\omega_{AB})\,, \tag{4.21}$$

$$\overline{d}_{0rrA} = \frac{1}{2}r^{-1}\partial_r\overline{L}_{0A}\,, \tag{4.22}$$

$$\overline{d}_{0rAB} = r^{-1}\check{\nabla}_{[A}\overline{L}_{B]0} - \frac{1}{2}r^{-1}\lambda_{[A}{}^C\omega_{B]C}\,, \tag{4.23}$$

$$
\begin{aligned}
(\partial_r +\ 3r^{-1})\overline{d}_{0r0r} &= \check{\nabla}^A\overline{d}_{0rrA} + \frac{1}{2}\lambda^{AB}\overline{d}_{rArB}\ , &(4.24)\\
2(\partial_r + r^{-1})\overline{d}_{0r0A} &= \check{\nabla}^B(\overline{d}_{0rAB} - \overline{d}_{rArB}) + \check{\nabla}_A\overline{d}_{0r0r} + 2r^{-1}\overline{d}_{0rrA} + 2\lambda_A{}^B\overline{d}_{0rrB}\ , &(4.25)\\
4(\partial_r - r^{-1})\breve{\overline{d}}_{0A0B} &= (\partial_r - r^{-1})\overline{d}_{rArB} + 2(\check{\nabla}_{(A}\overline{d}_{B)rr0})\breve{} + 4(\check{\nabla}_{(A}\overline{d}_{B)0r0})\breve{}\\
&\quad +3\lambda_{(A}{}^C\overline{d}_{B)C0r} + 3\overline{d}_{0r0r}\lambda_{AB}\ , &(4.26)\\
4(\partial_r + r^{-1})\overline{L}_{00} &= \lambda^{AB}\omega_{AB} - 4r\overline{d}_{0r0r} - 2\check{\nabla}^A\overline{L}_{0A}\ . &(4.27)
\end{aligned}
$$

Recall that "$\breve{}$" denotes the trace-free part. The r-dependent tensor field λ_{AB} on S^2 is the unique solution of

$$(\partial_r - r^{-1})\lambda_{AB} = -2\omega_{AB} \quad \text{with} \quad \lambda_{AB} = O(r^5)\ . \tag{4.28}$$

The solutions to the ODEs are uniquely determined by regularity conditions near the tip,

$$\overline{d}_{0r0r} = O(1), \quad \overline{d}_{0r0A} = O(r)\ , \quad \breve{\overline{d}}_{0A0B} = O(r^2)\ , \quad \overline{L}_{00} = O(1)\ . \tag{4.29}$$

There is no constraint for the tensor field $\omega_{AB} = \breve{\overline{L}}_{AB} = O(r^4)$ (for initial data which fall-off slower, the rescaled Weyl tensor will not be regular at i^-). As in the ordinary characteristic case discussed in Section 2.4, one can extract certain functions which form the free "reduced" data. A look at the constraint equations shows that there are various alternatives. Instead of ω_{AB}, one may, e.g., prescribe λ_{AB} (which will be $(\overline{\partial_0 g_{AB}})\breve{}$ in the emerging vacuum space-time), or, as in [57, 63], the *incoming radiation field* $\overline{d}_{rArB}$.

By all means, as in the space-like case, the freedom to specify initial data is the same as in the ordinary characteristic case, namely two real functions on $\mathbb{R}_+ \times S^2$. This was interpreted by Friedrich [52] as an indication that it is a quite general feature of vacuum space-times to possess "pieces of a smooth $\mathscr{I}$".

In [69] we have proved:

THEOREM 4.2.3 *Suppose we have been given a smooth one-parameter family of s_{AB}-traceless tensors $\omega_{AB}(r,x^A) = O(r^4)$ on the 2-sphere, where s_{AB} denotes the standard metric. A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma\rho}, \Theta, s)$ of the CWE (4.5)-(4.9) with $\lambda = 0$ to the future of C_{i^-}, smoothly extendable through C_{i^-}, which induces these data on C_{i^-} is a solution of the MCFE (3.3)-(3.8) in the $(R = 0, \overline{s} = -2, \kappa = 0, W^\lambda = 0, \hat{g} = \eta)$-wave-map gauge if and only if the fields $(\overline{g}_{\mu\nu}, \overline{L}_{\mu\nu}, \overline{d}_{\mu\nu\sigma\rho})$ have their usual algebraic properties and solve the constraint equations (4.19)-(4.28) with boundary conditions (4.29).*

4.3 Well-posedness result for the MCFE with data on C_{i^-}

Due to Theorem 4.2.3 we have a system of wave equations at our disposal which can be used to construct solutions of the MCFE with a regular C_{i^-}-cone. Now, in order to apply Dossa's theorem [41] (cf. Appendix A) and establish a well-posedness result for the asymptotic characteristic initial value problem with data on C_{i^-}, we need to make sure that the initial data for the CWE $(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma\rho}, \mathring{\Theta}, \mathring{s})$, which arise from the unconstrained seed data ω_{AB} as solutions of the constraint equations (4.19)-(4.28), are restrictions to the cone of smooth space-time fields. While this is trivial for $\mathring{g}_{\mu\nu} = \eta_{\mu\nu}$, $\mathring{\Theta} = 0$ and $\mathring{s} = -2$, it is an intricate issue for $\mathring{L}_{\mu\nu}$ and $\mathring{d}_{\mu\nu\sigma\rho}$.

One method to ensure that this is indeed the case consists of two main steps:

1. The first one, accomplished by Friedrich in [57], is to construct approximate solutions of the CFE. These are smooth tensor fields $g^{\mathrm{appr}}_{\mu\nu}$ and Θ^{appr} defined in some neighborhood of i^- which satisfy the CFE and induce the given data at all orders at i^-, i.e. up to terms which decay faster than any power of the Euclidean coordinate distance from i^- (note that the remaining fields $L^{\mathrm{appr}}_{\mu\nu}$, $d^{\mathrm{appr}}_{\mu\nu\sigma\rho}$ and s^{appr} are determined by $g^{\mathrm{appr}}_{\mu\nu}$ and Θ^{appr}). This is done by a computation of formal Taylor expansions of all the relevant fields and an application of Borel's summation lemma, starting from suitable initial data, namely the incoming radiation field ς defined below.

2. The restrictions of the fields $(g^{\mathrm{appr}}_{\mu\nu}, L^{\mathrm{appr}}_{\mu\nu}, d^{\mathrm{appr}}_{\mu\nu\sigma\rho}, \Theta^{\mathrm{appr}}, s^{\mathrm{appr}})$ obtained in step 1 to C_{i^-} form approximate solutions of the constraint equations (4.19)-(4.28). These differ from the exact solution $(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma\rho}, \mathring{\Theta}, \mathring{s})$ constructed from the same data ς by error terms which are $O(r^\infty)$, where r is an affine distance from the tip along the generators. Such tensor fields on the light-cone arise from smooth tensors in space-time. The exact solution $(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma\rho}, \mathring{\Theta}, \mathring{s})$ of the constraint equations constructed from the data ς thus arises indeed from smooth tensors in space-time. A precise formulation of this argument is the contents of [36], attached as Chapter 9.

Proceeding this way, we end up with the following result [36]:

THEOREM 4.3.1 *Let C_O be a light-cone in Minkowski space-time $(\mathbb{R}^{3+1}, \eta)$. Let, in manifestly flat coordinates y^μ, $\ell = \partial_0 + (y^i/|\vec{y}|)\partial_i$ denote the field of null tangents to C_O. Let $\tilde{d}_{\alpha\beta\gamma\delta}$ be a tensor with algebraic symmetries of the Weyl tensor and with vanishing η-traces. The incoming radiation field ς is defined as the pull-back of $\tilde{d}_{\alpha\beta\gamma\delta}\ell^\alpha\ell^\gamma$ to $C_O \setminus \{O\}$. Let, finally, ς_{ab} denote the components of ς in a frame parallel-propagated along the generators of C_O.*

Then, there exists a neighborhood $\mathscr{O}$ of O, a smooth metric g and a smooth function Θ such that C_O is the light-cone of O for g, Θ vanishes on C_O, with $\mathrm{d}\Theta$ nonzero on $\dot{J}^+(O)\cap\mathscr{O}\setminus\{O\}$, the function Θ has no zeros on $\mathscr{O}\cap I^+(O)$, i.e. $C_O = C_{i^-}$, and the metric $\Theta^{-2}g$ satisfies the vacuum Einstein equations there. Further, the tensor field $d_{\alpha\beta\gamma\delta} := \Theta^{-1}C_{\alpha\beta\gamma\delta}$ extends smoothly across C_{i^-}, and ς_{ab} are the frame components, in a g-parallel-propagated frame, of the pull-back to C_{i^-} of $d_{\alpha\beta\gamma\delta}\ell^\alpha\ell^\gamma$. The solution is locally unique up to isometries.

4.4 Smoothness of $\mathscr{I}$

So far we have discussed asymptotic Cauchy problems where data are prescribed in the unphysical space-time on (a part of) $\mathscr{I}$. In this section, we want to return to the standard characteristic Cauchy problem with a null cone given in the physical space-time, which, in the conformally rescaled space-time, intersects $\mathscr{I}^+$ in a smooth spherical cross-section. It will be convenient to denote the physical space-time by $(\mathscr{M}, g)$ and the unphysical one by $(\tilde{\mathscr{M}}, \tilde{g})$.

Recall Luk's result [67] that, assuming that the constraint equations can be integrated all the way to infinity (equivalently that the Raychaudhuri equation admits a global solution $\tau > 0$, cf. below), there exists a solution of the vacuum Einstein equations to the future of the whole cone and not just in some neighborhood to the future of its vertex. Nonetheless, the neighborhood where existence is predicted may shrink to zero when going up to null infinity along the cone. It would be desirable to have a result at hand where this does not happen and to construct classes of vacuum space-times this way which admit a piece of a smooth $\mathscr{I}^+$ à la Penrose. As mentioned before, such a result is available for the hyperboloidal Cauchy problem [1, 2, 49].

One goal of this Ph.D. project was to provide a setting in which a corresponding well-posedness result can be established in the characteristic case. As a first step, one needs to find conditions under which the characteristic wave-map gauge constraints for the Einstein equations admit global solutions on C_O. In [38], attached as Chapter 10, we provide sufficient conditions on the initial data such that this is the case.

As shown in Section 2.4, the constraint equations can be transformed into a linear, hierarchical ODE-system. For a global solution of the original equations to exist, though, the coefficients must be regular everywhere (except possibly at the vertex), which is the case if and only if the functions φ and ν^0 are non-vanishing (positive in the conventions of [16, 38]). Since ν^0 is in one-to-one correspondence with the gauge source function $\overline{W}^0$, its positivity is just a matter of gauge. However, the Raychaudhuri equation does impose proper restrictions on the initial data. It is shown in [38, Section 2.2] that a global solution exists by all means if $([\breve{g}], \kappa)$ satisfies

$$\int_0^\infty \Big(\int_0^r e^{H(\hat{r})}\mathrm{d}\hat{r}\Big)e^{-H(r)}|\sigma|^2(r)\,\mathrm{d}r \,<\, 2\;, \quad \text{where} \quad H(r,x^A) \,:=\, \int_0^r \kappa(\tilde{r},x^A)\,\mathrm{d}\tilde{r}\;. \tag{4.30}$$

A sufficient condition for the positivity of ν^0 is

$$\overline{W}^0 \,<\, r\varphi^{-2}\sqrt{\frac{\det\gamma}{\det s}}\gamma^{AB}s_{AB}\;, \tag{4.31}$$

which holds in particular for any $\overline{W}^0 \leq 0$.

It is expected for both the space-like and the characteristic Cauchy problem [1, 2, 12, 33] that even conformally smooth reduced data evolve into solutions which “generically” develop logarithmic terms at null infinity which destroy smoothness there. Although it might be feasible to establish a well-posedness result in the polyhomogeneous setting as well, it appears simpler and more in the spirit of Penrose’s original notion of asymptotic flatness to prove such a result for characteristic initial data in the smooth case. For this, one needs to make sure that the solution $\overline{g}$ of the Einstein wave-map gauge constraints on C_O is smooth and non-degenerate at conformal infinity, and, even more, that not just $\overline{g}$ but the traces of all the fields on C_O which appear in the CFE have smooth extensions at conformal infinity as made precise in [38, Definition 4.1]. This issue has been analyzed in detail in [38, 72], attached as Chapters 10 and 11. We will sum up the results in the remainder of this section.

Assuming that a global solution $\overline{g}$ to the constraint equations exists, it needs to be ensured that the conformally rescaled metric $\overline{\tilde{g}} = \overline{\Theta}^2\overline{g}$ with $\overline{\Theta} = x := 1/r$ has a smooth extension as a Lorentzian metric across $\{x = 0\}$. We have noted in Section 3.2 that the existence of a smooth conformal completion à la Penrose implies the existence of Bondi coordinates [83]. It follows from their existence that the physical metric needs to be smooth when rescaled with the conformal factor Θ [38]. Assuming also the connection coefficients $\overline{\Gamma}^r_{rA}$ to be smooth at $\{x = 0\}$ when transformed into the unphysical space-time,[1] this leads to a couple of asymptotic necessary and sufficient conditions to be imposed on the initial data $[\gamma]$ and the gauge functions κ and $\overline{W}^\lambda$, which have been worked out in [38, Sections 4 and 5] and [72, Section 3]:[2]

1. A representative of $[\gamma]$ has an asymptotic expansion of the form

$$\gamma \sim r^2\Big(s_{AB} + \sum_{n=1}^\infty h^{(n)}_{AB}r^{-n}\Big)\;, \tag{4.32}$$

   and corresponding expansions hold for its non-transverse derivatives.

[1] This is clearly necessary for the existence of a solution of the CFE which is smooth at $\mathscr{I}^+$.
[2] The result can easily be adapted to other gauge schemes where, e.g., the metric $g|_{C_O}$ is prescribed.

2. $\lim_{r\to\infty}(r^{-1}\varphi) > 0$ and $\lim_{r\to\infty}\nu^0 > 0$.

3. $\kappa = \mathcal{O}(r^{-3})$, $\overline{W}^0 = \mathcal{O}(r^{-1})$, $\overline{W}^A = \mathcal{O}(r^{-1})$, $\overline{W}^r = \mathcal{O}(r)$,[3] supplemented by certain conditions which need to be satisfied by the leading-order expansion coefficients of $\overline{W}^\lambda \sim \sum_k(\overline{W}^\lambda)_k r^{-k}$,

$$(\overline{W}^0)_2 = \big[\frac{1}{2}(\overline{W}^0)_1 + (\varphi_{-1})^{-2}\big]\tau_2\,, \tag{4.33}$$
$$(\overline{W}^A)_3 = f^A((\overline{W}^A)_2, (\overline{W}^A)_1, (\overline{W}^0)_2, (\overline{W}^0)_1)\,, \tag{4.34}$$
$$(\overline{W}^1)_2 = f((\overline{W}^1)_1, (\overline{W}^1)_0, (\overline{W}^1)_{-1})\,, \tag{4.35}$$

where

$$\varphi_{-1} := \lim_{r\to\infty}(r^{-1}\varphi) \quad \text{and} \quad \tau_2 := \lim_{r\to\infty}(r^2\tau - 2r)\,, \tag{4.36}$$

and where f and f^A are functions of the indicated expansion coefficients, whose specific form is not relevant here.[4]

4. The *no-logs-condition* is satisfied,

$$\breve{h}^{(2)}_{AB} = \frac{1}{2}(h^{(1)} + \tau_2)\breve{h}^{(1)}_{AB}\,. \tag{4.37}$$

There are several aspects which deserve a comment: This result is obtained by a thorough analysis of the asymptotic behavior of the solutions of Einstein's wave-map gauge constraints in vacuum. That the leading-order term in the expansion of $[\gamma]$ can be taken to be the standard metric (or rather $r^2 s_{AB}$) follows from the three facts that asymptotic flatness ensures that the leading order term in the expansion of γ needs to be non-degenerated (supposing that there are no coordinate singularities), that on S^2 any Riemannian metric is conformal to the standard metric, and that only the conformal class of γ matters. It is shown in [38, Section 2] that a sufficient condition on the initial data and the gauge source functions to satisfy the conditions in 2 is, apart from (4.30) (which implies $\varphi_{-1} > 0$) and (4.31),

$$(\overline{W}^0)_1 < 2(\varphi_{-1})^{-2}\,. \tag{4.38}$$

While the requirement of the existence of a globally positive φ (equivalently the existence of a global τ) with $\varphi_{-1} > 0$ is to exclude conjugate points up-to-and-including conformal infinity, the second condition in 2 and all the conditions in 3 merely concern the choice of the gauge functions κ and $\overline{W}^\lambda$; in other words, many of the logarithmic terms appearing in the asymptotic solutions of the constraint equations can be eliminated by an appropriate choice of coordinates. They are gauge artifacts. It is only the no-logs-condition (4.37) (together with the positivity requirements on φ and φ_{-1}) which imposes proper restrictions on the initial data. Indeed, these conditions are gauge-invariant [38, Section 6] and thus need to be satisfied in any regular adapted null coordinate system which covers the whole cone as a necessary condition for the emerging space-time to admit a "piece of a smooth $\mathscr{I}^+$".

The conditions on the gauge functions exclude, for instance, any $(\kappa = 0, \overline{W}^0 = 0)$-gauge as then (4.33) implies $\tau_2 = 0$, which, for $\kappa = 0$, holds only for flat data. This is the no-go result proved in [38, Section 3]: In harmonic coordinates adapted to the light-cone a non-flat metric will not be smooth at $\mathscr{I}$. A much more convenient gauge in this context is the *metric*

[3]We write $f = \mathcal{O}(r^N)$, $N \in \mathbb{N}$, if the function $F(x,\cdot) := x^N f(x^{-1},\cdot)$ is smooth at $x = 0$.

[4]The condition $\kappa = \mathcal{O}(r^{-3})$ implies that an affine parameter diverges as r goes to infinity so that "$r = \infty$" represents null infinity.

gauge: It is shown in [72, Section 4] that the gauge freedom to prescribe κ, $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$ can be replaced, at least for large r, by the freedom to prescribe φ, $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$. Choosing, for large r, the Minkowskian values

$$\varphi = r\;, \quad \nu_0 = 1\;, \quad \nu_A = 0\;, \quad \overline{g}_{00} = -1\;, \quad \overline{\partial_0 g_{0\mu}} = 0\;, \tag{4.39}$$

one verifies that all the above conditions 1-4 are satisfied if and only if the initial data γ are of the form

$$\gamma \sim r^2\Big(s_{AB} + \sum_{n=1}^{\infty} h^{(n)}_{AB} r^{-n}\Big) \quad \text{with} \quad \breve{h}^{(2)}_{AB} \;=\; \frac{1}{2} h^{(1)} \breve{h}^{(1)}_{AB}\;. \tag{4.40}$$

Even more, it is proved in [72, Section 5] that in the current metric gauge and under assumption (4.40), all the fields appearing in the MCFE and the GCFE are smooth at $\{x = 0\}$ in the unphysical space-time. The no-logs-condition (together with (4.32)) is therefore a necessary and sufficient condition on $([\gamma],\kappa)$ to obtain initial data for the CFE which are smooth at $\{x = 0\}$ in the metric gauge.

In the metric gauge, in Bondi coordinates, or, more general, in any gauge where $\kappa = \frac{r}{2}|\sigma|^2$, the no-logs-condition is equivalent to Bondi's *outgoing wave condition* [9], cf. [72, Section 4.5]. A more geometric interpretation of this condition is provided in [38, Section 6]. The conformal Weyl tensor satisfies $\overline{C}_{rAr}{}^B = O(r^{-4})$, and we have

$$\overline{C}_{rAr}{}^B \;=\; O(r^{-5}) \quad\Longleftrightarrow\quad (4.37) \text{ holds}\;. \tag{4.41}$$

In fact, the no-logs-condition (4.37) holds if and only if the Weyl tensor $\overline{C}_{\mu\nu\sigma}{}^\rho$ vanishes where the cone intersects $\mathscr{I}^+$ [72, Section 5.2], compare [33].

As mentioned above, characteristic initial data sets for the CFE admitting a smooth extension across $\{x = 0\}$ provide a promising starting point to establish a well-posedness result for Einstein's vacuum field equations which admit patches of a smooth $\mathscr{I}^+$. This has been achieved in [10] without any further restrictions, compare [66], so that the no-logs-condition in fact *characterizes* such space-times.

CHAPTER 5

The notion of mass in general relativity

"It is important to realize that in physics today, we have no knowledge of what energy is."
RICHARD P. FEYNMAN (1918-1988)

5.1 ADM and Bondi mass

The notion of energy plays a key role in many physical theories. In general relativity, though, it is somewhat more intricate to give a reasonable definition. This is owing to the fact that the gravitational field, i.e. the space-time itself, contributes to the energy, and generally there is no natural way to decompose it into a background field and a dynamical field, or to define a preferred coordinate system (cf. e.g. [86]). General relativity does not seem to be compatible with a notion of a locally well-defined energy density of the gravitational field. Nonetheless, there exists the notion of a *total energy* p^0 or, more general, of an *energy-momentum 4-vector* p^μ, of e.g. an isolated gravitational system as represented by a space-time which is asymptotically flat at null and spatial infinity. In fact, there are several approaches, including certain quasi-local notions (cf. [26] for an overview). We focus on the two most relevant ones, ADM and Bondi mass:[1]

- Consider an asymptotically Euclidean space-like hypersurface $\Sigma \subset \mathscr{M}$ representing an "instant of time" t. Then, one can define the notation of energy at that time at spatial infinity i^0. This is the so-called *ADM mass* m_{ADM} [4] (cf. [5, 61] where geometric approaches to the ADM mass are described).
- A second possibility is to define the mass of an (asymptotically) null hypersurface at a given moment of "retarded time" u at the cross-section where it intersects null infinity. This leads to the notion of the *Trautman-Bondi mass* m_{TB} [9, 78, 85] (cf. [83] for a geometric reformulation and [26] for alternative approaches).

The physical picture behind and the expected relation between these two notions of mass is as follows: The ADM mass m_{ADM} at i^0 is interpreted as the total energy available in

[1] More properly, it should be called *ADM* and *Bondi energy*, but we stick to the classical terminology used by Bondi *et al.*

the hypersurface Σ. Now, it is a peculiar feature of gravitational waves that they radiate away their source strength, that is the mass. By virtue of gravitational radiation and other radiation processes the system may loose energy. Cross sections of $\mathscr{I}$ represent the asymptotic properties of a radiating system at a retarded time u. The Bondi mass associated to such a section is interpreted as the remaining total energy of the system at the retarded time u after a loss of energy due to radiation which escaped to infinity up to that time. Indeed, it follows from the *Bondi-Sachs mass loss formula* [9, 78] that, in vacuum or for matter fields satisfying appropriate fall-off requirements, m_{TB} is monotonically decreasing in u, i.e. radiation always carries away energy from the system. Moreover, there are partial results [6] that ADM and Bondi mass differ precisely by the amount of energy radiated away. One therefore expects that, within an appropriate setting, $m_{\mathrm{TB}}(u) \to m_{\mathrm{ADM}}$ as u approaches i^0.

An issue of fundamental physical importance concerns the positivity of ADM and Bondi mass, especially since it is closely related to the stability of isolated systems. First of all, one would like to show that the total energy of an isolated system is positive. But then it still may happen that due to gravitational radiation the Bondi mass, even if initially positive, becomes negative at some later retarded time. So one would further like to prove that the energy radiated away is bounded by the total energy content of the system.

It turned out that proving positivity of ADM and Bondi mass is remarkably difficult, and it took decades to achieve that. Schoen and Yau [81, 82] (cf. Witten [88] for am alternative proof) were the first ones who accomplished a complete proof of the positivity of the ADM mass in a *non-singular* asymptotically flat space-time satisfying the *dominant energy condition*. A rigorous proof that the Bondi mass is positive has been given in [32]. However, the proofs are indirect and rely on ingenious PDE- or spinor-techniques, and an intuition what is going on there from a physical point of view is lacking (cf. e.g. [26] for an overview over various proofs).

In this Ph.D. project we study initial value problems and properties the emerging space-time will have which can be predicted directly from the prescribed data. In [37], attached as Chapter 12, we derive an expression for the Bondi mass of a globally smooth light-cone in terms of physically relevant fields, which can be easily calculated from the initial data. In addition, this expression will be manifestly positive-definite. We merely use elementary methods, so that we end up with a simple and direct proof of the positivity of the Bondi mass in space-times containing globally smooth light-cones. Since we want to generalize a formula in [37], we will provide some of the main steps of the proof in Section 5.3.

5.2 The classical definition of the Bondi mass

The classical definition of the Trautman-Bondi mass requires a smooth conformal completion at null infinity. It is defined in terms of certain asymptotic expansion coefficients of the metric in Bondi coordinates, which are assigned to sections of the conformal boundary. We assume 4 space-time dimensions and a vanishing cosmological constant λ.

Let us suppose we have been given a space-time $(\mathscr{M}, g)$, and that there exist coordinates such that the line-element takes the form

$$g_{\mathrm{Bo}} = -\frac{V}{r} e^{2\beta} \mathrm{d}u^2 - 2e^{2\beta} \mathrm{d}u \mathrm{d}r + r^2 h_{AB} (\mathrm{d}x^A - U^A \mathrm{d}u)(\mathrm{d}x^B - U^B \mathrm{d}u) \,, \tag{5.1}$$

where $\det h_{AB} = \det s_{AB}$, where $u \in (u_-, u_+)$ is the *retarded time* (the $(u = \mathrm{const})$-surfaces are null), where $r \in (R, \infty)$ is the *luminosity distance*, and where the x^A's, $A = 2, 3$, are local coordinates on $(u = \mathrm{const}, r = \mathrm{const}) \cong S^2$. One further demands the fields V, U^A, β

and h_{AB} to satisfy certain fall-off conditions which ensure asymptotic flatness and that the coordinates exhibit this asymptotic flatness. For $u = \text{const}$, one stipulates

$$\lim_{r\to\infty}(r^{-1}V) = 1 \;, \quad \lim_{r\to\infty}(rU^A) = 0 \;, \quad \lim_{r\to\infty}\beta = 0 \;, \quad \lim_{r\to\infty} h_{AB} = s_{AB} \;. \tag{5.2}$$

These coordinates are called *Bondi coordinates* and have been designed by Bondi [9] and Sachs [78] via distinguished families of outgoing null hypersurfaces to investigate gravitational radiation in the far field regime of an isolated system. They provide a special case of adapted null coordinates where the form (2.25) is preserved under the evolution in u. Their existence in asymptotically flat space-times follows from [83] for a smooth $\mathscr{I}$ and from [33] for a polyhomogeneous $\mathscr{I}$.

Along the $(u = \text{const}, x^A = \text{const}^A)$-null geodesics Einstein's vacuum equations imply transport equations, which, for h_{AB} given, determine asymptotic expansions of the fields V, U^A and β in terms of the luminosity distance r as $r \to \infty$. To avoid the appearance of logarithmic terms in these expansions Bondi and Sachs imposed an "outgoing wave condition". However, it has been shown in [33] that this is an unnecassary restriction, and that the logarithmic terms do not disturb when defining the Trautman-Bondi mass or deriving the Bondi-Sachs mass loss formula.

The asymptotic expansions involve certain integration constants, or rather functions, since they depend on the values of u and x^A. It has been observed by Bondi *et al.* [9] that in the static, axisymmetric case, where the integration functions are proper constants, they are related to certain multipole moments. He therefore proposed to ascribe, in the non-static case, a corresponding meaning to the mean values of these integration functions over the 2-sphere.

The (uu)-component of g admits the following expansion for large r,

$$(g_{\text{Bo}})_{uu} = -r^{-1}Ve^{2\beta} = -1 + \frac{2M}{r} + o(r^{-1}) \;. \tag{5.3}$$

The expansion coefficient $M = M(u, x^A)$, which arises as one of the integration functions when integrating Einstein's field equations, is called the *Bondi mass aspect*. The *Trautman-Bondi mass* is defined to be its mean value on (S^2, s_{AB}),

$$m_{\text{TB}}(u) := \frac{1}{4\pi}\int_{S^2} M \,\mathrm{d}\mu_s \;, \quad \text{where} \quad \mathrm{d}\mu_s = \sqrt{\det s_{AB}}\,\mathrm{d}\theta\mathrm{d}\phi \tag{5.4}$$

denotes the volume element of the unit round metric. In the static case, when no radiation occurs, it is indpendent of u and coincides with the ADM mass, $m_{\text{TB}} = M = m_{\text{ADM}}$.

5.3 An alternative approach to the Bondi mass

In this section we put more emphasis on a characteristic initial value problem. We present an approach to the Bondi mass which purely relies on the data given on some characteristic (initial) surface, the mass of which is to be determined. For this, observe that for the above derivation we actually do not need a space-time on which Bondi coordinates exist, but merely a characteristic surface on which we choose the gauge functions as required by Bondi coordinates. That is completely sufficient to ascribe a mass to an initial surface in the sense of Bondi *et al.*

Bondi coordinates, which we assume to exist at least for large r, say $r > r_0$, imply,

$$\varphi^{\text{Bo}} = r \;, \quad \overline{\partial_0 g^{\text{Bo}}_{rr}} = 0 \;, \quad \overline{\partial_0 g^{\text{Bo}}_{rA}} = 0 \;, \quad \overline{g}^{AB}_{\text{Bo}}\overline{\partial_0 g^{\text{Bo}}_{AB}} = 0 \;, \tag{5.5}$$

as follows from (5.1). This requires to take, for $r > r_0$,

$$\begin{aligned}
\kappa^{\rm Bo} &= \frac{1}{2} r(|\sigma^{\rm Bo}|^2 + 8\pi \overline{T}^{\rm Bo}_{rr}) \,, && (5.6)\\
\overline{V}^0_{\rm Bo} &= -\tau^{\rm Bo}\nu^0_{\rm Bo} \,, && (5.7)\\
\overline{V}^A_{\rm Bo} &= \overline{g}^{CD}_{\rm Bo}(\check{\Gamma}^{\rm Bo})^A_{CD} - \nu^0_{\rm Bo}\check{\nabla}^A\nu^{\rm Bo}_0 + \nu^0_{\rm Bo}(\partial_r + \tau^{\rm Bo})\nu^A_{\rm Bo} \,, && (5.8)\\
\overline{V}^r_{\rm Bo} &= \nu^0_{\rm Bo}\check{\nabla}_A\nu^A_{\rm Bo} - (\partial_r + \tau^{\rm Bo} + \nu^0_{\rm Bo}\partial_r\nu^{\rm Bo}_0)\overline{g}^{rr}_{\rm Bo} \,. && (5.9)
\end{aligned}$$

The characteristic wave-map gauge constraints (2.36)-(2.41) then become for $r > r_0$ (with $n = 3$ and $\lambda = 0$),

$$\begin{aligned}
\partial_r\tau^{\rm Bo} - \frac{1}{2} r(|\sigma^{\rm Bo}|^2 + 8\pi\overline{T}^{\rm Bo}_{rr})\tau^{\rm Bo} + \frac{1}{2}(\tau^{\rm Bo})^2 + |\sigma^{\rm Bo}|^2 &= -8\pi\overline{T}^{\rm Bo}_{rr} \,, && (5.10)\\
(\partial_r + \frac{r}{2}(|\sigma^{\rm Bo}|^2 + 8\pi\overline{T}^{\rm Bo}_{rr}))\nu^0 &= 0 \,, && (5.11)\\
(\partial_r + \tau^{\rm Bo})\xi^{\rm Bo}_A - 2\check{\nabla}_B\sigma^{\rm Bo}_A{}^B + \partial_A\tau^{\rm Bo} + r\partial_A(|\sigma^{\rm Bo}|^2 + 8\pi\overline{T}^{\rm Bo}_{rr}) &= -16\pi\overline{T}^{\rm Bo}_{rA} \,, && (5.12)\\
\partial_r\nu^A_{\rm Bo} + (\check{\nabla}^A + \xi^A_{\rm Bo})\nu^{\rm Bo}_0 &= 0 \,, && (5.13)\\
(\partial_r + \tau^{\rm Bo} + \frac{r}{2}(|\sigma^{\rm Bo}|^2 + 8\pi\overline{T}^{\rm Bo}_{rr}))\zeta^{\rm Bo} + \check{R}^{\rm Bo} - \frac{|\xi^{\rm Bo}|^2}{2} + \check{\nabla}_A\xi^A_{\rm Bo} &= 8\pi(\overline{g}^{AB}_{\rm Bo}\overline{T}^{\rm Bo}_{AB} - \overline{T}^{\rm Bo}) \,, && (5.14)\\
\overline{g}^{rr}_{\rm Bo} + (\tau^{\rm Bo})^{-1}(\zeta^{\rm Bo} - 2\nu^0_{\rm Bo}\check{\nabla}_A\nu^A_{\rm Bo}) &= 0 \,. && (5.15)
\end{aligned}$$

Note that the constraint equation (2.41) has become an algebraic equation for $\overline{g}^{rr}_{\rm Bo}$.

A comment is in order how to make sure that $\varphi = r$ as required by Bondi coordinates. In the case of a regular light-cone, it is shown in [27] that, in vacuum, the gauge choice (5.6) for κ can be made up to the tip of the cone. It follows from regularity conditions at the vertex as considered in [16, Section 4.5] that $\varphi = O(r)$ and $\partial_r\varphi = 1 + O(r)$ for small r, and we infer that $\varphi = r$ is the unique solution of (5.10) with $\tau = 2\partial_r \log\varphi$ as desired. In the general case, one easily transforms into a gauge where $\varphi = r$ by taking φ as the new r-coordinate, say for $r > r_0$ (cf. [72, Section 4]). This is always possible within the current setting where φ is, at least for large r, a strictly increasing function along the null geodesics generating the initial surface, and thus a reasonable r-coordinate.

We further need to assume

$$\lim_{r\to\infty}\nu^A_{\rm Bo} = 0 \,, \quad \lim_{r\to\infty}\nu^0_{\rm Bo} = 1 \,, \quad \lim_{r\to\infty}(r^{-2}\overline{g}^{\rm Bo}_{AB}) = s_{AB} \,. \qquad (5.16)$$

In fact, these are consequences of the asymptotic conditions (5.2). In terms of an initial value problem, where e.g. the conformal class $[\gamma]$ of $\check{g}$ is prescribed, it seems hard to control the asymptotic behavior of the metric components and thereby realize (5.16). However, it is only the current derivation of an alternative expression for the Bondi mass which relies on (5.16), the final gauge-independent formula (5.29) below will not rely on the conditions (5.16), which merely concern the gauge.

With (5.16) we solve the remaining equations (5.11)-(5.15). We assume the $T_{\mu\nu}$'s to be given functions of the coordinates, as it will be the case if, e.g., a space-time has already been constructed or the matter model is of a form which respects the hierarchical character of the constraint equations. Assuming further that for large r

$$\sigma^{\rm Bo}_A{}^B = (\sigma^{\rm Bo}_A{}^B)_2 r^{-2} + O(r^{-3}) \,, \qquad (5.17)$$

(corresponding assumptions on its first- and second-order non-transverse derivatives are tacitly assumed), as follows e.g. for initial data $[\gamma]$ of the form (4.32), and that

$$\overline{T}^{\rm Bo}_{rr} = O(r^{-4}) \,, \quad \overline{T}^{\rm Bo}_{rA} = O(r^{-3}) \,, \quad \overline{g}^{AB}_{\rm Bo}\overline{T}^{\rm Bo}_{AB} - \overline{T}^{\rm Bo} = O(r^{-4}) \,, \qquad (5.18)$$

we end up with

$$\begin{aligned}
\varphi^{\mathrm{Bo}} &= r \quad (\Longrightarrow \quad \tau^{\mathrm{Bo}} = 2r^{-1}) \,, && (5.19)\\
\nu^0_{\mathrm{Bo}} &= 1 + O(r^{-2}) \,, && (5.20)\\
\xi^{\mathrm{Bo}}_A &= 2\mathring{\nabla}_B(\sigma_A^{\mathrm{Bo}\,B})_2 r^{-1} + o(r^{-1}) \,, && (5.21)\\
\nu^A_{\mathrm{Bo}} &= s^{AC}\mathring{\nabla}_B(\sigma_C^{\mathrm{Bo}\,B})_2 r^{-2} + o(r^{-2}) \,, && (5.22)\\
\zeta^{\mathrm{Bo}} &= -2r^{-1} + \zeta_2^{\mathrm{Bo}} r^{-2} + o(r^{-3}) \,, && (5.23)\\
\overline{g}^{rr}_{\mathrm{Bo}} &= 1 + [\mathring{\nabla}^A\mathring{\nabla}_B(\sigma_A^{\mathrm{Bo}\,B})_2 - \frac{1}{2}\zeta_2^{\mathrm{Bo}}]r^{-1} + o(r^{-1}) \,, && (5.24)
\end{aligned}$$

whence

$$\overline{g}^{\mathrm{Bo}}_{00} = \overline{g}^{\mathrm{Bo}}_{AB}\nu^A_{\mathrm{Bo}}\nu^B_{\mathrm{Bo}} - (\nu_0^{\mathrm{Bo}})^2\overline{g}^{rr}_{\mathrm{Bo}} = -1 + \frac{1}{2}r^{-1}\underbrace{[\zeta_2^{\mathrm{Bo}} - 2\mathring{\nabla}^A\mathring{\nabla}_B(\sigma_A^{\mathrm{Bo}\,B})_2]}_{=4M} + o(r^{-1}) \,. \tag{5.25}$$

Here, $\mathring{\nabla}$ denotes the Levi-Civita connection of the standard metric. We stress that the no-logs-condition (4.37) does not need to be assumed, since logarithmic terms appear only in higher orders, i.e. the fields appearing in this section will be generally polyhomogeneous rather than smooth.

We obtain the following formula for the Bondi mass:

$$m_{\mathrm{TB}} = \frac{1}{4\pi}\int_{S^2} M \, d\mu_s = \frac{1}{16\pi}\int_{S^2}\zeta_2^{\mathrm{Bo}} \, d\mu_s \,. \tag{5.26}$$

We thus have expressed the Bondi mass in terms of an expansion coefficient of ζ rather than $\overline{g}_{uu}$, which depends heavily on the gauge, cf. equation (2.41). In contrast, an inspection of the wave-map gauge constraints (2.36)-(2.41) shows that the function ζ is completely independent of the choice of the gauge source functions W^λ, i.e. how the coordinates are chosen off the initial surface. Our formula for the Bondi mass remains equally valid in non-Bondi-coordinates as long as, for $r > r_0$,

$$\varphi = r \,. \tag{5.27}$$

To obtain an expression for the Bondi mass of the initial surface which is valid in arbitrary adapted null coordinates, we determine the behavior of ζ_2 under changes of coordinate transformations $r \mapsto \tilde{r}(r, x^A)$, which we assume to be of the asymptotic form

$$r(\tilde{r}, x^A) = r_{-1}(x^A)\tilde{r} + r_0(x^A) + O(\tilde{r}^{-1}) \,, \quad \text{with} \quad r_{-1} > 0 \tag{5.28}$$

(again, corresponding assumptions on the first- and second-order x^i-derivatives are tacitly assumed). If the coordinate transformation was not of this form the so-obtained coordinates would not be well-behaved at null infinity, cf. [38, Section 6.1].

We employ the relation (2.43) which tells us how ζ behaves under coordinate transformations. Denoting by f_n the coefficient of r^{-n} in the expansion of f, a lengthy computation reveals that

$$\begin{aligned}
\tilde\zeta(\tilde r) &= 2\tilde{\overline{g}}^{AB}\tilde{\overline{\Gamma}}^r_{AB} + \tilde\tau\tilde{\overline{g}}^{rr} = 2\overline{g}^{AB}\Big(\frac{\partial\tilde r}{\partial x^k}\frac{\partial x^i}{\partial\tilde x^A}\frac{\partial x^j}{\partial\tilde x^B}\overline{\Gamma}^k_{ij} + \frac{\partial\tilde r}{\partial r}\frac{\partial^2 r}{\partial\tilde x^A\partial\tilde x^B}\Big) + \tau\frac{\partial r}{\partial\tilde r}\frac{\partial\tilde r}{\partial x^i}\frac{\partial\tilde r}{\partial x^j}\overline{g}^{ij}\\
&= \frac{\partial\tilde r}{\partial r}\zeta + 2\overline{g}^{AB}\frac{\partial\tilde r}{\partial x^C}\tilde\Gamma^C_{AB} - 2\overline{g}^{AB}\frac{\partial\tilde r}{\partial r}\frac{\partial r}{\partial\tilde x^B}\xi_A + 4\overline{g}^{AB}\frac{\partial\tilde r}{\partial x^C}\frac{\partial r}{\partial\tilde x^B}\chi_A{}^C\\
&\quad +2\overline{g}^{AB}\frac{\partial\tilde r}{\partial r}\frac{\partial r}{\partial\tilde x^A}\frac{\partial r}{\partial\tilde x^B}\kappa + 2\overline{g}^{AB}\frac{\partial\tilde r}{\partial r}\frac{\partial^2 r}{\partial\tilde x^A\partial\tilde x^B} + \tau\frac{\partial r}{\partial\tilde r}\frac{\partial\tilde r}{\partial x^A}\frac{\partial\tilde r}{\partial x^B}\overline{g}^{AB}\\
&= \tilde r^{-1}(r_{-1})^{-2}\Big[\zeta_1 + 2(\varphi_{-1})^{-2}\Delta_s\log r_{-1}\Big] + \tilde r^{-2}(r_{-1})^{-3}\Big[\zeta_2 - r_0\zeta_1 + 2(\varphi_{-1})^{-2}\Delta_s r_0\\
&\quad +3(\varphi_{-1})^{-2}(r_{-1})^{-2}(2r_0-\tau_2)\mathring\nabla_A r_{-1}\mathring\nabla^A r_{-1} \; - 2(\varphi_{-1})^{-2}(r_{-1})^{-1}\mathring\nabla_A[(2r_0-\tau_2)\mathring\nabla^A r_{-1}]\\
&\quad -4r_{-1}(\varphi_{-1})^{-2}(\sigma_A{}^B)_2[\mathring\nabla^A\mathring\nabla_B(r_{-1})^{-1} - 2\mathring\nabla^A(r_{-1})^{-1}\mathring\nabla_B\log\varphi_{-1}]\Big] + O(\tilde r^{-3})\,.
\end{aligned}$$

If (r,x^A) denote Bondi coordinates on the cone, we have $\varphi^{\rm Bo}_{-1}=1$, $\zeta^{\rm Bo}_1=-2$, $\tau^{\rm Bo}_2=0$, $r^{\rm Bo}_{-1}=\tilde\varphi_{-1}$, $r^{\rm Bo}_0=\tilde\varphi_0$ and $(\sigma^{{\rm Bo}B}_A)_2=\tilde\varphi_{-1}(\tilde\sigma_A{}^B)_2$. That yields

$$\begin{aligned}
\tilde\zeta_1 &= -2(\tilde\varphi_{-1})^{-2}[1-\Delta_s\log\tilde\varphi_{-1}]\,,\\
(\tilde\varphi_{-1})^3\tilde\zeta_2 &= \zeta^{\rm Bo}_2 - \tilde\varphi_{-1}\Delta_s\tilde\tau_2 - 4(\tilde\varphi_{-1})^2(\tilde\sigma_A{}^B)_2\mathring\nabla^A\mathring\nabla_B(\tilde\varphi_{-1})^{-1} + \tilde\varphi_{-1}\tilde\tau_2(\Delta_s\log\tilde\varphi_{-1}-1)\,,
\end{aligned}$$

and thus

$$\zeta^{\rm Bo}_2 = (\tilde\varphi_{-1})^3[\tilde\zeta_2 - \frac{1}{2}\tilde\tau_2\tilde\zeta_1] - \tilde\varphi_{-1}[2\mathring\nabla^A(\tilde\xi_A)_1 + \Delta_s\tilde\tau_2] + \text{divergences}\,.$$

Denoting the $(\tilde r,\tilde x^A)$-coordinates by (r,x^A), equation (5.26) becomes

$$m_{\rm TB} = \frac{1}{16\pi}\int_{S^2}\Big((\varphi_{-1})^3[\zeta_2 - \frac{\tau_2}{2}\zeta_1] - \varphi_{-1}[2\mathring\nabla^A(\xi_A)_1 + \Delta_s\tau_2]\Big)d\mu_s\,. \tag{5.29}$$

This provides a formula for the Bondi mass in terms of the leading-order expansion coefficients of φ, ξ_A and ζ (recall that $\tau = 2\partial_r\log\varphi$) in adapted null coordinates, which does not rely on any further gauge condition. To make sense of (5.29) only the asymptotic behavior matters, in particular no assumptions need to be made at the tip of the cone.[2] We only need to assume that there are no conjugate points where the cone intersects the conformal boundary, i.e. the positivity of φ_{-1}. This approach to the Bondi mass does not involve any space-time constructions, one advantage being that no existence theorem for an associated space-time is needed.

A gauge choice for κ does not uniquely fix the parameterization of the null geodesics. For instance, for $\kappa=0$ the r-coordinate needs to be an affine parameter, but this still leaves the freedom of angle-dependent rescalings of r, and it has been shown in [37] that this additional freedom corresponds to the freedom to prescribe $\varphi_{-1}>0$. Another way of seeing this, which works for arbitrary κ, is to note that instead of κ the function φ may be regarded as a gauge function [72, Section 4]. Any strictly increasing φ (this is necessary in our current setting, at least asymptotically) determines κ, whereas a given κ does not uniquely determine φ, cf. equation (2.47). There remains the freedom to multiply φ by an arbitrary positive scaling function, which can be used to adjust $\varphi_{-1}>0$.

[2]Note, however, that in terms of an initial value problem some of the expansion coefficients in (5.29) appear as "integration functions" in the constraint ODEs, and are thus *globally* determined by the initial data and the gauge functions.

Imposing the gauge condition $\varphi_{-1} = 1$, the formula (5.29) takes the simple form

$$m_{\rm TB} \;=\; \frac{1}{16\pi}\int_{S^2}(\zeta_2+\tau_2)d\mu_s\,. \tag{5.30}$$

Taking, in addition, r as an affine parameter (equivalently $\kappa = 0$), this expression was the starting point in [37] to derive a manifestly positive expression for the Bondi mass of globally smooth light-cones. By that, we mean a regular light-cone which meets $\mathscr{I}^+$ in a spherical cross-section, with no conjugate points up-to-and-including conformal infinity, i.e. $\tau > 0$ and $\varphi_{-1} > 0$, and where (5.17) and (5.18) hold.

Integrating the Raychaudhuri equation (2.36) and the ζ-equation (2.40) with $\kappa = 0$ and performing appropriate limits, one derives expressions for τ_2 and ζ_2. Taking regularity at the vertex into account and using the Gauß-Bonnet theorem one ends up with the formula[3]

$$m_{\rm TB} = \frac{1}{16\pi}\int_0^\infty\int_{S^2}\Big(\frac{1}{2}|\xi|^2+8\pi(\overline{g}^{AB}\overline{T}_{AB}-\overline{T})+(|\sigma|^2+8\pi\overline{T}_{rr})e^{\int_r^\infty\frac{\tilde r\tau-2}{2\tilde r}\,d\tilde r}\Big)e^{-\int_r^\infty\frac{\tilde r\tau-2}{\tilde r}\,d\tilde r}d\mu_{\breve g}dr\,. \tag{5.31}$$

In vacuum, this provides a manifestly non-negative-definite formula for the Bondi mass of a globally smooth light-cone in terms of the physically relevant fields τ, σ and ξ. The vanishing of $m_{\rm TB}$ yields $|\sigma|^2 = 0 = \xi_A$, but the vanishing of $|\sigma|^2$ already implies that we are in the flat case [15].

For non-vanishing matter fields we assume the *dominant energy condition*. This implies $\overline{T}_{rr} = \overline{T}(\partial_r, \partial_r) \geq 0$. Consider the surface $\{u = 0, r = \text{const.}\} \cong S^2$ and denote by e_0 and e_1 two linearly independent null vectors which are orthogonal to $e_A = \partial_A$. Then,

$$\overline{g}^{AB}\overline{T}_{AB}-\overline{T} \;=\; -2\overline{g}^{-1}(e_0,e_1)\overline{T}(e_0,e_1) \;\geq\; 0$$

by the dominant energy condition, so that (5.31) is again manifestly non-negative. Moreover, $m_{TB} = 0$ yields $|\sigma|^2 = \xi_A = \overline{g}^{AB}\overline{T}_{AB} - \overline{T} = \overline{T}_{rr} = 0$. For many matter models the vanishing of $\overline{T}_{rr}$ implies vacuum [15], and we infer that m_{TB} is non-negative and vanishes precisely in the flat case.

The Bondi mass of a characteristic surface with interior boundary is considered in [37].

[3] For this argument to work it is crucial that the cone is regular at its tip and has no conjugate points up-to-and-including conformal infinity, meaning that $\tau > 0$ on $\mathbb{R}_{>0} \times S^2$ and $\varphi_{-1} > 0$ on S^2.

CHAPTER 6

Killing initial data (KIDs)

"Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect."
HERMANN K. H. WEYL (1885–1955)

6.1 Some preliminary considerations

Initial value problems provide a powerful tool to construct space-times, solutions to Einstein's field equations. In Section 4 we have seen how to construct such space-times with a specific asymptotic structure. Another important property of a space-time is that it possesses certain symmetries.

Configurations with symmetries play a distinguished role in physics, such as e.g. equilibrium states in thermodynamics. This is particularly true for general relativity, where it is e.g. expected that gravitational collapse leads to a *stationary* asymptotically flat (for $\lambda = 0$) vacuum space-time as the final state [62, 86]. This is one reason why one is interested in space-times which satisfy Einstein's field equations and admit *isometries*, i.e. diffeomorphisms $\phi : \mathscr{M} \to \mathscr{M}$ for which $\phi^* g = g$. Now, the isometry group of a pseudo-Riemannian manifold carries a natural manifold structure, and thus forms a Lie group. Since any element of the connected component of the identity of a Lie group belongs to a one-parameter subgroup, the action of a symmetry group can be studied by means of the generators of these subgroups, so-called *Killing vector fields* (cf. e.g. [28]). This is a vector field X fulfilling *Killing's equation*

$$\mathscr{L}_X g = 0 \quad \iff \quad \nabla_{(\mu} X_{\nu)} = 0 \,, \tag{6.1}$$

where $\mathscr{L}$ denotes the Lie derivative. Conversely, any Killing vector field generates a *local* isometry. To obtain a one-parameter group of isometries, though, one, in addition, needs to make sure that X is complete. The full isometry group of a space-time may also include discrete isometries which are not generated by Killing vector fields.

Killing vector fields form a Lie algebra w.r.t. to the usual commutator of vector fields as follows immediately from the relation $\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]$. A connected pseudo-Riemannian manifold admits $0 \leq k \leq \dim \mathscr{M}(\dim \mathscr{M}+1)/2$ linearly independent Killing vector fields.

For later purposes let us also introduce a generalization of Killing vector fields, so-called *conformal Killing vector fields* which satisfy the *conformal Killing equation*

$$\mathscr{L}_X g = \frac{2}{n+1} g \operatorname{div} X \quad \Longleftrightarrow \quad (\nabla_{(\mu} X_{\nu)})\check{} \equiv \nabla_{(\mu} X_{\nu)} - \frac{1}{n+1} g_{\mu\nu} \nabla_\alpha X^\alpha = 0 \,. \tag{6.2}$$

Conformal Killing vector fields arise as infinitesimal generators of conformal isometries, which are diffeomorphisms $\phi : \mathscr{M} \to \mathscr{M}$ such that $\phi^* g = \Omega^2 g$ for some $\Omega > 0$. Clearly, any Killing vector field is a conformal one. A connected pseudo-Riemannian manifold admits $0 \leq k \leq (\dim \mathscr{M}+1)(\dim \mathscr{M}+2)/2$ linearly independent conformal Killing vector fields, which constitute a Lie algebra which contains the Killing vector fields as a Lie subalgebra.

Another reason for the importance of Killing vector fields is that any one-parameter group of isometries gives rise to a conserved quantity for freely falling particles and light rays, and proves helpful when integrating the geodesic equation [86]. Moreover, the presence of an isometry may be employed to perform a dimensional reduction of the problem at hand. Apart from these "positive" properties Killing vector fields also play an important role by providing an obstruction in gluing initial data for the space-like Cauchy problem, cf. e.g. [29, 31] and the references given therein. Moreover, in the compact case, the non-existence of Killing vector fields characterizes the Marsden-Fischer linearization stability [68].

One therefore would like to construct in a systematic manner space-times with Killing vector fields in terms of initial value problems. In other words, one would like to infer just from the initial manifold and the data given there, that the emerging space-time contains one or several Killing vector fields. Such data will be called *Killing Initial Data (KIDs)*. Their specification has first been accomplished for the space-like Cauchy problem, cf. e.g. [8, 68]. The characteristic case is treated in [35], attached as Chapter 13. We shall restrict attention to the vacuum case.

The basic idea to derive the KID equations is as follows: We first observe that (6.1) implies a linear wave equation for X,

$$\Box_g X^\alpha = -R_\beta{}^\alpha X^\beta \,. \tag{6.3}$$

Given appropriate initial data $[X]$ on an initial surface Σ, (6.3) determines uniquely a candidate field X. In general, though, this will not be a Killing vector field. We set

$$A_{\mu\nu} := 2\nabla_{(\mu} X_{\nu)} \,, \tag{6.4}$$

and observe that (6.3) together with the vacuum Einstein equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ imply a linear, homogeneous wave equation which is satisfied by $A_{\mu\nu}$,

$$\Box_g A_{\mu\nu} = -2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} \,. \tag{6.5}$$

From these considerations it follows that a vacuum space-time contains a Killing vector field if and only if there exists a vector field X such that (6.3) holds with $[A_{\mu\nu}] = 0$. It is important to stress that the wave equations (6.3) and (6.5) are *linear*, so that a solution exists globally in the whole domain of dependence of Σ (cf. Appendix A), though non-trivial data may lead to the trivial solution away from a neighborhood of Σ. Next, we extract necessary and sufficient conditions which make sure that the data $[X]$ lead via (6.3) to a vector field in space-time with $[A_{\mu\nu}] = 0$ in order to characterize the existence of a non-trivial Killing vector field near Σ.

6.2 KIDs for the space-like Cauchy problem

Let us first consider the space-like case (cf. [8]). Consider a vacuum space-time $(\mathscr{M}, g)$ and a space-like hypersurface $\Sigma \subset \mathscr{M}$ given in adapted coordinates (t, x^i). One then needs to make sure that $\overline{A_{\mu\nu}} = 0$ and $\overline{\nabla_t A_{\mu\nu}} = 0$. For convenience, let us assume a gauge where $\overline{g}_{tt} = -1$, $\overline{g}_{ti} = 0$ and $\overline{\partial_t g_{t\mu}} = 0$.

First of all, one straightforwardly checks that the equations $\overline{A}_{\mu\nu} = 0$, $\overline{\nabla_t A_{ij}} = 0$ and (6.3) imply $\overline{\nabla_t A_{t\mu}} = 0$. One further finds that $\overline{A}_{\mu\nu} = 0$ is equivalent to

$$\overline{\nabla_t X^t} = 0 \, , \tag{6.6}$$
$$\overline{\nabla_t X^i} = \mathscr{D}^i \overline{X}^t \, , \tag{6.7}$$
$$\mathscr{D}_{(i} \overline{X}_{j)} + K_{ij} \overline{X}^t = 0 \, . \tag{6.8}$$

Recall that $\mathscr{D}$ denotes the covariant derivative of the induced Riemannian metric $h = h_{ij} \mathrm{d}x^i \mathrm{d}x^j$. Assuming $\overline{A}_{\mu\nu} = 0$, the equation $\overline{\nabla_t A_{ij}} = 0$ becomes

$$\mathscr{D}_i \mathscr{D}_j \overline{X}^t + \frac{1}{2} \overline{\partial^2_{tt} g_{ij}}\, \overline{X}^t + \mathscr{L}_{\overline{X}^k \partial_k} K_{ij} = 0 \, . \tag{6.9}$$

The second-order transverse derivative can be eliminated via the Einstein equations,

$$\overline{R}_{ij} = \frac{2}{n-1} \lambda \overline{g}_{ij} \iff \frac{1}{2} \overline{\partial^2_{tt} g_{ij}} = 2 K_{ik} K_j{}^k - K K_{ij} - {}^{(3)}R_{ij} + \frac{2}{n-1} \lambda \overline{g}_{ij} \, . \tag{6.10}$$

That yields

$$\mathscr{D}_i \mathscr{D}_j \overline{X}^t + \mathscr{L}_{\overline{X}^k \partial_k} K_{ij} = [{}^{(3)}R_{ij} + K K_{ij} - 2 K_{ik} K_j{}^k - \frac{2}{n-1} \lambda \overline{g}_{ij}] \overline{X}^t \, . \tag{6.11}$$

The proper KID equations are (6.8) and (6.11): If and only if we find a function N and a vector field Y on the Riemannian initial manifold (Σ, h_{ij}) with second fundamental form K_{ij} such that

$$\mathscr{D}_{(i} Y_{j)} + N K_{ij} = 0 \, , \tag{6.12}$$
$$\mathscr{D}_i \mathscr{D}_j N + \mathscr{L}_Y K_{ij} = [{}^{(3)}R_{ij} + K K_{ij} - 2 K_{ik} K_j{}^k - \frac{2}{n-1} \lambda h_{ij}] N \, , \tag{6.13}$$

the vacuum space-time contains a Killing vector field X. This is then obtained by solving (6.3) with initial data $\overline{X}^t = N$, $\overline{X}^i = Y^i$, $\overline{\nabla_t X^t} = 0$ and $\overline{\nabla_t X^i} = \mathscr{D}^i N$.

In terms of a Cauchy problem we are led to the result:

THEOREM 6.2.1 *Consider the tuple $(\Sigma, h_{ij}, K_{ij}, N, Y^i)$. There exists an (up to isometries) unique maximal globally hyperbolic vacuum space-time with a Killing vector field which contains the embedded initial manifold Σ as a Cauchy surface on which the push-forwards of the above data are induced, if and only if $(\Sigma, h_{ij}, K_{ij}, N, Y^i)$ satisfies the usual vacuum constraint equations (2.16)-(2.17) on Σ supplemented by the KID equations (6.12)-(6.13).*

6.3 KIDs for the characteristic Cauchy problem

Let us now pass to the characteristic case, cf. [35], attached as Chapter 13. As before, we consider for reasons of definiteness a light-cone C_O, regular at its vertex. For two null surfaces intersecting transversally we refer the reader to [35, Section 3], where we also analyze some special cases in detail such as bifurcate Killing horizons.

In the characteristic case it suffices to ensure that $\overline{A}_{\mu\nu}$ vanishes. It proves fruitful to define

$$S_{\mu\nu\sigma} := \nabla_\mu \nabla_\nu X_\sigma - R^\alpha{}_{\mu\nu\sigma} X_\alpha \,. \tag{6.14}$$

In a first step [35, Section 2.2] one establishes, employing regularity at the vertex and assuming $\overline{A}_{ij} = 0$ and (6.3), that the equations $\overline{S}_{ir0} = 0$ and $\tilde{S} := \overline{g}^{AB}(\overline{S}_{AB0} + \frac{1}{2}\overline{\Gamma}^0_{AB}\overline{A}_{00}) = 0$ are equivalent to $\overline{A}_{0\mu} = 0$. In a second step [35, Section 2.5] one shows that $\overline{A}_{ij} = 0$, (6.3) and the vacuum Einstein equations imply $\tau \overline{S}_{rr0} = 0$, $\overline{S}_{Ar0} = 0$ and $\tilde{S} = 0$. Noting that we have $\tau > 0$, at least sufficiently close to the vertex, we conclude that it remains to ensure $\overline{A}_{ij} = 0$,[1] and this equation does not involve any transverse derivative of X on C_O.

A subtlety in the light-cone case is that one needs to make sure that $\overline{X}$ is the restriction to C_O of a smooth space-time vector field, which is a non-trivial issue albeit necessary to apply Dossa's well-posedness result [41] to solve (6.3) (cf. Appendix A). However, this can be inferred from $\overline{A}_{ij} = 0$ [35, Section 2.4]: It follows from the fact that $\overline{X}$ can be written as the restriction to C_O of a solution of a certain system of differential equations, which reduces to an ODE-system along any geodesic issued from O, together with the property that solutions of ODEs depend smoothly upon their initial data.

An analysis of the KID equations $\overline{A}_{ij} = 0$ reveals that the equations $\overline{A}_{1i} = 0$ and $\overline{g}^{AB}\overline{A}_{AB} = 0$ determine candidate fields Y^μ on the cone (let us focus attention to the region near the tip where $\tau > 0$),

$$\begin{aligned}
(\partial_r - \kappa)(\nu_0 Y^0) &= 0\,, && (6.15)\\
\partial_r Y^A + [\tilde{\nabla}^A + \xi^A - \partial_r \overline{g}^{rA} - \kappa \overline{g}^{rA}](\nu_0 Y^0) &= 0\,, && (6.16)\\
\tau Y^r + \tilde{\nabla}_A Y^A - \frac{1}{2}(\zeta + \tau \overline{g}^{rr} + 2\tilde{\nabla}_A \overline{g}^{rA} + 2\overline{g}^{rA}\tilde{\nabla}_A)(\nu_0 Y^0) &= 0\,. && (6.17)
\end{aligned}$$

The candidate fields depend on the integration functions

$$c(x^A) = \lim_{r\to 0} \overline{Y}^0 \quad \text{and} \quad f^A(x^B) = \lim_{r\to 0}(Y^A - r^{-1}\mathring{\nabla}^A c) \tag{6.18}$$

arising when integrating (6.15) and (6.16).

Each of the candidate fields extends to a Killing vector field of the space-time if and only if it satisfies the *reduced KID equations* $\breve{\overline{A}}_{AB} = 0$ (reccall that "$\breve{\ }$" denotes the $\check{g}$-trace-free part)

$$(\tilde{\nabla}_{(A}Y_{B)})\breve{} + \sigma_{AB}Y^r - (\overline{g}^{rr}\sigma_{AB} + \breve{\overline{\Gamma}}{}^r_{AB})(\nu_0 Y^0) = 0\,. \tag{6.19}$$

In particular this equation implies that $\mathring{\nabla}_A c$ and f_A need to be conformal Killing one-forms on $(S^{n-1}, s_{AB}\mathrm{d}x^A\mathrm{d}x^B)$.

Let us formulate this result in terms of an initial value problem:

THEOREM 6.3.1 *For given data on a light-cone C_O, $(\check{g}, \kappa, Y^\mu)$ say, there exists an (up to isometries) locally unique globally hyperbolic vacuum space-time to the future of C_O with a Killing vector field (we restrict attention to the regime where $\tau > 0$) which induces the above data on C_O, if and only if the pair $(\check{g}, \kappa)$ satisfies the Raychaudhuri equation (2.36), and $(\check{g}, \kappa, Y^\mu)$ fulfills (6.15)-(6.19) once the vacuum wave-map gauge constraints have been solved.*[2]

REMARK 6.3.2 The equation (6.19) contains the transverse derivatives $(\overline{\partial_0 g_{AB}})\breve{}$, which need to be eliminated via the Einstein equations $\breve{\overline{R}}_{AB} = 0$, which provide a coupled ODE-system for $(\overline{\partial_0 g_{AB}})\breve{}$.

[1] If τ vanishes in certain regions of C_O, the equation needs to be supplemented by the condition $\overline{S}_{rr0} = 0$ there.

[2] Of course, one needs to make sure that Dossa's well-posedness result for the evolution equations is applicable, as it is e.g. the case if $\kappa = 0$ and $\check{g}$ is induced on C_O by a smooth space-time metric [27].

6.4 KIDs in Penrose's conformally rescaled space-times: Some preliminaries

In Section 4 we have discussed well-posedness results for asymptotic Cauchy problems which permit the construction of asymptotically flat or de Sitter space-times, solution to Einstein's vacuum field equations. In Section 6.2 and 6.3 we have derived the KID equations, by which the usual constraint equations need to be supplemented to end up with space-times which contain one or several Killing vector fields. In this section, we will combine both of them, i.e. we will construct asymptotically flat and asymptotically de Sitter space-times with Killing vector fields via asymptotic initial value problems.

As before, when dealing with the unphysical space-time we restrict attention to 4 dimensions, since the CFE and also the wave equations (6.25)-(6.28), which we shall derive below for the Killing vector field and which replace (6.4) in the unphysical space-time, provide a nice evolution system only in 4 dimensions, cf. [56, 70].

The first step to accomplish this is to derive a substitute to Killing's equation (6.1) in the unphysical space-time. It turns out, cf. [70, Section 3.1], attached as Chapter 14, that a vector field is a Killing vector field in the physical space-time $(\tilde{\mathscr{M}}, \tilde{g})$ if and only if its push-forward X is a conformal Killing vector field in the unphysical space-time $(\mathscr{M}, g = \Theta^2\tilde{g})$, i.e.

$$\mathscr{L}_X g = \frac{1}{2}\mathrm{div}X g \quad \Longleftrightarrow \quad \nabla_{(\mu}X_{\nu)} = \frac{1}{4}\nabla_\alpha X^\alpha g_{\mu\nu} \,, \tag{6.20}$$

and satisfies there the equation

$$\mathscr{L}_X \Theta = \frac{1}{4}\mathrm{div}X\Theta \quad \Longleftrightarrow \quad X^\mu\nabla_\mu\Theta = \frac{1}{4}\Theta\nabla_\mu X^\mu \,. \tag{6.21}$$

Any Killing vector field is a conformal Killing vector field of the conformally rescaled metric, that is where (6.20) comes from. Equation (6.21) makes sure that the conformal Killing vector field becomes a Killing vector field rather than just a conformal one if one goes back to the physical space-time. It will be essential that (6.20) and (6.21), which we call the *unphysical Killing equations*, make sense also where the conformal factor Θ vanishes.

To carry on in close analogy to the proceeding in Section 6.1, we need a wave equation which is necessarily satisfied by any solution of (6.20)-(6.21). Then, we can construct candidate fields which, for appropriately specified initial data, satisfy the unphysical Killing equations, and thus correspond to Killing vector fields of the original physical space-time.

Indeed, introducing the auxiliary scalar function

$$Y := \frac{1}{4}\mathrm{div}X \,, \tag{6.22}$$

we observe that the unphysical Killing equations imply a system of linear wave equations:

$$\Box_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y = 0 \,, \tag{6.23}$$

$$\Box_g Y + \frac{1}{6}X^\mu\nabla_\mu R + \frac{1}{3}RY = 0 \,. \tag{6.24}$$

For given initial data $[X]$ and $[Y]$, (6.23)-(6.24) determine a candidate field X. It remains to extract conditions which ensure that the so-obtained candidate field X satisfies the unphysical Killing equations. This is somewhat more involved than in Section 6.1, an intuitive reason being as follows. In asymptotic initial value problems the initial data (e.g., depending on the scheme, certain components of the rescaled Weyl tensor or the Schouten tensor) involve higher-order transverse derivatives of the metric than in the ordinary case. Since one should expect the existence of Killing vector fields to depend on the initial data, it cannot be sufficient to infer its existence just by ensuring that the unphysical Killing equations hold initially (including their transverse derivatives in the space-like case), which do *not* involve such higher-order transverse derivatives of the metric.

It is convenient to introduce the fields:

$$\phi := X^\mu\nabla_\mu\Theta - \Theta Y \,, \tag{6.25}$$

$$\psi := X^\mu\nabla_\mu s + sY - \nabla_\mu\Theta\nabla^\mu Y \,, \tag{6.26}$$

$$A_{\mu\nu} := 2\nabla_{(\mu}X_{\nu)} - 2Yg_{\mu\nu} \,, \tag{6.27}$$

$$B_{\mu\nu} := \mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y \,. \tag{6.28}$$

All these fields vanish identically whenever X satisfies the unphysical Killing equations and Y is given by (6.22). Conversely, assuming that the wave equations (6.23)-(6.24) and the MCFE (3.3)-(3.8) hold, one derives a linear, homogeneous system of wave equations which is fulfilled by the fields ϕ, ψ, $A_{\mu\nu}$, $\nabla_\sigma A_{\mu\nu}$ and $B_{\mu\nu}$ [70, Section 3.3]. We conclude that a space-time which solves the MCFE admits a vector field satisfying the unphysical Killing equations if and only if there exists a vector field X and a function Y which solve (6.23)-(6.24) with $[\phi] = [\psi] = [A_{\mu\nu}] = [\nabla_\sigma A_{\mu\nu}] = [B_{\mu\nu}] = 0$ (note that the vanishing of $A_{\mu\nu}$ implies (6.22)).

Again, all the relevant wave equations are *linear*, whence the Killing vector field will exist in the whole domain of dependence, though it may become trivial away from a neighborhood of the initial surface.

Remark 6.4.1 It is the $\overline{\phi} = 0$-condition which requires the vanishing of $\overline{X}^0$ on both a space-like and a characteristic $\mathscr{I}$. A vector field which satisfies the unphysical Killing equations cannot be time-like on $\mathscr{I}$. Even more, a closer analysis [70, Section 5.3] and [71, Section 4.3] shows that for $\lambda > 0$ there are no vacuum space-times which are stationary, while for $\lambda = 0$ Minkowski space-time is the only stationary vacuum space-time which admits a regular C_{i^-}-cone.[3]

6.5 KIDs on conformal boundaries: The space-like case

Let us first pay attention to the space-like case (cf. [71], attached as Chapter 15), where the task consists of characterizing initial data on the initial surface Σ for which

$$\overline{\phi} = \overline{\nabla_t\phi} = \overline{\psi} = \overline{\nabla_t\psi} = \overline{A}_{\mu\nu} = \overline{\nabla_t A_{\mu\nu}} = \overline{\nabla_t\nabla_t A_{\mu\nu}} = \overline{B}_{\mu\nu} = \overline{\nabla_t B_{\mu\nu}} = 0 \,.$$

Assuming (6.23)-(6.24) and the MCFE, it turns out [71, Sections 3.4 and 3.5] that some of these conditions are automatically fulfilled. The remaining ones are

$$\overline{\phi} = \overline{\nabla_t\phi} = \overline{\psi} = \overline{\nabla_t\psi} = \overline{A}_{\mu\nu} = \overline{\nabla_t A_{ij}} = \breve{\overline{B}}_{ij} = (\overline{\nabla_t B_{ij}})\breve{} = 0 \,. \tag{6.29}$$

As for KIDs in the physical space-time, some of these conditions involve undesired second-order transverse derivative of Y. They can be eliminated via (6.24).

A particularly interesting space-like hypersurface in a conformally rescaled space-time is the one, for $\lambda > 0$, which corresponds to $\mathscr{I}^-$. Let us therefore pay attention to this special

[3] By a "stationary space-time" we mean a space-time which admits a Killing vector field which is asymptotically time-like.

case. For convenience, we choose coordinates as in Section 4.2.2, equation (4.10). In this case, some more conditions are automatically satisfied and we are left with [71, Section 4.1]

$$\overline{\phi} = \overline{\nabla_t\phi} = \overline{\psi} = \overline{A}_{\mu\nu} = (\overline{\nabla_t B_{ij}})\breve{} = 0 \; . \tag{6.30}$$

The conditions $\overline{\phi} = \overline{\nabla_t\phi} = \overline{\psi} = \overline{A}_{\mu\nu} = 0$ are equivalent to

$$\begin{aligned} \overline{X}^t &= 0 \, , & (6.31)\\ \overline{\nabla_t X^t} &= \overline{Y} \, , & (6.32)\\ \overline{\nabla_t X^i} &= 0 & (6.33)\\ \overline{Y} &= \frac{1}{3}\mathscr{D}_i \overline{X}^i \, , & (6.34)\\ \overline{\nabla_t Y} &= 0 \, , & (6.35)\\ (\tilde{\nabla}_{(i}\overline{X}_{j)})\breve{} &= 0 \, . & (6.36) \end{aligned}$$

Assuming that (6.24) holds on Σ, we further find that $(\overline{\nabla_t B_{ij}})\breve{}$ can be written as

$$\mathscr{L}_{\overline{X}^k\partial_k} D_{ij} + \frac{1}{3} D_{ij}\mathscr{D}_k \overline{X}^k = 0 \, . \tag{6.37}$$

The proper KID equations are (6.36) and (6.37): If and only if there exists a vector field $\mathring{X} = \mathring{X}^k\partial_k$ on $\mathscr{I}^-$ such that (6.36) and (6.37) hold, the space-time contains a vector field X satisfying the unphysical Killing equations. It is obtained by setting $\overline{X}^i = \mathring{X}^i$ and $\overline{X}^t$, $\overline{\nabla_t X^\mu}$, $\overline{Y}$ and $\overline{\nabla_t Y}$ as required by (6.31)-(6.35), and solving (6.23)-(6.24) for these data.

In terms of an asymptotic Cauchy problem we are led to the following result:

THEOREM 6.5.1 *The tuple $(\Sigma, h_{ij}, D_{ij}, \mathring{X}^i)$, where (Σ, h_{ij}) is a Riemannian manifold and D_{ij} a symmetric tensor of valence two on Σ, defines, for $\lambda > 0$, an (up to isometries) unique, in the unphysical space-time maximal globally hyperbolic vacuum space-time with a smooth $\mathscr{I}^-$, represented by $\iota(\Sigma)$, with $\iota^* g_{ij}|_\Sigma = h_{ij}$ and $\iota^* d_{titj}|_\Sigma = D_{ij}$, which contains a Killing vector field X with $\overline{X}^i = \mathring{X}^i$, if and only if the symmetric 2-tensor D_{ij} is trace- and divergence free and $\mathring{X}$ is a conformal Killing vector field on (Σ, h_{ij}) which satisfies the reduced KID equations*

$$\mathscr{L}_{\mathring{X}} D + \frac{1}{3} D \,\mathrm{div}\mathring{X} = 0 \, . \tag{6.38}$$

6.6 KIDs on conformal boundaries: The characteristic case

The characteristic case is treated in [70], attached as Chapter 14. Here, we confine attention to a regular light-cone C_O. We need to analyze the conditions

$$\overline{\phi} = \overline{\psi} = \overline{A}_{\mu\nu} = \overline{\nabla_t A_{\mu\nu}} = \overline{B}_{\mu\nu} = 0 \, .$$

It is no surprise anymore that many of the conditions are automatically satisfied, supposing that the wave equations (6.23)-(6.24) hold, and that the vector field X is regular at the vertex of the cone [70, Sections 3.3 and 4.1]. The remaining conditions are

$$\overline{\phi} = \overline{\psi} = \overline{A}_{ij} = \overline{A}_{0r} = \overline{B}_{ij} = 0 \, . \tag{6.39}$$

As on light-cones in the physical space-time, cf. Section 6.3, the condition $\overline{A}_{0r} = 0$ is not needed on the closure of those sets on which τ is non-zero, such as close to the tip. The transverse derivative of Y appearing in $\overline{\psi}$ and $\overline{B}_{AB}$ can be eliminated via the wave equation (6.24) for Y.

We need to make sure that the initial data $\overline{X}$ and $\overline{Y}$ for (6.23)-(6.24) are restrictions to the cone of smooth space-time fields. Fortunately, it can be proved that all the fields solving $\overline{A}_{ij} = \overline{B}_{1i} = 0$ can be extended to smooth space-time fields by invoking a similar argument as in Section 6.3, cf. [70, Section 4.1]

Light-cones C_O of particular interest in the unphysical space-time are those where $O \in \mathscr{I}^-$ for $\lambda \geq 0$ as well as the C_{i^-}-light-cone for $\lambda = 0$ whose vertex is located at past time-like infinity. We focus on the latter one, since this is the one for which a well-posedness result for the vacuum Einstein equations is available, namely the one we discussed in Section 4.3. For convenience, we impose the gauge condition (4.18) of Section 4.2.3.

Then, some more of the necessary and sufficient conditions in (6.39) are automatically satisfied [70, Section 5.2], and it remains to analyze the KID equations (note that on the C_{i^-}-cone with regular vertex we have $\tau > 0$ globally)

$$\overline{\phi} = \overline{\psi} = \overline{A}_{rA} = \overline{A}_{AB} = \breve{\overline{B}}_{AB} = 0\,, \tag{6.40}$$

or, equivalently,

$$\begin{aligned} \overline{X}^0 &= 0\,, & (6.41)\\ \overline{X}^A &= d^A\,, & (6.42)\\ \overline{X}^r &= cr^2 - \frac{1}{2} r \mathscr{D}_A d^A\,, & (6.43)\\ \overline{Y} &= cr\,, & (6.44) \end{aligned}$$

with $\mathscr{D}_A c(x^B)$ and $d_A(x^B)$ being conformal Killing one-forms on $(S^2, s = s_{AB}\mathrm{d}x^A \mathrm{d}x^B)$, and

$$\mathscr{L}_d \lambda_{AB} - (\frac{1}{2} r \mathscr{D}_C d^C - cr^2)\partial_r \lambda_{AB} + (\frac{1}{2}\mathscr{D}_C d^C - 2cr)\lambda_{AB} \;=\; 0\,. \tag{6.45}$$

A few comments are in order: The candidate fields X and Y obtained from (6.41)-(6.44) with $\mathscr{D}_A c$ and d_A being conformal Killing one-forms on $(S^2, s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B)$ coincide with the Minkowskian candidate fields, whatever the initial data λ_{AB} have been taken to be. This is in contrast to KIDs on ordinary initial surfaces and also to KIDs on a space-like $\mathscr{I}^-$. Whether or not they extend to space-time Killing vector fields via (6.23)-(6.24) does, of course, depend on λ_{AB} via the *reduced KID equations* (6.45).

Let us formulate the result in terms of an initial value problem, as well:

THEOREM 6.6.1 *Consider the symmetric and s-trace-free tensor field $\lambda = \lambda_{AB}\mathrm{d}x^A\mathrm{d}x^B$ on a light-cone C_O constructed from the incoming radiation field ς_{ab} defined in Section 4.3. Then, there exists a locally unique vacuum space-time with a regular C_{i^-}-cone, represented by C_O, with $\overline{\partial_t g_{AB}} = \lambda_{AB}$ and which admits a non-trivial Killing vector field if and only if the reduced KID equations*

$$\mathscr{L}_d \lambda - (\frac{1}{2} r\,\mathrm{div} d - cr^2)\partial_r \lambda + (\frac{1}{2}\mathrm{div} d - 2cr)\lambda \;=\; 0 \tag{6.46}$$

are satisfied for a conformal Killing vector field d^A on $(S^2, s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B)$ and a function c on S^2 which is a linear combination of $\ell = 0, 1$-spherical harmonics.

IOP Publishing — Classical and Quantum Gravity

Class. Quantum Grav. **29** (2012) 145006 (27pp) doi:10.1088/0264-9381/29/14/145006

The many ways of the characteristic Cauchy problem

Piotr T Chruściel and Tim-Torben Paetz

Gravitational Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria

E-mail: Piotr.Chrusciel@univie.ac.at and Tim-Torben.Paetz@univie.ac.at

Received 20 March 2012, in final form 8 May 2012
Published 19 June 2012
Online at stacks.iop.org/CQG/29/145006

Abstract
We review various aspects of the characteristic initial-value problem for the Einstein equations, presenting new approaches to some of the issues arising.

PACS number: 04.20.Ex

1. Introduction

By now there exist four well-established ways of systematically constructing general solutions of the Einstein equations: by solving

(1) a spacelike Cauchy problem (see [1, 2] and references therein),
(2) a boundary-initial-value problem [3–5] (for further references, see [6]),
(3) a characteristic Cauchy problem on two transverse hypersurfaces or
(4) a characteristic Cauchy problem on the light-cone.

One can further consider mixtures of the above. The aim of this paper is to present some new approaches to the last two questions, and to review the existing ones.

To put things in perspective, recall that Einstein's equations by themselves do not have any type that lends itself directly into a known mathematical framework which would provide the existence and/or uniqueness of solutions [7]. The monumental discovery of Yvonne Choquet-Bruhat in 1952 [8] was that the imposition of wave equations on the coordinate functions led to a system where both existence and uniqueness could be proved. The constraint equations satisfied by the initial data on a spacelike hypersurface turned out to be both necessary and sufficient conditions for solving the problem. The constraint equations and the 'harmonicity conditions' became then the two standard notions in our understanding of the spacelike Cauchy problem.

In the early 1960s, there arose a strong interest in the characteristic initial-value problem because of attempts to formulate non-approximate notions of gravitational radiation in the nonlinear theory [9–13]. While those papers provided much insight into the problem at hand, it is widely recognized that the first mathematically satisfactory treatment of the Cauchy problem on two intersecting null hypersurfaces is due to Rendall [14], see also [15–25].

Rendall's initial data consist of a conformal class of a family of two-dimensional Riemannian metrics

$$\tilde{\gamma} := \gamma_{AB}(r, x^C)\, \mathrm{d}x^A\, \mathrm{d}x^B$$

on the null hypersurfaces, complemented by suitable data on the intersection. Here r is an affine parameter on the null geodesics threading the initial-data surfaces. Rendall uses the Raychaudhuri equation to compute the conformal factor Ω needed to determine the family of physically relevant data

$$\tilde{g} := g_{AB}(r, x^C)\, \mathrm{d}x^A\, \mathrm{d}x^B \equiv \Omega^2(r, x^C)\gamma_{AB}(r, x^C)\, \mathrm{d}x^A\, \mathrm{d}x^B \tag{1.1}$$

on the null hypersurfaces. The harmonicity conditions and the Einstein equations determine then uniquely the whole metric g to the future of the initial-data surfaces and near the intersection surface S. The reader will find more details in section 2.

Rendall's elegant approach works well in vacuum, and more generally for a class of matter fields that includes scalar, Maxwell or Yang–Mills fields. However, it appears awkward to use an unphysical family of conformal metrics as initial data, since the physically relevant, and geometrically natural, object is the family $\tilde{g}$. In this context, it appears appropriate to view the tensor field $\tilde{g}$ as an initial datum on the characteristic surfaces, with the Raychaudhuri equation playing the role of a constraint equation. The idea of prescribing $\tilde{\gamma}$ should then be viewed as a conformal ansatz for constructing solutions of this constraint equation.

The last point is only a question of interpretation. More importantly, Rendall's scheme does not work for e.g. the Einstein–Vlasov equations for particles with prescribed rest mass m (see [26] for an existence theorem for those equations with initial data on a spacelike hypersurface), because the energy–momentum tensor for the Vlasov field[1],

$$T_{\alpha\beta} = 8\pi \int_{\{g_{\rho\sigma}p^\rho p^\sigma = -m^2\}} f p_\alpha p_\beta\, \mathrm{d}\mu(p), \tag{1.2}$$

where $f = f(x, p)$ is the Vlasov distribution function and $\mathrm{d}\mu \equiv \mathrm{d}\mu(p)$ is the Riemannian measure induced on the 'mass-shell' $\{g_{\rho\sigma}p^\rho p^\sigma = -m^2\}$, depends explicitly upon all components of the metric. This leads to the need of reformulating the problem so that the whole metric tensor is allowed as part of the initial data on the characteristic surfaces. Such a method will be presented in section 3, after having reviewed Rendall's approach in section 2. We will do this both for data given on two transversally intersecting null hypersurfaces and on a light-cone.

We complement the above with a geometric formulation of the characteristic initial data in section 4, where we give geometric interpretations of n, out of $n + 1$, wave-map gauge constraint equations.

The bottom line of our analysis is that the gravitational characteristic initial data have to satisfy one single constraint equation, the Raychaudhuri equation. This raises the question, how to construct solutions thereof. In section 5, we present several methods to do this. In the short sections 5.1–5.4, we recall how this has already been done in the preceding sections. In section 5.5, we analyse the Hayward gauge condition $\kappa = \tau/(n - 1)$, which may be used as an alternative to an affine-parameterization gauge where the function κ vanishes.

It has been proposed to use the shear tensor σ as the free initial data for the gravitational field rather than $\tilde{\gamma}$. However, one defect is that it is not clear how to guarantee tracelessness of σ. We present in section 5.6 a tetrad formulation of the problem to get rid of this grievance. Finally, we adapt in section 5.7 to any dimensions an approach of Helmut Friedrich (originally

[1] We have included a factor 8π in the definition of $T_{\mu\nu}$, so that the Einstein equations read $S_{\mu\nu} = T_{\mu\nu}$, where $S_{\mu\nu}$ is the Einstein tensor.

developed in dimension 4, using spinors), where certain components of the Weyl tensor are used as unconstrained initial data for the gravitational field. Again, this requires to work in a null-frame formalism to take care of the tracelessness of the Weyl tensor.

In the case of a light-cone, the Hayward gauge leads us to the following issue: Under which conditions is the assumption, that the vertex is located at the origin $r = 0$ of the adapted coordinate system, consistent with regularity at the vertex? This question is considered in an appendix.

2. Rendall's approach

In this section, we review Rendall's approach to the characteristic initial-value problem. For definiteness, in the remainder of this section we will consider the vacuum Einstein equations; we comment at the end of this section on those energy–momentum tensors which are compatible with the analysis here.

Consider two smooth hypersurfaces N_a, $a = 1, 2$, in an $(n+1)$-dimensional manifold $\mathscr{M}$, with transverse intersection along a smooth submanifold S. Near the N_a's, one can choose adapted coordinates (x^1, x^2, x^A) so that N_1 coincides with the set $\{x^1 = 0\}$, while N_2 is given by $\{x^2 = 0\}$. The hypersurfaces N_a are supposed to be characteristic, which is equivalent to the requirement that, in the coordinates above, on N_1, the metric takes the form

$$g|_{N_1} = \overline{g}_{11}(\mathrm{d}x^1)^2 + 2\overline{g}_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 + 2\overline{g}_{1A}\,\mathrm{d}x^1\,\mathrm{d}x^A + \underbrace{\overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B}_{=:\tilde{g}}, \tag{2.1}$$

similarly on N_2. Here, and elsewhere, the terminology and notation of [27] are used; in particular, an overbar denotes restriction to the initial-data surface $N_1 \cup N_2$. Rendall assumes moreover that x^2 is an affine parameter along the curves $\{x^1 = 0, x^A = \mathrm{const}^A\}$, and that x^1 is an affine parameter along the curves $\{x^2 = 0, x^A = \mathrm{const}^A\}$.

On N_1 let

$$\tau \equiv \tfrac{1}{2}\overline{g}^{AB}\partial_2\overline{g}_{AB} \tag{2.2}$$

be the divergence scalar, and let

$$\sigma_{AB} \equiv \frac{1}{2}\partial_2\overline{g}_{AB} - \frac{1}{n-1}\tau\overline{g}_{AB} \tag{2.3}$$

be the trace-free part of $\partial_2\overline{g}_{AB}$, also known as the shear tensor. The vacuum Raychaudhuri equation

$$\partial_2\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0 \tag{2.4}$$

provides a constraint equation on the family of two-dimensional metrics $x^2 \mapsto \overline{g}_{AB}(x^2, x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$, where

$$|\sigma|^2 \equiv \sigma_A{}^B\sigma_B{}^A, \qquad \sigma_A{}^B \equiv \overline{g}^{BC}\sigma_{AC}.$$

Note that $\sigma_A{}^B$ depends only on the conformal class of $\overline{g}_{AB}$. As shown by Rendall, metrics satisfying the constraint (2.4) can be constructed by freely prescribing the family $x^2 \mapsto \gamma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$. Writing $\overline{g}_{AB} = \Omega^2\gamma_{AB}$, (2.4) becomes then a second-order ODE in x^2 for Ω,

$$0 = \partial_2^2\Omega + \frac{\Omega}{n-1}\Bigg[\frac{1}{2}\partial_2(\gamma^{AB}\partial_2\gamma_{AB}) + \underbrace{|\sigma|^2 + \frac{1}{4(n-1)}(\gamma^{AB}\partial_2\gamma_{AB})^2}_{=-\frac{1}{4}\partial_2\gamma^{AB}\partial_2\gamma_{AB}}\Bigg] + \frac{1}{n-1}\gamma^{AB}\partial_2\gamma_{AB}\partial_2\Omega. \tag{2.5}$$

This needs to be complemented by $\Omega|_S$ and $\partial_2\Omega|_S$.

Let us require all coordinate functions to satisfy the scalar wave equation $\Box_g x^\mu = 0$. Then the affine-parameterization condition $\Gamma^2_{22}|_{N_1} = 0$ can be rewritten as

$$\partial_2 \overline{g}_{12} = \tfrac{1}{2}\tau \overline{g}_{12}.$$

This equation determines the metric function g_{12} on N_1 with the freedom to prescribe g_{12} on S.

The equation $\overline{R}_{2A} = 0$ on N_1 takes the form

$$-\frac{1}{2}(\partial_2 + \tau)\xi_A + \tilde{\nabla}_B \sigma_A{}^B - \frac{n-2}{n-1}\partial_A \tau = 0, \tag{2.6}$$

where $\tilde{\nabla}$ is the covariant derivative operator of the metric $g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$. Here, using the assumption that all coordinate functions satisfy the wave equation, the covector ξ_A reads [27, equation (8.25)]

$$\xi_A := -2\overline{g}^{12}\partial_2 \overline{g}_{1A} + 4\overline{g}^{12}\overline{g}_{1B}\sigma_A{}^B + 2\overline{g}^{12}\overline{g}_{1A}\tau - \overline{g}_{AB}\overline{g}^{CD}\tilde{\Gamma}^B_{CD}, \tag{2.7}$$

where the $\tilde{\Gamma}^B_{CD}$s are the Christoffel symbols of the metric $\overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$. This provides an ODE for the metric functions $\overline{g}_{1A}$: indeed, one determines ξ_A by integrating (3.12), with the freedom to prescribe

$$\xi^{N_1}_A := \xi_A(x^2 = 0)$$

on S. (One should keep in mind that ξ_A here is unrelated to the corresponding field ξ_A on N_2, determined by an analogous equation where all quantities τ, σ, etc, are calculated using the fields on N_2.)

Then $\overline{g}_{1A}$ is found by integrating (2.7). Since the metric needs to be continuous, the metric component g_{1A} has to vanish at S; this requirement defines the integration constant. We further observe that by definition of ξ_A, the freedom to prescribe $\xi^{N_1}_A$ corresponds to the freedom of prescribing $\partial_2 g_{1A}$ on S.

The equation $\overline{g}^{AB}\overline{R}_{AB} = 0$ in vacuum takes the form

$$(\partial_2 + \tau)\zeta + \tilde{R} - \tfrac{1}{2}\overline{g}^{AB}\xi_A\xi_B + \overline{g}^{AB}\tilde{\nabla}_A\xi_B = 0, \tag{2.8}$$

where $\tilde{R}$ is the curvature scalar of $\tilde{g}$, and where

$$\zeta := (2\partial_2 + \tau)\overline{g}^{22}. \tag{2.9}$$

Taken together, those equations provide a second-order ODE for $\overline{g}^{22}$; integration requires the knowledge of $\overline{g}^{22}$ and $\partial_2\overline{g}^{22}$ on S. Employing the relation $\overline{g}^{22} = (\overline{g}^{12})^2(\overline{g}^{AB}\overline{g}_{1A}\overline{g}_{1B} - \overline{g}_{11})$, we observe that g^{22} has to vanish at S, while there remains the freedom of prescribing $\partial_2 g^{22}$, equivalently $\partial_2 g_{11}$, on S.

However, the validity of the harmonicity conditions implies certain constraints on S (see below): the value of $\partial_2 g_{11}$ at S is determined by equation (2.10*c*); similarly the function $\partial_2 g_{11}$ at S follows from (2.10*a*).

One has thus determined all the metric functions $g_{\mu\nu}$ on N_1; the procedure on N_2 is completely analogous. These are the data needed for the harmonically reduced Einstein equations, which form a well-posed evolutionary system for the metric.

However, not every solution of the equations constructed in this way will satisfy the vacuum Einstein equations: one still needs to make sure that the harmonicity conditions are satisfied. There is in fact one more subtlety, as one needs to verify that the parameter x^2 is indeed an affine parameter on the null geodesics threading N_1. It turns out [14] that all this will be verified provided three more conditions are imposed on S: if we write ν^+_A for what was $g_{1A}|_{N_1}$ so far, ν^-_A for $g_{2A}|_{N_2}$, the wave-coordinate conditions will hold if we require that on S,

$$\partial_1 g_{22}|_S = \overline{g}_{12}\frac{\partial_2\sqrt{\det g_{AB}}}{\sqrt{\det g_{AB}}}, \tag{2.10a}$$

$$\partial_1 \nu_A^- + \partial_2 \nu_A^+ = \frac{1}{\sqrt{\det g_{EF}}} g_{AB} \partial_C (\overline{g}_{12} \sqrt{\det g_{EF}} g^{BC}), \tag{2.10b}$$

$$\partial_2 g_{11}|_S = \overline{g}_{12} \frac{\partial_1 \sqrt{\det g_{AB}}}{\sqrt{\det g_{AB}}}. \tag{2.10c}$$

As already indicated above, the integration functions $\partial_1 g_{22}|_S$ and $\partial_2 g_{11}|_S$ cannot be freely specified but have to be chosen in such a way that equations (2.10*a*) and (2.10*c*) are fulfilled. Equation (2.10*b*) will be satisfied by exploiting the freedom in the choice of $\partial_1 \nu_A^-$ and $\partial_2 \nu_A^+$. So there remains the freedom to prescribe, say, $\partial_1 \nu_A^- - \partial_2 \nu_A^+$.

The constraint equation (2.10*b*) can be tied to a terminology introduced by Christodoulou [28] as follows: let L and $\underline{L}$ be two null normals to a codimension-2 spacelike hypersurface S satisfying

$$g(L, \underline{L}) = -2.$$

Christodoulou [28] defines the *torsion one-form* of S by the formula

$$\zeta(X) = \tfrac{1}{2} g(\nabla_X L, \underline{L}), \tag{2.11}$$

where $X \in TS$. Assuming that $g_{12}|_S$ is positive, we can choose $L = \sqrt{2\overline{g}^{12}}\, \partial_2$; then, on S, using the notation above, $\underline{L} = -\sqrt{2\overline{g}^{12}}\, \partial_1$ and (2.11) reads

$$\begin{aligned} \zeta_A &= \tfrac{1}{2} g(\nabla_A L, \underline{L}) = \tfrac{1}{2} \overline{g}^{12} \partial_A \overline{g}_{12} - \overline{\Gamma}^2_{2A} = \tfrac{1}{2}\big(\overline{\Gamma}^1_{1A} - \overline{\Gamma}^2_{2A}\big) \\ &= \tfrac{1}{2} \overline{g}^{12} (\partial_1 \overline{g}_{2A} - \partial_2 \overline{g}_{1A}) = \tfrac{1}{2} \overline{g}^{12} \big(\partial_1 \nu_A^- - \partial_2 \nu_A^+\big). \end{aligned} \tag{2.12}$$

So ζ_A contains precisely the information needed to determine $\partial_1 \nu_A^-$ and $\partial_2 \nu_B^+$ at S, after taking into account (2.10*b*).

Theorem 2.1 (Rendall)**.** *Consider two smooth hypersurfaces N_1 and N_2 in an $(n+1)$-dimensional manifold with transverse intersection along a smooth submanifold S in adapted null coordinates. Let γ_{AB} be a smooth family of Riemannian metrics on $N_1 \cup N_2$, continuous at S. Moreover, let Ω, $\partial_1 \Omega$, $\partial_2 \Omega$, f and f_A, $A = 3, \ldots, n+1$, be smooth fields on S, where we assume Ω and f to be nowhere vanishing on S. Then there exists an open neighbourhood U of S in the region $\{x^1 \geqslant 0, x^2 \geqslant 0\}$, a unique function Ω on $(N_1 \cup N_2) \cap U$ and a unique smooth Lorentz metric $g_{\mu\nu}$ on U such that*

(1) $g_{\mu\nu}$ satisfies the vacuum Einstein equations,
(2) $\bar{g}_{AB} = \Omega^2 \gamma_{AB}$,
(3) Ω induces the given data on S, $g_{12}|_S = f$ and $\zeta_A = f_A$.

This analysis of the constraints applies equally well to a light-cone with some minor modifications [27], where the wave equations for the coordinate functions are replaced by wave-map conditions. Moreover, there is no need to provide further initial data at the tip of the light-cone, as those are replaced by conditions arising from the requirement of regularity of the metric there; the reader is referred to [27] for a detailed discussion. An explicit parameterization of tensors $\tilde{g}$ which arise by the restriction of a smooth metric in normal coordinates has been given in [29]. The reader should keep in mind the serious difficulties with regularity of the metric at the vertex, when attempting to prove an existence theorem for the light-cone problem; see [30, 31] for results under restrictive conditions on the data.

The above extends easily to non-vacuum models with energy–momentum tensors of the form

$$\begin{aligned}
\overline{T}_{22} &= \overline{T}_{22}(\text{matter data}, \gamma_{AB}, \partial_i\gamma_{AB}, \Omega, \partial_2\Omega, \overline{g}_{12}, \partial_i\overline{g}_{12}, x^i), \quad i = 2, A,\\
\overline{T}_{2A} &= \overline{T}_{2A}(\text{matter data}, \gamma_{AB}, \partial_i\gamma_{AB}, \Omega, \partial_i\Omega, \overline{g}_{1i}, \partial_i\overline{g}_{12}, \partial_2\overline{g}_{1A}, \overline{\partial_1 g_{22}}, x^i),\\
\overline{T}_{12} &= \overline{T}_{12}(\text{matter data}, \gamma_{AB}, \partial_i\gamma_{AB}, \Omega, \partial_i\Omega, \overline{g}_{1\mu}, \partial_2\overline{g}_{1\mu}, \partial_A\overline{g}_{12}, \overline{\partial_1 g_{2i}}, x^i),
\end{aligned}$$

on the initial surface $\{x^1 = 0\}$, cf [27].

3. All components of the metric as initial data

Let ℓ^ν denote the field of null tangents to a characteristic hypersurface. In this section, we present a treatment of the characteristic Cauchy problem which applies to energy–momentum tensors of the form

$$T_{\mu\nu}\ell^\nu = T_\mu(g, \phi, \partial^\parallel g, \partial^\parallel \phi, x), \tag{3.1}$$

for some fields ϕ satisfying equations which, when the metric is considered as given, possess a well-posed characteristic Cauchy problem. Here the symbol $\partial^\parallel$ denotes derivatives in directions purely tangential to the initial-data surfaces. In particular, equation (3.1) includes the Einstein–Vlasov case.

As already discussed, in [14] the corresponding problem for the vacuum Einstein equations is solved using an affine parameterization of the generators and a wave-map ('harmonic') gauge. In Rendall's approach, some components of the metric are calculated by solving the characteristic harmonic gauge constraint equations, which form a hierarchical ODE system along the generators of the initial surface. For an energy–momentum tensor (1.2), this approach will generally lead instead to a quasi-linear PDE system for the metric components. To establish an existence result for that system might be intricate, if possible at all. It is in any case not obvious how to include an energy–momentum tensor (1.2) in this scheme, compare [32]. We circumvent the problem by using a gauge adapted to the initial data, where the metric tensor, and thereby (1.2), is fully given on the initial surface, while the *wave-gauge source vector* $\mathring{W}^\mu$ is computed from the values of the metric on the initial surface using the *Einstein wave-map-gauge* constraint equations of [27].

We start with an analysis of two intersecting hypersurfaces; the case of a light-cone will be covered in section 3.2.

3.1. Two transverse hypersurfaces

Consider two smooth hypersurfaces N_a, $a = 1, 2$, in an $(n + 1)$-dimensional manifold $\mathscr{M}$, with transverse intersection along a smooth submanifold S. As before, we choose adapted null coordinates (x^1, x^2, x^A) so that N_1 coincides with the set $\{x^1 = 0\}$, while N_2 is given by $\{x^2 = 0\}$. We use a 'generalized wave-map gauge' as in [27], with target metric $\hat{g}$ of the form

$$\hat{g} = 2\,dx^1\,dx^2 + \hat{g}_{AB}(x^1, x^2, x^C)\,dx^A\,dx^B.$$

Here $\hat{g}_{AB}$ is any family of Riemannian metrics on S parameterized by x^1 and x^2, smooth in all variables. The metric $\hat{g}$ is only introduced so that the harmonicity vector H^μ, defined in equation (3.4), is a vector field, and plays no significant role in what follows.

As gravitational initial data on the initial hypersurfaces, we prescribe all metric components $\overline{g}_{\mu\nu}$ in the coordinates above, as well as a connection coefficient κ; this needs to be supplemented by the initial data $\overline{\phi}$ for ϕ. For instance, in the Einstein–Vlasov case, the supplementary data will be a function $\overline{f}$ defined on the mass-shell $\{\overline{g}_{\mu\nu}p^\mu p^\nu = -m^2\}$, viewed as a subset of the pull-back of $T\mathscr{M}$ to the N_a's.

The hypersurfaces N_a are supposed to be characteristic, which is equivalent to the requirement that, in the coordinates above, on N_1 the metric takes the form

$$g|_{N_1} = \overline{g}_{11}(\mathrm{d}x^1)^2 + 2\overline{g}_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 + 2\overline{g}_{1A}\,\mathrm{d}x^1\,\mathrm{d}x^A + \underbrace{\overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B}_{=:\tilde{g}}, \tag{3.2}$$

similarly on N_2. Here, and elsewhere, the terminology and notation of [27] are used; in particular, an overbar denotes restriction to the initial-data surface $N_1 \cup N_2$. (Some obvious renamings need to be applied to the equations in [27], for instance, the variable u there is x^1 on N_1, and x^2 on N_2; the variable r there is x^2 on N_1 and x^1 on N_2.)

We want to apply Rendall's existence theorem [14] for an appropriately reduced system of equations. For this, the trace, $\overline{g}_{\mu\nu}$, of the metric on the initial surface $N_1 \cup N_2$ needs to be the restriction of a smooth Lorentzian spacetime metric. This will be the case if $\overline{g}_{12}$ is nowhere vanishing, if $\overline{g}_{AB}|_{N_a}$ is a family of Riemannian metrics and if $\overline{g}_{\mu\nu}$ is smooth on N_1 and N_2 and continuous across $S \equiv N_1 \cap N_2$. We therefore need to impose the following continuity conditions on S:

$$\lim_{x^2\to 0} g_{AB}|_{N_1} = \lim_{x^1\to 0} g_{AB}|_{N_2}, \tag{3.3a}$$

$$\lim_{x^2\to 0} g_{12}|N_1 = \lim_{x^1\to 0} g_{12}|N_2, \tag{3.3b}$$

$$\lim_{x^2\to 0} g_{1A}|N_1 = 0, \quad \lim_{x^1\to 0} g_{2A}|N_2 = 0, \tag{3.3c}$$

$$\lim_{x^2\to 0} g_{11}|N_1 = 0, \quad \lim_{x^1\to 0} g_{22}|N_2 = 0. \tag{3.3d}$$

Let H^μ be the *harmonicity vector*, defined as

$$H^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} - W^\lambda, \quad \text{with} \quad W^\lambda := \underbrace{g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}}_{=:\hat{W}^\lambda} + \mathring{W}^\lambda, \tag{3.4}$$

where $\mathring{W}^\lambda$ will be a vector depending only upon the coordinates, and determined by the initial data in a way to be described below. (In principle, $\mathring{W}^\lambda$ can be allowed to depend on the metric as well, but not on derivatives of the metric.) To obtain a well-posed system of evolution equations, we will impose the generalized wave-map gauge condition

$$H^\lambda = 0.$$

More precisely, we view the wave-map gauge constraints [27] as equations for the restriction $\overline{\mathring{W}^\mu}$ of $\mathring{W}^\mu$ to N_1 and N_2. We will solve those equations hierarchically. We emphasize that, assuming (3.1), all components of the energy–momentum tensor restricted to N_1 and N_2 are explicitly known since $\overline{g}_{\mu\nu}$ and $\overline{\phi}$ are.

We present the calculations on N_1; the equations on N_2 are obtained by interchanging index 1 with index 2 in all the formulae.

Let $S_{\mu\nu}$ denote the Einstein tensor. In the notation and terminology of [27], the first constraint, arising from the equation $\overline{S}_{22} \equiv \overline{R}_{22} = \overline{T}_{22}$ evaluated on N_1, reads (see [27, equation (6.11)])

$$-\partial_2\tau + \kappa\tau - |\sigma|^2 - \frac{\tau^2}{n-1} = \overline{T}_{22}, \tag{3.5}$$

where τ and σ are defined as in (2.2) and (2.3), respectively. Indeed, using the formulae in [27, appendix A], one finds

$$\begin{aligned}
\overline{\partial_1\Gamma^1_{22}} &= \partial_2\overline{\Gamma}^1_{12} + \left(\overline{\Gamma}^1_{12}\right)^2 - \overline{\Gamma}^1_{12}\overline{\Gamma}^2_{22} \qquad \Longrightarrow \\
\overline{S}_{22} &= \overline{\partial_1\Gamma^1_{22}} - \partial_2\left(\overline{\Gamma}^1_{12} - \overline{\Gamma}^A_{2A}\right) + \left(\overline{\Gamma}^1_{12} + \overline{\Gamma}^A_{2A}\right)\overline{\Gamma}^2_{22} - \left(\overline{\Gamma}^1_{12}\right)^2 - \overline{\Gamma}^A_{2B}\overline{\Gamma}^B_{2A} \\
&= -\partial_2\overline{\Gamma}^A_{2A} + \overline{\Gamma}^A_{2A}\overline{\Gamma}^2_{22} - \overline{\Gamma}^A_{2B}\overline{\Gamma}^B_{2A} \\
&= -\partial_2\tau + \tau\overline{\Gamma}^2_{22} - \chi_A{}^B\chi_B{}^A,
\end{aligned} \tag{3.6}$$

where

$$\chi_A{}^B := \tfrac{1}{2}\overline{g}^{BC}\partial_2\overline{g}_{AC}.$$

Here we adapt the point of view that the function κ is the value on N_1 of the Christoffel coefficient Γ^2_{22}, and is part of the initial data. Hence, we view (3.5) as a constraint equation linking $\overline{g}_{AB}$, κ, and the matter sources (if any).

In the region where τ has no zeros, equation (3.5) can be trivially solved for κ to give

$$\kappa = \frac{\partial_2\tau + \frac{1}{n-1}\tau^2 + |\sigma|^2 + \overline{T}_{22}}{\tau}. \tag{3.7}$$

It follows that κ does not need to be included as part of initial data when τ has no zeros, and then the constraint equation (3.5) can be replaced by the last equation, determining κ.

Equation (3.7) can still be used, by continuity, to determine κ on the closure of the set where τ has no zeros, keeping in mind that the requirement of smoothness of the function so determined imposes non-trivial constraints on the right-hand side. In any case, equation (3.7) does not make sense if there are open regions where τ vanishes. It seems therefore best to assume that κ is any smooth function on N_1 such that (3.5) holds, and view that last equation as a constraint equation relating κ, $\overline{g}_{AB}$ and its derivatives, and the matter fields; similarly on N_2.

It should be kept in mind that once a candidate solution of the Einstein equations has been constructed, one needs to verify that κ is indeed the value of Γ^2_{22} on N_1. We will return to this in (3.25).

We choose $\overline{\mathring{W}}{}^1$ to be

$$\overline{\mathring{W}}{}^1 := -\overline{\hat{W}}{}^1 - \overline{g}^{12}(2\kappa + \tau) - 2\partial_2\overline{g}^{12}. \tag{3.8}$$

Note that the right-hand side is known, so this defines $\overline{\mathring{W}}{}^1$. By definition, this is the ∂_1 component of $\mathring{W}^\mu$ in the coordinate system (x^1, x^2, x^A). We will define the remaining components of $\mathring{W}^\mu$ shortly, the resulting collection of functions transforming *by definition* as a vector when changing coordinates.

From [27, appendix A], one then finds

$$\overline{\Gamma}^2_{22} = \kappa - \tfrac{1}{2}\overline{g_{12}}\overline{H}^1, \tag{3.9}$$

and (3.6) together with (3.5) give

$$\overline{S}_{22} - \overline{T}_{22} = -\tfrac{1}{2}\overline{g_{12}}\overline{H}^1\tau. \tag{3.10}$$

The corresponding constraint equation on N_2 determines $\mathring{W}^2|_{N_2}$. We shall return to the question of continuity at S of $\mathring{W}^1|_{N_1\cup N_2}$ and of $\mathring{W}^2|_{N_1\cup N_2}$ shortly.

The next constraint equation follows from $\overline{S}_{2A} \equiv \overline{R}_{2A} = \overline{T}_{2A}$. From the formulae in [27, appendix A], we find

$$\overline{\partial_1\Gamma^1_{2A}} = \partial_A\overline{\Gamma}^1_{12} + \overline{\Gamma}^1_{12}\left(\overline{\Gamma}^1_{1A} - \overline{\Gamma}^2_{2A}\right) + \overline{\Gamma}^B_{12}\overline{\Gamma}^1_{AB} - \overline{\Gamma}^B_{2A}\overline{\Gamma}^1_{1B},$$

which gives

$$
\begin{aligned}
\overline{S}_{2A} &= \overline{\partial_1 \Gamma^1_{2A}} + \partial_2 \overline{\Gamma}^2_{2A} + \partial_B \overline{\Gamma}^B_{2A} - \partial_A \overline{\Gamma}^1_{12} - \partial_A \overline{\Gamma}^2_{22} - \partial_A \overline{\Gamma}^B_{2B} + \overline{\Gamma}^1_{1B}\overline{\Gamma}^B_{2A} \\
&\quad + \overline{\Gamma}^1_{12}\big(\overline{\Gamma}^2_{2A} - \overline{\Gamma}^1_{1A}\big) + \overline{\Gamma}^B_{2B}\overline{\Gamma}^2_{2A} + \overline{\Gamma}^B_{BC}\overline{\Gamma}^C_{2A} - \overline{\Gamma}^1_{AB}\overline{\Gamma}^B_{12} - \overline{\Gamma}^B_{AC}\overline{\Gamma}^C_{1B} \\
&= \partial_2 \overline{\Gamma}^2_{2A} + \partial_B \overline{\Gamma}^B_{2A} - \partial_A \overline{\Gamma}^2_{22} - \partial_A \overline{\Gamma}^B_{2B} + \overline{\Gamma}^B_{2B}\overline{\Gamma}^2_{2A} + \overline{\Gamma}^B_{BC}\overline{\Gamma}^C_{2A} - \overline{\Gamma}^B_{AC}\overline{\Gamma}^C_{2B} \\
&= (\partial_2 + \tau)\overline{\Gamma}^2_{2A} + \tilde{\nabla}_B \chi_A{}^B - \partial_A \overline{\Gamma}^2_{22} - \partial_A \tau,
\end{aligned}
\tag{3.11}
$$

where $\tilde{\nabla}$ is the covariant derivative associated with the Riemannian metric g_{AB}. That leads us to the equation

$$
-\frac{1}{2}(\partial_2 + \tau)\xi_A + \tilde{\nabla}_B \sigma_A{}^B - \frac{n-2}{n-1}\partial_A \tau - \partial_A \kappa = \overline{T}_{2A}, \tag{3.12}
$$

where the field ξ_A denotes the restriction of the (rescaled) Christoffel coefficient $-2\Gamma^2_{2A}$ to N_1. We determine ξ_A by integrating (3.12), with the freedom to prescribe

$$
\xi_A^{N_1} := \xi_A(x^2 = 0)
$$

on S. (One should keep in mind that ξ_A here is unrelated to the corresponding field ξ_A on N_2, determined by an analogous equation where all quantities τ, σ, etc, are calculated using the fields on N_2.) We then define $\overline{\mathring{W}}{}^A$ through the formula

$$
\overline{\mathring{W}}{}^A := \overline{g}^{AB}[\xi_B + 2\overline{g}^{12}(\partial_2 \overline{g}_{1B} - 2\overline{g}_{1C}\sigma_B{}^C - \overline{g}_{1B}\tau) - \overline{g}_{1B}(\overline{\mathring{W}}{}^1 + \overline{\hat{W}}{}^1)] + \overline{g}^{CD}\tilde{\Gamma}^A_{CD} - \overline{\hat{W}}{}^A; \tag{3.13}
$$

equivalently

$$
\begin{aligned}
\xi_A = &- 2\overline{g}^{12}\partial_2 \overline{g}_{1A} + 4\overline{g}^{12}\overline{g}_{1B}\sigma_A{}^B + 2\overline{g}^{12}\overline{g}_{1A}\tau + \overline{g}_{1A}(\overline{\mathring{W}}{}^1 + \overline{\hat{W}}{}^1) \\
&+ \overline{g}_{AB}(\overline{\mathring{W}}{}^B + \overline{\hat{W}}{}^B) - \overline{g}_{AB}\overline{g}^{CD}\tilde{\Gamma}^B_{CD}.
\end{aligned}
\tag{3.14}
$$

This has been chosen so that, using the formulae in [27, appendix A and section 9],

$$
\overline{S}_{2A} - \overline{T}_{2A} = -\tfrac{1}{2}(\partial_2 + \tau)(\overline{g}_{AB}\overline{H}^B + \overline{g}_{1A}\overline{H}^1) + \tfrac{1}{2}\partial_A(\overline{g}_{12}\overline{H}^1). \tag{3.15}
$$

Moreover, one finds (cf equation (10.35) in [27])

$$
\xi_A = -2\overline{\Gamma}^2_{2A} - \overline{g}_{AB}\overline{H}^B - \overline{g}_{1A}\overline{H}^1. \tag{3.16}
$$

On S, equation (3.13) takes the form

$$
\overline{\mathring{W}}{}^A|_S = \overline{g}^{AB}\big[\xi_B^{N_1} + 2\overline{g}^{12}\partial_2 \overline{g}_{1B}\big] + \overline{g}^{BC}(\tilde{\Gamma}^A_{BC} - \hat{\Gamma}^A_{BC}).
$$

Keeping in mind the corresponding equation on N_2,

$$
\overline{\mathring{W}}{}^A|_S = \overline{g}^{AB}\big[\xi_B^{N_2} + 2\overline{g}^{12}\partial_1 \overline{g}_{2B}\big] + \overline{g}^{BC}(\tilde{\Gamma}^A_{BC} - \hat{\Gamma}^A_{BC}),
$$

the requirement of continuity of $\mathring{W}^A|_{N_1 \cup N_2}$ leads to

$$
\xi_A^{N_1} - \xi_A^{N_2} = 2g^{12}(\partial_1 g_{2A} - \partial_2 g_{1A})|_S \equiv 4\zeta_A. \tag{3.17}
$$

Recall that the torsion one-form ζ_A has been defined in (2.12).

We continue with the equation $\overline{S}_{12} = \overline{T}_{12}$, or, equivalently,

$$
\overline{g}^{AB}\overline{R}_{AB} = -(2\overline{g}^{12}\overline{T}_{12} + \overline{g}^{22}\overline{T}_{22} + 2\overline{g}^{2A}\overline{T}_{2A}) = \overline{g}^{AB}\overline{T}_{AB} - \overline{T},
$$

which we handle in a manner similar to the previous equations. Using identities (10.33) and (a corrected version of[2]) (10.36) in [27], we find that on N_1, we have

$$
\overline{g}^{AB}\overline{R}_{AB} \equiv \big(\partial_2 + \overline{\Gamma}^2_{22} + \tau\big)\big(2\overline{g}^{AB}\overline{\Gamma}^2_{AB} + \tau\overline{g}^{22}\big) - 2\overline{g}^{AB}\overline{\Gamma}^2_{2A}\overline{\Gamma}^2_{2B} - 2\overline{g}^{AB}\tilde{\nabla}_A\overline{\Gamma}^2_{2B} + \tilde{R},
$$

[2] On the right-hand side of (10.36) in [27], in the conventions and notations there, a term $\tau\overline{g}^{11}/2$ is missing.

and we are led to the equation

$$(\partial_2 + \kappa + \tau)\zeta + \big(\tilde{\nabla}_A - \tfrac{1}{2}\xi_A\big)\xi^A + \tilde{R} = \overline{g}^{AB}\overline{T}_{AB} - \overline{T}, \tag{3.18}$$

with $\xi^A := \overline{g}^{AB}\xi_B$, and where the quantity ζ denotes the restriction of

$$2\big(\overline{g}^{AB}\overline{\Gamma}^2_{AB} + \tau\overline{g}^{22}\big)$$

to N_1. We integrate (3.18), viewed as a first-order ODE for ζ. The initial data on S are determined by the requirement of continuity of $\overline{\mathring{W}}{}^2$ at S, which we choose to be

$$\overline{\mathring{W}}{}^2 := \tfrac{1}{2}\zeta - \big(\partial_2 + \kappa + \tfrac{1}{2}\tau\big)\overline{g}^{22} - \overline{\hat{W}}{}^2. \tag{3.19}$$

Indeed, recall that $\overline{\mathring{W}}{}^2|_S$ has already been calculated algebraically when analysing the first constraint equation on N_2, in exactly the same way as we calculated $\overline{\mathring{W}}{}^1$ in the first step of the analysis above.

Similarly the initial data for the integration of the constraint which determines $\mathring{W}^1|_{N_2}$ are determined by the requirement of continuity of $\mathring{W}^1|_{N_1\cup N_2}$.

The choice (3.19) has been made so that

$$\begin{aligned}\overline{g}^{AB}\overline{R}_{AB} - \overline{g}^{AB}\overline{T}_{AB} + \overline{T} = {}& \big(\partial_2 + \kappa + \tau - \tfrac{1}{2}\overline{g}_{12}\overline{H}^1\big)(2\overline{H}^2 - \overline{g}_{12}\overline{g}^{22}\overline{H}^1) - \tfrac{1}{2}\zeta\overline{g}_{12}\overline{H}^1 \\ &+ \big(\tilde{\nabla}_A - \xi_A - \tfrac{1}{2}\overline{g}_{AB}\overline{H}^B - \tfrac{1}{2}\overline{g}_{1A}\overline{H}^1\big)(\overline{H}^A + \overline{g}_{1C}\overline{g}^{AC}\overline{H}^1).\end{aligned} \tag{3.20}$$

We note that our choice of $\overline{\mathring{W}}{}^2$ is equivalent to

$$\zeta = 2\overline{g}^{AB}\overline{\Gamma}^2_{AB} + \tau\overline{g}^{22} + \overline{g}_{12}\overline{g}^{22}\overline{H}^1 - 2\overline{H}^2. \tag{3.21}$$

Summarizing, given the fields κ, $\overline{g}_{\mu\nu}$ and $\overline{\phi}$ on $N_1 \cup N_2$, satisfying (3.17), and the sum $\xi_A^{N_1} + \xi_A^{N_2}$ on S, we have found a unique continuous vector field $\overline{\mathring{W}}$ on $N_1 \cup N_2$, smooth up-to-boundary on N_1 and N_2, so that (3.10), (3.15) and (3.20) hold on $N_1 \cup N_2$. Letting $\mathring{W}$ be any smooth vector field on $\mathscr{M}$ which coincides with $\overline{\mathring{W}}$ on $N_1 \cup N_2$, and assuming that the reduced Einstein equations (see (3.29) below) can be complemented by well-posed evolution equations for the matter fields, we obtain a metric, solution of the Cauchy problem for the reduced Einstein equations, in a future neighbourhood of S.

However, the metric so obtained will solve the full Einstein equations if and only if [27] H^μ vanishes on $N_1 \cup N_2$, so we need to ensure that this condition holds. Note that at this stage, a smooth metric g, satisfying the reduced Einstein equations, and a smooth vector field W^μ are known to the future of $N_1 \cup N_2$ in some neighbourhood of S, and thus H^μ is a known smooth vector field there.

By [27, section 7.6], $\overline{H}^1$ will vanish on N_1 if and only if $\overline{H}^1$ vanishes on S. Using (3.8) and (3.9) together with the equations in [27, appendix A], we find

$$H^1|_S = (g^{12})^2\partial_1 g_{22} + 2g^{12}\kappa + 2\partial_2 g^{12}. \tag{3.22}$$

We conclude that $H^1|_{N_1}$ will vanish if and only if the initial data $\overline{g}_{22}$ on N_2 have the property that the derivative $\partial_1 g_{22}$ on S satisfies

$$\partial_1 g_{22}|_S = 2(\partial_2 g_{12} - g_{12}\kappa_{N_1}) \quad\Longleftrightarrow\quad \Gamma^2_{22}|_S = \kappa_{N_1}, \tag{3.23}$$

where, to avoid ambiguities, we denote by κ_{N_a} the function κ associated with the hypersurface N_a, etc. Similarly, $H^2|_{N_2}$ will vanish if and only if we choose g_{11} on N_1 so that

$$\partial_2 g_{11}|_S = 2(\partial_1 g_{12} - g_{12}\kappa_{N_2}) \quad\Longleftrightarrow\quad \Gamma^1_{11}|_S = \kappa_{N_2}. \tag{3.24}$$

With those choices, we have $H^1|_S = H^2|_S = 0$, and the arguments in [27] show that $H^1|_{N_2} = H^2|_{N_1} = 0$ as well.

We continue with $\overline{H}^A$. Then by equation (3.16), we have

$$\begin{aligned}\xi_A^{N_1} &= -\big(2\Gamma^2_{2A} + g_{AB}H^B + g_{1A}H^1\big)\big|_S\\ &= -\big(\overline{g}^{12}\big(\partial_A\overline{g}_{12} - \overline{\partial_1 g_{2A}} + \partial_2\overline{g}_{1A}\big) + g_{AB}H^B + g_{1A}H^1\big)\big|_S,\\ \xi_A^{N_2} &= -\big(2\Gamma^1_{1A} + g_{AB}H^B + g_{1A}H^1\big)\big|_S\\ &= -\big(\overline{g}^{12}\big(\partial_A\overline{g}_{12} - \overline{\partial_2 g_{1A}} + \partial_1\overline{g}_{2A}\big) + g_{AB}H^B + g_{1A}H^1\big)\big|_S,\end{aligned}$$

and the conditions $H^A|_S = 0$ and $H^1|_S = 0$ determine $\xi_A^{N_a}$ in terms of the remaining data. Note that (3.17) is then automatically satisfied, and that we loose the freedom to prescribe $\xi_A^{N_1} + \xi_A^{N_2}$.

It remains to show that our choice of the parameterization of the null rays is consistent, i.e. we have to make sure that the relations $\Gamma^2_{22}|_{N_1} = \kappa_{N_1}$ and $\Gamma^1_{11}|_{N_2} = \kappa_{N_2}$ hold. This follows trivially from the vanishing of the wave-gauge vector H due to the identities

$$H^1|_{N_1} \equiv 2g^{12}\big(\kappa - \Gamma^2_{22}\big) \quad \text{and} \quad H^2|_{N_2} \equiv 2g^{12}\big(\kappa - \Gamma^1_{11}\big). \tag{3.25}$$

Similarly, the vanishing of $\overline{H}^1$ and $\overline{H}^A$ shows via (3.16) that the identification of ξ_A with the rescaled Christoffel coefficient $-2\overline{\Gamma}^2_{2A}$ on N_1 and $-2\overline{\Gamma}^1_{1A}$ on N_2 is consistent. The vanishing of $\overline{H}^1$ and $\overline{H}^2$ together with the identity (3.21) implies that on N_1 the field ζ indeed represents the value of $2\overline{g}^{AB}\overline{\Gamma}^2_{AB} + \tau\overline{g}^{22}$, and the corresponding field on N_2 represents the value of $2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}$ there.

In particular, the above provides a new and simple integration scheme for the vacuum Einstein equations, where all the metric functions are freely prescribable on $N_1 \cup N_2$:

Theorem 3.1. *Given any continuous functions* $(\kappa, \overline{g}_{\mu\nu})$ *on* $N_1 \cup N_2$ *such that*

$$g|_{N_1} = \overline{g}_{22}(\mathrm{d}x^2)^2 + 2\overline{g}_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 + 2\overline{g}_{2A}\,\mathrm{d}x^2\,\mathrm{d}x^A + \overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{3.26a}$$

$$g|_{N_2} = \overline{g}_{11}(\mathrm{d}x^1)^2 + 2\overline{g}_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 + 2\overline{g}_{1A}\,\mathrm{d}x^1\,\mathrm{d}x^A + \overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{3.26b}$$

smooth up-to-boundary on N_1 *and* N_2*, and satisfying (3.23)–(3.24) together with the vacuum constraint equations (here* $\kappa_{N_1} := \kappa|_{N_1}$*, etc)*

$$-\partial_2\tau_{N_1} + \kappa_{N_1}\tau_{N_1} - |\sigma_{N_1}|^2 - \frac{\tau_{N_1}^2}{n-1} = 0 \text{ on } N_1, \tag{3.27a}$$

$$-\partial_1\tau_{N_2} + \kappa_{N_2}\tau_{N_2} - |\sigma_{N_2}|^2 - \frac{\tau_{N_2}^2}{n-1} = 0 \text{ on } N_2, \tag{3.27b}$$

there exists a smooth metric defined on some neighbourhood of S, solution of the vacuum Einstein equations to the future of $N_1 \cup N_2$.

Note that all the conditions are necessary. To see this, let g be any metric solving the Einstein equations to the future of $N_1 \cup N_2$, with N_a characteristic. We can introduce adapted coordinates near $N_1 \cup N_2$ so that (3.26*a*) and (3.26*b*) hold. Constraints (3.27*a*) and (3.27*b*) follow then from the Einstein equations [27], while (3.23) and (3.24) follow from our calculations above.

Proof. While the main elements of the proof have already been given, to avoid ambiguities, we summarize the argument: let $(\kappa, \overline{g})$ be given as above. Set

$$\mathscr{M} := [0, \infty) \times [0, \infty) \times S,$$

where the first $[0, \infty)$ factor refers to the variable x^1, and the second to x^2. On $\mathscr{M}$ let $\hat{g}$ be the metric $\hat{g} = 2\mathrm{d}x^1\mathrm{d}x^2 + \phi_{AB}\mathrm{d}x^A\mathrm{d}x^B$, where $\phi_{AB}\mathrm{d}x^A\mathrm{d}x^B$ is a Riemannian metric on S. Let $\mathring{\overline{W}}{}^\mu$ be constructed as above. Let W^μ be any smooth extension of $\overline{W}^\mu$ to $\mathscr{M}$, and let g be the solution of the wave-map-reduced Einstein equations $R^{(H)}_{\alpha\beta} = 0$, with initial data $\overline{g}$, where

$$R^{(H)}_{\alpha\beta} := R_{\alpha\beta} - \tfrac{1}{2}(g_{\alpha\lambda}\hat{D}_\beta H^\lambda + g_{\beta\lambda}\hat{D}_\alpha H^\lambda), \tag{3.28}$$

with H^μ defined by (3.4), and where $\hat{D}$ is the Lévi-Cività covariant derivative in the metric $\hat{g}$. (It follows from [33, page 163] that $R^{(H)}_{\alpha\beta}$ is a quasi-linear, quasi-diagonal operator on g, tensor-valued, depending on $\hat{g}$, of the form

$$R^{(H)}_{\alpha\beta} \equiv -\tfrac{1}{2} g^{\lambda\mu}\hat{D}_\lambda \hat{D}_\mu g_{\alpha\beta} + \hat{f}[g, \hat{D}g]_{\alpha\beta}, \tag{3.29}$$

where $\hat{f}[g, \hat{D}g]_{\alpha\beta}$ is a tensor quadratic in $\hat{D}g$ with coefficients depending upon g, $\hat{g}$, $\mathring{W}$, $\hat{D}\hat{W}$ and $\hat{D}\mathring{W}$; the existence of solutions of this problem follows from [14].) As $\overline{H}^\mu = 0$ by construction, a standard argument shows that $H^\mu \equiv 0$, and so g is a solution of the vacuum Einstein equations in a suitable neighbourhood of S in $\mathscr{M}$. □

3.2. The light-cone

Now let us consider the same problem on a light-cone C_O with vertex O. We prescribe the metric functions $\overline{g}_{\mu\nu}$ on the cone in adapted null coordinates (cf [27]) as well as $\overline{\phi}$ and, if τ has zeros, κ (note that $\tau \equiv \frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB}$ has no zeros in a sufficiently small neighbourhood around the vertex). In this section, the notations and conventions from [27] are used again; in particular, the x^1-coordinate will be frequently denoted by r, and the light-cone is given as the surface $\{x^0 \equiv u = 0\}$.

For C_O to be a characteristic cone, we need to have $\overline{g}_{11} = 0 = \overline{g}_{1A}$ in our adapted coordinates. To end up with a Lorentzian metric, the component $\nu_0 \equiv \overline{g}_{01}$ has to be nowhere vanishing, while $\overline{g}_{AB}$ has to be a family of Riemannian metrics on S^{n-1}. We consider initial data which satisfy

$$\overline{g}_{00} = -1 + O(r^2), \quad \partial_r\overline{g}_{00} = O(r), \tag{3.30a}$$

$$\nu_0 = 1 + O(r^2), \quad \partial_r\nu_0 = O(r), \tag{3.30b}$$

$$\nu_A = O(r^3), \quad \partial_r\nu_A = O(r^2), \tag{3.30c}$$

$$\overline{g}_{AB} = r^2 s_{AB} + O_2(r^4), \quad \partial_r\partial_C\overline{g}_{AB} = 2r\partial_C s_{AB} + O_1(r^3), \tag{3.30d}$$

$$\partial_r^2\partial_C\partial_D\overline{g}_{AB} = 2\partial_C\partial_D s_{AB} + O(r^2), \tag{3.30e}$$

for small r, where $f = O_n(r^\alpha)$ means that $\partial_r^i\partial_A^\beta f = O(r^{\alpha-i})$ for $i + |\beta| \leqslant n$. The tensor s_{AB} denotes the round sphere metric.

These conditions ensure that the metric is of the same form near the vertex as in [27]. The assumptions concerning the derivatives, which are compatible with relations (4.41)–(4.51) in [27], are made to compute the behaviour of $\overline{\mathring{W}}$ near the vertex[3]: although we do not attempt to tackle the regularity problem at the vertex here, as a necessary condition we want to make sure that $\overline{\mathring{W}}$ remains bounded near the vertex, which in our adapted coordinates means

$$\overline{\mathring{W}}^0 = O(1), \quad \overline{\mathring{W}}^1 = O(1), \quad \overline{\mathring{W}}^A = O(r^{-1}).$$

In fact it turns out that with (3.30a)–(3.30e) and the subsequent assumptions on the target metric and the energy–momentum tensor, the vector $\overline{\mathring{W}}$ goes to zero.

We present the scheme for an arbitrary target metric $\hat{g}$ that satisfies the relations

$$\hat{\nu}_0 = 1 + O_1(r^2), \quad \hat{\nu}_A = O_1(r^3), \quad \overline{\hat{g}}_{00} = -1 + O(r^2), \tag{3.31a}$$

$$\partial_r\overline{\hat{g}}_{00} = O(r), \quad \overline{\hat{g}}_{AB} = r^2 s_{AB} + O_1(r^4), \tag{3.31b}$$

[3] It is conceivable that a larger class of initial data turns out to be compatible with regularity at the vertex.

$$\overline{\partial_0 \hat{g}_{11}} = O(r), \quad \overline{\partial_0 \hat{g}_{1A}} = O(r^2), \quad \overline{g}^{AB}\overline{\partial_0 \hat{g}_{AB}} = O(r). \tag{3.31c}$$

Again, these assumptions are to ensure that the behaviour of $\overline{\mathring{W}}$ can be determined at the vertex.

Additionally, we take a look at two particular target metrics: a Minkowski target $\hat{g} = \eta$ as in [27] and a target metric $\hat{g} = C$ which satisfies $\overline{C} = \overline{g}$ and which simplifies the expressions for the components of $\overline{\mathring{W}}$.

Let us now solve the constraint equations. The first constraint yields (supposing that τ has no zeros, the case where it does have zeros can be treated as the case of two transversally intersecting hypersurfaces)

$$\kappa = \frac{\partial_1 \tau + \frac{1}{n-1}\tau^2 + |\sigma|^2 + \overline{T}_{11}}{\tau} \tag{3.32}$$

and

$$\overline{\mathring{W}}^0 = -\overline{\hat{W}}^0 - \nu^0(2\kappa + \tau) - 2\partial_1 \nu^0.$$

If we assume[4]

$$\overline{T}_{11} = O(1), \quad \partial_A \overline{T}_{11} = O(1), \quad \partial_A \partial_B \overline{T}_{11} = O(1),$$

we obtain with our assumptions (3.30*a*)–(3.30*e*) and with assumptions (3.31*a*) and (3.31*b*) concerning the target metric

$$\kappa = O(r), \quad \partial_A \kappa = O(r), \quad \partial_A \partial_B \kappa = O(r),$$

and

$$\overline{\mathring{W}}^0 = O(r).$$

Let us write $\overset{\eta}{=}$ for an equality which holds when $\hat{g}$ is the Minkowski metric, with an obvious similar meaning for $\overset{C}{=}$. Then

$$\overline{\hat{W}}^0 \overset{\eta}{=} -r\overline{g}^{AB} s_{AB},$$

and also

$$\overline{\mathring{W}}^0 \overset{C}{=} 2\nu^0\left(\hat{\Gamma}^1_{11} - \kappa\right).$$

From the second constraint equation, one first determines ξ_A. Recall that this is a first-order ODE. The integration constant which arises is determined by the requirement of finiteness of ξ_A at the vertex (cf [27, section 9.2]),

$$\xi_A = 2\frac{e^{-\int_1^r (\tau - \frac{n-1}{\tilde{r}})\, d\tilde{r}}}{r^{n-1}} \int_0^r \tilde{r}^{n-1} e^{\int_1^{\tilde{r}} (\tau - \frac{n-1}{\tilde{\tilde{r}}})\, d\tilde{\tilde{r}}} \left(\tilde{\nabla}_B \sigma_A{}^B - \frac{n-2}{n-1}\partial_A \tau - \partial_A \kappa - \overline{T}_{1A}\right) d\tilde{r}. \tag{3.33}$$

If we assume

$$\overline{T}_{1A} = O(r), \qquad \partial_B \overline{T}_{1A} = O(r)$$

and employ (3.30*a*)–(3.31*c*), we find

$$\xi_A = O_1(r^2).$$

[4] These assumptions on the energy–momentum tensor, as well as those which will be made later, will hold for a tensor $T_{\mu\nu}$ which has bounded components in coordinates which are well behaved near the vertex; note that the (u, r, x^A) coordinates are singular at the vertex.

The function $\overline{\mathring{W}^A}$ can then be computed algebraically,

$$\overline{\mathring{W}^A} = \overline{g}^{AB}\xi_B + 2\nu^0\overline{g}^{AB}(\partial_1\nu_B - 2\nu_C\chi_B{}^C) - \nu_B\overline{g}^{AB}(\overline{\mathring{W}^0} + \overline{\hat{W}^0}) + \overline{g}^{BC}\tilde{\Gamma}^A_{BC} - \overline{\hat{W}^A}$$
$$= O(1),$$

where

$$\chi_A{}^B \equiv \tfrac{1}{2}\overline{g}^{BC}\partial_1\overline{g}_{AC}.$$

In particular,

$$\overline{\hat{W}^A} \overset{\eta}{=} -\frac{2}{r}\nu^0\overline{g}^{AB}\overline{g}_{0B} + \overline{g}^{BC}S^A_{BC},$$

and

$$\overline{\mathring{W}^A} \overset{C}{=} 2\overline{g}^{1A}\big(\hat{\Gamma}^1_{11} - \kappa\big) + \overline{g}^{AB}\big(2\hat{\Gamma}^1_{1B} + \xi_B\big).$$

The functions S^A_{BC} denote the Christoffel coefficients associated with the round sphere metric.

Finally, we have a first-order equation for

$$\zeta = (2\partial_1 + 2\kappa + \tau)\overline{g}^{11} + 2\overline{\mathring{W}^1} + 2\overline{\hat{W}^1}. \tag{3.34}$$

It reads

$$(\partial_1 + \kappa + \tau)\zeta + \tilde{R} + \overline{g}^{AB}\big(\tilde{\nabla}_A\xi_B - \tfrac{1}{2}\xi_A\xi_B\big) + \overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{1A}\overline{T}_{1A} + 2\nu^0\overline{T}_{01} = 0. \tag{3.35}$$

This can be integrated,

$$\zeta = \frac{e^{-\int_1^r(\kappa+\tau-\frac{n-1}{\tilde{r}})\,d\tilde{r}}}{r^{n-1}}\Bigg[c - \int_0^r \tilde{r}^{n-1}\, e^{\int_1^{\tilde{r}}(\kappa+\tau-\frac{n-1}{\tilde{\tilde{r}}})\,d\tilde{\tilde{r}}}\Bigg(\tilde{R} + \overline{g}^{AB}\tilde{\nabla}_A\xi_B$$
$$- \frac{1}{2}\overline{g}^{AB}\xi_A\xi_B + \overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{1A}\overline{T}_{1A} + 2\nu^0\overline{T}_{01}\Bigg)\,d\tilde{r}\Bigg],$$

where c is an integration constant.

Using again relations (3.30*a*)–(3.30*e*) and our assumptions on the target metric, we deduce that

$$\overline{g}^{11} = 1 + O(r^2), \qquad \partial_1\overline{g}^{11} = O(r),$$
$$\tilde{R} = (n-1)(n-2)r^{-2} + O(1).$$

Assuming that $\overline{T}_{01} = O(1)$, we find that a general solution ζ has a term of order $r^{-(n-1)}$ due to which $\overline{\mathring{W}^1}$ would not converge at the vertex. We thus set $c = 0$. That yields

$$\zeta = -(n-1)r^{-1} + O(1).$$

Inserting this result into (3.34), we end up with

$$\overline{\mathring{W}^1} = O(r).$$

For that we employed

$$\overline{\hat{W}^1} = -(n-1)r^{-1} + O(r).$$

In the special case of a Minkowski target, we have

$$\overline{\hat{W}^1} \overset{\eta}{=} \overline{\hat{W}^0} \overset{\eta}{=} -r\overline{g}^{AB}s_{AB}.$$

Moreover, we find

$$\overline{\mathring{W}^1} \overset{C}{=} \tfrac{1}{2}\zeta - \overline{g}^{AB}\hat{\Gamma}^1_{AB} - \tfrac{1}{2}\tau\overline{g}^{11} + \overline{g}^{11}\big(\hat{\Gamma}^1_{11} - \kappa\big).$$

Let us assume that the vector field $\overline{\mathring{W}^\lambda}$ can be extended to a smooth spacetime vector field $\mathring{W}^\lambda$ on the spacetime manifold $\mathscr{M}$. If we further assume, as in the case of two transversally intersecting null hypersurfaces, that the reduced Einstein equations can be complemented by well-posed evolution equations for the matter fields, for sufficiently well-behaved initial data, we obtain [34] a solution of the Cauchy problem in a future neighbourhood of the tip of the cone. The metric obtained this way solves the full Einstein equations if and only if H vanishes on C_O, as shown in sections 7.6, 9.3 and 11.3 of [27].

Let us assume now that the initial data $(\kappa, \overline{g}_{\mu\nu})$ and a target metric $\hat{g}$ have been specified. In order to prove that the wave-map gauge vector H^λ vanishes on the cone, one first establishes that it is bounded near the vertex. In our adapted coordinates, that means

$$\overline{H}^0 = O(1), \quad \overline{H}^1 = O(1), \quad \overline{H}^A = O(r^{-1}). \tag{3.36}$$

If we assume that those transverse derivatives which appear in the generalized wave-map gauge condition $\overline{H} = 0$ satisfy (compare [27] for a justification under the conditions there)[5]

$$\overline{\partial_0 g_{11}} = O(r), \quad \overline{\partial_0 g_{1A}} = O(r^2), \quad \overline{g}^{AB}\overline{\partial_0 g_{AB}} = O(r),$$

and the initial data fulfil, additional to (3.30a)–(3.30e), the relations

$$\partial_A \nu_0 = O(r^2), \quad \partial_B \nu_A = O(r^3),$$

then one finds (using (3.30a)–(3.31c)), say in vacuum,

$$\overline{H}^0 = O(r), \quad \overline{H}^1 = O(r), \quad \overline{H}^A = O(1),$$

which more than suffices for (3.36).

4. A geometric perspective

4.1. One constraint equation

Let us present a more geometric description of initial data on a characteristic surface.

A triple $(\mathscr{N}, \tilde{g}, \kappa)$ will be called a *characteristic initial-data set* if $\mathscr{N}$ is a smooth n-dimensional manifold, $n \geqslant 3$, equipped with a degenerate quadratic form $\tilde{g}$ of signature $(0, +, \ldots, +)$, as well as a connection form κ on the one-dimensional degeneracy bundle $\mathrm{Ker}\,\tilde{g}$, understood as a bundle above its own integral curves. The data are moreover required to satisfy a constraint equation, as follows.

We can always locally introduce an adapted coordinate system where $\mathrm{Ker}\,\tilde{g}$ is $\mathrm{Span}\,\partial_1$.[6] (There only remains the freedom of coordinate transformations of the form $(x^1, x^A) \mapsto (\bar{x}^1(x^1, x^A), \bar{x}^B(x^A))$.) Then the connection form κ reduces to one connection coefficient:

$$\nabla_{\partial_1}\partial_1 = \kappa\partial_1. \tag{4.1}$$

In this coordinate system, we have $\tilde{g} = \overline{g}_{AB}\,dx^A\,dx^B$. Denoting by $\overline{g}^{AB}$ the matrix inverse to $\overline{g}_{AB}$, set

$$\chi_B{}^A := \tfrac{1}{2}\overline{g}^{AC}\partial_1\overline{g}_{CB}, \qquad \tau := \chi_A{}^A. \tag{4.2}$$

Under redefinitions of the adapted coordinates, the field τ transforms as a covector, which leads to the natural covariant derivative operator

$$\nabla_1\tau := (\partial_1 - \kappa)\tau. \tag{4.3}$$

[5] Note that these transverse derivatives are obtained from the solution g of the reduced Einstein equations with initial data $\overline{g}$. Assumptions (3.36) are known to hold e.g. if one uses the wave-map gauge $\mathring{W}^\mu = 0$ near the vertex [27].

[6] This vector was denoted by ∂_r or ∂_1 in section 3.2, by ∂_1 in section 2 when considering the null hypersurface N_2 and by ∂_2 in section 2 when considering the null hypersurface N_1.

With those definitions, the *characteristic constraint equation* reads

$$\nabla_1\tau = -\chi_B{}^A\chi_A{}^B - \rho \iff \nabla_1\tau + \frac{1}{n-1}\tau^2 = -\sigma_B{}^A\sigma_A{}^B - \rho, \tag{4.4}$$

where ρ represents the component $T_{11}|_{\mathscr{N}}$ of the energy–momentum tensor of the associated spacetime $(\mathscr{M}, g)$ and, as before, σ is the trace-free part of χ.

A triple $(\mathscr{N}, \tilde{g}, \kappa)$ satisfying (4.2) with $\rho = 0$ will be called *vacuum characteristic initial data*.

An *initial-data set on a light-cone* is a characteristic data set where $\mathscr{N}$ is a star-shaped neighbourhood of the origin in $\mathbb{R}^n$ from which the origin has been removed, with the tangents to the half-rays from the origin lying in the kernel of $\tilde{g}$, and with $(\tilde{g}, \kappa)$ having specific behaviour at the origin as described e.g. in [27].

The reader is referred to [35] for a clear discussion of the geometry of null hypersurfaces, and to [36–38] for a further analysis of the objects involved.

4.2. Dim-$\mathscr{N}$ constraint equations

An alternative geometric point of view, closely related to that in [37], is a slight variation of the above, as follows: instead of considering a connection on the degeneracy bundle Ker $\widetilde{g}$, viewed as a bundle over the integral curves of Ker $\widetilde{g}$, one considers a connection on this bundle *viewed as a bundle over* $\mathscr{N}$. For this, one needs the connection coefficients κ and ξ_A, defined by the equations

$$\nabla_{\partial_1}\partial_1 = \kappa\partial_1, \quad \nabla_{\partial_A}\partial_1 = -\tfrac{1}{2}\xi_A\partial_1 + \chi_A{}^B\partial_B. \tag{4.5}$$

The coefficient κ satisfies the same constraint equation as before. The remaining coefficients ξ_A are obtained from (3.12), in notation adapted to the current setting:

$$-\frac{1}{2}(\partial_1 + \tau)\xi_A + \tilde{\nabla}_B\sigma_A{}^B - \frac{n-2}{n-1}\partial_A\tau - \partial_A\kappa = \overline{T}_{1A}. \tag{4.6}$$

The fact that this system of ODEs can be solved in a rather straightforward way (compare (3.33)) given κ and g_{AB} should not prevent one to view this equation as a constraint on the initial data.

A useful observation here is that in [37, appendix C], where it is shown that (4.4) and (4.6) can be obtained from the usual *vector constraint equation* on a spacelike hypersurface by a limiting process, when considering a family of spacelike hypersurfaces which become null in the limit.

The alert reader will note that the constraint equations (4.4) and (4.6) exhaust all tangential components of $T_{\mu\nu}\ell^\nu$. We are not aware of a geometric interpretation of equation (3.35), which involves the remaining, transverse, component of $T_{\mu\nu}\ell^\nu$. Pursuing the analogy with the spacelike Cauchy problem, one could be tempted to think of this equation as corresponding to the scalar spacelike constraint equation. However, this analogy is wrong since it is shown in [37, appendix C] that the scalar constraint equation and one of the vector constraint equations degenerate to the same single equation when a family of spacelike hypersurfaces degenerates to a characteristic one.

4.3. Uniqueness of solutions

Given a vacuum characteristic data set on a light-cone, or two vacuum characteristic data sets with a common boundary S (where some further data might have to be prescribed, as made clear in previous sections), one can impose various supplementary conditions to construct an associated spacetime metric. For example, one can redefine x^1 so that $\kappa = 0$ and impose wave-coordinate conditions or wave-map coordinate conditions in the light-cone case, to obtain the required spacetime metric, or one can prescribe the remaining metric functions as in section 3, with appropriate conditions at the tip of the light-cone or at the intersection surface. In [27, section 7.1], a scheme is presented where $g_{12}|_{\mathscr{N}}$ is prescribed, together with wave-map conditions. It is obvious that there exist further schemes which are mixtures of the above and which might be more appropriate for some specific physical situations, or for matter fields with exotic coupling to gravity.

Rendall's analysis, or that in [27], makes it clear that every vacuum characteristic data set as defined in section 4.1 leads to a unique, up to isometry, associated spacetime, either near the tip of the light-cone, or near the intersection surface S. Here uniqueness is understood locally, though again it is clear that unique maximal globally hyperbolic developments should exist in the current context.

5. Solving the constraint equation

There are several ways of solving (4.4). The aim of this section is to present those methods, in vacuum. One should keep in mind that some further specific hypotheses on the matter fields might have to be made in the schemes below for non-vacuum initial data.

5.1. Solving for κ

For any $\tilde{g}$ for which τ has no zeros, equations (4.3) and (4.4) can be solved algebraically for κ. This appears to be the most natural choice near the tip of a light-cone, where τ is nowhere vanishing.

5.2. τ and $[\overline{g}_{AB}]$ as free data

Another way of solving (4.4) is to prescribe $[\overline{g}_{AB}]$ and the mean null extrinsic curvature τ. Here one can simply choose τ to be nowhere vanishing such that (4.4) is solvable for κ. Regularity conditions on τ in the light-cone case are discussed in an appendix.

5.3. $\kappa = 0$

Rendall's proposal is to reparameterize the characteristic curves so that $\kappa = 0$; equation (4.4) can then be rewritten as a linear equation for a conformal factor, say Ω, such that $g_{AB} = \Omega^2\gamma_{AB}$, where $\gamma_{AB} = [\overline{g}_{AB}]$ is freely prescribed; compare (2.5).

5.4. κ and $[\overline{g}_{AB}]$ as free data

In some situations, it might be convenient not to assume that $\kappa = 0$, but retain a version of the conformal approach of Rendall. This requires only a few minor modifications in section 2: it suffices to replace (2.4) by (3.5), (2.6) by (3.12), (2.9) by (3.19) and (2.8) by (3.18) with $\overline{T}_{\mu\nu}$, $\overline{\mathring{W}^\mu}$ and $\overline{\hat{W}^\mu}$ set to zero. As a matter of course, the corresponding equations on N_2 have to be adjusted analogously.

By an appropriate choice of κ, i.e. by choosing an adapted parameterization of the null rays, equation (3.5), which determines τ, can sometimes be simplified; an example will be given in the following section.

5.5. $\kappa = \tau/(n-1)$

An elegant approach is due to Hayward [25] who, in space dimension $n = 3$, proposes to use a parameterization where $\kappa = \tau/(n-1)$. Then, in vacuum, (4.4) becomes a linear equation for τ, in terms of the trace-free part of χ which depends only upon the conformal class of $\tilde{g}$,

$$\partial_1 \tau + |\sigma|^2 = 0. \tag{5.1}$$

The solution τ can then be used to determine a conformal factor Ω^2 relating $\overline{g}_{AB}$ to a freely prescribed representative γ_{AB} of the conformal class,

$$\partial_1 \Omega - \frac{\Omega}{n-1}\left(\tau - \frac{1}{2}\gamma^{AB}\partial_1\gamma_{AB}\right) = 0.$$

In the case where data are given on a light-cone, one has to face the question of boundary conditions for (5.1), of the (necessary and/or sufficient) conditions on the data which will guarantee regularity at the vertex and a possible relation between those.

To address those questions, we start by comparing the $\kappa = \tau/(n-1)$-gauge with the geometric $\mathring{\kappa} = 0$ gauge. Those quantities computed in the latter gauge will be labelled by $\mathring{}$ in what follows. Both gauges are related by an angle-dependent rescaling of the coordinate r: using the transformation law of the Christoffel symbols, we find

$$\tau/(n-1) \;=\; \kappa \;=\; \Gamma^1_{11} \;=\; \frac{\partial r}{\partial \mathring{r}} \underbrace{\mathring{\Gamma}^1_{11}}_{=\mathring{\kappa}=0} + \frac{\partial r}{\partial \mathring{r}}\frac{\partial^2 \mathring{r}}{\partial r^2} \tag{5.2}$$

for $r = r(\mathring{r}, \mathring{x}^A)$. That yields

$$\mathring{r}(r) = \int^r e^{\frac{1}{n-1}\int^{r_1}\tau(r_2)\,dr_2}\, dr_1 \;=\; \int^r e^{-\frac{1}{n-1}\int^{r_1}\int^{r_2}|\sigma(r_3)|^2\,dr_3\,dr_2}\, dr_1, \tag{5.3}$$

where we have suppressed any angle dependence, and left unspecified any potential constants of integration. This defines the desired local diffeomorphism.

As an example (and to obtain some intuition for this gauge scheme), consider the flat case where $|\sigma|^2 \equiv 0$ and for which we can compute everything explicitly. There is no difficulty in determining the transformed data which satisfy $|\mathring{\sigma}|^2 \equiv 0$. The general solution of (5.1) is

$$\tau(r, x^A) = \tau_0(x^A).$$

Now we can explicitly compute (5.3),

$$\mathring{r}(r) = A^{(1)} + A^{(2)} e^{\frac{\tau_0}{n-1} r} \tag{5.4}$$

for some integration functions $A^{(i)}$, with $A^{(2)}$ and τ_0 nowhere vanishing since we seek a map $r \mapsto \mathring{r}$ which is a diffeomorphism on each generator. Then

$$\mathring{\tau}(\mathring{r}) = \left(\frac{\partial \mathring{r}}{\partial r}\right)^{-1} \tau(r(\mathring{r})) = \frac{n-1}{\mathring{r} - A^{(1)}}. \tag{5.5}$$

We choose, as usual, the affine parameter $\mathring{r}$ in such a way that $\{\mathring{r} = 0\}$ represents the vertex and such that $\mathring{\tau} = \frac{n-1}{\mathring{r}}$. This leads to $A^{(1)} = 0$. Consequently, we either have to place the vertex at $r = -\infty$ and choose a positive τ_0 or at $r = +\infty$ with a negative τ_0 (w.l.o.g. we shall prefer the first alternative). We conclude that in the $\kappa = \tau/(n-1)$-gauge, we need to prescribe initial data for all $r \in \mathbb{R}$.

The regularity condition for $\mathring{\tau}$ translated into the $\kappa = \tau/(n-1)$-gauge does not lead to any boundary conditions for τ, except for the requirement of constant sign. It determines instead the position of the vertex, which in the new coordinates is located at infinity.

Let us come back to the general case, which we tackle from the other side, namely by starting in the $\mathring{\kappa} = 0$-gauge. We use the identity $\frac{\partial^2 \mathring{r}}{\partial r^2} = -\left(\frac{\partial \mathring{r}}{\partial r}\right)^3 \frac{\partial^2 r}{\partial \mathring{r}^2}$ to rewrite (5.2),

$$\begin{aligned}
&\frac{\partial \mathring{r}}{\partial r}\frac{\mathring{\tau}}{n-1} = \frac{\tau}{n-1} = -\left(\frac{\partial \mathring{r}}{\partial r}\right)^2 \frac{\partial^2 r}{\partial \mathring{r}^2}\\
\Longleftrightarrow \quad & \frac{\partial^2 r}{\partial \mathring{r}^2} + \frac{\mathring{\tau}}{n-1}\frac{\partial r}{\partial \mathring{r}} = 0\\
\Longleftrightarrow \quad & r(\mathring{r}) = \int^{\mathring{r}} \mathrm{e}^{-\frac{1}{n-1}\int^{\mathring{r}_1}\mathring{\tau}(\mathring{r}_2)\,\mathrm{d}\mathring{r}_2}\,\mathrm{d}\mathring{r}_1.
\end{aligned} \tag{5.6}$$

This provides the inverse coordinate transformation.

Now, for a smooth metric, in adapted null coordinates which are constructed starting from normal coordinates, the generators are affinely parameterized and we have

$$\mathring{\tau} = \frac{n-1}{\mathring{r}} + O(\mathring{r}). \tag{5.7}$$

But this behaviour remains unchanged under all reparameterizations of the generators which preserve the affine parameterization as well as the position of the vertex. It follows that (5.7) holds for all smooth metrics in the gauge $\mathring{\kappa} = 0$.

From (5.6) and (5.7), we obtain

$$r(\mathring{r}) = A^{(1)} + A^{(2)} \log \mathring{r} + O(\mathring{r}^2), \quad A^{(2)} \neq 0 \;\; \forall x^A.$$

If we start in the $\mathring{\kappa} = 0$-gauge, with the vertex at $\mathring{r} = 0$, and transform into the $\kappa = \tau/(n-1)$-gauge, then, similarly to Minkowski spacetime, the vertex is shifted to, w.l.o.g., $r = -\infty$. Thus, spacetime regularity forces the vertex to be located at infinity in the $\kappa = \tau/(n-1)$-gauge.

5.6. The shear as free data

Following Christodoulou [28], we let the *second fundamental form* χ of a null hypersurface $\mathscr{N}$ with null tangent ℓ be defined as

$$\chi(X, Y) = g(\nabla_X \ell, Y), \tag{5.8}$$

where $X, Y \in T\mathscr{N}$. Choosing ℓ to be ∂_r, we then have, using [27, appendix A],

$$\begin{aligned}
\chi_{AB} &= \overline{g(\nabla_A \partial_r, \partial_B)} = \overline{g}_{\mu B}\overline{\Gamma}^\mu_{Ar} = \overline{g}_{CB}\overline{\Gamma}^C_{Ar} + \overline{g}_{uB}\overline{\Gamma}^u_{Ar}\\
&= \frac{1}{2}\partial_r \overline{g}_{AB},
\end{aligned} \tag{5.9a}$$

$$\chi_{rr} = \overline{g(\nabla_r \partial_r, \partial_r)} = 0, \tag{5.9b}$$

$$\chi_{Ar} = \overline{g(\nabla_A \partial_r, \partial_r)} = \overline{g}_{\mu r}\overline{\Gamma}^\mu_{Ar} = \overline{g}_{ur}\overline{\Gamma}^u_{rr} = 0. \tag{5.9c}$$

Let σ be the trace-free part of χ on the level sets of r:

$$\sigma_{AB} := \chi_{AB} - \frac{1}{n-1}\overline{g}^{CD}\chi_{CD}\overline{g}_{AB}\;;$$

σ is often called the *shear* tensor of $\mathscr{N}$. It has been proposed (cf, e.g., [28]) to consider σ as the free gravitational data at $\mathscr{N}$. There is an apparent problem with this proposal, because to define a trace-free tensor, one needs a conformal metric; but if a conformal class $[\widetilde{g}(r)]$ is given on $\mathscr{N}$, there does not seem to be any need to supplement this class with σ. This issue can be taken care of by working in a frame formalism, as follows.

Let $\mathscr{N}$ be an n-dimensional manifold threaded by a family of curves, which we call *characteristic curves* or *generators*. We assume moreover that each curve is equipped with a connection: if, in local coordinates, ∂_r is tangent to the curves, then we let κ denote the corresponding connection coefficient, as in (4.1).

Suppose, for the moment, that $\mathscr{N}$ is a characteristic hypersurface embedded as the submanifold $\{u = 0\}$ in a spacetime $\mathscr{M}$. Choose some local coordinates so that ∂_r is tangent to the characteristic curves of $\mathscr{N}$. Let e_a denote a basis of $T\mathscr{M}$ along $\mathscr{N}$ such that

$$\nabla_r e_a = 0. \tag{5.10}$$

Let S denote an $(n-1)$-dimensional submanifold of $\mathscr{N}$ (possibly, but not necessarily, its boundary) which intersects the generators transversally. We will further require on S that e_0 is null, and that for $a = 2, \ldots, n$, the family of vectors e_a is orthonormal. These properties will then hold along all those generators that meet S.

Let x^A be the local coordinates on S; we propagate those along the generators of $\mathscr{N}$ to a neighbourhood $\mathscr{U} \subset \mathscr{N}$ of S by requiring $\mathscr{L}_{\partial_r} x^A = 0$.

We choose $e_1 \sim \partial_r$ at S; equation (4.1) implies then that this will hold throughout $\mathscr{U}$:

$$e_1 = e_1{}^r \partial_r \text{ on } \mathscr{U}. \tag{5.11}$$

We choose the e_as, $a = 2, \ldots n$, to be tangent to S; since $T\mathscr{N}$ coincides with $e_0^\perp$, the e_as, $a = 2, \ldots n$, will remain tangent to $\mathscr{N}$:

$$e_a = e_a{}^r \partial_r + e_a{}^B \partial_B \text{ on } \mathscr{U}, \quad a = 2, \ldots, n. \tag{5.12}$$

On S, we choose the vector e_0 to be null, orthogonal to S, with

$$g(e_0, e_1) = 1. \tag{5.13}$$

Let $\{\theta^a\}_{a=0,1,\ldots,n}$, be a spacetime coframe, defined on $\mathscr{U} \subset \mathscr{N}$, dual to the frame $\{e_a\}_{a=0,1,\ldots,n}$. From what has been said, we have

$$g = \underbrace{\theta^0 \otimes \theta^1 + \theta^1 \otimes \theta^0 + \theta^2 \otimes \theta^2 + \cdots + \theta^n \otimes \theta^n}_{=:\eta_{ab}\theta^a\theta^b}. \tag{5.14}$$

By construction, the one-forms θ^a are covariantly constant along the generators of $\mathscr{N}$:

$$\nabla_r \theta^a = 0. \tag{5.15}$$

Here ∇ is understood as the covariant-derivative operator acting on one-forms.

Again by construction, θ^0 annihilates $T\mathscr{N}$; thus, $\theta^0 \sim \mathrm{d}u$ along $\mathscr{N}$:

$$\theta^0 = \theta^0{}_u \, \mathrm{d}u \text{ on } \mathscr{U}. \tag{5.16}$$

We further note that

$$\theta^a(\partial_r) = 0 \text{ on } \mathscr{U} \text{ for } a = 2, \ldots, n. \tag{5.17}$$

To see this, recall that ∂_r is orthogonal to ∂_A; hence,

$$0 = g(\partial_r, \partial_A) = \eta_{ab}\theta^a(\partial_r)\theta^b(\partial_B) = \sum_{a=2}^{N} \theta^a(\partial_r)\theta^a(\partial_B).$$

The result follows now from the fact that the $(n-1) \times (n-1)$ matrix $(\theta^a(\partial_B))_{a\geqslant 2}$ is non-degenerate.

We would like to calculate $\tilde{g} := \overline{g}_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B$ using the covectors θ^a. For this, note that (5.14) and (5.16) imply

$$\overline{g}_{AB} = \sum_{a=2}^{N} \theta^a(\partial_A)\theta^a(\partial_B). \tag{5.18}$$

Thus, to determine $\tilde{g}$ it suffices to know the components $(\theta^a{}_B := \theta^a(\partial_B))_{a\geqslant 2}$. Now, using (5.17) together with [27, appendix A], we have for $a \geqslant 2$

$$0 = \nabla_r \theta^a{}_B = \partial_r \theta^a{}_B - \overline{\Gamma}^\mu{}_{rB}\theta^a{}_\mu = \partial_r \theta^a{}_B - \tfrac{1}{2}\overline{g}^{AC}\partial_r g_{CB}\theta^a{}_A. \tag{5.19}$$

There holds thus the following evolution equation for $(\theta^a{}_B)_{a\geqslant 2}$:

$$\partial_r \theta^a{}_B - \overline{g}^{AC}\chi_{CB}\theta^a{}_A = 0, \tag{5.20}$$

where $\overline{g}^{AC}$ denotes the matrix inverse to $\sum_{a=2}^N \theta^a{}_A\theta^a{}_B$.

Let, as before, $\ell = \partial_r$ and define

$$B_{\mu\nu} := \nabla_\mu \ell_\nu. \tag{5.21}$$

Let B_{ab} denote the frame components of B:

$$B_{ab} = e_a{}^\mu e_b{}^\nu B_{\mu\nu} \quad\Longleftrightarrow\quad B_{\mu\nu} = \theta^a(\partial_\mu)\theta^b(\partial_\nu)B_{ab}. \tag{5.22}$$

It follows from definition (5.8) that χ encodes the information on components of B in directions tangent to $\mathscr{N}$:

$$\chi_{AB} = B_{AB}, \qquad \chi_{rr} = B_{rr} = 0, \qquad \chi_{Ar} = B_{Ar} = 0. \tag{5.23}$$

The key distinction between B and χ is that B has components along directions transverse to $\mathscr{N}$, while χ has not. Also note that for $a, b \geqslant 1$ the frame components B_{ab} only involve the coordinate components of $B_{\mu\nu}$ tangential to $\mathscr{N}$, so the frame formula

$$\chi_{ab} = B_{ab}, \; a, b \geqslant 1,$$

is geometrically sensible.

The last two equations in (5.23) give

$$\chi_{AC} = \sum_{a,b=2}^{n} \theta^a{}_A\theta^b{}_C B_{ab} \equiv \sum_{a,b=2}^{n} \theta^a{}_A\theta^b{}_C \chi_{ab}, \tag{5.24}$$

which allows us to rewrite (5.20) as

$$\partial_r \theta^a{}_B - \sum_{b,c=2}^{n} \overline{g}^{AC}\theta^b{}_B\theta^c{}_C\chi_{bc}\theta^a{}_A = 0. \tag{5.25}$$

Now, for $a, c \geqslant 2$,

$$\eta^{ac} = \overline{g}^{\mu\nu}\theta^a{}_\mu\theta^c{}_\nu = \overline{g}^{AC}\theta^a{}_A\theta^c{}_C, \tag{5.26}$$

which leads to

$$\partial_r \theta^a{}_B - \sum_{b,c=2}^{n} \eta^{ac}\theta^b{}_B\chi_{bc} = 0. \tag{5.27}$$

This equation leads naturally to the following picture, assuming for simplicity vacuum Einstein equations. Consider, first, two null transversely intersecting hypersurfaces $\mathscr{N}_1$ and $\mathscr{N}_2$, with $\mathscr{N}_1 \cap \mathscr{N}_2 = S$. For $a, b \geqslant 2$, let η^{ab} be 1 when a and b coincide, and 0 otherwise. In addition to κ, the gravitational data on $\mathscr{N} = \mathscr{N}_1 \cup \mathscr{N}_2$ can be encoded in a field of symmetric η-trace-free $(n-1)\times(n-1)$ matrices σ_{ab}, $a, b = 2, \ldots, n$.

$$\sigma_{ab} = \sigma_{ba}, \qquad \eta^{ab}\sigma_{ab} = 0.$$

Let τ be a solution of the equation

$$(\partial_r - \kappa)\tau + \frac{\tau^2}{n-1} + |\sigma|^2_\eta = 0, \qquad \text{where} \qquad |\sigma|^2_\eta := \eta^{ac}\eta^{bd}\sigma_{ab}\sigma_{cd}. \tag{5.28}$$

There remains the freedom to prescribe $\tau \equiv g^{AB}\chi_{AB} = \sum_{a,b=2}^{n} \eta^{ab}\chi_{ab}$ on S (one such function for each surface N_1 and N_2). Define

$$\chi_{ab} = \sigma_{ab} + \frac{\tau}{n-1}\eta_{ab}. \tag{5.29}$$

Solving (5.27) for $\theta^a{}_B$ along the generators of $\mathcal{N}_1$ and $\mathcal{N}_2$, we can calculate $\overline{g}_{AB}$ on $\mathcal{N}$ from (5.18), as long as the determinant of the matrix $(\theta^a{}_B)_{a\geqslant 2}$ does not vanish (which will be the case in a neighbourhood of S). Here one has the freedom of prescribing $\theta^a{}_B$ on S for $a \geqslant 2$. The characteristic constraint equation $R_{\mu\nu}\ell^\mu\ell^\nu = 0$ holds by construction. Indeed, the relation $\sigma_{ab} = e_a{}^A e_b{}^B \sigma_{AB}$, $a, b \geqslant 2$, can be justified in an analogous manner as equation (5.24). That gives, using (5.26), with $a, b, c, d \geqslant 2$,

$$\begin{aligned}|\sigma|^2_\eta \equiv \eta^{ac}\eta^{bd}\sigma_{ab}\sigma_{cd} &= \overline{g}^{AC}\theta^a{}_A\theta^c{}_C\overline{g}^{BD}\theta^b{}_B\theta^d{}_D e_a{}^E e_b{}^F\sigma_{EF}e_c{}^G e_d{}^H\sigma_{GH}\\ &= \overline{g}^{AC}\overline{g}^{BD}\sigma_{AB}\sigma_{CD} \equiv |\sigma|^2,\end{aligned}$$

and the assertion follows immediately.

We can now apply any of the methods described previously (e.g., Rendall's original method if $\kappa = 0$) to obtain a solution of the characteristic Cauchy problem to the future of $\mathcal{N}$.[7]

One should keep in mind the following: prescribing the data $e_a{}^A|_S$, or equivalently $\theta^a{}_B|_S$, determines the metric $g_{AB}|_S$ on S. There is a supplementary freedom of making an $O(n-1)$-rotation of the frame:

$$e_a{}^A|_S(x^A) \mapsto \omega^b{}_a(x^A)e_b{}^A|_S(x^A),$$

where the $\omega^b{}_a(x^A)$s are $O(n-1)$-matrices. Any such rotation needs to be reflected in the σ_{ab}s:

$$\sigma_{ab}(r, x^A) \mapsto \omega^c{}_a(x^A)\omega^d{}_b(x^A)\sigma_{cd}(r, x^A).$$

So in this construction, σ_{ab} undergoes gauge transformations which are constant along the generators, and are thus non-local in this sense.

In the case of a light-cone, the above construction can be implemented by first choosing an orthonormal coframe $\mathring{\phi}^a \equiv \mathring{\phi}^a{}_A\, dx^A$, $a \geqslant 2$, for the unit round metric $s_{AB}\, dx^A\, dx^B$ on S^{n-1}. The solutions $\theta^a := r\phi^a \equiv r\phi^a{}_A\, dx^A$, $a \geqslant 2$, of (5.27) are then chosen as the unique solutions asymptotic to $r\mathring{\phi}_a$. It would be of interest to settle the question, ignored here, of sufficient and necessary conditions on σ_{ab} so that the resulting initial data on the light-cone can be realized by restricting a smooth spacetime metric to the light-cone.

5.7. Friedrich's free data

In [19, 39], Friedrich proposes alternative initial data on $\mathcal{N}$, based on the identity[8]

$$\partial_r^2\overline{g}_{AB} - \kappa\partial_r\overline{g}_{AB} - \tfrac{1}{2}\overline{g}^{CD}\partial_r\overline{g}_{AC}\partial_r\overline{g}_{BD} = -2\overline{R}_{ArBr}; \tag{5.30}$$

equivalently

$$\partial_r B_{AB} - \kappa B_{AB} - \overline{g}^{CD}B_{AC}B_{BD} = -\overline{R}_{ArBr}. \tag{5.31}$$

Equation (5.30) shows that the component $\overline{R}_{ArBr}$ of the Riemann tensor can be calculated in terms of κ and the field $\overline{g}_{AB}$.

[7] Note that prescribing σ_{ab} and κ is equivalent to prescribing χ_{ab} as primary data. If we assume for simplicity that the η-trace of χ_{ab} is nowhere vanishing, one can then determine κ from (5.28) and continue as described in the paragraph following (5.29). One could also consider this procedure in an adapted frame: from χ_{AB} one determines successively $\overline{g}_{AB}$, τ and κ. However, since the $\overline{g}_{AB}$-trace of χ is then not known *a priori*, it is not clear how to satisfy constraint (5.28).

[8] This equation reduces to the usual Riccati equation (cf, e.g., [35]) satisfied by the null extrinsic curvature tensor when $\kappa = 0$. We are grateful to José-Maria Martín-Garcia for providing the general version of that equation.

Alternatively, given the fields $\overline{C}_{ArBr}$, $\rho \equiv \overline{T}_{rr}$ and κ, together with suitable boundary conditions, one can solve (5.30) to determine $\overline{g}_{AB}$. So in spacetime dimension 4, Friedrich [19] proposes to use frame components of $\overline{C}_{ArBr}$ as the free data on $\mathscr{N}$. There is, however, a problem, in that $\overline{C}_{ArBr}$ is traceless,

$$0 = \overline{g}^{\mu\nu}\overline{C}_{\mu r\nu r} = \overline{g}^{AC}\overline{C}_{ArCr}.$$

So this condition has to be built-in into the formalism. But, as in the previous section, the tracelessness condition does not seemingly make sense unless the inverse metric g^{AB}, or at least its conformal class, is known.

This issue can again be taken care of by a frame formalism, whatever the dimension, as follows: let the orthonormal frame e_a, $a = 0, \ldots, n$, and its dual coframe θ^a be as in the last section. The property that the frame is parallel along the generators implies

$$\partial_r e_a{}^B = -\Gamma^B_{rC} e_a{}^C = -\overline{g}^{BA} B_{AC} e_a{}^C \text{ for } a \geqslant 2. \tag{5.32}$$

We then have, for $a, b \geqslant 2$,

$$\begin{aligned}
\partial_r B_{ab} &= \partial_r\big(e_a{}^\mu e_b{}^\nu B_{\mu\nu}\big)\\
&= \partial_r\big(e_a{}^A\big) e_b{}^C B_{AC} + e_a{}^A \partial_r\big(e_b{}^C\big) B_{AC} + e_a{}^A e_b{}^C \partial_r B_{AC}\\
&= -\overline{g}^{AD} B_{DE} e_a{}^E e_b{}^C B_{AC} - e_a{}^A \overline{g}^{CD} B_{DE} e_b{}^E B_{AC}\\
&\quad + e_a{}^A e_b{}^C (\kappa B_{AC} + \overline{g}^{ED} B_{AE} B_{CD} - \overline{R}_{ArCr})\\
&= -e_a{}^A e_b{}^E \overline{g}^{CD} B_{AC} B_{DE} + \kappa B_{ab} - e_a{}^A e_b{}^B \overline{R}_{ArBr}.
\end{aligned} \tag{5.33}$$

Using

$$\overline{g}^{CD} = \eta^{cd} e_c{}^C e_d{}^D = \sum_{c,d=2}^{n} \eta^{cd} e_c{}^C e_d{}^D,$$

we conclude that

$$\partial_r B_{ab} = -\sum_{c,d=2}^{n} \eta^{cd} B_{ac} B_{db} + \kappa B_{ab} - e_a{}^A e_b{}^B \overline{R}_{ArBr}.$$

From the definition of the Weyl tensor in dimension $n+1$,

$$\begin{aligned}
C_{\mu\nu\sigma\rho} &:= R_{\mu\nu\sigma\rho} - \frac{1}{n-1}\big(g_{\mu\sigma}R_{\nu\rho} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho} + g_{\nu\rho}R_{\mu\sigma}\big)\\
&\quad + \frac{1}{n(n-1)} R(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}),
\end{aligned} \tag{5.34}$$

we find

$$\overline{C}_{ArBr} = \overline{R}_{ArBr} - \frac{1}{n-1}\overline{g}_{AB}\overline{R}_{rr}.$$

For $a, b \geqslant 2$, let

$$\psi_{ab} := e_a{}^A e_b{}^B \overline{C}_{ArBr}$$

represent the components of $\overline{C}_{ArBr}$ in the current frame. Then ψ_{ab} is symmetric, with vanishing η-trace. We finally obtain the following equation for $B_{ab} \equiv \chi_{ab}$, $a, b \geqslant 2$:

$$(\partial_r - \kappa)\chi_{ab} = -\sum_{c,d=2}^{n} \eta^{cd}\chi_{ac}\chi_{db} - \psi_{ab} - \frac{1}{n-1}\eta_{ab}\overline{T}_{rr}. \tag{5.35}$$

This equation shows that (κ, ψ_{ab}) can be used as the free data for the gravitational field: indeed, given κ, ψ_{ab} and the component $\overline{T}_{rr}$ of the energy–momentum tensor, we can integrate

(5.35) to obtain χ_{ab}. Note that by taking the η-trace of (5.35), one recovers constraint (5.28) (here with $\overline{T}_{rr}$ possibly non-vanishing).

In the case of two transverse hypersurfaces, the integration leaves the freedom of prescribing two tensors χ_{ab} on S, one corresponding to N_1 and another to N_2. Then one proceeds as in the previous section to construct the remaining data on the initial surfaces, keeping in mind the further freedom to choose $\theta^a{}_B|_S$, $a \geqslant 2$, on S.

On a light-cone, equation (5.35) should be integrated with vanishing data at the vertex.

Acknowledgments

PTC acknowledges useful discussions with Yvonne Choquet-Bruhat, Helmut Friedrich, Jacek Jezierski, Jerzy Kijowski and José Maria Martín-García.

Appendix. The expansion τ and the location of the vertex

The Hayward gauge condition of section 5.5 has led us to the interesting conclusion that in some gauge choices, the vertex of the light-cone will be located at infinity. This raises the following question: Under which conditions is the hypothesis, that the vertex is located at $r = 0$, consistent with natural boundary conditions at the tip of the light-cone?

We start with the derivation of a necessary condition which needs to be imposed on the behaviour of the initial data in the $\kappa = \tau/(n-1)$-gauge near the vertex in order to be compatible with regularity. It is known [27] that in a $\mathring{\kappa} = 0$-gauge arising from normal coordinates, the initial data have to be of the form

$$\mathring{\gamma}_{AB} = \mathring{r}^2 s_{AB} + h_{AB}, \quad \text{where } h_{AB} = O_1(\mathring{r}^4). \tag{A.1}$$

Recall that we decorate with a circle the quantities corresponding to the gauge $\mathring{\kappa} = 0$. We want to work out how such data look like in the $\kappa = \tau/(n-1)$-gauge. The coordinate transformation (5.6), which defines a local diffeomorphism as long as $A^{(2)}$ does not change sign, reads

$$\begin{aligned} r(\mathring{r}) &= A^{(1)} + A^{(2)} \log \mathring{r} + f_h \\ &= \log \mathring{r} + f_h, \end{aligned} \tag{A.2}$$

where $f_h = O_1(\mathring{r}^2)$ is determined by h_{AB} and where $A^{(i)}$ are integration functions. Here we have set $A^{(1)} = 0$ and $A^{(2)} = 1$, so that the r-coordinate in the $\kappa = \tau/(n-1)$-gauge is completely fixed, once $\mathring{r}$ has been chosen.

From (A.2), we extract the behaviour of the inverse transformation

$$\mathring{r}(r) = \mathrm{e}^r + g_h, \qquad g_h = O_1(\mathrm{e}^{3r})$$

(where the symbol O in connection with the r-coordinate refers to the limit $r \to -\infty$). Now we can compute the overall form of the initial data,

$$\gamma_{AB}(r) = \mathring{\gamma}_{AB}(\mathring{r}(r)) = \mathrm{e}^{2r} s_{AB} + k_{AB}, \quad \text{where } k_{AB} = O_1(\mathrm{e}^{4r}). \tag{A.3}$$

This implies

$$|\sigma|^2 = -\frac{1}{4}\left(\partial_1\gamma^{AB}\partial_1\gamma_{AB} + \frac{(\gamma^{AB}\partial_1\gamma_{AB})^2}{n-1}\right) = O(\mathrm{e}^{2r}). \tag{A.4}$$

Note that, in contrast to the $\mathring{\kappa} = 0$-gauge, τ remains bounded at the vertex for regular light-cone data of the form (A.3), due to (5.1) and (A.4).

Next, let us show that a bounded τ can only be compatible with regularity when the vertex is located at infinity. For definiteness, we consider the initial data $\overline{g}_{\mu\nu}$ with nowhere vanishing

$\tau \equiv \frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB}$, within the scheme of section 3. Then κ is computed algebraically via (3.7) (and depends on the initial data). Note that at this stage, τ is a known function of r which can be regarded as 'gauge part' of the initial data.

By calculations similar to those in (5.2) and (5.3), we can then obtain the coordinate $\mathring{r}$ relevant to the $\mathring{\kappa} = 0$-gauge:

$$\mathring{r}(r) = \int^r e^{\int^{r_1} \kappa(r_2)\,\mathrm{d}r_2}\,\mathrm{d}r_1, \tag{A.5}$$

and transform all the fields to this gauge.

We have the identity

$$\tau(r) = \frac{\partial \mathring{r}}{\partial r}\mathring{\tau}(\mathring{r}(r)), \quad \text{where} \quad \mathring{\tau} = \frac{n-1}{\mathring{r}} + O(\mathring{r}), \tag{A.6}$$

since we assume regular light-cone data. We consider the maximal range of $\mathring{r}$, near $\mathring{r} = 0$, where $\mathring{\tau}$ is positive. It follows from (A.5) that $\mathring{r} \mapsto r(\mathring{r})$ is monotone there. Let us assume that this function is strictly increasing (the decreasing case is handled in a similar way), and let (R_1, R_2) denote the corresponding range of r, with $-\infty \leqslant R_1 < R_2 \leqslant \infty$. Then τ is positive on (R_1, R_2) by (A.6). If we choose $R_1 < r_0 < R_2$ such that $r^{-1}(r_0) = \mathring{r}_0 > 0$, from the last equation we find, for some (x^B-dependent) constant A,

$$\int_{r_0}^{r(\mathring{r})} \tau\,\mathrm{d}\tilde{r} = \int_{\mathring{r}_0}^{\mathring{r}} \left(\frac{n-1}{\tilde{\mathring{r}}} + O(\tilde{\mathring{r}})\right)\mathrm{d}\tilde{\mathring{r}} = A + (n-1)\log\mathring{r} + O(\mathring{r}^2).$$

The right-hand side diverges to minus infinity at the vertex $\mathring{r} = 0$, which is mapped to R_1. This gives

$$\int_{R_1}^{r_0} \tau\,\mathrm{d}\tilde{r} = +\infty. \tag{A.7}$$

We conclude that any gauge in which τ is bounded will force the vertex to lie at infinity, for initial data which can be realized by a smooth spacetime metric.

As another application of (A.7), we reconsider the $\kappa = \tau/(n-1)$-gauge. Let us denote by $\tau_0(x^A)$ the integration function which arises in the associated constraint equation $\partial_1\tau + |\sigma|^2 = 0$. The exponential decay of $|\sigma|^2$ at a regular vertex, cf equation (A.4), is compatible with (A.7) only if τ_0 is bounded away from zero.

Finally, consider initial data on $(0, \infty)$ with

$$\overline{g}_{AB} = r^2 s_{AB} + O_1(r^4),$$

as in section 3.2. The function τ satisfies then

$$\tau = \frac{n-1}{r} + O(r), \tag{A.8}$$

which is, not unexpectedly, fully compatible with (A.7).

References

[1] Choquet-Bruhat Y and York J W 1979 The Cauchy problem *General Relativity and Gravitation—The Einstein Centenary* ed A Held (New York: Plenum) pp 99–160

[2] Bartnik R and Isenberg J 2004 The constraint equations *The Einstein Equations and the Large Scale Behavior of Gravitational Fields* (Basel: Birkhäuser) pp 1–38

[3] Friedrich H and Nagy G 1998 The initial boundary value problem for Einstein's vacuum field equation *Commun. Math. Phys.* **201** 619–55

[4] Kreiss H-O, Reula O, Sarbach O and Winicour J 2009 Boundary conditions for coupled quasilinear wave equations with application to isolated systems *Commun. Math. Phys.* **289** 1099–129

[5] Friedrich H 1995 Einstein equations and conformal structure: existence of anti-de Sitter-type space-times *J. Geom. Phys.* **17** 125–84

[6] Sarbach O and Tiglio M 2012 Continuum and discrete initial-boundary-value problems and Einstein's field equations *Living Rev. Rel.* to appear (arXiv:1203.6443 [gr-qc])

[7] Friedrich H and Rendall A 2000 The Cauchy problem for the Einstein equations *Einstein's Field Equations and Their Physical Implications* (*Lecture Notes in Phys.* vol 540) ed B Schmidt (Berlin: Springer) pp 127–223

[8] Fourès-Bruhat Y 1952 Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires *Acta Math.* **88** 141–225

[9] Newman E T and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients *J. Math. Phys.* **3** 566–78

[10] Bondi H, van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity: VII. Waves from axi-symmetric isolated systems *Proc. R. Soc.* A **269** 21–52

[11] Penrose R 1965 Zero rest mass fields including gravitation *Proc. R. Soc.* A **284** 159–203

[12] Sachs R K 1962 On the characteristic initial value problem in gravitational theory *J. Math. Phys.* **3** 908–14

[13] Dautcourt G 1963 Zum charakteristischen Anfangswertproblem der Einsteinschen Feldgleichungen *Ann. Phys., Lpz.* **12** 302–24

[14] Rendall A D 1990 Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations *Proc. R. Soc.* A **427** 221–39

[15] Cagnac F 1966 Problème de Cauchy sur les hypersurfaces caractéristiques des équations d'Einstein du vide *C. R. Acad. Sci. Paris* A-B **262** 1356–9

[16] Cagnac F 1966 Problème de Cauchy sur les hypersurfaces caractéristiques des équations d'Einstein du vide *C. R. Acad. Sci. Paris* A-B **262** 1488–91

[17] Christodoulou D and Müller zum Hagen M 1981 Problème de valeur initiale caractéristique pour des systèmes quasi linéaires du second ordre *C. R. Acad. Sci., Paris* I **293** 39–42

[18] Penrose R 1980 Null hypersurface initial data for classical fields of arbitrary spin and for general relativity *Gen. Rel. Grav.* **12** 225–64

[19] Friedrich H 1981 On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations *Proc. R. Soc.* A **375** 169–84

[20] Friedrich H 1981 The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system *Proc. R. Soc.* A **378** 401–21

[21] Damour T and Schmidt B 1990 Reliability of perturbation theory in general relativity *J. Math. Phys.* **31** 2441–53

[22] Müller zum Hagen H and Seifert H-J 1977 On characteristic initial-value and mixed problems *Gen. Rel. Grav.* **8** 259–301

[23] Caciotta G and Nicolò F 2005 Global characteristic problem for Einstein vacuum equations with small initial data: I. The initial data constraints *J. Hyp. Diff. Eqns* **2** 201–77 (arXiv:gr-qc/0409028)

[24] Caciotta G and Nicolò F 2010 On a class of global characteristic problems for the Einstein vacuum equations with small initial data *J. Math. Phys.* **51** 102503 (arXiv:gr-qc/0608038)

[25] Hayward S A 1993 The general solution to the Einstein equations on a null surface *Class. Quantum Grav.* **10** 773–8

[26] Choquet-Bruhat Y 1971 Problème de Cauchy pour le système intégro différentiel d'Einstein–Liouville *Ann. Inst. Fourier (Grenoble)* **21** 181–201

[27] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions *Ann. Henri Poincare* **12** 419–82 (arXiv:1006.4467 [gr-qc])

[28] Christodoulou D 2008 *The Formation of Black Holes in General Relativity* (Zürich: EMS Publishing House) (arXiv:0805.3880 [gr-qc])

[29] Chruściel P T and Jezierski J 2012 On free general relativistic initial data on the light cone *J. Geom. Phys.* **62** 578–93 (arXiv:1010.2098 [gr-qc])

[30] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations *Proc. Complex Analysis and Dynamical Systems IV (Nahariya, May 2009), Contemp. Math.* **554** 73–81 (arXiv:1006.5558 [gr-qc])

[31] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2010 An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations with near-round analytic data *Proc. Petrov 2010 Anniversary Symp. on General Relativity and Gravitation (Kazan)* (arXiv:1012.0777 [gr-qc])

[32] Tadmon C 2011 The Goursat problem for the Einstein–Vlasov system: I. The initial data constraints, arXiv:1109.6844 [gr-qc]

[33] Choquet-Bruhat Y 2009 *General Relativity and the Einstein Equations* (Oxford: Oxford University Press)

[34] Dossa M 2003 Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang–Mills–Higgs *Ann. Henri Poincare* **4** 385–411

[35] Galloway G J 2000 Maximum principles for null hypersurfaces and null splitting theorems *Ann. Henri Poincare* **1** 543–67

[36] Jezierski J, Kijowski J and Czuchry E 2000 Geometry of null-like surfaces in general relativity and its application to dynamics of gravitating matter *Rep. Math. Phys.* **46** 399–418

[37] Jezierski J, Kijowski J and Czuchry E 2002 Dynamics of a self-gravitating lightlike matter shell: a gauge-invariant Lagrangian and Hamiltonian description *Phys. Rev.* D **65** 064036

[38] Jezierski J 2004 Geometry of null hypersurfaces *Proc. 7th Hungarian Relativity Workshop, 2003, Relativity Today* ed I Racz (Budapest: Akademiai) (arXiv:gr-qc/0405108)

[39] Friedrich H 1986 On purely radiative space-times *Commun. Math. Phys.* **103** 35–65

Conformally covariant systems of wave equations and their equivalence to Einstein's field equations*

Tim-Torben Paetz†

Gravitational Physics, University of Vienna
Boltzmanngasse 5, 1090 Vienna, Austria

May 23, 2014

Abstract

We derive, in $3+1$ spacetime dimensions, two alternative systems of quasi-linear wave equations, based on Friedrich's conformal field equations. We analyse their equivalence to Einstein's vacuum field equations when appropriate constraint equations are satisfied by the initial data. As an application, the characteristic initial value problem for the Einstein equations with data on past null infinity is reduced to a characteristic initial value problem for wave equations with data on an ordinary light-cone.

PACs numbers: 04.20.Ex

Contents

*Preprint UWThPh-2013-15.
†E-mail: Tim-Torben.Paetz@univie.ac.at

1 Introduction

1.1 Asymptotic flatness

In general relativity there is the endeavour to characterize those spacetimes which one would regard as being "asymptotically flat", possibly merely in certain (null) directions. Spacetimes which possess this property would be well-suited to describe e.g. purely radiative spacetimes or isolated gravitational systems. However, due to the absence of a non-dynamical background field this is an intricate issue in general relativity. In [31, 32] (see e.g. [26] for an overview) R. Penrose proposed a geometric approach to resolve this problem: The starting point is a 3 + 1-dimensional spacetime $(\tilde{\mathscr{M}}, \tilde{g})$, the *physical spacetime*. It then proves fruitful to introduce a so-called *unphysical spacetime* $(\mathscr{M}, g)$ into which (a part of) $(\tilde{\mathscr{M}}, \tilde{g})$ is conformally embedded,

$$\tilde{g} \overset{\phi}{\mapsto} g := \Theta^2 \tilde{g} \;, \quad \tilde{\mathscr{M}} \overset{\phi}{\hookrightarrow} \mathscr{M} \;, \quad \Theta|_{\phi(\tilde{\mathscr{M}})} > 0 \;.$$

The part of $\partial\phi(\tilde{\mathscr{M}})$ where the conformal factor Θ vanishes can be interpreted as representing infinity of the original, physical spacetime, for the physical affine parameter diverges along null geodesics when approaching this part of the boundary. The subset $\{\Theta = 0 \ , \mathrm{d}\Theta \neq 0\} \subset \partial\phi(\tilde{\mathscr{M}})$ is called *Scri*, denoted by $\mathscr{I}$. Large classes of solutions of the Einstein equations (with vanishing cosmological constant) possess a $\mathscr{I}$ which forms a smooth null hypersurface in $(\mathscr{M}, g)$, on which null geodesics in $(\mathscr{M}, g)$ acquire end-points. The hypersurface $\mathscr{I}$ is therefore regarded as providing a representation of null infinity.

Penrose's proposal to distinguish those spacetimes which have an "asymptotically flat" structure [in certain null directions] is to require that the unphysical metric tensor g extends smoothly across [a part of] $\mathscr{I}$.[1] The idea is that such a smooth conformal extension is possible whenever the gravitational field has an appropriate "asymptotically flat" fall-off behaviour in these directions.

Null infinity can be split into two components, past and future null infinity $\mathscr{I}^-$ and $\mathscr{I}^+$, which are generated by the past and future endpoints of null geodesics in $\mathscr{M}$, respectively. If the spacetime is further supposed to be asymptotically flat in all spacelike directions, one may require the existence of a point i^0, representing *spacelike infinity*, where all the spacelike geodesics meet. However, i^0 cannot be assumed to be smooth (it cannot even assumed to be C^1 [3]).

In this work we are particularly interested in spacetimes (and the construction thereof) which, at sufficiently early times, possess a conformal infinity which is similar to that of Minkowski spacetime. By that we mean that (a part of) $(\tilde{\mathscr{M}}, \tilde{g})$ can be conformally mapped into an unphysical spacetime, where all timelike geodesics originate from one regular point, which represents *past timelike infinity*, denoted by i^-; moreover, we assume that, at least sufficiently close to i^-, a regular $\mathscr{I}^-$ exists and is generated by the null geodesics emanating from i^-, i.e. forms the future null cone of i^-, denoted by $C_{i^-} := \mathscr{I}^- \cup \{i^-\}$. By the term "regular" we mean that the conformally rescaled metric g, and also the rescaled Weyl tensor, admit smooth extensions. In fact, in $3+1$ dimensions the extendability assumption across $\mathscr{I}$ on the rescaled Weyl tensor is automatically satisfied in the current setting. At i^- this assumption will be dropped in Section 6. Purely radiative spacetimes are expected to possess such a conformal structure [20].

It is an important issue to understand the interplay between the geometric concept of asymptotic flatness and the Einstein equations, and whether all relevant physical systems are compatible with the notion of a regular conformal infinity. There are various results indicating that this is a reasonable concept, cf. [1, 2, 10, 21, 28, 32] and references given therein. An open issue is to characterize the set of asymptotically Euclidean initial data on a spacelike hypersurface which lead to solutions of Einstein's field equations which are "null asymptotically flat".

Since we have a characteristic initial value problem at C_{i^-} in mind, we want to avoid too many technical assumptions which might lead to a more reasonable (and rigid) notion of asymptotic flatness, asymptotic simplicity, etc. (cf. e.g. [27]). In a nutshell, we are concerned with solutions of the vacuum Einstein equations (with vanishing cosmological constant) which admit a regular null cone at past timelike infinity, at least near i^-.

[1] One may also think of weaker requirements here.

1.2 Conformal field equations

Due to the geometric construction outlined above, the asymptotic behaviour of the gravitational field can be analysed in terms of a local problem in a neighbourhood of $\mathscr{I}$ (as well as $i^{\pm}$ and i^0). However, the vacuum Einstein equations, regarded as equations for the unphysical metric g, are (formally) singular at conformal infinity (set $\Box_g := \nabla^\mu\nabla_\mu$),

$$\begin{aligned}&\tilde{R}_{\mu\nu}[\tilde{g}] = \lambda\tilde{g}_{\mu\nu} \quad \Longleftrightarrow \\ &R_{\mu\nu}[g] + 2\Theta^{-1}\nabla_\mu\nabla_\nu\Theta + g_{\mu\nu}\big(\Theta^{-1}\Box_g\Theta - 3\Theta^{-2}\nabla^\sigma\Theta\nabla_\sigma\Theta\big) = \lambda\Theta^{-2}g_{\mu\nu}\,,\end{aligned} \tag{1.1}$$

where the conformal factor Θ is assumed to be some given (smooth) function. The system (1.1) does therefore not seem to be convenient to study unphysical spacetimes $(\mathscr{M}, g)$ with $\Theta^{-2}g$ being a solution of the Einstein equations away from conformal infinity. Serendipitously, H. Friedrich [16, 17, 23] was able to extract a system, the *conformal field equations*, which does remain regular even if Θ vanishes, and which is equivalent to the vacuum Einstein equations wherever Θ is non-vanishing.

In a suitable gauge the propagational part of the conformal field equations implies, in $3+1$ dimensions, a symmetric hyperbolic system, the *reduced conformal field equations*. Thus equipped with some nice mathematical properties Friedrich's equations provide a powerful tool to analyse the asymptotic behaviour of those solutions of the Einstein equations which admit an appropriate conformal structure at infinity.

1.3 Characteristic initial value problems

The characteristic initial value problem in general relativity provides a tool to construct systematically general solutions of Einstein's field equations. An advantage in comparison with the spacelike Cauchy problem is that the constraint equations can be read as a hierarchical system of ODEs, which is much more convenient to deal with. In fact, one may think of several different types of (asymptotic) characteristic initial value problems, which we want to recall briefly.

One possibility is to take two transversally intersecting null hypersurfaces as initial surface. This problem was studied by Rendall [34] who established well-posedness results for quasi-linear wave equations as well as for symmetric hyperbolic systems in a neighbourhood of the cross-section of these hypersurfaces. Using a harmonic reduction of the Einstein equations he then applied his results to prove well-posedness for the Einstein equations.

Another approach is to prescribe data on a light-cone. There is a well-posedness result for quasi-linear wave equations near the tip of a cone available which is due to Cagnac [4] and Dossa [13]. A crucial assumption in their proof is that the initial data are restrictions to the light-cone of smooth[2] spacetime fields. Well-posedness of the Einstein equations was investigated in a series of papers [5–7] by Choquet-Bruhat, Chruściel and Martín-García, and by Chruściel [9]. The authors impose a wave-map gauge condition to obtain a system of wave equations to which the Cagnac-Dossa theorem is applied. A main

[2]There is a version for finite differentiability, but here we restrict attention to the smooth case.

difficulty, in the most comprehensive case treated in [9], is to make sure that the Cagnac-Dossa theorem is indeed applicable. For that one needs to make sure that the initial data for the reduced Einstein equations, which are constructed from suitable free data as solution of the constraint equations, can be extended to smooth spacetime fields. One then ends up with the result that these free data determine a unique solution (up to isometries) in some neighbourhood of the tip of the cone C_O, intersected with $J^+(C_O)$.

A third important case arises when the initial surface is, again, given by two transversally intersecting null hypersurfaces, but now in the unphysical spacetime and with one of the hypersurfaces belonging to $\mathscr{I}$. This issue was treated by Friedrich [18], who proved well-posedness for analytic data, and by Kánnár [28], who extended Friedrich's result to the smooth case. The basic idea for the proof is to exploit the fact that the reduced conformal field equations form a symmetric hyperbolic system to which Rendall's local existence result is applicable.

The case we have in mind is when the initial surface is given in the unphysical spacetime by the light-cone C_{i^-} emanating from past timelike infinity i^-. In order to construct systematically solutions of Einstein's field equations which are compatible with Penrose's notion of asymptotic flatness and a regular i^-, one would like to prescribe data on C_{i^-} and predict existence of a solution of Einstein's equations off C_{i^-} by solving an appropriate initial value problem. One way to establish well-posedness near the tip of the cone is to mimic the analysis in [5, 9]. To do that, one needs a system of wave equations which, when supplemented by an appropriate set of constrain equations, is equivalent to the vacuum Einstein equations wherever Θ is non-vanishing and which remains regular when Θ vanishes. Based on a conformal system of equations due to Choquet-Bruhat and Novello [8], such a regular system of wave equations was employed by Dossa [14] who states a well-posedness result for suitable initial data for which, however, it is not clear how they can be constructed, nor to what extent his system of wave equations is equivalent to the Einstein equations.

The purpose of this paper is to derive two such systems of wave equations in $3+1$-spacetime dimensions, which we will call *conformal wave equations*, and prove equivalence to Friedrich's conformal system for solutions of the characteristic initial value problem with initial surface C_{i^-} which satisfy certain constraint equations on C_{i^-}. Our first system will use the same set of unknowns as Friedrich's *metric conformal field equations* [23], while the second system will employ the Weyl and the Cotton tensor rather than the rescaled Weyl tensor (and might be advantageous in view of the construction of solutions with a rescaled Weyl tensor which diverges at i^-). The construction of initial data to which the Cagnac-Dossa theorem is applicable, and thus a well-posedness proof of the Cauchy problem with data on the C_{i^-}-cone, is accomplished in [12, 25].

Apart from the application to tackle the characteristic initial value problem with data on C_{i^-}, a regular system of wave equations might be interesting for numerics, as well [29].

1.4 Structure of the paper

In Section 2 we recall the metric conformal field equations and address the gauge freedom inherent to them. In Section 3 we derive the first system of conformal wave equations, (3.11)-(3.15), and prove equivalence to the conformal field equations and consistency with the gauge condition under the assumption that certain relations hold initially. In Section 4 we derive the constraint equations induced by the conformal field equations on C_{i^-} in adapted coordinates and imposing a generalized wave-map gauge condition. We then focus on the case of a light-cone with vertex at past timelike infinity to verify in Section 5 that the hypotheses needed for the equivalence theorem of Section 3 are indeed satisfied, supposing that the initial data fulfill the constraint equations (5.6)-(5.16). Our main result, Theorem 5.1, states that a solution of the characteristic initial value problem for the conformal wave equations, with initial data on C_{i^-} which have been constructed as solutions of the constraint equations, is also a solution of the conformal field equations in wave-map gauge and vice versa. In Section 6 we then derive an alternative system of wave equations, (6.9)-(6.14), and study equivalence to the conformal field equations, supposing that certain constraint equations, namely (6.52)-(6.65), are satisfied, cf. Theorem 6.5. In Section 7 we briefly compare both systems of wave equations and give a short summary. We conclude the article by reviewing some basic properties of cone-smooth functions, which are utilized to prove a lemma stated in Section 2.

Throughout this work we restrict attention to 3+1 dimensions, cf. footnote 7.

2 Friedrich's conformal field equations and gauge freedom

2.1 Metric conformal field equations (MCFE)

As indicated above, the vacuum Einstein equations themselves do not provide a nice evolution system near infinity and are therefore not suitable to tackle the issue at hand, namely to analyse existence of a solution to the future of C_{i^-}. Nonetheless, they permit a representation which does not contain factors of Θ^{-1} and which is regular everywhere [16, 17, 23]. Due to this property the Einstein equations are called *conformally regular*.

The curvature of a spacetime is measured by the Riemann curvature tensor $R_{\mu\nu\sigma}{}^{\rho}$, which can be decomposed into the trace-free *Weyl tensor* $W_{\mu\nu\sigma}{}^{\rho}$, invariant under conformal transformations, and a term which involves the *Schouten tensor* $L_{\mu\nu}$,

$$R_{\mu\nu\sigma}{}^{\rho} = W_{\mu\nu\sigma}{}^{\rho} + 2(g_{\sigma[\mu}L_{\nu]}{}^{\rho} - \delta_{[\mu}{}^{\rho}L_{\nu]\sigma}) \ . \tag{2.1}$$

The Schouten tensor is defined in terms of the Ricci tensor $R_{\mu\nu}$,

$$L_{\mu\nu} := \frac{1}{2}R_{\mu\nu} - \frac{1}{12}Rg_{\mu\nu} \ . \tag{2.2}$$

The Weyl tensor is usually considered to represent the radiation part of the gravitational field. Let us further define the *rescaled Weyl tensor*

$$d_{\mu\nu\sigma}{}^{\rho} := \Theta^{-1}W_{\mu\nu\sigma}{}^{\rho} \ , \tag{2.3}$$

as well as the scalar function ($\Box_g \equiv \nabla^{\mu}\nabla_{\mu}$)

$$s := \frac{1}{4}\Box_g\Theta + \frac{1}{24}R\Theta \ . \tag{2.4}$$

There exist different versions of the conformal field equations, depending on which fields are regarded as unknowns. Here we present the *metric conformal field equations (MCFE)* [23] which read in $3+1$ spacetime dimensions

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0 \,, \tag{2.5}$$

$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta \, d_{\nu\mu\sigma}{}^\rho \,, \tag{2.6}$$

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} \,, \tag{2.7}$$

$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta \,, \tag{2.8}$$

$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3 \,, \tag{2.9}$$

$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2\left(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}\right) \,. \tag{2.10}$$

The unknowns are $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$.

Friedrich has shown that the MCFE are equivalent to the vacuum Einstein equations,

$$\tilde{R}_{\mu\nu}[\tilde{g}] \;=\; \lambda \tilde{g}_{\mu\nu} \,, \quad \tilde{g}_{\mu\nu} \;=\; \Theta^{-2} g_{\mu\nu} \,,$$

in the region where Θ is non-vanishing. They give rise to a complicated and highly overdetermined PDE-system. It turns out that (2.9) is a consequence of (2.7) and (2.8) if it is known to hold at just one point (e.g. by an appropriate choice of the initial data). Moreover, Friedrich has separated constraint and evolution equations from the conformal field equations by working in a spin frame [16, 17] . In Sections 3.1, 4.2 and 4.3 we shall do the same (if the initial surface is C_{i^-}) in a coordinate frame and by imposing a generalized wave-map gauge condition.

A specific property in the $3+1$-dimensional case is that the contracted Bianchi identity is equivalent to the Bianchi identity. That is the reason why (2.5) implies hyperbolic equations; in higher dimensions this is no longer true [23]. The conformal field equations provide a nice, i.e. symmetric hyperbolic, evolution system only in $3+1$ dimensions.

Penrose proposed to distinguish asymptotically flat spacetimes by requiring the unphysical metric g to be smoothly extendable across $\mathscr{I}$. The Weyl tensor of g is known to vanish on $\mathscr{I}$ [32]. Since by definition $\mathrm{d}\Theta|_{\mathscr{I}} \neq 0$ the rescaled Weyl tensor can be smoothly continued across $\mathscr{I}$. However, there seems to be no reason why the same should be possible at i^- where $\mathrm{d}\Theta = 0$. When dealing with the MCFE, where the rescaled Weyl tensor is one of the unknowns, it is convenient to confine attention to the class of solutions with a regular i^- in the sense that both $g_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ are smoothly extendable across i^- (cf. Section 6 where this additional assumption is dropped).

2.2 Gauge freedom and conformal covariance inherent to the MCFE

The gauge freedom contained in the MCFE comes from the freedom to choose coordinates supplemented by the freedom to choose the conformal factor Θ relating the physical and the unphysical spacetime. Since Θ is regarded as an unknown rather than a gauge function, it remains to identify another function which reflects this gauge freedom. The most convenient choice is the Ricci scalar R:

Let us assume we have been given a smooth solution $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ of the MCFE. Then we can compute R. For a conformal rescaling $g \mapsto \phi^2 g$ for some $\phi > 0$, the Ricci scalars R and R^* of g and $\phi^2 g$, respectively, are related via

$$\phi R - \phi^3 R^* = 6\Box_g \phi \; . \tag{2.11}$$

Now, let us prescribe R^* and read (2.11) as an equation for ϕ. If we think of a characteristic initial value problem with data on a light-cone C_O (including the C_{i^-}-case) we are free to prescribe some $\mathring{\phi} > 0$ on C_O.[3][4] Supposing that $\mathring{\phi}$ is the restriction to C_O of a smooth spacetime function, the Cagnac-Dossa theorem [4, 13] tells us that there is a solution $\phi > 0$ with $\phi|_{C_O} = \mathring{\phi}$ in some neighbourhood of the tip of the cone. Due to the *conformal covariance* of the conformal field equations, the conformally rescaled fields

$$g^* = \phi^2 g \; , \tag{2.12}$$

$$\Theta^* = \phi\,\Theta \; , \tag{2.13}$$

$$s^* = \frac{1}{4}\Box_{g^*}\Theta^* + \frac{1}{24}R^*\Theta^* \; , \tag{2.14}$$

$$L^*_{\mu\nu} = \frac{1}{2}R^*_{\mu\nu}[g^*] - \frac{1}{12}R^* g^*_{\mu\nu} \; , \tag{2.15}$$

$$d^*_{\mu\nu\sigma}{}^{\rho} = \phi^{-1} d_{\mu\nu\sigma}{}^{\rho} \; , \tag{2.16}$$

provide another solution of the MCFE with Ricci scalar R^* which corresponds to the same physical solution $\tilde{g}_{\mu\nu}$. These considerations show that if we treat the conformal factor Θ as unknown, determined by the MCFE, the curvature scalar R of the unphysical spacetime can be arranged to take any preassigned form. The function R can therefore be regarded as a *conformal gauge source function* which can be chosen arbitrarily.

There remains the freedom to prescribe $\mathring{\phi}$ on C_O. On an ordinary cone with nowhere vanishing Θ this freedom can be employed to prescribe the initial data for the conformal factor, $\Theta|_{C_O}$ (it clearly needs to be the restriction to C_O of a smooth spacetime function). In this work we are particularly interested in the case where the vertex of the cone is located at past timelike infinity i^-, where, by definition, $\Theta = 0$ (note that this requires to take $\lambda = 0$). Then the gauge freedom to choose $\mathring{\phi}$ can be employed to prescribe the function s on C_{i^-}. To see that, let us assume we have been given a smooth solution $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^{\rho})$ of the MCFE to the future of C_{i^-}, at least in some neighbourhood of i^-, by which we also mean that the solution admits a smooth extension through C_{i^-}. (When Θ vanishes e.g. on one of two transversally intersecting null hypersurfaces one might put forward a similar argument.) In particular the function s is smooth. According to (2.9) (with $\lambda = 0$), it can be written away from C_{i^-} as

$$s = \frac{1}{2}\Theta^{-1}\nabla_\mu\Theta\nabla^\mu\Theta \; ,$$

with the right-hand side smoothly extendable across C_{i^-}. Under the conformal rescaling

$$\Theta \mapsto \Theta^* := \phi\,\Theta \; , \quad g_{\mu\nu} \mapsto g^*_{\mu\nu} := \phi^2 g_{\mu\nu} \; , \quad \phi > 0 \; , \tag{2.17}$$

[3]The positivity of ϕ at the vertex guarantees any solution of (2.11) to be positive sufficiently close to the vertex and thereby the positivity of Θ^* (in the C_{i^-}-case just off the cone).

[4]Since we are mainly interested in this case, we focus on an initial surface which is a cone. However, an analogous result can be obtained for two transversally intersecting null hypersurfaces.

the function s becomes

$$s^* = \phi^{-1}\Big(\frac{1}{2}\Theta\phi^{-2}\nabla^\mu\phi\nabla_\mu\phi + \phi^{-1}\nabla^\mu\Theta\nabla_\mu\phi + s\Big) \,. \tag{2.18}$$

Evaluation of this expression on C_{i^-} yields

$$\overline{\nabla^\mu\Theta\nabla_\mu\phi + \phi\, s - \phi^2 s^*} = 0 \,. \tag{2.19}$$

Here and henceforth we use an overbar to denote the restriction of a spacetime object to the initial surface. Note that $\overline{\nabla^\mu\Theta}$ is tangent to $\mathscr{I}$, so (2.19) does not involve transverse derivatives of ϕ on $\mathscr{I}$. Let us prescribe $\overline{s}^*$ (as a matter of course it needs to be the restriction of a smooth spacetime function) and assume for the moment that some positive solution of (2.19) exists,[5] which we denote by $\mathring{\phi}$. We take $\mathring{\phi}$ as initial datum for (2.11). We would like to have a $\mathring{\phi}$ which is the restriction to C_{i^-} of a smooth spacetime function, so that we can apply the Cagnac-Dossa theorem, which would supply us with a function ϕ solving (2.11) and satisfying $\phi|_{C^-} = \mathring{\phi}$. Via the conformal rescaling (2.12)-(2.16) we then would be led to a new solution of the MCFE with preassigned functions R^* and $\overline{s}^*$ which represents the same physical solution we started with.

The crucial point, which remains to be checked, is whether a solution of (2.19) exists with the desired properties. The following lemma, which is proven in Appendix A, shows that this is indeed the case (cf. [12, Appendix A] where an alternative proof is given).

Lemma 2.1 *Consider any smooth solution of the MCFE in $3+1$ dimensions in some neighbourhood $\mathscr{U}$ to the future of i^-, smoothly extendable through C_{i^-}, which satisfies*

$$s|_{i^-} \neq 0 \,. \tag{2.20}$$

Let $\overline{s}^*$ be the restriction of a smooth spacetime function on $\mathscr{U} \cap \partial J^+(i^-)$ with $\overline{s}^*|_{i^-} \neq 0$ and $\lim_{r\to 0}\partial_r \overline{s}^* = 0$.[6] Then (2.19) is a Fuchsian ODE and for every solution $\mathring{\phi}$ (note that the solution set is non-empty) it holds that

$$\mathrm{sign}(\mathring{\phi}|_{i^-}) = \mathrm{sign}(s|_{i^-})\mathrm{sign}(s^*|_{i^-}) \,, \tag{2.21}$$

and $\mathring{\phi}$ is the restriction to C_{i^-} of a smooth spacetime function. In particular, if $\mathrm{sign}(s|_{i^-}) = \mathrm{sign}(s^*|_{i^-})$ the function $\mathring{\phi}$ will be positive sufficiently close to i^-.

Remark 2.2 *Note that solutions with $s|_{i^-} = 0$ would satisfy $\mathrm{d}\Theta = 0$ on $\mathscr{I}^-$, which is why the corresponding class of solutions is not of physical interest.*

To sum it up, due the conformal covariance of the MCFE the functions R and $s|_{C_{i^-}}$ can and will be regarded as gauge source functions.

3 Conformal wave equations (CWE)

3.1 Derivation of the conformal wave equations

In this section we derive a system of wave equations from the MCFE (2.5)-(2.10). Recall that the unknowns are $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$, while the Ricci

[5] In case of a negative $\overline{s}^*$, the gauge transformation would change the sign of Θ.

[6] r is a suitable (e.g. an affine) parameter along the null geodesics emanating from i^-, see Section 4 and Appendix A for more details.

scalar R (and the function $\overline{s}$ or $\overline{\Theta}$, respectively, depending on the characteristic initial surface) are considered as gauge functions. The cosmological constant λ is allowed to be non-vanishing in this section.

Derivation of an appropriate second-order system

From (2.5) and (2.6) we obtain (with $\Box_g \equiv \nabla^\mu \nabla_\mu$)

$$\Box_g L_{\mu\nu} - R_{\mu\kappa} L_\nu{}^\kappa - R_{\alpha\mu\nu}{}^\kappa L_\kappa{}^\alpha - \nabla_\mu \nabla_\alpha L_\nu{}^\alpha = d_\mu{}^\alpha{}_\nu{}^\rho \nabla_\alpha \nabla_\rho \Theta \;.$$

Using the definition (2.2) of the Schouten tensor, together with the contracted Bianchi identity, we find

$$\nabla_\mu L_\nu{}^\mu = \frac{1}{6} \nabla_\nu R \;, \tag{3.1}$$

and thus

$$\Box_g L_{\mu\nu} - R_{\mu\kappa} L_\nu{}^\kappa - R_{\alpha\mu\nu}{}^\kappa L_\kappa{}^\alpha - \frac{1}{6} \nabla_\mu \nabla_\nu R \;\; = \;\; d_\mu{}^\alpha{}_\nu{}^\rho \nabla_\alpha \nabla_\rho \Theta \;.$$

We combine the right-hand side with (2.7), and employ (2.3) as well as (2.10) to transform the third term on the left-hand side to end up with a wave equation for the Schouten tensor (suppose for the time being that $g_{\mu\nu}$ is given, cf. below),

$$\Box_g L_{\mu\nu} - 4 L_{\mu\kappa} L_\nu{}^\kappa + g_{\mu\nu} |L|^2 + 2\Theta d_{\mu\alpha\nu}{}^\rho L_\rho{}^\alpha = \frac{1}{6} \nabla_\mu \nabla_\nu R \;, \tag{3.2}$$

where we have set

$$|L|^2 := L_\mu{}^\nu L_\nu{}^\mu \;.$$

Next, let us consider the function s. From (2.8), (3.1) and (2.7) we deduce the wave equation

$$\begin{aligned} \Box_g s &= -\nabla_\mu L_\nu{}^\mu \nabla^\nu \Theta - L^{\mu\nu} \nabla_\mu \nabla_\nu \Theta \\ &= \Theta |L|^2 - \frac{1}{6} \nabla_\nu R \, \nabla^\nu \Theta - \frac{1}{6} s R \;. \end{aligned} \tag{3.3}$$

The definition of s provides a wave equation for the conformal factor,

$$\Box_g \Theta \;\; = \;\; 4s - \frac{1}{6} \Theta R \;. \tag{3.4}$$

To obtain a wave equation for the rescaled Weyl tensor $d_{\mu\nu\sigma}{}^\rho$ *in 3+1 dimensions* one proceeds as follows: Due to its algebraic properties the rescaled Weyl tensor satisfies the relation

$$\epsilon_{\mu\nu}{}^{\alpha\beta} d_{\alpha\beta\lambda\rho} = \epsilon_{\lambda\rho}{}^{\alpha\beta} d_{\mu\nu\alpha\beta} \;,$$

where $\epsilon_{\mu\nu\sigma\rho}$ denotes the totally antisymmetric tensor. We conclude that (cf. [33])

$$\nabla_{[\lambda} d_{\mu\nu]\sigma\rho} \;\; = \;\; -\frac{1}{6} \epsilon_{\lambda\mu\nu\kappa} \epsilon^{\alpha\beta\gamma\kappa} \nabla_\alpha d_{\beta\gamma\sigma\rho} \; = \; \frac{1}{6} \epsilon_{\lambda\mu\nu}{}^\kappa \epsilon_{\sigma\rho}{}^{\beta\gamma} \nabla_\alpha d_{\beta\gamma\kappa}{}^\alpha \;. \tag{3.5}$$

This equation implies the equivalence[7]

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0 \quad \Longleftrightarrow \quad \nabla_{[\lambda} d_{\mu\nu]\sigma\rho} = 0 \;.$$

[7] We remark that this equivalence holds only in 4 dimensions. Any attempt to derive a wave equation for $d_{\mu\nu\sigma}{}^\rho$ in dimension $d \geq 5$ seems to lead to singular terms. Also, if one uses a different set of variables, like e.g. Cotton and Weyl tensor instead of $d_{\mu\nu\sigma}{}^\rho$, cf. Section 6, the derivation of a regular system of wave equations seems to be possible merely in the 4-dimensional case. This is in line with the observation that the conformal field equations provide a good evolution system only in 4 dimensions.

Equation (2.5) can therefore be replaced by

$$\nabla_{[\lambda}d_{\mu\nu]\sigma\rho} = 0\ . \tag{3.6}$$

Applying ∇^λ and commuting the covariant derivatives yields with (2.5)

$$\Box_g d_{\mu\nu\sigma\rho} + 2R_{\kappa\mu\nu}{}^{\alpha}d_{\sigma\rho\alpha}{}^{\kappa} + 2R_{\alpha[\mu}d_{\nu]}{}^{\alpha}{}_{\sigma\rho} + 2R_{\kappa[\mu|\sigma|}{}^{\alpha}d_{\nu]}{}^{\kappa}{}_{\alpha\rho} - 2R_{\alpha\rho\kappa[\mu}d_{\nu]}{}^{\kappa}{}_{\sigma}{}^{\alpha} = 0\ .$$

With (2.10) we end up with a wave equation for the rescaled Weyl tensor,

$$\begin{aligned} \Box_g d_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\kappa}{}^{\alpha}d_{\sigma\rho\alpha}{}^{\kappa} + 4\Theta d_{\sigma\kappa[\mu}{}^{\alpha}d_{\nu]\alpha\rho}{}^{\kappa} + 2g_{\sigma[\mu}d_{\nu]\alpha\rho\kappa}L^{\alpha\kappa} & \\ -2g_{\rho[\mu}d_{\nu]\alpha\sigma\kappa}L^{\alpha\kappa} + 2d_{\mu\nu\kappa[\sigma}L_{\rho]}{}^{\kappa} + 2d_{\sigma\rho\kappa[\mu}L_{\nu]}{}^{\kappa} - \frac{1}{3}Rd_{\mu\nu\sigma\rho} &= 0\ . \end{aligned} \tag{3.7}$$

It turns out that this equation does not take its simplest form yet. To see this let us exploit (3.6) again. Invoking the Bianchi identity and (2.6) we find

$$\begin{aligned} 0 &= \Theta\nabla_{[\lambda}d_{\mu\nu]\sigma\rho} \;=\; \nabla_{[\lambda}W_{\mu\nu]\sigma\rho} - (\nabla_{[\lambda}\Theta)d_{\mu\nu]\sigma\rho} \\ &= \frac{2}{3}\big(g_{\sigma\nu}\nabla_{[\lambda}L_{\mu]\rho} + g_{\mu\rho}\nabla_{[\lambda}L_{\nu]\sigma} + g_{\sigma\mu}\nabla_{[\nu}L_{\lambda]\rho} + g_{\lambda\rho}\nabla_{[\nu}L_{\mu]\sigma} \\ &\quad + g_{\sigma\lambda}\nabla_{[\mu}L_{\nu]\rho} + g_{\nu\rho}\nabla_{[\mu}L_{\lambda]\sigma}\big) - (\nabla_{[\lambda}\Theta)d_{\mu\nu]\sigma\rho} \\ &= g_{\rho[\lambda}d_{\mu\nu]\sigma}{}^{\alpha}\nabla_\alpha\Theta - g_{\sigma[\lambda}d_{\mu\nu]\rho}{}^{\alpha}\nabla_\alpha\Theta - (\nabla_{[\lambda}\Theta)d_{\mu\nu]\sigma\rho}\ . \end{aligned}$$

Applying ∇^λ and using (3.6), (2.7) and (3.4) we are led to

$$\begin{aligned} 0 &= 3\nabla^\lambda(g_{\rho[\lambda}d_{\mu\nu]\sigma}{}^{\alpha}\nabla_\alpha\Theta - g_{\sigma[\lambda}d_{\mu\nu]\rho}{}^{\alpha}\nabla_\alpha\Theta - \nabla_{[\lambda}\Theta\, d_{\mu\nu]\sigma\rho}) \\ &= 2d_{\mu\nu[\sigma}{}^{\alpha}\nabla_{\rho]}\nabla_\alpha\Theta + 2g_{\rho[\mu}d_{\nu]\lambda\sigma}{}^{\alpha}\nabla^\lambda\nabla_\alpha\Theta - 2g_{\sigma[\mu}d_{\nu]\lambda\rho}{}^{\alpha}\nabla^\lambda\nabla_\alpha\Theta \\ &\quad -\Box\Theta\, d_{\mu\nu\sigma\rho} - \nabla^\lambda\nabla_\nu\Theta\, d_{\lambda\mu\sigma\rho} - \nabla^\lambda\nabla_\mu\Theta\, d_{\nu\lambda\sigma\rho} \\ &= 2\Theta g_{\sigma[\mu}d_{\nu]\lambda\rho}{}^{\alpha}L_\alpha{}^\lambda - 2\Theta g_{\rho[\mu}d_{\nu]\lambda\sigma}{}^{\alpha}L_\alpha{}^\lambda + 2\Theta d_{\mu\nu\alpha[\sigma}L_{\rho]}{}^{\alpha} \\ &\quad +2\Theta d_{\sigma\rho\alpha[\mu}L_{\nu]}{}^{\alpha} + \frac{1}{6}\Theta Rd_{\mu\nu\sigma\rho}\ . \end{aligned}$$

This relation simplifies (3.7) significantly,

$$\Box_g d_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\kappa}{}^{\alpha}d_{\sigma\rho\alpha}{}^{\kappa} + 4\Theta d_{\sigma\kappa[\mu}{}^{\alpha}d_{\nu]\alpha\rho}{}^{\kappa} - \frac{1}{2}Rd_{\mu\nu\sigma\rho} = 0\ . \tag{3.8}$$

We have found a system of wave equations (3.2)-(3.4) and (3.8) for the fields $L_{\mu\nu}$, s, Θ and $d_{\mu\nu\sigma}{}^{\rho}$, assuming that $g_{\mu\nu}$ is given. Now, we drop this assumption, so first of all the system needs to be complemented by an equation for the metric tensor. Taking the trace of (2.10) yields

$$R_{\mu\nu}[g] = 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}\ . \tag{3.9}$$

However, the equations (3.2)-(3.4) and (3.8)-(3.9) do *not* form a system of wave equations yet: Equation (3.9) is not a wave equation due to the fact that the principal part of the Ricci tensor is not a d'Alembert operator. Moreover, the principal part of the wave-operator $\Box_g$ is not a d'Alembert operator when acting on tensors of valence ≥ 1 and when the metric tensor is part of the unknowns, for the corresponding expression contains second-order derivatives of the metric due to which the principal part is not $g^{\mu\nu}\partial_\mu\partial_\nu$ anymore. Consequently (3.2) and (3.8) are no wave equations, as well.

We need to impose an appropriate gauge condition to transform these equations into wave equations, which is accomplished subsequently.

Generalized wave-map gauge

Let us introduce the so-called *$\hat{g}$-generalized wave-map gauge* (cf. [5, 19, 22]), where $\hat{g}_{\mu\nu}$ denotes some *target metric*. For that we define the *wave-gauge vector*

$$H^\sigma := g^{\alpha\beta}(\Gamma^\sigma_{\alpha\beta} - \hat{\Gamma}^\sigma_{\alpha\beta}) - W^\sigma \ .$$

Herein $\hat{\Gamma}^\sigma_{\alpha\beta}$ are the Christoffel symbols of $\hat{g}_{\mu\nu}$. Moreover,

$$W^\sigma = W^\sigma(x^\mu, g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$$

is an arbitrary vector field, which is allowed to depend upon the coordinates, and possibly upon $g_{\mu\nu}$ as well as all the other fields which appear in the MCFE,[8] but not upon derivatives thereof. The freedom to prescribe W^σ reflects the freedom to choose coordinates off the initial surface. We then impose the *$\hat{g}$-generalized wave-map gauge condition*

$$H^\sigma = 0 \ .$$

The *reduced Ricci tensor* $R^{(H)}_{\mu\nu}$ is defined as

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma \ , \tag{3.10}$$

where $\hat{\nabla}$ denotes the covariant derivative associated to the target metric. The principal part of the reduced Ricci tensor *is* a d'Alembert operator.

Furthermore, we define a *reduced wave-operator* as follows: We observe that for any covector field v_λ we have

$$\begin{aligned}
\Box_g v_\lambda &= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda - g^{\mu\nu}(\partial_\mu\Gamma^\sigma_{\nu\lambda})v_\sigma + f_\lambda(g,\partial g, v, \partial v) \\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + (R_\lambda{}^\sigma - \partial_\lambda(g^{\mu\nu}\Gamma^\sigma_{\mu\nu}))v_\sigma + f_\lambda(g,\partial g, v, \partial v) \\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + (R_\lambda{}^\sigma - \partial_\lambda H^\sigma)v_\sigma + f_\lambda(g,\partial g, v, \partial v, \hat{g}, \partial\hat{g}, \partial^2\hat{g}, \partial W) \\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + (R^{(H)}_{\mu\lambda} + g_{\sigma[\lambda}\hat{\nabla}_{\mu]}H^\sigma)v^\mu + f_\lambda(g,\partial g, v, \partial v, \hat{g}, \partial\hat{g}, \partial^2\hat{g}, \partial W) \ .
\end{aligned}$$

Similarly, the action on a vector field v^λ yields

$$\Box_g v^\lambda = g^{\mu\nu}\partial_\mu\partial_\nu v^\lambda - (R^{\mu\lambda}_{(H)} + g^{\sigma[\lambda}\hat{\nabla}_\sigma H^{\mu]})v_\mu + f^\lambda(g,\partial g, v, \partial v, \hat{g}, \partial\hat{g}, \partial^2\hat{g}, \partial W) \ .$$

This motivates to define a *reduced wave-operator* $\Box^{(H)}_g$ via its action on (co)vector fields in the following way:

$$\begin{aligned}
\Box^{(H)}_g v_\lambda &:= \Box_g v_\lambda - g_{\sigma[\lambda}(\hat{\nabla}_{\mu]}H^\sigma)v^\mu + (2L_{\mu\lambda} - R^{(H)}_{\mu\lambda} + \frac{1}{6}Rg_{\mu\lambda})v^\mu \ , \\
\Box^{(H)}_g v^\lambda &:= \Box_g v^\lambda + g^{\sigma[\lambda}(\hat{\nabla}_\sigma H^{\mu]})v_\mu - (2L^{\mu\lambda} - R^{\mu\lambda}_{(H)} + \frac{1}{6}Rg^{\mu\lambda})v_\mu \ .
\end{aligned}$$

[8] I am grateful to L. Andersson for pointing that out. However, in view of the constraint equations we shall consider later on for convenience merely those W^σ's which depend just on the coordinates.

For arbitrary tensor fields we set

$$\begin{aligned}
\Box_g^{(H)} v_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots\beta_m} \;:=\; & \Box_g v_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots\beta_m} - \sum_i g_{\sigma[\alpha_i}(\hat\nabla_{\mu]}H^\sigma) v_{\alpha_1\ldots}{}^{\mu}{}_{\ldots\alpha_n}{}^{\beta_1\ldots\beta_m} \\
& + \sum_i (2L_{\mu\alpha_i} - R^{(H)}_{\mu\alpha_i} + \frac{1}{6} R g_{\mu\alpha_i}) v_{\alpha_1\ldots}{}^{\mu}{}_{\ldots\alpha_n}{}^{\beta_1\ldots\beta_m} \\
& + \sum_i g^{\sigma[\beta_i}(\hat\nabla_\sigma H^{\mu]}) v_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots}{}_{\mu}{}^{\ldots\beta_m} \\
& - \sum_i (2L^{\mu\beta_i} - R^{\mu\beta_i}_{(H)} + \frac{1}{6} R g^{\mu\beta_i}) v_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots}{}_{\mu}{}^{\ldots\beta_m} \;,
\end{aligned}$$

which is a proper wave-operator even if $g_{\mu\nu}$ is part of the unknowns since $L_{\mu\nu}$ and the gauge source function R are regarded as independent of $g_{\mu\nu}$. Note that the action of $\Box_g$ and $\Box_g^{(H)}$ coincides on scalars. Moreover, if $H^\sigma = 0$, and $L_{\mu\nu}$ and R are known to be the Schouten tensor and the Ricci scalar of $g_{\mu\nu}$, respectively, then the action of $\Box_g$ and $\Box_g^{(H)}$ coincides on all tensor fields.

Conformal wave equations

Let us reconsider the system (3.2), (3.3), (3.4), (3.8) and (3.9). We replace the Ricci tensor by the reduced Ricci tensor and the wave-operator by the reduced wave-operator to end up with a closed regular system of wave equations for $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$,

$$\Box_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \frac{1}{6}\nabla_\mu\nabla_\nu R \;, \tag{3.11}$$

$$\Box_g s = \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \frac{1}{6} sR \;, \tag{3.12}$$

$$\Box_g \Theta = 4s - \frac{1}{6}\Theta R \;, \tag{3.13}$$

$$\Box_g^{(H)} d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa + \frac{1}{2} R d_{\mu\nu\sigma\rho} \;, \tag{3.14}$$

$$R^{(H)}_{\mu\nu}[g] = 2L_{\mu\nu} + \frac{1}{6} R g_{\mu\nu} \;. \tag{3.15}$$

Henceforth the system (3.11)-(3.15) will be called *conformal wave equations* (CWE).

Remark 3.1 *Since R is regarded as a gauge degree of freedom and not as unknown, there is no need to worry about its second-order derivatives appearing in (3.11). Note, however, that, unlike W^σ, the gauge source function R cannot be allowed to depend upon the fields $L_{\mu\nu}$, $d_{\mu\nu\sigma\rho}$, Θ and s, due to the fact that (3.11) contains second-order derivatives of R. Since $\nabla g = 0$, R can in principle be allowed to depend upon $g_{\mu\nu}$.*

3.2 Consistency with the gauge condition

Let us analyse now consistency of the CWE with the gauge conditions we imposed. More concretely, we consider a characteristic initial value problem, where, for definiteness, we think of two transversally intersecting null hypersurfaces or a light-cone, and assume that we have been given initial data ($\mathring{g}_{\mu\nu}$, $\mathring{s}$,

$\mathring{\Theta}$, $\mathring{L}_{\mu\nu}$, $\mathring{d}_{\mu\nu\sigma}{}^{\rho}$). We further assume that there exists a smooth solution ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^{\rho}$) of the CWE with gauge source function R which induces these data. We aim to work out conditions, which need to be satisfied initially, which guarantee consistency with the gauge conditions in the sense that the solution implies $H^\sigma = 0$ and $R_g = R$, where $R_g := R[g]$ denotes the curvature scalar of $g_{\mu\nu}$. (Recall that there is, depending on the type of the characteristic initial surface, the additional gauge freedom to prescribe $\overline{\Theta}$ or $\overline{s}$, but here consistency is trivial.)

Let us outline the strategy. To make sure that H^σ and $R - R_g$ vanish we shall derive a linear, homogeneous system of wave equations for H^σ as well as some subsidiary fields, which is fulfilled by any solution of the CWE. We shall see that it is not necessary to regard $R - R_g$ as an unknown. We shall assume that all the fields which are regarded as unknowns in this set of equations vanish on the initial surface (in Section 5 these assumptions will be justified). Due to the uniqueness of solutions of wave equations, which is established by standard energy estimates, cf. e.g. [15], we then conclude that the trivial solution is the only one and that the fields involved need to vanish everywhere.

Some properties of solutions of the CWE

Let establish some properties of solutions of the CWE. First of all we show that the tensors $g_{\mu\nu}$ and $L_{\mu\nu}$ are symmetric, supposing that their initial data are (and that $d_{\mu\nu\sigma\rho}$ satisfies a certain symmetry property on the initial surface).

LEMMA 3.2 *Assume that the initial data on a characteristic initial surface S of some smooth solution of the CWE are such that $g_{\mu\nu}|_S$ is the restriction to S of a Lorentzian metric, that $L_{[\mu\nu]}|_S = 0$ and $d_{\mu\nu\sigma\rho}|_S = d_{\sigma\rho\mu\nu}$. Then the solution has the following properties:*

1. *$g_{\mu\nu}$ and $L_{\mu\nu}$ are symmetric tensors,*
2. *$d_{\mu\nu\sigma\rho} = d_{\sigma\rho\mu\nu}$.*

REMARK 3.3 *A priori it might happen that $g_{\mu\nu}$ becomes non-symmetric away from the initial surface. However, the lemma shows that the tensor $g_{\mu\nu}$ does indeed define a metric as long as it does not degenerate (i.e. at least sufficiently close to the vertex or the intersection manifold, respectively). Later on, the initial data will be constructed from certain free data such that all the hypotheses of Lemma 3.2 are satisfied, we thus will assume throughout that $g_{\mu\nu}$ and $L_{\mu\nu}$ have their usual symmetry properties.*

PROOF: Equation (3.14) yields[9]

$$\begin{aligned}\Box_g^{(H)}(d_{\mu\nu\sigma\rho} - d_{\sigma\rho\mu\nu}) \quad &= \quad 4\Theta[g^{[\alpha\beta]}d_{\sigma\beta\mu\kappa}d_{\rho\alpha\nu}{}^{\kappa} - g^{[\gamma\kappa]}d_{\sigma\beta\mu\kappa}d_{\rho}{}^{\beta}{}_{\nu\gamma}] \\ &+2\Theta g^{\alpha\beta}g^{\kappa\gamma}[d_{\rho\alpha\nu\gamma}(d_{\mu\kappa\sigma\beta} - d_{\sigma\beta\mu\kappa}) - d_{\sigma\kappa\mu\beta}(d_{\nu\alpha\rho\gamma} - d_{\rho\gamma\nu\alpha})] \\ &+\frac{1}{2}R(d_{\mu\nu\sigma\rho} - d_{\sigma\rho\mu\nu})\,. \end{aligned} \tag{3.16}$$

[9] The indices are raised and lowered as follows: $v^\mu := g^{\mu\nu}v_\nu$ and $w_\mu := g_{\mu\nu}w^\nu$. Note for this that $g_{\mu\nu}$ is non-degenerated sufficiently close to S. The definition of the Ricci tensor, which appears in (3.15), in terms of Christoffel symbols which in turn are expressed in terms of g make sense even if g is not symmetric.

From (3.11) and (3.15) we find

$$\begin{aligned}\Box_g^{(H)} L_{[\mu\nu]} &= 4g_{[\alpha\beta]}L_\mu{}^\alpha L_\nu{}^\beta - g_{[\mu\nu]}|L|^2 + \Theta g^{\rho\gamma}L_\rho{}^\sigma(d_{\nu\sigma\mu\gamma} - d_{\mu\gamma\nu\sigma}) \\ &\quad +2\Theta g^{\sigma\kappa}d_\mu{}^\rho{}_{\nu\sigma}L_{[\rho\kappa]} - 2\Theta g^{[\sigma\kappa]}d_{\mu\sigma\nu}{}^\rho L_{\rho\kappa}\ , \qquad (3.17)\\ R^{(H)}_{[\mu\nu]}[g_{(\sigma\rho)}, g_{[\sigma\rho]}] &= 2L_{[\mu\nu]} + \frac{1}{6}Rg_{[\mu\nu]}\ . \qquad (3.18)\end{aligned}$$

The equations (3.16)-(3.18) are to be read as a linear, homogeneous system of wave equations satisfied by $g_{[\mu\nu]}$, $L_{[\mu\nu]}$ and $d_{\mu\nu\sigma\rho} - d_{\sigma\rho\mu\nu}$, and with all the other fields regarded as being given. Since, by assumption, these fields vanish initially they have to vanish everywhere and the assertion follows. □

It is useful to derive some more properties of the tensor $d_{\mu\nu\sigma\rho}$. We emphasize that $d_{\mu\nu\sigma\rho}$ is assumed to be part of some given solution of the CWE and that, a priori, it neither needs to be the rescaled Weyl tensor nor does it need to have all its algebraic properties.

LEMMA 3.4 *Assume that $d_{\mu\nu\sigma\rho}$ belongs to a solution of the CWE (3.11)-(3.15) for which the hypotheses of Lemma 3.2 are fulfilled. Then the tensor $d_{\mu\nu\sigma\rho}$ has the following properties:*

(i) $d_{\mu\nu\sigma\rho} = d_{\sigma\rho\mu\nu}$,

(ii) $d_{\mu\nu\sigma\rho}$ *is anti-symmetric in its first two and last two indices,*

(iii) $d_{\mu\nu\sigma\rho}$ *satisfies the first Bianchi identity, i.e.* $d_{[\mu\nu\sigma]\rho} = 0$,

(iv) $d_{\mu\nu\sigma\rho}$ *is trace-free,*

supposing that (i)-(iv) hold initially.

REMARK 3.5 *The constraint equations we shall impose later on on the initial data guarantee that (i)-(iv) are initially satisfied. As for $g_{\mu\nu}$ and $L_{\mu\nu}$ we shall therefore use the implications of this lemma without mentioning it each time.*

PROOF: (i) This is part of the proof of Lemma 3.2.

(ii) Equation (3.14) implies a linear, homogeneous wave equation for $d_{(\mu\nu)\sigma\rho}$,

$$\Box_g^{(H)} d_{(\mu\nu)\sigma\rho} = \Theta d_{\sigma\rho\alpha}{}^\kappa d_{(\mu\nu)\kappa}{}^\alpha + \frac{1}{2}Rd_{(\mu\nu)\sigma\rho}\ ,$$

i.e. the tensor $d_{\mu\nu\sigma\rho}$ is antisymmetric in its first two (and therefore by (i) in its last two indices) since this is assumed to be initially the case.

(iii) Due to the (anti-)symmetry properties (i)-(ii), we find the following linear, homogeneous wave equation from (3.14),

$$\begin{aligned}\Box_g^{(H)} d_{[\mu\nu\sigma]\rho} &= \Theta d_{[\mu\nu|\kappa|}{}^\alpha d_{\sigma]\rho\alpha}{}^\kappa + 4\Theta d_{\kappa[\sigma\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa + \frac{1}{2}Rd_{[\mu\nu\sigma]\rho}\\ &= 2\Theta d_{\sigma\alpha\rho}{}^\kappa d_{[\kappa\mu\nu]}{}^\alpha + 2\Theta d_{\mu\alpha\rho}{}^\kappa d_{[\kappa\nu\sigma]}{}^\alpha + 2\Theta d_{\nu\alpha\rho}{}^\kappa d_{[\kappa\sigma\mu]}{}^\alpha \\ &\quad +\Theta d_{\mu\nu\kappa}{}^\alpha d_{[\alpha\sigma\rho]}{}^\kappa + \Theta d_{\nu\sigma\kappa}{}^\alpha d_{[\alpha\mu\rho]}{}^\kappa + \Theta d_{\sigma\mu\kappa}{}^\alpha d_{[\alpha\nu\rho]}{}^\kappa + \frac{1}{2}Rd_{[\mu\nu\sigma]\rho}\ .\end{aligned}$$

(iv) It remains to be shown that $d_{\mu\rho\sigma}{}^\rho = 0$. Employing the properties (i)-(iii) we conclude from (3.14) that

$$\Box_g^{(H)} d_{\mu\rho\sigma}{}^\rho = -2\Theta d_\sigma{}^\kappa{}_\mu{}^\alpha d_{\kappa\rho\alpha}{}^\rho + \frac{1}{2}Rd_{\mu\rho\sigma}{}^\rho\ ,$$

which is again a linear, homogeneous wave equation. □

Next, let us establish another important property:

LEMMA 3.6 *Assume that the hypotheses of Lemma 3.2 and 3.4 are satisfied and that, in addition, the trace*

$$L := L_\sigma{}^\sigma$$

of $L_{\mu\nu}$ coincides on the initial surface with one sixth of the gauge source function R, $\overline{L} = \frac{1}{6}\overline{R}$. Then

$$L = \frac{1}{6}R\,. \tag{3.19}$$

(This is what one would expect if $L_{\mu\nu}$ was the Schouten tensor and R the Ricci scalar.)

PROOF: We observe that in virtue of (3.11) the tracelessness of $d_{\mu\nu\sigma\rho}$ implies

$$\Box_g(L - \frac{1}{6}R) \quad = \quad 0\,.$$

and the assertion follows again from standard uniqueness results for linear wave equations. $\Box$

Gauge consistency

Let us return to the question of whether we have consistency with the gauge condition in the sense that a solution of the CWE satisfies $H^\sigma = 0$ and $R_g = R$. For that we assume that all the hypotheses of Lemma 3.2, 3.4 and 3.6 are fulfilled. We consider the identity

$$R_{\mu\nu} - \frac{1}{2}R_g g_{\mu\nu} \equiv R^{(H)}_{\mu\nu} - \frac{1}{2}R^{(H)}g_{\mu\nu} + g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma - \frac{1}{2}g_{\mu\nu}\hat{\nabla}_\sigma H^\sigma\,. \tag{3.20}$$

Invoking (3.15) and Lemma 3.6 we deduce that

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}R_g g_{\mu\nu} \;&=\; 2L_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu} + g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma - \frac{1}{2}g_{\mu\nu}\hat{\nabla}_\sigma H^\sigma \\ \overset{\text{Bianchi}}{\Longrightarrow} \qquad \nabla^\nu\hat{\nabla}_\nu H^\alpha + 2g^{\mu\alpha}\nabla_{[\sigma}\hat{\nabla}_{\mu]}H^\sigma &+ 4(\nabla^\nu L_\nu{}^\alpha - \frac{1}{6}\nabla^\alpha R) \;=\; 0\,. \end{aligned} \tag{3.21}$$

Be aware that at this stage it is not known whether $L_{\mu\nu}$ coincides with the Schouten tensor and thus satisfies the contracted Bianchi identity (3.1) such that the term in brackets in (3.21) drops out. That is the reason why we cannot immediately deduce $H^\sigma = 0$ as in [5] supposing that this is initially the case.

Given two covariant derivative operators ∇ and $\hat{\nabla}$ (associated to the metrics g and $\hat{g}$, respectively), there exists a tensor field $C^\sigma_{\mu\nu} = C^\sigma_{\nu\mu}$, which depends on g, ∂g, $\hat{g}$ and $\partial\hat{g}$, such that

$$\nabla_\mu v^\sigma - \hat{\nabla}_\mu v^\sigma = C^\sigma_{\mu\nu}v^\nu\,, \tag{3.22}$$

for any vector v^σ, and similar formulae hold for tensor fields of other types. Setting

$$\zeta_\mu := -4(\nabla_\nu L_\mu{}^\nu - \frac{1}{6}\nabla_\mu R)\,, \tag{3.23}$$

the equation (3.21) can therefore be written as

$$\Box_g H^\alpha \quad = \quad \zeta^\alpha + f^\alpha(g, \hat{g}; H, \nabla H)\,, \tag{3.24}$$

which is a linear wave equation satisfied by the wave-gauge vector H^σ.[10] In (3.24), as in what follows, the generic smooth field $f^\alpha(g,\hat{g};H,\nabla H)$, or more general $f_{\alpha_1\ldots\alpha_p}{}^{\beta_1\ldots\beta_q}(v_1,\ldots,v_m;w_1,\ldots,w_n)$, represents a sum of fields, each of which contains precisely one multiplicative factor from the set $\{w_i\}$ as well as further factors which may depend on the v_j's and also higher-order derivatives of the v_j's. The latter does not cause any problems since the v_j's will be regarded as given fields rather than unknowns of the system we are about to derive. In most cases we will therefore simply write $f_{\alpha_1\ldots\alpha_p}{}^{\beta_1\ldots\beta_q}(x;w_1,\ldots,w_n)$.

Taking the trace of (3.20) and inserting (3.15), yields (note that $L=R/6$)

$$R_g \quad\equiv\quad R^{(H)}+\hat{\nabla}_\sigma H^\sigma = R+\hat{\nabla}_\sigma H^\sigma\ . \tag{3.25}$$

The vanishing of H^σ would therefore immediately ensure that $R_g=R$.

The tensor $d_{\mu\nu\sigma}{}^\rho$ is supposed to be part of a solution of the CWE. Note, again, that at this stage it is by no means clear whether it, indeed, represents the rescaled Weyl tensor of $g_{\mu\nu}$ and Θ. As before, we denote by $W_{\mu\nu\sigma}{}^\rho$ the Weyl tensor associated to $g_{\mu\nu}$, defined via the decomposition

$$R_{\mu\nu\sigma\rho}=W_{\mu\nu\sigma\rho}+g_{\sigma[\mu}R_{\nu]\rho}-g_{\rho[\mu}R_{\nu]\sigma}-\frac{1}{3}R_g g_{\sigma[\mu}g_{\nu]\rho}\ . \tag{3.26}$$

As outlined above we want to derive a closed, linear, homogeneous system of wave equations for a certain set of fields in order to establish the vanishing of H^σ. First of all, we need a wave equation for ζ_μ. Making use of the Bianchi identity, (3.19) and (3.11), we obtain

$$\begin{aligned}
\Box_g\zeta_\mu \quad\equiv\quad & -4\nabla_\nu\Box_g L_\mu{}^\nu+\frac{2}{3}\Box_g\nabla_\mu R-8\nabla^\nu(W_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma)+8R_\nu{}^\kappa\nabla_\kappa L_\mu{}^\nu\\
& -4R_\nu{}^\kappa\nabla_\mu L_\kappa{}^\nu-R_\mu{}^\nu\zeta_\nu+\frac{1}{3}R_g\zeta_\mu-4R_\mu{}^\nu\nabla_\nu(L-\frac{1}{6}R)\\
& +\frac{4}{3}R_g\nabla_\mu(L-\frac{1}{6}R)+\frac{8}{3}L_\mu{}^\nu\nabla_\nu R_g-\frac{2}{3}L\nabla_\mu R_g\\
= \quad & (4L_\mu{}^\nu-R_\mu{}^\nu)\zeta_\nu+4(2L_{\nu\sigma}-R_{\nu\sigma}+\frac{1}{6}Rg_{\nu\sigma})(\nabla_\mu L^{\nu\sigma}-2\nabla^\sigma L_\mu{}^\nu)\\
& -8\nabla^\nu[(W_{\mu\sigma\nu}{}^\rho-\Theta d_{\mu\sigma\nu}{}^\rho)L_\rho{}^\sigma]+\frac{1}{3}(\zeta_\mu+8L_\mu{}^\nu\nabla_\nu-2L\nabla_\mu)(R_g-R)\\
& -4L_\nu{}^\lambda\nabla^\nu\nabla_{[\lambda}H_{\mu]}-4L_\mu{}^\lambda\nabla^\nu\nabla_{[\lambda}H_{\nu]}+f_\mu(x;H,\nabla H)\ .
\end{aligned} \tag{3.27}$$

We employ (3.10), (3.15), (3.25) and (3.24) to end up with

$$\begin{aligned}
\Box_g\zeta_\mu \quad=\quad & 4L_\mu{}^\nu\zeta_\nu-\frac{1}{6}R\zeta_\mu-8\nabla^\nu[(W_{\mu\sigma\nu}{}^\rho-\Theta d_{\mu\sigma\nu}{}^\rho)L_\rho{}^\sigma]-\frac{2}{3}L\nabla_\mu\nabla_\nu H^\nu\\
& +\frac{2}{3}L_\mu{}^\nu\nabla_\nu\nabla_\sigma H^\sigma-4L_\nu{}^\lambda\nabla^\nu\nabla_{[\lambda}H_{\mu]}+f_\mu(x;H,\nabla H)\ .
\end{aligned} \tag{3.28}$$

In order to get rid of the undesired second-order derivatives in H^σ, we introduce the tensor field

$$K_\mu{}^\nu := \nabla_\mu H^\nu \tag{3.29}$$

[10] Note that in this part the metric is regarded as being given, so $\Box_g$ is a wave-operator and there is no need to work with the reduced wave-operator $\Box_g^{(H)}$.

as another unknown for which we need to derive a wave equation, as well. We employ the fact that the right-hand side of (3.24) does not contain derivatives of ζ^α: Differentiating (3.24) we are straightforwardly led to the desired equation,

$$\begin{aligned}\Box_g K_{\mu\nu} &\equiv \nabla_\mu \Box_g H_\nu + R_\mu{}^\kappa \nabla_\kappa H_\nu + H^\kappa \nabla_\sigma R_{\kappa\nu\mu}{}^\sigma + 2R_{\kappa\nu\mu}{}^\sigma \nabla_\sigma H^\kappa \\ &= \nabla_\mu \zeta_\nu + f_{\mu\nu}(x; H, \nabla H, \nabla K) \ . \qquad (3.30)\end{aligned}$$

Moreover, (3.28) becomes a wave equation for ζ_μ,

$$\begin{aligned}\Box_g \zeta_\mu &= 4L_\mu{}^\nu \zeta_\nu - \frac{1}{6} R\zeta_\mu - 8\nabla^\nu [(W_{\mu\sigma\nu\rho} - \Theta d_{\mu\sigma\nu\rho})L^{\sigma\rho}] \\ &\quad + f_\mu(x; H, \nabla H, \nabla K) \ . \qquad (3.31)\end{aligned}$$

We observe that we need a wave equation for $W_{\mu\sigma\nu\rho} - \Theta d_{\mu\sigma\nu\rho}$ (actually just for its contraction with $L^{\sigma\rho}$, but for later purposes it is useful to show that $\Theta d_{\mu\sigma\nu\rho}$ coincides with the Weyl tensor, which would follow, supposing, as usual, that it is initially true). For this purpose let us introduce the tensor field $\zeta_{\mu\nu\sigma}$,

$$\zeta_{\mu\nu\sigma} := 4\nabla_{[\sigma} L_{\nu]\mu} \ .$$

Note that $\zeta_{[\mu\nu\sigma]} = 0$ for a symmetric $L_{\mu\nu}$.

Starting from the second Bianchi identity, we find with (3.26), (3.10), (3.15) and (3.25)

$$\begin{aligned}\nabla_\alpha W_{\mu\nu\sigma\rho} &\equiv -2\nabla_{[\mu} W_{\nu]\alpha\sigma\rho} + 2\nabla_{[\alpha} R_{\nu][\sigma} g_{\rho]\mu} - 2\nabla_{[\alpha} R_{\mu][\sigma} g_{\rho]\nu} - 2\nabla_{[\mu} R_{\nu][\sigma} g_{\rho]\alpha} \\ &\quad + \frac{2}{3} g_{\mu[\sigma} g_{\rho][\nu} \nabla_{\alpha]} R_g - \frac{1}{3} g_{\alpha[\sigma} g_{\rho]\nu} \nabla_\mu R_g \\ &= g_{\mu[\sigma} \zeta_{\rho]\alpha\nu} + g_{\nu[\sigma} \zeta_{\rho]\mu\alpha} - g_{\alpha[\sigma} \zeta_{\rho]\mu\nu} - 2\nabla_{[\mu} W_{\nu]\alpha\sigma\rho} \\ &\quad + \frac{2}{3} g_{\mu[\sigma} g_{\rho][\nu} \nabla_{\alpha]} \nabla_\kappa H^\kappa - \frac{1}{3} g_{\alpha[\sigma} g_{\rho]\nu} \nabla_\mu \nabla_\kappa H^\kappa + g_{\alpha[\sigma} \nabla_{\rho]} \nabla_{[\mu} H_{\nu]} \\ &\quad + g_{\mu[\sigma} \nabla_{\rho]} \nabla_{[\nu} H_{\alpha]} + g_{\nu[\sigma} \nabla_{\rho]} \nabla_{[\alpha} H_{\mu]} + f_{\alpha\mu\nu\sigma\rho}(x; H, \nabla H) \ . \qquad (3.32)\end{aligned}$$

Applying ∇^α yields

$$\begin{aligned}\Box_g W_{\mu\nu\sigma\rho} &= 2\nabla_{[\nu} \nabla^\alpha W_{\mu]\alpha\sigma\rho} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa + \frac{1}{3} RW_{\mu\nu\sigma\rho} \\ &\quad + 2(g_{\rho[\mu} W_{\nu]\alpha\sigma}{}^\kappa - g_{\sigma[\mu} W_{\nu]\alpha\rho}{}^\kappa) L_\kappa{}^\alpha - 2L_{[\mu}{}^\kappa W_{\nu]\kappa\sigma\rho} - 2L_{[\sigma}{}^\kappa W_{\rho]\kappa\mu\nu} \\ &\quad + \nabla_{[\sigma} \zeta_{\rho]\nu\mu} + g_{\sigma[\mu} \nabla^\alpha \zeta_{|\rho\alpha|\nu]} - g_{\rho[\mu} \nabla^\alpha \zeta_{|\sigma\alpha|\nu]} + \frac{1}{3} g_{\mu[\sigma} g_{\rho]\nu} \nabla_\kappa \Box_g H^\kappa \\ &\quad + \frac{1}{6} g_{\mu[\sigma} \nabla_{\rho]} \nabla_\nu \nabla_\alpha H^\alpha - \frac{1}{6} g_{\nu[\sigma} \nabla_{\rho]} \nabla_\mu \nabla_\alpha H^\alpha - \frac{1}{2} g_{\mu[\sigma} \nabla_{\rho]} \Box_g H_\nu \\ &\quad + \frac{1}{2} g_{\nu[\sigma} \nabla_{\rho]} \Box_g H_\mu + f_{\mu\nu\sigma\rho}(x; H, \nabla H, \nabla K) \ . \qquad (3.33)\end{aligned}$$

Before we manipulate this expression any further it is useful to compute

$$\begin{aligned}\nabla^\alpha \zeta_{\mu\nu\alpha} &\equiv 2\Box_g L_{\mu\nu} - 2\nabla_\nu \nabla_\alpha L_\mu{}^\alpha - 2R_{\alpha\nu\mu}{}^\kappa L_\kappa{}^\alpha - 2R_{\nu\kappa} L_\mu{}^\kappa \\ &= 2\Box_g^{(H)} L_{\mu\nu} + 2L_\alpha{}^\kappa W_{\mu\kappa\nu}{}^\alpha - 3L_\mu{}^\kappa R^{(H)}_{\nu\kappa} - L_\nu{}^\alpha R^{(H)}_{\mu\alpha} + g_{\mu\nu} L^{\alpha\kappa} R^{(H)}_{\alpha\kappa} \\ &\quad - \frac{1}{2} \nabla_\mu \nabla_\nu (R_g - \frac{1}{3} R) + LR^{(H)}_{\mu\nu} + \frac{1}{3} L_{\mu\nu} R_g - \frac{1}{3} LR_g g_{\mu\nu} \\ &\quad + \frac{1}{2} \nabla_\nu \nabla_\kappa \hat{\nabla}_\mu H^\kappa + \frac{1}{2} g_{\mu\kappa} \nabla_\nu \nabla^\alpha \hat{\nabla}_\alpha H^\kappa + f_{\mu\nu}(x; H, \nabla H) \\ &= 2(W_{\mu\alpha\nu}{}^\kappa - 2\Theta d_{\mu\alpha\nu}{}^\kappa) L_\kappa{}^\alpha + \frac{1}{2} \nabla_\nu \Box_g H_\mu \\ &\quad + f_{\mu\nu}(x; H, \nabla H, \nabla K) \ , \qquad (3.34)\end{aligned}$$

which follows from (3.10), (3.11), (3.15), (3.19), (3.21), (3.25) and (3.26). Due to the Bianchi identity, (3.10), (3.15) and (3.25), we also have

$$\begin{aligned}\nabla_\alpha W_{\mu\nu\sigma}{}^\alpha &\equiv -\nabla_{[\mu}R_{\nu]\sigma} - \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}R_g \\ &= \frac{1}{2}\zeta_{\sigma\mu\nu} - \frac{1}{2}\nabla_\sigma\nabla_{[\mu}H_{\nu]} - \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}\nabla_\kappa H^\kappa + f_{\mu\nu\sigma}(x;H,\nabla H)\ .\end{aligned} \tag{3.35}$$

Invoking (3.34) and (3.35) we rewrite (3.33) to obtain

$$\begin{aligned}\Box_g W_{\mu\nu\sigma\rho} &= \nabla_{[\sigma}\zeta_{\rho]\nu\mu} - \nabla_{[\mu}\zeta_{\nu]\sigma\rho} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa + \frac{1}{3}RW_{\mu\nu\sigma\rho} \\ &\quad -2L_{[\mu}{}^\kappa W_{\nu]\kappa\sigma\rho} - 2L_{[\sigma}{}^\kappa W_{\rho]\kappa\mu\nu} + 4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - \Theta d_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} \\ &\quad -4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - \Theta d_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho} + \frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\Box_g H_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\Box_g H_\rho \\ &\quad +\frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\Box_g H_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\Box_g H_\nu + \frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\nabla_\kappa\Box_g H^\kappa \\ &\quad +f_{\mu\nu\sigma\rho}(x;H,\nabla H,\nabla K)\ .\end{aligned} \tag{3.36}$$

We insert (3.24),

$$\begin{aligned}\Box_g W_{\mu\nu\sigma\rho} &= \nabla_{[\sigma}\zeta_{\rho]\nu\mu} - \nabla_{[\mu}\zeta_{\nu]\sigma\rho} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa \\ &\quad -2L_{[\mu}{}^\kappa W_{\nu]\kappa\sigma\rho} - 2L_{[\sigma}{}^\kappa W_{\rho]\kappa\mu\nu} + 4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - \Theta d_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} \\ &\quad -4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - \Theta d_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho} + \frac{1}{3}RW_{\mu\nu\sigma\rho} + \frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\nabla_\kappa\zeta^\kappa \\ &\quad +\frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\zeta_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\zeta_\rho + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\zeta_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\zeta_\nu \\ &\quad +f_{\mu\nu\sigma\rho}(x;H,\nabla H,\nabla K)\ .\end{aligned} \tag{3.37}$$

It proves useful to make the following definitions:

$$\varkappa_{\mu\nu\sigma} := \frac{1}{2}\zeta_{\mu\nu\sigma} - \nabla_\kappa\Theta d_{\nu\sigma\mu}{}^\kappa\ , \tag{3.38}$$

$$\Xi_{\mu\nu} := \nabla_\mu\nabla_\nu\Theta + \Theta L_{\mu\nu} - sg_{\mu\nu}\ . \tag{3.39}$$

We observe the relation

$$\nabla_\rho\zeta_{\mu\nu\sigma} = 2\nabla_\rho\varkappa_{\mu\nu\sigma} + 2\nabla^\kappa\Theta\nabla_\rho d_{\nu\sigma\mu\kappa} + 2\Xi_{\rho\kappa}d_{\nu\sigma\mu}{}^\kappa - 2L_\rho{}^\kappa\Theta d_{\nu\sigma\mu\kappa} + 2sd_{\nu\sigma\mu\rho}\ .$$

Then, due to the (anti-)symmetry properties of the tensor $d_{\mu\nu\sigma\rho}$ derived above, (3.37) yields

$$\begin{aligned}\Box_g W_{\mu\nu\sigma\rho} &= 2\nabla^\kappa\Theta\nabla_{[\sigma}d_{\rho]\kappa\nu\mu} - 2\nabla^\kappa\Theta\nabla_{[\mu}d_{\nu]\kappa\sigma\rho} + 2\nabla_{[\sigma}\varkappa_{\rho]\nu\mu} - 2\nabla_{[\mu}\varkappa_{\nu]\sigma\rho} \\ &\quad +2d_{\nu\mu[\rho}{}^\kappa\Xi_{\sigma]\kappa} - 2d_{\sigma\rho[\nu}{}^\kappa\Xi_{\mu]\kappa} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa \\ &\quad +\frac{1}{3}RW_{\mu\nu\sigma\rho} + 2L_{[\mu}{}^\kappa(\Theta d_{\nu]\kappa\sigma\rho} - W_{\nu]\kappa\sigma\rho}) - 2L_{[\sigma}{}^\kappa(\Theta d_{\rho]\kappa\nu\mu} - W_{\rho]\kappa\nu\mu}) \\ &\quad +4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - \Theta d_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} - 4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - \Theta d_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho} \\ &\quad +\frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\zeta_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\zeta_\rho + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\zeta_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\zeta_\nu \\ &\quad +\frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\nabla_\kappa\zeta^\kappa + 4sd_{\mu\nu\sigma\rho} + f_{\mu\nu\sigma\rho}(x;H,\nabla H,\nabla K)\ .\end{aligned} \tag{3.40}$$

On the other hand, in virtue of (3.13) and (3.14), we have

$$\begin{aligned}\Box_g(\Theta d_{\mu\nu\sigma\rho}) &\equiv d_{\mu\nu\sigma\rho}\Box_g\Theta + \Theta\Box_g d_{\mu\nu\sigma\rho} + 2\nabla^\kappa\Theta\nabla_\kappa d_{\mu\nu\sigma\rho}\\ &= 4s d_{\mu\nu\sigma\rho} + 2\nabla^\kappa\Theta\nabla_\kappa d_{\mu\nu\sigma\rho} + \Theta^2 d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta^2 d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa\\ &\quad+\frac{1}{3}R\Theta d_{\mu\nu\sigma\rho} + f_{\mu\nu\sigma\rho}(x;H,\nabla H)\;. \end{aligned}\tag{3.41}$$

Combining (3.40) and (3.41), and invoking (3.5), we are led to the wave equation

$$\begin{aligned}\Box_g(W_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\sigma\rho}) \;=\;& 2\nabla_{[\sigma}\varkappa_{\rho]\nu\mu} - 2\nabla_{[\mu}\varkappa_{\nu]\sigma\rho} + 2d_{\mu\nu[\sigma}{}^\kappa\Xi_{\rho]\kappa} + 2d_{\sigma\rho[\mu}{}^\kappa\Xi_{\nu]\kappa}\\ &+W_{\mu\nu\alpha}{}^\kappa(W_{\sigma\rho\kappa}{}^\alpha - \Theta d_{\sigma\rho\kappa}{}^\alpha) + \Theta d_{\sigma\rho\kappa}{}^\alpha(W_{\mu\nu\alpha}{}^\kappa - \Theta d_{\mu\nu\alpha}{}^\kappa)\\ &-4W_{\sigma\kappa[\mu}{}^\alpha(W_{\nu]\alpha\rho}{}^\kappa - \Theta d_{\nu]\alpha\rho}{}^\kappa) - 4(W_{\sigma\kappa[\mu}{}^\alpha - \Theta d_{\sigma\kappa[\mu}{}^\alpha)\Theta d_{\nu]\alpha\rho}{}^\kappa\\ &-2L_{[\mu}{}^\kappa(W_{\nu]\kappa\sigma\rho} - \Theta d_{\nu]\kappa\sigma\rho}) + 2L_{[\sigma}{}^\kappa(W_{\rho]\kappa\nu\mu} - \Theta d_{\rho]\kappa\nu\mu})\\ &+4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - \Theta d_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} - 4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - \Theta d_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho}\\ &+\frac{1}{3}R(W_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\sigma\rho}) - \frac{1}{2}\nabla^\kappa\Theta(\epsilon_{\kappa\sigma\rho}{}^\delta\epsilon_{\mu\nu}{}^{\beta\gamma} + \epsilon_{\kappa\mu\nu}{}^\delta\epsilon_{\sigma\rho}{}^{\beta\gamma})\nabla_\alpha d_{\beta\gamma\delta}{}^\alpha\\ &+\frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\zeta_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\zeta_\rho + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\zeta_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\zeta_\nu\\ &+\frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\nabla_\kappa\zeta^\kappa + f_{\mu\nu\sigma\rho}(x;H,\nabla H,\nabla K)\;, \end{aligned}\tag{3.42}$$

which is fulfilled by any solution of the CWE.

In order to end up with a homogeneous system of wave equations, it remains to derive wave equations for $\varkappa_{\mu\nu\sigma}$, $\Xi_{\mu\nu}$ and $\nabla_\rho d_{\mu\nu\sigma}{}^\rho$. Let us start with $\nabla_\rho d_{\mu\nu\sigma}{}^\rho$,

$$\begin{aligned}\Box_g\nabla_\rho d_{\mu\nu\sigma}{}^\rho &\equiv \nabla_\rho\Box_g d_{\mu\nu\sigma}{}^\rho - 4W_{\kappa\rho[\mu}{}^\alpha\nabla^\kappa d_{\nu]\alpha\sigma}{}^\rho + 2W_{\kappa\rho\sigma}{}^\alpha\nabla^\kappa d_{\mu\nu\alpha}{}^\rho\\ &\quad-2d_{\mu\nu\rho}{}^\alpha\nabla_{[\sigma}R_{\alpha]}{}^\rho - 2d_{\sigma\rho\nu}{}^\alpha\nabla_{[\mu}R_{\alpha]}{}^\rho + 2d_{\sigma\rho\mu}{}^\alpha\nabla_{[\nu}R_{\alpha]}{}^\rho\\ &\quad+2R_{\rho[\mu}\nabla^\alpha d_{\nu]\alpha\sigma}{}^\rho + R_\sigma{}^\rho\nabla_\alpha d_{\mu\nu\rho}{}^\alpha + 3R_\rho{}^\alpha\nabla_{[\mu}d_{\alpha\nu]\sigma}{}^\rho\\ &\quad-\frac{1}{2}d_{\mu\nu\sigma}{}^\alpha\nabla_\alpha R_g\\ &= 2d_{\mu\nu\rho}{}^\alpha\varkappa^\rho{}_{\sigma\alpha} - 4d_{\sigma\rho[\mu}{}^\alpha\varkappa^\rho{}_{\nu]\alpha} + (W_{\kappa\sigma\rho}{}^\alpha - \Theta d_{\kappa\sigma\rho}{}^\alpha)\nabla^\kappa d_{\mu\nu\alpha}{}^\rho\\ &\quad-4(W_{\kappa\rho[\mu}{}^\alpha - \Theta d_{\kappa\rho[\mu}{}^\alpha)\nabla^\kappa d_{\nu]\alpha\sigma}{}^\rho + \frac{1}{2}R^{\rho\alpha}\epsilon_{\mu\alpha\nu}{}^\delta\epsilon_{\sigma\rho}{}^{\beta\gamma}\nabla_\lambda d_{\beta\gamma\delta}{}^\lambda\\ &\quad+2R_{[\mu}{}^\alpha\nabla_{|\rho}d_{\sigma\alpha|\nu]}{}^\rho + \Theta d_{\mu\nu\kappa}{}^\alpha\nabla_\rho d_\alpha{}^\kappa{}_\sigma{}^\rho + 4\Theta d_\sigma{}^\kappa{}_{[\mu}{}^\alpha\nabla_{|\rho|}d_{\nu]\alpha\kappa}{}^\rho\\ &\quad+(R_\sigma{}^\alpha + \frac{1}{2}R\delta_\sigma{}^\alpha)\nabla_\rho d_{\mu\nu\alpha}{}^\rho + f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K)\;. \end{aligned}\tag{3.43}$$

The validity of the last equality follows from (3.10), (3.14), (3.15), (3.25) and (3.5). Note that to establish (3.5) one just needs the algebraic properties of $d_{\mu\nu\sigma}{}^\rho$ which are ensured by Lemma 3.4.

Next, let us derive a wave equation for $\Xi_{\mu\nu}$. With (3.11)-(3.13), (3.15), (3.10), (3.19) and (3.25) the following relation is verified,

$$\begin{aligned}\Box_g\Xi_{\mu\nu} &\equiv \nabla_\mu\nabla_\nu\Box_g\Theta + 2\nabla_{(\mu}R_{\nu)\kappa}\nabla^\kappa\Theta + 2R_{\kappa(\mu}\nabla_{\nu)}\nabla^\kappa\Theta + 2R_{\sigma\mu\nu}{}^\kappa\nabla^\sigma\nabla_\kappa\Theta\\ &\quad-\nabla_\kappa R_{\mu\nu}\nabla^\kappa\Theta + L_{\mu\nu}\Box_g\Theta + \Theta\Box_g L_{\mu\nu} + 2\nabla^\sigma\Theta\nabla_\sigma L_{\mu\nu} - g_{\mu\nu}\Box_g s\\ &= 2(2L_{(\mu}{}^\kappa\delta_{\nu)}{}^\sigma - g_{\mu\nu}L^{\sigma\kappa} - W_\mu{}^\sigma{}_\nu{}^\kappa)\Xi_{\sigma\kappa} + 2\Theta L_{\sigma\kappa}(W_\mu{}^\sigma{}_\nu{}^\kappa - \Theta d_\mu{}^\sigma{}_\nu{}^\kappa)\\ &\quad+4\nabla_{(\mu}\Upsilon_{\nu)} + \frac{1}{6}R\Xi_{\mu\nu} + f_{\mu\nu}(x;H,\nabla H,\nabla K)\;, \end{aligned}\tag{3.44}$$

where we have set

$$\Upsilon_\mu := \nabla_\mu s + L_{\mu\nu}\nabla^\nu\Theta \ . \tag{3.45}$$

Of course, we also need a wave equation for Υ_μ. Using again (3.11)-(3.13), (3.15) as well as (3.10) and (3.25) we find that

$$\begin{aligned}\Box_g\Upsilon_\mu &\equiv \nabla_\mu\Box_g s + R_\mu{}^\kappa\nabla_\kappa s + \Box_g L_{\mu\nu}\nabla^\nu\Theta + L_\mu{}^\nu\nabla_\nu\Box_g\Theta + L_\mu{}^\nu R_\nu{}^\kappa\nabla_\kappa\Theta \\ &\quad +2\nabla_\sigma L_{\mu\nu}\nabla^\sigma\nabla^\nu\Theta \\ &= 6L_\mu{}^\kappa\Upsilon_\kappa + 2\Theta L^{\rho\kappa}\varkappa_{\rho\kappa\mu} + 2\Xi_{\nu\sigma}\nabla^\sigma L_\mu{}^\nu - \frac{1}{6}\Xi_\mu{}^\nu\nabla_\nu R \\ &\quad + f_\mu(x;H,\nabla H,\nabla K) \ . \end{aligned} \tag{3.46}$$

Finally, let us derive a wave equation which is satisfied by $\varkappa_{\mu\nu\sigma} \equiv \frac{1}{2}\zeta_{\mu\nu\sigma} - \nabla_\kappa\Theta d_{\nu\sigma\mu}{}^\kappa$. The definition of the Weyl tensor (3.26) together with the Bianchi identities yield

$$\begin{aligned}\frac{1}{2}\Box_g\zeta_{\mu\nu\sigma} &\equiv 2\nabla_{[\sigma}\Box_g L_{\nu]\mu} - 2W_{\nu\sigma\kappa\rho}\nabla^\rho L_\mu{}^\kappa + 4W_{\mu\kappa\rho[\sigma}\nabla^\rho L_{\nu]}{}^\kappa - 2R_{\kappa[\nu}\nabla_{\sigma]}L_\mu{}^\kappa \\ &\quad +2R_{\kappa[\sigma}\nabla_{|\mu|}L_{\nu]}{}^\kappa - 2R_{\mu[\sigma}\nabla_{|\kappa|}L_{\nu]}{}^\kappa - 2R_{\rho\kappa}g_{\mu[\sigma}\nabla^\rho L_{\nu]}{}^\kappa + \frac{1}{6}R_g\zeta_{\mu\nu\sigma} \\ &\quad +\frac{2}{3}R_g g_{\mu[\sigma}\nabla^\kappa L_{\nu]\kappa} + 2L_\mu{}^\kappa\nabla_{[\nu}R_{\sigma]\kappa} + 2L_\nu{}^\kappa\nabla_{[\mu}R_{\kappa]\sigma} + 2L_\sigma{}^\kappa\nabla_{[\kappa}R_{\mu]\nu} \\ &= 2\zeta_{\mu\kappa[\sigma}L_{\nu]}{}^\kappa + 3\zeta_{\alpha[\nu\sigma}g_{\kappa]\mu}L^{\alpha\kappa} + 4L_\rho{}^\kappa\nabla_{[\nu}(\Theta d_{\sigma]\kappa\mu}{}^\rho) + 2\Theta\zeta_{\alpha\kappa[\nu}d_{\sigma]}{}^\kappa{}_\mu{}^\alpha \\ &\quad +4(W_\mu{}^\rho{}_{[\nu}{}^\kappa - \Theta d_\mu{}^\rho{}_{[\nu}{}^\kappa)\nabla_{|\kappa|}L_{\sigma]\rho} - \zeta_{\mu\alpha\kappa}W_\nu{}^\alpha{}_\sigma{}^\kappa + \frac{1}{3}L_{\mu[\nu}\nabla_{\sigma]}R \\ &\quad +\frac{1}{6}(R_{\sigma\nu\mu}{}^\kappa + 2g_{\mu[\nu}L_{\sigma]}{}^\kappa)\nabla_\kappa R + \frac{1}{12}R_g\zeta_{\mu\nu\sigma} + f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K) \ ,\end{aligned}$$

where the last equality follows from (3.10), (3.11), (3.15) and (3.25). We employ (3.13)-(3.15) and (3.10) to deduce that

$$\begin{aligned}\Box_g(\nabla_\kappa\Theta d_{\nu\sigma\mu}{}^\kappa) &\equiv d_{\nu\sigma\mu}{}^\kappa(\nabla_\kappa\Box_g\Theta + R_\kappa{}^\rho\nabla_\rho\Theta) + \nabla_\kappa\Theta\Box_g d_{\nu\sigma\mu}{}^\kappa + 2\nabla_\alpha\nabla_\kappa\Theta\nabla^\alpha d_{\nu\sigma\mu}{}^\kappa \\ &= 4\Upsilon_\kappa d_{\nu\sigma\mu}{}^\kappa - 2L_\kappa{}^\rho\nabla_\rho(\Theta d_{\nu\sigma\mu}{}^\kappa) + 2\Xi_{\lambda\kappa}\nabla^\lambda d_{\nu\sigma\mu}{}^\kappa + 2s\nabla_\kappa d_{\nu\sigma\mu}{}^\kappa \\ &\quad +\Theta(\frac{1}{2}\zeta_{\mu\lambda}{}^\alpha - \varkappa_{\mu\lambda}{}^\alpha)d_{\nu\sigma\alpha}{}^\lambda - \Theta(2\zeta_{\alpha\lambda[\sigma} - 4\varkappa_{\alpha\lambda[\sigma})d_{\nu]}{}^\lambda{}_\mu{}^\alpha \\ &\quad +\frac{1}{2}Rd_{\nu\sigma\mu}{}^\kappa\nabla_\kappa\Theta - \frac{1}{6}\Theta d_{\nu\sigma\mu}{}^\kappa\nabla_\kappa R + f_{\mu\nu\sigma}(x;H,\nabla H) \ .\end{aligned}$$

With (3.25) we are led to

$$\begin{aligned}\Box_g\varkappa_{\mu\nu\sigma} &= 4\nabla^\beta\Theta\big\{g_{\beta[\nu}d_{\sigma]\kappa\mu}{}^\alpha L_\alpha{}^\kappa - g_{\mu[\nu}d_{\sigma]\kappa\beta}{}^\alpha L_\alpha{}^\kappa - d_{\mu\beta\kappa[\nu}L_{\sigma]}{}^\kappa \\ &\qquad -d_{\nu\sigma\kappa[\mu}L_{\beta]}{}^\kappa - \frac{1}{12}Rd_{\nu\sigma\mu\beta}\big\} \\ &\quad +6\Theta L_\rho{}^\kappa\nabla_{[\nu}d_{\sigma\kappa]\mu}{}^\rho - 2\Xi_{\lambda\kappa}\nabla^\lambda d_{\nu\sigma\mu}{}^\kappa - 4\Upsilon_\kappa d_{\nu\sigma\mu}{}^\kappa \\ &\quad +4(W_\mu{}^\rho{}_{[\nu}{}^\kappa - \Theta d_\mu{}^\rho{}_{[\nu}{}^\kappa)\nabla_{|\kappa|}L_{\sigma]\rho} - \frac{1}{2}\zeta_{\mu\kappa}{}^\alpha(W_{\nu\sigma\alpha}{}^\kappa - \Theta d_{\nu\sigma\alpha}{}^\kappa) \\ &\quad -4\varkappa_{\mu\kappa[\nu}L_{\sigma]}{}^\kappa + 6\varkappa_{\alpha[\nu\sigma}g_{\kappa]\mu}L^{\alpha\kappa} + \Theta\varkappa_{\mu\lambda}{}^\alpha d_{\nu\sigma\alpha}{}^\lambda - 4\Theta\varkappa_{\alpha\lambda[\sigma}d_{\nu]}{}^\lambda{}_\mu{}^\alpha \\ &\quad -2s\nabla_\kappa d_{\nu\sigma\mu}{}^\kappa - \frac{1}{6}(W_{\nu\sigma\mu}{}^\kappa - \Theta d_{\nu\sigma\mu}{}^\kappa)\nabla_\kappa R + \frac{1}{6}R_g\varkappa_{\mu\nu\sigma} \\ &\quad +f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K) \ .\end{aligned} \tag{3.47}$$

The term in braces needs to be eliminated. To this end let us consider the expression (we use the implications of Lemma 3.4)

$$\begin{aligned}3\nabla^{\lambda}\nabla_{[\lambda}d_{\mu\nu]\sigma\rho} &= \Box_g d_{\mu\nu\sigma\rho} - \nabla_\nu\nabla_\lambda d_{\sigma\rho\mu}{}^{\lambda} + \nabla_\mu\nabla_\lambda d_{\sigma\rho\nu}{}^{\lambda} \\ &\quad - W_{\mu\nu\lambda}{}^{\kappa}d_{\sigma\rho\kappa}{}^{\lambda} + 2W_{\lambda\nu[\sigma}{}^{\kappa}d_{\rho]\kappa\mu}{}^{\lambda} - 2W_{\lambda\mu[\sigma}{}^{\kappa}d_{\rho]\kappa\nu}{}^{\lambda} \\ &\quad - d_{\sigma\rho[\mu}{}^{\kappa}R_{\nu]\kappa} - d_{\mu\nu[\sigma}{}^{\lambda}R_{\rho]\lambda} + g_{\mu[\sigma}d_{\rho]\kappa\nu\lambda}R^{\lambda\kappa} - g_{\nu[\sigma}d_{\rho]\kappa\mu\lambda}R^{\lambda\kappa}\,.\end{aligned}$$

We take (3.5), (3.10), (3.14) and (3.15) into account to rewrite this equation as

$$\begin{aligned}&2g_{\nu[\sigma}d_{\rho]\kappa\mu}{}^{\alpha}L_{\alpha}{}^{\kappa} - 2g_{\mu[\sigma}d_{\rho]\kappa\nu}{}^{\alpha}L_{\alpha}{}^{\kappa} - 2d_{\mu\nu\kappa[\sigma}L_{\rho]}{}^{\kappa} - 2d_{\sigma\rho\kappa[\mu}L_{\nu]}{}^{\kappa} - \frac{1}{6}Rd_{\mu\nu\sigma\rho} \\ &\quad\equiv 2(W_{\sigma\kappa[\mu}{}^{\alpha} - \Theta d_{\sigma\kappa[\mu}{}^{\alpha})d_{\nu]\alpha\rho}{}^{\kappa} - 2(W_{[\mu|\alpha\rho}{}^{\kappa} - \Theta d_{[\mu|\alpha\rho}{}^{\kappa})d_{\sigma\kappa|\nu]}{}^{\alpha} \\ &\quad\quad - (W_{\mu\nu\kappa}{}^{\alpha} - \Theta d_{\mu\nu\kappa}{}^{\alpha})d_{\sigma\rho\alpha}{}^{\kappa} + 2\nabla_{[\mu}\nabla_{|\lambda}d_{\sigma\rho|\nu]}{}^{\lambda} \\ &\quad\quad - \frac{1}{2}\epsilon_{\lambda\mu\nu}{}^{\kappa}\epsilon_{\sigma\rho}{}^{\beta\gamma}\nabla^{\lambda}\nabla_{\alpha}d_{\beta\gamma\kappa}{}^{\alpha} + f_{\mu\nu\sigma\rho}(x;H,\nabla H)\,. \qquad (3.48)\end{aligned}$$

Combining (3.48) with (3.47) and (3.5) yields a wave equation for $\varkappa_{\mu\nu\sigma}$,

$$\begin{aligned}\Box_g\varkappa_{\mu\nu\sigma} &= 4\nabla^{\beta}\Theta[(W_{\nu\kappa[\mu}{}^{\alpha} - \Theta d_{\nu\kappa[\mu}{}^{\alpha})d_{\beta]\alpha\sigma}{}^{\kappa} - (W_{\sigma\kappa[\mu}{}^{\alpha} - \Theta d_{\sigma\kappa[\mu}{}^{\alpha})d_{\beta]\alpha\nu}{}^{\kappa}] \\ &\quad + 2(W_{\mu\beta\kappa}{}^{\alpha} - \Theta d_{\mu\beta\kappa}{}^{\alpha})\nabla^{\beta}\Theta d_{\sigma\nu\alpha}{}^{\kappa} + 4\nabla^{\beta}\Theta\nabla_{[\beta}(\nabla_{\lambda}d_{|\sigma\nu|\mu]}{}^{\lambda}) \\ &\quad + \epsilon_{\lambda\mu\beta}{}^{\kappa}\epsilon_{\sigma\nu}{}^{\delta\gamma}\nabla^{\beta}\Theta\nabla^{\lambda}(\nabla_{\alpha}d_{\delta\gamma\kappa}{}^{\alpha}) + \Theta L_{\rho}{}^{\kappa}\epsilon_{\sigma\kappa\nu}{}^{\delta}\epsilon_{\mu}{}^{\rho\beta\gamma}\nabla_{\alpha}d_{\beta\gamma\delta}{}^{\alpha} \\ &\quad - 2\Xi_{\lambda\kappa}\nabla^{\lambda}d_{\nu\sigma\mu}{}^{\kappa} - 4\Upsilon_{\kappa}d_{\nu\sigma\mu}{}^{\kappa} + 4(W_{\mu}{}^{\rho}{}_{[\nu}{}^{\kappa} - \Theta d_{\mu}{}^{\rho}{}_{[\nu}{}^{\kappa})\nabla_{|\kappa|}L_{\sigma]\rho} \\ &\quad - 4\varkappa_{\mu\kappa[\nu}L_{\sigma]}{}^{\kappa} + 6\varkappa_{\alpha[\nu\sigma}g_{\kappa]\mu}L^{\alpha\kappa} + \frac{1}{2}\zeta_{\mu\kappa}{}^{\alpha}(W_{\nu\sigma\alpha}{}^{\kappa} - \Theta d_{\nu\sigma\alpha}{}^{\kappa}) \\ &\quad + \Theta\varkappa_{\mu\lambda}{}^{\alpha}d_{\nu\sigma\alpha}{}^{\lambda} + 4\Theta\varkappa_{\alpha\lambda[\nu}d_{\sigma]}{}^{\lambda}{}_{\mu}{}^{\alpha} - \frac{1}{6}(W_{\nu\sigma\mu}{}^{\kappa} - \Theta d_{\nu\sigma\mu}{}^{\kappa})\nabla_{\kappa}R \\ &\quad - 2s\nabla_{\kappa}d_{\nu\sigma\mu}{}^{\kappa} + \frac{1}{6}R_g\varkappa_{\mu\nu\sigma} + f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K)\,. \qquad (3.49)\end{aligned}$$

The equations (3.24), (3.30), (3.31), (3.42), (3.43), (3.44), (3.46) and (3.49) form a closed, linear, homogeneous system of linear wave equations satisfied by the fields H^{σ}, $K_{\mu\nu}$, ζ_{μ}, $W_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\sigma\rho}$, $\nabla_{\rho}d_{\mu\nu\sigma}{}^{\rho}$, $\Xi_{\mu\nu}$, Υ_{μ} and $\varkappa_{\mu\nu\sigma}$, with all other quantities regarded as being given. An application of standard uniqueness results, cf. e.g. [15], establishes that all the fields vanish, supposing that this is initially the case. In particular this guarantees consistency with the gauge condition, i.e. $H^{\sigma} = 0$ and, by (3.9), $R_g = R$, for solutions of the CWE. In fact we have proven more, and that will be of importance in the next section.

3.3 Equivalence issue between the CWE and the MCFE

Recall the CWE (3.11)-(3.15) and the MCFE (2.5)-(2.10). Let us tackle the equivalence issue between them. A look at the derivation of the CWE reveals that any solution of the MCFE which satisfies the gauge condition $H^{\sigma} = 0$ will be a solution of the CWE with gauge source function $R = R_g$. The other direction is the more interesting albeit more involved one. We therefore devote ourselves subsequently to the issue whether (or rather under which conditions) a solution of the CWE is also a solution of the MCFE. We shall demonstrate that a solution of the CWE is a solution of the MCFE supposing that it satisfies certain relations on the initial surface. In fact, most of the work has already been done in the previous section.

We have the following intermediate result; we emphasize that the conformal factor is allowed to have zeros, or vanish, on the initial surface:

THEOREM 3.7 *Assume we have been given data ($\mathring{g}_{\mu\nu}$, $\mathring{s}$, $\mathring{\Theta}$, $\mathring{L}_{\mu\nu}$, $\mathring{d}_{\mu\nu\sigma}{}^{\rho}$) on a characteristic initial surface S (for definiteness we think either of two transversally intersecting null hypersurfaces or a light-cone) and a gauge source function R, such that $\mathring{g}_{\mu\nu}$ is the restriction to S of a Lorentzian metric, $\mathring{L}_{\mu\nu}$ is symmetric, $\mathring{L}_{\mu}{}^{\mu} \equiv \mathring{L} = \overline{R}/6$, and such that $\mathring{d}_{\mu\nu\sigma}{}^{\rho}$ satisfies all the algebraic properties of the Weyl tensor (cf. the assumptions of Lemma 3.4). Suppose further that there exists a solution ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^{\rho}$) of the CWE (3.11)-(3.15) with gauge source function R which induces the above data on S and fulfills the following conditions:*

1. *The MCFE (2.5)-(2.8) are fulfilled on S;*
2. *equation (2.9) holds at one point on S;*
3. *the Weyl tensor $W_{\mu\nu\sigma}{}^{\rho}[g]$ coincides on S with $\mathring{\Theta}\mathring{d}_{\mu\nu\sigma}{}^{\rho}$;*
4. *the wave-gauge vector H^{σ} and its covariant derivative $K_{\mu}{}^{\sigma} \equiv \nabla_{\mu} H^{\sigma}$ vanish on S;*
5. *the covector field $\zeta_{\mu} \equiv -4(\nabla_{\nu} L_{\mu}{}^{\nu} - \frac{1}{6}\nabla_{\mu} R)$ vanishes on S.*

Then

a) $H^{\sigma} = 0$ and $R_g = R$;

b) $L_{\mu\nu}$ is the Schouten tensor of $g_{\mu\nu}$;

c) $\Theta d_{\mu\nu\sigma}{}^{\rho}$ is the Weyl tensor of $g_{\mu\nu}$;

d) ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^{\rho}$) solves the MCFE (2.5)-(2.10) with $H^{\sigma} = 0$ and $R_g = R$.

The conditions 1-5 are necessary for d) to be true.

PROOF: The conditions 1 and 3-5 make sure that the fields H^{σ}, $K_{\mu\nu}$, ζ_{μ}, $W_{\mu\nu\sigma\rho} - \Theta d_{\mu\nu\sigma\rho}$, $\nabla_{\rho} d_{\mu\nu\sigma}{}^{\rho}$, $\Xi_{\mu\nu}$, Υ_{μ} and $\varkappa_{\mu\nu\sigma}$ vanish on S. In the previous section we have seen that they provide a solution of the closed, linear, homogeneous system of wave equations (3.24), (3.30), (3.31), (3.42), (3.43), (3.44), (3.46) and (3.49), so that all these fields need to vanish identically. In particular that implies $H^{\sigma} = 0$, that $\Theta d_{\mu\nu\sigma}{}^{\rho}$ is the Weyl tensor of $g_{\mu\nu}$, and that (2.5)-(2.8) hold. The vanishing of H^{σ} guarantees that the Ricci tensor coincides with the reduced Ricci tensor and by (3.25) that R is the curvature scalar R_g of $g_{\mu\nu}$. Equation (3.15) then tells us that $L_{\mu\nu}$ is the Schouten tensor. Hence (2.10) is an identity and automatically satisfied. To establish (2.9), it suffices to check that it is satisfied at one point, which is ensured by condition 2. □

In the following we shall investigate to what extent the conditions 1-5 are satisfied if the fields $\mathring{g}_{\mu\nu}$, $\mathring{L}_{\mu\nu}$, $\mathring{d}_{\mu\nu\sigma}{}^{\rho}$, $\mathring{\Theta}$ and $\mathring{s}$ are constructed as solutions of the constraint equations induced by the MCFE on the initial surface.

4 Constraint equations induced by the MCFE on the C_{i^-}-cone

4.1 Adapted null coordinates and another gauge freedom

The aim of this section is to derive the set of constraint equations induced by the MCFE,

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0 \,, \tag{4.1}$$
$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta \, d_{\nu\mu\sigma}{}^\rho \,, \tag{4.2}$$
$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} \,, \tag{4.3}$$
$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta \,, \tag{4.4}$$
$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = 0 \,, \tag{4.5}$$
$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2\left(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}\right) \,, \tag{4.6}$$

on the initial surface S, where we assume henceforth

$$\lambda = 0 \,. \tag{4.7}$$

By *constraint equations* we mean intrinsic equations on the initial surface which determine the fields $g_{\mu\nu}|_S$, $L_{\mu\nu}|_S$, $d_{\mu\nu\sigma}{}^\rho|_S$, $\Theta|_S$ and $s|_S$ starting from suitable free "reduced" data. We shall do this in adapted null coordinates and imposing a generalized wave-map gauge condition. To avoid too many case distinctions we shall derive them in the case where the initial surface is the light-cone C_{i^-} on which the conformal factor Θ vanishes (this requires (4.7), cf. (2.9) evaluated on C_{i^-}), which is completely sufficient for our purposes.

Adapted null coordinates (u, r, x^A) are defined in such a way that $\{x^0 \equiv u = 0\} = \mathscr{I}^- \equiv C_{i^-} \setminus \{i^-\}$, $x^1 \equiv r > 0$ parameterizes the null rays emanating from i^-, and x^A, $A = 2, 3$, are local coordinates on the level sets $\{r = \mathrm{const}, u = 0\} \cong S^2$ (note that these coordinates are singular at the tip, see [5] for more details).

First we shall sketch how the constraint equations are obtained in a generalized wave-map gauge with arbitrary gauge functions. We shall write them down explicitly in a specific gauge afterwards.

We use the same notation as in [5], i.e. $\nu_0 := \overline{g}_{01}$, $\nu_A := \overline{g}_{0A}$. The function $\chi_A{}^B := \frac{1}{2}\overline{g}^{BC}\partial_1 \overline{g}_{AC}$ denotes the *null second fundamental form*, the function τ, which describes the *expansion* of the cone, its trace, and the *shear tensor* $\sigma_A{}^B$ its traceless part. The symbols $\tilde\nabla_A$, $\tilde\Gamma^C_{AB}$ and $\tilde R_{AB}$ refer to the r-dependent Riemannian metric $\tilde g := \overline{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B$.

The equation (4.12) below together with regularity conditions at the tip of the cone imply that $\tilde g$ is conformal to the the standard metric s_{AB} on the 2-sphere S^2. It therefore makes sense to take as reduced data the $\tilde g$ -trace-free part of L_{AB} on C_{i^-} (which coincides with its s-trace-free part). It will be denoted by $\breve{\overline{L}}_{AB} =: \omega_{AB}$.

The field ω_{AB} is an r-dependent tensor on S^2. Here and in what follows $\breve{.}$ denotes the $\tilde g$-trace-free part of the corresponding 2-tensor on S^2. As before, overlining is used to indicate restriction to the initial surface. The gauge degrees of freedom are comprised by R, W^λ, $\overline{s}$ (cf. Sections 2 and 3.1) and κ. The function κ is given by

$$\kappa := \nu^0 \partial_1 \nu_0 - \frac{1}{2}\tau - \frac{1}{2}\nu_0 (\overline{g}^{\mu\nu}\overline{\Gamma}^0_{\mu\nu} + \overline{W}^0) \,, \tag{4.8}$$

where $\nu^0 := \overline{g}^{01} = (\nu_0)^{-1}$. It reflects the freedom to parameterize the null geodesics generating the initial surface [5]; the choice $\kappa = 0$ corresponds to an affine parameterization.

4.2 Constraint equations in a generalized wave-map gauge

We show that, in the case where the initial surface is C_{i^-}, the constraint equations form a hierarchical system of algebraic equations and ODEs along the generators of C_{i^-}. In doing so, we merely consider those gauge choices W^λ which depend just upon the coordinates and none of the fields appearing in the CWE (cf. footnote 11). To derive the constraint equations we assume we have been given a smooth solution of the MCFE in a generalized wave-map gauge $H^\sigma = 0$, smoothly extendable through C_{i^-} We then evaluate the MCFE on C_{i^-} and eliminate the transverse derivatives. For this we shall assume that the solution satisfies $s_{i^-} \neq 0$, which implies that $\overline{s}^{-1}$ and $(\overline{\partial_0\Theta})^{-1}$ exist near i^- (the existence of the latter one follows e.g. from (4.10) below). The function τ^{-1} needs to exist anyway close to i^- [5]. It should be emphasized that, on a light-cone, the initial data for the ODEs cannot be specified freely but follow from regularity conditions at the vertex. For sufficiently regular gauges the behaviour of the relevant fields near the vertex is computed in [5]. When stating this behaviour below we shall always tacitly assume that the gauge is sufficiently regular.

In the following we shall frequently make use of the formulae (A.8)-(A.25) in [5] for the Christoffel symbols computed in adapted null coordinates on a cone.

We consider (4.3) for $(\mu\nu) = (10), (AB)$ on C_{i^-}, where we take the $\overline{g}^{AB}$-trace of the latter equation,

$$\partial_1\overline{\partial_0\Theta} + (\kappa - \nu^0\partial_1\nu_0)\overline{\partial_0\Theta} = \nu_0\overline{s}\,, \tag{4.9}$$

$$\overline{s} = \frac{1}{2}\tau\nu^0\overline{\partial_0\Theta} \tag{4.10}$$

(note that $\overline{H}^0 = 0$ implies $\kappa = \overline{\Gamma}^1_{11}$ [5]). Differentiating (4.10) and inserting the result into (4.9) we obtain an equation for τ,

$$\partial_1\tau - (\kappa + \partial_1 \log|\overline{s}|)\tau + \frac{1}{2}\tau^2 = 0\,. \tag{4.11}$$

The boundary behaviour is given by $\tau = 2r^{-1} + O(r)$ [5].

Due to our assumption $s_{i^-} \neq 0$ the (AB)-component of (4.3), together with (4.10), provides an equation for $\overline{g}_{AB}$ (at least sufficiently close to the vertex),

$$\overline{s}(\partial_1\overline{g}_{AB} - \tau\overline{g}_{AB}) = 0 \quad \Longleftrightarrow \quad \sigma_{AB} = 0\,. \tag{4.12}$$

The boundary condition is $\overline{g}_{AB} = r^2 s_{AB} + O(r^4)$ [5], with s_{AB} the round sphere metric.

Using the definition of $L_{\mu\nu}$, which can be recovered from (4.6), as well as (4.12), we find that

$$\overline{L}_{11} \equiv -\frac{1}{2}\big(\partial_1\tau - \overline{\Gamma}^1_{11}\tau + \chi_A{}^B\chi_B{}^A\big) = -\frac{1}{2}\partial_1\tau + \frac{1}{2}\kappa\tau - \frac{1}{4}\tau^2\,. \tag{4.13}$$

The gauge condition $\overline{H}^0 = 0$ provides an equation for ν_0,[11]

$$\partial_1\nu^0 + \nu^0(\frac{1}{2}\tau + \kappa) + \frac{1}{2}\overline{V}^0 = 0 \; . \tag{4.14}$$

Here we have set

$$V^\lambda := g^{\mu\nu}\hat{\Gamma}^\lambda_{\mu\nu} + W^\lambda \; .$$

The boundary condition is $\nu_0 = 1 + O(r^2)$ [5]. Equation (4.10) then determines $\overline{\partial_0\Theta}$. The function $\overline{\partial_0 g_{11}}$ is computed from $\kappa = \overline{\Gamma}^1_{11}$,

$$\overline{\partial_0 g_{11}} = 2\partial_1\nu_0 - 2\nu_0\kappa \; . \tag{4.15}$$

We remark that the values of certain transverse derivatives are needed on the way to derive the constraint equations. As a matter of course the constraint equations themselves will not involve any transverse derivatives, for they are not part of the characteristic initial data for the CWE.

Let us introduce the field

$$\xi_A \; := \; -2\nu^0\partial_1\nu_A + 4\nu^0\nu_B\chi_A{}^B + \nu_A\overline{V}^0 + \overline{g}_{AB}\overline{V}^B - \overline{g}_{AD}\overline{g}^{BC}\tilde{\Gamma}^D_{BC} \; . \tag{4.16}$$

In a generalized wave-map gauge we have [5]

$$\xi_A \; = \; -2\overline{\Gamma}^1_{1A} \; . \tag{4.17}$$

Invoking (4.10) and (4.12), equation (4.3) with $(\mu\nu) = (0A)$ can be written as an equation for ξ_A,

$$\xi_A \; = \; 2\partial_A \log|\overline{\partial_0\Theta}| - 2\nu^0\partial_A\nu_0 \; . \tag{4.18}$$

The definition of ξ_A can then be employed to compute ν_A,

$$\nu^0\partial_1\nu_A - \tau\nu^0\nu_A - \frac{1}{2}\nu_A\overline{V}^0 - \frac{1}{2}\overline{g}_{AB}\overline{V}^B + \frac{1}{2}\overline{g}_{AD}\overline{g}^{BC}\tilde{\Gamma}^D_{BC} + \frac{1}{2}\xi_A = 0 \; . \tag{4.19}$$

The boundary condition is given by $\nu_A = O(r^3)$ [5]. The equation $\xi_A = -2\overline{\Gamma}^1_{1A}$ then determines $\overline{\partial_0 g_{1A}}$ algebraically,

$$\overline{\partial_0 g_{1A}} = (\partial_A + \xi_A)\nu_0 + (\partial_1 - \tau)\nu_A \; . \tag{4.20}$$

From (4.9), (4.10) and (4.18) we obtain the relation

$$\partial_1\xi_A \; = \; \partial_A(\tau - 2\kappa) \; ,$$

which yields

$$\begin{aligned} \overline{L}_{1A} \;&\equiv\; \frac{1}{2}(\partial_1 + \tau)\overline{\Gamma}^1_{1A} + \frac{1}{2}\tilde{\nabla}_B\chi_A{}^B - \frac{1}{2}\partial_A\overline{\Gamma}^1_{11} - \frac{1}{2}\partial_A\tau \\ &=\; -\frac{1}{4}\tau\xi_A - \frac{1}{2}\partial_A\tau \; . \end{aligned} \tag{4.21}$$

[11] Recall that we assume W^λ to depend just upon the coordinates, otherwise one would have to be careful here and specify upon which components of which fields W^λ is allowed to depend in order to get the hierarchical system we are about to derive.

We define the function

$$\zeta \;:=\; 2(\partial_1 + \kappa + \frac{1}{2}\tau)\overline{g}^{11} + 2\overline{V}^1 \;. \tag{4.22}$$

For a solution which satisfies the generalized wave-map gauge condition $H^\sigma = 0$ it holds [5] that

$$\zeta \;=\; 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11} \;. \tag{4.23}$$

We find that

$$\begin{aligned} \overline{g}^{AB}\overline{R}_{ACB}{}^{C} &\equiv \tilde{R} - \frac{1}{2}\overline{g}^{1A}\partial_A\tau + \tau\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \frac{1}{2}\tau\overline{g}^{1A}\overline{\Gamma}^1_{1A} + \frac{1}{2}\tau^2\overline{g}^{11} \\ &= \tilde{R} - \frac{1}{2}\overline{g}^{1A}(\partial_A + \frac{1}{2}\xi_A)\tau + \frac{1}{2}\tau\zeta \;. \end{aligned} \tag{4.24}$$

On the other hand, the $\overline{g}^{AB}\overline{R}_{ACB}{}^{C}$-part of (4.6) yields (we set $\xi^A := \overline{g}^{AB}\xi_B$)

$$\begin{aligned} \overline{g}^{AB}\overline{R}_{ACB}{}^{C} &= \overline{g}^{1A}\overline{L}_{1A} + 2\overline{g}^{AB}\overline{L}_{AB} \\ &= (\tilde{\nabla}^A - \frac{1}{2}\xi^A - \frac{1}{4}\tau\overline{g}^{1A})\xi_A - \frac{1}{2}\overline{g}^{1A}\partial_A\tau + (\partial_1 + \tau + \kappa)\zeta \\ &\quad + \tilde{R} - \frac{1}{3}R \;, \end{aligned} \tag{4.25}$$

where we took into account that

$$2\overline{g}^{AB}\overline{L}_{AB} \;\equiv\; (\partial_1 + \tau + \kappa)\zeta + (\tilde{\nabla}_A - \frac{1}{2}\xi_A)\xi^A + \tilde{R} - \frac{1}{3}\overline{R} \;. \tag{4.26}$$

Combining (4.24) and (4.25), we end up with an equation for ζ,

$$(\partial_1 + \frac{1}{2}\tau + \kappa)\zeta + (\tilde{\nabla}_A - \frac{1}{2}\xi_A)\xi^A - \frac{1}{3}\overline{R} \;=\; 0 \;, \tag{4.27}$$

where the boundary condition is $\zeta + 2r^{-1} = O(1)$. Then (4.26) becomes

$$\overline{g}^{AB}\overline{L}_{AB} \;=\; \frac{1}{4}\tau\zeta + \frac{1}{2}\tilde{R} \;. \tag{4.28}$$

The definition (4.22) of ζ can be employed to compute $\overline{g}_{00}$, since $\overline{g}_{00} = \overline{g}^{AB}\nu_A\nu_B - (\nu_0)^2\overline{g}^{11}$. The boundary condition is [5] $\overline{g}^{11} = 1 + O(r^2)$. The equation $\zeta = 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}$ can then be read as an equation for $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$,

$$\overline{g}^{AB}\overline{\partial_0 g_{AB}} = 2\tilde{\nabla}^A\nu_A - \nu_0(\tau\overline{g}^{11} + \zeta) \;. \tag{4.29}$$

An expression for $\overline{L}_{01}$ follows from the relation $g^{\mu\nu}L_{\mu\nu} = \frac{1}{6}R$, which yields

$$\begin{aligned} \overline{L}_{01} &= -\frac{1}{2}\nu^A(\partial_A + \frac{1}{2}\xi_A)\tau + \frac{1}{4}\nu_0\overline{g}^{11}[\partial_1\tau - \kappa\tau + \frac{1}{2}\tau^2] \\ &\quad - \frac{1}{8}\nu_0(\tau\zeta + 2\tilde{R}) + \frac{1}{12}\nu_0\overline{R} \;, \end{aligned} \tag{4.30}$$

where $\nu^A := \overline{g}^{AB}\nu_B$. On the other hand we have

$$\begin{aligned} 2\overline{L}_{01} &\equiv \overline{R}_{01} - \frac{1}{6}\nu_0\overline{R} \\ &\equiv \overline{\partial_0\Gamma^0_{01}} - \partial_1\overline{\Gamma}^0_{00} + (\tilde{\nabla}_A + \frac{1}{2}\tau\nu^0\nu_A)\overline{\Gamma}^A_{01} + (\nu^0\partial_1\nu_0 - \kappa + \tau)\overline{\Gamma}^1_{01} \\ &\quad - (\partial_1 - \nu^0\partial_1\nu_0 + \kappa + \frac{1}{2}\tau)\overline{\Gamma}^A_{0A} - \frac{1}{6}\nu_0\overline{R} \;. \end{aligned}$$

Combining this with the gauge condition $\overline{\partial_0 H^0} = 0$ one determines $\overline{\partial_0 g_{01}}$ and $\overline{\partial^2_{00} g_{11}}$, with boundary condition $\overline{\partial_0 g_{01}} = O(r)$ [5].

Note that up to this stage the initial data ω_{AB} have not entered yet, i.e. all the field components computed so far have a pure gauge-character. Note further that $(\overline{\partial_0 g_{AB}})\breve{}$ can be computed in terms of $\omega_{AB} \equiv \breve{\overline{L}}_{AB} = \frac{1}{2}\breve{\overline{R}}_{AB}$ and those quantities computed so far. (Recall that $(\overline{\partial_0 g_{AB}})\breve{}$ denotes the trace-free part of $\overline{\partial_0 g_{AB}}$ with respect to $\overline{g}_{AB}$, and note that $(.)\breve{}$ always refers to the two free angular indices.)

Equation (4.6) with $(\mu\nu\sigma\kappa) = (0ABC)$, contracted with $\overline{g}^{AB}$, gives an equation for $\overline{L}_{0A}$,

$$\begin{aligned}\overline{L}_{0A} &= -\overline{g}_{AC}\overline{g}^{BD}(\partial_B\overline{\Gamma}^C_{0D} - \overline{\partial_0\Gamma^C_{BD}} + \overline{\Gamma}^\alpha_{0D}\overline{\Gamma}^C_{\alpha B} - \overline{\Gamma}^\alpha_{BD}\overline{\Gamma}^C_{\alpha 0}) \\ &\quad -\nu^0\nu_A\nu^B\overline{L}_{1B} + \nu^B\overline{L}_{AB} + 2\nu^0\nu_A\overline{L}_{01} - \frac{1}{6}\nu_A\tilde{R} \qquad (4.31)\end{aligned}$$

(the right-hand side contains only known quantities). From the definition of $\overline{L}_{0A}$ and the gauge condition $\overline{\partial_0 H^C} = 0$ one then computes $\overline{\partial_0 g_{0A}}$ and $\overline{\partial^2_{00} g_{1A}}$. The relevant boundary condition is $\overline{\partial_0 g_{0A}} = O(r^2)$. The $\tilde{g}$-trace-free part of (4.6) for $(\mu\nu\sigma\kappa) = (0A0B)$ yields $(\overline{\partial^2_{00} g_{AB}})\breve{}$.

The 10 independent components of the rescaled Weyl tensor in adapted null coordinates are

$$\overline{d}_{0101}\ ,\quad \overline{d}_{011A}\ ,\quad \overline{d}_{010A}\ ,\quad \overline{d}_{01AB}\ ,\quad \breve{\overline{d}}_{1A1B}\ ,\quad \breve{\overline{d}}_{0A0B}\ .$$

The $\tilde{g}$-trace-free part of (4.2) with $(\mu\nu\sigma) = (A1B)$ determines $\breve{\overline{d}}_{1A1B}$,

$$\begin{aligned}\breve{\overline{d}}_{1A1B} &= \nu_0(\overline{\partial_0\Theta})^{-1}\Big[(\partial_1 - \frac{1}{2}\tau)\omega_{AB} + \overline{L}_{11}\breve{\overline{\Gamma}}^1_{AB} - \tilde{\nabla}_A\overline{L}_{1B} + \frac{1}{2}\xi_B\overline{L}_{1A} \\ &\quad + \frac{1}{2}\overline{g}_{AB}(\tilde{\nabla}^C - \frac{1}{2}\xi^C)\overline{L}_{1C}\Big]\ . \qquad (4.32)\end{aligned}$$

All the remaining components of the rescaled Weyl tensor can be determined from (4.1). We will be rather sketchy here. For $(\mu\nu\sigma) = (1A1)$ one finds

$$\overline{\nabla_1 d_{011A}} + \nu^B\overline{\nabla_1 d_{1A1B}} - \nu_0\overline{g}^{CD}\overline{\nabla_C d_{1A1D}} = 0\ , \qquad (4.33)$$

which is an ODE for $\overline{d}_{011A}$, since the term $\overline{g}^{AB}\overline{d}_{1ABC}$, which appears when expressing the covariant derivatives in terms of partial derivatives and connection coefficients, can be written as

$$\overline{g}^{AB}\overline{d}_{1ABC} = \nu^0\overline{d}_{011C} - \overline{g}^{1B}\overline{d}_{1B1C}\ .$$

Any bounded solution of the MCFE satisfies $\overline{d}_{011A} = O(r)$ for small r.

For $(\mu\nu\sigma) = (AB1)$ one obtains an ODE for $\overline{d}_{01AB}$,

$$\overline{\nabla_1 d_{01AB}} + \nu^C\overline{\nabla_1 d_{1CAB}} - \nu_0\overline{g}^{CD}\overline{\nabla_D d_{1CAB}} = 0\ , \qquad (4.34)$$

the boundary condition is given by the requirement $\overline{d}_{01AB} = O(r^2)$. Note for this that $\overline{d}_{1ABC}$ and $\overline{d}_{0[AB]1}$, both of which are hidden in the covariant derivatives appearing in (4.34), can be expressed in terms of $\overline{d}_{1A1B}$ and $\overline{d}_{011A}$. Indeed, symmetries of the rescaled Weyl tensor imply that

$$\begin{aligned}\overline{d}_{1ABC} &= 2\overline{g}^{EF}\overline{d}_{1EF[C}\overline{g}_{B]A} = 2(\nu^0\overline{d}_{011[C} - \overline{g}^{1D}\overline{d}_{1D1[C})\overline{g}_{B]A}\ , \\ 2\overline{d}_{0[AB]1} &= -\overline{d}_{01AB}\ .\end{aligned}$$

The $(\mu\nu\sigma) = (101)$-component of (4.1) can be employed to determine $\overline{d}_{0101}$,

$$\overline{\nabla_1 d_{0101}} + \nu^C \overline{\nabla_1 d_{011C}} - \nu_0 \overline{g}^{CD} \overline{\nabla_C d_{011D}} = 0 \,. \tag{4.35}$$

For that purpose one needs to express $\overline{g}^{AB}\overline{d}_{0AB1}$ in terms of known components and $\overline{d}_{0101}$,

$$\overline{g}^{AB}\overline{d}_{0AB1} \quad = \quad \overline{g}^{1A}\overline{d}_{011A} - \nu^0 \overline{d}_{0101} \,.$$

The boundary condition for bounded solutions is $\overline{d}_{0101} = O(1)$.

The function $\overline{d}_{010A}$ is obtained from (4.1) with $(\mu\nu\sigma) = (0A1)$,

$$\overline{\nabla_1 d_{010A}} + \nu^C \overline{\nabla_1 d_{0A1C}} - \nu_0 \overline{g}^{CD} \overline{\nabla_C d_{0A1D}} = 0 \,, \tag{4.36}$$

and $\overline{d}_{010A} = O(r)$. To obtain the desired ODE one needs to use the following relations, which, again, follow from the symmetry properties of the rescaled Weyl tensor:

$$\begin{aligned}
\overline{g}^{AB}\overline{d}_{0AB1} &= \overline{g}^{1A}\overline{d}_{011A} - \nu^0 \overline{d}_{0101} \,, \\
2\nu^0 \overline{d}_{0(AB)1} &= \overline{g}^{11}\overline{d}_{1A1B} - 2\overline{g}^{1C}\overline{d}_{1(AB)C} - \overline{g}^{CD}\overline{d}_{CABD} \,, \\
\overline{g}^{AB}\overline{g}^{CD}\overline{d}_{CABD} &= -2\overline{g}^{AB}(\overline{g}^{1C}\overline{d}_{1ABC} + \nu^0 \overline{d}_{0AB1}) \,, \\
\overline{g}^{CD}\overline{d}_{CABD} &= \frac{1}{2}\overline{g}_{AB}\overline{g}^{CD}\overline{g}^{EF}\overline{d}_{CEFD} \,, \\
\overline{d}_{ABCD} &= \overline{g}^{EF}(\overline{g}_{C[B}\overline{d}_{A]EFD} - \overline{g}_{D[B}\overline{d}_{A]EFC}) \,, \\
\overline{g}^{AB}\overline{d}_{0ABC} &= -\nu^0 \overline{d}_{010C} - \overline{g}^{11}\overline{d}_{011C} - \overline{g}^{1B}(\overline{d}_{01BC} - \overline{d}_{0(BC)1} - \overline{d}_{0[BC]1}) \,, \\
\overline{d}_{0ABC} &= 2\overline{g}^{EF}\overline{d}_{0EF[C}\overline{g}_{B]A} \,.
\end{aligned}$$

To gain an equation for $\breve{\overline{d}}_{0A0B}$ we observe that due to the tracelessness of the rescaled Weyl tensor we have

$$\begin{aligned}
0 &= \overline{g}^{\mu\nu}\overline{\nabla_0 d_{\mu AB\nu}} - \frac{1}{2}\overline{g}_{AB}\overline{g}^{CD}\overline{g}^{\mu\nu}\overline{\nabla_0 d_{\mu CD\nu}} \\
&= 2\nu^0 \overline{\nabla_0 \breve{d}_{0(AB)1}} - \overline{g}^{11}\overline{\nabla_0 \breve{d}_{1A1B}} + 2(\overline{g}^{1C}\overline{\nabla_0 d_{1(AB)C}})\breve{} \,.
\end{aligned}$$

Two of the transverse derivatives can be eliminated via the following relations,

$$\begin{aligned}
0 = \nu_0 \overline{\nabla_\rho \breve{d}_{1(AB)}{}^\rho} \quad &\equiv \quad -\overline{\nabla_0 \breve{d}_{1A1B}} + \overline{\nabla_1 \breve{d}_{0(AB)1}} - \nu_0 \overline{g}^{11}\overline{\nabla_1 \breve{d}_{1A1B}} \\
&\qquad -(\nu^C \overline{\nabla_1 d_{1(AB)C}})\breve{} + \nu^C \overline{\nabla_C \breve{d}_{1A1B}} + \nu_0 (\overline{g}^{CD}\overline{\nabla_D d_{1(AB)C}})\breve{} \,, \\
0 = \nu_0 \overline{\nabla_\rho d_{ABC}{}^\rho} \quad &\equiv \quad \overline{\nabla_0 d_{ABC1}} + \overline{\nabla_1 d_{ABC0}} + \nu_0 \overline{g}^{11}\overline{\nabla_1 d_{ABC1}} \\
&\qquad -\nu^D \overline{\nabla_1 d_{ABCD}} - \nu^D \overline{\nabla_D d_{ABC1}} + \nu_0 \overline{g}^{DE}\overline{\nabla_E d_{ABCD}} \,,
\end{aligned}$$

so that we end up with an expression for $\overline{\nabla_0 \breve{d}_{0(AB)1}}$. The trace-free and symmetrized part of equation (4.1) with $(\mu\nu\sigma) = (0AB)$ reads

$$\begin{aligned}
0 \quad = \quad & \nu^0 \overline{\nabla_0 \breve{d}_{0(AB)1}} + \nu^0 \overline{\nabla_1 \breve{d}_{0AB0}} + \overline{g}^{11}\overline{\nabla_1 \breve{d}_{0(AB)1}} + \overline{g}^{1C}\overline{\nabla_C \breve{d}_{0(AB)1}} \\
& + (\overline{g}^{1C}\overline{\nabla_1 d_{0(AB)C}})\breve{} + (\overline{g}^{CD}\overline{\nabla_D d_{0(AB)C}})\breve{} \,,
\end{aligned} \tag{4.37}$$

which thus provides an ODE for $\breve{\overline{d}}_{0A0B}$ with boundary condition $\breve{\overline{d}}_{0A0B} = O(r^2)$.

Finally, one determines $\overline{L}_{00}$ from equation (4.2) with $(\mu\nu\sigma) = (100)$ and the contracted Bianchi identity (3.1),

$$2\nu^0\overline{\nabla_1 L_{00}} + \overline{g}^{11}\overline{\nabla_1 L_{01}} + 2\overline{g}^{1A}\overline{\nabla_{(1} L_{A)0}} + \overline{g}^{AB}\overline{\nabla_A L_{0B}} - \frac{1}{6}\overline{\partial_0 R} = (\nu^0)^2\overline{\partial_0\Theta}\,\overline{d}_{0101} \,. \tag{4.38}$$

The boundary condition, satisfied by any bounded solution, is $\overline{L}_{00} = O(1)$.

4.3 Constraint equations in the $(R = 0, \overline{s} = -2, \kappa = 0, \hat{g} = \eta)$-wave-map gauge

To simplify computations significantly let us choose a specific gauge. The CWE take their simplest form if we impose the gauge condition

$$R \,=\, 0 \,, \tag{4.39}$$

which we shall do henceforth. Moreover, we assume the wave-map gauge condition and an affinely parameterized cone, meaning that

$$\kappa = 0 \quad \text{and} \quad W^\sigma = 0 \,. \tag{4.40}$$

Furthermore, we set

$$\overline{s} = -2 \,, \tag{4.41}$$

(the negative sign of $\overline{s}$ makes sure that Θ will be positive inside the cone), and use a Minkowski target $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. This way many of the constraint equations can be solved explicitly. From now on all equalities are meant to hold in this particular gauge, if not stated otherwise.

The relevant boundary conditions for the ODEs, which follow from regularity conditions at the vertex, have been specified in the previous section. Recall that the free initial data are given by the $\tilde{g}$-trace-free tensor ω_{AB} and that we treat the case where the initial surface is C_{i^-}, i.e. we have

$$\overline{\Theta} = 0 \,. \tag{4.42}$$

Regularity for the Schouten tensor requires $\omega_{AB} = O(r^2)$. However, regularity for the rescaled Weyl tensor requires the stronger condition (cf. (4.46) below)

$$\omega_{AB} = O(r^4) \,. \tag{4.43}$$

Many of the above equations can be solved straightforwardly, we just present the results,

$$\overline{g}_{\mu\nu} = \eta_{\mu\nu} \,, \quad \overline{L}_{1\mu} = 0 \,, \quad \overline{g}^{AB}\overline{L}_{AB} = 0 \,, \quad \overline{L}_{0A} = \frac{1}{2}\tilde{\nabla}^B\overline{\partial_0 g_{AB}} \,. \tag{4.44}$$

Note that $L_{\mu\nu}$ is trace-free as required by Lemma 3.6. On the way to compute these quantities we have found

$$\begin{gathered}\tau = 2/r \,, \quad \overline{\partial_0\Theta} = -2r \,, \quad \overline{\partial_0 g_{1\mu}} = 0 \,, \quad \overline{g}^{AB}\overline{\partial_0 g_{AB}} = 0 \,, \\ \xi_A = 0 \,, \quad \zeta = -2/r \,, \\ (\partial_1 - r^{-1})\overline{\partial_0 g_{AB}} = -2\omega_{AB} \;\text{ with }\; \overline{\partial_0 g_{AB}} = O(r^3) \,.\end{gathered} \tag{4.45}$$

We further obtain (note that $\overline{\Gamma}^B_{0A} = \frac{1}{2}\overline{g}^{BC}\overline{\partial_0 g_{AC}}$)

$$
\begin{aligned}
\overline{d}_{1A1B} &= -\frac{1}{2}\partial_1(r^{-1}\omega_{AB}) \;, && (4.46)\\
(\partial_1 + 3r^{-1})\overline{d}_{011A} &= \tilde{\nabla}^B\overline{d}_{1A1B} \;, && (4.47)\\
(\partial_1 + 3r^{-1})\overline{d}_{0101} &= \tilde{\nabla}^A\overline{d}_{011A} - \frac{1}{2}\overline{\partial_0 g^{AB}}\overline{d}_{1A1B} \;, && (4.48)\\
(\partial_1 + r^{-1})\overline{d}_{01AB} &= 2\tilde{\nabla}_{[A}\overline{d}_{B]110} - 2\overline{\Gamma}^C_{0[A}\overline{d}_{B]11C} \;, && (4.49)\\
(\partial_1 + r^{-1})\overline{d}_{010A} &= \frac{1}{2}\tilde{\nabla}^B(\overline{d}_{01AB} - \overline{d}_{1A1B}) + \frac{1}{2}\tilde{\nabla}_A\overline{d}_{0101} + r^{-1}\overline{d}_{011A} \\
&\quad + 2\overline{\Gamma}^B_{0A}\overline{d}_{011B} \;, && (4.50)
\end{aligned}
$$

with $\overline{d}_{011A} = O(r)$, $\overline{d}_{0101} = O(1)$, $\overline{d}_{01AB} = O(r^2)$ and $\overline{d}_{010A} = O(r)$. To derive (4.46)-(4.50) we have used the following relations, which follow from algebraic symmetry properties of the Weyl tensor,

$$
\begin{aligned}
\overline{d}_{ABCD} &= -2\overline{g}_{A[C}\overline{g}_{D]B}d_{0101} \;, && (4.51)\\
2d_{0[AB]1} &= -d_{01AB} \;, && (4.52)\\
2\overline{d}_{0(AB)1} &= \overline{d}_{1A1B} - \overline{g}_{AB}\overline{d}_{0101} \;, && (4.53)\\
\overline{d}_{1ABC} &= -2\overline{d}_{011[B}\overline{g}_{C]A} \;, && (4.54)\\
\overline{d}_{0ABC} &= 2\overline{d}_{010[B}\overline{g}_{C]A} + 2\overline{d}_{011[B}\overline{g}_{C]A} \;. && (4.55)
\end{aligned}
$$

Before we proceed let us establish some relations:

Lemma 4.1 *(i)* $(\overline{g}^{CD}\overline{\partial_0 g_{AC}}\,\overline{\partial_0 g_{BD}})\breve{} = 0$,

(ii) $(\overline{g}^{CD}\overline{\partial_0 g_{C(A}}\omega_{B)D})\breve{} = 0$,

(iii) $(\overline{g}^{CD}\overline{\partial_0 g_{C(A}}\,\overline{d}_{B)1D1})\breve{} = 0$.

Proof: This follows from the constraint equations (4.45)-(4.46), together with the $\tilde{g}$-tracelessness of $\overline{\partial_0 g_{AB}}$. □

The lemma can be employed to simplify the ODE which determines $\breve{\overline{d}}_{0A0B}$,

$$
\begin{aligned}
2(\partial_1 - r^{-1})\breve{\overline{d}}_{0A0B} &= 3(\partial_1 - r^{-1})\breve{\overline{d}}_{0(AB)1} - (\partial_1 - r^{-1})\overline{d}_{1A1B} \\
&\quad + (\tilde{\nabla}^C\overline{d}_{1(AB)C})\breve{} + 2(\tilde{\nabla}^C\overline{d}_{0(AB)C})\breve{} - (\overline{\partial_0 g^{CD}}\overline{d}_{ACBD})\breve{} \\
&\quad + [2\overline{\Gamma}^C_{0(A}(\overline{d}_{B)C01} - \overline{d}_{B)01C} + \frac{1}{2}\overline{d}_{B)1C1})]\breve{} \\
&= \frac{1}{2}(\partial_1 - r^{-1})\overline{d}_{1A1B} + (\tilde{\nabla}_{(A}\overline{d}_{B)110})\breve{} + 2(\tilde{\nabla}_{(A}\overline{d}_{B)010})\breve{} \\
&\quad + 3\overline{\Gamma}^C_{0(A}\overline{d}_{B)C01} + \frac{3}{2}\overline{d}_{0101}\overline{\partial_0 g_{AB}} \;, && (4.56)
\end{aligned}
$$

with $\breve{\overline{d}}_{0A0B} = O(r^2)$. Finally, one shows that the missing component of the Schouten tensor satisfies

$$
2(\partial_1 + r^{-1})\overline{L}_{00} = \frac{1}{2}\omega^{AB}\overline{\partial_0 g_{AB}} - 2r\overline{d}_{0101} - \tilde{\nabla}^A\overline{L}_{0A} \;, \qquad (4.57)
$$

with $\overline{L}_{00} = O(1)$.

We aim now to find explicit solutions to some of the remaining ODEs (4.47)-(4.50). The key observation to solve (4.47) is that, due to (4.45), we have

$$\overline{d}_{1A1B} = -\frac{1}{2}r^{-1}\partial_1\Big(\omega_{AB} + \frac{1}{2}r^{-1}\overline{\partial_0 g_{AB}}\Big) . \tag{4.58}$$

Hence we find

$$\begin{aligned}\partial_1(r^3\overline{d}_{011A}) &= -\frac{1}{2}\partial_1\Big(r^2\tilde{\nabla}^B\omega_{AB} + r\overline{L}_{0A}\Big)\\ \overset{\overline{d}_{011A}=O(r)}{\Longrightarrow}\quad \overline{d}_{011A} &= -\frac{1}{2}r^{-1}\tilde{\nabla}^B\omega_{AB} - \frac{1}{2}r^{-2}\overline{L}_{0A} \qquad (4.59)\\ &\overset{(4.45)}{=} \frac{1}{2}r^{-1}\partial_1\overline{L}_{0A} . \qquad (4.60)\end{aligned}$$

The equations (4.45) and (4.60) can be used to rewrite (4.49),

$$\begin{aligned}\partial_1(r\overline{d}_{01AB}) &= \partial_1\tilde{\nabla}_{[A}\overline{L}_{B]0} - r\overline{\Gamma}^C_{0[A}\partial_{|1|}(r^{-1}\omega_{B]C})\\ &= \partial_1(\tilde{\nabla}_{[A}\overline{L}_{B]0} - \overline{\Gamma}^C_{0[A}\omega_{B]C})\\ \overset{\overline{d}_{01AB}=O(r^2)}{\Longrightarrow}\quad \overline{d}_{01AB} &= r^{-1}\tilde{\nabla}_{[A}\overline{L}_{B]0} - r^{-1}\overline{\Gamma}^C_{0[A}\omega_{B]C} . \qquad (4.61)\end{aligned}$$

The constraint equations in the $(R = 0, \overline{s} = -2, \kappa = 0, \hat{g} = \eta)$-wave-map gauge are summed up in (5.6)-(5.16) below.

5 Applicability of Theorem 3.7 on the C_{i^-}-cone

Let us suppose we have been given initial data $\omega_{AB} \equiv \check{\mathring{L}}_{AB}$ on C_{i^-}, supplemented by a gauge choice for R, $\mathring{s}$, W^σ and κ. Then we solve the hierarchical system of constraint equations derived above; the solutions are denoted by $\mathring{g}_{\mu\nu}$, $\mathring{L}_{\mu\nu}$ and $\mathring{d}_{\mu\nu\sigma\rho}$. Let us further assume that there exists a smooth solution of the CWE in some neighbourhood to the future of i^-, smoothly extendable through C_{i^-}, which induces the data $\overline{\Theta} = 0$, $\overline{s} = \mathring{s}$, $\overline{g}_{\mu\nu} = \mathring{g}_{\mu\nu}$, $\overline{L}_{\mu\nu} = \mathring{L}_{\mu\nu}$ and $\overline{d}_{\mu\nu\sigma\rho} = \mathring{d}_{\mu\nu\sigma\rho}$ on C_{i^-}. The purpose of this section is to investigate to what extent the hypotheses made in Theorem 3.7 are satisfied in the case of initial data which have been constructed as a solution of the constraint equations. For convenience and to make computations significantly easier we shall not do it in an arbitrary generalized wave-map gauge but prefer to work within the specific gauge (4.39)-(4.41).

5.1 $(R = 0, \overline{s} = -2, \kappa = 0, \hat{g} = \eta)$-wave-map gauge

We restrict attention to the $\kappa = 0$-wave-map gauge with $W^\sigma = 0$; moreover, we set $R = 0$ and $\mathring{s} = -2$, and use a Minkowski target $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. All equalities are meant to hold in this specific gauge. For reasons of clarity let us recall the CWE in an $(R = 0)$-gauge, where they take their simplest form,

$$\Box^{(H)}_g L_{\mu\nu} = 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma , \tag{5.1}$$

$$\Box_g s = \Theta|L|^2 , \tag{5.2}$$

$$\Box_g\Theta = 4s , \tag{5.3}$$

$$\Box^{(H)}_g d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa , \tag{5.4}$$

$$R^{(H)}_{\mu\nu}[g] = 2L_{\mu\nu} . \tag{5.5}$$

The constraint equations, from which the initial data for the CWE are determined from given free data $\omega_{AB} \equiv \breve{\mathring{L}}_{AB} = O(r^4)$ read:

$$
\begin{aligned}
\mathring{g}_{\mu\nu} &= \eta_{\mu\nu} \;, && (5.6)\\
\mathring{L}_{1\mu} &= 0 \;, \quad \mathring{L}_{0A} \;=\; \frac{1}{2}\tilde{\nabla}^B\lambda_{AB} \;, \quad \mathring{g}^{AB}\mathring{L}_{AB} \;=\; 0 \;, && (5.7)\\
\mathring{d}_{1A1B} &= -\frac{1}{2}\partial_1(r^{-1}\omega_{AB}) \;, && (5.8)\\
\mathring{d}_{011A} &= \frac{1}{2}r^{-1}\partial_1\mathring{L}_{0A} \;, && (5.9)\\
\mathring{d}_{01AB} &= r^{-1}\tilde{\nabla}_{[A}\mathring{L}_{B]0} - \frac{1}{2}r^{-1}\lambda_{[A}{}^{C}\omega_{B]C} \;, && (5.10)\\
(\partial_1 + \;3r^{-1})\mathring{d}_{0101} &= \tilde{\nabla}^A\mathring{d}_{011A} + \frac{1}{2}\lambda^{AB}\mathring{d}_{1A1B} \;, && (5.11)\\
2(\partial_1 + r^{-1})\mathring{d}_{010A} &= \tilde{\nabla}^B(\mathring{d}_{01AB} - \mathring{d}_{1A1B}) + \tilde{\nabla}_A\mathring{d}_{0101} + 2r^{-1}\mathring{d}_{011A}\\
&\quad +2\lambda_A{}^B\mathring{d}_{011B} \;, && (5.12)\\
4(\partial_1 - r^{-1})\breve{\mathring{d}}_{0A0B} &= (\partial_1 - r^{-1})\mathring{d}_{1A1B} + 2(\tilde{\nabla}_{(A}\mathring{d}_{B)110})\breve{} + 4(\tilde{\nabla}_{(A}\mathring{d}_{B)010})\breve{}\\
&\quad +3\lambda_{(A}{}^{C}\mathring{d}_{B)C01} + 3\mathring{d}_{0101}\lambda_{AB} \;, && (5.13)\\
4(\partial_1 + r^{-1})\mathring{L}_{00} &= \lambda^{AB}\omega_{AB} - 4r\mathring{d}_{0101} - 2\tilde{\nabla}^A\mathring{L}_{0A} \;, && (5.14)
\end{aligned}
$$

with

$$
\mathring{d}_{0101} = O(1) \;, \quad \mathring{d}_{010A} = O(r) \;, \quad \breve{\mathring{d}}_{0A0B} = O(r^2) \;, \quad \mathring{L}_{00} = O(1) \;, \qquad (5.15)
$$

and where λ_{AB} is the unique solution of

$$
(\partial_1 - r^{-1})\lambda_{AB} = -2\omega_{AB} \quad \text{with} \quad \lambda_{AB} = O(r^5) \;. \qquad (5.16)
$$

Note that the expansion τ satisfies

$$
\tau \;=\; 2/r \;. \qquad (5.17)
$$

All the other components of $\mathring{g}_{\mu\nu}$, $\mathring{L}_{\mu\nu}$ and $\mathring{d}_{\mu\nu\sigma\rho}$ follow from their usual symmetry properties which they are required to satisfy.

5.2 Vanishing of $\overline{H}^\sigma$

Inserting the definition of the reduced Ricci tensor (3.10) equation (5.5) becomes

$$
R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma = 2L_{\mu\nu} \;. \qquad (5.18)
$$

Utilizing the constraint equations (5.6) and the identities [5]

$$
\begin{aligned}
\overline{R}_{11} &\equiv -\partial_1\tau + \tau\overline{\Gamma}^1_{11} - |\sigma|^2 - \frac{1}{2}\tau^2 \;=\; \tau\overline{\Gamma}^1_{11} \;,\\
\overline{\Gamma}^1_{11} &\equiv \kappa - \frac{1}{2}\nu_0\overline{H}^0 \;=\; -\frac{1}{2}\overline{H}^0 \;,
\end{aligned}
$$

the latter one follows from the definitions of H^σ and κ, we conclude that the solution satisfies the ODE

$$
\hat{\nabla}_1\overline{H}^0 + \frac{1}{2}\tau\overline{H}^0 = 0 \quad \Longleftrightarrow \quad (\partial_1 + r^{-1})\overline{H}^0 = 0 \;. \qquad (5.19)
$$

For any regular solution of the CWE the function $\overline{H}^0$ will be bounded near the vertex. We observe that

$$\overline{H}^0 \;=\; 0 \tag{5.20}$$

is the only solution of (5.19) where this is the case. Then we immediately obtain

$$\overline{\Gamma}^1_{11} \;=\; \kappa \;=\; 0\;. \tag{5.21}$$

Recall the definition of the field ξ_A, which vanishes in our gauge,

$$\xi_A \;\equiv\; -2\nu^0\partial_1\nu_A + 4\nu^0\nu_B\chi_A{}^B + \nu_A\overline{V}^0 + \overline{g}_{AB}\overline{V}^B - \overline{g}_{AD}\overline{g}^{BC}\tilde{\Gamma}^D_{BC} \;=\; 0\;.$$

From the constraint equations, (5.18) and the identities [5]

$$\begin{aligned}\overline{R}_{1A} \;&\equiv\; (\partial_1+\tau)\overline{\Gamma}^1_{1A} + \tilde{\nabla}_B\chi_A{}^B - \partial_A\overline{\Gamma}^1_{11} - \partial_A\tau \;=\; (\partial_1+\tau)\overline{\Gamma}^1_{1A}\;, &(5.22)\\ \xi_A \;&\equiv\; -2\overline{\Gamma}^1_{1A} - \overline{H}_A - \nu_A\overline{H}^0 \;=\; -2\overline{\Gamma}^1_{1A} - \overline{H}_A\;, &(5.23)\end{aligned}$$

we find that $\overline{H}_A := \overline{g}_{AB}\overline{H}^B$ fulfills the ODE

$$\partial_1\overline{H}_A \;=\; 0\;.$$

Any regular solution necessarily satisfies $\overline{H}_A = O(r)$ and we infer

$$\overline{H}^A = 0 \quad \text{and} \quad \overline{\Gamma}^1_{1A} = 0\;. \tag{5.24}$$

We have introduced the function

$$\zeta \;\equiv\; 2(\partial_1 + \kappa + \frac{1}{2}\tau)\overline{g}^{11} + 2\overline{V}^1 \;=\; -\tau\;.$$

From (5.18), the constraint equation $\overline{g}^{AB}\overline{L}_{AB} = 0$ and the identities [5]

$$\begin{aligned}\overline{g}^{AB}\overline{R}_{AB} \;&\equiv\; 2(\partial_1 + \overline{\Gamma}^1_{11} + \tau)[\underbrace{(\partial_1 + \overline{\Gamma}^1_{11} + \frac{1}{2}\tau)\overline{g}^{11} + \overline{g}^{\mu\nu}\overline{\Gamma}^1_{\mu\nu}]}_{\equiv\overline{g}^{AB}\overline{\Gamma}^1_{AB}+\frac{1}{2}\tau g^{11}}\\ &\quad + \tilde{R} - 2\overline{g}^{AB}\overline{\Gamma}^1_{1A}\overline{\Gamma}^1_{1B} - 2\overline{g}^{AB}\tilde{\nabla}_A\overline{\Gamma}^1_{1B}\\ &=\; 2(\partial_1+\tau)[\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \frac{1}{2}\tau] + \frac{1}{2}\tau^2\;, &(5.25)\\ \zeta \;&\equiv\; 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11} + \nu_0\overline{g}^{11}\overline{H}^0 - 2\overline{H}^1\\ &=\; 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau - 2\overline{H}^1\;, &(5.26)\end{aligned}$$

we deduce that

$$(\partial_1 + r^{-1})\overline{H}^1 \;=\; 0\;.$$

Our solution is supposed to be regular at i^-, whence $\overline{H}^1 = O(1)$ and we conclude

$$\overline{H}^1 = 0 \quad \text{and} \quad \overline{g}^{AB}\overline{\Gamma}^1_{AB} = -\tau\;. \tag{5.27}$$

Altogether we have proven that

$$\overline{H}^\sigma = 0\;. \tag{5.28}$$

Note that once we know the values of the wave-gauge vector on C_{i^-}, we can compute the values of certain components of the transverse derivative of the metric on C_{i^-}. More concretely, we find that the solution satisfies

$$\overline{\partial_0 g_{11}} = 0 \,, \quad \overline{\partial_0 g_{1A}} = 0 \,, \quad \overline{g}^{AB}\overline{\partial_0 g_{AB}} = 0 \,.$$

We also have

$$\begin{aligned}
\overline{R}_{AB} &\equiv \overline{\partial_\alpha \Gamma^\alpha_{AB}} - \partial_A \overline{\Gamma}^\alpha_{\alpha B} + \overline{\Gamma}^\alpha_{AB}\overline{\Gamma}^\beta_{\beta\alpha} - \overline{\Gamma}^\alpha_{\beta A}\overline{\Gamma}^\beta_{\alpha B} \\
&= \tilde{R}_{AB} - \frac{1}{4}\tau^2 \overline{g}_{AB} - \frac{1}{2}(\partial_1 - \tau)\overline{\partial_0 g_{AB}} + \overline{\partial_0 \Gamma^0_{AB}} - \frac{1}{2}\tau \overline{g}_{AB}\overline{\Gamma}^0_{00} \\
&= -(\partial_1 - r^{-1})\overline{\partial_0 g_{AB}} \,,
\end{aligned}$$

where we employed the relation

$$\overline{\partial_0 \Gamma^0_{AB}} = \frac{1}{2}\tau \overline{g}_{AB}\overline{\partial_0 g_{01}} - \frac{1}{2}\partial_1 \overline{\partial_0 g_{AB}} \,.$$

The vanishing of $\overline{H}^\sigma$ implies via (5.18) and (5.6)

$$\overline{R}_{AB} \, = \, 2\overline{L}_{AB} \, = 2\omega_{AB} \,,$$

and thus by (5.16)

$$(\partial_1 - r^{-1})(\lambda_{AB} - \overline{\partial_0 g_{AB}}) = 0 \,.$$

For initial data of the form $\omega_{AB} = O(r^4)$ we have $\lambda_{AB} = O(r^5)$. Since regularity requires [5] $\overline{\partial_0 g_{AB}} = O(r^3)$, we discover the expected relation

$$\lambda_{AB} \, = \, \overline{\partial_0 g_{AB}} \,.$$

5.3 Vanishing of $\overline{\nabla_\mu H^\sigma}$ and $\overline{\zeta}_\mu$

We know that the wave-gauge vector satisfies the wave equation (3.21),

$$\nabla^\nu \hat{\nabla}_\nu H^\alpha + 2g^{\mu\alpha}\nabla_{[\sigma}\hat{\nabla}_{\mu]}H^\sigma + 4\nabla^\nu L_\nu{}^\alpha = 0 \,. \tag{5.29}$$

Let us first consider the $\alpha = 0$-component evaluated on $\mathscr{I}^-$,

$$(\partial_1 + r^{-1})\overline{\partial_0 H^0} + 2\overline{\partial_0 L_{11}} = 0 \,. \tag{5.30}$$

We need to show that the source term vanishes. Equation (5.1) provides an expression for $\overline{\partial_0 L_{11}}$,

$$\overline{\Box^{(H)}_g L_{11}} = 0 \quad \Longleftrightarrow \quad (\partial_1 + r^{-1})\overline{\partial_0 L_{11}} = 0 \,. \tag{5.31}$$

Any regular solution satisfies $\overline{\partial_0 L_{11}} = \overline{\nabla_0 L_{11}} = O(1)$. There is precisely one bounded solution of (5.31), which is

$$\overline{\partial_0 L_{11}} = 0 \,. \tag{5.32}$$

The function $\overline{\nabla_0 H^0} = \overline{\partial_0 H^0}$ needs to be bounded as well, and the only bounded solution of (5.30) is

$$\overline{\partial_0 H^0} = 0 \,. \tag{5.33}$$

Taking the trace of (5.18) then shows that the curvature scalar vanishes initially,

$$\overline{R}_g = 0\ . \tag{5.34}$$

Using (5.33) as well as the relation $\overline{R}_{01} = 2\overline{L}_{01} = 0$, which follows from (5.18), one verifies that

$$\overline{\partial_0 g_{01}} = 0 \quad \text{and} \quad \overline{\partial^2_{00} g_{11}} = 0\ .$$

The $\alpha = A$-component of (5.29) yields

$$(\partial_1 + 2r^{-1})\overline{\partial_0 H^A} + 2\overline{g}^{AB}(\overline{\partial_0 L_{1B}} + \partial_1 \overline{L}_{0B} + \tau \overline{L}_{0B} + \tilde{\nabla}^C \omega_{BC}) = 0\ . \tag{5.35}$$

We employ (5.1) to compute the source term,

$$\overline{\Box^{(H)}_g L_{1A}} = 0 \quad \Longleftrightarrow \quad 2\partial_1 \overline{\partial_0 L_{1A}} - \tau \tilde{\nabla}^B \omega_{AB} - \tau^2 \overline{L}_{0A} = 0\ . \tag{5.36}$$

Equation (5.16) implies

$$2\tilde{\nabla}^B \omega_{AB} = -\tilde{\nabla}^B \partial_1 \lambda_{AB} + \tau \overline{L}_{0A} = -2\partial_1 \overline{L}_{0A} - \tau \overline{L}_{0A}\ . \tag{5.37}$$

From (5.36) and (5.37) we derive the ODE

$$\partial_1(\overline{\partial_0 L_{1A}} + r^{-1}\overline{L}_{0A}) = 0\ . \tag{5.38}$$

For any sufficiently regular solution we have $\overline{\partial_0 L_{1A}} = \overline{\nabla_0 L_{1A}} = O(r)$. Since the initial data satisfy $\omega_{AB} = O(r^4)$, we have $\overline{L}_{0A} = O(r^2)$ by (5.36). We then conclude from (5.38) that

$$\overline{\partial_0 L_{1A}} = -r^{-1}\overline{L}_{0A} = -\frac{1}{4}\tau \tilde{\nabla}^B \lambda_{AB}\ . \tag{5.39}$$

With (5.6), (5.37) and (5.39) equation (5.35) becomes

$$(\partial_1 + 2r^{-1})\overline{\partial_0 H^A} = 0\ . \tag{5.40}$$

Any solution which is regular at i^- fulfills $\overline{\partial_0 H^A} = \overline{\nabla_0 H^A} = O(r^{-1})$. The ODE (5.40) admits precisely one such solution, namely

$$\overline{\partial_0 H^A} = 0\ . \tag{5.41}$$

We have

$$\begin{aligned} \tilde{\nabla}^B \lambda_{AB} &= 2\overline{L}_{0A} = \overline{R}_{0A} = \frac{1}{2}\overline{\partial^2_{00} g_{1A}} - \frac{1}{2}(\partial_1 - \tau)\overline{\partial_0 g_{0A}} + \frac{1}{2}\tilde{\nabla}^B \lambda_{AB}\ , \\ 0 &= \overline{g}_{AB}\overline{\partial_0 H^B} = \overline{\partial^2_{00} g_{1A}} + (\partial_1 + \tau)\overline{\partial_0 g_{0A}} + \tilde{\nabla}^B \lambda_{AB}\ . \end{aligned}$$

The combination of both equations yields

$$\partial_1 \overline{\partial_0 g_{0A}} + \tilde{\nabla}^B \lambda_{AB} = 0 \quad \text{and} \quad \overline{\partial^2_{00} g_{1A}} = -\tau \overline{\partial_0 g_{0A}}\ . \tag{5.42}$$

Utilizing the previous results of this section the $\alpha = 1$-component of (5.29) can be written in our gauge as

$$(\partial_1 + r^{-1})\overline{\partial_0 H^1} + \underbrace{2(\partial_1 + \tau)\overline{L}_{00} + 2\tilde{\nabla}^A \overline{L}_{0A} - \overline{g}^{AB}\overline{\partial_0 L_{AB}}}_{=:f} = 0\ , \tag{5.43}$$

where we took into account that owing to Lemma 3.6 we have

$$0 = \overline{\partial_0 L} = 2\overline{\partial_0 L_{01}} + \overline{g}^{AB}\overline{\partial_0 L_{AB}} - \omega^{AB}\lambda_{AB}\ . \tag{5.44}$$

We show that the source f vanishes. To do that we compute the $\tilde{g}$-trace of the $(\mu\nu) = (AB)$-component of (5.1) on $\mathscr{I}^-$. With (5.16) we obtain

$$\begin{aligned}
&\overline{g}^{AB}\overline{\Box_g^{(H)} L_{AB}} = 2\overline{L}_A{}^B\overline{L}_B{}^A \quad \Longleftrightarrow \\
&2(\partial_1 + r^{-1})(\overline{g}^{AB}\overline{\partial_0 L_{AB}}) - 2\lambda^{AB}(\partial_1 - r^{-1})\omega_{AB} \\
&\qquad + 2\tau\tilde{\nabla}^A\overline{L}_{0A} + \tau^2\overline{L}_{00} + 2|\omega|^2 = 0\ ,
\end{aligned} \tag{5.45}$$

where we have set $|\omega|^2 := \omega_A{}^B\omega_B{}^A$.

As another intermediate step it is useful to derive a second-order equation for $\overline{L}_{00}$, so let us differentiate (5.14) with respect to r,

$$(4\partial^2_{11} + 2\tau\partial_1 - \tau^2)\overline{L}_{00} = 8\overline{d}_{0101} - 4r(\partial_1 + 3r^{-1})\overline{d}_{0101} - 2\partial_1\tilde{\nabla}^A\overline{L}_{0A} + \partial_1(\lambda^{AB}\omega_{AB})\ .$$

With (5.8), (5.9) (5.11), (5.16) and again (5.14) that yields

$$2(\partial^2_{11} + 3r^{-1}\partial_1 + r^{-2})\overline{L}_{00} = \lambda^{AB}(\partial_1 - r^{-1})\omega_{AB} - |\omega|^2 - 2\tilde{\nabla}^A\partial_1\overline{L}_{0A}\ . \tag{5.46}$$

Let us return to the source term f in (5.43). It satisfies the ODE

$$\begin{aligned}
2(\partial_1 + r^{-1})f &= 4\partial^2_{11}\overline{L}_{00} + 6\tau\partial_1\overline{L}_{00} - 2\tau\tilde{\nabla}^A\overline{L}_{0A} + 4\tilde{\nabla}^A\partial_1\overline{L}_{0A} \\
&\quad -2(\partial_1 + r^{-1})(\overline{g}^{AB}\overline{\partial_0 L_{AB}}) \\
&\overset{(5.45)}{=} 4\partial^2_{11}\overline{L}_{00} + 6\tau\partial_1\overline{L}_{00} + \tau^2\overline{L}_{00} + 4\tilde{\nabla}^A\partial_1\overline{L}_{0A} \\
&\quad -2\lambda^{AB}(\partial_1 - r^{-1})\omega_{AB} + 2|\omega|^2 \\
&\overset{(5.46)}{=} 0\ .
\end{aligned}$$

We conclude that

$$f \equiv 2(\partial_1 + \tau)\overline{L}_{00} + 2\tilde{\nabla}^A\overline{L}_{0A} - \overline{g}^{AB}\overline{\partial_0 L_{AB}} = c(x^A)r^{-1} \tag{5.47}$$

for some angle-dependent function c. Regularity at i^- implies $\overline{L}_{00} = O(1)$ and $\partial_1\overline{L}_{00} = \overline{\nabla_1 L_{00}} = O(1)$. Furthermore, we have (note that $\lambda^{AB}\omega_{AB} = O(r^5)$)

$$\begin{aligned}
O(1) &= \overline{\nabla^A L_{0A}} = \tilde{\nabla}^A\overline{L}_{0A} - \frac{1}{2}\lambda^{AB}\omega_{AB} + \tau\overline{L}_{00}\ , \\
O(1) &= \overline{g}^{AB}\overline{\nabla_0 L_{AB}} = \overline{g}^{AB}\overline{\partial_0 L_{AB}} - \lambda^{AB}\omega_{AB} \\
\Longrightarrow &\quad \tilde{\nabla}^A\overline{L}_{0A} + \tau\overline{L}_{00} = O(1)\ , \quad \overline{g}^{AB}\overline{\partial_0 L_{AB}} = O(1)\ .
\end{aligned}$$

Therefore the problematic r^{-1}-term in the expansion of f needs to vanish, and we conclude $c = 0$. Then (5.43) enforces $\overline{\partial_0 H^1}$ to vanish in order to be bounded, i.e. altogether we have proven that

$$\overline{\nabla_\mu H^\nu} = 0\ . \tag{5.48}$$

Recall that $\zeta_\mu \equiv -4(\nabla_\nu L_\mu{}^\nu - \nabla_\mu R/6) = -4\nabla_\nu L_\mu{}^\nu$. If we evaluate (5.29) on $\mathscr{I}^-$ (which, as a matter of course, is to be read as an equation for ζ_μ) and insert (5.48), we immediately observe that

$$\overline{\zeta}_\mu = 0\ . \tag{5.49}$$

5.4 Vanishing of $\overline{W}_{\mu\nu\sigma\rho}$

We want to show that the Weyl tensor $W_{\mu\nu\sigma\rho}$ of $g_{\mu\nu}$ vanishes on C_{i^-}, and thus coincides there with the tensor $\Theta d_{\mu\nu\sigma\rho}$. The 10 independent components are

$$\overline{W}_{0101}\ , \quad \overline{W}_{011A}\ , \quad \overline{W}_{010A}\ , \quad \overline{W}_{01AB}\ , \quad \breve{\overline{W}}_{1A1B}\ , \quad \breve{\overline{W}}_{0A0B}\ .$$

Due to the vanishing of $\overline{H}^\sigma$, $\overline{\nabla_\mu H^\sigma}$ and $\overline{R}_g$, (5.5) tells us that the tensor $L_{\mu\nu}$ coincides on C_{i^-} with the Schouten tensor. We thus have the formula:

$$\overline{W}_{\mu\nu\sigma\rho} = \overline{R}_{\mu\nu\sigma\rho} - 2(\overline{g}_{\sigma[\mu}\overline{L}_{\nu]\rho} - \overline{g}_{\rho[\mu}\overline{L}_{\nu]\sigma})\ . \tag{5.50}$$

The following list of Christoffel symbols, or rather of their transverse derivatives, will be useful:

$$\begin{aligned}
\overline{\partial_0\Gamma^0_{01}} &= \overline{\partial_0\Gamma^1_{11}} = 0\ ,\\
\overline{\partial_0\Gamma^0_{0A}} &\overset{(5.42)}{=} -\frac{1}{2}(\partial_1+\tau)\overline{\partial_0 g_{0A}}\ ,\\
\overline{\partial_0\Gamma^0_{AB}} &= -\frac{1}{2}\partial_1\lambda_{AB}\ ,\\
\overline{\partial_0\Gamma^1_{1A}} &= \frac{1}{2}\partial_1\overline{\partial_0 g_{0A}}\ ,\\
\overline{\partial_0\Gamma^1_{AB}} &= \frac{1}{2}\tau\overline{g}_{AB}\overline{\partial_0 g_{00}} + \tilde\nabla_{(A}\overline{\partial_{|0}g_{0|B)}} - \frac{1}{2}\overline{\partial^2_{00}g_{AB}} - \frac{1}{2}\partial_1\lambda_{AB}\ ,\\
\overline{\partial_0\Gamma^C_{0A}} &= \frac{1}{2}\overline{g}^{CD}\overline{\partial^2_{00}g_{AD}} - \frac{1}{2}\lambda_A{}^D\lambda_D{}^C + \overline{g}^{CD}\tilde\nabla_{[A}\overline{\partial_{|0}g_{0|D]}}\ ,\\
\overline{\partial_0\Gamma^C_{1A}} &= \frac{1}{2}\partial_1\lambda_A{}^C\ ,\\
\overline{\partial_0\Gamma^C_{AB}} &= \frac{1}{2}\tau\overline{g}_{AB}\overline{g}^{CD}\overline{\partial_0 g_{0D}} + \tilde\nabla_{(A}\lambda_{B)}{}^C - \frac{1}{2}\tilde\nabla^C\lambda_{AB}\ ,\\
\overline{\partial^2_{00}\Gamma^0_{AB}} &\overset{(5.42)}{=} \frac{1}{2}\tau\overline{g}_{AB}\overline{\partial^2_{00}g_{01}} - \tau\tilde\nabla_{(A}\overline{\partial_{|0}g_{0|B)}} - \frac{1}{2}\partial_1\overline{\partial^2_{00}g_{AB}}\ .
\end{aligned}$$

We compute the relevant components of the Riemann tensor $R_{\mu\nu\sigma}{}^\rho \equiv \partial_\nu\Gamma^\rho_{\mu\sigma} - \partial_\mu\Gamma^\rho_{\nu\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\rho_{\nu\alpha} - \Gamma^\alpha_{\nu\sigma}\Gamma^\rho_{\mu\alpha}$,

$$\overline{R}_{0101} = 0\ , \quad \overline{R}_{011A} = 0\ , \quad \overline{R}_{01AB} = 0\ , \quad \overline{R}_{1A1B} = 0\ , \tag{5.51}$$

$$\overline{R}_{010A} = \frac{1}{2}(\partial_1-\tau)\overline{\partial_0 g_{0A}} - \frac{1}{2}\overline{\partial^2_{00}g_{1A}} \overset{(5.42)}{=} -\frac{1}{2}\tilde\nabla^B\lambda_{AB}\ , \tag{5.52}$$

$$\breve{\overline{R}}_{0A0B} = (\tilde\nabla_{(A}\overline{\partial_{|0}g_{0|B)}})\breve{} - \frac{1}{2}(\overline{\partial^2_{00}g_{AB}})\breve{}\ . \tag{5.53}$$

Next, we determine the independent components of the Weyl tensor on $\mathscr{I}^-$ via (5.50) and by taking into account the values we have found for $\overline{L}_{\mu\nu}$,

$$\overline{W}_{0101} = 0\ , \quad \overline{W}_{011A} = 0\ , \quad \overline{W}_{010A} = 0\ , \tag{5.54}$$

$$\overline{W}_{01AB} = 0\ , \quad \overline{W}_{1A1B} = 0\ , \tag{5.55}$$

$$\breve{\overline{W}}_{0A0B} = \omega_{AB} + (\tilde\nabla_{(A}\overline{\partial_{|0}g_{0|B)}})\breve{} - \frac{1}{2}(\overline{\partial^2_{00}g_{AB}})\breve{}\ . \tag{5.56}$$

It remains to determine $\overline{\partial^2_{00}g_{AB}}$. Note that according to (5.18) the vanishing of $\overline{H}^\sigma$ and $\overline{\nabla_\mu H^\sigma}$ implies

$$\begin{aligned}
& \overline{\partial_0 R_{AB}} = 2\overline{\partial_0 L_{AB}}\\
\Longrightarrow\quad & \overline{\Box_g R_{AB}} = 2\overline{\Box_g L_{AB}} = 2\overline{\Box^{(H)}_g L_{AB}} \overset{(5.1)}{=} 8\omega_{AC}\omega_B{}^C - 2\overline{g}_{AB}|\omega|^2\ .
\end{aligned}$$

A rather lengthy computation, which uses (5.16), reveals that this is equivalent to (set $\Delta_{\tilde{g}} := \tilde{\nabla}^A \tilde{\nabla}_A$)

$$
\begin{aligned}
&(\partial_1 - r^{-1})\overline{\partial_0 R_{AB}} - 2(\partial_1 - r^{-1})(\omega_{C(A}\lambda_{B)}{}^C) + 2\tau\tilde{\nabla}_{(A}\overline{L}_{B)0} \\
&\qquad + (\partial^2_{11} - \tau\partial_1 + \Delta_{\tilde{g}})\omega_{AB} + \overline{g}_{AB}(\frac{1}{2}\tau^2\overline{L}_{00} + |\omega|^2) \;=\; 0\,.
\end{aligned}
$$

We take its traceless part and invoke Lemma 4.1,

$$
(\partial_1 - r^{-1})\breve{(\overline{\partial_0 R_{AB}})} + 2\tau\breve{(\tilde{\nabla}_{(A}\overline{L}_{B)0})} + (\partial^2_{11} - \tau\partial_1 + \Delta_{\tilde{g}})\omega_{AB} \;=\; 0\,. \tag{5.57}
$$

Let us compute $\partial_0 R_{AB}$ on $\mathscr{I}^-$, which is done by using (5.16), (5.42) and the above formulae for the u-differentiated Christoffel symbols,

$$
\begin{aligned}
\overline{\partial_0 R_{AB}} \;=\;& \overline{\partial^2_{00}\Gamma^0_{AB}} + \partial_1\overline{\partial_0\Gamma^1_{AB}} + \tilde{\nabla}_C\overline{\partial_0\Gamma^C_{AB}} - \tilde{\nabla}_A\overline{\partial_0\Gamma^C_{BC}} - \tilde{\nabla}_A\overline{\partial_0\Gamma^1_{1B}} \\
& - \tilde{\nabla}_A\overline{\partial_0\Gamma^0_{0B}} - \overline{\Gamma}^0_{BC}\overline{\partial_0\Gamma^C_{0A}} - \overline{\Gamma}^0_{AC}\overline{\partial_0\Gamma^C_{B0}} - \overline{\Gamma}^C_{B0}\overline{\partial_0\Gamma^0_{AC}} - \overline{\Gamma}^C_{0A}\overline{\partial_0\Gamma^0_{BC}} \\
& - \overline{\Gamma}^1_{BC}\overline{\partial_0\Gamma^C_{1A}} - \overline{\Gamma}^1_{AC}\overline{\partial_0\Gamma^C_{B1}} + \overline{\Gamma}^0_{AB}\overline{\partial_0\Gamma^\mu_{\mu 0}} + \overline{\Gamma}^1_{AB}\overline{\partial_0\Gamma^\mu_{\mu 1}} \\
\;=\;& -(\partial_1 - r^{-1})\overline{\partial^2_{00}g_{AB}} + (\partial_1 - r^{-1})\omega_{AB} - \frac{1}{2}(\Delta_{\tilde{g}} - \frac{1}{2}\tau^2)\lambda_{AB} \\
& - \tau\tilde{\nabla}_{(A}\overline{\partial_{|0}g_{0|B)}} - \frac{1}{2}\tau\lambda_A{}^C\lambda_{BC} - 2\lambda_{(A}{}^C\omega_{B)C} + f(r,x^C)\overline{g}_{AB}\,.
\end{aligned} \tag{5.58}
$$

The traceless part of $\overline{\partial_0 R_{AB}}$ reads

$$
\begin{aligned}
\breve{(\overline{\partial_0 R_{AB}})} \;=\;& -(\partial_1 - r^{-1})\breve{(\overline{\partial^2_{00}g_{AB}})} + (\partial_1 - r^{-1})\omega_{AB} - \frac{1}{2}(\Delta_{\tilde{g}} - \frac{1}{2}\tau^2)\lambda_{AB} \\
& - \tau\breve{(\tilde{\nabla}_{(A}\overline{\partial_{|0}g_{0|B)}})}\,.
\end{aligned} \tag{5.59}
$$

Next, we apply $2(\partial_1 - r^{-1})$ to the expression (5.56) which we have found for $\breve{\overline{W}}_{0A0B}$. With (5.59) and (5.42) we end up with

$$
\begin{aligned}
2(\partial_1 - r^{-1})\breve{\overline{W}}_{0A0B} \;=\;& (\partial_1 - r^{-1})\big[2\omega_{AB} + 2\breve{(\tilde{\nabla}_{(A}\overline{\partial_{|0}g_{0|B)}})} - \breve{(\overline{\partial^2_{00}g_{AB}})}\big] \\
\;=\;& (\partial_1 - r^{-1})\omega_{AB} - 4\breve{(\tilde{\nabla}_{(A}\overline{L}_{B)0})} + \breve{(\overline{\partial_0 R_{AB}})} \\
& + \frac{1}{2}(\Delta_{\tilde{g}} - 2r^{-2})\lambda_{AB}\,.
\end{aligned} \tag{5.60}
$$

On the other hand, from the Bianchi identity $\nabla_{[\mu}R_{iA]B}{}^\mu = 0$, $i = 0, 1$, we infer

$$
\breve{(\overline{\nabla_\mu W_{i(AB)}{}^\mu})} + \frac{1}{2}\breve{(\overline{\nabla_i R_{AB}})} - \frac{1}{2}\breve{(\overline{\nabla_{(A}R_{B)i}})} \;=\; 0\,.
$$

Employing further the tracelessness of the Weyl tensor,

$$
g^{\mu\nu}\nabla_0 W_{\mu AB\nu} = 0 \quad\Longrightarrow\quad 2\breve{(\overline{\nabla_0 W_{0(AB)1}})} = \breve{(\overline{\nabla_0 W_{1A1B}})}\,,
$$

we obtain with $\overline{R}_{\mu\nu} = 2\overline{L}_{\mu\nu}$, $\overline{R}_g = 0$, Lemma 4.1 and since the other components of the Weyl tensor are already known to vanish initially,

$$
2(\partial_1 - r^{-1})\breve{\overline{W}}_{0A0B} \;=\; (\partial_1 - r^{-1})\omega_{AB} + \breve{(\overline{\partial_0 R_{AB}})} - 2\breve{(\tilde{\nabla}_{(A}\overline{L}_{B)0})}\,. \tag{5.61}
$$

Combining (5.60) and (5.61) we are led to

$$
(\Delta_{\tilde{g}} - \frac{1}{2}\tau^2)\lambda_{AB} - 4\breve{(\tilde{\nabla}_{(A}\overline{L}_{B)0})} \;=\; 0\,. \tag{5.62}
$$

We apply $(\partial_1 + r^{-1})$ and use (5.16) to conclude that

$$(\Delta_{\tilde{g}} - \frac{1}{2}\tau^2)\omega_{AB} + 2(\partial_1 + r^{-1})(\tilde{\nabla}_{(A}\overline{L}_{B)0})\breve{} \;=\; 0 \;, \tag{5.63}$$

which will prove to be a useful relation. Next we apply $(\partial_1 - r^{-1})$ to (5.60). With (5.16), (5.57), (5.62) and (5.63) we end up with

$$\begin{aligned} 2(\partial_1 - r^{-1})^2\breve{\overline{W}}_{0A0B} &= (\partial^2_{11} - 2r^{-1}\partial_1 + 2r^{-2})\omega_{AB} - 4(\partial_1 - r^{-1})(\tilde{\nabla}_{(A}\overline{L}_{B)0})\breve{} \\ &\quad -(\Delta_{\tilde{g}} - 2r^{-1})(\omega_{AB} + r^{-1}\lambda_{AB}) + (\partial_1 - r^{-1})(\overline{\partial_0 R_{AB}})\breve{} \\ &= 0 \\ \Longrightarrow \quad \breve{\overline{W}}_{0A0B} &= c_{AB}(x^C)r^2 + d_{AB}(x^C)r = c_{AB}(x^C)r^2 \;, \end{aligned}$$

for any regular solution satisfies $\breve{\overline{W}}_{0A0B} = O(r^2)$ in adapted coordinates.

We have $\omega_{AB} = O(r^4)$ and $\lambda_{AB} = O(r^5) = \overline{\partial_0 g_{AB}}$. A regular solution satisfies $O(r^2) = (\overline{\nabla_{(A}L_{B)0}})\breve{} = \tilde{\nabla}_{(A}\overline{L}_{B)0}$. Similarly, we have $O(r^2) = \overline{\nabla_0 R_{AB}} = \overline{\partial_0 R_{AB}} - 2\overline{\Gamma}^C_{0(A}\overline{R}_{B)C}$, which implies $(\overline{\partial_0 R_{AB}})\breve{} = O(r^2)$, so the right-hand side of (5.61) is $O(r^2)$, consequently $\breve{\overline{W}}_{0A0B} = O(r^3)$, whence $c_{AB} = 0$ and

$$\breve{\overline{W}}_{0A0B} = 0 \;.$$

5.5 Validity of equation (2.9) on C_{i^-}

We need to show that (2.9) holds at at least one point. In fact, since $\overline{\Theta}$ vanishes and $\overline{\nabla_\mu \Theta}$ is null, one immediately observes that it is satisfied on the whole initial surface C_{i-}.

5.6 Vanishing of $\overline{\Upsilon}_\mu$

Using the constraint equations (5.6) it is easily checked that the components $\mu = 1, A$ of $\overline{\Upsilon}_\mu \equiv \overline{\nabla_\mu s} + \overline{L}_\mu{}^\nu \overline{\nabla_\nu \Theta}$ vanish. To show that also the $\mu = 0$-component vanishes, we need to compute the value of the transverse derivative of s on $\mathscr{I}^-$, which is accomplished via the CWE (5.2),

$$\overline{\Box_g s} \;=\; 0 \quad \Longleftrightarrow \quad (\partial_1 + r^{-1})\overline{\partial_0 s} \;=\; 0 \;.$$

The function $\overline{\partial_0 s}$ is bounded. Thus

$$\overline{\partial_0 s} \;=\; 0 \;, \tag{5.64}$$

and the vanishing of $\overline{\Upsilon}_\mu$ is ensured.

5.7 Vanishing of $\overline{\Xi}_{\mu\nu}$

We consider

$$\overline{\Xi}_{\mu\nu} \equiv \overline{\nabla_\mu \nabla_\nu \Theta + \Theta\, L_{\mu\nu} - s\, g_{\mu\nu}} = \overline{\partial_\mu \partial_\nu \Theta} - \overline{\Gamma}^0_{\mu\nu}\overline{\partial_0 \Theta} + 2\overline{g}_{\mu\nu} \;.$$

First of all we need to determine the value of $\overline{\partial_0 \Theta}$, which is not part of the initial data. It can be derived from the CWE. Evaluation of (5.3) on $\mathscr{I}^-$ gives

$$\overline{\Box_g \Theta} = 4\overline{s} \quad \Longleftrightarrow \quad (\partial_1 + r^{-1})\overline{\partial_0 \Theta} = -4 \;.$$

For any sufficiently regular solution of the CWE the function $\overline{\partial_0\Theta}$ is bounded near the vertex, and there is precisely one such solution,

$$\overline{\partial_0\Theta} \;=\; -2r\;. \tag{5.65}$$

One straightforwardly checks that $\overline{\Xi}_{\mu\nu}=0$ for $(\mu\nu)\neq(00)$. To determine $\overline{\Xi}_{00}$ we need to compute the second-order transverse derivative of Θ first. This is done via the CWE (5.3),

$$\overline{\partial_0\Box_g\Theta} = 4\overline{\partial_0 s} \overset{(5.64)}{=} 0 \quad\Longleftrightarrow\quad (\partial_1 + r^{-1})\overline{\partial^2_{00}\Theta} - 2r^{-1} = 0\;,$$

where we took into account that $\overline{\partial_0 g_{1\mu}}=0$, $\overline{g}^{AB}\overline{\partial_0 g_{AB}}=0$, $\overline{\partial^2_{00}g_{11}}=0$, as well as the formulae for the u-differentiated Christoffel symbols. The general solution of the ODE is $\overline{\partial^2_{00}\Theta}=2+cr^{-1}$. For any sufficiently regular solution $\overline{\partial^2_{00}\Theta}=\overline{\nabla_0\nabla_0\Theta}$ is bounded, and we conclude

$$\overline{\partial^2_{00}\Theta} \;=\; 2\;,$$

which guarantees the vanishing of $\overline{\Xi}_{00}$.

5.8 Vanishing of $\overline{\varkappa}_{\mu\nu\sigma}$

Recall the definition of the tensor

$$\varkappa_{\mu\nu\sigma} \equiv 2\nabla_{[\sigma}L_{\nu]\mu} - \nabla_\kappa\Theta d_{\nu\sigma\mu}{}^{\kappa}\;.$$

Due to the symmetries $\varkappa_{\mu(\nu\sigma)}=0$, $\varkappa_{[\mu\nu\sigma]}=0$ and $\varkappa_{\nu\mu}{}^{\nu}=0$ (since $\overline{\zeta}_\mu=0$ and $L=0$) its independent components on the initial surface are

$$\overline{\varkappa}_{11A}\;,\quad \overline{\varkappa}_{A1B}\;,\quad \overline{\varkappa}_{01A}\;,\quad \overline{\varkappa}_{ABC}\quad \overline{\varkappa}_{00A}\;,\quad \overline{\varkappa}_{A0B}\;.$$

We find (recall that $\overline{L}_{1\mu}=0$ and $\overline{L}_{0A}=\frac{1}{2}\tilde{\nabla}^B\lambda_{AB}$),

$$\begin{aligned}
\overline{\varkappa}_{11A} &= 0\;,\\
\overline{\varkappa}_{A1B} &= -(\partial_1 - r^{-1})\omega_{AB} - 2r\overline{d}_{1A1B} \overset{(5.8)}{=} 0\;,\\
\overline{\varkappa}_{01A} &= -\partial_1\overline{L}_{0A} + 2r\overline{d}_{011A} \overset{(4.59)}{=} 0\;,\\
\overline{\varkappa}_{ABC} &= 2\tilde{\nabla}_{[C}\omega_{B]A} - \tau\overline{g}_{A[B}\overline{L}_{C]0} - 2r\overline{d}_{1ABC}\\
&= 2\tilde{\nabla}_{[C}\omega_{B]A} - 2\tilde{\nabla}_D\omega_{[B}{}^{D}\overline{g}_{C]A} \overset{\mathrm{tr}(\omega)=0}{=} 0\;,
\end{aligned}$$

where the first equal sign in the last line follows from (4.54), (5.9), (5.6) and (5.16).

To prove the vanishing of the remaining components,

$$\begin{aligned}
\overline{\varkappa}_{A0B} &= \tilde{\nabla}_B\overline{L}_{0A} - \frac{1}{2}\lambda_B{}^C\omega_{AC} + \frac{1}{2}\tau\overline{g}_{AB}\overline{L}_{00} - \overline{\nabla_0 L_{AB}} + 2r\overline{d}_{0BA1}\;,\\
\overline{\varkappa}_{00A} &= \tilde{\nabla}_A\overline{L}_{00} - \lambda_A{}^B\overline{L}_{0B} - \overline{\nabla_0 L_{0A}} + 2r\overline{d}_{010A}\;,
\end{aligned}$$

is somewhat more involved as it requires the knowledge of certain transverse derivatives of $L_{\mu\nu}$ on $\mathscr{I}^-$. These can be extracted from (5.1),

$$\begin{aligned}
\overline{\Box_g L_{AB}} &= \overline{\Box^{(H)}_g L_{AB}} \;=\; 4\omega_{AC}\omega_B{}^C - \overline{g}_{AB}|\omega|^2\;,\\
\overline{\Box_g L_{0A}} &= \overline{\Box^{(H)}_g L_{0A}} \;=\; 4\omega_A{}^B\overline{L}_{0B}\;.
\end{aligned}$$

We employ the facts, established above, that the Weyl tensor vanishes on C_{i^-} and that $L_{\mu\nu}$ coincides there with the Schouten tensor, to compute the action of $\Box_g$ on L_{AB} and L_{0A},

$$\begin{aligned}
\overline{\Box_g L_{AB}} &= 2(\partial_1 - r^{-1})\overline{\nabla_0 L_{AB}} + \partial_1(\partial_1-\tau)\omega_{AB} + (\Delta_{\tilde g} - \frac{1}{2}\tau^2)\omega_{AB} \\
&\quad +2\tau\tilde\nabla_{(A}\overline{L}_{B)0} - \tau\lambda_{(A}{}^C\omega_{B)C} + \frac{1}{2}\tau^2\overline{g}_{AB}\overline{L}_{00} \\
&\overset{(5.63)}{=} 2(\partial_1 - r^{-1})\overline{\nabla_0 L_{AB}} + \partial_1(\partial_1-\tau)\omega_{AB} - 2(\partial_1 - r^{-1})(\tilde\nabla_{(A}\overline{L}_{B)0})\breve{} \\
&\quad +\tau\overline{g}_{AB}\tilde\nabla^C\overline{L}_{0C} - \tau\lambda_{(A}{}^C\omega_{B)C} + \frac{1}{2}\tau^2\overline{g}_{AB}\overline{L}_{00}\ , \\
\overline{\Box_g L_{0A}} &= 2\partial_1\overline{\nabla_0 L_{0A}} + (\partial_1 + r^{-1})(\partial_1 - r^{-1})\overline{L}_{0A} - 2\omega_A{}^B\overline{L}_{0B} \\
&\quad +(\Delta_{\tilde g} - r^2)\overline{L}_{0A} + \tau\tilde\nabla_A\overline{L}_{00} - \lambda_B{}^C\tilde\nabla^B\omega_{AC} - \tau\lambda_A{}^B\overline{L}_{0B} \\
&\overset{(5.16)}{=} 2\partial_1\overline{\nabla_0 L_{0A}} - (\partial_1 - r^{-1})\tilde\nabla^B\omega_{AB} - 2\omega_A{}^B\overline{L}_{0B} \\
&\quad +(\Delta_{\tilde g} + r^{-2})\overline{L}_{0A} + \tau\tilde\nabla_A\overline{L}_{00} - \lambda_B{}^C\tilde\nabla^B\omega_{AC} - \tau\lambda_A{}^B\overline{L}_{0B}\ .
\end{aligned}$$

With these expressions, Lemma 4.1, (5.6)-(5.16) and (4.51)-(4.55) we find

$$\begin{aligned}
2(\partial_1 - r^{-1})\overline{\varkappa}_{A0B} &= 2(\partial_1 - r^{-1})\tilde\nabla_B\overline{L}_{A0} - \omega_A{}^C(\partial_1 - r^{-1})\lambda_{BC} + \tau\overline{g}_{AB}\partial_1\overline{L}_{00} \\
&\quad -\lambda_B{}^C(\partial_1-\tau)\omega_{AC} + 4r\partial_1\overline{d}_{0BA1} - 2(\partial_1 - r^{-1})\overline{\nabla_0 L_{AB}} \\
&= 2(\partial_1 - r^{-1})\tilde\nabla_{[B}\overline{L}_{A]0} + \overline{g}_{AB}(\partial_1 + 3r^{-1})\tilde\nabla^C\overline{L}_{C0} \\
&\quad -\lambda_B{}^C\partial_1\omega_{AC} + \tau\lambda_{[B}{}^C\omega_{A]C} - 2\omega_{AC}\omega_B{}^C + \overline{g}_{AB}|\omega|^2 \\
&\quad +\tau\overline{g}_{AB}(\partial_1 + r^{-1})\overline{L}_{00} - 2r\overline{g}_{AB}(\partial_1+\tau)\overline{d}_{0101} + 2r\partial_1\overline{d}_{01AB} \\
&= -(\partial_1 + r^{-1})(\lambda_{(A}{}^C\omega_{B)C})\breve{} - 4(\omega_{CA}\omega_B{}^C)\breve{} \\
&= 0\ ,
\end{aligned}$$

as well as

$$\begin{aligned}
2\partial_1\overline{\varkappa}_{00A} &= 2\partial_1\tilde\nabla_A\overline{L}_{00} + 4(\omega_A{}^B + r^{-1}\lambda_A{}^B)\overline{L}_{0B} + 2\lambda_A{}^B\tilde\nabla^C\omega_{BC} - 2\partial_1\overline{\nabla_0 L_{0A}} \\
&\quad +4r(\partial_1 + r^{-1})\overline{d}_{010A} \\
&= \frac{1}{2}\tilde\nabla_A(\omega_{BC}\lambda^{BC}) + 2(r^{-1}\lambda_A{}^B - \omega_A{}^B)\overline{L}_{0B} - \tilde\nabla_A\tilde\nabla^B\overline{L}_{0B} - \lambda_C{}^B\tilde\nabla^C\omega_{AB} \\
&\quad +2\lambda_A{}^B\tilde\nabla^C\omega_{BC} - (\partial_1 - r^{-1})\tilde\nabla^B\omega_{AB} + (\Delta_{\tilde g} + r^{-2})\overline{L}_{0A} \\
&\quad +2r\tilde\nabla^B\overline{d}_{01AB} - 2r\tilde\nabla^B\overline{d}_{1A1B} + 4\overline{d}_{011A} + 4r\lambda_A{}^B\overline{d}_{011B} \\
&= -\tilde\nabla^B(\lambda_{C(A}\omega_{B)}{}^C)\breve{} - r^{-2}\overline{L}_{0A} - 2\tilde\nabla_{[A}\tilde\nabla_{B]}\overline{L}_0{}^B \\
&= 0\ .
\end{aligned}$$

Due to regularity we have $\overline{\varkappa}_{A0B} = O(r^2)$ and $\overline{\varkappa}_{00A} = O(r)$, so the only remaining possibilities are

$$\overline{\varkappa}_{A0B} = 0 \quad \text{and} \quad \overline{\varkappa}_{00A} = 0\ .$$

5.9 Vanishing of $\overline{\nabla_\rho d_{\mu\nu\sigma}{}^\rho}$

The independent components of $\overline{\nabla_\rho d_{\mu\nu\sigma}{}^\rho}$, which by Lemma 3.4 is antisymmetric in its first two indices, trace-free and satisfies the first Bianchi identity, are

$$\overline{\nabla_\rho d_{0A0}{}^\rho}\ , \quad \overline{\nabla_\rho d_{0A1}{}^\rho}\ , \quad \overline{\nabla_\rho d_{0AB}{}^\rho}\ , \quad \overline{\nabla_\rho d_{1A1}{}^\rho}\ , \quad \overline{\nabla_\rho d_{1AB}{}^\rho}\ , \quad \overline{\nabla_\rho d_{ABC}{}^\rho}\ .$$

We need to show that they vanish altogether. Let us start with those components which do not involve transverse derivatives. Then their vanishing follows immediately from the constraint equations (5.6)-(5.16) and (4.51)-(4.55),

$$\begin{aligned}\overline{\nabla_\rho d_{0A1}{}^\rho} &= -(\partial_1 + r^{-1})\overline{d}_{010A} + \frac{1}{2}\tilde{\nabla}^B\overline{d}_{01AB} - \frac{1}{2}\tilde{\nabla}^B\overline{d}_{1A1B} + \frac{1}{2}\tilde{\nabla}_A\overline{d}_{0101} \\ &\quad + r^{-1}\overline{d}_{011A} + \lambda_A{}^B\overline{d}_{011B} = 0\,, \\ \overline{\nabla_\rho d_{1A1}{}^\rho} &= -(\partial_1 + 3r^{-1})\overline{d}_{011A} + \tilde{\nabla}^B\overline{d}_{1A1B} = 0\,.\end{aligned}$$

To determine the remaining components we first of all need to compute the transverse derivatives. This is done by evaluating the CWE (5.4) on C_{i^-},

$$\overline{\Box_g d_{\mu\nu\sigma\rho}} = \overline{\Box_g^{(H)} d_{\mu\nu\sigma\rho}} = 0\,. \tag{5.66}$$

Moreover, we will exploit the Lemmas 3.4 and 4.1, the fact that the Weyl tensor vanishes on C_{i^-}, and that $L_{\mu\nu}$ coincides there with the Schouten tensor, i.e. that (4.6) holds initially.

Invoking (5.6)-(5.16) and (4.51)-(4.55) we compute

$$\begin{aligned}(\partial_1 - r^{-1})\overline{\nabla_\rho d_{1AB}{}^\rho} &= -(\partial_1 - r^{-1})\overline{\nabla_0 d_{1A1B}} - \frac{1}{2}(\partial_1 - r^{-1})^2\overline{d}_{1A1B} \\ &\quad + 2\tau(\tilde{\nabla}_{(A}\overline{d}_{B)110})\breve{} - (\tilde{\nabla}_{(A}\tilde{\nabla}^C\overline{d}_{B)1C1})\breve{}\,.\end{aligned} \tag{5.67}$$

With (5.51) we further find

$$\begin{aligned}\overline{\Box_g d_{1A1B}} &= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{1A1B}} + (\partial_{11}^2 - 2r^{-1}\partial_1)\overline{d}_{1A1B} + \Delta_{\tilde{g}}\overline{d}_{1A1B} \\ &\quad + 2\tau\tilde{\nabla}^C\overline{d}_{1(AB)C} + \frac{1}{2}\tau^2\overline{g}^{CD}\overline{d}_{ACBD} + 2\tau\tilde{\nabla}_{(A}\overline{d}_{B)101} + \frac{1}{2}\tau^2\overline{g}_{AB}\overline{d}_{0101} \\ &= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{1A1B}} + (\Delta_{\tilde{g}} + \partial_{11}^2 - 2r^{-1}\partial_1)\overline{d}_{1A1B} - 4\tau(\tilde{\nabla}_{(A}\overline{d}_{B)110})\breve{}\,.\end{aligned}$$

The transverse derivative in (5.67) is eliminated via $\overline{\Box_g d_{1A1B}} = 0$,

$$(\partial_1 - r^{-1})\overline{\nabla_\rho d_{1AB}{}^\rho} = \frac{1}{2}\Delta_{\tilde{g}}\overline{d}_{1A1B} - r^{-2}\overline{d}_{1A1B} - (\tilde{\nabla}_{(A}\tilde{\nabla}^C\overline{d}_{B)1C1})\breve{}\,.$$

We need an expression for the $\Delta_{\tilde{g}}$-term, which can be derived from (5.8), (5.63), (5.6) and (5.16) as follows,

$$\begin{aligned}\Delta_{\tilde{g}}\overline{d}_{1A1B} &= -\frac{1}{2}r^{-1}(\partial_1 + r^{-1})\Delta_{\tilde{g}}\omega_{AB} \\ &= \frac{1}{2}r^{-1}(\partial_1 + r^{-1})[2(\partial_1 + r^{-1})(\tilde{\nabla}_{(A}\overline{L}_{B)0})\breve{} - \frac{1}{2}\tau^2\omega_{AB}] \\ &= 2r^{-2}\overline{d}_{1A1B} + 2(\tilde{\nabla}_{(A}\tilde{\nabla}^C\overline{d}_{B)1C1})\breve{}\,.\end{aligned} \tag{5.68}$$

Plugging that in we are led to the ODE

$$(\partial_1 - r^{-1})\overline{\nabla_\rho d_{1AB}{}^\rho} = 0\,.$$

For any sufficiently regular solution of the CWE we have $\overline{\nabla_\rho d_{1AB}{}^\rho} = O(r^2)$ and hence

$$\overline{\nabla_\rho d_{1AB}{}^\rho} = 0\,.$$

To show that the other components of $\nabla_\rho d_{\mu\nu\sigma}{}^\rho$ vanish initially, we proceed in a similar manner. In particular, we shall make extensively use of the constraint equations (5.6)-(5.16) (and also of their non-integrated counterparts (4.46)-(4.50)), (4.51)-(4.55) and of the expressions (5.51)-(5.53) we computed for the components of the Riemann tensor.

Let us establish the vanishing of $\overline{\nabla_\rho d_{ABC}{}^\rho}$. By (5.66) we have $\overline{\Box_g d_{1ABC}} = 0$ on $\mathscr{I}^-$ with

$$\begin{aligned}
\overline{\Box_g d_{1ABC}} &= 2(\partial_1-\tau)\overline{\nabla_0 d_{1ABC}} + (\partial^2_{11} - 4r^{-1}\partial_1 + r^{-2})\overline{d}_{1ABC} + \Delta_{\tilde g}\overline{d}_{1ABC}\\
&\quad -\tau\tilde\nabla^D\overline{d}_{DABC} + \tau\tilde\nabla_A\overline{d}_{10BC} + 2\tau\tilde\nabla_{[B}\overline{d}_{C]0A1} + 2\tau\tilde\nabla_{[B}\overline{d}_{C]1A1}\\
&\quad +2\tilde\nabla_D(\lambda_{[B}{}^D\overline{d}_{C]1A1}) - \tau\lambda_{[B}{}^D\overline{d}_{C]1AD} - \tau\lambda_{[B}{}^D\overline{d}_{C]DA1}\\
&\quad -\frac{1}{2}\tau^2\overline{d}_{0ABC} - \tau^2\overline{g}_{A[B}\overline{d}_{C]010} - \tau^2\overline{g}_{A[B}\overline{d}_{C]110} - \tau\lambda_{A[B}\overline{d}_{C]110}\\
&= 2(\partial_1-\tau)\overline{\nabla_0 d_{1ABC}} + 2\overline{g}_{A[B}(\partial^2_{11} - 5r^{-2})\overline{d}_{C]110} + 2\overline{g}_{A[B}\Delta_{\tilde g}\overline{d}_{C]110}\\
&\quad -3\tau\overline{g}_{A[B}\tilde\nabla_{C]}\overline{d}_{0101} - \tau\tilde\nabla_A\overline{d}_{01BC} + \tau\tilde\nabla_{[B}\overline{d}_{C]A01} + \tau\tilde\nabla_{[B}\overline{d}_{C]1A1}\\
&\quad +2\tilde\nabla_D(\lambda_{[B}{}^D\overline{d}_{C]1A1}) - 2\tau\overline{g}_{A[B}\lambda_{C]}{}^D\overline{d}_{011D} - 2\tau\lambda_{A[B}\overline{d}_{C]110}\,. \qquad (5.69)
\end{aligned}$$

We determine

$$\begin{aligned}
\overline{\nabla_\rho d_{ABC}{}^\rho} &= \overline{\nabla_0 d_{ABC1}} + 2\overline{g}_{C[A}(\partial_1 + r^{-1})\overline{d}_{B]010}\\
&\quad -2\overline{g}_{C[A}\tilde\nabla_{B]}\overline{d}_{0101} - \overline{g}_{C[A}\lambda_{B]}{}^D\overline{d}_{011D} + \lambda_{C[A}\overline{d}_{B]110}\\
&= \overline{\nabla_0 d_{ABC1}} + \overline{g}_{C[A}\tilde\nabla^D\overline{d}_{B]D01} + \overline{g}_{C[A}\tilde\nabla^D\overline{d}_{B]11D} + \tau\overline{g}_{C[A}\overline{d}_{B]110}\\
&\quad -\overline{g}_{C[A}\tilde\nabla_{B]}\overline{d}_{0101} + \overline{g}_{C[A}\lambda_{B]}{}^D\overline{d}_{011D} + \lambda_{C[A}\overline{d}_{B]110}\,.
\end{aligned}$$

Due to the constraint equations that yields

$$\begin{aligned}
&2(\partial_1-\tau)\overline{\nabla_\rho d_{CBA}{}^\rho}\\
&\quad= 2(\partial_1-\tau)\overline{\nabla_0 d_{1ABC}} + 2\overline{g}_{A[B}\tilde\nabla^D(\partial_{|1|} - \tau)\overline{d}_{C]1D1}\\
&\qquad -2\overline{g}_{A[B}\tilde\nabla^D(\partial_{|1|} + r^{-1})\overline{d}_{C]D01} - 2\tau\overline{g}_{A[B}(\partial_{|1|} + 3r^{-1})\overline{d}_{C]110}\\
&\qquad +2\overline{g}_{A[B}\tilde\nabla_{C]}(\partial_1 + 3r^{-1})\overline{d}_{0101} - 2\overline{g}_{A[B}\lambda_{C]}{}^D(\partial_1 + 3r^{-1})\overline{d}_{011D}\\
&\qquad -2\lambda_{A[B}(\partial_{|1|} + 3r^{-1})\overline{d}_{C]110} + 3\tau\overline{g}_{A[B}\tilde\nabla^D\overline{d}_{C]D01} + 4\tau^2\overline{g}_{A[B}\overline{d}_{C]110}\\
&\qquad -3\tau\overline{g}_{A[B}\tilde\nabla_{C]}\overline{d}_{0101} + 4\tau\overline{g}_{A[B}\lambda_{C]}{}^D\overline{d}_{011D} + 4\tau\lambda_{A[B}\overline{d}_{C]110}\\
&\qquad + \underbrace{4\omega_{A[B}\overline{d}_{C]110} + 4\overline{g}_{A[B}\omega_{C]}{}^D\overline{d}_{011D}}_{=0}\\
&\quad= 2(\partial_1-\tau)\overline{\nabla_0 d_{1ABC}} + 2\overline{g}_{A[B}\tilde\nabla^D(\partial_{|1|} - 2\tau)\overline{d}_{C]1D1} + 4\tau\overline{g}_{A[B}\lambda_{C]}{}^D\overline{d}_{011D}\\
&\qquad +4\tau\lambda_{A[B}\overline{d}_{C]110} - 3\tau\overline{g}_{A[B}\tilde\nabla_{C]}\overline{d}_{0101} + 3\tau\overline{g}_{A[B}\tilde\nabla^D\overline{d}_{C]D01} + \frac{7}{2}\tau^2\overline{g}_{A[B}\overline{d}_{C]110}\\
&\qquad +\overline{g}_{A[B}\tilde\nabla_{C]}(\lambda^{DE}\overline{d}_{1D1E}) + 2\overline{g}_{A[B}\Delta_{\tilde g}\overline{d}_{C]110} + \overline{g}_{A[B}\tilde\nabla^D(\overline{d}_{C]1F1}\lambda_D{}^F)\\
&\qquad -\overline{g}_{A[B}\tilde\nabla^D(\lambda_{C]}{}^E\overline{d}_{1D1E})\underbrace{-2\overline{g}_{A[B}\lambda_{C]}{}^E\tilde\nabla^D\overline{d}_{1D1E} - 2\lambda_{A[B}\tilde\nabla^D\overline{d}_{C]1D1}}_{=0}\,.
\end{aligned}$$

With $\overline{\Box_g d_{1ABC}} = 0$ we eliminate the transverse derivative. Employing further

(5.8), (5.9) and (5.16) we end up with

$$\begin{aligned}
&2(\partial_1 - 2r^{-1})\overline{\nabla_\rho d_{CBA}{}^\rho} \\
&\quad = -\frac{1}{4}\tau^2(\partial_1 - r^{-1})(\overline{g}_{A[B}\tilde\nabla^D\omega_{C]D} - \tilde\nabla_{[B}\omega_{C]A}) \\
&\qquad +\tau\tilde\nabla_A\overline{d}_{01BC} - \tau\tilde\nabla_{[B}\overline{d}_{C]A01} + 3\tau\overline{g}_{A[B}\tilde\nabla^D\overline{d}_{C]D01} \\
&\qquad +6\tau\overline{g}_{A[B}\lambda_{C]}{}^D\overline{d}_{011D} + 6\tau\lambda_{A[B}\overline{d}_{C]110} \\
&\qquad -\overline{g}_{A[B}\tilde\nabla^D(\lambda_{C]}{}^E\overline{d}_{1D1E}) + \overline{g}_{A[B}\tilde\nabla_{C]}(\lambda^{DE}\overline{d}_{1D1E}) - \tilde\nabla_D(\lambda_{[B}{}^D\overline{d}_{C]1A1}) \\
&\qquad +\overline{g}_{A[B}\tilde\nabla^D(\overline{d}_{C]1E1}\lambda_D{}^E) - \tilde\nabla_D(\lambda_{[B}{}^D\overline{d}_{C]1A1}) \\
&\quad = 0\,,
\end{aligned} \tag{5.70}$$

since the terms in each line add up to zero, as one checks e.g. by introducing an orthonormal frame for $\tilde g$. By regularity we have $\overline{\nabla_\rho d_{ABC}{}^\rho} = O(r^3)$, so (5.70) enforces

$$\overline{\nabla_\rho d_{ABC}{}^\rho} = 0\,.$$

To check the vanishing of $\overline{\nabla_\rho d_{0A0}{}^\rho}$ we start with the relation $\overline{\Box_g d_{010A}} = 0$, and compute

$$\begin{aligned}
\overline{\Box_g d_{010A}} &= 2\overline{L}_0{}^B\overline{d}_{0A1B} + 2\overline{L}_0{}^B\overline{d}_{01AB} + 2\overline{L}_{0A}\overline{d}_{0101} + 2\partial_1\overline{\nabla_0 d_{010A}} - \frac{1}{2}\tau\lambda_A{}^B\overline{d}_{010B} \\
&\quad +(\Delta_{\tilde g} + \partial_{11}^2 - \frac{5}{4}\tau^2)\overline{d}_{010A} - \lambda_B{}^C\tilde\nabla^B\overline{d}_{C10A} + \lambda_B{}^C\tilde\nabla^B\overline{d}_{01AC} \\
&\quad +\frac{1}{2}\lambda_B{}^C\lambda^{BD}\overline{d}_{1CAD} - \frac{1}{4}|\lambda|^2\overline{d}_{011A} + (\tau\lambda_A{}^B + \lambda_A{}^C\lambda_C{}^B)\overline{d}_{011B} \\
&\quad -\tau\tilde\nabla^B\overline{d}_{0A0B} - \frac{1}{2}\tau\lambda^{BC}\overline{d}_{0BAC} + \tau\tilde\nabla_A\overline{d}_{0101} + \lambda_A{}^B\tilde\nabla_B\overline{d}_{0101} \\
&= 2\partial_1\overline{\nabla_0 d_{010A}} + \tilde\nabla_A\tilde\nabla^B\overline{d}_{011B} - \frac{1}{2}\Delta_{\tilde g}\overline{d}_{011A} + \frac{1}{4}\tilde\nabla^B(\lambda_A{}^C\overline{d}_{1B1C}) \\
&\quad +\frac{1}{4}\tilde\nabla_A(\lambda^{BC}\overline{d}_{1B1C}) - \frac{1}{2}\tilde\nabla^B(\partial_1 - 5r^{-1})\overline{d}_{1A1B} - \tau\tilde\nabla^B\overline{d}_{01AB} \\
&\quad -2\omega_A{}^B\overline{d}_{011B} + \lambda_A{}^C\tilde\nabla^B\overline{d}_{1B1C} - \tau\tilde\nabla^B\overline{d}_{0A0B} - \frac{3}{4}\tilde\nabla^B(\lambda_B{}^C\overline{d}_{1A1C}) \\
&\quad +\frac{3}{2}\tilde\nabla^B(\lambda_B{}^C\overline{d}_{01AC}) + \frac{3}{2}\tilde\nabla_B(\lambda_A{}^B\overline{d}_{0101}) + \Delta_{\tilde g}\overline{d}_{010A} - \frac{9}{8}\tau^2\overline{d}_{011A} \\
&\quad -\frac{3}{4}\tau^2\overline{d}_{010A} - \tau\lambda_A{}^B\overline{d}_{011B} + \underbrace{\frac{3}{2}\lambda_A{}^B\lambda_B{}^C\overline{d}_{011C} - \frac{3}{4}|\lambda|^2\overline{d}_{011A}}_{=0}\,,
\end{aligned}$$

as follows from the constraint equations. We have

$$\overline{\nabla_\rho d_{0A0}{}^\rho} = \overline{\nabla_0 d_{010A}} + (\partial_1 + \tau)\overline{d}_{010A} + \tilde\nabla^B\overline{d}_{0A0B} - \frac{1}{2}\lambda_A{}^B\overline{d}_{011B}\,,$$

which implies, again via the constraint equations,

$$\begin{aligned}
2\partial_1\overline{\nabla_\rho d_{0A0}{}^\rho} &= 2\partial_1\overline{\nabla_0 d_{010A}} + 2\tilde\nabla^B(\partial_1 - r^{-1})\breve{\overline{d}}_{0A0B} - \tilde\nabla_A(\partial_1 + 3r^{-1})\overline{d}_{0101} \\
&\quad +\tau\tilde\nabla_A\overline{d}_{0101} - \tau\tilde\nabla^B\overline{d}_{0A0B} - \lambda_A{}^B(\partial_1 + 3r^{-1})\overline{d}_{011B} - \tau^2\overline{d}_{010A} \\
&\quad +2\tau\lambda_A{}^B\overline{d}_{011B} + 2\omega_A{}^B\overline{d}_{011B} + 2(\partial_1 + r^{-1})^2\overline{d}_{010A} \\
&= 2\partial_1\overline{\nabla_0 d_{010A}} - \frac{9}{8}\tau^2\overline{d}_{011A} - \frac{3}{4}\tau^2\overline{d}_{010A} + \Delta_{\tilde g}\overline{d}_{010A} - \frac{1}{2}\Delta_{\tilde g}\overline{d}_{011A} \\
&\quad +\frac{3}{2}\tilde\nabla^B(\lambda_{(A}{}^C\overline{d}_{B)C01}) + \frac{3}{2}\tilde\nabla_B(\lambda_A{}^B\overline{d}_{0101}) + \tilde\nabla^B(\lambda_{[A}{}^C\overline{d}_{B]1C1}) \\
&\quad +\tilde\nabla_A\tilde\nabla^B\overline{d}_{011B} - \tau\tilde\nabla^B\overline{d}_{0A0B} - \tau\tilde\nabla^B\overline{d}_{01AB} - 2\omega_A{}^B\overline{d}_{011B} \\
&\quad -\frac{1}{2}\tilde\nabla^B(\partial_1 - 5r^{-1})\overline{d}_{1A1B} - \tau\lambda_A{}^B\overline{d}_{011B} + \lambda_A{}^C\tilde\nabla^B\overline{d}_{1B1C}\ .
\end{aligned}$$

Combining these results we end up with

$$2\partial_1\overline{\nabla_\rho d_{0A0}{}^\rho} = \underbrace{\frac{1}{2}\tilde\nabla^B(\lambda_{(A}{}^C\overline{d}_{B)1C1}) - \frac{1}{4}\tilde\nabla_A(\lambda^{BC}\overline{d}_{1B1C})}_{=0} - \frac{3}{2}\underbrace{\tilde\nabla^B(\lambda_{[B}{}^C\overline{d}_{A]C01})}_{=0}\ ,$$

and, as regularity requires $\overline{\nabla_\rho d_{0A0}{}^\rho} = O(r)$, that gives

$$\overline{\nabla_\rho d_{0A0}{}^\rho} = 0\ .$$

To continue, we analyse the vanishing of $\overline{\nabla_\rho d_{0AB}{}^\rho}$. We have

$$\begin{aligned}
\overline{\Box_g d_{0AB1}} &= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{0AB1}} - 2\overline{L}_0{}^C\overline{d}_{1BAC} - 2\overline{L}_{0A}\overline{d}_{011B} \\
&\quad +(\partial_1 - \tau)\partial_1\overline{d}_{0AB1} + \Delta_{\tilde g}\overline{d}_{0AB1} - \lambda_{AC}\tilde\nabla^C\overline{d}_{011B} - \tau\tilde\nabla_A\overline{d}_{011B} \\
&\quad +\tau\tilde\nabla_B\overline{d}_{010A} - \tau\tilde\nabla^C\overline{d}_{0ABC} - \lambda^{CD}\tilde\nabla_D\overline{d}_{1BAC} - \frac{1}{2}\tau\lambda^{CD}\overline{d}_{ACBD} \\
&\quad -\frac{1}{2}\tau\lambda_A{}^C\overline{d}_{1B1C} + \frac{1}{2}\tau^2\overline{d}_{01AB} + \frac{1}{2}\tau(\tau\overline{g}_{AB} + \lambda_{AB})\overline{d}_{0101} \\
&\quad -\tau\underbrace{\lambda_{[A}{}^C\overline{d}_{B]C01}}_{=0} + \underbrace{\frac{1}{4}|\lambda|^2\overline{d}_{1A1B} - \frac{1}{2}\lambda_A{}^C\lambda_C{}^D\overline{d}_{1B1D}}_{=0} \\
&= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{0AB1}} - 2\overline{g}_{AB}\overline{L}_0{}^C\overline{d}_{011C} - 4\overline{L}_{0[A}\overline{d}_{B]110} + \frac{1}{2}\Delta_{\tilde g}\overline{d}_{1A1B} \\
&\quad +\frac{1}{2}(\partial_1 - \tau)\partial_1\overline{d}_{1A1B} - \frac{1}{2}\overline{g}_{AB}\tilde\nabla^C\tilde\nabla^D\overline{d}_{1C1D} + \frac{1}{2}\overline{g}_{AB}\omega^{CD}\overline{d}_{1C1D} \\
&\quad -\frac{1}{4}\overline{g}_{AB}\lambda^{CD}(\partial_1 - 2\tau)\overline{d}_{1C1D} - \frac{1}{2}\overline{g}_{AB}\Delta_{\tilde g}\overline{d}_{0101} - \tilde\nabla_{[A}\tilde\nabla^C\overline{d}_{B]1C1} \\
&\quad -\tau\tilde\nabla_B\overline{d}_{A110} + \omega_{[A}{}^C\overline{d}_{B]1C1} - \frac{1}{2}\lambda_{[A}{}^C(\partial_1 - 2\tau)\overline{d}_{B]1C1} - \frac{1}{2}\Delta_{\tilde g}\overline{d}_{01AB} \\
&\quad -2\lambda_{C[A}\tilde\nabla^C\overline{d}_{B]110} - \tau\tilde\nabla_{(A}\overline{d}_{B)110} - 2\tau\tilde\nabla_{[A}\overline{d}_{B]010} - \frac{1}{2}\tau\lambda_A{}^C\overline{d}_{1B1C} \\
&\quad +\tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{010C} + \frac{5}{2}\tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{011C} - \overline{g}_{AB}\lambda^{CD}\tilde\nabla_D\overline{d}_{011C}\ ,
\end{aligned}$$

which vanishes owing to (5.66). Moreover,

$$\begin{aligned}
\overline{\nabla_\rho d_{0AB}{}^\rho} &= \overline{\nabla_0 d_{0AB1}} + (\partial_1 - r^{-1})\overline{d}_{0AB0} + (\partial_1 - r^{-1})\overline{d}_{0AB1} + \tilde\nabla^C\overline{d}_{0ABC} \\
&\quad + \frac{1}{2}\lambda^{CD}\overline{d}_{ACBD} + \frac{1}{2}\lambda_A{}^C\overline{d}_{01BC} + \frac{1}{2}\lambda_B{}^C\overline{d}_{0A1C} - \frac{1}{2}\tau\overline{d}_{01AB} \\
&= \overline{\nabla_0 d_{0AB1}} + \frac{1}{4}(\partial_1 - r^{-1})\overline{d}_{1A1B} - \frac{1}{4}\lambda_A{}^C\overline{d}_{1B1C} + \tilde\nabla_{[A}\overline{d}_{B]010} \\
&\quad + \frac{1}{2}\tilde\nabla_{(A}\overline{d}_{B)110} - \frac{1}{2}\overline{g}_{AB}\tilde\nabla^C\overline{d}_{010C} - \frac{3}{4}\overline{g}_{AB}\tilde\nabla^C\overline{d}_{011C} + \frac{1}{4}\lambda_{[A}{}^C\overline{d}_{B]C01}\,,
\end{aligned}$$

and thus

$$\begin{aligned}
&2(\partial_1 - r^{-1})\overline{\nabla_\rho d_{0AB}{}^\rho} \\
&\quad= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{0AB1}} + \frac{1}{2}(\partial_1 - r^{-1})^2\overline{d}_{1A1B} - \frac{1}{2}\lambda_A{}^C(\partial_1 - \tau)\overline{d}_{1B1C} \\
&\qquad + \omega_A{}^C\overline{d}_{1B1C} + 2\tilde\nabla_{[A}(\partial_1 + r^{-1})\overline{d}_{B]010} + \tilde\nabla_{(A}(\partial_1 + 3r^{-1})\overline{d}_{B)110} \\
&\qquad - \overline{g}_{AB}\tilde\nabla^C(\partial_1 + r^{-1})\overline{d}_{010C} - \frac{3}{2}\overline{g}_{AB}\tilde\nabla^C(\partial_1 + 3r^{-1})\overline{d}_{011C} - 2\tau\tilde\nabla_{[A}\overline{d}_{B]010} \\
&\qquad + \frac{1}{2}\lambda_{[A}{}^C(\partial_1 + r^{-1})\overline{d}_{B]C01} - \frac{3}{2}r^{-1}\lambda_{[A}{}^C\overline{d}_{B]C01} + \tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{010C} \\
&\qquad - 2\tau\tilde\nabla_{(A}\overline{d}_{B)110} + 3\tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{011C} - \omega_{[A}{}^C\overline{d}_{B]C01} \\
&\quad= 2(\partial_1 - r^{-1})\overline{\nabla_0 d_{0AB1}} + \frac{1}{2}(\partial_1 - r^{-1})^2\overline{d}_{1A1B} - \frac{1}{2}\lambda_A{}^C(\partial_1 - \tau)\overline{d}_{1B1C} \\
&\qquad + \omega_A{}^C\overline{d}_{1B1C} + \tilde\nabla_B\tilde\nabla^C\overline{d}_{1A1C} - \overline{g}_{AB}\tilde\nabla^C\tilde\nabla^D\overline{d}_{1C1D} - \tau\tilde\nabla_B\overline{d}_{A110} \\
&\qquad - \tau\tilde\nabla_{(A}\overline{d}_{B)110} + \frac{5}{2}\tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{011C} + 2(\tilde\nabla_{[A}\lambda_{B]}{}^C - \overline{g}_{AB}\overline{L}_0{}^C)\overline{d}_{011C} \\
&\qquad + \tilde\nabla_{[A}\tilde\nabla^C\overline{d}_{B]C01} - \frac{3}{2}\lambda_{[A}{}^C\tilde\nabla_{B]}\overline{d}_{C110} - 2\tau\tilde\nabla_{[A}\overline{d}_{B]010} + \tau\overline{g}_{AB}\tilde\nabla^C\overline{d}_{010C} \\
&\qquad - \frac{1}{2}\lambda_{C[A}\tilde\nabla^C\overline{d}_{B]110} - \overline{g}_{AB}\lambda_C{}^D\tilde\nabla^C\overline{d}_{011D} - \frac{1}{2}\overline{g}_{AB}\Delta_{\tilde g}\overline{d}_{0101} \\
&\qquad - \frac{1}{4}\underbrace{\lambda_C{}^D\lambda_{[A}{}^C\overline{d}_{B]1D1}}_{=0} - \underbrace{\omega_{[A}{}^C\overline{d}_{B]C01}}_{=0} - \frac{3}{2}r^{-1}\underbrace{\lambda_{[A}{}^C\overline{d}_{B]C01}}_{=0}\,.
\end{aligned}$$

Using the formula (5.68) we derived for $\Delta_{\tilde g}\overline{d}_{1A1B}$, we conclude that

$$\begin{aligned}
&2(\partial_1 - r^{-1})\overline{\nabla_\rho d_{0AB}{}^\rho} \\
&\quad= 2\{(\tilde\nabla_C\lambda_{[A}{}^C)\overline{d}_{B]110} + (\tilde\nabla_{[A}\lambda_{B]}{}^C)\overline{d}_{011C}\} - \frac{1}{2}(\partial_1 - 3r^{-1})\{(\lambda_{(A}{}^C\overline{d}_{B)1C1})\check{}\} \\
&\qquad + \frac{3}{2}\{\lambda_{C[A}\tilde\nabla^C\overline{d}_{B]110} - \lambda_{[A}{}^C\tilde\nabla_{B]}\overline{d}_{C110}\} + \{\tilde\nabla_{[A}\tilde\nabla^C\overline{d}_{B]C01} + \frac{1}{2}\Delta_{\tilde g}\overline{d}_{01AB}\} \\
&\quad= 0\,,
\end{aligned}$$

since the terms in each of the braces add up to zero (recall Lemma 4.1). Taking further into account that regularity yields $\overline{\nabla_\rho d_{0AB}{}^\rho} = O(r^2)$, we deduce that

$$\overline{\nabla_\rho d_{0AB}{}^\rho} = 0\,.$$

5.10 Main result

By way of summary we end up with the following result:

Theorem 5.1 *Let us suppose we have been given a smooth one-parameter family of s-traceless tensors $\omega_{AB}(r,x^A)=O(r^4)$ on the 2-sphere, where s denotes the standard metric. Let λ_{AB} be the unique solution of the equation*

$$(\partial_1 - r^{-1})\lambda_{AB} = -2\omega_{AB}\ , \tag{5.71}$$

with $\lambda_{AB}=O(r^5)$. A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^{\rho}, \Theta, s)$ of the CWE (5.1)-(5.5) to the future of C_{i^-}, smoothly extendable through C_{i^-}, with initial data $(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma}{}^{\rho}, \mathring{\Theta}=0, \mathring{s}=-2)$ and with $\breve{\mathring{L}}_{AB}=\omega_{AB}$, is a solution of the MCFE (4.1)-(4.6) with $\lambda=0$ in the

$$(R=0, \overline{s}=-2, \kappa=0, \hat{g}_{\mu\nu}=\eta_{\mu\nu})\text{-wave-map gauge}\ ,$$

if and only if the initial data have their usual algebraic properties and solve the constraint equations (5.6)-(5.14) with boundary conditions (5.15).

The function Θ is positive in the interior of C_{i^-} and sufficiently close to i^-, and $\mathrm{d}\Theta\neq 0$ on $C_{i^-}\setminus\{i^-\}$.

Remark 5.2 *Note that regularity for the rescaled Weyl tensor implies that the initial data necessarily need to satisfy $\omega_{AB}(r,x^A)=O(r^4)$, cf. equation (5.8).*

Proof: The previous computations show that Theorem 3.7 is applicable. The positivity of Θ inside the cone simply follows from (4.3) and the negativity of s near the vertex as one might check using e.g. normal coordinates.

Concerning the "only if"-part: That the constraint equations (5.6)-(5.14) are satisfied by any solution of the MCFE in the $(R=0, \overline{s}=-2, \kappa=0, \hat{g}_{\mu\nu}=\eta_{\mu\nu})$-wave-map gauge and with $\overline{\Theta}=0$ follows directly from their derivation. □

6 Alternative system of conformal wave equations (CWE2)

Instead of a wave equation for the rescaled Weyl tensor $d_{\mu\nu\sigma}{}^{\rho}$, it might be advantageous in certain situations to work with the Weyl tensor itself, which we denote here by $C_{\mu\nu\sigma}{}^{\rho}$, as unknown. The Weyl tensor is a more physical quantity (it is conformally invariant and thus coincides with the physical Weyl tensor) and can be expressed in terms of the metric even on null and timelike infinity. We shall see that proceeding this way it becomes necessary to regard the Cotton tensor as another unknown, so that the system of wave equations we are about to derive might be somewhat more complicated. An advantage is that we just need to require the metric to be regular at i^- rather than the metric and the rescaled Weyl tensor, so the alternative system might be useful to find a more general class of solutions (cf. the discussion in Section 7.1).

Since many of the computations which need to be done to derive the alternative system of wave equations (6.9)-(6.14) and prove Theorem 6.5 are very similar to the ones we did for the CWE involving the rescaled Weyl tensor, the computations are partially even more compressed than in the previous part.

6.1 Derivation

The *Cotton tensor* in 4-spacetime dimensions is defined as

$$\xi_{\mu\nu\sigma} := 2\nabla_{[\sigma}R_{\nu]\mu} + \frac{1}{3}g_{\mu[\sigma}\nabla_{\nu]}R = 4\nabla_{[\sigma}L_{\nu]\mu} .$$

It is manifestly antisymmetric in its last two indices. Moreover, the Bianchi identities imply the following properties,

$$\begin{aligned} \xi_{[\mu\nu\sigma]} &= 0 , && (6.1)\\ \xi_{\rho\nu}{}^{\rho} &= 0 , && (6.2)\\ \nabla^{\rho}\xi_{\rho\nu\sigma} &= 0 , && (6.3)\\ \xi_{\mu\nu\sigma} &= -2\nabla_{\alpha}C^{\alpha}{}_{\mu\nu\sigma} . && (6.4) \end{aligned}$$

Using the wave equation (3.2) for the Schouten tensor (written in terms of $C_{\mu\nu\sigma}{}^{\rho}$ rather than $\Theta d_{\mu\nu\sigma}{}^{\rho}$) one further verifies the relation

$$2L_{\alpha\sigma}C_{\mu}{}^{\alpha}{}_{\nu}{}^{\sigma} + \nabla^{\sigma}\xi_{\mu\nu\sigma} = 0 , \tag{6.5}$$

which expresses the vanishing of the *Bach tensor*.

The second Bianchi identity implies,

$$2\nabla_{[\alpha}C_{\mu\nu]\sigma}{}^{\rho} = g_{\sigma[\mu}\xi^{\rho}{}_{\alpha\nu]} + \delta_{[\mu}{}^{\rho}\xi_{|\sigma|\nu\alpha]} . \tag{6.6}$$

(In particular one recovers (6.4) for $\rho = \alpha$.) Contracting (6.6) with ∇^{α} we find a wave equation for the Weyl tensor[12]

$$\begin{aligned} \Box_g C_{\mu\nu\sigma\rho} &\overset{(6.5)}{=} 2\nabla^{\alpha}\nabla_{[\nu}C_{\mu]\alpha\sigma\rho} + 2g_{\sigma[\mu}C_{\nu]\alpha\rho\beta}L^{\alpha\beta} - 2g_{\rho[\mu}C_{\nu]\alpha\sigma\beta}L^{\alpha\beta} - \nabla_{[\sigma}\xi_{\rho]\mu\nu}\\ &\overset{(6.4)}{=} C_{\mu\nu\alpha}{}^{\kappa}C_{\sigma\rho\kappa}{}^{\alpha} - 4C_{\sigma\kappa[\mu}{}^{\alpha}C_{\nu]\alpha\rho}{}^{\kappa} - 2C_{\sigma\rho\kappa[\mu}L_{\nu]}{}^{\kappa} - 2C_{\mu\nu\kappa[\sigma}L_{\rho]}{}^{\kappa}\\ &\quad -\nabla_{[\sigma}\xi_{\rho]\mu\nu} - \nabla_{[\mu}\xi_{\nu]\sigma\rho} + \frac{1}{3}RC_{\mu\nu\sigma\rho} . \end{aligned} \tag{6.7}$$

We observe that the Cotton tensor is needed to eliminate the disturbing second-order derivatives of $C_{\mu\nu\sigma\rho}$.

Finally, we derive a wave equation for the Cotton tensor $\xi_{\mu\nu\sigma}$ by employing the wave equation (3.2) for the Schouten tensor, the Bianchi identity and (6.4):

$$\begin{aligned} \Box_g\xi_{\mu\nu\sigma} &\equiv 4\nabla_{[\sigma}\Box_g L_{\nu]\mu} + 8g_{\mu[\nu}L_{|\alpha|}{}^{\kappa}\nabla^{\alpha}L_{\sigma]\kappa} - 16L_{[\nu}{}^{\kappa}\nabla_{\sigma]}L_{\mu\kappa}\\ &\quad +2\xi_{\kappa\sigma\nu}L_{\mu}{}^{\kappa} + 4\xi_{\mu\kappa[\sigma}L_{\nu]}{}^{\kappa} + C_{\nu\sigma\alpha}{}^{\kappa}\xi_{\mu\kappa}{}^{\alpha} + 8C_{\alpha[\sigma|\mu|}{}^{\kappa}\nabla^{\alpha}L_{\nu]\kappa}\\ &\quad -\frac{2}{3}R\nabla_{[\nu}L_{\sigma]\mu} + \frac{2}{3}L_{\mu[\nu}\nabla_{\sigma]}R + \frac{2}{3}g_{\mu[\nu}L_{\sigma]\kappa}\nabla^{\kappa}R\\ &= 4\xi_{\kappa\alpha[\nu}C_{\sigma]}{}^{\alpha}{}_{\mu}{}^{\kappa} + C_{\nu\sigma\alpha}{}^{\kappa}\xi_{\mu\kappa}{}^{\alpha} - 4\xi_{\mu\kappa[\nu}L_{\sigma]}{}^{\kappa} + 6g_{\mu[\nu}\xi^{\kappa}{}_{\sigma\alpha]}L_{\kappa}{}^{\alpha}\\ &\quad +8L_{\alpha\kappa}\nabla_{[\nu}C_{\sigma]}{}^{\alpha}{}_{\mu}{}^{\kappa} + \frac{1}{6}R\xi_{\mu\nu\sigma} - \frac{1}{3}C_{\nu\sigma\mu}{}^{\kappa}\nabla_{\kappa}R . \end{aligned} \tag{6.8}$$

Combining these results with the equations we found for Θ, s, $g_{\mu\nu}$ and $L_{\mu\nu}$, we end up with an alternative system of conformal wave equations (of course

[12]Recall that $\Box_g$, acting on higher valence tensors, is not a wave-operator if the metric field belongs to the unknowns.

we need to replace $\Box_g$ by $\Box_g^{(H)}$, cf. Section 3.1),

$$\begin{aligned}
\Box_g^{(H)} L_{\mu\nu} &= 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2C_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \frac{1}{6}\nabla_\mu\nabla_\nu R\,, &(6.9)\\
\Box_g s &= \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \frac{1}{6}sR\,, &(6.10)\\
\Box_g\Theta &= 4s - \frac{1}{6}\Theta R\,, &(6.11)\\
\Box_g^{(H)} C_{\mu\nu\sigma\rho} &= C_{\mu\nu\alpha}{}^\kappa C_{\sigma\rho\kappa}{}^\alpha - 4C_{\sigma\kappa[\mu}{}^\alpha C_{\nu]\alpha\rho}{}^\kappa - 2C_{\sigma\rho\kappa[\mu}L_{\nu]}{}^\kappa - 2C_{\mu\nu\kappa[\sigma}L_{\rho]}{}^\kappa\\
&\quad -\nabla_{[\sigma}\xi_{\rho]\mu\nu} - \nabla_{[\mu}\xi_{\nu]\sigma\rho} + \frac{1}{3}RC_{\mu\nu\sigma\rho}\,, &(6.12)\\
\Box_g^{(H)}\xi_{\mu\nu\sigma} &= 4\xi_{\kappa\alpha[\nu}C_{\sigma]}{}^\alpha{}_\mu{}^\kappa + C_{\nu\sigma\alpha}{}^\kappa\xi_{\mu\kappa}{}^\alpha - 4\xi_{\mu\kappa[\nu}L_{\sigma]}{}^\kappa + 6g_{\mu[\nu}\xi^\kappa{}_{\sigma\alpha]}L_\kappa{}^\alpha\\
&\quad +8L_{\alpha\kappa}\nabla_{[\nu}C_{\sigma]}{}^\alpha{}_\mu{}^\kappa + \frac{1}{6}R\xi_{\mu\nu\sigma} - \frac{1}{3}C_{\nu\sigma\mu}{}^\kappa\nabla_\kappa R\,, &(6.13)\\
R^{(H)}_{\mu\nu}[g] &= 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}\,. &(6.14)
\end{aligned}$$

Remark 6.1 *Note that (6.9) and (6.12)-(6.14) do not involve the functions s and Θ, so they form a closed system of wave equations for $g_{\mu\nu}$, $L_{\mu\nu}$, $\xi_{\mu\nu\sigma}$ and $C_{\mu\nu\sigma\rho}$. Once a solution has been constructed, it remains to solve the linear wave equations (6.10) and (6.11) for s and Θ.*

We want to investigate under which conditions a solution of the system (6.9)-(6.14), which we denote henceforth by CWE2, provides a solution of the MCFE.

6.2 Some properties of the CWE2 and gauge consistency

First of all we want to establish consistency with the gauge conditions $H^\sigma = 0$ and $R = R_g$. To do that we assume that there are smooth fields $g_{\mu\nu}$, s, Θ, $C_{\mu\nu\sigma}{}^\rho$, $L_{\mu\nu}$ and $\xi_{\mu\nu\sigma}$ which solve the CWE2. We aim to derive necessary and sufficient conditions on the initial surface which guarantee the vanishing of H^σ and $R - R_g$. For definiteness we, again, think of the case where the initial surface consists of either two transversally intersecting null hypersurfaces or a light-cone. The strategy will be the same as for the CWE, which is to derive a homogeneous system of wave equations for H^σ as well as some subsidiary fields, and infer the desired result from standard uniqueness results for wave equations by making the assumption, which will be analysed afterwards, that all the fields involved vanish initially.

However, let us first derive some properties of solutions of the CWE2.

Lemma 6.2 *Assume that the initial data on a characteristic initial surface S of some smooth solution of the CWE2 are such that $g_{\mu\nu}|_S$ is the restriction to S of a Lorentzian metric, that $L_{[\mu\nu]}|_S = 0$ and $C_{\mu\nu\sigma\rho}|_S = C_{\sigma\rho\mu\nu}$. Then $g_{\mu\nu}$ and $L_{\mu\nu}$ are symmetric and $C_{\mu\nu\sigma\rho} = C_{\sigma\rho\mu\nu}$.*

Proof: Equation (6.9) yields (cf. footnote 9)

$$\begin{aligned}
\Box_g^{(H)}(C_{\mu\nu\sigma\rho} - C_{\sigma\rho\mu\nu}) &= \frac{1}{3}R(C_{\mu\nu\sigma\rho} - C_{\sigma\rho\mu\nu})\\
&\quad +4g^{\alpha\beta}g^{\kappa\gamma}[(C_{[\mu|\beta\sigma\kappa|} - C_{\sigma\kappa[\mu|\beta|})C_{\nu]\alpha\rho\gamma} + (C_{\rho\alpha[\nu|\gamma|} - C_{[\nu|\gamma\rho\alpha|})C_{\mu]\kappa\sigma\beta}]\\
&\quad -4(g^{\alpha\beta}g^{[\kappa\gamma]} + g^{\gamma\kappa}g^{[\alpha\beta]})C_{\nu\alpha\rho\gamma}C_{\mu\beta\sigma\kappa}\,. &(6.15)
\end{aligned}$$

From (6.9) and (6.14) we further find

$$\Box_g^{(H)} L_{[\mu\nu]} \;=\; 4g_{[\alpha\beta]}L_\mu{}^\alpha L_\nu{}^\beta - g_{[\mu\nu]}|L|^2 + g^{\rho\gamma}L_\rho{}^\sigma(C_{\nu\sigma\mu\gamma} - C_{\mu\gamma\nu\sigma}) + 2g^{\sigma\kappa}C_\mu{}^\rho{}_{\nu\sigma}L_{[\rho\kappa]} - 2g^{[\sigma\kappa]}C_{\mu\sigma\nu}{}^\rho L_{\rho\kappa} \;, \tag{6.16}$$

$$R^{(H)}_{[\mu\nu]}[g_{(\sigma\rho)}, g_{[\sigma\rho]}] \;=\; 2L_{[\mu\nu]} + \frac{1}{6}Rg_{[\mu\nu]} \;. \tag{6.17}$$

The equations (6.15)-(6.17) are to be read as a linear, homogeneous system of wave equations satisfied by $g_{[\mu\nu]}$, $L_{[\mu\nu]}$ and $C_{\mu\nu\sigma\rho} - C_{\sigma\rho\mu\nu}$ with all the other fields being given. Hence if we assume these fields to vanish initially they will vanish everywhere. □

The lemma shows that the tensor $g_{\mu\nu}$ determines indeed a metric as long as it does not degenerate. We will only care about initial data for which the assumptions of this lemma hold.

In analogy to Lemma 3.4 one could show that $C_{\mu\nu\sigma\rho}$ is anti-symmetric in its first two and last two indices and satisfies $C_{[\mu\nu\sigma]\rho} = 0$, and that $\xi_{\mu\nu\sigma}$ is anti-symmetric in its last two indices and fulfills $\xi_{[\mu\nu\sigma]} = 0$, supposing that this is initially the case. However, these properties will follow a posteriori anyway, so it is not necessary to prove them here. Due to the appearance of first-order derivatives on the right-hand side of the wave equations for $C_{\mu\nu\sigma\rho}$ and $\xi_{\mu\nu\sigma}$, it is not possible to establish tracelessness of $C_{\mu\nu\sigma\rho}$ and $\xi_{\mu\nu\sigma}$ at this stage in a manner it was possible for the CWE (where it simplified the subsequent computations), since this would require to have something like the second Bianchi identity; also these properties can be, again, inferred a posteriori, once we know that $C_{\mu\nu\sigma\rho}$ and $\xi_{\mu\nu\sigma}$ are Weyl and Cotton tensor of $g_{\mu\nu}$, respectively.

Gauge consistency

Similarly to what we did in Section 3.2, one proceeds to verify the formulae

$$R_{\mu\nu} - \frac{1}{2}R_g g_{\mu\nu} = 2L_{\mu\nu} - (L + \frac{1}{6}R)g_{\mu\nu} + g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma - \frac{1}{2}g_{\mu\nu}\hat{\nabla}_\sigma H^\sigma \;, \tag{6.18}$$

$$\nabla^\nu\hat{\nabla}_\nu H^\alpha + 2g^{\mu\alpha}\nabla_{[\sigma}\hat{\nabla}_{\mu]}H^\sigma + 4\nabla^\nu L_\nu{}^\alpha - 2\nabla^\alpha L - \frac{1}{3}\nabla^\alpha R = 0 \;, \tag{6.19}$$

$$\Box_g H^\alpha = \zeta^\alpha + f^\alpha(x; H, \nabla H) \;, \quad \zeta_\mu := -4\nabla_\kappa L_\mu{}^\kappa + 2\nabla_\mu L + \frac{1}{3}\nabla_\mu R \;, \tag{6.20}$$

$$\Box_g K_{\mu\nu} = \nabla_\mu\zeta_\nu + f_{\mu\nu}(x; H, \nabla H, \nabla K) \;, \quad K_{\mu\nu} := \nabla_\mu H_\nu \;, \tag{6.21}$$

$$R_g = 2L + \frac{2}{3}R + \hat{\nabla}_\sigma H^\sigma \;. \tag{6.22}$$

From (6.9) we derive a wave equation for $L - R/6$,

$$\Box_g(L - \frac{1}{6}R) \;=\; -2C_{\mu\sigma}{}^{\mu\rho}L_\rho{}^\sigma \;=\; 2(W_{\mu\sigma}{}^{\mu\rho} - C_{\mu\sigma}{}^{\mu\rho})L_\rho{}^\sigma \;. \tag{6.23}$$

The tensors $L_{\mu\nu}$, $C_{\mu\nu\sigma}{}^\rho$ and $\xi_{\mu\nu\sigma}$ are supposed to be part of the given solution of the CWE2; we stress that it is by no means clear, whether they, indeed, represent the Schouten, Weyl and Cotton tensor of $g_{\mu\nu}$, respectively. We denote by $W_{\mu\nu\sigma}{}^\rho$ the Weyl tensor associated to $g_{\mu\nu}$, while we define the tensor $\zeta_{\mu\nu\sigma}$ to be

$$\zeta_{\mu\nu\sigma} := 4\nabla_{[\sigma}L_{\nu]\mu} \;.$$

Since we do not know at this stage whether the source term in (6.23) vanishes, we have no analogue of Lemma 3.6. It is not possible to conclude that $L - \frac{1}{6}R$ vanishes as we did for the CWE, supposing that it vanishes initially. In fact that is the reason for the modified definition of ζ_μ in (6.20).

Note that once we have established $L = \frac{1}{6}R$ and $H^\sigma = 0$, (6.22) implies $R = R_g$. For (6.23) to be part of a homogeneous system of wave equations, we regard $W_{\mu\nu\sigma\rho} - C_{\mu\nu\sigma\rho}$ as another unknown and show that it satisfies an appropriate homogeneous wave equation (for later purposes this is more advantageous than to derive a wave equation for the traces $C_{\mu\sigma}{}^{\mu\rho}$).

From (6.18) and (6.22) we find for the Weyl tensor, cf. (3.32) and (3.36) (since we do not know yet whether $L - R/6$ vanishes, the formulae differ slightly),

$$\begin{aligned}
\nabla_\alpha W_{\mu\nu\sigma\rho} \;=\;& g_{\mu[\sigma}\zeta_{\rho]\alpha\nu} + g_{\nu[\sigma}\zeta_{\rho]\mu\alpha} - g_{\alpha[\sigma}\zeta_{\rho]\mu\nu} - 2\nabla_{[\mu}W_{\nu]\alpha\sigma\rho} \\
&+ g_{\mu[\sigma}\nabla_{\rho]}\nabla_{[\nu}H_{\alpha]} + g_{\nu[\sigma}\nabla_{\rho]}\nabla_{[\alpha}H_{\mu]} + g_{\alpha[\sigma}\nabla_{\rho]}\nabla_{[\mu}H_{\nu]} \\
&+ \frac{4}{3}g_{\mu[\sigma}g_{\rho][\nu}\nabla_{\alpha]}(L - \frac{1}{6}R) - \frac{2}{3}g_{\alpha[\sigma}g_{\rho]\nu}\nabla_\mu(L - \frac{1}{6}R) \\
&+ \frac{2}{3}g_{\mu[\sigma}g_{\rho][\nu}\nabla_{\alpha]}\nabla_\kappa H^\kappa - \frac{1}{3}g_{\alpha[\sigma}g_{\rho]\nu}\nabla_\mu\nabla_\kappa H^\kappa + f_{\alpha\mu\nu\sigma\rho}(x; H, \nabla H)\;, \\
\Box_g W_{\mu\nu\sigma\rho} \;=\;& \nabla_{[\sigma}\zeta_{\rho]\nu\mu} + 2\nabla_{[\nu}\nabla^\alpha W_{\mu]\alpha\sigma\rho} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa \\
&+ 2(g_{\rho[\mu}W_{\nu]\alpha\sigma}{}^\kappa - g_{\sigma[\mu}W_{\nu]\alpha\rho}{}^\kappa)L_\kappa{}^\alpha - 2L_{[\mu}{}^\kappa W_{\nu]\kappa\sigma\rho} - 2L_{[\sigma}{}^\kappa W_{\rho]\kappa\mu\nu} \\
&+ g_{\sigma[\mu}\nabla^\alpha\zeta_{|\rho\alpha|\nu]} - g_{\rho[\mu}\nabla^\alpha\zeta_{|\sigma\alpha|\nu]} + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\Box_g H_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\Box_g H_\nu \\
&+ \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\nabla_\rho\nabla_\alpha H^\alpha - \frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\nabla_\sigma\nabla_\alpha H^\alpha - \frac{1}{3}g_{\mu[\sigma}\nabla_{\rho]}\nabla_\nu\nabla_\kappa H^\kappa \\
&+ \frac{1}{3}g_{\nu[\sigma}\nabla_{\rho]}\nabla_\mu\nabla_\kappa H^\kappa + \frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\Box_g\nabla_\kappa H^\kappa + \frac{2}{3}g_{\mu[\sigma}g_{\rho]\nu}\Box_g(L - \frac{1}{6}R) \\
&+ \frac{4}{3}g_{\alpha[\sigma}g_{\rho][\mu}\nabla_{\nu]}\nabla^\alpha(L - \frac{1}{6}R) + \frac{1}{3}RW_{\mu\nu\sigma\rho} + f_{\mu\nu\sigma\rho}(x; H, \nabla H, \nabla K)\;.
\end{aligned}$$

We further have (cf. (3.34) and (3.35))

$$\begin{aligned}
\nabla^\alpha\zeta_{\mu\nu\alpha} \;=\;& 2(W_{\mu\alpha\nu}{}^\kappa - 2C_{\mu\alpha\nu}{}^\kappa)L_\kappa{}^\alpha + \frac{1}{2}\nabla_\nu\Box_g H_\mu - \nabla_\mu\nabla_\nu(L - \frac{1}{6}R) \\
&- (\frac{5}{3}R_g - 6L - \frac{2}{3}R)L_{\mu\nu} + (\frac{2}{3}R_g - 2L - \frac{1}{3}R)Lg_{\mu\nu} \\
&+ f_{\mu\nu}(x; H, \nabla H, \nabla K)\;, \qquad (6.24) \\
\nabla_\alpha W^\alpha{}_{\mu\nu\sigma} \;=\;& -\frac{1}{2}\zeta_{\mu\nu\sigma} + \frac{1}{2}\nabla_\mu\nabla_{[\nu}H_{\sigma]} + \frac{1}{6}g_{\mu[\nu}\nabla_{\sigma]}(R_g - R) \\
&+ f_{\mu\nu\sigma}(x; H, \nabla H)\;, \qquad (6.25)
\end{aligned}$$

which yields with (6.20) and (6.22)

$$\begin{aligned}
\Box_g W_{\mu\nu\sigma\rho} \;=\;& \nabla_{[\sigma}\zeta_{\rho]\nu\mu} - \nabla_{[\mu}\zeta_{\nu]\sigma\rho} + W_{\mu\nu\alpha}{}^\kappa W_{\sigma\rho\kappa}{}^\alpha - 4W_{\sigma\kappa[\mu}{}^\alpha W_{\nu]\alpha\rho}{}^\kappa \\
&- 2L_{[\mu}{}^\kappa W_{\nu]\kappa\sigma\rho} - 2L_{[\sigma}{}^\kappa W_{\rho]\kappa\mu\nu} + 4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - C_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} \\
&- 4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - C_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho} + \frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\zeta_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\zeta_\rho + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\zeta_\mu \\
&- \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\zeta_\nu - \frac{8}{3}g_{\sigma[\mu}L_{\nu]\rho}(L - \frac{1}{6}R) + \frac{8}{3}g_{\rho[\mu}L_{\nu]\sigma}(L - \frac{1}{6}R) + \frac{1}{3}RW_{\mu\nu\sigma\rho} \\
&+ \frac{4}{3}Lg_{\sigma[\mu}g_{\nu]\rho}(L - \frac{1}{6}R) + \frac{1}{3}g_{\mu[\sigma}g_{\rho]\nu}\Box_g(R_g - R) + f_{\mu\nu\sigma\rho}(x; H, \nabla H, \nabla K)\;.
\end{aligned}$$

The first term in the last line is disturbing. However, invoking (6.22) and (6.23) we find the relation

$$\begin{aligned}\Box_g(R_g - R) &= 2\Box_g(L - \frac{1}{6}R) + \Box_g\hat{\nabla}_\sigma H^\sigma \\ &= 4(W_{\mu\sigma}{}^{\mu\rho} - C_{\mu\sigma}{}^{\mu\rho})L_\rho{}^\sigma + f(x; H, \nabla H, \nabla K)\ .\end{aligned}$$

Combining with (6.12) we end up with the wave equation

$$\begin{aligned}\Box_g(W_{\mu\nu\sigma\rho} - C_{\mu\nu\sigma\rho}) \;=\; & -\nabla_{[\sigma}(\zeta_{\rho]\mu\nu} - \xi_{\rho]\mu\nu}) - \nabla_{[\mu}(\zeta_{\nu]\sigma\rho} - \xi_{\nu]\sigma\rho}) \\ & +(W_{\mu\nu\alpha}{}^\kappa - C_{\mu\nu\alpha}{}^\kappa)W_{\sigma\rho\kappa}{}^\alpha + C_{\mu\nu\alpha}{}^\kappa(W_{\sigma\rho\kappa}{}^\alpha - C_{\sigma\rho\kappa}{}^\alpha) \\ & -4(W_{\sigma\kappa[\mu}{}^\alpha - C_{\sigma\kappa[\mu}{}^\alpha)W_{\nu]\alpha\rho}{}^\kappa - 4C_{\sigma\kappa[\mu}{}^\alpha(W_{\nu]\alpha\rho}{}^\kappa - C_{\nu]\alpha\rho}{}^\kappa) \\ & -2(W_{\sigma\rho\kappa[\mu} - C_{\sigma\rho\kappa[\mu})L_{\nu]}{}^\kappa - 2(W_{\mu\nu\kappa[\sigma} - C_{\mu\nu\kappa[\sigma})L_{\rho]}{}^\kappa \\ & +4L_\kappa{}^\alpha(W_{\rho\alpha[\mu}{}^\kappa - C_{\rho\alpha[\mu}{}^\kappa)g_{\nu]\sigma} - 4L_\kappa{}^\alpha(W_{\sigma\alpha[\mu}{}^\kappa - C_{\sigma\alpha[\mu}{}^\kappa)g_{\nu]\rho} \\ & +\frac{1}{2}g_{\rho[\mu}\nabla_{\nu]}\zeta_\sigma - \frac{1}{2}g_{\sigma[\mu}\nabla_{\nu]}\zeta_\rho + \frac{1}{2}g_{\nu[\sigma}\nabla_{\rho]}\zeta_\mu - \frac{1}{2}g_{\mu[\sigma}\nabla_{\rho]}\zeta_\nu \\ & +\frac{4}{3}(L - \frac{1}{6}R)(Lg_{\sigma[\mu}g_{\nu]\rho} - 2g_{\sigma[\mu}L_{\nu]\rho} + 2g_{\rho[\mu}L_{\nu]\sigma}) + \frac{R}{3}(W_{\mu\nu\sigma\rho} - C_{\mu\nu\sigma\rho}) \\ & +\frac{4}{3}g_{\mu[\sigma}g_{\rho]\nu}(W_{\kappa\alpha}{}^{\kappa\beta} - C_{\kappa\alpha}{}^{\kappa\beta})L_\beta{}^\alpha + f_{\mu\nu\sigma\rho}(x; H, \nabla H, \nabla K)\ , \qquad (6.26)\end{aligned}$$

which is fulfilled by any solution of the CWE2.

To end up with a homogeneous system we need to derive wave equations for ζ_μ and $\zeta_{\mu\nu\sigma} - \xi_{\mu\nu\sigma}$. Let us start with ζ_μ. In close analogy to (3.27) and (3.28) we find with (6.9), (6.23), (6.25), (6.18) and (6.22),

$$\begin{aligned}\Box_g\zeta_\mu \;\equiv\; & -4\nabla_\kappa\Box_g L_\mu{}^\kappa - 8W_{\alpha\kappa\mu}{}^\rho\nabla^\alpha L_\rho{}^\kappa - 4R_{\kappa\rho}\nabla_\mu L^{\rho\kappa} + 8R_{\alpha\rho}\nabla^\alpha L_\mu{}^\rho \\ & -4L^{\rho\kappa}\nabla_\mu R_{\rho\kappa} + 4L^{\rho\kappa}\nabla_\rho R_{\mu\kappa} + \frac{2}{3}R_\mu{}^\kappa\nabla_\kappa R - R_\mu{}^\nu\zeta_\nu + \frac{1}{3}R_g\zeta_\mu \\ & +2L_\mu{}^\rho\nabla_\rho R_g + \frac{2}{3}R_g\nabla_\mu(L - \frac{1}{6}R) + 2\nabla_\mu\Box_g(L + \frac{1}{6}R) \\ =\; & -8\nabla^\nu[(W_{\mu\sigma\nu}{}^\rho - C_{\mu\sigma\nu}{}^\rho)L_\rho{}^\sigma] + 4\nabla_\mu[(W_{\alpha\sigma}{}^{\alpha\rho} - C_{\alpha\sigma}{}^{\alpha\rho})L_\rho{}^\sigma] \\ & -\frac{8}{3}L_\mu{}^\nu\nabla_\nu(L - \frac{1}{6}R) + \frac{4}{9}R\nabla_\mu(L - \frac{1}{6}R) - \frac{2}{3}(L - \frac{1}{6}R)\nabla_\mu R \\ & +(4L_\mu{}^\nu - R_\mu{}^\nu)\zeta_\nu + \frac{1}{3}(R_g - R)\zeta_\mu + f_\mu(x; H, \nabla H, \nabla K)\ . \qquad (6.27)\end{aligned}$$

Finally, let us establish a wave equation which is satisfied by $\zeta_{\mu\nu\sigma} - \xi_{\mu\nu\sigma}$. The definition of the Weyl tensor together with the Bianchi identities yield

$$\begin{aligned}\Box_g\zeta_{\mu\nu\sigma} \;\equiv\; & 4\nabla_{[\sigma}\Box_g L_{\nu]\mu} - 4W_{\nu\sigma\kappa\rho}\nabla^\rho L_\mu{}^\kappa + 8W_{\mu\kappa\rho[\sigma}\nabla^\rho L_{\nu]}{}^\kappa - 4R_{\kappa[\nu}\nabla_{\sigma]}L_\mu{}^\kappa \\ & +4R_{\kappa[\sigma}\nabla_{|\mu|}L_{\nu]}{}^\kappa - 4R_{\mu[\sigma}\nabla_{|\kappa|}L_{\nu]}{}^\kappa - 4R_{\rho\kappa}g_{\mu[\sigma}\nabla^\rho L_{\nu]}{}^\kappa + \frac{1}{3}R_g\zeta_{\mu\nu\sigma} \\ & +\frac{4}{3}R_g g_{\mu[\sigma}\nabla^\kappa L_{\nu]\kappa} + 4L_\mu{}^\kappa\nabla_{[\nu}R_{\sigma]\kappa} + 4L_\nu{}^\kappa\nabla_{[\mu}R_{\kappa]\sigma} + 4L_\sigma{}^\kappa\nabla_{[\kappa}R_{\mu]\nu} \\ \overset{(6.9)}{=}\; & 4L_\mu{}^\kappa\zeta_{\kappa\nu\sigma} - 4L_\mu{}^\kappa\nabla_{[\sigma}R_{\nu]\kappa} + 16L_{\kappa[\nu}\nabla_{\sigma]}L_\mu{}^\kappa - 4R_{\kappa[\nu}\nabla_{\sigma]}L_\mu{}^\kappa \\ & -8L_\rho{}^\kappa g_{\mu[\nu}\nabla_{\sigma]}L_\kappa{}^\rho - 4R_{\rho\kappa}g_{\mu[\sigma}\nabla^\rho L_{\nu]}{}^\kappa + 4R_{\kappa[\sigma}\nabla_{|\mu|}L_{\nu]}{}^\kappa \\ & -4R_{\mu[\sigma}\nabla_{|\kappa|}L_{\nu]}{}^\kappa + 4L_\nu{}^\kappa\nabla_{[\mu}R_{\kappa]\sigma} + 4L_\sigma{}^\kappa\nabla_{[\kappa}R_{\mu]\nu} + \frac{4}{3}R_g g_{\mu[\sigma}\nabla^\kappa L_{\nu]\kappa} \\ & +8L_\rho{}^\kappa\nabla_{[\nu}C_{\sigma]\kappa\mu}{}^\rho + 4\zeta_{\alpha\kappa[\nu}C_{\sigma]}{}^\kappa{}_\mu{}^\alpha - 2\zeta_{\mu\alpha\kappa}W_\nu{}^\alpha{}_\sigma{}^\kappa + \frac{1}{3}R_{\sigma\nu\mu}{}^\kappa\nabla_\kappa R \\ & +8(W_\mu{}^\rho{}_{[\nu}{}^\kappa - C_\mu{}^\rho{}_{[\nu}{}^\kappa)\nabla_{|\kappa|}L_{\sigma]\rho} + \frac{1}{3}R_g\zeta_{\mu\nu\sigma} + f_{\mu\nu\sigma}(x; H, \nabla H, \nabla K)\ .\end{aligned}$$

Using the relations (6.18), (6.22) and $\zeta_{[\mu\nu\sigma]}=0$ one then shows

$$\begin{aligned}\Box_g\zeta_{\mu\nu\sigma} &= -4\zeta_{\mu\kappa[\nu}L_{\sigma]}{}^\kappa + 6g_{\mu[\nu}\zeta^\kappa{}_{\sigma\alpha]}L_\kappa{}^\alpha + 8L_\rho{}^\kappa\nabla_{[\nu}C_{\sigma]\kappa\mu}{}^\rho + 4\zeta_{\alpha\kappa[\nu}C_{\sigma]}{}^\kappa{}_\mu{}^\alpha \\ &\quad -2\zeta_{\mu\alpha\kappa}W_\nu{}^\alpha{}_\sigma{}^\kappa + 8(W_\mu{}^\rho{}_{[\nu}{}^\kappa - C_\mu{}^\rho{}_{[\nu}{}^\kappa)\nabla_{|\kappa|}L_{\sigma]\rho} - 2L_{\mu[\nu}\zeta_{\sigma]} \\ &\quad +\frac{1}{3}W_{\sigma\nu\mu}{}^\kappa\nabla_\kappa R - \frac{1}{6}(R-2R_g)\zeta_{\mu\nu\sigma} + f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K)\,.\end{aligned}$$

Combining with (6.13) we infer that $\zeta_{\mu\nu\sigma}-\xi_{\mu\nu\sigma}$ fulfills the wave equation,

$$\begin{aligned}&\Box_g(\zeta_{\mu\nu\sigma}-\xi_{\mu\nu\sigma}) \\ &\quad= 6g_{\mu[\nu}(\zeta^\kappa{}_{\sigma\alpha]} - \xi^\kappa{}_{\sigma\alpha]})L_\kappa{}^\alpha - 4(\zeta_{\mu\kappa[\nu} - \xi_{\mu\kappa[\nu})L_{\sigma]}{}^\kappa + 4(\zeta_{\alpha\kappa[\nu} - \xi_{\alpha\kappa[\nu})C_{\sigma]}{}^\kappa{}_\mu{}^\alpha \\ &\qquad +\xi_{\mu\kappa}{}^\alpha(W_{\nu\sigma\alpha}{}^\kappa - C_{\nu\sigma\alpha}{}^\kappa) + 8(W_\mu{}^\rho{}_{[\nu}{}^\kappa - C_\mu{}^\rho{}_{[\nu}{}^\kappa)\nabla_{|\kappa|}L_{\sigma]\rho} - 2L_{\mu[\nu}\zeta_{\sigma]} \\ &\qquad -\frac{1}{3}(W_{\nu\sigma\mu}{}^\kappa - C_{\nu\sigma\mu}{}^\kappa)\nabla_\kappa R + (\zeta_{\mu\kappa}{}^\alpha - \xi_{\mu\kappa}{}^\alpha)W_{\nu\sigma\alpha}{}^\kappa + \frac{1}{6}R(\zeta_{\mu\nu\sigma}-\xi_{\mu\nu\sigma}) \\ &\qquad +4L_{\mu[\nu}\nabla_{\sigma]}(L-\frac{1}{6}R) + \frac{8}{3}(L-\frac{1}{6}R)g_{\mu[\sigma}\nabla^\kappa L_{\nu]\kappa} + \frac{2}{9}(L-\frac{1}{6}R)g_{\mu[\nu}\nabla_{\sigma]}R \\ &\qquad +\frac{2}{3}(L-\frac{1}{6}R)\zeta_{\mu\nu\sigma} + f_{\mu\nu\sigma}(x;H,\nabla H,\nabla K)\,. \end{aligned} \tag{6.28}$$

The equations (6.20), (6.21), (6.23), (6.26), (6.27) and (6.28) form a closed, linear, homogeneous system of wave equations satisfied by H^σ, $K_{\mu\nu}$, $L-R/6$, $W_{\mu\nu\sigma\rho}-C_{\mu\nu\sigma\rho}$, ζ_μ and $\zeta_{\mu\nu\sigma}-\xi_{\mu\nu\sigma}$, with $g_{\mu\nu}$, $L_{\mu\nu}$, etc. regarded as being given. An application of standard uniqueness results for wave equations, cf. e.g. [15], establishes that all the fields vanish identically, supposing that this is initially the case. In particular this guarantees the vanishing of H^σ and, via (6.22), of R_g-R, and therefore consistency of the CWE2 with the gauge condition.

Moreover, the computations above reveal that the solution satisfies certain relations expected from the derivation of the CWE2; e.g. it follows from (6.14) that $L_{\mu\nu}$ is the Schouten tensor of $g_{\mu\nu}$ if $H^\sigma=0$ and $R_g=R$.

PROPOSITION 6.3 *Let us assume we have been given data ($\mathring{g}_{\mu\nu}$, $\mathring{s}$, $\mathring{\Theta}$, $\mathring{L}_{\mu\nu}$, $\mathring{C}_{\mu\nu\sigma}{}^\rho$, $\mathring{\xi}_{\mu\nu\sigma}$) on an initial surface S (for definiteness we think either of two transversally intersecting null hypersurfaces or a light-cone) and a gauge source function R, such that $\mathring{g}_{\mu\nu}$ is the restriction to S of a Lorentzian metric, $\mathring{L}_{\mu\nu}$ is symmetric and $\mathring{L}=\overline{R}/6$. Suppose further that there exists a smooth solution ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $C_{\mu\nu\sigma}{}^\rho$, $\xi_{\mu\nu\sigma}$) of the CWE2 (6.9)-(6.14) with gauge source function R which induces the above data on S and fulfills the following conditions:*

1. $\overline{H}^\sigma[g]=0$,
2. $\overline{K}_\mu{}^\sigma[g]=0$, *where* $K_\mu{}^\sigma\equiv\nabla_\mu H^\sigma$,
3. $\overline{W}_{\mu\nu\sigma}{}^\rho[g]=\overline{C}_{\mu\nu\sigma}{}^\rho$,
4. $\overline{\zeta}_{\mu\nu\sigma}[g,L]=\overline{\xi}_{\mu\nu\sigma}$, *where* $\zeta_{\mu\nu\sigma}\equiv 4\nabla_{[\sigma}L_{\nu]\mu}$,
5. $\overline{\zeta}_\mu=0$, *where* $\zeta_\mu\equiv -4\nabla_\kappa L_\mu{}^\kappa+2\nabla_\mu L+\frac{1}{3}\nabla_\mu R$.

Then

a) $H^\sigma=0$ *and* $R_g=R$,

b) $C_{\mu\nu\sigma}{}^\rho$ *is the Weyl tensor of* $g_{\mu\nu}$, *c) $L_{\mu\nu}$ is the Schouten tensor of $g_{\mu\nu}$,*

d) $\xi_{\mu\nu\sigma}$ is the Cotton tensor of $g_{\mu\nu}$.

The validity of the assumptions 1-5 will be the subject of Section 6.5.

6.3 Equivalence issue between the CWE2 and the MCFE

We devote ourselves now to the issue to what extent and under which conditions a solution of the CWE2 is also a solution of the MCFE. It turns out that this issue is somewhat more intricate than for the CWE due to the change of variables. Note that at this stage the cosmological constant λ does not need to vanish.

A subsidiary system

Recall the MCFE,

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0 \,, \tag{6.29}$$
$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta \, d_{\nu\mu\sigma}{}^\rho \,, \tag{6.30}$$
$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} \,, \tag{6.31}$$
$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta \,, \tag{6.32}$$
$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3 \,, \tag{6.33}$$
$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2\left(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}\right) \,. \tag{6.34}$$

The MCFE are equivalent to the following system, supposing that $\Theta > 0$,

$$\nabla_\rho C_{\nu\mu\sigma}{}^\rho = \nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} \,, \tag{6.35}$$
$$\Theta(\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma}) = \nabla_\rho \Theta \, C_{\nu\mu\sigma}{}^\rho \,, \tag{6.36}$$
$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} \,, \tag{6.37}$$
$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta \,, \tag{6.38}$$
$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3 \,, \tag{6.39}$$
$$R_{\mu\nu\sigma}{}^\kappa[g] = C_{\mu\nu\sigma}{}^\kappa + 2\left(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}\right) \,. \tag{6.40}$$

This can be seen as follows: Suppose we have a solution of (6.35)-(6.40), then we obtain a solution of (6.29)-(6.34) by identifying $d_{\mu\nu\sigma}{}^\rho$ with $\Theta^{-1} C_{\mu\nu\sigma}{}^\rho$ and vice versa (hence the system (6.35)-(6.40) is also equivalent to the vacuum Einstein equations for $\Theta > 0$). In fact, a solution of (6.29)-(6.34) provides a solution of (6.35)-(6.40) for any Θ since the identification of $C_{\mu\nu\sigma}{}^\rho$ with $\Theta d_{\mu\nu\sigma}{}^\rho$ is possible, even where $\Theta = 0$.

We elaborate in somewhat more detail on the characteristic initial value problem for an initial surface S for which the set $\{\overline{\Theta} = 0\}$ is non-empty. Since we are mainly interested in a light-cone with $\overline{\Theta} = 0$ everywhere we specialise to the case $S = C_{i^-}$ (we then need to assume $\lambda = 0$). Let us assume we have been given free initial data $\omega_{AB} \equiv \breve{\overline{L}}_{AB}$ on C_{i^-}, and that the fields $\mathring{g}_{\mu\nu}$, $\mathring{\Theta} = 0$, $\mathring{s}$, $\mathring{L}_{\mu\nu}$, $\mathring{C}_{\mu\nu\sigma}{}^\rho = 0$ and $\mathring{\xi}_{\mu\nu\sigma}$ have been constructed by solving the constraint equations to be derived below (cf. Section 6.4). Let us further assume that there exists a smooth solution of the system (6.35)-(6.40) to the future of S which induces these data on S and which satisfies $s|_{i^-} \neq 0$. Then Θ has no zeroes inside the cone and sufficiently close to the vertex. Moreover, cf. the proof of Lemma A.5 in Appendix A, $d\Theta \neq 0$ on $\mathscr{I}^-$ and $d\Theta|_{i^-} = 0$. Since the tensor $C_{\mu\nu\sigma}{}^\rho$ vanishes on C_{i^-} the field $C_{\mu\nu\sigma}{}^\rho/\Theta$ can be smoothly continued across $\mathscr{I}^-$ (though not necessarily across i^-). The solution at hand thus solves (6.29)-(6.34) (except possibly at i^-) when identifying $C_{\mu\nu\sigma}{}^\rho/\Theta$ with $d_{\mu\nu\sigma}{}^\rho$, smoothly continued across $\mathscr{I}^-$.

The system (6.35)-(6.40) is not regular for $\Theta = 0$, and thus does not provide a good evolution system. However, it turns out that it is equivalent to the CWE2, when the latter system is supplemented by the constraint equations, and thus provides a useful tool to solve the equivalence issue between the MCFE and the CWE2. The only grievance (or possibly advantage, we will come back to this issue later) is that we do not know how $d_{\mu\nu\sigma}{}^\rho$ behaves near the vertex, in particular it is by no means clear whether it can be continuously continued across past timelike infinity at all. Nevertheless, the solution provides a solution of the MCFE up to and excluding the vertex, which induces the free initial data ω_{AB} on C_{i^-}, and it provides a solution of the vacuum Einstein equations inside the cone, at least near i^-.

Equivalence of the CWE2 and the subsidiary system

In this section we address the equivalence issue between the CWE2 (6.9)-(6.14) and the subsidiary system (6.35)-(6.40) we just introduced and which, once we have constructed a solution thereof, provides a solution of the MCFE (6.29)-(6.34), with the possible exception of the vertex of the cone C_{i^-}. For that we shall demonstrate that a solution of the CWE2 is a solution of the subsidiary system supposing that certain relations are satisfied on the initial surface, namely the constraint equations, cf. the next section. The other direction follows from the derivation of the CWE2. As initial surface we have, as before, two transversally intersecting null hypersurfaces or a light-cone in mind.

Recall the CWE2 (6.9)-(6.14). We assume we have been given a smooth solution $(g_{\mu\nu}, L_{\mu\nu}, C_{\mu\nu\sigma}{}^\rho, \xi_{\mu\nu\sigma}, \Theta, s)$ with all the hypotheses of Proposition 6.3 being satisfied. Then $L_{\mu\nu}$, $C_{\mu\nu\sigma}{}^\rho$ and $\xi_{\mu\nu\sigma}$ are the Schouten, Weyl and Cotton tensor of $g_{\mu\nu}$, respectively. The equations (6.35) and (6.40) are thus identities and automatically satisfied. Recall that it suffices for (6.39) to be satisfied at just one point. Let us derive a homogeneous system of wave equations which establishes the validity of the remaining equations, (6.36)-(6.38).

It is convenient to make the following definitions:

$$\begin{aligned}
\Lambda_{\sigma\nu\mu} &:= \frac{1}{2}\Theta\xi_{\sigma\nu\mu} + \nabla_\rho\Theta C_{\mu\nu\sigma}{}^\rho \;, \\
\Xi_{\mu\nu} &:= \nabla_\mu\nabla_\nu\Theta + \Theta L_{\mu\nu} - s g_{\mu\nu} \;, \\
\Upsilon_\mu &:= \nabla_\mu s + L_{\mu\nu}\nabla^\nu\Theta \;.
\end{aligned}$$

Computations similar to the ones which led us to (3.44) and (3.46) (now with H^σ and $K_{\mu\nu}$ vanishing) reveal that, because of (6.9)-(6.11), we have

$$\Box_g\Xi_{\mu\nu} = 2\Xi_{\sigma\kappa}(2L_{(\mu}{}^\kappa\delta_{\nu)}{}^\sigma - g_{\mu\nu}L^{\sigma\kappa} - C_\mu{}^\sigma{}_\nu{}^\kappa) + 4\nabla_{(\mu}\Upsilon_{\nu)} + \frac{1}{6}R\Xi_{\mu\nu} \;, \tag{6.41}$$

$$\Box_g\Upsilon_\mu = 6L_\mu{}^\kappa\Upsilon_\kappa + 2L^{\rho\kappa}\Lambda_{\rho\kappa\mu} + 2\Xi_\nu{}^\sigma\nabla_\sigma L_\mu{}^\nu - \frac{1}{6}\Xi_\mu{}^\nu\nabla_\nu R \;. \tag{6.42}$$

Furthermore, in virtue of (6.11)-(6.13) and (6.6) we find that

$$\begin{aligned}\Box_g\Lambda_{\sigma\nu\mu} &= s\xi_{\sigma\nu\mu} - 2L_{\rho\kappa}\nabla^\kappa\Theta\, C_{\mu\nu\sigma}{}^\rho - 2\nabla^\rho\Theta C_{\sigma\rho\kappa[\mu}L_{\nu]}{}^\kappa - 2\nabla^\rho\Theta C_{\mu\nu\kappa[\sigma}L_{\rho]}{}^\kappa \\ &\quad +\nabla^\rho\Theta(\nabla_{[\rho}\xi_{\sigma]\nu\mu} + \nabla_{[\mu}\xi_{\nu]\rho\sigma}) + \nabla^\rho\Theta\nabla_\sigma\xi_{\rho\nu\mu} + 4\Upsilon_\rho C_{\mu\nu\sigma}{}^\rho \\ &\quad +2\Xi_{\kappa\rho}\nabla^\kappa C_{\mu\nu\sigma}{}^\rho + 4C_\sigma{}^\kappa{}_{[\mu}{}^\alpha\Lambda_{|\kappa\alpha|\nu]} - C_{\mu\nu\alpha}{}^\kappa\Lambda_{\sigma\kappa}{}^\alpha + \frac{1}{3}R\Lambda_{\sigma\nu\mu}\ .\end{aligned}$$

We observe the relation

$$\begin{aligned}&2\nabla^\rho\Theta(\nabla_{[\rho}\xi_{\sigma]\nu\mu} + \nabla_{[\mu}\xi_{\nu]\rho\sigma}) \\ &\quad= 4\nabla^\rho\Theta(\nabla_{[\rho}\nabla_{\mu]}L_{\nu\sigma} - \nabla_{[\rho}\nabla_{\nu]}L_{\mu\sigma} + \nabla_{[\sigma}\nabla_{\nu]}L_{\mu\rho} - \nabla_{[\sigma}\nabla_{\mu]}L_{\nu\rho}) \\ &\quad= 4\nabla^\rho\Theta(C_{\mu\nu[\sigma}{}^\kappa L_{\rho]\kappa} - C_{\sigma\rho[\mu}{}^\kappa L_{\nu]\kappa})\ ,\end{aligned}$$

which yields

$$\begin{aligned}\Box_g\Lambda_{\sigma\nu\mu} &= 2\Xi_{\kappa\rho}\nabla^\kappa C_{\mu\nu\sigma}{}^\rho + \xi^\rho{}_{\mu\nu}\Xi_{\sigma\rho} + 2L_\sigma{}^\rho\Lambda_{\rho\nu\mu} + 4C_\sigma{}^\kappa{}_{[\mu}{}^\alpha\Lambda_{|\kappa\alpha|\nu]} \\ &\quad -C_{\mu\nu\alpha}{}^\kappa\Lambda_{\sigma\kappa}{}^\alpha + 4\Upsilon_\rho C_{\mu\nu\sigma}{}^\rho + \nabla_\sigma(\xi_{\rho\nu\mu}\nabla^\rho\Theta) + \frac{1}{3}R\Lambda_{\sigma\nu\mu}\ . \qquad (6.43)\end{aligned}$$

It remains to derive a wave equation for $\xi_{\rho\nu\mu}\nabla^\rho\Theta$ which follows from (6.11), (6.13) and (6.6),

$$\begin{aligned}\Box_g(\xi_{\rho\nu\mu}\nabla^\rho\Theta) &= \xi^\rho{}_{\nu\mu}\nabla_\rho\Box_g\Theta + 2\Xi^{\kappa\rho}(\nabla_\kappa\xi_{\rho\nu\mu} + 2L_\kappa{}^\delta C_{\mu\nu\delta\rho}) - 4L^{\kappa\rho}\nabla_\kappa\Lambda_{\rho\nu\mu} \\ &\quad +\nabla^\kappa\Theta(4L^{\delta\rho}\nabla_\delta C_{\mu\nu\rho\kappa} + 4L_\kappa{}^\rho\xi_{\rho\nu\mu} + \Box_g\xi_{\kappa\nu\mu}) + \frac{1}{6}R\xi_{\rho\nu\mu}\nabla^\rho\Theta \\ &= 4\xi^\rho{}_{\nu\mu}\Upsilon_\rho + 2\Xi^{\kappa\rho}(\nabla_\kappa\xi_{\rho\nu\mu} + 2L_\kappa{}^\delta C_{\mu\nu\delta\rho}) - 4L^{\kappa\rho}\nabla_\kappa\Lambda_{\rho\nu\mu} \\ &\quad -(\xi_{\kappa\beta}{}^\alpha\nabla^\kappa\Theta)C_{\mu\nu\alpha}{}^\beta + 4\xi^{\alpha\beta}{}_{[\mu}\Lambda_{|\alpha\beta|\nu]} - \frac{1}{3}\Lambda_{\rho\nu\mu}\nabla^\rho R \\ &\quad +\frac{1}{2}R\,\xi_{\rho\nu\mu}\nabla^\rho\Theta\ . \qquad (6.44)\end{aligned}$$

The equations (6.41)-(6.44) form a closed, linear, homogeneous system of wave equations for the fields $\Xi_{\mu\nu}$, Υ_μ, $\Lambda_{\sigma\nu\mu}$ and $\xi_{\rho\nu\mu}\nabla^\rho\Theta$. If we assume that the equations (6.36)-(6.38) are initially satisfied and that $\overline{\xi_{\rho\nu\mu}\nabla^\rho\Theta} = 0$, we have vanishing initial data, and standard uniqueness results for wave equations can be applied (cf. e.g. [15]) to conclude that (6.36)-(6.38) are fulfilled.

As an extension of Proposition 6.3 we have proven the following result (note that the cosmological constant λ is allowed to be non-vanishing):

Theorem 6.4 *Let us assume we have been given data ($\mathring{g}_{\mu\nu}$, $\mathring{s}$, $\mathring{\Theta}$, $\mathring{L}_{\mu\nu}$, $\mathring{C}_{\mu\nu\sigma}{}^\rho$, $\mathring{\xi}_{\mu\nu\sigma}$) on a characterteristic initial surface S (for definiteness we think either of two transversally intersecting null hypersurfaces or a light-cone) and a gauge source function R, such that $\mathring{g}_{\mu\nu}$ is the restriction of a Lorentzian metric, $\mathring{L}_{\mu\nu}$ is symmetric and $\mathring{L} = \overline{R}/6$. Suppose further that there exists a smooth solution ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $C_{\mu\nu\sigma}{}^\rho$, $\xi_{\mu\nu\sigma}$) of the CWE2 (6.9)-(6.14) with gauge source function R which induces the above data on S and satisfies the following conditions (since it is the case of physical relevance we assume $\Theta \neq 0$ away from S; later on we shall consider only initial data where this is automatically the case, at least sufficiently close to S):*

1. *The equations (6.36)-(6.39) are satisfied on S (it suffices if (6.39) holds at just one point on S).*
2. *The Weyl tensor of $g_{\mu\nu}$ coincides on S with $C_{\mu\nu\sigma}{}^{\rho}$.*
3. *The relation $\xi_{\mu\nu\sigma} = 4\nabla_{[\sigma}L_{\nu]\mu}$ holds on S.*
4. *The covector field $\zeta_{\mu} \equiv -4\nabla_{\kappa}L_{\mu}{}^{\kappa} + 2\nabla_{\mu}L + \frac{1}{3}\nabla_{\mu}R$ vanishes on S.*
5. *The tensor field $\xi_{\rho\nu\mu}\nabla^{\rho}\Theta$ vanishes on S.*
6. *The wave-gauge vector H^{σ} and its covariant derivative $K_{\mu}{}^{\sigma} \equiv \nabla_{\mu}H^{\sigma}$ vanish on S.*

Then:

a) *$H^{\sigma} = 0$ and $R_g = R$.*

b) *The fields $C_{\mu\nu\sigma}{}^{\rho}$, $L_{\mu\nu}$ and $\xi_{\mu\nu\sigma}$ are the Weyl, Schouten and Cotton tensor of $g_{\mu\nu}$, respectively.*

c) *Set $d_{\mu\nu\sigma}{}^{\rho} := \Theta^{-1}C_{\mu\nu\sigma}{}^{\rho}$ where $\Theta \neq 0$. The tensor $d_{\mu\nu\sigma}{}^{\rho}$ extends to the set $\{\overline{\Theta} = 0, \overline{\mathrm{d}\Theta} \neq 0\} \subset S$. Moreover, the tuple $(g_{\mu\nu}\ ,\ L_{\mu\nu}\ ,\ \Theta\ ,\ s\ ,\ d_{\mu\nu\sigma}{}^{\rho})$ solves the MCFE (6.29)-(6.34) in the $(H^{\sigma} = 0, R_g = R)$-gauge.*

The conditions 1-6 are necessary for c) to be fulfilled.

We shall investigate next to what extent the conditions 1-6 are satisfied if the initial data are constructed as solutions of the constraint equations induced by the MCFE on the initial surface.

6.4 Constraint equations on C_{i^-} in terms of Weyl and Cotton tensor

Generalized wave-map gauge

The aim of this section is to determine the constraint equations induced by the MCFE on the fields $g_{\mu\nu}$, $L_{\mu\nu}$, Θ, s, $C_{\mu\nu\sigma\rho}$ and $\xi_{\mu\nu\sigma}$. For this purpose we assume we have been given some smooth solution $(g_{\mu\nu}, L_{\mu\nu}, \Theta, s, d_{\mu\nu\sigma\rho})$ of the MCFE. For simplicity and to avoid an exhaustive case-by-case analysis we shall restrict attention, as for the CWE, to the case where the initial surface is $S = C_{i^-}$. This requires to assume

$$\lambda = 0 .$$

As a matter of course the constraints for $g_{\mu\nu}$, $L_{\mu\nu}$, Θ and s are the same as before, cf. Section 4.2. The Weyl tensor vanishes on $\mathscr{I}$ [32],

$$\overline{C}_{\mu\nu\sigma}{}^{\rho} = 0 . \tag{6.45}$$

It thus remains to determine the constraint equations for $\xi_{\mu\nu\sigma}$. In adapted null coordinates the independent components of the Cotton tensor are

$$\overline{\xi}_{00A}\ ,\quad \overline{\xi}_{01A}\ ,\quad \overline{\xi}_{11A}\ ,\quad \overline{\xi}_{A0B}\ ,\quad \overline{\xi}_{A1B}\ ,\quad \overline{\xi}_{ABC}\ .$$

We have

$$\begin{aligned}\overline{\xi}_{01A} &= 2(\overline{\nabla_A L_{01}} - \overline{\nabla_1 L_{0A}})\ , &(6.46)\\ \overline{\xi}_{11A} &= 2(\overline{\nabla_A L_{11}} - \overline{\nabla_1 L_{1A}})\ , &(6.47)\\ \overline{\xi}_{A1B} &= 2(\overline{\nabla_B L_{1A}} - \overline{\nabla_1 L_{AB}})\ , &(6.48)\\ \overline{\xi}_{ABC} &= 2(\overline{\nabla_C L_{AB}} - \overline{\nabla_B L_{AC}})\ , &(6.49)\end{aligned}$$

and no transverse derivatives of $L_{\mu\nu}$ are involved. The remaining components follow from (6.30),

$$\begin{aligned}\overline{\xi}_{00A} &= 2\nu^0\overline{\partial_0\Theta}\,\overline{d}_{010A}\ , &(6.50)\\ \overline{\xi}_{A0B} &= 2\nu^0\overline{\partial_0\Theta}\,\overline{d}_{0BA1}\ . &(6.51)\end{aligned}$$

$(R=0,\overline{s}=-2,\kappa=0,\hat{g}=\eta)$-wave-map gauge

To make computations easier we restrict attention to the $(R=0,\overline{s}=-2,\kappa=0,\hat{g}=\eta)$-wave-map gauge. Henceforth all equalities are meant to hold in this particular gauge. As free initial data we take the s-trace-free tensor $\omega_{AB}=\breve{\overline{L}}_{AB}$. The constraint equations for $\mathring{g}_{\mu\nu}$, $\mathring{L}_{\mu\nu}$ and $\mathring{C}_{\mu\nu\sigma\rho}$ read (cf. (5.6)-(5.16))

$$\mathring{g}_{\mu\nu}=\eta_{\mu\nu}\ ,\quad \mathring{C}_{\mu\nu\sigma\rho}=0\ , \tag{6.52}$$

$$\mathring{L}_{1\mu}=0\ ,\quad \mathring{L}_{0A}=\tfrac{1}{2}\tilde{\nabla}^B\lambda_{AB}\ ,\quad \mathring{g}^{AB}\mathring{L}_{AB}=0\ , \tag{6.53}$$

$$4(\partial_1+r^{-1})\mathring{L}_{00}=\lambda^{AB}\omega_{AB}-4r\rho-2\tilde{\nabla}^A\mathring{L}_{0A}\ , \tag{6.54}$$

where

$$\begin{aligned}(\partial_1-r^{-1})\lambda_{AB} &= -2\omega_{AB}\ , &(6.55)\\ (\partial_1+\ 3r^{-1})\rho &= \frac{1}{2}r^{-1}\tilde{\nabla}^A\partial_1\mathring{L}_{0A}-\frac{1}{4}\lambda^{AB}\partial_1(r^{-1}\omega_{AB})\ . &(6.56)\end{aligned}$$

The relevant boundary conditions are

$$\mathring{L}_{00}=O(1)\ ,\quad \lambda_{AB}=O(r^3)\ ,\quad \rho=O(1)\ . \tag{6.57}$$

The equations (6.46)-(6.51) yield

$$\begin{aligned}\mathring{\xi}_{01A} &= -2\partial_1\mathring{L}_{0A}\ =\ \mathring{g}^{BC}\mathring{\xi}_{BAC}\ , &(6.58)\\ \mathring{\xi}_{11A} &= 0\ , &(6.59)\\ \mathring{\xi}_{A1B} &= -2r\partial_1(r^{-1}\omega_{AB})\ , &(6.60)\\ \mathring{\xi}_{ABC} &= 4\tilde{\nabla}_{[C}\omega_{B]A}-4r^{-1}\mathring{g}_{A[B}\mathring{L}_{C]0}\ , &(6.61)\\ \mathring{\xi}_{00A} &= -4r\mathring{d}_{010A}\ ,\ \text{i.e.} &(6.62)\\ \partial_1\mathring{\xi}_{00A} &= \tilde{\nabla}^B(\lambda_{[A}{}^C\omega_{B]C})-2\tilde{\nabla}^B\tilde{\nabla}_{[A}\mathring{L}_{B]0}+\frac{1}{2}\tilde{\nabla}^B\mathring{\xi}_{A1B}\\ &\quad -2r\tilde{\nabla}_A\rho+r^{-1}\mathring{\xi}_{01A}+\lambda_A{}^B\mathring{\xi}_{01B}\ , &(6.63)\\ \mathring{\xi}_{A0B} &= 2r\mathring{g}_{AB}\mathring{d}_{0101}-2r\mathring{d}_{1A1B}-2r\mathring{d}_{01AB}\\ &= \lambda_{[A}{}^C\omega_{B]C}-2\tilde{\nabla}_{[A}\mathring{L}_{B]0}+2r\rho\,\mathring{g}_{AB}-\frac{1}{2}\mathring{\xi}_{A1B}\ , &(6.64)\end{aligned}$$

with boundary condition

$$\mathring{\xi}_{00A}=O(r)\ . \tag{6.65}$$

We employed the $d_{\mu\nu\sigma\rho}$-constraints (5.8)-(5.12) to derive the expressions for $\mathring{\xi}_{00A}$ and $\mathring{\xi}_{A0B}$ (recall that $\overline{\partial_0\Theta}=-2r$, cf. Section 4.3).

Using the constraints for $\xi_{\mu\nu\sigma}$, one may rewrite the equation for ρ,

$$8(\partial_1+\ 3r^{-1})\rho \;=\; r^{-1}\lambda^{AB}\mathring{\xi}_{A1B}-2r^{-1}\tilde{\nabla}^A\mathring{\xi}_{01A}\ . \tag{6.66}$$

Note that for the Cotton tensor to be regular at i^- the initial data necessarily need to satisfy $\omega_{AB}=O(r^3)$, cf. (6.60).

6.5 Applicability of Theorem 6.4 on the C_{i^-}-cone

Let us assume we have been given initial data $\omega_{AB}=O(r^3)$ on C_{i^-}, such that a smooth solution of the CWE2 exists in some neighbourhood to the future of i^-, smoothly extendable through C_{i^-}, which induces the prescribed data $\mathring{\Theta}=0$, $\mathring{s}=-2$, $\mathring{C}_{\mu\nu\sigma}{}^{\rho}=0$, $\mathring{g}_{\mu\nu}=\eta_{\mu\nu}$, $\mathring{L}_{\mu\nu}$ and $\mathring{\xi}_{\mu\nu\sigma}$ on C_{i^-}, the last two fields determined from the hierarchical system of constraint equations (6.52)-(6.64). We want to investigate to what extent the hypotheses of Theorem 6.4 are satisfied under these assumptions.

For convenience let us recall the CWE2 in an $(R=0)$-gauge,

$$\begin{aligned}
\Box_g^{(H)}L_{\mu\nu} &= 4L_{\mu\kappa}L_\nu{}^\kappa-g_{\mu\nu}|L|^2-2C_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma\ , &(6.67)\\
\Box_g s &= \Theta|L|^2\ , &(6.68)\\
\Box_g\Theta &= 4s\ , &(6.69)\\
\Box_g^{(H)}C_{\mu\nu\sigma\rho} &= C_{\mu\nu\alpha}{}^\kappa C_{\sigma\rho\kappa}{}^\alpha-4C_{\sigma\kappa[\mu}{}^\alpha C_{\nu]\alpha\rho}{}^\kappa-2C_{\sigma\rho\kappa[\mu}L_{\nu]}{}^\kappa-2C_{\mu\nu\kappa[\sigma}L_{\rho]}{}^\kappa\\
&\quad-\nabla_{[\sigma}\xi_{\rho]\mu\nu}-\nabla_{[\mu}\xi_{\nu]\sigma\rho}\ , &(6.70)\\
\Box_g^{(H)}\xi_{\mu\nu\sigma} &= 4\xi_{\kappa\alpha[\nu}C_{\sigma]}{}^\alpha{}_\mu{}^\kappa+C_{\nu\sigma\alpha}{}^\kappa\xi_{\mu\kappa}{}^\alpha-4\xi_{\mu\kappa[\nu}L_{\sigma]}{}^\kappa+6g_{\mu[\nu}\xi^\kappa{}_{\sigma\alpha]}L_\kappa{}^\alpha\\
&\quad+8L_{\alpha\kappa}\nabla_{[\nu}C_{\sigma]}{}^\alpha{}_\mu{}^\kappa\ , &(6.71)\\
R_{\mu\nu}^{(H)}[g] &= 2L_{\mu\nu}\ . &(6.72)
\end{aligned}$$

Vanishing of $\overline{H}^\sigma$

This can be shown in exactly the same manner as for the CWE, Section 5.2.

Vanishing of $\overline{\partial_0 H^\sigma}$ and $\overline{\zeta}_\mu$

We know that the wave-gauge vector fulfills the wave equation (6.19) with $R=0$,

$$\nabla^\nu\hat{\nabla}_\nu H^\alpha+2g^{\mu\alpha}\nabla_{[\sigma}\hat{\nabla}_{\mu]}H^\sigma+4\nabla^\nu L_\nu{}^\alpha-2\nabla^\alpha L=0\ . \tag{6.73}$$

As for the CWE the vanishing of $\overline{\partial_0H^0}$ and $\overline{\partial_0H^A}$ follows from (6.73) with $\alpha=0,A$ by taking regularity at the vertex into account. Taking the trace of the restriction of (6.72) to the initial surface then shows that the curvature scalar vanishes initially, $\overline{R}_g=0$.

The $\alpha=1$-component of (6.73) can be written as

$$(\partial_1+r^{-1})\overline{\partial_0H^1}+2(\partial_1+\tau)\overline{L}_{00}+2\tilde{\nabla}^A\overline{L}_{0A}-\overline{g}^{AB}\overline{\partial_0L_{AB}}=0\ , \tag{6.74}$$

where we used that $\overline{\partial_0L_{11}}=0$, cf. (5.32), and that

$$\overline{\partial_0L}\;=\;\overline{\partial_0(g^{\mu\nu}L_{\mu\nu})}\;=\;2\overline{\partial_0L_{01}}+\overline{g}^{AB}\overline{\partial_0L_{AB}}-\lambda^{AB}\overline{L}_{AB}\ . \tag{6.75}$$

We observe that, although we do not know yet whether $\overline{\partial_0 L}$ vanishes, equation (6.74) coincides with (5.43) of Section 5.3, and thus the vanishing of $\overline{\partial_0 H^1}$ can be established by proceeding in exactly the same manner as for the CWE; one first shows that the source terms in (6.74) vanishes and then utilizes regularity to deduce the desired result. Altogether we have

$$\overline{\nabla_\mu H^\nu} = 0 \ . \tag{6.76}$$

Inserting the definition (6.20) of ζ_μ into (6.73) yields

$$\overline{\zeta}_\mu = 0 \ . \tag{6.77}$$

Vanishing of $\overline{\xi_{\rho\nu\mu}\nabla^\rho\Theta}$

Since $\overline{\Theta} = 0$, it suffices to show that $\overline{\xi}_{1\mu\nu} = 0$. Invoking the symmetries of $\overline{\xi}_{\mu\nu\sigma}$, we deduce from the constraint equations (6.58)-(6.64) that

$$\begin{aligned}
\overline{\xi}_{101} &\equiv \overline{g}^{AB}\overline{\xi}_{A1B} = 0 \ , \\
\overline{\xi}_{10A} &\equiv \overline{g}^{BC}\overline{\xi}_{BAC} - \overline{\xi}_{01A} - \overline{\xi}_{11A} = 0 \ , \\
\overline{\xi}_{11A} &= 0 \ , \\
\overline{\xi}_{1AB} &\equiv -2\overline{\xi}_{[AB]1} = 0 \ .
\end{aligned}$$

Vanishing of $\overline{\zeta_{\mu\nu\sigma} - \xi_{\mu\nu\sigma}}$

We need to show that

$$\overline{\xi}_{\mu\nu\sigma} = \overline{\zeta}_{\mu\nu\sigma} \equiv 4\overline{\nabla_{[\sigma}L_{\nu]\mu}} \ .$$

For the components $\overline{\xi}_{01A}$, $\overline{\xi}_{11A}$, $\overline{\xi}_{A1B}$ and $\overline{\xi}_{ABC}$ this follows straightforwardly from the constraint equations (6.58)-(6.61). The remaining independent components $\overline{\xi}_{00A}$ and $\overline{\xi}_{A0B}$ are determined by (6.63) and (6.64), respectively. We observe that $\overline{\zeta}_{00A} - \overline{\xi}_{00A}$ and $\overline{\zeta}_{A0B} - \overline{\xi}_{A0B}$ satisfy the same equations as the components $\overline{\varkappa}_{00A}/2$ and $\overline{\varkappa}_{A0B}/2$ in Section 5.8, so one just needs to repeat the computations carried out there to accomplish the proof that $\overline{\xi}_{\mu\nu\sigma} = \overline{\zeta}_{\mu\nu\sigma}$.

Vanishing of $\overline{W}_{\mu\nu\sigma}{}^\rho$

In the same manner as for the CWE, Section 5.4, one shows that the Weyl tensor $W_{\mu\nu\sigma}{}^\rho$ of $g_{\mu\nu}$ vanishes initially.

Validity of the equations (6.36)-(6.39) on C_{i^-}

The validity of (6.36) on C_{i^-} follows from the vanishing of $\overline{\Theta}$ and $\overline{C}_{\mu\nu\sigma\rho}$. The computation which shows the vanishing of (6.37)-(6.39) is identical to the one we did for the CWE, cf. Sections 5.5-5.7.

6.6 Main result concerning the CWE2

We end up with the following result, which is in close analogy with Theorem 5.1:

THEOREM 6.5 *Let us suppose we have been given a smooth one-parameter family of s-traceless tensors $\omega_{AB}(r,x^A)=O(r^3)$ on the 2-sphere, where s denotes the standard metric. A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, C_{\mu\nu\sigma}{}^\rho, \xi_{\mu\nu\sigma}, \Theta, s)$ of the CWE2 (6.67)-(6.72) to the future of C_{i^-}, smoothly extendable through C_{i^-}, with initial data $(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{C}_{\mu\nu\sigma}{}^\rho, \mathring{\xi}_{\mu\nu\sigma}, \mathring{\Theta}=0, \mathring{s}=-2)$, where $\breve{\mathring{L}}_{AB}=\omega_{AB}$, provides a solution*

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho = \Theta^{-1}C_{\mu\nu\sigma}{}^\rho, \Theta, s)$$

of the MCFE (6.29)-(6.34) with $\lambda=0$, smoothly continued across $\mathscr{I}^-$, in a neighbourhood of i^- intersected with $J^+(i^-)$, with the possible exception of i^- itself, in the

$$(R=0, \overline{s}=-2, \kappa=0, \hat{g}_{\mu\nu}=\eta_{\mu\nu})\text{-wave-map gauge}$$

if and only if the initial data have their usual symmetry properties and satisfy the constraint equations (6.52)-(6.56) and (6.58)-(6.64) with boundary conditions (6.57) and (6.65),[13]

REMARK 6.6 *Note that regularity for the Cotton tensor implies that the initial data necessarily need to satisfy $\omega_{AB}(r,x^A)=O(r^3)$, cf. equation (6.60).*

7 Conclusions and outlook

Let us finish by briefly comparing the two systems of wave equations, CWE and CWE2, which we have studied here, and by summarizing the results we have established for them.

7.1 Comparison of both systems CWE & CWE2

It might be advantageous in certain situations that the Schouten, Weyl and Cotton tensor, which appear in the CWE2-system, can be directly expressed in terms of the metric. In contrast, the rescaled Weyl tensor, which is an unknown of the CWE, can be defined on $\mathscr{I}$ in terms of the metric and the conformal factor only via a limiting process from the inside.

Once a smooth solution of the CWE has been constructed (we think of a characteristic Cauchy problem with data on C_{i^-}), it is, as a matter of course, known that the rescaled Weyl tensor is regular at i^-. Since both, Θ and $d\Theta$, vanish at i^- the same conclusion cannot be straightforwardly drawn for a solution of CWE2, even if one takes initial data $\omega_{AB}=O(r^4)$. Note for this that the constraint equations for CWE2 are somewhat "weaker" than the constraints for the CWE involving the rescaled Weyl tensor, which is due to the fact that the Cotton tensor has less independent components than the rescaled Weyl tensor. It is the $\breve{d}_{0A0B}$-constraint which has no equivalent in the CWE2-system. Thus it seems to be plausible that the conclusions are weaker, too. One has no control how $C_{\mu\nu\sigma\rho}/\Theta$ behaves near the vertex. It seems to be hopeless to catch the behaviour of $d_{\mu\nu\sigma}{}^\rho$ when approaching i^- in terms of the initial data on C_{i^-}.

However, this can be seen as an advantage as well, for there seems to be no reason why the rescaled Weyl tensor should be regular at i^-. It might be more

[13]Note that if $s|_{i^-}<0$ then Θ is positive in the interior of C_{i^-} and sufficiently close to i^- and $d\Theta\neq 0$ on $C_{i^-}\setminus\{i^-\}$ near i^-, so a solution of the CWE2 provides a solution of the MCFE in $J^+(i^-)\setminus\{i^-\}$ sufficiently close to i^-.

sensible to assume just the unphysical metric to be regular there. Note, however, that for analytic data the rescaled Weyl tensor will be regular at i^- [24], while for smooth data this is an open issue. The CWE2 might be predestined to construct solutions of the Einstein equations with a rescaled Weyl tensor which cannot be extended across i^-, supposing of course that such solutions do exist at all. In fact, we have seen that any smooth solution of the CWE (supplemented by the constraint equations) necessarily requires initial data $\omega_{AB} = O(r^4)$, while for the CWE2 we just needed to require $\omega_{AB} = O(r^3)$. So if one is able to construct a solution of the CWE2 from free initial data ω_{AB} which are properly $O(r^3)$, the corresponding solution of the MCFE will lead to a rescaled Weyl tensor which could not be regular at i^-.

7.2 Summary and outlook

Both CWE and CWE2 have been extracted from the MCFE by imposing a generalized wave-map gauge condition. Similar to Friedrich's reduced conformal field equations, they provide, in $3+1$ dimensions, a well behaved system of evolution equations. The main object of this paper was to investigate the issue under which conditions a solution of the CWE/CWE2 is also a solution of the MCFE. Since, roughly speaking, the CWE/CWE2 have been derived from the MCFE by differentiation, one needs to make sure, on characteristic initial surfaces, that the MCFE are initially satisfied, as made rigorous by Theorems 3.7 and 6.4.

One would like to construct the initial data for the CWE/CWE2 in such a way, that all the hypothesis in these theorems are fulfilled. The expectation is that this is the case whenever the data are constructed from suitable free "reduced" data by solving a set of constraint equations induced by the MCFE on the initial surface, which is a hierarchical system of algebraic equations and ODEs, as typical for characteristic initial value problems for Einstein's vacuum field equations. In this work, we have restricted attention to the C_{i^-}-cone, which requires $\lambda = 0$, and, for computational purposes, to a specific gauge, and showed that this is indeed the case, cf. Theorems 5.1 and 6.5.

Analogous results should be expected to hold for e.g. a light-cone for $\lambda \geq 0$ whose vertex is located at $\mathscr{I}^-$, or for two transversally intersecting null hypersurfaces, one of which belongs to $\mathscr{I}^-$ for $\lambda = 0$, or where the intersection manifold is located at $\mathscr{I}^-$ for $\lambda > 0$, and also for such surfaces with a vertex, or intersection manifold, located in the physical space-time. Furthermore, any generalized wave-map gauge with sufficiently well-behaved gauge functions should lead to the same conclusions. We will not work out the details here. It should also be clear that results similar to Theorems 5.1 and 6.5 can be established with initial data of finite differentiability.

The equivalence issue between CWE/CWE2 and MCFE is also of relevance for spacelike Cauchy problems. This has been analysed in [30]. It is shown there that, roughly speaking, a solution of the CWE is a solution of the MCFE if the MCFE and their transverse derivatives are satisfied on the initial surface. As in the characteristic case, it should be expected that this can be guaranteed whenever the initial data are constructed as solutions of an appropriate set of constraint equations. In [30] this has been proved to be the case if the initial surface is a spacelike $\mathscr{I}^-$ (here one needs to assume $\lambda > 0$).

Acknowledgements I am grateful to my advisor Piotr T. Chruściel for numerous stimulating discussions and suggestions, and for reading a first draft of this article. Moreover, it is a pleasure to thank Helmut Friedrich for his valuable input to rewrite the conformal field equations as a system of wave equations. This work was supported in part by the Austrian Science Fund (FWF) P 24170-N16.

A Cone-smoothness and proof of Lemma 2.1

In order to prove Lemma 2.1 we need some facts about *cone-smooth* functions. Therefore let us briefly review the notion of cone-smoothness as well as some basic properties of cone-smooth functions. For the details we refer the reader to [9, 11].

We denote by $\{y^0 \equiv t, y^i\}$ coordinates in a 4-dimensional spacetime for which

$$C_O := \left\{ y^\mu \in \mathbb{R}^4 : y^0 = \sqrt{\sum_i (y^i)^2} \right\} .$$

is the light-cone emanating from a point O. Such coordinates exist at least sufficiently close to the vertex. Adapted null coordinates are denoted by $\{x^0 \equiv u, x^1 \equiv r, x^A\}$. Both coordinate systems are related via a transformation of the form (cf. [11]),

$$y^0 = x^1 - x^0 \ , \quad y^i = x^1 \Theta^i(x^A) \quad \text{with} \quad \sum_{i=1}^3 [\Theta^i(x^A)]^2 = 1 \ .$$

Definition A.1 ([11]) *A function φ defined on C_O is said to belong to $\mathscr{C}^k(C_O)$, $k \in \mathbb{N} \cup \{\infty\}$, if it can be written as $\hat{\varphi} + r\check{\varphi}$ with $\hat{\varphi}$ and $\check{\varphi}$ being $\mathscr{C}^k$-functions of y^i. If $k = \infty$ the function φ is called cone-smooth.*

Remark A.2 *We are particularly interested in the cone-smooth case $k = \infty$.*

Proposition A.3 ([11]) *Let $\varphi : C_O \to \mathbb{R}$ be a function and let $k \in \mathbb{N} \cup \{\infty\}$. The following statements are equivalent:*

(i) The function φ can be extended to a $\mathscr{C}^k$ function on $\mathbb{R}^4$.

(ii) $\varphi \in \mathscr{C}^k(C_O)$.

(iii) The function φ admits an expansion of the form ($\varphi_{i_1 \dots i_p}, \varphi'_{i_1 \dots i_{p-1}} \in \mathbb{R}$)

$$\varphi = \sum_{p=0}^k \varphi_p r^p + o_k(r^k) \ \text{ where } \ \varphi_p := \varphi_{i_1 \dots i_p} \Theta^{i_1} \dots \Theta^{i_p} + \varphi'_{i_1 \dots i_{p-1}} \Theta^{i_1} \dots \Theta^{i_{p-1}} \ .$$

Lemma A.4 ([9]) *Let $\varphi \in \mathscr{C}^k(C_O)$ with $k \in \mathbb{N} \cup \{\infty\}$. Then*

(i) $\exp(\varphi) \in \mathscr{C}^k(C_O)$, and

(ii) $r^{-1} \int_0^r \varphi(\hat{r}, x^A) \, d\hat{r} \in \mathscr{C}^k(C_O)$.

If, in addition, $\varphi(0) = 0$, then *(iii)* $\int_0^r \hat{r}^{-1}\varphi(\hat{r},x^A)\,\mathrm{d}\hat{r} \in \mathscr{C}^k(C_O)$.

LEMMA A.5 *Consider any smooth solution of the MCFE in 4 spacetime dimensions in some neighbourhood $\mathscr{U}$ to the future of i^-, smoothly extendable through C_{i^-}, which satisfies*

$$s_{i^-} := s|_{i^-} \neq 0 \; . \tag{A.1}$$

Let ρ be any function on $\mathscr{U}\cap\partial J^+(i^-)$ with $\rho_{i^-} := \rho|_{i^-} \neq 0$ and $\lim_{r\to 0}\partial_r\rho = 0$ which can be extended to a smooth spacetime function. Then the equation

$$\overline{\nabla_\mu\Theta\nabla^\mu\mathring{\phi} + \mathring{\phi}\, s - \mathring{\phi}^2\rho} = 0 \tag{A.2}$$

is a Fuchsian ODE for $\mathring{\phi}$ and any solution satisfies (set $\mathring{\phi}_{i^-} := \mathring{\phi}|_{i^-}$)

$$\mathrm{sign}(\mathring{\phi}_{i^-}) = \mathrm{sign}(s_{i^-})\mathrm{sign}(\rho_{i^-}) \tag{A.3}$$

(in particular $\mathring{\phi}_{i^-} \neq 0$), and is the restriction to C_{i^-} of a smooth spacetime function.

PROOF: We assume a sufficiently regular gauge so that the regularity conditions (4.41)-(4.51) in [5] hold. Evaluation of the MCFE (2.7) on $\mathscr{I}^-$ in coordinates adapted to the cone implies the relation

$$\overline{g}^{AB}\overline{\nabla_A\nabla_B\Theta} = 2\overline{s} \quad \Longleftrightarrow \quad \tau\overline{\partial_0\Theta} = 2\nu_0\overline{s} \; . \tag{A.4}$$

The notation is introduced at the beginning of Section 4. The expansion τ of the light-cone satisfies

$$\tau = \frac{2}{r} + O(r) \; , \quad \partial_1\tau = -\frac{2}{r^2} + O(1) \; .$$

Moreover, regularity requires

$$\nu_0 = 1 + O(r^2) \; , \;\; \partial_1\nu_0 = O(r) \; , \;\; \overline{s} = O(1) \; , \;\; \partial_1\overline{s} = O(1) \; .$$

Hence

$$\overline{\partial_0\Theta} = s_{i^-}r + O(r^2) \quad \text{and} \quad \partial_1\overline{\partial_0\Theta} = s_{i^-} + O(r) \; . \tag{A.5}$$

The r-component of the MCFE (2.8) yields

$$\partial_1\overline{s} + \nu^0\overline{L}_{11}\overline{\partial_0\Theta} = 0 \quad \Longrightarrow \quad \partial_1\overline{s}|_{i^-} = 0$$

due to regularity (note that $\overline{L}_{11} = O(1)$), i.e.

$$\overline{s} = s_{i^-} + O(r^2) \; .$$

In adapted null coordinates (A.2) reads

$$\nu^0\overline{\partial_0\Theta}\partial_1\mathring{\phi} + \overline{s}\mathring{\phi} - \rho\mathring{\phi}^2 = 0 \; , \tag{A.6}$$

i.e., since $\mathrm{d}\Theta|_{\mathscr{I}^-} \neq 0$, (A.2) is a Fuchsian ODE for $\mathring{\phi}$ along the null geodesics emanating from i^-. By assumption, the functions $\overline{s}$ and ρ are cone-smooth. In [9] it is shown that ν^0 and $r\tau$ are cone-smooth. That implies that the function

$$\psi := \frac{\overline{\partial_0\Theta}}{r} \overset{(A.4)}{=} \frac{2\nu_0\overline{s}}{r\tau} = s_{i^-} + O(r^2)$$

is cone-smooth, as well (note that $(r\tau)|_{i^-} \neq 0$). Since we have assumed $s_{i^-} \neq 0$, the function ψ has no zeros near i^-, so ψ^{-1} exists near i^- and is cone-smooth. The ODE (A.6) thus takes the form

$$r\partial_1 \mathring{\phi} + \hat{\omega}\mathring{\phi} - \omega\mathring{\phi}^2 = 0 \,, \tag{A.7}$$

where the functions $\hat{\omega} := \nu_0 s\psi^{-1} = 1 + O(r^2)$ and $\omega := \nu_0\rho\psi^{-1} = \frac{\rho_{i^-}}{s_{i^-}} + O(r^2)$ are cone-smooth and non-vanishing near the tip of the cone.[14]

Let $\varepsilon > 0$ be sufficiently small. We introduce the function

$$\gamma := e^{-\int_\varepsilon^r \hat{r}^{-1}\hat{\omega}\,\mathrm{d}\hat{r}}\,\mathring{\phi}^{-1} \,, \tag{A.8}$$

so that (A.7) becomes

$$r^2\partial_1\gamma + \zeta = 0 \,, \tag{A.9}$$

where

$$\zeta := \varepsilon\omega e^{-\int_\varepsilon^r \hat{r}^{-1}(\hat{\omega}-1)\,\mathrm{d}\hat{r}} = \underbrace{\varepsilon\frac{\rho_{i^-}}{s_{i^-}} e^{\int_0^\varepsilon \hat{r}^{-1}(\hat{\omega}-1)\,\mathrm{d}\hat{r}}}_{=:c} + O(r^2) \tag{A.10}$$

is cone-smooth by Lemma A.4 and has a sign near the tip,

$$\mathrm{sign}(\zeta_{i^-}) \;=\; \mathrm{sign}(s_{i^-})\mathrm{sign}(\rho_{i^-}) \,.$$

Consequently,

$$r\gamma \;=\; -r\int \hat{r}^{-2}\zeta\,\mathrm{d}\hat{r} \;=\; c + \hat{c}r + O(r^2) \tag{A.11}$$

is cone-smooth and has a sign as follows immediately from the expansions in Proposition A.3 and term-by-term integration. The constant $\hat{c}$ can be regarded as representing the, possibly x^A-dependent, integration function. We conclude that the function

$$\mathring{\phi} \quad = \quad \varepsilon e^{-\int_\varepsilon^r \hat{r}^{-1}(\hat{\omega}-1)\,\mathrm{d}\hat{r}}\,(r\gamma)^{-1} = \frac{s_{i^-}}{\rho_{i^-}} + O(r) \tag{A.12}$$

is cone-smooth and has a sign near the vertex of the cone,

$$\mathrm{sign}(\mathring{\phi}_{i^-}) \;=\; \mathrm{sign}(s_{i^-})\mathrm{sign}(\rho_{i^-})$$

(Note that there remains a gauge freedom to choose $\partial_1\mathring{\phi}|_{i^-}$.) □

References

[1] M. T. Anderson, P. T. Chruściel, *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions*, Comm. Math. Phys. **260** (2005) 557–577.

[14] The absence of first-order terms is due to the vanishing of $\partial_1\overline{s}|_{i^-}$ and $\partial_1\rho|_{i^-}$. In fact, a term proportional to r in ω would produce logarithmic terms in the expansion of ζ, (A.10), and thereby in the expansions of $r\gamma$, (A.11), and $\mathring{\phi}$, (A.12).

[2] L. Andersson, P.T. Chruściel, *On asymptotic behaviour of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissertationes Math. **355** (1996) 1–100.

[3] A. Ashtekar, *Asymptotic structure of the gravitational field at spatial infinity*, in: *General relativity and gravitation – One hundred years after the birth of Albert Einstein, Volume 2*, A. Held (ed.), New York: Plenum Press, 1980, 37–70.

[4] F. Cagnac, *Problème de Cauchy sur un conoïde caractéristique pour des equations quasi-lineaires*, Ann. Mat. Pura Appl. (4) **129** (1981) 13–41.

[5] Y. Choquet-Bruhat, P. T. Chruściel, J. M. Martín-García, *The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions*, Ann. Henri Poincaré **12** (2011) 419-482.

[6] Y. Choquet-Bruhat, P. T. Chruściel, J. M. Martín-García, *An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations*, Contemp. Math. **554** (2011) 73—81.

[7] Y. Choquet-Bruhat, P. T. Chruściel, J. M. Martín-García, *An existence theorem for the Cauchy problem on the light-cone for the vacuum Einstein equations with near-round analytic data*, Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki **153**:3 (2011) 115–138.

[8] Y. Choquet-Bruhat, M. Novello, *Système conforme régulier pour les èquations d'Einstein*, C.R. Acad. Sci. Paris, **t.305**, Serie II, (1987) 155-160.

[9] P. T. Chruściel, *The existence theorem for the general relativistic Cauchy problem on the light-cone*, (2012), arXiv:1209.1971 [gr-qc].

[10] P. T. Chruściel, E. Delay: *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. France **94** (2003) 1–103.

[11] P. T. Chruściel, J. Jezierski, *On free general relativistic initial data on the light cone*, J. Geom. Phys. **62** (2012) 578593.

[12] P. T. Chruściel, T.-T. Paetz, *Solutions of the vacuum Einstein equations with initial data on past null infinity*, Class. Quantum Grav. **30** (2013) 235037.

[13] M. Dossa, *Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique*, Ann. Inst. H. Poincaré Phys. Thér. (1997), no. 1 37–107.

[14] M. Dossa, *Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang-Mills-Higgs*, Ann. Henri Poincaré **4** (2003) 385–411.

[15] F. G. Friedlander: *The wave-equation on a curved space-time*, Cambridge University Press, Cambridge, 1975.

[16] H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. R. Soc. Lond. A **375** (1981) 169–184.

[17] H. Friedrich, *The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system*, Proc. R. Soc. Lond. A **378** (1981) 401–421.

[18] H. Friedrich, *On the existence of analytic null asymptotically flat solutions of Einstein's vacuum field equations*, Proc. R. Soc. Lond. A **381** (1982) 361–371.

[19] H. Friedrich, *On the hyperbolicity of Einstein's and other gauge field equations*, Comm. Math. Phys. **100** (1985) 525–543.

[20] H. Friedrich, *On purely radiative space-times* , Comm. Math. Phys. **103** (1986) 35–65.

[21] H. Friedrich, *On the existence of n-geodesically complete or future complete solutions of Einstein's equations with smooth asymptotic structure*, Comm. Math. Phys. **107** (1986) 587–609.

[22] H. Friedrich, *Hyperbolic reductions for Einstein's equations*, Class. Quantum Grav. **13** (1996) 1451–1469.

[23] H. Friedrich, *Conformal Einstein evolution*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg, Springer, 2002, 1–50.

[24] H. Friedrich, *Conformal Structures of Static Vacuum Data*, Comm. Math. Phys. **321** (2013) 419–482.

[25] H. Friedrich, *The Taylor expansion at past time-like infinity*, Comm. Math. Phys. **324** (2013) 263–300.

[26] R. Geroch, *Asymptotic structure of space-time*, in: *Asymptotic structure of space-time*, F. P. Esposito, L. Witten (eds.), New York and London, Plenum Press, 1977, 1–107.

[27] R. Geroch, G. T. Horowitz, *Asymptotically simple does not imply asymptotically Minkowskian*, Phys. Rev. Lett. **40** (1978) 203–206.

[28] J. Kánnár, *On the existence of C^∞ solutions to the asymptotic characteristic initial value problem in general relativity*, Proc. R. Soc. Lond. A **452** (1996) 945–952.

[29] H.-O. Kreiss, O. E. Ortiz, *Some mathematical and numerical questions connected with first and second order time-dependent systems of partial differential equations*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg, Springer, 2002, 359–370.

[30] T.-T. Paetz, *Killing Initial Data on spacelike conformal boundaries*, (2014), arXiv:1403.2682 [gr-qc].

[31] R. Penrose, *Asymptotic properties of fields and space-time*, Phys. Rev. Lett. **10** (1963) 66–68.

[32] R. Penrose, *Zero rest-mass fields including gravitation: Asymptotic behavior*, Proc. R. Soc. Lond. A **284** (1965) 159–203.

[33] R. Penrose, W. Rindler, *Spinors and space-time Vol. 1*, Cambridge University Press, 1984.

[34] A. D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. R. Soc. Lond. A **427** (1990) 221–239.

OPEN ACCESS
IOP Publishing — Classical and Quantum Gravity

Class. Quantum Grav. **30** (2013) 235037 (14pp) doi:10.1088/0264-9381/30/23/235037

Solutions of the vacuum Einstein equations with initial data on past null infinity

Piotr T Chruściel and Tim-Torben Paetz

Gravitational Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: Piotr.Chrusciel@univie.ac.at and Tim-Torben.Paetz@univie.ac.at

Received 1 July 2013, in final form 18 September 2013
Published 8 November 2013
Online at stacks.iop.org/CQG/30/235037

Abstract
We prove existence of vacuum space-times with freely prescribable cone-smooth initial data on past null infinity.

PACS numbers: 04.20.Ex, 04.20.Ha

1. Introduction

A question of interest in general relativity is the construction of large classes of space-times with controlled global properties. A flagship example of this line of enquiries is the Christodoulou–Klainerman theorem [3] of nonlinear stability of Minkowski space-time. Because this theorem carries only limited information on the asymptotic behaviour of the resulting gravitational fields, and applies only to near-Minkowskian configurations in any case, it is of interest to construct space-times with better understood global properties. One way of doing this is to carry out the construction starting from initial data at the future null cone, $\mathscr{I}^-$, of past timelike infinity i^-. An approach to this has been presented in [10], but an existence theorem for the problem is still lacking. The purpose of this work is to fill this gap.

In order to present our result some terminology and notation is needed: Let C_O denote the (future) light-cone of the origin O in Minkowski space-time (throughout this work, by 'light-cone of a point O' we mean the subset of a spacetime $\mathscr{M}$ covered by future directed null geodesics issued from O). Let, in manifestly flat coordinates y^μ, $\ell = \partial_0 + (y^i/|\vec{y}|)\partial_i$ denote the field of null tangents to C_O. Let $\tilde{d}_{\alpha\beta\gamma\delta}$ be a tensor with algebraic symmetries of the Weyl tensor and with vanishing η-traces, where η denotes the Minkowski metric. Let ς be the pull-back of

$$\tilde{d}_{\alpha\beta\gamma\delta}\ell^\alpha\ell^\gamma$$

to $C_O \setminus \{O\}$. Let, finally, ς_{ab} denote the components of ς in a frame parallel-propagated along the generators of C_O. We prove the following:

Theorem 1.1. *Let C_O be the light-cone of the origin O in Minkowski space-time. For any ς as above there exists a neighbourhood $\mathscr{O}$ of O, a smooth metric g and a smooth function Θ such that C_O is the light-cone of O for g, Θ vanishes on C_O, with $\nabla\Theta$ nonzero on $\dot{J}^+(O)\cap\mathscr{O}\setminus\{O\}$, the function Θ has no zeros on $\mathscr{O}\cap I^+(O)$, and the metric $\Theta^{-2}g$ satisfies the vacuum Einstein equations there. Further, the tensor field*

$$d_{\alpha\beta\gamma\delta} := \Theta^{-1}C_{\alpha\beta\gamma\delta},$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor of g, extends smoothly across $\{\Theta = 0\}$, and ς_{ab} are the frame components, in a g-parallel-propagated frame, of the pull-back to C_O of $d_{\alpha\beta\gamma\delta}\ell^\alpha\ell^\gamma$. The solution is unique up to isometry.

1.1. Strategy of the proof

The starting point of our analysis are the conformal field equations of Friedrich. The task consists of constructing initial data, for those equations, which arise as the restriction to the future light-cone $\mathscr{I}^-$ of past timelike infinity i^- of tensors which are smooth in the unphysical space-time. We then use a system of conformally invariant wave equations of [13] to obtain a space-time with a metric solving the vacuum Einstein equations to the future of i^-.

Now, some of Friedrich's conformal equations involve only derivatives tangent to $\mathscr{I}^-$, and have therefore the character of *constraint equations*. Those equations form a set of PDEs with a specific hierarchical structure, so that solutions can be obtained by integrating ODEs along the generators of $\mathscr{I}^-$. This implies that the constraint equations can be solved in a straightforward way in coordinates adapted to $\mathscr{I}^-$ in terms of a subset of the fields on the light-cone. However, there arise serious difficulties when attempting to show that solutions of the conformal constraint equations can be realized by smooth space-time tensors. These difficulties lie at the heart of the problem at hand. To be able to handle this issue, we note that ς determines the *null data* of [12]. These null data are used in [12] to construct smooth tensor fields satisfying Friedrich's equations up to terms which decay faster than any power of the Euclidean coordinate distance from i^-, similarly for their derivatives of any order; such error terms are said to be $O(|y|^\infty)$. For fields on the light-cone, the notation $O(r^\infty)$ is defined similarly, where r is an affine distance from the vertex along the generators, with derivatives only in directions tangent to the light-cone. In particular the *approximate solution* so obtained solves the constraint equations up to error terms of order $O(r^\infty)$. Using a comparison argument, we show that the approximate fields differ, on C_O, from the exact solution of the constraints by terms which are $O(r^\infty)$. But tensor fields on the light-cone which decay to infinite order in adapted coordinates arise from smooth tensors in space-time, which implies that the solution of the constraint equations arises indeed from a smooth tensor in space-time. As already indicated, this is what is needed to be able to apply the existence theorems for systems of wave equations in [7], provided such a system is at disposal. This last element of our proof is provided by the system of wave equations of [13], and the results on propagation of constraints for this system established there.[1]

2. From approximate solutions to solutions

Recall Friedrich's system of conformally-regular equations (see [11] and references therein)

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0, \tag{2.1}$$

[1] Compare [8], where a system based on the equations of Choquet-Bruhat and Novello [2] is used.

$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta\, d_{\nu\mu\sigma}{}^\rho, \tag{2.2}$$

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu}, \tag{2.3}$$

$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta, \tag{2.4}$$

$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = 0, \tag{2.5}$$

$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}). \tag{2.6}$$

Here Θ is the conformal factor relating the physical metric $\Theta^{-2}\tilde{g}_{\mu\nu}$ with the unphysical metric $g_{\mu\nu}$, the fields $d_{\mu\nu\sigma}{}^\kappa$ and $L_{\alpha\beta}$ encode the information about the unphysical Riemann tensor as made explicit in (2.6), while the trace of (2.3) can be viewed as the definition of s.

We wish to construct solutions of (2.1)–(2.6) with initial data on a light-cone C_{i^-}, emanating from a point i^-, with Θ vanishing on C_{i^-} and with $s(i^-) \neq 0$. (The actual value of $s(i^-)$ can be changed by constant rescalings of the conformal factor Θ and of the field $d_{\alpha\beta\gamma}{}^\delta$. For definiteness we will choose $s(i^-) = -2$.) As explained in [9], such solutions lead to vacuum space-times, where past timelike infinity is the point i^- and where past null infinity $\mathscr{I}^-$ is $C_{i^-} \setminus \{i^-\}$.

We will present two methods of doing this: while the second one is closely related to the classical one in [2], the advantage of the first one is that it allows in principle a larger class of initial data, see remark 2.6 below.

Let, then, a 'target metric' $\hat{g}$ be given and let the operator $\hat{\nabla}$ denote its covariant derivative with associated Christoffel symbols $\hat{\Gamma}^\sigma_{\alpha\beta}$. Set

$$H^\sigma := g^{\alpha\beta}\big(\Gamma^\sigma_{\alpha\beta} - \hat{\Gamma}^\sigma_{\alpha\beta}\big). \tag{2.7}$$

Consider the system of wave equations which [13, section 6.1] follows from (2.1)–(2.6) when H^σ vanishes:

$$\Box_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa} L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2C_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \tfrac{1}{6}\nabla_\mu \nabla_\nu R, \tag{2.8}$$

$$\Box_g s = \Theta |L|^2 - \tfrac{1}{6}\nabla_\kappa R\, \nabla^\kappa \Theta - \tfrac{1}{6} s R, \tag{2.9}$$

$$\Box_g \Theta = 4s - \tfrac{1}{6}\Theta R, \tag{2.10}$$

$$\begin{aligned}\Box_g^{(H)} C_{\mu\nu\sigma\rho} = {}& C_{\mu\nu\alpha}{}^\kappa C_{\sigma\rho\kappa}{}^\alpha - 4C_{\sigma\kappa[\mu}{}^\alpha C_{\nu]\alpha\rho}{}^\kappa - 2C_{\sigma\rho\kappa[\mu} L_{\nu]}{}^\kappa - 2C_{\mu\nu\kappa[\sigma} L_{\rho]}{}^\kappa \\ & - \nabla_{[\sigma}\xi_{\rho]\mu\nu} - \nabla_{[\mu}\xi_{\nu]\sigma\rho} + \tfrac{1}{3} R C_{\mu\nu\sigma\rho},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Box_g^{(H)} \xi_{\mu\nu\sigma} = {}& 4\xi_{\kappa\alpha[\nu} C_{\sigma]}{}^\alpha{}_\mu{}^\kappa + C_{\nu\sigma\alpha}{}^\kappa \xi_{\mu\kappa}{}^\alpha - 4\xi_{\mu\kappa[\nu} L_{\sigma]}{}^\kappa + 6g_{\mu[\nu}\xi^\kappa{}_{\sigma\alpha]} L_\kappa{}^\alpha \\ & + 8L_{\alpha\kappa} \nabla_{[\nu} C_{\sigma]}{}^\alpha{}_\mu{}^\kappa + \tfrac{1}{6} R\xi_{\mu\nu\sigma} - \tfrac{1}{3} C_{\nu\sigma\mu}{}^\kappa \nabla_\kappa R,\end{aligned} \tag{2.12}$$

$$R^{(H)}_{\mu\nu}[g] = 2L_{\mu\nu} + \tfrac{1}{6} R g_{\mu\nu}. \tag{2.13}$$

Here $R^{(H)}_{\mu\nu}[g]$ is defined as

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu} \hat{\nabla}_{\nu)} H^\sigma. \tag{2.14}$$

Further, the field $\xi_{\mu\nu\sigma}$ above will, in the final space-time, be the Cotton tensor, related to the Schouten tensor $L_{\mu\nu}$ as

$$\xi_{\mu\nu\sigma} = 4\nabla_{[\sigma} L_{\nu]\mu} = 2\nabla_{[\sigma} R_{\nu]\mu} + \tfrac{1}{3} g_{\mu[\sigma} \nabla_{\nu]} R.$$

Finally, the operator $\Box_g^{(H)}$ is defined as

$$\Box_g^{(H)} v_{\alpha_1\ldots\alpha_n} := \Box_g v_{\alpha_1\ldots\alpha_n} - \sum_i g_{\sigma[\alpha_i}(\hat{\nabla}_{\mu]}H^\sigma)v_{\alpha_1\ldots}{}^{\mu}{}_{\ldots\alpha_n}$$
$$+ \sum_i \left(2L_{\mu\alpha_i} - R^{(H)}_{\mu\alpha_i} + \frac{1}{6}Rg_{\mu\alpha_i}\right) v_{\alpha_1\ldots}{}^{\mu}{}_{\ldots\alpha_n}, \tag{2.15}$$

with $\Box_g = \nabla^\mu\nabla_\mu$, where in the sums in (2.15) the index μ occurs as the i'th index on $v_{\alpha_1\ldots\alpha_n}$.

Some comments concerning (2.15) are in order. First, if g solves Friedrich's equations (2.1)–(2.6) in the gauge $H^\sigma = 0$, then $\Box_g^{(H)} = \Box_g$, so one may wonder why we are not simply using $\Box_g$. The issue is that the operator $\Box_g$ on tensor fields of nonzero valence contains second-order derivatives of the metric, so that the principal part of a system of equations obtained by replacing $\Box_g^{(H)}$ by $\Box_g$ in (2.8)–(2.13) will *not* be diagonal. This could be cured by adding equations obtained by differentiating (2.13), which is not convenient as it leads to further constraints. Instead, one observes [13, section 3.1] that the second derivatives of the metric appearing in $\Box_g$ can be eliminated in terms of the remaining fields above. For example, for a covector field v,

$$\begin{aligned}
\Box_g v_\lambda &= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda - g^{\mu\nu}\big(\partial_\mu\Gamma^\sigma_{\nu\lambda}\big)v_\sigma + f_\lambda(g, \partial g, v, \partial v)\\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + \big(R_\lambda{}^\sigma - \partial_\lambda\big(g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\big)\big)v_\sigma + f_\lambda(g, \partial g, v, \partial v)\\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + (R_\lambda{}^\sigma - \partial_\lambda H^\sigma)v_\sigma + f_\lambda(g, \partial g, v, \partial v, \hat{g}, \partial\hat{g}, \partial^2\hat{g})\\
&= g^{\mu\nu}\partial_\mu\partial_\nu v_\lambda + \big(R^{(H)}_{\mu\lambda} + g_{\sigma[\lambda}\hat{\nabla}_{\mu]}H^\sigma\big)v^\mu + f_\lambda(g, \partial g, v, \partial v, \hat{g}, \partial\hat{g}, \partial^2\hat{g}),
\end{aligned}$$

with f_λ changing from line to line. This leads to the definition

$$\Box_g^{(H)} v_\lambda := \Box_g v_\lambda - g_{\sigma[\lambda}(\hat{\nabla}_{\mu]}H^\sigma)v^\mu + \left(2L_{\mu\lambda} - R^{(H)}_{\mu\lambda} + \tfrac{1}{6}Rg_{\mu\lambda}\right)v^\mu, \tag{2.16}$$

consistently with (2.15).

An identical calculation shows that the operator (2.15) has the properties just described for higher-valence covariant tensor fields.

It follows from the above that the principal part of $\Box_g^{(H)}$ is $g^{\mu\nu}\partial_\mu\partial_\nu$. This implies that the principal part of (2.8)–(2.13) is diagonal, with principal symbol equal to $g^{\mu\nu}p_\mu p_\nu$ times the identity matrix. In particular, we can use [7] to find solutions of our equations whenever suitably regular initial data are at disposal.

Let $(x^0, x^1 \equiv r, x^A)$ be coordinates adapted to the light-cone C_{i^-} of i^- as in [1, section 4], and let κ measure how the coordinate x^1 differs from an affine parameter along the generators of the light-cone of i^-:

$$\nabla_1\partial_1|_{C_{i^-}} = \kappa\partial_1.$$

There are various gauge freedoms in the equations above. To get rid of this we can, and will, impose

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu}, \quad R = 0, \quad H^\sigma = 0, \quad \kappa = 0, \quad s|_{C_{i^-}} = -2. \tag{2.17}$$

The condition $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ is a matter of choice. The conditions $R = 0$ and $H^\sigma = 0$ are classical, and can be realized by solving wave equations. The condition $\kappa = 0$ is a choice of parameterization of the generators of C_{i^-}. The fact that s can be made a negative constant on C_{i^-} is justified in the appendix, see remark A.3. As already pointed out, the value $s = -2$ is a matter of convenience, and can be achieved by a constant rescaling of Θ and of the field $d_{\alpha\beta\gamma}{}^\delta$.

Consider the set of fields

$$\Psi = (g_{\mu\nu}, L_{\mu\nu}, C_{\mu\nu\sigma}{}^\rho, \xi_{\mu\nu\sigma}, \Theta, s). \tag{2.18}$$

We will denote by

$$\mathring{\Psi} := (\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{C}_{\mu\nu\sigma}{}^{\rho}, \mathring{\xi}_{\mu\nu\sigma}, \mathring{\Theta}, \mathring{s}) \tag{2.19}$$

the (characteristic) initial data for Ψ defined along C_{i^-}.

Set

$$\omega_{AB} \equiv \breve{L}_{AB} := \mathring{L}_{AB} - \tfrac{1}{2}\mathring{g}^{CD}\mathring{L}_{CD}\mathring{g}_{AB}, \tag{2.20}$$

and define λ_{AB} to be the solution of the equation

$$(\partial_1 - r^{-1})\lambda_{AB} = -2\omega_{AB} \tag{2.21}$$

satisfying $\lambda_{AB} = O(r^3)$.[2] The following can be derived [13, sections 4.2, 4.3 & 6.4] from (2.1)–(2.6) and the gauge conditions (2.17):

$$\mathring{g}_{\mu\nu} = \eta_{\mu\nu}, \tag{2.22}$$

$$\mathring{L}_{1\mu} = 0, \quad \mathring{L}_{0A} = \tfrac{1}{2}\mathrm{D}^B\lambda_{AB}, \quad \mathring{g}^{AB}\mathring{L}_{AB} = 0, \tag{2.23}$$

$$\mathring{C}_{\mu\nu\sigma\rho} = 0, \tag{2.24}$$

$$\mathring{\xi}_{11A} = 0, \tag{2.25}$$

$$\mathring{\xi}_{A1B} = -2r\partial_1(r^{-1}\omega_{AB}), \tag{2.26}$$

$$\mathring{\xi}_{ABC} = 4\mathrm{D}_{[C}\omega_{B]A} - 4r^{-1}\mathring{g}_{A[B}\mathring{L}_{C]0}, \tag{2.27}$$

$$\mathring{\xi}_{01A} = \mathring{g}^{BC}\mathring{\xi}_{BAC}, \tag{2.28}$$

$$\partial_1\mathring{\xi}_{00A} = \mathrm{D}^B(\lambda_{[A}{}^C\omega_{B]C}) - 2\mathrm{D}^B\mathrm{D}_{[A}\mathring{L}_{B]0} + \tfrac{1}{2}\mathrm{D}^B\overline{\xi}_{A1B} - 2r\mathrm{D}_A\rho + r^{-1}\mathring{\xi}_{01A} + \lambda_A{}^B\mathring{\xi}_{01B}, \tag{2.29}$$

$$\mathring{\xi}_{A0B} = \lambda_{[A}{}^C\omega_{B]C} - 2\mathrm{D}_{[A}\mathring{L}_{B]0} + 2r\mathring{g}_{AB}\rho - \tfrac{1}{2}\mathring{\xi}_{A1B}, \tag{2.30}$$

$$4(\partial_1 + r^{-1})\mathring{L}_{00} = \lambda^{AB}\omega_{AB} - 2\mathrm{D}^A\mathring{L}_{0A} - 4r\rho, \tag{2.31}$$

with $\mathring{\xi}_{00A} = O(r)$, $\mathring{L}_{00} = O(1)$, and where ρ is the unique bounded solution of

$$(\partial_1 + 3r^{-1})\rho = \tfrac{1}{2}r^{-1}\mathrm{D}^A\partial_1\mathring{L}_{0A} - \tfrac{1}{4}\lambda^{AB}\partial_1(r^{-1}\omega_{AB}). \tag{2.32}$$

Here, and elsewhere, the symbol D_A denotes the covariant derivative of $\mathring{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$.

Let s_{AB} denote the unit round metric on S^2. We will need the following result [13, theorem 6.5]:

Theorem 2.1. *Consider a set of smooth fields Ψ defined in a neighbourhood $\mathcal{U}$ of i^- and satisfying (2.8)–(2.13) in $I^+(i^-)$. Define the data (2.19) by restriction of Ψ to C_{i^-}, suppose that $\mathring{\Theta} = 0$ and $\mathring{s} = -2$. Then the fields*

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^{\rho} = \Theta^{-1}C_{\mu\nu\sigma}{}^{\rho}, \Theta, s)$$

solve on $\mathcal{D}^+(\dot{J}^+(i^-)\cap\mathcal{U})$ the conformal field equations (2.1)–(2.6) in the gauge (2.17), with the conformal factor Θ positive on $I^+(i^-)$ sufficiently close to i^-, with $\mathrm{d}\Theta \neq 0$ on $C_{i^-}\setminus\{i^-\}$ near i^-, and with $\mathring{C}_{\mu\nu\sigma}{}^{\rho} = 0$, if and only if (2.21)–(2.32) hold with ρ and $r^{-3}\lambda_{AB}$ bounded.

[2] When $\mathring{L}_{AB}$ arises from the restriction to the light-cone of a bounded space-time tensor, it holds that $\omega_{AB} = O(r^2)$ or better. We will only consider such initial data here, then there exists a unique solution of (2.21) satisfying $\lambda_{AB} = O(r^3)$
.

Remark 2.2. It follows from (2.26) that a necessary condition for existence of solutions as in the theorem is $\omega_{AB} = O(r^3)$.

Remark 2.3. Note that solutions of the ODEs (2.29) and (2.31) are rendered unique by the conditions $\mathring{\xi}_{00A} = O(r)$ and $L_{00} = O(1)$, which follow from regularity of the fields at the vertex.

We use overlining to denote *restriction to the* η*-light-cone* of i^-. Consider a set of fields $(\tilde{L}_{\mu\nu}, \tilde{\xi}_{\mu\nu\rho})$ defined in a neighbourhood of i^-, and set

$$\omega_{AB} := \overline{\tilde{L}}_{AB} - \tfrac{1}{2}\overline{\tilde{g}}^{CD}\overline{\tilde{L}}_{CD}\overline{\tilde{g}}_{AB}. \tag{2.33}$$

We will say that $(\tilde{L}_{\mu\nu}, \tilde{\xi}_{\mu\nu\rho})$ provides an *approximate solution of the constraint equations* if (2.21)–(2.32) hold up to $O(r^\infty)$ error terms. Thus it must hold that

$$\overline{\tilde{L}}_{1\mu} = O(r^\infty), \quad \overline{\tilde{L}}_{0A} = \tfrac{1}{2}\mathrm{D}^B\lambda_{AB} + O(r^\infty), \quad \overline{\tilde{g}}^{AB}\overline{\tilde{L}}_{AB} = O(r^\infty), \tag{2.34}$$

$$\overline{\tilde{\xi}}_{11A} = O(r^\infty), \tag{2.35}$$

$$\overline{\tilde{\xi}}_{A1B} = -2r\partial_1(r^{-1}\omega_{AB}) + O(r^\infty), \tag{2.36}$$

$$\overline{\tilde{\xi}}_{ABC} = 4\mathrm{D}_{[C}\omega_{B]A} - 4r^{-1}\overline{\tilde{g}}_{A[B}\overline{\tilde{L}}_{C]0} + O(r^\infty), \tag{2.37}$$

$$\overline{\tilde{\xi}}_{01A} = \overline{\tilde{g}}^{BC}\overline{\tilde{\xi}}_{BAC} + O(r^\infty), \tag{2.38}$$

$$\begin{aligned}\partial_1\overline{\tilde{\xi}}_{00A} = {}& \mathrm{D}^B(\lambda_{[A}{}^C\omega_{B]C}) - 2\mathrm{D}^B\mathrm{D}_{[A}\overline{\tilde{L}}_{B]0} + \tfrac{1}{2}\mathrm{D}^B\overline{\tilde{\xi}}_{A1B} \\ & - 2r\mathrm{D}_A\tilde{\rho} + r^{-1}\overline{\tilde{\xi}}_{01A} + \lambda_A{}^B\overline{\tilde{\xi}}_{01B} + O(r^\infty),\end{aligned} \tag{2.39}$$

$$\overline{\tilde{\xi}}_{A0B} = \lambda_{[A}{}^C\omega_{B]C} - 2\mathrm{D}_{[A}\overline{\tilde{L}}_{B]0} + 2r\overline{\tilde{g}}_{AB}\tilde{\rho} - \tfrac{1}{2}\overline{\tilde{\xi}}_{A1B} + O(r^\infty), \tag{2.40}$$

$$4(\partial_1 + r^{-1})\overline{\tilde{L}}_{00} = \lambda^{AB}\omega_{AB} - 2\mathrm{D}^A\overline{\tilde{L}}_{0A} - 4r\tilde{\rho} + O(r^\infty), \tag{2.41}$$

with $\overline{\tilde{\xi}}_{00A} = O(r)$, $\overline{\tilde{L}}_{00} = O(1)$, where $\tilde{\rho}$ is a bounded solution of

$$(\partial_1 + 3r^{-1})\tilde{\rho} = \tfrac{1}{2}r^{-1}\mathrm{D}^A\partial_1\overline{\tilde{L}}_{0A} - \tfrac{1}{4}\lambda^{AB}\partial_1(r^{-1}\omega_{AB}) + O(r^\infty), \tag{2.42}$$

and where λ_{AB} is the solution of (2.21) satisfying $\lambda_{AB} = O(r^3)$, or differs from that solution by $O(r^\infty)$ terms.

Our first main result is the following:

Theorem 2.4. *Let $\tilde{g}_{\mu\nu}$ be a smooth metric defined near i^- such that for small r we have*

$$\overline{\tilde{g}_{\mu\nu} - \eta_{\mu\nu}} = O(r^\infty).$$

Let $\tilde{L}_{\mu\nu}$ be the Schouten tensor of $\tilde{g}_{\mu\nu}$, let $\tilde{\xi}_{\alpha\beta\gamma}$ be the Cotton tensor of $\tilde{g}_{\mu\nu}$ and let $\tilde{C}_{\alpha\beta\gamma\beta}$ be its Weyl tensor. Assume that $(\tilde{L}_{\mu\nu}, \tilde{\xi}_{\mu\nu\rho})$ solves the approximate constraint equations.

Then there exist smooth fields $(g_{\mu\nu}, L_{\mu\nu}, C_{\mu\nu\sigma}{}^\rho, \Theta, s)$ defined in a neighbourhood of i^- such that the fields

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho = \Theta^{-1}C_{\mu\nu\sigma}{}^\rho, \Theta, s)$$

solve the conformal field equations (2.1)–(2.6) in $I^+(i^-)$, satisfy the gauge conditions (2.17), with

$$\overline{\Theta} = 0, \quad \overline{C}_{\mu\nu\sigma}{}^\rho = 0, \quad \breve{L}_{AB} = \omega_{AB}, \tag{2.43}$$

with the conformal factor Θ positive on $I^+(i^-)$ sufficiently close to i^-, and with $\mathrm{d}\Theta \neq 0$ on $C_{i^-} \setminus \{i^-\}$ near i^-.

Proof. We will apply theorem 2.1 to a suitable evolution of the initial data. For this we need to correct Ψ by smooth fields so that the restriction to the light-cone of the new Ψ satisfies the constraint equations as needed for that theorem. Subsequently, we define new fields

$$\check{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \check{L}_{\mu\nu} = \tilde{L}_{\mu\nu} + \delta L_{\mu\nu}, \quad \check{\xi}_{\mu\nu\sigma} = \tilde{\xi}_{\mu\nu\sigma} + \delta\xi_{\mu\nu\sigma},$$

as follows.

We let $\delta g_{\mu\nu}$ be any smooth tensor field defined in a neighbourhood of i^- which is $O(|y|^\infty)$ and which satisfies

$$\overline{\delta g_{\mu\nu}} = \overline{\eta_{\mu\nu} - \tilde{g}_{\mu\nu}}.$$

Indeed, it follows from e.g. [4, equations (C4)–(C5)] that the y–coordinates components $\overline{\delta g_{\mu\nu}}$ of g are $O(r^\infty)$, and existence of their smooth extensions follows from [5, lemma A.1]. This extension procedure will be used extensively from now on without further reference.

To continue, we let $\delta\xi_{\alpha\beta\gamma}$ be any smooth tensor defined in a neighbourhood of i^-, with y-coordinate-components which are $O(|y|^\infty)$, such that

(1) $\overline{\delta\xi}_{11A} = -\overline{\tilde{\xi}}_{11A}$;
(2) $\overline{\delta\xi}_{A1B} = -\overline{\tilde{\xi}}_{A1B} - 2r\partial_1(r^{-1}\omega_{AB})$
(recall that ω_{AB} has been defined in (2.33));
(3) $\overline{\delta\xi}_{ABC} = -\overline{\tilde{\xi}}_{ABC} + 4\mathrm{D}_{[C}\omega_{B]A} - 4r^{-1}\overline{\tilde{g}}_{A[B}\overline{\tilde{L}}_{C]0}$;
(4) $\overline{\delta\xi}_{01A} = -\overline{\tilde{\xi}}_{01A} + \overline{\tilde{g}}^{BC}\overline{\tilde{\xi}}_{BAC}$;
(5) $\overline{\delta\xi}_{00A}$ is the solution vanishing at $r = 0$ of the ODE

$$\partial_1(\overline{\delta\xi}_{00A} + \overline{\tilde{\xi}}_{00A}) = \mathrm{D}^B(\lambda_{[A}{}^C\omega_{B]C}) - 2\mathrm{D}^B\mathrm{D}_{[A}\overline{\tilde{L}}_{B]0} + \tfrac{1}{2}\mathrm{D}^B\overline{\tilde{\xi}}_{A1B} \\ -2r\mathrm{D}_A\check{\rho} + r^{-1}\overline{\tilde{\xi}}_{01A} + \lambda_A{}^B\overline{\tilde{\xi}}_{01B}, \tag{2.44}$$

where $\check{\rho}$ is the unique bounded solution of

$$(\partial_1 + 3r^{-1})\check{\rho} = \tfrac{1}{2}r^{-1}\mathrm{D}^A\partial_1\overline{\tilde{L}}_{0A} - \tfrac{1}{4}r^{-1}\lambda^{AB}\partial_1(r^{-1}\omega_{AB}) \tag{2.45}$$

(it follows from [4, appendix B] that $\overline{\delta\xi}_{00A}$ is $O(r^\infty)$);
(6) $\overline{\delta\xi}_{A0B} = -\overline{\tilde{\xi}}_{A0B} + \lambda_{[A}{}^C\omega_{B]C} - 2\mathrm{D}_{[A}\overline{\tilde{L}}_{B]0} + 2r\overline{\tilde{g}}_{AB}\check{\rho} - \tfrac{1}{2}\overline{\tilde{\xi}}_{A1B}$.

Finally, we let $\delta L_{\mu\nu}$ be any smooth tensor field defined in a neighbourhood of i^-, the y-components of which are $O(|y|^\infty)$, such that:

(1) $\overline{\delta L}_{1\mu} = -\overline{\tilde{L}}_{1\mu}$;
(2) $\overline{\delta L}_{0A} = -\overline{\tilde{L}}_{0A} + \tfrac{1}{2}\mathrm{D}^B\lambda_{AB}$;
(3) $\delta L_{AB} = f\eta_{AB}$, where f is any smooth function defined in a neighbourhood of i^- which is $O(|y|^\infty)$ such that

$$2\overline{f} \equiv \overline{\eta}^{AB}\overline{\delta L}_{AB} = -\overline{\tilde{g}}^{AB}\overline{\tilde{L}}_{AB}$$

(we emphasise that the η_{AB}-trace-free part of L_{AB} coincides thus with the η_{AB}-trace-free part of $\tilde{L}_{AB}$);
(4) $\overline{\delta L}_{00}$ is the solution vanishing at $r = 0$ of the system of ODEs

$$4(\partial_1 + r^{-1})(\overline{\delta L}_{00} + \overline{\tilde{L}}_{00}) = \lambda^{AB}\omega_{AB} - 2\mathrm{D}^A\overline{\tilde{L}}_{0A} - 4r\check{\rho} \tag{2.46}$$

(it follows from [4, appendix B] that $\overline{\delta L}_{00}$ is $O(r^\infty)$).

Let

$$(g_{\mu\nu}, L_{\mu\nu}, C_{\mu\nu\sigma}{}^{\rho}, \xi_{\mu\nu\sigma}, \Theta, s)$$

be a solution of (2.8)–(2.13) with initial data

$$(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{C}_{\mu\nu\sigma}{}^{\rho}, \mathring{\xi}_{\mu\nu\sigma}, \mathring{\Theta}, \mathring{s}) := \overline{(\eta_{\mu\nu}, \check{L}_{\mu\nu}, 0, \check{\xi}_{\mu\nu\sigma}, 0, -2)}\ .$$

A solution exists by [7, théorème 2]. It follows by construction that the hypotheses of theorem 2.1 hold, and the theorem is proved. □

An alternative way of obtaining solutions of our problem proceeds via the following system of conformal wave equations:

$$\Box_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \tfrac{1}{6}\nabla_\mu\nabla_\nu R, \tag{2.47}$$

$$\Box_g s = \Theta|L|^2 - \tfrac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \tfrac{1}{6}sR, \tag{2.48}$$

$$\Box_g \Theta = 4s - \tfrac{1}{6}\Theta R, \tag{2.49}$$

$$\Box_g^{(H)} d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa + \tfrac{1}{2}Rd_{\mu\nu\sigma\rho}, \tag{2.50}$$

$$R^{(H)}_{\mu\nu}[g] = 2L_{\mu\nu} + \tfrac{1}{6}Rg_{\mu\nu}. \tag{2.51}$$

Any solution of the conformal field equations (2.1)–(2.6) in the gauge

$$(R = 0, \hat{g}_{\mu\nu} = \eta_{\mu\nu}, H^\mu = 0, \kappa = 0, \mathring{s} = -2) \tag{2.52}$$

necessarily satisfies [13, sections 4.2 & 4.3]:

$$\mathring{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \mathring{L}_{1\mu} = 0, \quad \mathring{L}_{0A} = \tfrac{1}{2}\mathrm{D}^B\lambda_{AB}, \quad \mathring{g}^{AB}\mathring{L}_{AB} = 0, \tag{2.53}$$

$$\mathring{d}_{1A1B} = -\tfrac{1}{2}\partial_1(r^{-1}\omega_{AB}), \tag{2.54}$$

$$\mathring{d}_{011A} = \tfrac{1}{2}r^{-1}\partial_1\mathring{L}_{0A}, \tag{2.55}$$

$$\mathring{d}_{01AB} = r^{-1}\mathrm{D}_{[A}\mathring{L}_{B]0} - \tfrac{1}{2}r^{-1}\lambda_{[A}{}^C\omega_{B]C}, \tag{2.56}$$

$$(\partial_1 + 3r^{-1})\mathring{d}_{0101} = \mathrm{D}^A\mathring{d}_{011A} + \tfrac{1}{2}\lambda^{AB}\mathring{d}_{1A1B}, \tag{2.57}$$

$$2(\partial_1 + r^{-1})\mathring{d}_{010A} = \mathrm{D}^B(\mathring{d}_{01AB} - \mathring{d}_{1A1B}) + \mathrm{D}_A\mathring{d}_{0101} + 2r^{-1}\mathring{d}_{011A} + 2\lambda_A{}^B\mathring{d}_{011B}, \tag{2.58}$$

$$\begin{aligned} 4(\partial_1 - r^{-1})\check{\mathring{d}}_{0A0B} = {} & (\partial_1 - r^{-1})\mathring{d}_{1A1B} + 2(\mathrm{D}_{(A}\mathring{d}_{B)110})\check{} + 4(\mathrm{D}_{(A}\mathring{d}_{B)010})\check{} \\ & + 3\lambda_{(A}{}^C\mathring{d}_{B)C01} + 3\mathring{d}_{0101}\lambda_{AB}, \end{aligned} \tag{2.59}$$

$$4(\partial_1 + r^{-1})\mathring{L}_{00} = \lambda^{AB}\omega_{AB} - 2\mathrm{D}^A\mathring{L}_{0A} - 4r\mathring{d}_{0101}. \tag{2.60}$$

We have the following result [13, theorem 5.1]:

Theorem 2.5. *A smooth solution*

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \Theta, s)$$

of the system (2.47)–(2.51), with initial data

$$(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma}{}^\rho, \mathring{\Theta} = 0, \mathring{s} = -2)$$

on C_{i^-}, solves on $\mathcal{D}^+(\dot{J}^+(i^-))$ the conformal field equations (2.1)–(2.6) in the gauge (2.52), with Θ positive on $I^+(i^-)$ sufficiently close to i^-, and with $\mathrm{d}\Theta \neq 0$ on $C_{i^-}\setminus\{i^-\}$ near i^-, if and only if (2.53)–(2.60) hold with $\omega_{AB}(r, x^A)$ and $\lambda_{AB}(r, x^A)$ defined by (2.20)–(2.21).

Remark 2.6. It follows from (2.54) that a necessary condition for existence of solutions as in theorem 2.5 is $\omega_{AB} = O(r^4)$. Note that this is stronger than what is needed in theorem 2.4, see remark 2.2. It would be of interest to clarify the question of existence of data needed for theorem 2.1 with $\omega_{AB} = O(r^3)$ properly.

Remark 2.7. Note that the solutions of the ODEs (2.57)–(2.60) are rendered unique by the conditions $\mathring{d}_{0101} = O(1)$, $\mathring{d}_{010A} = O(r)$, $\check{\mathring{d}}_{0A0B} = O(r^2)$ and $\mathring{L}_{00} = O(1)$, which follow from regularity of the fields at the vertex.

A smooth metric $\tilde{g}$ will be called an *approximate solution of the constraint equations* (2.53)–(2.60) if $\tilde{C}^\alpha{}_{\beta\gamma\delta} = \tilde{\Theta}\tilde{d}^\alpha{}_{\beta\gamma\delta}$ for some smooth function $\tilde{\Theta}$ vanishing on C_{i^-} and for some smooth tensor $\tilde{d}^\alpha{}_{\beta\gamma\delta}$, where $\tilde{C}^\alpha{}_{\beta\gamma\delta}$ is the Weyl tensor of $\tilde{g}$, and if (2.53)–(2.60) hold on the light-cone of i^- up to terms which are $O(r^\infty)$, where $\tilde{L}_{\mu\nu}$ is the Schouten tensor of $\tilde{g}$, and where ω_{AB} and λ_{AB} are, possibly up to $O(r^\infty)$ terms, given by (2.20)–(2.21).

Our second main result is the following:

Theorem 2.8. *Let $\tilde{g}_{\mu\nu}$ be an approximate solution of the constraint equations (2.53)–(2.60) defined near i^-. Then there exist smooth fields*

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho = \Theta^{-1}C_{\mu\nu\sigma}{}^\rho, \Theta, s)$$

defined in a neighbourhood of i^- which solve the conformal field equations (2.1)–(2.6) in $I^+(i^-)$, satisfy the gauge conditions (2.17), with

$$\overline{\Theta} = 0, \quad \overline{C}_{\mu\nu\sigma}{}^\rho = 0, \quad \check{\overline{L}}_{AB} = \omega_{AB}, \tag{2.61}$$

with the conformal factor Θ positive on $I^+(i^-)$ sufficiently close to i^-, and with $\mathrm{d}\Theta \neq 0$ on $C_{i^-} \setminus \{i^-\}$ near i^-.

Proof. We will apply theorem 2.5 to a suitable evolution of the initial data. For this we need to correct $(\tilde{g}_{\mu\nu}, \tilde{L}_{\mu\nu}, \tilde{d}_{\mu\nu\sigma\rho})$ by smooth fields so that the new initial data on the light-cone satisfy the constraint equations as needed for that theorem. The construction of the new fields

$$\check{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \check{L}_{\mu\nu} = \tilde{L}_{\mu\nu} + \delta L_{\mu\nu}, \quad \check{d}_{\mu\nu\sigma\rho} = \tilde{d}_{\mu\nu\sigma\rho} + \delta d_{\mu\nu\sigma\rho}, \tag{2.62}$$

is essentially identical to that of the new fields of the proof of theorem 2.4, the reader should have no difficulties filling-in the details. We emphasise that the trace-free part of δL_{AB} is chosen to be zero, hence the trace-free part of $\check{L}_{AB}$ coincides with the trace-free part of $\tilde{L}_{AB}$ on the light-cone.

Once the fields (2.62) have been constructed, we let

$$(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma\rho}, \Theta, s)$$

be a solution of (2.47)–(2.51) with initial data

$$(\mathring{g}_{\mu\nu}, \mathring{L}_{\mu\nu}, \mathring{d}_{\mu\nu\sigma\rho}, \mathring{\Theta}, \mathring{s}) := \overline{(\eta_{\mu\nu}, \check{L}_{\mu\nu}, \check{d}_{\mu\nu\sigma\rho}, 0, -2)}.$$

A solution exists by [7, théorème 2]. It follows by construction that the hypotheses of theorem 2.5 hold, and the theorem is proved. □

3. Proof of theorem 1.1

We are ready now to prove theorem 1.1. Let ς be the cone-smooth tensor field of the statement of the theorem. Thus, there exists a smooth tensor field $\tilde{d}_{\alpha\beta\gamma\delta}$ with the algebraic symmetries of the Weyl tensor so that ς is the pull-back of

$$\tilde{d}_{\alpha\beta\gamma\delta}\ell^\alpha\ell^\gamma \tag{3.1}$$

to $C_O \setminus \{O\}$.

Let $\tilde{\psi}_{MNPQ}$ be a totally-symmetric two-index spinor associated to $\tilde{d}_{\alpha\beta\gamma\delta}$ in the usual way [12, section 3] (compare [14]). Set $\theta^0 = \mathrm{d}t$, $\theta^1 = \mathrm{d}r$. Let γ be a generator of C_0, and let θ^2, θ^3 be a pair of covector fields so that $\{\theta^\mu\}$ forms an orthonormal basis of $T^*\mathscr{M}$ over γ and which are η-parallel propagated along γ. Then ς can be written as

$$\varsigma = \varsigma_{ab}\theta^a\theta^b,$$

with a, b running over $\{2, 3\}$. By construction, the coordinate components ς_{AB} of ς coincide with the coordinate components $\overline{\tilde{d}_{1A1B}}$ of the restriction to the light-cone of $\tilde{d}_{1A1B}$, and thus define a unique field ω_{AB} by integrating (2.54) with the boundary condition $\omega_{AB} = O(r^2)$.

Let the basis $\{e_\mu\}$ be dual to $\{\theta^\mu\}$, set $m = e_2 + \sqrt{-1}e_3$. Then the radiation field ψ_0 of [12, equation (5.3)], defined using $\tilde{\psi}_{MNPQ}$, equals

$$\psi_0 = \varsigma_{ab}m^a m^b.$$

(Under a rotation of $\{e_3, e_4\}$ the field ψ_0 changes by a phase, and defines thus a section of a spin-weighted bundle over $C_O \setminus \{O\}$.) Conversely, any radiation field ψ_0 of [12] arises from a unique cone-smooth ς_{ab} as above.

It has been shown in [12, propositions 8.1 & 9.1] that the radiation field ψ_0, hence ς, defines a smooth Lorentzian metric $\tilde{\tilde{g}}$ such that the resulting collection of fields $(\tilde{\tilde{g}}_{\mu\nu}, \tilde{\tilde{L}}_{\mu\nu}, \tilde{\tilde{d}}_{\mu\nu\sigma\rho}, \tilde{\tilde{\Theta}}, \tilde{\tilde{s}})$ satisfies (2.1)–(2.6) up to error terms which are $O(|y|^\infty)$, with $\tilde{\tilde{g}}_{\mu\nu}|_{C_O} = \eta_{\mu\nu}$. The construction in [12] is such that the field ς calculated from the field $\tilde{d}_{\alpha\beta\gamma\delta}$ of (3.1) coincides with the field ς calculated from the field $\tilde{\tilde{d}}_{\alpha\beta\gamma\delta}$ associated with the metric $\tilde{\tilde{g}}$. Hence the fields ω_{AB} associated with $\tilde{d}_{\alpha\beta\gamma\delta}$ and $\tilde{\tilde{d}}_{\alpha\beta\gamma\delta}$ are identical. The conclusion follows now from theorem 2.8.

4. Alternative data at $\mathscr{I}^-$

Recall that there are many alternative ways to specify initial data for the Cauchy problem for the vacuum Einstein equations on a (usual) light-cone, cf e.g. [6]. Similarly there are many ways to provide initial data on a light-cone emanating from past timelike infinity. In theorem 1.1 some components of the rescaled Weyl tensor $d_{\mu\nu\sigma\rho}$ have been prescribed as free data. As made clear in the proof of that theorem, this is equivalent to providing some components of the rescaled Weyl spinor ψ_{MNPQ}, providing thus an alternative equivalent prescription. Our theorems 2.4 and 2.8 use instead the components (2.33) of the rescaled Schouten tensor $\tilde{L}_{\mu\nu}$. These components are related directly to the free data of theorem 1.1 via the constraint equation (2.54). It is clear that further possibilities exist. Which of these descriptions of the degrees of freedom of the gravitational field at large retarded times is most useful for physical applications remains to be seen.

Acknowledgments

Useful discussions with Helmut Friedrich are acknowledged. Supported in part by the Austrian Science Fund (FWF): P 24170-N16.

Appendix. The $\bar{s} = -2$ gauge

We start with some terminology. We say that a function f defined on a space-time neighbourhood of the origin is $o_m(|y|^k)$ if f is C^m and if for $0 \leqslant \ell \leqslant m$ we have

$$\lim_{|y|\to 0} |y|^{\ell-k}\partial_{\mu_1}\ldots\partial_{\mu_\ell} f = 0,$$

where $|y| := \sqrt{\sum_{\mu=0}^n (y^\mu)^2}$.

A similar definition will be used for functions defined in a neighbourhood of O on the future light-cone

$$C_O = \{y^0 = |\vec{y}|\}.$$

For this, we parameterize C_O by coordinates $\vec{y} = (y^i) \in \mathbb{R}^n$, and we say that a function f defined on a neighbourhood of O within C_O is $o_m(r^k)$ if f is a C^m function of the coordinates y^i and if for $0 \leqslant \ell \leqslant m$ we have $\lim_{r\to 0} r^{\ell-k}\partial_{\mu_1}\ldots\partial_{\mu_\ell} f = 0$, where

$$r := |\vec{y}| \equiv \sqrt{\sum_{i=1}^{n}(y^i)^2}.$$

We further set

$$\Theta^i := \frac{y^i}{r}.$$

A function φ defined on C_O will be said to be C^k*-cone-smooth* if there exists a function f on space-time of differentiability class C^k such that φ is the restriction of f to C_O. We will simply say *cone-smooth* if $k = \infty$.

The following lemma will be used repeatedly:

Lemma A.1 (lemma A.1 in [5])**.** *Let $k \in \mathbb{N}$. A function φ defined on a light-cone C_O is the trace $\overline{f}$ on C_O of a C^k space-time function f if and only if φ admits an expansion of the form*

$$\varphi = \sum_{p=0}^{k} f_p r^p + o_k(r^k), \tag{A.1}$$

with

$$f_p \equiv f_{i_1\ldots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + f'_{i_1\ldots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}}, \tag{A.2}$$

where $f_{i_1\ldots i_p}$ and $f'_{i_1\ldots i_{p-1}}$ are numbers.

The claim remains true with $k = \infty$ if (A.1) holds for all $k \in \mathbb{N}$.

Coefficients f_p of the form (A.2) will be said to be *admissible*.

One of the elements needed for the construction in [12] is provided by the following result:

Proposition A.2. *Let p be a point in a smooth space-time $(\mathscr{M}, g)$, $\mathscr{U}$ a neighbourhood of p, and $S_{\mu\nu}[g]$ the trace-free part of the Ricci tensor of g. Let ℓ^ν denote the field of null directions tangent to $\partial J^+(p)\cap\mathscr{U}$. Let $a > 0$ be a real number and let β be a one-form at p. Then, replacing $\mathscr{U}$ by a smaller neighbourhood of p if necessary, there exists a unique smooth function θ defined on $\mathscr{U}$ satisfying*

$$\theta = a \text{ and } d\theta = \beta \text{ at } p, \qquad \ell^\mu\ell^\nu S_{\mu\nu}[\theta^2 g] = 0 \text{ on } \partial J^+(p)\cap\mathscr{U}, \tag{A.3}$$

and such that

$$\text{the Ricci scalar of } \theta^2 g \text{ vanishes on } J^+(p)\cap\mathscr{U}. \tag{A.4}$$

Remark A.3. It follows from (2.4) multiplied by $\nabla^\mu\Theta$ that s is constant on $\partial J^+(p)\cap\mathscr{U}$ when the gauge (A.3) has been chosen.

Proof. In dimension n let $g' = \phi^{4/(n-2)}g$, then

$$R'_{\mu\nu} = R_{\mu\nu} - 2\phi^{-1}\nabla_\mu\nabla_\nu\phi + \frac{2n}{n-2}\phi^{-2}\nabla_\mu\phi\nabla_\nu\phi - \frac{2}{n-2}\phi^{-1}(\nabla^\sigma\nabla_\sigma\phi + \phi^{-1}|d\phi|^2)g_{\mu\nu}.$$

So, in dimension $n = 4$, and with $\phi = \theta$ we obtain

$$S'_{\mu\nu} = S_{\mu\nu} - 2\theta^{-1}\left(\nabla_\mu\nabla_\nu\theta - \tfrac{1}{4}\Delta\theta g_{\mu\nu}\right) + 4\theta^{-2}\left(\nabla_\mu\theta\nabla_\nu\theta - \tfrac{1}{4}|\nabla\theta|^2 g_{\mu\nu}\right).$$

We overline restrictions of space-time functions to the light-cone. The equation $\ell^\mu\ell^\nu S_{\mu\nu}[\theta^2 g] = 0$ takes thus the form

$$\overline{\ell^\mu\ell^\nu\nabla_\mu\nabla_\nu\theta - 2\theta^{-1}(\ell^\mu\nabla_\mu\theta)^2} = \frac{1}{2}\overline{\theta\ell^\mu\ell^\nu S_{\mu\nu}}.$$

In coordinates adapted to the light-cone as in [1, appendix A], so that $\ell^\mu\partial_\mu = \partial_r$, with r an affine parameter, this reads

$$\overline{\theta}^{-1}\partial_r^2\overline{\theta} - 2\overline{\theta}^{-2}(\partial_r\overline{\theta})^2 = \tfrac{1}{2}\overline{S}_{11}.$$

Setting

$$\varphi := \frac{\partial_r\overline{\theta}}{\overline{\theta}},$$

this can be rewritten as

$$\partial_r\varphi = \varphi^2 + \tfrac{1}{2}\overline{S}_{11}. \tag{A.5}$$

It is useful to introduce some notation. As in [1], we underline the components of a tensor in the coordinates y^μ, thus:

$$\underline{S_{\mu\nu}} = S\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\mu}\right), \qquad S_{\mu\nu} = S\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\mu}\right),$$

etc, where $(x^\mu) := (y^0 - |\vec{y}|, |\vec{y}|, x^A)$, with x^A being any local coordinates on S^2. We write interchangeably x^1 and r.

The initial data for φ are

$$\varphi(0) = \frac{1}{a}(\underline{\beta_0} + \underline{\beta_i}\Theta^i). \tag{A.6}$$

To show that $\overline{\theta}$ is cone-smooth, it suffices to prove that

$$\psi := r\varphi$$

is C^k-cone-smooth for all k, as follows immediately from the expansions of lemma A.1, together with integration term-by-term in the formula

$$\ln\left(\frac{\overline{\theta}}{a}\right) = \int_0^r \varphi,$$

compare [4, lemma B.1].

We shall proceed by induction. So suppose that ψ is C^k-cone-smooth. The result is true for $k = 0$ since every solution of (A.5) is continuous in all variables, and $r\varphi$ tends to zero as r tends to zero, uniformly in $\Theta^i \in S^2$.

It follows that the source term $\overline{S}_{11}$ in (A.5) can be written as

$$\overline{S}_{11} = r^{-2}\underbrace{\overline{r^2\underline{S_{00}} + 2t\underline{S_{0i}}y^i + \underline{S_{ij}}y^i y^j}}_{=:\chi} := r^{-2}\overline{\chi},$$

where χ is a smooth function on space-time. We thus have

$$\partial_r\varphi = \varphi^2 + \tfrac{1}{2}\overline{S}_{11} = r^{-2}\big(\psi^2 + \tfrac{1}{2}\overline{\chi}\big). \tag{A.7}$$

The function ψ is C^k-cone-smooth and $O(r)$, and can thus be written in the form (A.1)–(A.2),

$$\psi = \sum_{p=1}^{k} f_p r^p + o_k(r^k).$$

Squaring we obtain

$$\psi^2 = \sum_{p=2}^{k+1} f'_p r^p + o_k(r^{k+1}),$$

for some new admissible coefficients f'_p. The function $\overline{\chi}$ is C^{k+1}-cone-smooth and $O(r^2)$, and can thus be written in the form (A.1) and (A.2) with k replaced by $k+1$ there,

$$\overline{\chi} = \sum_{p=2}^{k+1} f''_p r^p + o_{k+1}(r^{k+1}).$$

Hence

$$\partial_r \varphi = \sum_{p=2}^{k-1} f'''_p r^{p-2} + o_k(r^{k-1}) \tag{A.8}$$

for some admissible coefficients f'''_p. Integration gives

$$\varphi = \frac{1}{a}(\underline{\beta_0} + \underline{\beta_i}\Theta^i) + \sum_{p=2}^{k} \frac{1}{p-1} f'''_p r^{p-1} + o_k(r^k), \tag{A.9}$$

and thus

$$\psi = r\varphi = \frac{1}{a}(\underline{\beta_0}r + \underline{\beta_i}y^i) + \sum_{p=2}^{k+1} \frac{1}{p-1} f'''_p r^p + o_k(r^{k+1}). \tag{A.10}$$

Differentiating $r\varphi$ with respect to r and using (A.8) we further obtain

$$\partial_r \psi = \partial_r \left(\frac{1}{a}(\underline{\beta_0}r + \underline{\beta_i}y^i) + \sum_{p=2}^{k+1} \frac{1}{p-1} f'''_p r^p \right) + o_k(r^k). \tag{A.11}$$

Let $X = X^A\partial_A$ be any vector field on S^2, then $X(\varphi)$ solves the equation obtained by differentiating (A.7),

$$\partial_r X(\varphi) = 2\varphi X(\varphi) + \tfrac{1}{2} X(\overline{S}_{11}). \tag{A.12}$$

Equivalently,

$$X(r\varphi)(r, x^A) = \mathrm{e}^{2\int_0^r \varphi(\tilde{r},x^A)\,\mathrm{d}\tilde{r}} \left(X(r\varphi(0, x^A)) + \frac{1}{2} r \int_0^r \mathrm{e}^{-2\int_0^{\hat{r}} \varphi(\tilde{r},x^A)\,\mathrm{d}\tilde{r}} X(\overline{S}_{11})(\hat{r}, x^A)\,\mathrm{d}\hat{r} \right).$$

The right-hand side is C^k-cone-smooth. We conclude that ψ is C^{k+1}-cone-smooth. This finishes the induction, and proves that $\overline{\theta}$ is cone-smooth.

The existence and uniqueness of a solution θ of (A.4) which equals $\overline{\theta}$ on C_O follows now from [7, théorème 2]. □

References

[1] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions *Ann. Henri Poincaré* **12** 419–82 (arXiv:1006.4467 [gr-qc])

[2] Choquet-Bruhat Y and Novello M 1987 Système conforme régulier pour les équations d'Einstein *C. R. Acad. Sci., Paris* **205** 155–60

[3] Christodoulou D and Klainermann S 1993 *Nonlinear Stability of Minkowski Space* (Princeton, NJ: Princeton University Press)

[4] Chruściel P T 2012 The existence theorem for the Cauchy problem on the light-cone for the vacuum Einstein equations arXiv:1209.1971 [gr-qc]

[5] Chruściel P T and Jezierski J 2012 On free general relativistic initial data on the light cone *J. Geom. Phys.* **62** 578–93 (arXiv:1010.2098 [gr-qc])

[6] Chruściel P T and Paetz T-T 2012 The many ways of the characteristic Cauchy problem *Class. Quantum Grav.* **29** 145006 (arXiv:1203.4534 [gr-qc])

[7] Dossa M 1997 Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique *Ann. Inst. Henri Poincaré Phys. Théor.* **66** 37–107

[8] Dossa M 2003 Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang—Mills–Higgs *Ann. Henri Poincaré* **4** 385–411

[9] Friedrich H 1981 On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations *Proc. R. Soc. Lond.* A **375** 169–84

[10] Friedrich H 1986 On purely radiative space-times *Commun. Math. Phys.* **103** 35–65

[11] Friedrich H 2002 Conformal Einstein evolution *The Conformal Structure of Space-Time* (*Lecture Notes in Physics* vol 604) (Berlin: Springer) pp 1–50 (arXiv:gr-qc/0209018)

[12] Friedrich H 2013 The Taylor expansion at past time-like infinity arXiv:1306.5626 [gr-qc]

[13] Paetz T-T 2013 Conformally covariant systems of wave equations and their equivalence to Einstein's field equations arXiv:1306.6204 [gr-qc]

[14] Penrose R and Rindler W 1984 *Spinors and Spacetime I: Two-Spinor Calculus and Relativistic Fields* (Cambridge, MA: Cambridge University Press)

Characteristic initial data and smoothness of Scri. I. Framework and results*

Piotr T. Chruściel† and Tim-Torben Paetz‡

Gravitational Physics, University of Vienna
Boltzmanngasse 5, 1090 Vienna, Austria

May 26, 2014

Abstract

We analyze the Cauchy problem for the vacuum Einstein equations with data on a complete light-cone in an asymptotically Minkowskian space-time. We provide conditions on the free initial data which guarantee existence of global solutions of the characteristic constraint equations. We present necessary-and-sufficient conditions on characteristic initial data in 3+1 dimensions to have no logarithmic terms in an asymptotic expansion at null infinity.

PACS: 02.30.Hq, 02.30.Mv, 04.20.Ex, 04.20.Ha

Contents

*Preprint UWThPh-2014-1.

†Email Piotr.Chrusciel@univie.ac.at, URL http://homepage.univie.ac.at/piotr.chrusciel

‡Email Tim-Torben.Paetz@univie.ac.at

1 Introduction

An issue of central importance in general relativity is the understanding of gravitational radiation. This has direct implications for the soon-expected direct detection of gravitational waves. The current main effort in this topic appears to be a mixture of numerical modeling and approximation methods. From this perspective there does not seem to be a need for a better understanding of the exact properties of the gravitational field in the radiation regime. However, as observations and numerics will have become routine, solid theoretical foundations for the problem will become necessary.

Now, a generally accepted framework for describing gravitational radiation seems to be the method of conformal completions of Penrose. Here a key hypothesis is that a suitable conformal rescaling of the space-time metric becomes smooth on a new manifold with boundary $\mathscr{I}^+$. One then needs to face the question, if and how such space-times can be constructed. Ultimately one would like to isolate the class of initial data, on a spacelike slice extending to spatial infinity, the evolution of which admits a Penrose-type conformal completion at infinity, and show that the class is large enough to model all physical processes at hand. Direct attempts to carry this out (see [17, 22, 23] and references therein) have not been successful so far. Similarly, the asymptotic behaviour of the gravitational field established in [3, 9, 19–21, 24] is inconclusive as far as the smoothness of the conformally rescaled metric at $\mathscr{I}^+$ is concerned. The reader is referred to [18] for an extensive discussion of the issues arising.

On the other hand, clear-cut constructions have been carried-out in less demanding settings, with data on characteristic surfaces as pioneered by Bondi et al. [5], or with initial data with hyperboloidal asymptotics. It has been found [1, 2, 11, 30] that both generic Bondi data and generic hyperboloidal data, constructed out of conformally smooth seed data, will *not* lead to space-times with a smooth conformal completion. Instead, a *polyhomogeneous* asymptotics of solutions of the relevant constraint equations was obtained, with logarithmic terms appearing in asymptotic expansions of the fields.

The case for the necessity of a polyhomogeneous-at-best framework, as resulting from the above work, is not waterproof: In both cases it is not clear whether initial data with logarithmic terms can arise from evolution of a physical system which is asymptotically flat in spacelike directions. There is a further issue with the Bondi expansions, because the framework of Bondi et al. [5, 28] does not provide a well posed system of evolution equations for the problem at hand.

The aim of this work is to rederive the existence of obstructions to smoothness of the metric at $\mathscr{I}^+$ in a framework in which the evolution problem for the Einstein vacuum equations is well posed and where free initial data are given on a light-cone extending to null infinity, or on two characteristic hypersurfaces one of which extends to infinity, or in a mixed setting where part of the data are prescribed on a spacelike surface and part on a characteristic one extending to infinity. This can be viewed as a revisiting of the Bondi-type setting in a framework where an associated space-time is guaranteed to exist.

One of the attractive features of the characteristic Cauchy problem is that one can explicitly provide an exhaustive class of freely prescribable initial data. By "exhaustive class" we mean that the map from the space of free initial data to the set of solutions is surjective, where "solution" refers to that part of space-time which is covered by the domain of dependence of the smooth part of the light-cone, or of the smooth part of the null hypersurfaces issuing normally from a smooth submanifold of codimension two.[1] There is, moreover, considerable flexibility in prescribing characteristic initial data [12]. In this work we will concentrate on the following approaches:

1. The free data are a triple $(\mathscr{N}, [\gamma], \kappa)$, where $\mathscr{N}$ is a n-dimensional manifold, $[\gamma]$ is a conformal class of symmetric two-covariant tensors on $\mathscr{N}$ of signature $(0, +, \dots, +)$, and κ is a field of connections on the bundles of tangents to the integral curves of the kernel of γ.[2]

2. Alternatively, the data are a triple $(\mathscr{N}, \breve{g}, \kappa)$, where $\breve{g}$ is a field of symmetric two-covariant tensors on $\mathscr{N}$ of signature $(0, +, \dots, +)$, and κ is a field of connections on the bundles of tangents to the integral curves of the characteristic direction of $\breve{g}$. [3] The pair $(\breve{g}, \kappa)$ is further required to satisfy the constraint equation

$$\partial_r \tau - \kappa\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0 \,, \tag{1.1}$$

   where τ is the divergence and σ is the shear (see Section 2.2 for details), which will be referred to as the *Raychaudhuri equation*.

3. Alternatively, the connection coefficient κ and all the components of the space-time metric are prescribed on $\mathscr{N}$, subject to the Raychaudhuri constraint equation. Here $\mathscr{N}$ is viewed as the hypersurface $\{u = 0\}$ in the

[1]This should be contrasted with the spacelike Cauchy problem, where no exhaustive method for constructing non-CMC initial data sets is known. It should, however, be kept in mind that the spacelike Cauchy problem does not suffer from the serious problem of formation of caustics, inherent to the characteristic one.

[2]Recall that a connection ∇ on each such bundle is uniquely described by writing $\nabla_r\partial_r = \kappa\partial_r$, in a coordinate system where ∂_r is in the kernel of γ. Once the associated space-time has been constructed we will also have $\nabla_r\partial_r = \kappa\partial_r$, where ∇ now is the covariant derivative operator associated with the space-time metric.

[3]We will often write $(\breve{g}, \kappa)$ instead of $(\mathscr{N}, \breve{g}, \kappa)$, with $\mathscr{N}$ being implicitly understood, when no precise description of $\mathscr{N}$ is required.

space-time to-be-constructed, and thus all metric components $g_{\mu\nu}$ are prescribed at $u = 0$ in a coordinate system $(x^\mu) = (u, x^i)$, where (x^i) are local coordinates on $\mathscr{N}$.

4. Finally, schemes where tetrad components of the conformal Weyl tensor are used as free data are briefly discussed.

In the first two cases, to obtain a well posed evolution problem one needs to impose gauge conditions; in the third case, the initial data themselves determine the gauge, with the "gauge-source functions" determined from the initial data.

The aim of this work is to analyze the occurrence of log terms in the asymptotic expansions as r goes to infinity for initial data sets as above. The gauge choice $\kappa = O(r^{-3})$ below (in particular the gauge choice $\kappa = \frac{r}{2}|\sigma|^2$, on which we focus in part II), ensures that affine parameters along the generators of $\mathscr{N}$ diverge as r goes to infinity (cf. [26, Appendix B]), so that in the associated space-time the limit $r \to \infty$ will correspond to null geodesics approaching a (possibly non-smooth) null infinity.

It turns out that the simplest choice of gauge conditions, namely $\kappa = 0$ and harmonic coordinates, is *not compatible* with smooth asymptotics at the conformal boundary at infinity: we prove that the *only* vacuum metric, constructed from characteristic Cauchy data on a light-cone, and which has a smooth conformal completion in this gauge, is Minkowski space-time.

It should be pointed out, that the observation that some sets of harmonic coordinates are problematic for an analysis of null infinity has already been made in [4, 6]. Our contribution here is to make a precise *no-go* statement, without approximation procedures or supplementary assumptions.

One way out of the problem is to replace the harmonic-coordinates condition by a wave-map gauge with non-vanishing gauge-source functions. This provides a useful tool to isolate those log terms which are gauge artifacts, in the sense that they can be removed from the solution by an appropriate choice of the gauge-source functions. There remain, however, some logarithmic coefficients which cannot be removed in this way. We identify those coefficients, and show that the requirement that these coefficients do not vanish is gauge-independent. In part II of this work we show that the logarithmic coefficients are non-zero for generic initial data. The equations which lead to vanishing logarithmic coefficients will be referred to as the *no-logs-condition*.

It is expected that for generic initial data sets, as considered here, the space-times obtained by solving the Cauchy problem will have a polyhomogeneous expansion at null infinity. There are, however, no theorems in the existing mathematical literature which guarantee existence of a polyhomogeneous $\mathscr{I}^+$ when the initial data have non-trivial log terms.

The situation is different when the no-logs-condition is satisfied. In part II of this work we show that the resulting initial data lead to smooth initial data for Friedrich's conformal field equations [14] as considered in [13]. This implies that the no-logs-condition provides a necessary-and-sufficient condition for the evolved space-time to posses a smooth $\mathscr{I}^+$. For initial data close enough to Minkowskian ones, solutions global to the future are obtained.

It may still be the case that the logarithmic expansions are irrelevant as far as our understanding of gravitational radiation is concerned, either because they never arise from the evolution of isolated physical systems, or because their occurrence prevents existence of a sufficiently long evolution of the data, or because all essential physical issues are already satisfactorily described by smooth conformal completions. While we haven't provided a definite answer to those questions, we hope that our results here will contribute to resolve the issue.

If not explicitly stated otherwise, all manifolds, fields, and expansion coefficients are assumed to be smooth.

2 The characteristic Cauchy problem on a light-cone

In this section we will review some facts concerning the characteristic Cauchy problem. Most of the discussion applies to any characteristic surface. We concentrate on a light-cone, as in this case all the information needed is contained in the characteristic initial data together with the requirement of the smoothness of the metric at the vertex. The remaining Cauchy problems mentioned in the Introduction will be discussed in Sections 7 below.

2.1 Gauge freedom

2.1.1 Adapted null coordinates

Our starting point is a C^∞-manifold $\mathscr{M} \cong \mathbb{R}^{n+1}$ and a future light-cone $C_O \subset \mathscr{M}$ emanating from some point $O \in \mathscr{M}$. We make the assumption that the subset C_O can be *globally* represented in suitable coordinates (y^μ) by the equation of a Minkowskian cone, i.e.

$$C_O = \{(y^\mu) : y^0 = \sqrt{\sum_{i=1}^{n}(y^i)^2}\} \subset \mathscr{M} \ .$$

Given a $C^{1,1}$-Lorentzian space-time such a representation is always possible in some neighbourhood of the vertex. However, since caustics may develop along the null geodesics which generate the cone, it is a geometric restriction to assume the existence of a Minkowskian representation globally.

A treatment of the characteristic initial value problem at hand is easier in coordinates x^μ adapted to the geometry of the light-cone [8, 27]. We consider space-time-dimensions $n+1 \geq 3$. It is standard to construct a set of coordinates $(x^\mu) \equiv (u, r, x^A)$, $A = 2, \ldots, n$, so that $C_O \setminus \{0\} = \{u = 0\}$. The x^A's denote local coordinates on the level sets $\Sigma_r := \{r = \text{const}, u = 0\} \cong S^{n-1}$, and are constant along the generators. The coordinate r induces, by restriction, a parameterization of the generators and is chosen so that the point O is approached when $r \to 0$. The general form of the trace $\overline{g}$ on the cone C_O of the space-time metric g reduces in these *adapted null coordinates* to

$$\overline{g} = \overline{g}_{00}\mathrm{d}u^2 + 2\nu_0\mathrm{d}u\mathrm{d}r + 2\nu_A\mathrm{d}u\mathrm{d}x^A + \check{g} \ , \tag{2.1}$$

where

$$\nu_0 := \overline{g}_{01} \ , \quad \nu_A := \overline{g}_{0A} \ ,$$

and where

$$\breve{g} = \breve{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B := \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

is a degenerate quadratic form induced by g on C_O which induces on each slice Σ_r an r-dependent Riemannian metric $\breve{g}_{\Sigma_r}$ (coinciding with $\breve{g}(r,\cdot)$ in the coordinates above).[4]

The components $\overline{g}_{00}$, ν_0 and ν_A are gauge-dependent quantities. In particular, ν_0 changes sign when u is replaced by $-u$. Whenever useful and/or relevant, we will assume that ∂_r is future-directed and ∂_u is past-directed, which corresponds to requiring that $\nu_0 > 0$.

The quadratic form $\breve{g}$ is intrinsically defined on C_O, independently of the choice of the parameter r and of how the coordinates are extended off the cone.

Throughout this work an overline denotes the restriction of space-time objects to C_O.

The restriction of the inverse metric to the light-cone takes the form

$$\overline{g}^{\#} = 2\nu^0\partial_u\partial_r + \overline{g}^{11}\partial_r\partial_r + 2\overline{g}^{1A}\partial_r\partial_A + \overline{g}^{AB}\partial_A\partial_B \ ,$$

where

$$\nu^0 := \overline{g}^{01} = (\nu_0)^{-1} \ , \ \nu^A := \overline{g}^{AB}\nu_B \ , \ \overline{g}^{1A} = -\nu^0\nu^A \ , \ \overline{g}^{11} = (\nu^0)^2(\nu^A\nu_A - \overline{g}_{00}) \ ,$$

and where $\overline{g}^{AB}$ is the inverse of $\overline{g}_{AB}$. The coordinate transformation relating the two coordinate systems (y^μ) and (x^μ) takes the form

$$u = \hat{r} - y^0 \ , \quad r = \hat{r} \ , \quad x^A = \mu^A(y^i/\hat{r}) \ , \quad \text{with} \quad \hat{r} := \sqrt{\sum_i (y^i)^2} \ .$$

The inverse transformation reads

$$y^0 = r - u \ , \quad y^i = r\Theta^i(x^A) \ , \quad \text{with} \quad \sum_i (\Theta^i)^2 = 1 \ .$$

Adapted null coordinates are singular at the vertex of the cone C_O and C^∞ elsewhere. They are convenient to analyze the initial data constraints satisfied by the trace $\overline{g}$ on the light-cone. Note that the space-time metric g will in general not be of the form (2.1) away from C_O. We further remark that adapted null coordinates are not uniquely fixed, for there remains the freedom to redefine the coordinate r (the only restriction being that r is strictly increasing on the generators and that $r = 0$ at the vertex; compare Section 2.2 below), and to choose local coordinates on S^{n-1}.

2.1.2 Generalized wave-map gauge

Let us be given an auxiliary Lorentzian metric $\hat{g}$. A standard method to establish existence, and well-posedness, results for Einstein's vacuum field equations $R_{\mu\nu} = 0$ is a "hyperbolic reduction" where the Ricci tensor is replaced by the *reduced Ricci tensor in $\hat{g}$-wave-map gauge*,

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma \ . \tag{2.2}$$

[4]The degenerate quadratic form denoted here by $\breve{g}$ has been denoted by $\tilde{g}$ in [8, 12]. However, here we will use $\tilde{g}$ to denote the conformally rescaled unphysical metric, as done in most of the literature on the subject.

Here

$$H^\lambda := \Gamma^\lambda - \hat\Gamma^\lambda - W^\lambda \ , \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} \ , \quad \hat\Gamma^\lambda := g^{\alpha\beta}\hat\Gamma^\lambda_{\alpha\beta} \ . \tag{2.3}$$

We use the hat symbol "ˆ" to indicate quantities associated with the *target metric* $\hat g$, while $W^\lambda = W^\lambda(x^\mu, g_{\mu\nu})$ denotes a vector field which is allowed to depend upon the coordinates and the metric g, but not upon derivatives of g.

The *wave-gauge vector* H^λ has been chosen of the above form [8, 15, 16] to remove some second-derivatives terms in the Ricci tensor, so that the *reduced vacuum Einstein equations*

$$R^{(H)}_{\mu\nu} = 0 \tag{2.4}$$

form a system of quasi-linear wave equations for g.

Any solution of (2.4) will provide a solution of the vacuum Einstein equations provided that the so-called *$\hat g$-generalized wave-map gauge condition*

$$H^\lambda = 0 \tag{2.5}$$

is satisfied. In the context of the characteristic initial value problem, the "gauge condition" (2.5) is satisfied by solutions of the reduced Einstein equations if it is satisfied on the initial characteristic hypersurfaces.

The vector field W^λ reflects the freedom to choose coordinates off the cone. Its components can be freely specified, or chosen to satisfy ad hoc equations. Indeed, by a suitable choice of coordinates the gauge source functions W^λ can locally be given any preassigned form, and conversely the W^λ's can be used to determine coordinates by solving wave equations, given appropriate initial data on the cone.

In most of this work we will use a Minkowski target in adapted null coordinates, that is

$$\hat g = \eta \equiv -\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2 s_{AB}\mathrm{d}x^A\mathrm{d}x^B \ , \tag{2.6}$$

where s is the unit round metric on the sphere S^{n-1}.

2.2 The first constraint equation

Set $\ell \equiv \ell^\mu\partial_\mu \equiv \partial_r$. The Raychaudhuri equation $\overline{R}_{\mu\nu}\ell^\mu\ell^\nu \equiv \overline{R}_{11} = 0$ provides a constraining relation between the connection coefficient κ and other geometric objects on C_O, as follows: Recall that the *null second fundamental form* of C_O is defined as

$$\chi_{ij} := \frac{1}{2}(\mathcal{L}_\ell \check g)_{ij} \ ,$$

where $\mathcal{L}$ denotes the Lie derivative. In the adapted coordinates described above we have

$$\chi_{AB} = -\overline{\Gamma}^0_{AB}\nu_0 = \frac{1}{2}\partial_r \overline{g}_{AB} \ , \quad \chi_{11} = 0 \ , \quad \chi_{1A} = 0 \ .$$

The null second fundamental form is sometimes called *null extrinsic curvature* of the initial surface C_O, which is misleading since only objects intrinsic to C_O are involved in its definition.

The *mean null extrinsic curvature* of C_O, or the *divergence* of C_O, which we denote by τ and which is often denoted by θ in the literature, is defined as a trace of χ:

$$\tau := {\chi_A}^A \equiv \overline{g}^{AB}\chi_{AB} \equiv \frac{1}{2}\overline{g}^{AB}\partial_r\overline{g}_{AB} \equiv \partial_r \log\sqrt{\det \check g_{\Sigma_r}} \ . \tag{2.7}$$

It measures the rate of change of area along the null geodesic generators of C_O. Its traceless part,

$$\begin{aligned} \sigma_A{}^B &:= \chi_A{}^B - \frac{1}{n-1}\delta_A{}^B\tau \equiv \overline{g}^{BC}\chi_{AC} - \frac{1}{n-1}\delta_A{}^B\tau & (2.8) \\ &= \frac{1}{2}\gamma^{BC}(\partial_r\gamma_{AC})\breve{}\,, & (2.9) \end{aligned}$$

is known as the *shear* of C_O. In (2.9) the field γ is any representative of the conformal class of $\breve{g}_{\Sigma_r}$, which is sometimes regarded as the free initial data. The addition of the "$\breve{}$"-symbol to a tensor w_{AB} denotes "the trace-free part of":

$$\breve{w}_{AB} := w_{AB} - \frac{1}{n-1}\gamma_{AB}\gamma^{CD}w_{CD}\,. \tag{2.10}$$

We set

$$|\sigma|^2 \quad := \quad \sigma_A{}^B\sigma_B{}^A = -\frac{1}{4}(\partial_r\gamma^{AB})\breve{}(\partial_r\gamma_{AB})\breve{}\,. \tag{2.11}$$

We observe that the shear $\sigma_A{}^B$ depends merely on the conformal class of $\breve{g}_{\Sigma_r}$. This is not true for τ, which is instead in one-to-one correspondence with the conformal factor relating $\breve{g}_{\Sigma_r}$ and γ.

Imposing the generalized wave-map gauge condition $H^\lambda = 0$, the wave-gauge constraint equation induced by $\overline{R}_{11} = 0$ reads [8, equation (6.13)],

$$\partial_r\tau - \underbrace{\Big(\nu^0\partial_r\nu_0 - \frac{1}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0) - \frac{1}{2}\tau\Big)}_{=:\kappa}\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0\,. \tag{2.12}$$

Under the allowed changes of the coordinate r, $r \mapsto \overline{r}(r, x^A)$, with $\partial\overline{r}/\partial r > 0$, $\overline{r}(0, x^A) = 0$, the tensor field g_{AB} transforms as a scalar,

$$\overline{g}_{AB}(\overline{r}, x^C) = g_{AB}(r(\overline{r}, x^C), x^C)\,, \tag{2.13}$$

the field κ changes as a connection coefficient

$$\bar{\kappa} = \frac{\partial r}{\partial \overline{r}}\kappa + \frac{\partial \overline{r}}{\partial r}\frac{\partial^2 r}{\partial \overline{r}^2}\,, \tag{2.14}$$

while τ and σ_{AB} transform as one-forms:

$$\overline{\tau} = \frac{\partial r}{\partial \overline{r}}\tau\,, \quad \overline{\sigma}_{AB} = \frac{\partial r}{\partial \overline{r}}\sigma_{AB}\,. \tag{2.15}$$

The freedom to choose κ is thus directly related to the freedom to reparameterize the generators of C_O. Geometrically, κ describes the acceleration of the integral curves of ℓ, as seen from the identity $\nabla_\ell \ell^\mu = \kappa\ell^\mu$. The choice $\kappa = 0$ corresponds to the requirement that the coordinate r be an affine parameter along the rays. For a given κ the first constraint equation splits into an equation for τ and, once this has been solved, an equation for ν_0.

Once a parameterization of generators has been chosen, we see that the metric function ν_0 is largely determined by the choice of the gauge-source function $\overline{W}^0$ and, in fact, the remaining gauge-freedom in ν_0 can be encoded in $\overline{W}^0$.

2.3 The wave-map gauge characteristic constraint equations

Here we present the whole hierarchical ODE-system of Einstein wave-map gauge constraints induced by the vacuum Einstein equations in a generalized wave-map gauge (cf. [8] for details) for given initial data $([\gamma],\kappa)$ and gauge source-functions $\overline{W}^\lambda$.

The equation (2.12) induced by $\overline{R}_{11}=0$ leads to the equations

$$\partial_r\tau-\kappa\tau+|\sigma|^2+\frac{\tau^2}{n-1} = 0\,, \tag{2.16}$$

$$\partial_r\nu^0+\frac{1}{2}(\overline{W}^0+\overline{\hat{\Gamma}}^0)+\nu^0(\frac{1}{2}\tau+\kappa) = 0\,. \tag{2.17}$$

Equation (2.16) is a Riccati differential equation for τ along each null ray, for $\kappa=0$ it reduces to the standard form of the Raychaudhuri equation. Equation (2.17) is expressed in terms of

$$\nu^0:=\frac{1}{\nu_0}$$

rather than of ν_0, as then it becomes linear. Our aim is to analyze the asymptotic behavior of solutions of the constraints, for this it turns out to be convenient to introduce an auxiliary positive function φ, defined as

$$\tau=(n-1)\partial_r\log\varphi\,, \tag{2.18}$$

which transforms (2.16) into a second-order *linear* ODE,

$$\partial_r^2\varphi-\kappa\partial_r\varphi+\frac{|\sigma|^2}{n-1}\varphi=0\,. \tag{2.19}$$

The function φ is essentially a rewriting of the conformal factor Ω relating $\breve{g}$ and the initial data γ, $\overline{g}_{AB}=\Omega^2\gamma_{AB}$:

$$\Omega=\varphi\left(\frac{\det s}{\det\gamma}\right)^{1/(2n-2)}\,. \tag{2.20}$$

Here $s=s_{AB}\mathrm{d}x^A\mathrm{d}x^B$ denotes the standard metric on S^{n-1}. The initial data symmetric tensor field $\gamma=\gamma_{AB}dx^Adx^B$ is assumed to form a one-parameter family of Riemannian metrics $r\mapsto\gamma(r,x^A)$ on S^{n-1}.

The boundary conditions at the vertex O of the cone for the ODEs occurring in this work follow from the requirement of regularity of the metric there. When imposed, they guarantee that (2.17) and (2.19), as well as all the remaining constraint equations below, have unique solutions. The relevant conditions at the vertex have been computed in regular coordinates and then translated into adapted null coordinates in [8] for a natural family of gauges.

For ν^0 and φ the boundary conditions read

$$\begin{cases}\lim_{r\to0}\nu^0=1\,,\\ \lim_{r\to0}\varphi=0\,,\quad \lim_{r\to0}\partial_r\varphi=1\,.\end{cases}$$

The Einstein equations $\overline{R}_{1A}=0$ imply the equations [8, Equation (9.2)] (compare [12, Equation (3.12)])

$$\frac{1}{2}(\partial_r+\tau)\xi_A-\check{\nabla}_B\sigma_A{}^B+\frac{n-2}{n-1}\partial_A\tau+\partial_A\kappa=0\,, \tag{2.21}$$

where $\check\nabla$ denotes the Riemannian connection defined by $\check g_{\Sigma_r}$, and

$$\xi_A := -2\overline{\Gamma}^1_{1A} .$$

When $\overline{H}^0 = 0$ one has $\overline{H}^A = 0$ if and only if

$$\begin{aligned}\xi_A \quad = \quad & -2\nu^0\partial_r\nu_A + 4\nu^0\nu_B\chi_A{}^B + \nu_A(\overline{W}^0 + \overline{\hat\Gamma}^0) + \overline{g}_{AB}(\overline{W}^B + \overline{\hat\Gamma}^B) \\ & -\gamma_{AB}\gamma^{CD}\check\Gamma^B_{CD} .\end{aligned} \tag{2.22}$$

Here $\check\Gamma^B_{CD}$ are the Christoffel symbols associated to the metric $\check g_{\Sigma_r}$.

Given fields κ and $\overline{g}_{AB} = g_{AB}|_{u=0}$ satisfying the Raychaudhuri constraint equation, the equations (2.21) and (2.22) can be read as hierarchical linear first-order PDE-system which successively determines ξ_A and ν_A by solving ODEs. The boundary conditions at the vertex are

$$\lim_{r\to 0}\nu_A = 0 = \lim_{r\to 0}\xi_A .$$

The remaining constraint equation follows from the Einstein equation $\overline{g}^{AB}\overline{R}_{AB} = 0$ [8, Equations (10.33) & (10.36)],

$$(\partial_r + \tau + \kappa)\zeta + \check R - \frac{1}{2}\xi_A\xi^A + \check\nabla_A\xi^A = 0 , \tag{2.23}$$

where we have set $\xi^A := \overline{g}^{AB}\xi_B$. The function $\check R$ is the curvature scalar associated to $\check g_{\Sigma_r}$. The auxiliary function ζ is defined as

$$\zeta := (2\partial_r + \tau + 2\kappa)\overline{g}^{11} + 2\overline{W}^1 + 2\overline{\hat\Gamma}^1 , \tag{2.24}$$

and satisfies, if $\overline{H}^\lambda = 0$, the relation $\zeta = 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}$. The term $\overline{\hat\Gamma}^1$ depends upon the target metric chosen, and with our current Minkowski target $\hat g = \eta$ we have

$$\overline{\hat\Gamma}^1 = \overline{\hat\Gamma}^0 = -r\overline{g}^{AB}s_{AB} . \tag{2.25}$$

Taking the relation

$$\overline{g}^{11} = (\nu^0)^2(\nu^A\nu_A - \overline{g}_{00}) \tag{2.26}$$

into account, the definition (2.24) of ζ becomes an equation for $\overline{g}_{00}$ once ζ has been determined. The boundary conditions for (2.23) and (2.24) are

$$\lim_{r\to 0}\overline{g}^{11} = 1 , \quad \lim_{r\to 0}(\zeta + 2r^{-1}) = 0 .$$

2.4 Global solutions

A prerequisite for obtaining asymptotic expansions is existence of solutions of the constraint equations defined for all r. The question of globally defined data becomes trivial when all metric components are prescribed on C_O: Then the only condition is that τ, as calculated from $\overline{g}_{AB}$, is strictly positive. Now, as is well known, and will be rederived shortly in any case, negativity of τ implies formation of conjugate points in finite affine time, or geodesic incompleteness of the generators. In this work we will only be interested in light-cones C_O which are globally smooth (except, of course, at the vertex), and extending all the way to conformal infinity. Such cones have complete generators without conjugate points, and so τ must remain positive. But then one can solve algebraically the Raychaudhuri equation to globally determine κ.

We note that the function τ depends upon the choice of parameterisation of the generators, but its sign does not, hence the above discussion applies regardless of that choice. Recall that we assume that the tip of the cone corresponds to $r \to 0$ and that the condition that $\kappa = O(r^{-3})$ ensures that an affine parameter along the generators tends to infinity for $r \to \infty$, so that the parameterization of r covers the whole cone from O to null infinity.

In some situations it might be convenient to request that κ vanishes, or takes some prescribed value. In this case the Raychaudhuri equation becomes an equation for the function φ, and the question of its global positivity arises.

Recall that the initial conditions for φ at the vertex are $\varphi(0) = 0$ and $\partial_r\varphi(0) = 1$, and so both $\partial_r\varphi$ and φ are positive near zero. Now, (2.19) with $\kappa = 0$ shows that φ is concave as long as it is non-negative; equivalently, $\partial_r\varphi$ is non-increasing in the region where $\varphi > 0$. An immediate consequence of this is that if $\partial_r\varphi$ becomes negative at some $r_0 > 0$, then it stays so, with φ vanishing for some $r_0 < r_1 < \infty$, i.e. after some finite affine parameter time. We recover the result just mentioned, that negativity of $\partial_r\varphi$ indicates incompleteness, or occurrence of conjugate points, or both. In the first case the solution will not be defined for all affine parameters r, in the second C_O will fail to be smooth for $r > r_1$ by standard results on conjugate points. Since the sign of $\partial_r\varphi$ is invariant under orientation-preserving reparameterisations, we conclude that:

PROPOSITION 2.1 *Globally smooth and null-geodesically-complete light-cones must have $\partial_r\varphi$ positive.*

A set of conditions guaranteeing global existence of positive solutions of the Raychaudhuri equation, viewed as an equation for φ, has been given in [8, Theorem 7.3]. Here we shall give an alternative simpler criterion, as follows:

Suppose, first, that $\kappa = 0$. Integration of (2.19) gives

$$\partial_r\varphi(r, x^A) = 1 - \frac{1}{n-1}\int_0^r \big(\varphi|\sigma|^2\big)(\tilde r, x^A)\,\mathrm{d}\tilde r \le 1 \qquad (2.27)$$

as long as φ remains positive. Since $\varphi(0) = 0$, we see that we always have

$$\varphi(r, x^A) \le r$$

in the region where φ is positive, and in that region it holds

$$\begin{aligned}\partial_r\varphi(r, x^A) &\ge 1 - \frac{1}{n-1}\int_0^r \tilde r\,|\sigma(\tilde r, x^A)|^2 \mathrm{d}\tilde r\\ &\ge 1 - \frac{1}{n-1}\int_0^\infty \tilde r\,|\sigma(\tilde r, x^A)|^2 \mathrm{d}\tilde r\ .\end{aligned}$$

This implies that φ is strictly increasing if

$$\int_0^\infty r|\sigma|^2\,\mathrm{d}r < n-1\ . \qquad (2.28)$$

Since φ is positive for small r it remains positive as long as $\partial_r\varphi$ remains positive, and so global positivity of φ is guaranteed whenever (2.28) holds.

A rather similar analysis applies to the case $\kappa \neq 0$, in which we set

$$H(r,x^A) := \int_0^r \kappa(\tilde r, x^A)\mathrm{d}\tilde r \; . \tag{2.29}$$

Let

$$\varphi(r) = \mathring{\varphi}(s(r)) \; , \quad \text{where} \quad s(r) := \int_0^r e^{H(\hat r)}\mathrm{d}\hat r \; , \tag{2.30}$$

the x^A-dependence being implicit. The function $s(r)$ is strictly increasing with $s(0) = 0$. If we assume that κ is continuous in r with $\kappa(0) = 0$, defined for all r and, e.g.,

$$\int_0^\infty \kappa > -\infty \; , \tag{2.31}$$

then $\lim_{r\to\infty} s(r) = +\infty$, and thus the function $r \mapsto s(r)$ defines a differentiable bijection from $\mathbb{R}^+$ to itself. Consequently, a differentiable inverse function $s \mapsto r(s)$ exists, and is smooth if κ is.

Expressed in terms of (2.30), (2.19) becomes

$$\partial_s^2\mathring{\varphi}(s) + e^{-2H(r(s))}\frac{|\sigma|^2(r(s))}{n-1}\mathring{\varphi}(s) = 0 \; . \tag{2.32}$$

A global solution $\varphi > 0$ of (2.19) exists if and only if a global solution $\mathring{\varphi} > 0$ of (2.32) exists. It follows from the considerations above (note that $\mathring{\varphi}(s=0) = 0$ and $\partial_s\mathring{\varphi}(s=0) = 1$) that a sufficient condition for global existence of positive solutions of (2.32) is

$$\begin{aligned} &\int_0^\infty s e^{-2H(r(s))}|\sigma|^2(r(s))\,\mathrm{d}s < n-1 \\ &\Longleftrightarrow \int_0^\infty \Big(\int_0^r e^{H(\hat r)}\mathrm{d}\hat r\Big)e^{-H(r)}|\sigma|^2(r)\,\mathrm{d}r < n-1 \; . \end{aligned} \tag{2.33}$$

Consider now the question of positivity of ν^0. In the $\kappa = 0$-wave-map gauge with Minkowski metric as a target we have (see [10, Equation (4.7)])

$$\nu^0(r,x^A) = \frac{\varphi^{-(n-1)/2}(r,x^A)}{2}\int_0^r \Big(\hat r \varphi^{(n-1)/2}\overline{g}^{AB}s_{AB}\Big)(\hat r, x^A)\,\mathrm{d}\hat r \; . \tag{2.34}$$

In an s-orthonormal coframe $\theta^{(A)}$, $\overline{g}^{AB}s_{AB}$ is the sum of the diagonal elements $\overline{g}^{(A)(A)} = \overline{g}^\sharp(\theta^{(A)},\theta^{(A)})$, $A = 1,\ldots,n-1$, where $\overline{g}^\sharp$ the scalar product on $T^*\Sigma_r$ associated to $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$, each of which is positive in Riemannian signature. Hence

$$\overline{g}^{AB}s_{AB} > 0 \; .$$

So, for globally positive φ we obtain a globally defined strictly positive ν^0, hence also a globally defined strictly positive $\nu_0 \equiv 1/\nu^0$.

When $\kappa \neq 0$, and allowing further a non-vanishing W^0, we find instead

$$\begin{aligned} \nu^0(r,x^A) \;&=\; \frac{\big(e^{-H}\varphi^{-(n-1)/2}\big)(r,x^A)}{2} \times \\ &\int_0^r \Big(e^H\varphi^{(n-1)/2}(\hat r\overline{g}^{AB}s_{AB} - \overline{W}^0)\Big)(\hat r, x^A)\,\mathrm{d}\hat r \; , \end{aligned} \tag{2.35}$$

with H as in (2.29). If $\overline{W}^0 = 0$ we obtain positivity as before. More generally, we see that a necessary-and-sufficient condition for positivity of ν^0 is positivity of the integral in the last line of (2.35) for all r. This will certainly be the case if the gauge-source function $\overline{W}^0$ satisfies

$$\overline{W}^0 < r\overline{g}^{AB}s_{AB} = r\varphi^{-2}\Big(\frac{\det\gamma}{\det s}\Big)^{1/(n-1)}\gamma^{AB}s_{AB} \ . \tag{2.36}$$

Summarising we have proved:

PROPOSITION 2.2 *1. Solutions of the Raychaudhuri equation with prescribed κ and σ are global when (2.31) and (2.33) hold, and lead to globally positive functions φ and τ.*

2. Any global solution of the Raychaudhuri equation with $\varphi > 0$ leads to a globally defined positive function ν_0 when the gauge source function $\overline{W}^0$ satisfies (2.36). This condition will be satisfied for any $\overline{W}^0 \le 0$.

2.5 Positivity of φ_{-1} and $(\nu^0)_0$

For reasons that will become clear in Section 4, we are interested in fields φ and ν_0 which, for large r, take the form

$$\varphi(r,x^A) = \varphi_{-1}(x^A)r + o(r) \ , \quad \nu^0(r,x^A) = (\nu^0)_0(x^A) + o(1) \ , \tag{2.37}$$

with φ_{-1} and $(\nu^0)_0$ positive. The object of this section is to provide conditions which guarantee existence of such expansions, assuming a global positive solution φ.

Let us further assume that $e^{-2H}\varphi|\sigma|^2$ is continuous in r with

$$\int_0^\infty \big(e^{-2H}\varphi|\sigma|^2\big)\big|_{r=r(s)}\mathrm{d}s = \int_0^\infty e^{-H}\varphi|\sigma|^2\mathrm{d}r < \infty \ .$$

Integration of (2.32) and de l'Hospital rule at infinity give

$$\mathring{\varphi}_{-1} := \lim_{s\to\infty}\frac{\mathring{\varphi}(s)}{s} = \lim_{s\to\infty}\partial_s\mathring{\varphi}(s) = 1 - \frac{1}{n-1}\int_0^\infty e^{-H}\varphi|\sigma|^2\mathrm{d}r \ . \tag{2.38}$$

This will be strictly positive if e.g. (2.33) holds, as

$$\begin{aligned}\int_0^r e^{H(\tilde r)}\mathrm{d}\tilde r - \varphi(r) &= \int_0^r (e^{H(\tilde r)} - \partial_{\tilde r}\varphi(\tilde r))\mathrm{d}\tilde r \\ &= \int_0^{r(s)} \underbrace{(1-\partial_{\tilde s}\mathring{\varphi}(\tilde s))}_{\ge 0 \text{ by } (2.27)}\mathrm{d}\tilde s \ \ge\ 0 \ ,\end{aligned}$$

and thus by (2.38) and (2.33)

$$\mathring{\varphi}_{-1} \ \ge\ 1 - \frac{1}{n-1}\int_0^\infty \Big(\int_0^r e^{H(\hat r)}\mathrm{d}\hat r\Big)e^{-H(r)}|\sigma|^2(r)\,\mathrm{d}r \ >\ 0 \ .$$

One can now use (2.38) to obtain (2.37) if we assume that the integral of κ over r converges:

$$\forall x^A \qquad -\infty < \beta(x^A) := \int_0^\infty \kappa(r,x^A)\mathrm{d}r < \infty \ , \tag{2.39}$$

so that

$$\int_0^r \kappa(s,\cdot)\mathrm{d}s = \beta(\cdot) + o(1) \ . \tag{2.40}$$

Indeed, it follows from (2.39) that there exists a constant C such that the parameter s defined in (2.30) satisfies

$$C^{-1} \le \frac{\partial s}{\partial r} \le C \ , \quad C^{-1} r \le s \le Cr \ , \quad \lim_{r\to\infty} \frac{\partial s}{\partial r} = e^{\beta} \ . \tag{2.41}$$

We then have

$$\begin{aligned}\varphi_{-1} &= \lim_{r\to\infty} \frac{\varphi(r)}{r} = \lim_{s\to\infty} \frac{\mathring{\varphi}(s)}{r(s)} = \lim_{s\to\infty} \frac{\partial_s \mathring{\varphi}(s)}{\partial_s r(s)} = e^{-\beta} \mathring{\varphi}_{-1} \\ &= e^{-\beta} \left(1 - \frac{1}{n-1} \int_0^\infty e^{-H} \varphi |\sigma|^2 \mathrm{d}r \right) \ . \end{aligned} \tag{2.42}$$

We have proved:

PROPOSITION 2.3 *Suppose that (2.31), (2.33) and (2.39) hold. Then the function φ is globally positive with $\varphi_{-1} > 0$.*

Consider, next, the asymptotic behaviour of ν_0. In addition to (2.39), we assume now that $\varphi = \varphi_{-1} r + o(r)$, for some function of the angles φ_{-1}, and that there exists a bounded function of the angles α such that

$$r \overline{g}^{AB} s_{AB} - \overline{W}^0 = \frac{\alpha}{r} + o(r^{-1}) \ . \tag{2.43}$$

Passing to the limit $r \to \infty$ in (2.35) one obtains

$$\nu^0(r, x^A) \quad = \quad \frac{\alpha(x^A)}{n-1} + o(1) \ .$$

We see thus that

$$(\nu^0)_0 > 0 \quad \Longleftrightarrow \quad \alpha > 0 \ , \qquad (\nu_0)_0 > 0 \quad \Longleftrightarrow \quad \alpha < \infty \ .$$

REMARK 2.4 Note that (2.39) and (2.43) will hold with smooth functions α and β when the a priori restrictions (4.8)-(4.10), discussed below, are satisfied and when both φ and φ_{-1} are positive. Recall also that if $\overline{W}^0 \le 0$ (in particular, if $\overline{W}^0 \equiv 0$), then the condition $\alpha \ge 0$ follows from the fact that both s_{AB} and $\overline{g}_{AB}$ are Riemannian.

So far we have justified the expansion (2.37). For the purposes of Section 3 we need to push the expansion one order further. This is the contents of the following:

PROPOSITION 2.5 *Suppose that there exists a Riemannian metric $(\gamma_{AB})_{-2} \equiv (\gamma_{AB})_{-2}(x^C)$ and a tensor field $(\gamma_{AB})_{-1} \equiv (\gamma_{AB})_{-1}(x^C)$ on S^{n-1} such that for large r we have*

$$\gamma_{AB} = r^2 (\gamma_{AB})_{-2} + r(\gamma_{AB})_{-1} + o(r) \ , \tag{2.44}$$

$$\partial_r \big(\gamma_{AB} - r^2 (\gamma_{AB})_{-2} - r(\gamma_{AB})_{-1} \big) = o(1) \ , \tag{2.45}$$

$$\int_0^r \kappa(s, x^A)\mathrm{d}s = \beta_0(x^A) + \beta_1(x^A) r^{-2} + o(r^{-2}) \ . \tag{2.46}$$

Assume moreover that φ exists for all r, with $\varphi > 0$. Then:

1. *There exist bounded functions of the angles $\varphi_{-1} \geq 0$ and φ_0 such that*

$$\varphi(r) = \varphi_{-1} r + \varphi_0 + O(r^{-1}) . \tag{2.47}$$

2. *If, in addition, ν_0 exists for all r, if it holds that $\varphi_{-1} > 0$ and if $\overline{W}^0$ takes the form $\overline{W}^0(r, x^A) = (\overline{W}^0)_1(x^A) r^{-1} + o(r^{-1})$ with*

$$(\overline{W}^0)_1 < s_{AB}(\overline{g}^{AB})_2 = (\varphi_{-1})^{-2}\Big(\frac{\det \gamma_{-2}}{\det s}\Big)^{1/(n-1)} \gamma_{-2}^{AB} s_{AB} , \tag{2.48}$$

then

$$0 < (\nu_0)_0 < \infty .$$

Remark 2.6 If the space-time is not vacuum, then (2.32) becomes

$$\partial_s^2 \mathring{\varphi}(s) + e^{-2H(r(s))} \frac{\big(|\sigma|^2 + \overline{R}_{rr}\big)(r(s))}{n-1} \mathring{\varphi}(s) = 0 . \tag{2.49}$$

and the conclusions of Proposition 2.5 remain unchanged if we assume in addition that

$$\overline{R}_{rr} = O(r^{-4}) . \tag{2.50}$$

Proof: From (2.11) one finds

$$|\sigma|^2 = O(r^{-4}) .$$

We have already seen that

$$\mathring{\varphi} = \mathring{\varphi}_{-1} s + o(s) .$$

Plugging this in the second term in (2.32) and integrating shows that

$$\partial_s \mathring{\varphi}(s) = \mathring{\varphi}_{-1} + O(s^{-2}) , \quad \mathring{\varphi}(s) = \mathring{\varphi}_{-1} s + \mathring{\varphi}_0 + O(s^{-1}) .$$

A simple analysis of the equation relating r with s gives now

$$\partial_r \varphi(r) = \varphi_{-1} + O(r^{-2}) , \quad \varphi(r) = \varphi_{-1} r + \varphi_0 + O(r^{-1}) .$$

This establishes point 1.

When φ_{-1} is positive one finds that (2.43) holds, and from what has been said the result follows. □

3 A no-go theorem for the ($\kappa = 0$, $\overline{W}^0 = 0$)-wave-map gauge

Rendall's proposal, to solve the characteristic Cauchy problem using the ($\kappa = 0$, $\overline{W}^\mu = 0$)-wave-map gauge, has been adopted by many authors. The object of this section is to show that, in $3+1$ dimensions, this approach will always lead to logarithmic terms in an asymptotic expansion of the metric *except for the Minkowski metric*. This makes clear the need to allow non-vanishing gauge-source functions $\overline{W}^\mu$.

More precisely, we prove (compare [6]):

THEOREM 3.1 *Consider a four-dimensional vacuum space-time $(\mathscr{M}, g)$ which has a conformal completion at future null infinity $(\mathscr{M} \cup \mathscr{I}^+, \tilde{g})$ with a C^3 conformally rescaled metric, and suppose that there exists a point $O \in \mathscr{M}$ such that $\overline{C}_O \setminus \{O\}$, where $\overline{C}_O$ denotes the closure of C_O in $\mathscr{M} \cup \mathscr{I}^+$, is a smooth hypersurface in the conformally completed space-time. If the metric g has no logarithmic terms in its asymptotic expansion for large r in the $\overline{W}^0 = 0$ wave-map gauge, where r is an affine parameter on the generators of C_O, then $(\mathscr{M}, g)$ is the Minkowski space-time.*

PROOF: Let $S \subset \mathscr{I}^+$ denote the intersection of $\overline{C}_O$ with $\mathscr{I}^+$. Elementary arguments show that $\overline{C}_O$ intersects $\mathscr{I}^+$ transversally and that S is diffeomorphic to S^2. Introduce near S coordinates so that S is given by the equation $\{u = 0 = x\}$, where x is an $\tilde{g}$-affine parameter along the generators of $\overline{C}_O$, with $x = 0$ at S, while the x^A's are coordinates on S in which the metric induced by $\breve{g}$ is manifestly conformal to the round-unit metric $s_{AB}\mathrm{d}x^A\mathrm{d}x^B$ on S^2. (Note that this construction might lead to the loss of one derivative of the metric.) The usual calculation shows that the g-affine parameter r along the generators of $\overline{C}_O$ equals $a(x^A)/x$ for some positive function of the angles $a(x^A)$. Discarding strictly positive conformal factors, we conclude that for large r the tensor field $\breve{g}$ is conformal to a tensor field $\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ satisfying

$$\gamma_{AB} = r^2\big(s_{AB} + (\gamma_{AB})_{-1}r^{-1} + o(r^{-1})\big) \ , \tag{3.1}$$

$$\partial_r\big(\gamma_{AB} - r^2 s_{AB} - r(\gamma_{AB})_{-1}\big) = o(1) \ . \tag{3.2}$$

The result follows now immediately from [7] and from our next Theorem 3.2. □

THEOREM 3.2 *Suppose that the space-dimension n equals three. Let $r|\sigma|$, $r\overline{W}^0$ and $r^2\overline{R}_{\mu\nu}\ell^\mu\ell^\nu$ be bounded for small r. Suppose that $\gamma_{AB}(r, x^A)$ is positive definite for all $r > 0$ and admits the expansion (3.1)-(3.2), for large r with the coefficients in the expansion depending only upon x^C. Assume that the first constraint equation (2.19) with $\kappa = 0$ and*

$$0 \le \overline{R}_{\mu\nu}\ell^\mu\ell^\nu = O(r^{-4})$$

has a globally defined positive solution satisfying $\varphi(0) = 0$, $\partial_r\varphi(0) = 1$, $\varphi > 0$, and $\varphi_{-1} > 0$. Then there are no logarithmic terms in the asymptotic expansion of ν^0 in a gauge $\kappa = 0$ and $\overline{W}^0 = o(r^{-2})$ (for large r) if and only if

$$\sigma \equiv 0 \equiv \overline{R}_{\mu\nu}\ell^\mu\ell^\nu \ .$$

PROOF OF THEOREM 3.2: At the heart of the proof lies the following observation:

LEMMA 3.3 *In space-dimension n, suppose that $\kappa = 0$ and set*

$$\Psi = r^2 \exp\Big(\int_0^r \big(\frac{\tau + \tau_1}{2} - \frac{n-1}{r}\big)\,\mathrm{d}r\Big) \ . \tag{3.3}$$

with $\tau_1 \equiv (n-1)/r$ We have $\tau = (n-1)r^{-1} + \tau_2 r^{-2} + o(r^{-2})$, where

$$\tau_2 := -\lim_{r\to\infty} r^2\Psi^{-1} \times \int_0^r (|\sigma|^2 + \overline{R}_{\mu\nu}\ell^\mu\ell^\nu)\Psi\,\mathrm{d}r \ , \tag{3.4}$$

provided that the limit exists.

PROOF: Let $\delta\tau = \tau - \tau_1$. It follows from the Raychaudhuri equation with $\kappa = 0$ that $\delta\tau$ satisfies the equation

$$\frac{\mathrm{d}\delta\tau}{\mathrm{d}r} + \frac{\tau + \tau_1}{2}\delta\tau = -|\sigma|^2 - 8\pi\overline{T}_{rr} \; .$$

Solving, one finds

$$\begin{aligned} \delta\tau &= -\Psi^{-1}\int_0^r (|\sigma|^2 + 8\pi\overline{T}_{rr})\Psi \, \mathrm{d}r \\ &= \frac{\tau_2}{r^2} + o(r^{-2}) \; , \end{aligned}$$

as claimed. □

Let us return to the proof of Theorem 3.2. Proposition 2.5 and Remark 2.6 show that

$$\varphi(r, x^A) = \varphi_{-1}(x^A)r + \varphi_0(x^A) + o(r^{-1}) \; , \tag{3.5}$$

$$\tau \equiv 2\partial_r \log\varphi = 2r^{-1} - 2\varphi_0(\varphi_{-1})^{-1}r^{-2} + o(r^{-2}) \; . \tag{3.6}$$

Recall, next, the solution formula (2.34) for the constraint equation (2.17) with $\kappa = 0$ and $n = 3$:

$$\nu^0(r, x^A) = \frac{1}{2\varphi(r, x^A)}\int_0^r \varphi\left(s\overline{g}^{AB}s_{AB} - \overline{W}^0\right)(s, x^A)\, \mathrm{d}s \; . \tag{3.7}$$

From (3.5)-(3.6) one finds

$$\overline{g}^{AB} = r^{-2}(\varphi_{-1})^{-2}[s^{AB} + r^{-1}(\tau_2 s^{AB} - \breve{\gamma}^{AB}_{-1}) + o(r^{-1})] \; , \tag{3.8}$$

with

$$\breve{\gamma}^{AB}_{-1} := s^{AC}s^{BD}[(\gamma_{CD})_{-1} - \frac{1}{2}s_{CD}s^{EF}(\gamma_{EF})_{-1}] \; .$$

Inserting this into (3.7), and assuming that $\overline{W}^0 = o(r^{-2})$, one finds for large r

$$\nu^0 = (\varphi_{-1})^{-2} + \frac{1}{2}\tau_2(\varphi_{-1})^{-2}\frac{\ln r}{r} + O(r^{-1}) \; , \tag{3.9}$$

with the coefficient of the logarithmic term vanishing if and only if $\tau_2 = 0$ when a bounded positive coefficient φ_{-1} exists. One can check that the hypotheses of Lemma 3.3 are satisfied, and the result follows. □

4 Preliminaries to solve the constraints asymptotically

4.1 Notation and terminology

Consider a metric which has a smooth, or polyhomogeneous, conformal completion at infinity *à la Penrose*, and suppose that the closure (in the completed space-time) $\overline{\mathscr{N}}$ of a null hypersurface $\mathscr{N}$ of O meets $\mathscr{I}^+$ in a smooth sphere. One can then introduce Bondi coordinates (u, r, x^A) near $\mathscr{I}^+$, with $\overline{\mathscr{N}} \cap \mathscr{I}^+$ being the level set of a Bondi retarded coordinate u (see [29] in the smooth case, and [11, Appendix B] in the polyhomogeneous case). The resulting Bondi area coordinate r behaves as $1/\Omega$, where Ω is the compactifying factor. If one uses Ω as one of the coordinates near $\mathscr{I}^+$, say x, and chooses $1/x$ as a parameter along the generators of $\mathscr{N}$, one is led to an asymptotic behaviour of the metric which is captured by the following definition:

DEFINITION 4.1 We say that a smooth metric tensor $\overline{g}_{\mu\nu}$ defined on a null hypersurface $\mathscr{N}$ given in adapted null coordinates has a *smooth conformal completion at infinity* if the unphysical metric tensor field $\overline{\tilde{g}}_{\mu\nu}$ obtained via the coordinate transformation $r \mapsto 1/r =: x$ and the conformal rescaling $\overline{g} \mapsto \overline{\tilde{g}} \equiv x^2\overline{g}$ is, as a Lorentzian metric, smoothly extendable across $\{x = 0\}$. We will say that $\overline{g}_{\mu\nu}$ is *polyhomogeneous* if the conformal extension obtained as above is polyhomogeneous at $\{x = 0\}$, see Appendix A.

The components of a smooth tensor field on $\mathscr{N}$ will be said to be *smooth at infinity*, respectively *polyhomogeneous at infinity*, whenever they admit, in the (x, x^A)-coordinates, a smooth, respectively polyhomogeneous, extension in the conformally rescaled space-time across $\{x = 0\}$.

REMARK 4.2 The reader is warned that the definition contains an implicit restriction, that $\mathscr{N}$ *is a smooth hypersurface in the conformally completed space-time.* In the case of a light-cone, this excludes existence of points which are conjugate to O both in $\mathscr{M}$ and on $\overline{C}_O \cap \mathscr{I}^+$.

We emphasise that Definition 4.1 concerns only fields on $\mathscr{N}$, and *no assumptions are made concerning existence, or properties, of an associated space-time.* In particular there *might not* be an associated space-time; and if there is one, it *might or might not* have a smooth completion through a conformal boundary at null infinity.

The conditions of the definition are both conditions on the metric and on the coordinate system. While the definition restricts the class of parameters r, there remains considerable freedom, which will be exploited in what follows. It should be clear that the existence of a coordinate system as above on a globally-smooth light-cone is a necessary condition for a space-time to admit a smooth conformal completion at null infinity, for points O such that $\overline{C_O} \cap \mathscr{I}^+$ forms a smooth hypersurface in the conformally completed space-time.

Consider a real-valued function

$$f : (0, \infty) \times S^{n-1} \longrightarrow \mathbb{R} , \quad (r, x^A) \longmapsto f(r, x^A) .$$

If this function admits an asymptotic expansion in terms of powers of r (whether to finite or arbitrarily high order) we denote by f_n, or $(f)_n$, the coefficient of r^{-n} in the expansion.

We will write $f = \mathcal{O}(r^N)$ (or $f = \mathcal{O}(x^{-N})$, $x \equiv 1/r$), $N \in \mathbb{Z}$ if the function

$$F(x, \cdot) := x^N f(x^{-1}, \cdot) \tag{4.1}$$

is smooth at $x = 0$. We emphasize that this is a restriction on f for large r, and the condition does not say anything about the behaviour of f near the vertex of the cone (whenever relevant), where r approaches zero.

We write

$$f(r, x^A) \sim \sum_{k=-N}^{\infty} f_k(x^A) r^{-k}$$

if the right-hand side is the asymptotic expansion at $x = 0$ of the function $x \mapsto r^{-N} f(r,\cdot)|_{r=1/x}$, compare Appendix A.

The next lemma summarizes some useful properties of the symbol $\mathcal{O}$:

LEMMA 4.3 *Let $f = \mathcal{O}(r^N)$ and $g = \mathcal{O}(r^M)$ with $N, M \in \mathbb{Z}$.*

1. *f can be asymptotically expanded as a power series starting from r^N,*

$$f(r, x^A) \sim \sum_{k=-N}^{\infty} f_k(x^A) r^{-k}$$

for some suitable smooth functions $f_k : S^{n-1} \to \mathbb{R}$.

2. *The n-th order derivative, $n \geq 0$, satisfies*

$$\partial_r^n f(r, x^A) = \begin{cases} \mathcal{O}(r^{N-n}) & \text{for } N < 0 \\ \mathcal{O}(r^{N-n}) & \text{for } N \geq 0 \text{ and } N - n \geq 0 \\ \mathcal{O}(r^{N-n-1}) & \text{for } N \geq 0 \text{ and } N - n \leq -1 \end{cases}$$

as well as

$$\partial_A^n f(r, x^B) = \mathcal{O}(r^N) \ .$$

3. *$f^n g^m = \mathcal{O}(r^{nN+mM})$ for all $n, m \in \mathbb{Z}$.*

4.2 Some a priori restrictions

In order to solve the constraint equations asymptotically and derive necessary-and-sufficient conditions concerning smoothness of the solutions at infinity in adapted coordinates, it is convenient to have some a priori knowledge regarding the lowest admissible orders of certain functions appearing in these equations, and to exclude the appearance of logarithmic terms in the expansions of fields such as ξ_A and $\overline{W}^\lambda$. Let us therefore derive the necessary restrictions on the metric, the gauge source functions, etc. needed to end up with a trace of a metric on the light-cone which admits a smooth conformal completion at infinity.

4.2.1 Non-vanishing of φ and ν^0

As described above, the Einstein wave-map gauge constraints can be represented as a system of *linear* ODEs for φ, ν^0, ν_A and $\overline{g}^{11}$, so that existence and uniqueness (with the described boundary conditions) of global solutions is guaranteed if the coefficients in the relevant ODEs are globally defined. Indeed, we have to make sure that the resulting symmetric tensor field $\overline{g}_{\mu\nu}$ does not degenerate, so that it represents a regular Lorentzian metric in the respective adapted null coordinate system. In a setting where the starting point are conformal data $\gamma_{AB}(r,\cdot)\mathrm{d}x^A\mathrm{d}x^B$ which define a Riemannian metric for all $r > 0$, this will be the case if and only if φ and ν^0 are nowhere vanishing, in fact *strictly positive* in our conventions,

$$\varphi > 0 \ , \ \nu^0 > 0 \quad \forall\, r > 0 \ . \tag{4.2}$$

4.2.2 A priori restrictions on $\overline{g}_{\mu\nu}$

Assume that $\overline{g}_{\mu\nu}$ admits a smooth conformal completion in the sense of Definition 4.1. Then its conformally rescaled counterpart $\overline{\tilde{g}}_{\mu\nu} \equiv x^2\overline{g}_{\mu\nu}$ satisfies

$$\overline{\tilde{g}}_{\mu\nu} = \mathcal{O}(1) \quad \text{with} \quad \overline{\tilde{g}}_{0x}|_{x=0} > 0 \;, \quad \det \overline{\tilde{g}}_{AB}|_{x=0} > 0 \;. \tag{4.3}$$

This imposes the following restrictions on the admissible asymptotic form of the components $g_{\mu\nu}$ in adapted null coordinates $(u, r \equiv 1/x, x^A)$:

$$\nu_0 \;=\; \mathcal{O}(1) \;, \quad \nu_A \;=\; \mathcal{O}(r^2) \;, \quad \overline{g}_{00} \;=\; \mathcal{O}(r^2) \;, \quad \overline{g}_{AB} \;=\; \mathcal{O}(r^2) \;, \tag{4.4}$$

with

$$(\nu_0)_0 > 0 \quad \text{and} \quad (\det \check{g}_{\Sigma_r})_{-4} > 0 \;. \tag{4.5}$$

Moreover,

$$\tau \;\equiv\; \frac{1}{2}\overline{g}^{AB}\partial_r\overline{g}_{AB} \;=\; \frac{n-1}{r} + \mathcal{O}(r^{-2}) \;, \tag{4.6}$$

and (recall that $\tau = (n-1)\partial_r \log\varphi$)

$$\varphi \;=\; \varphi_{-1}r + \mathcal{O}(1) \quad \text{for some positive function } \varphi_{-1} \text{ on } S^{n-1}. \tag{4.7}$$

Indeed assuming that φ_{-1} vanishes for some x^A, the function φ does not diverge as r goes to infinity along some null ray Υ emanating from O, i.e. $\varphi|_\Upsilon = \mathcal{O}(1)$ and $\det \check{g}_{\Sigma_r}|_\Upsilon \equiv (\varphi^{2(n-1)} \det s)|_\Upsilon = \mathcal{O}(1)$, which is incompatible with (4.5).

The assumptions $\varphi(r, x^A) > 0$ and $\varphi_{-1}(x^A) > 0$ imply the non-existence of conjugate points on the light-cone up-to-and-including conformal infinity.

4.2.3 A priori restrictions on gauge source functions

Assume that there exists a smooth conformal completion of the metric, as in Definition 4.1. We wish to find the class of gauge functions κ and $\overline{W}^\mu$ which are compatible with this asymptotic behaviour.

The relation $\overline{g}_{AB} = \mathcal{O}(r^2)$ together with $\partial_r = -x^2\partial_x$ and the definition (2.8) implies

$$\sigma_A{}^B = \mathcal{O}(r^{-2}), \quad |\sigma|^2 = \mathcal{O}(r^{-4}). \tag{4.8}$$

Using the estimate (4.7) for τ and the Raychaudhuri equation (2.16) we find

$$\kappa = \mathcal{O}(r^{-3}) \;, \tag{4.9}$$

where cancellations in both the leading and the next-to-leading terms in (2.16) have been used. Then (2.17), (4.4), (4.7) and (4.9) imply

$$\overline{W}^0 = \mathcal{O}(r^{-1}) \;. \tag{4.10}$$

Similarly to $\kappa = \overline{\Gamma}^r_{rr}$, ξ_A corresponds to the restriction to C_O of certain connection coefficients (cf. [8, 12])

$$\xi_A = -2\overline{\Gamma}^r_{rA} \;.$$

We will use this equation to determine the asymptotic behaviour of ξ_A; the main point is to show that there needs to exist a gauge in which ξ_A has no logarithmic terms. We note that the argument here requires assumptions about the whole space-time metric and some of its derivatives transverse to the characteristic initial surface, rather than on $\overline{g}_{AB}$.

A necessary condition for the space-time metric to be smoothly extendable across $\mathscr{I}^+$ is that the Christoffel symbols of the unphysical metric $\tilde{g}$ in coordinates $(u, x \equiv 1/r, x^A)$ are smooth at $\mathscr{I}^+$, in particular

$$\overline{\tilde{\Gamma}}^x_{xA} = \mathcal{O}(1) \; . \tag{4.11}$$

The formula for the transformation of Christoffel symbols under conformal rescalings of the metric, $\tilde{g} = \Theta^2 g$, reads

$$\tilde{\Gamma}^\rho_{\mu\nu} \quad = \quad \Gamma^\rho_{\mu\nu} + \frac{1}{\Theta}\left(\delta_\nu{}^\rho \partial_\mu \Theta + \delta_\mu{}^\rho \partial_\nu \Theta - g_{\mu\nu} g^{\rho\sigma} \partial_\sigma \Theta\right) \; ,$$

and shows that (4.11) is equivalent to

$$\overline{\Gamma}^x_{xA} = \mathcal{O}(1) \; , \quad \text{or} \quad \overline{\Gamma}^r_{rA} = \mathcal{O}(1) \; ; \tag{4.12}$$

the second equation is obtained from the first one using the transformation law of the Christoffel symbols under the coordinate transformation $x \mapsto r \equiv 1/x$. Hence $\xi_A = \mathcal{O}(1)$. Inspection of the leading-order terms in (2.21) leads now to

$$\xi_A = \mathcal{O}(r^{-1}) \; . \tag{4.13}$$

One can insert all this into (2.22), viewed as an equation for $\overline{W}^A$, to obtain

$$\overline{W}^A = \mathcal{O}(r^{-1}) \; .$$

We note the formula

$$\zeta = \overline{2 g^{AB} \Gamma^r_{AB} + \tau g^{rr}}$$

which allows one to relate ζ to the Christoffel symbols of g, and hence also to those of $\tilde{g}$. However, when relating $\overline{\tilde{\Gamma}}^x_{AB}$ and $\overline{\Gamma}^r_{AB}$ derivatives of the conformal factor Θ appear which are transverse to the light-cone and whose expansion is a priori not clear. Therefore this formula cannot be used to obtain information about ζ in a direct way, and one has to proceed differently. Assuming, from now on, that we are in space-dimension three, it will be shown in part II of this work that the above a priori restrictions and the constraint equation (2.23) *imply* that the auxiliary function ζ has the asymptotic behaviour

$$\zeta = \mathcal{O}(r^{-1}) \; . \tag{4.14}$$

It then follows from (2.24) and (4.4) that

$$\overline{W}^1 = \mathcal{O}(r) \; . \tag{4.15}$$

This is our final condition on the gauge functions. To summarize, necessary conditions for existence of both a smooth conformal completion of the metric $\overline{g}$ and of smooth extensions of the connection coefficients $\overline{\Gamma}^1_{1A}$ are

$$\xi_A = \mathcal{O}(r^{-1}) \; , \quad \overline{W}^0 = \mathcal{O}(r^{-1}) \; , \quad \overline{W}^A = \mathcal{O}(r^{-1}) \; . \tag{4.16}$$

Moreover,

$$\text{if } \zeta = \mathcal{O}(r^{-1}) \text{ then } \overline{W}^1 = \mathcal{O}(r) \; . \tag{4.17}$$

5 Asymptotic expansions

We have seen in Section 4.2.2 that existence of a smooth completion at null infinity requires $g_{AB} = \mathcal{O}(r^2)$ with $(\det \overline{g}_{AB})_{-4} > 0$, and thus $\varphi = \mathcal{O}(r)$ with $\varphi_{-1} > 0$. But then

$$\frac{1}{\sqrt{\det \gamma}}\gamma_{AB} = \varphi^{-2}\frac{1}{\sqrt{\det s}}g_{AB} = \mathcal{O}(1) .$$

Since only the conformal class of γ_{AB} matters, we see that there is no loss of generality to assume that $\gamma_{AB} = \mathcal{O}(r^2)$, with $(\det \gamma_{AB})_{-4} \neq 0$; this is convenient because then γ_{AB} and $\overline{g}_{AB}$ will display similar asymptotic behaviour. Moreover, since any Riemannian metric on the 2-sphere is conformal to the standard metric $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$, in the case of smooth conformal completions we may without loss of generality require the initial data γ to be of the form, for large r,

$$\gamma_{AB} \sim r^2\Big(s_{AB} + \sum_{n=1}^{\infty} h^{(n)}_{AB} r^{-n}\Big) , \tag{5.1}$$

for some smooth tensor fields $h^{(n)}_{AB}$ on S^2. (Recall that the symbol "$\sim$" has been defined in Section 4.1.) If the initial data γ_{AB} are not directly of the form (5.1), they can either be brought to (5.1) via an appropriate choice of coordinates and conformal rescaling, or they lead to a metric $\overline{g}_{\mu\nu}$ which is not smoothly extendable across $\mathscr{I}^+$.

In the second part of this work [26] the following theorem will be proved:

THEOREM 5.1 *Consider the characteristic initial value problem for Einstein's vacuum field equations in four space-time dimensions with smooth conformal data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ and gauge functions κ and $\overline{W}^\lambda$ on a cone C_O which has smooth closure in the conformally completed space-time. The following conditions are necessary-and-sufficient for the trace of the metric $g = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ on C_O, obtained as solution to Einstein's wave-map characteristic vacuum constraint equations (2.19) and (2.21)-(2.24), to admit a smooth conformal completion at infinity and for the connection coefficients $\overline{\Gamma}^r_{rA}$ to be smooth at $\mathscr{I}^+$, in the sense of Definition 4.1, when imposing a generalized wave-map gauge condition $H^\lambda = 0$:*

(i) There exists a one-parameter family $\varkappa = \varkappa(r)$ of Riemannian metrics on S^2 such that $\varkappa$ satisfies (5.1) and is conformal to γ (in particular we may assume γ itself to be of the form (5.1)).

(ii) The functions φ, ν^0, φ_{-1} and $(\nu_0)_0$ have no zeros on $C_O \setminus \{0\}$ and S^2, respectively, with the non-vanishing of $(\nu^0)_0$ being equivalent to

$$(\overline{W}^0)_1 \quad < \quad 2(\varphi_{-1})^{-2} . \tag{5.2}$$

(iii) The gauge functions satisfy $\kappa = \mathcal{O}(r^{-3})$, $\overline{W}^0 = \mathcal{O}(r^{-1})$, $\overline{W}^A = \mathcal{O}(r^{-1})$,

$\overline{W}^1 = \mathcal{O}(r)$ *and, setting* $\overline{W}_A := \overline{g}_{AB}\overline{W}^A$,

$$
\begin{aligned}
(\overline{W}^0)_2 &= \Big[\frac{1}{2}(\overline{W}^0)_1 + (\varphi_{-1})^{-2}\Big]\tau_2 \;, && (5.3)\\
(\overline{W}_A)_1 &= 4(\sigma_A{}^B)_2\mathring{\nabla}_A \log\varphi_{-1} - (\check{\varphi}_{-1})^{-2}[(\nu_0)_2(\overline{W}_A)_{-1} + (\nu_0)_1(\overline{W}_A)_0] \\
&\quad -\mathring{\nabla}_A\tau_2 - \frac{1}{2}(w_A{}^B)_1(w_B{}^C)_1(\overline{W}_C)_{-1} - \frac{1}{2}(w_A{}^B)_2(\overline{W}_B)_{-1} \\
&\quad -(w_A{}^B)_1\Big[(\overline{W}_B)_0 + (\check{\varphi}_{-1})^2(\nu_0)_1(\overline{W}_B)_{-1}\Big] \;, && (5.4)\\
(\overline{W}^1)_2 &= \frac{\zeta_2}{2} + (\varphi_{-1})^{-2}\tau_2 + \frac{\tau_2}{4}\check{R}_2 + \frac{\tau_2}{2}(\overline{W}^1)_1 + \Big[\frac{\tau_3}{4} + \frac{\kappa_3}{2} - \frac{(\tau_2)^2}{8}\Big](\overline{W}^1)_0 \\
&\quad \Big[\frac{1}{48}(\tau_2)^3 - \frac{1}{8}\tau_2\tau_3 - \frac{1}{4}\tau_2\kappa_3 + \frac{1}{6}\tau_4 + \frac{1}{3}\kappa_4\Big](\overline{W}^1)_{-1} \;, && (5.5)
\end{aligned}
$$

where $\mathring{\nabla}$ *is the covariant derivative operator of the unit round metric on the sphere* $s_{AB}\mathrm{d}x^A\mathrm{d}x^B$*,* $\check{R}_2$ *is the* r^{-2}*-coefficient of the scalar curvature* $\check{R}$ *of the metric* $\check{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$*,* $\check{\varphi}_{-1} := [(\varphi_{-1})^{-2} - \frac{1}{2}(\overline{W}^0)_1]^{-1/2}$*, and the expansion coefficients* $(w_A{}^B)_n$ *are defined using*

$$
w_A{}^B := \Big[\frac{r}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0) - 1\Big]\delta_A{}^B + 2r\chi_A{}^B \;.
$$

(iv) The <u>*no-logs-condition*</u> *is satisfied:*

$$
(\sigma_A{}^B)_3 = \tau_2(\sigma_A{}^B)_2 \;. \qquad (5.6)
$$

Remark 5.2 If any of the equations (5.3)-(5.6) fail to hold, the resulting characteristic initial data sets will have a *polyhomogeneous* expansion in terms of powers of r.

Remark 5.3 Theorem 5.1 is independent of the particular setting used (and remains also valid when the light-cone is replaced by one of two transversally intersecting null hypersurfaces meeting $\mathscr{I}^+$ in a sphere), cf. Section 7: As long as the generalized wave-map gauge condition is imposed one can always compute $\overline{W}^\lambda$, τ, σ etc. and check the validity of (5.3)-(5.6), whatever the prescribed initial data sets are. Some care is needed when the Minkowski target is replaced by some other target metric, cf. [26].

All the conditions in (ii) and (iii) which involve κ or $\overline{W}^\lambda$ can always be satisfied by an appropriate choice of coordinates. Equivalently, those logarithmic terms which appear if these conditions are not satisfied are pure gauge artifacts.

Recall that to solve the equation for ξ_A both κ and φ need to be known. This requires a choice of the κ-gauge. Since the choice of $\overline{W}^0$ does not affect the ξ_A-equation, there is no gauge-freedom left in that equation and if the no-logs-condition (5.6) does not hold there is no possibility to get rid of the log terms that arise in this equation. (In Section 6 we will return to the question, whether (5.6) can be satisfied by a choice of κ.) Similarly there is no gauge-freedom left when the equation for ζ is integrated but, due to the special structure of the asymptotic expansion of its source term, no new log terms arise in the expansion of ζ.

The no-logs-condition involves two functions, φ_{-1} and φ_0, which are globally determined by the gauge function κ and the initial data γ, cf. (2.19). The dependence of these functions on the gauge and on the initial data is rather intricate. Thus the question arises for which class of initial data one can find a function $\kappa = \mathcal{O}(r^{-3})$, such that the no-logs-condition holds, and accordingly what the geometric restrictions are for this to be possible. This issue will be analysed in part II of this work, using a gauge scheme adjusted to the initial data so that all globally defined integration functions can be computed explicitly.

6 The no-logs-condition

6.1 Gauge-independence

In this section we show gauge-independence of (5.6). It is shown in paper II [26] that (5.6) arises from integration of the equation for ξ_A, where the gauge-functions $\overline{W}^\mu$ do not occur. Equation (5.6) is therefore independent of those functions. So the only relevant freedom is that of rescaling the r-coordinate parameterizing the null rays. We therefore need to compute how (5.6) transforms under rescalings of r. For this we consider a smooth coordinate transformation

$$r \mapsto \tilde{r} = \tilde{r}(r, x^A) \; . \tag{6.1}$$

Under (6.1) the function φ transforms as a scalar. We have seen above that a necessary condition for the metric to be smoothly extendable across $\mathscr{I}^+$ is that φ has the asymptotic behaviour

$$\varphi(r, x^A) \;=\; \varphi_{-1}(x^A)r + \varphi_0 + \mathcal{O}(r^{-1}) \; , \quad \text{with} \quad \varphi_{-1} > 0 \; . \tag{6.2}$$

The transformed φ thus takes the form

$$\begin{aligned}
\tilde{\varphi}(\tilde{r}, x^A) \;&=\; \varphi(r(\tilde{r}), x^A) \;=\; \varphi_{-1}(x^A)r(\tilde{r}) + \varphi_0 + O(r(\tilde{r})^{-1}) \; , \\
\partial_{\tilde{r}}\tilde{\varphi}(\tilde{r}, x^A) \;&=\; \frac{\partial r}{\partial \tilde{r}}\partial_r \varphi(r(\tilde{r}), x^A) \;=\; \frac{\partial r}{\partial \tilde{r}}\varphi_{-1}(x^A)r(\tilde{r}) + \frac{\partial r}{\partial \tilde{r}}O(r(\tilde{r})^{-2}) \; .
\end{aligned}$$

If we require $\tilde{\varphi}$ to be of the form (6.2) as well, it is easy to check that we must have

$$\begin{aligned}
r(\tilde{r}, x^A) \;&=\; r_{-1}(x^A)\tilde{r} + r_0 + O(\tilde{r}^{-1}) \quad \text{and} && (6.3) \\
\partial_{\tilde{r}} r(\tilde{r}, x^A) \;&=\; r_{-1}(x^A) + O(\tilde{r}^{-2}) \; , \quad \text{with} \quad r_{-1} > 0 \; . && (6.4)
\end{aligned}$$

We have:

PROPOSITION 6.1 *The no-logs-condition (5.6) is invariant under the coordinate transformations (6.3)-(6.4).*

PROOF: For the transformation behavior of the expansion coefficients we obtain

$$\begin{aligned}
& \varphi_{-1} \;=\; (r_{-1})^{-1}\tilde{\varphi}_{-1} \; , \quad \varphi_0 \;=\; \tilde{\varphi}_0 - r_0(r_{-1})^{-1}\tilde{\varphi}_{-1} \\
\Longrightarrow \quad & \tau_2 \;=\; -2(\varphi_{-1})^{-1}\varphi_0 \;=\; r_{-1}\tilde{\tau}_2 + 2r_0 \; .
\end{aligned}$$

Moreover, with (6.3)-(6.4) we find

$$\begin{aligned}
\tilde{\sigma}_A{}^B \;&=\; \frac{\partial r}{\partial \tilde{r}}\sigma_A{}^B = \big[r_{-1} + O(\tilde{r}^{-2})\big]\big[(\sigma_A{}^B)_2 r(\tilde{r})^{-2} + (\sigma_A{}^B)_3 r(\tilde{r})^{-3} + \mathcal{O}(r(\tilde{r})^{-4})\big] \\
&=\; (r_{-1})^{-1}(\sigma_A{}^B)_2\tilde{r}^{-2} + \big[(r_{-1})^{-2}(\sigma_A{}^B)_3 - 2r_0(r_{-1})^{-2}(\sigma_A{}^B)_2\big]\,\tilde{r}^{-3} + O(\tilde{r}^{-4}) \\
\Longrightarrow \quad & (\sigma_A{}^B)_2 = r_{-1}(\tilde{\sigma}_A{}^B)_2 \; , \\
& (\sigma_A{}^B)_3 = (r_{-1})^2(\tilde{\sigma}_A{}^B)_3 + 2r_0 r_{-1}(\tilde{\sigma}_A{}^B)_2 \; .
\end{aligned}$$

Hence

$$(\sigma_A{}^B)_3 - \tau_2(\sigma_A{}^B)_2 = (r_{-1})^2[(\tilde\sigma_A{}^B)_3 - \tilde\tau_2(\tilde\sigma_A{}^B)_2] \ .$$

□

Although the No-Go Theorem 3.1 shows that the ($\kappa = 0$, $\overline{W}^\lambda = 0$)-wave-map gauge invariably produces logarithmic terms except in the flat case, one can decide whether the logarithmic terms can be transformed away by checking (5.6) using this gauge, or in fact any other. In the $([\gamma], \kappa)$ scheme this requires to determine τ_2 by solving the Raychaudhuri equation, which makes this scheme not practical for the purpose. In particular, it is not a priori clear within this scheme whether *any* initial data satisfying this condition exist unless both $(\sigma^A{}_B)_2$ and $(\sigma^A{}_B)_3$ vanish. On the other hand, in any gauge scheme where $\check g$ is prescribed on the cone, the no-logs-condition (5.6) is a straightforward condition on the asymptotic behaviour of the metric.

Let us assume that (5.6) is violated for say $\kappa = 0$. We know that the metric cannot have a smooth conformal completion at infinity in an adapted null coordinate system arising from the $\kappa = 0$-gauge via a transformation which *is not* of the asymptotic form (6.3)-(6.4). On the other hand if the transformation *is* of the form (6.3), then the no-logs-condition will also be violated in the new coordinates. We conclude that we cannot have a smooth conformal completion in *any* adapted null coordinate system. That yields

THEOREM 6.2 *Consider initial data γ on a light-cone C_O in a $\kappa = 0$-gauge with asymptotic behaviour $\gamma_{AB} \sim r^2(s_{AB} + \sum_{n=1}^\infty h^{(n)}_{AB} r^{-n})$. Assume that φ, ν^0 and φ_{-1} are strictly positive on $C_O \setminus \{O\}$ and S^2, respectively. Then there exist a gauge w.r.t. which the trace $\overline{g}$ of the metric on the cone admits a smooth conformal completion at infinity and where the connection coefficients $\overline{\Gamma}^r_{rA}$ are smooth at $\mathscr{I}^+$ (in the sense of Definition 4.1) if and only if the no-logs-condition (5.6) holds in one (and then any) coordinate system related to the original one by a coordinate transformation of the form (6.3)-(6.4).*

6.2 Geometric interpretation

Here we provide a geometric interpretation of the no-logs-condition (5.6) in terms of the conformal Weyl tensor. This ties our results with the analysis in [1] (compare also Section 7.4).

For this purpose let us consider the components of the conformal Weyl tensor, $C_{rAr}{}^B$, on the cone. To end up with smooth initial data for the conformal fields equations we need to require the rescaled Weyl tensor $\overline{\tilde d}_{rAr}{}^B = \overline{\Theta}^{-1}\overline{\tilde C}_{rAr}{}^B = \overline{\Theta}^{-1}\overline{C}_{rAr}{}^B$ to be smooth at $\mathscr{I}^+$, which is equivalent to

$$\overline{C}_{rAr}{}^B = \mathcal{O}(r^{-5}) \ . \tag{6.5}$$

In particular the $\overline{C}_{rAr}{}^B$-components of the Weyl tensor need to vanish one order faster than naively expected from the asymptotic behavior of the metric. In adapted null coordinates and in vacuum we have, using the formulae of [8,

Appendix A],

$$\begin{aligned}
\overline{C}_{rAr}{}^{B} &= \overline{R}_{rAr}{}^{B} = -\partial_r\overline{\Gamma}^{B}_{rA} + \overline{\Gamma}^{B}_{rA}\overline{\Gamma}^{r}_{rr} - \overline{\Gamma}^{B}_{rC}\overline{\Gamma}^{C}_{rA} \\
&= -(\partial_r - \kappa)\chi_A{}^B - \chi_A{}^C\chi_C{}^B \\
&= -\frac{1}{2}(\partial_r\tau - \kappa\tau + \frac{1}{2}\tau^2)\delta_A{}^B - (\partial_r + \tau - \kappa)\sigma_A{}^B - \sigma_A{}^C\sigma_C{}^B \\
&= \frac{1}{2}|\sigma|^2\delta_A{}^B - (\partial_r + \tau - \kappa)\sigma_A{}^B - \sigma_A{}^C\sigma_C{}^B \ .
\end{aligned}$$

Assuming, for definiteness, that $\kappa = \mathcal{O}(r^{-3})$ and $\overline{g}_{AB} = \mathcal{O}(r^2)$ with $(\det \overline{g}_{AB})_{-4} > 0$ we have

$$\begin{aligned}
\overline{C}_{rAr}{}^{B} &= \Big((\sigma_A{}^B)_3 - \tau_2(\sigma_A{}^B)_2 + \frac{1}{2}(\sigma_C{}^D)_2(\sigma_D{}^C)_2\delta_A{}^B - (\sigma_A{}^C)_2(\sigma_C{}^B)_2\Big)r^{-4} \\
&\quad +\mathcal{O}(r^{-5}) \ .
\end{aligned}$$

As an s-symmetric, trace-free tensor $(\sigma_A{}^C)_2$ has the property

$$(\sigma_A{}^C)_2(\sigma_C{}^B)_2 = \frac{1}{2}(\sigma_D{}^C)_2(\sigma_C{}^D)_2\delta_A{}^B \ ,$$

i.e.

$$\overline{C}_{rAr}{}^{B} = \big[(\sigma_A{}^B)_3 - \tau_2(\sigma_A{}^B)_2\big]r^{-4} + \mathcal{O}(r^{-5}) \ ,$$

and (6.5) holds if and only if the no-logs-condition is satisfied.

7 Other settings

We pass now to the discussion, how to modify the above when other data sets are given, or Cauchy problems other than a light-cone are considered.

7.1 Prescribed $(\check{g}_{AB}, \kappa)$

In this setting the initial data are a symmetric degenerate twice-covariant tensor field $\check{g}$, and a connection κ on the family of bundles tangent to the integral curves of the kernel of $\check{g}$, satisfying the Raychaudhuri constraint (2.16).

Recall that so far we have mainly been considering a characteristic Cauchy problem where $([\gamma], \kappa)$ are given. There (2.19) was used to solve for the conformal factor relating $\check{g}$ and γ:

$$\check{g} \equiv \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \varphi^2\Big(\frac{\det s_{CD}}{\det \gamma_{EF}}\Big)^{\frac{1}{n-1}}\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B \ . \tag{7.1}$$

But then a pair $(\check{g}, \kappa)$ satisfying (2.16) is obtained.

So, in fact, prescribing the pair $(\check{g}, \kappa)$ satisfying (2.16) can be viewed a special case of the $([\gamma], \kappa)$-prescription, where one sets $\gamma := \check{g}$. Indeed, when $\check{g}$ and κ are suitably regular at the vertex, uniqueness of solutions of (2.19) with the boundary conditions $\varphi(0) = 0$ and $\partial_r\varphi(0) = 1$ shows that

$$\varphi = \Big(\frac{\det \overline{g}_{EF}}{\det s_{CD}}\Big)^{\frac{1}{2(n-1)}} \quad \Longleftrightarrow \quad \overline{g}_{AB} \equiv \gamma_{AB} \quad \Longleftrightarrow \quad \check{g} \equiv \gamma \ . \tag{7.2}$$

In particular all the results so far apply to this case.

If τ is nowhere vanishing, as necessary for a smooth null-geodesically complete light-cone extending to null infinity, then (2.16) can be algebraically solved for κ, so that the constraint becomes trivial.

7.2 Prescribed $(\overline{g}_{\mu\nu}, \kappa)$

In this approach one prescribes all metric functions $\overline{g}_{\mu\nu}$ on the initial characteristic hypersurface, together with the connection coefficient κ, subject to the Raychaudhuri equation (2.16). Equation (2.12) relating κ and ν_0 becomes an algebraic equation for the gauge-source function $\overline{W}^0$, while the equations $\overline{R}_{rA} = 0 = \overline{g}^{AB}\overline{R}_{AB}$ become algebraic equations for $\overline{W}^A$ and $\overline{W}^r$.

In four space-time dimensions, a smooth conformal completion at null infinity will exist if and only if $r^{-2}\overline{g}_{\mu\nu}$ can be smoothly extended as a Lorentzian metric across $\mathscr{I}^+$ and no logarithmic terms appear in the asymptotic expansion of $\overline{\Gamma}^r_{rA}$; this last fact is equivalent to (5.6). To see this, note that since the equations for $\overline{W}^\mu$ are algebraic, no log terms arise in these fields as long as no log terms appear in the remaining fields appearing in the constraint equation. Similarly no log terms arise in the ζ-equation. The only possible source of log terms is thus the ξ_A-equation, and the appearance of log terms there is excluded precisely by the no-logs-condition. The existence of an associated space-time with a "piece of smooth $\mathscr{I}^+$" follows then from the analysis of the initial data for Friedrich's conformal equations in part II of this work, together with the analysis in [13].

We conclude that (5.6) is again a necessary-and-sufficient condition for existence of a smooth $\mathscr{I}^+$ for the current scheme in space-time dimension four.

7.3 Frame components of σ as free data

In this section we consider as free data the components χ_{ab} in an adapted parallel-propagated frame as in [12, Section 5.6]. We will assume that

$$\chi^a{}_b = \frac{1}{r}\delta^a{}_b + \mathcal{O}(r^{-2}) \ , \quad a, b \in \{2, 3\} \ . \tag{7.3}$$

There are actually at least two schemes which would lead to this form of $\chi^a{}_b$: One can e.g. prescribe any $\chi^a{}_b$ satisfying (7.3) such that $\chi^2{}_2 + \chi^3{}_3 = \chi_{22} + \chi_{33}$ has no zeros, define $\sigma_{ab} = \chi_{ab} - \frac{1}{2}(\chi^2{}_2 + \chi^3{}_3)\delta_{ab}$, and solve algebraically the Raychaudhuri equation for κ. Another possibility is to prescribe directly a symmetric trace-free tensor σ_{ab} in the $\kappa = 0$ gauge, use the Raychaudhuri equation to determine τ, and construct χ_{ab} using

$$\chi^a{}_b = \frac{\tau}{2}\delta^a{}_b + \sigma^a{}_b \ , \quad a, b \in \{2, 3\} \ . \tag{7.4}$$

The asymptotics (7.3) will then hold if $\sigma^a{}_b$ is taken to be $\mathcal{O}(r^{-2})$.

Given χ_{ab}, the tensor field $\check{g}$ is obtained by setting

$$\check{g} = \big(\theta^2{}_A\theta^2{}_B + \theta^3{}_A\theta^3{}_B\big)\mathrm{d}x^A\mathrm{d}x^B \ , \tag{7.5}$$

where the co-frame coefficients $\theta^a{}_A$ are solutions of the equation [12]

$$\partial_r\theta^a{}_A = \chi^a{}_b\theta^b{}_A \ , \quad a, b \in \{2, 3\} \ . \tag{7.6}$$

Assuming (7.3), one finds that solutions of (7.6) have an asymptotic expansion for $\theta^a{}_A$ without log terms:

$$\theta^a{}_A = r\varphi^a{}_A + \mathcal{O}(1) \ , \quad a, b \in \{2, 3\} \tag{7.7}$$

for some globally determined functions $\varphi^a{}_A$. If the determinant of the two-by-two matrix $(\varphi^a{}_A)$ does not vanish, one obtains a tensor field $\breve{g}$ to which our previous considerations apply. This leads again to the no-logs conditions (5.6).

Writing, as usual,

$$\sigma_{ab} = (\sigma_{ab})_2 r^{-2} + (\sigma_{ab})_3 r^{-3} + \mathcal{O}(r^{-4}) , \quad a, b \in \{2, 3\} , \tag{7.8}$$

the no-logs-condition rewritten in terms of σ_{ab} read

$$(\sigma_{ab})_3 = \tau_2 (\sigma_{ab})_2 , \quad a, b \in \{2, 3\} . \tag{7.9}$$

7.4 Frame components of the Weyl tensor as free data

Let $C_{\alpha\beta\gamma\delta}$ denote the space-time Weyl tensor. For $a, b \geq 2$ let

$$\psi_{ab} := e_a{}^A e_b{}^B \overline{C}_{ArBr}$$

represent the components of $\overline{C}_{ArBr}$ in a parallelly-transported adapted frame, as in Section 7.3. The tensor field ψ_{ab} is symmetric, with vanishing η-trace, and we have in space-time dimension four (cf., e.g., [12, Section 5.7])

$$(\partial_r - \kappa)\chi_{ab} \quad = \quad -\sum_{c=2}^{3} \chi_{ac}\chi_{cb} - \psi_{ab} - \frac{1}{2}\eta_{ab}\overline{T}_{rr} . \tag{7.10}$$

Given (κ, ψ_{ab}), we can integrate this equation in vacuum to obtain the tensor field χ_{ab} needed in Section 7.3. However, this approach leads to at least two difficulties: First, it is not clear under which conditions on ψ_{ab} the solutions will exist for all values of r. Next, it is not clear that the global solutions will have the desired asymptotics. We will not address these questions but, taking into account the behaviour of the Weyl tensor under conformal transformations, we will assume that

$$\kappa = \mathcal{O}(r^{-3}) , \quad \psi_{ab} = \mathcal{O}(r^{-4}) , \tag{7.11}$$

and that the associated tensor field χ_{ab} exists globally and satisfies (7.3). The no-logs-condition will then hold if and only if

$$\psi_{ab} = \mathcal{O}(r^{-5}) \qquad \Longleftrightarrow \qquad (\psi_{ab})_4 = 0 . \tag{7.12}$$

Note that one can reverse the procedure just described: given χ_{ab} we can use (7.10) to determine ψ_{ab}. Assuming (7.3), the no-logs-condition will hold if and only if the ψ_{ab}-components of the Weyl tensor vanish one order faster than naively expected from the asymptotic behaviour of the metric (cf. Section 6.2).

Equation (7.12) is the well-known starting point of the analysis in [25], and has also been obtained previously as a necessary condition for existence of a smooth $\mathscr{I}$ in the analysis of the hyperboloidal Cauchy problem [1]. It is therefore not surprising that it reappears in the analysis of the characteristic Cauchy problem. However, as pointed out above, a satisfactory treatment of the problem using ψ_{ab} as initial data requires further work.

7.5 Characteristic surfaces intersecting transversally

Consider two characteristic surfaces, say $\mathscr{N}_1$ and $\mathscr{N}_2$, intersecting transversally along a smooth submanifold S diffeomorphic to S^2. Assume moreover that the initial data on $\mathscr{N}_1$ (in any of the versions just discussed) are such that the metric $\overline{g}_{\mu\nu}$ admits a smooth conformal completion across the sphere $\{x = 0\}$, as in Definition 4.1. The no-logs-condition (5.6) remains unchanged. Indeed, the only difference is the integration procedure for the constraint equations: while on the light-cone we have been integrating from the tip of the light-cone, on $\mathscr{N}_1$ we integrate from the intersection surface S. This leads to the need to provide supplementary date at S which render the solutions unique. Hence the asymptotic values of the fields, which arise from the integration of the constraints, will also depend on the supplementary data at S.

7.6 Mixed spacelike-characteristic initial value problem

Consider a mixed initial value problem, where the initial data set consists of:

1. A spacelike initial data set $(\mathscr{S}, {}^3g, K)$, where ${}^3g$ is a Riemannian metric on $\mathscr{S}$ and K is a symmetric two-covariant tensor field on $\mathscr{S}$. The three-dimensional manifold $\mathscr{S}$ is supposed to have a compact smooth boundary S diffeomorphic to S^2, and the fields $({}^3g, K)$ are assumed to satisfy the usual vacuum Einstein constraint equations.

2. A hypersurface $\mathscr{N}_1$ with boundary S equipped with a characteristic initial data set, in any of the configurations discussed so far. Here $\mathscr{N}_1$ should be thought of as a characteristic initial data surface emanating from S in the outgoing direction.

3. The data on $\mathscr{S}$ and $\mathscr{N}_1$ satisfy a set of "corner conditions" at S, to be defined shortly.

The usual evolution theorems for the spacelike general relativistic initial value problem provide a unique future maximal globally hyperbolic vacuum development $\mathscr{D}^+$ of $(\mathscr{S}, {}^3g, K)$. Since $\mathscr{S}$ has a boundary, $\mathscr{D}^+$ will also have a boundary. Near S, the null part of the boundary of $\partial\mathscr{D}^+$ will be a smooth null hypersurface emanating from S, say $\mathscr{N}_2$, generated by null geodesics normal to S and "pointing towards $\mathscr{S}$" at S. In particular the space-time metric on $\mathscr{D}^+$ induces characteristic initial data on $\mathscr{N}_2$. In fact, all derivatives of the metric, both in directions tangent and transverse to $\mathscr{N}_2$, will be determined on $\mathscr{N}_2$ by the initial data set $(\mathscr{S}, {}^3g, K)$. This implies that the characteristic initial data needed on $\mathscr{N}_1$, as well as their derivatives in directions tangent to $\mathscr{N}_1$, are determined on S by $(\mathscr{S}, {}^3g, K)$. These are the "corner conditions" which have to be satisfied by the data on $\mathscr{N}_1$ at S, with these data being arbitrary otherwise. The corner conditions can be calculated algebraically in terms of the fields $({}^3g, K)$, the gauge-source functions W^μ, and the derivatives of those fields, at S, using the vacuum Einstein equations.

One can use now the Cauchy problem discussed in Section 7.5 to obtain the metric to the future of $\mathscr{N}_1 \cup \mathscr{N}_2$, and the discussion of the no-logs-condition given in Section 7.5 applies.

Acknowledgments Supported in part by the Austrian Science Fund (FWF): P 24170-N16. Parts of this material are based upon work supported by the National Science Foundation under Grant No. 0932078 000, while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the fall semester of 2013.

A Polyhomogeneous functions

A function f defined on an open set $\mathscr{U}$ with smooth boundary $\partial\mathscr{U} = \{x = 0\}$ is said to be *polyhomogeneous* at $x = 0$ if $f \in C^\infty(\mathscr{U})$ and if there exist integers N_i, real numbers n_i, and functions $f_{ij} \in C^\infty(\overline{\mathscr{U}})$ such that

$$\forall m \in \mathbb{N}, \quad \exists N(m) \in \mathbb{N}, \quad f - \sum_{i=0}^{N(m)} \sum_{j=0}^{N_i} f_{ij} x^{n_i} \ln^j x \in C^m(\overline{\mathscr{U}}) \, . \tag{A.1}$$

We will then write

$$f \sim \sum_{i,j} f_{ij} x^{n_i} \ln^j x \, .$$

We will similarly write

$$f \sim \sum_i f_i x^i$$

when (A.1) holds with no logarithmic terms.

References

[1] L. Andersson and P.T. Chruściel, *Hyperboloidal Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity*, Phys. Rev. Lett. **70** (1993), no. 19, 2829–2832.

[2] L. Andersson, P.T. Chruściel, and H. Friedrich, *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einsteins field equations*, Commun. Math. Phys. **149** (1992), 587–612.

[3] L. Bieri and N. Zipser, *Extensions of the stability theorem of the Minkowski space in general relativity*, AMS/IP Studies in Advanced Mathematics, vol. 45, American Mathematical Society, Providence, RI, 2009, pp. xxiv+491.

[4] L. Blanchet, *Radiative gravitational fields in general relativity. II. Asymptotic behaviour at future null infinity*, Proc. Roy. Soc. London Ser. A **409** (1987), 383–399.

[5] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, *Gravitational waves in general relativity VII: Waves from axi–symmetric isolated systems*, Proc. Roy. Soc. London A **269** (1962), 21–52.

[6] Y. Choquet-Bruhat, *Un théorème d'instabilité pour certaines équations hyperboliques non linéares*, C.R. Acad. Sci. Paris **276** (1973), 281–284.

[7] Y. Choquet-Bruhat, P.T. Chruściel, and J.M. Martín-García, *The light-cone theorem*, Class. Quantum Grav. **26** (2009), 135011 (22 pp), arXiv:0905.2133 [gr-qc].

[8] ______, *The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions*, Ann. H. Poincaré **12** (2011), 419–482, arXiv:1006.4467 [gr-qc].

[9] D. Christodoulou and S. Klainermann, *Nonlinear stability of Minkowski space*, Princeton University Press, Princeton, 1993.

[10] P.T. Chruściel, *The existence theorem for the Cauchy problem on the light-cone for the vacuum Einstein equations*, Forum for Mathematics Sigma **in press** (2012), arXiv:1209.1971 [gr-qc].

[11] P.T. Chruściel, M.A.H. MacCallum, and D. Singleton, *Gravitational waves in general relativity. XIV: Bondi expansions and the "polyhomogeneity" of Scri*, Philos. Trans. Roy. Soc. London Ser. A **350** (1995), 113–141, arXiv:gr-qc/9305021.

[12] P.T. Chruściel and T.-T. Paetz, *The many ways of the characteristic Cauchy problem*, Class. Quantum Grav. **29** (2012), 145006, 27 pp., arXiv:1203.4534 [gr-qc].

[13] P.T. Chruściel, T.-T. Paetz, and R. Tagné Wafo, *The Neighborhood Theorem for the characteristic Cauchy problem for Einstein-type wave equations*, (2014), in preparation.

[14] H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. Roy. Soc. London Ser. A **375** (1981), 169–184.

[15] ______, *On the hyperbolicity of Einstein's and other gauge field equations*, Commun. Math. Phys. **100** (1985), 525–543.

[16] ______, *Hyperbolic reductions for Einstein's equations*, Class. Quantum Grav. **13** (1996), 1451–1469.

[17] ______, *Conformal Einstein evolution*, The conformal structure of space-time, Lecture Notes in Phys., vol. 604, Springer, Berlin, 2002, arXiv:gr-qc/0209018, pp. 1–50.

[18] H. Friedrich, *Conformal structures of static vacuum data*, Commun. Math. Phys. **321** (2013), 419–482.

[19] S. Klainerman and F. Nicolò, *On local and global aspects of the Cauchy problem in general relativity*, Class. Quantum Grav. **16** (1999), R73–R157.

[20] ______, *The evolution problem in general relativity*, Progress in Mathematical Physics, vol. 25, Birkhäuser, Boston, MA, 2003.

[21] ______, *Peeling properties of asymptotically flat solutions to the Einstein vacuum equations*, Class. Quantum Grav. **20** (2003), 3215–3257.

[22] J.A. Valiente Kroon, *On the existence and convergence of polyhomogeneous expansions of zero-rest-mass fields*, Class. Quantum Grav. **17** (2000), 4365–4375, arXiv:gr-qc/0005087.

[23] ______, *A new class of obstructions to the smoothness of null infinity*, Commun. Math. Phys. **244** (2004), 133–156, arXiv:gr-qc/0211024.

[24] H. Lindblad and I. Rodnianski, *Global existence for the Einstein vacuum equations in wave coordinates*, Commun. Math. Phys. **256** (2005), 43–110, arXiv:math.ap/0312479.

[25] E.T. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, Jour. Math. Phys. **3** (1962), 566–578.

[26] T.-T. Paetz, *Characteristic initial data and smoothness of scri. II. Asymptotic expansions and construction of conformally smooth data sets*, (2014), arXiv:1403.3560 [gr-qc].

[27] A.D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. Roy. Soc. London A **427** (1990), 221–239.

[28] R.K. Sachs, *Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time*, Proc. Roy. Soc. London A **270** (1962), 103–126.

[29] L.A. Tamburino and J.H. Winicour, *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150** (1966), 1039–1053.

[30] R.J. Torrence and W.E. Couch, *Generating fields from data on $\mathscr{H}^- \cup \mathscr{H}^+$ and $\mathscr{H}^- \cup \mathscr{I}^-$*, Gen. Rel. Grav. **16** (1984), 847–866.

Characteristic initial data and smoothness of Scri. II. Asymptotic expansions and construction of conformally smooth data sets*

Tim-Torben Paetz†

Gravitational Physics, University of Vienna
Boltzmanngasse 5, 1090 Vienna, Austria

March 17, 2014

Abstract

We derive necessary-and-sufficient conditions on characteristic initial data for Friedrich's conformal field equations in $3+1$ dimensions to have no logarithmic terms in an asymptotic expansion at null infinity.

PACS: 02.30.Hq, 02.30.Mv, 04.20.Ex, 04.20.Ha

Contents

*Preprint UWThPh-2014-9.
†Email Tim-Torben.Paetz@univie.ac.at

1 Introduction

In this work we continue the work initiated in [7] to analyze the occurrence of logarithmic terms in the asymptotic expansion of the metric tensor and some other fields at null infinity. In part I of this work, where we also described the setting, it has been shown that the harmonic coordinate condition is not compatible with a smooth asymptotic structure at the conformal boundary at infinity, but has to be replaced by a wave-map gauge condition with non-vanishing gauge-source functions.

However, it is expected [5, 12] (compare also [1]) that even for smooth initial data the asymptotic expansion of the space-time metric at null infinity will generically be polyhomogeneous and involve logarithmic terms which do not have their origin in an inconvenient choice of coordinates. One main object of this note, treated in Section 3, is to study thoroughly the asymptotic behavior of solutions of the Einstein's vacuum constraint equations and analyze under which conditions a smooth conformal completion of the *restriction of the space-time metric to the characteristic initial surface* across null infinity is possible. As announced in [7, Theorem 5.1] we intend to provide necessary-and-sufficient conditions on the initial data and the gauge source functions which permit such extensions. In doing so it will become manifest that many, though not all, of the logarithmic terms which arise at infinity are gauge artifacts. The remaining non-gauge logarithmic terms can be eliminated by imposing restrictions on the asymptotic behavior of the initial data, captured by what we call *no-logs-condition*.

In Section 5, we will show that solutions of the characteristic vacuum constraint equations satisfying the no-logs-condition lead to smooth initial data for Friedrich's conformal field equations. The data will be computed in a new gauge scheme developed in Section 4 and will provide the basis to solve the evolution problem and construct space-times with a "piece of smooth $\mathscr{I}^+$".

In Section 2 we give a summary of [7] where we briefly describe the framework and recall the most important definitions and results of part I. Finally, in Appendix A our proceeding in Section 3-5 to solve the constraint equations in terms of polyhomogeneous expansions will be rigorously justified, while in Appendix B we compare the peculiarities of different gauge schemes.

2 Preliminaries

We use all the notation, terminology and conventions introduced in part I [7]. For the convenience of the reader, though, let us briefly recall the most essential ingredients and definitions of our framework.

2.1 Notation

Consider a smooth function

$$f : (0,\infty) \times S^2 \longrightarrow \mathbb{R} , \quad (r, x^A) \longmapsto f(r, x^A) .$$

If this function permits an asymptotic expansion as a power series in r, we denote by f_n, or $(f)_n$, the coefficient of r^{-n} in the corresponding expansion. Be aware that sometimes a lower index might denote both the component of a vector, and the n-th order term in an expansion of the corresponding object. If both indices need to appear simultaneously we use brackets and place the index corresponding to the n-th order expansion term outside the brackets. We write

$$f(r, x^A) \sim \sum_{k=-N}^{\infty} f_k(x^A) r^{-k}$$

if the right-hand side is the polyhomogeneous expansion at $x = 0$ of the function $x \mapsto r^{-N} f(r, \cdot)|_{r=1/x}$. Moreover, we write $f = \mathcal{O}(r^N)$ (or $f = \mathcal{O}(x^{-N})$, $x \equiv 1/r$), $N \in \mathbb{Z}$, if the function $x \mapsto r^{-N} f(r, \cdot)|_{r=1/x}$ is smooth at $x = 0$.

2.2 Setting

We consider a 3+1-dimensional C^∞-manifold $\mathscr{M}$. For definiteness we take as initial surface either a globally smooth light-cone $C_O \subset \mathscr{M}$ or two null hypersurfaces $\mathscr{N}_1$, $\mathscr{N}_2 \subset \mathscr{M}$ intersecting transversally along a smooth submanifold $S \cong S^2$. Suppose that the closure (in the completed space-time) $\overline{\mathscr{N}}$ of $\mathscr{N} \in \{C_O \setminus \{O\}, \mathscr{N}_1\}$ meets $\mathscr{I}^+$ transversally in a smooth spherical cross-section.

We introduce *adapted null coordinates* $(u \equiv x^0, r \equiv x^1, x^A)$ on $\mathscr{N}$ (i.e. $\mathscr{N} = \{u = 0\}$, where r parameterizes the null rays generating $\mathscr{N}$, and (x^A) are local coordinates on $\Sigma_r \equiv \{r = \text{const}, u = 0\} \cong S^2$). Then the trace $\overline{g}$ of the metric on $\mathscr{N}$ becomes

$$\overline{g} = \overline{g}_{00}\mathrm{d}u^2 + 2\nu_0 \mathrm{d}u\mathrm{d}r + 2\nu_A \mathrm{d}u\mathrm{d}x^A + \check{g} \;, \tag{2.1}$$

where

$$\check{g} = \check{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B := \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{2.2}$$

is a degenerate quadratic form induced by g on $\mathscr{N}$ which induces on each slice Σ_r an r-dependent Riemannian metric $\check{g}_{\Sigma_r}$ (coinciding with $\check{g}(r,\cdot)$ in the coordinates above). While the components $\overline{g}_{00}$, ν_0 and ν_A depend upon the choice of coordinates off $\mathscr{N}$, the quadratic form $\check{g}$ is intrinsically defined on $\mathscr{N}$.

In fact, we will be interested merely in the asymptotic behavior of the restriction of the space-time metric to $\mathscr{N}$. A regular light-cone or two transversally intersecting null hypersurfaces with appropriately specified initial data, though, guarantee that the vacuum constraint equations have unique solutions.

Throughout this work we use an overline to denote a space-time object restricted to $\mathscr{N}$. The symbol " ˇ " will be used to denote objects associated with the Riemannian metric $\check{g}_{\Sigma_r}$.

Definition 2.1 (cf. [7]) We say that a smooth metric tensor $\overline{g}_{\mu\nu}$ defined on a null hypersurface $\mathscr{N}$ given in adapted null coordinates has a *smooth conformal completion at infinity* if the unphysical metric tensor $\overline{\tilde{g}}_{\mu\nu}$ obtained via the coordinate transformation $r \mapsto 1/r =: x$ and the conformal rescaling $\overline{g} \mapsto \overline{\tilde{g}} \equiv x^2\overline{g}$ is, as a Lorentzian metric, smoothly extendable at $\{x = 0\}$.

The components of a smooth tensor field on $\mathscr{N}$ will be said to be *smooth at infinity* whenever they admit a smooth expansion in the conformally rescaled space-time at $\{x = 0\}$ and expressed in the (x, x^A)-coordinates.

As remarked in [7], Definition 2.1 concerns only fields on $\mathscr{N}$ and is not tied to the existence of an associated space-time. Moreover, it concerns both conditions on the metric and on the coordinate system.

The Einstein equations split into a set of evolution equations and a set of constraint equations which need to be satisfied on the initial surface. In the characteristic case the data for the evolution equations are provided by the trace of the metric on the initial surface. Due to the constraints not all of its components can be prescribed freely. There are various ways of choosing free data [6, 7]. Here we focus on Rendall's scheme [10], where the free data are formed by the conformal class $[\gamma]$ of the tensor $\check{g}_{\Sigma_r}$ (and the function κ introduced below).[1] By choosing a representative they can be viewed as a one-parameter family, parameterized by r, of Riemannian metrics $\gamma(r,\cdot)$ on S^2. A major advantage of this scheme in particular in view of Section 4 is that it permits a separation of physical and gauge degrees of freedom. Some comments on how things change for other approaches to prescribe characteristic initial data are given in [7], cf. Remark 3.4.

Einstein's vacuum constraint equations in a $\hat{g}$-generalized wave-map gauge are obtained from Einstein's vacuum equations assuming that the *wave-gauge vector*

$$H^\lambda := \Gamma^\lambda - \hat{\Gamma}^\lambda - W^\lambda = 0 \;, \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} \;, \quad \hat{\Gamma}^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta} \;. \tag{2.3}$$

[1] In the case of two transversally intersecting null hypersurfaces these data need to be supplemented by corresponding data on $\mathscr{N}_2$ and certain data on the intersection manifold S.

vanishes. We use the hat-symbol "ˆ" to indicate objects associated with some *target metric* $\hat{g}$, which we assume for convenience to be of the form

$$\overline{\hat{g}}_{1i} = 0\,, \quad \overline{\hat{g}}_{AB} = r^2 s_{AB} + \mathcal{O}(1)\,, \quad \hat{\nu}_0 = 1 + \mathcal{O}(r^{-3})\,, \quad \hat{\nu}_A = \mathcal{O}(r^{-2})\,,$$
$$\overline{\hat{g}}_{00} = -1 + \mathcal{O}(r^{-2})\,, \quad \overline{\partial_0 \hat{g}_{1i}} = \mathcal{O}(r^{-3})\,, \quad \overline{\partial_0 \hat{g}_{AB}} = \mathcal{O}(r^{-1}) \tag{2.4}$$

on $\mathscr{N}$, where $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$ is the unit round metric on the sphere S^2. By $W^\lambda = W^\lambda(x^\mu, g_{\mu\nu})$ we denote the components of a vector field, the *gauge source functions*, which can be arbitrarily prescribed. They reflect the freedom to choose coordinates off the initial surface, and thus allow us to analyze smoothness of the metric tensor at infinity in arbitrary coordinates.

For given initial data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ the wave-map gauge constraints form a hierarchical system of ODEs along the null generators of the cone (cf. [2]):

$$\partial^2_{rr}\varphi - \kappa\partial_r\varphi + \frac{1}{2}|\sigma|^2\varphi = 0\,, \tag{2.5}$$
$$(\partial_r + \frac{1}{2}\tau + \kappa)\nu^0 + \frac{1}{2}(\overline{W}^0 + \overline{\hat{\Gamma}}^0) = 0\,, \tag{2.6}$$
$$(\partial_r + \tau)\xi_A - 2\check{\nabla}_B \sigma_A{}^B + \partial_A\tau + 2\partial_A\kappa = 0\,, \tag{2.7}$$
$$2\nu^0(\partial_r\nu_A - 2\nu_B\chi_A{}^B) - \nu_A(\overline{W}^0 + \overline{\hat{\Gamma}}^0) - \overline{g}_{AB}(\overline{W}^B + \overline{\hat{\Gamma}}^B) + \gamma_{AB}\gamma^{CD}\check{\Gamma}^B_{CD} + \xi_A = 0\,, \tag{2.8}$$
$$(\partial_r + \tau + \kappa)\zeta + \check{R} - \frac{1}{2}\xi_A\xi^A + \check{\nabla}_A\xi^A = 0\,, \tag{2.9}$$
$$(2\partial_r + \tau + 2\kappa)\overline{g}^{rr} + 2\overline{W}^r + 2\overline{\hat{\Gamma}}^r - \zeta = 0\,, \tag{2.10}$$

where τ and $\sigma_A{}^B$ denote the expansion and the shear of $\mathscr{N}$, respectively,

$$\kappa = \overline{\Gamma}^r_{rr}\,, \quad \xi_A = -2\overline{\Gamma}^r_{rA}\,, \quad \zeta = 2\overline{g}^{AB}\overline{\Gamma}^r_{AB} + \tau\overline{g}^{rr}\,. \tag{2.11}$$

Apart from W^λ the function κ turns out to be another gauge degree of freedom, reflecting the freedom to choose r, which can be prescribed conveniently.

Integrating these equations successively one determines all the components of $\overline{g}$ (note that $\nu_0 = (\nu^0)^{-1}$ and $\overline{g}_{00} = \overline{g}^{AB}\nu_A\nu_B - (\nu_0)^2\overline{g}^{rr}$). The relevant boundary conditions follow either from regularity conditions at the vertex [2] when $\mathscr{N}$ is a cone, or from the remaining data specified on $\mathscr{N}_2$ and S (cf. e.g. [6, 10]) in the case of two characteristic surfaces intersecting transversally.

Our ultimate goal is to find necessary-and-sufficient conditions on the initial data given on $\mathscr{N}$, such that the resulting space-time has a smooth conformal completion at infinity *à la Penrose*. This requires the following ingredients:

1. To exclude the appearance of conjugate points or coordinate singularities on $\mathscr{N}$, the constraint equations need to admit a non-degenerated global solution $\overline{g}$ on $\mathscr{N}$. This will be the case if and only if the functions φ and ν^0 are of constant sign,
$$\varphi \neq 0\,, \quad \nu^0 \neq 0 \quad \text{on } \mathscr{N}\,. \tag{2.12}$$
2. The metric $\overline{g}$ needs to be smoothly extendable as a Lorentzian metric, which means that the functions φ_{-1} and $(\nu^0)_0$ need to have a sign
$$\varphi_{-1} \neq 0\,, \quad (\nu^0)_0 \neq 0 \quad \text{on } S^2\,. \tag{2.13}$$
This assumption excludes conjugate points at the intersection of $\mathscr{N}$ with $\mathscr{I}^+$.

3. The components of $\overline{g}$ need to be smooth at $\mathscr{I}^+$. For this one has to make sure that their asymptotic expansions contain no logarithmic terms and have the correct order in terms of powers of r.

4. All the fields which appear in Friedrich's conformal field equations (which provide an evolution system which, in contrast to Einstein's field equations, is well-behaved at $\mathscr{I}^+$) need to be smooth at $\mathscr{I}^+$.

5. Finally, an appropriate well-posedness result for the conformal field equations is needed.

Point 1 and 2 have been addressed in [7], cf. Proposition 2.2 below. Point 3 will be the subject of Section 3, while point 4 will be investigated in Section 5. Point 5 will be addressed elsewhere.

From now on we shall consider exclusively conformal data $\gamma_{AB}(r,\cdot)dx^A dx^B$ and gauge functions κ and $\overline{W}^\lambda$ for which (2.12) and (2.13) hold. Let us summarize some of the results established in [7] (adapted to the smooth setting on which we focus here) which provide sufficient conditions such that (2.12) and (2.13) hold in the case where $\mathscr{N}$ represents a regular light-cone C_O:[2]

Proposition 2.2 *1. Solutions of the Raychaudhuri equation (2.5) with prescribed $\kappa = \mathcal{O}(r^{-3})$ and ${\sigma_A}^B = \mathcal{O}(r^{-2})$ lead to a globally positive φ on $C_O \setminus \{O\}$ with $\varphi_{-1} > 0$ on S^2 if*

$$\int_0^\infty \Big(\int_0^r e^{H(\hat{r})} \mathrm{d}\hat{r} \Big) e^{-H(r)} |\sigma|^2(r)\, \mathrm{d}r < 2 \;, \quad \textit{where } H(r, x^A) := \int_0^r \kappa(\tilde{r}, x^A) d\tilde{r} \;.$$

2. Assuming a Minkowski target and

$$\overline{W}^0 = \mathcal{O}(r^{-1}) \;\; \textit{with} \;\; \overline{W}^0 < r\varphi^{-2} \sqrt{\frac{\det\gamma}{\det s}} \gamma^{AB} s_{AB} \;\; \textit{and} \;\; (\overline{W}^0)_1 < 2(\varphi_{-1})^{-2} \;,$$

any positive solution φ of (2.5) leads to a globally defined positive function ν_0 on C_O with $0 < (\nu_0)_0 < \infty$.

2.3 A priori restrictions

Before we analyze thoroughly the asymptotic behavior of the vacuum Einstein constraint equations and derive necessary-and-sufficient conditions concerning smoothness of the solutions at infinity it is convenient to have some a priori knowledge regarding the lowest admissible orders of the gauge functions, and to exclude the appearance of log terms in the expansion of "auxiliary" fields such as $\xi_A = \overline{\Gamma}^r_{rA}$. In [7] we have shown that the following equations need to be necessarily fulfilled in some adapted null coordinate system to end up with a trace of a metric on $\mathscr{N}$ which admits a smooth conformal completion and infinity, and connection coefficients which are smooth at $\mathscr{I}^+$:

$$\kappa = \mathcal{O}(r^{-3}) \;, \quad \overline{W}^0 = \mathcal{O}(r^{-1}) \;, \quad \xi_A = \mathcal{O}(r^{-1}) \;, \quad \overline{W}^A = \mathcal{O}(r^{-1}) \;. \tag{2.14}$$

Moreover, we may assume the initial data to be of the asymptotic form

$$\gamma_{AB} \sim r^2 \Big(s_{AB} + \sum_{n=1}^\infty h^{(n)}_{AB} r^{-n} \Big) \;, \tag{2.15}$$

[2] In the conventions of [7] the functions φ, ν^0, φ_{-1} and $(\nu^0)_0$ need to be positive.

for some smooth tensor fields $h^{(n)}_{AB}$ on S^2. If γ is not of the form (2.15), it can either be brought to it via a conformal rescaling and an suitable choice of r, or it leads to a metric $\overline{g}$ which does not have a smooth conformal completion at $\mathscr{I}^+$.

At this stage we do not know whether a space-time which admits a conformal completion at infinity is compatible with polyhomogeneous rather than smooth expansions of the functions ζ and $\overline{W}^r$. It will turn out that this is not the case. However, we note that it follows from the constraint equations and the above a priori restrictions that

$$\text{if } \zeta = \mathcal{O}(r^{-1}) \text{ then } \overline{W}^r = \mathcal{O}(r). \tag{2.16}$$

3 Asymptotic solutions of Einstein's characteristic vacuum constraint equations

It is useful to introduce some notation first: Let w_{AB} be a rank-2-tensor on the initial surface $\mathscr{N}$. We denote by $\breve{w}_{AB}$, or $(w_{AB})\breve{}$, its trace-free part w.r.t. the metric $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$. Consider now the expansion coefficients $(w_{AB})_n$ at infinity, which are tensors on S^2. We denote by $(\breve{w}_{AB})_n$, or $(w_{AB})\breve{}_n$, the trace-free part w.r.t. the unit round metric $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$. Finally, we set

$$|w|^2 := \overline{g}^{AC}\overline{g}^{BD}w_{AB}w_{CD}\ , \quad \text{and} \quad |w_n|^2 := s^{AC}s^{BD}(w_{AB})_n(w_{CD})_n\ .$$

Let us make the convention to raise indices of the expansion coefficients $h^{(n)}_{AB}$ with the standard metric. Moreover, we set

$$h^{(n)} \quad := \quad s^{AB}h^{(n)}_{AB}\ .$$

A ring $\mathring{}$ on a covariant derivative operator or a connection coefficient indicates that the corresponding object is associated to $s = s_{AB}\mathrm{d}x^A\mathrm{d}x^B$.

3.1 Asymptotic solutions of the constraint equations

The object of this section is twofold: First of all we will show that the Einstein wave-map gauge constraints (2.5)-(2.10) can be solved asymptotically in terms of polyhomogeneous expansions at infinity of the solution. This is done by rewriting the equations in a form to which Appendix A applies. The second aim is to make some general considerations concerning the appearance of logarithmic terms in the asymptotic solutions of (2.5)-(2.10) for initial data of the form (2.15). We want to extract necessary-and-sufficient conditions leading to the trace $\overline{g}$ of a metric on $\mathscr{N}$ which admits a smooth conformal completion at infinity in the sense of Definition 2.1.

Our starting point are initial data γ with an asymptotic behavior of the form (2.15) and gauge functions

$$\kappa = \mathcal{O}(r^{-3})\ , \quad \overline{W}^0 = \mathcal{O}(r^{-1})\ , \quad \overline{W}^A = \mathcal{O}(r^{-1})\ , \quad \overline{W}^r = \mathcal{O}(r)\ , \tag{3.1}$$

for which φ, ν^0, φ_{-1} and $(\nu^0)_0$ have a sign on $\mathscr{N}$ and S^2, respectively. We further require that the asymptotic expansion of ξ_A contains no logarithmic terms, i.e. $\xi_A = \mathcal{O}(r^{-1})$. A violation of one of these assumptions would not be compatible with a space-time which admits a smooth conformal completion at infinity as follows from the a priori restriction and the following fact: The considerations below reveal that (2.15), $\kappa = \mathcal{O}(r^{-3})$ and $\xi_A = \mathcal{O}(r^{-1})$ *imply* $\zeta = \mathcal{O}(r^{-1})$, and that (2.16) applies. So we are imposing no restrictions when assuming $\overline{W}^r = \mathcal{O}(r)$.

Consider the shear tensor,

$$\sigma_A{}^B = \frac{1}{2}\overline{g}^{BC}(\partial_r\overline{g}_{AC})\breve{} = \frac{1}{2}\gamma^{BC}(\partial_r\gamma_{AC})\breve{}\,, \tag{3.2}$$

whose asymptotic expansion we express in terms of the expansion coefficients of the initial data γ

$$\sigma_A{}^B = -\frac{1}{2}\breve{h}^{(1)B}_A r^{-2} + \Big(\frac{1}{2}h^{(1)}\breve{h}^{(1)B}_A - \breve{h}^{(2)B}_A\Big)r^{-3} + \mathcal{O}(r^{-4})\,, \tag{3.3}$$

$$|\sigma|^2 = \frac{1}{4}|\breve{h}^{(1)}|^2 r^{-4} + \Big(\breve{h}^{(1)B}_A\breve{h}^{(2)A}_B - \frac{1}{2}h^{(1)}|\breve{h}^{(1)}|^2\Big)r^{-5} + \mathcal{O}(r^{-6})\,. \tag{3.4}$$

A global solution, and thereby also the value of the "asymptotic integration functions", to each of the constraint ODEs is determined by regularity conditions at the vertex O of a light-cone, or by the data on the intersection manifold S for two intersecting characteristic surfaces. However, the integration functions, which depend on the initial data γ, the gauge functions and the boundary conditions at the vertex or the intersection manifold, appear difficult to control.

3.1.1 Expansion of φ

We start with the constraint equation (2.5) for the function φ,

$$\partial^2_{rr}\varphi - \kappa\partial_r\varphi + \frac{1}{2}|\sigma|^2\varphi = 0\,. \tag{3.5}$$

In order to enable an easier comparison to Appendix A we make the transformation $r \mapsto 1/r \equiv x$, with φ, κ and $|\sigma|^2$ treated as scalars, and rewrite the ODE as a first-order system. The equation then reads with $\varphi^{(1)} := \varphi$ and $\varphi^{(2)} := x\partial_x\varphi$,

$$\left[x\partial_x + \begin{pmatrix} 0 & -1 \\ \frac{|\sigma|^2}{2x} & 1+\frac{\kappa}{x}\end{pmatrix}\right]\begin{pmatrix}\varphi^{(1)} \\ \varphi^{(2)}\end{pmatrix} = 0\,,$$

or, when the leading order term is diagonalized,

$$\left[x\partial_x + \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\kappa}{x} - \frac{|\sigma|^2}{2x} & -\frac{|\sigma|^2}{\sqrt{2}x} \\ \frac{|\sigma|^2}{2^{3/2}x} - \frac{\kappa}{\sqrt{2}x} & \frac{|\sigma|^2}{2x}\end{pmatrix}}_{=\mathcal{O}(x)}\right]\begin{pmatrix}\tilde{\varphi}^{(1)} \\ \tilde{\varphi}^{(2)}\end{pmatrix} = 0\,,$$

with

$$\begin{pmatrix}\tilde{\varphi}^{(1)} \\ \tilde{\varphi}^{(2)}\end{pmatrix} := \begin{pmatrix} 0 & -\sqrt{2} \\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^{(1)} \\ \varphi^{(2)}\end{pmatrix}\,.$$

The results in Appendix A (we have, in the notation used there, $\lambda_1 = -1$, $\lambda_2 = 0$, and, since there is no source, $\hat{\ell} = -1$) shows that this ODE can be solved asymptotically via a polyhomogeneous expansion with $\tilde{\varphi}^{(n)} = O(x^{-1})$. It also reveals that the coefficients $(\tilde{\varphi}^{(n)})_{\lambda_n}$, $n = 1, 2$, i.e. $(\tilde{\varphi}^{(1)})_{-1}$ and $(\tilde{\varphi}^{(2)})_0$, can be regarded as integrations functions, and that logarithmic terms do not appear if and only if (cf. condition (A.19))

$$\Big[\Big(\frac{|\sigma|^2}{2^{3/2}x} - \frac{\kappa}{\sqrt{2}x}\Big)\tilde{\varphi}^{(1)} - \frac{|\sigma|^2}{2x}\tilde{\varphi}^{(2)}\Big]_0 = 0 \; . \tag{3.6}$$

Recall that $\cdot_n$ denotes the r^{-n}-term (x^n-term) in the asymptotic expansion of the corresponding field. Using $\kappa = \mathcal{O}(r^{-3})$ and $|\sigma|^2 = \mathcal{O}(r^{-4})$ we observe that (3.6) holds automatically. In particular we have $\varphi = \mathcal{O}(r)$. Furthermore, since

$$\begin{aligned} (\tilde{\varphi}^{(1)})_{-1} &= -\sqrt{2}(\varphi^{(2)})_{-1} = \sqrt{2}\varphi_{-1} \; , \\ (\tilde{\varphi}^{(2)})_0 &= (\varphi^{(1)})_0 + (\varphi^{(2)})_0 = \varphi_0 \; , \end{aligned}$$

the coefficients φ_{-1} and φ_0 can be identified as the integration functions. As indicated above, though being left undetermined by the equation itself, they have global character.

In the following we shall set for convenience

$$\sigma_n := (|\sigma|^2)_n \; . \tag{3.7}$$

Inserting the expansion $\varphi \sim \sum_{n=\hat{\ell}}^{\infty} \varphi_n r^{-n}$ into (3.5) and equating coefficients gives the expansion coefficients φ_n by a hierarchy of equations,

$$\varphi_1 = \Big(\frac{1}{2}\kappa_3 - \frac{1}{4}\sigma_4\Big)\varphi_{-1} \; , \tag{3.8}$$

$$\varphi_2 = \Big(\frac{1}{6}\kappa_4 - \frac{1}{12}\sigma_5\Big)\varphi_{-1} - \frac{1}{12}\sigma_4\varphi_0 \; , \tag{3.9}$$

while

$$\begin{aligned} \tau \equiv 2\partial_r \log\varphi &= 2r^{-1} - 2\varphi_0(\varphi_{-1})^{-1}r^{-2} \\ &\quad + [2(\varphi_0)^2(\varphi_{-1})^{-2} + \sigma_4 - 2\kappa_3]r^{-3} + \mathcal{O}(r^{-4}) \; . \end{aligned} \tag{3.10}$$

Consider the conformal factor relating the r-dependent Riemannian metric $\check{g}_{\Sigma_r}$ and γ, $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \Omega^2\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$. We find

$$\det\gamma = \det s\Big[r^4 + h^{(1)}r^3 + \Big(h^{(2)} + \frac{1}{2}(h^{(1)})^2 - \frac{1}{2}|h^{(1)}|^2\Big)r^2\Big] + \mathcal{O}(r) \; , \tag{3.11}$$

$$\begin{aligned} \Omega \equiv \varphi\Big(\frac{\det s}{\det\gamma}\Big)^{1/4} &= \varphi_{-1} - \frac{1}{4}\varphi_{-1}\Big(2\tau_2 + h^{(1)}\Big)r^{-1} \\ &\quad + \frac{1}{4}\varphi_{-1}\Big[2\kappa_3 + \frac{1}{4}(h^{(1)})^2 + \frac{1}{4}|h^{(1)}|^2 - h^{(2)} + \frac{1}{2}\tau_2 h^{(1)}\Big]r^{-2} + \mathcal{O}(r^{-3}) \; . \end{aligned} \tag{3.12}$$

We conclude that

$$\begin{aligned} \overline{g}_{AB} &= (\varphi_{-1})^2\Big[s_{AB}r^2 + (\check{h}^{(1)}_{AB} - \tau_2 s_{AB})r \\ &\quad + \check{h}^{(2)}_{AB} - (\tau_2 + \frac{1}{2}h^{(1)})\check{h}^{(1)}_{AB} + [\frac{1}{4}(\tau_2)^2 + \kappa_3 + \frac{1}{2}\sigma_4]s_{AB}\Big] + \mathcal{O}(r^{-1}) \; . \end{aligned} \tag{3.13}$$

To sum it up, (2.15) and $\kappa = \mathcal{O}(r^{-3})$ imply that no logarithmic terms appear in the conformal factor relating γ_{AB} and $\overline{g}_{AB}$, the latter one thus being smoothly extendable at $\mathscr{I}^+$ as a Riemannian metric on S^2 whenever a global solution of the Raychaudhuri equation exists on $\mathscr{N}$ with $\varphi_{-1} \neq 0$.

3.1.2 Expansion of ν^0

We consider the constraint equation (2.6) which determines ν^0,

$$\partial_r \nu^0 + \nu^0(\frac{1}{2}\tau + \kappa) + \frac{1}{2}(\overline{W}^0 + \overline{\hat{\Gamma}}^0) \quad = \quad 0 \;, \tag{3.14}$$

where $\overline{W}^0 = \mathcal{O}(r^{-1})$ and, using (2.4),

$$\begin{aligned} \overline{\hat{\Gamma}}^0 &= 2\nu^0(\hat{\nu}^0 \partial_r \hat{\nu}_0 - \hat{\kappa}) - \hat{\nu}^0 \Omega^{-2} \gamma^{AB} \hat{\chi}_{AB} \\ &= \mathcal{O}(r^{-3})\nu^0 - 2(\varphi_{-1})^{-2} r^{-1} - 2(\varphi_{-1})^{-2}\tau_2 r^{-2} + \mathcal{O}(r^{-3}) \;. \end{aligned} \tag{3.15}$$

Again, we express the ODE by $x \equiv 1/r$. Its asymptotic form reads

$$\begin{aligned} &[x\partial_x - 1 - \frac{\tau_2}{2}x + \mathcal{O}(x^2)]\nu^0 \\ &= \underbrace{\frac{1}{2}(\overline{W}^0)_1 - (\varphi_{-1})^{-2}}_{=:-\Phi^{-2}} + [\frac{1}{2}(\overline{W}^0)_2 - (\varphi_{-1})^{-2}\tau_2]x + \mathcal{O}(x^2) \;. \end{aligned} \tag{3.16}$$

We want to make sure that the asymptotic solution of ν^0 can be written as power series. In the notation of Appendix A we have $\lambda = 1$ and $\hat{\ell} = 0$, and condition (A.10) which characterizes the absence of logarithmic terms reads

$$(\overline{W}^0)_2 \quad = \quad \Big[\frac{1}{2}(\overline{W}^0)_1 + (\varphi_{-1})^{-2}\Big]\tau_2 \;. \tag{3.17}$$

Assuming (3.17) we insert the expansion $\nu^0 \sim \sum_{n=\hat{\ell}}^{\infty}(\nu^0)_n r^{-n}$ into (3.14) to obtain the expansion coefficients $(\nu^0)_n$ in terms of κ, $\overline{W}^0$, φ and γ,

$$\nu^0 \quad = \quad \Phi^{-2} - D^{(\nu_0)}\Phi^{-4} r^{-1} + \mathcal{O}(r^{-2}) \;, \tag{3.18}$$

where $D^{(\nu_0)}$ denotes the globally defined integration function.

As mentioned above, for $\overline{g}_{\mu\nu}$ to have a smooth conformal completion at infinity the function $(\nu^0)_0$ needs to be of constant sign which is the case if and only if the gauge source function is chosen so that (compare Proposition 2.2)

$$(\overline{W}^0)_1 \neq 2(\varphi_{-1})^{-2} \;\; \forall x^A \;. \tag{3.19}$$

For the inverse of ν^0 we then find

$$\nu_0 \quad = \quad \Phi^2 + D^{(\nu_0)} r^{-1} + \mathcal{O}(r^{-2}) \;. \tag{3.20}$$

Note that in the special case where $(\overline{W}^0)_1 = 0$ we have $\Phi = \varphi_{-1}$ and the positivity of $(\nu^0)_0$ follows from that of φ_{-1}.

Since $(\overline{W}^0)_1$ can be prescribed arbitrarily and the value of φ_{-1} does not depend on that choice, (3.19) is not a geometric restriction. Similarly, (3.17) can be fulfilled by an appropriate choice of $(\overline{W}^0)_2$. The gauge freedom associated with the choice of $\overline{W}^0$ can be used to control the behavior of ν^0 and to get rid of the log terms in its asymptotic expansion (only the two leading order terms in the expansion of $\overline{W}^0$ are affected, compare the discussion in Section 3.3).

3.1.3 Expansion of ξ_A

The connection coefficients $\xi_A = -2\overline{\Gamma}^r_{rA}$ are determined by (2.7),

$$(\partial_r + \tau)\xi_A - 2\check{\nabla}_B \sigma_A{}^B + \partial_A \tau + 2\partial_A \kappa \quad = \quad 0 \ . \tag{3.21}$$

To compute the covariant derivative of $\sigma_A{}^B$ associated to $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$, we first determine the asymptotic form of the Christoffel symbols,

$$\check{\Gamma}^C_{AB} \ = \ \frac{1}{2}\gamma^{CD}(2\partial_{(A}\gamma_{B)D} - \partial_D\gamma_{AB}) + 2\delta_{(B}{}^C\partial_{A)}\log\Omega - \gamma^{CD}\gamma_{AB}\partial_D\log\Omega \ = \ \mathcal{O}(1) \ ,$$

with

$$(\check{\Gamma}^C_{AB})_0 \quad = \quad \mathring{\Gamma}^C_{AB} + 2\delta_{(B}{}^C\mathring{\nabla}_{A)}\log\varphi_{-1} - s_{AB}\mathring{\nabla}^C\log\varphi_{-1} \ , \tag{3.22}$$

$$(\check{\Gamma}^C_{AB})_1 \quad = \quad \mathring{\nabla}_{(A}\check{h}^{(1)C}_{B)} - \frac{1}{2}\mathring{\nabla}^C\check{h}^{(1)}_{AB} - \delta_{(A}{}^C\mathring{\nabla}_{B)}\tau_2 + \frac{1}{2}s_{AB}\mathring{\nabla}^C\tau_2$$
$$-(s^{CD}h^{(1)}_{AB} - h^{(1)CD}s_{AB})\mathring{\nabla}_D\log\varphi_{-1} \ . \tag{3.23}$$

Invoking (3.3) that yields

$$\check{\nabla}_B\sigma_A{}^B \quad = \quad \Xi^{(2)}_A r^{-2} + \Xi^{(3)}_A r^{-3} + \mathcal{O}(r^{-4}) \ , \tag{3.24}$$

where

$$\Xi^{(2)}_A = -\frac{1}{2}\mathring{\nabla}_B\check{h}^{(1)B}_A - \check{h}^{(1)B}_A\mathring{\nabla}_B\log\varphi_{-1} = \mathring{\nabla}_B(\sigma_A{}^B)_2 + 2(\sigma_A{}^B)_2\mathring{\nabla}_B\log\varphi_{-1} \ , \tag{3.25}$$

$$\Xi^{(3)}_A = \mathring{\nabla}_B(\sigma_A{}^B)_3 + 2(\sigma_A{}^B)_3\mathring{\nabla}_B\log\varphi_{-1} - (\sigma_A{}^B)_2\mathring{\nabla}_B\tau_2 + \frac{1}{2}\mathring{\nabla}_A\sigma_4 \ . \tag{3.26}$$

Substituting now the coefficients by their asymptotic expansions we observe that (3.21) has the asymptotic structure,

$$(\partial_r + 2r^{-1} + \tau_2 r^{-2} + \mathcal{O}(r^{-3}))\xi_A$$
$$= (2\Xi^{(2)}_A - \partial_A\tau_2)r^{-2} + [2\Xi^{(3)}_A - \partial_A(\tau_3 + 2\kappa_3)]r^{-3} + \mathcal{O}(r^{-4}) \ .$$

Nicely enough, the ODEs for ξ_A, $A = 2, 3$, are decoupled. For comparison with the formulae in Appendix A we rewrite them in terms of $x \equiv 1/r$,

$$(x\partial_x - 2 - \tau_2 x + \mathcal{O}(x^2))\xi_A$$
$$= (\partial_A\tau_2 - 2\Xi^{(2)}_A)x - [2\Xi^{(3)}_A - \partial_A(\tau_3 + 2\kappa_3)]x^2 + \mathcal{O}(x^3) \ .$$

Appendix A tells us (with $\lambda = 2$ and $\hat{\ell} = 1$) that there are no logarithmic terms in the expansion of ξ_A if and only if (A.10) holds,

$$\tau_2(2\Xi^{(2)}_A - \partial_A\tau_2) = 2\Xi^{(3)}_A - \mathring{\nabla}_A(\tau_3 + 2\kappa_3) \ . \tag{3.27}$$

The asymptotic expansion (3.10) of τ implies

$$\tau_3 = \frac{1}{2}(\tau_2)^2 + \sigma_4 - 2\kappa_3 \ , \tag{3.28}$$

such that (3.27) can be written as

$$2\tau_2\Xi^{(2)}_A = 2\Xi^{(3)}_A - \mathring{\nabla}_A\sigma_4 \ . \tag{3.29}$$

Note that κ_3, on which we have not imposed conditions yet, drops out, so there is no gauge freedom left which could be appropriately adjusted to fulfill this equation. The impact of (3.29) will be analyzed in Section 3.2, where it becomes manifest that it *does* impose geometric restrictions on the initial data. We refer to (3.29) as *no-logs-condition*.

Whenever the no-logs-condition holds, which we assume henceforth, the covector field ξ_A can be expanded as a power series,

$$\xi_A = (2\Xi^{(2)}_A - \mathring{\nabla}_A\tau_2)r^{-1} + C^{(\xi_B)}_A r^{-2} + \mathcal{O}(r^{-3}) \; . \tag{3.30}$$

The coefficients $C^{(\xi_B)}_A = C^{(\xi_B)}_A(x^C)$, $A = 2,3$, represent globally defined integration functions.

3.1.4 Expansion of ν_A

We analyze (2.8) to compute the asymptotic behavior of ν_A,

$$\big[2\partial_r - \nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}{}^0)\big]\nu_A - 4\nu_B\chi_A{}^B + \nu_0[\xi_A - \overline{g}_{AB}(\overline{W}^B + \overline{\hat{\Gamma}}{}^B) + \gamma_{AB}\gamma^{CD}\check{\Gamma}^B_{CD}] = 0 \; . \tag{3.31}$$

Here

$$\begin{aligned}
\nu_0\overline{g}_{AB}\overline{\hat{\Gamma}}{}^B &= 2\overline{g}_{AB}\overline{\hat{\Gamma}}{}^B_{01} - 2\overline{g}_{AB}\nu^C\hat{\chi}_C{}^B + \nu_0\gamma_{AB}\gamma^{CD}(\hat{\mathring{\Gamma}}{}^B_{CD} - \overline{\hat{g}}{}^{1B}\hat{\chi}_{CD}) \\
&= -\hat{\tau}\nu_A + v_A{}^B\nu_B + \nu_0\gamma_{AB}\gamma^{CD}\mathring{\Gamma}^B_{CD} + \mathcal{O}(r^{-2}) \; ,
\end{aligned}$$

with $v_C{}^B = \mathcal{O}(r^{-2}) \in \mathrm{Mat}(2,2)$, as follows from (2.4). Recall that $\overline{W}^0 = \mathcal{O}(r^{-1})$ and $\overline{W}^A = \mathcal{O}(r^{-1})$, or, by (3.13), $\overline{W}_A := \overline{g}_{AB}\overline{W}^B = \mathcal{O}(r)$. We determine the asymptotic expansions of the coefficients involved,

$$\begin{aligned}
\nu_B\chi_A{}^B &\equiv \frac{1}{2}\nu_B\overline{g}^{BC}\partial_r\overline{g}_{AC} = \nu_A\partial_r\log\Omega + \frac{1}{2}\nu_B\gamma^{BC}\partial_r\gamma_{AC} \\
&= \nu_A r^{-1} + \frac{1}{2}(\tau_2\nu_A - \check{h}^{(1)B}_A\nu_B)r^{-2} + \mathcal{O}(r^{-3}) \; .
\end{aligned} \tag{3.32}$$

For the source term involving Christoffel symbols of the metrics s_{AB} and $\overline{g}_{AB} = \Omega^2\gamma_{AB}$ we obtain,

$$\gamma_{AB}\gamma^{CD}(\mathring{\Gamma}^B_{CD} - \check{\Gamma}^B_{CD}) = -\mathring{\nabla}_B\check{h}^{(1)B}_A r^{-1} + \mathcal{O}(r^{-2}) \; . \tag{3.33}$$

Combining this with what we found for ν_0, $\overline{\hat{\Gamma}}{}^0$ and ξ_A (assuming that the no-logs-condition holds) the ODE for ν_A takes the asymptotic form

$$\begin{aligned}
\partial_r\nu_A + w_A{}^B\nu_B = \frac{\Phi^2}{2}(\overline{W}_A)_{-1}r + \frac{1}{2}\Big[\Phi^2(\overline{W}_A)_0 + D^{(\nu_0)}(\overline{W}_A)_{-1}\Big] + \frac{1}{2}\Big[(\nu_0)_2(\overline{W}_A)_{-1} \\
+ D^{(\nu_0)}(\overline{W}_A)_0 + \Phi^2\big(\mathring{\nabla}_A\tau_2 + 2\check{h}^{(1)B}_A\mathring{\nabla}_B\log\varphi_{-1} + (\overline{W}_A)_1\big)\Big]r^{-1} + \mathcal{O}(r^{-2}) \; ,
\end{aligned}$$

where

$$\begin{aligned}
w_A{}^B &= \Big[\frac{\hat{\tau}}{2} - \frac{1}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}{}^0)\Big]\delta_A{}^B - 2\chi_A{}^B \\
&= \Big[[\Phi^{-2}D^{(\nu_0)} - \frac{1}{2}\tau_2]\delta_A{}^B - 2(\sigma_A{}^B)_2\Big]r^{-2} + \mathcal{O}(r^{-3}) \in \mathrm{Mat}(2,2) \; .
\end{aligned}$$

Note that this first-order system is *not* decoupled. In terms of $x \equiv 1/r$ the equation becomes (set $\tilde{w}_A{}^B(x, x^A) = -x^{-1} w_A{}^B(x^{-1}, x^A) = \mathcal{O}(x)$)

$$\begin{aligned} x\partial_x \nu_A + \tilde{w}_A{}^B \nu_B &= -\frac{1}{2}\Phi^2(\overline{W}_A)_{-1}x^{-2} - \frac{1}{2}\Big[\Phi^2(\overline{W}_A)_0 + D^{(\nu_0)}(\overline{W}_A)_{-1}\Big]x^{-1} \\ -\frac{1}{2}\Big[(\nu_0)_2(\overline{W}_A)_{-1} &+ D^{(\nu_0)}(\overline{W}_A)_0 + \Phi^2\big(\mathring{\nabla}_A\tau_2 + 2\breve{h}^{(1)B}_A\mathring{\nabla}_B\log\varphi_{-1} + (\overline{W}_A)_1\big)\Big] + \mathcal{O}(x)\;. \end{aligned}$$

Again, we consult Appendix A; inspection of (A.19) tells us (with $\lambda_1 = \lambda_2 = 0$, $\hat{\ell} = -2$) that no logarithmic terms appear whenever the source satisfies,

$$\begin{aligned} (\overline{W}_A)_1 &= 4(\sigma_A{}^B)_2\mathring{\nabla}_B\log\varphi_{-1} - \mathring{\nabla}_A\tau_2 - \Phi^{-2}(\nu_0)_2(\overline{W}_A)_{-1} \\ &\quad -\Phi^{-2}D^{(\nu_0)}(\overline{W}_A)_0 - \frac{1}{2}[(\tilde{w}_A{}^B)_1(\tilde{w}_B{}^C)_1 + (\tilde{w}_A{}^C)_2](\overline{W}_C)_{-1} \\ &\quad -(\tilde{w}_A{}^B)_1\Big[(\overline{W}_B)_0 + \Phi^{-2}D^{(\nu_0)}(\overline{W}_B)_{-1}\Big]\;. \end{aligned} \tag{3.34}$$

We express (3.34) in terms of $\overline{W}^A = \overline{g}^{AB}\overline{W}_B$. It can be solved for $(\overline{W}^A)_1$, and thus provides a condition on $(\overline{W}^A)_1$ once $(\overline{W}^A)_{-1}$ and $(\overline{W}^A)_0$ have been chosen.

Assuming that the gauge functions $\overline{W}^A$ have an asymptotic behavior which fulfills (3.34), any solution of (3.31) has the asymptotic form $\nu_A \sim \sum_{n=\hat{\ell}}^{\infty}(\nu_A)_n r^{-n}$. For suitable, globally defined integration functions $D_A^{(\nu_A)}$ we find,

$$\begin{aligned} \nu_A &= \frac{\Phi^2}{4}(\overline{W}_A)_{-1}r^2 + \Big[\frac{\Phi^2}{2}[(\overline{W}_A)_0 + (\sigma_A{}^B)_2(\overline{W}_B)_{-1} + \frac{\tau_2}{4}(\overline{W}_A)_{-1}] \\ &\quad +\frac{1}{4}D^{(\nu_0)}(\overline{W}_A)_{-1}\Big]r + D_A^{(\nu_A)} + \mathcal{O}(r^{-1})\;. \end{aligned} \tag{3.35}$$

3.1.5 Expansion of ζ

We consider the ODE (2.9) which determines the auxiliary function ζ,

$$(\partial_r + \tau + \kappa)\zeta + \check{R} - \frac{1}{2}\xi_A\xi^A + \check{\nabla}_A\xi^A = 0\;. \tag{3.36}$$

It remains to compute the source terms. From (3.13) and (3.30) we find

$$\frac{1}{2}\overline{g}^{AB}\xi_A\xi_B = \mathcal{O}(r^{-4}) \tag{3.37}$$

$$\overline{g}^{AB}\check{\nabla}_A\xi_B = (\varphi_{-1})^{-2}\mathring{\nabla}^A(\xi_A)_1 r^{-3} + \mathcal{O}(r^{-4})\;, \tag{3.38}$$

with

$$(\xi_A)_1 = 2(\check{\nabla}_B\sigma_A{}^B)_2 - \mathring{\nabla}_A\tau_2\;. \tag{3.39}$$

LEMMA 3.1 *The curvature scalar* $\check{R}$ *satisfies*

$$\check{R} = \check{R}_2 r^{-2} + [\tau_2\check{R}_2 - (\varphi_{-1})^{-2}\mathring{\nabla}^A(\xi_A)_1]r^{-3} + \mathcal{O}(r^{-4})\;,$$

with $\check{R}_2 = 2(\varphi_{-1})^{-2}(1 - \Delta_s\log\varphi_{-1})$.

Since the computation of the curvature scalar is elementary though somewhat lengthy we leave the proof of Lemma 3.1 to the reader.

Putting everything together, we observe that the ζ-constraint is of the form

$$\Big[\partial_r + \frac{2}{r} + \tau_2 r^{-2} + \mathcal{O}(r^{-3})\Big]\zeta \quad = \quad -\check{R}_2 r^{-2} - \tau_2 \check{R}_2 r^{-3} + \mathcal{O}(r^{-4}) \ .$$

We express the ODE in terms of $x \equiv 1/r$,

$$\big[x\partial_x - 2 - \tau_2 x + \mathcal{O}(x^2)\big]\,\zeta \quad = \quad \check{R}_2 x + \tau_2 \check{R}_2 x^2 + \mathcal{O}(x^3) \ .$$

Comparison with Appendix A tells us (with $\lambda = 2$ and $\hat{\ell} = 1$) that, due to the specific form of the source, condition (A.10), which excludes the appearance of logarithmic terms, is automatically satisfied, and the asymptotic expansion of ζ reads

$$\zeta = -\check{R}_2 r^{-1} + C^{(\zeta)} r^{-2} + \mathcal{O}(r^{-3}) \ , \tag{3.40}$$

where $C^{(\zeta)} = C^{(\zeta)}(x^A)$ can be identified with the integration function. It thus *follows* that $\zeta = \mathcal{O}(r^{-1})$, supposing that $\kappa = \mathcal{O}(r^{-3})$, that γ has the asymptotic form (2.15) with $\varphi_{-1} \neq 0$, and that the no-logs-condition holds.

3.1.6 Expansion of $\overline{g}^{rr}$

Let us compute $\overline{g}^{rr}$ which satisfies (2.10),

$$(\partial_r + \frac{1}{2}\tau + \kappa)\overline{g}^{rr} = \frac{1}{2}\zeta - \overline{W}^r - \overline{\hat{\Gamma}}^r \ . \tag{3.41}$$

Recall that now where we have established (3.40), it follows from the a priori restrictions that $\overline{W}^r = \mathcal{O}(r)$. We further have,

$$\begin{aligned} \overline{\hat{\Gamma}}^r \quad &= \quad 2\nu^0 \hat{\Gamma}^r_{0r} + \overline{g}^{rr}\hat{\kappa} - \overline{g}^{rA}\hat{\xi}_A + \overline{g}^{AB}\hat{\Gamma}^r_{AB} \\ &= \quad \hat{\kappa}\overline{g}^{rr} - 2(\varphi_{-1})^{-2} r^{-1} - 2(\varphi_{-1})^{-2}\tau_2 r^{-2} + \mathcal{O}(r^{-3}) \ , \end{aligned}$$

so the ODE for $\overline{g}^{rr}$ is of the form

$$(\partial_r + r^{-1} + \mathcal{O}(r^{-2}))\overline{g}^{rr} = \mathcal{O}(r) \ ,$$

or, expressed in terms of the $x \equiv 1/r$-coordinate,

$$(x\partial_x - 1 + f^*)\overline{g}^{rr} = f^{**} \ , \tag{3.42}$$

where

$$\begin{aligned} f^* \quad &= \quad -\frac{1}{2}\tau_2 x - (\frac{1}{2}\tau_3 + \kappa_3 + \hat{\kappa}_3)x^2 - (\frac{1}{2}\tau_4 + \kappa_4 + \hat{\kappa}_4)x^3 + \mathcal{O}(x^4) \ , \\ f^{**} \quad &= \quad (\overline{W}^r)_{-1} x^{-2} + (\overline{W}^r)_0 x^{-1} + (\overline{W}^r)_1 + \frac{1}{2}\check{R}_2 - 2(\varphi_{-1})^{-2} \\ &\quad + \Big[(\overline{W}^r)_2 - \frac{1}{2}\zeta_2 - 2(\varphi_{-1})^{-2}\tau_2\Big] x + \mathcal{O}(x^2) \ . \end{aligned}$$

We analyze the occurrence of logarithmic terms in the asymptotic solution of (3.42). In the notation of Appendix A we have $\lambda = 1$ and $\hat{\ell} = -2$. The considerations made there show that the asymptotic expansion of $\overline{g}^{rr}$ is $\mathcal{O}(r^2)$ if and only if the following condition is fulfilled (cf. (A.10)),

$$f_1^*(\overline{g}^{rr})_0 + f_2^*(\overline{g}^{rr})_{-1} + f_3^*(\overline{g}^{rr})_{-2} = f_1^{**} \,. \tag{3.43}$$

The expansion coefficients $(\overline{g}^{11})_i$ can be derived from (A.9),

$$\begin{aligned}
(\overline{g}^{rr})_{-2} &= -\frac{1}{3}f_{-2}^{**} \,, \\
(\overline{g}^{rr})_{-1} &= -\frac{1}{2}f_{-1}^{**} - \frac{1}{6}f_1^* f_{-2}^{**} \,, \\
(\overline{g}^{rr})_0 &= -f_0^{**} - \frac{1}{3}f_2^* f_{-2}^{**} - \frac{1}{2}f_1^* f_{-1}^{**} - \frac{1}{6}(f_1^*)^2 f_{-2}^{**} \,.
\end{aligned}$$

A straightforward calculation reveals that (3.43) is equivalent to

$$\begin{aligned}
(\overline{W}^r)_2 &= \frac{\zeta_2}{2} + (\varphi_{-1})^{-2}\tau_2 + \frac{\tau_2}{4}\check{R}_2 + \frac{\tau_2}{2}(\overline{W}^r)_1 + \Big[\frac{\tau_3}{4} + \frac{\kappa_3 + \hat{\kappa}_3}{2} - \frac{(\tau_2)^2}{8}\Big](\overline{W}^r)_0 \\
&\quad \Big[\frac{1}{48}(\tau_2)^3 - \frac{\tau_2}{4}(\frac{\tau_3}{2} + \kappa_3 + \hat{\kappa}_3) + \frac{\tau_4}{6} + \frac{1}{3}(\kappa_4 + \hat{\kappa}_4)\Big](\overline{W}^r)_{-1} \,. \qquad (3.44)
\end{aligned}$$

By an appropriate choice of the gauge source function $\overline{W}^r$, or merely the expansion coefficient $(\overline{W}^r)_2$, one can always arrange that (3.44) holds and thereby get rid of all logarithmic terms in the expansion of $\overline{g}^{rr}$. In that case

$$\begin{aligned}
\overline{g}^{rr} &= -\frac{1}{3}(\overline{W}^r)_{-1}r^2 + \mathcal{O}(r) \\
\Longrightarrow \quad \overline{g}_{00} &= \Phi^4\Big[\frac{1}{16}(\varphi_{-1})^2 s_{AB}(\overline{W}^A)_1(\overline{W}^B)_1 + \frac{1}{3}(\overline{W}^r)_{-1}\Big]r^2 + \mathcal{O}(r) \,.
\end{aligned}$$

The integration function is represented by $(\overline{g}_{00})_1$, and its explicit form is not relevant here.

3.2 The no-logs-condition

The no-logs-condition (3.29)

$$2\tau_2[\check{\nabla}_B\sigma_A{}^B]_2 \;=\; 2[\check{\nabla}_B\sigma_A{}^B]_3 - \mathring{\nabla}_A\sigma_4 \,, \tag{3.45}$$

needs to be imposed to exclude logarithmic terms in the asymptotic expansion of ξ_A. Let us rewrite and simplify it. In (3.25) and (3.26) we have computed $[\tilde{\nabla}_B\sigma_A{}^B]_2$ and $[\tilde{\nabla}_B\sigma_A{}^B]_3$. Plugging this in, we observe that the $\mathring{\nabla}_A\sigma_4$-terms cancel out in (3.45), which becomes

$$\mathring{\nabla}_B[\tau_2(\sigma_A{}^B)_2 - (\sigma_A{}^B)_3] + 2[\tau_2(\sigma_A{}^B)_2 - (\sigma_A{}^B)_3\mathring{\nabla}_B]\mathring{\nabla}_B \log\varphi_{-1} = 0 \,. \tag{3.46}$$

This can be written as a divergence,

$$\mathring{\nabla}_B\big[(\varphi_{-1})^2(\tau_2(\sigma_A{}^B)_2 - (\sigma_A{}^B)_3)\big] = 0 \,. \tag{3.47}$$

One can view this equation as a linear first-order PDE-system on S^2. Note that by definition of $\sigma_A{}^B$ the expansion coefficients $(\sigma_A{}^B)_n$ are s-traceless tensors on S^2, whence the term in squared brackets in (3.47) is traceless. It is known (cf. e.g. [9]) that for any smooth source v_A a PDE-system of the form $\mathring{\nabla}_B\zeta_A{}^B = v_A$, with $\zeta_A{}^B$ a traceless tensor on the unit sphere S^2, admits precisely one smooth solution. In our case the source of the PDE vanishes, so we conclude that the no-logs-condition (3.45) is equivalent to (recall our assumption $\varphi_{-1} \neq 0$)

$$(\sigma_A{}^B)_3 = \tau_2(\sigma_A{}^B)_2 \quad \Longleftrightarrow \quad \breve{h}^{(2)}_{AB} = \frac{1}{2}(h^{(1)} + \tau_2)\breve{h}^{(1)}_{AB} \,. \tag{3.48}$$

Since $\xi_A = -2\overline{\Gamma}^r_{rA}$, determined by (2.7), has a geometric meaning one should expect the no-logs-condition to be gauge-invariant. In [7] it is shown that this is indeed the case. Although the ($\kappa = 0$, $\overline{W}^0 = 0$)-wave-map gauge invariably produces logarithmic terms except for the flat case [7], one can decide whether they can be transformed away or not by checking (3.48).

In the current scheme, where conformal data γ are prescribed on $\mathscr{N}$, together with the gauge functions κ and W^λ, this requires to determine τ_2 by solving the Raychaudhuri equation, which makes this scheme not practical for the purpose. In particular, it is not a priori clear within this scheme whether *any* initial data satisfying this condition exist unless both $(\sigma^A{}_B)_2$ and $(\sigma^A{}_B)_3$ vanish ($\Longleftrightarrow$ $\breve{h}^{(1)}_{AB} = 0 = \breve{h}^{(2)}_{AB}$). On the other hand, in gauge schemes where $\breve{g}$ is prescribed, (3.48) is a straightforward condition on its asymptotic behavior (cf. Section 4 where a related scheme is used).

In [7] we also provide a geometric interpretation of the no-logs-condition (3.48) via the conformal Weyl tensor. Moreover, (3.48) can be related to Bondi's "outgoing wave condition", cf. Section 4.5, Remark 4.1.

3.3 Summary and discussion

The subsequent theorem summarizes our analysis of the asymptotic behavior of solutions to Einstein's vacuum constraint equations ((i) and (ii) follow from [7]):

THEOREM 3.2 *Consider the characteristic initial value problem for Einstein's vacuum field equations with smooth conformal data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ and gauge functions κ and $\overline{W}^\lambda$, all defined on a smooth characteristic surface $\mathscr{N}$ meeting $\mathscr{I}^+$ transversally in a smooth spherical cross-section (and supplemented by certain data on the intersection manifold S if $\mathscr{N}$ is one of two transversally intersecting null hypersurfaces). The following conditions are necessary-and-sufficient for the trace of the metric $g = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ on $\mathscr{N}$, obtained as solution to the characteristic wave-map vacuum constraint equations (2.5)-(2.10), to admit a smooth conformal completion at infinity, and for the connection coefficients $\overline{\Gamma}^r_{rA}$ to be smooth at $\mathscr{I}^+$ (in the sense of Definition 2.1), when imposing a generalized wave-map gauge condition $H^\lambda = 0$:*

(i) There exists a one-parameter family $\varkappa = \varkappa(r)$ of Riemannian metrics on S^2 such that $\varkappa$ satisfies (2.15) and is conformal to γ (in particular we may assume γ itself to be of the form (2.15)).

(ii) The functions φ, ν^0, φ_{-1} and $(\nu_0)_0$ have no zeros on $\mathscr{N}$ and S^2, respectively (the non-vanishing of $(\nu^0)_0$ is equivalent to (3.19)).

(iii) The gauge source functions are chosen in such a way that $\kappa = \mathcal{O}(r^{-3})$, $\overline{W}^0 = \mathcal{O}(r^{-1})$, $\overline{W}^A = \mathcal{O}(r^{-1})$, $\overline{W}^r = \mathcal{O}(r)$, and such that the conditions (3.17), (3.34) and (3.44) hold.

(iv) The no-logs-condition is satisfied, i.e.

$$(\sigma_A{}^B)_3 = \tau_2(\sigma_A{}^B)_2 \quad\Longleftrightarrow\quad \breve{h}^{(2)}_{AB} = \frac{1}{2}(h^{(1)} + \tau_2)\breve{h}^{(1)}_{AB} \, . \tag{3.49}$$

REMARK 3.3 For further reference let us explicitly list the conditions (3.17), (3.34) and (3.44) in the case where the gauge source functions satisfy $\overline{W}^0 = \mathcal{O}(r^{-2})$, $\overline{W}^A = \mathcal{O}(r^{-3})$, $\overline{W}^r = \mathcal{O}(r^{-2})$:

$$(\overline{W}^0)_2 = \tau_2(\varphi_{-1})^{-2} \, , \tag{3.50}$$

$$(\overline{W}^A)_3 = -(\varphi_{-1})^{-2}\mathring{\nabla}^A\tau_2 + 4(\varphi_{-1})^{-3}(\sigma_B{}^A)_2\mathring{\nabla}^B\varphi_{-1} \, , \tag{3.51}$$

$$(\overline{W}^r)_2 = \frac{1}{2}\zeta_2 + \tau_2\Big[(\varphi_{-1})^{-2} + \frac{1}{4}\check{R}_2 + \frac{1}{2}(\overline{W}^r)_1\Big] \, . \tag{3.52}$$

If, in addition, the no-logs-condition (3.49) is fulfilled then the leading order terms of the non-vanishing metric components restricted to the cone read

$$\overline{g}_{00} = -(1+\Delta_s\varphi_{-1}) + D^{(\overline{g}_{00})}r^{-1} + \mathcal{O}(r^{-2}) \, , \tag{3.53}$$

$$\nu_0 = (\varphi_{-1})^2 + D^{(\nu_0)}r^{-1} + \mathcal{O}(r^{-2}) \, , \tag{3.54}$$

$$\nu_A = D^{(\nu_A)}_A + \mathcal{O}(r^{-1}) \, ,$$

$$\overline{g}_{AB} = (\varphi_{-1})^2 s_{AB}r^2 + (\varphi_{-1})^2[\breve{h}^{(1)}_{AB} - \tau_2 s_{AB}]r \tag{3.55}$$

$$-\frac{1}{2}\tau_2\breve{h}^{(1)}_{AB} + (\frac{1}{4}(\tau_2)^2 + \kappa_3 + \frac{1}{2}\sigma_4)s_{AB} + \mathcal{O}(r^{-1}) \, . \tag{3.56}$$

The coefficients denoted by D have a global character in that they are globally defined by the initial data and the gauge functions.

REMARK 3.4 For definiteness we have restricted attention to the setting where the initial data are provided by the conformal data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ together with the gauge functions κ and W^λ. Theorem 3.2, though, is quite independent of the particular setting that has been chosen. This is discussed in [7].

It is shown in Appendix A that when analyzing the asymptotic behavior of a linear first-order Fuchsian ODE-system whose indicial matrix has only integer eigenvalues one has to distinguish two different cases. If the indicial matrix cannot be diagonalized, or if it can be but the condition (A.20) of Appendix A is violated, then the appearance of logarithmic terms always depends on the boundary conditions and thereby on the globally defined integration functions. If, however, (A.20) is fulfilled their appearance depends exclusively on the asymptotic behavior of the coefficients in the corresponding ODE. In fact, all the Einstein wave-map gauge constraints are of the latter type.[3]

Due to this property and since many of the constraint equations have a source term which involves a gauge freedom, many, though not all, logarithmic terms arising can be eliminated for given data γ by a carefully adjusted leading-order-term-behavior of the gauge functions. These are precisely the conditions of Theorem 3.2 which involve a gauge source function κ or $\overline{W}^\lambda$. Logarithmic terms which appear if these conditions are violated are pure gauge artifacts.

This does not apply to the ξ_A- and the ζ-equation. Recall that to solve the equation for ξ_A both κ and φ need to be known. This requires a choice of the

[3] Since the integration functions of some constraint equations appear as coefficients in other ones, the no-logs-condition *does* depend on the integration functions φ_{-1} and φ_0.

κ-gauge. Since the choice of $\overline{W}^0$ does not affect the ξ_A-equation, there is no gauge-freedom left. If the no-logs-condition (3.49), which is gauge-invariant [7], does not hold, there is no possibility to get rid of the log terms that arise there, whatever κ has been chosen to be. Similarly, there is no gauge-freedom left when the equation for ζ is integrated but, due to the special asymptotic structure of its source term, no new log terms arise in the expansion of ζ.

The ξ_A-equation is the reason why the conditions on $\overline{W}^\lambda$ have to be supplemented by the no-logs-condition (3.49) which involves two integration functions, φ_{-1} and φ_0, globally determined by the initial data γ and the gauge function κ. A decisive grievance is that, at least in our current setting, their dependence on γ and κ is very intricate. Thus the question arises for which class of initial data one finds a function $\kappa = \mathcal{O}(r^{-3})$, such that the no-logs-condition holds, and accordingly what the geometric restrictions are for this to be possible. The only obvious fact is that (3.49) will be satisfied for sure whenever $\breve{h}^{(1)}_{AB} = \breve{h}^{(2)}_{AB} = 0$.

For a "generic" choice of γ and κ one should expect that the expansion coefficient $\tau_2 = -2\varphi_0(\varphi_{-1})^{-1}$ will not vanish. Equation (3.17) then shows that a $\overline{W}^0 = 0$-gauge (in particular the harmonic gauge) is not adequate for our purposes, since logarithmic terms can only be removed by a non-vanishing gauge source function $\overline{W}^0 \neq 0$. This is illustrated best by the no-go result [6, Theorem 3.1]. In order to fulfill (3.17) one needs a gauge choice for $\overline{W}^0$ which depends on the globally defined integration functions $(\varphi_{-1}, \varphi_0)$. This indicates the need of an initial data-dependent gauge choice to get rid of the logarithmic terms. (Note that the higher-order terms in the expansion of $\overline{W}^0$ do not affect the appearance of log terms.) We shall address this issue in the next sections.

4 Metric gauge

4.1 Introduction

Up to now we investigated the overall form of the asymptotic expansions of the trace of the metric on $\mathscr{N}$. It turned out that it is not possible to manifestly eliminate all logarithmic terms just by imposing restrictions on the asymptotic behavior of the gauge functions. Instead, an additional gauge-independent "no-logs-condition" (3.49), which depends on some of the globally defined integration functions, needs to be fulfilled to ensure expansions at conformal infinity in terms of power series. Nonetheless, the current gauge scheme is unsatisfactory in that it seems hopeless to characterize those classes of initial data which satisfy (3.49).

In the following we will modify the scheme to allow for a better treatment of these problems at hand. We develop a gauge scheme which is adjusted to the initial data in such a way that we can solve at least some of the constraint equations analytically, so that the values of the troublesome functions φ_{-1} and φ_0, which appeared hitherto as "integration functions" in the asymptotic solutions and which are related to the appearance of log terms via (3.49), can be computed explicitly. In view of the computation of all the fields appearing in the conformal field equations we will choose a gauge in which the components of the metric tensor take preferably simple values (at the expense of more complicated gauge functions κ and W^λ). As before we shall adopt Rendall's point of view [10] and regard $[\gamma]$ (together with a choice of κ) as the free "physical" data.

4.2 Gauge scheme

The trace of certain components of the vacuum Einstein equations on $\mathscr{N}$ together with the gauge condition $\overline{H}^\lambda = 0$ can be used to determine the metric and some of its transverse derivatives on $\mathscr{N}$. Similarly, the remaining components of the Einstein equations on $\mathscr{N}$ as well as their transverse derivatives, $\overline{\partial_0^n R_{\mu\nu}} = 0$, $n \in \mathbb{N}$, in combination with the gauge condition $H^\lambda = 0$, provide a way to determine higher-order transverse derivatives of the metric on $\mathscr{N}$ when necessary. To compute all the fields which appear in the conformal field equations, such as e.g. the rescaled Weyl tensor, this is what needs to be done.

However, it is very convenient to exploit the gauge freedom contained in the vector field W^λ to prescribe certain metric components and transverse derivatives thereof on $\mathscr{N}$, and treat the corresponding equations as equations for W^λ. Indeed, proceeding this way the computations below can be significantly simplified. It turns out that the most convenient way here is a mixture of schemes described in [6]: We still regard the conformal class $[\gamma]$ of $\check{g}_{\Sigma_r}$ as the free "non-gauge"-initial data.[4] Recall that up to this stage we have regarded κ and W^λ as gauge degrees of freedom. Now, instead of $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$ we shall show that it is possible to prescribe the functions

$$\overline{g}_{0\mu} \quad \text{and} \quad \overline{\partial_0 g_{0\mu}} \quad \text{with} \quad \nu_0 = \overline{g}_{0r} \neq 0 \, .$$

Of course there are some restrictions coming from regularity when $\mathscr{N}$ is a light-cone, and from the requirement of the metric to be continuous at the intersection manifold when $\mathscr{N}$ is one of two intersecting null surfaces. Since we are mainly interested in prescribing these function for large values of r, this issue can be ignored for our purposes. Some comments are given in the course of this section.

Let us now explain how the above gauge scheme can be realized: First of all we solve the Raychaudhuri equation to compute $\overline{g}_{AB}$, where, as before, we assume that a global solution τ exists. Then we compute $\overline{W}^\lambda$ from the constraint equations (2.6), (2.8) and (2.10), a procedure which was introduced in [6] as an alternative scheme for integrating the null constraint equations. In order to make sure that we may prescribe $\overline{\partial_0 g_{0\mu}}$ rather than $\overline{\partial_0 W^\lambda}$ we need to analyze the remaining Einstein equations on $\mathscr{N}$ (those components which have not already been used to derive the Einstein wave-map gauge constraints in [2]).

We impose a wave-map gauge condition $H^\lambda = 0$ with arbitrarily prescribed gauge functions κ and W^λ. From $\overline{H}^\lambda = 0$ we obtain algebraic equations for $\overline{\partial_0 g_{rr}}$, $\overline{\partial_0 g_{rA}}$ and $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$,

$$\begin{aligned}
\overline{\partial_0 g_{rr}} &= \tau\nu_0 + (\nu_0)^2(\overline{\hat{\Gamma}}^0 + \overline{W}^0) \, , && (4.1)\\
\overline{g}^{AB}\overline{\partial_0 g_{rB}} &= \check{\nabla}^A \nu_0 - \partial_r \nu^A - \nu_0 \overline{g}^{CD}\check{\Gamma}^A_{CD} - \overline{g}^{rA}(\overline{\partial_0 g_{rr}} - \tau\nu_0) \\
&\quad + \nu_0(\overline{\hat{\Gamma}}^A + \overline{W}^A) \, , && (4.2)\\
\overline{g}^{AB}\overline{\partial_0 g_{AB}} &= 2\check{\nabla}_A \nu^A - (2\partial_r + 2\tau - \nu^0 \overline{\partial_0 g_{rr}})(\nu_0 \overline{g}^{rr}) - 2\nu_0(\overline{\hat{\Gamma}}^r + \overline{W}^r) && (4.3)
\end{aligned}$$

Recall that

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \, . \quad (4.4)$$

[4] We do not regard e.g. $\overline{g}_{\mu\nu}$ as the free data as in [6] since we want to separate gauge degrees of freedom and physical initial data.

Einstein equations $\breve{\overline{R}}_{AB}=0$: Using the formulae in [2, Appendix A] (which we shall make extensively use of for all the subsequent computations) we find

$$\begin{aligned}
\breve{\overline{R}}_{AB} &= (\overline{\partial_0\Gamma^0_{AB}})\breve{} - \frac{1}{2}\nu^0(\partial_r-\tau)(\overline{\partial_0 g_{AB}})\breve{} - \frac{1}{2}\nu^0\overline{g}_{AB}\sigma^{CD}(\overline{\partial_0 g_{CD}})\breve{} \\
&\quad +2\nu^0\sigma_{(B}{}^C(\overline{\partial_{|0|}g_{A)C}})\breve{} - (\nu^0)^2\overline{\partial_0 g_{0r}}\sigma_{AB} + \text{known quantities} \\
&= -\nu^0\partial_r(\overline{\partial_0 g_{AB}})\breve{} + \frac{1}{2}(2\tau\nu^0 + \overline{W}^0 + \overline{\hat{\Gamma}}^0)(\overline{\partial_0 g_{AB}})\breve{} + 2\nu^0\sigma_{(A}{}^C(\overline{\partial_{|0|}g_{B)C}})\breve{} \\
&\quad +\text{known quantities}\,,
\end{aligned}$$

where we used that

$$\begin{aligned}
\overline{\partial_0\Gamma^0_{AB}} &= \frac{1}{2}(\nu^0)^2\overline{\partial_0 g_{rr}}\,\overline{\partial_0 g_{AB}} + (\nu^0)^2\overline{\partial_0 g_{0r}}\chi_{AB} - \frac{1}{2}\nu^0\partial_r\overline{\partial_0 g_{AB}} \\
&\quad +\text{known quantities}\,. \qquad (4.5)
\end{aligned}$$

Note that $\overline{\partial_0 g_{0r}}$, which appears in an intermediate step, cancels out at the end. We observe that $\breve{\overline{R}}_{AB}=0$ provides a coupled linear ODE-system for $(\overline{\partial_0 g_{AB}})\breve{}$.

The relevant boundary condition if $\mathscr{N}$ is a light-cone is [2, Section 4.5] $\lim_{r\to 0}(\overline{\partial_0 g_{AB}})\breve{}=0$, in the case of two transversally intersecting characteristic surfaces the boundary condition is determined by the shear of $\mathscr{N}_2$.

Einstein equation $\overline{R}_{0r}=0$: We have

$$\begin{aligned}
\overline{R}_{0r} &= -\overline{\partial_0\Gamma^r_{rr}} - \overline{\partial_0\Gamma^A_{rA}} + \text{known quantities} \\
&= \frac{1}{2}\nu^0\overline{\partial^2_{00}g_{rr}} - \nu^0(\partial_r-\kappa)\overline{\partial_0 g_{0r}} + \text{known quantities}\,,
\end{aligned}$$

when taking into account that

$$\begin{aligned}
\overline{\partial_0\Gamma^r_{rr}} &= \nu^0(\partial_r\overline{\partial_0 g_{0r}} - \frac{1}{2}\overline{\partial^2_{00}g_{rr}}) - (\nu^0)^2(\partial_r\nu_0 - \frac{1}{2}\overline{\partial_0 g_{rr}})\overline{\partial_0 g_{0r}} + \text{known quantities}\,, \\
\overline{\partial_0\Gamma^A_{rB}} &= \text{known quantities}\,.
\end{aligned}$$

Moreover, employing

$$\begin{aligned}
\overline{\partial_0\Gamma^0_{0r}} &= \frac{1}{2}\nu^0\overline{\partial^2_{00}g_{rr}} - \frac{1}{2}(\nu^0)^2\overline{\partial_0 g_{rr}}\,\overline{\partial_0 g_{0r}} + \text{known quantities}\,, \\
\overline{\partial_0\Gamma^0_{ir}} &= \text{known quantities}\,.
\end{aligned}$$

we find that

$$\begin{aligned}
\overline{\partial_0 H^0} &= \overline{\partial_0 g^{\mu\nu}}(\overline{\Gamma}^0_{\mu\nu} - \overline{\hat{\Gamma}}^0_{\mu\nu}) + \overline{g}^{\mu\nu}(\overline{\partial_0\Gamma^0_{\mu\nu}} - \overline{\partial_0\hat{\Gamma}^0_{\mu\nu}}) - \overline{\partial_0 W^0} \\
&= 2\nu^0\overline{\partial_0\Gamma^0_{0r}} + \overline{g}^{rr}\overline{\partial_0\Gamma^0_{rr}} + 2\overline{g}^{rA}\overline{\partial_0\Gamma^0_{rA}} + \overline{g}^{AB}\overline{\partial_0\Gamma^0_{AB}} - 2(\nu^0)^3\overline{\partial_0 g_{rr}}\,\overline{\partial_0 g_{0r}} \\
&\quad -2(\nu^0)^2\overline{\partial_0 g_{0r}}\hat{\Gamma}^0_{0r} - \overline{\partial_0 W^0} + \text{known quantities} \\
&= (\nu^0)^2\overline{\partial^2_{00}g_{rr}} - [2(\nu^0)^2(\tau + \hat{\Gamma}^0_{0r}) + 3\nu^0(\overline{W}^0 + \overline{\hat{\Gamma}}^0)]\overline{\partial_0 g_{0r}} - \overline{\partial_0 W^0} \\
&\quad +\text{known quantities}\,.
\end{aligned}$$

Consequently, the gauge condition $\overline{\partial_0 H^0}=0$ can be used to rewrite the Einstein equation $\overline{R}_{0r}=0$ as a linear ODE for $\overline{\partial_0 g_{0r}}$, or, depending on the setting, as an algebraic equation for the gauge source function $\overline{\partial_0 W^0}$,

$$\Big[\partial_r - \tau - \kappa - \hat{\Gamma}^0_{0r} - \frac{3}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0)\Big]\overline{\partial_0 g_{0r}} = \frac{1}{2}(\nu_0)^2\overline{\partial_0 W^0} + \text{known quantities}\,. \qquad (4.6)$$

The boundary condition if $\mathscr{N}$ is a light-cone is [2, Section 4.5] $\lim_{r\to 0}\overline{\partial_0 g_{0r}}=0$; in the case of two characteristic surfaces intersecting transversally it is determined by $\lim_{u\to 0}\partial_u g_{ur}|_{\mathscr{N}_2}$.

Once $\overline{\partial_0 g_{0r}}$ (or $\overline{\partial_0 W^0}$) has been determined, we obtain $\overline{\partial^2_{00} g_{rr}}$ algebraically from the gauge condition $\overline{\partial_0 H^0}=0$.

Einstein equations $\overline{R}_{0A}=0$: We find

$$\begin{aligned}\overline{R}_{0A} &= -\overline{\partial_0\Gamma^r_{rA}}-\overline{\partial_0\Gamma^B_{AB}}+\nu^0\chi_A{}^B\overline{\partial_0 g_{0B}}+\text{known quantities}\\ &= \frac{1}{2}\nu^0\overline{\partial^2_{00}g_{rA}}-\frac{1}{2}\nu^0\partial_r\overline{\partial_0 g_{0A}}+\nu^0\chi_A{}^B\overline{\partial_0 g_{0B}}+\text{known quantities}\,,\end{aligned}$$

where we used that

$$\begin{aligned}\overline{\partial_0\Gamma^r_{rA}} &= -\nu^0\chi_A{}^B\overline{\partial_0 g_{0B}}+\frac{1}{2}\nu^0(\partial_r\overline{\partial_0 g_{0A}}-\overline{\partial^2_{00}g_{rA}})+\text{known quantities}\,,\\ \overline{\partial_0\Gamma^C_{AB}} &= \nu^0\chi_{AB}\overline{g}^{CD}\overline{\partial_0 g_{0D}}+\text{known quantities}\,.\end{aligned}$$

From

$$\begin{aligned}\overline{\partial_0\Gamma^A_{0r}} &= -\frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{rr}}\,\overline{\partial_0 g_{0B}}+\frac{1}{2}\overline{g}^{AB}(\overline{\partial^2_{00}g_{rB}}+\partial_r\overline{\partial_0 g_{0B}})+\text{known quantities}\,,\\ \overline{\partial_0\Gamma^A_{rr}} &= \text{known quantities}\,,\end{aligned}$$

we obtain for the angle-components of the u-differentiated wave-gauge vector,

$$\begin{aligned}\overline{\partial_0 H^A} &= \overline{\partial_0 g^{\mu\nu}}(\overline{\Gamma}^A_{\mu\nu}-\hat{\Gamma}^A_{\mu\nu})+\overline{g}^{\mu\nu}(\overline{\partial_0\Gamma^A_{\mu\nu}}-\overline{\partial_0\hat{\Gamma}^A_{\mu\nu}})-\overline{\partial_0 W^A}\\ &= -(\nu^0)^2\overline{\partial_0 g_{rr}}\overline{g}^{AB}\overline{\partial_0 g_{0B}}-2\nu^0\overline{g}^{BC}\overline{\partial_0 g_{0C}}(\chi_B{}^A-\hat{\chi}_B{}^A)+2\nu^0\overline{\partial_0\Gamma^A_{0r}}\\ &\quad+\overline{g}^{ij}\overline{\partial_0\Gamma^A_{ij}}-\overline{\partial_0 W^A}+\text{known quantities}\\ &= \nu^0\overline{g}^{AB}\overline{\partial^2_{00}g_{rB}}+\nu^0\overline{g}^{AB}\partial_r\overline{\partial_0 g_{0B}}-(\tau\nu^0+2\overline{W}^0+2\overline{\hat{\Gamma}}^0)\overline{g}^{AB}\overline{\partial_0 g_{0B}}\\ &\quad-2\nu^0\overline{g}^{BC}\overline{\partial_0 g_{0C}}(\chi_B{}^A-\hat{\chi}_B{}^A)-\overline{\partial_0 W^A}+\text{known quantities}\,.\end{aligned}$$

The gauge condition $\overline{\partial_0 H^A}=0$ can be used to rewrite $\overline{R}_{0A}=0$ as a coupled linear ODE-system for $\overline{\partial_0 g_{0A}}$ or, depending on the setting, as an algebraic equation for $\overline{\partial_0 W^A}$,

$$\begin{aligned}&\Big[\partial_r-\frac{3}{2}\tau+\frac{1}{2}\hat{\tau}-\nu_0(\overline{W}^0+\overline{\hat{\Gamma}}^0)\Big]\overline{\partial_0 g_{0A}}-(2\sigma_A{}^B-\overline{g}_{AD}\overline{g}^{BC}\hat{\sigma}_C{}^D)\overline{\partial_0 g_{0B}}\\ &\qquad\qquad=\frac{1}{2}\nu_0 g_{AB}\overline{\partial_0 W^B}+\text{known quantities}\,.\end{aligned}\tag{4.7}$$

In the light-cone-case [2, Section 4.5] as well as in the case of two transversally intersecting null hypersurfaces the boundary condition is $\lim_{r\to 0}\overline{\partial_0 g_{0A}}=0$.

The gauge condition $\overline{\partial_0 H^A}=0$ then determines $\overline{\partial^2_{00}g_{rA}}$ algebraically.

Einstein equation $\overline{R}_{00}=0$: Finally, we have

$$\begin{aligned}\overline{R}_{00} &= \nu^0\Big(\frac{1}{2}\partial_r+\kappa+\frac{\tau}{2}-\nu^0\partial_r\nu_0\Big)\overline{\partial_0 g_{00}}-\overline{\partial_0\Gamma^r_{0r}}-\overline{\partial_0\Gamma^A_{0A}}+\text{known quantities}\\ &= \frac{1}{2}\nu^0\tau\overline{\partial_0 g_{00}}-\frac{1}{2}\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}}+\text{known quantities}\,.\end{aligned}$$

For that calculation we used that

$$\begin{aligned}
\overline{\partial_0\Gamma^r_{0r}} &= -\frac{1}{2}(\nu^0)^2\overline{\partial_0 g_{rr}}\,\overline{\partial_0 g_{00}} + \frac{1}{2}\nu^0\partial_r\overline{\partial_0 g_{00}} + \text{known quantities}\,, \\
\overline{\partial_0\Gamma^A_{0B}} &= \frac{1}{2}\overline{g}^{AC}\overline{\partial^2_{00}g_{BC}} + \text{known quantities}\,.
\end{aligned}$$

Taking into account that

$$\overline{\partial_0\Gamma^r_{AB}} = (\nu^0)^2\chi_{AB}\overline{\partial_0 g_{00}} - \frac{1}{2}\nu^0\overline{\partial^2_{00}g_{AB}} + \text{known quantities}\,,$$

we compute the r-component of the u-differentiated wave-gauge vector,

$$\begin{aligned}
\overline{\partial_0 H^r} &= \overline{\partial_0 g^{\mu\nu}}(\overline{\Gamma}^r_{\mu\nu} - \hat{\Gamma}^r_{\mu\nu}) + \overline{g}^{\mu\nu}(\overline{\partial_0\Gamma^r_{\mu\nu}} - \overline{\partial_0\hat{\Gamma}^r_{\mu\nu}}) - \overline{\partial_0 W^r} \\
&= -(\nu^0)^2\Big(\kappa - \hat{\kappa} + \frac{1}{2}\nu^0\overline{\partial_0 g_{rr}}\Big)\overline{\partial_0 g_{00}} + 2\nu^0\overline{\partial_0\Gamma^r_{0r}} + \overline{g}^{ij}\overline{\partial_0\Gamma^r_{ij}} \\
&\quad -\overline{\partial_0 W^r} + \text{known quantities} \\
&= (\nu^0)^2\Big[\partial_r - \frac{\tau}{2} - \kappa + \hat{\kappa} - \frac{3}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0)\Big]\overline{\partial_0 g_{00}} - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}} \\
&\quad -\overline{\partial_0 W^r} + \text{known quantities}\,.
\end{aligned}$$

Once again we exploit the gauge condition, here $\overline{\partial_0 H^r} = 0$, to rewrite $\overline{R}_{00} = 0$ as a linear ODE for $\overline{\partial_0 g_{00}}$, or, alternatively, an algebraic equation for $\overline{\partial_0 W^r}$

$$\Big[\partial_r - \tau - \kappa + \hat{\kappa} - \frac{3}{2}\nu_0(\overline{W}^0 + \overline{\hat{\Gamma}}^0)\Big]\overline{\partial_0 g_{00}} = (\nu_0)^2\overline{\partial_0 W^r} + \text{known quantities}\,. \quad (4.8)$$

For both light-cones [2, Section 4.5] and transversally intersecting null hypersurfaces, the boundary condition is $\lim_{r\to 0}\overline{\partial_0 g_{00}} = 0$. The gauge condition $\overline{\partial_0 H^r} = 0$ then determines $\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}}$ algebraically.

The gauge scheme we want to use here now works as follows: We prescribe γ, κ, $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$. The Raychaudhuri equation and the vacuum Einstein wave-map gauge constraints then determine $\overline{g}_{AB}$ and $\overline{W}^\lambda$. We then solve the hierarchical system of equations derived above to compute $\overline{\partial_0 W^\lambda}$ from (4.6), (4.7) and (4.8). We choose a smooth space-time vector field W^λ which induces the computed values for $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$ on $\mathscr{N}$ (in the light-cone case it is a non-trivial issue whether such an extension off the cone exists near the tip, we return to this issue below). Then we solve the reduced Einstein equations (or the conformal field equations, cf. Section 5). It is well-known that the solution induces the prescribed data for the metric tensor on the initial surface and satisfies $H^\lambda = 0$ with the prescribed source vector field W^λ, and thus solves the full Einstein equations. But then it follows from the equations (4.6)-(4.8), together with the relevant boundary conditions at the vertex or the intersection manifold, respectively, that the desired values for $\overline{\partial_0 g_{0\mu}}$ are induced, as well.

Some comments in the light-cone-case concerning regularity at the vertex are in order: What we have described above is exactly the strategy on two transverse characteristic surfaces (some care is needed to obtain a W^λ which is continuous at the intersection manifold, cf. [6, Section 3.1]). On a null cone there is a difficulty related to the behavior near the tip: In [3] it has been shown that if the metric $\overline{g}$ is the restriction to the cone of a smooth space-time metric, then $\overline{W}^\lambda$ does admit a smooth extension (and the known well-posedness result is applicable). However, we also want to prescribe $\overline{\partial_0 g_{0\mu}}$, whence $\overline{\partial_0 W^\lambda}$ needs to be of a specific form on the cone, as well. It seems difficult to ensure that $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$ arise from a smooth vector field W^λ in space-time if one proceeds this way. To avoid dealing with the issue of extendability near the tip of the cone, the gauge scheme will be somewhat modified:

Since we are merely interested in the asymptotic behavior of the fields, it suffices for our purposes to prescribe $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$ just for large r, say $r > r_2$. For small r, say $r < r_1 < r_2$ we can use a conventional scheme where for instance γ, κ and W^λ are prescribed with e.g. $W^\lambda = 0$. Then the regularity issue near the tip of the cone is well-understood [3]. A smooth transition in the regime $r_1 < r < r_2$ is obtained via cut-off functions:

Let $\chi \in C^\infty(\mathbb{R})$ be any non-negative non-increasing function satisfying $\chi(r) = 0$ for $r \geq r_2$ and $\chi = 1$ for $r \leq r_1$. Let $(\overline{g}_{0\mu})^\circ$ and $(\overline{\partial_0 g_{0\mu}})^\circ$ denote the values of $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$ computed from (γ, κ) for say $W^\lambda = 0$, and denote by $(\overline{g}_{0\mu})^\dagger$ and $(\overline{\partial_0 g_{0\mu}})^\dagger$ those values we want realize for $r > r_2$. We then set

$$
\begin{aligned}
\overline{g}_{0\mu} &= (1-\chi)(\overline{g}_{0\mu})^\dagger + \chi(\overline{g}_{0\mu})^\circ , &(4.9)\\
\overline{\partial_0 g_{0\mu}} &= (1-\chi)(\overline{\partial_0 g_{0\mu}})^\dagger + \chi(\overline{\partial_0 g_{0\mu}})^\circ . &(4.10)
\end{aligned}
$$

For these new values for $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$ we go through the above scheme again. By construction we still obtain the desired values for $\overline{g}_{0\mu}$ and $\overline{\partial_0 g_{0\mu}}$ for $r > r_2$, but now we also have $\overline{W}^\lambda = 0 = \overline{\partial_0 W^\lambda}$ for $r < r_1$, and the extension problem becomes trivial: Since there are no difficulties in extending a vector field defined along the light-cone to a neighborhood of the cone away from the tip, we conclude that $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$ arise from the restriction to the cone of some smooth vector field W^λ.

The coordinate r which parameterizes the null rays generating the initial surface is determined to a large extent by the choice of κ. We would like to choose an r for which the expansion on $\mathscr{N}$ takes the form $\tau = 2/r$. On a light-cone this is straightforward, we simply choose $\kappa = \frac{r}{2}|\sigma|^2$. It follows from [3] that this choice can be made up to the vertex, and it follows from the Raychaudhuri equation and regularity at the vertex that τ takes the desired form.

If $\mathscr{N}$ is one of two transversally intersecting characteristic surfaces we cannot achieve $\tau = 2/r$ globally, for the expansion needs to be regular on the intersection manifold. We would like to construct a κ such that $\tau = 2/r$ for large r. Let us therefore again modify our gauge scheme slightly to bypass this issue:

Instead of κ (supplemented by the initial data for the φ-equation $\varphi|_S \neq 0$ and $\partial_r\varphi|_S$ in the case of two characteristic surfaces, cf. [6, 10]), we prescribe the function φ on $\mathscr{N}$. An analysis of the Raychaudhuri equation (2.5) shows that, *for given* κ, the existence of a nowhere vanishing φ on $\mathscr{N}$ (which can then without loss of generality be taken to be positive), and which further satisfies $\varphi_{-1} > 0$ *implies* $\partial_r\varphi > 0$ on $\mathscr{N}$.[5] When prescribing φ rather than κ we therefore assume that φ is a strictly increasing, positive function with $\varphi_{-1} > 0$. Indeed, the function κ is then determined algebraically via the φ-equation (cf. [6]).

In this section we have established the following setting: On $\mathscr{N}$ we regard the conformal class of the family $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ of Riemannian metrics as

[5] Indeed, we observe that for $\kappa = 0$ any globally positive φ is concave in r. Together with the required positivity of φ_{-1} this implies that $\partial_r\varphi$ needs to be positive for all r. The case $\kappa \neq 0$ can be reduced to $\kappa = 0$, cf. [7, Section 2.4].

the "physical" initial data (in the case of two transversally intersecting null hypersurfaces supplemented by data on $\mathscr{N}_2$ and S), while the functions

$$\varphi \,, \quad \overline{g}_{0\mu} \,, \quad \overline{\partial_0 g_{0\mu}} \,, \tag{4.11}$$

rather than κ, $\overline{W}^\mu$ and $\overline{\partial_0 W^\mu}$, are regarded as gauge functions on $\mathscr{N}$, at least for $r > r_2$ (recall that φ , $\partial_r\varphi$ and ν^0 need to be positive).

4.3 Metric gauge and some conventions

We now impose specific values for the gauge functions (4.11). Guided by their Minkowskian values we set, for $r > r_2$,

$$\varphi = r \,, \quad \nu_0 = 1 \,, \quad \nu_A = 0 \,, \quad \overline{g}_{00} = -1 \,, \quad \overline{\partial_0 g_{0\mu}} = 0 \,. \tag{4.12}$$

Moreover, we assume a Minkowski target for $r > r_2$,

$$\hat{g} = \eta \equiv -\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2 s_{AB}\mathrm{d}x^A\mathrm{d}x^B \,. \tag{4.13}$$

From now on everything will be expressed in this gauge, which we call *metric gauge*. Although it may not explicitly be mentioned each time, since (4.12) and (4.13) are merely assumed to hold for $r > r_2$ we will tacitly assume henceforth that all the equations are meant to hold in this regime.

The relation between metric gauge and harmonic coordinates is discussed in Appendix B, where we address the "paradox" that in the metric gauge we do not need to impose conditions on γ while in the harmonic gauge we do need to do it to make sure that the constraint equations admit a global solution.

We have not discussed here what in this gauge scheme (which may be used on $\mathscr{N}_2$ as well) the free data on the intersection manifold are, and we leave it to the reader to work this out. We are interested in the asymptotic regime, and the main object of this section was to show that the choice (4.12) can be made without any geometric restrictions for large r, if one does not prescribe κ and W^λ anymore, but treats them as unknowns determined by the constraints.

4.4 Solution of the constraint equations in the metric gauge

We solve the Einstein vacuum constraints in the metric gauge for κ and $\overline{W}^\lambda$, where we assume the initial data γ to be of the form (2.15). Recall our convention that all equations are meant to hold for $r > r_2$.

Equation (2.5) yields with $\varphi = r$

$$\begin{aligned} \kappa &= (\partial_r\varphi)^{-1}\Big(\partial^2_{rr}\varphi + \frac{1}{2}|\sigma|^2\varphi\Big) = \frac{1}{2}r|\sigma|^2 && (4.14)\\ &= \frac{1}{2}\sigma_4 r^{-3} + \mathcal{O}(r^{-4}) = \frac{1}{8}|\breve{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4}) \,. && (4.15) \end{aligned}$$

We further note that $\varphi = r$ implies

$$\tau = 2/r \,, \tag{4.16}$$

as desired, in particular $\tau_2 = 0$. We emphasize that the argument used for the No-Go Theorem in [7], that the coefficient τ_2 vanishes only for Minkowski data, does not apply when $\kappa \neq 0$. In our case κ does not vanish unless $|\sigma|^2 \equiv 0$.

It follows from (3.13), (4.12) and (4.15) that

$$\overline{g}_{AB} = \varphi^2\sqrt{\frac{\det s}{\det\gamma}}\gamma_{AB} = r^2 s_{AB} + r\check{h}^{(1)}_{AB} + \frac{1}{4}|\check{h}^{(1)}|^2 s_{AB} - (\sigma_A{}^C)_3 s_{BC} + \mathcal{O}(r^{-1})\ . \tag{4.17}$$

Moreover,

$$\begin{aligned}\overline{W}^0 &= -(2\partial_r+\tau+2\kappa)\nu^0 - \overline{\hat{\Gamma}}{}^0 = -2r^{-1} - r|\sigma|^2 + r\overline{g}^{AB}s_{AB} &(4.18)\\ &= \frac{1}{4}|\check{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4})\ , &(4.19)\end{aligned}$$

where we used that

$$\overline{\hat{\Gamma}}{}^0 = -r\overline{g}^{AB}s_{AB} = -2r^{-1} - \frac{1}{2}|\check{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4})\ . \tag{4.20}$$

The equation (2.7) which determines ξ_A cannot be solved in explicit form. The asymptotic analysis of its solutions has already been done in Section 3.1.3. The asymptotic expansion of ξ_A does not contain logarithmic terms if and only if the no-logs-condition is fulfilled, which in the metric gauge where $\tau_2 = 0$ reduces to

$$(\sigma_A{}^B)_3 = 0\ . \tag{4.21}$$

Assuming (4.21) one finds

$$\xi_A = -\mathring{\nabla}^B\check{h}^{(1)}_{AB}r^{-1} + C^{(\xi_B)}_A + \mathcal{O}(r^{-3})\ . \tag{4.22}$$

Now we can solve (2.8) for $\overline{W}^A$,

$$\overline{W}^A = \xi^A + \overline{g}^{CD}\check{\Gamma}^A_{CD} - \hat{\Gamma}^A = \mathcal{O}(r^{-4})\ , \tag{4.23}$$

where we used

$$\hat{\Gamma}^A = \overline{g}^{CD}\mathring{\Gamma}^A_{CD} = s^{CD}\mathring{\Gamma}^A_{CD}r^{-2} - \check{h}^{(1)CD}\mathring{\Gamma}^A_{CD}r^{-3} + \mathcal{O}(r^{-4})\ . \tag{4.24}$$

Note that the two leading-order terms in (4.23) cancel out.

The ζ-equation (2.9) cannot be solved analytically, its asymptotic solution, which has been determined in Section 3.1.5, does not involve log terms,

$$\zeta = -2r^{-1} + C^{(\zeta)}r^{-2} + \mathcal{O}(r^{-3})\ . \tag{4.25}$$

Finally, we solve the constraint equation (2.10) for $\overline{W}^r$,

$$\begin{aligned}\overline{W}^r &= \frac{1}{2}\zeta - \big(\partial_1 + \frac{1}{2}\tau + \kappa\big)\overline{g}^{rr} - \overline{\hat{\Gamma}}{}^r \\ &= \frac{1}{2}\zeta - r^{-1} - \frac{r}{2}|\sigma|^2 + r\overline{g}^{AB}s_{AB} = \frac{1}{2}C^{(\zeta)} + \mathcal{O}(r^{-3})\ , &(4.26)\end{aligned}$$

as

$$\overline{\hat{\Gamma}}{}^r = -r\overline{g}^{AB}s_{AB} = -2r^{-1} - \frac{1}{2}|\check{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4})\ . \tag{4.27}$$

4.5 Overview of the metric gauge I

In the metric gauge one treats, at least for large r, some of the metric components as gauge degrees of freedom rather than κ, $\overline{W}^\lambda$ and $\overline{\partial_0 W^\lambda}$. This provides the decisive advantage that one obtains more explicit solutions of the constraint equations since most of them are algebraic equations rather than ODEs in this setting, so that the asymptotic expansions contain less integration functions, whose values are not explicitly known. Moreover, the computations of Schouten tensor, Weyl tensor etc. (as needed in Section 5 for Friedrich's equations) will be simplified significantly by the fact that several metric components take their Minkowskian values.

We have seen in Section 3 that many log terms are produced due to a bad choice of coordinates. In the metric gauge all the gauge-dependent conditions of Theorem 3.2 (cf. Remark 3.3 and compare (4.29)-(4.34) below with (3.50)-(3.56)), are satisfied and we are left with the gauge-invariant no-logs-condition, which can be expressed as an explicit condition on γ in this gauge,

$$0 = (\sigma_A{}^B)_3 = \frac{1}{2}h^{(1)}\breve{h}^{(1)B}_A - \breve{h}^{(2)B}_A \ . \tag{4.28}$$

Consequently we can freely prescribe all the $h^{(n)}_{AB}$'s except for $\breve{h}^{(2)}_{AB}$ which is determined by (4.28).

It is easy to find sufficient conditions on the initial data γ such that the no-logs-condition (3.49) in its general form is fulfilled (just take $\breve{h}^{(1)}_{AB} = \breve{h}^{(2)}_{AB} = 0$), i.e. Theorem 3.2 shows that there is a large class of initial data for which the metric admits a smooth conformal completion at infinity. To find necessary conditions on γ is more involved, since the expansion coefficient τ_2 is in general not explicitly known. In the metric gauge, though, τ_2 *is* explicitly known, whence it is easily possible to *characterize* the initial data sets completely which lead to the restriction of a metric to $\mathscr{N}$ which admits a smooth conformal completion at infinity. Moreover, it becomes obvious that for "generic" initial data the no-logs-condition will be violated.

Note that we have not studied yet the asymptotic behavior of the unknowns of the conformal field equations which also involve transverse derivatives of the metric. The gauge functions are more complicated in the metric gauge and contain certain integration functions, and one might wonder whether log-terms arise at some later stage. Luckily, it turns out (cf. Section 5) that this is not the case, and that the no-logs-condition is necessary-and-sufficient to guarantee smoothness of all the relevant fields at conformal infinity .

Of course, once we have determined the gauge functions it is possible to return to the original viewpoint and regard them as the relevant gauge degrees of freedom. This leads to an *initial data-dependent gauge*. Assuming that the initial data γ satisfy the no-logs-condition (4.28) we give their values for $r > r_2$ (the functions $\overline{\partial_0 W^\lambda}$ will be computed in Section 4.6):

$$\begin{aligned}
\kappa &= \frac{1}{2}r|\sigma|^2 = \frac{1}{8}|\breve{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4}) \ , &(4.29)\\
\overline{W}^0 &= -r|\sigma|^2 - \frac{2}{r} - \overline{\hat{\Gamma}}{}^0 = \frac{1}{4}|\breve{h}^{(1)}|^2 r^{-3} + \mathcal{O}(r^{-4}) \ , &(4.30)\\
\overline{W}^A &= \xi^A + \overline{g}^{CD}\breve{\Gamma}^A_{CD} - \hat{\Gamma}^A = \mathcal{O}(r^{-4}) \ , &(4.31)\\
\overline{W}^r &= \frac{1}{2}\zeta - r^{-1} - \frac{1}{2}r|\sigma|^2 - \overline{\hat{\Gamma}}{}^r = \mathcal{O}(r^{-2}) \ , &(4.32)
\end{aligned}$$

with ξ_A and ζ given by (4.22) and (4.25), respectively. The restrictions to $\mathscr{N}$ of the metric components then take the form (again for $r > r_2$):

$$\overline{g}_{00} = -1 \,, \quad \nu_0 = 1 \,, \quad \nu_A = 0 \,, \quad \overline{g}_{rr} = \overline{g}_{rA} = 0 \,, \tag{4.33}$$

$$\overline{g}_{AB} = r^2 s_{AB} + r\breve{h}^{(1)}_{AB} + \tfrac{1}{4}|\breve{h}^{(1)}|^2 s_{AB} + \mathcal{O}(r^{-1}) \,. \tag{4.34}$$

Remark 4.1 The asymptotic expansion of $\overline{g}_{AB}$ depends only on the gauge choice for κ. Our κ, equation (4.29), coincides with the κ used to define Bondi coordinates (where one also requires $\tau = 2/r$). The no-logs-condition which, in a $\kappa = \frac{r}{2}|\sigma|^2$-gauge is equivalent to the absence of s-trace-free terms in $(\overline{g}_{AB})_0$,

$$(\overline{g}_{AB}\breve{)}_0 = 0 \,,$$

recovers the "outgoing wave condition" imposed a priori by Bondi et al. and by Sachs to inhibit the appearance of logarithmic terms (cf. e.g. [5]).

For an arbitrary $\kappa = \mathcal{O}(r^{-3})$ the correct generalization of this condition, equivalent to the no-logs-condition (3.49), is (cf. (3.13)),

$$(\overline{g}_{AB}\breve{)}_0 = \tau_2 s_{AC}(\sigma_B{}^C)_2 = -\frac{1}{2}\tau_2\breve{h}^{(1)}_{AB} \,.$$

4.6 Transverse derivatives of the metric on $\mathscr{N}$

In this section we compute the $\overline{\partial_0 W^\lambda}$'s and certain transverse derivatives of the metric on $\mathscr{N}$ (the asymptotic expansions thereof) in the metric gauge using the vacuum Einstein equations $R_{\mu\nu} = 0$ and assuming that the no-logs-condition is fulfilled, so that (4.29)-(4.34) and $\overline{\partial_0 g_{0\mu}} = 0$ hold. As before, all equalities are meant to be valid for $r > r_2$, even if this is not mentioned wherever relevant.

It follows from (4.1)-(4.3) that

$$\overline{\partial_0 g_{rr}} = -r|\sigma|^2 = -2\kappa \,, \tag{4.35}$$

$$\overline{\partial_0 g_{rA}} = \xi_A \,, \tag{4.36}$$

$$\overline{g}^{AB}\overline{\partial_0 g_{AB}} = -\zeta - \tau \,. \tag{4.37}$$

If we plug in the values for $\overline{g}_{\mu\nu}$ and $\overline{\partial_0 g_{\mu\nu}}$ we find from [2, Appendix A] the following expressions for the Christoffel symbols restricted to $\mathscr{N}$, which we shall made frequently use of:

$$\overline{\Gamma}^\mu_{00} = \overline{\Gamma}^0_{rr} = \overline{\Gamma}^0_{rA} = \overline{\Gamma}^C_{rr} = 0 \,, \; \overline{\Gamma}^r_{rr} = -\overline{\Gamma}^0_{0r} = -\overline{\Gamma}^r_{0r} = \kappa \,, \tag{4.38}$$

$$\overline{\Gamma}^0_{0A} = \overline{\Gamma}^r_{0A} = -\overline{\Gamma}^r_{rA} = \tfrac{1}{2}\xi_A \,, \quad \overline{\Gamma}^C_{0r} = \tfrac{1}{2}\xi^C \,, \; \overline{\Gamma}^C_{rA} = \chi_A{}^C \,, \; \overline{\Gamma}^0_{AB} = -\chi_{AB} \,, \tag{4.39}$$

$$\overline{\Gamma}^r_{AB} = -\tfrac{1}{2}\overline{\partial_0 g_{AB}} - \chi_{AB} \,, \; \overline{\Gamma}^C_{0A} = \tfrac{1}{2}\overline{g}^{CD}\overline{\partial_0 g_{AD}} \,, \; \overline{\Gamma}^C_{AB} = \check{\Gamma}^C_{AB} \,. \tag{4.40}$$

Einstein equations $\breve{\overline{R}}_{AB} = 0$: From (4.4) we obtain

$$\begin{aligned}\overline{R}_{AB} = {} & \check{R}_{AB} + \overline{\partial_0\Gamma^0_{AB}} + (\partial_r + \overline{\Gamma}^C_{rC})\overline{\Gamma}^r_{AB} + \overline{\Gamma}^0_{AB}(\overline{\Gamma}^0_{00} + \overline{\Gamma}^r_{0r} + \overline{\Gamma}^C_{0C}) \\ & -2\overline{\Gamma}^C_{0(A}\overline{\Gamma}^0_{B)C} - 2\overline{\Gamma}^r_{rA}\overline{\Gamma}^r_{rB} - 2\overline{\Gamma}^C_{r(A}\overline{\Gamma}^r_{B)C} \,.\end{aligned}$$

First of all we have to determine the u-differentiated Christoffel symbol,

$$\overline{\partial_0\Gamma^0_{AB}} = -(\frac{1}{2}\partial_r + \kappa)\overline{\partial_0 g_{AB}} - 2\kappa\chi_{AB} + \check{\nabla}_{(A}\xi_{B)} \,. \tag{4.41}$$

Employing (4.38)-(4.40) and the relation $\check{R}_{AB} = \frac{1}{2}\check{R}\overline{g}_{AB}$ we obtain after some simplifications that

$$\begin{aligned}\breve{\overline{R}}_{AB} &= -\Big(\partial_r - \frac{1}{r} + \frac{r}{2}|\sigma|^2\Big)(\overline{\partial_0 g_{AB}})\breve{} + 2\sigma_{(A}{}^C(\overline{\partial_{|0|}g_{B)C}})\breve{} - (\partial_r\sigma_{AB})\breve{} \\ &\quad + \Big(\frac{\zeta}{2} + \frac{1}{r} - \frac{r}{2}|\sigma|^2\Big)\sigma_{AB} - \frac{1}{2}(\xi_A\xi_B)\breve{} + (\check{\nabla}_{(A}\xi_{B)})\breve{}\,, \qquad (4.42)\end{aligned}$$

which should vanish in vacuum. The equations $\breve{\overline{R}}_{AB} = 0$ form a closed linear ODE-system for $(\overline{\partial_0 g_{AB}})\breve{}$. Since it is generally hopeless to look for an analytic solution, we content ourselves with computing the asymptotic solution. For this it is convenient to rewrite the matrix equation (4.42) as a vector equation,

$$\left[\partial_r + \begin{pmatrix} \big(\frac{r}{2}|\sigma|^2 - \frac{1}{r}\big) - 2\sigma_2{}^2 & -2\sigma_2{}^3 & 0 \\ -\sigma_3{}^2 & \big(\frac{r}{2}|\sigma|^2 - \frac{1}{r}\big) & -\sigma_2{}^3 \\ 0 & -2\sigma_3{}^2 & \big(\frac{r}{2}|\sigma|^2 - \frac{1}{r}\big) - 2\sigma_3{}^3 \end{pmatrix}\right] \begin{pmatrix} (\overline{\partial_0 g_{22}})\breve{} \\ (\overline{\partial_0 g_{23}})\breve{} \\ (\overline{\partial_0 g_{33}})\breve{} \end{pmatrix} = \begin{pmatrix} q_{22} \\ q_{23} \\ q_{33} \end{pmatrix}\,,$$

where

$$q_{AB} := -(\partial_r\sigma_{AB})\breve{} + \Big(\frac{\zeta}{2} + \frac{1}{r} - \frac{r}{2}|\sigma|^2\Big)\sigma_{AB} - \frac{1}{2}(\xi_A\xi_B)\breve{} + (\check{\nabla}_{(A}\xi_{B)})\breve{} = \mathcal{O}(r^{-1})\,.$$

In order to permit an easier comparison with Appendix A, we apply the transformation $r \mapsto x := 1/r$ (with all quantities treated as scalars). Then the ODE adopts the following asymptotic form,

$$\left[x\partial_x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + M\right] \begin{pmatrix} (\overline{\partial_0 g_{22}})\breve{} \\ (\overline{\partial_0 g_{23}})\breve{} \\ (\overline{\partial_0 g_{33}})\breve{} \end{pmatrix} = \mathcal{O}(1)\,,$$

where $M = \mathrm{Mat}(3,3) = \mathcal{O}(x)$. In the notation of Appendix A we observe that $\lambda = -1$ and $\hat{\ell} = -1$, and no logarithmic terms appear in the asymptotic solution. Moreover, it is $\mathcal{O}(r)$ and the integration functions are represented by the leading order terms,

$$(\overline{\partial_0 g_{AB}})\breve{} = D_{AB}r + \mathcal{O}(1)\,.$$

Note that D_{AB} is symmetric, $D_{AB} = D_{BA}$, and s-trace-free, $s^{AB}D_{AB} = 0$.

Einstein equation $\overline{R}_{0r} = 0$:

$$\overline{R}_{0r} = \partial_r\overline{\Gamma}^r_{0r} + \check{\nabla}_A\overline{\Gamma}^A_{0r} - \overline{\partial_0\Gamma^r_{rr}} - \overline{\partial_0\Gamma^A_{rA}} + \overline{\Gamma}^0_{0r}\overline{\Gamma}^A_{0A} + \overline{\Gamma}^r_{0r}(\overline{\Gamma}^0_{0r} + \overline{\Gamma}^A_{rA}) - \overline{\Gamma}^A_{rB}\overline{\Gamma}^B_{0A}\,.$$

We determine the u-differentiated Christoffel symbols involved,

$$\begin{aligned} \overline{\partial_0\Gamma^r_{rr}} &= -\frac{1}{2}\overline{\partial^2_{00}g_{rr}} + \frac{r^2}{2}|\sigma|^4 - \frac{1}{2}|\sigma|^2 - \frac{r}{2}\partial_r|\sigma|^2\,, && (4.43)\\ \overline{\partial_0\Gamma^C_{rA}} &= \frac{1}{2}\xi_A\xi^C + \frac{1}{2}\partial_r(\overline{g}^{BC}\overline{\partial_0 g_{AB}}) + \overline{g}^{BC}\check{\nabla}_{[A}\xi_{B]} && (4.44)\\ \Longrightarrow\quad \overline{\partial_0\Gamma^A_{rA}} &= \frac{1}{2}\xi_A\xi^A - \frac{1}{2}\partial_r\zeta + \frac{1}{r^2}\,. \end{aligned}$$

Note that $\overline{\partial^2_{00}g_{rr}}$ is the only unknown at this stage. We obtain

$$\begin{aligned}\overline{R}_{0r} &= \frac{1}{2}\overline{\partial^2_{00}g_{rr}} + \frac{1}{2}\Big(\partial_r + r^{-1} + \frac{r}{2}|\sigma|^2\Big)\zeta - \frac{r^2}{4}|\sigma|^4 - \frac{1}{2}|\sigma|^2 \\ &\quad + \frac{1}{2}(\check{\nabla}_A - \xi_A)\xi^A - \frac{1}{2}\sigma^{AB}\overline{\partial_0 g_{AB}}\,,\end{aligned}$$

which, again, should vanish in vacuum. With (2.9) that yields

$$\overline{\partial^2_{00}g_{rr}} = \frac{1}{r}\zeta + \check{R} + \frac{1}{2}\xi_A\xi^A + \frac{r^2}{2}|\sigma|^4 + |\sigma|^2 + \sigma^{AB}\overline{\partial_0 g_{AB}} = \mathcal{O}(r^{-3})\,. \quad (4.45)$$

The leading-order term

$$\Xi := (\overline{\partial^2_{00}g_{rr}})_3 = \zeta_2 + [D_A{}^B - 2\mathring{\nabla}_A\mathring{\nabla}^B](\sigma_B{}^A)_2\,. \quad (4.46)$$

depends on certain integration functions and is not explicitly known. We choose this special notation, though, since it will appear several times in the leading-order terms of other expansions.

The gauge condition then provides an algebraic equation for $\overline{\partial_0 W^0}$,

$$\begin{aligned}0 = \overline{\partial_0 H^0} &= -\overline{\partial_0 g_{rr}\Gamma^0_{00}} - 2\overline{\partial_0 g_{rr}\Gamma^0_{0r}} - 2\overline{g}^{AB}\overline{\partial_0 g_{rB}\Gamma^0_{0A}} \\ &\quad + \overline{\partial_0 g^{AB}}(\overline{\Gamma}^0_{AB} + rs_{AB}) + 2\overline{\partial_0\Gamma^0_{0r}} + \overline{\partial_0\Gamma^0_{rr}} + \overline{g}^{AB}\overline{\partial_0\Gamma^0_{AB}} - \overline{\partial_0 W^0} \\ &= \overline{\partial^2_{00}g_{rr}} + rs_{AB}\overline{\partial_0 g^{AB}} + \check{\nabla}_A\xi^A - 2\xi_A\xi^A - \frac{1}{r^2} - \frac{3}{2}r^2|\sigma|^4 \\ &\quad - \frac{3}{2}|\sigma|^2 - \frac{r}{2}\partial_r|\sigma|^2 + \frac{1}{2}(\partial_r + r|\sigma|^2)\zeta - \overline{\partial_0 W^0}\,,\end{aligned}$$

where we used that

$$\overline{\partial_0\Gamma^0_{rr}} = \frac{r^2}{2}|\sigma|^4 - \frac{1}{2}|\sigma|^2 - \frac{r}{2}\partial_r|\sigma|^2\,, \quad (4.47)$$

$$\overline{\partial_0\Gamma^0_{0r}} = \frac{1}{2}\overline{\partial^2_{00}g_{rr}} - \frac{r^2}{2}|\sigma|^4 - \frac{1}{2}\xi_A\xi^A\,. \quad (4.48)$$

Inserting (4.45) and using again (2.9) we deduce that

$$\begin{aligned}\overline{\partial_0 W^0} &= -\Big(\frac{1}{2}\partial_r + \frac{1}{r}\Big)\zeta + (rs_{AB} - \sigma_{AB})\overline{\partial_0 g^{AB}} - \frac{1}{r^2} - \frac{r}{2}\partial_r|\sigma|^2 \\ &\quad - \frac{1}{2}|\sigma|^2 - r^2|\sigma|^4 - \xi_A\xi^A \quad (4.49) \\ &= \mathcal{O}(r^{-3})\,. \quad (4.50)\end{aligned}$$

Einstein equations $\overline{R}_{0A} = 0$:

$$\begin{aligned}\overline{R}_{0A} &= \partial_r\overline{\Gamma}^r_{0A} + \check{\nabla}_B\overline{\Gamma}^B_{0A} - \overline{\partial_0\Gamma^r_{rA}} - \overline{\partial_0\Gamma^B_{AB}} + \overline{\Gamma}^0_{0A}\overline{\Gamma}^B_{0B} + (\overline{\Gamma}^r_{rr} + \overline{\Gamma}^B_{rB})\overline{\Gamma}^r_{0A} \\ &\quad + \overline{\Gamma}^B_{0A}\overline{\Gamma}^r_{rB} - \overline{\Gamma}^0_{AB}\overline{\Gamma}^B_{00} - 2\overline{\Gamma}^r_{rA}\overline{\Gamma}^r_{0r} - \overline{\Gamma}^r_{AB}\overline{\Gamma}^B_{0r} - \overline{\Gamma}^B_{rA}\overline{\Gamma}^r_{0B}\,.\end{aligned}$$

For the u-differentiated Christoffel symbols we find,

$$\overline{\partial_0\Gamma^r_{rA}} = -\frac{1}{2}\overline{\partial^2_{00}g_{rA}} - \chi_A{}^B\xi_B - \frac{r}{2}\partial_A|\sigma|^2 - \frac{1}{2}r|\sigma|^2\xi_A\,, \quad (4.51)$$

$$\begin{aligned}\overline{\partial_0\Gamma^C_{AB}} &= \overline{g}^{CD}\check{\nabla}_{(A}\overline{\partial_{|0|}g_{B)D}} - \frac{1}{2}\check{\nabla}^C\overline{\partial_0 g_{AB}} + \frac{1}{2}\xi^C\overline{\partial_0 g_{AB}} + \xi^C\chi_{AB} \quad (4.52) \\ \Longrightarrow\quad \overline{\partial_0\Gamma^B_{AB}} &= \chi_A{}^B\xi_B + \frac{1}{2}\xi^B\overline{\partial_0 g_{AB}} - \frac{1}{2}\partial_A\zeta\,.\end{aligned}$$

We insert these expressions into the formula for $\overline{R}_{0A}$ to obtain,

$$\begin{aligned}\overline{R}_{0A} &= \frac{1}{2}(\partial_r + r^{-1})\xi_A + \frac{1}{2}(\check{\nabla}^B - \xi^B)\overline{\partial_0 g_{AB}} + \frac{r}{2}(\partial_A + \frac{1}{2}\xi_A)|\sigma|^2 + \frac{1}{2}\overline{\partial^2_{00} g_{rA}} \\ &\quad + \frac{1}{2}(\partial_A - \frac{1}{2}\xi_A)\zeta \,,\end{aligned}$$

which should vanish in vacuum, i.e.

$$\begin{aligned}\overline{\partial^2_{00} g_{rA}} &= (\xi^B - \check{\nabla}^B)\overline{\partial_0 g_{AB}} - (\partial_r + \frac{1}{r} - \frac{\zeta}{2})\xi_A - r(\partial_A + \frac{\xi_A}{2})|\sigma|^2 - \partial_A\zeta \quad (4.53) \\ &= -\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}) \,.\end{aligned}$$

We employ the gauge condition $\overline{\partial_0 H^A} = 0$ to compute $\overline{\partial_0 W^A}$. This requires the knowledge of further u-differentiated Christoffel symbols,

$$\overline{\partial_0 \Gamma^C_{0r}} = \frac{1}{2}\overline{g}^{CD}\overline{\partial^2_{00} g_{rD}} + \frac{r}{2}|\sigma|^2\xi^C + \frac{1}{2}\xi_D\overline{\partial_0 g^{CD}} \,, \tag{4.54}$$

$$\overline{\partial_0 \Gamma^C_{rr}} = \overline{g}^{CD}\partial_r\xi_D - \frac{1}{2}r|\sigma|^2\xi^C + \frac{r}{2}\check{\nabla}^C|\sigma|^2 \,. \tag{4.55}$$

We then obtain

$$\begin{aligned}0 = \overline{\partial_0 H^C} &= -\overline{\partial_0 g_{rr}}\overline{\Gamma}^C_{00} - 2\overline{\partial_0 g_{rr}}\overline{\Gamma}^C_{0r} - 2\overline{g}^{AB}\overline{\partial_0 g_{rB}}\overline{\Gamma}^C_{0A} - 2\overline{g}^{AB}\overline{\partial_0 g_{rB}}(\overline{\Gamma}^C_{rA} - \hat{\Gamma}^C_{rA}) \\ &\quad + \overline{\partial_0 g^{AB}}(\overline{\Gamma}^C_{AB} - \hat{\Gamma}^C_{AB}) + 2\overline{\partial_0 \Gamma^C_{0r}} + \overline{\partial_0 \Gamma^C_{rr}} + \overline{g}^{AB}\overline{\partial_0 \Gamma^C_{AB}} - \overline{\partial_0 W^C} \\ &= \Big(\partial_r + \frac{3}{r} + \frac{3}{2}r|\sigma|^2\Big)\xi^C + \overline{g}^{CD}\overline{\partial^2_{00} g_{rD}} + \frac{1}{2}(\check{\nabla}^C - \xi^C)\zeta + \frac{r}{2}\check{\nabla}^C|\sigma|^2 \\ &\quad + \overline{\partial_0 g^{AB}}(\check{\Gamma}^C_{AB} - \mathring{\Gamma}^C_{AB}) - (\check{\nabla}_A - 2\xi_A)\overline{\partial_0 g^{AC}} - \overline{\partial_0 W^C} \,.\end{aligned}$$

Combining this with the expression we found for $\overline{\partial^2_{00} g_{rA}}$ yields

$$\begin{aligned}\overline{\partial_0 W^A} &= (\overline{\partial_0 g^{AB}} - 2\sigma^{AB})\xi_B - \frac{r}{2}(\check{\nabla}^A - 2\xi^A)|\sigma|^2 - \frac{1}{2}\check{\nabla}^A\zeta \\ &\quad + \overline{\partial_0 g^{BC}}(\check{\Gamma}^A_{BC} - \mathring{\Gamma}^A_{BC}) \quad (4.56) \\ &= \mathcal{O}(r^{-4}) \,. \quad (4.57)\end{aligned}$$

Einstein equation $\overline{R}_{00} = 0$: The (00)-component of the Ricci tensor satisfies

$$\overline{R}_{00} = (\partial_r + 2\overline{\Gamma}^r_{rr} + \overline{\Gamma}^A_{rA})\overline{\Gamma}^r_{00} - \overline{\partial_0 \Gamma^r_{0r}} - \overline{\partial_0 \Gamma^A_{0A}} - \overline{\Gamma}^r_{0r}\overline{\Gamma}^r_{0r} - 2\overline{\Gamma}^r_{0A}\overline{\Gamma}^A_{0r} - \overline{\Gamma}^A_{0B}\overline{\Gamma}^B_{0A} \,.$$

The u-differentiated Christoffel symbols appearing in this expression satisfy,

$$\overline{\partial_0 \Gamma^r_{0r}} = \frac{1}{2r}\zeta + \frac{1}{2}\check{R} - \frac{1}{4}\xi_A\xi^A - \frac{r^2}{4}|\sigma|^4 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}\sigma^{AB}\overline{\partial_0 g_{AB}} \,, \tag{4.58}$$

$$\overline{\partial_0 \Gamma^A_{0A}} = \frac{1}{2}\overline{g}^{AB}\overline{\partial^2_{00} g_{AB}} + \frac{1}{2}\overline{\partial_0 g^{AB}}\,\overline{\partial_0 g_{AB}} - \frac{1}{2}\xi_A\xi^A \,, \tag{4.59}$$

and we are led to

$$\overline{R}_{00} = -\frac{1}{2}\overline{g}^{AB}\overline{\partial^2_{00} g_{AB}} - \Big(\frac{1}{2}\sigma^{AB} + \frac{1}{4}\overline{\partial_0 g^{AB}}\Big)\overline{\partial_0 g_{AB}} - \frac{1}{2r}\zeta - \frac{1}{2}\check{R} + \frac{1}{4}\xi_A\xi^A - \frac{1}{2}|\sigma|^2 \,,$$

which should vanish in vacuum, solving for $\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}}$ we are led to

$$\begin{aligned}
\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}} &= -\big(\sigma^{AB}+\frac{1}{2}\overline{\partial_0 g^{AB}}\big)\overline{\partial_0 g_{AB}} - \frac{1}{r}\zeta - \check{R} + \frac{1}{2}\xi_A\xi^A - |\sigma|^2 && (4.60)\\
&= \frac{1}{2}|D|^2 r^{-2} + \mathcal{O}(r^{-3})\ . && (4.61)
\end{aligned}$$

To compute $\overline{\partial_0 W^r}$ we employ the gauge condition

$$\begin{aligned}
0=\overline{\partial_0 H^r} &= -\overline{\partial_0 g_{rr}}(\overline{\Gamma}^r_{00}-\overline{\Gamma}^r_{rr}) - 2\overline{g}^{AB}\overline{\partial_0 g_{rB}}\overline{\Gamma}^r_{0A} - 2\overline{g}^{AB}\overline{\partial}_0 g_{rB}\overline{\Gamma}^r_{rA}\\
&\quad +\overline{\partial_0 g^{AB}}(\overline{\Gamma}^r_{AB}-\hat{\Gamma}^r_{AB}) + 2\overline{\partial_0\Gamma^r_{0r}} + \overline{\partial_0\Gamma^r_{rr}} + \overline{g}^{AB}\overline{\partial_0\Gamma^r_{AB}} - \overline{\partial_0 W^r}\\
&= \frac{1}{2}(\check{\nabla}_A-\xi_A)\xi^A - \frac{1}{2}\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}} - \frac{1}{r^2} - (1+\frac{r}{2}\partial_r)|\sigma|^2 - \frac{3}{4}r^2|\sigma|^4\\
&\quad -\frac{\zeta}{2}(\frac{1}{r}-\frac{r}{2}|\sigma|^2) + (rs_{AB}-\frac{1}{2}\sigma_{AB}-\frac{1}{2}\overline{\partial_0 g_{AB}})\overline{\partial_0 g^{AB}} - \overline{\partial_0 W^r} && (4.62)
\end{aligned}$$

For this calculation we used that

$$\overline{g}^{AB}\overline{\partial_0\Gamma^r_{AB}} = \check{\nabla}_A\xi^A + \frac{1}{2}(\partial_r+\frac{2}{r}+r|\sigma|^2)\zeta + \frac{1}{r^2} - |\sigma|^2 - \sigma^{AB}\overline{\partial_0 g_{AB}} - \frac{1}{2}\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}}\ .$$

We solve (4.62) for $\overline{\partial_0 W^r}$, and insert the expression we found for $\overline{g}^{AB}\overline{\partial^2_{00}g_{AB}}$,

$$\begin{aligned}
\overline{\partial_0 W^r} &= \frac{1}{2}(\check{\nabla}_A-\frac{3}{2}\xi_A)\xi^A + \frac{1}{4}r|\sigma|^2\zeta + \frac{1}{2}\check{R} - \frac{1}{r^2} - \frac{1}{2}|\sigma|^2 - \frac{r}{2}\partial_r|\sigma|^2\\
&\quad -\frac{3}{4}r^2|\sigma|^4 - (\frac{1}{4}\overline{\partial_0 g_{AB}} + \sigma_{AB} - rs_{AB})\overline{\partial_0 g^{AB}} && (4.63)\\
&= \frac{1}{4}|D|^2 r^{-2} + \mathcal{O}(r^{-3})\ . && (4.64)
\end{aligned}$$

Einstein equations $(\overline{\partial_0 R_{AB}})\breve{}=0$ Assuming the gauge condition $H^\lambda=0$ the Einstein equations $(\overline{\partial_0 R_{AB}})\breve{}=0$ provide a linear ODE-system for $(\overline{\partial^2_{00}g_{AB}})\breve{}$. We have

$$\overline{\partial_0 R_{AB}} = \overline{\partial_\lambda\partial_0\Gamma^\lambda_{AB}} - \partial_A\overline{\partial_0\Gamma^\lambda_{\lambda B}} + \overline{\Gamma}^\lambda_{\rho\lambda}\overline{\partial_0\Gamma^\rho_{AB}} + \overline{\Gamma}^\rho_{AB}\overline{\partial_0\Gamma^\lambda_{\rho\lambda}} - \overline{\Gamma}^\lambda_{\rho A}\overline{\partial_0\Gamma^\rho_{\lambda B}} - \overline{\Gamma}^\rho_{\lambda B}\overline{\partial_0\Gamma^\lambda_{\rho A}}\ .$$

The right-hand side contains several first-order u-differentiated Christoffel symbols and one of second-order, $\overline{\partial^2_{00}\Gamma^0_{AB}}$, which we have not determined yet,

$$\begin{aligned}
\overline{\partial_0\Gamma^0_{00}} &= \overline{\partial^2_{00}g_{0r}}\ , && (4.65)\\
\overline{\partial_0\Gamma^0_{0A}} &= \frac{1}{2}\overline{\partial^2_{00}g_{rA}} - \frac{1}{2}\xi^B\overline{\partial_0 g_{AB}} + \frac{r}{2}|\sigma|^2\xi_A && (4.66)\\
&= -\frac{1}{2}\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2})\ , && (4.67)\\
\overline{\partial_0\Gamma^0_{rA}} &= -\chi_A{}^B\xi_B - \frac{r}{2}(\partial_A+\xi_A)|\sigma|^2 && (4.68)\\
&= -2\mathring{\nabla}_B(\sigma_A{}^B)_2 r^{-2} + \mathcal{O}(r^{-3})\ , && (4.69)\\
\overline{\partial_0\Gamma^r_{AB}} &= -\frac{1}{2}\overline{\partial^2_{00}g_{AB}} - \frac{1}{2}(\partial_r+r|\sigma|^2)\overline{\partial_0 g_{AB}} + \check{\nabla}_{(A}\xi_{B)} - r|\sigma|^2\chi_{AB} && (4.70)\\
&= -\frac{1}{2}\overline{\partial^2_{00}g_{AB}} - \frac{1}{2}D_{AB} + \mathcal{O}(r^{-1})\ , && (4.71)\\
\overline{\partial_0\Gamma^C_{0A}} &= -\frac{1}{2}\xi_A\xi^C + \frac{1}{2}\overline{\partial_0 g^{CD}}\overline{\partial_0 g_{AD}} + \frac{1}{2}\overline{g}^{CD}\overline{\partial^2_{00}g_{AD}} && (4.72)\\
&= \frac{1}{2}g^{CD}\partial^2_{00}g_{AD} - \frac{1}{2}D_{AD}D^{CD}r^{-2} + \mathcal{O}(r^{-3})\ . && (4.73)
\end{aligned}$$

To calculate the two-times u-differentiated Christoffel symbol we use that

$$\overline{\partial^2_{00}g^{\mu\nu}} = -2\overline{g}^{\nu\sigma}\overline{\partial_0 g^{\mu\rho}\partial_0 g_{\rho\sigma}} - \overline{g}^{\mu\rho}\overline{g}^{\nu\sigma}\overline{\partial^2_{00}g_{\rho\sigma}} \,,$$

whence

$$\begin{aligned}
\overline{\partial^2_{00}g^{00}} &= -\overline{\partial^2_{00}g_{rr}} + 2\xi_A\xi^A + 2r^2|\sigma|^4 && (4.74)\\
&= -\Xi r^{-3} + \mathcal{O}(r^{-4}) \,, && (4.75)\\
\overline{\partial^2_{00}g^{0r}} &= -\overline{\partial^2_{00}g_{0r}} - \overline{\partial^2_{00}g_{rr}} + \underbrace{2\xi_A\xi^A + 2r^2|\sigma|^4}_{=\mathcal{O}(r^{-4})} \,, && (4.76)\\
\overline{\partial^2_{00}g^{0C}} &= -\overline{g}^{CD}\overline{\partial^2_{00}g_{rD}} - 2r|\sigma|^2\xi^C - 2\xi_D\overline{\partial_0 g^{CD}} && (4.77)\\
&= \mathring{\nabla}_B D^{BC}r^{-3} + \mathcal{O}(r^{-4}) \,. && (4.78)
\end{aligned}$$

That yields

$$\begin{aligned}
\overline{\partial^2_{00}\Gamma^0_{AB}} &= \Big(\frac{1}{2}\overline{\partial^2_{00}g_{rr}} - r^2|\sigma|^4 - \xi_C\xi^C\Big)(\overline{\partial_0 g_{AB}} + 2\chi_{AB}) + \chi_{AB}\overline{\partial^2_{00}g_{0r}}\\
&\quad -\Big(\frac{1}{2}\partial_r + r|\sigma|^2\Big)\overline{\partial^2_{00}g_{AB}} - r|\sigma|^2\partial_r\overline{\partial_0 g_{AB}} + 2r|\sigma|^2\check{\nabla}_{(A}\xi_{B)}\\
&\quad -\xi^C(2\check{\nabla}_{(A}\overline{\partial_{|0|}g_{B)C}} - \check{\nabla}_C\overline{\partial_0 g_{AB}}) + \check{\nabla}_{(A}\overline{\partial^2_{|00|}g_{B)r}}\\
&= \chi_{AB}\overline{\partial^2_{00}g_{0r}} - \Big(\frac{1}{2}\partial_r + r|\sigma|^2\Big)\overline{\partial^2_{00}g_{AB}} - \mathring{\nabla}_{(A}\mathring{\nabla}_{|C|}D_{B)}{}^C r^{-1} + \mathcal{O}(r^{-2}) \,.
\end{aligned}$$

Now all quantities have been determined which are needed to compute $(\overline{\partial_0 R_{AB}})\breve{}$ (the $\overline{\partial^2_{00}g_{0r}}$'s cancel out); we are just interested in the asymptotic behavior,

$$\begin{aligned}
(\overline{\partial_0 R_{AB}})\breve{} &= -\big(\partial_r - r^{-1} + r|\sigma|^2\big)(\overline{\partial^2_{00}g_{AB}})\breve{} + 2\sigma_{(A}{}^C(\overline{\partial^2_{|00|}g_{B)C}})\breve{}\\
&\quad - \frac{1}{2}(\Delta_s - 2)D_{AB}r^{-1} + \mathcal{O}(r^{-2}) \,,
\end{aligned}$$

and this expression should vanish in vacuum.

We need to check whether there are logarithmic terms in the asymptotic expansion of $(\overline{\partial^2_{00}g_{AB}})\breve{}$, solution to $(\overline{\partial_0 R_{AB}})\breve{} = 0$. However, this ODE is of exactly the same form as the ODE (4.42) for $\overline{\partial_0 g_{AB}}$. Therefore an identical argument applies and leads to the conclusion that the solution can be asymptotically expanded as a power series,

$$(\partial^2_{00}g_{AB})\breve{} = C_{AB}r + \mathcal{O}(1) \,, \qquad (4.79)$$

where the symmetric and s-trace-free tensor C_{AB} can be identified with the integration functions.

For completeness let us also compute

$$\begin{aligned}
\overline{\partial_0\Gamma^r_{0A}} &= \frac{1}{2}\overline{\partial^2_{00}g_{rA}} - \frac{1}{2}\xi^B\overline{\partial_0 g_{AB}} + \frac{r}{2}|\sigma|^2\xi_A && (4.80)\\
&= -\frac{1}{2}\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}) \,. && (4.81)
\end{aligned}$$

4.7 Overview of the metric gauge II

We give an overview of the values of all those objects we have computed so far in the metric gauge. Recall the values for the gauge functions κ, $\overline{W^\lambda}$ and $\overline{\partial_0 W^\lambda}$ as given by (4.29)-(4.32), (4.49), (4.56) and (4.63), needed to realize these values. Recall further that equality is meant to hold for $r > r_2$.

Metric components

$$
\begin{array}{ll}
\overline{g}_{00} = -1, & \overline{g}^{00} = 0, \\
\overline{g}_{0r} = 1, & \overline{g}^{0r} = 1, \\
\overline{g}_{0A} = 0, & \overline{g}^{0A} = 0, \\
\overline{g}_{rr} = 0, & \overline{g}^{rr} = 1, \\
\overline{g}_{rA} = 0, & \overline{g}^{rA} = 0, \\
\overline{g}_{AB} = r^2\sqrt{\frac{\det s}{\det \gamma}}\gamma_{AB} = r^2 s_{AB} + \mathcal{O}(r), & \overline{g}^{AB} = r^{-2}\sqrt{\frac{\det \gamma}{\det s}}\gamma^{AB} = r^{-2}s^{AB} + \mathcal{O}(r^{-3}).
\end{array}
$$

First-order u-derivatives of the metric components

$$
\begin{array}{ll}
\overline{\partial_0 g_{00}} = 0, & \overline{\partial_0 g^{00}} = r|\sigma|^2 = \mathcal{O}(r^{-3}), \\
\overline{\partial_0 g_{0r}} = 0, & \overline{\partial_0 g^{0r}} = r|\sigma|^2 = \mathcal{O}(r^{-3}), \\
\overline{\partial_0 g_{0A}} = 0, & \overline{\partial_0 g^{0A}} = -\xi^A = \mathcal{O}(r^{-3}), \\
\overline{\partial_0 g_{rr}} = -r|\sigma|^2 = \mathcal{O}(r^{-3}), & \overline{\partial_0 g^{rr}} = r|\sigma|^2 = \mathcal{O}(r^{-3}), \\
\overline{\partial_0 g_{rA}} = \xi_A = \mathcal{O}(r^{-1}), & \overline{\partial_0 g^{rA}} = -\xi^A = \mathcal{O}(r^{-3}), \\
\overline{\partial_0 g_{AB}} = D_{AB} r + \mathcal{O}(1), & \overline{\partial_0 g^{AB}} = -D^{AB} r^{-3} + \mathcal{O}(r^{-4}).
\end{array}
$$

Second-order u-derivatives of the metric components

$$
\begin{aligned}
\overline{\partial^2_{00} g_{rr}} &= \Xi r^{-3} + \mathcal{O}(r^{-4}) \,, \\
\overline{\partial^2_{00} g_{rA}} &= -\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}) \,, \\
\overline{g}^{AB}\overline{\partial^2_{00} g_{AB}} &= \frac{1}{2}|D|^2 r^{-2} + \mathcal{O}(r^{-3}) \,.
\end{aligned}
$$

Asymptotic behavior of the Christoffel symbols (cf. p. 27)

$$
\begin{array}{ll}
\overline{\Gamma}^0_{00} = 0, & \overline{\Gamma}^r_{rr} = \mathcal{O}(r^{-3}), \\
\overline{\Gamma}^0_{0r} = \mathcal{O}(r^{-3}), & \overline{\Gamma}^r_{rA} = \frac{1}{2}\mathring{\nabla}_B h^{(1)B}_A r^{-1} + \mathcal{O}(r^{-2}), \\
\overline{\Gamma}^0_{0A} = -\frac{1}{2}\mathring{\nabla}_B \check{h}^{(1)B}_A r^{-1} + \mathcal{O}(r^{-2}), & \overline{\Gamma}^r_{AB} = -(s_{AB} + \frac{1}{2}D_{AB})r + \mathcal{O}(1), \\
\overline{\Gamma}^0_{rr} = 0, & \overline{\Gamma}^C_{00} = 0, \\
\overline{\Gamma}^0_{rA} = 0, & \overline{\Gamma}^C_{0r} = -\frac{1}{2}\mathring{\nabla}^B h^{(1)C}_B r^{-3} + \mathcal{O}(r^{-4}), \\
\overline{\Gamma}^0_{AB} = -r s_{AB} + \mathcal{O}(1), & \overline{\Gamma}^C_{0A} = \frac{1}{2}D_A{}^C r^{-1} + \mathcal{O}(r^{-2}), \\
\overline{\Gamma}^r_{00} = 0, & \overline{\Gamma}^C_{rr} = 0, \\
\overline{\Gamma}^r_{0r} = \mathcal{O}(r^{-3}), & \overline{\Gamma}^C_{rA} = \frac{1}{r}\delta_A{}^C + \mathcal{O}(r^{-2}), \\
\overline{\Gamma}^r_{0A} = -\frac{1}{2}\mathring{\nabla}_B \check{h}^{(1)B}_A r^{-1} + \mathcal{O}(r^{-2}), & \overline{\Gamma}^C_{AB} = \mathring{\Gamma}^C_{AB} + \mathcal{O}(r^{-1}).
\end{array}
$$

First-order u-derivatives of the Christoffel symbols

We give a list of the asymptotic behavior of all the u-differentiated Christoffel symbols crucial e.g. for the computation of the Weyl tensor, which can be straightforwardly derived from our previous results,

$$
\begin{aligned}
&\overline{\partial_0\Gamma^0_{00}} = \text{not needed}, && \overline{\partial_0\Gamma^r_{rr}} = -\tfrac{1}{2}\Xi r^{-3} + \mathcal{O}(r^{-4}),\\
&\overline{\partial_0\Gamma^0_{0r}} = \tfrac{1}{2}\Xi r^{-3} + \mathcal{O}(r^{-4}), && \overline{\partial_0\Gamma^r_{rA}} = \tfrac{1}{2}\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}),\\
&\overline{\partial_0\Gamma^0_{0A}} = -\tfrac{1}{2}\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}), && \overline{\partial_0\Gamma^r_{AB}} = -\tfrac{1}{2}C_{AB} r + \mathcal{O}(1),\\
&\overline{\partial_0\Gamma^0_{rr}} = \tfrac{3}{8}|\breve{h}^{(1)}|^2 r^{-4} + \mathcal{O}(r^{-5}), && \overline{\partial_0\Gamma^C_{00}} = \text{not needed},\\
&\overline{\partial_0\Gamma^0_{rA}} = \mathring{\nabla}_B h^{(1)B}_A r^{-2} + \mathcal{O}(r^{-3}), && \overline{\partial_0\Gamma^C_{0r}} = -\tfrac{1}{2}\mathring{\nabla}_B D^{BC} r^{-3} + \mathcal{O}(r^{-4}),\\
&\overline{\partial_0\Gamma^0_{AB}} = -\tfrac{1}{2}D_{AB} + \mathcal{O}(r^{-1}), && \overline{\partial_0\Gamma^C_{0A}} = \tfrac{1}{2}C_A{}^C r^{-1} + \mathcal{O}(r^{-2}),\\
&\overline{\partial_0\Gamma^r_{00}} = \text{not needed}, && \overline{\partial_0\Gamma^C_{rr}} = \mathring{\nabla}^B h^{(1)C}_B r^{-4} + \mathcal{O}(r^{-5}),\\
&\overline{\partial_0\Gamma^r_{0r}} = \tfrac{1}{2}\Xi r^{-3} + \mathcal{O}(r^{-4}), && \overline{\partial_0\Gamma^C_{rA}} = -\tfrac{1}{2}D_A{}^C r^{-2} + \mathcal{O}(r^{-3}),\\
&\overline{\partial_0\Gamma^r_{0A}} = -\tfrac{1}{2}\mathring{\nabla}_B D_A{}^B r^{-1} + \mathcal{O}(r^{-2}), && \overline{\partial_0\Gamma^C_{AB}} = \big[\mathring{\nabla}_{(A}D_{B)}{}^C - \tfrac{1}{2}\mathring{\nabla}^C D_{AB}\big] r^{-1} + \mathcal{O}(r^{-2}).
\end{aligned}
$$

5 Asymptotic Expansions of the unknowns of the CFE on $\mathscr{N}$

5.1 Conformal field equations (CFE)

As indicated in the introduction, one would like to establish an existence theorem for the characteristic initial value problem for the vacuum Einstein equations which guarantees a "piece of a smooth $\mathscr{I}^+$". This global existence problem, where one needs to control the limiting behavior near null infinity, can be transformed into a local one via a conformal rescaling $g \mapsto \tilde{g} = \Theta^2 g$, where the conformal factor Θ has to vanish on the hypersurface $\mathscr{I}^+$ which represents (future) null infinity (with $\mathrm{d}\Theta \neq 0$ there). Henceforth we use ˜ to label objects related to the unphysical space-time metric $\tilde{g}$.

The Einstein equations, regarded as equations for $\tilde{g}$ and Θ, become singular wherever Θ vanishes, and are therefore not suitable to tackle the issue of proving existence of a solution near $\mathscr{I}^+$. However, H. Friedrich was able to find a representation of the vacuum Einstein equations which remains regular even where the conformal factor vanishes, cf. e.g. [8]. These *conformal field equations (CFE)* treat Θ as an unknown rather than a gauge function. Its gauge freedom is reflected in the freedom to prescribe the curvature scalar $\tilde{R}$. This still leaves the freedom to prescribe Θ on the initial surface. We shall take

$$\overline{\Theta} = 1/r = x \quad \text{for } r > r_2 \tag{5.1}$$

as initial data on $\mathscr{N}$. The CFE give rise to a complicated and highly overdetermined system of PDEs, which, in 4 space-time dimensions, can be represented as a symmetric hyperbolic system, supplemented by certain constraint equations.

5.1.1 Metric conformal field equations (MCFE)

There exist different versions of the CFE depending on which fields are regarded as unknowns. Let us first introduce the *metric conformal field equations (MCFE)* [8]. Beside $\tilde{g}$ and Θ the unknowns are the Schouten tensor,

$$\tilde{L}_{\mu\nu} = \frac{1}{2}\tilde{R}_{\mu\nu} - \frac{1}{12}\tilde{R}\tilde{g}_{\mu\nu}\,,$$

the rescaled Weyl tensor,

$$\tilde{d}_{\mu\nu\sigma}{}^{\rho} = \Theta^{-1}\tilde{C}_{\mu\nu\sigma}{}^{\rho} = \Theta^{-1}C_{\mu\nu\sigma}{}^{\rho}\,,$$

and the scalar function (set $\Box_{\tilde g} = \tilde g^{\mu\nu}\tilde\nabla_\mu\tilde\nabla_\nu$)

$$\tilde s = \frac{1}{4}\Box_{\tilde g}\Theta + \frac{1}{24}\tilde R\Theta \ .$$

The MCFE read

$$\tilde R_{\mu\nu\sigma}{}^\kappa[\tilde g] = \Theta \tilde d_{\mu\nu\sigma}{}^\kappa + 2(\tilde g_{\sigma[\mu}\tilde L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa \tilde L_{\nu]\sigma}) \tag{5.2}$$
$$\tilde\nabla_\rho \tilde d_{\mu\nu\sigma}{}^\rho = 0 \ , \tag{5.3}$$
$$\tilde\nabla_\mu \tilde L_{\nu\sigma} - \tilde\nabla_\nu \tilde L_{\mu\sigma} = \tilde\nabla_\rho\Theta\, \tilde d_{\nu\mu\sigma}{}^\rho \ , \tag{5.4}$$
$$\tilde\nabla_\mu\tilde\nabla_\nu\Theta = -\Theta\tilde L_{\mu\nu} + \tilde s\tilde g_{\mu\nu} \ , \tag{5.5}$$
$$\tilde\nabla_\mu \tilde s = -\tilde L_{\mu\nu}\tilde\nabla^\nu\Theta \ , \tag{5.6}$$
$$2\Theta\tilde s - \tilde\nabla_\mu\Theta\tilde\nabla^\mu\Theta = 0 \ , \ . \tag{5.7}$$

Equation (5.7) is a consequence of (5.5) and (5.6) if it is known to hold at one point (e.g. by an appropriate choice of the initial data).

5.1.2 General conformal field equations (GCFE)

Consider any frame field $e_k = e^\mu{}_k\partial_\mu$ such that the $\tilde g(e_i, e_k) \equiv \tilde g_{ik}$'s are constants. The *general conformal field equations (GCFE)* [8] for the unknowns

$$e^\mu{}_k \ , \quad \tilde\Gamma_i{}^k{}_j \ , \quad \tilde d_{ijk}{}^l \ , \quad \tilde L_{ij} \ , \quad \Theta \ , \quad \tilde s$$

read (from now on Latin indices are used to denote frame-components)

$$[e_p, e_q] = (\tilde\Gamma_p{}^l{}_q - \tilde\Gamma_q{}^l{}_p)e_l \ , \tag{5.8}$$
$$e_{[p}(\tilde\Gamma_{q]}{}^i{}_j) - \tilde\Gamma_k{}^i{}_j\tilde\Gamma_{[p}{}^k{}_{q]} + \tilde\Gamma_{[p}{}^i{}_{|k|}\tilde\Gamma_{q]}{}^k{}_j = \delta_{[p}{}^i\tilde L_{q]j} - \tilde g_{j[p}\tilde L_{q]}{}^i - \frac{1}{2}\Theta\tilde d_{pqj}{}^i \ , \tag{5.9}$$
$$\tilde\nabla_i \tilde d_{pqj}{}^i = 0 \ , \tag{5.10}$$
$$\tilde\nabla_i\tilde L_{jk} - \tilde\nabla_j\tilde L_{ik} = \tilde\nabla_l\Theta\tilde d_{jik}{}^l \ , \tag{5.11}$$
$$\tilde\nabla_i\tilde\nabla_j\Theta = -\Theta\tilde L_{ij} + s\tilde g_{ij} \ , \tag{5.12}$$
$$\tilde\nabla_i\tilde s = -\tilde L_{ij}\tilde\nabla^j\Theta \ , \tag{5.13}$$
$$2\Theta\tilde s - \tilde\nabla_j\Theta\tilde\nabla^j\Theta = 0 \ , \tag{5.14}$$

where the $\tilde\Gamma_i{}^j{}_k$'s denote the Levi-Civita connection coefficients in the frame e_k. Again, (5.14) suffices to be satisfied at just one point.

5.2 Asymptotic behavior of the fields appearing in the CFE

In this section we analyze the asymptotic behavior of the restriction to $\mathscr{N}$ of the fields $\tilde g_{\mu\nu}$, $e^\mu{}_k$, $\tilde\Gamma^\sigma_{\mu\nu}$, $\tilde d_{\mu\nu\sigma}{}^\rho$, $\tilde L_{\mu\nu}$, Θ and $\tilde s$ and prove that they are smooth at infinity in the metric gauge and taking (5.1) when constructed as solutions of the constraint equations induced by the wave-map gauge reduced vacuum Einstein equations, supposing that the initial data γ are of the form (2.15), and the no-logs-condition (3.49) hold. As we shall show that the above fields are smooth at $\mathscr{I}^+$ w.r.t the e_k-frame if and only if they have this property in the coordinate frame defined by $\{u, x, x^A\}$, we shall end up with the result that (2.15) and (3.49) lead to smooth initial data for both MCFE and GCFE.

Using our previous results summarized in Section 4.7 most of the computations will be straightforward.

5.2.1 Asymptotic behavior of the metric tensor

It follows from (4.33)-(4.34) that[6]

$$\overline{\tilde{g}}_{00} = -x^2, \quad \overline{\tilde{g}}_{0x} = -1 \;, \quad \overline{\tilde{g}}_{0A} = \overline{\tilde{g}}_{xx} = \overline{\tilde{g}}_{xA} = 0 \;, \quad \overline{\tilde{g}}_{AB} = s_{AB} + \mathcal{O}(x) \;,$$

i.e. $\overline{\tilde{g}}$ has a smooth conformal completion at conformal infinity.

5.2.2 Asymptotic behavior of the Weyl tensor

Note that the rescaled Weyl tensor $\overline{\tilde{d}}_{\mu\nu\sigma}{}^{\rho} = \mathcal{O}(1)$ will be smooth at $\mathscr{I}^+$ if and only if the Weyl tensor is smooth at $\mathscr{I}^+$ and vanishes there, i.e. $\overline{\tilde{C}}_{\mu\nu\sigma}{}^{\rho} = \overline{C}_{\mu\nu\sigma}{}^{\rho} = \mathcal{O}(x)$.

In vacuum we have $R_{\mu\nu} = 0$, and the Weyl tensor coincides with the Riemann tensor,

$$\overline{C}_{\mu\nu\rho}{}^{\sigma} \;=\; \overline{R}_{\mu\nu\rho}{}^{\sigma} \;=\; \overline{\partial_\nu \Gamma^\sigma_{\mu\rho}} - \overline{\partial_\mu \Gamma^\sigma_{\nu\rho}} + \overline{\Gamma}^\alpha_{\mu\rho}\overline{\Gamma}^\sigma_{\alpha\nu} - \overline{\Gamma}^\alpha_{\nu\rho}\overline{\Gamma}^\sigma_{\alpha\mu} \;.$$

Due to its algebraic symmetries it suffices to consider the components

$$\begin{aligned} \overline{C}_{0r0}{}^{0} &= \mathcal{O}(r^{-3}) \;, \quad \overline{C}_{0rA}{}^{0} = \mathcal{O}(r^{-3}) \;, \quad \overline{C}_{0A0}{}^{0} = \mathcal{O}(r^{-1}) \;, \\ \overline{C}_{0A0}{}^{B} &= \mathcal{O}(r^{-1}) \;, \quad \overline{C}_{AB0}{}^{0} = \mathcal{O}(r^{-1}) \;, \quad \overline{C}_{rAB}{}^{0} = \mathcal{O}(r^{-3}) \;, \end{aligned}$$

as follows from the formulae in Section 4.7. Remarkably all the leading order terms which would induce terms of constant order after carrying out the coordinate transformation $r \mapsto x := 1/r$ cancel out, in particular those involving some of the integration constants whose explicit values are not known,

$$\begin{aligned} \overline{C}_{0x0}{}^{0} &= \mathcal{O}(x) \;, \quad \overline{C}_{0xA}{}^{0} = \mathcal{O}(x) \;, \quad \overline{C}_{0A0}{}^{0} = \mathcal{O}(x) \;, \\ \overline{C}_{0A0}{}^{B} &= \mathcal{O}(x) \;, \quad \overline{C}_{AB0}{}^{0} = \mathcal{O}(x) \;, \quad \overline{C}_{xAB}{}^{0} = \mathcal{O}(x) \;. \end{aligned}$$

(Recall that $\mathcal{O}(x)$ has been defined in Section 2.1.) To establish that $\overline{C}_{rAB}{}^{0} = \mathcal{O}(r^{-3})$ rather than $\overline{C}_{rAB}{}^{0} = \mathcal{O}(r^{-2})$ one needs to employ the no-logs-condition and is led to the geometric interpretation described in [6, Section 6.2]. No further condition on the initial data needs to be imposed.

5.2.3 Asymptotic behavior of $\overline{\partial_0 \Theta}$ and $\overline{\partial^2_{00}\Theta}$

To compute the remaining fields on $\mathscr{N}$ we first need to determine the trace of the first- and second-order u-derivative of the conformal factor Θ on $\mathscr{N}$. However, the values of Θ away from $\mathscr{N}$ depend on the unphysical curvature scalar $\tilde{R}$, which is treated as a conformal gauge source function in the CFE [8]. We impose the gauge condition

$$\overline{\tilde{R}} \;=\; \mathcal{O}(1) \;, \quad \overline{\tilde{\nabla}_0 \tilde{R}} = \mathcal{O}(1) \tag{5.15}$$

(which is no restriction since $\tilde{R}$ needs to be smooth at $\mathscr{I}^+$ anyway). The Ricci scalars of $\tilde{g} = \Theta^2 g$ and g are related via

$$\tilde{R} \;\;=\;\; \Theta^{-2}(R - 6\Theta^{-1}g^{\rho\sigma}\partial_\rho\partial_\sigma\Theta + 6\Theta^{-1}g^{\rho\sigma}\Gamma^\alpha_{\rho\sigma}\partial_\alpha\Theta) \;. \tag{5.16}$$

[6] While up to now we used the variable $x = 1/r$ mainly as an auxiliary quantity to facilitate the comparison of the constraint equations with the formulae in Appendix A, we shall use it henceforth as a coordinate.

With $R=0$ that yields an ODE for $\overline{\partial_0\Theta}$ on $\mathscr{N}$ where $\overline{\Theta}=1/r$,

$$
\begin{aligned}
\overline{\tilde{R}} &= 6r^2\Big[-2r\Big(\partial_r+r^{-1}+\frac{r}{2}|\sigma|^2\Big)\overline{\partial_0\Theta}+|\sigma|^2+\frac{1}{2}r^{-1}\overline{g}^{AB}\overline{\partial_0 g_{AB}}\Big] \\
&= -12r^3\Big(\partial_r+r^{-1}+\mathcal{O}(r^{-3})\Big)\overline{\partial_0\Theta}+\mathcal{O}(r^{-1})\;. \qquad (5.17)
\end{aligned}
$$

Employing (5.15) it takes the form

$$
r\partial_r\overline{\partial_0\Theta}+[1+\mathcal{O}(r^{-2})]\overline{\partial_0\Theta} \;=\; \mathcal{O}(r^{-2})\;. \qquad (5.18)
$$

Appendix A tells us (with $\lambda=\hat{\ell}=1$) that the asymptotic expansion does not contain logarithmic terms and is of the form

$$
\overline{\partial_0\Theta}=\mathcal{O}(r^{-1})=\mathcal{O}(x)\;. \qquad (5.19)
$$

The second-order u-derivative of the conformal factor can be computed as follows: From (5.16) we deduce with $\overline{\partial_0 R}=0$ that

$$
\begin{aligned}
\overline{\partial_0\tilde{R}} &= 6\overline{\partial_0(\Theta^{-3}g^{\rho\sigma}\Gamma^{\alpha}_{\rho\sigma}\partial_\alpha\Theta-\Theta^{-3}g^{\rho\sigma}\partial_\rho\partial_\sigma\Theta)} \\
&= -3r\overline{\tilde{R}}\overline{\partial_0\Theta}-12r^3\Big(\partial_r+r^{-1}+\mathcal{O}(r^{-3})\Big)\overline{\partial^2_{00}\Theta}+\mathcal{O}(1)\;.
\end{aligned}
$$

Taking the gauge condition (5.15) into account this ODE for $\overline{\partial^2_{00}\Theta}$ becomes

$$
r\partial_r\overline{\partial^2_{00}\Theta}+[1+\mathcal{O}(r^{-2})]\overline{\partial^2_{00}\Theta} \;=\; \mathcal{O}(r^{-2})\;, \qquad (5.20)
$$

which is of the same form as (5.19). Hence

$$
\overline{\partial^2_{00}\Theta}=\mathcal{O}(r^{-1})=\mathcal{O}(x)\;. \qquad (5.21)
$$

5.2.4 Asymptotic behavior of the Christoffel symbols

We have computed the restriction to $\mathscr{N}$ of the Christoffel symbols in adapted null coordinates (u,r,x^A) by imposing the metric gauge condition, cf. Section 4.7. Using the well-known behavior of Christoffel symbols under coordinate transformations we determine their asymptotic behavior in the $(u,x=1/r,x^A)$-coordinates,

$$
\begin{array}{ll}
\overline{\Gamma}^0_{00}=0\;, & \overline{\Gamma}^x_{xx}=-2x^{-1}+\mathcal{O}(x)\;, \\
\overline{\Gamma}^0_{0x}=\mathcal{O}(x)\;, & \overline{\Gamma}^x_{xA}=\mathcal{O}(x)\;, \\
\overline{\Gamma}^0_{0A}=\mathcal{O}(x)\;, & \overline{\Gamma}^x_{AB}=(s_{AB}+\frac{1}{2}D_{AB})x+\mathcal{O}(x^2)\;, \\
\overline{\Gamma}^0_{xx}=0\;, & \overline{\Gamma}^C_{00}=0\;, \\
\overline{\Gamma}^0_{xA}=0\;, & \overline{\Gamma}^C_{0x}=\mathcal{O}(x)\;, \\
\overline{\Gamma}^0_{AB}=-x^{-1}s_{AB}+\mathcal{O}(1)\;, & \overline{\Gamma}^C_{0A}=\frac{1}{2}D_A{}^Cx+\mathcal{O}(x^2)\;, \\
\overline{\Gamma}^x_{00}=0\;, & \overline{\Gamma}^C_{xx}=0\;, \\
\overline{\Gamma}^x_{0x}=\mathcal{O}(x^3)\;, & \overline{\Gamma}^C_{xA}=-\delta_A{}^Cx^{-1}+\mathcal{O}(1)\;, \\
\overline{\Gamma}^x_{0A}=\mathcal{O}(x^3)\;, & \overline{\Gamma}^C_{AB}=\mathring{\Gamma}^C_{AB}+\mathcal{O}(x)\;.
\end{array}
$$

We compute the trace of the Christoffel symbols on $\mathscr{N}$ associated to the unphysical metric $\tilde{g}$. The transformation law for Christoffel symbols under a conformal rescaling $g\mapsto\Theta^2 g$ of the metric reads,

$$
\tilde{\Gamma}^\rho_{\mu\nu} \;=\; \Gamma^\rho_{\mu\nu}+\frac{1}{\Theta}\left(\delta_\nu{}^\rho\partial_\mu\Theta+\delta_\mu{}^\rho\partial_\nu\Theta-g_{\mu\nu}g^{\rho\sigma}\partial_\sigma\Theta\right)\;, \qquad (5.22)
$$

which yields on $\mathscr{N}$, where $\overline{\Theta} = x$ and $\overline{\partial_0\Theta} = \mathcal{O}(x)$,

$$\begin{gathered}
\overline{\tilde{\Gamma}}^0_{00} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^0_{0x} = \mathcal{O}(x)\ ,\quad \overline{\tilde{\Gamma}}^0_{0A} = \mathcal{O}(x)\ ,\quad \overline{\tilde{\Gamma}}^0_{xx} = 0\ ,\quad \overline{\tilde{\Gamma}}^0_{xA} = 0\ ,\\
\overline{\tilde{\Gamma}}^0_{AB} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^x_{00} = \mathcal{O}(x^2)\ ,\quad \overline{\tilde{\Gamma}}^x_{0x} = \mathcal{O}(x)\ ,\quad \overline{\tilde{\Gamma}}^x_{0A} = \mathcal{O}(x^3)\ ,\\
\overline{\tilde{\Gamma}}^x_{xx} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^x_{xA} = \mathcal{O}(x)\ ,\quad \overline{\tilde{\Gamma}}^x_{AB} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^C_{00} = 0\ ,\quad \overline{\tilde{\Gamma}}^C_{0x} = \mathcal{O}(x)\ ,\\
\overline{\tilde{\Gamma}}^C_{0A} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^C_{xx} = 0\ ,\quad \overline{\tilde{\Gamma}}^C_{xA} = \mathcal{O}(1)\ ,\quad \overline{\tilde{\Gamma}}^C_{AB} = \mathcal{O}(1)\ .
\end{gathered}$$

The Christoffel symbols are smooth without any further restrictions on γ.

5.2.5 Asymptotic behavior of the Schouten tensor

From now all tensors will be expressed in terms of the coordinates (u, x, x^A). We compute the Schouten tensor $\tilde{L}_{\mu\nu} = \frac{1}{2}\tilde{R}_{\mu\nu} - \frac{1}{12}\tilde{R}\tilde{g}_{\mu\nu}$, restricted to $\mathscr{N}$, for the conformally rescaled metric $\tilde{g} = \Theta^2 g$. The transformation law for the Ricci tensor under conformal rescalings of the metric reads,

$$\tilde{L}_{\mu\nu} \;=\; L_{\mu\nu} - \Theta^{-1}(\partial_\mu\partial_\nu\Theta - \Gamma^\alpha_{\mu\nu}\partial_\alpha\Theta) + 2\Theta^{-2}(\partial_\mu\Theta\partial_\nu\Theta)\breve{}\,. \qquad (5.23)$$

With $\overline{L}_{\mu\nu} = 0$ we obtain on $\mathscr{N}$

$$\overline{\tilde{L}}_{\mu\nu} \;=\; 2x^{-2}\overline{\partial_\mu\Theta}\,\overline{\partial_\nu\Theta} - x^{-1}\overline{\partial_\mu\partial_\nu\Theta} + (x^{-1}\overline{\Gamma}^0_{\mu\nu} + \overline{g}_{\mu\nu})\overline{\partial_0\Theta} + x^{-1}\overline{\Gamma}^x_{\mu\nu} - \frac{1}{2}x^2\overline{g}_{\mu\nu}\ .$$

Assuming (5.15), so that (5.19) and (5.21) hold, we find

$$\begin{aligned}
\overline{\tilde{L}}_{00} &= 2x^{-2}\overline{\partial_0\Theta}\,\overline{\partial_0\Theta} - \overline{\partial_0\Theta} + \frac{1}{2}x^2 - x^{-1}\overline{\partial^2_{00}\Theta} \;=\; \mathcal{O}(1)\ ,\\
\overline{\tilde{L}}_{0x} &= \frac{1}{2} - \partial_x(x^{-1}\overline{\partial_0\Theta}) + \mathcal{O}(1)\overline{\partial_0\Theta} + \mathcal{O}(x^2) \;=\; \mathcal{O}(1)\ ,\\
\overline{\tilde{L}}_{0A} &= -x^{-1}\partial_A\overline{\partial_0\Theta} + \mathcal{O}(1)\overline{\partial_0\Theta} + \mathcal{O}(x^2) \;=\; \mathcal{O}(1)\ ,\\
\overline{\tilde{L}}_{xx} &= \mathcal{O}(1)\ ,\\
\overline{\tilde{L}}_{xA} &= \mathcal{O}(1)\ ,\\
\overline{\tilde{L}}_{AB} &= \mathcal{O}(x^{-1})\overline{\partial_0\Theta} + \mathcal{O}(1) \;=\; \mathcal{O}(1)\ ,
\end{aligned}$$

We conclude that the trace of the Schouten tensor on $\mathscr{N}$ is smooth at conformal infinity.

5.2.6 Asymptotic behavior of the function $\tilde{s}$

Let us determine the asymptotic behavior of the function $\tilde{s} \equiv \frac{1}{4}\Box_{\tilde{g}}\Theta + \frac{1}{24}\tilde{R}\Theta$ on $\mathscr{N}$. Using (5.16) with $\overline{R} = 0$ and (5.22) we find that

$$\begin{aligned}
\overline{\tilde{s}} &= \frac{1}{4}x^{-2}\overline{g}^{\mu\nu}\partial_\mu\partial_\nu\Theta - \frac{1}{4}x^{-2}\overline{g}^{\mu\nu}\tilde{\Gamma}^\kappa_{\mu\nu}\overline{\partial_\kappa\Theta} + \frac{1}{24}\overline{\tilde{R}}x\\
&= \frac{1}{24}x^{-1}\overline{R} + \frac{1}{4}x^{-2}\overline{g}^{\mu\nu}(\overline{\Gamma}^\alpha_{\mu\nu} - \overline{\tilde{\Gamma}}^\alpha_{\mu\nu})\partial_\alpha\Theta\\
&= \frac{1}{2}x^{-3}\overline{g}^{\alpha\beta}\overline{\partial_\alpha\Theta}\,\overline{\partial_\beta\Theta}\ ,
\end{aligned}$$

which recovers (5.6), and which yields with (5.19) that

$$\overline{\overline{s}} = \frac{1}{2}x - x^{-1}\overline{\partial_0 \Theta} = \mathcal{O}(1) , \tag{5.24}$$

i.e. the function $\overline{\overline{s}}$ is smooth at conformal infinity.

Asymptotic behavior of the frame field

The GCFE (5.8)-(5.14) require a frame field $(e^\mu{}_k)$ w.r.t which the metric tensor is constant. In our coordinates the trace of $\tilde g$ on $\mathscr{N}$ looks as follows (recall that equality is meant to hold for $r > r_2$):

$$\overline{\tilde g} = -x^2 \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}x + \tilde g_{AB}\mathrm{d}x^A \mathrm{d}x^B \quad \text{with} \quad \tilde g_{AB} = x^2 \overline{g}_{AB} = \mathcal{O}(1) .$$

We deduce that we may take $(e^\mu{}_k)$ to be of the following form on $\mathscr{N}$:

$$\begin{aligned} \overline{e}_0 &= \partial_u - \frac{x^2}{4}\partial_x , \\ \overline{e}_x &= \partial_x , \\ \overline{e}_{\tilde A} &= e^A{}_{\tilde A}\partial_A \quad \text{with} \quad e^A{}_{\tilde A} = \mathcal{O}(1) \quad \tilde A = 2,3 , \end{aligned}$$

and its dual

$$\begin{aligned} \overline{\Theta}^0 &= \mathrm{d}u , \\ \overline{\Theta}^x &= \mathrm{d}x + \frac{x^2}{4}\mathrm{d}u , \\ \overline{\Theta}^{\tilde A} &= \hat e^{\tilde A}{}_A \mathrm{d}x^A \quad \text{with} \quad \hat e^{\tilde A}{}_A = \mathcal{O}(1) \quad \tilde A = 2,3 . \end{aligned}$$

All the relevant fields are smooth at $\mathscr{I}^+$ w.r.t. this frame if and only if they have this property w.r.t. the coordinate frame defined by the adapted coordinates $\{u, x, x^A\}$, which we have shown to be the case.

5.3 Main result

Consider a space-time $(\mathscr{M}, g)$ which admits a smooth conformal completion at infinity *à la Penrose*, and consider a null hypersurface $\mathscr{N} \subset \mathscr{M}$ whose closure in the conformally completed space-time $\mathscr{M} \cup \mathscr{I}^+$ is smooth and meets $\mathscr{I}^+$ in a smooth spherical cross-section. It follows from the considerations in [11], cf. [7], that then one can introduce Bondi coordinates near $\overline{\mathscr{N}} \cap \mathscr{I}^+$ w.r.t. which $\mathscr{N}$ intersects $\mathscr{I}^+$ at the surface $\{u = 0\}$, and in which all the fields appearing in the CFE are smooth at $\mathscr{I}^+$ in the sense of Definition 2.1.

The existence of adapted null coordinates in which the unknowns of the CFE are smooth at $\mathscr{I}^+$ is thus a necessary condition for the existence of a space-time which admits a smooth conformal completion *à la Penrose*. We are led to the following result:

THEOREM 5.1 *A necessary-and-sufficient condition for the restrictions to $\mathscr{N}$ of all the fields appearing in the GCFE (5.8)-(5.14), or the MCFE (5.2)-(5.7), constructed from initial data $\gamma = \gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$ of the form (2.15) to be smooth at $\mathscr{I}^+$, is that the initial data γ satisfy the no-logs-condition (3.49) in some (and then all) adapted null coordinate systems. In that case the metric gauge provides a gauge choice where smoothness at $\mathscr{I}^+$ holds.*

Acknowledgments I am grateful to my advisor Piotr T. Chruściel for many valuable comments and suggestions, and for reading a first draft of this article. Supported in part by the Austrian Science Fund (FWF): P 24170-N16.

A Asymptotic solutions of Fuchsian ODEs

The main object of this appendix is to justify rigorously our use of expansions as asymptotic solutions to Einstein's characteristic constraint equations. Smoothness of these solutions at infinity is a crucial aspect of the analysis, which is why we derive necessary-and-sufficient conditions for the asymptotic expansions to involve no logarithmic terms. To do that we shall proceed as follows: Instead of using r as the independent variable, we introduce $x := 1/r$ as a new variable and study the transformed ODE near $x = 0$. For this we make Taylor expansions of the coefficients appearing in the ODE at $x = 0$, and write down the formal polyhomogeneous solutions. The Borel summation lemma guarantees that there exists a function whose polyhomogeneous expansion coincides with the formal series. This function will be shown to approximate the exact solution around $x = 0$, from which we eventually conclude that the formal polyhomogeneous solution is in fact an expansion of the exact solution at $x = 0$.

To illustrate the procedure, we first show how it works for linear first-order scalar equations in full generality. We then show how this adapts to linear first-order systems, under conditions corresponding to those that arise in the main text, in order to avoid an uninteresting discussion of several special cases. Every dependence on further variables, which we assume to have compact support, will be suppressed for convenience.

For the definition of polyhomogeneous functions we refer the reader to [7, Appendix A].

A.1 Formal solutions

A.1.1 Scalar equation $x\partial_x f + hf = g$

We consider the ODE

$$x\partial_x f + hf = g \ , \tag{A.1}$$

where $x^{-\ell} g = O(1)$, $\ell \in \mathbb{Z}$, and $h = O(1)$ (which clearly includes those cases where h has a zero of any order at $x = 0$) are assumed to be smooth functions on some interval $[0, x_0)$.

We represent $x^{-\ell} g$ and h via their Taylor expansions at $x = 0$,

$$g \sim \sum_{n=\ell}^{\infty} g_n x^n \ , \quad h \sim \sum_{n=0}^{\infty} h_n x^n$$

(the symbol $\sim$ has been defined in Section 2.1). We define the *indicial exponent* to be

$$\lambda := -h(0) = -h_0 \ . \tag{A.2}$$

When considering functions which further depend on angular variables, we will always make the supplementary hypothesis that

$$h_0 \text{ is angle-independent.} \tag{A.3}$$

1st case: $\lambda \notin \mathbb{Z}$. We make the ansatz

$$f \sim x^\lambda \sum_{n=\ell}^{\infty} f_{n+\lambda}^{(0)} x^n + \sum_{n=\ell}^{\infty} f_n^{(1)} x^n =: x^\lambda f^{(0)} + f^{(1)} . \tag{A.4}$$

Here we use $f^{(a)}$ as short form for the corresponding *formal* power series (in Section A.2 we shall use Borel summation to obtain a proper function from these formal expansions). In the course of this appendix it will become clear that any solution of (A.1) admits an expansion of the form (A.4), so this ansatz is not restrictive. It follows from (A.1) that $f^{(1)}$ needs to satisfy for any $N \in \mathbb{N}$

$$\begin{aligned} &\sum_{n=\ell}^{N} n f_n^{(1)} x^n + \sum_{n=\ell}^{N} \sum_{k=0}^{n-\ell} h_k f_{n-k}^{(1)} x^n = \sum_{n=\ell}^{N} g_n x^n + o(x^N) \\ \iff \quad &(n-\lambda) f_n^{(1)} = g_n - \sum_{k=1}^{n-\ell} h_k f_{n-k}^{(1)} \quad \text{for } n = \ell, \ell+1, \ldots \end{aligned} \tag{A.5}$$

Since $n - \lambda \neq 0$ by assumption, this defines a unique formal solution $f^{(1)}$ by solving the equations hierarchically. The formal series $f^{(0)}$ needs to satisfy

$$\begin{aligned} &x^\lambda \sum_{n=\ell}^{N} (\lambda + n) f_{n+\lambda}^{(0)} x^n + x^\lambda \sum_{n=\ell}^{N} \sum_{k=0}^{n-\ell} h_k f_{n-k+\lambda}^{(0)} x^n = o(x^{\lambda+N}) \\ \iff \quad &n f_{n+\lambda}^{(0)} = - \sum_{k=1}^{n-\ell} h_k f_{n-k+\lambda}^{(0)} \quad \text{for } n = \ell, \ell+1, \ldots \end{aligned} \tag{A.6}$$

We observe that $f_{n+\lambda}^{(0)} = 0$ for $n < 0$, while $f_\lambda^{(0)}$ can be chosen arbitrarily. Once this has been done, (A.6) determines the higher-order coefficients. Hence, our ansatz leads to a formal solution, where $f_\lambda^{(0)}$ can be thought of as representing the integration constant, or function if angular variables are involved. For convenience we will just speak of an integration constant in what follows.

2nd case: $\lambda \in \mathbb{Z}$. We start with the (again non-restrictive) ansatz

$$f \sim \sum_{n=\hat{\ell}}^{\infty} f_n^{(0)} x^n + \log x \sum_{n=\hat{\ell}}^{\infty} f_n^{(1)} x^n =: f^{(0)} + f^{(1)} \log x , \tag{A.7}$$

where we have set $\hat{\ell} := \min(\lambda, \ell)$. Inserting (A.7) into (A.1) yields

$$\begin{aligned} \log x \quad &\sum_{n=\hat{\ell}}^{N} n f_n^{(1)} x^n + \sum_{n=\hat{\ell}}^{N} f_n^{(1)} x^n + \log x \sum_{n=\hat{\ell}}^{N} \sum_{k=0}^{n-\hat{\ell}} h_k f_{n-k}^{(1)} x^n + \sum_{n=\hat{\ell}}^{N} n f_n^{(0)} x^n \\ &+ \sum_{n=\hat{\ell}}^{N} \sum_{k=0}^{n-\hat{\ell}} h_k f_{n-k}^{(0)} x^n = \sum_{n=\hat{\ell}}^{N} g_n x^n + o(x^N \log x) \end{aligned}$$

$$\iff \qquad (n-\lambda) f_n^{(1)} + \sum_{k=1}^{n-\hat{\ell}} h_k f_{n-k}^{(1)} = 0 \tag{A.8}$$

$$\text{and} \qquad f_n^{(1)} + (n-\lambda) f_n^{(0)} + \sum_{k=1}^{n-\hat{\ell}} h_k f_{n-k}^{(0)} = g_n \quad \text{for any } n \geq \hat{\ell} . \tag{A.9}$$

If $n = \hat{\ell}$, (A.8) is understood as $(\hat{\ell} - \lambda) f^{(0)}_{\hat{\ell}} = 0$, for $\hat{\ell} \le n < \lambda$ it leads to $f^{(1)}_n = 0$. Then (A.9) with $\hat{\ell} \le n < \lambda$ determines the coefficients $f^{(0)}_n$. If $n = \lambda$ (A.8), holds automatically, while (A.9) determines $f^{(1)}_\lambda$. The coefficient $f^{(0)}_\lambda$ can always be chosen arbitrarily. Once this has been done, all coefficients $f^{(1)}_n$ and $f^{(0)}_n$ with $\lambda > n$ are determined by the (A.8) and (A.9), respectively. This way we obtain a formal solution with one free parameter, $f^{(0)}_\lambda$, which can be regarded as integration constant.

These considerations also reveal that the formal solution (A.7) contains no logarithmic terms if and only if $f^{(1)}_\lambda = 0$ or, equivalently,

$$\sum_{k=1}^{\lambda-\hat{\ell}} h_k f^{(0)}_{\lambda-k} = g_\lambda \ . \tag{A.10}$$

Indeed the vanishing of $f^{(1)}_\lambda$ enforces by (A.8) that all the $f^{(1)}_n$'s are zero for $n > \lambda$ as well, while those with $n < \lambda$ have to vanish anyway. Note that (A.10) is in fact a condition relating h and g, since the $f^{(0)}_{\lambda-k}$'s are determined hierarchically from (A.9) (with $f^{(1)}_n = 0$), it does not depend on the boundary conditions as captured by the integration constant.

A.1.2 ODE-system $x\partial_x f + hf = g$

Let us now consider a first-order linear ODE-system,

$$x\partial_x f + hf = g \ , \tag{A.11}$$

where $h = O(1) \in \mathrm{Mat}(n,n)$ and $g = O(x^\ell)$, $\ell \in \mathbb{Z}$. The components of h and $x^{-\ell} g$ are assumed to be smooth functions on the interval $[0, x_0)$. For convenience, and because it suffices for our purposes, we focus on the case $n = 2$. Only at some points we add a comment how the general case looks like.

Again, we represent $x^{-\ell} g$ and h via their Taylor expansions at $x = 0$,

$$g \sim \sum_{n=\ell}^{\infty} g_n x^n \ , \quad g_n \in \mathbb{R}^2 \ , \quad h \sim \sum_{n=0}^{\infty} h_n x^n \ , \quad h_n \in \mathrm{Mat}(2,2) \ .$$

There exists a change of basis matrix $T \in \mathrm{GL}(2)$ such that $T h_0 T^{-1} =: h_0^J$ adopts Jordan normal form. Hence, it suffices to study the system

$$x\partial_x Tf + \Big[h_0^J + \sum_{n=1}^{\infty} T h_n T^{-1} x^n\Big] Tf = Tg \ ,$$

or, by relabeling the symbols,

$$x\partial_x f + \Big(\sum_{n=0}^{\infty} h_n x^n\Big) f = g \ , \tag{A.12}$$

with either

$$h_0 = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \quad \text{or} \quad h_0 = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \ . \tag{A.13}$$

We mentioned that a dependence on additional variables is permitted. However, as in the scalar case, we assume

$$\text{the } \textit{indicial matrix } h_0 \text{ is angle-independent,} \tag{A.14}$$

and thus is a truly constant matrix. In addition, since this covers all cases we are interested in with regard to the main text, we assume that

$$\lambda, \lambda_i \in \mathbb{Z} \ . \tag{A.15}$$

1st case: $h_0 = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$. W.l.o.g. we assume $\lambda_1 \leq \lambda_2$. Furthermore, we define

$$\hat{\ell} := \min(\lambda_1, \ell) \ .$$

We make the ansatz (again, later on it will be shown that any solution of (A.11) admits an expansion of the form (A.16))

$$f_i \sim \sum_{k=0}^{2} \log^k x \sum_{n=\hat{\ell}}^{\infty} (f_i^{(k)})_n x^n \ =: \ f_i^{(0)} + f_i^{(1)} \log x + f_i^{(2)} \log^2 x \ , \tag{A.16}$$

where the symbols appearing in this definition are to be understood in the same way as above. The upper index in brackets displays the order of the log term. Whenever useful, the coefficients $f_k^{(a)}$ with $k < \hat{\ell}$ are defined as to be zero.

We insert (A.16) into (A.12). The coefficients need to satisfy the following set of equations (for $n \geq \hat{\ell}$):

$$\begin{aligned}
&(1) \quad (n - \lambda_i)(f_i^{(2)})_n + F_i[(f^{(2)})_k, k < n] = 0 \ , \quad i = 1, 2 \ , \\
&(2) \quad 2(f_i^{(2)})_n + (n - \lambda_i)(f_i^{(1)})_n + G_i[(f^{(1)})_k, k < n] = 0 \ , \quad i = 1, 2 \ , \\
&(3) \quad (f_i^{(1)})_n + (n - \lambda_i)(f_i^{(0)})_n + H_i[(f^{(0)})_k, k < n] = (g_i)_n \ , \quad i = 1, 2 \ ,
\end{aligned}$$

where F_i, G_i and H_i are multi-linear functions of the indicated quantities. The explicit form of H_i, which will be needed later on, is

$$H_i = \sum_{k=1}^{n-\hat{\ell}} \Big[h_k (f^{(0)})_{n-k} \Big]_i \ , \tag{A.17}$$

analogously for the other functions. Note that H_i generally depends on both components of $f^{(0)}$ since there is no need for the h_k's, $k \geq 1$, to be diagonal.

A solution to the equations (1)-(3) can be constructed as follows: We describe the case $\lambda_1 < \lambda_2$, the case $\lambda_1 = \lambda_2$ can be treated similarly.

$\mathbf{n} < \lambda_\mathbf{1}$: We have to choose $(f_i^{(2)})_n = 0$ and $(f_i^{(1)})_n = 0$ to fulfill (1) and (2). The coefficients $(f_i^{(0)})_n$ will be generally non-zero and are determined by (3).

$\mathbf{n} = \lambda_\mathbf{1}$: Choose $(f_i^{(2)})_{\lambda_1} = 0$ and $(f_2^{(1)})_{\lambda_1} = 0$. The first component of (2) (i.e. the one with $i = 1$) is automatically satisfied, while the first component of (3) determines $(f_1^{(1)})_{\lambda_1}$. The coefficient $(f_1^{(0)})_{\lambda_1}$ is free to choose, while $(f_2^{(0)})_{\lambda_1}$ follows from the second component of (3) (the one with $i = 2$).

$\lambda_\mathbf{1} < \mathbf{n} < \lambda_\mathbf{2}$: (1) still requires $(f_i^{(2)})_n = 0$. The coefficients $(f_i^{(1)})_n$ are determined by (2), while the $(f_i^{(0)})_n$'s are determined by (3).

$\mathbf{n} = \lambda_2$: Set $(f_1^{(2)})_{\lambda_2} = 0$. The second component of (1) holds automatically, no matter what the value of $(f_2^{(2)})_{\lambda_2}$ is. The coefficient $(f_1^{(1)})_{\lambda_2}$ is determined by the first component of (2). The coefficient $(f_2^{(2)})_{\lambda_2}$ follows from (2). The coefficient $(f_2^{(1)})_{\lambda_2}$ is determined by the second component of (3). The coefficient $(f_1^{(0)})_{\lambda_2}$ follows from (3), while $(f_2^{(0)})_{\lambda_2}$ can be chosen arbitrarily.

$\mathbf{n} > \lambda_2$: All coefficients $(f_i^{(j)})_n = 0$ are determined by (1)-(3).

We remark that for $\lambda_1 = \lambda_2$ the coefficients which can be viewed as integration constants are $(f_i^{(0)})_\lambda$, $i = 1, 2$. Moreover, this case implies $f^{(2)} = 0$.

Consequently, the ansatz (A.16) leads to a formal solution of (A.12) with two free parameters, which can be considered as integration constants $(f_i^{(0)})_{\lambda_i}$.

In fact a similar ansatz, namely

$$f_i \quad \sim \quad \sum_{k=0}^{N} \sum_{n=\hat{\ell}}^{\infty} (f_i^{(k)})_n x^n \log^k x \;, \tag{A.18}$$

leads to a formal solution of the corresponding N-dimensional system with $h_0 = \operatorname{diag}(-\lambda_1, \ldots, -\lambda_N)$. The integration constants can be identified with $(f_i^{(0)})_{\lambda_i}$.

Logarithmic terms do not appear in (A.16) if and only if

$$\sum_{k=1}^{\lambda_i - \hat{\ell}} [h_k (f^{(0)})_{\lambda_i - k}]_i = (g_i)_{\lambda_i} \;, \quad i = 1, 2 \;. \tag{A.19}$$

Here one has to distinguish two cases: If

$$\text{(A.19) is independent of } (f_1^{(0)})_{\lambda_1} \;, \tag{A.20}$$

then (A.19) is, as in the scalar case, a condition involving exclusively g and h, i.e. it just concerns the equations itself and is independent of the boundary conditions. In particular this case occurs for $\lambda_1 = \lambda_2$. Otherwise the appearance of log terms depends on the value of the integration constant $(f_1^{(0)})_{\lambda_1}$.

2nd case: $h_0 = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$. We use the same ansatz (A.16) as in the 1st case, and with $\hat{\ell} := \min(\lambda, \ell)$. We insert this ansatz into (A.12) to end up with the following relations among the coefficients ($n \geq \hat{\ell}$):

(1) $(n - \lambda)(f_i^{(2)})_n + \delta^1{}_i (f_2^{(2)})_n + F_i[(f^{(2)})_k, k < n] = 0 \;, \; i = 1, 2,$

(2) $2(f_i^{(2)})_n + (n - \lambda)(f_i^{(1)})_n + \delta^1{}_i (f_2^{(1)})_n + G_i[(f^{(1)})_k, k < n] = 0 \;, \; i = 1, 2,$

(3) $(f_i^{(1)})_n + (n - \lambda)(f_i^{(0)})_n + \delta^1{}_i (f_2^{(0)})_n + H_i[(f^{(0)})_k, k < n] = (g_i)_n \;, \; i = 1, 2,$

where F_i, G_i and H_i are again multi-linear functions of the indicated quantities whose explicit formulae look similar to (A.17). We describe how one obtains a solution of these equations:

$\mathbf{n} < \lambda$: All coefficients are determined by (1)-(3), in particular $(f_i^{(1)})_n = (f_i^{(2)})_n = 0$.

$\mathbf{n} = \lambda$: Equation (1) is fulfilled iff $(f_2^{(2)})_\lambda = 0$. The second component of (2) holds since we have chosen $(f_2^{(2)})_\lambda = 0$. The first component of (2) enforces

$2(f_1^{(2)})_\lambda + (f_2^{(1)})_\lambda = 0$. In order to satisfy the second component of (3) the yet unspecified $(f_2^{(1)})_\lambda$ (equivalently $(f_1^{(2)})_\lambda$) has to be chosen such that

$$(f_2^{(1)})_\lambda + H_2[(f^{(0)})_k, k < \lambda] = (g_2)_\lambda \;, \tag{A.21}$$

whatever is taken for $(f_1^{(1)})_\lambda$ and $(f_2^{(0)})_\lambda$. The first component of (3) can be fulfilled by an appropriate choice of $(f_1^{(1)})_\lambda$, and is independent of $(f_1^{(0)})_\lambda$,

$$(f_1^{(1)})_\lambda + (f_2^{(0)})_\lambda + H_1[(f^{(0)})_k, k < \lambda] = (g_1)_\lambda \;. \tag{A.22}$$

$\mathbf{n} > \lambda$: All coefficients $(f_i^{(j)})_n$ are uniquely determined by (1)-(3).

This way we get a formal solution for any choice of $(f_i^{(0)})_\lambda$, $i = 1, 2$, which may be regarded as representing the integration constants.

The above algorithm shows that, in the current setting, logarithms are absent if and only if

$$(f_i^{(1)})_\lambda = 0 \;. \tag{A.23}$$

According to (A.21), $(f_2^{(1)})_\lambda$ vanishes iff $H_2[(f^{(0)})_k, k < \lambda] = (g_2)_\lambda$, i.e. iff h and g satisfy appropriate relations. However, (A.22) shows that the vanishing of $(f_1^{(1)})_\lambda$ depends on the integration constant $(f_2^{(0)})_\lambda$, i.e. on the boundary conditions, and thus cannot be guaranteed to hold generally. Only specific boundary data lead to solutions without logarithmic terms.

A.2 Borel summation

We have seen that there exist formal solutions of the ODE $x\partial_x f + hf = g$ (in one or two dimensions) of the form

$$f_{\text{formal}} = \varkappa^{(0)} f^{(0)} + \varkappa^{(1)} f^{(1)} + \varkappa^{(2)} f^{(2)} \;, \quad \varkappa^{(i)} \in \{\log x, \log^2 x, x^\lambda, 0, 1\} \tag{A.24}$$

where the form of $\varkappa^{(i)}$ depends on the specific value of the indicial exponent λ. However, in any case all the $f^{(i)}$'s are formal power series. We can therefore appeal to the *Borel Summation Lemma* (see, e.g., [4, Appendix D]) which states that for every formal power series f one can find a smooth function $\hat{f}$ whose Taylor expansion, around $x = 0$ say, coincides with f. Applied to our case that means that there are smooth functions $\hat{f}^{(i)}$, $i = 0, 1, 2$, such that

$$\hat{f} := \varkappa^{(0)} \hat{f}^{(0)} + \varkappa^{(1)} \hat{f}^{(1)} + \varkappa^{(2)} \hat{f}^{(2)} \sim f_{\text{formal}} \;. \tag{A.25}$$

Finally we emphasize that the integration constants are $(f^{(0)})_\lambda$ in the scalar case, and $(f_i^{(0)})_{\lambda_i}$ or $(f_i^{(0)})_\lambda$, respectively, in the 2-dimensional case, so that the corresponding expansion coefficients of $\hat{f}$ can be specified arbitrarily. This will be crucial for the subsequent argument.

A.3 Approximation of the exact solution

In the final step we will show that $\hat{f}$ approximates the exact solution f to arbitrary high order in x; equivalently, they have the same polyhomogeneous expansions. We denote by c (possibly supplemented by some index) a generic positive constant, while C is supposed to be a constant with a specific value. If angular variables are involved c and C will still supposed to be constant. We consider the ODE

$$x\partial_x f + hf = g\ , \quad h = O(1) \in \mathrm{Mat}(n,n)\ , \quad 0 < x < x_0\ , \tag{A.26}$$

where, as before, the components of h and $x^{-\ell}g$ are smooth functions on $[0, x_0)$. Set

$$\hat{g} := x\partial_x \hat{f} + h\hat{f}\ . \tag{A.27}$$

By construction of $\hat{f}$ the Taylor expansion of $g - \hat{g}$ at $x = 0$ is zero,

$$\delta g := g - \hat{g} \sim 0\ , \quad \text{i.e.} \tag{A.28}$$
$$\forall\ N \in \mathbb{N}\ \exists\ c^{(N)} > 0 : \|\delta g\| \le c^{(N)} x^N\ \ \forall\ x < x_0\ , \tag{A.29}$$

where $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Set $\delta f := f - \hat{f}$, then

$$x\partial_x \delta f + h\delta f = \delta g\ . \tag{A.30}$$

We want to show that for a given f we can adjust the initial conditions of $\hat{f}$ (equivalently, of the formal solution), such that δf is a solution of the ODE and satisfies $\|\delta f\| \le c^{(N)}x^N$ for all N. We find

$$\begin{aligned} |x\partial_x\|\delta f\|^2| &= 2|\,\langle \delta f, x\partial_x \delta f\rangle\,| = 2|\,\langle \delta f, \delta g - h\delta f\rangle\,| \\ &\le 2|\,\langle \delta f, \delta g\rangle\,| + 2|\,\langle \delta f, h\delta f\rangle\,| \le (1 + 2\|h\|)\|\delta f\|^2 + \|\delta g\|^2\ . \end{aligned}$$

Setting $\psi := \|\delta f\|^2$, we thus have

$$\begin{aligned} \pm x\partial_x \psi &\le (1 + 2\|h\|)\psi + \|\delta g\|^2 \\ \Longrightarrow \quad \pm\partial_x(\chi^{\pm 1}\psi) &\le \frac{\|\delta g\|^2}{x}\chi^{\pm 1}\ , \end{aligned}$$

where

$$\chi := \exp\left(\int_x^{x_0} \frac{1 + 2\|h\|}{y}\,\mathrm{d}y\right)\ .$$

The function χ satisfies the inequality (with $h_0 \equiv h(0)$)

$$\chi \le \exp\Big(\int_x^{x_0} \frac{1 + 2\|h_0\|}{y}\,\mathrm{d}y\Big) \cdot \underbrace{\exp\Big(2\int_x^{x_0} \frac{\|h - h_0\|}{y}\,\mathrm{d}y\Big)}_{=:\hat{\chi}=\mathcal{O}(1)} = x^{-\mu}x_0^{\mu}\hat{\chi}\ , \tag{A.31}$$

where

$$\mu := 1 + 2\|h_0\| \ge 1\ ,$$

and $\hat{\chi} > 0$ is a smooth function bounded away from zero in $[0, x_0)$,

$$\hat{\chi}^{\pm 1} \le c \tag{A.32}$$
$$\Longrightarrow \quad \pm\partial_x(\chi^{\pm 1}\psi) \le c\,\|\delta g\|^2 x_0^{-\mu} x^{\mu - 1} \le c^{(N)}x^N\ . \tag{A.33}$$

The inequality with the minus sign yields

$$\begin{aligned} (\chi^{-1}\psi)(x) &\le (\chi^{-1}\psi)(x_0) + \int_x^{x_0} c^{(N)}y^N\,\mathrm{d}y \le c \\ &\overset{(A.31)\ \&\ (A.32)\,`+\text{'}}{\Longrightarrow} \quad \psi \le c\chi \le c\,x^{-\mu} \\ &\Longrightarrow \quad \|\delta f\| \le c\,x^{-\mu/2}\ . \end{aligned} \tag{A.34}$$

We conclude that

$$\|x\partial_x \delta f + h_0 \delta f\| = \|\delta g - (h - h_0)\delta f\| \le c\, x^{-\mu/2+1} \,. \tag{A.35}$$

This leads us to the study of the ODE

$$x\partial_x \tilde{f} + h_0 \tilde{f} = \tilde{g} \tag{A.36}$$

with source $\tilde{g}$ fulfilling $\|\tilde{g}\| \le c\, x^{1-\mu/2}$. The general solution to this equation is

$$\tilde{f}(x) = x^{-h_0} x_0^{h_0} \tilde{f}(x_0) - x^{-h_0} \int_x^{x_0} y^{h_0 - \mathbf{1}} \tilde{g}(y)\, \mathrm{d}y \,. \tag{A.37}$$

Here $\mathbf{1}$ denotes the n-dimensional identity matrix. When writing x^{h_0} for a matrix h_0 we mean $\exp(h_0 \log x)$. In the following we will distinguish two cases, depending on whether h_0 can be diagonalized or not.

1st case: We assume that h_0 can be diagonalized. Clearly this case includes the 1-dim. case. In fact, let us focus for the time being on that case and return to the general case later. Equation (A.37) then implies

$$\begin{aligned} |\delta f| \quad &\le \quad c^{(1)} x^{-h_0} + c^{(2)} x^{-h_0} \int_x^{x_0} y^{h_0-1}(|\delta g| + c^{(3)} y |\delta f|)\mathrm{d}y \\ &\overset{(A.34)}{\le} \quad c^{(1)} x^{-h_0} + c^{(2)} x^{-h_0} \int_x^{x_0} y^{h_0-\mu/2}\mathrm{d}y \,. \end{aligned}$$

Replacing μ by a slightly larger number if necessary, we may assume w.l.o.g. $h_0 - \mu/2 \neq -1$. Then

$$|\delta f| \le c^{(1)} x^{-h_0} + c^{(2)} x^{-h_0} \left[\frac{y^{h_0-\mu/2+1}}{h_0 - \mu/2 + 1} \right]_x^{x_0} = c^{(1)} x^{-h_0} + c^{(2)} x^{-\mu/2+1} \,.$$

Suppose that $\mu/2 - 1 < h_0$, then the inequality $|\delta f| \le c\, x^{-h_0}$ follows. If $\mu/2 - 1 > h_0$, we can merely conclude $|\delta f| \le c\, x^{-\mu/2+1}$. However, this improves the estimate in (A.34) by a factor x. Repeating the whole procedure k-times until $\mu/2 - 1 - k < h_0$ we finally end up with the estimate

$$|x^{h_0} \delta f| \quad \le \quad c \quad \text{for all } x \in (0, x_0) \,,$$

which is independent of the specific relation between μ and h_0. Since

$$|\partial_x (x^{h_0} \delta f)| = x^{h_0-1} |x\partial_x \delta f + h_0 \delta f| = x^{h_0-1} |\delta g - (h - h_0)\delta f| \le c \,,$$

$x^{h_0}\delta f$ can be continued to a continuous function on $[0, x_0)$. Hence, multiplying (A.37) (with $\tilde{f} = \delta f$) by x^{h_0}, we observe that $\delta F := x^{h_0}\delta f$ is continuous even at $x = 0$. Performing the limit $x_0 \to 0$, we find

$$\begin{aligned} \delta F \quad &= \quad C + \int_0^x y^{h_0-1}(\delta g - (h - h_0)\delta f)\, \mathrm{d}y \\ \Longrightarrow \quad \partial_x \delta F \quad &= \quad x^{h_0-1}\delta g - \frac{h - h_0}{x} x^{h_0} \delta f \,, \end{aligned}$$

for a suitable constant C. We read off that the function δF is in fact continuously differentiable at $x = 0$. Then, by Taylor's theorem,

$$\begin{aligned} \delta F &= (\partial_x \delta F)_0 + O(x) = C + O(x) \\ \Longrightarrow \quad \delta f &= x^{-h_0} C + O(x^{1-h_0}) \,. \end{aligned}$$

Recall that in the polyhomogeneous expansion of $\hat{f}$ the coefficient $\hat{f}^{(0)}_\lambda$, with $\lambda \equiv -h_0$, can be chosen freely. We choose it such that $\delta f_\lambda \equiv (x^{-\lambda}\delta f)_0$ vanishes, leading to

$$\begin{aligned} \delta f &= x^{-h_0} \int_0^x y^{h_0 - 1}(\delta g - O(y)\delta f)\,\mathrm{d}y && \text{(A.38)} \\ \Longrightarrow \quad |\delta f| &\leq c\, x^{-h_0+1} \,. && \text{(A.39)} \end{aligned}$$

Inserting (A.39) into (A.38) improves the estimate in each step by a factor of x. Repeating this as many times as necessary (there is no disturbing term anymore which is proportional to x^{-h_0} and prohibits the improvement of the estimate), we eventually end up with the desired result,

$$|\delta f| \leq c^{(N)} x^N \quad \text{for all } N, \text{ i.e.} \quad \delta f \sim 0 \,. \tag{A.40}$$

When dealing with higher-dim. systems we can proceed in a similar manner (note that we have derived the formal solution only for $\lambda_i \in \mathbb{Z}$). We give a sketch. Denote by $-\lambda_1, \dots, -\lambda_n$ the eigenvalues of h_0 and assume w.l.o.g. $\lambda_1 \leq \dots \leq \lambda_n$. Equation (A.37) provides the estimate (for $\tilde{g} = \delta g - (h - h_0)\delta f$)

$$|\delta f_i| \quad \leq \quad c^{(1)} x^{\lambda_i} + c^{(2)} x^{\lambda_i} \int_x^{x_0} y^{-\lambda_i - 1}(|\delta g_i| + c^{(3)} y \|\delta f\|)\mathrm{d}y \,. \tag{A.41}$$

Note that the integrand depends on $\|\delta f\|$ and not just on $|\delta f_i|$, because the higher-order terms in the expansion of h do not need to be diagonal.

Again we assume without restriction $\mu/2 + \lambda_i \neq 1$. We conclude from (A.34) and (A.41) that if $\mu/2 - 1 < -\lambda_1$ then $|\delta f_1| \leq c\, x^{\lambda_1}$. Otherwise $|\delta f_i| \leq c\, x^{-\mu/2+1}$ for all i, and thus $\|\delta f\| \leq c\, x^{-\mu/2+1}$. This can be repeated until $\mu/2 - 1 - k_1 < -\lambda_1$, which means $|\delta f_1| \leq c\, x^{\lambda_1}$. By adjusting the initial conditions via the integration constant $(\hat{f}^{(0)}_1)_{\lambda_1}$ appearing in the expansion of $\hat{f}$ one achieves that the λ_1-th order term in the polyhomogeneous expansion of δf_1 vanishes. Then one proceeds in the same way until $\mu/2 - 1 - k_2 < -\lambda_2$. To continue this process the integration constant $(\hat{f}^{(0)}_2)_{\lambda_2}$ has to be chosen suitably. And so on. Eventually one obtains $\|\delta f\| \leq c\, x^{\lambda_n + 1}$ and a formula analog to (A.38) for higher dimensions. One then straightforwardly establishes the desired estimate,

$$\|\delta f\| \leq c^{(N)} x^N \quad \text{for all } N, \text{ i.e.} \quad \delta f \sim 0 \,. \tag{A.42}$$

2nd case: It remains to deal with the case where h_0 cannot be diagonalized. For reasons of simplicity we restrict attention again to the two-dimensional case. The matrix h_0 can be brought into Jordan normal form:

$$h_0 = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \,. \tag{A.43}$$

Note that we have derived the formal solution only for $\lambda \in \mathbb{Z}$, which we assume here, as well. First, we compute x^{h_0},

$$x^{h_0} \equiv e^{h_0 \log x} \equiv \sum_{m=0}^{\infty} \frac{h_0^m \log^m x}{m!} = x^{-\lambda} \begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix} .$$

From equation (A.37) we find

$$\delta f(x) = x^{\lambda} x_0^{-\lambda} \begin{pmatrix} 1 & \log(x_0/x) \\ 0 & 1 \end{pmatrix} \delta f(x_0) - x^{\lambda} \int_x^{x_0} y^{-\lambda-1} \begin{pmatrix} 1 & \log(y/x) \\ 0 & 1 \end{pmatrix} \tilde{g}(y)\, \mathrm{d}y . \tag{A.44}$$

The second component of δf, δf_2, fulfills the integral equation

$$\delta f_2(x) = x^{\lambda} x_0^{-\lambda} \delta f_2(x_0) - x^{\lambda} \int_x^{x_0} y^{-\lambda-1} \tilde{g}_2(y)\, \mathrm{d}y . \tag{A.45}$$

Recall that $\tilde{g}_i \equiv \delta g_i - [(h-h_0)\delta f]_i$. As before, we may assume w.l.o.g. $\mu/2-1 \neq -\lambda$, and deduce

$$\begin{aligned} |\delta f_2| &\leq c^{(1)} x^{\lambda} + c^{(2)} x^{\lambda} \int_x^{x_0} y^{-\lambda-1} (|\delta g_2| + c^{(3)} y \|\delta f\|) \mathrm{d}y \\ &\leq c^{(1)} x^{\lambda} + c^{(2)} x^{-\mu/2+1} \end{aligned} \tag{A.46}$$

by using (A.34). Next, we consider the first component of (A.44), which, once we have an estimate for δf_2, supplies one for δf_1:

$$\begin{aligned} \delta f_1(x) &= x^{\lambda} x_0^{-\lambda} [\delta f_1(x_0) + \log(x_0/x) \delta f_2(x_0)] \\ &\quad - x^{\lambda} \int_x^{x_0} y^{-\lambda-1} [\tilde{g}_1(y) + \tilde{g}_2(y) \log(y/x)]\, \mathrm{d}y \\ &= -\delta f_2(x) \log x + x^{\lambda} x_0^{-\lambda} [\delta f_1(x_0) + \log x_0 \delta f_2(x_0)] \\ &\quad - x^{\lambda} \int_x^{x_0} y^{-\lambda-1} [\tilde{g}_1(y) + \tilde{g}_2(y) \log y]\, \mathrm{d}y . \end{aligned} \tag{A.47}$$

That yields (w.l.o.g. we assume $x_0 < 1$)

$$|\delta f_1(x)| \leq |\delta f_2(x) \log x| + c^{(1)} x^{\lambda} - c^{(2)} x^{\lambda} \int_x^{x_0} \|\delta f\|\, y^{-\lambda} \log y\, \mathrm{d}y . \tag{A.48}$$

The estimate (A.34) supplemented by the one for $|\delta f_2|$, (A.46), implies

$$\begin{aligned} |\delta f_1(x)| &\leq |\delta f_2 \log x| + c^{(1)} x^{\lambda} + \left| x^{\lambda} \Big[y^{-\lambda-\mu/2+1} (c^{(2)} + c^{(3)} \log y) \Big]_x^{x_0} \right| \\ &\leq c^{(1)} x^{\lambda} |\log x| + c^{(2)} x^{-\mu/2+1} |\log x| . \end{aligned}$$

If $\mu/2 - 1 < -\lambda$ we are immediately led to $|\delta f_2(x)| \leq c\, x^{\lambda}$ and $|\delta f_1(x)| \leq c\, x^{\lambda} |\log x|$, whence $\|\delta f\| \leq c x^{\lambda} |\log x|$.

If the reverse inequality holds, we find $|\delta f_2(x)| \leq c\, x^{-\mu/2+1}$ and $|\delta f_1(x)| \leq c\, x^{-\mu/2+1} |\log x|$. Combined, that gives $\|\delta f(x)\| \leq c\, x^{-\mu/2+1} |\log x|$. Repeating this procedure k-times as long as $\mu/2 - k > 1 - \lambda$ the estimates for δf_i improve in each round by a factor of x, accompanied possibly by the appearance of higher order powers of $\log x$,

$$|\delta f_2(x)| \leq c\, x^{-\mu/2+k} |\log x|^{k-1} , \quad |\delta f_1(x)| \leq c\, x^{-\mu/2+k} |\log x|^{k} ,$$

with k a positive integer. Anyway, the log terms do not cause any troubles here, because as soon as $k > \mu/2 + \lambda - 1$ becomes true, the inequalities

$$|\delta f_2| \le cx^\lambda \ , \quad |\delta f_1| \le cx^\lambda |\log x| \quad \Longrightarrow \quad \|\delta f(x)\| \le c\, x^\lambda |\log x| \tag{A.49}$$

replace the estimates with μ. This is owing to the fact that a term of the form x^q, $q > 0$, kills any log term, $x^q \log^m x \to 0$ if $x \to 0$.

We proceed in a similar way as above to show that one can improve the estimates by adjusting the integration constants contained in $\hat{f}$. Recall that

$$\delta F_2(x) := x^{-\lambda} \delta f_2(x) = x_0^{-\lambda} \delta f_2(x_0) - \int_x^{x_0} y^{-\lambda-1} \tilde{g}_2(y)\, \mathrm{d}y \ . \tag{A.50}$$

From the preceding considerations we conclude that the integrand is $O(\log y)$. Thus we can perform the limit $x \to 0$. The function δF_2 is continuous on $[0, x_0)$ and we rewrite it as

$$\delta F_2(x) \ = \ C + \int_0^x y^{-\lambda-1} \tilde{g}_2(y)\, \mathrm{d}y \ = \ C + O(x|\log x|) \ . \tag{A.51}$$

By choosing the value of $(\hat{f}_2^{(0)})_\lambda$ appropriately (this coefficient was free to choose in our analysis above) one achieves that C vanishes,

$$|\delta f_2| \le c\, x^{\lambda+1} |\log x| \ . \tag{A.52}$$

Combined with $|\delta f_1| \le cx^\lambda |\log x|$, this inequality can be used to improve the estimate for $|\delta f_1|$: Starting from (A.48) one establishes

$$|\delta f_1| \le cx^\lambda \quad \overset{(A.52)}{\Longrightarrow} \quad \|\delta f\| \le cx^\lambda \ . \tag{A.53}$$

It follows from (A.47) that there exists a constant C with

$$\lim_{x \to 0} x^{-\lambda} \delta f_1(x) = C \ , \tag{A.54}$$

so one can perform the limit $x_0 \to 0$,

$$\begin{aligned} x^{-\lambda} \delta f_1(x) \ &= \ C - x^{-\lambda} \delta f_2(x) \log x + \int_0^x y^{-\lambda-1} \left[\tilde{g}_1(y) + \tilde{g}_2(y) \log y\right] \mathrm{d}y \\ &= \ C + O(x|\log x|) \ . \end{aligned} \tag{A.55}$$

We choose the (yet unspecified) value $(\hat{f}_1^{(0)})_\lambda$ in the expansion of $\hat{f}_1$ such that C vanishes. Then

$$|\delta f_1| \ \le \ cx^{\lambda+1} |\log x| \ , \tag{A.56}$$

and (A.51) yields

$$|\delta f_2| \ \le \ cx^{\lambda+1} \ , \tag{A.57}$$

i.e. we have improved (A.49) by a factor of x.

If we continue this process, an analysis of (cf. (A.45) and (A.47)),

$$\begin{aligned} \delta f_2(x) \ &= \ x^\lambda \int_0^x y^{-\lambda-1} \tilde{g}_2(y)\, \mathrm{d}y \ , \\ \delta f_1(x) \ &= \ -\delta f_2(x) \log x + x^\lambda \int_0^x y^{-\lambda-1} \left[\tilde{g}_1(y) + \tilde{g}_2(y) \log y\right] \mathrm{d}y \end{aligned}$$

reveals that the estimates improve in each step by a factor of x, and, since the log term in δf_1 does not matter in the end, we arrive at

$$|\delta f_i| \leq c^{(N)} x^N \quad \text{for all } N \quad \Longleftrightarrow \quad \|\delta f\| \leq c^{(N)} x^N \quad \text{for all } N \; . \qquad \text{(A.58)}$$

Summarizing, we have proved:

Theorem A.1 *Consider the linear ODE*

$$x\partial_x f + hf = g \quad on \quad (0, x_0) \; ,$$

where $x^{-\ell} g = O(1)$, $\ell \in \mathbb{Z}$, *and* $h = O(1)$ *are assumed to be smooth maps on* $[0, x_0)$, *and where* f *and* g *have values in* $\mathbb{R}$ *or* $\mathbb{R}^2$, *with* h *having values in the space of corresponding linear maps. Let* $\hat{f}$ *be a solution of the ODE specified by boundary conditions, that is by a choice of the expansion coefficients* $\hat{f}_\lambda$ *in the scalar case, and* $(\hat{f}_i)_{\lambda_i}$ *or* $(\hat{f}_i)_\lambda$, *respectively, in the 2-dimensional case. Denote by* g_{formal} *and* h_{formal} *the Taylor expansions of* g *and* h, *respectively, at* $x = 0$. *Then there exists a formal solution* f_{formal} *of*

$$x\partial_x f + h_{formal} f = g_{formal} \; ,$$

such that f_{formal} *is the polyhomogeneous expansion of* $\hat{f}$ *at* $x = 0$,

$$\hat{f} \sim f_{formal} \; .$$

We have further indicated that the theorem remains true in arbitrary dimensions if h_0 is a diagonal matrix with integer entries.

B Relation between $\kappa = 0$- and $\kappa = \frac{r}{2}|\sigma|^2$-gauge

The metric gauge turned out to be very convenient to construct data for the CFE which are smooth at $\mathscr{I}^+$: Global existence and positivity of φ and ν^0 are implicitly contained in the gauge condition, while the no-logs-condition reduces to a simple algebraic condition on the expansion coefficients of γ.

Using an affine parameterization, i.e. a $\kappa = 0$-gauge, we have to *assume* that the initial data $[\gamma]$ are chosen in such a way that the function φ is positive on $\mathscr{N}$ and that φ_{-1} is positive on S^2. These assumptions *do* impose geometric restrictions on γ in that they exclude data producing conjugate points or even space-time singularities on the initial surface. On the other hand, the gauge choice $\kappa = \frac{r}{2}|\sigma|^2$, on a light-cone say, *implies* that the Raychaudhuri equation admits a global solution with all the required properties without any additional assumptions. To resolve this "paradox" it is useful to understand in which way a $\kappa = \frac{r}{2}|\sigma|^2$-gauge is related to other choices of the r-coordinate parameterizing the null rays generating the cone, such as affine parameterizations. A similar problem arises for ν^0, and can be resolved in the same way.

If the gauge function κ depends on the initial data, the physical/geometrical interpretation of the parameter r (its deviation from an affine one) w.r.t. which the γ is given, depends on γ itself. Due to this "implicit definition" of r, the choice $\kappa = \frac{r}{2}|\sigma|^2$ conceals geometric restrictions, which we want to discuss now.

We consider initial data $\mathring{\gamma}$ given in a $\mathring{\kappa}$-gauge on an initial surface $\mathscr{N}$, which we assume for definiteness to be a light-cone, and analyze under which conditions a transformation to a $\kappa = \frac{r}{2}|\sigma|^2$-gauge is possible, by which we mean that both solutions differ by a coordinate change only. The relevant coordinate transformations $(\mathring{r}, \mathring{x}^A) \mapsto (r, x^A)$ are angle-dependent transformations of the r-coordinate

$$r = r(\mathring{r}, \mathring{x}^A)\,, \quad x^A = \mathring{x}^A\,.$$

It follows from the transformation behavior of connection coefficients that

$$\kappa = \overline{\Gamma}^r_{rr} = \underbrace{\overline{\Gamma}^{\mathring{r}}_{\mathring{r}\mathring{r}}}_{=\mathring{\kappa}} \frac{\partial \mathring{r}}{\partial r} - \left(\frac{\partial \mathring{r}}{\partial r}\right)^2 \frac{\partial^2 r}{\partial \mathring{r}^2}\,. \tag{B.1}$$

The function $|\sigma|^2$ which contains partial derivatives of r (cf. (3.2)) transforms as

$$|\sigma|^2(r) = \left(\frac{\partial \mathring{r}}{\partial r}\right)^2 |\mathring{\sigma}|^2(\mathring{r}(r))\,. \tag{B.2}$$

With $\kappa = \frac{r}{2}|\sigma|^2$ (B.1) becomes

$$\frac{\partial^2 r}{\partial \mathring{r}^2} - \mathring{\kappa}\frac{\partial r}{\partial \mathring{r}} + \frac{1}{2} r|\mathring{\sigma}|^2 \;=\; 0\,.$$

We observe that $r(\mathring{r}, \mathring{x}^A)$ and $\mathring{\varphi}(\mathring{r}, \mathring{x}^A)$ satisfy the same ODE. Imposing the boundary conditions $r|_{\mathring{r}=0} = 0$ and $\partial_{\mathring{r}} r|_{\mathring{r}=0} = 1$, we conclude that

$$r(\mathring{r}, \mathring{x}^A) = \mathring{\varphi}(\mathring{r}, \mathring{x}^A)\,. \tag{B.3}$$

Since $g_{AB}(r) = \mathring{g}_{AB}(r(\mathring{r}))$ the function $\varphi \equiv (\frac{\det \tilde{g}_{\Sigma_r}}{\det s})^{1/4}$ transforms as a scalar, and

$$r = \mathring{\varphi}(\mathring{r}(r, x^A), x^A) = \varphi(r, x^A)\,, \tag{B.4}$$

as expected and required by the metric gauge.

To transform from a $\mathring{\kappa}$-gauge to a $\kappa = \frac{r}{2}|\sigma|^2$-gauge one simply identifies $\mathring{\varphi}$ as the new r-coordinate. However, this is only possible when $(\mathring{r}, \mathring{x}^A) \mapsto (r = \mathring{\varphi}, x^A = \mathring{x}^A)$ defines a diffeomorphism. Globally this happens if and only if $\mathring{\varphi}$ is a strictly increasing function, which is equivalent to the existence of a global solution to the Raychaudhuri equation. Another requirement on r should be that

$$\lim_{\mathring{r}\to\infty} r = \infty \quad \Longleftrightarrow \quad \lim_{\mathring{r}\to\infty} \mathring{\varphi} = \infty \quad \Longleftrightarrow \quad \mathring{\varphi}_{-1} > 0\,.$$

This derivation clarifies in which way the assumptions on $\mathring{\varphi}$ in say a $\mathring{\kappa} = 0$-gauge enter: Prescribing smooth data in a $\kappa = \frac{r}{2}|\sigma|^2$-gauge one implicitly excludes data violating these assumptions and thereby the existence of conjugate points up-to-and-including conformal infinity. In this work, though, we are only interested in those cases where $\mathring{\varphi}$ *is* strictly increasing with $\mathring{\varphi}_{-1} > 0$, in which case a transition from a $\mathring{\kappa} = 0$- to a $\kappa = \frac{r}{2}|\sigma|^2$-gauge is always possible.

Let us take a look at the reversed direction. It follows from (B.1) and the requirement $\partial_r \mathring{r}|_{r=0} = 1$ that

$$\frac{r}{2}|\sigma|^2 = -\left(\frac{\partial \mathring{r}}{\partial r}\right)^2 \frac{\partial^2 r}{\partial \mathring{r}^2} = \left(\frac{\partial \mathring{r}}{\partial r}\right)^{-1} \frac{\partial^2 \mathring{r}}{\partial r^2} \quad \Longleftrightarrow \quad \frac{\partial \mathring{r}}{\partial r} = e^{\int_0^r \frac{\hat{r}}{2}|\sigma|^2 \mathrm{d}\hat{r}} > 0\,,$$

which defines a diffeomorphism. Consequently, a transformation from a $\kappa = \frac{r}{2}|\sigma|^2$-gauge to a $\mathring{\kappa} = 0$-gauge is possible without any restrictions.

We conclude that the metric gauge is a reasonable and convenient gauge condition whenever the light-cone is supposed to be globally smooth.

References

[1] Y. Choquet-Bruhat, *Un théorème d'instabilité pour certaines équations hyperboliques non linéares*, C.R. Acad. Sci. Paris **276** (1973), 281–284.

[2] Y. Choquet-Bruhat, P.T. Chruściel, and J.M. Martín-García, *The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions*, Ann. Henri Poincaré **12** (2011), 419–482.

[3] P.T. Chruściel, *The existence theorem for the general relativistic Cauchy problem on the light-cone*, (2012), arXiv:1209.1971 [gr-qc].

[4] P.T. Chruściel, and J. Jezierski, *On free general relativistic initial data on the light cone*, J. Geom. Phys. **62** (2012) 578-593.

[5] P.T. Chruściel, and M.A.H. MacCallum, D. Singleton: *Gravitational Waves in General Relativity. XIV: Bondi Expansions and the "Polyhomogeneity" of Scri*, Phil. Trans. Royal Soc. of London A **350**, 113–141 (1995).

[6] P.T. Chruściel and T.-T. Paetz, *The many ways of the characteristic Cauchy problem*, Class. Quantum Grav. **29** (2012), 145006.

[7] P.T. Chruściel, and T.-T. Paetz, *Characteristic initial data and smoothness of Scri. I. Framework and results*, preprint (2014).

[8] H. Friedrich, *Conformal Einstein Evolution*, in: *The Conformal Structure of Space-Time – Geometry, Analysis, Numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg, Springer, 2002, pp. 1–50.

[9] J. Jezierski, *'Peeling property' for linearized gravity in null coordinates*, Class. Quantum Grav. **19** (2002), 2463-2490.

[10] A.D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. Roy. Soc. London A **427** (1990), 221-239.

[11] L.A. Tambourino, and J.H. Winicour, *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150**, 1039–1053 (1966).

[12] R.J. Torrence, and W.E. Couch, *Generating fields from data on $\mathscr{H}^-\cup\mathscr{H}^+$ and $\mathscr{H}^-\cup\mathscr{I}^-$*, Gen. Rel. Grav. **16** (1984), 847–866.

Fast Track Communications

IOP Publishing

Classical and Quantum Gravity

Class. Quantum Grav. **31** (2014) 102001 (9pp)

doi:10.1088/0264-9381/31/10/102001

Fast Track Communications

The mass of light-cones

Piotr T Chruściel and Tim-Torben Paetz

Gravitational Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: piotr.chrusciel@univie.ac.at and Tim-Torben.Paetz@univie.ac.at

Received 30 January 2014
Accepted for publication 8 April 2014
Published 29 April 2014

Abstract

We give an elementary proof of positivity of total gravitational energy in space-times containing complete smooth light-cones.

Keywords: Bondi mass, complete light-cones, positivity of total gravitational energy

PACS numbers: 04.20.Cv, 04.20.Ex, 04.20.Ha

One of the deepest questions arising in general relativity is that of positivity of total energy. Years of attempts by many authors have led to an affirmative answer in the milestone papers of Schoen and Yau [13, 14] and Witten [17]; see also [6, 10, 15]. These proofs use sophisticated PDE techniques, with positivity resulting from the analysis of solutions of seemingly unrelated partial differential equations. The aim of this communication is to show that an elementary direct proof of positivity can be given for a large class of space-times, namely those containing globally smooth light-cones. As a bonus, our proof gives an explicit positive-definite expression for the mass, equation (31), in terms of physically relevant fields, such as the shear of the light-cone.

Thus, consider a globally smooth, null-geodesically complete light-cone in an asymptotically Minkowskian space-time. The formula (47) below for the mass in this context has been derived by Bondi *et al* [2, 12], compare [16]. We show how to rewrite this formula in terms of geometric data on the light-cone, equation (37) below. The constraint equations induced by Einstein's field equations on the light-cone are then used to obtain our manifestly positive mass formula (31) by elementary manipulations.

The initial data on the light-cone comprise a pair $(\mathscr{N}, \check{g})$, where $\mathscr{N} = \mathbb{R}^3 \setminus \{0\}$ and $\check{g}$ is a smooth field of symmetric two-covariant tensors on $\mathscr{N}$ of signature $(0, +, +)$ such that $\check{g}(\partial_r, \cdot) = 0$. [1] The vertex O of the light-cone $C_O := \mathscr{N} \cup \{O\}$ is located at the origin of $\mathbb{R}^3$, and the half-rays issued from the origin correspond to the generators of C_O. For simplicity

[1] To avoid an ambiguity in notation we write $\check{g}$ for what was denoted by $\tilde{g}$ in [4], as $\tilde{g}$ is usually used for the conformally rescaled metric when discussing $\mathscr{I}^+$.

we assume throughout that the initial data lead to a smooth space-time metric, cf [5]. The requirements of regularity at the origin, asymptotic flatness, and global smoothness, lead to the following restrictions on $\check{g}$.

Letting (r, x^A), $A \in \{2, 3\}$, denote spherical coordinates on $\mathbb{R}^3$, and writing $s_{AB}\, dx^A\, dx^B$ for the unit round metric on S^2, regularity conditions at the vertex imply that the coordinate r can be chosen so that for small r we have

$$\check{g} \equiv \overline{g}_{AB}\, dx^A\, dx^B = r^2(s_{AB} + h_{AB})\, dx^A\, dx^B, \tag{1}$$

$$h_{AB} = O(r^2),\ \partial_C h_{AB} = O(r^2),\ \partial_r h_{AB} = O(r); \tag{2}$$

see [4, section 4.5] for a detailed discussion, including properties of higher derivatives; the latter are assumed implicitly whenever needed below. Here, and elsewhere, an overline denotes a space-time object restricted to the light-cone.

Existence of a Penrose-type conformal completion implies that the coordinate r can be chosen so that for large r we have

$$\overline{g}_{AB} = r^2(\overline{g}_{AB})_{-2} + r(\overline{g}_{AB})_{-1} + \psi_{AB}, \tag{3}$$

$$\psi_{AB} = O(1),\ \partial_C\psi_{AB} = O(1),\ \partial_r\psi_{AB} = O(r^{-2}), \tag{4}$$

$$\partial_r^2\psi_{AB} = O(r^{-3}),\ \partial_C\partial_D\psi_{AB} = O(1), \tag{5}$$

$$\partial_r\partial_C\psi_{AB} = O(r^{-2}),\ \partial_r\partial_C\partial_D\psi_{AB} = O(r^{-2}), \tag{6}$$

$$\partial_r^2\partial_C\psi_{AB} = O(r^{-3}),\ \partial_r^2\partial_C\partial_D\psi_{AB} = O(r^{-3}), \tag{7}$$

for some smooth tensors $(\overline{g}_{AB})_{-i} = (\overline{g}_{AB})_{-i}(x^C)$, $i = 1, 2$, on S^2.

Equations (3)–(7) are necessary for existence of a smooth $\mathscr{I}^+$, but certainly not sufficient: our conditions admit initial data sets which might lead to a polyhomogeneous but *not* smooth $\mathscr{I}^+$; see [1, 7, 8, 11].

We denote by τ the *divergence*, sometimes called *expansion*, of the light-cone:

$$\tau := \chi_A{}^A,\ \text{where}\ \chi_A{}^B := \tfrac{1}{2}\overline{g}^{BC}\partial_r\overline{g}_{AC}. \tag{8}$$

Conditions (1)–(7) imply

$$\tau = 2r^{-1} + \tau_2 r^{-2} + O(r^{-3})\ \text{for large}\ r, \tag{9}$$

$$\tau = 2r^{-1} + O(r)\ \text{for small}\ r. \tag{10}$$

In particular τ is positive in both regions. Now, standard arguments show that if τ becomes negative somewhere, then the light-cone will either fail to be globally smooth, or the space-time will not be null-geodesically complete. So our requirement of global smoothness of the light-cone together with completeness of generators imposes the condition[2]

$$\tau > 0. \tag{11}$$

We further require

$$\det(\overline{g}_{AB})_{-2} > 0, \tag{12}$$

which excludes conjugate points at the intersection of the light-cone with $\mathscr{I}^+$. Both conditions will be assumed to hold from now on.

[2] The Raychaudhuri equation (13) below with $\kappa = 0$ implies that τ is monotonous non-increasing, which yields (11) directly in any case after noting that the sign of τ is invariant under orientation-preserving changes of parametrization of the generators.

The inequality (11) implies that the connection coefficient κ, defined through the equation

$$\nabla_{\partial_r}\partial_r = \kappa\partial_r,$$

and measuring thus how the parameter r differs from an affine-one, can be algebraically calculated from the Raychaudhuri equation:

$$\begin{aligned}&\partial_r\tau - \kappa\tau + \frac{\tau^2}{2} + |\sigma|^2 + 8\pi T_{rr}|_{\mathscr{N}} = 0\\ \Longleftrightarrow \quad &\kappa = \frac{1}{\tau}\left(\partial_r\tau + \frac{\tau^2}{2} + |\sigma|^2 + 8\pi T_{rr}|_{\mathscr{N}}\right).\end{aligned} \tag{13}$$

Here σ is the shear of the light-cone:

$$\sigma_A{}^B = \chi_A{}^B - \tfrac{1}{2}\tau\delta_A{}^B, \tag{14}$$

which satisfies

$$\sigma_A{}^B = O(r) \text{ for small } r, \quad \sigma_A{}^B = O(r^{-2}) \text{ for large } r. \tag{15}$$

Assuming that

$$T_{rr}|_{\mathscr{N}} = O(r^{-4}) \text{ for large } r, \tag{16}$$

we find from (13) and our previous hypotheses

$$\kappa = O(r) \text{ for small r}, \quad \kappa = O(r^{-3}) \text{ for large } r. \tag{17}$$

For the proof of positivity of the Trautman–Bondi mass it will be convenient to change r to a new coordinate so that $(\overline{g}_{AB})_{-2}$ in (3) is the unit round sphere metric and that the resulting κ vanishes (i.e. the new coordinate r will be an affine parameter along the generators of the light-cone). Denoting momentarily the new coordinate by $r_{\rm as}$, and using the fact that every metric on S^2 is conformal to the unit round metric s_{AB}, the result is achieved by setting

$$r_{\rm as}(r, x^A) = \Theta(x^A)\int_0^r e^{H(\hat{r},x^A)}\,d\hat{r}, \tag{18}$$

$$H(r, x^A) = -\int_r^\infty \kappa(\tilde{r}, x^A)\,d\tilde{r}, \tag{19}$$

$$\Theta(x^A) = \left(\frac{\det(\overline{g}_{AB})_{-2}}{\det s_{AB}}\right)^{1/4}. \tag{20}$$

The functions $r \mapsto r_{\rm as}(r, x^A)$ are strictly increasing with $r_{\rm as}(0, x^A) = 0$. Equation (17) shows that there exists a constant C such that for all r we have

$$e^{-C} \leqslant e^H \leqslant e^C, \tag{21}$$

which implies that $\lim_{r\to\infty} r_{\rm as}(r, x^A) = +\infty$. We conclude that for each x^A the function $r \mapsto r_{\rm as}(r, x^A)$ defines a smooth bijection from $\mathbb{R}^+$ to itself. Consequently, smooth inverse functions $r_{\rm as} \mapsto r(r_{\rm as}, x^A)$ exist.

We have normalized the affine parameter $r_{\rm as}$ so that

$$r_{\rm as}(r, x^A) = \Theta(x^A)r + (r_{\rm as})_\infty(x^A) + O(r^{-1}), \tag{22}$$

for large r, which implies

$$r_{\rm as}(r, x^A) = (r_{\rm as})_0(x^A)r + O(r^3) \text{ for small } r, \tag{23}$$

where

$$(r_{\rm as})_\infty(x^A) = \Theta(x^A)\int_0^\infty (e^{H(r,x^A)} - 1)\,dr, \tag{24}$$

$$(r_{\rm as})_0(x^A) = \Theta(x^A)\, e^{-\int_0^\infty \kappa(\tilde r, x^A)\mathrm{d}\tilde r}. \tag{25}$$

After some obvious redefinitions, for $r_{\rm as}$ large (3) becomes

$$\overline{g}_{AB} = r_{\rm as}^2 s_{AB} + r_{\rm as}(\overline{g}_{AB})_{-1} + \psi_{AB}. \tag{26}$$

The boundary conditions (4)–(7) remain unchanged when r is replaced by $r_{\rm as}$ there. On the other hand, (1)–(2) will not be true anymore. However, we note for further use that the coordinate transformation (18) preserves the behaviour of τ and σ near the vertex. Indeed, inserting $r = r(r_{\rm as})$ into (3) and using the definitions (8) and (14) one finds that (9)–(10) and (15) continue to hold with r replaced by $r_{\rm as}$.

A key role in what follows will be played by the equation [4, equations (10.33) and (10.36)], [3]

$$(\partial_r + \tau + \kappa)\zeta + \check{R} - \tfrac{1}{2}|\xi|^2 + \overline{g}^{AB}\check{\nabla}_A\xi_B = S, \tag{27}$$

where $|\xi|^2 := \overline{g}^{AB}\xi_A\xi_B$. In coordinates adapted to the light-cone as in [4] the space-time formula for the auxiliary function ζ is

$$\zeta = (2g^{AB}\Gamma^r_{AB} + \tau g^{rr})|_{\mathscr{N}}, \tag{28}$$

and ζ is in fact the divergence of the family of suitably normalized null generators normal to the spheres of constant r and transverse to $\mathscr{N}$. Here $\check{\nabla}$ denotes the Levi-Civita connection of $\check{g}$ viewed as a metric on S^2 (more precisely, an r-dependent family of metrics). The symbol $\check{R}$ denotes the curvature scalar of $\check{g}$. The connection coefficients $\xi_A \equiv -2\Gamma^r_{rA}|_{\mathscr{N}}$ are determined by [4, equation (9.2)]:

$$\tfrac{1}{2}(\partial_r + \tau)\xi_A - \check{\nabla}_B\sigma_A{}^B + \tfrac{1}{2}\partial_A\tau + \partial_A\kappa = -8\pi T_{rA}|_{\mathscr{N}}. \tag{29}$$

Finally,

$$\begin{aligned} S &:= 8\pi(g^{AB}T_{AB} - g^{\mu\nu}T_{\mu\nu})|_{\mathscr{N}} \\ &= -8\pi\left(g^{rr}T_{rr} + 2g^{rA}T_{rA} + 2g^{ur}T_{ur}\right)|_{\mathscr{N}}, \end{aligned} \tag{30}$$

with $T_{ur} = T(\partial_u, \partial_r)$, where ∂_u is transverse to $\mathscr{N}$. The first equality makes it clear that S does not depend upon the choice of coordinates away from $\mathscr{N}$. In a coordinate system where $\overline{g}^{rr} = \overline{g}^{rA} = 0$ we have $S = -16\pi g^{ur}T_{ur}|_{\mathscr{N}}$ which, with our signature $(-,+,+,+)$, is non-negative for matter fields satisfying the dominant energy condition when both ∂_r and ∂_u are causal future pointing, as will be assumed from now on.

Letting $\mathrm{d}\mu_{\check g} = \sqrt{\det \overline{g}_{AB}}\,\mathrm{d}x^2\,\mathrm{d}x^3$, we derive below the following surprising formula for the Trautman–Bondi [2, 12, 16] mass $m_{\rm TB}$ of complete light-cones:

$$m_{\rm TB} = \frac{1}{16\pi}\int_0^\infty \int_{S^2}\left(\frac{1}{2}|\xi|^2 + S + (|\sigma|^2 + 8\pi T_{rr}|_{\mathscr{N}})e^{\int_r^\infty \frac{\tilde r\tau - 2}{2\tilde r}\mathrm{d}\tilde r}\right)\mathrm{d}\mu_{\check g}\,\mathrm{d}r \tag{31}$$

The coordinate r here is an affine parameter along the generators normalised so that $r = 0$ at the vertex, with (26) holding for large r. Positivity of $m_{\rm TB}$ obviously follows in vacuum. For matter fields satisfying the dominant energy condition we have $S \geqslant 0$, $T_{rr} \geqslant 0$ and positivity again follows.

Note that since $m_{\rm TB}$ decreases when sections of $\mathscr{I}^+$ are moved to the future, (31) provides an *a priori* bound on the integrals appearing there both on $\mathscr{N}$ and for all later light-cones, which is likely to be useful when analysing the global behaviour of solutions of the Einstein equations.

[3] On the right-hand side of the second equality in [4, equation (10.36)] a term $\tau\overline{g}^{11}/2$ is missing.

To prove (31), we assume (16). We change coordinates via (18), and use now the symbol r for the coordinate r_{as}. Thus $\kappa = 0$, we have (3) with $(g_{AB})_{-2} = s_{AB}$, and further (4)–(7), (9)–(10) and (15) hold. For r large one immediately obtains

$$\sqrt{\det \breve{g}_{AB}} = r^2\sqrt{\det s_{AB}}(1 - \tau_2 r^{-1} + O(r^{-2})). \tag{32}$$

Let us further assume that, again for large r,

$$T_{rA}|_{\mathcal{N}} = O(r^{-3}), \quad S = O(r^{-4}). \tag{33}$$

It then follows from (29) and our remaining hypotheses that ξ_A satisfies

$$\xi_A = (\xi_A)_1 r^{-1} + o(r^{-1}) \text{ and } \partial_B \xi_A = O(r^{-1}), \tag{34}$$

for some smooth covector field $(\xi_A)_1$ on S^2. An analysis of (27) gives

$$\zeta(r, x^A) = -2r^{-1} + \zeta_2(x^A)r^{-2} + o(r^{-2}), \tag{35}$$

with a smooth function ζ_2. Regularity at the vertex requires that for small r

$$\xi_A = O(r^2), \quad \zeta = O(r^{-1}), \tag{36}$$

where r is the original coordinate which makes the initial data manifestly regular at the vertex. Using the transformation formulae for connection coefficients one checks that this behaviour is preserved under (23).

We will show shortly that *if* the light-cone data arise from a space-time with a smooth conformal completion at null infinity $\mathscr{I}^+$, and *if* the light-cone intersects $\mathscr{I}^+$ in a smooth cross-section S, then the Trautman–Bondi mass of S equals

$$m_{\mathrm{TB}} = \frac{1}{16\pi}\int_{S^2}(\zeta_2 + \tau_2)\,\mathrm{d}\mu_s, \tag{37}$$

where $\mathrm{d}\mu_s = \sqrt{\det s_{AB}}\,\mathrm{d}x^2\mathrm{d}x^3$. Note that this justifies the use of (37) as the *definition* of mass of an initial data set on a light-cone with complete generators, regardless of any space-time assumptions.

It follows from (9), (32) and (35) that for large r we have

$$\begin{aligned}\int_{S^2}\zeta\,\mathrm{d}\mu_{\breve{g}} &= \int_{S^2}(-2r + \zeta_2 + o(1))(1 - \tau_2 r^{-1} + O(r^{-2}))\,\mathrm{d}\mu_s \\ &= -8\pi r + \int_{S^2}(\zeta_2 + 2\tau_2)\,\mathrm{d}\mu_s + o(1).\end{aligned} \tag{38}$$

This allows us to rewrite (37) as

$$16\pi m_{\mathrm{TB}} = \lim_{r\to\infty}\left(\int_{S^2}\zeta\,\mathrm{d}\mu_{\breve{g}} + 8\pi r\right) - \int_{S^2}\tau_2\,\mathrm{d}\mu_s. \tag{39}$$

To establish (31), first note that from (27) with $\kappa = 0$ and the Gauss–Bonnet theorem we have, using $\partial_r\sqrt{\det \breve{g}_{AB}} = \tau\sqrt{\det \breve{g}_{AB}}$,

$$\partial_r\int_{S^2}\zeta\,\mathrm{d}\mu_{\breve{g}} = -8\pi + \int_{S^2}\left(\frac{1}{2}|\xi|^2 + S\right)\mathrm{d}\mu_{\breve{g}}. \tag{40}$$

Integrating in r and using (33)–(36) one obtains

$$\lim_{r\to\infty}\left(\int_{S^2}\zeta\,\mathrm{d}\mu_{\breve{g}} + 8\pi r\right) = \int_0^\infty\int_{S^2}\left(\frac{1}{2}|\xi|^2 + S\right)\mathrm{d}\mu_{\breve{g}}\,\mathrm{d}r.$$

Next, let $\tau_1 := 2/r$, $\delta\tau := \tau - \tau_1$. It follows from the Raychaudhuri equation with $\kappa = 0$ that $\delta\tau$ satisfies the equation

$$\frac{\mathrm{d}\delta\tau}{\mathrm{d}r} + \frac{\tau + \tau_1}{2}\delta\tau = -|\sigma|^2 - 8\pi T_{rr}|_{\mathcal{N}}.$$

Letting

$$\Psi = \exp\left(\int_0^r \frac{\tilde r \tau - 2}{2\tilde r}\,\mathrm{d}\tilde r\right), \tag{41}$$

and using (9)–(10) and (15)–(16) one finds

$$\begin{aligned}\delta\tau(r) &= -r^{-2}\Psi^{-1}\int_0^r (|\sigma|^2 + 8\pi T_{rr}|_{\mathscr N})\Psi r^2 \mathrm{d}r\\ &= \tau_2 r^{-2} + o(r^{-2}),\end{aligned} \tag{42}$$

where

$$\tau_2 = -\lim_{r\to\infty}\Psi^{-1}\int_0^r (|\sigma|^2 + 8\pi T_{rr}|_{\mathscr N})\Psi r^2\,\mathrm{d}r \leqslant 0. \tag{43}$$

Inserting this into (39) gives (31) after noting that

$$\mathrm{d}\mu_{\check g} = \mathrm{e}^{-\int_r^\infty \frac{\tilde r\tau - 2}{\tilde r}\mathrm{d}\tilde r} r^2 \mathrm{d}\mu_s. \tag{44}$$

To continue, suppose that $m_{\rm TB}$ vanishes. It then follows from (43) that $T_{rr}|_{\mathscr N} = 0 = \sigma$. In vacuum this implies [3] that the metric is flat to the future of the light-cone. In fact, for many matter models the vanishing of T_{rr} on the light-cone implies the vanishing of $T_{\mu\nu}$ to the future of the light-cone [3], and the same conclusion can then be obtained.

It remains to establish (37). We decorate with a symbol 'Bo' all fields arising in Bondi coordinates. Consider characteristic data in Bondi coordinates, possibly defined only for large values of $r_{\rm Bo}$. The space-time metric on $\mathscr N = \{u^{\rm Bo} = 0\}$ can be written as

$$\overline g = \overline g^{\rm Bo}_{00}\mathrm{d}u_{\rm Bo}^2 + 2\nu_0^{\rm Bo}\mathrm{d}u_{\rm Bo}\mathrm{d}r_{\rm Bo} + 2\nu_A^{\rm Bo}\mathrm{d}u_{\rm Bo}\mathrm{d}x^A_{\rm Bo} + \check g^{\rm Bo}. \tag{45}$$

Under the usual asymptotic conditions on $\nu_A^{\rm Bo}$ and $\nu_0^{\rm Bo}$ one has (see, e.g., [7, 12])

$$\overline g^{\rm Bo}_{00} = -1 + \frac{2M\big(x^A_{\rm Bo}\big)}{r_{\rm Bo}} + O\big(r_{\rm Bo}^{-2}\big), \tag{46}$$

and the Bondi mass is then defined as

$$m_{\rm TB} = \frac{1}{4\pi}\int_{S^2} M\,\mathrm{d}\mu_s. \tag{47}$$

In Bondi coordinates (28) becomes

$$\zeta^{\rm Bo} = 2\left(\frac{\check\nabla^A\nu_A^{\rm Bo}}{\nu_0^{\rm Bo}} - \frac{(\overline g^{\rm Bo})^{rr}}{r_{\rm Bo}}\right), \tag{48}$$

which allows us to express $(\overline g^{\rm Bo})^{rr}$ in terms of $\zeta^{\rm Bo}$, leading eventually to

$$\begin{aligned}\overline g^{\rm Bo}_{00} &= (\check g^{\rm Bo})^{AB}\nu_A^{\rm Bo}\nu_B^{\rm Bo} - \big(\nu_0^{\rm Bo}\big)^2(\overline g^{\rm Bo})^{rr}\\ &= -1 + \frac{\zeta_2^{\rm Bo} - 2\mathring\nabla^A\big(\nu_A^{\rm Bo}\big)_0}{2r_{\rm Bo}} + O\big(r_{\rm Bo}^{-2}\big),\end{aligned} \tag{49}$$

where $\mathring\nabla$ is the Levi-Civita connection of the metric $s_{AB}\mathrm{d}x^A\mathrm{d}x^B$, and $(\nu_A^{\rm Bo})_0$ is the r-independent coefficient in an asymptotic expansion of $\nu_A^{\rm Bo}$. Comparing with (46), we conclude that

$$m_{\rm TB} = \frac{1}{16\pi}\int_{S^2}\zeta_2^{\rm Bo}\,\mathrm{d}\mu_s. \tag{50}$$

To finish the calculation we need to relate $\zeta_2^{\rm Bo}$ to the characteristic data. In Bondi coordinates we have

$$\tau_{\rm Bo} = \frac{2}{r_{\rm Bo}},$$

and the Raychaudhuri equation implies that

$$\kappa^{\mathrm{Bo}} = r_{\mathrm{Bo}} \frac{|\sigma^{\mathrm{Bo}}|^2 + 8\pi \overline{T}_{rr}}{2}.$$

The equation $\kappa \equiv \overline{\Gamma}^r_{rr} = 0$ together with the usual transformation law for connection coefficients gives the equation

$$\begin{aligned} &\partial_{r_{\mathrm{Bo}}}\left(\frac{\partial r_{\mathrm{Bo}}}{\partial r}\right) + \kappa^{\mathrm{Bo}} \frac{\partial r_{\mathrm{Bo}}}{\partial r} = 0 \\ \Longrightarrow \quad &\frac{\partial r}{\partial r_{\mathrm{Bo}}} = \mathrm{e}^{-\int_{r_{\mathrm{Bo}}}^{\infty} \kappa^{\mathrm{Bo}}} = 1 + O(r_{\mathrm{Bo}}^{-2}), \end{aligned}$$

where we have used the asymptotic condition $\lim_{r\to\infty} \frac{\partial r_{\mathrm{Bo}}}{\partial r} = 1$. Hence

$$r = \int_0^{r_{\mathrm{Bo}}} \mathrm{e}^{-\int_{r_{\mathrm{Bo}}}^{\infty} \kappa^{\mathrm{Bo}}} = r_{\mathrm{Bo}} + \underbrace{\int_0^{\infty} \left(\mathrm{e}^{-\int_{r_{\mathrm{Bo}}}^{\infty} \kappa^{\mathrm{Bo}}} - 1\right)}_{=:-(r_{\mathrm{Bo}})_0} + O(r_{\mathrm{Bo}}^{-1}).$$

To continue, we note that $x^A_{\mathrm{Bo}} = x^A$ and

$$\overline{g}^{\mathrm{Bo}}_{AB}(r_{\mathrm{Bo}}, x^A) = \overline{g}_{AB}\big(r(r_{\mathrm{Bo}}, x^A), x^A\big),$$

which implies

$$\begin{aligned} \frac{2}{r_{\mathrm{Bo}}} = \tau^{\mathrm{Bo}} = \tau\, \partial_{r_{\mathrm{Bo}}} r &= \left(\frac{2}{r} + \frac{\tau_2}{r^2} + O(r^{-3})\right)(1 + O(r^{-2})) \\ &= \frac{2}{r} + \frac{\tau_2}{r^2} + O(r^{-3}), \end{aligned}$$

equivalently

$$r_{\mathrm{Bo}} = r - \frac{\tau_2}{2} + O(r^{-1}) \implies (r_{\mathrm{Bo}})_0 = -\frac{\tau_2}{2}. \tag{51}$$

We are ready now to transform ζ as given by (28) to the new coordinate system:

$$\begin{aligned} \zeta^{\mathrm{Bo}} &= 2(\overline{g}^{\mathrm{Bo}})^{AB} (\overline{\Gamma}^{\mathrm{Bo}})^{r_{\mathrm{Bo}}}_{AB} + \tau^{\mathrm{Bo}} (\overline{g}^{\mathrm{Bo}})^{r_{\mathrm{Bo}} r_{\mathrm{Bo}}} \\ &= 2(\overline{g}^{\mathrm{Bo}})^{AB} \left(\frac{\partial r_{\mathrm{Bo}}}{\partial x^k} \frac{\partial x^i}{\partial x^A_{\mathrm{Bo}}} \frac{\partial x^j}{\partial x^B_{\mathrm{Bo}}} \overline{\Gamma}^k_{ij} + \frac{\partial r_{\mathrm{Bo}}}{\partial r} \frac{\partial^2 r}{\partial x^A_{\mathrm{Bo}} \partial x^B_{\mathrm{Bo}}}\right) + \tau \frac{\partial r}{\partial r_{\mathrm{Bo}}} \frac{\partial r_{\mathrm{Bo}}}{\partial x^i} \frac{\partial r_{\mathrm{Bo}}}{\partial x^j} \overline{g}^{ij} \\ &= \frac{\partial r_{\mathrm{Bo}}}{\partial r} \zeta + 2 \frac{\partial r_{\mathrm{Bo}}}{\partial r} \Delta_{\breve{g}} r + O(r_{\mathrm{Bo}}^{-3}), \end{aligned}$$

where $\Delta_{\breve{g}}$ is the Laplace operator of the two-dimensional metric $\breve{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B$. From this one easily obtains

$$\zeta_2^{\mathrm{Bo}} = \zeta_2 + \tau_2 + \Delta_s \tau_2. \tag{52}$$

Inserting (52) into (50) one obtains (37), which completes the proof.

In fact, the calculations just made show that the integral (37) is invariant under changes of the coordinate r of the form $r \mapsto r + r_0(x^A) + O(r^{-1})$.

Let S be a section of $\mathscr{I}^+$ arising from a smooth light-cone as above, and let S' be any section of $\mathscr{I}^+$ contained entirely in the past of S. The Trautman–Bondi mass loss formula shows that $m_{\mathrm{TB}}(S')$ will be larger than or equal to $m_{\mathrm{TB}}(S)$ [9, section 8.1]. So, our formula (31) establishes positivity of m_{TB} for all such sections S'.

In many cases the past limit of m_{TB} is the ADM mass, and one expects this to be true quite generally for asymptotically Minkowskian space-times. Whenever this and (31) hold, we obtain an elementary proof of non-negativity of the ADM mass.

It is tempting to use a density argument to remove the hypothesis of non-existence of conjugate point precisely at $\mathscr{I}^+$ along some generators; such generators will be referred to as *asymptotically singular*. For this one could first rewrite (31) as

$$m_{\rm TB} = \frac{1}{16\pi}\int_0^\infty \int_{S^2}\left(\frac{1}{2}|\xi|^2 + S + (|\sigma|^2 + 8\pi T_{rr}|_{\mathcal{N}})\,e^{\int_r^\infty \frac{\tilde r\tau-2}{2\tilde r}d\tilde r}\right) r^2\,e^{-\int_r^\infty \frac{\tilde r\tau-2}{\tilde r}d\tilde r}\,d\mu_s\,dr \qquad (53)$$

where we have used (44). One might then consider an increasing sequence of tensors σ_i which converge to σ as $i\to\infty$ so that $|\sigma_i|^2$ converges to $|\sigma|^2$ from below, and such that for each i the associated solution τ_i of the Raychaudhuri equation leads to a metric satisfying (12). It is easy to see that τ_i is then decreasing to zero along the asymptotically singular generators as i tends to infinity, monotonically in i, which leads to an infinite integral $-\int_r^\infty \frac{\tilde r\tau-2}{\tilde r}d\tilde r$ on those generators. It is however far from clear if and when this divergence leads to a finite volume integral after integrating over the generators, and we have not been able to conclude along those lines.

Suppose, finally, that instead of a complete light-cone we have a smooth characteristic hypersurface $\mathcal{N}$ with an interior boundary S_0 diffeomorphic to S^2, and intersecting $\mathscr{I}^+$ transversally in a smooth cross-section S as before. Let r be an affine parameter on the generators chosen so that $S_0 = \{r = r_0\}$ for some $r_0 > 0$ and such that (3)–(7) hold. The calculations above give the following formula for $m_{\rm TB}$:

$$m_{\rm TB} = \text{r.h.s. of (53)} + \frac{1}{16\pi}\left(8\pi r_0 + \int_{r=r_0}\left(\zeta + (2r^{-1}-\tau)\,e^{\int_{r_0}^\infty \frac{r\tau-2}{2r}dr}\right)d\mu_{\breve g}\right), \qquad (54)$$

with the range $[0,\infty)$ replaced by $[r_0,\infty]$ in the first integral symbol appearing in (53). (For outgoing null hypersurfaces issued from any sphere of symmetry in the domain of outer-communications in Schwarzschild, the term multiplying the exponential in (54) and the right-hand side of (31) vanish, while the remaining terms add to the usual mass parameter m.)

Equation (54) leads to the following interesting inequality for *space-times containing white hole regions*. We will say that a surface S is *smoothly visible* from $\mathscr{I}^+$ if the null hypersurface generated by the family of outgoing null geodesics is smooth in the conformally rescaled space-time. Assume, then, that S is smoothly visible and *weakly past outer trapped*:

$$\zeta|_S \geqslant 0. \qquad (55)$$

Smooth visibility implies that $\tau > 0$ everywhere, in particular on S. By a translation of the affine parameter r of the generators of $\mathcal{N}$ we can achieve

$$r_0 = \frac{2}{\sup_S \tau}. \qquad (56)$$

All terms in (54) are non-negative now, leading to the interesting inequality

$$m_{\rm TB} \geqslant \frac{1}{\sup_S \tau}. \qquad (57)$$

Further, equality implies that S and T_{rr} vanish along $\mathcal{N}$, and that we have

$$\xi = \sigma = 0 \text{ along } \mathcal{N}, \quad \zeta|_S = 0, \quad \tau|_S \equiv \frac{2}{r_0} = \frac{1}{m_{\rm TB}}. \qquad (58)$$

Integrating the Raychaudhuri equation gives

$$\tau = \frac{2}{r}, \quad r \geqslant r_0. \qquad (59)$$

The equations $\tau\bar g_{AB} = 2\chi_{AB} = \partial_r \bar g_{AB}$ together with the asymptotic behaviour of the metric imply

$$g_{AB} = r^2 s_{AB}, \quad r \geqslant r_0. \qquad (60)$$

It follows that the outgoing null hypersurface issued from S can be isometrically embedded in the Schwarzschild space-time with $m = m_{\mathrm{TB}}$ as a null hypersurface emanating from a spherically symmetric cross-section of the past event horizon.

Time-reversal and our result provide, of course, an inequality between the mass of the past directed null hypersurface issuing from a weakly future outer trapped surface S and $\sup_S \zeta$ in black hole space-times.

We finish this communication by noting that formulae such as (54), together with monotonicity of mass, might be useful in a stability analysis of black hole solutions, by choosing the boundary to lie on the initial data surface, for then one obtains an *a priori* L^2-weighted bound on $|\sigma|^2$ and $|\xi|^2$ on all corresponding null hypersurfaces.

Acknowledgments

Supported in part by the Austrian Science Fund (FWF): P 24170-N16. PTC is grateful to the Stanford Mathematics Research Center for hospitality and support during work on this communication.

References

[1] Andersson L and Chruściel P T 1993 Hyperboloidal Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity *Phys. Rev. Lett.* **70** 2829–32
[2] Bondi H, van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity VII: Waves from axi–symmetric isolated systems *Proc. R. Soc. Lond.* A **269** 21–52
[3] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2009 The light-cone theorem *Class. Quantum Grav.* **26** 135011 (arXiv:0905.2133 [gr-qc])
[4] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions *Ann. Henri Poincaré* **12** 419–82 (arXiv:1006.4467 [gr-qc])
[5] Chruściel P T 2012 The existence theorem for the Cauchy problem on the light-cone for the vacuum Einstein equations *Forum for Mathematics Sigma* at press (arXiv:1209.1971 [gr-qc])
[6] Chruściel P T, Jezierski J and Łęski S 2004 The Trautman–Bondi mass of hyperboloidal initial data sets *Adv. Theor. Math. Phys.* **8** 83–139 (arXiv:gr-qc/0307109)
[7] Chruściel P T, MacCallum M A H and Singleton D 1995 Gravitational waves in general relativity. XIV: Bondi expansions and the 'polyhomogeneity' of Scri *Phil. Trans. R. Soc.* A **350** 113–41 (arXiv:gr-qc/9305021)
[8] Chruściel P T and Paetz T-T 2014 Light-cone initial data and smoothness of Scri. I. Formalism and results arXiv:1403.3558 [gr-qc]
[9] Jezierski J 1998 Bondi mass in classical field theory *Acta Phys. Pol.* B **29** 667–743
[10] Ludvigsen M and Vickers J A G 1982 A simple proof of the positivity of Bondi mass *J. Phys. A: Math. Gen.* **15** L67–70
[11] Paetz T-T 2014 Characteristic initial data and smoothness of scri. II. Asymptotic expansions and construction of conformally smooth data sets arXiv:1403.3560 [gr-qc]
[12] Sachs R K 1962 Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time *Proc. Roy. Soc. Lond.* A **270** 103–26
[13] Schoen R and Yau S-T 1979 On the proof of the positive mass conjecture in general relativity *Commun. Math. Phys.* **65** 45–76
[14] Schoen R and Yau S-T 1981 Proof of the positive mass theorem II *Commun. Math. Phys.* **79** 231–60
[15] Schoen R and Yau S-T 1982 Proof that the Bondi mass is positive *Phys. Rev. Lett.* **48** 369–71
[16] Trautman A 1958 Radiation and boundary conditions in the theory of gravitation *Bull. Acad. Pol. Sci. Série sci. Math. Astron. Phys.* **VI** 407–12
[17] Witten E 1981 A simple proof of the positive energy theorem *Commun. Math. Phys.* **80** 381–402

OPEN ACCESS
IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **30** (2013) 235036 (24pp) doi:10.1088/0264-9381/30/23/235036

KIDs like cones

Piotr T Chruściel and Tim-Torben Paetz

Gravitational Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: Piotr.Chrusciel@univie.ac.at and Tim-Torben.Paetz@univie.ac.at

Received 31 May 2013, in final form 6 August 2013
Published 8 November 2013
Online at stacks.iop.org/CQG/30/235036

Abstract
We analyze vacuum Killing Initial Data on characteristic Cauchy surfaces. A general theorem on existence of Killing vectors in the domain of dependence is proved, and some special cases are analyzed in detail, including the case of bifurcate Killing horizons.

PACS numbers: 04.20.Cv, 04.20.Ex

1. Introduction

Killing Initial Data (KIDs) are defined as initial data on a Cauchy surface for a spacetime Killing vector field. Vacuum KIDs on spacelike hypersurfaces are well understood (see [1, 10] and references therein). In the spacelike case they play a significant role by providing an obstruction to gluing initial data sets [4, 6].

The question of KIDs on light-cones has been recently raised in [14]. The object of this note is to analyze this, as well as KIDs on characteristic surfaces intersecting transversally. It turns out that the situation in the light-cone case is considerably simpler than for the spacelike Cauchy problem, which explains our title.

For definiteness we assume the Einstein vacuum equations, in dimensions $n+1$, $n \geqslant 3$, possibly with a cosmological constant,

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \qquad \lambda \in \mathbb{R}. \tag{1.1}$$

Similar results can be proved for Einstein equations with matter fields satisfying well-behaved evolution equations.

1.1. Light-cone

Consider the (future) light-cone C_O issued from a point O in an $(n+1)$-dimensional spacetime $(\mathscr{M}, g)$, $n \geqslant 3$; by this we mean the subset of $\mathscr{M}$ covered by future-directed null geodesics issued from O. (We expect that our results remain true for $n = 2$; this requires a more careful analysis of some of the equations arising, which we have not attempted to carry out.) Let

$(x^\mu) = (x^0, x^i) = (x^0, r, x^A)$ be a coordinate system such that x^0 vanishes on C_O. In the theorem that follows the initial data for the sought-for Killing vector field are provided by a spacetime vector field $\overline{Y}$ which is defined on C_O only.[1] We will need to differentiate $\overline{Y}$ in directions tangent to C_O, for this we need a covariant derivative operator which involves only derivatives tangent to the characteristic hypersurfaces. In a coordinate system such that the hypersurface under consideration is given by the equation $x^0 = 0$, for the first derivatives the usual spacetime covariant derivative $\nabla_i \overline{Y}^\mu$ applies. However, the tensor of second spacetime covariant derivatives involves the undefined fields $\nabla_0 \overline{Y}^\mu$. To avoid this we set, on the hypersurface $\{x^0 = 0\}$,[2]

$$\mathrm{D}_i \overline{Y}_\mu := \partial_i \overline{Y}_\mu - \Gamma^\nu_{\mu i} \overline{Y}_\nu \equiv \nabla_i \overline{Y}_\mu, \tag{1.2}$$

$$\begin{aligned}\mathrm{D}_i \mathrm{D}_j \overline{Y}_\mu &:= \partial_i \mathrm{D}_j \overline{Y}_\mu - \Gamma^k_{ij} \mathrm{D}_k \overline{Y}_\mu - \Gamma^\nu_{i\mu} \mathrm{D}_j \overline{Y}_\nu \\ &\equiv \partial_i \nabla_j \overline{Y}_\mu - \Gamma^k_{ij} \nabla_k \overline{Y}_\mu - \Gamma^\nu_{i\mu} \mathrm{D}_j \overline{Y}_\nu,\end{aligned} \tag{1.3}$$

with an obvious similar formula for $\mathrm{D}_i \mathrm{D}_j \overline{Y}^\mu$. When the restriction $\overline{\nabla_0 Y_\mu}$ to the hypersurface $\{x^0 = 0\}$ of the x^0-derivative is defined we have

$$\mathrm{D}_i \mathrm{D}_j \overline{Y}_\mu = \nabla_i \nabla_j \overline{Y}_\mu + \Gamma^0_{ij} \overline{\nabla_0 Y_\mu}.$$

Clearly, $\mathrm{D}_i \mathrm{D}_j \overline{Y}_0$ coincides with $\nabla_i \nabla_j Y_0|_{\{x^0=0\}}$ when $\overline{Y}^\mu$ is the restriction to the hypersurface $\{x^0 = 0\}$ of a Killing vector field Y^μ, as then $\nabla_0 Y_0 = 0$. This is of key importance for our equations below.

In the adapted null coordinates of [2] we have $\Gamma^0_{1i} = 0$ on $\{x^0 = 0\}$ (see [2, appendix A]), so in these coordinates $\mathrm{D}_i \mathrm{D}_j$ differs from $\nabla_i \nabla_j$ only when $i, j \in \{2, \ldots, n\}$.

Our main result is:

Theorem 1.1. *Let $\overline{Y}$ be a continuous vector field defined along C_O in a vacuum spacetime $(\mathscr{M}, g)$, smooth on $C_O \setminus \{O\}$. There exists a smooth vector field X satisfying the Killing equations on $D^+(C_O)$ and coinciding with $\overline{Y}$ on C_O if and only if on C_O it holds that*

$$\mathrm{D}_i \overline{Y}_j + \mathrm{D}_j \overline{Y}_i = 0, \tag{1.4}$$

$$\mathrm{D}_1 \mathrm{D}_1 \overline{Y}_0 = R_{011}{}^\mu \overline{Y}_\mu. \tag{1.5}$$

Furthermore, (1.5) is not needed on the closure of the set on which the divergence τ of C_O is non-zero.

While this is not necessary, the analysis of KIDs on light-cones thus can be split into two cases: the first is concerned with the region sufficiently close to the tip of the cone where the expansion τ has no zeros. Once a spacetime with Killing field has been constructed near the vertex, the initial value problem for the remaining part of the cone can be reduced to a characteristic initial value problem with two transversally intersecting null hypersurfaces, which will be addressed in theorem 1.2. From this point of view the key restriction for light-cones is (1.4).

The proof of theorem 1.1 can be found in section 2.5. In order to prove it we will first establish some intermediate results, theorems 2.1 and 2.5 below, which require

[1] Given a smooth vector field Y^μ defined in a spacetime neighborhood of a hypersurface $\{f = 0\}$ we will write $\overline{Y}^\mu = Y^\mu|_{f=0}$, but at this point of the discussion $\overline{Y}^\mu$ is simply a vector field defined along the surface $\{f = 0\}$, it being irrelevant whether or not $\overline{Y}^\mu$ arises by restriction of a smooth spacetime vector field. On the other hand, that last question will become a central issue in the proof of theorem 2.5.

[2] We use the following conventions on indices: Greek indices are for spacetime tensors and coordinates, small Latin letters shall be used for tensors and coordinates on the light-cone or the characteristic surfaces, and capital Latin letters for tensors or coordinates in the hypersurfaces of spacetime co-dimension two foliating the characteristic surfaces.

further hypotheses. It is somewhat surprising that these additional conditions turn out to be automatically satisfied.

Equations (1.4) provide thus necessary-and-sufficient conditions for the existence, to the nearby future of O, of a Killing vector field. They can be viewed as the light-cone equivalent of the spacelike KID equations, keeping in mind that (1.5) should be added when C_O contains open subsets on which τ vanishes. As made clear by the definitions, (1.4) and (1.5) involve only the derivatives of $\overline{Y}$ in directions tangent to C_O.

We shall see in section 2.6 that some of the equations (1.4) can be integrated to determine $\overline{Y}$ in terms of data at O. Once this has been done, we are left with the trace-free part of $\mathrm{D}_A\overline{Y}_B + \mathrm{D}_B\overline{Y}_A = 0$ as the 'reduced KID equations'.

It should be kept in mind that a Killing vector field satisfies an overdetermined system of second-order ODEs which can be integrated along geodesics starting from O, see (2.56) below. This provides both X, and its restriction $\overline{X}$ to C_O, in a neighborhood of O, given the free data $X^\alpha|_O$ and $\nabla^{[\alpha}X^{\beta]}|_O$. We will see in the course of the proof of theorem 2.1 how such a scheme ties-in with the statement of the theorem, cf in particular section 2.3.

In section 2.6 we give a more explicit form of the KID equations (1.4) on a cone and discuss some special cases.

In section 3.3 we show how bifurcate Killing horizons arise from totally geodesic null surfaces normal to a spacelike submanifold S of co-dimension two, and how isometries of S propagate to the spacetime.

1.2. Two intersecting null hypersurfaces

Throughout, we employ the symbol $\breve{(\cdot)}$ to denote the trace-free part of the field $(\cdot)$ with respect to $\tilde{g} = \overline{g}_{AB}\,dx^A dx^B$. Further, an overbar denotes restriction to the initial surface.

In what follows the coordinates x^A are assumed to be constant on the generators of the null hypersurfaces. The analogue of theorem 1.1 for two intersecting hypersurfaces reads:

Theorem 1.2. *Consider two smooth null hypersurfaces $N_1 = \{x^1 = 0\}$ and $N_2 = \{x^2 = 0\}$ in an $(n+1)$-dimensional vacuum spacetime $(\mathscr{M}, g)$, with transverse intersection along a smooth $(n-1)$-dimensional submanifold S. Let $\overline{Y}$ be a continuous vector field defined on $N_1 \cup N_2$ such that $\overline{Y}|_{N_1}$ and $\overline{Y}|_{N_2}$ are smooth. There exists a smooth vector field X satisfying the Killing equations on $D^+(N_1 \cup N_2)$ and coinciding with $\overline{Y}$ on $N_1 \cup N_2$ if and only if on N_1 it holds that*

$$\mathrm{D}_2\overline{Y}_2 = 0, \tag{1.6}$$

$$\mathrm{D}_{(2}\overline{Y}_{A)} = 0, \tag{1.7}$$

$$(\mathrm{D}_{(A}\overline{Y}_{B)})\breve{} = 0, \tag{1.8}$$

$$R_{122}{}^{\mu}\overline{Y}_\mu - \mathrm{D}_2\mathrm{D}_2\overline{Y}_1 = 0, \tag{1.9}$$

where D is the analogue on N_1 of the derivative operator (1.2) and (1.3), with identical corresponding conditions on N_2, and on S one needs further to assume that

$$(\mathrm{D}_1\overline{Y}_2 + \mathrm{D}_2\overline{Y}_1)|_S = 0, \tag{1.10}$$

$$g^{AB}\mathrm{D}_A\overline{Y}_B|_S = 0, \tag{1.11}$$

$$\partial_i(g^{AB}\mathrm{D}_A\overline{Y}_B)|_S = 0, \quad i = 1, 2, \tag{1.12}$$

$$R_{21A}{}^{\mu}\overline{Y}_\mu - \mathrm{D}_A\mathrm{D}_{[1}\overline{Y}_{2]}|_S = 0. \tag{1.13}$$

Similarly to theorem 1.1, (1.9) can be replaced by the requirement that $\overline{g}^{AB}\mathrm{D}_A\overline{Y}_B = 0$ on regions where the divergence of N_1 is non-zero. An identical statement applies to N_2.

Theorem 1.2 is proved in section 3. As before, (1.6)–(1.13) provide necessary-and-sufficient conditions for the existence, to the future of S, of a Killing vector field. Hence they provide the equivalent of the spacelike KID equations in the current setting. Note that in (1.6)–(1.13) the derivative D coincides with ∇.

2. The light-cone case

2.1. Adapted null coordinates

We use local coordinates $(x^0 \equiv u, x^1 \equiv r, x^A)$ adapted to the light-cone as in [2], in the sense that the cone is given by $C_O = \{x^0 = 0\}$. Further, the coordinate x^1 parameterizes the null geodesics emanating to the future from the vertex of the cone, while the x^A's are local coordinates on the level sets $\{x^0 = 0,\ x^1 = \text{const}\} \cong S^{n-1}$, and are constant along the generators. Then the metric takes the following form on C_O:

$$g|_{C_O} = \overline{g}_{00}(\,\mathrm{d}x^0)^2 + 2\nu_0\,\mathrm{d}x^0\,\mathrm{d}x^1 + 2\nu_A\,\mathrm{d}x^0\,\mathrm{d}x^A + \overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B. \tag{2.1}$$

We stress that we *do not* assume that this form of the metric is preserved under differentiation in the x^0–direction, i.e. we do not impose any gauge condition off the cone. On C_O the inverse metric reads

$$g^\sharp|_{C_O} = \overline{g}^{11}\partial_1^2 + 2\overline{g}^{1A}\partial_r\partial_A + 2\nu^0\partial_0\partial_r + \overline{g}^{AB}\partial_A\partial_B, \tag{2.2}$$

with

$$\nu^0 = \frac{1}{\nu_0}, \quad \overline{g}^{1A} = -\nu^0\overline{g}^{AB}\nu_B, \quad \overline{g}^{11} = (\nu^0)^2(-\overline{g}_{00} + \overline{g}^{AB}\nu_A\nu_B). \tag{2.3}$$

2.2. A weaker result

We start with a weaker version of theorem 2.5 which, moreover, assumes that the vector field $\overline{Y}$ there is the restriction to the light-cone of some smooth vector field Y:

Theorem 2.1. *Let Y be a smooth vector field defined in a neighborhood of C_O in a vacuum spacetime $(\mathscr{M}, g)$. There exists a smooth vector field X satisfying the Killing equations on $D^+(C_O)$ and coinciding with Y on C_O if and only if the equations*

$$\mathrm{D}_i\overline{Y}_j + \mathrm{D}_j\overline{Y}_i = 0, \tag{2.4}$$

$$R_{011}{}^{\mu}\overline{Y}_\mu - \mathrm{D}_1\mathrm{D}_1\overline{Y}_0 = 0, \tag{2.5}$$

$$R_{01A}{}^{\mu}\overline{Y}_\mu - \mathrm{D}_A\mathrm{D}_1\overline{Y}_0 = 0, \tag{2.6}$$

$$g^{AB}(R_{0AB}{}^{\mu}\overline{Y}_\mu - \mathrm{D}_A\mathrm{D}_B\overline{Y}_0) = 0, \tag{2.7}$$

are satisfied by the restriction $\overline{Y}$ of Y to C_O.

Proof. To prove necessity, let X be a smooth vector field satisfying the Killing equations on $D^+(C_O)$:

$$\underbrace{\nabla_\mu X_\nu + \nabla_\nu X_\mu}_{=:A_{\mu\nu}} = 0, \tag{2.8}$$

the tangent components of which give (2.4). It further easily follows from (2.8) that X satisfies

$$\nabla_\mu \nabla_\nu X_\sigma = R_{\alpha\mu\nu\sigma} X^\alpha, \tag{2.9}$$

and (2.5)–(2.7) similarly follow; for (2.7) the equation $\overline{A}_{00} = 0$ is used.

To prove sufficiency, by contracting (2.9) one finds

$$\Box X^\sigma = -R^\sigma{}_\alpha X^\alpha \tag{2.10}$$

(which equals $-\lambda X^\sigma$ under (1.1)). So, should a solution X of our problem exist, it will necessarily satisfy the wave equation (2.10).

Now, it follows from e.g. [7, théorème 2] that for any smooth vector field Y defined on $\mathscr{M}$ there exists a smooth vector field X on $\mathscr{M}$ solving (2.10) to the future of O, such that,

$$\overline{X}^\mu = \overline{Y}^\mu. \tag{2.11}$$

Here, and elsewhere, overlining denotes restriction to C_O. Further $X|_{D^+(C_O)}$ is uniquely defined by $Y|_{C_O}$.

Applying $\Box$ to (2.8) leads to the identity

$$\Box A_{\mu\nu} = 2\nabla_{(\mu} \Box X_{\nu)} + 4R_{\kappa(\mu\nu)}{}^\alpha \nabla^\kappa X_\alpha + 2X_\alpha \nabla^\kappa R_{\kappa(\mu\nu)}{}^\alpha + 2R_{\alpha(\mu} \nabla^\alpha X_{\nu)}. \tag{2.12}$$

When X^μ solves (2.10) this can be rewritten as a homogeneous linear wave equation for the tensor field $A_{\mu\nu}$,

$$\Box A_{\mu\nu} = -2R_\mu{}^\alpha{}_\nu{}^\kappa A_{\alpha\kappa} + 2R_{(\mu}{}^\alpha A_{\nu)\alpha} - 2\mathscr{L}_X R_{\mu\nu}, \tag{2.13}$$

if one notes that, under (1.1) the last term $-2\mathscr{L}_X R_{\mu\nu}$ equals $-2\lambda A_{\mu\nu}$ (and, in fact, cancels with the before-last one, though this cancellation is irrelevant for the current discussion). It follows from uniqueness of solutions of (2.13) that a solution X of (2.10) will satisfy the Killing equation on $D^+(C_O)$ if and only if

$$\overline{A}_{\mu\nu} = 0. \tag{2.14}$$

But by (2.4) we already have

$$\overline{A}_{ij} = 0, \tag{2.15}$$

so it remains to show that the equations $\overline{A}_{0\mu} = 0$ hold. (Annoyingly, these equations involve the derivatives $\overline{\partial_0 X_\mu}$ which *cannot* be expressed as *local* expressions involving only the initial data $\overline{X} = \overline{Y}$.) The theorem follows now directly from lemma 2.4 below. $\Box$

Definition 2.2. *It is convenient to introduce, for a given vector field X, the tensor field*

$$S^{(X)}_{\mu\nu\sigma} := \nabla_\mu \nabla_\nu X_\sigma - R^\alpha{}_{\mu\nu\sigma} X_\alpha. \tag{2.16}$$

Whenever it is clear from the context which vector field is meant we will suppress its appearance and simply write $S_{\mu\nu\sigma}$.

Using the algebraic symmetries of the Riemann tensor we find:

Lemma 2.3. *It holds that:*

(i) $2S_{\alpha\beta\gamma} = 2\nabla_{(\alpha} A_{\beta)\gamma} - \nabla_\gamma A_{\alpha\beta}$,
(ii) $2S_{\alpha(\beta\gamma)} = \nabla_\alpha A_{\beta\gamma}$,
(iii) $S_{[\alpha\beta]\gamma} = 0$.

Lemma 2.4. *Suppose that* $\overline{A}_{ij} = 0$ *and* $\Box X = -\lambda X$.

(1) (2.5) is equivalent to $\overline{A}_{01} = 0$.
(2) If (2.5) holds, then (2.6) is equivalent to $\overline{A}_{0A} = 0$.
(3) If (2.5) and (2.6) hold, then (2.7) is equivalent to $\overline{A}_{00} = 0$.

Proof. It turns out to be convenient to consider the identity

$$\nabla^\mu A_{\mu\nu} \equiv \underbrace{\nabla^\mu(\nabla_\mu X_\nu}_{=-R_{\nu\mu}X^\mu} + \nabla_\nu X_\mu) = -R_{\nu\mu}X^\mu + \nabla^\mu\nabla_\nu X_\mu = \nabla_\nu\nabla^\mu X_\mu = \tfrac{1}{2}\nabla_\nu A^\mu{}_\mu.$$

Thus, it holds that

$$g^{\alpha\beta}(2\nabla_\alpha A_{\beta\nu} - \nabla_\nu A_{\alpha\beta}) = 0. \tag{2.17}$$

In adapted null coordinates (2.17) implies

$$\begin{aligned} 0 = {} & 2\nu^0(\overline{\nabla_0 A_{1\mu}} + \nabla_1\overline{A}_{0\mu} - \nabla_\mu\overline{A}_{01}) + 2\overline{g}^{1B}(\nabla_1\overline{A}_{B\mu} + \nabla_B\overline{A}_{1\mu} - \nabla_\mu\overline{A}_{1B}) \\ & + \overline{g}^{11}(2\nabla_1\overline{A}_{1\mu} - \nabla_\mu\overline{A}_{11}) + \overline{g}^{BC}(2\nabla_B\overline{A}_{C\mu} - \nabla_\mu\overline{A}_{BC}). \end{aligned} \tag{2.18}$$

Due to lemma 2.3 we have

$$\nabla_0 A_{ij} = \nabla_i A_{j0} + \nabla_j A_{i0} - 2S_{ij0}. \tag{2.19}$$

When $(ij) = (11)$ that yields

$$\nabla_0 A_{11} - 2\nabla_1 A_{01} = -2S_{110}. \tag{2.20}$$

Inserting into (2.18) with $\mu = 1$, after some simplifications one obtains

$$4\nu^0\nabla_1\overline{A}_{01} + 2\overline{g}^{1B}\nabla_B\overline{A}_{11} - \overline{g}^{BC}(\nabla_1\overline{A}_{BC} - 2\nabla_B\overline{A}_{C1}) + \overline{g}^{11}\nabla_1\overline{A}_{11} = 4\nu^0\overline{S}_{110}.$$

Using the vanishing of the $\overline{\Gamma}^0_{i1}$'s [2, appendix A] and the $\overline{A}_{ij}$'s, this becomes a linear homogeneous ODE for $\overline{A}_{01}$; in the notation of the last reference (where, in particular, τ denotes the divergence of C_O):[3]

$$(2\partial_r + \tau - 2\nu^0\partial_r\nu_0)\overline{A}_{01} = 2\overline{S}_{110}. \tag{2.21}$$

If $\overline{A}_{01} = 0$, the vanishing of $\overline{S}_{110}$ immediately follows.

To prove the reverse implication, for definiteness we assume here and in what follows a coordinate system as in [2, section 4.5]. In this coordinate system τ behaves as $(n-1)/r$ for small r, ν_0 satisfies $\nu_0 = 1 + O(r^2)$, and (2.21) is a Fuchsian ODE with the property that every solution which is $o(r^{-(n-1)/2})$ for small r is identically zero, see appendix A. As $\overline{A}_{01}$ is bounded, when $\overline{S}_{110}$ vanishes we conclude that

$$\overline{A}_{01} = 0. \tag{2.22}$$

This proves point 1 of the lemma.

Next, (2.18) with $\mu = D$ reads

$$\begin{aligned} 0 = {} & 2\nu^0(\overline{\nabla_0 A_{1D}} + \nabla_1\overline{A}_{0D} - \nabla_D\overline{A}_{01}) + 2\overline{g}^{1B}(\nabla_1\overline{A}_{BD} + \nabla_B\overline{A}_{1D} - \nabla_D\overline{A}_{1B}) \\ & + \overline{g}^{11}(2\nabla_1\overline{A}_{1D} - \nabla_D\overline{A}_{11}) + \overline{g}^{BC}(2\nabla_B\overline{A}_{CD} - \nabla_D\overline{A}_{BC}). \end{aligned} \tag{2.23}$$

Using (2.19) with $(ij) = (A1)$,

$$\nabla_0 A_{1A} = \nabla_A A_{01} + \nabla_1 A_{0A} - 2S_{A10}, \tag{2.24}$$

to eliminate $\overline{\nabla_0 A_{1D}}$ from (2.23), and invoking (2.22), on C_O one obtains a system of Fuchsian radial ODEs for $\overline{A}_{0D}$,

$$\left(2\partial_r + \frac{n-3}{n-1}\tau + 2\kappa - 2\nu^0\partial_r\nu_0\right)\overline{A}_{0B} - 2\sigma_B{}^C\overline{A}_{0C} = 2\overline{S}_{B10}, \tag{2.25}$$

with zero being the unique solution with the required behavior at $r = 0$ when $\overline{S}_{B10} = 0$:

$$\overline{A}_{0B} = 0. \tag{2.26}$$

This proves point 2 of the lemma.

[3] Throughout we shall make extensively use of the formulae for the Christoffel symbols in adapted null coordinates computed in [2, appendix A]. Apart from the vanishing of the $\overline{\Gamma}^0_{i1}$'s the expressions for $\overline{\Gamma}^0_{01}$, $\overline{\Gamma}^1_{11} = \kappa$, $\overline{\Gamma}^A_{1B}$ and $\overline{\Gamma}^0_{AB}$ will be often used.

Let us finally turn attention to (2.18) with $\mu = 0$:

$$0 = 2\nu^0\nabla_1\overline{A}_{00} + 2\overline{g}^{1B}(\nabla_1\overline{A}_{B0} + \nabla_B\overline{A}_{10} - \overline{\nabla_0 A_{1B}})$$
$$+ \overline{g}^{11}(2\nabla_1\overline{A}_{10} - \overline{\nabla_0 A_{11}}) + \overline{g}^{BC}(2\nabla_B\overline{A}_{C0} - \overline{\nabla_0 A_{BC}}). \tag{2.27}$$

The transverse derivatives $\overline{\nabla_0 A_{1i}}$ can be eliminated using (2.20) and (2.24),

$$2\nu^0\nabla_1\overline{A}_{00} + 4\overline{g}^{1B}\overline{S}_{B10} + 2\overline{g}^{11}\overline{S}_{110} + \overline{g}^{BC}(2\nabla_B\overline{A}_{0C} - \overline{\nabla_0 A_{BC}}) = 0. \tag{2.28}$$

The remaining one, $\overline{g}^{AB}\overline{\nabla_0 A_{AB}}$, fulfils the following equation on C_O, which follows from (2.19),

$$\overline{g}^{AB}\overline{\nabla_0 A_{AB}} = 2\overline{g}^{AB}\nabla_A\overline{A}_{0B} - 2\overline{g}^{AB}\overline{S}_{AB0} = 2\overline{g}^{AB}\nabla_A\overline{A}_{0B} - 2\tilde{S} - \tau\nu^0\overline{A}_{00},$$

where we have set

$$\tilde{S} := \overline{g}^{AB}\overline{S}_{AB0} - \tfrac{1}{2}\tau\nu^0\overline{A}_{00}. \tag{2.29}$$

Note that $\tilde{S}$ is the negative of the left-hand side of (2.7), and we want to show that the vanishing of $\tilde{S}$ is equivalent to that of $\overline{A}_{00}$. Equation (2.28) with $\overline{A}_{ij} = 0$ and $\overline{A}_{0i} = 0$ (i.e. $\overline{S}_{i10} = 0$) yields

$$\nu^0(2\partial_r + \tau + 4\kappa - 4\nu^0\partial_r\nu_0)\overline{A}_{00} = -2\tilde{S}. \tag{2.30}$$

For $\tilde{S} = 0$ this is again a Fuchsian radial ODE for $\overline{A}_{00}$, with the only regular solution $\overline{A}_{00} = 0$, and the lemma is proved. □

2.3. The free data for X

Let us explore the nature of (1.4). Making extensive use of [2, appendix A], and of the notation there (thus $\kappa \equiv \overline{\Gamma}^1_{11}$, $\xi_A \equiv -2\overline{\Gamma}^1_{1A}$, while $\chi_A{}^B = \overline{\Gamma}^B_{1A}$ denotes the null second fundamental form of C_O), we find

$$\overline{A}_{11} = 2(\partial_r - \kappa)\overline{X}_1, \tag{2.31}$$

$$\overline{A}_{1A} = \partial_r\overline{X}_A - 2\chi_A{}^B\overline{X}_B + (\partial_A + \xi_A)\overline{X}_1, \tag{2.32}$$

$$\overline{A}_{AB} = 2\tilde{\nabla}_{(A}\overline{X}_{B)} + 2\chi_{AB}\overline{X}^1 - \nu^0(2\tilde{\nabla}_{(A}\nu_{B)} - \overline{\partial_0 g_{AB}})\overline{X}_1. \tag{2.33}$$

For definiteness, in the discussion that follows we continue to assume a coordinate system as in [2, section 4.5], in particular $\kappa = 0$ and

$$\chi_A{}^B = \frac{1}{r}\delta_A{}^B + O(r), \quad \xi_A = O(r^2), \tag{2.34}$$

$$\tau = \frac{n-1}{r} + O(r), \quad \partial_r\left(\tau - \frac{n-1}{r}\right) = O(1), \quad \partial_A\tau = O(r), \tag{2.35}$$

$$\sigma_A{}^B = O(r), \quad \partial_r\sigma_A{}^B = O(1), \quad \partial_C\sigma_A{}^B = O(r). \tag{2.36}$$

Under (1.4) the left-hand sides of (2.31)–(2.33) vanish. Hence, we can determine $\overline{X}_1$ by integrating (2.31),

$$\overline{X}_1(r, x^A) = c(x^A), \tag{2.37}$$

for some function of the angles.

We continue by integrating (2.32). This is a Fuchsian ODE for $\overline{X}_B$, the solutions of which are of the form

$$\overline{X}_A = r\mathring{\mathscr{D}}_A c + f_A(x^B)r^2 + O(r^3), \tag{2.38}$$

where $\mathring{\mathscr{D}}$ is the covariant derivative operator of the unit round metric s on S^{n-1} where $f_A(x^B)$ is an integration function.

In a neighborhood of O, where τ does not vanish, the component $\overline{X}^1$ can be algebraically determined from the equation $\overline{A}_A{}^A = 0$, leading to

$$\overline{X}^1 = -\frac{1}{n-1}\Delta_s c - \frac{r}{n-1} s^{AB} \mathring{\mathscr{D}}_A f_B + O(r^2), \tag{2.39}$$

where Δ_s is the Laplace operator of the metric s.

The equation $\breve{\overline{A}}_{AB} = 0$, where $\breve{\overline{A}}_{AB}$ denotes the $\tilde{g}$-trace free part of $\overline{A}_{AB}$, imposes the relations

$$(\mathring{\mathscr{D}}_A \mathring{\mathscr{D}}_B c)\breve{} = 0, \tag{2.40}$$

$$(\mathring{\mathscr{D}}_{(A} f_{B)})\breve{} = 0, \tag{2.41}$$

with $(\breve{\cdot})$ denoting here the trace-free part with respect to the metric s.

We wish, now, to relate the values of c and f_A to the values of the vector field $\overline{X}$ at the vertex, under the supplementary assumption that $\overline{X}$ is the restriction to C_O of a differentiable vector field defined in spacetime. Following [2], we denote by y^μ normal coordinates centered at O. Given the coordinates y^μ the coordinates x^α can be obtained by setting

$$x^0 = r - y^0, \quad x^1 = r, \quad x^A = \mu^A\left(\frac{y^i}{r}\right), \tag{2.42}$$

for some functions μ^A, so that the x^A form local coordinates on S^{n-1}, and

$$r := \left\{\sum_i (y^i)^2\right\}^{\frac{1}{2}}. \tag{2.43}$$

We underline the components of tensor fields in the y^α-coordinates, in particular

$$\underline{X}_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} X_\mu, \quad X_\alpha = \frac{\partial y^\mu}{\partial x^\alpha}\underline{X}_\mu, \quad \underline{X}^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} X^\mu, \quad X^\alpha = \frac{\partial x^\alpha}{\partial y^\mu}\underline{X}^\mu. \tag{2.44}$$

For vector fields such that $\underline{X}^\mu$ is continuous, we obtain

$$\overline{X}_1(0) = X_1(0) = \underline{X}_0(0) + \sum_i \underline{X}_i(0)\frac{y^i}{r} = -\underline{X}^0(0) + \sum_i \underline{X}^i(0)\frac{y^i}{r}. \tag{2.45}$$

Thus, for such vector fields, $\overline{X}_1(0)$ is a linear combination of $\ell = 0$ and $\ell = 1$ spherical harmonics, and contains the whole information about $\underline{X}^\alpha(0)$. We conclude that

$$c(x^A) = -\underline{X}^0(0) + \sum_i \underline{X}^i(0)\frac{y^i}{r}. \tag{2.46}$$

Equation (2.40) will be satisfied if and only if c is of the form (2.46), which can be seen by noting that (2.46) provides a family of solutions of (2.40) with the maximal possible dimension.

To determine f_A when $\underline{X}^\mu$ is differentiable at the origin we Taylor expand $\underline{X}$ there,

$$\underline{X}_\mu = \underline{X}_\mu(0) + y^j \underline{\partial_j X_\mu}(0) + y^0 \underline{\partial_0 X_\mu}(0) + O(|y|^2).$$

so that

$$\overline{X}_A = \frac{\partial y^i}{\partial x^A}\underline{X}_i = \frac{\partial y^i}{\partial x^A}\left(\underline{X}_i(0) + y^j \underline{\partial_j X_i}(0) + y^0 \underline{\partial_0 X_i}(0)\right) + O(r^2), \tag{2.47}$$

which determines f_A in terms of $\underline{\partial_\mu X_i}(0)$. Equation (2.41), which is the conformal Killing vector field equation on S^{n-1}, will be satisfied under the hypotheses of theorem 2.1 if and only if $\underline{\partial_i X_j}(0)$ is anti-symmetric.

2.4. A second intermediate result

As a next step toward the proof of theorem 1.1, we drop in theorem 2.1 the assumption of $\overline{Y}$ being the restriction of a smooth spacetime vector field:

Theorem 2.5. *Let $\overline{Y}$ be a vector field defined along C_O in a vacuum spacetime $(\mathscr{M}, g)$. There exists a smooth vector field X satisfying the Killing equations on $D^+(C_O)$ and coinciding with $\overline{Y}$ on C_O if and only if on C_O it holds that*

$$\mathrm{D}_i\overline{Y}_j + \mathrm{D}_j\overline{Y}_i = 0, \tag{2.48}$$

$$R_{011}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_1\mathrm{D}_1\overline{Y}_0 = 0, \tag{2.49}$$

$$R_{01A}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_A\mathrm{D}_1\overline{Y}_0 = 0, \tag{2.50}$$

$$g^{AB}(R_{0AB}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_A\mathrm{D}_B\overline{Y}_0) = 0. \tag{2.51}$$

Proof of theorem 2.5. We wish to apply theorem 2.1. The crucial step is to construct the vector field Y needed there. For further reference we note that (2.51) will not be needed for this construction.

In the argument that follows we shall ignore the distinction between $\overline{X}$ and $\overline{Y}$ whenever it does not matter.

By hypothesis it holds that

$$\overline{A}_{ij} = 0, \tag{2.52}$$

$$\overline{S}_{i10} = 0. \tag{2.53}$$

We define an anti-symmetric tensor $\overline{F}_{\mu\nu}$ via

$$\begin{aligned} \overline{F}_{ij} &:= \nabla_{[i}\overline{X}_{j]}, \\ -\overline{F}_{0i} = \overline{F}_{i0} &:= \nabla_i\overline{X}_0. \end{aligned}$$

Then

$$\begin{aligned} \overline{F}_{1i} &\equiv \tfrac{1}{2}\nabla_1\overline{X}_i - \tfrac{1}{2}\overline{\nabla_i X_1} \equiv \nabla_1\overline{X}_i - \tfrac{1}{2}\overline{A}_{1i} \\ &= \nabla_1\overline{X}_i. \end{aligned}$$

Moreover,

$$\begin{aligned} 2\nabla_1\overline{F}_{ij} &\equiv \nabla_1\nabla_i\overline{X}_j - \nabla_1\nabla_j\overline{X}_i \\ &\equiv \nabla_i\nabla_1\overline{X}_j - \nabla_j\nabla_1\overline{X}_i + \overline{R}_{1ij}{}^{\alpha}\overline{X}_{\alpha} + \overline{R}_{j1i}{}^{\alpha}\overline{X}_{\alpha} \\ &\equiv \nabla_i\overline{A}_{1j} - \nabla_j\overline{A}_{1i} - \overline{\nabla_i\nabla_j X_1} + \overline{\nabla_j\nabla_i X_1} - \overline{R}_{ij1}{}^{\alpha}\overline{X}_{\alpha} \\ &\equiv \nabla_i\overline{A}_{1j} - \nabla_j\overline{A}_{1i} - 2\overline{R}_{ij1}{}^{\alpha}\overline{X}_{\alpha}. \end{aligned}$$

With (2.52) that gives

$$\overline{\nabla_1 F_{ij}} = -\overline{R}_{ij1}{}^{\alpha}\overline{X}_{\alpha}.$$

Further,

$$\begin{aligned} \nabla_1\overline{F}_{i0} &\equiv \nabla_1\nabla_i\overline{X}_0 = \nabla_i\nabla_1\overline{X}_0 + \overline{R}_{1i0}{}^{\alpha}\overline{X}_{\alpha} = \overline{R}_{01i}{}^{\mu}\overline{X}_{\mu} + \overline{R}_{1i0}{}^{\alpha}\overline{X}_{\alpha} \\ &= -\overline{R}_{i01}{}^{\mu}\overline{X}_{\mu}. \end{aligned}$$

To sum it up, (2.52) and (2.53) imply that the equations

$$\nabla_1\overline{X}_{\mu} = \overline{F}_{1\mu}, \tag{2.54}$$

$$\nabla_1\overline{F}_{\mu\nu} = \overline{R}^{\alpha}{}_{1\mu\nu}\overline{X}_{\alpha} \tag{2.55}$$

hold on C_O,

Let $\mathring{X}^\mu = \overline{X}^\mu|_O$ and $\mathring{F}_{\mu\nu} = \overline{F}_{\mu\nu}|_O$ be the initial data at O needed for solving those equations. These data can be calculated as follows: (2.40) and (2.41) show that $\mathring{\mathscr{D}}_A c$ and f_A are conformal Killing fields on the standard sphere (S^{n-1}, s). It follows from [13, proposition 3.2] (a detailed exposition can be found in [12, proposition 2.5.1]) that c is a linear combination of the first two spherical harmonics, so that $\mathring{X}^\mu$ can be read off from c using (2.46). Similarly (2.47) can be used to read off $\mathring{F}_{\mu\nu}$ from f_A.

We conclude that, in coordinates adapted to C_O as in (2.42), under the hypotheses of theorem 2.5 the desired Killing vector X is a solution of the following problem:

$$\begin{cases} \nabla_1 \overline{X}_\mu = \overline{F}_{1\mu}, & \text{on } C_O; \\ \nabla_1 \overline{F}_{\alpha\beta} = R_{\gamma 1 \alpha\beta}\overline{X}^\gamma, & \text{on } C_O; \\ \overline{X}^\mu = \mathring{X}^\mu, & \text{at } O; \\ \overline{F}_{\mu\nu} = \mathring{F}_{\mu\nu}, & \text{at } O; \\ \Box X^\mu = -\lambda X^\mu, & \text{on } D^+(C_O); \\ X^\mu = \overline{X}^\mu, & \text{on } C_O. \end{cases} \tag{2.56}$$

Note that the first four equations above determine uniquely the initial data $\overline{X}^\mu$ on C_O needed to obtain a unique solution of the wave equation for X^μ.

Now, we claim that there exists a smooth vector field Y^μ defined near O so that $\overline{X}^\mu$ is the restriction of Y^μ to the light-cone. To see this, let $\mathring{\ell}^\mu$ be given and define $(x^\mu(s), Z^\mu(s), F_{\alpha\beta}(s))$ as the unique solution of the problem

$$\begin{cases} \dfrac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds} = 0, \\ \dfrac{dZ_\mu}{ds} - \Gamma^\alpha_{\mu\beta} Z_\alpha \dfrac{dx^\beta}{ds} = F_{\alpha\mu}\dfrac{dx^\alpha}{ds}, \\ \dfrac{dF_{\alpha\beta}}{ds} - \Gamma^\mu_{\alpha\gamma}\dfrac{dx^\gamma}{ds}F_{\mu\beta} - \Gamma^\mu_{\beta\gamma}\dfrac{dx^\gamma}{ds}F_{\mu\alpha} = R_{\gamma\delta\alpha\beta}Z^\gamma\dfrac{dx^\delta}{ds}, \\ x^\mu(0) = 0, \quad \dfrac{dx^\mu}{ds}(0) = \mathring{\ell}^\mu, \\ F_{\mu\nu}(0) = \mathring{F}_{\mu\nu}, \\ Z^\mu(0) = \mathring{X}^\mu. \end{cases} \tag{2.57}$$

For initial values such that $x^\mu(1)$ is defined, set

$$Y^\mu|_{x^\mu(1)} = Z^\mu|_{s=1}. \tag{2.58}$$

It follows from smooth dependence of solutions of ODEs upon initial data that Y^μ is smooth in all initial variables, in particular in $\mathring{\ell}^\mu$. If the x^μ's are normal coordinates centered at O, then $x^\mu(s=1) = \mathring{\ell}^\mu$, which implies that (2.58) defines a smooth vector field in a neighborhood of O. It then easily follows that the restriction of Y^μ to C_O equals $\overline{X}^\mu$, as defined by the first four equations in (2.56).

The hypotheses of theorem 2.1 are now satisfied, and theorem 2.5 is proved. □

2.5. *Proof of theorem 1.1*

To prove theorem 1.1 we will use theorem 2.1, together with some of the ideas of the proof of theorem 2.5. We need to show that (1.4) together with the Einstein equations imply both the existence of a smooth extension Y of $\overline{Y}$, and that (2.5)–(2.7) hold.

2.5.1. Properties of $S_{\mu\nu\sigma}$. Recall the definition

$$S_{\mu\nu\sigma} \equiv \nabla_\mu \nabla_\nu X_\sigma - R^\alpha{}_{\mu\nu\sigma} X_\alpha , \tag{2.59}$$

and lemma 2.3.

In the context of theorem 1.1, only those components of the tensor field $S_{\alpha\beta\gamma}$ which *do not* involve ∂_0-derivatives of X are *a priori* known. One easily checks:

Lemma 2.6. *The components*

$$\overline{S}_{ij\mu} \text{ with } (ij) \neq (AB) \tag{2.60}$$

of the restriction to C_O of $S_{\mu\nu\sigma}$ can be algebraically $\overline{X}_\sigma \equiv \overline{Y}_\sigma$, $D_i\overline{X}_\sigma \equiv D_i\overline{Y}_\sigma$ and $D_iD_j\overline{X}_\sigma \equiv D_iD_j\overline{Y}_\sigma$.

We wish, next, to calculate $\nabla^\alpha S_{\alpha\beta\gamma}$ and $\nabla^\gamma S_{\alpha\beta\gamma}$. This requires the knowledge of $\nabla_0 X_\mu$, of $\nabla_0\nabla_0 X_\mu$, and even of $\nabla_0\nabla_0\nabla_0 X_\mu$ in some equations. For this, let X be any extension of $\overline{X}$ from the light-cone to a punctured neighborhood $\mathscr{O} \setminus \{O\}$ of O, so that the transverse derivatives appearing in the following equations are defined. X is assumed to be smooth on its domain of definition, and we emphasize that we do not make any hypotheses on the behavior of the extension X as the tip O of the light-cone is approached. As will be seen, the transverse derivatives of X on C_O drop out from those final formulae which are relevant for us.

We will make use several times of

$$\nabla_\alpha R^\alpha{}_{\beta\gamma\delta} = 0,$$

which is a standard consequence of the second Bianchi identity when the Ricci tensor is proportional to the metric.

We start with $\nabla^\alpha S_{\alpha\beta\gamma}$. Two commutations of derivatives allow us to rewrite the first term in the divergence of $S_{\alpha\beta\gamma}$ over the first index as

$$\begin{aligned}\nabla^\alpha\nabla_\alpha\nabla_\beta X_\gamma &= \nabla^\alpha(\nabla_\beta\nabla_\alpha X_\gamma + R_\gamma{}^\sigma{}_{\alpha\beta} X_\sigma)\\ &= \nabla_\beta\nabla^\alpha\nabla_\alpha X_\gamma + R^{\alpha\sigma}{}_{\alpha\beta}\nabla_\sigma X_\gamma + 2R_\gamma{}^\sigma{}_{\alpha\beta}\nabla^\alpha X_\sigma\\ &= \nabla_\beta \Box X_\gamma + R^\sigma{}_\beta \nabla_\sigma X_\gamma + 2R_\gamma{}^\sigma{}_{\alpha\beta}\nabla^\alpha X_\sigma .\end{aligned} \tag{2.61}$$

Hence, since $R_{\alpha\beta} = \lambda g_{\alpha\beta}$, and using the first Bianchi identity in the second line

$$\begin{aligned}\nabla^\alpha S_{\alpha\beta\gamma} &= \nabla_\beta \Box X_\gamma + \lambda\nabla_\beta X_\gamma + 2R_\gamma{}^\sigma{}_{\alpha\beta}\nabla^\alpha X_\sigma - R^\sigma{}_{\alpha\beta\gamma}\nabla^\alpha X_\sigma\\ &= \nabla_\beta(\Box X_\gamma + \lambda X_\gamma) + R_\gamma{}^{\sigma\alpha}{}_\beta A_{\alpha\sigma} .\end{aligned} \tag{2.62}$$

Similarly,

$$\begin{aligned}\nabla^\alpha S_{\beta\gamma\alpha} &= \nabla^\alpha(\nabla_\beta\nabla_\gamma X_\alpha - R^\sigma{}_{\beta\gamma\alpha}X_\sigma)\\ &= \nabla_\beta\nabla^\alpha\nabla_\gamma X_\alpha + R_\gamma{}^{\sigma\alpha}{}_\beta\nabla_\sigma X_\alpha + R_\alpha{}^{\sigma\alpha}{}_\beta\nabla_\gamma X_\sigma - R^\sigma{}_{\beta\gamma\alpha}\nabla^\alpha X_\sigma\\ &= \nabla_\beta(\nabla_\gamma\nabla^\alpha X_\alpha + R^\sigma{}_\gamma X_\sigma) + R_\gamma{}^{\sigma\alpha}{}_\beta\nabla_\sigma X_\alpha + R^\sigma{}_\beta\nabla_\gamma X_\sigma - R^\sigma{}_{\beta\gamma\alpha}\nabla^\alpha X_\sigma\\ &= \tfrac{1}{2}\nabla_\beta\nabla_\gamma A^\alpha{}_\alpha + \lambda A_{\beta\gamma} .\end{aligned} \tag{2.63}$$

Now, on C_O and in coordinates adapted to the cone

$$\begin{aligned}\overline{\nabla^\alpha S_{\alpha\beta\gamma}} &= \nu^0(\overline{\nabla_0 S_{1\beta\gamma}} + \nabla_1\overline{S}_{0\beta\gamma}) + \overline{g}^{1A}(\nabla_1\overline{S}_{A\beta\gamma} + \nabla_A\overline{S}_{1\beta\gamma})\\ &\quad + \overline{g}^{11}\nabla_1\overline{S}_{1\beta\gamma} + \overline{g}^{AB}\nabla_A\overline{S}_{B\beta\gamma} ,\end{aligned} \tag{2.64}$$

while

$$\begin{aligned}\overline{\nabla^\alpha S_{\beta\gamma\alpha}} &= \nu^0(\overline{\nabla_0 S_{\beta\gamma 1}} + \nabla_1\overline{S}_{\beta\gamma 0}) + \overline{g}^{1B}(\nabla_1\overline{S}_{\beta\gamma B} + \nabla_B\overline{S}_{\beta\gamma 1})\\ &\quad + \overline{g}^{11}\nabla_1\overline{S}_{\beta\gamma 1} + \overline{g}^{BC}\nabla_B\overline{S}_{\beta\gamma C} .\end{aligned} \tag{2.65}$$

In order to handle undesirable terms such as $\nabla_0 S_{\beta\gamma 1}$ we write

$$\begin{aligned}\nabla_0\nabla_\alpha\nabla_\beta X_\gamma &= \nabla_\alpha\nabla_0\nabla_\beta X_\gamma + R_\beta{}^\sigma{}_{0\alpha}\nabla_\sigma X_\gamma + R_\gamma{}^\sigma{}_{0\alpha}\nabla_\beta X_\sigma\\ &= \nabla_\alpha(\nabla_\beta\nabla_0 X_\gamma + R_\gamma{}^\sigma{}_{0\beta}X_\sigma) + R_\beta{}^\sigma{}_{0\alpha}\nabla_\sigma X_\gamma + R_\gamma{}^\sigma{}_{0\alpha}\nabla_\beta X_\sigma\\ &= \nabla_\alpha(\nabla_\beta(A_{0\gamma} - \nabla_\gamma X_0) + R_\gamma{}^\sigma{}_{0\beta}X_\sigma)\\ &\quad + R_\beta{}^\sigma{}_{0\alpha}\nabla_\sigma X_\gamma + R_\gamma{}^\sigma{}_{0\alpha}\nabla_\beta X_\sigma.\end{aligned} \tag{2.66}$$

2.5.2. *Analysis of condition (2.5).*

Lemma 2.7. *Assume that* $\overline{A}_{1i} = 0$ *and* $\breve{\overline{A}}_{AB} = 0$. *Then, in vacuum,*

$$\left(\partial_r + \frac{2}{n-1}\tau - \kappa\right)\partial_r(\overline{g}^{AB}\overline{A}_{AB}) = 2\tau\nu^0\overline{S}_{110}. \tag{2.67}$$

Proof. By lemma 2.3 the vanishing of $\overline{A}_{1i}$ implies

(i) $\overline{S}_{111} = 0$,
(ii) $0 = \overline{S}_{11A}$, as well as all permutations thereof.

Consider (2.63) with $(\beta\gamma) = (11)$. Setting $a := \overline{g}^{AB}\overline{A}_{AB}$ we find

$$\begin{aligned}\overline{\nabla^\alpha S_{11\alpha}} &= \tfrac{1}{2}\nabla_1\nabla_1\overline{A}_\alpha{}^\alpha + \lambda\overline{A}_{11} = \nu^0\nabla_1\nabla_1\overline{A}_{01} + \tfrac{1}{2}\overline{g}^{AB}\nabla_1\nabla_1\overline{A}_{AB}\\ &= \nu^0\nabla_1\nabla_1\overline{A}_{01} + \tfrac{1}{2}(\partial_r - \kappa)\partial_r a.\end{aligned} \tag{2.68}$$

Due to lemma 2.3 we have

$$\begin{aligned}2\chi^{AB}\overline{S}_{1AB} &= \chi^{AB}\nabla_1\overline{A}_{AB} = \chi^{AB}(\partial_r\overline{A}_{AB} - 2\chi_A{}^C\overline{A}_{BC})\\ &= \frac{1}{n-1}\chi^{AB}[\partial_r(a\overline{g}_{AB}) - 2a\chi_{AB}]\\ &= \frac{1}{n-1}\tau\partial_r a.\end{aligned}$$

Using (2.65) with $(\beta\gamma) = (11)$, as well as the last equation, and employing again lemma 2.3 we obtain, on C_O,

$$\begin{aligned}\overline{\nabla^\alpha S_{11\alpha}} &= \nu^0(\overline{\nabla_0 S_{111}} + \nabla_1\overline{S}_{110}) + \overline{g}^{1B}(\nabla_1\overline{S}_{11B} + \nabla_B\overline{S}_{111})\\ &\quad + \overline{g}^{11}\nabla_1\overline{S}_{111} + \overline{g}^{BC}\nabla_B\overline{S}_{11C}\\ &= \nu^0(\overline{\nabla_0 S_{111}} + \nabla_1\overline{S}_{110}) + \overline{g}^{BC}\nabla_B\overline{S}_{11C}\\ &= \nu^0(\overline{\nabla_0 S_{111}} + \nabla_1\overline{S}_{110}) - 2\chi^{AB}\overline{S}_{1AB} + \tau\nu^0\overline{S}_{110}\\ &= \nu^0(\overline{\nabla_0 S_{111}} + \nabla_1\overline{S}_{110}) - \frac{1}{n-1}\tau\partial_r a + \tau\nu^0\overline{S}_{110}.\end{aligned} \tag{2.69}$$

Using (2.66) we find

$$\begin{aligned}\overline{\nabla_0 S_{111}} &= \overline{\nabla_0\nabla_1\nabla_1 X_1} = \nabla_1\nabla_1\overline{A}_{01} - \nabla_1\overline{S}_{110} + \overline{R}_{011}{}^\mu\overline{A}_{1\mu}\\ &= \nabla_1\nabla_1\overline{A}_{01} - \nabla_1\overline{S}_{110}.\end{aligned} \tag{2.70}$$

Equating (2.68) with (2.69), and using the last equation we end up with (2.67). □

Corollary 2.8. *In a region where the divergence* τ *does not vanish (in particular, near the vertex),* $\overline{A}_{ij} = 0$ *implies, in vacuum,* $\overline{S}_{110} = 0$.

2.5.3. Analysis of condition (2.6).

Lemma 2.9. *Assume that $\overline{A}_{ij} = 0$ and $\overline{S}_{110} = 0$. Then, in vacuum, $\overline{S}_{A10} = 0$.*

Proof. From (2.66) we obtain

$$\nabla_0 S_{A11} = \nabla_0 \nabla_A \nabla_1 X_1 = \nabla_A \nabla_1 A_{01} - \nabla_A S_{110} + R_1{}^\sigma{}_{0A} A_{\sigma 1}.$$

This allows us to rewrite (2.65) with $(\beta\gamma) = (A1)$ on C_O as

$$\begin{aligned}\overline{\nabla^\alpha S_{A1\alpha}} = \nu^0(\nabla_A\nabla_1\overline{A}_{01} - \nabla_A\overline{S}_{110} + \overline{R}_1{}^\sigma{}_{0A}\overline{A}_{\sigma 1} + \nabla_1\overline{S}_{A10})\\ +\overline{g}^{1B}(\nabla_1\overline{S}_{A1B} + \nabla_B\overline{S}_{A11}) + \overline{g}^{11}\nabla_1\overline{S}_{A11} + \overline{g}^{BC}\nabla_B\overline{S}_{A1C}.\end{aligned} \tag{2.71}$$

Combining with (2.63), which reads with $(\beta\gamma) = (A1)$

$$\nabla^\alpha S_{A1\alpha} = \tfrac{1}{2}\nabla_A\nabla_1 A^\alpha{}_\alpha + \lambda A_{1A},$$

we obtain on the initial surface

$$\begin{aligned}-\nu^0\nabla_1\overline{S}_{A10} = \nu^0(\nabla_A\nabla_1\overline{A}_{01} - \nabla_A\overline{S}_{110} + \overline{R}_1{}^j{}_{0A}\overline{A}_{j1}) + \overline{g}^{1B}(\nabla_1\overline{S}_{A1B} + \nabla_B\overline{S}_{A11})\\ +\overline{g}^{11}\nabla_1\overline{S}_{A11} + \overline{g}^{BC}\nabla_B\overline{S}_{A1C} - \tfrac{1}{2}\nabla_A\nabla_1\overline{A}^\alpha{}_\alpha - \lambda\overline{A}_{1A}.\end{aligned} \tag{2.72}$$

Assuming $\overline{A}_{ij} = 0$, lemma 2.3 shows that $\overline{S}_{i11} = 0 = \overline{S}_{1i1} = \overline{S}_{11i} = \overline{S}_{A1B}$. This allows us to rewrite the right-hand side of (2.72) as

$$\begin{aligned}\text{rhs} &= \nu^0\nabla_A\nabla_1\overline{A}_{01} - \nu^0\nabla_A\overline{S}_{110} + \overline{g}^{BC}\nabla_B\overline{S}_{A1C} + \overline{g}^{1B}(\nabla_1\overline{S}_{A1B} + \nabla_B\overline{S}_{A11}) - \tfrac{1}{2}\nabla_A\nabla_1\overline{A}^\alpha{}_\alpha\\ &= -\nu^0\nabla_A\overline{S}_{110} + \overline{g}^{BC}\nabla_B\overline{S}_{A1C} - \tfrac{1}{2}\overline{g}^{BC}\nabla_A\nabla_1\overline{A}_{BC} - \overline{g}^{1B}(\nabla_B\overline{S}_{A11} - \nabla_A\nabla_1\overline{A}_{1B}).\end{aligned}$$

Now, using in addition that $\overline{S}_{110} = 0$,

$$\begin{aligned}\nabla_B\overline{S}_{A11} - \nabla_A\nabla_1\overline{A}_{1B} &= \tfrac{1}{2}\overline{\nabla_A(\nabla_B A_{11} - 2\nabla_1 A_{1B})}\\ &= -\nabla_A\overline{S}_{11B} = -\nu^0\chi_{AB}\overline{S}_{110} = 0.\end{aligned}$$

Hence,

$$\begin{aligned}\text{rhs} &= -\nu^0\nabla_A\overline{S}_{110} + \overline{g}^{BC}\nabla_B\overline{S}_{A1C} - \tfrac{1}{2}\overline{g}^{BC}\nabla_A\nabla_1\overline{A}_{BC}\\ &= -\nu^0\nabla_A\overline{S}_{110} + \overline{g}^{BC}(\nabla_B\overline{S}_{A1C} - \nabla_A\overline{S}_{B1C})\\ &= -\nu^0\nabla_A\overline{S}_{110} - \overline{g}^{BC}(\underbrace{2\chi_{[B}{}^D\overline{S}_{A]DC}}_{=0\text{ by lemma 2.3}} - \nu^0\chi_{BC}\overline{S}_{1A0} + \nu^0\chi_{AC}\overline{S}_{1B0})\\ &= -\nu^0\nabla_A\overline{S}_{110} + \tau\nu^0\overline{S}_{1A0} - \nu^0\chi_A{}^B\overline{S}_{1B0}\\ &= \tau\nu^0\overline{S}_{A10} + \nu^0\chi_A{}^B\overline{S}_{B10},\end{aligned}$$

and thus, again due to $\overline{S}_{110} = 0$ and lemma 2.3,

$$(\partial_r + \tau - \nu^0\partial_r\nu_0)\overline{S}_{A10} = 0. \tag{2.73}$$

But zero is the only solution of this equation which is $o(r^{-(n-1)})$, and to be able to conclude that

$$\overline{S}_{A10} = 0 \tag{2.74}$$

we need to check the behavior of $\overline{S}_{A10}$ at the vertex. For definiteness we assume a coordinate system as in [2, section 4.5]. Now, by definition,

$$\begin{aligned}\overline{S}_{A10} &= \overline{\nabla_A\nabla_1 X_0 - R_{\mu A10}X^\mu}\\ &= \overline{\partial_A(\partial_r X_0 - \Gamma^\mu_{10}X_\mu) - \Gamma^\mu_{A1}\nabla_\mu X_0 - \Gamma^\mu_{A0}\nabla_1 X_\mu - R_{\mu A10}X^\mu}.\end{aligned}$$

From (2.37)–(2.39) we find

$$\overline{X}_1,\ \partial_i\overline{X}_1 = O(1), \quad \overline{X}_0,\ \partial_i\overline{X}_0,\ \partial_A\partial_r\overline{X}_0 = O(1), \tag{2.75}$$

$$\overline{X}_A,\ \partial_B\overline{X}_A = O(r),\ \ \partial_r\overline{X}_A = O(1). \tag{2.76}$$

Using the formulae from [2, appendix A] one obtains

$$\overline{S}_{A10} = -\underbrace{\overline{\Gamma^0_{A1}}}_{=0}\,\overline{\nabla_0 X_0} + O(r^{-1}) = O(r^{-1}),$$

which implies that (2.74) holds, and lemma 2.9 is proved. □

2.5.4. Proof of theorem 1.1. We are ready now to prove our main result.

Proof of theorem 1.1. By assumption, using obvious notation, $\overline{A}^{(\overline{Y})}_{ij} = 0$. When τ does not vanish corollary 2.8 applies and shows that $\overline{S}^{(\overline{Y})}_{110} = 0$. Otherwise, $\overline{S}^{(\overline{Y})}_{110} = 0$ holds by hypothesis and lemma 2.9 shows that $\overline{S}^{(\overline{Y})}_{A10}$ vanishes as well. In the proof of theorem 2.5 we have shown that $\overline{A}^{(\overline{Y})}_{ij} = 0$ and $\overline{S}^{(\overline{Y})}_{i10} = 0$ suffice to make sure that $\overline{Y}$ is the restriction to C_O of a smooth spacetime vector field Y. Then, due to the Cagnac–Dossa theorem [7, théorème 2], there exists a smooth vector field X with $\overline{X} = \overline{Y}$ which solves $\Box X = -\lambda X$. The assertions of theorem 1.1 follow now from theorem 2.1, whose remaining hypotheses are satisfied by lemma 2.11 below. □

2.5.5. Analysis of condition (2.7). A straightforward application of lemma 2.3 yields

Lemma 2.10. *Assume that* $\overline{A}_{i\mu} = 0$. *Then*

(i) $\overline{S}_{ijk} = 0$,
(ii) $\overline{S}_{110} = \overline{S}_{101} = \overline{S}_{011} = 0$,
(iii) $\overline{S}_{A10} = \overline{S}_{A01} = \overline{S}_{1A0} = \overline{S}_{0A1} = \overline{S}_{10A} = \overline{S}_{01A} = 0$.

Lemma 2.11. *Consider a smooth vector field* X *in a vacuum spacetime* $(\mathscr{M}, g)$ *which satisfies* $\overline{A}_{i\mu} = 0$ *on* C_O *and* $\Box X + \lambda X = 0$. *Then*

$$\tilde{S} := \overline{g}^{AB}\overline{S}_{AB0} - \tfrac{1}{2}\tau\nu^0\overline{A}_{00} = \overline{g}^{AB}(\mathrm{D}_A\mathrm{D}_B\overline{X}_0 - \overline{R}_{0AB}{}^{\mu}\overline{X}_\mu) = 0. \tag{2.77}$$

Proof. Equation (2.62) yields with $\overline{A}_{i\mu} = 0$, $\Box X + \lambda X = 0$ and in vacuum

$$\overline{g^{AB}\nabla^\alpha S_{\alpha AB}} = -(\nu^0)^2\overline{R}_{11}\overline{A}_{00} = -\lambda(\nu^0)^2\underbrace{\overline{g}_{11}}_{=0}\overline{A}_{00} = 0.$$

On the other hand, (2.64) gives with lemma 2.3 and 2.10 on C_O,

$$\begin{aligned}\overline{g}^{AB}\overline{\nabla^\alpha S_{\alpha AB}} &= \nu^0\overline{g}^{AB}(\overline{\nabla_0 S_{1AB}} + \nabla_1\overline{S}_{0AB}) + \overline{g}^{1C}\overline{g}^{AB}(\nabla_1\overline{S}_{CAB} + \nabla_C\overline{S}_{1AB})\\ &\quad + \overline{g}^{11}\overline{g}^{AB}\nabla_1\overline{S}_{1AB} + \overline{g}^{AB}\overline{g}^{CD}\nabla_C\overline{S}_{DAB}\\ &= \nu^0\overline{g}^{AB}(\overline{\nabla_0 S_{1AB}} + \nabla_1\overline{S}_{0AB}) + \overline{g}^{AB}\overline{g}^{CD}\nabla_C\overline{S}_{DAB}\\ &= \nu^0\overline{g}^{AB}(\overline{\nabla_0 S_{1AB}} + \nabla_1\overline{S}_{0AB}) + \nu^0\overline{g}^{AB}(\tau\overline{S}_{0AB} + 2\chi_A{}^D\overline{S}_{D(0B)})\\ &= \nu^0\overline{g}^{AB}(\overline{\nabla_0 S_{1AB}} - \nabla_1\overline{S}_{AB0} + \nabla_1\nabla_A\overline{A}_{0B})\\ &\quad - \tau\nu^0\overline{g}^{AB}\overline{S}_{AB0} + (\nu^0)^2(\tau^2 + |\chi|^2)\overline{A}_{00}.\end{aligned}$$

Moreover, from (2.66) and lemma 2.3 we deduce that, on C_O,

$$\begin{aligned}\overline{g}^{AB}\overline{\nabla_0 S_{1AB}} &= \overline{g}^{AB}\overline{\nabla_0\nabla_1\nabla_A X_B}\\ &= \overline{g}^{AB}\overline{\nabla_1\nabla_A A_{0B}} - \overline{g}^{AB}\nabla_1\overline{S}_{AB0} + \overline{g}^{AB}\overline{R}_{01A}{}^{\mu}\overline{A}_{B\mu}\\ &= \overline{g}^{AB}\overline{\nabla_1\nabla_A A_{0B}} - \overline{g}^{AB}\nabla_1\overline{S}_{AB0}.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= 2\overline{g}^{AB}\nabla_1\nabla_A\overline{A}_{0B} - 2\overline{g}^{AB}\nabla_1\overline{S}_{AB0} - \tau\overline{g}^{AB}\overline{S}_{AB0} + \nu^0(\tau^2+|\chi|^2)\overline{A}_{00}\\ &= 2\overline{g}^{AB}\nabla_A\nabla_1\overline{A}_{0B} - 2\nu^0\overline{R}_{11}\overline{A}_{00} - 2\overline{g}^{AB}\nabla_1\overline{S}_{AB0} - \tau\overline{g}^{AB}\overline{S}_{AB0}\\ &\quad + \nu^0(\tau^2+|\chi|^2)\overline{A}_{00}\\ &= -2\overline{g}^{AB}\nabla_1\overline{S}_{AB0} - \tau\overline{g}^{AB}\overline{S}_{AB0} + \nu^0(2\tau\nabla_1+\tau^2-|\chi|^2)\overline{A}_{00}\\ &= \tau\nu^0\left(\partial_r + \tfrac{1}{2}\tau - 2\overline{\Gamma}^0_{01}\right)\overline{A}_{00} - \nu^0(\partial_r\tau - \kappa\tau + |\chi|^2)\overline{A}_{00} - 2\left(\partial_r + \tfrac{1}{2}\tau - \overline{\Gamma}^0_{10}\right)\tilde{S}.\end{aligned}$$

Using the vacuum constraint [2] $0 = \lambda\overline{g}_{11} = \overline{R}_{11} = -\partial_r\tau + \kappa\tau - |\chi|^2$, we obtain

$$0 = -2\left(\partial_r + \tfrac{1}{2}\tau + \kappa - \nu^0\partial_r\nu_0\right)\tilde{S} - \tau\nu^0\left(\partial_r + \tfrac{1}{2}\tau + 2\kappa - 2\nu^0\partial_r\nu_0\right)\overline{A}_{00}.$$

We employ (2.30),

$$\tilde{S} = -\nu^0\left(\partial_r + \tfrac{1}{2}\tau + 2\kappa - 2\nu^0\partial_r\nu_0\right)\overline{A}_{00},$$

which holds since all the hypotheses of lemma 2.4 are fulfilled, to end up with

$$(\partial_r + \tau + \kappa - \nu^0\partial_r\nu_0)\tilde{S} = 0.$$

Regularity at O in coordinates as in [2, section 4.5] gives $\tilde{S} = O(r^{-1})$, which implies that $\tilde{S} = 0$ is the only possibility. □

2.6. Analysis of the KID equations in some special cases

2.6.1. KID equations. Theorem 1.1 shows that a vacuum spacetime emerging as solution of the characteristic initial value problem with data on a light-cone possesses a Killing field if and only if the conformal class $\gamma_{AB} = [\overline{g}_{AB}]$ of g_{AB}, which together with κ describes the free data on the light-cone, is such that, in the region where τ has no zeros, the KID *equations* $\overline{A}_{ij} = 0$ admit a non-trivial solution $\overline{Y}$. Written as equations for the vector field $\overline{Y}$, they read (we use the formulae from [2, appendix A])

$$(\partial_r - \kappa + \nu^0\partial_r\nu_0)\overline{Y}^0 = 0, \tag{2.78}$$

$$\partial_r\overline{Y}^A + (\tilde{\nabla}^A\nu_0 - \partial_r\overline{g}^{1A} + \kappa\nu^A + \nu_0\xi^A + \nu_0\tilde{\nabla}^A)\overline{Y}^0 = 0, \tag{2.79}$$

$$\tau\overline{Y}^1 + \tilde{\nabla}_A\overline{Y}^A - \tfrac{1}{2}\nu_0(\zeta + \tau\overline{g}^{11} + 2\overline{g}^{1A}\tilde{\nabla}_A)\overline{Y}^0 = 0, \tag{2.80}$$

$$(\tilde{\nabla}_{(A}\overline{Y}_{B)})\breve{} + \sigma_{AB}\overline{Y}^1 - \nu_0\big(\overline{g}^{11}\sigma_{AB} + \breve{\overline{\Gamma}}{}^1_{AB)}\big)\overline{Y}^0 = 0, \tag{2.81}$$

where $\sigma_A{}^B$ denotes the trace-free part of $\chi_A{}^B$, $\xi^A := \overline{g}^{AB}\xi_B$ and

$$\zeta := 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}, \tag{2.82}$$

$$\xi_A := -2\overline{\Gamma}^1_{1A}. \tag{2.83}$$

The analysis of these equations is identical to that of their covariant counterpart, already discussed in section 2.3. The first three equations, arising from $\overline{A}_{1i} = 0$ and $\overline{g}^{AB}\overline{A}_{AB} = 0$ determine a class of candidate fields (depending on the integration functions $c(x^A)$ and $f_A(x^B)$, with $\mathring{\mathscr{D}}_A c$ and f_A being conformal Killing fields on (S^{n-1}, s). Note that it is crucial for the expansion τ to be non-vanishing in order for $\overline{g}^{AB}\overline{A}_{AB} = 0$ to provide an algebraic equation for $\overline{Y}^1$. Regardless of whether τ has zeros or not, we can determine $\overline{Y}^1$ by integrating radially (1.5), compare remark 3.2 below.

2.6.2. Killing vector fields tangent to spheres. Let us consider the special case where the spacetime admits a Killing field X with the property that $\overline{X}^0 = \overline{X}^1 = 0$ on C_O. The KID equations for the candidate field $\overline{Y}$ (2.78)–(2.81) then reduce to

$$\begin{aligned}\partial_r Y^A &= 0,\\ \tilde{\nabla}_{(A}Y_{B)} &= 0,\end{aligned}$$

which leads us to the following corollary of theorem 1.1.

Corollary 2.12. *Consider initial data $\bar{g}_{AB}(r, x^C)\, \mathrm{d}x^A\, \mathrm{d}x^B$ for the vacuum Einstein equations (cf, e.g., [5]) on a light-cone C_O. In the resulting vacuum spacetime there exists a Killing field X with $\overline{X}^0 = \overline{X}^1 = 0$ on C_O defined on a neighborhood of the vertex O if and only if the family of Riemannian manifolds*

$$(S^{n-1}, g_{AB}(r, \cdot)\, \mathrm{d}x^A\, \mathrm{d}x^B)$$

admits an r-independent Killing field $f^A = f^A(x^B)$.

2.6.3. Killing vector fields tangent to the light-cone. Let us now restrict attention to those Killing fields which are tangent to the cone C_O, i.e. we assume

$$\overline{X}^0 = 0. \tag{2.84}$$

We start by noting that in the coordinates of (2.42) we have

$$\underline{X}_\mu = \omega_{\mu\nu}y^\nu + O(|y|^2),$$

for an anti-symmetric matrix $\omega_{\mu\nu}$. Hence, quite generally,

$$X^0 = \frac{\partial x^0}{\partial y^\mu}\underline{X}^\mu = -\underline{X}^0 + \frac{y^i}{r}\underline{X}^i = \omega_{0i}y^i + \frac{y^i}{r}(\omega_{ij}y^j - \omega_{0i}y^0) + O(|y|^2) = O(|y|^2), \tag{2.85}$$

$$X^1 = \frac{\partial r}{\partial y^\mu}\underline{X}^\mu = \omega_{i0}\frac{y^i}{r}y^0 + O(|y|^2). \tag{2.86}$$

Thus (2.84) does not impose any restrictions on $\omega_{\mu\nu}$, and we have

$$\overline{X}^1 = \omega_{i0}y^i + O(r^2). \tag{2.87}$$

Next, under (2.84) the KID equations (2.78)–(2.81) for the candidate field $\overline{Y}$ become

$$\partial_r\overline{Y}^A = 0, \tag{2.88}$$

$$\tau\overline{Y}^1 + \tilde{\nabla}_A\overline{Y}^A = 0, \tag{2.89}$$

$$(\tilde{\nabla}_{(A}\overline{Y}_{B)})\breve{} + \sigma_{AB}\overline{Y}^1 = 0, \tag{2.90}$$

or, equivalently (note that $\partial_r\tilde{\Gamma}^B_{AB} = \partial_A\tau$)

$$\overline{Y}^A = f^A(x^B), \tag{2.91}$$

$$\partial_r(\tau\overline{Y}^1) + f^A\partial_A\tau = 0, \tag{2.92}$$

$$\tilde{\nabla}_{(A}f_{B)} + \chi_{AB}\overline{Y}^1 = 0, \tag{2.93}$$

where we have set $f_A := \bar{g}_{AB}f^B$. Equations (2.91)–(2.93) provide thus a relatively simple form of the necessary-and-sufficient conditions for existence of Killing vectors tangent to C_O.

If we choose a gauge where $\tau = (n-1)/r$ (cf e.g. [5]), the last three equations become

$$\overline{Y}^A = f^A(x^B), \tag{2.94}$$

$$\overline{Y}^1 = -\frac{r}{n-1}\tilde{\nabla}_A f^A = -\frac{r}{n-1}\mathring{\mathscr{D}}_A f^A, \tag{2.95}$$

$$(\tilde{\nabla}_{(A} f_{B)})\breve{} = \sigma_{AB}\frac{r}{n-1}\mathring{\mathscr{D}}_C f^C. \tag{2.96}$$

Note that there are no non-trivial Killing vectors tangent to all generators of the cone, $\overline{Y}^A = 0$, as (2.95) gives then $\overline{Y}^1 = 0$. This should be contrasted with a similar question for intersecting null hypersurfaces, see section 3.3.

3. Two intersecting null hypersurfaces

3.1. An intermediate result

In analogy with the light-cone case let us first prove an intermediate result.

Theorem 3.1. *Consider two smooth null hypersurfaces $N_1 = \{x^1 = 0\}$ and $N_2 = \{x^2 = 0\}$ in an $(n+1)$-dimensional vacuum spacetime $(\mathscr{M}, g)$, with transverse intersection along a smooth submanifold S. Let $\overline{Y}$ be a vector field defined on $N_1 \cup N_2$. There exists a smooth vector field X satisfying the Killing equations on $D^+(N_1 \cup N_2)$ and coinciding with $\overline{Y}$ on $N_1 \cup N_2$ if and only if on N_1 it holds that*

$$\mathrm{D}_2\overline{Y}_2 = 0, \tag{3.1}$$

$$\mathrm{D}_{(2}\overline{Y}_{A)} = 0, \tag{3.2}$$

$$\mathrm{D}_{(A}\overline{Y}_{B)} = 0, \tag{3.3}$$

$$R_{122}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_2\mathrm{D}_2\overline{Y}_1 = 0, \tag{3.4}$$

$$R_{12A}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_A\mathrm{D}_2\overline{Y}_1 = 0, \tag{3.5}$$

$$g^{AB}(R_{1AB}{}^{\mu}\overline{Y}_{\mu} - \mathrm{D}_A\mathrm{D}_B\overline{Y}_1) = 0, \tag{3.6}$$

where D *is the analogue on N_1 of the derivative operator (1.2)–(1.3); similarly on N_2; while on S one needs further to assume that*

$$(\mathrm{D}_1 Y_2 + \mathrm{D}_2 Y_1)|_S = 0. \tag{3.7}$$

Proof. The proof is essentially identical to the proof of theorem 2.1. The candidate field is constructed as a solution of the wave equation (2.10); the delicate question of regularity of $\overline{Y}$ needed at the vertex in the cone case does not arise. Existence of the solution in $J^+(N_1 \cup N_2)$ follows from [11].

The main difference is that one cannot invoke regularity at the vertex to deduce the vanishing of, say on N_2, $A_{2\mu}|_{N_2}$ from the equations which correspond to (2.21), (2.25) and (2.30). Instead, one needs further to require (3.7) as well as

$$(\nabla_2 Y_A + \nabla_A Y_2)|_S = 0 \quad \text{and} \quad \nabla_2 Y_2|_S = 0.$$

However, the last two conditions follow from (3.1) and (3.2) on N_1. □

3.2. Proof of theorem 1.2

We prove now our main result for transversally intersecting null hypersurfaces.

Proof of theorem 1.2. We want to show that (1.6)–(1.13) imply that all the remaining assumptions of theorem 3.1, namely (3.1)–(3.7), are satisfied. The conditions (3.1), (3.2), (3.4) and (3.7) follow trivially.

Lemma 2.7, adapted to the intersecting null hypersurfaces-setting, tells us that $g^{AB}\nabla_{(A}Y_{B)}$ vanishes on $N_1 \cup N_2$ due to (1.11) and (1.12). Hence (3.3) is fulfilled.

The analogue of lemma 2.9 for two intersecting null hypersurfaces requires, in addition to (1.13), the vanishing of

$$\mathrm{D}_A \mathrm{D}_{(1}\overline{Y}_{2)}|_S = 0. \tag{3.8}$$

Both (1.13) and (3.8) together imply vanishing initial data for the analogue of the ODE (2.73) in the current setting. Equation (3.8) follows from (1.7), the corresponding equation on N_2, and (1.10). Thus (3.5) is fulfilled.

A straigthforward adaptation of lemma 2.11 to the current setting shows that, say on N_2, $g^{AB}S_{AB2} - \frac{1}{2}\tau g^{12}A_{22}$ vanishes, supposing that it vanishes on S. Using lemma 2.3 (i) we find

$$\begin{aligned}\left(g^{AB}S_{AB2} - \tfrac{1}{2}\tau g^{12}A_{22}\right)|_S &= g^{AB}\left(\nabla_{(A}A_{B)2} - \tfrac{1}{2}\nabla_2 A_{AB}\right) - \tfrac{1}{2}\tau g^{12}A_{22}\\ &= 0,\end{aligned}$$

because of (1.6)–(1.12). Hence also (3.6) is fulfilled. □

Remark 3.2. While τ has no zeros near the tip of a light-cone, for two transversally intersecting null hypersurfaces the expansion τ may vanish even near the intersection. In that case the trace of (3.3) on, say, N_1 will fail to provide an algebraic equation for $\overline{X}^2$. Also, corollary 2.8 cannot be applied to deduce the vanishing of $\overline{S}_{221}$, equivalently, the validity of (3.4), in the regions where τ vanishes. Instead one can use the second-order ODE (3.4) to find a candidate for $\overline{X}^2$, and then lemma 2.7 guarantees that the trace of the left-hand side of (3.3) vanishes when $g^{AB}A_{AB}|_S = 0 = \partial_2(g^{AB}A_{AB})|_S$.

3.3. Bifurcate horizons

A key notion to the understanding of the geometry of stationary black holes is that of a *bifurcation surface*. This is a smooth submanifold S of co-dimension two on which a Killing vector X vanishes, with S forming a transverse intersection of two smooth null hypersurfaces so that X is tangent to the generators of each. In our context this would correspond to a KID which vanishes on S, and is tangent to the null generators of the two characteristic hypersurfaces emanating normally from S. In coordinates adapted to one of the null hypersurfaces, so that the hypersurface is given by the equation $x^1 = 0$, we have $\overline{X} = \overline{X}^2\partial_2$, and $\overline{X}^\flat = \overline{g}_{12}\overline{X}^2\mathrm{d}x^1$. Then (2.84) holds, and therefore also (2.88)–(2.90) (which correspond to (3.2) and (3.3)). Equations (2.89)–(2.90) show that this is only possible if $\tau = \sigma_{AB} = 0$, which implies that translations along the generators of the light-cone are isometries of the $(n-1)$-dimensional metric $\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$. Equivalently, N_1 and N_2 have vanishing null second fundamental forms, which provides a necessary condition for a bifurcate horizon.

Assuming vacuum (as everywhere else in this work), this condition turns out to be sufficient. Let ζ_A be the torsion one-form of S (see, e.g., [3], or (B.4) below). We can use theorem 2.1 to prove (compare [8, proposition B.1] in dimension 3 + 1 and [9, end of section 2] in higher dimensions):

Theorem 3.3. *Within the setup of theorem 2.1, suppose that the null second fundamental forms of the hypersurfaces N_a, $a = 1, 2$, vanish. Then:*

(i) There exists a Killing vector field X defined on $D^+(N_1 \cup N_2)$ which vanishes on S and is null on $N_1 \cup N_2$.
(ii) Furthermore, any Killing vector $\hat{Y} = \hat{Y}^A\partial_A$ of the metric induced by g on S extends to a Killing vector X of g on $D^+(N_1 \cup N_2)$ if and only if the $\hat{Y}$-Lie derivative of the torsion one-form of S is exact.

Remarks 3.4.

(1) Killing vectors as above would exist to the past of S if the past-directed null hypersurfaces emanating from S also had vanishing second null fundamental forms. However, this does not need to be the case, a vacuum example is provided by suitable Robinson– Trautman spacetimes.
(2) Concerning point 2. of the theorem, when the $\hat{Y}$-Lie derivative of ζ is merely closed the argument of the proof below provides one-parameter families of Killing vectors defined on domains of dependence $D^+(\mathscr{O})$ of simply connected subsets $\mathscr{O}$ of S. It would be of interest to find out whether or not the resulting locally defined Killing vectors can be patched together to a global one when S is not simply connected.

Proof.

(1) In coordinates adapted to the null hypersurfaces the condition that a Killing vector X is tangent to the generators is equivalent to

$$X^\mu|_S = 0, \quad X^1|_{N_1} = X^2|_{N_2} = 0, \quad X^A|_{N_1\cup N_2} = 0, \tag{3.9}$$

$$\Longleftrightarrow X_\mu|_S = 0, \quad X_2|_{N_1} = X_1|_{N_2} = 0, \quad X_A|_{N_1\cup N_2} = 0. \tag{3.10}$$

For simplicity we assume that the generators of the two null hypersurfaces are affinely parameterized, i.e. $\kappa_{N_1} = \kappa_{N_2} = 0$. By hypothesis we have

$$\tau_{N_1} = \tau_{N_2} = \sigma^{N_1}_{AB} = \sigma^{N_2}_{AB} = 0. \tag{3.11}$$

The KID equations (1.6)–(1.13) for the candidate field $\overline{Y}$ reduce to

$$\partial_2\partial_2\overline{Y}_1 - 2\Gamma^1_{12}\partial_2\overline{Y}_1 + \big((\Gamma^1_{12})^2 - \partial_2\Gamma^1_{12}\big)\overline{Y}_1|_{N_1} = 0, \tag{3.12}$$

$$\partial_1\partial_1\overline{Y}_2 - 2\Gamma^2_{12}\partial_1\overline{Y}_2 + \big((\Gamma^2_{12})^2 - \partial_1\Gamma^2_{12}\big)\overline{Y}_2|_{N_2} = 0, \tag{3.13}$$

$$(\partial_1\overline{Y}_2 + \partial_2\overline{Y}_1)|_S = 0, \tag{3.14}$$

$$\partial_A(\partial_1\overline{Y}_2 - \partial_2\overline{Y}_1)|_S = 0. \tag{3.15}$$

Since $\overline{Y}_\mu|_S = 0$ we need non-trivial initial data $\partial_2\overline{Y}_1|_S$ and $\partial_1\overline{Y}_2|_S$ for the ODEs (3.12) and (3.13) for $\partial_2\overline{Y}_1|_{N_1}$ and $\partial_1\overline{Y}_2|_{N_2}$, respectively.
Using the formulae in [2, appendix A], (3.12)–(3.15) can be rewritten as

$$\partial_2\partial_2\overline{Y}^2|_{N_1} = 0, \tag{3.16}$$

$$\partial_1\partial_1\overline{Y}^1|_{N_2} = 0, \tag{3.17}$$

$$(\partial_1\overline{Y}^1 + \partial_2\overline{Y}^2)|_S = 0, \tag{3.18}$$

$$\partial_A(\partial_1\overline{Y}^1 - \partial_2\overline{Y}^2)|_S = 0. \tag{3.19}$$

Hence there remains the freedom to prescribe a constant $c \neq 0$ for $\frac{1}{2}(\partial_1 \overline{Y}^1 - \partial_2 \overline{Y}^2)|_S$. (The constant c reflects the freedom of scaling the Killing vector field by a constant, and is related to the surface gravity of the horizon; we will return to this shortly.) By (3.18) one needs to choose $\partial_1 \overline{Y}^1|_S = c$ and $\partial_2 \overline{Y}^2|_S = -c$. Together with $\overline{Y}^1|_S = \overline{Y}^2|_S = 0$ the functions $\overline{Y}^2|_{N_1}$ and $\overline{Y}^1|_{N_2}$ are then determined by (3.16) and (3.17), and existence of the desired Killing vector follows from theorem 2.1.

(2) By assumption we have, using obvious notation,

$$\chi^{N_1}_{AB} = \chi^{N_2}_{AB} = 0 \quad \Longleftrightarrow \quad \partial_2 g_{AB}|_{N_1} = \partial_1 g_{AB}|_{N_2} = 0. \tag{3.20}$$

Now, the flow of a spacetime Killing vector field which is tangent to S preserves S. This implies that the bundle of null vectors normal to S is invariant under the flow. Equivalently, the image by the flow of a null geodesic normal to S will be a one-parameter family of null geodesics normal to S. This is possible only if the Killing vector field is tangent to both N_1 and N_2. It thus suffices to consider candidate Killing vector fields $\overline{Y}$ which satisfy, in our adapted coordinates,

$$\overline{Y}^1|_{N_1} = \overline{Y}^2|_{N_2} = 0 \quad \Longleftrightarrow \quad \overline{Y}_2|_{N_1} = \overline{Y}_1|_{N_2} = 0. \tag{3.21}$$

To continue, we need a simple form of the KID equations (1.6)–(1.13), assuming (3.20) and (3.21), and supposing again that the generators of the two null hypersurfaces are affinely parameterized, i.e. $\kappa_{N_1} = \kappa_{N_2} = 0$. Using the notation $\hat{Y} \equiv Y^A|_S \partial_A$ we find:

$$\partial_2 \overline{Y}^A|_{N_1} = 0, \quad \partial_1 \overline{Y}^A|_{N_2} = 0, \tag{3.22}$$

$$(\tilde{\nabla}_{(A} \overline{Y}_{B)})\breve{\ }|_{N_1 \cup N_2} = 0, \tag{3.23}$$

$$\partial_2 \partial_2 \overline{Y}^2|_{N_1} = 0, \quad \partial_1 \partial_1 \overline{Y}^1|_{N_2} = 0, \tag{3.24}$$

$$(\partial_1 \overline{Y}^1 + \partial_2 \overline{Y}^2 + g^{12} \mathscr{L}_{\hat{Y}} g_{12})|_S = 0, \tag{3.25}$$

$$\tilde{\nabla}_A \overline{Y}^A|_S = 0, \tag{3.26}$$

$$[\partial_A(\partial_1 \overline{Y}^1 - \partial_2 \overline{Y}^2) + 2\mathscr{L}_{\hat{Y}} \zeta_A]|_S = 0. \tag{3.27}$$

(The fields $g_{12}|_S$ and

$$\zeta_A = \tfrac{1}{2}(\Gamma^1_{1A} - \Gamma^2_{2A})|_S \tag{3.28}$$

are part of the free initial data on S [11]; compare [5].) The first-order equations above are straightforward; some details of the derivation of the remaining equations above will be given shortly. Before passing to that, we observe that (3.22)–(3.23) and (3.26), together with (3.20), are equivalent to the requirement that $\hat{Y}|_S$ is a Killing vector field of $(S, g_{AB}|_S)$, and that $\overline{Y}^A = \hat{Y}^A$ on $N_1 \cup N_2$, i.e. that $\overline{Y}^A$ is independent of the coordinates x^1 and x^2. Supposing further that $\mathscr{L}_{\hat{Y}} \zeta_A$ is exact, the remaining equations (3.24), (3.25) and (3.27) can be used to determine $\overline{Y}^1$ and $\overline{Y}^2$ on $N_1 \cup N_2$. (As such, on each connected component of S the difference $(\partial_1 \overline{Y}^1 - \partial_2 \overline{Y}^2)|_S$ is determined up to an additive constant by (3.27), which reflects the freedom of adding a Killing vector field which vanishes on S and is tangent to the null geodesics generating both initial surfaces.) The existence of a Killing vector field X on the spacetime which coincides with $\hat{Y}$ on S follows now from theorem 1.2.

In appendix B it is shown that the KID equations (3.22)–(3.27) are invariant under affine reparameterizations of generators. We also show there that the freedom to choose an affine parameter on N_1 and N_2 can be employed to prescribe g_{12} on S, and also to add arbitrary gradients to ζ. In particular exactness of ζ, and thereby solvability of the KID equations, is independent of the gauge, as one should expect.

Let us pass to some details of the derivation of the second-order equations above. The calculation uses extensively

$$0 = \Gamma^{\mu}_{22}|_{N_1} = \Gamma^{A}_{B2}|_{N_1},$$

similarly on N_2. First, let us compute $R_{122}{}^{A}|_{N_1}$ and $R_{211}{}^{A}|_{N_2}$, as needed to evaluate (1.9). Making use of [2, appendix A], we find

$$\begin{aligned}\partial_1\Gamma^{A}_{22}|_{N_1} &= g^{2A}\partial_2\partial_2 g_{12} + g^{AB}\partial_2(\partial_1 g_{2B} - \partial_B g_{12})\\ &= g^{2A}\partial_2\partial_2 g_{12} + \partial_2\big(g_{12}g^{AB}\xi^{N_1}_{B}\big) + g^{AB}\partial_2\partial_2 g_{1B},\\ \Gamma^{A}_{12}|_{N_1} &= \tfrac{1}{2}g_{12}g^{AB}\xi^{N_1}_{B} + g^{AB}\partial_2 g_{1B} + g^{2A}\partial_2 g_{12},\end{aligned}$$

and thus

$$R_{122}{}^{A}|_{N_1} = (\partial_2 + g^{12}\partial_2 g_{12})\Gamma^{A}_{12} - \partial_1\Gamma^{A}_{22} = \tfrac{1}{2}g_{12}g^{AB}\partial_2\xi^{N_1}_{B},$$

where, using the notation in [5], $\xi^{N_1}_{A} = -2\Gamma^{2}_{2A}|_{N_1}$ and $\xi^{N_2}_{A} = -2\Gamma^{1}_{1A}|_{N_2}$. Now for $\chi^{N_1}_{AB} = \chi^{N_2}_{AB} = 0$ the vacuum characteristic constraint equations [2] imply $\partial_2\xi^{N_1}_{A} = 0$ and $\partial_1\xi^{N_2}_{A} = 0$, hence

$$R_{122}{}^{A}|_{N_1} = R_{211}{}^{A}|_{N_2} = 0.$$

Further simple calculations lead to (3.24).

Next, on S, we consider the KID equation

$$0 = \nabla_A\nabla_1\overline{Y}^1 - R_{\mu A1}{}^{1}\overline{Y}^{\mu} = \nabla_A\nabla_1\overline{Y}^1 + R^{1}{}_{1BA}\overline{Y}^{B}. \tag{3.29}$$

Again on S it holds that

$$\begin{aligned}\nabla_1\overline{Y}^{\mu}\partial_{\mu} &= \partial_1\overline{Y}^1\partial_1 + \Gamma^{\mu}_{1A}\overline{Y}^{A}\partial_{\mu},\\ \nabla_{\mu}\overline{Y}^1 dx^{\mu} &= \partial_1\overline{Y}^1 dx^1 + \Gamma^{1}_{\mu A}\overline{Y}^{A}dx^{\mu},\\ \nabla_A\nabla_1\overline{Y}^1 &= \partial_A\big(\partial_1\overline{Y}^1 + \Gamma^{1}_{1B}\overline{Y}^{B}\big) + \Gamma^{1}_{A\mu}\nabla_1\overline{Y}^{\mu} - \Gamma^{\mu}_{A1}\nabla_{\mu}\overline{Y}^1\\ &= \partial_A\partial_1\overline{Y}^1 + \Gamma^{1}_{1B}\partial_A\overline{Y}^{B} + \underbrace{\big(\partial_A\Gamma^{1}_{1B} + \Gamma^{1}_{A\mu}\Gamma^{\mu}_{1B} - \Gamma^{\mu}_{A1}\Gamma^{1}_{\mu B}\big)}_{=\partial_B\Gamma^{1}_{1A} - R^{1}{}_{1BA}}\overline{Y}^{B}.\end{aligned}$$

Inserting into (3.29) gives the following equation on S, in coordinates adapted to N_2:

$$0 = \nabla_A\nabla_1\overline{Y}^1 - R_{\mu A1}{}^{1}\overline{Y}^{\mu} = \partial_A\partial_1\overline{Y}^1 + \Gamma^{1}_{1B}\partial_A\overline{Y}^{B} + \partial_B\Gamma^{1}_{1A}\overline{Y}^{B}. \tag{3.30}$$

The analogous formula in coordinates adapted to N_1 reads

$$0 = \nabla_A\nabla_2\overline{Y}^2 - R_{\mu A2}{}^{2}\overline{Y}^{\mu} = \partial_A\partial_2\overline{Y}^2 + \Gamma^{2}_{2B}\partial_A\overline{Y}^{B} + \partial_B\Gamma^{2}_{2A}\overline{Y}^{B}. \tag{3.31}$$

Subtracting we obtain (3.27).

From the discussion so far it should be clear that the conditions are necessary. This concludes the proof. □

As an example, suppose that $\hat{Y}$ is a Killing vector on S and that the torsion one-form is invariant under the flow of $\hat{Y}$. It follows from the equations above that we can reparameterize the initial data surfaces so that $g_{12} = 1$ on S, with the torsion remaining invariant in the new gauge. Then $\overline{Y}^1 = \overline{Y}^2 = 0$ and $\overline{Y}^A = \hat{Y}^A$ provides a solution of the KID equations on $N_1 \cup N_2$.

It is of interest to relate the constant c, arising in the paragraph following (3.19), to the surface gravity (which we denote by $\kappa_{\mathscr{H}}$ here); this will also prove in which sense the seemingly coordinate-dependent derivatives $\partial_1 X^1|_S = -\partial_2 X^2|_S$ are in fact geometric invariants. In the process we recover the well-known fact, that surface gravity is constant on bifurcate horizons. We have

$$\kappa^2_{\mathscr{H}} = -\tfrac{1}{2}(\nabla^{\mu}X^{\nu})(\nabla_{\mu}X_{\nu})|_{N_1\cup N_2} = -(\nabla_1 X^1)(\nabla_2 X^2)|_{N_1\cup N_2}. \tag{3.32}$$

On, say, N_1 we have due to (3.9) and (3.16)–(3.19)

$$\nabla_2 X^2|_{N_1} = \partial_2 X^2 = -c, \tag{3.33}$$

while $\nabla_1 X^1|_{N_1}$ can be computed from (2.10),

$$\Box X^1|_{N_1} = -R^1{}_\alpha X^\alpha = -g^{12} R_{22} X^2 = 0 \iff \partial_2 \nabla_1 X^1|_{N_1} = 0 \iff \nabla_1 X^1|_{N_1} = c,$$

where we used (3.9), the vanishing of χ_{AB}, and $\nabla_1 X^1|_S = c$. Hence

$$\kappa_{\mathscr{H}} = |c|. \tag{3.34}$$

Acknowledgment

This work was supported in part by the Austrian Science Fund (FWF): P 24170-N16.

Appendix A. Fuchsian ODEs

Since it appears difficult to find an adequate reference, we describe here the main property of first-order Fuchsian ODEs used in our work.

Consider a first order system of equations for a set of fields $\phi = (\phi^I)$, $I = 1, \dots, N$, of the form

$$r\partial_r \phi = A(r)\phi + F(r, \phi), \tag{A.1}$$

for some smooth map $F = (F^I)$ with $F(0, 0) = 0$, $\partial_\phi F(r, 0) = 0$, where $A(r)$ is a smooth map with values in $N \times N$ matrices. For our purposes it is sufficient to consider the case where

$$A(0) = \lambda \mathrm{Id},$$

where Id is the $N \times N$ identity matrix. It holds that *the only solution of (A.1) such that* $\lim_{r\to 0} r^{-\lambda}\phi(r) = 0$ is $\phi(r) = 0$ for all r.

Appendix B. Gauge-dependence of the torsion one-form

In this appendix we consider the question of gauge-independence in point (ii) of theorem 3.3. Indeed, even within the gauge conditions imposed so far, that x^1 and x^2 are affine parameters on the relevant characteristic surfaces, there is some gauge freedom left concerning the gravitational initial data. The point is that we can rescale the affine parameters x^2 on N_1 and x^1 on N_2,

$$x^2 \mapsto \check{x}^2 = e^{-f^+(x^B)} x^2, \quad x^1 \mapsto \check{x}^1 = e^{-f^-(x^B)} x^1, \tag{B.1}$$

with some functions $f^\pm$ defined on S. Under (B.1), the metric on N_1 becomes

$$\begin{aligned} g|_{N_1} &= \overline{g}_{11}(\mathrm{d}x^1)^2 + 2(\overline{g}_{12}\,\mathrm{d}x^2 + \overline{g}_{1A}\,\mathrm{d}x^A)\,\mathrm{d}x^1 + \overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B \\ &= e^{2f^-}\overline{g}_{11}(\mathrm{d}\check{x}^1)^2 + 2(e^{f^+}\overline{g}_{12}\,\mathrm{d}\check{x}^2 + (\overline{g}_{1A} + \check{x}^2 e^{f^+}\overline{g}_{12}\partial_A f^+)\,\mathrm{d}x^A)e^{f^-}\,\mathrm{d}\check{x}^1 \\ &= e^{2f^-}\overline{g}_{11}(\mathrm{d}\check{x}^1)^2 + 2(\underbrace{e^{(f^+ + f^-)}\overline{g}_{12}}_{\bar{\check{g}}_{12}}\,\mathrm{d}\check{x}^2 + \underbrace{e^{f^-}(\overline{g}_{1A} + x^2\overline{g}_{12}\partial_A f^+)}_{=:\bar{\check{g}}_{1A}}\,\mathrm{d}x^A)\,\mathrm{d}\check{x}^1 \\ &\quad + \overline{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \end{aligned} \tag{B.2}$$

with a similar formula on N_2. This leads to

$$\check{g}_{12}|_S = e^{(f^+ + f^-)} g_{12}|_S, \tag{B.3}$$

as well as, using [5, equation (2.12)],

$$\begin{aligned}\check\zeta_A &= \frac{1}{2}\check g^{12}\left(\frac{\partial \check g_{1A}}{\partial \check x^2}-\frac{\partial \check g_{2A}}{\partial \check x^1}\right)\Big|_S\\
&= \frac{1}{2}e^{-(f^+ + f^-)}g^{12}\left(e^{f^+}\frac{\partial(e^{f^-}(\overline{g}_{1A}+x^2\overline{g}_{12}\partial_A f^+))}{\partial x^2}-\frac{\partial \check g_{2A}}{\partial \check x^1}\right)\Big|_S\\
&= \zeta_A + \frac{1}{2}(\partial_A f^+ - \partial_A f^-)|_S. \end{aligned} \tag{B.4}$$

Letting $\check x^A = x^A$, in the new coordinates the Killing vector becomes

$$\begin{aligned} Y &= Y^\mu \partial_{x^\mu} = Y^\mu \frac{\partial \check x^\nu}{\partial x^\mu}\partial_{\check x^\nu}\\
&= Y^\mu \frac{\partial(e^{-f^-}x^1)}{\partial x^\mu}\partial_{\check x^1} + Y^\mu \frac{\partial(e^{-f^+}x^2)}{\partial x^\mu}\partial_{\check x^2} + Y^\mu \frac{\partial \check x^A}{\partial x^\mu}\partial_{\check x^A}\\
&= (e^{-f^-}Y^1 - \check x^1 Y^A \partial_A f^-)\partial_{\check x^1} + (e^{-f^+}Y^2 - \check x^2 Y^A\partial_A f^+)\partial_{\check x^2} + Y^A\partial_{\check x^A}. \end{aligned}\tag{B.5}$$

Invariance of (3.22)–(3.24) and (3.26) is clear. One can further check invariance of (3.25) (recall that $\hat Y \equiv Y^A\partial_A|_S$):

$$\begin{aligned}&(\partial_{\check x^1}\check Y^1 + \partial_{\check x^2}\check Y^2 + \check g^{12}\mathscr{L}_{\hat Y}\check g_{12})|_S\\
&\quad = (\partial_{\check x^1}(e^{-f^-}Y^1 - \check x^1 Y^A\partial_A f^-) + \partial_{\check x^2}\check Y^2 + e^{-(f^+ + f^-)}g^{12}\mathscr{L}_{\hat Y}(e^{f^+ + f^-}g_{12}))|_S\\
&\quad = (\partial_1 Y^1 + \partial_2 Y^2 - \hat Y^A\partial_A(f^+ + f^-) + \hat Y^A\partial_A(f^+ + f^-) + g^{12}\mathscr{L}_{\hat Y}g_{12})|_S\\
&\quad = (\partial_1 Y^1 + \partial_2 Y^2 + g^{12}\mathscr{L}_{\hat Y}g_{12})|_S. \end{aligned}\tag{B.6}$$

As such, on S the first two-terms in (3.27) transform as

$$\begin{aligned}\partial_{\check x^A}(\partial_{\check x^1}\check Y^1 - \partial_{\check x^2}\check Y^2) &= \frac{\partial x^\mu}{\partial \check x^A}\partial_\mu(\partial_{\check x^1}\check Y^1 - \partial_{\check x^2}\check Y^2)\\
&= \partial_A(\partial_1 Y^1 - \partial_2 Y^2 + \hat Y^B\partial_B(f^- - f^+)).\end{aligned}\tag{B.7}$$

Equation (B.4) can be rewritten as

$$\check\zeta = \zeta + \tfrac{1}{2}d(f^+ - f^-). \tag{B.8}$$

Thus

$$2\mathscr{L}_{\hat Y}\check\zeta = \mathscr{L}_{\hat Y}(2\zeta + d(f^+ - f^-)) = 2\mathscr{L}_{\hat Y}\zeta + d(\mathscr{L}_{\hat Y}(f^+ - f^-)),$$

which shows that the one-form

$$d(\partial_1 Y^1 - \partial_2 Y^2) + 2\mathscr{L}_{\hat Y}\zeta \tag{B.9}$$

is invariant under changes of the affine parameters, as desired.

We end this paper by deriving the behavior of $\zeta_A \equiv \frac{1}{2}(\Gamma^1_{1A} - \Gamma^2_{2A})|_S$ under arbitrary coordinate transformations which preserve the adapted null coordinates conditions,

$$\check x^1 = e^{-f^+(x^\mu)}x^1, \qquad \check x^2 = e^{-f^-(x^\mu)}x^2, \qquad \check x^A = x^A. \tag{B.10}$$

We set

$$f_0^\pm(x^A) := f^\pm(x^1 = 0, x^2 = 0, x^A).$$

Then

$$\begin{aligned}\left(\check\Gamma^1_{1A} - \check\Gamma^2_{2A}\right)|_S &= \Gamma^\sigma_{\mu\nu}\left(\frac{\partial \check x^1}{\partial x^\sigma}\frac{\partial x^\mu}{\partial \check x^1} - \frac{\partial \check x^2}{\partial x^\sigma}\frac{\partial x^\mu}{\partial \check x^2}\right)\frac{\partial x^\nu}{\partial \check x^A} + \frac{\partial \check x^1}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial \check x^1\partial \check x^A} - \frac{\partial \check x^2}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial \check x^2\partial \check x^A}\\
&= \Gamma^1_{1B}\frac{\partial \check x^1}{\partial x^1}\frac{\partial x^1}{\partial \check x^1}\frac{\partial x^B}{\partial \check x^A} - \Gamma^2_{2B}\frac{\partial \check x^2}{\partial x^2}\frac{\partial x^2}{\partial \check x^2}\frac{\partial x^B}{\partial \check x^A} + \frac{\partial \check x^1}{\partial x^1}\frac{\partial^2 x^1}{\partial \check x^1\partial \check x^A} - \frac{\partial \check x^2}{\partial x^2}\frac{\partial^2 x^2}{\partial \check x^2\partial \check x^A}\end{aligned}$$

$$= \Gamma^1_{1A} e^{-f_0^+} \frac{\partial x^1}{\partial \check{x}^1} - \Gamma^2_{2A} e^{-f_0^-} \frac{\partial x^2}{\partial \check{x}^2} + e^{-f_0^+} \partial_A \frac{\partial x^1}{\partial \check{x}^1} - e^{-f_0^-} \partial_A \frac{\partial x^2}{\partial \check{x}^2}$$

$$= \Gamma^1_{1A} - \Gamma^2_{2A} + \partial_A (f_0^+ - f_0^-),$$

i.e. (B.8) holds under coordinate transformations of the form (B.10).

If we assume S to be compact, there is a natural way to fix the gauge: according to the Hodge decomposition theorem ζ can be uniquely written as the sum of an exact one-form, a dual exact one-form and a harmonic one-form. The considerations above show that the first term has a pure gauge character and can be transformed away, while the remaining part has an intrinsic meaning. In particular, if ζ is exact, the remaining gauge freedom can be employed to transform it to zero.

References

[1] Beig R and Chruściel P T 1997 Killing initial data *Class. Quantum Grav.* **14** A83–A92

[2] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions *Ann. Henri Poincaré* **12** 419–82 (arXiv:1006.4467 [gr-qc])

[3] Christodoulou D and Klainermann S 1995 *On the Global Nonlinear Stability of Minkowski Space* (Princeton, NJ: Princeton University Press)

[4] Chruściel P T and Delay E 2003 On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications *Mém. Soc. Math. France* **94** 103 (arXiv:gr-qc/0301073v2)

[5] Chruściel P T and Paetz T-T 2012 The many ways of the characteristic Cauchy problem *Class. Quantum Grav.* **29** 145006 (arXiv:1203.4534 [gr-qc])

[6] Corvino J and Schoen R M 2006 On the asymptotics for the vacuum Einstein constraint equations *J. Differ. Geom.* **73** 185–217 (arXiv:gr-qc/0301071)

[7] Dossa M 1997 Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique *Ann. Inst. Henri Poincaré Phys. Théor.* **66** 37–107

[8] Friedrich H, Rácz I and Wald R M 1999 On the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon *Commun. Math. Phys.* **204** 691–707 (arXiv:gr-qc/9811021)

[9] Hollands S, Ishibashi A and Wald R M 2007 A higher dimensional stationary rotating black hole must be axisymmetric *Commun. Math. Phys.* **271** 699–722 (arXiv:gr-qc/0605106)

[10] Maerten D 2004 Killing initial data revisited *J. Math. Phys.* **45** 2594–9

[11] Rendall A D 1990 Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations *Proc. R. Soc.* A **427** 221–39

[12] Semmelmann U Conformal Killing forms on Riemannian manifolds *Habilitationsschrift* www.igt.uni-stuttgart.de/LstGeo/Semmelmann/Publikationen/killing20.pdf

[13] Semmelmann U 2003 Conformal Killing forms on Riemannian manifolds *Math. Z.* **245** 503–27

[14] Tilquin A and Schucker T 2012 Maximal symmetry at the speed of light arXiv:1210.1468[astro-ph.CO]

IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. **31** (2014) 085007 (29pp) doi:10.1088/0264-9381/31/8/085007

KIDs prefer special cones

Tim-Torben Paetz

Gravitational Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: Tim-Torben.Paetz@univie.ac.at

Received 14 November 2013, revised 17 February 2014
Accepted for publication 27 February 2014
Published 1 April 2014

Abstract
As complement to Chruściel and Paetz (2013 *Class. Quantum Grav.* **30** 235036) we analyse Killing Initial Data (KID) on characteristic Cauchy surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. As an application, we derive and discuss the KID equations on a light-cone with vertex at past timelike infinity.

Keywords: characteristic initial value problem, Killing initial data, light-cone with vertex at past timelike infinity, conformal field equations
PACS numbers: 04.20.Ex, 04.20.Ha

1. Introduction

Gaining a better insight into properties and peculiarities of space-times which represent (physically meaningful) solutions to Einstein's field equations belongs to the core of the analysis of general relativity. One question of interest concerns the existence of space-times which possess certain symmetry groups, mathematically expressed via a Lie algebra of Killing vector fields on that space-time. A fundamental issue in this context is to systematically construct such space-times in terms of an initial value problem. By that it is meant to supplement the usual constraint equations, which need to be satisfied by a suitably specified set of initial data, by some further equations which make sure that the emerging space-time contains one or several Killing vector fields. In vacuum, such Killing Initial Data (KIDs) are well-understood in the spacelike case as well as in the characteristic case, cf [1, 3, 10] and references therein. In this article we would like to complement the analysis of the characteristic case given in [3] to space-times satisfying Friedrich's conformal field equations, and in particular to analyse the case where the initial surface is a light-cone with vertex at past timelike infinity.

In a first step, section 3, we translate the Killing equation into the unphysical, conformally rescaled space-time. The so-obtained 'unphysical Killing equations' constitute the main focus of our subsequent analysis. Assuming the validity of the conformal field equations, recalled in section 2, we will derive necessary-and-sufficient conditions on a characteristic initial surface which guarantee the existence of a vector field satisfying the unphysical Killing equations, cf theorem 3.4.[1] In section 4 we then restrict attention to four space-time dimensions (it will be indicated that the higher dimensional case is more intricate). As in [3] we shall see that many of the hypotheses appearing in theorem 3.4 are automatically satisfied. The remaining 'KID equations' are collected in theorem 4.4 (cf proposition 4.9) for a light-cone, and in theorem 4.13 for two characteristic hypersurfaces intersecting transversally.

In section 5 we then apply theorem 4.4 to the 'special cone' C_{i^-} whose vertex is located at past timelike infinity (assuming the cosmological constant to be zero). As for 'ordinary cones' treated in [3] it turns out that some of the KID equations determine a class of candidate fields on the initial surface while the remaining 'reduced KID equations' provide restrictions on the initial data to make sure that one of these candidate fields does indeed extend to a space-time vector field satisfying the unphysical Killing equations. However, contrary to the 'ordinary case', and this explains our title, on C_{i^-} the candidate fields can be explicitly computed, and, besides, the reduced KID equations can be given in terms of explicitly known quantities. The main result for the C_{i^-}-cone is the contents of theorem 5.1.

Finally, in appendix A we recall a result on Fuchsian ODEs which will be of importance in the main part, in appendix B we review conformal Killing vector fields on the round 2-sphere.

2. Setting

Our analysis will be carried out in the so-called *unphysical space-time* $(\mathscr{M}, g, \Theta)$, related to the physical space-time $(\tilde{\mathscr{M}}, \tilde{g})$, $\tilde{g}$ being a solution to Einstein's field equations, via a conformal rescaling,

$$\tilde{g} \overset{\phi}{\mapsto} g := \Theta^2 \tilde{g}, \quad \tilde{\mathscr{M}} \overset{\phi}{\hookrightarrow} \mathscr{M}, \quad \Theta|_{\phi(\tilde{\mathscr{M}})} > 0.$$

The part of $\partial\phi(\tilde{\mathscr{M}})$ on which the conformal factor Θ vanishes represents 'infinity' in the physical space-time.

In $(\mathscr{M}, g, \Theta)$ Einstein's vacuum field equations with cosmological constant λ are replaced by *Friedrich's conformal field equations* (cf e.g. [7]), which read in $d \geqslant 4$ space-time dimensions

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0, \tag{2.1}$$

$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \Theta^{d-4} \nabla_\rho \Theta\, d_{\nu\mu\sigma}{}^\rho, \tag{2.2}$$

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu}, \tag{2.3}$$

$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta, \tag{2.4}$$

$$(d-1)(2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta) = \lambda, \tag{2.5}$$

$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta^{d-3} d_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}), \tag{2.6}$$

with $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ regarded as unknowns. The trace of (2.3) can be read as the definition of the function s,

$$s := \frac{1}{d} g^{\mu\nu} \nabla_\mu \nabla_\nu \Theta + \frac{1}{2d(d-1)} R\Theta. \tag{2.7}$$

[1] This issue has already been analysed in [8]. However, it is claimed there that regularity of the principal part of a wave equation suffices to guarantee uniqueness of solutions, and counter-examples of this assertion can be easily constructed. For instance, let Θ be the unique solution of the wave-equation $\Box_g \Theta = 1$ which vanishes on the initial surface which we assume to be a light-cone C_O, i.e. $\Theta|_{C_O} = 0$. Then $\Theta|_{I^+(O)} > 0$, at least sufficiently close to O. Consider the non-regular wave-equation $\Box_g f - \frac{1}{\Theta} f = 0$. For given initial data $f|_{C_O} = 0$ there exist at least three solutions: $f = 0, \pm\Theta$.

The tensor field $L_{\mu\nu}$ is the Schouten tensor

$$L_{\mu\nu} := \frac{1}{d-2}R_{\mu\nu} - \frac{1}{2(d-1)(d-2)}Rg_{\mu\nu}, \tag{2.8}$$

while

$$d_{\mu\nu\sigma}{}^{\rho} := \Theta^{3-d}C_{\mu\nu\sigma}{}^{\rho} \tag{2.9}$$

is a rescaling of the conformal Weyl tensor $C_{\mu\nu\sigma}{}^{\rho}$.

The conformal field equations are equivalent to the vacuum Einstein equations where Θ is positive, but remain regular even where Θ vanishes. The Ricci scalar R turns out to be a conformal gauge source function which reflects the freedom to choose the conformal factor Θ. It can be prescribed arbitrarily.

In (2.1)–(2.6) the fields s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^{\rho}$ are treated as independent of $g_{\mu\nu}$ and Θ. However, once a solution has been constructed they are related to them via (2.7)–(2.9). When talking about a solution of the conformal field equations we therefore just need to specify the pair $(g_{\mu\nu}, \Theta)$.

The conformal field equations imply a wave equation for the Schouten tensor, which will be of importance later on: One starts by taking the divergence of (2.2). Using then (2.1), (2.3), (2.6) and the tracelessness of the rescaled Weyl tensor one finds (cf [11] where the four-dimensional case is treated in detail)

$$\begin{aligned}\Box_g L_{\mu\nu} &= -2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}L_{\alpha\beta} + g_{\mu\nu}|L|^2 + \frac{1}{2(d-1)}\nabla_\mu\nabla_\nu R + \frac{1}{d-1}RL_{\mu\nu}\\ &\quad + (d-4)\big[L_{\mu}{}^{\alpha}L_{\nu\alpha} + \Theta^{d-5}\nabla_\alpha\Theta\nabla_\beta\Theta d_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\big],\end{aligned}$$

with $|L|^2 := L_{\mu}{}^{\nu}L_{\nu}{}^{\mu}$. Supposing that Θ has no zeros, or that we are in the four-dimensional case, we can use (2.2) to rewrite this as

$$\begin{aligned}\Box_g L_{\mu\nu} &= -2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}L_{\alpha\beta} + g_{\mu\nu}|L|^2 + \frac{1}{2(d-1)}\nabla_\mu\nabla_\nu R + \frac{1}{d-1}RL_{\mu\nu}\\ &\quad + (d-4)\big[L_{\mu}{}^{\alpha}L_{\nu\alpha} + 2\Theta^{-1}\nabla^{\alpha}\Theta\nabla_{[\alpha}L_{\mu]\nu}\big].\end{aligned} \tag{2.10}$$

3. KID equations in the unphysical space-time

3.1. The Killing equation in terms of a conformally rescaled metric

Lemma 3.1. *A vector field $\tilde{X}$ is a Killing vector field in the physical space-time $(\tilde{\mathscr{M}}, \tilde{g})$ if and only if its push-forward $X := \phi_*\tilde{X}$ is a conformal Killing vector field in the unphysical space-time $(\mathscr{M}, g, \Theta)$ and satisfies there the equation $X^\kappa\nabla_\kappa\Theta = \frac{1}{d}\Theta\nabla_\kappa X^\kappa$.*

Proof. By definition $\tilde{X}$ is a Killing field if and only if (set $\tilde{X}_\mu := \tilde{g}_{\mu\nu}\tilde{X}^\nu$ and $X_\mu := g_{\mu\nu}X^\nu$)

$$\begin{aligned}&\tilde{\nabla}_{(\mu}\tilde{X}_{\nu)} = 0\\ \Longleftrightarrow\ &\tilde{\nabla}_{(\mu}(\Theta^{-2}X_{\nu)}) = 0\\ \Longleftrightarrow\ &\nabla_{(\mu}(\Theta^{-2}X_{\nu)}) + 2\Theta^{-2}X_{(\mu}\nabla_{\nu)}\log\Theta = g_{\mu\nu}\Theta^{-2}X_\kappa\nabla^\kappa\log\Theta\\ \Longleftrightarrow\ &\nabla_{(\mu}X_{\nu)} = g_{\mu\nu}\Theta^{-1}X_\kappa\nabla^\kappa\Theta\\ \Longleftrightarrow\ &\nabla_{(\mu}X_{\nu)} = \frac{1}{d}\nabla_\kappa X^\kappa g_{\mu\nu} \quad \& \quad X^\kappa\nabla_\kappa\Theta = \frac{1}{d}\Theta\nabla_\kappa X^\kappa\end{aligned} \tag{3.1}$$

(note that $\Theta|_{\phi(\tilde{\mathscr{M}})} > 0$). □

Remark 3.2. The conditions (3.1), which replace the Killing equation in the unphysical space-time, make sense also where Θ is vanishing, supposing that g can be smoothly extended across $\{\Theta = 0\}$ (note that the conformal Killing equation induces a linear symmetric hyperbolic system of propagation equations for $\phi_*\tilde{X}$ which then implies that $\phi_*\tilde{X}$ is smoothly extendable across the conformal boundary [6]).

Remark 3.3. We will refer to (3.1) as the *unphysical Killing equations*.

The main object of this work is to extract necessary-and-sufficient conditions on a characteristic initial surface which ensure the existence of some vector field X which fulfils the unphysical Killing equations, so that its pull-back is a Killing vector field of the physical space-time.

3.2. Necessary conditions for the existence of Killing vector fields

Let us first derive some implications of the unphysical Killing equations (3.1) under the hypothesis that the conformal field equations (2.1)–(2.6) are satisfied.

From the conformal Killing equation we first derive a system of wave equations for X and for the function

$$Y := \frac{1}{d}\nabla_\kappa X^\kappa \tag{3.2}$$

(set $\Box_g := g^{\mu\nu}\nabla_\mu\nabla_\nu$):

$$\Box_g X_\mu + R_\mu{}^\nu X_\nu + (d-2)\nabla_\mu Y = 0, \tag{3.3}$$

$$\Box_g Y + \frac{1}{d-1}\left(\frac{1}{2}X^\mu\nabla_\mu R + RY\right) = 0. \tag{3.4}$$

With (3.1) and (2.7) we find

$$\begin{aligned} 0 &= \Box_g(X^\mu\nabla_\mu\Theta - \Theta Y) \\ &\equiv \Box_g X^\mu\nabla_\mu\Theta + X^\mu\nabla_\mu\Box_g\Theta + X^\mu R_\mu{}^\nu\nabla_\nu\Theta + 2\nabla_\nu X_\mu\nabla^\mu\nabla^\nu\Theta \\ &\quad - Y\Box_g\Theta - \Theta\Box_g Y - 2\nabla_\mu\Theta\nabla^\mu Y \\ &= d(X^\mu\nabla_\mu s + sY - \nabla_\mu\Theta\nabla^\mu Y). \end{aligned} \tag{3.5}$$

We set

$$A_{\mu\nu} := 2\nabla_{(\mu}X_{\nu)} - 2Yg_{\mu\nu}. \tag{3.6}$$

Using the second Bianchi identity, (2.8), (3.3) and (3.4) we obtain

$$\begin{aligned} \Box_g A_{\mu\nu} &\equiv 2\nabla_{(\mu}\Box_g X_{\nu)} + 2R_{\kappa(\mu}\nabla^\kappa X_{\nu)} - 2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} - 4R_{\mu\nu}Y \\ &\quad + 2X^\kappa\nabla_{(\mu}R_{\nu)\kappa} - 2X^\kappa\nabla_\kappa R_{\mu\nu} - 2\Box_g Y g_{\mu\nu} \\ &= 2R_{(\mu}{}^\kappa A_{\nu)\kappa} - 2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} - 2(d-2)(\mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y). \end{aligned} \tag{3.7}$$

(Recall that

$$\mathscr{L}_X L_{\mu\nu} \equiv X^\kappa\nabla_\kappa L_{\mu\nu} + 2L_{\kappa(\mu}\nabla_{\nu)}X^\kappa.\ )$$

Hence the conformal Killing equation for X, which is $A_{\mu\nu} = 0$, implies

$$B_{\mu\nu} := \mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y = 0. \tag{3.8}$$

3.3. KID equations on a characteristic initial surface

3.3.1. First main result. We are now in a position to formulate our first main result. Here and in the following we use an overbar to denote restriction to the initial surface.

Theorem 3.4. *Assume we have been given, in dimension $d \geqslant 4$, an'unphysical' space-time $(\mathscr{M}, g, \Theta)$, with (g, Θ) a smooth solution of the conformal field equations (2.1)–(2.6). Assume further that Θ is bounded away from zero if $d \geqslant 5$. Consider some characteristic initial surface $N \subset \mathscr{M}$ (for definiteness we think of a light-cone or two transversally intersecting null hypersurfaces). Then there exists a vector field $\hat{X}$ satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(N)$ (i.e. representing a Killing field of the physical space-time) if and only if there exists a vector field X and a function Y which fulfil the following equations (recall the definitions (2.7) and (2.8) for s and $L_{\mu\nu}$, respectively)*

(i) $\Box_g X_\mu + R_\mu{}^\nu X_\nu + (d-2)\nabla_\mu Y = 0$,
(ii) $\Box_g Y + \frac{1}{d-1}\big(\frac{1}{2}X^\mu\nabla_\mu R + RY\big) = 0$,
(iii) $\overline{\phi} = 0$ *with* $\phi = X^\mu\nabla_\mu\Theta - \Theta Y$,
(iv) $\overline{\psi} = 0$ *with* $\psi = X^\mu\nabla_\mu s + sY - \nabla_\mu\Theta\nabla^\mu Y$,
(v) $\overline{A}_{\mu\nu} = 0$ *with* $A_{\mu\nu} = 2\nabla_{(\mu}X_{\nu)} - 2Yg_{\mu\nu}$,
(vi) $\breve{\overline{B}}_{\mu\nu} := \overline{B}_{\mu\nu} - \frac{1}{d}\overline{g}_{\mu\nu}\overline{B}_\alpha{}^\alpha = 0$ *with* $B_{\mu\nu} = \mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y$.

Proof. '$\Longrightarrow$': Follows from the considerations above if one takes $X = \hat{X}$ and $Y = \frac{1}{d}\nabla_\kappa\hat{X}^\kappa$.

'$\Longleftarrow$': We will derive a homogeneous system of wave equations from which we conclude the vanishing of $A_{\mu\nu}$ and ϕ as well as the relation $Y = \frac{1}{d}\nabla_\kappa X^\kappa$, which imply that X satisfies (3.1). Since by assumption (3.3) and (3.4) hold we can repeat the steps which led us to (3.7),

$$\Box_g A_{\mu\nu} = 2R_{(\mu}{}^\kappa A_{\nu)\kappa} - 2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} - 2(d-2)B_{\mu\nu}. \tag{3.9}$$

With (i), (ii) and the definition (2.7) of s we find

$$\begin{aligned}\Box_g\phi &\equiv \Box_g X_\mu\nabla^\mu\Theta + X^\mu\nabla_\mu\Box_g\Theta + X^\mu R_\mu{}^\nu\nabla_\nu\Theta + 2\nabla_\mu X_\nu\nabla^\mu\nabla^\nu\Theta \\ &\quad - Y\Box_g\Theta - \Theta\Box_g Y - 2\nabla_\mu\Theta\nabla^\mu Y \\ &= d\psi - \frac{1}{2(d-1)}R\phi + A_{\mu\nu}\nabla^\mu\nabla^\nu\Theta.\end{aligned} \tag{3.10}$$

We use (i), (ii), (2.7), (2.8) and the conformal field equations (2.3) & (2.4) (which imply $\Box_g s = \Theta|L|^2 - \frac{1}{2(d-1)}(sR + \nabla^\mu\Theta\nabla_\mu R)$) to obtain

$$\begin{aligned}\Box_g\psi &\equiv \Box_g X_\mu\nabla^\mu s + X^\mu\nabla_\mu\Box_g s + X^\mu R_\mu{}^\nu\nabla_\nu s + A_{\mu\nu}\nabla^\mu\nabla^\nu s \\ &\quad + 3Y\Box_g s + s\Box_g Y + 2\nabla_\nu s\nabla^\nu Y - \nabla_\mu\Box_g\Theta\nabla^\mu Y - 2R_\mu{}^\nu\nabla_\nu\Theta\nabla^\mu Y \\ &\quad - \nabla^\mu\Theta\nabla_\mu\Box_g Y - 2\nabla_\mu\nabla_\nu\Theta\nabla^\mu\nabla^\nu Y \\ &= |L|^2\phi + A_{\mu\nu}(\nabla^\mu\nabla^\nu s - 2\Theta L_\kappa{}^\mu L^{\nu\kappa}) + 2\Theta L^{\mu\nu}B_{\mu\nu} \\ &\quad + \frac{1}{2(d-1)}\big(A_{\mu\nu}\nabla^\mu R\nabla^\nu\Theta - \nabla^\mu R\nabla_\mu\phi - R\psi\big).\end{aligned} \tag{3.11}$$

As an immediate consequence of the first Bianchi identity we observe the identity

$$\tfrac{1}{2}\nabla_\mu A_{\nu\kappa} + \nabla_{[\nu}A_{\kappa]\mu} \equiv \nabla_\mu\nabla_\nu X_\kappa + R_{\nu\kappa\mu}{}^\alpha X_\alpha - 2\nabla_{(\mu}Yg_{\nu)\kappa} + \nabla_\kappa Yg_{\mu\nu}. \tag{3.12}$$

Another useful relation which follows from the Bianchi identities is

$$\begin{aligned}2L^{\alpha\beta}(\nabla_\beta\nabla_{[\alpha}A_{\nu]\mu} - \nabla_\mu\nabla_{[\alpha}A_{\nu]\beta}) &= 2L^{\alpha\beta}X^\kappa\nabla_\kappa R_{\alpha(\mu\nu)\beta} \\ &\quad + L^{\alpha\beta}(R_{\beta\nu\mu}{}^\kappa A_{\alpha\kappa} - R_{\mu\beta\alpha}{}^\kappa A_{\nu\kappa}) + 4L^{\alpha\beta}(R_{\alpha(\mu\nu)}{}^\kappa\nabla_{[\beta}X_{\kappa]} + R_{(\mu|\beta\alpha|}{}^\kappa\nabla_{\nu)}X_\kappa) \\ &\quad - 2g_{\mu\nu}L^{\alpha\beta}\nabla_\alpha\nabla_\beta Y + 4L_{(\mu}{}^\beta\nabla_{\nu)}\nabla_\beta Y - \frac{1}{d-1}R\nabla_\mu\nabla_\nu Y\,.\end{aligned} \tag{3.13}$$

Employing (i), (ii), the conformal field equations, the wave equation for the Schouten tensor (2.10) as well as the identities (3.12) and (3.13) a tedious computation reveals that

$$\begin{aligned}\Box_g B_{\mu\nu} \equiv{}& 2(g_{\mu\nu}L^{\alpha\beta} - R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta})B_{\alpha\beta} - 2R_{(\mu}{}^{\kappa}B_{\nu)\kappa} + \frac{2}{d-1}RB_{\mu\nu}\\ &+ 2L^{\alpha\beta}(\nabla_\beta\nabla_{[\alpha}A_{\nu]\mu} - \nabla_\mu\nabla_{[\alpha}A_{\nu]\beta})\\ &+ (\nabla_{(\mu}A_{|\alpha\beta|} + 2\nabla_{[\alpha}A_{\beta](\mu})(2\nabla^{\alpha}L_{\nu)}{}^{\beta} - \frac{1}{4(d-1)}\delta_{\nu)}{}^{\alpha}\nabla^{\beta}R)\\ &+ A^{\alpha\beta}[\nabla_\alpha\nabla_\beta L_{\mu\nu} - 2L_{(\mu}{}^{\kappa}R_{\nu)\alpha\kappa\beta} + 2L_{\mu\alpha}R_{\nu\beta} + L_{\alpha}{}^{\kappa}(2R_{\mu\beta\nu\kappa} + R_{\nu\beta\mu\kappa})\\ &- 2g_{\mu\nu}L_{\alpha\kappa}L_{\beta}{}^{\kappa}] + |L|^2A_{\mu\nu} + L^{\alpha\beta}R_{\mu\alpha\beta}{}^{\kappa}A_{\nu\kappa} - \frac{1}{d-1}RL_{(\mu}{}^{\kappa}A_{\nu)\kappa}\\ &+ (d-4)[2L_{(\mu}{}^{\beta}B_{\nu)\beta} - L_{\mu}{}^{\alpha}L_{\nu}{}^{\beta}A_{\alpha\beta} - \Theta^{d-5}\nabla_\alpha\Theta d_{\kappa\mu\nu}{}^{\alpha}(\nabla^{\kappa}\phi - \nabla_\beta\Theta A^{\kappa\beta})]\\ &- (d-4)\Theta^{d-6}\phi\nabla_\alpha\Theta\nabla_\beta\Theta d_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\\ &+ (d-4)\Theta^{-1}\nabla^{\alpha}\Theta[2\nabla_{[\alpha}B_{\mu]\nu} + (\nabla_{[\mu}A_{|\nu\kappa|} + 2\nabla_{[\nu}A_{\kappa][\mu})L_{\alpha]}{}^{\kappa}].\end{aligned} \tag{3.14}$$

While the before-last line contains negative powers of Θ merely in five dimensions, the last line contains such powers in any dimension $d \geqslant 5$. This is the point where our assumption enters that Θ is bounded away from zero for $d \geqslant 5$, since this ensures that (3.14) is a regular equation also in higher dimensions.

Note that the right-hand side of (3.14) involves second-order derivatives of $A_{\mu\nu}$, which is why we regard $\nabla_\sigma A_{\mu\nu}$ as another unknown for which we derive a wave equation. However, since the right-hand side of (3.9) does not involve derivatives, such a wave equation is easily obtained by differentiation (and, once again, the second Bianchi identity),

$$\begin{aligned}\Box_g\nabla_\sigma A_{\mu\nu} ={}& 2\nabla_\sigma(R_{(\mu}{}^{\kappa}A_{\nu)\kappa} - R_{\mu}{}^{\alpha}{}_{\nu}{}^{\kappa}A_{\alpha\kappa}) + 2A_{\alpha(\mu}(\nabla_{\nu)}R_{\sigma}{}^{\alpha} - \nabla^{\alpha}R_{\nu)\sigma})\\ &- 4R_{\sigma\kappa(\mu}{}^{\alpha}\nabla^{\kappa}A_{\nu)\alpha} + R_{\alpha\sigma}\nabla^{\alpha}A_{\mu\nu} - 2(d-2)\nabla_\sigma B_{\mu\nu}.\end{aligned} \tag{3.15}$$

In the current setting the equations (3.9)–(3.11), (3.14) and (3.15) form a closed homogeneous system of regular wave equations for $A_{\mu\nu}$, ϕ, ψ, $B_{\mu\nu}$ and $\nabla_\sigma A_{\mu\nu}$. The assumptions (iii)–(v) assure that the first three fields vanish initially. By (ii) and (v) we have

$$\overline{B}_{\alpha}{}^{\alpha} = \frac{1}{2(d-1)}\overline{X^{\mu}\nabla_{\mu}R} + 2\overline{L^{\mu\nu}\nabla_{\mu}X_{\nu}} + \overline{\Box_g Y} = \overline{L}^{\mu\nu}\overline{A}_{\mu\nu} = 0,$$

which, together with (vi), implies

$$\overline{B}_{\mu\nu} = 0. \tag{3.16}$$

It remains to verify the vanishing of $\overline{\nabla_\sigma A_{\mu\nu}}$. This follows from lemma 3.5 below, together with (i), (ii), (v) and (3.16). We thus have vanishing initial data for the homogeneous system of wave equations (3.9)–(3.11), (3.14) and (3.15). It follows from [5] in the light-cone-case and from [12] in the case of two characteristic hypersurfaces intersecting transversally that there exists a unique solution, whence all the fields involved need to vanish identically.

It is important to note that we have treated X and Y as independent so far. The vanishing of $A_{\mu\nu}$ and ϕ implies that the unphysical Killing equations (3.1) hold for X only once we have shown that $Y = \frac{1}{d}\nabla_\kappa X^\kappa$. Fortunately we have

$$0 = A_{\alpha}{}^{\alpha} = 2\nabla_\kappa X^\kappa - 2dY, \tag{3.17}$$

and the theorem is proved. □

3.3.2. Adapted null coordinates. Before we state and prove lemma 3.5, which is needed to complete our proof of theorem 3.4, it is useful, also with regard to later purposes, to introduce *adapted null coordinates* on light-cones and on transversally intersecting null hypersurfaces. We will be rather sketchy here, the details can be found e.g. in [2, 12].

First we consider a light-cone $C_O \subset \mathscr{M}$ with vertex $O \in \mathscr{M}$ in a d-dimensional space-time $(\mathscr{M}, g)$. We use coordinates $(x^0 = 0, x^1 = r, x^A)$, $A = 2, \ldots, d-1$, adapted to C_O in the sense that $C_O \setminus \{O\} = \{x^0 = 0\}$, r parameterizes the null geodesics generating the cone, and the x^A's are local coordinates on the level sets $\{x^0 = 0, r = \text{const}\} \cong S^{d-2}$. On C_O the metric then reads

$$\overline{g} = \overline{g}_{00}(\mathrm{d}x^0)^2 + 2\nu_0 \mathrm{d}x^0\,\mathrm{d}x^1 + 2\nu_A \mathrm{d}x^0\,\mathrm{d}x^A + \overline{g}_{AB}\mathrm{d}x^A\,\mathrm{d}x^B.$$

Note that these coordinates are singular at the vertex of the cone. Moreover, we stress that we do not impose any gauge condition off the cone. The inverse metric takes the form

$$\overline{g}^\sharp = 2\nu^0 \partial_0\partial_1 + \overline{g}^{11}\partial_1^2 + 2\overline{g}^{1A}\partial_1\partial_A + \overline{g}^{AB}\partial_A\partial_B,$$

with

$$\nu^0 = (\nu_0)^{-1}, \quad \overline{g}^{1A} = -\nu^0\overline{g}^{AB}\nu_B, \quad \overline{g}^{11} = (\nu^0)^2(\overline{g}^{AB}\nu_A\nu_B - \overline{g}_{00}).$$

It is customary to introduce the following quantities:

$$\chi_A{}^B := \tfrac{1}{2}\overline{g}^{BC}\partial_1\overline{g}_{AC} \quad \textit{null second fundamental form},$$
$$\tau := \chi_A{}^A \quad \textit{expansion},$$
$$\sigma_A{}^B := \chi_A{}^B - \frac{\tau}{d-2}\delta_A{}^B \quad \textit{shear tensor}.$$

Next, let us consider two smooth hypersurfaces N_a, $a = 1, 2$, with transverse intersection along a smooth submanifold S. Then, near the N_a's one can introduce coordinates (x^1, x^2, x^A), $A = 3, \ldots, d$, such that $N_a = \{x^a = 0\}$. On N_1 the coordinate x^2 parameterizes the null geodesics $\{x^1 = 0, x^A = \text{const}^A\}$ generating N_1 and vice versa. Since the hypersurfaces are required to be characteristic the metric takes there the specific form, on N_1 say,

$$g|_{N_1} = \overline{g}_{11}(\mathrm{d}x^1)^2 + 2\overline{g}_{12}\mathrm{d}x^1\,\mathrm{d}x^2 + 2\overline{g}_{1A}\mathrm{d}x^1\,\mathrm{d}x^A + \overline{g}_{AB}\mathrm{d}x^A\,\mathrm{d}x^B,$$

similarly on N_2. The quantities τ, $\sigma_A{}^B$, $\chi_A{}^B$ are defined on N_1 and N_2 analogous to the light-cone-case.

3.3.3. Some useful relations. In this section we consider a light-cone. However, we note that exactly the same relations hold in the case of two intersecting null hypersurfaces.

Recall that the wave equations for X and Y, (3.3)–(3.4), imply the wave equation (3.9) for $A_{\mu\nu}$. One straightforwardly verifies that in adapted null coordinates

$$\begin{aligned}\overline{\Box_g A_{\mu\nu}} = {}& 2\nu^0(\overline{\nabla_1\nabla_0 A_{\mu\nu}} + \overline{R}_{01(\mu}{}^\alpha\overline{A}_{\nu)\alpha}) + \overline{g}^{11}\overline{\nabla_1\nabla_1 A_{\mu\nu}} \\ & + 2\overline{g}^{1A}(\overline{\nabla_1\nabla_A A_{\mu\nu}} + \overline{R}_{A1(\mu}{}^\alpha\overline{A}_{\nu)\alpha}) + \overline{g}^{AB}\overline{\nabla_A\nabla_B A_{\mu\nu}}.\end{aligned} \tag{3.18}$$

We equate the trace of (3.9) on the initial surface with (3.18). Making use of the formulae for the Christoffel symbols in adapted null coordinates in [2, appendix A], an elementary calculation yields the following set of equations where f, f_A and f_{AB} denote generic (multi-linear) functions which vanish whenever their arguments vanish:

$\underline{(\mu\nu) = (11)}$:

$$\left(\partial_1 + \frac{\tau}{2} - \overline{\Gamma}^0_{01} - 2\overline{\Gamma}^1_{11}\right)\overline{\nabla_0 A_{11}} = (\overline{R}_{11} + |\chi|^2)\overline{A}_{01} - (d-2)\nu_0\overline{B}_{11} + f(\overline{A}_{ij}) \tag{3.19}$$

$\underline{(\mu\nu) = (1A)}$:

$$\left(\partial_1 + \frac{d-4}{2(d-2)}\tau - \nu^0\partial_1\nu_0\right)\overline{\nabla_0 A_{1A}} - \sigma_A{}^B\overline{\nabla_0 A_{1B}} = \frac{1}{2}(\overline{R}_{11} + |\chi|^2)\overline{A}_{0A}$$
$$+ (\overline{R}_{1A1}{}^B + \chi_C{}^B\chi_A{}^C)\overline{A}_{0B} - (d-2)\nu_0\overline{B}_{1A} + f_A(\overline{A}_{ij}, \overline{A}_{01}, \overline{\nabla_0 A_{11}}) \tag{3.20}$$

$\underline{(\mu\nu) = (AB)}$:

$$\left(\partial_1 + \frac{d-6}{2(d-2)}\tau - \overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{AB}} - 2\sigma_{(A}{}^C\overline{\nabla_0 A_{B)C}}$$
$$= -\nu^0(\overline{R}_{1A1B} + \chi_{AC}\chi_B{}^C)\overline{A}_{00} - (d-2)\nu_0\overline{B}_{AB} + f_{AB}(\overline{A}_{ij}, \overline{A}_{0i}, \overline{\nabla_0 A_{1i}}) \tag{3.21}$$

$\underline{(\mu\nu) = (01)}$:

$$\left(\partial_1 + \frac{\tau}{2} + \overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\right)\overline{\nabla_0 A_{01}} = -(d-2)\nu_0\overline{B}_{01} + f(\overline{A}_{\mu\nu}, \overline{\nabla_0 A_{ij}}) \tag{3.22}$$

$\underline{(\mu\nu) = (0A)}$:

$$\left(\partial_1 + \frac{d-4}{2(d-2)}\tau - 2\overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{0A}} - \sigma_A{}^B\overline{\nabla_0 A_{0B}} = -(d-2)\nu_0\overline{B}_{0A} + f_A(\overline{A}_{\mu\nu}, \overline{\nabla_0 A_{ij}}, \overline{\nabla_0 A_{01}}) \tag{3.23}$$

$\underline{(\mu\nu) = (00)}$:

$$\left(\partial_1 + \frac{\tau}{2} - 3\overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{00}} = -(d-2)\nu_0\overline{B}_{00} + f(\overline{A}_{\mu\nu}, \overline{\nabla_0 A_{ij}}, \overline{\nabla_0 A_{0i}}). \tag{3.24}$$

The Christoffel symbols appearing in these equations satisfy

$$\overline{\Gamma}^0_{01} = \tfrac{1}{2}\nu^0\overline{\partial_0 g_{11}}, \qquad \overline{\Gamma}^1_{11} = \nu^0\partial_1\nu_0 - \tfrac{1}{2}\nu^0\overline{\partial_0 g_{11}}. \tag{3.25}$$

3.3.4. An auxiliary lemma.

Lemma 3.5. *Assume that the wave equations for X and Y, (3.3) and (3.4), are fulfilled. Assume further that $\overline{A}_{\mu\nu} = 0 = \overline{B}_{\mu\nu}$ on either a light-cone or two transversally intersecting null hypersurfaces. Then $\overline{\nabla_\sigma A_{\mu\nu}} = 0$.*

Proof. We start with the light-cone case. By assumption the equations (3.19)–(3.24) hold. Invoking $\overline{A}_{\mu\nu} = 0 = \overline{B}_{\mu\nu}$ they become

$$\left(\partial_1 + \frac{\tau}{2} - \overline{\Gamma}^0_{01} - 2\overline{\Gamma}^1_{11}\right)\overline{\nabla_0 A_{11}} = 0, \tag{3.26}$$

$$\left(\partial_1 + \frac{d-4}{2(d-2)}\tau - \nu^0\partial_1\nu_0\right)\overline{\nabla_0 A_{1A}} - \sigma_A{}^B\overline{\nabla_0 A_{1B}} = f_A(\overline{\nabla_0 A_{11}}), \tag{3.27}$$

$$\left(\partial_1 + \frac{d-6}{2(d-2)}\tau - \overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{AB}} - 2\sigma_{(A}{}^C\overline{\nabla_0 A_{B)C}} = f_{AB}(\overline{\nabla_0 A_{1i}}), \tag{3.28}$$

$$\left(\partial_1 + \frac{\tau}{2} + \overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\right)\overline{\nabla_0 A_{01}} = f(\overline{\nabla_0 A_{ij}}), \tag{3.29}$$

$$\left(\partial_1 + \frac{d-4}{2(d-2)}\tau - 2\overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{0A}} - \sigma_A{}^B\overline{\nabla_0 A_{0B}} = f_A(\overline{\nabla_0 A_{ij}}, \overline{\nabla_0 A_{01}}), \tag{3.30}$$

$$\left(\partial_1 + \frac{\tau}{2} - 3\overline{\Gamma}^0_{01}\right)\overline{\nabla_0 A_{00}} = f(\overline{\nabla_0 A_{ij}}, \overline{\nabla_0 A_{0i}}). \tag{3.31}$$

Taking the behaviour of the metric components at the tip of the cone into account, cf the formulae (4.41)–(4.51) in [2], we have

$$\tau = \frac{d-2}{r} + O(r), \quad \sigma_A{}^B = O(r), \quad \nu_0 = 1 + O(r^2), \quad \partial_1 \nu_0 = O(r), \tag{3.32}$$

and it follows from [2, appendix A] that

$$\overline{\Gamma}^0_{01} = O(r), \quad \overline{\Gamma}^1_{11} = O(r). \tag{3.33}$$

With (3.32)–(3.33) we observe that the equations (3.26)–(3.31) form a hierarchical system of Fuchsian ODEs which can be solved step-by-step. Existence of a regular conformal Killing field X requires the tensor field $A_{\mu\nu}$ to be regular, as well. Then $\overline{\nabla_0 A_{\mu\nu}}$ needs to show the following behaviour near the vertex:

$$\overline{\nabla_0 A_{11}} = O(1), \quad \overline{\nabla_0 A_{1A}} = O(r), \quad \overline{\nabla_0 A_{AB}} = O(r^2), \tag{3.34}$$

$$\overline{\nabla_0 A_{01}} = O(1), \quad \overline{\nabla_0 A_{0A}} = O(r), \quad \overline{\nabla_0 A_{00}} = O(1). \tag{3.35}$$

Standard results on Fuchsian ODEs (cf e.g. [3, appendix A]) imply that the only solution of (3.26)–(3.35) is provided by $\overline{\nabla_0 A_{\mu\nu}} = 0$.

In the case of two transversally intersecting null hypersurfaces one can derive the same hierarchical system of ODEs on N_1 and N_2, respectively, which now is a system of regular ODEs. The assumption $\overline{A}_{\mu\nu} = 0$ implies $\nabla_\sigma A_{\mu\nu}|_S = 0$. We thus have vanishing initial data for the ODEs and the unique solutions are $\overline{\nabla_1 A_{\mu\nu}}|_{N_1} = 0$ and $\overline{\nabla_2 A_{\mu\nu}}|_{N_2} = 0$. □

3.4. A special case: $\Theta = 1$

Let us briefly analyse the implications of theorem 3.4 in the special case where the conformal factor Θ is identical to one,

$$\Theta = 1$$

(note that thereby the gauge freedom to prescribe the Ricci scalar is lost). Then the unphysical space-time can be identified with the physical space-time. The conformal field equations (2.1)–(2.6) imply the equations

$$s = \frac{1}{2(d-1)}\lambda,$$
$$L_{\mu\nu} = s g_{\mu\nu} \quad \Longleftrightarrow \quad R_{\mu\nu} = \lambda g_{\mu\nu},$$

i.e. in particular the vacuum Einstein equations hold.

Let us analyse the conditions (i)–(vi) of theorem 3.4 in this setting: condition (iii) gives $\overline{Y} = 0$, which we take as initial data for the wave equation (ii) which then implies $Y = 0$, i.e. X needs to be divergence-free, as desired. We observe that (iv) is automatically satisfied. Moreover,

$$B_{\mu\nu} = \mathscr{L}_X L_{\mu\nu} = s\mathscr{L}_X g_{\mu\nu} = 2s\nabla_{(\mu}X_{\nu)},$$

so (vi) follows from (v). To sum it up, the hypotheses of theorem 3.4 are satisfied if and only if there is a vector field X which satisfies

$$\Box_g X_\mu + \lambda X_\mu = 0,$$
$$\overline{\nabla_{(\mu}X_{\nu)}} = 0.$$

This was the starting point of the analysis in [3].

4. KID equations in four dimensions

Theorem 3.4 can be applied to dimensions $d \geqslant 5$ only when the conformal factor Θ is bounded away from zero. In fact, this situation is rather uninteresting since then there is no need to pass to a conformally rescaled space-time (or to put it differently, it is just a matter of gauge to set $\Theta = 1$). One reason why we included this case, though, was to emphasize that there arise difficulties when one tries to go from four to higher dimensions (which is in line with the observation that the conformal field equations provide a good evolution system only in four space-time dimensions). Another reason was to be able to consider the limiting case $\Theta = 1$ in any dimension $d \geqslant 4$ where the unphysical space-time can be identified with the physical space-time, and to compare the resulting equations with those in [3]. This is also a reason why we avoided to make the common gauge choice $R = 0$: $\Theta = 1$ is compatible with $R = 0$ solely when the cosmological constant vanishes. Henceforth we restrict attention to $d = 4$ space-time dimensions.

4.1. A stronger version of theorem 3.4 for light-cones

A more careful analysis of the computations made in the proof of lemma 3.5 will lead us to a refinement of theorem 3.4. We first treat the light-cone case. We will assume the vanishing of $\overline{A}_{ij}$ and $\overline{B}_{ij}$ together with the validity of the wave equations (3.3) & (3.4) for X and Y, and explore the consequences concerning the vanishing of other components of these tensors, including certain transverse derivatives thereof.

Indeed as a straightforward consequence of theorem 3.4 and lemmas 4.2 and 4.3 below we establish the following result.

Theorem 4.1. *Assume that we have been given a* $3 + 1$*-dimensional space-time* $(\mathscr{M}, g, \Theta)$*, with* (g, Θ) *a smooth solution of the conformal field equations (2.1)–(2.6). Let* $C_O \subset \mathscr{M}$ *be a light-cone. Then there exists a vector field* $\hat{X}$ *satisfying the unphysical Killing equations (3.1) on* $\mathrm{D}^+(C_O)$ *if and only if there exists a pair* (X, Y)*,* X *a vector field and* Y *a function, which fulfils the following conditions:*

(i) $\Box_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y = 0$,
(ii) $\Box_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} RY = 0$,
(iii) $\overline{\phi} = 0$ *with* $\phi = X^\mu \nabla_\mu \Theta - \Theta Y$,
(iv) $\overline{\psi} = 0$ *with* $\psi = X^\mu \nabla_\mu s + sY - \nabla_\mu \Theta \nabla^\mu Y$,
(v) $\overline{A}_{ij} = 0$ *with* $A_{\mu\nu} = 2\nabla_{(\mu} X_{\nu)} - 2Y g_{\mu\nu}$,
(vi) $\overline{A}_{01} = 0$,
(vii) $\overline{B}_{ij} = 0$ *with* $B_{\mu\nu} = \mathscr{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y$.

In that case one may take $\hat{X} = X$ *and* $\nabla_\kappa \hat{X}^\kappa = 4Y$. *The condition (vi) is not needed on the closure of those sets where* τ *is non-zero.*

4.1.1. Vanishing of $\overline{A}_{0\mu}$.

We take the trace of (3.12) which together with the wave equation (3.3) for X implies the relation

$$\nabla_\nu A_\mu{}^\nu - \tfrac{1}{2} \nabla_\mu A_\nu{}^\nu = 0. \tag{4.1}$$

On the initial surface that yields in adapted null coordinates,

$$\begin{aligned} 0 = \nu^0 (2\overline{\nabla_{(0} A_{1)\mu}} - \overline{\nabla_\mu A_{01}}) + \overline{g}^{11} \big(\nabla_1 \overline{A}_{\mu 1} - \tfrac{1}{2} \overline{\nabla_\mu A_{11}}\big) \\ + \overline{g}^{1A} (2\nabla_{(1} \overline{A}_{A)\mu} - \overline{\nabla_\mu A_{1A}}) + \overline{g}^{AB} \big(\nabla_A \overline{A}_{\mu B} - \tfrac{1}{2} \overline{\nabla_\mu A_{AB}}\big). \end{aligned} \tag{4.2}$$

In the following we shall always assume that (3.3) and (3.4) hold, and thereby in particular (4.2) and (3.19)–(3.21).

With the assumptions $\overline{A}_{ij} = 0$ and $\overline{B}_{11} = 0$ equation (3.19) becomes

$$\big(\partial_1 + \tfrac{1}{2}\tau - \overline{\Gamma}^1_{11} - \nu^0\partial_1\nu_0\big)\overline{\nabla_0 A_{11}} = -\overline{A}_{01}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau,$$

where we have fallen back on the identity [2]

$$\overline{R}_{11} \equiv -\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau - |\chi|^2. \tag{4.3}$$

Moreover, we deduce from the $\mu = 1$-component of (4.2) that

$$\tau\overline{A}_{01} + \overline{\nabla_0 A_{11}} = 0, \tag{4.4}$$

which leads us to an ODE satisfied by $\overline{A}_{01}$,

$$\tau\big(\partial_1 + \tfrac{1}{2}\tau - \nu^0\partial_1\nu_0\big)\overline{A}_{01} = 0. \tag{4.5}$$

Since τ has no zeros sufficiently close to the vertex it follows from regularity, which requires $\overline{A}_{01}$ to be bounded near the vertex, that $\overline{A}_{01} = 0$ (and thus $\overline{\nabla_0 A_{11}} = 0$) in that region. Even more, $\overline{A}_{01} = 0$ will automatically vanish on the closure of those sets on which τ is non-zero.

Next we assume $\overline{A}_{ij} = 0$, $\overline{A}_{01} = 0$, $\overline{\nabla_0 A_{11}} = 0$, $\overline{B}_{1A} = 0$. Then, due to (3.20), (4.3) and the identity

$$\overline{R}_{1A1}{}^B \equiv -\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\chi_A{}^B - \chi_A{}^C\chi_C{}^B \tag{4.6}$$

we have

$$(\partial_1 - \nu^0\partial_1\nu_0)\overline{\nabla_0 A_{1A}} - \sigma_A{}^B\overline{\nabla_0 A_{1B}} = -\overline{A}_{0A}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau - \overline{A}_{0B}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\sigma_A{}^B.$$

With the current assumptions the $\mu = A$-component of (4.2) can be written as

$$\overline{\nabla_0 A_{1A}} + \big(\partial_1 + \tau - \overline{\Gamma}^0_{01}\big)\overline{A}_{0A} = 0, \tag{4.7}$$

whence

$$(\partial_1 + \tau - \nu^0\partial_1\nu_0)\big[\big(\partial_1 - \overline{\Gamma}^0_{01}\big)\overline{A}_{0A} - \sigma_A{}^B\overline{A}_{0B}\big] = 0. \tag{4.8}$$

Regularity requires $\overline{A}_{0A} = O(r)$, and $\overline{A}_{0A} = 0$ is the only solution of (4.8) with this property. Of course, $\overline{\nabla_0 A_{1A}}$ will then vanish as well.

In the final step we assume (in addition to (3.3) and (3.4)) $\overline{A}_{ij} = 0$, $\overline{A}_{0i} = 0$, $\overline{\nabla_0 A_{1i}} = 0$ and $\overline{g}^{AB}\overline{B}_{AB} = 0$. Taking the $\overline{g}_{AB}$-trace of (3.21) and using again (4.3) we find

$$\big(\partial_1 + \tfrac{1}{2}\tau + \overline{\Gamma}^1_{11} - \nu^0\partial_1\nu_0\big)\big(\overline{g^{AB}\nabla_0 A_{AB}}\big) = \nu^0\overline{A}_{00}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau.$$

From the $\mu = 0$-component of (4.2) we derive the equation

$$\nu^0\big(\partial_1 + \tau + 2\overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\big)\overline{A}_{00} - \tfrac{1}{2}\overline{g^{AB}\nabla_0 A_{AB}} = 0,$$

and end up with an ODE satisfied by $\overline{A}_{00}$,

$$\big(\partial_1 + \tau + \overline{\Gamma}^1_{11} - \nu^0\partial_1\nu_0\big)\big[\nu^0\big(\partial_1 + \tfrac{1}{2}\tau + 2\overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\big)\overline{A}_{00}\big] = 0. \tag{4.9}$$

Regularity requires $\overline{A}_{00} = O(1)$, which leads to $\overline{A}_{00} = 0$, which in turn implies $\overline{g^{AB}\nabla_0 A_{AB}} = 0$. Assuming $\overline{B}_{AB} = 0$ we further have $\overline{\nabla_0 A_{AB}} = 0$ due to (3.21).

Altogether we have proved the lemma.

Lemma 4.2. *Assume that (3.3) and (3.4) hold, and that* $\overline{A}_{ij} = \overline{A}_{01} = 0 = \overline{B}_{ij}$. *Then* $\overline{A}_{0\mu} = 0$ *and* $\overline{\nabla_0 A_{ij}} = 0$. *On the closure of those sets where* τ *is non-zero, in particular sufficiently close to the vertex of the cone, the assumption* $\overline{A}_{01} = 0$ *is not needed, but follows from the remaining hypotheses.*

4.1.2. Vanishing of $\overline{B}_{0\mu}$. By the second Bianchi identity we have

$$\nabla_\nu B_\mu{}^\nu - \tfrac{1}{2}\nabla_\mu B_\nu{}^\nu \equiv A_{\alpha\beta}\big(\nabla^\alpha L_\mu{}^\beta - \tfrac{1}{2}\nabla_\mu L^{\alpha\beta}\big) + L_\mu{}^\kappa(\Box_g X_\kappa + R_\kappa{}^\alpha X_\alpha + 2\nabla_\kappa Y) + \tfrac{1}{2}\nabla_\mu\big(\Box_g Y + \tfrac{1}{6}X^\nu\nabla_\nu R + \tfrac{1}{3}RY\big).$$

Assuming the wave equations (3.3) and (3.4) for X and Y as well as $\overline{A}_{\mu\nu} = 0$ this induces on the initial surface the relation

$$\overline{\nabla_\nu B_\mu{}^\nu} - \tfrac{1}{2}\overline{\nabla_\mu B_\nu{}^\nu} = 0. \tag{4.10}$$

As for $A_{\mu\nu}$, equation (4.2), we deduce that in adapted coordinates we have

$$0 = \nu^0(2\overline{\nabla_{(0}B_{1)\mu}} - \overline{\nabla_\mu B_{01}}) + \overline{g}^{11}\big(\nabla_1\overline{B}_{\mu 1} - \tfrac{1}{2}\overline{\nabla_\mu B_{11}}\big) + \overline{g}^{1A}(2\nabla_{(1}\overline{B}_{A)\mu} - \overline{\nabla_\mu B_{1A}}) + \overline{g}^{AB}\big(\nabla_A\overline{B}_{\mu B} - \tfrac{1}{2}\overline{\nabla_\mu B_{AB}}\big). \tag{4.11}$$

Recall that (3.3), (3.4) and the conformal field equations imply a wave equation (3.14) which is satisfied by $B_{\mu\nu}$. Assuming $\overline{A_{\mu\nu}} = 0$, $\overline{\nabla_0 A_{ij}} = 0$ and $\overline{B}_{ij} = 0$, evaluation on the initial surface yields

$$\overline{\Box_g B_{ij}} = 2(\overline{g}_{ij}\overline{L}^{\alpha\beta} - \overline{R}_i{}^\alpha{}_j{}^\beta)\overline{B}_{\alpha\beta} - 2\overline{R}_{(i}{}^\alpha\overline{B}_{j)\alpha} + 2\overline{L}^{\alpha\beta}(\overline{\nabla_\beta\nabla_{[\alpha}A_{j]i}} - \overline{\nabla_i\nabla_{[\alpha}A_{j]\beta}}). \tag{4.12}$$

In adapted null coordinates we have, as for the corresponding expression (3.18) for $A_{\mu\nu}$,

$$\overline{\Box_g B_{ij}} = 2\nu^0(\overline{\nabla_1\nabla_0 B_{ij}} + \overline{R}_{01(i}{}^\alpha\overline{B}_{j)\alpha}) + \overline{g}^{11}\overline{\nabla_1\nabla_1 B_{ij}} + 2\overline{g}^{1A}(\overline{\nabla_1\nabla_A B_{ij}} + \overline{R}_{A1(i}{}^\alpha\overline{B}_{j)\alpha}) + \overline{g}^{AB}\overline{\nabla_A\nabla_B B_{ij}}. \tag{4.13}$$

Moreover, we have seen that (3.3) & (3.4) imply that the wave equation (3.15) is satisfied by $\nabla_\sigma A_{\mu\nu}$. Assuming $\overline{A_{\mu\nu}} = 0$ and $\overline{\nabla_0 A_{ij}} = 0$ we compute its trace on the initial surface,

$$\overline{\Box_g\nabla_0 A_{ij}} = 2\nu^0(\overline{R}_{1(i} + 2\nu^0\overline{R}_{011(i)}\overline{\nabla_{|0|}A_{j)0}} - 2\overline{R}_i{}^\alpha{}_j{}^\kappa\overline{\nabla_0 A_{\alpha\kappa}} - 4\overline{\nabla_0 B_{ij}}. \tag{4.14}$$

In adapted null coordinates and with the current assumptions the left-hand side becomes

$$\overline{\Box_g\nabla_0 A_{ij}} = 2\nu^0\overline{\nabla_1\nabla_0\nabla_0 A_{ij}} + 2\nu^0\overline{R}_{01(i}{}^\mu\overline{\nabla_{|0|}A_{j)\mu}} + 2\overline{g}^{1A}\overline{\nabla_1\nabla_A\nabla_0 A_{ij}} + 2\overline{g}^{1A}\overline{R}_{A1(i}{}^\mu\overline{\nabla_{|0|}A_{j)\mu}} + \overline{g}^{AB}\overline{\nabla_A\nabla_B\nabla_0 A_{ij}}. \tag{4.15}$$

Recall that $\overline{A}_{\mu\nu} = 0$ suffices to establish $\overline{B}_\alpha{}^\alpha = 0$. In that case $\overline{B}_{ij} = 0$ implies

$$\overline{B}_{01} = 0, \tag{4.16}$$

and, by (3.22),

$$\overline{\nabla_0 A_{01}} = 0.$$

As for $A_{\mu\nu}$, the $\mu = 1$-component of (4.11) yields

$$\tau\overline{B}_{01} + \overline{\nabla_0 B_{11}} = 0 \quad\Longrightarrow\quad \overline{\nabla_0 B_{11}} = 0. \tag{4.17}$$

The $(ij) = (11)$-component of (4.14) reads

$$\overline{\Box_g\nabla_0 A_{11}} = 0 \overset{(4.15)}{\Longrightarrow} \big(\partial_1 + \tfrac{1}{2}\tau - 2\nu^0\partial_1\nu_0\big)\overline{\nabla_0\nabla_0 A_{11}} = 0 \Longrightarrow \overline{\nabla_0\nabla_0 A_{11}} = 0$$

by regularity.

At this stage we can and will assume $\overline{A}_{\mu\nu} = \overline{\nabla_0 A_{ij}} = \overline{\nabla_0 A_{01}} = \overline{\nabla_0\nabla_0 A_{11}} = \overline{B}_{ij} = \overline{B}_{01} = \overline{\nabla_0 B_{11}} = 0$. With (3.23) we then find for the $(ij) = (1A)$-components of (4.12)

$$\overline{\Box_g B_{1A}} = 2\nu^0\overline{R}_{1A1}{}^B\overline{B}_{0B} + \tfrac{1}{2}(\nu^0)^2\overline{R}_{11}\overline{\nabla_0\nabla_0 A_{1A}} + \tfrac{1}{2}\tau(\nu^0)^2\overline{R}_{11}\overline{\nabla_0 A_{0A}} + \tfrac{1}{2}(\nu^0)^2\overline{R}_{11}\sigma_A{}^B\overline{\nabla_0 A_{0B}}.$$

The $(ij) = (1A)$-components of (4.13) read

$$\overline{\Box_g B_{1A}} = 2\nu^0(\partial_1 - \nu^0\partial_1\nu_0)\overline{\nabla_0 B_{1A}} - 2\nu^0\sigma_A{}^B\overline{\nabla_0 B_{1B}} - \nu^0(|\chi|^2\overline{B}_{0A} + 2\chi_A{}^C\chi_C{}^B\overline{B}_{0B}).$$

Equating both expressions for $\overline{\Box_g B_{1A}}$ and using (4.6) we deduce that

$$2(\partial_1 - \nu^0\partial_1\nu_0)\overline{\nabla_0 B_{1A}} - 2\sigma_A{}^B\overline{\nabla_0 B_{1B}} = -2\overline{B}_{0B}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\chi_A{}^B + |\chi|^2\overline{B}_{0A} + \tfrac{1}{2}\nu^0\overline{R}_{11}\overline{\nabla_0\nabla_0 A_{1A}}$$
$$+ \tfrac{1}{2}\tau\nu^0\overline{R}_{11}\overline{\nabla_0 A_{0A}} + \tfrac{1}{2}\nu^0\overline{R}_{11}\sigma_A{}^B\overline{\nabla_0 A_{0B}}. \tag{4.18}$$

Evaluation of (4.14) for $(ij) = (1A)$ leads to

$$\overline{\Box_g\nabla_0 A_{1A}} = \nu^0\overline{R}_{11}\overline{\nabla_0 A_{0A}} + 2\nu^0\overline{R}_{1A1}{}^B\overline{\nabla_0 A_{0B}} - 4\overline{\nabla_0 B_{1A}},$$

while (4.15) becomes

$$\overline{\Box_g\nabla_0 A_{1A}} = 2\nu^0\big(\partial_1 + \overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\big)\overline{\nabla_0\nabla_0 A_{1A}} - 2\nu^0\sigma_A{}^B\overline{\nabla_0\nabla_0 A_{1B}}$$
$$- \nu^0|\chi|^2\overline{\nabla_0 A_{0A}} - 2\nu^0\chi_A{}^C\chi_C{}^B\overline{\nabla_0 A_{0B}}.$$

Using (4.3) and (4.6) we end up with

$$2\big(\partial_1 + \overline{\Gamma}^1_{11} - 2\nu^0\partial_1\nu_0\big)\overline{\nabla_0\nabla_0 A_{1A}} - 2\sigma_A{}^B\overline{\nabla_0\nabla_0 A_{1B}}$$
$$= -2\overline{\nabla_0 A_{0A}}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau - 2\overline{\nabla_0 A_{0B}}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\sigma_A{}^B - 4\nu_0\overline{\nabla_0 B_{1A}}. \tag{4.19}$$

The $\mu = A$-components of (4.11) give, again in close analogy to the corresponding equations for $A_{\mu\nu}$,

$$\big(\partial_1 + \tau - \overline{\Gamma}^0_{01}\big)\overline{B}_{0A} + \overline{\nabla_0 B_{1A}} = 0. \tag{4.20}$$

By (3.23) we have

$$\big(\partial_1 - 2\overline{\Gamma}^0_{01}\big)\overline{\nabla_0 A_{0A}} - \sigma_A{}^B\overline{\nabla_0 A_{0B}} = -2\nu_0\overline{B}_{0A}. \tag{4.21}$$

Taking the behaviour of the metric components at the vertex into account, cf [2, section 4.5], we observe that the ODE-system (4.18)–(4.21) for $\overline{B}_{0A}$, $\overline{\nabla_0 B_{1A}}$, $\overline{\nabla_0 A_{0A}}$ and $\overline{\nabla_0\nabla_0 A_{1A}}$ is of the form

$$\left[\partial_1 + \begin{pmatrix} 2r^{-1} + O(r) & 1 & 0 & 0 \\ -2r^{-2} + O(1) & O(r) & O(r^{-1}) & O(1) \\ 2 + O(r^2) & 0 & O(r) & 0 \\ 0 & 2 + O(r^2) & -2r^{-2} + O(1) & O(r) \end{pmatrix}\right]\begin{pmatrix} \overline{B}_{0A} \\ \overline{\nabla_0 B_{1A}} \\ \overline{\nabla_0 A_{0A}} \\ \overline{\nabla_0\nabla_0 A_{1A}} \end{pmatrix} = 0,$$

where each matrix entry is actually a 2×2-matrix. Regularity requires

$$\overline{B}_{0A}, \overline{\nabla_0 B_{1A}}, \overline{\nabla_0 A_{0A}}, \overline{\nabla_0\nabla_0 A_{1A}} = O(r).$$

But then a necessary condition for (4.20) to be satisfied is

$$\overline{B}_{0A} = O(r^2), \tag{4.22}$$

whence it follows from (4.21) and (4.19) that

$$\overline{\nabla_0 A_{0A}} = O(r^3), \quad \overline{\nabla_0\nabla_0 A_{1A}} = O(r^2). \tag{4.23}$$

Setting $\tilde{\overline{B}}_{0A} := r^{-2}\overline{B}_{0A}$, $\widetilde{\overline{\nabla_0 B_{1A}}} = r^{-1}\overline{\nabla_0 B_{1A}}$, $\widetilde{\overline{\nabla_0 A_{0A}}} := r^{-3}\overline{\nabla_0 A_{0A}}$ and $\widetilde{\overline{\nabla_0\nabla_0 A_{1A}}} = r^{-2}\overline{\nabla_0\nabla_0 A_{1A}}$ the ODE-system adopts the form

$$\left[\partial_1 + r^{-1}\begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & -2 & 2 \end{pmatrix} + M\right] v = 0, \quad v := \begin{pmatrix} \tilde{\overline{B}}_{0A} \\ \widetilde{\overline{\nabla_0 B_{1A}}} \\ \widetilde{\overline{\nabla_0 A_{0A}}} \\ \widetilde{\overline{\nabla_0\nabla_0 A_{1A}}} \end{pmatrix} = O(1),$$

where $M = O(r)$ is some matrix. Setting

$$\tilde{v} := T^{-1}v = O(1), \tag{4.24}$$

where

$$T := \begin{pmatrix} 0 & -1/2 & -1/3 & 0 \\ 0 & 1/2 & 2/3 & 0 \\ -1 & -1/2 & 2/3 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

is the change of basis matrix which transforms the leading order matrix to Jordan normal form, we end up with the Fuchsian ODE-system

$$\partial_1 \tilde{v} + r^{-1}\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\tilde{v} + \tilde{M}\tilde{v} = 0, \quad \tilde{M} := T^{-1}MT = O(r). \tag{4.25}$$

In appendix A it is shown that any solution of (4.25) which is $O(1)$ needs to vanish identically (take, in the notation used there, $\lambda = -1$). Hence $\overline{B}_{0A} = \overline{\nabla_0 B_{1A}} = \overline{\nabla_0 A_{0A}} = \overline{\nabla_0\nabla_0 A_{1A}} = 0$, which we can and will assume in the subsequent computations.

The $\overline{g}_{AB}$-trace of the $(ij) = (AB)$-component of (4.12) reads

$$\overline{g}^{AB}\overline{\Box_g B_{AB}} = \tfrac{1}{2}(\nu^0)^2\overline{R}_{11}\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} - \tfrac{1}{2}\tau(\nu^0)^3\overline{R}_{11}\overline{\nabla_0 A_{00}}.$$

For the corresponding component of (4.13) we find

$$\overline{g}^{AB}\overline{\Box_g B_{AB}} \equiv 2\nu^0\big(\partial_1 + \tfrac{1}{2}\tau - \overline{\Gamma}^0_{01}\big)(\overline{g}^{AB}\overline{\nabla_0 B_{AB}}) + 2(\nu^0)^2|\chi|^2\overline{B}_{00},$$

and thus

$$\begin{aligned}&\big(\partial_1 + \tfrac{1}{2}\tau - \overline{\Gamma}^0_{01}\big)(\overline{g}^{AB}\overline{\nabla_0 B_{AB}}) + \nu^0|\chi|^2\overline{B}_{00} \\ &\quad = \tfrac{1}{4}\nu^0\overline{R}_{11}\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} - \tfrac{1}{4}\tau(\nu^0)^2\overline{R}_{11}\overline{\nabla_0 A_{00}}.\end{aligned} \tag{4.26}$$

From (4.14) we deduce

$$\overline{g}^{AB}\overline{\Box_g \nabla_0 A_{AB}} = -2(\nu^0)^2\overline{R}_{11}\overline{\nabla_0 A_{00}} - 4\overline{g}^{AB}\overline{\nabla_0 B_{AB}},$$

while from (4.15) we obtain

$$\overline{g}^{AB}\overline{\Box_g \nabla_0 A_{AB}} = 2\nu^0\big(\partial_1 + \tfrac{1}{2}\tau - 2\overline{\Gamma}^0_{01}\big)(\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}}) + 2(\nu^0)^2|\chi|^2\overline{\nabla_0 A_{00}}.$$

Invoking (4.3) we are led to the equation

$$\big(\partial_1 + \tfrac{1}{2}\tau - 2\overline{\Gamma}^0_{01}\big)(\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}}) = \nu^0\overline{\nabla_0 A_{00}}\big(\partial_1 - \overline{\Gamma}^1_{11}\big)\tau - 2\nu_0\overline{g}^{AB}\overline{\nabla_0 B_{AB}}. \tag{4.27}$$

The $\mu = 0$-component of (4.11) reads

$$(\partial_1 + \tau - 2\overline{\Gamma}^0_{01})\overline{B}_{00} - \tfrac{1}{2}\nu_0\overline{g}^{AB}\overline{\nabla_0 B_{AB}} = 0. \tag{4.28}$$

Recall that by (3.24) we have

$$\big(\partial_1 + \tfrac{1}{2}\tau - 3\overline{\Gamma}^0_{01}\big)\overline{\nabla_0 A_{00}} + 2\nu_0\overline{B}_{00} = 0. \tag{4.29}$$

Using again the results of [2, section 4.5] we find that the ODE-system (4.26)–(4.29) for $\overline{B}_{00}$, $\overline{g}^{AB}\overline{\nabla_0 B_{AB}}$, $\overline{\nabla_0 A_{00}}$ and $\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}}$ is of the form

$$\left[\partial_1 + \begin{pmatrix} 2r^{-1}+O(r) & -\frac{1}{2}+O(r^2) & 0 & 0 \\ 2r^{-2}+O(1) & r^{-1}+O(r) & O(r^{-1}) & O(1) \\ 2+O(r^2) & 0 & r^{-1}+O(r) & 0 \\ 0 & 2+O(r^2) & 2r^{-2}+O(1) & r^{-1}+O(r) \end{pmatrix}\right]\begin{pmatrix} \overline{B}_{00} \\ \overline{g}^{AB}\overline{\nabla_0 B_{AB}} \\ \overline{\nabla_0 A_{00}} \\ \overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} \end{pmatrix} = 0.$$

Due to regularity we have

$$\overline{B}_{00}\,,\ \overline{g}^{AB}\overline{\nabla_0 B_{AB}},\ \overline{\nabla_0 A_{00}},\ \overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} = O(1).$$

Even more, from (4.28) we conclude that

$$\overline{B}_{00} = O(r). \tag{4.30}$$

From (4.29) and (4.27) we then deduce

$$\overline{\nabla_0 A_{00}} = O(r^2), \quad \overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} = O(r). \tag{4.31}$$

In terms of the rescaled fields $\widetilde{\overline{g^{AB}\nabla_0\nabla_0 A_{AB}}} = r^{-1}\overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}}$ and $\tilde{\overline{B}}_{00} := r^{-1}\overline{B}_{00}$, $\widetilde{\overline{\nabla_0 A_{00}}} := r^{-2}\overline{\nabla_0 A_{00}}$ the ODE-system takes the form

$$\left[\partial_1 + r^{-1}\begin{pmatrix} 3 & -\frac{1}{2} & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix} + M\right] v = 0, \quad v := \begin{pmatrix} \tilde{\overline{B}}_{00} \\ \overline{g^{AB}\nabla_0 B_{AB}} \\ \widetilde{\overline{\nabla_0 A_{00}}} \\ \widetilde{\overline{g^{AB}\nabla_0\nabla_0 A_{AB}}} \end{pmatrix} = O(1),$$

with $M = O(r)$ being some matrix. The change of basis matrix

$$T := \begin{pmatrix} 0 & -1/3 & -1/3 & 0 \\ 0 & -2/3 & 0 & 0 \\ 1/\sqrt{5} & 2/3 & 4/3 & 0 \\ 2/\sqrt{5} & 8/3 & 0 & 1 \end{pmatrix},$$

transforms the indicial matrix to Jordan normal form, and we end up with another Fuchsian ODE-system,

$$\partial_1\tilde{v} + r^{-1}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\tilde{v} + \tilde{M}\tilde{v} = 0, \tag{4.32}$$

where $\tilde{v} := T^{-1}v = O(1)$ and $\tilde{M} := T^{-1}MT = O(r)$. Again, lemma A.1 in appendix A (with $\lambda = -1$) implies $\tilde{v} = 0$, and thus $\overline{B}_{00} = \overline{g}^{AB}\overline{\nabla_0 B_{AB}} = \overline{\nabla_0 A_{00}} = \overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} = 0$.

In this section we have proved:

Lemma 4.3. *Assume that (3.3) and (3.4) hold, and that $\overline{A}_{\mu\nu} = 0 = \overline{B}_{ij} = \overline{\nabla_0 A_{ij}}$. Then $\overline{B}_{0\mu} = 0$, $\overline{\nabla_0 B_{1i}} = \overline{g}^{AB}\overline{\nabla_0 B_{AB}} = 0$, $\overline{\nabla_0 A_{0\mu}} = 0$ and $\overline{\nabla_0\nabla_0 A_{1i}} = \overline{g}^{AB}\overline{\nabla_0\nabla_0 A_{AB}} = 0$.*

4.1.3. The (proper) KID equations. The conditions (iv), (vi) and (vii) in theorem 4.1 are not intrinsic in the sense that they involve transverse derivatives of X and Y which are not part of the initial data for the wave equations (i) and (ii). However, they can be eliminated via these wave equations. In fact, this is crucial if one wants to check for a certain candidate field defined only on the initial surface whether it extends to a vector field satisfying the unphysical Killing equations or not. In essence this is what we will do next.

We have

$$\overline{\Box_g Y} \equiv 2\nu^0\left(\nabla_1 + \tfrac{1}{2}\tau\right)\overline{\nabla_0 Y} + \overline{g}^{ij}\mathrm{D}_i\mathrm{D}_j\overline{Y}, \tag{4.33}$$

where D_i is the derivative operator introduced in [3],

$$\mathrm{D}_i\overline{Y} := \nabla_i\overline{Y},$$
$$\mathrm{D}_i\overline{X}_\mu := \nabla_i\overline{X}_\mu,$$
$$\mathrm{D}_i\mathrm{D}_j\overline{Y} := \partial_i\mathrm{D}_j\overline{Y} - \overline{\Gamma}^k_{ij}\mathrm{D}_k Y,$$
$$\mathrm{D}_i\mathrm{D}_j\overline{X}_\mu := \partial_i\mathrm{D}_j\overline{X}_\mu - \overline{\Gamma}^k_{ij}\mathrm{D}_k\overline{Y}_\mu - \overline{\Gamma}^\nu_{i\mu}\mathrm{D}_j\overline{Y}_\nu,$$

i.e. one simply removes the transverse derivatives which would appear in the corresponding expressions with covariant derivatives. Since the action of ∇_i and D_i coincides in many cases relevant to us one may often use them interchangeably. Nevertheless, we shall use D_i consistently whenever derivatives of X or Y appear to stress that no transverse derivatives of these fields are involved.

By (ii) and (4.33) the function $\Upsilon := \overline{\partial_0 Y}$ (note that Υ is not a scalar) satisfies the ODE

$$\left(\partial_1 + \tfrac{\tau}{2} - \overline{\Gamma}^0_{01}\right)\Upsilon - \overline{\Gamma}^i_{01}\nabla_i\overline{Y} + \tfrac{1}{2}\nu_0\left(\overline{g}^{ij}\mathrm{D}_i\mathrm{D}_j\overline{Y} + \tfrac{1}{6}\overline{X}^\mu\overline{\nabla_\mu R} + \tfrac{1}{3}\overline{RY}\right) = 0. \tag{4.34}$$

Regularity requires $\Upsilon = O(1)$.

It is useful to make the following definition

$$S_{\mu\nu\sigma} := \nabla_\mu\nabla_\nu X_\sigma - R_{\sigma\nu\mu}{}^\kappa X_\kappa - 2\nabla_{(\mu}Y g_{\nu)\sigma} + \nabla_\sigma Y g_{\mu\nu}. \tag{4.35}$$

It follows from the identity (3.12) that

$$S_{\mu\nu\sigma} = \nabla_{(\mu}A_{\nu)\sigma} - \tfrac{1}{2}\nabla_\sigma A_{\mu\nu}. \tag{4.36}$$

Note that this implies the useful relations

$$2S_{\mu(\nu\sigma)} = \nabla_\mu A_{\nu\sigma}, \tag{4.37}$$

$$S_{[\mu\nu]\sigma} = 0. \tag{4.38}$$

Recall that (4.4) is a consequence of (i) and (v). Hence

$$\overline{S}_{110} = \nabla_1\overline{A}_{01} - \tfrac{1}{2}\overline{\nabla_0 A_{11}} = \left(\partial_1 + \tfrac{1}{2}\tau - \nu^0\partial_1\nu_0\right)\overline{A}_{01}. \tag{4.39}$$

Taking regularity into account we conclude that

$$\overline{A}_{01} = 0 \quad\Longleftrightarrow\quad \overline{S}_{110} = 0.$$

This leads us to the following stronger version of theorem 4.1:

Theorem 4.4. *Assume that we have been given a* $3+1$*-dimensional space-time* $(\mathscr{M}, g, \Theta)$*, with* (g, Θ) *being a smooth solution of the conformal field equations (2.1)–(2.6). Let* $\mathring{X}$ *be a vector field and* $\mathring{Y}$ *a function defined on a light-cone* $C_O \subset \mathscr{M}$*. Then there exists a smooth vector field* X *with* $\overline{X} = \mathring{X}$ *and* $\overline{\nabla_\kappa X^\kappa} = 4\mathring{Y}$ *satisfying the unphysical Killing equations (3.1) on* $\mathrm{D}^+(C_O)$ *(i.e. representing a Killing field of the physical space-time) if and only if*

(a) the conditions (iii) and (v) in theorem 4.1 hold,
(b) $\overline{\psi}^{\mathrm{intr}} := \mathring{X}^\mu\overline{\nabla_\mu s} + \overline{s}\mathring{Y} - \overline{\nabla^i\Theta}\mathrm{D}_i\mathring{Y} - \nu^0\Upsilon\nabla_1\overline{\Theta} = 0,$
(c) $\overline{S}_{110} \equiv \mathrm{D}_1\mathrm{D}_1\mathring{X}_0 - \overline{R}_{011}{}^\kappa\mathring{X}_\kappa - 2\nu_0\mathrm{D}_1\mathring{Y} = 0,$
(d) $\overline{B}_{1i} \equiv \mathring{X}^\kappa\nabla_\kappa\overline{L}_{1i} + 2\overline{L}_{\kappa(1}\mathrm{D}_{i)}\mathring{X}^\kappa + \mathrm{D}_1\mathrm{D}_i\mathring{Y} = 0,$
(e) $\overline{B}^{\mathrm{intr}}_{AB} := \mathring{X}^\kappa\nabla_\kappa\overline{L}_{AB} + 2\overline{L}_{\kappa(A}\mathrm{D}_{B)}\mathring{X}^\kappa + \mathrm{D}_A\mathrm{D}_B\mathring{Y} + \nu^0\Upsilon\chi_{AB} = 0,$
(f) $\mathring{X}$ *and* $\mathring{Y}$ *are restrictions to the light-cone of smooth space-time fields.*

The function Υ *is the unique solution of*

$$\left(\partial_1 + \frac{\tau}{2} - \overline{\Gamma}^0_{01}\right)\Upsilon - \overline{\Gamma}^i_{01}\mathrm{D}_i\mathring{Y} + \frac{1}{2}\nu_0\left(\overline{g}^{ij}\mathrm{D}_i\mathrm{D}_j\mathring{Y} + \frac{1}{6}\mathring{X}^\mu\overline{\nabla_\mu R} + \frac{1}{3}\overline{R}\mathring{Y}\right) = 0 \tag{4.40}$$

which is bounded near the tip of the cone. The condition (c) is not needed on the closure of those sets on which the expansion τ *is non-zero.*

Proof. It needs to be shown that $(\mathring{X}, \mathring{Y})$ extends to a pair (X, Y) satisfying (i)–(vii) in theorem 4.1. From the considerations above it becomes clear that (a)–(e) do imply (i)–(vii) in theorem 4.1 if $\mathring{X}$ and $\mathring{Y}$ can be extended to smooth solutions of the wave equations (3.3) and (3.4) for X and Y. However, this follows from [5] and (f). $\square$

Remark 4.5. The conditions (a)–(e) will be called *(proper)* [2] *KID equations (cf proposition* 4.9 below which shows that condition (f) is not needed).

Remark 4.6. Theorem 4.4 can e.g. be applied to a light-cone with vertex at past timelike infinity for vanishing cosmological constant (this is done in section 5), or to light-cones with vertex on past null infinity for vanishing or positive cosmological constant.

4.1.4. Extendability of the candidate fields. A drawback of theorem 4.4 is the condition (f): it is a non-trivial issue to make sure that the candidate fields $\mathring{X}$ and $\mathring{Y}$ which are constructed from (a subset of) (a)–(e) are restrictions to the light-cone of smooth space-time fields. (Nonetheless we shall see in section 5 that (f) becomes trivial on the C_{i^-}-cone.) We therefore aim to prove that (f) follows directly and without any restrictions from the KID equations (a)–(e).

Since the validity of (f) is non-trivial only in some neighbourhood of the vertex of the cone, we can and will assume in this section that the expansion τ has no zeros.

The procedure will be in close analogy to [3, section 2.5]. First of all we shall compute the divergence $\nabla^\gamma S_{\alpha\beta\gamma}$ which contains certain transverse derivatives of $\mathring{X}$ and $\mathring{Y}$ (which eventually drop out from the relevant formulae). For these expressions to make sense let X and Y be any smooth extensions of $\mathring{X}$ and $\mathring{Y}$ from the cone C_O to a punctured neighbourhood of O. We stress that no assumptions are made concerning the behaviour of X and Y as the tip of the cone is approached.

By (4.35) and the second Bianchi identity we have

$$\nabla^\sigma S_{\mu\nu\sigma} \equiv \tfrac{1}{2}\nabla_\mu\nabla_\nu A_\sigma{}^\sigma + 2B_{\mu\nu} + \tfrac{1}{6}RA_{\mu\nu} + (B_\sigma{}^\sigma - L^{\alpha\beta}A_{\alpha\beta})g_{\mu\nu}. \tag{4.41}$$

In adapted null coordinates the trace of the left-hand side on the cone reads,

$$\overline{\nabla^\sigma S_{\mu\nu\sigma}} = \nu^0(\overline{\nabla_0 S_{\mu\nu1}} + \nabla_1\overline{S}_{\mu\nu0}) + \overline{g}^{1A}(\nabla_1\overline{S}_{\mu\nu A} + \nabla_A\overline{S}_{\mu\nu1}) + \overline{g}^{11}\nabla_1\overline{S}_{\mu\nu1} + \overline{g}^{AB}\nabla_A\overline{S}_{\mu\nu B}. \tag{4.42}$$

The undesirable transverse derivatives which appear in $\overline{\nabla_0 S_{\mu\nu1}}$ can be eliminated via

$$\begin{aligned}\nabla_0\nabla_\mu\nabla_\nu X_\sigma &= \nabla_\mu\nabla_\nu A_{0\sigma} - \nabla_\mu\nabla_\nu\nabla_\sigma X_0 + 2g_{0\sigma}\nabla_\mu\nabla_\nu Y + \nabla_\mu(R_{0\nu\sigma}{}^\kappa X_\kappa)\\ &\quad + R_{0\mu\nu}{}^\kappa\nabla_\kappa X_\sigma + R_{0\mu\sigma}{}^\kappa\nabla_\nu X_\kappa.\end{aligned} \tag{4.43}$$

Lemma 4.7. *Assume* $\overline{A}_{ij} = 0$. *Then*

$$2\overline{B}_{11} = \tau\nu^0\overline{S}_{110}.$$

Proof. Equation (4.41) with $(\mu\nu) = (11)$ yields

$$\overline{\nabla^\sigma S_{11\sigma}} = \nu^0\nabla_1\nabla_1\overline{A}_{01} + 2\overline{B}_{11}. \tag{4.44}$$

Note that it follows from (4.36) that the vanishing of $\overline{A}_{1i}$ implies the vanishing of $\overline{S}_{11i}$ and all permutations thereof. Due to (4.42) we further have $\overline{S}_{1AB} = \overline{S}_{A1B} = 0$. From (4.42) we then obtain with $(\mu\nu) = (11)$

$$\overline{\nabla^\sigma S_{11\sigma}} = \nu^0\overline{\nabla_0 S_{111}} + \nu^0\nabla_1\overline{S}_{110} - 2\chi^{AB}\overline{S}_{1AB} + \tau\nu^0\overline{S}_{110}, \tag{4.45}$$

while (4.43) gives

$$\overline{\nabla_0 S_{111}} = \overline{\nabla_0\nabla_1\nabla_1 X_1} = \nabla_1\nabla_1\overline{A}_{01} - \nabla_1\overline{S}_{110}.$$

Equating (4.44) with (4.45) yields the desired result. □

[2] In the sense that they do not involve transverse derivatives of X or Y.

Lemma 4.8. *Assume* $\overline{A}_{ij} = 0$ *and* $\overline{S}_{110} = 0$*. Then*

$$2\nu_0\overline{B}_{1A} = (\partial_1 + \tau - \nu^0\partial_1\nu_0)\overline{S}_{A10}.$$

Proof. From the $(\mu\nu) = (A1)$-components of (4.41) we deduce

$$\overline{\nabla^\sigma S_{A1\sigma}} = \tfrac{1}{2}\nabla_A\nabla_1\overline{A}_\sigma{}^\sigma + 2\overline{B}_{1A}. \tag{4.46}$$

It follows from (4.43) that

$$\overline{\nabla_0 S_{A11}} = \overline{\nabla_0\nabla_A\nabla_1 X_1} = \nabla_A\nabla_1\overline{A}_{01} - \nabla_A\overline{S}_{110} = \nabla_A\nabla_1\overline{A}_{01} + 2\chi_A{}^B\overline{S}_{B10}.$$

Recall that $\overline{S}_{11i}$ as well as all permutations thereof vanish, and that $\overline{S}_{A1B} = \overline{S}_{1AB} = 0$. Equation (4.42) with $(\mu\nu) = (A1)$ and (4.37) then yield

$$\begin{aligned}\overline{\nabla^\sigma S_{A1\sigma}} &= \nu^0\overline{\nabla_0 S_{A11}} + \nu^0\nabla_1\overline{S}_{A10} + \overline{g}^{1B}\nabla_B\overline{S}_{A11} + \overline{g}^{BC}\nabla_C\overline{S}_{A1B}\\ &= \nu^0\nabla_A\nabla_1\overline{A}_{01} + 2\nu^0\chi_A{}^B\overline{S}_{B10} + \nu^0\nabla_1\overline{S}_{A10} + \tfrac{1}{2}\overline{g}^{1B}\overline{\nabla_B\nabla_A A_{11}} + \overline{g}^{BC}\nabla_C\overline{S}_{A1B}.\end{aligned} \tag{4.47}$$

Combining (4.46) and (4.47) and invoking again (4.37) we obtain

$$2\overline{B}_{1A} = \nu^0\nabla_1\overline{S}_{A10} + 2\nu^0\chi_A{}^B\overline{S}_{B10} + 2\overline{g}^{BC}\nabla_{[C}\overline{S}_{A]1B} + \overline{g}^{1B}\big(\tfrac{1}{2}\overline{\nabla_B\nabla_A A_{11}} - \nabla_A\nabla_1\overline{A}_{1B}\big). \tag{4.48}$$

Since

$$\tfrac{1}{2}\overline{\nabla_B\nabla_A A_{11}} - \nabla_A\nabla_1\overline{A}_{1B} = -\nabla_A\overline{S}_{11B} = 0,$$

and

$$2\overline{g}^{BC}\nabla_{[C}\overline{S}_{A]1B} = \tau\nu^0\overline{S}_{A10} - \nu^0\chi_A{}^B\overline{S}_{B10} + \underbrace{2\overline{g}^{BC}\chi_{[A}{}^D\overline{S}_{C]DB}}_{=0 \text{ by } (4.36)}$$

the lemma is proved. □

As in [3] one checks via the formulae in [2, section 4.5] and assuming

$$\mathring{X}_1,\ \partial_i\mathring{X}_1 = O(1), \quad \mathring{X}_0,\ \partial_i\mathring{X}_0,\ \partial_A\partial_1\mathring{X}_0 = O(1), \tag{4.49}$$

$$\mathring{X}_A,\ \partial_B\mathring{X}_A = O(r), \quad \partial_1\mathring{X}_A = O(1), \quad \mathring{Y},\ \partial_i\mathring{Y} = O(1), \tag{4.50}$$

that $\overline{S}_{A10}$ needs to exhibit the following behaviour near the tip of the cone:

$$\overline{S}_{A10} = O(r^{-1}). \tag{4.51}$$

Note that (4.49)–(4.50) are necessarily satisfied by any pair $(\mathring{X}, \mathring{Y}) = (\overline{X}, \tfrac{1}{4}\overline{\mathrm{div}X})$ with X a smooth vector field.

It now follows immediately from lemmas 4.7 and 4.8 that for any vector field $\mathring{X}$ and any function $\mathring{Y}$ which satisfy $\overline{A}_{ij} = 0$ and $\overline{B}_{1i} = 0$ the equation

$$\overline{S}_{i10} = 0 \tag{4.52}$$

holds sufficiently close to the vertex of the cone where τ has no zeros.

Let us define an antisymmetric tensor field $\mathring{F}_{\mu\nu}$ via

$$\mathring{F}_{ij} := \nabla_{[i}\mathring{X}_{j]}, \tag{4.53}$$

$$\mathring{F}_{i0} := \nabla_i\mathring{X}_0 - \overline{g}_{0i}\mathring{Y}. \tag{4.54}$$

We also define a covector field $\mathring{H}_\mu$,

$$\mathring{H}_i := \nabla_i\mathring{Y}, \tag{4.55}$$

$$\mathring{H}_0 := 0. \tag{4.56}$$

In the following computations we assume

$$\overline{S}_{i10} = 0 = \overline{A}_{ij} = \overline{B}_{1i}. \tag{4.57}$$

Then, due to the first Bianchi identity,

$$\begin{aligned}
\mathring{F}_{1i} &\equiv \nabla_1 \mathring{X}_i - \tfrac{1}{2}\overline{A}_{1i} - \mathring{Y}\overline{g}_{1i} = \nabla_1 \mathring{X}_i,\\
\nabla_1 \mathring{F}_{ij} &\equiv \nabla_{[i}\overline{A}_{j]1} - 2\overline{g}_{1[i}\mathring{H}_{j]} - \overline{R}_{ij1}{}^{\alpha}\mathring{X}_\alpha = -\overline{R}_{ij1}{}^{\alpha}\mathring{X}_\alpha,\\
\nabla_1 \overline{F}_{i0} &\equiv \overline{S}_{i10} - \overline{R}_{i01}{}^{\alpha}\mathring{X}_\alpha + \nu_0 \mathring{H}_i - \overline{g}_{1i}\overline{\nabla_0 Y} = \nu_0\mathring{H}_i - \overline{R}_{i01}{}^{\alpha}\mathring{X}_\alpha.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\nabla_1 \mathring{H}_i &\equiv \overline{B}_{1i} - \overline{\mathscr{L}_X L_{1i}} \equiv \overline{B}_{1i} - \overline{L}_{(1}{}^{j}\overline{A}_{i)j} - \mathring{X}^{\alpha}\overline{\nabla_\alpha L_{1i}} - 2\overline{L}_{(1}{}^{\alpha}\mathring{F}_{i)\alpha} - 2\overline{L}_{1i}\mathring{Y}\\
&= -\mathring{X}^{\alpha}\overline{\nabla_\alpha L_{1i}} - 2\overline{L}_{(1}{}^{\alpha}(\mathring{F}_{i)\alpha} + \overline{g}_{i)\alpha}\mathring{Y}),\\
\nabla_1 \mathring{H}_0 &\equiv -\overline{\Gamma}^i_{01}\mathring{H}_i.
\end{aligned}$$

Therefore the candidate fields $\mathring{X}$ and $\mathring{Y}$ solving (a)–(e) in theorem 4.4 form a solution of the following problem on C_O,

$$\begin{cases}
\nabla_1 \mathring{X}_\mu = \mathring{F}_{1\mu} + \overline{g}_{1\mu}\mathring{Y},\\
\nabla_1 \mathring{F}_{\mu\nu} = 2\overline{g}_{1[\nu}\mathring{H}_{\mu]} - \overline{R}_{\mu\nu 1}{}^{\alpha}\mathring{X}_\alpha,\\
\nabla_1 \mathring{Y} = \mathring{H}_1,\\
\nabla_1 \mathring{H}_\mu = -\mathring{X}^{\alpha}\overline{\nabla_\alpha L_{1\mu}} - 2\overline{L}_{(1}{}^{\alpha}(\mathring{F}_{\mu)\alpha} + \overline{g}_{\mu)\alpha}\mathring{Y})\\
\qquad - \overline{g}_{1\mu}\nu^0[\overline{\Gamma}^i_{01}\mathring{H}_i - \mathring{X}^{\alpha}\overline{\nabla_\alpha L_{01}} - 2\overline{L}_{(1}{}^{\alpha}(\mathring{F}_{0)\alpha} + \overline{g}_{0)\alpha}\mathring{Y})],
\end{cases} \tag{4.58}$$

which is uniquely defined by the values of $\mathring{X}_\mu$, $\mathring{F}_{\mu\nu}$, $\mathring{Y}$ and $\mathring{H}_\mu$ at the vertex of the cone.

We want to show that the fields which solve (4.58) are restrictions to the cone of smooth space-time fields: Given any vector ℓ^μ in the tangent space at O define $(x^\mu(s), X^\mu(s), F_{\mu\nu}(s), Y(s), H_\mu(s))$ as the unique solution of the problem

$$\begin{cases}
\dfrac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds} = 0,\\
\dfrac{dX_\mu}{ds} - \Gamma^\alpha_{\mu\beta}X_\alpha\dfrac{dx^\beta}{ds} = F_{\alpha\mu}\dfrac{dx^\alpha}{ds} + g_{\alpha\mu}Y\dfrac{dx^\alpha}{ds},\\
\dfrac{dF_{\mu\nu}}{ds} - \Gamma^\alpha_{\mu\gamma}F_{\alpha\nu}\dfrac{dx^\gamma}{ds} - \Gamma^\alpha_{\nu\gamma}F_{\mu\alpha}\dfrac{dx^\gamma}{ds} = 2g_{\gamma[\nu}H_{\mu]}\dfrac{dx^\gamma}{ds} - R_{\mu\nu\gamma}{}^{\alpha}X_\alpha\dfrac{dx^\gamma}{ds},\\
\dfrac{dY}{ds} = H_\alpha\dfrac{dx^\alpha}{ds},\\
\dfrac{dH_\mu}{ds} - \Gamma^\alpha_{\mu\beta}H_\alpha\dfrac{dx^\beta}{ds} = \{-X^\alpha\nabla_\alpha L_{\gamma\mu} - 2L_{(\gamma}{}^{\alpha}(F_{\mu)\alpha} + g_{\mu)\alpha}Y)\\
\qquad -g_{\gamma\mu}\nu^0[\Gamma^i_{01}H_i - X^\alpha\nabla_\alpha L_{01} - 2L_{(1}{}^{\alpha}(F_{0)\alpha} + g_{0)\alpha}Y)]\}\dfrac{dx^\gamma}{ds},\\
x^\mu(0) = 0, \quad \dfrac{dx^\mu}{ds}(0) = \ell^\mu,
\end{cases} \tag{4.59}$$

for given initial data $(X^\mu(0), F_{\mu\nu}(0), Y(0), H_\mu(0))$. As in [3, section 2.4] the system (4.59), together with the property that solutions of ODEs depend smoothly upon initial data, and that the trace of solutions of (4.59) on C_O solve (4.58), shows that the fields solving (4.58) are restrictions to the cone of smooth space-time fields. We have proved:

Proposition 4.9. *The condition (f) in theorem 4.4 can be removed.*

4.2. A stronger version of theorem 3.4 for two transversally intersecting null hypersurfaces

4.2.1. Stronger version. We want to establish the analogues of lemmas 4.2 and 4.3 for two transversally intersecting null hypersurfaces.

Lemma 4.10. *Assume that the wave equations (3.3) and (3.4) for X and Y hold, and that, on N_1, $\overline{A}_{2\mu} = \overline{A}_{AB} = 0 = \overline{B}_{22} = \overline{B}_{2A} = \overline{B}_{AB}$, similarly on N_2. Furthermore, we assume that $\nabla_{[1}A_{2]A}|_S = 0$. Then, on N_1, $\overline{A}_{11} = \overline{A}_{1A} = 0$ and $\overline{\nabla_1 A_{22}} = \overline{\nabla_1 A_{2A}} = \overline{\nabla_1 A_{AB}} = 0$, and a corresponding statement holds on N_2. On the closure of those sets where τ is non-zero the assumption $\overline{A}_{12} = 0$ is not needed but follows from the remaining assumptions, supposing that $A_{12}|_S = 0$.*

Proof. We can repeat most of the steps which were necessary to prove lemma 4.2. The only difference is that the ODEs are not of Fuchsian type anymore, but regular ones. To make sure that all the fields involved vanish on $N_1 \cup N_2$ we therefore need to make sure that we have vanishing initial data on S. This is the case if, on S,

$$A_{11} = A_{22} = A_{1A} = A_{2A} = \nabla_1 A_{2A} = \nabla_2 A_{1A} = g^{AB}\nabla_1 A_{AB} = g^{AB}\nabla_2 A_{AB} = 0.$$

Observing that the analogue of (4.7) for light-cones holds, i.e.

$$\overline{\nabla_{(1}A_{2)A}} = 0$$

this is an obvious consequence of the hypotheses made above. □

In analogy to lemma 4.3 we have

Lemma 4.11. *Assume that (3.3) and (3.4) hold, and that $\overline{A}_{\mu\nu} = 0$. Moreover, assume that, on N_1, $\overline{B}_{22} = \overline{B}_{2A} = \overline{B}_{AB} = 0$ and $\overline{\nabla_1 A_{22}} = \overline{\nabla_1 A_{2A}} = \overline{\nabla_1 A_{AB}} = 0$, similarly on N_2. Then, on N_1, $\overline{B}_{1\mu} = 0$, $\overline{\nabla_1 B_{22}} = \overline{\nabla_1 B_{2A}} = \overline{g}^{AB}\overline{\nabla_1 B_{AB}} = 0$, $\overline{\nabla_1 A_{1\mu}} = 0$ and $\overline{\nabla_1\nabla_1 A_{22}} = \overline{\nabla_1\nabla_1 A_{2A}} = \overline{g}^{AB}\overline{\nabla_1\nabla_1 A_{AB}} = 0$, and similar conclusions can be drawn on N_2.*

Proof. Again, we just need to make sure that all the initial data for the ODEs vanish on S. For all the field components involving covariant derivatives of $A_{\mu\nu}$ this follows directly from the vanishing of $\overline{A}_{\mu\nu}$. The vanishing of those field components involving (covariant derivatives of) $B_{\mu\nu}$ follows from the same fact, since, by (3.9), they can be expressed in terms of $A_{\mu\nu}$ and covariant derivatives thereof. □

Altogether we have proved

Theorem 4.12. *Assume that we have been given a $3 + 1$ dimensional space-time $(\mathscr{M}, g, \Theta)$, with (g, Θ) a smooth solution of the conformal field equations. Let $N_a \subset \mathscr{M}$, $a = 1, 2$, be two transversally intersecting null hypersurfaces with transverse intersection along a smooth two-dimensional submanifold S. Then there exists a vector field $\hat{X}$ satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(N_1 \cup N_2)$ if and only if there exists a pair (X, Y), X a vector field and Y a function, which fulfils the following conditions:*

(a) the conditions (i)–(iv) in theorem 3.4 hold,
(b) $\overline{A}_{12} = 0$ with $A_{\mu\nu} \equiv 2\nabla_{(\mu}X_{\nu)} - 2Y g_{\mu\nu}$,
(c) $\overline{A}_{AB} = 0 = \overline{A}_{22}|_{N_1} = \overline{A}_{2A}|_{N_1} = \overline{A}_{11}|_{N_2} = \overline{A}_{1A}|_{N_2}$,
(d) $\overline{\nabla_{[1}A_{2]A}}|_S = 0$,
(e) $\overline{B}_{AB} = 0 = \overline{B}_{22}|_{N_1} = \overline{B}_{2A}|_{N_1} = \overline{B}_{11}|_{N_2} = \overline{B}_{1A}|_{N_2}$ with $B_{\mu\nu} \equiv \mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y$.

In that case one may take $\hat{X} = X$ and $\nabla_\kappa \hat{X}^\kappa = 4Y$. The condition (b) is not needed on the closure of those sets where τ is non-zero, supposing that $A_{12}|_S = 0$.

4.2.2. The (proper) KID equations. Again, we would like to replace the non-intrinsic conditions (b), (e) and $\overline{\psi} = 0$ by conditions which do not involve transverse derivatives of X and Y. For the latter two this can be done as in the light-cone case. We just note that the ODEs for Υ_{N_a}, $a = 1, 2$, corresponding to (4.34), need to be supplemented by the boundary condition $\Upsilon_{N_a}|_S = \overline{\partial_a Y}$. To replace (b) one needs to take into account that, due to (4.39), we have

$$\overline{A}_{12} = 0 \quad \Longleftrightarrow \quad A_{12}|_S = 0 = S_{221}|_{N_1} = S_{112}|_{N_2}.$$

Furthermore, (b) and (c) imply

$$S_{A12}|_S = 2\nabla_{(A}A_{1)2} - \nabla_2 A_{1A} = 2\nabla_{[1}A_{2]A},$$

i.e. (d) can be replaced by the condition

$$\begin{aligned} 0 = S_{A12}|_S &\equiv 2\nabla_A\nabla_1 X_2 - 2R_{21A}{}^{\kappa}X_{\kappa} - 4\nabla_{(A}Y g_{1)2} + 2\nabla_2 Y g_{A1} \\ &= 2\nabla_A\nabla_1 X_2 - 2R_{21A}{}^{\kappa}X_{\kappa} - 2g_{12}\nabla_A Y. \end{aligned}$$

As a direct consequence of theorem (4.12) we end up with the result:

Theorem 4.13. *Assume we have been given a $3+1$-dimensional space-time $(\mathscr{M}, g, \Theta)$, with (g, Θ) a smooth solution of the conformal field equations. Let $\mathring{X}$ be a vector field and $\mathring{Y}$ a function defined on two transversally intersecting null hypersurfaces $N_a \subset \mathscr{M}$, $a = 1, 2$, with transverse intersection along a smooth two-dimensional submanifold S. Then there exists a smooth vector field X with $\overline{X} = \mathring{X}$ and $\overline{\nabla_\kappa X^\kappa} = 4\mathring{Y}$ satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(N_1 \cup N_2)$ (i.e. representing a Killing field of the physical space-time) if and only if the KID equations are fulfilled (we suppress the dependence of D_i on N_a):*

(i) $\mathring{X}^\mu\overline{\nabla_\mu\Theta} - \overline{\Theta}\mathring{Y} = 0,$

(ii) $\mathring{X}^\mu\overline{\nabla_\mu s} + \overline{s}\mathring{Y} - \overline{\nabla^2\Theta}\mathrm{D}_2\mathring{Y} - \overline{\nabla^A\Theta}\mathrm{D}_A\mathring{Y} - \Upsilon_{N_1}\overline{g}^{12}\nabla_1\overline{\Theta}|_{N_1} = 0,$
$\mathring{X}^\mu\overline{\nabla_\mu s} + \overline{s}\mathring{Y} - \overline{\nabla^1\Theta}\mathrm{D}_1\mathring{Y} - \overline{\nabla^A\Theta}\mathrm{D}_A\mathring{Y} - \Upsilon_{N_2}\overline{g}^{12}\nabla_2\overline{\Theta}|_{N_2} = 0,$

(iii) $\mathrm{D}_{(A}\mathring{X}_{B)} - \mathring{Y}\overline{g}_{AB} = 0,$
$\mathrm{D}_2\mathring{X}_2|_{N_1} = \mathrm{D}_{(2}\mathring{X}_{A)}|_{N_1} = 0,$
$\mathrm{D}_1\mathring{X}_1|_{N_2} = \mathrm{D}_{(1}\mathring{X}_{A)}|_{N_2} = 0,$

(iv) $\mathrm{D}_2\mathrm{D}_2\mathring{X}_1 - \overline{R}_{122}{}^{\kappa}\mathring{X}_\kappa - 2\overline{g}_{12}\mathrm{D}_2\mathring{Y}|_{N_1} = 0,$
$\mathrm{D}_1\mathrm{D}_1\mathring{X}_2 - \overline{R}_{211}{}^{\kappa}\mathring{X}_\kappa - 2\overline{g}_{12}\mathrm{D}_1\mathring{Y}|_{N_2} = 0,$

(v) $\mathring{X}^\kappa\nabla_\kappa\overline{L}_{2i} + 2\overline{L}_{\kappa(2}\mathrm{D}_{i)}\mathring{X}^\kappa + \mathrm{D}_2\mathrm{D}_i\mathring{Y}|_{N_1} = 0,\ i = 2, A,$
$\mathring{X}^\kappa\nabla_\kappa\overline{L}_{1i} + 2\overline{L}_{\kappa(1}\mathrm{D}_{i)}\mathring{X}^\kappa + \mathrm{D}_1\mathrm{D}_i\mathring{Y}|_{N_2} = 0,\ i = 1, A,$

(vi) $\mathring{X}^\kappa\nabla_\kappa\overline{L}_{AB} + 2\overline{L}_{\kappa(A}\mathrm{D}_{B)}\mathring{X}^\kappa + \mathrm{D}_A\mathrm{D}_B\mathring{Y} + \Upsilon_{N_a}\overline{g}^{12}\chi^{N_a}_{AB}|_{N_a} = 0,\ a = 1, 2,$

(vii) $\mathrm{D}_{(1}\mathring{X}_{2)} - \mathring{Y}\overline{g}_{12}|_S = 0,$

(viii) $2\mathrm{D}_A\mathrm{D}_1\mathring{X}_2 - 2\overline{R}_{21A}{}^{\kappa}\mathring{X}_\kappa - 2\overline{g}_{12}\mathrm{D}_A\mathring{Y}|_S = 0,$

where Υ_{N_1} is given by $\Upsilon_{N_1}|_S = \mathrm{D}_1\mathring{Y}$ and

$$\begin{aligned} &\left(\partial_2 + \frac{\tau_{N_1}}{2} - \overline{\Gamma}^1_{12}\right)\Upsilon_{N_1} - \overline{\Gamma}^2_{12}\mathrm{D}_2\mathring{Y} - \overline{\Gamma}^A_{12}\mathrm{D}_A\mathring{Y} \\ &\quad + \frac{1}{2}\overline{g}_{12}\left(\overline{g}^{22}\mathrm{D}_2\mathrm{D}_2\mathring{Y} + 2\overline{g}^{2A}\mathrm{D}_2\mathrm{D}_A\mathring{Y} + \overline{g}^{AB}\mathrm{D}_A\mathrm{D}_B\mathring{Y} + \frac{1}{6}\mathring{X}^\mu\overline{\nabla_\mu R} + \frac{1}{3}\overline{R}\mathring{Y}\right) = 0, \end{aligned}$$

similarly on N_2.

The condition (iv) is not needed on the closure of those sets on which the expansion τ is non-zero.

Proof. Once (i)–(viii) have been solved one uses the solutions $\mathring{X}$ and $\mathring{Y}$ as initial data for the wave equations (3.3) and (3.4). A solution exists due to [12], and the rest follows from the considerations above. □

Remark 4.14. As in [3] one could replace the condition $\overline{g}^{AB}\mathrm{D}_{(A}\mathring{X}_{B)} - 2\mathring{Y} = 0$ of (iii) by certain conditions on S if one makes sure that (iv) holds regardless of the (non-)vanishing of τ.

Remark 4.15. Theorem 4.13 can e.g. be applied to two null hypersurfaces intersecting transversally with one of them being part of $\mathscr{I}^-$.

5. KID equations on the light-cone C_{i^-}

Let us analyse now in detail the case where the initial surface is the light-cone C_{i^-} with vertex at past timelike infinity i^- in $3+1$-space-time dimensions (note that this requires a vanishing cosmological constant λ). In particular that means

$$\overline{\Theta} = 0. \tag{5.1}$$

That the corresponding initial value problem is well-posed for suitably prescribed data has been shown in [4]. Our aim is to apply theorem 4.4 and analyse the KID equations in this special case.

5.1. Gauge freedom and constraint equations

To make computations as easy as possible it is useful to impose a convenient gauge condition. We will adopt the gauge scheme explained in [11, section 2.2 & 4.1], where the reader is referred to for further details. Let us start with a brief overview over the relevant gauge degrees of freedom.

The freedom to choose the conformal factor Θ, regarded as an unknown in the conformal field equations (2.1)–(2.6), is comprised in the freedom to prescribe the Ricci scalar R and the function $\overline{s}$, where the latter one needs to be the restriction to C_{i^-} of a smooth function, non-vanishing at i^- (which ensures $\mathrm{d}\Theta|_{\mathscr{I}^-} \neq 0$).

As above, we will choose adapted null coordinates $(x^0 = u, x^1 = r, x^A)$, $A = 2, 3$, on C_{i^-}. The freedom to choose coordinates off the cone is reflected in the freedom to prescribe an arbitrary vector field W^σ for the $\hat{g}$*-generalized wave-map gauge condition*

$$H^\sigma := g^{\alpha\beta}\big(\Gamma^\sigma_{\alpha\beta} - \hat{\Gamma}^\sigma_{\alpha\beta}\big) - W^\sigma = 0,$$

where $\hat{g}$ denotes some target metric. The choice $W^\sigma = 0$ is called *wave-map gauge*.

This still leaves the freedom to parameterize the null geodesics generating C_{i^-}, due to which it is possible to additionally prescribe the function

$$\kappa := \nu^0\partial_1\nu_0 - \tfrac{1}{2}\tau - \tfrac{1}{2}\nu_0\big(\overline{g^{\mu\nu}\hat{\Gamma}^0_{\mu\nu}} + \overline{W}^0\big).$$

The choice $\kappa = 0$ corresponds to an affine parameterization. Moreover, when $H^\sigma = 0$ it holds that

$$\kappa = \overline{\Gamma}^1_{11}.$$

Henceforth we choose as in [4, 11]

$$R = 0, \quad \overline{s} = -2, \quad W^\sigma = 0, \quad \kappa = 0, \quad \hat{g} = \eta, \tag{5.2}$$

where

$$\eta := -(\mathrm{d}u)^2 + 2\,\mathrm{d}u\,\mathrm{d}r + r^2 s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$$

denotes the Minkowski metric in adapted null coordinates, with $s = s_{AB}\,dx^A\,dx^B$ being the standard metric on S^2.

Let us assume we have been given a smooth solution (g, Θ) of the conformal field equations (2.1)–(2.6) in the $(R = 0, \bar{s} = -2, \kappa = 0, \hat{g} = \eta)$-wave-map gauge.[3] It is shown in [11, section 4] that then the following equations are valid on C_{i^-},

$$\overline{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \overline{L}_{1\mu} = 0, \quad \overline{L}_{AB} = \omega_{AB}, \quad \overline{L}_{0A} = \tfrac{1}{2}\tilde{\nabla}_B\lambda_A{}^B, \tag{5.3}$$

$$\overline{\partial_0\Theta} = -2r, \quad \overline{\partial_0 g_{1\mu}} = 0, \tag{5.4}$$

$$\tau = 2/r, \quad \xi_A := -2\overline{\Gamma}^1_{1A} = 0, \quad \zeta := 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau = -2/r, \tag{5.5}$$

$$(\partial_1 - r^{-1})\lambda_{AB} = -2\omega_{AB}\,, \quad \lambda_A{}^A = \omega_A{}^A = 0, \tag{5.6}$$

where $\lambda_{AB} := \overline{\partial_0 g_{AB}} = O(r^3)$. The operator $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{g} := \overline{g}_{AB}dx^A dx^B$. The s-trace-free tensor $\omega_{AB} = O(r^2)$ may be regarded as representing the free initial data in the corresponding characteristic initial value problem [4, 11].

For convenience we give a list of the Christoffel symbols in adapted null coordinates on C_{i^-}, which are easily obtained from (5.3)–(5.6) and the formulae in [2, appendix A],

$$\overline{\Gamma}^0_{0\mu} = \overline{\Gamma}^\mu_{01} = \overline{\Gamma}^\mu_{11} = \overline{\Gamma}^0_{1A} = \overline{\Gamma}^1_{0A} = \overline{\Gamma}^1_{1A} = 0,$$
$$\overline{\Gamma}^1_{00} = \frac{1}{2}\overline{\partial_0 g_{00}}, \quad \overline{\Gamma}^C_{00} = \overline{g}^{CD}\overline{\partial_0 g_{0D}}, \quad \overline{\Gamma}^0_{AB} = -\frac{1}{r}\overline{g}_{AB}, \quad \overline{\Gamma}^C_{1A} = \frac{1}{r}\delta_A{}^C,$$
$$\overline{\Gamma}^C_{0A} = \frac{1}{2}\lambda_A{}^C, \quad \overline{\Gamma}^1_{AB} = -\frac{1}{r}\overline{g}_{AB} - \frac{1}{2}\lambda_{AB}, \quad \overline{\Gamma}^C_{AB} = \tilde{\Gamma}^C_{AB} = S^C_{AB}.$$

5.2. Analysis of the KID equations

5.2.1. The conditions $\overline{\phi} = 0$, $\overline{\psi}^{\mathrm{intr}} = 0$, $\overline{A}_{ij} = 0$ *and* $\overline{S}_{110} = 0$. With $\overline{\Theta} = 0$ and $\overline{\partial_0\Theta} = -2r$ it immediately follows that (recall that ϕ has been defined in theorem 4.1)

$$\overline{\phi} = 0 \quad \Longleftrightarrow \quad \mathring{X}^0 = 0, \tag{5.7}$$

i.e. any vector field satisfying the unphysical Killing equations necessarily needs to be tangent to C_{i^-}.

Taking further into account that $\bar{s} = -2$ and $\nu_0 = 1$ we obtain (recall that $\overline{\psi}^{\mathrm{intr}}$ has been defined in theorem 4.4)

$$\overline{\psi}^{\mathrm{intr}} = 0 \quad \Longleftrightarrow \quad (\partial_1 - r^{-1})\mathring{Y} = 0 \quad \Longleftrightarrow \quad \mathring{Y} = c(x^A)r, \tag{5.8}$$

for some angle-dependent function c. The condition $\overline{A}_{11} = 0$ is then automatically fulfilled. Furthermore, one readily checks that (we denote by $\mathscr{D}$ the Levi-Civita connection associated to the standard metric on S^2)

$$\overline{A}_{1A} = 0 \quad \Longleftrightarrow \quad \partial_1\mathring{X}^A = 0 \quad \Longleftrightarrow \quad \mathring{X}^A = d^A(x^B), \tag{5.9}$$

$$\overline{g}^{AB}\overline{A}_{AB} = 0 \quad \Longleftrightarrow \quad \mathring{X}^1 = -\tfrac{1}{2}r\mathscr{D}_A d^A + cr^2, \tag{5.10}$$

$$\breve{\overline{A}}_{AB} = 0 \quad \Longleftrightarrow \quad d^A \text{ is a conformal Killing field on } (S^2, s_{AB}). \tag{5.11}$$

Here and in what follows $\breve{.}$ denotes the s_{AB}- (equivalently the $\overline{g}_{AB}$-) trace-free part of the corresponding rank-2 tensor field.

Since $\tau = 2/r > 0$ the condition $\overline{S}_{110} = 0$ holds automatically for all $r > 0$.

[3] In fact it is not necessary here to require the rescaled Weyl tensor to be regular at i^-.

5.2.2. The conditions $\overline{B}_{1i} = 0$ *and* $\overline{B}^{\mathrm{intr}}_{AB} = 0$. First we solve (4.40) for Υ, which in our gauge becomes

$$(\partial_1 + r^{-1})\Upsilon + \tfrac{1}{2}r^{-2}\Delta_s \mathring{Y} + r^{-1}\partial_1\mathring{Y} = 0, \tag{5.12}$$

where we have set $\Delta_s := s^{AB}\mathscr{D}_A\mathscr{D}_B$. With $\mathring{Y} = cr$ and $\Upsilon = O(1)$ we obtain as the unique solution of (5.12)

$$\Upsilon = -\tfrac{1}{2}(\Delta_s + 2)c. \tag{5.13}$$

For $\overline{B}_{1i}$ we find

$$\begin{aligned}\overline{B}_{11} &\equiv \mathring{X}^\kappa \nabla_\kappa \overline{L}_{11} + 2\overline{L}_{\kappa(1}\mathrm{D}_{1)}\mathring{X}^\kappa + \mathrm{D}_1\mathrm{D}_1\mathring{Y} \\ &= 0, \\ \overline{B}_{1A} &\equiv \mathring{X}^\kappa \nabla_\kappa \overline{L}_{1A} + 2\overline{L}_{\kappa(1}\mathrm{D}_{A)}\mathring{X}^\kappa + \mathrm{D}_1\mathrm{D}_A\mathring{Y} \\ &= \omega_{AB}\partial_1\mathring{X}^B + \partial_A(\partial_1 - r^{-1})\mathring{Y} \\ &= 0\end{aligned}$$

without any further restrictions on $\mathring{X}$, $\mathring{Y}$ or the initial data ω_{AB}. It remains to determine $\overline{B}^{\mathrm{intr}}_{AB}$,

$$\begin{aligned}\overline{B}^{\mathrm{intr}}_{AB} &= \mathring{X}^i\nabla_i\overline{L}_{AB} + 2\overline{L}_{0(A}\mathrm{D}_{B)}\mathring{X}^0 + 2\overline{L}_{C(A}\mathrm{D}_{B)}\mathring{X}^C + \mathrm{D}_A\mathrm{D}_B\mathring{Y} + r^{-1}\overline{g}_{AB}\Upsilon \\ &= \mathring{X}^1\partial_1\omega_{AB} + \mathring{X}^C\tilde{\nabla}_C\omega_{AB} + 2\omega_{C(A}\tilde{\nabla}_{B)}\mathring{X}^C + \tilde{\nabla}_A\tilde{\nabla}_B\mathring{Y} + \tfrac{1}{2}\lambda_{AB}\partial_1\mathring{Y} + r^{-1}\overline{g}_{AB}(\partial_1\mathring{Y} + \Upsilon).\end{aligned}$$

We first compute its trace,

$$\overline{g}^{AB}\overline{B}^{\mathrm{intr}}_{AB} = 2\omega^{AB}(\tilde{\nabla}_A\mathring{X}_B)\breve{} + \Delta_{\tilde{g}}\mathring{Y} + 2r^{-1}\partial_1\mathring{Y} + 2r^{-1}\Upsilon = 0,$$

again without any further restrictions. For its traceless part we find

$$\begin{aligned}\breve{\overline{B}}^{\mathrm{intr}}_{AB} &= \mathring{X}^1\partial_1\omega_{AB} + \mathring{X}^C\tilde{\nabla}_C\omega_{AB} + 2\omega_{C(A}\tilde{\nabla}_{B)}\mathring{X}^C - \overline{g}_{AB}\omega^{CD}(\tilde{\nabla}_C\mathring{X}_D)\breve{} + (\tilde{\nabla}_A\tilde{\nabla}_B\mathring{Y})\breve{} + \tfrac{1}{2}\lambda_{AB}\partial_1\mathring{Y} \\ &= \mathscr{L}_d\omega_{AB} - \tfrac{1}{2}r\partial_1\omega_{AB}\mathscr{D}_C d^C + cr^2\partial_1\omega_{AB} + \tfrac{1}{2}c\lambda_{AB} + r(\mathscr{D}_A\mathscr{D}_B c)\breve{}.\end{aligned}$$

Recall that regularity of the metric requires $\omega_{AB} = O(r^2)$ and $\lambda_{AB} = O(r^3)$, in particular $\mathscr{L}_d\omega_{AB} = O(r^2)$. Hence $\breve{\overline{B}}^{\mathrm{intr}}_{AB} = 0$ if and only if

$$\mathring{\nabla}_A c \text{ is a conformal Killing field on } (S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B), \tag{5.14}$$

$$\mathscr{L}_d\omega_{AB} - \tfrac{1}{2}r\partial_1\omega_{AB}\mathring{\nabla}_C d^C + cr^2\partial_1\omega_{AB} + \tfrac{1}{2}c\lambda_{AB} = 0. \tag{5.15}$$

5.2.3. Summary. By way of summary the conditions (a)–(f) in theorem 4.4 hold if and only if

$$\mathring{X}^0 = 0, \tag{5.16}$$

$$\mathring{X}^A = d^A, \tag{5.17}$$

$$\mathring{X}^1 = -\tfrac{1}{2}r\mathscr{D}_A d^A + cr^2, \tag{5.18}$$

$$\mathring{Y} = cr, \tag{5.19}$$

such that

$$\mathscr{D}_A c \text{ and } d_A \text{ are conformal Killing fields on } (S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B), \tag{5.20}$$

$$\mathscr{L}_d\omega_{AB} - \tfrac{1}{2}r\mathscr{D}_C d^C\partial_1\omega_{AB} + cr^2\partial_1\omega_{AB} + \tfrac{1}{2}c\lambda_{AB} = 0. \tag{5.21}$$

In section 4.1.4 we have shown that solutions of the KID equations are restrictions to the light-cone of smooth space-time fields. On C_{i^-} this turns out to be a trivial issue anyway: the candidate fields satisfying (5.16)–(5.20) are explicitly known[4] and coincide *independently of the choice of initial data* ω_{AB}, with the restriction to C_{i^-} of the Minkowskian Killing fields.

While in the Minkowski case $\omega_{AB} = 0$ every candidate field does extend to a Killing vector field, equation (5.21) provides an obstruction equation for non-flat data. We call (5.21) the *reduced KID equations*.

As a corollary of theorem 4.4 we obtain:

Theorem 5.1. *Assume that we have been given a 3 + 1-dimensional 'unphysical' space-time* $(\mathscr{M}, g, \Theta)$ *which contains a regular C_{i^-}-cone (the cosmological constant λ thus needs to vanish) and where (g, Θ) is a smooth solution of the conformal field equations in the $(R = 0, \bar{s} = -2, \kappa = 0, \hat{g} = \eta)$-wave-map gauge. Then there exists a smooth vector field X satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(C_{i^-})$ (i.e. representing a Killing field of the physical space-time) if and only if there exist a function c and a vector field d^A on S^2 with $\mathscr{D}_A c$ and d_A conformal Killing fields on $(S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B)$ such that the reduced KID equations*

$$\mathscr{L}_d \omega_{AB} - \tfrac{1}{2} r \partial_1 \omega_{AB} \mathscr{D}_C d^C + c r^2 \partial_1 \omega_{AB} + \tfrac{1}{2} c \lambda_{AB} = 0 \tag{5.22}$$

are satisfied on C_{i^-} (recall that λ_{AB} is the unique solution of $(\partial_1 - r^{-1})\lambda_{AB} = -2\omega_{AB}$ with $\lambda_{AB} = O(r^3)$).

The Killing field satisfies

$$\overline{X}^0 = 0, \quad \overline{X}^A = d^A, \quad \overline{X}^1 = -\tfrac{1}{2} r \mathscr{D}_A d^A + c r^2, \quad \overline{\nabla_\mu X^\mu} = 4cr. \tag{5.23}$$

Remark 5.2. The reduced KID equations (5.22) can be replaced by one of their equivalents (i)–(iii) in lemma 5.3.

5.3. Analysis of the reduced KID equations

5.3.1. Equivalent representations of the reduced KID equations. We provide some alternative formulations of the reduced KID equations.

Lemma 5.3. *The reduced KID equations (5.22) are equivalent to each of the following equations:*

(i) $\mathscr{L}_d \lambda_{AB} - (\tfrac{1}{2} r \mathscr{D}_C d^C - c r^2)\partial_1 \lambda_{AB} + (\tfrac{1}{2}\mathscr{D}_C d^C - 2cr)\lambda_{AB} = 0,$

(ii) $(\partial_1 - r^{-1})\mathscr{L}_d \omega_{AB} - \tfrac{1}{2} r \partial^2_{11} \omega_{AB} \mathscr{D}_C d^C + c r^2 \partial_1 (\partial_1 + r^{-1})\omega_{AB} = 0,$

(iii) $2\mathscr{L}_d \overline{L}_{0A} + (1 - r\partial_1)\overline{L}_{0A} \mathscr{D}_B d^B + r \omega_A{}^C \mathscr{D}_C \mathscr{D}_B d^B + 2cr^2 \partial_1 \overline{L}_{0A} - (2\omega_{AB} + r^{-1}\lambda_{AB})\mathscr{D}^B c = 0$ *(recall that* $\overline{L}_{0A} = \tfrac{1}{2}\tilde{\nabla}_B \lambda_A{}^B$*).*

Proof. (i) Applying $(\partial_1 - r^{-1})$ to (i) yields (5.22), equivalence follows from regularity.

(ii) Applying $(\partial_1 - r^{-1})$ to (5.22) yields (ii), equivalence follows from regularity.

(iii) We use the fact that on $(S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B)$ the equations $w_{AB} = 0$ and $\mathscr{D}^B w_{AB} = 0$ with w_{AB} trace-free are equivalent. Taking the divergence of (i) and invoking the conformal Killing equation for d^A complete the equivalence proof. □

Both ω_{AB} or λ_{AB} may be regarded as the freely prescribable initial data. So (i) and (ii) in lemma 5.3 provide formulations of the reduced KID equations which involve exclusively explicitly known quantities for all admissible initial data. In the case of an ordinary cone, treated in [3], this is not possible: For generic KIDs there, neither the candidate fields nor all the relevant metric components can be computed analytically.

[4] The function c satisfies the equation $\mathscr{D}_A(\Delta_s + 2)c = 0$ and can thus be written as linear combination of $\ell = 0, 1$ spherical harmonics. Conformal Killing fields on the round 2-sphere are discussed in appendix B.

5.3.2. Some special cases. We continue with a brief discussion of some special cases. There exists a vector field X satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(C_{i^-})$ with

(1) $\overline{\nabla_\mu X^\mu} = 0 \iff \exists$ a conformal Killing field d^A on $(S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B)$ with $\mathscr{L}_d\omega_{AB} = \frac{1}{2}r\partial_1\omega_{AB}\mathscr{D}_C d^C$,
(2) $\overline{X}^1 = 0 \iff \exists$ a Killing field d^A on $(S^2, s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B)$ with $\mathscr{L}_d\omega_{AB} = 0$,
(3) $\overline{X}^A = 0 \iff \partial_1(\partial_1 + r^{-1})\omega_{AB} = 0 \overset{\omega_{AB}=O(r^2)}{\iff} \omega_{AB} = 0$
($\omega_{AB} \equiv \breve{\overline{L}}_{AB} = O(r^2)$ is a necessary condition for the Schouten tensor to be regular at i^-).

The third case shows that the property $\overline{X}^A = 0$ is compatible only with the Minkowski case (supposing that i^- is a regular point). In the non-flat case any non-trivial vector field satisfying the unphysical Killing equations has a non-trivial component $\overline{X}^A = d^A \not\equiv 0$. Since

$$\overline{g}_{\mu\nu}\overline{X}^\mu\overline{X}^\nu = \overline{g}_{AB}\overline{X}^A\overline{X}^B = r^2 s_{AB}d^A d^B,$$

we see that there are no non-trivial vector fields satisfying the unphysical Killing equations which are null on C_{i^-}. To put it differently, possibly apart from certain directions determined by the zeros of d^A, any isometry of a non-flat, asymptotically flat vacuum space-time is necessarily spacelike sufficiently close to $\mathscr{I}^-$. This leads to the following version of a classical result of Lichnerowicz [9]:

Theorem 5.4. *Minkowski space-time is the only stationary vacuum space-time which admits a regular C_{i^-}-cone.*

5.3.3. Structure of the solution space. Let X and $\hat{X}$ be two distinct non-trivial solutions of the unphysical Killing equations (3.1). Since solutions of these equations form a Lie algebra, $\hat{\hat{X}} := [X, \hat{X}]$ is another, possibly trivial solution. We have

$$\overline{\hat{\hat{X}}}{}^0 = \overline{[X, \hat{X}]}{}^0 = 0,$$
$$\overline{\hat{\hat{X}}}{}^A = \overline{[X, \hat{X}]}{}^A = [d, \hat{d}]^A,$$
$$\overline{\hat{\hat{X}}}{}^1 = \overline{[X, \hat{X}]}{}^1 = -\tfrac{1}{2}r\mathscr{D}_B[d, \hat{d}]^B + r^2\big(d^B\mathscr{D}_B\hat{c} - \hat{d}^B\mathscr{D}_B c + \tfrac{1}{2}c\mathscr{D}_B\hat{d}^B - \tfrac{1}{2}\hat{c}\mathscr{D}_B d^B\big).$$

Hence, by their derivation, the reduced KID equations are fulfilled with

$$\hat{\hat{d}}{}^A = [d, \hat{d}]^A,$$
$$\hat{\hat{c}} = d^B\mathscr{D}_B\hat{c} - \hat{d}^B\mathscr{D}_B c + \tfrac{1}{2}c\mathscr{D}_B\hat{d}^B - \tfrac{1}{2}\hat{c}\mathscr{D}_B d^B,$$

and $\hat{\hat{d}}{}^A$ and $\mathscr{D}_A\hat{\hat{c}}$ are conformal Killing fields on the standard 2-sphere. Indeed, via the relation $\mathscr{L}_{[d,\hat{d}]}\lambda_{AB} = [\mathscr{L}_d, \mathscr{L}_{\hat{d}}]\lambda_{AB}$, this can be straightforwardly checked. We refer the reader to appendix B where the conformal Killing fields on the standard 2-sphere are explicitly given.

Let us consider for the moment flat initial data $\lambda_{AB} = 0$ which generate Minkowski space-time. Then one has ten independent isometries.

- The four *translations* are generated by the tuples $(c, d^A = 0)$ with c being a spherical harmonic function of degree $\ell = 0$ or 1.
- The three *rotations* are generated by the tuples $(c = 0, d^A)$ with d^A being a Killing field on $(S^2, s_{AB}\mathrm{d}x^A\mathrm{d}x^B)$.
- The three *boosts* are generated by the tuples $(c = 0, d^A = \mathscr{D}^A f)$ with f being a spherical harmonic function of degree $\ell = 1$.

We have already seen above that translations (in the above sense) cannot exists in the non-flat case $\lambda_{AB} \neq 0$ if the Schouten tensor is assumed to be regular at i^-.

Proposition 5.5. *Minkowski space-time is the only vacuum space-time with a regular C_{i^-}-cone which admits translational Killing vector fields.*

This is linked with another observation. Since, in the non-flat case, any non-trivial Killing field of the physical space-time (i.e. a vector field satisfying the unphysical Killing equations) has a non-trivial d^A, for a given d^A there can be at most one c such that (c, d^A) solves the reduced KID equations. Now the standard 2-sphere admits six independent conformal Killing vector fields d^A. We thus have:

Proposition 5.6. *Any non-flat vacuum space-time with a regular C_{i^-}-cone admits at most six independent Killing vector fields.*

Now let us assume that there are two distinct rotations, i.e. two Killing fields $d^{(1)}$ and $d^{(2)}$ on $(S^2, s_{AB}\mathrm{d}x^A\mathrm{d}x^B)$ such that $(c = 0, d = d^{(i)})$, $i = 1, 2$, solves the reduced KID equations. Then $(c = 0, d = d^{(3)})$ with $d^{(3)} = [d^{(1)}, d^{(2)}]$ provides another independent, non-trivial solution of the reduced KID equations. Altogether we have

$$\mathscr{L}_{d^{(i)}}\lambda_{AB} = 0, \quad i = 1, 2, 3 \quad \Longrightarrow \quad \lambda_{AB} \propto s_{AB} \quad \Longrightarrow \quad \lambda_{AB} = 0,$$

since λ_{AB} is trace-free. This recovers the well-known fact that two rotational symmetries imply Minkowski space-time.

Acknowledgments

It is a pleasure to thank my advisor Piotr T Chruściel for various valuable comments and for reading a first draft of this article. Supported in part by the Austrian Science Fund (FWF): P 24170-N16.

Appendix A. Fuchsian ODEs

As we have not been able to find an adequate reference, we state and prove here a key result about Fuchsian ODEs which is used in our work.

Lemma A.1. *Let $a > 0$, and for $r \in (0, a)$ consider a first-order ODE-system of the form*

$$\partial_r\phi = r^{-1}A\phi + M(r)\phi, \tag{A.1}$$

for a set of fields $\phi = (\phi^I)$, $I = 1, \ldots, N$, where A is an $N \times N$-matrix, and where $M(r)$ is a continuous map on $[0, a)$ with values in $N \times N$-matrices which satisfies $r\|M(r)\|_{op} = o(1)$. Let λ denote the smallest number so that

$$\langle\phi, A\phi\rangle \leqslant \lambda\|\phi\|^2.$$

Suppose that there exists $\epsilon > 0$ such that

$$\phi = O(r^{\lambda+\epsilon}).$$

Then

$$\phi \equiv 0.$$

Proof. The proof is done by a simple energy estimate. Set

$$\langle\phi, \psi\rangle := \sum_I \phi^I\psi^I, \quad \|\phi\|^2 := \langle\phi, \phi\rangle\,,$$

then for any $k \in \mathbb{R}$

$$\begin{aligned}\partial_r(r^{-2k}\|\phi\|^2) &= 2r^{-2k}\phi\partial_r\phi - 2kr^{-2k-1}\|\phi\|^2\\ &= 2r^{-2k-1}(\langle\phi, A\phi\rangle + r\,\langle\phi, M(r)\phi\rangle - k\|\phi\|^2)\\ &\leqslant 2r^{-2k-1}(\lambda - k + r\|M(r)\|_{op})\|\phi\|^2.\end{aligned}$$

Applying $\int_{r_0}^r$ yields (assume $r_0 < r$)

$$\begin{aligned}r^{-2k}\|\phi(r)\|^2 &\leqslant r_0^{-2k}\|\phi(r_0)\|^2 + 2\int_{r_0}^r(\lambda - k + \tilde{r}\|M(\tilde{r})\|_{op})\tilde{r}^{-2k-1}\|\phi\|^2\mathrm{d}\tilde{r}\\ &\leqslant r_0^{-2k}\|\phi(r_0)\|^2 + 2\Big(\lambda - k + \sup_{0<\tilde{r}<r}(\tilde{r}\|M(\tilde{r})\|_{op})\Big)\int_{r_0}^r\tilde{r}^{-2k-1}\|\phi\|^2\mathrm{d}\tilde{r}.\end{aligned}$$

Due to our assumption $\phi = O(r^{\lambda+\varepsilon})$ any $\lambda < k_0 < \lambda + \varepsilon$ satisfies $r^{-2k_0}\|\phi\|^2 = O(r^{2\delta})$, where $\delta := \lambda - k_0 + \varepsilon > 0$. We then take the limit $r_0 \to 0$,

$$\begin{aligned}r^{-2k_0}\|\phi(r)\|^2 &\leqslant 2\Big(\lambda - k_0 + \sup_{0<\tilde{r}<r}(\tilde{r}\|M(\tilde{r})\|_{op})\Big)\int_0^r\tilde{r}^{-2k_0-1}\|\phi\|^2\mathrm{d}\tilde{r}\\ &\leqslant 0 \quad \text{for sufficiently small } r.\end{aligned}$$

Thus ϕ vanishes for small r, but then it needs to vanish for all r. □

Appendix B. Conformal Killing fields on the round 2-sphere

We consider the 2-sphere equipped with the standard metric

$$s = s_{AB}\mathrm{d}x^A\,\mathrm{d}x^B = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2.$$

It admits the maximal number of independent conformal Killing vector fields, which is 6. There are three independent Killing vector fields,

$$\begin{aligned}K_{(1)} &= \partial_\varphi,\\ K_{(2)} &= \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi,\\ K_{(3)} &= \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi,\end{aligned}$$

and three independent conformal Killing fields which are not Killing fields,

$$\begin{aligned}C_{(1)} &= \sin\theta\partial_\theta,\\ C_{(2)} &= \cos\theta\cos\varphi\partial_\theta - \sin^{-1}\theta\sin\varphi\partial_\varphi,\\ C_{(3)} &= \cos\theta\sin\varphi\partial_\theta + \sin^{-1}\theta\cos\varphi\partial_\varphi.\end{aligned}$$

All the $C_{(i)}$'s turn out to be gradients of $\ell = 1$-spherical harmonics,

$$\begin{aligned}C^A_{(1)} &= \mathscr{D}^A c_{(1)}, \quad \text{where} \quad c_{(1)} = \cos\theta,\\ C^A_{(2)} &= \mathscr{D}^A c_{(2)}, \quad \text{where} \quad c_{(2)} = \sin\theta\cos\varphi,\\ C^A_{(3)} &= \mathscr{D}^A c_{(3)}, \quad \text{where} \quad c_{(3)} = \sin\theta\sin\varphi.\end{aligned}$$

Moreover,

$$\mathscr{D}_A C^A_{(i)} = \Delta_s c_{(i)} = -2c_{(i)}, \quad i = 1, 2, 3.$$

The conformal Killing fields satisfy the commutation relations

$$\begin{aligned}[K_{(i)}, K_{(j)}] &= \sum_k \varepsilon_{ijk}K_{(k)},\\ [C_{(i)}, C_{(j)}] &= -\sum_k \varepsilon_{ijk}K_{(k)},\\ [K_{(i)}, C_{(j)}] &= \sum_k \varepsilon_{ijk}C_{(k)},\end{aligned}$$

i.e. they form a Lie algebra isomorphic to the Lie algebra $so(3, 1)$ of the Lorenz group in four dimensions. The Killing fields form a Lie subalgebra.

References

[1] Beig R and Chruściel P T 1997 Killing initial data *Class. Quantum Grav.* **14** A83–92
[2] Choquet-Bruhat Y, Chruściel P T and Martín-García J M 2011 The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions *Ann. Henri Poincaré* **12** 419–82
[3] Chruściel P T and Paetz T-T 2013 KIDs like cones *Class. Quantum Grav.* **30** 235036
[4] Chruściel P T and Paetz T-T 2013 Solutions of the vacuum Einstein equations with initial data on past null infinity *Class. Quantum Grav.* **30** 235037
[5] Dossa M 1997 Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoide caractéristique *Ann. Inst. Henri Poincaré (A) Phys. Théor.* **66** 37–107
[6] Friedrich H 1986 Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant *J. Geom. Phys.* **3** 101–17
[7] Friedrich H 2002 Conformal Einstein evolution *The Conformal Structure of Space-time—Geometry, Analysis, Numerics* ed J Frauendiener and H Friedrich (Berlin: Springer) pp 1–50
[8] Kánnár J 2000 On Killing vectors in initial value problems for asymptotically flat space-times *Class. Quantum Grav.* **17** 4981–90
[9] Lichnerowicz A 1955 *Théories Relativistes de la Gravitation Et de L'électromagnétisme: Relativité Générale Et Théories Unitaires* (Paris: Masson et Cie)
[10] Moncrief V 1975 Spacetime symmetries and linearization stability of the Einstein equations: I *J. Math. Phys.* **16** 493–8
[11] Paetz T-T 2013 Conformally covariant systems of wave equations and their equivalence to Einstein's field equations arXiv:1306.6204 [gr-qc]
[12] Rendall A D 1990 Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations *Proc. R. Soc. Lond.* A **427** 221–39

Killing Initial Data on spacelike conformal boundaries*

Tim-Torben Paetz†

Gravitational Physics, University of Vienna
Boltzmanngasse 5, 1090 Vienna, Austria

March 12, 2014

Abstract

We analyze Killing Initial Data on Cauchy surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. As an application, we derive the KID equations on a spacelike $\mathscr{I}^-$.

PACs number: 04.20.Ex, 04.20.Ha

Contents

*Preprint UWThPh-2013-8.
†E-mail: Tim-Torben.Paetz@univie.ac.at

1 Introduction

Symmetries are of utmost importance in physics, and so is the construction of space-times $(\tilde{\mathscr{M}}, \tilde{g})$ satisfying Einstein's field equations in general relativity which possess k-parameter groups of isometries, $1 \leq k \leq 10$ when $\dim\tilde{\mathscr{M}} = 4$, generated by so-called Killing vector fields. Indeed, such space-times can be systematically constructed in terms of an initial value problem when the usual constraint equations, which are required to be fulfilled by appropriately prescribed initial data, are supplemented by certain additional equations, the Killing Initial Data (KID) equations.

The KID equations have been derived on spacelike as well as characteristic initial surfaces (cf. [1, 3] and references therein). In [10] the same issue was analyzed for characteristic surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. In particular, for vanishing cosmological constant, the KID equations on a light-cone with vertex at past timelike infinity have been derived there. The aim of this work is to carry out the corresponding analysis on spacelike hypersurfaces in conformally rescaled vacuum space-times. As a special case we shall derive the KID equations on $\mathscr{I}^-$ supposing that the cosmological constant is positive so that $\mathscr{I}^-$ is a spacelike hypersurface.

In Section 2 we recall the conformal field equations, discuss their gauge freedom and derive the constraint equations induced on $\mathscr{I}^-$. Well-posedness of the Cauchy problem for the conformal field equations with data on $\mathscr{I}^-$ was shown in [4], we shall provide an alternative proof based on results proved in Appendix A by using a system of wave equations.

The "unphysical Killing equations", introduced in [10] replace, and are in fact equivalent to, the original-space-time Killing equations in the unphysical space-time. Employing results in [10] we derive in Section 3 necessary-and-sufficient conditions on a spacelike hypersurface in a space-time satisfying the conformal field equations which guarantee existence of a vector field fulfilling these equations (cf. Theorem 3.3). Similar to the proceeding in [3, 10] we first derive an intermediate result, Theorem 3.1, with a couple of additional hypotheses, which then are shown to be automatically satisfied.

In Section 4 we apply Theorem 3.3 to the special case where the spacelike hypersurface is $\mathscr{I}^-$. We shall see that some of the KID equations determine a set of candidate fields on $\mathscr{I}^-$. Whether or not these fields extend to vector fields satisfying the unphysical Killing equations depends on the remaining "reduced KID equations". As for a light-cone with vertex at past timelike infinity it turns out that the KID equations adopt at infinity a significantly simpler form as compared to "ordinary" Cauchy surfaces (cf. Theorem 4.1).

2 Setting

2.1 Conformal field equations

In 3+1 dimensions Friedrich's *metric conformal field equations (MCFE)* (cf. [5])[1]

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0\ , \tag{2.1}$$
$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta\, d_{\nu\mu\sigma}{}^\rho\ , \tag{2.2}$$
$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu}\ , \tag{2.3}$$
$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta\ , \tag{2.4}$$
$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3\ , \tag{2.5}$$
$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}) \tag{2.6}$$

form a closed system of equations for the unknowns $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$. The tensor field $L_{\mu\nu}$ denotes the Schouten tensor,

$$L_{\mu\nu} = \frac{1}{2} R_{\mu\nu} - \frac{1}{12} R g_{\mu\nu}\ , \tag{2.7}$$

while

$$d_{\mu\nu\sigma}{}^\rho = \Theta^{-1} C_{\mu\nu\sigma}{}^\rho \tag{2.8}$$

is a rescaling of the conformal Weyl tensor $C_{\mu\nu\sigma}{}^\rho$. The function s is defined as

$$s = \frac{1}{4} \Box_g \Theta + \frac{1}{24} R\Theta\ . \tag{2.9}$$

Friedrich has shown that the MCFE are equivalent to Einstein's vacuum field equations with cosmological constant λ in regions where the conformal factor Θ, relating the "unphysical" metric $g = \Theta^2 g_{\mathrm{phys}}$ with the physical metric g_{phys}, is positive. Their advantage lies in the property that they remain regular even where Θ vanishes.

The system (2.1)-(2.6) treats s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ as independent of $g_{\mu\nu}$ and Θ. However, once a solution of the MCFE has been given these fields are related to $g_{\mu\nu}$ and Θ via (2.7)-(2.9). A solution of the MCFE is thus completely determined by the pair $(g_{\mu\nu}, \Theta)$.

[1] It is indicated in [10] that things are considerably different in higher dimensions, which is why we restrict attention to 4 dimensions from the outset.

2.2 Gauge freedom

2.2.1 Conformal factor

Let $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^{\rho})$ be some smooth solution of the MCFE.[2] From $g_{\mu\nu}$ we compute R. Let us then conformally rescale the metric, $g \mapsto \phi^2 g$, for some positive function $\phi > 0$. The Ricci scalars R and R^* of g and $\phi^2 g$, respectively, are related via (set $\Box_g := g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\phi R - \phi^3 R^* = 6\Box_g \phi \,. \tag{2.10}$$

Now, let us *prescribe* R^* and read (2.10) as an equation for ϕ. When dealing with a Cauchy problem with data on some spacelike hypersurface $\mathcal{H}$ (including $\mathscr{I}^-$ for $\lambda > 0$) we are free to prescribe functions $\phi|_{\mathcal{H}} =: \mathring{\phi} > 0$ and $\partial_0\phi|_{\mathcal{H}} =: \mathring{\psi}$ on $\mathcal{H}$.[3] Throughout $x^0 \equiv t$ denotes a time-coordinate so that ∂_0 is transverse to $\mathcal{H}$. According to standard results there exists a unique solution $\phi > 0$ in some neighborhood of $\mathcal{H}$ which induces the above data on $\mathcal{H}$. The MCFE are conformally covariant, meaning that the conformally rescaled fields

$$\begin{aligned} g^* &= \phi^2 g \,, && (2.11)\\ \Theta^* &= \phi\,\Theta \,, && (2.12)\\ s^* &= \frac{1}{4}\Box_{g^*}\Theta^* + \frac{1}{24}R^*\Theta^* \,, && (2.13)\\ L^*_{\mu\nu} &= \frac{1}{2}R^*_{\mu\nu}[g^*] - \frac{1}{12}R^* g^*_{\mu\nu} \,, && (2.14)\\ d^*_{\mu\nu\sigma}{}^{\rho} &= \phi^{-1} d_{\mu\nu\sigma}{}^{\rho} \,, && (2.15) \end{aligned}$$

provide another solution of the MCFE, now with Ricci scalar R^*, which represents the same physical solution: If the conformal factor Θ is treated as an unknown, determined by the MCFE, the unphysical Ricci scalar R can be arranged to adopt any preassigned form, it represents a *conformal gauge source function.*

There remains the gauge freedom to prescribe the functions $\mathring{\phi}$ and $\mathring{\psi}$ on $\mathcal{H}$. On an ordinary hypersurface, where Θ has no zeros, this freedom can be used to prescribe $\Theta|_{\mathcal{H}}$ and $\partial_0\Theta|_{\mathcal{H}}$. A main object of this work is to treat the case $\mathcal{H} = \mathscr{I}^-$, where, by definition, $\Theta = 0$ (and $\mathrm{d}\Theta \neq 0$). We shall show that in this situation the gauge freedom allows one to prescribe the function s on $\mathscr{I}^-$ and to make conformal rescalings of the induced metric on $\mathscr{I}^-$.

To see this we consider a smooth solution of the MCFE to the future of $\mathscr{I}^-$. Now (2.5) and $\mathrm{d}\Theta|_{\mathscr{I}^-} \neq 0$ enforce $\overline{g}^{00} < 0$ (hence, as is well known, $\mathscr{I}^-$ must be spacelike when $\lambda > 0$). Due to (2.5), the function s can be written away from $\mathscr{I}^-$ as

$$s = \frac{1}{2}\Theta^{-1}\nabla_\mu\Theta\nabla^\mu\Theta + \frac{1}{6}\Theta^{-1}\lambda \,,$$

and the right-hand side is smoothly extendable at $\mathscr{I}^-$. A conformal rescaling

$$\Theta \mapsto \Theta^* := \phi\,\Theta \,, \quad g_{\mu\nu} \mapsto g^*_{\mu\nu} := \phi^2 g_{\mu\nu} \,, \quad \phi > 0 \,, \tag{2.16}$$

[2] For convenience we restrict attention throughout to the smooth case, though similar results can be obtained assuming finite differentiability.

[3] The positivity-assumption on $\mathring{\phi}$ makes sure that the solution of (2.10) is positive sufficiently close to $\mathcal{H}$ and thereby that the new conformal factor Θ^* is positive as well (in the $\mathscr{I}^-$-case just off the initial surface).

maps the function s to

$$s^* = \phi^{-1}\Big(\frac{1}{2}\Theta\phi^{-2}\nabla^\mu\phi\nabla_\mu\phi + \phi^{-1}\nabla^\mu\Theta\nabla_\mu\phi + s\Big) . \tag{2.17}$$

The trace of this equation on $\mathscr{I}^-$ is

$$\overline{\nabla^\mu\Theta\nabla_\mu\phi + \phi\, s - \phi^2 s^*} = 0 , \tag{2.18}$$

or, in coordinates adapted to $\mathscr{I}^-$, i.e. for which $\mathscr{I}^- = \{x^0 \equiv t = 0\}$ locally,

$$\overline{g^{0\mu}\nabla_0\Theta\nabla_\mu\phi + \phi\, s - \phi^2 s^*} = 0 . \tag{2.19}$$

Here and henceforth we use overlining to denote restriction to the initial surface. Let us prescribe $\overline{s}^*$. We choose any $\mathring{\phi} > 0$ to conformally rescale the induced metric on $\mathscr{I}^-$. Then we solve (2.19) for $\mathring{\psi} \equiv \overline{\nabla_0\phi}$ (recall that$\overline{\nabla_0\Theta}$ and $\overline{g}^{00}$ are not allowed to have zeros on $\mathscr{I}^-$). We take the so-obtained functions $\mathring{\phi} > 0$ and $\mathring{\psi}$ as initial data for (2.10).

By way of summary, the conformal covariance of the MCFE comprises a gauge freedom due to which the functions R and $s|_{\mathscr{I}^-}$ can be regarded as gauge source functions, and due to which only the conformal class of the induced metric on $\mathscr{I}^-$ matters.

2.2.2 Coordinates

It is well-known (cf. e.g. [2]) that the freedom to choose coordinates near a spacelike hypersurface $\mathcal{H} = \{x^0 = 0\}$ with induced Riemannian metric h_{ij} can be employed to prescribe

$$\overline{g}^{00} < 0 \quad \text{and} \quad \overline{g}^{0i} . \tag{2.20}$$

Equivalently, one may prescribe

$$\overline{g}_{00} \quad \text{and} \quad \overline{g}_{0i} \quad \text{such that} \quad \overline{g}_{00} - \overline{h}^{ij}\overline{g}_{0i}\overline{g}_{0j} < 0 . \tag{2.21}$$

The remaining freedom to choose coordinates off the initial surface is comprised in the *$\hat{g}$-generalized wave-map gauge condition*

$$H^\sigma = 0 \tag{2.22}$$

with

$$H^\sigma := g^{\alpha\beta}(\Gamma^\sigma_{\alpha\beta} - \hat{\Gamma}^\sigma_{\alpha\beta}) - W^\sigma \tag{2.23}$$

being the *generalized wave-gauge vector*. Here $\hat{g}_{\mu\nu}$ denotes some *target metric*, $\hat{\Gamma}^\sigma_{\alpha\beta}$ are the Christoffel symbols of $\hat{g}_{\mu\nu}$. More precisely, the gauge freedom is captured by the vector field

$$W^\sigma = W^\sigma(x^\mu, g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \hat{g}_{\mu\nu})$$

which can be arbitrarily prescribed. In fact, within our setup, it can be allowed to depend upon the coordinates, and possibly upon $g_{\mu\nu}$ as well as all other fields which appear in the MCFE, but not upon derivatives thereof.

2.2.3 Realization of the gauge scheme

Given some smooth solution of the MCFE and a new choice of gauge functions R, $\overline{s}$, W^σ, $\overline{g}_{0\mu}$, as well as a conformal factor $\Omega > 0$ by which one wants to rescale the induced metric $\overline{g}_{ij}$, a transformation into the new gauge is realized as follows:

In the first step we set $\mathring{\phi} := \Omega$ and solve (2.19) for $\mathring{\psi} \equiv \overline{\nabla_0 \phi}$, which gives us the relevant initial data for (2.10) which we then solve. This way $\overline{s}$ and R take their desired values, and a new representative $\Omega^2 \overline{g}_{ij}$ of the conformal class of the induced metric on $\mathscr{I}^-$ is selected. Then the coordinates are transformed in such a way that the metric takes the prescribed values for $\overline{g}_{0\mu}$ on $\mathscr{I}^-$. Finally we just need to solve another wave equation to obtain $H^\sigma = 0$ for the given vector field W^σ.

2.3 Constraint equations in the $(R = 0, W^\lambda = 0, \overline{s} = 0, \overline{g}_{00} = -1, \overline{g}_{0i} = 0, \hat{g}_{\mu\nu} = \overline{g}_{\mu\nu})$-wave map gauge

In the following we aim to derive the constraint equations for the fields $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^\rho$ on $\mathscr{I}^-$ as well as their transverse derivatives induced by the MCFE on a spacelike $\mathscr{I}^-$ in adapted coordinates $(x^0 = t, x^i)$ with $\mathscr{I}^- = \{t = 0\}$. The surface $\mathscr{I}^-$ is characterized by

$$\overline{\Theta} = 0 \quad \text{and} \quad \overline{\mathrm{d}\Theta} \neq 0 \,. \tag{2.24}$$

Note that for $\mathscr{I}^-$ to be spacelike a positive cosmological constant $\lambda > 0$ is required. The constraint equations will be relevant for the derivation of the KID equations in Section 4.

To simplify computations we make the specific gauge choice

$$R = 0\,, \quad \overline{s} = 0\,, \quad \overline{g}_{00} = -1\,, \quad \overline{g}_{0i} = 0\,, \quad W^\sigma = 0\,, \quad \hat{g}_{\mu\nu} = \overline{g}_{\mu\nu}\,. \tag{2.25}$$

(Note that the target metric is taken to be $\overline{g}_{\mu\nu}$ for all t.) We shall show that appropriate data to solve the constraint equations are $\overline{g}_{ij}$ and $\overline{d}_{0i0j}$, where the latter field needs to satisfy a vector and a scalar constraint equation.

Let us start with a list of all the Christoffel symbols in adapted coordinates

$$\overline{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ij}\,, \quad \overline{\Gamma}^0_{ij} = \tfrac{1}{2}\overline{\partial_0 g_{ij}}\,, \quad \overline{\Gamma}^0_{0i} = 0\,, \tag{2.26}$$

$$\overline{\Gamma}^0_{00} = -\tfrac{1}{2}\overline{\partial_0 g_{00}}\,, \quad \overline{\Gamma}^k_{00} = \overline{g}^{kl}\overline{\partial_0 g_{0l}}\,, \quad \overline{\Gamma}^k_{0i} = \tfrac{1}{2}\overline{g}^{kl}\overline{\partial_0 g_{il}}\,, \tag{2.27}$$

where the $\tilde{\Gamma}^k_{ij}$'s denote the Christoffel symbols of the Riemannian metric $\tilde{g} = \overline{g}_{ij}\mathrm{d}x^i \mathrm{d}x^j$. Throughout we shall use $\tilde{\ }$ to denote fields such as the Riemann tensor, the Levi-Civita connection etc. associated to $\tilde{g}$.

Evaluation of (2.5) on $\mathscr{I}^-$ gives

$$\overline{\nabla_0 \Theta} = \sqrt{\lambda/3}\,. \tag{2.28}$$

The $(\mu\nu) = (00)$-component of (2.3) implies

$$\overline{\nabla_0 \nabla_0 \Theta} = 0\,, \tag{2.29}$$

while the $(\mu\nu) = (ij)$-components of (2.3) yield

$$0 = \overline{\nabla_i \nabla_j \Theta} = -\overline{\Gamma}^0_{ij}\overline{\nabla_0 \Theta} = -\sqrt{\frac{\lambda}{12}}\,\overline{\partial_0 g_{ij}}\,. \tag{2.30}$$

We compute the $(\mu\nu\sigma\kappa) = (ikjk)$-components of (2.6),

$$\overline{R}_{ikj}{}^{k} = \overline{L}_{ij} + g_{ij}\overline{g}^{kl}\overline{L}_{kl} \ ,$$

where

$$\overline{R}_{ikj}{}^{k} \ = \ \partial_k\overline{\Gamma}^k_{ij} - \partial_i\overline{\Gamma}^k_{jk} + \overline{\Gamma}^\alpha_{ij}\overline{\Gamma}^k_{\alpha k} - \overline{\Gamma}^\alpha_{ik}\overline{\Gamma}^k_{j\alpha} \ = \ \tilde{R}_{ikj}{}^{k} \ = \ \tilde{R}_{ij} \ .$$

Hence

$$\overline{L}_{ij} \ = \ \tilde{R}_{ij} - \frac{1}{4}\overline{g}_{ij}\tilde{R} \ = \ \tilde{L}_{ij} \ , \tag{2.31}$$

where $\tilde{L}_{ij}$ is the *Schouten tensor* of $\tilde{g}$. The gauge conditions (2.25) imply

$$0 = \frac{1}{6}\overline{R} = \overline{g}^{\mu\nu}\overline{L}_{\mu\nu} = \overline{g}^{ij}\overline{L}_{ij} - \overline{L}_{00} = \frac{1}{4}\tilde{R} - \overline{L}_{00} \ . \tag{2.32}$$

From the $\mu = i$-component of (2.4) we deduce

$$\overline{L}_{0i} = 0 \ . \tag{2.33}$$

Next, we employ the wave-map gauge condition to obtain

$$\begin{aligned} 0 \ &= \ \overline{H}^k = g^{\alpha\beta}(\Gamma^k_{\alpha\beta} - \hat{\Gamma}^k_{\alpha\beta}) = -\overline{\Gamma}^k_{00} = -\overline{g}^{kl}\overline{\partial_0 g_{0l}} \ , \\ 0 \ &= \ \overline{H}^0 = g^{\alpha\beta}(\Gamma^0_{\alpha\beta} - \hat{\Gamma}^0_{\alpha\beta}) = -\overline{\Gamma}^0_{00} = \frac{1}{2}\overline{\partial_0 g_{00}} \ . \end{aligned}$$

Altogether we have found that

$$\overline{\partial_0 g_{\mu\nu}} \ = \ 0 \ . \tag{2.34}$$

Thus (2.26)-(2.27) simplify to

$$\overline{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ij} \ , \quad \overline{\Gamma}^0_{ij} = \overline{\Gamma}^0_{0i} = \overline{\Gamma}^0_{00} = \overline{\Gamma}^k_{00} = \overline{\Gamma}^k_{0i} = 0 \ . \tag{2.35}$$

We have

$$\begin{aligned} \overline{R}_{ij} \ &\equiv \ \overline{\partial_\mu\Gamma^\mu_{ij}} - \overline{\partial_i\Gamma^\mu_{j\mu}} + \overline{\Gamma}^\alpha_{ij}\overline{\Gamma}^\mu_{\alpha\mu} - \overline{\Gamma}^\alpha_{i\mu}\overline{\Gamma}^\mu_{j\alpha} \\ &= \ \tilde{R}_{ij} + \overline{\partial_0\Gamma^0_{ij}} = \tilde{R}_{ij} + \frac{1}{2}\overline{\partial_0\partial_0 g_{ij}} \ . \end{aligned}$$

Hence

$$\overline{\partial_0\partial_0 g_{ij}} \ = \ 4\overline{L}_{ij} - 2\tilde{R}_{ij} \ = \ 2\tilde{R}_{ij} - g_{ij}\tilde{R} \ . \tag{2.36}$$

If we evaluate the $\mu = 0$-component of (2.4) on $\mathscr{I}^-$ we are led to,

$$\overline{\nabla_0 s} = \overline{L}_{00}\overline{\nabla_0\Theta} = \sqrt{\frac{\lambda}{48}}\,\tilde{R} \ . \tag{2.37}$$

The $(\mu\nu\sigma) = (0i0)$-components of (2.2) yield

$$\overline{\nabla_0 L_{0i}} = \nabla_i\overline{L}_{00} = \frac{1}{4}\tilde{\nabla}_i\tilde{R} \ . \tag{2.38}$$

Moreover, for $(\mu\nu\sigma)=(jki)$ we obtain

$$\overline{d}_{0ijk} \;=\; \sqrt{\frac{12}{\lambda}}\,\tilde{\nabla}_{[k}\overline{L}_{j]i} \;=\; \sqrt{\frac{3}{\lambda}}\tilde{C}_{ijk}\;, \tag{2.39}$$

where $\tilde{C}_{ijk}$ is the *Cotton tensor* of $\tilde{g}$. For $(\mu\nu\sigma)=(0ji)$ we find

$$\overline{\nabla_0 L_{ij}} \;=\; -\sqrt{\lambda/3}\,\overline{d}_{0i0j}\;. \tag{2.40}$$

The gauge condition $R=0$ together with the tracelessness of the rescaled Weyl tensor then imply

$$0 \;=\; \overline{g}^{\mu\nu}\overline{\nabla_0 L_{\mu\nu}} \;=\; \overline{g}^{ij}\overline{\nabla_0 L_{ij}} - \overline{\nabla_0 L_{00}} \;=\; -\overline{\nabla_0 L_{00}}\;. \tag{2.41}$$

Via the second Bianchi identity the $(\mu\nu\sigma)=(0ij)$-components of (2.1) become

$$\overline{\nabla_0 d_{0i0j}} \;=\; -\tilde{\nabla}^k\overline{d}_{0ijk} \;=\; -\sqrt{\frac{3}{\lambda}}\tilde{\nabla}^k\tilde{C}_{ijk} \;=\; \sqrt{\frac{3}{\lambda}}\tilde{B}_{ij}\;, \tag{2.42}$$

where $\tilde{B}_{ij}$ denotes the *Bach tensor* of $\tilde{g}$. The $(\mu\nu\sigma)=(kji)$-components give

$$\overline{\nabla_0 d_{0ijk}} \;=\; -\tilde{\nabla}^l\overline{d}_{jkil} \;=\; 2\tilde{\nabla}_{[j}\overline{d}_{k]0i0} - 2g_{i[j}\tilde{\nabla}^l\overline{d}_{k]0l0}\;. \tag{2.43}$$

Here we used that due to the algebraic symmetries of the rescaled Weyl tensor

$$\begin{aligned}\overline{d}_{ijkl} \;&=\; 2\overline{g}^{mn}(\overline{g}_{k[i}\overline{d}_{j]mln} - \overline{g}_{l[i}\overline{d}_{j]mkn} - \overline{g}_{k[i}g_{j]l}\overline{g}^{pq}\overline{d}_{pmqn})\\ &=\; 2(\overline{g}_{k[i}\overline{d}_{j]0l0} - \overline{g}_{l[i}\overline{d}_{j]0k0})\;.\end{aligned} \tag{2.44}$$

The $(\mu\nu\sigma)=(0i0)$-components of (2.1) imply a *vector constraint* for $\overline{d}_{0i0j}$,

$$\tilde{\nabla}^j\overline{d}_{0i0j} \;=\; 0\;. \tag{2.45}$$

(A *"scalar constraint"*, which has already been used in the derivation of the constraint equations, is simply given by the tracelessness-requirement on the rescaled Weyl tensor,

$$\overline{g}^{ij}\overline{d}_{0i0j} \;=\; \overline{g}^{\mu\nu}\overline{d}_{0\mu0\nu} \;=\; 0\;.) \tag{2.46}$$

To sum it up, we have the following analogue of a result of Friedrich [4]: The free data can be identified with a Riemannian metric $h_{ij}:=\overline{g}_{ij}$ and a symmetric tensor field $D_{ij}:=\overline{d}_{0i0j}$ on $\mathscr{I}^-$ satisfying

$$h^{ij}D_{ij}=0 \quad\text{and}\quad \tilde{\nabla}^jD_{ij}=0 \tag{2.47}$$

(that these are indeed the free data follows e.g. from the considerations in Appendix A). Then the MCFE enforce on $\mathscr{I}^-$ in the $(R=0,\overline{s}=0,\overline{g}_{00}=-1,\overline{g}_{0i}=0,\hat{g}_{\mu\nu}=\overline{g}_{\mu\nu})$-wave-map gauge,

$$\overline{g}_{00}=-1\;,\quad \overline{g}_{0i}=0\;,\quad \overline{g}_{ij}=h_{ij}\;,\quad \overline{\partial_0 g_{\mu\nu}}=0\;, \tag{2.48}$$

$$\overline{\Theta}=0\;,\quad \overline{\partial_0\Theta}=\sqrt{\tfrac{\lambda}{3}}\;, \tag{2.49}$$

$$\overline{s}=0\;,\quad \overline{\partial_0 s}=\sqrt{\tfrac{\lambda}{48}}\,\tilde{R}\;, \tag{2.50}$$

$$\overline{L}_{ij}=\tilde{L}_{ij}\;,\quad \overline{L}_{0i}=0\;,\quad \overline{L}_{00}=\tfrac{1}{4}\tilde{R}\;, \tag{2.51}$$

$$\overline{\partial_0 L_{ij}}=-\sqrt{\tfrac{\lambda}{3}}\,D_{ij}\;,\quad \overline{\partial_0 L_{0i}}=\tfrac{1}{4}\tilde{\nabla}_i\tilde{R}\;,\quad \overline{\partial_0 L_{00}}=0\;, \tag{2.52}$$

$$\overline{d}_{0i0j}=D_{ij}\;,\quad \overline{d}_{0ijk}=\sqrt{\tfrac{3}{\lambda}}\tilde{C}_{ijk}\;, \tag{2.53}$$

$$\overline{\partial_0 d_{0i0j}}=\sqrt{\tfrac{3}{\lambda}}\tilde{B}_{ij}\;,\quad \overline{\partial_0 d_{0ijk}}=2\tilde{\nabla}_{[j}D_{k]i}\;. \tag{2.54}$$

Note that due to (2.35) the actions of ∇_0 and ∂_0, as well as ∇_i and $\tilde{\nabla}_i$, respectively, coincide on $\mathscr{I}^-$, so we can use them interchangeably.

We have seen in Section 2.2 (cf. also [4]) that there remains a gauge freedom to conformally rescale the induced metric on $\mathscr{I}^-$. Due to this freedom the pairs (h_{ij}, D_{ij}) and $(\Omega^2 h_{ij}, \Omega^{-1} D_{ij})$, with Ω some positive function, generate the same physical space-times. With regard to the constraint equations we note that $\Omega^{-1} D_{ij}$ is trace- and divergence-free w.r.t. $\Omega^2 h_{ij}$ whenever D_{ij} is w.r.t. h_{ij}.

In the following we shall write $[h_{ij}, D_{ij}]$ if this gauge freedom is left unspecified and if we merely want to refer to the conformal classes of h_{ij} and D_{ij}.

2.4 Well-posedness of the Cauchy problem on a spacelike $\mathscr{I}^-$

In [9] a system of conformal wave equations (CWE) has been derived from the MCFE. In Appendix A it is shown that a solution of the CWE, equations (A.1)-(A.5), is a solution of the MCFE if and only if the constraint equations (2.47)-(2.54) are satisfied. Using standard well-posedness results about wave equations we thereby recover a result due to Friedrich [4] who proved well-posedness of the Cauchy problem on $\mathscr{I}^-$ (Friedrich used a representation of the MCFE as a symmetric hyperbolic system, in some situations, however, it might be advantageous to deal with a system of wave equations instead [6]). We restrict attention to the smooth case (for a version with finite differentiability see [4]):

Theorem 2.1 *Let $\mathcal{H}$ be a 3-dimensional smooth manifold. Let h_{ij} be a smooth Riemannian metric and let D_{ij} be a smooth symmetric, trace- and divergence-free tensor field on $\mathcal{H}$. Moreover, assume a positive cosmological constant $\lambda > 0$. Then there exists an (up to isometries) unique smooth space-time $(\mathscr{M}, g, \Theta)$ with the following properties:*

(i) $(\mathscr{M}, g, \Theta)$ *satisfies the MCFE (2.1)-(2.6),*

(ii) $\Theta|_{\mathcal{H}} = 0$ *and* $\mathrm{d}\Theta|_{\mathcal{H}} \neq 0$*, i.e.* $\mathcal{H} = \mathscr{I}^-$ *(and Θ has no zeros away from and sufficiently close to $\mathcal{H}$),*

(iii) $g_{ij}|_{\mathcal{H}} = h_{ij}$, $d_{0i0j}|_{\mathcal{H}} = D_{ij}$.

The isometry class of the space-time does not change if the initial data are replaced by $(\hat{h}_{ij}, \hat{D}_{ij})$ with $[\hat{h}_{ij}, \hat{D}_{ij}] = [h_{ij}, D_{ij}]$.

Remark 2.2 De Sitter space-time is obtained for $\mathcal{H} = S^3$, $h_{ij} = s_{ij}$ and $D_{ij} = 0$, where $s = s_{ij}\mathrm{d}x^i\mathrm{d}x^j$ denotes the round sphere metric, cf. Section 4.3.2

3 KID equations

3.1 Unphysical Killing equations

In [10] it is shown that the appropriate substitute for the Killing equation in the unphysical, conformally rescaled space-time is provided by the *unphysical Killing equations*

$$\nabla_{(\mu} X_{\nu)} = \frac{1}{4} \nabla^\sigma X_\sigma \, g_{\mu\nu} \quad \& \quad X^\sigma \nabla_\sigma \Theta = \frac{1}{4} \Theta \nabla_\sigma X^\sigma \, . \tag{3.1}$$

A vector field X_{phys} is a Killing field in the physical space-time $(\mathscr{M}_{\mathrm{phys}}, g_{\mathrm{phys}})$ if and only if its push-forward $X := \phi_* X_{\mathrm{phys}}$ satisfies (3.1) in the unphysical space-time $(\phi(\mathscr{M}_{\mathrm{phys}}) \subset \mathscr{M}, g = \phi(g_{\mathrm{phys}}) = \Theta^2 g_{\mathrm{phys}})$, where ϕ defines the conformal rescaling. The unphysical Killing equations remain regular even where the conformal factor Θ vanishes.

In what follows we shall derive necessary-and-sufficient conditions on a spacelike initial surface which guarantee the existence of a vector field X which satisfies the unphysical Killing equations.

3.2 KID equations on a Cauchy surface

Necessary conditions on a vector field X to satisfy the unphysical Killing equations are that the following wave equations are fulfilled [10],

$$\begin{aligned} \Box_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y &= 0 \,, && (3.2)\\ \Box_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} R Y &= 0 \,, && (3.3) \end{aligned}$$

where we have set

$$Y := \frac{1}{4} \nabla_\sigma X^\sigma \,. \tag{3.4}$$

It proves fruitful to make the following definitions:

$$\begin{aligned} \phi &:= X^\mu \nabla_\mu \Theta - \Theta Y \,, && (3.5)\\ \psi &:= X^\mu \nabla_\mu s + sY - \nabla_\mu \Theta \nabla^\mu Y \,, && (3.6)\\ A_{\mu\nu} &:= 2\nabla_{(\mu} X_{\nu)} - 2Y g_{\mu\nu} \,, && (3.7)\\ B_{\mu\nu} &:= \mathscr{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y \,. && (3.8) \end{aligned}$$

All these fields need to vanish whenever X is a solution of (3.1) [10].

The equations (3.2) and (3.3) together with the MCFE imply that the following system of wave equations is satisfied by the fields ϕ, ψ, $A_{\mu\nu}$, $\nabla_\sigma A_{\mu\nu}$ and $B_{\mu\nu}$ (cf. [10]):

$$\begin{aligned} \Box_g A_{\mu\nu} &= 2R_{(\mu}{}^\kappa A_{\nu)\kappa} - 2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} - 4B_{\mu\nu} \,, && (3.9)\\ \Box_g \phi &= d\psi - \frac{1}{6} R\phi + A_{\mu\nu} \nabla^\mu \nabla^\nu \Theta \,, && (3.10)\\ \Box_g \psi &= |L|^2 \phi + A_{\mu\nu} (\nabla^\mu \nabla^\nu s - 2\Theta L_\kappa{}^\mu L^{\nu\kappa}) + 2\Theta L^{\mu\nu} B_{\mu\nu} \\ &\quad + \frac{1}{6} \big(A_{\mu\nu} \nabla^\mu R \nabla^\nu \Theta - \nabla^\mu R \nabla_\mu \phi - R\psi \big) \,, && (3.11)\\ \Box_g B_{\mu\nu} &\equiv 2(g_{\mu\nu} L^{\alpha\beta} - R_\mu{}^\alpha{}_\nu{}^\beta) B_{\alpha\beta} - 2R_{(\mu}{}^\kappa B_{\nu)\kappa} + \frac{2}{3} R B_{\mu\nu} \\ &\quad + 2L^{\alpha\beta} (\nabla_\beta \nabla_{[\alpha} A_{\nu]\mu} - \nabla_\mu \nabla_{[\alpha} A_{\nu]\beta}) \\ &\quad + (\nabla_{(\mu} A_{|\alpha\beta|} + 2\nabla_{[\alpha} A_{\beta](\mu})(2\nabla^\alpha L_{\nu)}{}^\beta - \frac{1}{12} \delta_{\nu)}{}^\alpha \nabla^\beta R) \\ &\quad + A^{\alpha\beta} [\nabla_\alpha \nabla_\beta L_{\mu\nu} - 2L_{(\mu}{}^\kappa R_{\nu)\alpha\kappa\beta} + 2L_{\mu\alpha} R_{\nu\beta} + L_\alpha{}^\kappa (2R_{\mu\beta\nu\kappa} + R_{\nu\beta\mu\kappa}) \\ &\quad - 2g_{\mu\nu} L_{\alpha\kappa} L_\beta{}^\kappa] + |L|^2 A_{\mu\nu} + L^{\alpha\beta} R_{\mu\alpha\beta}{}^\kappa A_{\nu\kappa} - \frac{1}{3} R L_{(\mu}{}^\kappa A_{\nu)\kappa} \,, && (3.12)\\ \Box_g \nabla_\sigma A_{\mu\nu} &= 2\nabla_\sigma (R_{(\mu}{}^\kappa A_{\nu)\kappa} - R_\mu{}^\alpha{}_\nu{}^\kappa A_{\alpha\kappa}) + 2A_{\alpha(\mu} (\nabla_{\nu)} R_\sigma{}^\alpha - \nabla^\alpha R_{\nu)\sigma}) \\ &\quad - 4R_{\sigma\kappa(\mu}{}^\alpha \nabla^\kappa A_{\nu)\alpha} + R_{\alpha\sigma} \nabla^\alpha A_{\mu\nu} - 4\nabla_\sigma B_{\mu\nu} \,. && (3.13) \end{aligned}$$

In close analogy to [10, Theorem 3.4] we immediately obtain the following result:

Theorem 3.1 *Assume we have been given, in* $3+1$ *dimensions, an "unphysical" space-time* $(\mathscr{M}, g, \Theta)$, *with* (g, Θ) *a smooth solution of the MCFE (2.1)-(2.6). Consider a spacelike hypersurface* $\mathcal{H} \subset \mathscr{M}$. *Then there exists a vector field* $\hat{X}$ *satisfying the unphysical Killing equations (3.1) on* $\mathrm{D}^+(\mathcal{H})$ *(and thus corresponding to a Killing vector field of the physical space-time) if and only if there exists a pair* (X, Y), X *a vector field and* Y *a function, which fulfills the following equations:*

(i) $\Box_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y = 0$,

(ii) $\Box_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} R Y = 0$,

(iii) $\overline{\phi} = 0$ *and* $\overline{\partial_0 \phi} = 0$,

(iv) $\overline{\psi} = 0$ *and* $\overline{\partial_0 \psi} = 0$,

(v) $\overline{A}_{\mu\nu} = 0$, $\overline{\nabla_0 A_{\mu\nu}} = 0$ *and* $\overline{\nabla_0 \nabla_0 A_{\mu\nu}} = 0$,

(vi) $\overline{B}_{\mu\nu} = 0$ *and* $\overline{\nabla_0 B_{\mu\nu}} = 0$.

Moreover, $\overline{\hat{X}} = \overline{X}$, $\overline{\nabla_0 \hat{X}} = \overline{\nabla_0 X}$, $\overline{\nabla_\mu \hat{X}^\mu} = \frac{1}{4}\overline{Y}$ *and* $\overline{\nabla_0 \nabla_\mu \hat{X}^\mu} = \frac{1}{4}\overline{\nabla_0 \hat{Y}}$.

3.3 A special case: $\Theta = 1$

Let us briefly discuss the case where the conformal factor Θ is identical to one,

$$\Theta = 1 ,$$

so that the unphysical space-time can be identified with the physical one. Then the MCFE imply

$$s = \frac{1}{6}\lambda , \quad L_{\mu\nu} = s g_{\mu\nu} , \quad R_{\mu\nu} = \lambda g_{\mu\nu} ,$$

i.e. the vacuum Einstein equations hold. We consider the conditions (i)-(vi) of Theorem 3.1 in this setting. Condition (iii) is equivalent to $\overline{Y} = 0$ and $\overline{\partial_0 Y} = 0$, which provide the initial data for the wave equation (ii). The only solution is $Y = 0$, i.e. X needs to be a Killing field, as desired. Condition (iv) is then automatically satisfied. Since

$$B_{\mu\nu} = \mathscr{L}_X L_{\mu\nu} = s \mathscr{L}_X g_{\mu\nu} = 2s \nabla_{(\mu} X_{\nu)} , \tag{3.14}$$

the validity of (vi) follows from (v), and we are left with the conditions

$$\Box_g X_\mu + \lambda X_\mu = 0 , \tag{3.15}$$
$$\overline{\nabla_{(\mu} X_{\nu)}} = 0 , \tag{3.16}$$
$$\overline{\nabla_0 \nabla_{(\mu} X_{\nu)}} = 0 , \tag{3.17}$$
$$\overline{\nabla_0 \nabla_0 \nabla_{(\mu} X_{\nu)}} = 0 . \tag{3.18}$$

Note that $\overline{B}_{\mu\nu} = 0$ due to (3.14) and (3.16), so that (3.15)-(3.17) imply via the trace of (3.9) on $\mathcal{H}$ the validity of (3.18).

The equations (3.15)-(3.17) form a possible starting point to derive the KID equations on Cauchy surfaces in space-times satisfying the vacuum Einstein equations (cf. [1, 8]).

3.4 A stronger version of Theorem 3.1

Let us now investigate to what extent the conditions (iii)-(vi) in Theorem 3.1 imply each other. For this purpose we choose adapted coordinates $(x^0 \equiv t, x^i)$ in the sense that the initial surface is (locally) given by the set $\{x^0 = 0\}$ and that, on $\mathcal{H}$, the metric takes the form

$$g|_{\mathcal{H}} = -(\mathrm{d}t)^2 + \overline{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j = -(\mathrm{d}t)^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j \;, \tag{3.19}$$

with h_{ij} some Riemannian metric. Moreover, we denote by f, f_i and f_{ij} generic functions which depend on the indicated fields (and possibly spatial derivatives thereof) and vanish whenever all their arguments vanish. The symbol $\breve{.}$ is used to denote the h-trace-free part of the corresponding 2-rank tensor on $\mathcal{H}$, i.e.

$$\breve{v}_{ij} := v_{ij} - \frac{1}{3}h_{ij}h^{kl}v_{kl} \;. \tag{3.20}$$

We start with the identity [10]

$$\nabla_\nu A_\mu{}^\nu - \frac{1}{2}\nabla_\mu A_\nu{}^\nu \;\equiv\; \Box_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y \;. \tag{3.21}$$

Because of (3.2) the right-hand side vanishes and we obtain

$$\overline{\nabla_0 A_{00}} \;=\; 2\overline{g}^{kl}\nabla_k\overline{A}_{0l} - \overline{g}^{kl}\overline{\nabla_0 A_{kl}} \;=\; -\overline{g}^{kl}\overline{\nabla_0 A_{kl}} + f(\overline{A}_{\mu\nu}) \;, \tag{3.22}$$

$$\overline{\nabla_0 A_{0i}} \;=\; \frac{1}{2}\nabla_i\overline{A}_{00} + \overline{g}^{kl}\nabla_k\overline{A}_{il} - \frac{1}{2}\overline{g}^{kl}\nabla_i\overline{A}_{kl} \;=\; f_i(\overline{A}_{\mu\nu}) \;, \tag{3.23}$$

$$\begin{aligned}\overline{\nabla_0\nabla_0 A_{00}} &= 2\overline{g}^{kl}\overline{\nabla_0\nabla_k A_{0l}} - \overline{g}^{kl}\overline{\nabla_0\nabla_0 A_{kl}} \\ &= 2\overline{g}^{kl}\nabla_k\overline{\nabla_0 A_{0l}} - \overline{g}^{kl}\overline{\nabla_0\nabla_0 A_{kl}} + f(\overline{A}_{\mu\nu}) \;,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\overline{\nabla_0\nabla_0 A_{0i}} &= \frac{1}{2}\overline{\nabla_0\nabla_i A_{00}} + \overline{g}^{kl}\overline{\nabla_0\nabla_k A_{il}} - \frac{1}{2}\overline{g}^{kl}\overline{\nabla_0\nabla_i A_{kl}} \\ &= \frac{1}{2}\nabla_i\overline{\nabla_0 A_{00}} + \overline{g}^{kl}\nabla_k\overline{\nabla_0 A_{il}} - \frac{1}{2}\overline{g}^{kl}\nabla_i\overline{\nabla_0 A_{kl}} + f_i(\overline{A}_{\mu\nu})\end{aligned} \tag{3.25}$$

We further have the identity [10]

$$\begin{aligned}\nabla_\nu B_\mu{}^\nu - \frac{1}{2}\nabla_\mu B_\nu{}^\nu \equiv{}& A_{\alpha\beta}(\nabla^\alpha L_\mu{}^\beta - \frac{1}{2}\nabla_\mu L^{\alpha\beta}) \\ &+L_\mu{}^\kappa(\Box_g X_\kappa + R_\kappa{}^\alpha X_\alpha + 2\nabla_\kappa Y) + \frac{1}{2}\nabla_\mu(\Box_g Y + \frac{1}{6}X^\nu\nabla_\nu R + \frac{1}{3}RY) \;.\end{aligned}$$

With (3.2) and (3.3) we deduce

$$\begin{aligned}\overline{\nabla_0 B_{00}} &= 2\overline{g}^{kl}\nabla_k\overline{B}_{0l} - \overline{g}^{kl}\overline{\nabla_0 B_{kl}} + f(\overline{A}_{\mu\nu}) \\ &= -\overline{g}^{kl}\overline{\nabla_0 B_{kl}} + f(\overline{A}_{\mu\nu}, \overline{B}_{\mu\nu}) \;,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\overline{\nabla_0 B_{0i}} &= \frac{1}{2}\nabla_i\overline{B}_{00} + \overline{g}^{kl}\nabla_k\overline{B}_{il} - \frac{1}{2}\overline{g}^{kl}\nabla_i\overline{B}_{kl} + f_i(\overline{A}_{\mu\nu}) \\ &= f_i(\overline{A}_{\mu\nu}, \overline{B}_{\mu\nu}) \;.\end{aligned} \tag{3.27}$$

Evaluation of (3.9) on the initial surface gives with (2.26)-(2.27)

$$\overline{\nabla_0\nabla_0 A_{ij}} \;=\; 4\overline{B}_{ij} - \overline{g}^{kl}\overline{\Gamma}^0_{kl}\overline{\nabla_0 A_{ij}} + f_{ij}(\overline{A}_{\mu\nu}) \;, \tag{3.28}$$

$$\overline{\nabla_0\nabla_0 A_{0i}} \;=\; 4\overline{B}_{0i} - \overline{g}^{kl}\overline{\Gamma}^0_{kl}\overline{\nabla_0 A_{0i}} + f_i(\overline{A}_{\mu\nu}) \;, \tag{3.29}$$

$$\overline{\nabla_0\nabla_0 A_{00}} \;=\; 4\overline{B}_{00} - \overline{g}^{kl}\overline{\Gamma}^0_{kl}\overline{\nabla_0 A_{00}} + f(\overline{A}_{\mu\nu}) \;. \tag{3.30}$$

From the definition of $B_{\mu\nu}$ we obtain with (3.3) (set $B := g^{\mu\nu}B_{\mu\nu}$)

$$\begin{aligned} \overline{B} &\equiv \overline{L}^{\mu\nu}\overline{A}_{\mu\nu} + \overline{\Box_g Y} + \frac{1}{6}\overline{X}^{\mu}\overline{\nabla_\mu R} + \frac{1}{3}\overline{RY} \\ &= \overline{L}^{\mu\nu}\overline{A}_{\mu\nu} \;, \end{aligned} \tag{3.31}$$

$$\begin{aligned} \overline{\nabla_0 B} &\equiv \overline{\nabla_0(L^{\mu\nu}A_{\mu\nu})} + \overline{\nabla_0(\Box_g Y + \frac{1}{6}X^{\mu}\nabla_\mu R + \frac{1}{3}RY)} \\ &= \overline{\nabla_0(L^{\mu\nu}A_{\mu\nu})} \;. \end{aligned} \tag{3.32}$$

We use the equations (3.22)-(3.32) to establish a stronger version of Theorem 3.1. Let us assume that

$$\overline{A}_{\mu\nu} = 0 \;, \quad \overline{\nabla_0 A_{ij}} = 0 \;, \quad \breve{\overline{B}}_{ij} = 0 \;, \quad (\overline{\nabla_0 B_{ij}})\breve{} = 0 \;. \tag{3.33}$$

Then by (3.22) and (3.23) we have $\overline{\nabla_0 A_{\mu\nu}} = 0$. From (3.31) and (3.32) we deduce $\overline{B} = \overline{\nabla_0 B} = 0$. The equations (3.24), (3.28) and (3.30) yield the system

$$\begin{aligned} \overline{\nabla_0\nabla_0 A_{00}} &= -\overline{g}^{ij}\overline{\nabla_0\nabla_0 A_{ij}} \;, \\ \overline{g}^{ij}\overline{\nabla_0\nabla_0 A_{ij}} &= 4\overline{g}^{ij}\overline{B}_{ij} \overset{\overline{B}=0}{=} 4\overline{B}_{00} \;, \\ \overline{\nabla_0\nabla_0 A_{00}} &= 4\overline{B}_{00} \;, \end{aligned}$$

from which we conclude $\overline{\nabla_0\nabla_0 A_{00}} = \overline{g}^{ij}\overline{\nabla_0\nabla_0 A_{ij}} = \overline{B}_{00} = 0$. From (3.25) and the trace-free part of (3.28) we then deduce $\overline{\nabla_0\nabla_0 A_{\mu\nu}} = 0$, and the equations (3.29) and (3.31) imply $\overline{B}_{\mu\nu} = 0$. Moreover, invoking (3.26) and (3.32) yields

$$\begin{aligned} \overline{\nabla_0 B_{00}} &= -\overline{g}^{ij}\overline{\nabla_0 B_{ij}} \;, \\ 0 = \overline{\nabla_0 B} &= \overline{g}^{ij}\overline{\nabla_0 B_{ij}} - \overline{\nabla_0 B_{00}} \;, \end{aligned}$$

i.e. $\overline{\nabla_0 B_{00}} = \overline{g}^{ij}\overline{\nabla_0 B_{ij}} = 0$. The equation (3.27) then completes the proof that $\overline{\nabla_0 B_{\mu\nu}} = 0$.

We end up with the result

THEOREM 3.2 *Assume we have been given, in $3+1$ dimensions, an "unphysical" space-time $(\mathscr{M}, g, \Theta)$, with (g, Θ) a smooth solution of the MCFE (2.1)-(2.6). Consider a spacelike hypersurface $\mathcal{H} \subset \mathscr{M}$. Then there exists a vector field $\hat{X}$ satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(\mathcal{H})$ if and only if there exists a pair (X, Y), X a vector field and Y a function, which fulfills the KID equations, i.e.*

(a) equations (i)-(iv) of Theorem 3.1,

(b) $\overline{A}_{\mu\nu} = 0$ and $\overline{\nabla_0 A_{ij}} = 0$ with $A_{\mu\nu} \equiv 2\nabla_{(\mu}X_{\nu)} - 2Y g_{\mu\nu}$,

(c) $\breve{\overline{B}}_{ij} = 0$ and $(\overline{\nabla_0 B_{ij}})\breve{} = 0$ with $B_{\mu\nu} \equiv \mathscr{L}_X L_{\mu\nu} + \nabla_\mu\nabla_\nu Y$.

Moreover, $\overline{\hat{X}} = \overline{X}$, $\overline{\nabla_0 \hat{X}} = \overline{\nabla_0 X}$, $\overline{\nabla_\mu \hat{X}^\mu} = \frac{1}{4}\overline{Y}$ and $\overline{\nabla_0\nabla_\mu \hat{X}^\mu} = \frac{1}{4}\overline{\nabla_0 \hat{Y}}$.

3.5 The (proper) KID equations

We want to replace the equations $\overline{\partial_0 \psi} = 0$ and $(\overline{\nabla_0 B_{ij}})\breve{} = 0$ appearing in Theorem 3.2 by intrinsic equations on $\mathcal{H}$ in the sense that they involve at most first-order transverse derivatives of X and Y, which belong to the freely prescribable initial data for the wave equations (3.2) and (3.3). The higher-order derivatives appearing can be eliminated via (3.3) which implies

$$\overline{\nabla_0\nabla_0 Y} \quad = \quad \overline{g}^{kl}\overline{\nabla_k\nabla_l Y} + \frac{1}{6}\overline{X^\mu}\overline{\nabla_\mu R} + \frac{1}{3}\overline{RY}\,. \tag{3.34}$$

We are straightforwardly led to

THEOREM 3.3 *Assume that we have been given a $3+1$-dimensional space-time $(\mathscr{M}, g, \Theta)$, with (g,Θ) being a smooth solution of the MCFE. Let $\mathring{X}$ and $\mathring{\Lambda}$ be spacetime vector fields, and $\mathring{Y}$ and $\mathring{\Upsilon}$ be functions defined along a spacelike hypersurface $\mathcal{H}\subset\mathscr{M}$. Then there exists a smooth space-time vector field X with $\overline{X}=\mathring{X}$, $\overline{\nabla_0 X}=\mathring{\Lambda}$, $\overline{\nabla_\mu X^\mu}=\frac{1}{4}\mathring{Y}$ and $\overline{\nabla_0\nabla_\mu X^\mu}=\frac{1}{4}\mathring{\Upsilon}$ satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(\mathcal{H})$ (and thus corresponding to a Killing field of the physical space-time) if and only if in the adapted coordinates (3.19):*

(i) $\overline{\phi}\equiv \mathring{X}^\mu\overline{\nabla_\mu\Theta}-\overline{\Theta}\mathring{Y}=0$,
$\overline{\partial_0\phi}\equiv \mathring{\Lambda}^\mu\overline{\nabla_\mu\Theta}+\mathring{X}^\mu\overline{\nabla_\mu\nabla_0\Theta}-\overline{\Theta}\mathring{\Upsilon}-\overline{\nabla_0\Theta}\mathring{Y}=0$,

(ii) $\overline{\psi}\equiv \mathring{X}^\mu\overline{\nabla_\mu s}+\overline{s}\mathring{Y}-\overline{\nabla^i\Theta}\tilde{\nabla}_i\mathring{Y}+\overline{\nabla_0\Theta}\mathring{\Upsilon}=0$,
$\overline{\partial_0\psi}^{\mathrm{intr}}:=\mathring{\Lambda}^\mu\overline{\nabla_\mu s}+\mathring{X}^\mu\overline{\nabla_\mu\nabla_0 s}+\overline{\nabla_0 s}\mathring{Y}+(\overline{s}+\overline{\nabla_0\nabla_0\Theta})\mathring{\Upsilon}-\overline{\nabla^i\nabla_0\Theta}\tilde{\nabla}_i\mathring{Y}+$
$\overline{\nabla_0\Theta}(\Delta_h\mathring{Y}-\overline{\Gamma}^k_{0k}\mathring{\Upsilon}+\frac{1}{6}\mathring{X}^\mu\overline{\nabla_\mu R}+\frac{1}{3}\overline{R}\mathring{Y})-\overline{\nabla^k\Theta}(\tilde{\nabla}_k\mathring{\Upsilon}-\overline{\Gamma}^i_{0k}\tilde{\nabla}_i\mathring{Y})=0$,

(iii) $\overline{A}_{ij}\equiv 2\nabla_{(i}\mathring{X}_{j)}-2\mathring{Y}\overline{g}_{ij}=0$,
$\overline{A}_{0i}\equiv\mathring{\Lambda}_i+\nabla_i\mathring{X}_0=0$,
$\overline{A}_{00}\equiv 2\mathring{\Lambda}_0+2\mathring{Y}=0$,
$\overline{\nabla_0 A_{ij}}\equiv 2\tilde{\nabla}_{(i}\mathring{\Lambda}_{j)}-2\overline{\Gamma}^k_{0(i}\nabla_k\mathring{X}_{j)}-2\overline{\Gamma}^0_{ij}\mathring{\Lambda}_0+2\overline{R}_{0(ij)}{}^\mu\mathring{X}_\mu-2\mathring{\Upsilon}\overline{g}_{ij}=0$,

(iv) $\breve{\overline{B}}_{ij}\equiv(\mathring{X}^\mu\overline{\nabla_\mu L_{ij}}+2\overline{L}_{\mu(i}\nabla_{j)}\mathring{X}^\mu+\tilde{\nabla}_i\tilde{\nabla}_j\mathring{Y}-\overline{\Gamma}^0_{ij}\mathring{\Upsilon})\breve{}=0$,
$(\overline{\nabla_0 B^{\mathrm{intr}}_{ij}})\breve{}:=[\overline{\mathscr{L}_{\mathring{X}}\nabla_0 L_{ij}}+2\overline{L}_{\mu(i}(\partial_{j)}\mathring{\Lambda}^\mu+\overline{\Gamma}^\mu_{j\alpha}\mathring{\Lambda}^\alpha-\overline{\Gamma}^k_{0j}\nabla_k\mathring{X}^\mu)+2\overline{L}_{k(i}\overline{R}_{j)\mu 0}{}^k\mathring{X}^\mu+$
$\tilde{\nabla}_i\tilde{\nabla}_j\mathring{\Upsilon}-\overline{\Gamma}^0_{ij}(\Delta_h\mathring{Y}+\frac{1}{6}\mathring{X}^\mu\overline{\nabla_\mu R}+\frac{1}{3}\overline{R}\mathring{Y})-2\overline{\Gamma}^k_{0(i}\tilde{\nabla}_{j)}\tilde{\nabla}_k\mathring{Y}+(\overline{R}_{0ij}{}^0+\overline{\Gamma}^k_{0i}\overline{\Gamma}^0_{jk}+$
$\overline{\Gamma}^0_{ij}\overline{\Gamma}^k_{0k})\mathring{\Upsilon}+(\overline{R}_{0ij}{}^k\ -\tilde{\nabla}_i\overline{\Gamma}^k_{0j})\tilde{\nabla}_k\mathring{Y}]\breve{}=0$.

PROOF: Assume that there exist fields $\mathring{X}$, $\mathring{\Lambda}$, $\mathring{Y}$ and $\mathring{\Upsilon}$ which satisfy (i)-(iv). These fields provide the initial data for the wave equations (3.2) and (3.3) for X and Y. A solution exists due to standard results. Once (3.2) and (3.3) are satisfied the considerations above reveal that (i)-(iv) are equivalent to (a)-(c) of Theorem 3.2, i.e. all the hypotheses of Theorem 3.2 hold and we are done. From the derivation of (i)-(iv) it follows that these conditions are necessary, as well. □

REMARK 3.4 We call the equations in (i)-(iv) the *(proper) KID equations on $\mathcal{H}$*.

4 KID equations on a spacelike $\mathscr{I}^-$

4.1 Derivation of the (reduced) KID equations

Let us restrict now attention to space-times which contain a spacelike $\mathscr{I}^-$, which we take henceforth as initial surface (recall that this requires a positive cosmological constant λ). We impose the $(R=0, \overline{s}=0, \overline{g}_{00}=-1, \overline{g}_{0i}=0, \hat{g}_{\mu\nu}=\overline{g}_{\mu\nu})$-wave-map gauge condition introduced in Section 2.3. Recall that the freely prescribable data on $\mathscr{I}^-$ for the Cauchy problem are the conformal class of a Riemannian metric h_{ij} and a symmetric, trace- and divergence-free tensor D_{ij}. The MCFE then imply the constraint equations (2.48)-(2.54) on $\mathscr{I}^-$. In Appendix A it is shown that a solution to the MCFE further satisfies

$$\overline{\nabla_0\nabla_0\Theta}=0\,, \quad \overline{R}_{0ij}{}^k=0\,. \tag{4.1}$$

We are now ready to evaluate the conditions (i)-(iv) of Theorem 3.3.

The condition (i) becomes

$$\mathring{X}^0=0\,, \quad \mathring{\Lambda}^0=\mathring{Y}\,. \tag{4.2}$$

Then condition (ii) is satisfied iff (set $\Delta_{\tilde{g}} := \overline{g}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j$)

$$\mathring{\Upsilon}=0\,, \quad \mathring{X}^i\tilde{\nabla}_i\tilde{R}+2\tilde{R}\mathring{Y}+4\Delta_{\tilde{g}}\mathring{Y}=0\,. \tag{4.3}$$

The condition $\overline{A}_{\mu\nu}=0$ requires

$$\begin{aligned} \mathring{\Lambda}^i &= 0\,, && (4.4)\\ \mathring{Y} &= \frac{1}{3}\tilde{\nabla}_i\mathring{X}^i\,, && (4.5)\\ (\tilde{\nabla}_{(i}\mathring{X}_{j)})\breve{} &= 0\,. && (4.6) \end{aligned}$$

The condition $\overline{\nabla_0 A_{ij}}=0$ is then automatically fulfilled.

We reconsider the second condition in (4.3). Observe that (4.5), (4.6) and the second Bianchi identity imply the relation

$$\begin{aligned} 0 = \tilde{\nabla}^i\tilde{\nabla}^j\overline{A}_{ij} &= \tilde{\nabla}_i\Delta_{\tilde{g}}\mathring{X}^i+\Delta_{\tilde{g}}\mathring{Y}+\frac{1}{2}\mathring{X}^i\tilde{\nabla}_i\tilde{R}+\underbrace{\tilde{R}_{jk}\tilde{\nabla}^j\mathring{X}^k}_{=\tilde{R}\mathring{Y}}\\ &= 4\Delta_{\tilde{g}}\mathring{Y}+\mathring{X}^i\tilde{\nabla}_i\tilde{R}+2\tilde{R}\mathring{Y}\,, \end{aligned}$$

i.e. (4.3) follows from (4.5) and (4.6).

We have

$$\begin{aligned} \breve{\overline{B}}_{ij} &= (\mathring{X}^k\tilde{\nabla}_k\tilde{L}_{ij}+2\tilde{L}_{k(i}\tilde{\nabla}_{j)}\mathring{X}^k+\tilde{\nabla}_i\tilde{\nabla}_j\mathring{Y})\breve{}\\ &= \mathscr{L}_{\mathring{X}^k\partial_k}\breve{\tilde{L}}_{ij}+(\tilde{\nabla}_i\tilde{\nabla}_j\mathring{Y})\breve{}, \end{aligned}$$

and

$$\begin{aligned} (\overline{\nabla_0 B^{\mathrm{intr}}_{ij}})\breve{} &= -\sqrt{\frac{\lambda}{3}}(D_{ij}\mathring{Y}+\mathring{X}^k\tilde{\nabla}_kD_{ij}+2D_{k(i}\tilde{\nabla}_{j)}\mathring{X}^k)\\ &= -\sqrt{\frac{\lambda}{3}}(\mathscr{L}_{\mathring{X}^k\partial_k}D_{ij}+D_{ij}\mathring{Y})\,. \end{aligned}$$

We observe that due to the second Bianchi identity and (4.5)

$$\begin{aligned} \tilde{\nabla}_i\tilde{\nabla}^k\overline{A}_{jk} &= \mathscr{L}_{\mathring{X}^k\partial_k}\tilde{R}_{\mu\nu}+\tilde{\nabla}_i\tilde{\nabla}_j\mathring{Y}+2\mathring{X}^k\tilde{\nabla}_{[i}\tilde{R}_{j]k}+\Delta_{\tilde{g}}\tilde{\nabla}_i\mathring{X}_j\\ &\quad +2\tilde{R}_i{}^k{}_j{}^l\tilde{\nabla}_k\mathring{X}_l-2\tilde{R}_{ij}\mathring{Y}-\tilde{R}_i{}^k\overline{A}_{jk}\,. \end{aligned}$$

Symmetrizing this expression, taking its traceless part and taking $\overline{A}_{ij} = 0$ into account we end up with

$$\mathscr{L}_{\mathring{X}^k\partial_k}\breve{\tilde{L}}_{\mu\nu} + (\tilde{\nabla}_i\tilde{\nabla}_j\mathring{Y})\breve{} = 0 \;,$$

i.e. $\breve{\overline{B}}_{ij}$ holds automatically, as well.

THEOREM 4.1 *Assume we have been given a $3+1$-dimensional "unphysical" space-time $(\mathscr{M}, g, \Theta)$, with $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^{\rho})$ a smooth solution of the MCFE with $\lambda > 0$ in the $(R = 0, \overline{s} = 0, \overline{g}_{00} = -1, \overline{g}_{0i} = 0, \hat{g}_{\mu\nu} = \overline{g}_{\mu\nu})$-wave-map gauge. Then there exists a smooth vector field X satisfying the unphysical Killing equations (3.1) on $\mathrm{D}^+(\mathscr{I}^-)$ (and thus corresponding to a Killing vector field of the physical space-time) if and only if there exists a conformal Killing vector field $\mathring{X}$ on $(\mathscr{I}^-, \tilde{g} = h_{ij}\mathrm{d}x^i\mathrm{d}x^j)$ such that the reduced KID equations*

$$\mathscr{L}_{\mathring{X}} D_{ij} + \frac{1}{3}D_{ij}\tilde{\nabla}_k\mathring{X}^k \quad = \quad 0 \tag{4.7}$$

hold (recall that the symmetric, trace- and divergence-free tensor field $D_{ij} = \overline{d}_{0i0j}$ belongs to the freely prescribable initial data). In that case X satisfies

$$\overline{X}^0 = 0 \;, \quad \overline{X^i} = \mathring{X}^i \;, \quad \overline{\nabla_0 X^0} = \frac{1}{3}\tilde{\nabla}_i\mathring{X}^i \;, \quad \overline{\nabla_0 X^i} = 0 \;. \tag{4.8}$$

REMARK 4.2 Note that, in contrast to the $\lambda = 0$-case treated in [10], the candidate fields, i.e. the conformal Killing fields on $\mathscr{I}^-$, do depend here on the initial data $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

REMARK 4.3 For initial data with $D_{ij} = 0$ the reduced Killing equations (4.7) are always satisfied, and each candidate field, i.e. each conformal Killing field on the initial manifold, extends to a Killing field of the physical space-time.

In terms of an initial value problem Theorem 2.1 and 4.1 state that given a Riemannian manifold $(\mathcal{H}, h)$ and a symmetric, trace- and divergence-free tensor field D_{ij} there exists an (up to isometries) unique evolution into a space-time manifold $(\mathscr{M}, g, \Theta)$ with $\mathcal{H} = \mathscr{I}^-$, $\overline{g}_{ij} = h_{ij}$ and $\overline{d}_{0i0j} = D_{ij}$ which fulfills the MCFE and contains a vector field satisfying the unphysical Killing equations (3.1) if and only if there exists a conformal Killing vector field $\mathring{X}$ on $(\mathcal{H}, h)$ such that the reduced KID equations (4.7) hold.

4.2 Properties of the reduced KID equations

We compute how the reduced KID equations (4.7) behave under conformal transformations. For this consider the conformally rescaled metric $\tilde{\tilde{g}} := \Omega^2\tilde{g}$ with Ω some positive function. Expressed in terms of $\tilde{\tilde{g}}$ (4.7) becomes

$$\mathscr{L}_{\mathring{X}}(\Omega^{-1}D_{ij}) + \frac{1}{3}(\Omega^{-1}D_{ij})\tilde{\tilde{\nabla}}_k\mathring{X}^k \quad = \quad 0 \;, \tag{4.9}$$

i.e. they are conformally covariant in the following sense:

LEMMA 4.4 *The pair $(\tilde{g}_{ij}, D_{ij})$ is a solution of the reduced KID equations (4.7) if and only if the conformally rescaled pair $(\Omega^2\tilde{g}_{ij}, \Omega^{-1}D_{ij})$, with Ω some positive function, is a solution of these equations.*

This is consistent with the observation that conformal rescalings of the initial data do not change the isometry class of the emerging space-time.

4.3 Some special cases

Let us finish by taking a look at some special cases:

4.3.1 Compact initial manifolds

We consider a *compact* initial manifold $(\mathscr{I}^-, \tilde{g})$ and assume that it admits a conformal Killing field $\mathring{X}$. Then there exists (cf. e.g. [7]) a positive function Ω such that the conformally rescaled metric $\tilde{\tilde{g}} = \Omega^2 \tilde{g}$ has one of the following properties:

- Either $(\mathscr{I}^-, \tilde{\tilde{g}}) = (S^3, s_{ij}\mathrm{d}x^i\mathrm{d}x^j)$ is the standard 3-sphere,
- or $\mathring{X}$ is a Killing vector field w.r.t. $\tilde{\tilde{g}}$.

If $(\mathscr{I}^-, \tilde{\tilde{g}})$ is the round 3-sphere all the conformal Killing fields are explicitly known. In the second case where $\mathring{X}$ is a Killing vector field w.r.t. $\tilde{\tilde{g}}$ the equation (4.9) simplifies to

$$\mathscr{L}_{\mathring{X}}(\Omega^{-1}D_{ij}) \quad = \quad 0\,. \tag{4.10}$$

That implies:

Lemma 4.5 *Consider a solution of the vacuum Einstein equations which admits a compact spacelike $\mathscr{I}^-$ and has a non-trivial Killing field. If $(\mathscr{I}^-, \tilde{g})$ is not conformal to a standard 3-sphere, then there exists a choice of conformal factor so that space-time Killing vector corresponds to a Killing field (rather than a conformal Killing field) of $(\mathscr{I}^-, \tilde{g})$.*

4.3.2 Maximally symmetric space-times

Let us consider the case where the initial manifold admits the maximal number of conformal Killing vector fields. Clearly this is a prerequisite to obtain a maximally symmetric space-time once the evolution problem has been solved. A connected 3-dimensional Riemannian manifold $(\mathcal{H}, h)$ admits at most 10 linearly independent conformal Killing vector fields. If equality is attained, $(\mathcal{H}, h)$ is known to be locally conformally flat [12].

Let us first consider the compact case. We use a classical result due to Kuiper (cf. [7]):

Theorem 4.6 *For any n-dimensional, simply connected, conformally flat Riemann manifold $(\mathcal{H}, h)$, there exists a conformal immersion $(\mathcal{H}, h) \hookrightarrow (S^n, s = s_{ij}\mathrm{d}x^i\mathrm{d}x^j)$, the so-called developing map, which is unique up to composition with Möbius transformations. If $\mathcal{H}$ is compact this map defines a conformal diffeomorphism from $(\mathcal{H}, h)$ onto (S^n, s).*

Since only the conformal class of the initial manifold matters we thus may assume $(\mathcal{H}, h)$ for compact $\mathcal{H}$ to be the standard 3-sphere from the outset. To end up with a maximally symmetric physical space-time containing 10 independent Killing fields one needs to make sure that each of the conformal Killing fields extends to a space-time vector field satisfying the unphysical Killing equations (3.1). In other words one needs to choose D_{ij} such that the reduced KID equations (4.7) hold for each and every conformal Killing field on (S^3, s). Via a stereographic projection onto Euclidean space one shows that this is only possible when D_{ij} is proportional to the round sphere metric. But D_{ij} is traceless, and thus needs to vanish. For data $(\mathcal{H}, h) = (S^3, s)$ and $D_{ij} = 0$ one ends up with de Sitter space-time. This is in accordance with the fact that de Sitter space-time is (up to isometries) the unique maximally symmetric, complete space-time with positive scalar curvature.

The non-compact case is somewhat more involved since the developing map does in general not define a global conformal diffeomorphism into (S^n, s). For convenience let us therefore make some simplifying assumptions on $(\mathcal{H}, h)$ which allow us to apply a result by Schoen & Yau [11] (we restrict attention to 3 dimensions when stating it):

THEOREM 4.7 *Let $(\mathcal{H}, h)$ be a complete, simply connected, conformally flat 3-dimensional Riemannian manifold and $\Phi : \mathcal{H} \hookrightarrow S^3$ its developing map. Assume that $|R(h)|$ is bounded on $\mathcal{H}$ and that $d(\mathcal{H}) < \frac{1}{3}$.*[4] *Then Φ is one-to-one and gives a conformal diffeomorphism from $\mathcal{H}$ onto a simply connected domain of S^3.*

We conclude, again, that the emerging space-time will be maximally symmetric iff $D_{ij} = 0$, and will be (isometric to) a part of de Sitter space-time.

4.3.3 Non-existence of stationary space-times

For $(\mathscr{M}, g, \Theta)$ to contain a timelike isometry there must exist a vector field X satisfying the unphysical Killing equations (3.1) which is null on $\mathscr{I}^-$ (it cannot be timelike since $\overline{X}^0 = 0$),

$$0 = \overline{g}_{\mu\nu}\overline{X}^\mu\overline{X}^\nu = h_{ij}\overline{X}^i\overline{X}^j \quad \Longrightarrow \quad \overline{X}^i = 0 \, .$$

But then the preceding considerations show that $\overline{X}^\mu = \overline{\nabla_0 X^\mu} = \overline{Y} = \overline{\nabla_0 Y} = 0$, and solving the wave equations for X and Y, (3.2) and (3.3), yields that X vanishes identically. It follows that there is no vacuum space-time with $\lambda > 0$ which is stationary near $\mathscr{I}^-$. (Compare [4, Section 4].)

Acknowledgements I would like to thank my advisor Piotr T. Chruściel for useful discussions, comments and for reading a first draft of this article. Moreover, I am grateful to Helmut Friedrich for pointing out reference [11] to me. Supported in part by the Austrian Science Fund (FWF): P 24170-N16.

[4]For the definition of the invariant $d(\mathcal{H})$ in terms of the minimal Green's function for the conformal Laplacian we refer the reader to [11].

A Equivalence between the CWE and the MCFE

A.1 Conformal wave equations (CWE)

In [9] the MCFE (2.1)-(2.6) have been rewritten as a system of conformal wave equations (CWE),

$$\begin{aligned}
\Box_g^{(H)} L_{\mu\nu} &= 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \frac{1}{6}\nabla_\mu\nabla_\nu R\,, && \text{(A.1)}\\
\Box_g s &= \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R\,\nabla^\kappa\Theta - \frac{1}{6}sR\,, && \text{(A.2)}\\
\Box_g \Theta &= 4s - \frac{1}{6}\Theta R\,, && \text{(A.3)}\\
\Box_g^{(H)} d_{\mu\nu\sigma\rho} &= \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa + \frac{1}{2}R d_{\mu\nu\sigma\rho}\,, && \text{(A.4)}\\
R^{(H)}_{\mu\nu}[g] &= 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}\,. && \text{(A.5)}
\end{aligned}$$

Here

$$R^{(H)}_{\mu\nu} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma\,, \tag{A.6}$$

denotes the reduced Ricci tensor. The reduced wave-operator $\Box_g^{(H)}$ (which is needed to obtain a PDE-system with a diagonal principal part) is defined via its action on covector fields v_λ,

$$\Box_g^{(H)} v_\lambda \;:=\; \Box_g v_\lambda - g_{\sigma[\lambda}(\hat{\nabla}_{\mu]}H^\sigma)v^\mu + (2L_{\mu\lambda} - R^{(H)}_{\mu\lambda} + \frac{1}{6}Rg_{\mu\lambda})v^\mu\,, \tag{A.7}$$

and similar formulae hold for higher-valence covariant tensor fields.

In the following we want to show that a solution of the CWE in the gauge (2.25) is a solution of the MCFE if and only if the constraint equations (2.47)-(2.54) hold on $\mathscr{I}^-$,

$$h^{ij}D_{ij} = 0\,, \quad \tilde{\nabla}^j D_{ij} = 0\,, \tag{A.8}$$

$$\overline{g}_{00} = -1\,, \quad \overline{g}_{0i} = 0\,, \quad \overline{g}_{ij} = h_{ij}\,, \quad \overline{\partial_0 g_{\mu\nu}} = 0\,, \tag{A.9}$$

$$\overline{\Theta} = 0\,, \quad \overline{\partial_0\Theta} = \sqrt{\tfrac{\lambda}{3}}\,, \tag{A.10}$$

$$\overline{s} = 0\,, \quad \overline{\partial_0 s} = \sqrt{\tfrac{\lambda}{48}}\,\tilde{R}\,, \tag{A.11}$$

$$\overline{L}_{ij} = \tilde{L}_{ij}\,, \quad \overline{L}_{0i} = 0\,, \quad \overline{L}_{00} = \tfrac{1}{4}\tilde{R}\,, \tag{A.12}$$

$$\overline{\partial_0 L_{ij}} = -\sqrt{\tfrac{\lambda}{3}}\,D_{ij}\,, \quad \overline{\partial_0 L_{0i}} = \tfrac{1}{4}\tilde{\nabla}_i\tilde{R}\,, \quad \overline{\partial_0 L_{00}} = 0\,, \tag{A.13}$$

$$\overline{d}_{0i0j} = D_{ij}\,, \quad \overline{d}_{0ijk} = \sqrt{\tfrac{3}{\lambda}}\tilde{C}_{ijk}\,, \tag{A.14}$$

$$\overline{\partial_0 d_{0i0j}} = \sqrt{\tfrac{3}{\lambda}}\tilde{B}_{ij}\,, \quad \overline{\partial_0 d_{0ijk}} = 2\tilde{\nabla}_{[j}D_{k]i}\,. \tag{A.15}$$

A.2 An intermediate result

In close analogy to [9, Theorem 3.7] one establishes the following result:

Theorem A.1 *Assume we have been given data ($\mathring{g}_{\mu\nu}$, $\mathring{K}_{\mu\nu}$, $\mathring{s}$, $\mathring{S}$, $\mathring{\Theta}$, $\mathring{\Omega}$, $\mathring{L}_{\mu\nu}$, $\mathring{M}_{\mu\nu}$, $\mathring{d}_{\mu\nu\sigma}{}^\rho$, $\mathring{D}_{\mu\nu\sigma}{}^\rho$) on a spacelike hypersurface $\mathcal{H}$ and a gauge source function*

R, such that $\mathring{g}_{\mu\nu}$ is the restriction to $\mathcal{H}$ of a Lorentzian metric, $\mathring{K}_{\mu\nu}$, $\mathring{L}_{\mu\nu}$ and $\mathring{M}_{\mu\nu}$ are symmetric, $\mathring{L} \equiv \mathring{L}_\mu{}^\mu = \overline{R}/6$, $\mathring{M}_\mu{}^\mu = \overline{\partial_0 R}/6$, and such that $\mathring{d}_{\mu\nu\sigma}{}^\rho$ and $\mathring{D}_{\mu\nu\sigma}{}^\rho$ satisfy all the algebraic properties of the Weyl tensor. Suppose further that there exists a solution ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^\rho$) of the CWE (A.1)-(A.5) with gauge source function R which induces the above data on $\mathcal{H}$,

$$\overline{g}_{\mu\nu} = \mathring{g}_{\mu\nu}\ , \quad \overline{s} = \mathring{s}\ , \quad \overline{\Theta} = \mathring{\Theta}\ , \quad \overline{L}_{\mu\nu} = \mathring{L}_{\mu\nu}\ , \quad \overline{d}_{\mu\nu\sigma}{}^\rho = \mathring{d}_{\mu\nu\sigma}{}^\rho\ ,$$
$$\overline{\partial_0 g_{\mu\nu}} = \mathring{K}_{\mu\nu}\ , \quad \overline{\partial_0 s} = \mathring{S}\ , \quad \overline{\partial_0 \Theta} = \mathring{\Omega}\ , \quad \overline{\partial_0 L_{\mu\nu}} = \mathring{M}_{\mu\nu}\ , \quad \overline{\partial_0 d_{\mu\nu\sigma}{}^\rho} = \mathring{D}_{\mu\nu\sigma}{}^\rho\ ,$$

and fulfills the following conditions:

1. *The MCFE (2.1)-(2.4) and their covariant derivatives are fulfilled on $\mathcal{H}$;*
2. *equation (2.5) holds at one point on $\mathcal{H}$;*
3. *$\overline{W}_{\mu\nu\sigma}{}^\rho[g] = \overline{\Theta}\,\overline{d}_{\mu\nu\sigma}{}^\rho$ and $\overline{\nabla_0 W_{\mu\nu\sigma}{}^\rho}[g] = \overline{\nabla_0(\Theta\, d_{\mu\nu\sigma}{}^\rho)}$;*
4. *the wave-gauge vector H^σ and its first- and second-order covariant derivatives $\nabla_\mu H^\sigma$ and $\nabla_\mu\nabla_\nu H^\sigma$ vanish on $\mathcal{H}$;*
5. *the covector field $\zeta_\mu \equiv -4(\nabla_\nu L_\mu{}^\nu - \frac{1}{6}\nabla_\mu R)$ and its covariant derivative $\nabla_\nu\zeta_\mu$ vanish on $\mathcal{H}$.*

Then

a) *$H^\sigma = 0$ and $R_g = R$ (where R_g denotes the Ricci scalar of $g_{\mu\nu}$);*

b) *$L_{\mu\nu}$ is the Schouten tensor of $g_{\mu\nu}$;*

c) *$\Theta d_{\mu\nu\sigma}{}^\rho$ is the Weyl tensor of $g_{\mu\nu}$;*

d) *($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^\rho$) solves the MCFE (2.1)-(2.6) in the ($H^\sigma = 0$, $R_g = R$)-gauge.*

The conditions 1.-5. are also necessary for d) to be true.

A.3 Applicability of Theorem A.1 on $\mathscr{I}^-$

We now consider the case where $\mathcal{H} = \mathscr{I}^-$. Using the gauge (2.25) we want to show that the hypotheses of Theorem A.1 are fulfilled by any tuple ($g_{\mu\nu}$, s, Θ, $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^\rho$) which satisfies the constraint equations (A.8)-(A.15) and the CWE.

For $R = 0$ the CWE reduce to

$$\begin{aligned}
\Box_g^{(H)} L_{\mu\nu} &= 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma\ , && \text{(A.16)}\\
\Box_g s &= \Theta|L|^2\ , && \text{(A.17)}\\
\Box_g \Theta &= 4s\ , && \text{(A.18)}\\
\Box_g^{(H)} d_{\mu\nu\sigma\rho} &= \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa\ , && \text{(A.19)}\\
R^{(H)}_{\mu\nu}[g] &= 2L_{\mu\nu}\ . && \text{(A.20)}
\end{aligned}$$

First of all note that $\overline{L} = 0 = \overline{R}/6$ and $\overline{\partial_0 L} = 0 = \overline{\partial_0 R}/6$, as required. Moreover (A.10) implies that (2.5) is satisfied on $\mathscr{I}^-$, i.e. it remains to verify that the hypotheses 1. and 3.-5. in Theorem A.1 are fulfilled.

Recall that in our gauge the only non-vanishing Christoffel symbols on $\mathscr{I}^-$ are $\overline{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ij}$, and that this implies that the action of ∇_0 and ∂_0 as well as the action of ∇_i and $\tilde{\nabla}_i$ coincides on $\mathscr{I}^-$.

A.3.1 Vanishing of $\overline{H}$, $\overline{\nabla H}$ and $\overline{\nabla\nabla H}$

We have

$$\overline{H}^0 \equiv \overline{g}^{\mu\nu}(\overline{\Gamma}^0_{\mu\nu} - \hat{\Gamma}^0_{\mu\nu}) = 0 \,, \tag{A.21}$$
$$\overline{H}^i \equiv \overline{g}^{\mu\nu}(\overline{\Gamma}^i_{\mu\nu} - \hat{\Gamma}^i_{\mu\nu}) = 0 \,. \tag{A.22}$$

Equation (A.20) can be written as

$$R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma = 2L_{\mu\nu} \,. \tag{A.23}$$

Invoking $\overline{H}^\sigma = 0$ that gives

$$\begin{aligned}
\overline{R}_{00} + \overline{\partial_0 H^0} &= 2\overline{L}_{00} \,, \\
\overline{R}_{0i} - \frac{1}{2}\overline{g}_{ij}\overline{\partial_0 H^j} &= 2\overline{L}_{0i} \,, \\
\overline{R}_{ij} &= 2\overline{L}_{ij} \,.
\end{aligned}$$

On the other hand, with (A.9) we find

$$\begin{aligned}
\overline{R}_{00} &= -\overline{\partial_0\Gamma^k_{0k}} = -\frac{1}{2}\overline{g}^{kl}\overline{\partial_0\partial_0 g_{kl}} \,, \\
\overline{R}_{0i} &= -\overline{\partial_0\Gamma^k_{ik}} = 0 \,, \\
\overline{R}_{ij} &= \overline{\partial_0\Gamma^0_{ij}} + \tilde{R}_{ij} = \frac{1}{2}\overline{\partial_0\partial_0 g_{ij}} + \tilde{R}_{ij} \,.
\end{aligned}$$

Taking (A.12) into account, we conclude that

$$\overline{\partial_0\partial_0 g_{ij}} = 2\tilde{R}_{ij} - \overline{g}_{ij}\tilde{R} \,, \tag{A.24}$$

as well as $\overline{\partial_0 H^\sigma} = 0$, and we end up with

$$\overline{\nabla_\mu H^\sigma} = 0 \,. \tag{A.25}$$

Note that this implies

$$0 = \overline{\partial_0 H^0} = \overline{g}^{\mu\nu}\overline{\partial_0\Gamma^0_{\mu\nu}} = \frac{1}{2}\overline{\partial_0\partial_0 g_{00}} - \frac{1}{2}\tilde{R} \,, \tag{A.26}$$
$$0 = \overline{\partial_0 H^k} = \overline{g}^{\mu\nu}\overline{\partial_0\Gamma^k_{\mu\nu}} = -\overline{g}^{kl}\overline{\partial_0\partial_0 g_{0l}} \,, \tag{A.27}$$

i.e.

$$\overline{\partial_0\partial_0 g_{00}} = \tilde{R} \,, \quad \overline{\partial_0\partial_0 g_{0i}} = 0 \,. \tag{A.28}$$

We give a list of the transverse derivatives of the Christoffel symbols on $\mathscr{I}^-$,

$$\overline{\partial_0\Gamma^0_{00}} = -\tfrac{1}{2}\tilde{R} \,, \quad \overline{\partial_0\Gamma^0_{ij}} = \overline{g}_{jk}\overline{\partial_0\Gamma^k_{0i}} = \tilde{R}_{ij} - \tfrac{1}{2}\overline{g}_{ij}\tilde{R} \,, \tag{A.29}$$
$$\overline{\partial_0\Gamma^0_{0i}} = \overline{\partial_0\Gamma^k_{00}} = \overline{\partial_0\Gamma^k_{ij}} = 0 \,. \tag{A.30}$$

Using (A.23) that yields with $\overline{H}^\sigma = 0 = \overline{\nabla_\mu H^\sigma}$ the relation

$$\overline{\partial_0 R_{\mu\nu}} - \overline{g}_{\sigma(\mu}\overline{\partial_{\nu)}\partial_0 H^\sigma} = 2\overline{\partial_0 L_{\mu\nu}} \,, \tag{A.31}$$

and thus

$$\begin{aligned}\overline{\partial_0 R_{00}} + \overline{\partial_0\partial_0 H^0} &= 2\overline{\partial_0 L_{00}}\ ,\\ \overline{\partial_0 R_{0i}} - \frac{1}{2}\overline{g}_{ij}\overline{\partial_0\partial_0 H^j} &= 2\overline{\partial_0 L_{0i}}\ ,\\ \overline{\partial_0 R_{ij}} &= 2\overline{\partial_0 L_{ij}}\ .\end{aligned}$$

We compute

$$\begin{aligned}\overline{\partial_0 R_{00}} &= -\overline{\partial_0\partial_0\Gamma^k_{0k}} \;=\; -\frac{1}{2}\overline{g}^{kl}\overline{\partial_0\partial_0\partial_0 g_{kl}}\ ,\\ \overline{\partial_0 R_{0i}} &= \underbrace{\tilde{\nabla}_k\overline{\partial_0\Gamma^k_{0i}}}_{=0} - \overline{\partial_0\partial_0\Gamma^k_{ik}} \;=\; -\frac{1}{2}\tilde{\nabla}_i(\overline{g}^{kl}\overline{\partial_0\partial_0 g_{kl}}) \;=\; \frac{1}{2}\tilde{\nabla}_i\tilde{R}\ ,\\ \overline{\partial_0 R_{ij}} &= \overline{\partial_0\partial_0\Gamma^0_{ij}} \;=\; \frac{1}{2}\overline{\partial_0\partial_0\partial_0 g_{ij}}\ .\end{aligned}$$

From (A.13) we deduce that

$$\overline{\partial_0\partial_0\partial_0 g_{ij}} \;=\; -4\sqrt{\frac{\lambda}{3}}\,D_{ij}\ , \tag{A.32}$$

from which we obtain $\overline{\partial_0\partial_0 H^\sigma} = 0$, and thus

$$\overline{\nabla_\mu\nabla_\nu H^\sigma} \;=\; 0\ . \tag{A.33}$$

A.3.2 Vanishing of $\overline{\zeta}$ and $\overline{\nabla\zeta}$

In our gauge we have

$$\zeta_\mu = -4\nabla_\alpha L_\mu{}^\alpha\ . \tag{A.34}$$

We invoke (A.12) and (A.13) to obtain

$$\overline{\zeta}_0 \;=\; 4\overline{\partial_0 L_{00}} - 4\overline{g}^{kl}\nabla_k\overline{L}_{0l} \;=\; 0\ , \tag{A.35}$$

$$\overline{\zeta}_i \;=\; 4\overline{\partial_0 L_{0i}} - 4\overline{g}^{kl}\tilde{\nabla}_k\overline{L}_{il} \;=\; 0\ . \tag{A.36}$$

The computation of $\overline{\nabla_0\zeta_\mu}$ requires the knowledge of certain second-order transverse derivatives of $\overline{L}_{\mu\nu}$ which we compute from the CWE (A.16). Since $\overline{H}^\sigma = 0 = \overline{\nabla_\mu H^\sigma}$ we have

$$\begin{gathered}\overline{\Box_g L_{\mu\nu}} = \overline{\Box^{(H)}_g L_{\mu\nu}} = 4\overline{L}_{\mu\kappa}\overline{L}_\nu{}^\kappa - \overline{g}_{\mu\nu}|\overline{L}|^2 \quad\Longleftrightarrow\\ \overline{\nabla_0\nabla_0 L_{\mu\nu}} = \Delta_{\tilde{g}}\overline{L}_{\mu\nu} - 4\overline{L}_{\mu\kappa}\overline{L}_\nu{}^\kappa + \overline{g}_{\mu\nu}[\overline{L}_k{}^l\overline{L}_l{}^k + (\overline{L}_{00})^2]\ ,\end{gathered}$$

whence

$$\overline{\nabla_0\nabla_0 L_{00}} \;=\; \frac{1}{4}\Delta_{\tilde{g}}\tilde{R} - |\tilde{R}|^2 + \frac{1}{2}\tilde{R}^2\ , \tag{A.37}$$

$$\overline{\nabla_0\nabla_0 L_{0i}} \;=\; 0\ . \tag{A.38}$$

From

$$\overline{\nabla_0\zeta_\mu} \;=\; 4\overline{\nabla_0\nabla_0 L_{0\mu}} - 4\overline{g}^{kl}\tilde{\nabla}_k\overline{\nabla_0 L_{l\mu}} - 4\overline{R}_{0k\mu}{}^l\overline{L}_l{}^k - 4\overline{R}_{00}\overline{L}_{0\mu}$$

and

$$\overline{R}_{0k0}{}^{l} = -\tilde{R}_k{}^l + \frac{1}{2}\delta_k{}^l \tilde{R} \,, \tag{A.39}$$

$$\overline{R}_{0ki}{}^{l} = 0 \,, \tag{A.40}$$

we conclude that

$$\overline{\nabla_0 \zeta_0} = \frac{3}{2}\tilde{R}^2 - 4|\tilde{R}|^2 - 4\overline{R}_{0k0}{}^l \overline{L}_l{}^k = 0 \,, \tag{A.41}$$

$$\overline{\nabla_0 \zeta_i} = 4\sqrt{\frac{\lambda}{3}}\tilde{\nabla}^j D_{ij} - 4\overline{R}_{0ki}{}^l \overline{L}_l{}^k = 0 \,. \tag{A.42}$$

A.3.3 Validity of the MCFE (2.1)-(2.4) and their transverse derivatives on $\mathscr{I}^-$

The independent components of $\overline{\nabla_\rho d_{\mu\nu\sigma}{}^\rho}$, which is antisymmetric in its first two indices, trace-free and satisfies the first Bianchi identity, are

$$\overline{\nabla_\rho d_{ijk}{}^\rho} \quad \text{and} \quad \overline{\nabla_\rho d_{0ij}{}^\rho}$$

(similarly for its transverse derivatives).

It follows from (2.44), (A.14), (A.15) and (A.8) that

$$\overline{\nabla_\rho d_{ijk}{}^\rho} = \tilde{\nabla}^l \overline{d}_{ijkl} - \overline{\nabla_0 d_{0kji}} = 2\overline{g}_{k[i}\tilde{\nabla}^l D_{j]l} = 0 \,, \tag{A.43}$$

$$\overline{\nabla_\rho d_{0ij}{}^\rho} = \nabla^k \overline{d}_{0ijk} + \overline{\nabla_0 d_{0i0j}} = 0 \,. \tag{A.44}$$

We consider the corresponding transverse derivatives. With (A.39) and (A.40) we find

$$\overline{\nabla_0\nabla_\rho d_{ijk}{}^\rho} = \overline{\nabla_0\nabla_0 d_{0kij}} + \tilde{\nabla}^l\overline{\nabla_0 d_{ijkl}} - 2\overline{R}_{0[j|0}{}^l\overline{d}_{0|i]kl} + \overline{R}_{0k0}{}^l\overline{d}_{0lij} + \overline{R}_{00}\overline{d}_{0kji} \,,$$

$$\overline{\nabla_0\nabla_\rho d_{0ij}{}^\rho} = \overline{\nabla_0\nabla_0 d_{0i0j}} + \tilde{\nabla}^k\overline{\nabla_0 d_{0ijk}} - \overline{R}_0{}^k{}_0{}^l\overline{d}_{ikjl} + \overline{R}_{0j0}{}^k D_{ik} - \overline{R}_{00} D_{ij} \,.$$

The second-order transverse derivatives of the rescaled Weyl tensor follow from the CWE (A.19),

$$\overline{\Box_g d_{\mu\nu\sigma\rho}} = \overline{\Box_g^{(H)} d_{\mu\nu\sigma\rho}} = 0 \quad \Longleftrightarrow \quad \overline{\nabla_0\nabla_0 d_{\mu\nu\sigma\rho}} = \Delta_{\tilde{g}}\overline{d}_{\mu\nu\sigma\rho} \,,$$

hence

$$\overline{\nabla_0\nabla_0 d_{0ijk}} = \Delta_{\tilde{g}}\overline{d}_{0ijk} = \sqrt{\frac{3}{\lambda}}\Delta_{\tilde{g}}\tilde{C}_{ijk} \,, \tag{A.45}$$

$$\overline{\nabla_0\nabla_0 d_{0i0j}} = \Delta_{\tilde{g}}\overline{d}_{0i0j} = \Delta_{\tilde{g}} D_{ij} \,. \tag{A.46}$$

The Bianchi identities together with the identity

$$\tilde{R}_{ijkl} \equiv 2\overline{g}_{i[k}\tilde{R}_{l]j} - 2\overline{g}_{j[k}\tilde{R}_{l]i} - \tilde{R}\overline{g}_{i[k}\overline{g}_{l]j} \,, \tag{A.47}$$

which holds in 3 dimensions, imply the following relations for Cotton and Bach tensor,

$$\tilde{C}_{[ijk]} = \tilde{C}^j{}_{ij} = \tilde{\nabla}^k\tilde{C}_{kij} = 0 \,,$$

$$\tilde{\nabla}_{[i}\tilde{C}_{j]kl} = \tilde{\nabla}_{[l}\tilde{C}_{k]ji} + \tilde{R}_{ij[l}{}^m\tilde{L}_{k]m} + \tilde{R}_{kl[i}{}^m\tilde{L}_{j]m} \,,$$

$$\tilde{\nabla}^j\tilde{B}_{ij} = \tilde{R}^{kl}\tilde{C}_{kli} \,,$$

$$\tilde{\nabla}_{[i}\tilde{B}_{j]k} = -\frac{1}{2}\Delta_{\tilde{g}}\tilde{C}_{kji} + \tilde{R}_{[j}{}^l\tilde{C}_{i]kl} - \frac{1}{2}\tilde{R}_k{}^l\tilde{C}_{lij} - g_{k[i}\tilde{C}^l{}_{j]}{}^m\tilde{R}_{lm} + \frac{1}{4}\tilde{R}\tilde{C}_{kji} \,.$$

With (2.44), (A.8), (A.14) and (A.15) we then obtain

$$\begin{aligned}\overline{\nabla_0\nabla_\rho d_{ijk}{}^\rho} &= \sqrt{\frac{3}{\lambda}}\Big(2\overline{g}_{k[i}\tilde{\nabla}^l\tilde{B}_{j]l} - 2\tilde{\nabla}_{[i}\tilde{B}_{j]k} - (\Delta_{\tilde{g}} - \frac{\tilde{R}}{2})\tilde{C}_{kji} + 2\tilde{R}_{[j}{}^l\tilde{C}_{i]kl} - \tilde{R}_k{}^l\tilde{C}_{lij}\Big) \\ &= 0\,, \\ \overline{\nabla_0\nabla_\rho d_{0ij}{}^\rho} &= 0\,.\end{aligned}$$

Set

$$\Xi_{\mu\nu} := \nabla_\mu\nabla_\nu\Theta + \Theta L_{\mu\nu} - s g_{\mu\nu}\,. \tag{A.48}$$

To compute $\overline{\Xi}_{00}$ we need to know the value of $\overline{\nabla_0\nabla_0\Theta}$ which can be determined from the CWE (A.18),

$$\overline{\Box_g\Theta} = 4\overline{s} \quad\Longleftrightarrow\quad \overline{\nabla_0\nabla_0\Theta} = 0\,. \tag{A.49}$$

Invoking (A.10)-(A.11) we then find

$$\overline{\Xi}_{ij} = 0\,, \tag{A.50}$$

$$\overline{\Xi}_{0i} = \overline{\nabla_i\nabla_0\Theta} = 0\,, \tag{A.51}$$

$$\overline{\Xi}_{00} = \overline{\nabla_0\nabla_0\Theta} = 0\,. \tag{A.52}$$

To calculate the transverse derivative of $\Xi_{\mu\nu}$ on $\mathscr{I}^-$ we need to determine the third-order transverse derivative of Θ

$$\overline{\nabla_0\Box_g\Theta} = 4\overline{\nabla_0 s} \quad\Longleftrightarrow\quad \overline{\nabla_0\nabla_0\nabla_0\Theta} = -\sqrt{\frac{\lambda}{12}}\tilde{R}\,. \tag{A.53}$$

One then straightforwardly verifies with (A.39) and the constraint equations

$$\begin{aligned}\overline{\nabla_0\Xi_{ij}} &= \overline{\nabla_i\nabla_j\nabla_0\Theta} + \overline{R}_{0i0j}\overline{\nabla_0\Theta} + \overline{L}_{ij}\overline{\nabla_0\Theta} - \overline{\nabla_0 s}\overline{g}_{ij} = 0\,, \\ \overline{\nabla_0\Xi_{0i}} &= \overline{\nabla_i\nabla_0\nabla_0\Theta} + \overline{L}_{0i}\overline{\nabla_0\Theta} = 0\,, \\ \overline{\nabla_0\Xi_{00}} &= \overline{\nabla_0\nabla_0\nabla_0\Theta} + \overline{L}_{00}\overline{\nabla_0\Theta} + \overline{\nabla_0 s} = 0\,.\end{aligned}$$

Set

$$\Upsilon_\mu := \nabla_\mu s + L_{\mu\nu}\nabla^\nu\Theta\,. \tag{A.54}$$

We observe that by (A.10)-(A.12)

$$\overline{\Upsilon}_0 = \overline{\nabla_0 s} - \overline{L}_{00}\overline{\nabla_0\Theta} = 0\,, \tag{A.55}$$

$$\overline{\Upsilon}_i = 0\,. \tag{A.56}$$

To compute the corresponding transverse derivatives on $\mathscr{I}^-$ we first of all need to calculate $\overline{\nabla_0\nabla_0 s}$, which follows from (A.17),

$$\overline{\Box_g s} = 0 \quad\Longleftrightarrow\quad \overline{\nabla_0\nabla_0 s} = 0;\,. \tag{A.57}$$

Employing further the constraint equations and (A.49) we then deduce

$$\begin{aligned}\overline{\nabla_0\Upsilon_0} &= \overline{\nabla_0\nabla_0 s} - \overline{\nabla_0 L_{00}\nabla_0\Theta} - \overline{L_{00}\nabla_0\nabla_0\Theta} = 0\,, \\ \overline{\nabla_0\Upsilon_i} &= \nabla_i\overline{\nabla_0 s} - \overline{\nabla_0 L_{0i}\nabla_0\Theta} + \overline{L}_{ij}\nabla^j\overline{\nabla_0\Theta} = 0\,.\end{aligned}$$

Set

$$\varkappa_{\mu\nu\sigma} := 2\nabla_{[\sigma}L_{\nu]\mu} - \nabla_\rho\Theta\, d_{\nu\sigma\mu}{}^\rho\,. \tag{A.58}$$

Due to the symmetries $\varkappa_{\mu(\nu\sigma)} = 0$, $\varkappa_{[\mu\nu\sigma]} = 0$ and $\varkappa_{\nu\mu}{}^{\nu} = 0$ (since $\overline{\zeta}_\mu = 0$ and $L = 0$) its independent components on the initial surface are

$$\overline{\varkappa}_{ijk} \quad \text{and} \quad \overline{\varkappa}_{ij0} \; .$$

Since also $\overline{\nabla_0\zeta_\mu} = 0$ an analogous statement holds true for $\overline{\nabla_0\varkappa_{\mu\nu\sigma}}$. We find with (A.10) and (A.12)-(A.14)

$$\overline{\varkappa}_{ijk} \;=\; 2\nabla_{[k}\overline{L}_{j]i} - \overline{\nabla_0\Theta}\,\overline{d}_{0ijk} \;=\; 0 \; , \tag{A.59}$$

$$\overline{\varkappa}_{ij0} \;=\; 2\overline{\nabla_{[0}L_{j]i}} + \overline{\nabla_0\Theta}\,\overline{d}_{0i0j} \;=\; 0 \; . \tag{A.60}$$

Before we proceed let us first determine the second-order transverse derivative of L_{ij} on $\mathscr{I}^-$. From the CWE (A.16) we obtain

$$\begin{aligned}\overline{\Box_g L_{ij}} = \overline{\Box_g^{(H)} L_{ij}} 4\tilde{L}_{ik}\tilde{L}_j{}^k - \overline{g}_{ij}|\tilde{L}|^2 - \overline{g}_{ij}(\overline{L}_{00})^2 \quad &\Longleftrightarrow \\ \overline{\nabla_0\nabla_0 L_{ij}} = \Delta_{\tilde{g}}\tilde{L}_{ij} - 4\tilde{L}_{ik}\tilde{L}_j{}^k + \overline{g}_{ij}(|\tilde{L}|^2 + \frac{1}{16}\tilde{R}^2) \; .&\end{aligned} \tag{A.61}$$

For the transverse derivatives we then find with (A.39), (A.40) and (A.49) and the constraint equations

$$\begin{aligned}\overline{\nabla_0\varkappa_{ijk}} &= 2\tilde{\nabla}_{[k}\overline{\nabla_{|0|}L_{j]i}} - \overline{\nabla_0\Theta}\,\overline{\nabla_0 d_{0ijk}} \;=\; 0 \; , \\ \overline{\nabla_0\varkappa_{ij0}} &= \overline{\nabla_0\nabla_0 L_{ij}} - \tilde{\nabla}_j\overline{\nabla_0 L_{0i}} - \overline{R}_{0j0}{}^k\overline{L}_{ik} - \overline{R}_{0i0j}\overline{L}_{00} + \overline{\nabla_0\Theta}\,\overline{\nabla_0 d_{0i0j}} \\ &= \overline{\nabla_0\nabla_0 L_{ij}} - \frac{1}{4}\tilde{\nabla}_i\tilde{\nabla}_j\tilde{R} + \tilde{L}_j{}^k\tilde{L}_{ik} - \frac{1}{16}\tilde{R}^2\overline{g}_{ij} + \tilde{B}_{ij} \;=\; 0 \; ,\end{aligned}$$

where we have used that

$$\tilde{B}_{ij} = -\Delta_{\tilde{g}}\tilde{L}_{ij} + \frac{1}{4}\tilde{\nabla}_i\tilde{\nabla}_j\tilde{R} - \overline{g}_{ij}|\tilde{L}|^2 + 3\tilde{L}_{ik}\tilde{L}_j{}^k \; . \tag{A.62}$$

A.3.4 Vanishing of $\overline{W}_{\mu\nu\sigma}{}^{\rho} - \overline{\Theta}\,\overline{d}_{\mu\nu\sigma}{}^{\rho}$ and $\overline{\nabla_0(W_{\mu\nu\sigma}{}^{\rho} - \Theta\, d_{\mu\nu\sigma}{}^{\rho})}$

The independent components of the conformal Weyl tensor in adapted coordinates are

$$\overline{W}_{0ij}{}^k \quad \text{and} \quad \overline{W}_{0i0}{}^j \; .$$

Using the definition of the Weyl tensor

$$W_{\mu\nu\sigma}{}^{\rho} \equiv R_{\mu\nu\sigma}{}^{\rho} - 2\left(g_{\sigma[\mu}L_{\nu]}{}^{\rho} - \delta_{[\mu}{}^{\rho}L_{\nu]\sigma}\right)$$

we observe that by (A.39), (A.40) and (A.12) we have

$$\begin{aligned}\overline{W}_{0ij}{}^k &= \overline{g}_{ij}\overline{L}_0{}^k - \delta_i{}^k\overline{L}_{0j} \;=\; 0 \; , \\ \overline{W}_{0i0}{}^j &= \overline{R}_{0i0}{}^j + \overline{L}_i{}^j - \delta_i{}^j\overline{L}_{00} \;=\; 0 \; .\end{aligned}$$

To derive expressions for the transverse derivatives recall the formulae (2.35), (A.29)-(A.30) for the Christoffel symbols and their transverse derivatives on $\mathscr{I}^-$. Since, by (A.24), (A.27) and (A.32), we further have

$$\begin{aligned}\overline{\partial_0\partial_0\Gamma^k_{ij}} &= \frac{1}{2}\overline{g}^{kl}(\tilde{\nabla}_i\overline{\partial_0\partial_0 g_{jl}} + \tilde{\nabla}_j\overline{\partial_0\partial_0 g_{il}} - \tilde{\nabla}_l\overline{\partial_0\partial_0 g_{ij}}) \\ &= 2\tilde{\nabla}_{(i}\tilde{R}_{j)}{}^k - \delta_{(i}{}^k\tilde{\nabla}_{j)}\tilde{R} - \tilde{\nabla}^k\tilde{R}_{ij} + \frac{1}{2}\overline{g}_{ij}\tilde{\nabla}^k\tilde{R} \; , \\ \overline{\partial_0\partial_0\Gamma^j_{i0}} &= \frac{1}{2}\overline{g}^{jk}(\overline{\partial_0\partial_0\partial_0 g_{ik}} + \tilde{\nabla}_i\overline{\partial_0\partial_0 g_{0k}} - \tilde{\nabla}_k\overline{\partial_0\partial_0 g_{0i}}) \;=\; -2\sqrt{\frac{\lambda}{3}}D_i{}^j \; ,\end{aligned}$$

we find that

$$\begin{aligned}
\overline{\nabla_0 R_{0ij}{}^k} = \overline{\partial_0 R_{0ij}{}^k} &= \tilde\nabla_i \overline{\partial_0\Gamma^k_{0j}} - \overline{\partial_0\partial_0\Gamma^k_{ij}} \\
&= -\tilde\nabla_j \tilde R_i{}^k + \frac{1}{2}\delta_i{}^k\tilde\nabla_j\tilde R + \tilde\nabla^k\tilde R_{ij} - \frac{1}{2}\overline{g}_{ij}\tilde\nabla^k\tilde R \,, \\
\overline{\nabla_0 R_{0i0}{}^j} = \overline{\partial_0 R_{0i0}{}^j} &= -\overline{\partial_0\partial_0\Gamma^j_{i0}} = 2\sqrt{\frac{\lambda}{3}} D_i{}^j \,.
\end{aligned}$$

Hence

$$\begin{aligned}
\overline{\nabla_0 W_{0ij}{}^k} &= \overline{\nabla_0 R_{0ij}{}^k} + \overline{g}_{ij}\overline{\nabla_0 L_0{}^k} - \delta_i{}^k\overline{\nabla_0 L_{0j}} = \tilde\nabla^k\tilde L_{ij} - \tilde\nabla_j\tilde L_i{}^k = \tilde C_{ij}{}^k \,, \\
\overline{\nabla_0 W_{0i0}{}^j} &= \overline{\nabla_0 R_{0i0}{}^j} + \overline{\nabla_0 L_i{}^j} - \delta_i{}^j\overline{\nabla_0 L_{00}} = \sqrt{\frac{\lambda}{3}} D_i{}^j \,,
\end{aligned}$$

and we end up with

$$\begin{aligned}
\overline{\nabla_0(W_{0ij}{}^k - \Theta d_{0ij}{}^k)} &= \overline{\nabla_0 W_{0ij}{}^k} - \overline{\nabla_0\Theta}\,\overline{d}_{0ij}{}^k = 0 \,, \\
\overline{\nabla_0(W_{0i0}{}^j - \Theta d_{0i0}{}^j)} &= \overline{\nabla_0 W_{0i0}{}^j} - \overline{\nabla_0\Theta}\,\overline{d}_{0i0}{}^j = 0 \,,
\end{aligned}$$

which completes the proof that Theorem A.1 is applicable supposing that the initial data for the CWE satisfy the constraint equations (A.8)-(A.15) on $\mathscr{I}^-$.

Theorem A.2 *Let us suppose we have been given a Riemannian metric h_{ij} and a smooth tensor field D_{ij} on $\mathscr{I}^-$. A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \Theta, s)$ of the CWE (A.16)-(A.20) to the future of $\mathscr{I}^-$ with initial data*

$$\begin{aligned}
(\overline{g}_{\mu\nu} = \mathring{g}_{\mu\nu},\ \overline{\partial_0 g_{\mu\nu}} = \mathring{K}_{\mu\nu},\ \overline{L}_{\mu\nu} = \mathring{L}_{\mu\nu},\ \overline{\partial_0 L_{\mu\nu}} = \mathring{M}_{\mu\nu},\ \overline{d}_{\mu\nu\sigma}{}^\rho = \mathring{d}_{\mu\nu\sigma}{}^\rho, \\
\overline{\partial_0 d_{\mu\nu\sigma}{}^\rho} = \mathring{D}_{\mu\nu\sigma}{}^\rho,\ \overline{\Theta} = \mathring{\Theta} = 0,\ \overline{\partial_0\Theta} = \mathring{\Omega},\ \overline{s} = \mathring{s} = 0, \overline{\partial_0 s} = \mathring{S})
\end{aligned}$$

where $\mathring{g}_{ij} = h_{ij}$ and the trace- and divergence-free part of $\mathring{d}_{0i0j} = D_{ij}$ are the free data, is a solution of the MCFE (2.1)-(2.6) in the

$$(R = 0, \overline{s} = 0, \overline{g}_{00} = -1, \overline{g}_{0i} = 0, \hat g_{\mu\nu} = \mathring{g}_{\mu\nu})\text{-wave-map gauge}$$

if and only if the initial data have their usual algebraic properties and solve the constraint equations (A.8)-(A.15). The function Θ is positive in some neighborhood to the future of $\mathscr{I}^-$, and $\mathrm{d}\Theta \neq 0$ on $\mathscr{I}^-$.

References

[1] R. Beig and P. T. Chruściel, *Killing Initial Data*, Class. Quantum Grav. **14** (1997), A83–A92.

[2] P. T. Chruściel, *An introduction to the Cauchy problem for the Einstein equations*, lecture notes, Roscoff (2010), `http://homepage.univie.ac.at/piotr.chrusciel/teaching/Cauchy/Roscoff.pdf`.

[3] P. T. Chruściel and T.-T. Paetz, *KIDs like cones*, Class. Quantum Grav. **30** (2013) 235036.

[4] H. Friedrich, *Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant*, J. Geom. Phys. **3** (1986) 101-117.

[5] H. Friedrich, *Conformal Einstein evolution*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg, Springer, 2002, pp. 1-50.

[6] H.-O. Kreiss and O. E. Ortiz, *Some mathematical and numerical questions connected with first and second order time-dependent systems of partial differential equations*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg, Springer, 2002, pp. 359-370.

[7] W. Kühnel and H.-B. Rademacher, *Conformal transformations of pseudo-Riemannian manifolds*, (2007), `http://www.igt.uni-stuttgart.de/LstDiffgeo/Kuehnel/preprints/surv4.pdf`.

[8] V. Moncrief, *Spacetime symmetries and linearization stability of the Einstein equations I*, J. Math. Phys **16** (1975) 493-498.

[9] T.-T. Paetz, *Conformally covariant systems of wave equations and their equivalence to Einstein's field equations*, (2013), arXiv:1306.6204 [gr-qc].

[10] T.-T. Paetz, *KIDs prefer special cones*, (2013), arXiv:1311.3692 [gr-qc].

[11] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92**, (1981) 47-71.

[12] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Habilitationsschrift (2001), `http://www.igt.uni-stuttgart.de/LstGeo/Semmelmann/Publikationen/killing20.pdf`.

CHAPTER 16

Conclusions and outlook

"To doubt everything or to believe everything are two equally convenient truths; both dispense with the necessity of reflection."
J. Henri Poincaré (1854–1912)

16.1 Conclusions

In this thesis we have analyzed and solved several issues related to the initial value problem in general relativity. We contributed to a systematic construction of space-times with specific physically reasonable properties in the sense that they possess certain symmetries, a certain (positive) mass, and/or a certain asymptotic structure.

First of all, we have developed new ways to integrate Einstein's wave-map gauge constraints for the characteristic initial value problem. The so-obtained flexibility is advantageous in various situations, for instance to prove well-posedness results for a larger class of matter models. We have used it to eliminate all the gauge-dependent logarithmic terms appearing in the asymptotic expansions of the solutions of the constraint equations at null infinity. It turned out, though, that some logarithmic terms cannot be removed by coordinate transformations. This led us to a gauge-independent no-logs-condition which characterizes the occurrence of log terms on the initial surface, and is strongly expected to characterize characteristic initial data leading to space-times which admit a "piece of a smooth $\mathscr{I}^{+}$". The construction of such space-times which are (at least in certain null directions) asymptotically flat in the sense of Penrose will be crucial to get a better understanding of the compatibility of Penrose's geometric notion of asymptotic flatness with Einstein's field equations.

Another approach to construct asymptotically flat (or de Sitter) space-times is via asymptotic initial value problems where (parts of the) data are prescribed on past null infinity. We focused attention to those cases where the initial surface is $\mathscr{I}^{-}$. For positive cosmological constant λ this is a space-like hypersurface, whereas for $\lambda = 0$ it is a null hypersurface, which we assumed to form a light-cone whose vertex, representing past time-like infinity, is a regular point as well. To analyze the corresponding Cauchy problems, we extracted a system of wave equations from Friedrich's conformal field equations. We proved that the wave equations are equivalent to the conformal field equations, supposing that they are supplemented by an appropriate set of constraint equations on $\mathscr{I}^-$. In the characteristic case we used these wave equations together with Dossa's well-posedness result for such systems to prove well-posedness of the conformal field equations with data on the C_{i^-}-cone near i^-. This result permits a systematic construction of solutions to Einstein's vacuum equations which extend arbitrarily far into the past and, at sufficiently early times, have an asymptotic structure resembling the Minkowskian one. Moreover, it provides a first step towards the construction of Friedrich's purely radiative space-times. We were further be able to use the system of wave equations to provide an alternative proof of Friedrich's well-posedness result for the conformal fields equations with data on a space-like $\mathscr{I}^-$.

The analysis of the asymptotic behavior of solutions of the wave-map gauge constraints led us to a new formula for the Trautman-Bondi mass, which we expressed in terms of the data prescribed on the associated null hypersurface. Assuming this null hypersurface to be a globally smooth light-cone, we have rewritten this formula as a manifestly positive expression, and thereby obtained an unexpected and direct proof of the positivity of the Bondi mass.

Finally, we studied the construction of Killing vector fields in terms of initial value problems, i.e. the issue which additional equations, the so-called KID equations, the data need to fulfill, so that the emerging space-time contains a Killing vector field. Since symmetries play an utmost important role in physics and Killing vector fields generate (local) isometries, it is of great relevance to have such results available. The analysis had only been done for the space-like case before, while we did it for the characteristic case. Moreover, we did the same analysis for the asymptotic Cauchy problems. We first established a substitute for Killing's equation in Penrose's conformally rescaled space-times, which we then analyzed thoroughly to derive the KID equations. We have studied them in detail on a space-like $\mathscr{I}^-$ and a characteristic $\mathscr{I}^-$.

16.2 Outlook and open issues

There are several open issues closely related to the problems which have been considered in the course of this Ph.D. project, which we briefly discuss here:

- We have explained that one main motivation for analyzing the numerous possibilities to integrate Einstein's wave-map gauge constraints in [34] was to provide the basis to include larger classes of matter models for which well-posedness results can be derived, such as above all the Einstein-Vlasov system. While an approach to this has been given in [14], a well-posedness result with non-vanishing Vlasov matter up to the vertex of the cone is still lacking.

- The considerations in [38, 72] concerning the construction of smooth initial data for the CFE up-to-and-including conformal infinity are just a first step towards the construction of space-times which admit a piece of a smooth $\mathscr{I}^+$. An appropriate well-posedness result for smooth data for the evolution equations has been established in [10]. It might be interesting to investigate, whether a corresponding result can be obtained in the polyhomogeneous setting, as well, i.e. without imposing the no-logs-condition.

- Another question of interest is to establish the existence of a vacuum development which includes at least a piece of a smooth or polyhomogeneous $\mathscr{I}^+$ when data a prescribed on the C_{i^-}-cone. This would provide a way to construct space-times with a smooth i^- and $\mathscr{I}^-$, a regular i^0 and a regular $\mathscr{I}^+$, though it is hardly predictable what difficulties one might encounter near i^0, where the metric tensor cannot be assumed to be smooth, and whether one is (generically) led to a smooth or rather to a polyhomogeneous $\mathscr{I}^+$. This would be a decisive step towards the construction of Friedrich's purely radiative space-times, for which it is not conclusively clear yet whether the very concept of such a space-time is appropriate in that a sufficiently large class of solutions is compatible with all its requirements.

- Asymptotic initial value problems provide a tool to construct systematically space-times which are asymptotically flat or de Sitter (similarly, asymptotic initial boundary value problems can be employed to construct asymptotically anti-de Sitter space-times). However, no exhaustive knowledge is gained how restrictive such asymptotic conditions on space-times are. The analysis of the asymptotic appearance of logarithmic terms when integrating the constraint equations on a light-cone led to the no-logs-condition as the only relevant obstruction. Nonetheless, this condition shows that, as in the hyperboloidal case [1, 2], initial data which are smooth at conformal infinity are not generic. Further investigations are necessary. In particular, it is an open issue to characterize the set of asymptotically Euclidean space-like initial data sets, the evolution of which lead to (null) asymptotically flat solutions of Einstein's field equations (compare [55] for an approach to this issue, where i^0 is blown up to a cylinder).
  While the result in [53] shows that small perturbations of de Sitter initial data still lead to a regular $\mathscr{I}$, the result on the non-linear stability of the Minkowski space-time [23] give less hints in this respect. However, it is known that in the stationary case smooth conformal extensions through $\mathscr{I}$ are admitted [40]. Moreover, Chruściel and Delay [29] modified Corvino's gluing technique [39] to construct large classes of asymptotically flat space-time from asymptotically Euclidean initial data sets.
  So before a conclusion can be drawn whether a smooth or a polyhomogeneous $\mathscr{I}$ should be required, the space-like Cauchy problem needs to be better understood. For instance, it is not clear at all to what extent a smooth or polyhomogeneous $\mathscr{I}$ is compatible with space-times which are asymptotically flat in spatial directions, i.e. arises from the evolution of asymptotically Euclidean initial data sets.

- Further open problems regard well-posedness results for the CFE with data on light-cone with vertex at $\mathscr{I}^-$ for $\lambda \geq 0$ or for two transversally intersecting characteristic surfaces whose intersection manifold is located at $\mathscr{I}^-$ for $\lambda > 0$, and to analyze which are the freely prescribable data in these cases.

- Similarly, it might be of interest to derive the KID equations for these initial surfaces.

- Another task would be to generalize the KID equations to include certain matter models.

- It would also be of interest to construct KIDs for which the corresponding Killing vector fields have specific properties, so that the emerging space-time is, e.g., stationary, static, axisymmetric etc.

- The formula for the Bondi mass derived in [37] could be generalized to include a cosmological constant.

- An explicit computation, if possible, of the *Bondi news function*, or rather of $\overline{\partial_u m_{\mathrm{TB}}}$, in terms of characteristic initial data would provide some information on the amount of energy radiated away by the system. Moreover, it might be possible to establish necessary and/or sufficient conditions on the null data this way to produce space-times where no gravitational energy is radiated away (at least for a certain retarded time interval).

- It is an open issue to derive – using elementary methods – a positive mass theorem for a characteristic hypersurface with an interior boundary diffeomorphic to S^2 and intersecting $\mathscr{I}^+$ in a smooth spherical cross-section. Starting from the corresponding expression for the Bondi mass [37, Equation (54)], which is not manifestly positive, this requires to gain a better understanding of the implications of the boundary terms (cf. [37] for partial results on this issue).

- The formula for the Bondi mass in [37] is derived in the physical space-time. It might be auspicious to rewrite it in terms of unphysical fields in the conformally rescaled space-time to obtain an expression which remains regular on the C_{i^-}-cone. It then could be compared with the ADM mass computed at i^0.

- An analysis of the asymptotic behavior of solutions of the Einstein wave-map gauge constraints in a space-time with negative cosmological constant may shed some light on how to satisfy the corner conditions arising at the intersection of the light-cone with $\mathscr{I}$, when reflective boundary conditions are imposed there. This would be particularly interesting, since it is not clear how they can be fulfilled in the space-like case [58], compare [30] where partial results are given.

- A technical issue of rather mathematical relevance would be to derive all the above results under lower regularity assumptions.

APPENDIX A

Mathematical preliminaries

"Mathematics is the queen of sciences [...]. She often condescends to render service to astronomy and other natural sciences [...]."
CARL FRIEDRICH GAUSS (1777–1855)

A.1 Well-posedness results for wave equations

In this appendix we have collected some well-posedness results for (quasi-)linear wave equations on which many results obtained in the main text are based. We restrict attention to the smooth case and assume a dimension of space $n \geq 1$.

We start with a classical result for a space-like Cauchy problem, cf. e.g. [84]:

THEOREM A.1.1 *Consider a quasi-linear system of wave equations for a collection of functions v on $\mathbb{R}^{n+1}$,*

$$A^{\mu\nu}(x, v, \partial v)\partial_\mu \partial_\nu v + f(x, v, \partial v) = 0 \,, \tag{A.1}$$

where A and f are smooth functions of the indicated variables, and where the quadratic form A has Lorentzian signature. Consider further a space-like (w.r.t. A) hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, on which smooth Cauchy data are given,

$$v|_\Sigma = \phi \,, \quad \partial_t v|_\Sigma = \psi \,. \tag{A.2}$$

Then, there exist unique smooth functions v in some neighborhood of Σ which solve (A.3) and induce the prescribed data. The solution depends in a continuous manner on the initial data.

The following result for a Cauchy problem on two characteristic hypersurfaces intersecting transversally is due to Rendall [76]:

THEOREM A.1.2 *Consider a quasi-linear system of wave equations for a collection of functions v on $\mathbb{R}^{n+1}$,*

$$A^{\mu\nu}(x, v)\partial_\mu \partial_\nu v + f(x, v, \partial v) = 0 \,, \tag{A.3}$$

where A and f are smooth functions of the indicated variables, and where the quadratic form A has Lorentzian signature. Consider further continuous initial data ϕ on two transversally intersecting characteristic (w.r.t. A) hypersurfaces $N_1, N_2 \subset \mathbb{R}^{n+1}$,

$$v|_{N_1 \cup N_2} = \phi\,, \tag{A.4}$$

whose restrictions to N_1 and N_2 are smooth.

Then there exist unique smooth functions v in some neighborhood to the future of the intersection manifold which solve (A.3) and induce the prescribed data. The solution depends continuously on the initial data.

A well-posedness result for quasi-linear wave equations with data on a light-cone goes back to work of Cagnac [11] and Dossa [41], compare [16] for an English summary of their result.

THEOREM A.1.3 *Consider a quasi-linear system of wave equations for a collection of functions v on $\mathbb{R}^{n+1}$,*

$$A^{\mu\nu}(x,v)\partial_\mu\partial_\nu v + f(x,v,\partial v) \;=\; 0\,, \tag{A.5}$$

where A and f are smooth functions of the indicated variables, and where the quadratic form A has Lorentzian signature and takes the Minkowskian values for $x = 0$ and $v = 0$. Consider initial data ϕ on a light-cone $C_O \subset \mathbb{R}^{n+1}$ (w.r.t. A),

$$v|_{C_O} = \phi \quad \text{with} \quad \phi(O) = 0\,, \tag{A.6}$$

and assume that ϕ is the trace on C_O of a smooth function.

Then, there exist unique smooth functions v in some neighborhood to the future of O which solve (A.5), which can be extended to smooth functions in $\mathbb{R}^{n+1}$, and which induce the prescribed data on C_O. The solution depends continuously on the initial data.

For all of the corresponding Cauchy problems for *linear* systems of wave equations it is possible to establish global existence results, though it appears partly difficult to find adequate references, cf. e.g. [13, 45, 62]:

THEOREM A.1.4 *Consider a linear system of wave equations for a collection of functions $v = (v_i)$ on a space-time $(\mathscr{M}, g)$,*

$$g^{\mu\nu}(x)\partial_\mu\partial_\nu v_i + \sum_j (a_{ij})^\mu(x)\partial_\mu v_j + \sum_j b_{ij}(x)v_j \;=\; f_i(x)\,, \tag{A.7}$$

where a_{ij}, b_{ij} and f_i are smooth on $\mathscr{M}$.

1. *Consider smooth initial data ϕ and ψ on a space-like hypersurface $\Sigma \subset \mathscr{M}$*

$$v|_\Sigma = \phi\,, \quad \partial_t v|_\Sigma = \psi\,. \tag{A.8}$$

Then, there exist unique smooth functions v in $D(\Sigma)$ which solve (A.7) and induce the data ϕ and ψ on Σ.

2. *Consider initial data ϕ on two null hypersurfaces $N_1 \cup N_2 \subset \mathscr{M}$ intersecting transversally along a smooth submanifold,*

$$v|_{N_1 \cup N_2} = \phi\,. \tag{A.9}$$

Assume that the ϕ's are smooth on N_1 and N_2 and continuous at the intersection manifold. Then, there exist unique smooth functions v in $D^+(N_1 \cup N_2)$ which solve (A.7) and induce the data ϕ on $N_1 \cup N_2$.

3. *Consider initial data ϕ on a light-cone $C_O \subset \mathscr{M}$*

$$v|_{C_O} = \phi \,. \tag{A.10}$$

Assume that the ϕ's are traces on C_O of smooth functions. Then, there exist unique smooth functions v in $D^+(C_O)$ which solve (A.7) and induce the data ϕ on C_O.

In each case, the solution depends in a continuous manner on the initial data.

Bibliography

[1] L. Andersson, P.T. Chruściel: *On asymptotic behaviour of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissertationes Math. **355** (1996) 1–100.

[2] L. Andersson, P.T. Chruściel, H. Friedrich: *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations*, Comm. Math. Phys. **149** (1992) 587–612.

[3] V.I. Arnol'd: *Ordinary differential equations*, Berlin, Heidelberg: Springer, 1992.

[4] R. Arnowitt, S. Deser, C.W. Misner: *The dynamics of general relativity*, in: *Gravitation: An introduction to current research*, L. Witten (ed.), NewYork: Wiley, 1962, 227–265.

[5] A. Ashtekar: *Asymptotic structure of the gravitational field at spatial infinity*, in: *General relativity and gravitation – One hundred years after the birth of Albert Einstein, Volume 2*, A. Held (ed.), New York: Plenum Press, 1980, 37–70.

[6] A. Ashtekar, A. Magnon-Ashtekar: *Energy-momentum in general relativity*, Phys. Rev. Lett. **43** (1979) 181–184.

[7] R. Bartnik, J. Isenberg: *The constraint equations*, in: *The Einstein equations and the large scale behavior of gravitational fields*, P.T. Chruściel, H. Friedrich (eds.), Basel: Birkhäuser (2004) 1–38.

[8] R. Beig, P.T. Chruściel: *Killing Initial Data*, Class. Quantum Grav. **14** (1997) A83–A92.

[9] H. Bondi, M.G.J. van der Burg, A.W.K. Metzner: *Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems*, Proc. R. Soc. Lond. A **269** (1962) 21–52.

[10] A. Cabet, P.T. Chruściel, R. Tagne-Wafo: *On the characteristic initial value problem for nonlinear symmetric hyperbolic systems, including Einstein equations*, (2014), arXiv:1406.3009 [gr-qc].

[11] F. Cagnac: *Problème de Cauchy sur un conoïde caractéristique pour des équations quasi-linéaires*, Ann. Mat. Pura Appl. (4) **129** (1981) 13–41.

[12] Y. Choquet-Bruhat: *Un théorème d'instabilité pour certaines équations hyperboliques non linéares*, C. R. Acad. Sci. Paris **276** (1973) 281–284.

[13] Y. Choquet-Bruhat: *General relativity and the Einstein equations*, Oxford: Oxford University Press, 2009.

[14] Y. Choquet-Bruhat, P.T. Chruściel: *Cauchy problem with data on a characteristic cone for the Einstein-Vlasov equations*, (2013), arXiv:1206.0390 [gr-qc].

[15] Y. Choquet-Bruhat, P.T. Chruściel, J.M. Martín-García: *The light-cone theorem*, Class. Quantum Grav. **26** (2009) 135011.

[16] Y. Choquet-Bruhat, P.T. Chruściel, J.M. Martín-García: *The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions*, Ann. Henri Poincaré **12** (2011) 419–482.

[17] Y. Choquet-Bruhat, P.T. Chruściel, J.M. Martín-García: *An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations*, Contemp. Math. **554** (2011) 73—81.

[18] Y. Choquet-Bruhat, P.T. Chruściel, J.M. Martín-García: *An existence theorem for the Cauchy problem on the light-cone for the vacuum Einstein equations with near-round analytic data*, Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki **153**:3 (2011) 115–138.

[19] Y. Choquet-Bruhat, R. Geroch: *Global aspects of the Cauchy problem in general relativity*, Comm. Math. Phys. **14** (1969) 329–335.

[20] Y. Choquet-Bruhat, M. Novello: *Système conforme régulier pour les équations d'Einstein*, C. R. Acad. Sci. Paris **305** (1987) 155–160.

[21] Y. Choquet-Bruhat, J.W. York: *The Cauchy problem*, in: *General relativity and gravitation – One hundred years after the birth of Albert Einstein, Volume 1*, A. Held (ed.), New York: Plenum Press, 1980, 99–172.

[22] D. Christodoulou: *The formation of black holes in general relativity*, Monographs in Mathematics, Zürich: European Mathematical Society, 2009.

[23] D. Christodoulou, S. Klainermann: *The global nonlinear stability of the Minkowski space*, Princeton: Princeton University Press, 1993.

[24] P.T. Chruściel: *Black holes*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg: Springer, 2002, 61–102.

[25] P.T. Chruściel: *An introduction to the Cauchy problem for the Einstein equations*, lecture notes, Roscoff (2010), http://homepage.univie.ac.at/piotr.chrusciel/teaching/Cauchy/Roscoff.pdf.

[26] P.T. Chruściel: *Lectures on energy in general relativity*, lecture notes, Kraków (2010), revisions and additions, Vienna (2013), http://homepage.univie.ac.at/piotr.chrusciel/teaching/Energy/Energy.pdf.

[27] P.T. Chruściel: *The existence theorem for the general relativistic Cauchy problem on the light-cone*, (2012), arXiv:1209.1971 [gr-qc].

[28] P.T. Chruściel: *Lorentzian geometry and Einstein equations*, in preparation.

[29] P.T. Chruściel, E. Delay: *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. France **94** (2003) 1–103.

[30] P.T. Chruściel, E. Delay: *Gluing constructions for asymptotically hyperbolic manifolds with constant scalar curvature* , Communications in Analysis and Geometry **17** (2009) 343–381.

[31] P.T. Chruściel, J. Isenberg, D. Pollack: *Gluing initial data sets for general relativity*, Phys. Rev. Lett. **93** (2004), 081101.

[32] P.T. Chruściel, J. Jezierski, S. Łęski: *The Trautman-Bondi mass of hyperboloidal initial data sets*, Adv. Theor. Math. Phys. **8** (2004) 83–139.

[33] P.T. Chruściel, M.A.H. MacCallum, D. Singleton: *Gravitational waves in general relativity. XIV. Bondi expansions and the "polyhomogeneity" of Scri*, Phil. Trans. R. Soc. Lond. A **350** (1995) 113–141.

[34] P.T. Chruściel, T.-T. Paetz: *The many ways of the characteristic Cauchy problem*, Class. Quantum Grav. **29** (2012) 145006.

[35] P.T. Chruściel, T.-T. Paetz: *KIDs like cones* , Class. Quantum Grav. **30** (2013) 235036.

[36] P.T. Chruściel, T.-T. Paetz: *Solutions of the vacuum Einstein equations with initial data on past null infinity* , Class. Quantum Grav. **30** (2013) 235037.

[37] P.T. Chruściel, T.-T. Paetz: *The mass of light-cones*, Class. Quantum Grav. **31** (2014), 102001.

[38] P.T. Chruściel, T.-T. Paetz: *Characteristic initial data and smoothness of Scri. I. Framework and results*, (2014), arXiv:1403.3558 [gr-qc].

[39] J. Corvino: *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000) 137–189.

[40] S. Dain: *Initial data for stationary spacetimes near space-like infinity*, Class. Quantum Grav. **18** (2001), 4329-4338.

[41] M. Dossa: *Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un conoïde caractéristique*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997) 37–107.

[42] M. Dossa: *Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang-Mills-Higgs*, Ann. Henri Poincaré **4** (2003) 385–411.

[43] A. Einstein: *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Physik **354** (1916) 769–822.

[44] Y. Fourès-Bruhat: *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. **88** (1952) 141–225.

[45] F.G. Friedlander: *The wave-equation on a curved space-time*, Cambridge: Cambridge University Press, 1975.

[46] H. Friedrich: *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. R. Soc. Lond. A **375** (1981) 169–184.

[47] H. Friedrich: *The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system*, Proc. R. Soc. Lond. A **378** (1981) 401–421.

[48] H. Friedrich: *On the existence of analytic null asymptotically flat solutions of Einstein's vacuum field equations*, Proc. R. Soc. Lond. A **381** (1982) 361–371.

[49] H. Friedrich: *Cauchy problems for the conformal vacuum field equations in general relativity*, Comm. Math. Phys. **91** (1983) 445–472.

[50] H. Friedrich: *On the hyperbolicity of Einstein's and other gauge field equations*, Comm. Math. Phys. **100** (1985) 525–543.

[51] H. Friedrich: *On purely radiative space-times*, Comm. Math. Phys. **103** (1986) 35–65.

[52] H. Friedrich: *Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant*, J. Geom. Phys. **3** (1986) 101–117.

[53] H. Friedrich: *On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Comm. Math. Phys. **107** (1986) 587–609.

[54] H. Friedrich: *Hyperbolic reductions for Einstein's equations*, Class. Quantum Grav. **13** (1996) 1451–1469.

[55] H. Friedrich: *Gravitational fields near space-lie and null infinity*, J. Geom. Phys. **24** (1998) 83–163.

[56] H. Friedrich: *Conformal Einstein evolution*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg: Springer, 2002, 1–50.

[57] H. Friedrich: *The Taylor expansion at past time-like infinity*, Comm. Math. Phys. **324** (2013) 263–300.

[58] H. Friedrich: *On the AdS stability problem*, Class. Quantum Grav. **31** (2014) 105001.

[59] H. Friedrich, G. Nagy: *The initial boundary value problem for Einstein's vacuum field equation*, Comm. Math. Phys. **201** (1999) 619–655.

[60] H. Friedrich, A.D. Rendall: *The Cauchy problem for the Einstein equations*, in: *Einstein's field equations and their physical implications*, B. Schmidt (ed.), Berlin, Heidelberg: Springer, 2000, 127–224.

[61] R. Geroch: *Asymptotic structure of space-time*, in: *Asymptotic structure of space-time*, F. P. Esposito, L. Witten (eds.), New York: Plenum Press, 1977, 1–105.

[62] S.W. Hawking, G.F.R. Ellis: *The large scale structure of space-time*, Cambridge: Cambridge University Press, 1973.

[63] J. Kánnár: *On the existence of C^∞ solutions to the asymptotic characteristic initial value problem in general relativity*, Proc. R. Soc. Lond. A **452** (1996) 945–952.

[64] S. Klainerman, I. Rodnianski: *On the formation of trapped surfaces*, (2009), arXiv:0912.5097 [gr-qc].

[65] H.-O. Kreiss, O.E. Ortiz: *Some mathematical and numerical questions connected with first and second order time-dependent systems of partial differential equations*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg: Springer, 2002, 359–370.

[66] J. Li, X.-P. Zhu: *On the local extension of the future null infinity*, (2014), arXiv:1406.0048 [gr-qc].

[67] J. Luk: *On the local existence for the characteristic initial value problem in general relativity*, Int. Math. Res. Notices **2012** (2012) 4625–4678.

[68] V. Moncrief: *Spacetime symmetries and linearization stability of the Einstein equations I*, J. Math. Phys. **16** (1975) 493–498.

[69] T.-T. Paetz: *Conformally covariant systems of wave equations and their equivalence to Einstein's field equations*, (2013), arXiv:1306.6204 [gr-qc].

[70] T.-T. Paetz: *KIDs prefer special cones* , Class. Quantum Grav. **31** (2014) 085007.

[71] T.-T. Paetz: *Killing Initial Data on space-like conformal boundaries*, (2014), arXiv:1403.2682 [gr-qc].

[72] T.-T. Paetz: *Characteristic initial data and smoothness of Scri. II. Asymptotic expansions and construction of conformally smooth data sets*, (2014), arXiv:1403.3560 [gr-qc].

[73] R. Penrose: *Asymptotic properties of fields and space-time*, Phys. Rev. Lett. **10** (1963) 66–68.

[74] R. Penrose: *Zero rest-mass fields including gravitation: Asymptotic behavior*, Proc. R. Soc. Lond. A **284** (1965) 159–203.

[75] F. Pretorius: *Evolution of binary black hole spacetimes*, Phys. Rev. Lett. **95** (2005) 121101.

[76] A.D. Rendall: *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. R. Soc. Lond. A **427** (1990) 221–239.

[77] H. Ringström: *On the topology and future stability of the universe*, Oxford: Oxford University Press, 2013.

[78] R.K. Sachs: *Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time*, Proc. R. Soc. Lond. A **270** (1962) 103–126.

[79] R.K. Sachs: *On the characteristic initial value problem in gravitational theory*, J. Math. Phys. **3** (1962) 908–914.

[80] O. Sarbach, M. Tiglio: *Continuum and discrete initial-boundary value problems and Einstein's field equations*, Living Rev. Relativity **15** (2012), http://www.livingreviews.org/lrr-2012-9.

[81] R. Schoen, S.-T. Yau: *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), 45–76.

[82] R. Schoen, S.-T. Yau: *Proof of the positive mass theorem II*, Comm. Math. Phys. **79** (1981), 231–260.

[83] L.A. Tamburino, J.H. Winicour: *Gravitational fields in finite and conformal Bondi frames*, Phys. Rev. **150** (1966), 1039–1053.

[84] M.E. Taylor: *Partial differential equations III*, Berlin, Heidelberg: Springer, 1996.

[85] A. Trautman: *Radiation and boundary conditions in the theory of gravitation*, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. **VI** (1958) 407–412.

[86] R.M. Wald: *General relativity*, Chicago and London: The University of Chicago Press, 1984.

[87] J.H. Winicour: *Characteristic evolution and matching*, Living Rev. Relativity **15** (2012), http://www.livingreviews.org/lrr-2012-2.

[88] E. Witten: *A new proof of the positivity energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.

"The most incomprehensible thing about the world is that it is comprehensible."
(Albert Einstein)

Abstract

Initial value problems provide an extremely powerful tool to construct in a systematic manner general solutions to Einstein's field equations. Due to the Lorentzian geometry of a space-time there are in fact various possibilities of choosing an initial surface, such as e.g. characteristic ones, on which the main focus of this thesis lies. Now, aside from predicting existence and uniqueness (up to isometries) of solutions to these initial value problems for appropriately prescribed data sets, one would also like to say something about the properties of the so-emerging space-times; or, to put it differently, one would like to construct space-times via initial value problems which exhibit specific, physically relevant properties.

One such property is that the space-time extends arbitrarily far, at least in certain (null) directions, and that the gravitational field shows there a specific "asymptotically flat or de Sitter"-like fall-off behavior. Such solutions to Einstein's field equations constitute well-suited candidates to model isolated gravitational systems and purely radiative space-times. Another such property is that the space-time admits certain isometry groups. These play a fundamental role in physics, since they e.g. give rise to conserved quantities, and are typical features of the preferred states of a system. Finally, the Bondi mass is an important quantity to capture the energy content of a space-time, while its dependence on the retarded time measures the loss of energy due to radiation escaping to infinity. Its positivity is an important physical expectation, though the proofs of the available positive mass theorems confirming this lack in a physical intuition where this positivity originates from.

This thesis is concerned with the construction of solutions to Einstein's field equations from space-like and in particular from characteristic surfaces such that some insights regarding these properties can be gained just from an analysis of the initial data. More concretely, we construct vacuum space-times via "asymptotic initial value problems" which possess a past-asymptotic structure similar to the Minkowskian one by using a novel system of wave equations. We investigate under which conditions on the initial data the prospective vacuum space-time admits a Killing vector field (that is a vector field generating a local isometry), and analyze the existence of vacuum space-times which have both a Killing vector field and a past-asymptotically flat (or de Sitter) structure. Moreover, we characterize null initial data sets for Einstein's field equations which are "smooth at infinity", and thus provide the basis to construct asymptotically flat vacuum space-times from ordinary (non-asymptotic) null surfaces. We further present a manifestly positive-definite formula for the Bondi mass of a globally smooth light-cone just in terms of the initial data given there. It is derived by elementary methods, whereby we obtain a direct, simple positivity proof in this setting.

Zusammenfassung

Anfangswertprobleme stellen ein äußerst mächtiges Werzeug dar, um auf systematische Art und Weise allgemeine Lösungen der Einsteinschen Feldgleichungen zu konstruieren. Die Lorentzsche Geometrie einer Raumzeit bietet vielfältige Möglichkeiten eine Anfangsfläche auszuwählen, wie beispielsweise charakteristische Anfangsflächen, auf denen das Hauptaugenmerk dieser Arbeit liegt. Nun möchte man nicht nur Existenz und Eindeutigkeit (bis auf Isometrien) von Lösungen zu solch einem Anfangswertproblem sicherstellen, sondern vielmehr möchte man auch etwas zu den Eigenschaften der so entstehenden Raumzeiten wissen. Das Ziel besteht also darin, über Anfangswertprobleme Raumzeiten zu konstruieren, welche spezifische, physkalische Eigenschaften aufweisen.

Eine dieser Eigenschaften ist, dass die Raumzeit zumindest in gewissen (Null-)Richtungen beliebig weit ausgedehnt ist, und dass das Gravitationsfeld dort ein spezifisches "asymptotisch flaches oder de Sitter"-artiges Abfallverhalten zeigt. Derartige Lösungen der Einsteinschen Feldgleichungen bilden geeignete Kandidaten um isolierte gravitierende Systeme und sogenannte *purely radiative space-times* zu modellieren. Eine weitere Eigenschaft ist, dass die Raumzeit gewisse Isometriegruppen besitzt. Diese spielen eine fundamentale Rolle in der Physik, da sie z.B. die Existenz von Erhaltungsgrößen anzeigen, und typische Merkmale von ausgezeichneten Zuständen eines System sind. Schließlich ist die Bondi-Masse eine wichtige Größe um den Energieinhalt einer Raumzeit zu erfassen; ihre Abhängigkeit von der retardierten Zeit misst den Energieverlust durch Strahlung, die ins Unendliche entkommt. Die physikalische Erwartung, dass diese Masse positiv sein muss, wird durch die verfügbaren positiven Massentheoreme bestätigt. Allerdings geben deren Beweise keinerlei physikalische Einsicht darin, warum das so ist.

Die vorliegende Arbeit beschäftigt sich mit der Konstruktion von Lösungen zu den Einsteinschen Feldgleichungen über raumartige und insbesondere charakteristische Anfangswertprobleme, so dass nur über eine Analyse der Anfangsdaten etwas zu den gerade beschriebenen Eigenschaften ausgesagt werden kann. Genauer gesagt, werden wir ein neuartiges System von Wellengleichungen benutzen, um Vakuum-Raumzeiten über "asymptotische Anfangswertprobleme" zu konstruieren, die zu hinreichend frühen Zeiten eine asymptotische Struktur ähnlich der der Minkowski Raumzeit besitzen. Wir untersuchen weiterhin welche Bedingungen Anfangsdaten erfüllen müssen, damit die aus ihnen entstehende Raumzeit Killingfelder enthält (d.h. Vektorfelder, die lokale Isometrien generieren), und analysieren die Existenz von Vakuum-Raumzeiten, die sowohl Killingfelder als auch eine asymptotisch flache- (oder de Sitter-) artige Struktur besitzen. Zudem beschreiben wir Nullanfangsdaten, die einerseits mit den Feldgleichungen kompatibel und andererseits "glatt im Unendlichen" sind, und daher die Basis für die Konstruktion von asymptotisch flachen Vakuum-Raumzeiten von gewöhnlichen (nicht-asymptotischen) Nullflächen aus bieten. Schließlich präsentieren wir eine Formel für die Bondi-Masse eines global glatten Lichtkegels, ausgedrückt nur durch die dort gegebenen Anfangsdaten, die unmittelbar deren Positivität zeigt und durch elementare Methoden hergeleitet werden kann.

Curriculum vitae

Personal information

Surname, first name:	Paetz, Tim-Torben
Date of birth:	20th February 1984
Place of birth:	Braunschweig, Germany
E-mail:	tim-torben.paetz@univie.ac.at

Education

since 03/2011	**Gravitational Physics, University of Vienna, Vienna, Austria** Ph.D. student in physics, thesis title: "On characteristic initial value problems in general relativity"
10/2007 - 06/2008	**University of Warwick, Coventry, U.K.** Exchange student at the Mathematics Institute
10/2004-09/2010	**University of Göttingen, Göttingen, Germany** Diploma student in physics, thesis title: "An Analysis of the 'Thermal-Time Concept' of Connes and Rovelli"
1996-2003	**Gymnasium Martino-Katharineum, Braunschweig, Germany** General qualification for university entrance

Selected conferences and summer schools

27/02/ – 01/03/2014	4th Central European Relativity Seminar, University of Vienna, Vienna, Austria
11/2013	Program associate for the semester program "Mathematical General Relativity", Mathematical Sciences Research Institute, Berkeley, CA, U.S.
07/07/ – 13/07/2013	20th International Conference on General Relativity and Gravitation University of Warsaw, Warsaw, Poland
07/02/ – 09/02/2013	3rd Central European Relativity Seminar, Max-Planck-Institute for Gravitational Physics, Golm, Germany
07/10/ – 21/10/2012	Trimester in Conformal and Kähler Geometry, Institut Henri Poincaré, Paris, France
29/07/ – 04/08/2012	Mathematical Aspects of General Relativity, Mathematisches Forschungsinstitut Oberwolfach, Germany
11/06/ – 29/06/2012	ICTP–ESF School and Conference on Geometric Analysis, International Centre for Theoretical Physics, Trieste, Italy
02/02/ – 04/02/2012	2nd Central European Relativity Seminar, University of Cracow, Cracow, Poland
27/01/ – 29/01/2011	1st Central European Relativity Seminar, University of Vienna, Vienna, Austria
29/07/ – 31/07/2009	Algebraic Quantum Field Theory – The First 50 Years, Göttingen, Germany