universität
wien

MASTERARBEIT

Titel der Masterarbeit

Spherically Symmetric, Static Solutions of the Einstein-Vlasov System with Cosmological Constant

Verfasser

Maximilian Thaller

angestrebter akademischer Grad

Master of Science (MSc)

Wien, 2014

Studienkennzahl lt. Studienblatt:	A 066 876
Studienrichtung lt. Studienblatt:	Masterstudium Physik
Betreuer:	Univ.-Prof. Piotr T. Chruściel, MSc PhD

Acknowledgments

First of all I thank my supervisor David Fajman for the proposition of the well-thought-out topic, the plenty ideas he shared with me in the course of countless pleasant discussions and his continuous encouragement. Moreover, I want to express deep gratitude to Håkan Andréasson who provided important ideas for the proofs in section 5.1 and made many other valuable contributions to the project. I am grateful to Walter Simon and Robert Beig for several discussions on static solutions. I thank Piotr Chruściel for many helpful discussions on the topic and in particular for the suggestion to study the case of negative cosmological constants. I am indebted to Greg Galloway for sharing his ideas and suggestions concerning the global solutions which are presented in section 5.4 of this work. Finally I want to thank my parents for making my studies possible.

Abstract

The Einstein-Vlasov system of equations describes a universe that contains matter consisting of freely falling particles of equal mass. Vlasov matter is used to model for example globular clusters, galaxies or galaxy clusters.

A detailed derivation of the static Einstein-Vlasov system in spherical symmetry is given and using the up-to-date notations important existing results on the system with cosmological constant $\Lambda = 0$ are reviewed in detail. These results include existence of regular solutions, solutions with a Schwarzschild singularity at the center, boundedness of the support of the matter quantities, and bounds on the ratio $\frac{m(r)}{r}$, also known as Buchdahl inequality. Numerical calculations illustrate important classes of static solutions.

For small positive cosmological constants we construct spherically symmetric, static solutions with matter quantities of bounded support and a regular center. In the vacuum region these solutions coincide with the Schwarzschild-de Sitter metric. For arbitrary negative cosmological constants we prove the existence of globally regular solutions via an energy estimate. The matter quantities of these solutions again are shown to have bounded support and the metric coincides with the Schwarzschild-Anti de Sitter metric beyond the matter regions. For both, positive and negative cosmological constants, we construct non-vacuum solutions that have a Schwarzschild singularity at the center. These solutions describe Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter spaces with immersed shells of Vlasov matter surrounding the black hole.

Finally we construct Penrose diagrams showing the maximal analytic extensions of the obtained metrics. By virtue of the matter balls and shells several classes of spacetimes with interesting topologies and properties occur.

Zusammenfassung

Das Einstein-Vlasov Gleichungssystem beschreibt ein Universum, welches Materie enthält, die sich aus im freien Fall befindlichen Teilchen von gleicher Masse zusammensetzt. Man verwendet das Vlasov Materiemodell um beispielsweise Kugelsternhaufen, Galaxien oder Galaxiehaufen zu modellieren.

Als erstes wird eine detaillierte Herleitung des Einstein-Vlasov Systems in Kugelsymmetrie präsentiert. Anschließend werden ausgewählte bereits bekannte Ergebnisse zum System ohne kosmologische Konstante Λ diskutiert, die für spätere Beweise zum System mit nicht verschwindender kosmologischer Konstante wichtig sind. Diese Ergebnisse umfassen die Existenz regulärer Lösungen des Systems, Lösungen mit einem schwarzen Loch im Zentrum, Beschränktheit des Trägers der Materiegrößen und obere Schranken für den Ausdruck $\frac{m(r)}{r}$, also eine verallgemeinerte Version der Buchdahl Ungleichung. Numerische Berechnungen veranschaulichen wichtige Klassen dieser statischen Lösungen.

Für kleine positive kosmologische Konstanten konstruieren wir kugelsymmetrische, statische Lösungen mit regulärem Zentrum. Die Materiegrößen dieser Lösungen haben beschränkten Träger. In der Vakuumregion jenseits des Trägers der Materiegrößen sind diese Lösungen durch die Schwarzschild-de Sitter Metrik gegeben. Für beliebige negative kosmologische Konstanten beweisen wir die Existenz von Lösungen, die für alle Radien einer gewissen Regularität genügen, mittels einer Energieabschätzung. Auch für diesen Fall zeigen wir, dass der Träger der Materiegrößen beschränkt ist. Jenseits dieses Trägers ist die Metrik durch die Schwarzschild-Anti de Sitter Metrik gegeben. Sowohl für positive, als auch für negative kosmologische Konstanten konstruieren wir Lösungen, die ein schwarzes Loch beschreiben, welches von Materie umgeben ist. Man kann diese Lösungen als Schwarzschild-de Sitter beziehungsweise Schwarzschild-Anti de Sitter Raumzeiten mit eingebetteten Kugelschalen aus Vlasov-Materie auffassen.

Schließlich stellen wir in Penrose-Diagrammen die maximale analytische Fortsetzung der Metriken, die wir in den vorhergehenden Abschnitten als Lösungen des Einstein-Vlasov Gleichungssystems konstruieren, dar. Die Kugeln und Kugelschalen aus Vlasov-Materie ermöglichen die Konstruktion einiger beachtenswerter Klassen von Raumzeiten mit interessanten Eigenschaften und Topologien.

Contents

1 Introduction

1.1 The cosmological constant

The notion of the cosmological constant Λ has undergone a changeful history. An overview on this history and also the subsequent considerations can be found in the review article [19] in more detail. The cosmological constant was originally introduced by Einstein himself to modify his field equations such that the theory fits observations of that time, that is to say the existence of a static, matter filled universe.

On very large scales the universe can be considered as being approximately spatially homogeneous and isotropic. Such a universe can be described by a Friedman-Robertson-Walker metric. In this geometry the relative size of the spatial sections of the spacetime, i.e. hypersurfaces of constant time coordinate, are characterized by a scale factor $R(t)$. This scale factor is a function of the time variable and may describe expanding or contracting behavior. In addition, in a rough approach one can model the matter contained in the universe as an ideal fluid characterized by an energy density ϱ and an isotropic pressure p. The corresponding energy momentum tensor is given by

$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{1.1}$$

where u_μ is the four velocity field of the ideal fluid. Under these assumptions Einstein's equations (3.1) without cosmological constant reduce to the two Friedman equations

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varrho - \frac{k}{a^2 R_0^2}, \tag{1.2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\varrho + 3p), \tag{1.3}$$

where G is the gravitational constant, k a curvature parameter taking values in $\{-1, 0, +1\}$, and $R(t)$ the aforementioned scaling factor. Also the Hubble parameter $H := \dot{R}/R$ is introduced. The dot denotes the derivative with respect to the time coordinate t. At the time of the introduction of general relativity one was interested in finding static solutions to be in accordance with the astronomical data as they were available and understood back then. But the Friedman equations (1.2) and (1.3) do not have static solutions, characterized by $\dot{R} = 0$, as can be seen by the following argument. A positive energy density ϱ implies a positive spatial curvature ($k = 1$) provided it is appropriately tuned. If p is also non-negative which is true for an ideal fluid and for most reasonable matter models, like the Vlasov model, equation (1.3) implies that $\ddot{R}$ will never vanish. This already implies that $\dot{R}$ cannot be zero for all t. But the Friedman equations do admit static solutions if a cosmological constant Λ is introduced even though the obtained solutions turned out to be not stable. These solutions are referred to as the *Einstein static universe*. In fact, the left hand side of the Einstein equations with cosmological constant (3.1) is the most general local, coordinate-invariant, divergenceless, symmetric two-index tensor that can be constructed solely from the metric and its first and second derivative.

Later, in the 1920s, it was discovered by Hubble that the universe is actually expanding. This made the cosmological constant needless concerning its original purpose since solutions of the Einstein equations without cosmological constant describing an expanding universe had already been discovered by Friedman and Lemaître. Even though it lost its original purpose, by no means the notion of a cosmological constant has become obsolete. First, it is far from evident that the expansion of the universe is fully determined by other mechanisms like presence of matter and curvature (the terms on the right hand side of (1.2)) and not also by a *bare* cosmological constant Λ_0. Moreover, the cosmological constant leads to vacuum solutions of the Einstein equations that expand or contract. These are the de Sitter and Anti de Sitter metric, respectively, playing an important role in this work. Furthermore, as meanwhile discovered by particle theorists, the cosmological constant can be interpreted as energy density of the vacuum. This mechanism can be illustrated by the following example. Consider a scalar field ϕ with potential energy $V(\phi)$. For this matter model the energy momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right)g_{\mu\nu} - V(\phi)g_{\mu\nu}. \tag{1.4}$$

In this setting, the configuration with the lowest energy density will be one without any contribution of kinetic energy. This means $\partial_\mu\phi = 0$ and the energy momentum tensor is given by $T_{\mu\nu} = -V(\phi_0)g_{\mu\nu}$ where ϕ_0 denotes the value of ϕ that minimizes V. The configuration on hand is understood as vacuum and one can write the energy momentum tensor in the form $T^{\text{vac}}_{\mu\nu} = -\varrho_{\text{vac}}g_{\mu\nu}$. In this example we have $\varrho_{\text{vac}} = V(\phi_0)$. This corresponds to the energy momentum tensor of an ideal fluid (1.1) with equation of state $p_{\text{vac}} = -\varrho_{\text{vac}}$. In the Einstein equations the effect of this effective energy momentum tensor is equivalent to that of a cosmological constant $\Lambda = 8\pi G\varrho_{\text{vac}}$. This can easily be seen by moving $\Lambda g_{\mu\nu}$ to the right hand side in the Einstein equations (3.1). There are also other effects, like for example vacuum fluctuations as predicted by quantum mechanics, that can be described by a cosmological constant. In fact, the cosmological constant can be understood as a combination of several constants resulting from different effects. Because of this variety of contributions, it is difficult to make a reliable theoretical prediction for the magnitude of Λ.

Every contribution to this assembled cosmological constant can be translated into an energy density as described above. It is important to note that the different contributions can still be distinguished by their behaviors as the universe expands. For example the vacuum energy density stays constant whereas the energy densities of matter or curvature decrease with specific rates. These differences in behavior can be exploited to determine the individual contributions using cosmological models and astronomical observations. Thereby the different contributions to the cosmological constant belong to the set of the so called *cosmological parameters*. Cosmological models determined by these parameters can be compared to observations and measurements. This yields constraints on combinations of these parameters. By a combination of various tests the range of possible values for the cosmological constant can be confined. Around the year 2000 large progress concerning such tests has been made.

An eligible phenomenon are Ia supernovae occurring when a white dwarf crosses the Chandrasekhar limit and explodes. For several reasons these supernovae are particularly suitable for astronomic observations. Since Ia supernovae always occur under similar circumstances they are of nearly uniform intrinsic luminosity which makes it possible to determine their distance and redshift. Supernovae happen relatively rarely, within a certain galaxy they can only be observed a few times in a century. But since they use to be extremely bright, approximately as bright as the host galaxy, a sufficient amount of them can be observed. Also the observation of supernovae in huge distances (i.e. high redshifts) is possible. A further difficulty is that the peak luminosities of different supernovae show still a scatter of approximately 40% due to different chemical compositions of the exploding star. Fortunately the observed differences in peak luminosity are very closely correlated with observed differences in the shapes of their light curves. Moreover the measurements of the light curves can be influenced by galactic dust or gravitational lensing. But still the observation of Ia supernovae provides an auspicious possibility to constrain the cosmological constant. In 2011 the physicists Perlmutter, Riess, and Schmidt were awarded with the Nobel prize for the discovery of the accelerating expansion of the universe through observations of distant supernovae [45]. In their works [28, 38, 44] the methods and difficulties of their measurements, the obtained data, and its interpretation are presented.

The *cosmic microwave background (CMB)* is another source of information. It originates from a very early stage of the development of the universe, the so called *epoch of recombination*. During this epoch the universe became cool enough such that protons and electrons combined to uncharged atoms that are transparent for light. Measurements show that the CMB is almost isotropic all over the sky, observed fluctuations are about 0.1%. But these fluctuations carry information about the statistical properties of the energy density in the universe at the epoch of recombination. By means of the cosmological parameters these fluctuations can be modeled and a fit of the parameters to the observations is possible. This yields further constraints on the value of the cosmological constant. An example for observed data and their interpretation can be found in [26].

As already mentioned various effects can be described by a single cosmological constant. In a common approach one distinguishes between a part caused by the matter density in the universe and the vacuum energy. The aforementioned observational tests yield constraints on combinations of these two parameters. So it is very useful to have constraints only on the mass density as well to get further information about the cosmological constant understood as vacuum energy. This is for example done by estimating the mass of a galaxy cluster, dividing it by its luminosity and extrapolating it to the universe [27]. One can also measure the matter density by means of the baryon density [22]. Lots of problems concerning the determination of the cosmological constant are still under discussion or unsolved, but research on this field is very active and it is strongly believed that the cosmological constant plays an important role in the dynamics of the universe.

1.2 Spacetimes with Vlasov matter

For the description of the dynamics of systems of stars the collisionless Boltzmann equation provides a suitable model. In the non-relativistic case it describes a collection of uncharged particles with equal mass that are moving through space under the influence of a gravitational potential U [17]. The matter is represented by a density function $f(t, x, p)$ depending on the time t, the position variables $x = (x^1, x^2, x^3)$ and the momentum variables $p = (p^1, p^2, p^3)$. This density function describes how many particles with the momentum p can be found at the position x at the time t. The gravitational potential U is caused by the whole of the present particles described by f via the Poisson equation

$$\Delta U(t, x) = 4\pi G \varrho(t, x), \quad \varrho(t, x) = \int_{\mathbb{R}^3} f(t, x, p) \mathrm{d}^3 p, \tag{1.5}$$

where G is the gravitational constant. Due to the low density of galaxies collisions between stars can be neglected and the only interaction is caused by the gravitational potential [17].

The time evolution of such a system serves as model for stellar dynamics and is therefore of great relevance. The questions of global existence and stability of solutions to the initial value problem of the Vlasov-Poisson system are already well understood. Pfaffelmoser [29] first established the global existence of classical solutions for the Vlasov-Poisson system with general initial data. His proof was considerably simplified by Schaeffer [42] and Horst [24]. Also the problem of static solutions is of great interest. In spherical symmetry the existence of solution was shown by Batt, Faltenbacher, and Horst [15]. In [13] Andréasson and Rein numerically construct disc solutions to the axially symmetric Vlasov-Poisson system. These solutions are a good model to describe disc galaxies. In this context the *rotation curves* are of particular interest. A rotation curve depicts the magnitude of the orbital velocities of visible stars or gas particles in the galaxy versus their radial distance to the center. Observations [18, 40] show that these curves rise steeply at the center of the galaxy and are approximately flat apart from the inner region. Andréasson and Rein calculated curves that have exactly this shape, thus match the observations. This is remarkable because these rotation curves could be constructed without introducing dark matter which was not possible until this work. In fact these rotation curves are one reason for the introduction of the notion of dark matter.

A full relativistic description of a spacetime that contains collisionless matter is provided by the Einstein-Vlasov system. The Vlasov-Poisson system is the Newtonian limit of the Einstein-Vlasov system [31]. Even though the analysis of the Einstein-Vlasov system is much more complicated several results on the initial value problem have been achieved. Rein and Rendall have proven the global existence of solutions of the spherically symmetric Einstein-Vlasov system for small initial data [32]. Rein, Rendall, and Schaeffer proved in the spherically symmetric setting that if singularities occur this happens first in the center of symmetry [35]. For outgoing matter the global existence of regular solutions has been established by Andréasson, Kunze, and Rein [5]. Also on the gravitational collapse and the formation of black holes several results are available, cf. the review article [7] and the references therein.

However, in the context of this work static solutions are at the center of interest. The existent proofs of existence of solutions use an ansatz for the matter distribution f fulfilling the Vlasov equation and show that corresponding solution to the Einstein equations exist. Basically three classes of ansätze are relevant. The most simple one is called *isotropic*. The isotropic ansatz leads to matter configurations where the momentum is equal in every direction. The corresponding solutions are very similar to solutions for ideal fluids [33, 37]. The *polytropic* ansatz, however, admits solutions that have different tangential and radial pressure. The *anisotropic* ansatz extends the class of possible matter configurations even further. Whereas the isotropic ansatz only leads to solutions where all of the matter is located in a ball, the anisotropic ansatz admits configurations where several shells of matter separated by vacuum are on hand. For the isotropic ansatz Rein and Rendall established global existence of regular, static solutions to the Einstein-Vlasov system [33]. The matter quantities belonging to these solutions were shown to be of compact support. Rein generalized the proof of existence to the anisotropic ansatz and the proof of boundedness of the support of the matter quantities to the polytropic ansatz [34]. In this work also spherically symmetric solutions are constructed that contain a Schwarzschild singularity at the center. Such solutions describe a situation where a black hole is surrounded by a shell of Vlasov matter. They are candidates for the final state of the gravitational collapse of a matter cloud. In [36] the existence of solutions with bounded support of the matter quantities was proved also for the anisotropic ansatz for f via a perturbation argument. Ramming and Rein present in [30] a simplified proof of finite extension of the support of the matter quantities for a large class of matter models.

Also the Einstein-Vlasov system in axially symmetry is an interesting subject of research.

Andréasson, Kunze, and Rein showed the existence of non-vacuum solutions to this system [9]. In [12] the same authors prove the existence of solutions with non-zero angular momentum. This provides mathematical models for rotating, general relativistic, and asymptotically flat non-vacuum spacetimes. All of the mentioned existence results are valid for the Einstein-Vlasov system *without* cosmological constant. For the system with cosmological constant the literature provides *no existence results for static solutions.* With [10] a detailed review article on both, static and dynamic solutions of the Einstein-Vlasov system is available.

An important feature of non-vacuum solutions of the Einstein equations are Buchdahl type inequalities. These inequalities give bounds on the expression $\frac{m(r)}{r}$ where $m(r)$ is the so called *quasilocal mass* and r is the radius. One can think of the quasilocal mass as the mass of the matter that is inside a ball of radius r. In [2] Andréasson proves for uncharged matter and without cosmological constant that the quantity $\frac{2m(r)}{r}$ is bounded by $\frac{8}{9}$ if the underlying matter model satisfies certain conditions. Vlasov matter fulfills these conditions. Constructing solutions of infinitely thin matter shells Andréasson also showed that this inequality is sharp. Karageorgis and Stalker developed a method for a simpler proof of this Buchdahl inequality [25]. Andréasson generalized their proof to the case of charged matter [3]. Together with Böhmer he proves a Buchdahl inequality for static matter objects with positive cosmological constant. Later Andréasson, Böhmer, and Mussa prove a Buchdahl inequality in a setting with positive cosmological constant and static objects of charged matter [11]. But it is important to note that the existence of non-vacuum solutions of the Einstein equations with positive cosmological constant is just assumed and not established. The primal proof of existence is given in this work.

1.3 Outline of the work

This work is organized as follows. In section 2 the Valsov matter model is introduced and important properties of the Vlasov equation and the corresponding energy momentum tensor are pointed out. In section 3 the complete Einstein-Vlasov system is considered. By virtue of the symmetry assumptions it can be shown that two of the Einstein equations are already equivalent to the full system of equations. The energy momentum tensor gives rise to matter quantities that form the right hand side of the remaining Einstein equations. Finally a transformation of the momentum variables is performed and the Vlasov equation is solved using the method of characteristics. This is done by choosing a suitable ansatz for the matter distribution function f that only depends on certain quantities that in turn are conserved along the characteristic curves of the Vlasov equation. By virtue of certain boundary conditions the system of equations reduces to a single ordinary differential equation that can be solved in the subsequent sections. Section 4 is dedicated to results for the Einstein-Vlasov system without cosmological constant. These results are already known in the literature, but nevertheless it is useful to collect them in a compact form and present the proofs using consistent and up-to-date notation and variables. First the proof of global existence of spherically symmetric, static solutions is given. Then finiteness of the ADM mass, boundedness of the support of the matter quantities, the Buchdahl inequality, and solutions with a black hole at the center surrounded by a finitely extended shell of Vlasov matter are discussed. Finally the discussed classes of solutions are illustrated by numerically obtained graphs using the methods of [1]. A detailed knowledge of solutions for vanishing cosmological constant is of great importance because the proofs concerning the system with non-vanishing cosmological constant make use of them at various places.

In section 5 results for the Einstein-Vlasov system with non vanishing cosmological constant are presented. All of these results have been developed in the course of this work and are the outcome of a collaboration with Håkan Andréasson and David Fajman. First we prove the global existence of spherically symmetric, static solutions for small cosmological constants with regular center. The proof of this assertion extends to several steps. For the first instance we prove local existence using a contraction argument. Then a continuation criterion is established. In the next step it is shown that the solution exists until a certain radius that lies already beyond the support of the matter. So a vacuum region is on hand. In this vacuum region the metric coincides with the Schwarzschild-de Sitter metric. A step showing that the metric can be continued with the Schwarzschild-de Sitter metric and the matter distribution with constant zero ends the proof. Secondly we prove global existence of solutions for arbitrary negative cosmological constants using an energy estimate similar to [34]. It is also shown that the matter quantities of these solutions are of bounded support. Thereafter, we construct solutions with a Schwarzschild singularity at the center for both, positive and negative cosmological constants. The introduction of a cosmological constant changes the large scale structure of the obtained spacetimes fundamentally. That is why we consider the maximal analytic extensions of the obtained metrics in the last part of the section. A straightforward procedure to construct Penrose diagrams showing the corresponding spacetimes is discussed.

2 The Vlasov matter model

2.1 Motivation

The Vlasov matter model describes a large collection of identical particles that are moving freely falling through space. The mass of these particles is normalized to $m = 1$. Collisions between the particles are not occurring and the only modeled interaction is due to gravitational effects. The Vlasov matter model is used to describe galaxies or even galaxy clusters [17]. In fact the density of galaxies in a galaxy cluster is higher than the density of stars in a galaxy. The fractional volume occupied by galaxies in a galaxy cluster is approximately 10^{-3} whereas the occupied volume of stars in a galaxy only is about 10^{-22}, cf. [17]. Collisions between two galaxies have been observed, just as deformed galaxies whose shape is the outcome of a collision. When a collision between galaxies occurs the individual stars collide almost never due to the galaxy's low density. The galaxies merely move through one another, get deformed, and change their trajectories.

The collection of particles is described by a continuous density function f. This density function generates the gravitational effects that influence the trajectory of a single star. Here a fundamental difference between the kinetic theory used to describe matter on a macroscopic scale and on a microscopic scale becomes apparent. If one considers for example molecules in a box the following behavior will occur. Since the forces the particles are affecting on one another with is of rather short range, most of the time the molecules move on straight lines without any influence of the other particles. The mechanism that makes a particle change its trajectory is collisions, i.e. scattering with other particles. In galactic dynamics, however, the stars are influencing each other on a much larger range. In addition their density is very small. So compared to the influence the whole collection of stars exerts on the trajectory of a certain particle single stars do not come to effect any more. This justifies that one uses a continuous density function f satisfying the

Vlasov equation to describe galaxies, cf. [17]. In general relativity gravity is modeled as curvature of spacetime caused by an energy momentum tensor. This mechanism is described by Einstein's field equations (3.1). Via the energy momentum tensor the Vlasov equation couples to the Einstein equations. In the following the relevant quantities are introduced and this coupling is described in a detailed way.

2.2 The Vlasov equation

We consider a Lorentzian manifold $\mathscr{M}$ equipped with local coordinates x^α, $\alpha = 0, \dots, 3$, and the tangent bundle $T\mathscr{M}$. The canonical momentum variables shall be denoted by p^α. Since the rest mass of all particles shall be conserved and normalized to one the four momentum of each particle fulfills $g_{\alpha\beta}p^\alpha p^\beta = -m^2 = -1$. We define the matter distribution function $f : \mathscr{P} \to \mathbb{R}^+$ on the mass shell

$$\mathscr{P} = \{(x^\alpha, p^\alpha) \in T\mathscr{M} : g_{\alpha\beta}p^\alpha p^\beta = -1,\ p^\alpha \text{ future directed}\}. \tag{2.1}$$

The geodesics of the metric $g_{\alpha\beta}$ are the projections onto spacetime of the curves in $T\mathscr{M}$ defined in local coordinates by

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = p^\alpha, \quad \frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = -\Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma, \tag{2.2}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, cf. [32]. Solutions of the system (2.2) are the integral curves of the vector field $\mathscr{X} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$. This vector field $\mathscr{X}$ is called the *geodesic spray*. $\mathscr{X}$ is tangent to $\mathscr{P}$, cf. [39, 41]. For this reason there are coordinates such that the geodesic spray can be expressed in a basis (x^μ, p^j), where $\mu = 0, \dots, 3$ and $j = 1, \dots, 3$. The Vlasov equation for the non-negative density function f is $\mathscr{X}f = 0$. In the aforementioned coordinates it reads

$$p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^j_{\alpha\beta}p^\alpha p^\beta \frac{\partial f}{\partial p^j} = 0. \tag{2.3}$$

Furthermore denoting the fiber of $\mathscr{P}$ over a point $x \in \mathscr{M}$ by $\mathscr{P}_x$ we define the particle current density N^α and the energy momentum tensor $T^{\mu\nu}$ by

$$N^\alpha(x) = -\int_{\mathbb{R}^3} f p^\alpha\ \mu_{\mathscr{P}_x}, \tag{2.4}$$

$$T^{\mu\nu}(x) = -\int_{\mathbb{R}^3} f p^\mu p^\nu\ \mu_{\mathscr{P}_x}. \tag{2.5}$$

The measure $\mu_{\mathscr{P}_x}$ on the fiber $\mathscr{P}_x$ of the mass shell is given by

$$\mu_{\mathscr{P}_x} = \frac{\sqrt{|g_{\mu\nu}|}}{-p_0}\mathrm{d}p^1 \wedge \mathrm{d}p^2 \wedge \mathrm{d}p^3. \tag{2.6}$$

It is constructed in the following way [39]. The metric g on $\mathscr{M}$ induces a metric $\tilde{\rho}$ on $T\mathscr{M}$ as one identifies $T_q\mathscr{M}$ and $T_wT_q\mathscr{M}$. Thus $\tilde{\rho}(\partial_{p^\mu}, \partial_{p^\nu})|_q = g_{\mu\nu}(q)$ for $q \in \mathscr{M}$. One calculates the metric ρ that is induced on $\mathscr{P}_x$ by $\tilde{\rho}$ and the desired volume element can be calculated from the formula

$$\mu_{\mathscr{P}_x} = \sqrt{|\det \rho_{ij}|}\mathrm{d}p^1 \wedge \mathrm{d}p^2 \wedge \mathrm{d}p^3. \tag{2.7}$$

The Vlasov matter model describes well behaved matter in the sense that the energy momentum tensor (2.5) shows some suitable properties. It is not hard to see, that T is divergence free, i.e. $\nabla_\mu T^{\mu\nu} = 0$, cf. [10]. This of course is a necessary property for the energy momentum tensor to be compatible with the Einstein equations since the Einstein tensor is divergence free due to the Bianchi identities. The fact that T is divergence free can be interpreted as conservation of energy. In local coordinates one can also calculate $\nabla_\alpha N^\alpha = 0$ which expresses the conservation of number of particles.

In virtue of the Vlasov equation (2.3) a number of inequalities hold. First, if V^α is a future-directed causal vector then we have $N_\alpha V^\alpha \leq 0$. Equality holds if and only if $f = 0$ at the given point, cf. [10]. Hence the particle current density N^α is always future directed and timelike if there are particles at the respective point. Moreover, if V^μ and W^ν are future directed timelike vectors we have $T_{\mu\nu} V^\mu W^\nu \geq 0$. This is called the dominant energy condition. It implies the weak energy condition. Moreover, if X^α is a spacelike vector, then $T_{\mu\nu} X^\mu X^\nu \geq 0$. This is called the non-negative pressure condition. Finally, a well-posedness theorem for the Cauchy problem of the Einstein Vlasov system holds, cf. [39, 20].

3 Derivation of the system of equations

3.1 The Einstein equations

In this section the relevant equations and quantities will be introduced and a suitable set of coordinates will be chosen that simplifies the analysis of the Einstein-Vlasov system. In view of the assumption of spherical symmetry the system will be further simplified. Finally, using boundary conditions for the metric coefficients and a suitable ansatz for the matter distribution f we derive a single ordinary differential equation that will be analyzed and solved in the subsequent sections. The solution of this equation will imply the solution of the full system.

Einstein's field equations read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.1}$$

G is the gravitational constant, c is the speed of light, Λ is the cosmological constant and $G_{\mu\nu}$ is the Einstein tensor. In the following we choose units such that $G = c = 1$. The Einstein tensor is defined to be

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{3.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar of $g_{\mu\nu}$. They can be calculated from the Riemann tensor that in turn is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3.3}$$

where ∇ denotes the covariant derivative and $[\cdot,\cdot]$ the Lie bracket. Its components are given by

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \mathrm{d}x^\alpha (R(\partial_\gamma, \partial_\delta)\partial_\beta) \\ &= \partial_\gamma \Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\nu\gamma} \Gamma^\nu_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\nu\delta} \Gamma^\nu_{\beta\gamma} \end{aligned} \tag{3.4}$$

Then the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R are defined as

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}, \quad R = g^{\alpha\beta} R_{\alpha\beta}. \tag{3.5}$$

In the appendix, section A, the Einstein tensor G is calculated using the ansatz (3.6) for the metric.

In this work we will be interested in spherically symmetric solutions of the Einstein equations. This suggests the ansatz

$$\mathrm{d}s^2 = -e^{2\mu(r,t)}\mathrm{d}t^2 + e^{2\lambda(r,t)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\mathrm{d}\varphi^2\right) \tag{3.6}$$

for the metric g, cf. [32].

By means of the energy momentum tensor $T^{\mu\nu}$, defined in equation (2.5), we can define matter quantities that will appear in the tt-, rr- and $\vartheta\vartheta$-component of the Einstein equations. Given the vector fields $z_0 = e^{-\mu}\partial_t$ and $z_1 = e^{-\lambda}\partial_r$ we define the matter quantities

$$\varrho := T_{\alpha\beta} z_0^\alpha z_0^\beta, \tag{3.7}$$

$$p := T_{\alpha\beta} z_1^\alpha z_1^\beta, \tag{3.8}$$

$$p_T := T_{\vartheta\vartheta}. \tag{3.9}$$

The quantity ϱ can be understood as energy density, p as radial pressure, and p_T as tangential pressure. Using these quantities and the expressions for the entries of the Einstein tensor derived in the appendix, section A, one obtains for the first three diagonal entries of the Einstein equations

$$e^{-2\lambda}\left(2r\lambda' - 1\right) + 1 - r^2\Lambda = 8\pi r^2 \varrho, \tag{3.10}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 + r^2\Lambda = 8\pi r^2 p, \tag{3.11}$$

$$e^{-2\lambda}\left(\mu'' \left(\mu + \frac{1}{r}\right)(\mu' - \lambda')\right) = 8\pi p_T. \tag{3.12}$$

Note that equations (3.10) and (3.11) are already equivalent to the whole set of Einstein equations (3.1) since there are only two unknown functions left, μ and λ. We state this as a theorem and prove it following [32] where the statement is shown for the case where $\Lambda = 0$.

Theorem 3.1. *Let λ, μ be a C^2-solution of the reduced system defined by the equations (3.10), (3.11) and the matter quantities (3.7) and (3.8). Let f be a C^1-solution of the Vlasov equation (2.3). Then all Einstein equations are satisfied.*

Proof. Let $E_{\alpha\beta} = G_{\alpha\beta} + g_{\alpha\beta}\Lambda - 8\pi T_{\alpha\beta}$. Note that $E_{\alpha\beta}$ is symmetric. Then the Einstein equations are equivalent to $E_{\alpha\beta} = 0$. We prove this using the equation $\nabla_\alpha E^{\alpha\beta} = 0$. This equation is equivalent to $\nabla_\alpha G^{\alpha\beta} = 0$ since $\nabla_\alpha T^{\alpha\beta} = 0$ and $\nabla_\alpha g^{\alpha\beta} = 0$. Of course, the latter equation only holds if the metric is C^3. But it is possible to approximate μ and λ by C^3-functions and obtain the desired equations [32]. We calculate the $\nabla_\alpha E^{\alpha t}$ component and obtain

$$\partial_t E^{tt} + \partial_r E^{tr} + E^{tt}(2\dot{\mu} + \dot{\lambda}) + E^{tr}\left(3\mu' + \lambda' + \frac{2}{r}\right) + E^{rr} e^{2(\lambda-\mu)}\dot{\lambda} = 0 \tag{3.13}$$

Since we take the first two Einstein equations (3.10) and (3.11) for granted, we have $E^{tt} = E^{rr} = 0$. We plug this into equation (3.13) and obtain

$$\partial_r(r^2 E^{tr}) + (3\mu' + \lambda')(r^2 E^{tr}) = 0. \tag{3.14}$$

This implies that E^{tr} is zero. Next we consider the component $\nabla_\alpha E^{\alpha r}$. One obtains

$$\partial_t E^{tr} + \partial_r E^{rr} + E^{tt} e^{2(\mu-\lambda)}\mu' + E^{tr}(3\dot\lambda + \dot\mu) + E^{rr}\left(\mu' + 2\lambda' + \frac{2}{r}\right) - \frac{e^{-2\lambda}}{r} g_{AB}E^{AB} = 0, \quad (3.15)$$

where $A, B \in \{\vartheta, \varphi\}$. Taking into account that $E^{tt} = E^{rr} = E^{tr} = 0$ one is left with $g_{AB}E^{AB} = 0$. Symmetries now imply that $E^{AB} = 0$ and also that E^{tA} and E^{rA} are zero. □

3.2 Coordinate transformation

It will be easier to work on the mass shell with an orthonormal frame $\{e_i\}$ instead of the coordinate basis $\{\partial_i\}$. So we perform a change of variables on $\mathscr{P}$. The metric (3.6) expressed in Cartesian coordinates, $x^0 = t$, $x^1 = r\sin\vartheta\cos\varphi$, $x^2 = r\sin\vartheta\sin\varphi$, and $x^3 = r\cos\vartheta$, reads

$$\mathrm{d}s^2 = -e^{2\mu}\mathrm{d}t^2 + \left(\delta_{ab} + \frac{e^{2\lambda}-1}{r^2}\delta_{ac}\delta_{bd}x^c x^d\right)\mathrm{d}x^a\mathrm{d}x^b. \quad (3.16)$$

We define the coefficients

$$e_i^a := \delta_i^a + \frac{e^{-\lambda}-1}{r^2}x^a\delta_{ib}x^b. \quad (3.17)$$

Here i is a frame index and a is a coordinate index, i.e. $e_i = e_i^a\partial_a$. This gives a new basis $(\partial_t, e_1, e_2, e_3)$ of the tangent space $T_q\mathscr{M}$ (at a point $q \in \mathscr{M}$). The vectors e_i are orthonormal with respect to g, thus

$$g(e_i, e_j) = \delta_{ij}. \quad (3.18)$$

The components p^a of the momentum vector p are given by $p^a = e_i^a v^i$, where

$$v^a = p^a + \frac{e^{\lambda}-1}{r^2}x^a\delta_{bc}x^b p^c. \quad (3.19)$$

Note that on the mass shell $\mathscr{P}$ we have

$$\left(p^0\right)^2 = -\frac{1}{g_{00}}(1 + g_{ab}e_i^a e_j^b v^i v^j) \quad \Rightarrow \; p^0 = e^{-\mu}\sqrt{1+|v|^2}. \quad (3.20)$$

We also introduce the abbreviations $|v| = \sqrt{\delta_{ij}v^i v^j}$ and $v_r = \frac{\delta_{ij}v^i x^j}{r}$.

In order to write down the Vlasov equation (2.3) in the new coordinates (x^α, v^a) we have to replace the p's by v's and the Christoffel symbols by the connection coefficients $\tilde\Gamma_{abc}$ with respect to the orthonormal frame $\{e_i\}$. They are given by the formula

$$\tilde\Gamma_{abc} = \frac{1}{2}\left[e_b(g_{ac}) + e_c(g_{ba}) - e_a(g_{cb}) + \gamma^d_{cb}g_{ad} + \gamma^d_{ac}g_{bd} - \gamma^d_{ba}g_{cd}\right] \quad (3.21)$$

where γ^a_{bc} are the commutation functions which are defined by

$$[e_b, e_c] = \gamma^a_{bc}e_a, \quad (3.22)$$

cf. [21]. The vectors e_α are the basis vectors of the tangent space $T_q\mathscr{M}$, $q \in \mathscr{M}$, thus $e_\alpha \in (\partial_t, e_1, e_2, e_3)$.

Furthermore, we impose several symmetry assumptions which lead to a more simplified form of the Vlasov equation. We assume f to be static, i.e. $\partial_t f = 0$, and spherically symmetric, i.e. $f(t, A\vec{x}, A\vec{v}) = f(t, \vec{x}, \vec{v})$, where $\vec{x} = \left(x^1, x^2, x^3\right)$, $\vec{v} = \left(v^1, v^2, v^3\right)$, for all $A \in SO(3)$. As carried out in section C of the appendix, one obtains

$$\frac{v^a}{\sqrt{1+|v|^2}}\frac{\partial f}{\partial x^a} - \sqrt{1+|v|^2}\mu' \frac{x^a}{r}\frac{\partial f}{\partial v^a} = 0 \tag{3.23}$$

for the Vlasov equation in the frame.

Finally, we give explicit expressions for the matter quantities ϱ and p in terms of the coordinates (v^a). We always use the metric in Cartesian coordinates (3.16). The energy density ϱ is defined to be $\varrho = T_{\alpha\beta} z_0^\alpha z_0^\beta$. Thus we write

$$\begin{aligned}\varrho &= -e^{2\mu} \int_{\mathbb{R}^3} \frac{f(x^\alpha, v^a(p^a))p^0 p^0 \sqrt{|g|}}{g_{00}p^0} |Dp^a(v^a)| \, \mathrm{d}v^1 \mathrm{d}v^2 \mathrm{d}v^3 \\ &= \int_{\mathbb{R}^3} f(x^\alpha, v^a(p^a))\sqrt{1+|v|^2} \, \mathrm{d}v^1 \mathrm{d}v^2 \mathrm{d}v^3. \end{aligned} \tag{3.24}$$

$|Dp^a(v^a)|$ is the determinant of the Jacobi matrix of the map $p^a(v^a) = e_i^a v^i = v^a + \frac{e^{-\lambda}-1}{r^2}\delta_{ij}x^i v^j x^a$ which is needed because of the change of variables from p^a to v^a in the integral. The radial pressure p is defined as $p = T_{\alpha\beta} z_1^\alpha z_1^\beta$ where the vector z_1 is given by $z_1 = e^{-\lambda}\partial_r$. We express z_1 in terms of the basis $(\partial_t, e_1, e_2, e_3)$:

$$\begin{aligned} z_1 = e^{-\lambda}\partial_r &= e^{-\lambda}\left(\frac{\partial r}{\partial x}\partial_x + \frac{\partial r}{\partial y}\partial_y + \frac{\partial r}{\partial z}\partial_z\right) = e^{-\lambda}\left(\frac{x}{r}\partial_x + \frac{y}{r}\partial_y + \frac{z}{r}\partial_z\right) \\ &= e^{-\lambda}\left(\frac{\partial x}{\partial r}(c_1^1 e_1 + c_1^2 e_2 + c_1^3 e_3) + \frac{\partial y}{\partial r}(c_2^1 e_1 + c_2^2 e_2 + c_2^3 e_3) + \frac{\partial z}{\partial r}(c_3^1 e_1 + c_3^2 e_2 + c_3^3 e_3)\right) \\ &= e^{-\lambda}\frac{e^\lambda}{r} x^i e_i = \frac{x^i}{r} e_i. \end{aligned} \tag{3.25}$$

The coefficients c_i^j are given in (C.3) – (C.8) in the appendix. Remember that in the $(\partial_t, e_1, e_2, e_3)$-frame $g_{ab} = \delta_{ab}$ holds. So we have

$$\begin{aligned} p = T_{\alpha\beta} z_1^\alpha z_2^\beta &= T_{ij}\frac{x^i x^j}{r^2} = T^{ab}\frac{g_{ia}g_{jb}x^i x^j}{r^2} \\ &= -\int_{\mathbb{R}^3} \frac{f(x^\alpha, p^a(v^a))\sqrt{|g|}|Dp^a(v^a)|}{g_{00}p^0} v^a v^b \frac{g_{ia}g_{jb}x^i x^j}{r^2} \mathrm{d}v^1 \mathrm{d}v^2 \mathrm{d}v^3 \\ &= \int_{\mathbb{R}^3} \frac{f(x^\alpha, v^a(p^a))}{\sqrt{1+|v|^2}} v_r^2 \mathrm{d}v^1 \mathrm{d}v^2 \mathrm{d}v^3. \end{aligned} \tag{3.26}$$

In a similar way one finds the analogue expression for the tangential pressure p_T, namely

$$p_T(r) = \frac{1}{2}\int_{\mathbb{R}^3} \left|\frac{x \times v}{r}\right|^2 f(t, x, v) \mathrm{d}v^1 \mathrm{d}v^2 \mathrm{d}v^3. \tag{3.27}$$

In the next subsection we will derive an ordinary differential equation by virtue of the choice of a suitable ansatz for the matter distribution f. For given initial conditions a solution of this ODE will imply a solution of the full system (3.10), (3.11), (3.23), (3.26), (3.24).

3.3 Conserved quantities and reduction of the problem

By virtue of theorem 3.1 is is sufficient to consider the reduced system

$$\frac{v^a}{\sqrt{1+|v|^2}}\frac{\partial f}{\partial x^a} - \sqrt{1+|v|^2}\mu'\frac{x^a}{r}\frac{\partial f}{\partial v^a} = 0, \tag{3.28}$$

$$e^{-2\lambda}(2r\lambda' - 1) + 1 - r^2\Lambda = 8\pi r^2 \varrho, \tag{3.29}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 + r^2\Lambda = 8\pi r^2 p, \tag{3.30}$$

where $|v| = \sqrt{\delta_{ij}v^iv^j}$, and $v_r = \frac{\delta_{ij}v^iv^j}{r}$ and the matter quantities read

$$\varrho = \int_{\mathbb{R}^3} f(t,x,v)\sqrt{1+|v|^2}\, \mathrm{d}v^1\mathrm{d}v^2\mathrm{d}v^3, \tag{3.31}$$

$$p = \int_{\mathbb{R}^3} \frac{f(t,x,v)}{\sqrt{1+|v|^2}} v_r^2\, \mathrm{d}v^1\mathrm{d}v^2\mathrm{d}v^3. \tag{3.32}$$

We consider the characteristic system of the Vlasov equation (3.28) in v-coordinates:

$$\frac{\partial X}{\partial s} = \dot{X} = \frac{V}{\sqrt{1+V^2}}, \tag{3.33}$$

$$\frac{\partial V}{\partial s} = \dot{V} = -\sqrt{1+V^2}\mu'(r)\frac{X}{r}. \tag{3.34}$$

The solutions (X,V) of the characteristic system are called *characteristic curves*. The characteristic curves are the integral curve of the geodesic spray $\mathscr{X}$. Along these curves the matter distribution function f is constant. There are quantities E and L, given by

$$E := e^{\mu(r)}\varepsilon := e^{\mu(r)}\sqrt{1+v^2}, \quad L := x^2v^2 - (x\cdot v)^2 = |x \times v|^2, \tag{3.35}$$

that are conserved along the characteristics. This can easily be seen:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}s} &= e^{\mu(r)}\mu'(r)\frac{x\cdot\dot{x}}{r}\sqrt{1+v^2} + \frac{v\cdot\dot{v}e^{\mu(r)}}{\sqrt{1+v^2}} \\
&= e^{\mu(r)}\mu'(r)\frac{v\cdot x}{r} - \frac{v\cdot x}{r}e^{\mu(r)}\mu'(r) = 0, \\
\frac{\mathrm{d}L}{\mathrm{d}s} &= 2x\cdot\dot{x}v^2 + 2x^2v\cdot\dot{v} - 2(x\cdot v)(\dot{x}\cdot v + x\cdot\dot{v}) \\
&= \frac{2x\cdot vv^2}{\sqrt{1+v^2}} - \frac{2x^2v\cdot x\mu'(r)\sqrt{1+v^2}}{r} - 2(x\cdot v)\left(\frac{v^2}{\sqrt{1+v^2}} - \frac{x^2}{r}\mu'(r)\sqrt{1+v^2}\right) = 0.
\end{aligned}$$

Certain values of the quantities E and L determine a certain characteristic curve. So every function $\Phi(E,L)$ depending only on these quantities solves already the Vlasov equation. One could ask the question whether this method already yields all possible solutions of the Vlasov equation. In the non-relativistic case, thus for the Vlasov-Poisson system, there is a theorem, called Jeans' theorem, stating exactly this [17]. But this does not hold for the Einstein-Vlasov system. In the relativistic setting Schaeffer found a class of counter examples to this theorem

[43]. In the literature three types of ansätze for the matter distribution are important. First, the *isotropic* ansatz stating that $\Phi(E,L)$ is actually not depending on L, thus

$$f(t,x,v) = \Phi(E), \tag{3.36}$$

where $E > 0$. The second class of ansätze is called *polytropic*. They are of the form

$$f(t,x,v) = \Phi(E,L) = \phi(E)L^{\ell} \tag{3.37}$$

with $\phi \in L^{\infty}((0,\infty))$, $E > 0$ and $\ell > -\frac{1}{2}$. The third, most general class is called *anisotropic* and reads

$$f(t,x,v) = \Phi(E,L) = \phi(E)[L - L_0]_+^{\ell} \tag{3.38}$$

where $\phi \in L^{\infty}((0,\infty))$, $\ell > -\frac{1}{2}$, $E > 0$ and $L_0 \geq 0$. L_0 can be interpreted as a cutoff for L from below.

The theorems stating that the matter quantities are of bounded support require a more specific ansatz for f, namely a certain form of $\phi(E)$ [33, 34, 36, 30]. This form is given by $\phi(e) = \tilde{\phi}(1 - E/E_0)$, where $\tilde{\phi} : \mathbb{R} \to [0,\infty)$ is measurable, $\tilde{\phi}(\eta) = 0$ for $\eta < 0$ a.e. on some interval $[0,\eta_1]$ with $\eta_1 > 0$ and E_0 is some prescribed cutoff energy. Moreover, it is required that there exists $\gamma > -1$ such that for every compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that $\tilde{\phi}(\eta) \leq C\eta^{\gamma}$, $\eta \in K$.

We choose an ansatz that is as general as possible, thus anisotropic, but that meets the assumptions of all theorems relevant for the proof concerning solutions to the Einstein-Vlasov system with cosmological constant. To simplify calculations we define $y := \ln(E_0) - \mu$ as in [30] so that $e^{\mu} = E_0/e^{y}$. In this work we will treat the ansatz

$$\begin{aligned} f(t,x,v) &= \Phi(E,L) = c_0\phi\left(1 - \frac{E}{E_0}\right)[L - L_0]_+^{\ell} = c_0\phi\left(1 - \varepsilon e^{-y}\right)[L - L_0]_+^{\ell}, \\ \phi(\eta) &= [\eta]_+^{k}, \end{aligned} \tag{3.39}$$

where $k \geq 0$, $\ell > -\frac{1}{2}$ fulfill the inequality $k < 3\ell + 7/2$ and $c_0, E_0 > 0$, $L_0 \geq 0$. The cutoff energy cannot be chosen arbitrarily but is determined by the initial value y_0 of y and the boundary conditions. For the construction of globally regular solutions L_0 has to be sufficiently small to ensure finite support of the matter quantities [36]. When considering solutions with a black hole at the center, there are positive lower bounds on L_0.

In the next step, we want to express ϱ and p as integrals over E and L instead of v. Thus we make a transformation of variables. The spherical coordinates in v-space shall be denoted by r_v, ϑ_v, φ_v. The coordinate system is chosen in a way that $\vec{x}$ is parallel to the v^3-axis. We introduce the map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $(r_v, \vartheta_v) \mapsto (\varepsilon, L)$. The Jacobi determinant takes the form

$$|\mathscr{J}_{\psi}| = \frac{2r^2 r_v^3}{\sqrt{1 + r_v^2}} \cos\vartheta_v \sin\vartheta_v. \tag{3.40}$$

We calculate

$$\begin{aligned} \varrho(r) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \Phi\left(e^{-y}\varepsilon(r_v), L(x, r_v, \vartheta_v)\right)\sqrt{1 + r_v^2}\, r_v^2 \sin\vartheta_v \, \mathrm{d}r_v \mathrm{d}\vartheta_v \mathrm{d}\varphi_v \\ &= 2\pi \int_0^{\pi}\int_0^{\infty} \Phi\left(e^{-y}\varepsilon(r_v), L(r_v, \vartheta_v)\right)\frac{(1 + r_v^2)}{2r^2 r_v \cos\vartheta_v}|\mathscr{J}_{\psi}| \, \mathrm{d}r_v \mathrm{d}\vartheta_v. \end{aligned} \tag{3.41}$$

Note that

$$\frac{(1+r_v^2)}{2r^2 r_v \cos\vartheta_v} = \frac{\varepsilon^2}{2rx\cdot v} = \frac{\varepsilon^2}{2r^2\sqrt{\varepsilon^2-1-L/r^2}}. \tag{3.42}$$

After adjusting the domains of integration, one obtains

$$\varrho(r) = \frac{2\pi}{r^2}\int_{\varepsilon=1}^{\infty}\int_{L=0}^{r^2(\varepsilon^2-1)} \Phi\left(e^{-y}\varepsilon, L\right)\frac{\varepsilon^2}{\sqrt{\varepsilon^2-1-L/r^2}}\,\mathrm{d}L\mathrm{d}\varepsilon. \tag{3.43}$$

In an analogous way, one calculates

$$p(r) = \frac{2\pi}{r^2}\int_{\varepsilon=1}^{\infty}\int_{L=0}^{r^2(\varepsilon^2-1)} \Phi\left(e^{-y}\varepsilon, L\right)\sqrt{\varepsilon^2-1-L/r^2}\,\mathrm{d}L\mathrm{d}\varepsilon. \tag{3.44}$$

As already mentioned above, one is not able to treat the system when Φ is depending on E and L in a fully general way. We are assuming the ansatz (3.39) and simplify the expressions for ϱ and p further. This yields

$$\varrho(r) = G_\phi(r, y(r)), \qquad p(r) = H_\phi(r, y(r)), \tag{3.45}$$

where

$$G_\phi(r,y) = c_\ell c_0 r^{2\ell}\int_{\sqrt{1+L_0/r^2}}^{\infty}\phi\left(1-\varepsilon e^{-y}\right)\varepsilon^2\left(\varepsilon^2-\left(1+\frac{L_0}{r^2}\right)\right)^{\ell+\frac{1}{2}}\mathrm{d}\varepsilon, \tag{3.46}$$

$$H_\phi(r,y) = \frac{c_\ell c_0}{2\ell+3} r^{2\ell}\int_{\sqrt{1+L_0/r^2}}^{\infty}\phi\left(1-\varepsilon e^{-y}\right)\left(\varepsilon^2-\left(1+\frac{L_0}{r^2}\right)\right)^{\ell+\frac{3}{2}}\mathrm{d}\varepsilon. \tag{3.47}$$

The constant c_ℓ is given by

$$c_\ell = 2\pi\int_0^1 \frac{s^\ell}{\sqrt{1-s}}\mathrm{d}s. \tag{3.48}$$

Lemma 3.2. *The functions $G_\phi(r,y)$ and $H_\phi(r,y)$ defined in (3.46) and (3.47), respectively, have the following properties.*

(i) $G_\phi(r,y)$ and $H_\phi(r,y)$ are continuously differentiable in r and y.

(ii) The functions $G_\phi(r,y)$ and $H_\phi(r,y)$ and the partial derivatives $\partial_y G_\phi(r,y)$ and $\partial_y H_\phi(r,y)$ are increasing both in r and y.

(iii) There is vacuum, i.e. $p(r) = \varrho(r) = 0$, if and only if $e^{-y(r)}\sqrt{1+L_0/r^2} \geq 1$, in particular there is vacuum if $y(r) \leq 0$.

Proof. The differentiability can be seen via a change of variables in the integral in (3.46) and (3.47) given by $E = \varepsilon e^{-y}E_0$. For G_ϕ one finds

$$G_\phi(r, y(r)) = c_\ell c_0 r^{2\ell}\left(\frac{e^y}{E_0}\right)^{4+2\ell}\tilde{g}_\phi\left(e^{-y}\sqrt{1+L_0/r^2}E_0\right) \tag{3.49}$$

with

$$\tilde{g}_\phi(t) = \int_t^\infty \phi\left(1-\frac{E}{E_0}\right)E^2\left(E^2-t^2\right)^{\ell+\frac{1}{2}}\mathrm{d}E. \tag{3.50}$$

In analogy to the arguments given in [32] we consider

$$\frac{1}{\Delta t}\left(\tilde{g}_\phi(t-\Delta t)-\tilde{g}(t)\right)=\frac{1}{\Delta t}\int_{t-\Delta t}^{t}\phi\left(1-\frac{E}{E_0}\right)E^2\left(E^2-(t-\Delta t)^2\right)^{\ell+\frac{1}{2}}\mathrm{d}E$$
$$+\int_t^\infty \phi\left(1-\frac{E}{E_0}\right)\frac{E^2}{\Delta t}\left(\left(E^2-(t-\Delta t)^2\right)^{\ell+\frac{1}{2}}-\left(E^2-t^2\right)^{\ell+\frac{1}{2}}\right)\mathrm{d}E \quad (3.51)$$

Obviously, the first integral converges to zero as Δt goes to zero. The second term has a limit as $\Delta t \to 0$ by Lebesgue's theorem. Thus, the function $\tilde{g}$ is left-differentiable with

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\tilde{g}(t)=-(2\ell+1)t\int_t^\infty \phi\left(1-\frac{E}{E_0}\right)E^2\left(E^2-t^2\right)^{\ell-\frac{1}{2}}\mathrm{d}E. \quad (3.52)$$

Again by Lebesgue's theorem this function is continuous, and a continuous and continuously left-differentiable function is continuously differentiable. For H_ϕ the calculation can be done analogously.

The monotonicity can be seen directly from the structure of G_ϕ and H_ϕ. The last statement becomes obvious if one performs the aforementioned change of variables in the integrals in (3.46) and (3.47), cf. [34]. □

The remaining problem is to prove that the Einstein equations have solutions being compatible with the chosen ansatz of the matter distribution f. By virtue of the upper simplifications and assumptions we now derive a single ODE that will be discussed and solved in the following sections. The Einstein equations take the form

$$e^{-2\lambda}(2r\lambda'-1)+1-r^2\Lambda=8\pi r^2 G_\phi(r,y(r)) \quad (3.53)$$
$$e^{-2\lambda}(2r\mu'+1)-1+r^2\Lambda=8\pi r^2 H_\phi(r,y(r)) \quad (3.54)$$

We integrate equation (3.53):

$$\int_0^r\left(-\frac{\mathrm{d}}{\mathrm{d}s}\left(se^{-2\lambda}\right)+1-s^2\Lambda\right)\mathrm{d}s=\int_0^r 8\pi s^2 G_\phi(r,y(r))\mathrm{d}s$$
$$\Leftrightarrow \quad e^{-2\lambda(r)}=1-\frac{r^2\Lambda}{3}-\frac{8}{\pi}r\int_0^r s^2 G_\phi(s,y(s))\mathrm{d}s. \quad (3.55)$$

We now plug this into the second Einstein equation (3.54) and obtain

$$\left(1-\frac{r^2\Lambda}{3}-\frac{8\pi}{r}\int_0^r s^2 G_\phi(s,y(s))\mathrm{d}s\right)(2r\mu'+1)-1+r^2\Lambda=8\pi r^2 H_\phi(r,y(r)). \quad (3.56)$$

Since $\mu'=-y'$ one obtains the differential equation for y

$$y'(r)=-\frac{4\pi}{1-\frac{\Lambda r^2}{3}-\frac{8\pi}{r}\int_0^r s^2 G_\phi(s,y(s))\mathrm{d}s}\times\left(rH_\phi(r,y(r))-\frac{r\Lambda}{12\pi}+\frac{1}{r^2}\int_0^r s^2 G_\phi(s,y(s))\mathrm{d}s\right). \quad (3.57)$$

A solution to (3.57) yields a solution to the full system (3.28)–(3.32) and as stated in theorem 3.1, a solution (μ, λ) to (3.29) and (3.30) already implies a solution of all Einstein equations. This equation is analyzed and solved in the remainder of this work.

We conclude this section by stating a useful differential equation for p relating the matter quantities ϱ, p, and p_T and the metric coefficient μ, the so called Tolman-Oppenheimer-Volkov equation. It is the subject of the following lemma.

Lemma 3.3. *Let μ, λ be a solution of the system (3.29), (3.30) and let ϱ and p be given via (3.45) and the quantity p_T by (3.27). Then the Tolman-Oppenheimer-Volkov equation (TOV equation) holds:*

$$p'(r) = -\mu'(p(r) + \varrho(r)) - \frac{2}{r}(p(r) - p_T(r)). \tag{3.58}$$

Proof. In a first step, one checks by direct calculation that

$$p_T(r) = (\ell+1)p(r) + \frac{c_\ell}{2} L_0 r^{2\ell-2} e^{-(2l+2)\mu} k_\phi \left(e^\mu \sqrt{1 + \frac{L_0}{r^2}} \right), \tag{3.59}$$

where

$$k_\phi(t) = \int_t^\infty \phi(E) \left(E^2 - t^2\right)^{\ell+\frac{1}{2}} \mathrm{d}E, \quad t > 0. \tag{3.60}$$

Using (3.47) and lemma 3.2 one obtains the relation

$$p'(r) = \frac{2\ell}{r} p(r) - (2\ell+4)\mu'(r)p(r) - c_\ell \left(1 + \frac{L_0}{r^2}\right) r^{2\ell} e^{-(2\ell+2)\mu} k_\phi \left(e^\mu \sqrt{1 + \frac{L_0}{r^2}} \right) \mu'$$
$$+ c_\ell r^{2\ell-3} L_0 e^{-(2l+2)\mu} k_\phi \left(e^\mu \sqrt{1 + \frac{L_0}{r^2}} \right) \tag{3.61}$$

which, after some further simplifications, leads to the Tolman-Oppenheimer-Volkov equation. □

4 The System with vanishing cosmological constant

In section 5 the existence of solutions of the Einstein-Vlasov system with non-vanishing cosmological constant Λ will be proved in various settings. For some of these proofs the existence of a solution in the corresponding setting with $\Lambda = 0$ will be crucial. In the course of the discussion in a setting with non-vanishing cosmological constant, the corresponding solution with $\Lambda = 0$ will be referred to as a *background solution.* Also certain properties of these solutions will be of major importance. Most of the needed results are already known in the literature. The purpose of this section is to present the relevant results for the $\Lambda = 0$ setting using a consistent, up-to-date notation. If needed, some minor adaptions and extensions will be made as well.

4.1 Existence results

This subsection is devoted to the proof of existence of static solutions of the Einstein-Vlasov system in spherical symmetry. We consider equation (3.57) with vanishing cosmological constant,

$$y'(r) = -\frac{4\pi}{1 - \frac{8\pi}{r}\int_0^r s^2 G(r, \mu(r))\mathrm{d}s} \left(rH(r, y(r)) + \frac{1}{r^2}\int_0^r s^2 G(s, y(s))\mathrm{d}s \right), \tag{4.1}$$

and prove the global in r existence of a solution y. The solutions that will be constructed in the following are asymptotically flat. This will be guaranteed by the boundary conditions

$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \mu(r) = 0. \tag{4.2}$$

Solutions with a regular center will be of special interest. The regular center at $r = 0$ is guaranteed by the boundary condition

$$\lambda(0) = 0. \tag{4.3}$$

The proof of existence divides into two steps. First, local existence is shown, i.e. that a unique solution exists on a small interval $[0, \delta]$. The solution can be extended onto a maximal interval of existence $[0, R_c)$. In the next step one shows that this solution can be extended to the whole $\mathbb{R}_+$–axis. The radius R_c is characterized by a *continuation criterion*. We have $R_c = \infty$ or either

$$\limsup_{r\to R_c} \frac{8\pi}{r} \int_0^r s^2 G_\phi(s, \mu(s))\mathrm{d}s = 1 \quad \text{or} \quad \limsup_{r\to R_c} \mu(r) = \infty. \tag{4.4}$$

So in particular if μ and λ stay bounded on an interval $[0, R)$ the solution already exists beyond R. For the first step we formulate the following lemma.

Lemma 4.1. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let G_ϕ and H_ϕ be defined by equations (3.46) and (3.47), respectively. Then for every $y_0 \in \mathbb{R}$ there is a $\delta > 0$ such that there exists a unique solution $y \in C^2([0, \delta])$ of equation (4.1) with initial value $y(0) = y_0$.*

Proof. The proof follows [33] and uses a contraction argument. Integrating equation (4.1) with respect to r we obtain the fixed point problem

$$y(r) = (Ty)(r) \tag{4.5}$$

with the operator

$$(Ty)(r) = y_0 - \int_0^r \frac{4\pi}{1 - \frac{8\pi}{s}\int_0^s \sigma^2 G(\sigma, y(\sigma))\mathrm{d}\sigma} \left(sH(s, y(s)) + \frac{1}{s^2}\int_0^s \sigma^2 G(\sigma, y(\sigma))\mathrm{d}\sigma \right) \mathrm{d}s. \tag{4.6}$$

This operator is considered on the set

$$M = \Big\{ u : [0, \delta] \to \mathbb{R} \,\Big|\, u(0) = y_0, y_0 \le u(r) \le y_0 + 1, \\ \frac{8\pi}{r} \int_0^r s^2 G(s, u(s))\mathrm{d}s \le \frac{1}{2}, r \in [0, \delta] \Big\}. \tag{4.7}$$

By direct calculation it can be shown that for δ small enough

(i) $u \equiv y_0 \in M$,

(ii) $u \in M \Rightarrow Tu \in M$ and

(iii) $\exists a \in (0, 1)\, \forall u, v \in M : \|Tu - Tv\|_{\infty,\delta} \le a\|u - v\|_{\infty,\delta}$, where $\|\cdot\|_{\infty,\delta} = \sup_{r\in[0,\delta]}(\cdot)$.

Thus T acts as a contraction on the set M. Then from Banach's fixed point theorem it follows that there exists a unique fixed point $y \in M$ of T. Differentiation of (4.5) with respect to r yields that y solves equation (4.1) on the interval $[0, \delta]$. The steps (i) – (iii) are carried out in detail for the more general case of a non vanishing cosmological constant in section 5.1.

Away from the singularity $r = 0$, standard existence and uniqueness results are applied to extend y to a maximal solution on an interval $[0, R_c)$. The asserted regularity of the solution can be seen as follows. Obviously, the boundary condition (4.3) at $r = 0$ is satisfied. The regularity of the functions G_ϕ and H_ϕ implies that $y \in C^2((0, R_c))$, cf. [34], and it can be shown that the second derivative continuously extends to $r = 0$ and $y'(0) = 0$. □

For the proof of global existence we follow the argument given in [34]. The third matter quantity p_T, defined in (3.9), becomes important. The next lemma states existence of solutions from a minimum radius r_0 onwards. This is for technical reasons. Of course we are interested in solutions that are defined on the whole of $[0, \infty)$. So when solutions with a regular center are considered, we always have $r_0 = 0$. But later, in subsection 4.3, solutions will be constructed that have a black hole at the center. These black holes will be encased by a vacuum region and there will be a certain radius at which a solution of the non-vacuum Einstein equations will be glued to the inner vacuum solution. Also in the proof of theorem 4.5 the existence of a solution of equation (4.1) starting at $r_0 > 0$ will be needed. In view of this procedure we prove the existence of non-vacuum solutions on the interval $[r_0, \infty)$.

Lemma 4.2. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let G_ϕ and H_ϕ be defined by equations (3.46) and (3.47), respectively. Then for every $r_0 \geq 0$ and $\lambda_0 \geq 0$, $\mu_0 \in \mathbb{R}$ with $\lambda_0 = 0$ if $r_0 = 0$ there exists a unique solution $\lambda, \mu \in C^1([r_0, \infty))$ of the Einstein equations (3.53) and (3.54) with $\lambda(0) = 0$ and $\mu(0) = \mu_0$.*

Proof. In the case $r_0 > 0$ the existence of a local solution of equation (4.1) follows from the regularity of its right hand side. Otherwise, the existence of a solution to equation (4.1) on an interval $[0, \delta]$ is established by lemma 4.1. Let $[0, R_c)$ be the maximum interval that the solution exists on. In the following we show that both metric coefficients, μ and λ, stay bounded on the interval $[0, R_c)$ following the arguments in the corresponding proof in [34]. This already implies that $R_c = \infty$ due to the continuation criterion.

Integration of the first Einstein equation (3.29) with $\Lambda = 0$ yields

$$e^{-2\lambda(r)} = 1 - \frac{8\pi}{r} \int_0^r s^2 \varrho(s) \mathrm{d}s. \tag{4.8}$$

If one plugs this into equation (3.30) one obtains

$$\mu'(r) = 4\pi e^{2\lambda(r)} \left(p(r) + \frac{1}{r^2} \int_0^r s^2 \varrho(s) \mathrm{d}s \right) =: 4\pi e^{2\lambda(r)} \left(p(r) + w(r) \right), \tag{4.9}$$

where $w(r)$ has been defined implicitly. By adding equations (3.29) and (3.30) we have

$$(\lambda' + \mu')(r) = 4\pi r e^{2\lambda(r)} (p(r) + \varrho(r)). \tag{4.10}$$

We define an energy by the quantity $e^{\lambda+\mu}(p+w)$ and wish to establish a differential inequality for this energy. Since for small radii the existence of a solution is already shown it suffices to consider the inequality on the interval $[\tilde{r}_0, R_c)$ where $\tilde{r}_0 := \max\{r_0, R_c/2\}$. This avoids the blow-up at $r = 0$ of certain summands in the course of the estimate. The estimate will allow to deduce that $\lambda + \mu$ stays bounded for all $r \in [r_0, \infty)$. Using the TOV equation (3.58) one finds

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\lambda+\mu}(p+w)\right) = e^{\lambda+\mu}\left(-\frac{2p}{r} + \frac{2p_T}{r} - \frac{2w}{r} + \frac{\varrho}{r}\right) \leq e^{\lambda+\mu}\left(\frac{2p_T}{r} + \frac{\varrho}{r}\right). \tag{4.11}$$

We can drop the terms $-2p/r$ and $-2w/r$ due to their sign. Now we assume $R_c < \infty$. From its definition we see that w is an increasing function. Thus there exists $r_1 \in (\tilde{r}_0, R_c)$ and a constant $C > 0$ such that $w(r) \geq C$ for all $r \in [r_1, R_c)$. Moreover, we observe that ϱ and p_T are bounded on $[r_1, R_c)$. This follows from the structure of p_T, cf. equations (3.27) and (3.59), and lemma 3.2. So we may write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\lambda+\mu}(p+w)\right) &\leq C_1 e^{\lambda+\mu} \leq C_2 (p+w) e^{\lambda+\mu} \\ \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}\ln\left(e^{\lambda+\mu}(p+w)\right) &\leq \frac{C_1}{p+w} \leq C_2 \\ \Rightarrow \quad e^{\lambda+\mu} &\leq C_3. \end{aligned} \tag{4.12}$$

It follows that $\mu + \lambda$ is bounded on $[r_1, R_c)$. We have $\mu'(r) > 0$ as can be easily seen from equation (4.1). This implies $\mu(r) > \mu_0$. Moreover, equation (4.8) implies that $\lambda > 0$. So we can conclude that both, μ and λ, stay bounded on R_c. But this is a contradiction to the definition of R_c by the continuation criterion. Hence $R_c = \infty$. □

4.2 Bounded support of the matter quantities

In this subsection we show that there exist solutions where the corresponding matter quantities have bounded support. These solutions describe finitely extended objects of Vlasov matter. Being able to refer to a background solution having compactly supported matter quantities is crucial for the proof of existence of solutions with positive cosmological constants.

The reasoning will go as follows. The ansatz (3.39) for the matter distribution function f has been chosen in a way such that there is vacuum if $y \leq 0$, cf. lemma 3.2, (iii). First we prove a *compact support lemma* stating that y will have a single zero if $y(0) = y_0 > 0$. Afterwards, this lemma will be applied to the Einstein-Vlasov system. Since with this methods compact support can only be proved for the polytropic case (3.37), the anisotropic case has to be covered separately. This can done by a perturbation argument [36]. So we start by stating the compact support lemma.

Lemma 4.3. *Let $y \in C^1([0,\infty))$ with $y(0) = \mathring{y} \in (0, y_{\max})$ satisfy the estimate*

$$y'(r) \leq -\frac{m(r)}{r^2} \quad \text{on } [0,\infty) \tag{4.13}$$

where

$$m(r) = m(r,y) := 4\pi \int_0^r s^{2+2\ell} g(y(s))\mathrm{d}s, \tag{4.14}$$

$g \in C((-\infty, y_{\max}))$ is increasing with $g(y) = 0$ for $y \leq 0$ and $g(y) > 0$ for $y > 0$, and $\ell > -1/2$. Let g satisfy the estimate

$$g(y) \geq c y^{n+\ell} \quad \text{for} \quad 0 < y < y^* \tag{4.15}$$

with parameters $c > 0$, $y^* > 0$, and $0 < n < 3 + \ell$. Then the function y has a unique zero.

Proof. The proof follows [30]. Since y is decreasing, the limit $y_\infty := \lim_{r\to\infty} y(r) \in [-\infty,\infty)$ exists. We need to show that $y_\infty < 0$. Since y is monotonically decreasing this will allow to deduce the existence of a unique zero of y. The proof works with a contradiction argument. Assume $y_\infty > 0$. By the monotonicity of y we have $y(r) \geq y_\infty$ on $[0,\infty)$ and by the monotonicity of g:

$$m(r) \geq 4\pi g(y_\infty) \int_0^r s^{2+2\ell} \mathrm{d}s = \frac{4\pi}{2\ell+3} g(y_\infty) r^{2\ell+3}. \tag{4.16}$$

We put this into the estimate (4.13) for y' and integrate with respect to r. This yields already the desired contradiction:

$$\int_0^r y'(s)\mathrm{d}s \leq -\int_0^r \frac{m(s)}{s^2}\mathrm{d}s \quad \Leftrightarrow \quad y(r) \leq y^0 - Cr^{2+2\ell} \overset{r\to\infty}{\longrightarrow} -\infty. \tag{4.17}$$

C denotes a positive constant which may depend on all the parameters but never on r.

So we have shown that $y_\infty \leq 0$. The remaining task is to exclude $y_\infty = 0$. We observe that $m(r)$ is increasing in r and positive for $r > 0$. Hence $m(r) \geq m(1) =: m_1 > 0$ for $r \geq 1$ and

$$y(r) = -\int_r^\infty y'(s)\mathrm{d}s \geq m_1 \int_r^\infty \frac{\mathrm{d}s}{s^2} = \frac{m_1}{r} \quad \text{for } r \geq 1. \tag{4.18}$$

Moreover, since g is increasing and y decreasing, we have

$$m(r) \geq 4\pi g(y(r)) \int_0^r s^{2+2\ell}\mathrm{d}s = \frac{4\pi}{2\ell+3} r^{2\ell+3} g(y(r)). \tag{4.19}$$

Hence

$$y'(r) \leq -\frac{4\pi}{2\ell+3} r^{2\ell+1} g(y(r)), \quad r > 0. \tag{4.20}$$

We consider the integral

$$\int_{y(r)}^{\mathring{y}} \frac{\mathrm{d}\eta}{g(\eta)} = -\int_0^r \frac{1}{g(y(s))} y'(s)\mathrm{d}s \geq \int_0^r \frac{4\pi}{2\ell+3} s^{2\ell+1}\mathrm{d}s = \frac{4\pi}{(2\ell+3)(2\ell+2)} r^{2\ell+2} \tag{4.21}$$

where a simple change of variables was performed. Now we take $r > 0$ sufficiently large so that $0 < y(r) < y^*$. Recall that we are assuming $y(r) \to 0$ as $r \to \infty$. This assumptions will now lead to a contradiction. By the growth assumption (4.15) on g and (4.21) we have

$$C_1 r^{2\ell+2} \leq \int_{y(r)}^{y(r_0)} \frac{\mathrm{d}\eta}{g(\eta)} \leq \int_{y(r)}^{y^*} \frac{\mathrm{d}\eta}{g(\eta)} + C_2 \leq \frac{1}{c} \int_{y(r)}^{y^*} \frac{\mathrm{d}\eta}{\eta^{n+1}} + C_2. \tag{4.22}$$

We estimate the left hand side from below using (4.18), multiply the resulting estimate by $y(r)^{2\ell+2}$, and compute the integral. At the last step the cases $n+\ell \neq 1$ and $n+\ell = 1$ need to be distinguished. We find

$$\begin{aligned} c_1 m_1^{2\ell+2} &\leq \frac{1}{c}\int_{y(r)}^{y^*} \frac{\mathrm{d}\eta}{\eta^{n+\ell}} y^{2\ell+2} + C_2 y^{2\ell+2} \\ &= C_2 y^{2\ell+2} + \frac{y^{2\ell+2}}{c} \left\{ \begin{array}{ll} \ln\left(\frac{y^*}{y(r)}\right); & n+\ell = 1 \\ \frac{1}{1-n-\ell}\left((y^*)^{1-n-\ell} - (y(r))^{1-n-\ell}\right); & n+\ell \neq 1 \end{array} \right. . \end{aligned} \tag{4.23}$$

Note that C_1 is determined by the right hand side of equation (4.21). In particular the constants C_1 and C_2 are strictly positive. So are the expressions m_1, $2\ell+2>0$, and $\ell+3-n$. Having this in mind one observes that the right hand side of (4.23) goes to zero as r goes to infinity while the left hand side stays strictly larger than zero. This yields the desired contradiction. ☐

This lemma enables us to prove bounded support of the matter quantities of solutions to the Einstein-Vlasov system. We formulate this as a theorem.

Theorem 4.4. *Let Φ be a polytropic ansatz of the form (3.37) with the properties described in subsection 3.3. Then for any $\mathring{y}>0$ the unique solution of the reduced equation (4.1) with $y(0)=\mathring{y}$ has a unique zero $R_0>0$. By*

$$f(t,x,v)=\phi\left(1-e^{-y}\sqrt{1+|v|^2}\right)|x\times v|^{2\ell} \tag{4.24}$$

a static, spherically symmetric solution to the Einstein-Vlasov system is defined. This solution is compactly supported, and its spatial support is the ball with radius R_0 centered at the origin. The parameter $\mathring{y}$ is related to the metric quantity μ via $\mathring{y}=\mu(R_0)-\mu(0)$. Moreover, $\varrho,p\in C(\mathbb{R}^3)\cap C^1(B_{R_0}(0))$.

Proof. The unique solution of equation (4.1) with $y(0)=\mathring{y}$ exists by virtue of lemma 4.2. We want to apply the compact support lemma to show that y has a unique zero R_0. The crucial assumption that has to be checked is (4.15). We want to identify $G_\phi(r,y(s))=r^{2\ell}g(y(s))$ where G_ϕ is defined in (3.46). The other assumptions then follow by means of equation (4.1). So we perform a change of variables given by $\eta=1-\varepsilon e^{-y}$ in the integral in the definition of G_ϕ and obtain

$$G_\phi(r,y(r))=r^{2\ell}g(y(r))=r^{2\ell}c_0c_\ell e^{3y}\int_0^{1-e^{-y}}\phi(\eta)(1-\eta)^2\left(e^{2y}(1-\eta)^2-1\right)^{\ell+\frac{1}{2}}\mathrm{d}\eta. \tag{4.25}$$

Using the explicit form of ϕ given by $\phi(\eta)=[\eta]_+^k$ and estimating $\eta\le 1-e^{-y}$ one obtains

$$c_0c_\ell e^{\left(\frac{3}{2}+\ell\right)y}\int_0^{1-e^{-y}}\eta^k\left(1-\eta-e^{-y}\right)^{\ell+\frac{1}{2}}\mathrm{d}\eta. \tag{4.26}$$

For y sufficiently small we observe $1-e^{-y}\ge y/2$ and $e^y\ge 1/2$. So we can estimate all factors in front of the integral by a constant C. Now we perform a change of variables given by $s=\frac{\eta}{1-e^{-y}}$. This yields

$$g(y)\ge C\left(1-e^{-y}\right)^{k+\ell+\frac{3}{2}}\int_0^1 s^k(1-s)^{\ell+\frac{1}{2}}\mathrm{d}s\ge Cy^{k+\ell+\frac{3}{2}} \tag{4.27}$$

being equivalent to (4.15). So the assumptions of lemma 4.3 are fulfilled. The compact support lemma and lemma 3.2, (iii) imply the assertion. ☐

As mentioned at the beginning of this subsection these results do not cover the most general ansatz for the matter distribution f with a non zero cutoff angular momentum L_0. But such ansätze are necessary to obtain solutions with vacuum at the center surrounded by Vlasov matter, cf. [1] and subsection 4.5. Such configurations are called *shells*. In [36] Rein proved the existence of such shell solutions. The proof relies on theorem 4.4 and includes small L_0 via a perturbation argument. The exact statement is the subject of the following theorem.

Theorem 4.5. *There exists a static, spherically symmetric solution $f, \varrho, p, \lambda, \mu$ of the Einstein-Vlasov system (3.28) – (3.32) with vanishing cosmological constant where f is given by the ansatz (3.39) and ϱ and p depend on μ via (3.45) in a neighborhood of their support. $\lambda, \mu \in C^2([0,\infty)) \cap C^2(\mathbb{R}^3)$ satisfy the boundary conditions (4.2) and (4.3). Furthermore $\varrho, p \in C^1([0,\infty)) \cap C^1(\mathbb{R}^3)$ with $\operatorname{supp}(\varrho) = \operatorname{supp}(p) = [R_i, R_0]$ for some $0 \leq R_i < R_0 < \infty$, where $R_i > 0$ provided $L_0 > 0$. The ADM mass M is finite.*

Proof. The proof follows [36]. Consider first the case $L_0 = 0$. As established by theorem 4.4 there exists a solution y of equation (4.1) with $e^{-y(0)} < 1$ and $e^{-y(R)} > 1$ for some $R > R_0 > 0$ where R_0 is the bound of the support of the matter quantities as introduced in theorem 4.4. We choose R such that $1 < e^{-y(R)} < 2$. For any $L_0 > 0$ we define

$$R_i = \sqrt{\frac{L_0}{e^{-2y(0)} - 1}}. \tag{4.28}$$

Then

$$e^{-y(0)}\sqrt{1 + \frac{L_0}{R_i^2}} = 1 \quad \text{and} \quad e^{-y(0)}\sqrt{1 + \frac{L_0}{r^2}} > 1 \text{ for } r \in [0, R_i). \tag{4.29}$$

Due to lemma 3.2, (iii), $y_{L_0}(r) \equiv y(0)$ solves equation (4.1) on $[0, R_i]$ with $\varrho_{L_0}(r) = p_{L_0}(r) = 0$. Note that $\varrho_{L_0}(r)$ and $p_{L_0}(r)$ are given by y_{L_0} via (3.45). By lemma 4.2 y_{L_0} can be extended as a solution of equation (4.1) for $r \geq R_i$. We want to show that $e^{-y_{L_0}(R)} > 1$ for $L_0 > 0$ small because then the support of the matter quantities is bounded by R. We may assume that $y_{L_0}(R) > y(R) - 1$ since otherwise we are done. By monotonicity,

$$y(0) \geq y(r), \quad y_{L_0}(r) > y(R) - 1, \quad r \in [0, R]. \tag{4.30}$$

In the remainder of the proof one compares the solution y_{L_0} in the setting with $L_0 > 0$ with the solution y for $L_0 = 0$. The latter one can be seen as a *background solution*. By virtue of the structure of G_ϕ and H_ϕ given in equations (3.46) and (3.47), the differential equation (4.1) for y, and Grönwall's inequality one finds an upper bound on the difference of y and y_{L_0}:

$$|y_{L_0}(r) - y(r)| \leq C\sqrt{L_0}, \quad r \in [0, R], \tag{4.31}$$

where $C > 0$ is a constant depending on the background solution y and R, but not on r or L_0. See [36] for details of the estimates. So by choosing L_0 small enough one can attain $e^{-y_{L_0}(R_0)} > 1$. Together with the monotonicity of y this implies the assertion. □

Remark 4.6. *One can prescribe a distinct value for the ADM mass $M > 0$ or $R_0 > 0$ or $R_i > 0$. For this purpose one has to rescale the matter distribution function f. If $f(x, v)$ is a static solution of the Vlasov equation, so is $f_a(x, v) = a^2 f(ax, v)$ for any $a > 0$. The rescaled function f_a has spatial support $[a^{-1}R_i, a^{-1}R]$ and ADM mass*

$$M = \frac{1}{a}\int f(x, v)\sqrt{1 + |v|^2}\mathrm{d}v\mathrm{d}x. \tag{4.32}$$

So by rescaling a given solution we can get any prescribed value for the ADM mass or the inner or the outer radius.

4.3 Black holes

An interesting modification of the results discussed above are static, spherically symmetric solutions of the Einstein-Vlasov system that are not regular on $[0, \infty)$ but that have a black hole at the center. This subsection will be devoted to the construction of such solutions. The configuration we will be obtaining is the following. In a ball of radius r_+ around the center we will have vacuum. This vacuum region will be surrounded by a shell of Vlasov matter supported on (r_+, R_0). In the outer region of this shell, thus for $r > R_0$, there will be vacuum again.

It is well known that if there is vacuum on hand, the spherically symmetric, asymptotically flat solution of the Einstein equations with vanishing cosmological constant is given by the Schwarzschild metric. The corresponding coefficients μ_{vac} and λ_{vac} are given by

$$e^{2\mu_{\text{vac}}(r)} = e^{-2\lambda_{\text{vac}}(r)} = 1 - \frac{2M_0}{r}, \quad r > 2M_0, \tag{4.33}$$

where M_0 is the mass parameter of the black hole. This solution describes a black hole of mass M_0 that is hidden behind an event horizon at $r = 2M_0$. At this event horizon we have a coordinate singularity and we consider the solution for $r > 2M_0$. In section 5.4 the global structure of the spacetimes corresponding to solutions constructed in this work will be discussed, i.e. analytic extensions of the metric beyond the black hole horizon. The existence of solutions as described above is stated by the following theorem.

When dealing with solutions containing a black hole we drop the variable y and work with μ instead. The reason for this is that not in every vacuum region an *unshifted* Schwarzschild solution can be glued to the non-vacuum solution as can be seen later, cf. remark 4.8.

Theorem 4.7. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) with $E_0 = 1$ and $L_0 > 0$. Moreover, let $L_0 > 16M_0^2 > 0$. Then there exists a static solution (f, λ, μ) of the spherically symmetric Einstein-Vlasov system (3.28) – (3.32) such that μ and λ are given by* (4.33) *for $2M_0 < r < r_+$ and $\varrho(r) = p(r) = f(x, v) = 0$ for $2M_0 < r < r_+$ or $r > R_0$ and*

$$0 < 4\pi \int_{r_+}^{R_0} r^2 \varrho(r) \mathrm{d}r < \infty \tag{4.34}$$

where

$$r_+ = \frac{L_0 + \sqrt{L_0^2 - 16M_0^2 L_0}}{4M_0} > 4M_0 \tag{4.35}$$

and $R_0 > r_+$. Furthermore, $\lambda, \mu \in C^2((2M_0, \infty)), \varrho, p \in C^1((2M_0, \infty))$, and the spacetime is asymptotically flat.

Proof. On the interval $(2M_0, r_+)$ the Einstein-Vlasov system is solved by the Schwarzschild metric (4.33) and the vanishing matter distribution $f(x, v) \equiv 0$. By virtue of lemma 3.2, (iii) for $E_0 = 1$ the ansatz (3.39) for the matter distribution f yields vacuum if $e^{\mu(r)}\sqrt{1 + L_0/r^2} \geq 1$. We consider this inequality for $e^{\mu(r)} = \sqrt{1 - 2M_0/r}$. It is fulfilled on the interval $[r_-, r_+]$ where

$$r_\pm = \frac{L_0 \pm \sqrt{L_0^2 - 16M_0^2 L_0}}{4M_0}. \tag{4.36}$$

Since $L_0 > 16M_0^2$ by assumption this interval is well defined and not empty. We have $2M_0 < r_- < r_+$ and on (r_-, r_+) the ansatz Φ given in (3.39) for f is consistent with $f(x, v) \equiv 0$. For

$r \geq r_+$ we want to continue $\mu(r)$ as a solution of the non-vacuum Einstein-Vlasov system. So we impose the initial conditions

$$\mu_0 = \mu_{\mathrm{vac}}(r_+) = \sqrt{1 - \frac{2M_0}{r_+}}, \quad \lambda_0 = \lambda_{\mathrm{vac}}(r_+) = \sqrt{1 - \frac{2M_0}{r_+}}. \tag{4.37}$$

A solution of the Einstein-Vlasov system fulfilling these initial conditions exists by means of theorem 4.2. Note that $\lambda_0 > 0$. The transition from μ_{vac} to the non-vacuum solution at r_+ is C^2.

Next, one has to check that the constructed solution is not vacuum everywhere. Therefore, one calculates

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\mu(r)}\sqrt{1+\frac{L_0}{r^2}}\right)\bigg|_{r=r_+} < 0, \tag{4.38}$$

which implies that $\varrho(r) > 0$ and $p(r) > 0$ in a right neighborhood of r_+. It remains to show that there exists $R_0 > r_+$ such that $f(r, v) = 0$ for all $r > R_0$. It suffices to show that $\lim_{r\to\infty} > 0$. To prove this we consider a corresponding equation to (4.1). We define

$$M(r) = M_0 + m(r) = M_0 + 4\pi \int_{r_+}^{r} s^2 \varrho(s)\mathrm{d}s. \tag{4.39}$$

The Einstein equations imply

$$\mu'(r) = \frac{1}{1 - \frac{2M(r)}{r}}\left(4\pi r p(r) + \frac{M(r)}{r^2}\right) \geq \frac{M_0}{r(r-2M_0)} + \frac{m(r)}{r^2}. \tag{4.40}$$

Integration of equation (4.40) yields

$$\mu(r) \geq \int_{r_+}^{\infty} \frac{m(s)}{s^2}\mathrm{d}s > 0 \tag{4.41}$$

as desired. Since μ is monotonically increasing there exists a single zero which we call R_0. So we have obtained a solution of (3.28) – (3.32) that for $r \in (2M_0, r_+]$ is given by the Schwarzschild metric with mass parameter M_0, that contains a matter shell supported on (r_+, R_0), and that is again given by a Schwarzschild metric with mass parameter $M := M(R_0)$ for $r > R_0$. □

Remark 4.8. *In the coordinates on hand it is not possible to have a Schwarzschild solution without any shift both, outside and inside of the matter shell. In the construction that is described in theorem 4.7 by setting $E_0 = 1$ the parameters were chosen in a way that μ does not tend to zero as r goes to infinity. But this only corresponds to a rescaling of the time coordinate and the solution is asymptotically flat, nevertheless.*

4.4 The Buchdahl inequality

At the investigation of the Einstein-Vlasov system with non-vanishing cosmological constant an upper bound for the mass term $\frac{2M(r)}{r}$ will turn out to be of great help. In the case of globally regular solutions $M(r)$ equals the quasilocal mass $m(r)$ given by

$$m(r) = 4\pi \int_0^r s^2 \varrho(s)\mathrm{d}s. \tag{4.42}$$

Such a bound has been proved by Andréasson, cf. [2]. He also shows that for Vlasov matter the estimate is sharp. But we also wish to dispose of a bound of the corresponding quantity for solutions with a Schwarzschild singularity at the center as described in subsection 4.3. In this context, the quantity $M(r)$ reads

$$M(r) = M_0 + 4\pi \int_{2M_0}^{r} s^2 \varrho(s) \mathrm{d}s, \quad r > 2M_0, \tag{4.43}$$

where M_0 is the mass parameter of the black hole. For this purpose we extend the result in [25] or [3] for the charged case to situations with Schwarzschild singularity at the center. We state the following lemma.

Lemma 4.9. *Let $M_0 \geq 0$ and let $\lambda, \mu \in C^2((2M_0, \infty))$ be a solution of the spherically symmetric, static Einstein equations (3.10) – (3.12) with a Schwarzschild singularity with mass parameter M_0 at the center. Let the matter quantities $\varrho, p, p_T \in L^1_{loc}((2M_0, \infty))$ be zero on $\left(2M_0, \frac{9}{4}M_0\right]$, fulfill the energy condition*

$$p + 2p_T \leq \varrho, \tag{4.44}$$

and the generalized TOV equation

$$p'(r) = -\mu'(\varrho + p) - \frac{2}{r}(p - p_T) \tag{4.45}$$

a.e. in their domain of definition. Then the inequality

$$\frac{2M(r)}{r} \leq \frac{8}{9} \tag{4.46}$$

holds for all $r \in \left[\frac{9}{4}M_0, \infty\right)$ where $M(r)$ is given in (4.43).

Remark 4.10. *If one wants to consider a spacetime without Schwarzschild singularity, thus the case $M_0 = 0$, the lemma is stating that $\frac{2m(r)}{r} \leq \frac{8}{9}$ for all $r \in [0, \infty)$ where $m(r)$ is the quasilocal mass given by (4.42).*

Proof. For the proof of the lemma we apply techniques that are already used in [25] to prove the Buchdahl inequality for globally regular solutions without Schwarzschild singularity. At first we introduce the variables

$$x = \frac{2(M_0 + m(r))}{r}, \quad y = 8\pi r^2 p(r). \tag{4.47}$$

Note that $x < 1$ and $y \geq 0$. The first inequality must hold true since otherwise the lapse function e^{μ} would not stay bounded. We integrate the Einstein equation (3.10) over the interval $\left(\frac{9M_0}{4}, r\right)$ and obtain

$$e^{-2\lambda} = 1 - \frac{9M_0}{4r}\left(1 - e^{-2\lambda_0}\right) - \frac{8\pi}{r}\int_{\frac{9M_0}{4}}^{r} s^2 \varrho(s) \mathrm{d}s, \tag{4.48}$$

where $\lambda_0 = \lambda\left(\frac{9M_0}{4}\right)$. Since we have vacuum on $\left(2M_0, \frac{9M_0}{4}\right)$, on this interval the metric is given by the Schwarzschild metric and one can compute λ_0 explicitly. One finds that $e^{-2\lambda} = 1 - \frac{2(M_0 + m(r))}{r}$. We plug this into the other Einstein equation (3.11) and obtain the differential equation

$$\mu'(r) = \frac{1}{1 - \frac{2(M_0+m(r))}{r}}\left(4\pi r p + \frac{M_0 + m(r)}{r^2}\right). \tag{4.49}$$

Now we introduce $\beta = 2\ln(r)$ and want to consider the curve $\left(x\left(e^{\beta/2}\right), y\left(e^{\beta/2}\right)\right)$ parameterized by β in $[0,1)\times[0,\infty)$. In the following a dot denotes the derivative with respect to β. Using the Einstein equations and the TOV equation (4.45) one checks that x and y satisfy the equations

$$8\pi r^2 \varrho = 2\dot{x} + x, \tag{4.50}$$
$$8\pi r^2 p = y, \tag{4.51}$$
$$8\pi r^2 p_T = \frac{x+y}{2(1-x)}\dot{x} + \dot{y} + \frac{(x+y)^2}{4(1-x)}. \tag{4.52}$$

By virtue of these equations (4.50) – (4.52) the condition $p + 2p_T \leq \varrho$ can be written in the form

$$(3x-2+y)\dot{x} + 2(1-x)\dot{y} \leq -\frac{\alpha(x,y)}{2}, \quad \alpha = 3x^2 - 2x + y^2 + 2y. \tag{4.53}$$

Next, one defines the quantity

$$w(x,y) = \frac{(3(1-x)+1+y)^2}{1-x} \tag{4.54}$$

and calculates the derivative with respect to β,

$$\dot{w} = \frac{4-3x+y}{(1-x)^2}((3x-2+y)\dot{x} + 2(1-x)\dot{y}) \leq -\frac{4-3x+y}{2(1-x)^2}\alpha(x,y), \tag{4.55}$$

depending on x and y. Since $0 \leq x < 1$ and $y \leq 0$ this quantity is decreasing along the curve (x,y) whenever $\alpha > 0$. Define the set

$$E = \{(x,y)\,|\,0 \leq x \leq 1, y \geq 0, \alpha(x,y) \leq 0\}. \tag{4.56}$$

So all parts of the curve $(x(\beta), y(\beta))$ where w is not decreasing are lying in E. This implies

$$w \leq \max_E w(x,y). \tag{4.57}$$

We consider partial derivatives of w with respect to x and y, respectively. We have

$$\frac{\partial w}{\partial x} = \frac{-9(1-x)^2 + 1 + 2y + y^2}{(1-x)^2}, \tag{4.58}$$
$$\frac{\partial w}{\partial y} = \frac{2(3(1-x)+1+y}{1-x}. \tag{4.59}$$

So there are no stationary points of w in the interior of E. Thus, the maximum is attained at the boundary ∂E of E. This maximum can be found with the method of Lagrange multipliers. Thus we consider the function $\tilde{w}(x,y,\lambda) = w(x,y) + \lambda\alpha(x,y)$ and the system of equations

$$\partial_x \tilde{w}(x,y,\lambda) = 0, \quad \partial_y \tilde{w}(x,y,\lambda) = 0, \quad \partial_\lambda \tilde{w}(x,y,\lambda) = 0. \tag{4.60}$$

Eliminating the Lagrange multiplier λ in the first two equations yields the system

$$(1-x)\left[6\left(3(1-x) + 2\frac{(3x-1)(1-x)}{1+y}\right) + 4(3x-1)\right] = 0, \tag{4.61}$$
$$3x^2 - 2x + y^2 + 2y = 0. \tag{4.62}$$

We calculate y from (4.62) taking the solution larger than zero and plug it into equation (4.61). One is left with a single equation for x that is solved by $x = 0$. Then equation (4.62) yields that also $y = 0$. So we have

$$w(x,y) = \frac{\left(3\left(1 - \frac{2(M_0+m(r))}{r}\right) + 8\pi r^2 p(r)\right)^2}{1 - \frac{2(M_0+m(r))}{r}} \leq w(0,0) = 16. \tag{4.63}$$

Since $p > 0$ one can omit the term containing p and some further simplifications yield

$$\frac{2(M_0 + m(r))}{r} \leq \frac{8}{9} \tag{4.64}$$

which was the assertion of this lemma. □

4.5 Numerical results

In this subsection examples of static solutions of the spherically symmetric Einstein-Vlasov system to different parameter choices and boundary conditions are presented. The data has been calculated numerically using methods similar to those of [1]. The purpose of this discussion is to illustrate the families of solutions that occur and how the parameters for example in the ansatz Φ of the matter distribution function f affect the solution. We discuss different classes of matter configurations as well as solutions with a black hole at the center.

For the different numerical calculations we choose for f an ansatz of the form (3.39). It is advantageous for the calculations to introduce a different variable u for the metric coefficient μ, given by $u = e^{-y}$. The differential equation that has to be integrated reads

$$u'(r) = \frac{u(r)}{1 - \frac{8\pi}{r}\int_0^r s^2 \tilde{G}_\phi(s, u(s))\mathrm{d}s}\left(4\pi r \tilde{H}_\phi(r, u(r)) + \frac{4\pi}{r^2}\int_0^r s^2 \tilde{G}_\phi(s, u(s))\mathrm{d}s\right), \tag{4.65}$$

where $\tilde{G}_\phi(r, u(r)) = G_\phi(r, -\ln(u))$ and $\tilde{H}_\phi(r, u(r)) = H_\phi(r, -\ln(u))$ and G_ϕ and H_ϕ are defined in equations (3.46) and (3.47), respectively. The matter quantities ϱ and p and the quasilocal mass $m(r)$ have been calculated using Simpson's rule and the integration of equation (4.65) was performed using the Euler method. Also a variable step width has been used like in [6]. Given a value for the second derivative $\varrho''(r)$ of the energy density and a maximum step width $h_{\max}$ the variable step width h_n is calculated by $h_n = h_{\max}/\ln(e + \varrho''(r_n))$.

The quantity $a(r)$ defined by $a(r) = u(r)\sqrt{1 + \frac{L_0}{r^2}}$ plays an important role. As stated by lemma 3.2, (iii) we have vacuum if and only if $a(r) \geq 1$. The behavior of this quantity depends strongly on the choice of L_0. If one chooses $L_0 = 0$ the quantity $a(r)$ will be smaller than 1 until a certain radius R_0 and will stay larger forever after, since $u'(r) > 0$ by means of equation (4.65). But if $L_0 > 0$ there will be a region $[0, r_+]$ where $a(r) \geq 1$. In that region there is vacuum, i.e. $\varrho = 0$ and u stays constant. So the numerical integration of equation (4.65) has to start not until $r = r_+$. As soon as $u(r)$ exceeds the value 1 no matter can appear any more. So the numerical integration can be stopped and u can be continued by a Schwarzschild solution.

In the following we will distinguish several types of solutions by the shape of the energy density and its support. Essentially there are two classes of solutions. We call them *shells* and *non-shells* and they are characterized as follows. Eventually we will discuss an example for a solution with a black hole at the center as constructed in subsection 4.3.

One obtains non-shell solutions by setting $L_0 = 0$. All ansätze with $L_0 = 0$ lead to energy densities ϱ whose support is an interval $[0, R_0]$. If the solution is isotropic, i.e. $\ell = L_0 = 0$, then ϱ is strictly decreasing on its support $[0, R_0]$. Figure 1 shows an energy density in this setting. The configuration just described is the most easy one to handle. Actually, the matter behaves

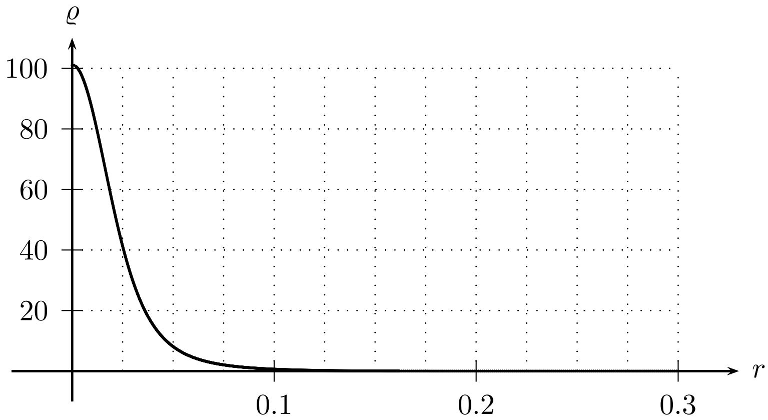

Figure 1: Energy density ϱ in the isotropic case, i.e. $k = 0$, $\ell = 0$, $L_0 = 0$, $y(0) = 0.4$. The matter is strongly peaked around $r = 0$. The support of ϱ is bounded by $R_0 \approx 0.162$.

like an ideal fluid only admitting ball configurations.

In the situation where $L_0 = 0$ but $\ell \neq 0$ the energy density ϱ and the radial pressure p are not monotonous any more. In fact, one observes the occurrence of peaks. It is important to note that these peaks are never separated by vacuum regions. Even though ϱ might become very small it always stays strictly positive on the interval $[0, R_0]$. Moreover, one observes that the smaller one chooses the initial value u_0 of u the more peaks one will obtain [1]. This also holds for the shell case which will be discussed later. A situation of a non-shell with two peaks is shown in figure 2. Even though for small r there is a region that happens to resemble a vacuum region the energy density ϱ is strictly positive until the end of its support at $r = R_0$. It is also remarkable that the size of the peaks tends to decay very fast for larger radii. Furthermore, the solution shows a so called *tail*, i.e. after the last peak there is a large region filled with a little matter.

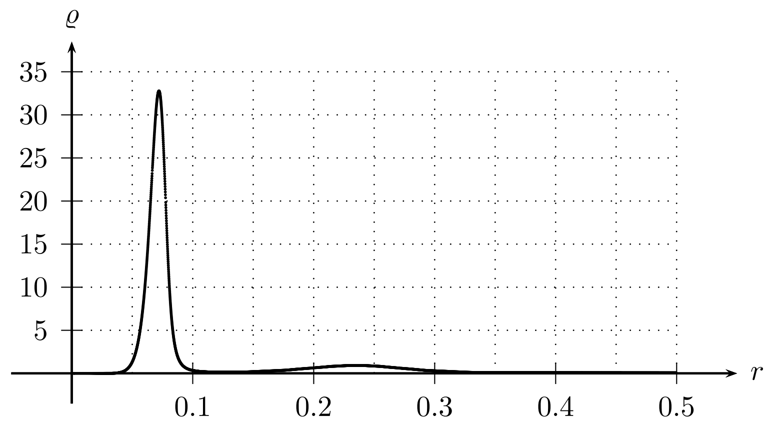

Figure 2: A non-shell with two peaks and a tail, $k = 0$, $\ell = 5.5$, $L_0 = 0$, $y(0) = 0.1$. After the second peak there is still a large area with a little matter, a so called tail. The support of ϱ ends only at $R_0 \approx 2.486$.

Setting $L_0 > 0$ leads to shell solutions. As already mentioned a vacuum region around $r = 0$ occurs. The support of the energy density ϱ will be contained in an interval $[r_+, R_0]$ where $r_+ > 0$. However, also within this interval the matter regions may be separated by vacuum areas. Thus the support of ϱ is not connected any more. Figure 3 shows a pure shell with a single peak. A shell with two peaks that are separated by vacuum is depicted in figure 4. Of course, the peaks of shells are not necessarily separated by vacuum.

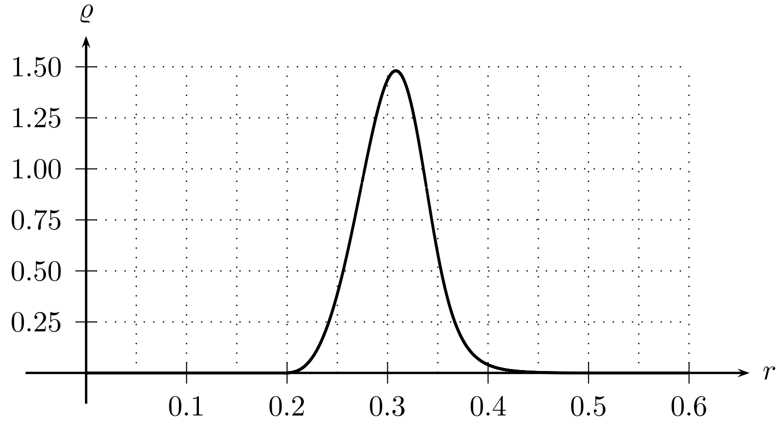

Figure 3: A pure shell with $k = 0$, $\ell = 1.5$, $L_0 = 0.2$, $y(0) = 0.4$. The energy density ϱ is supported on $(0.198, 0.506)$.

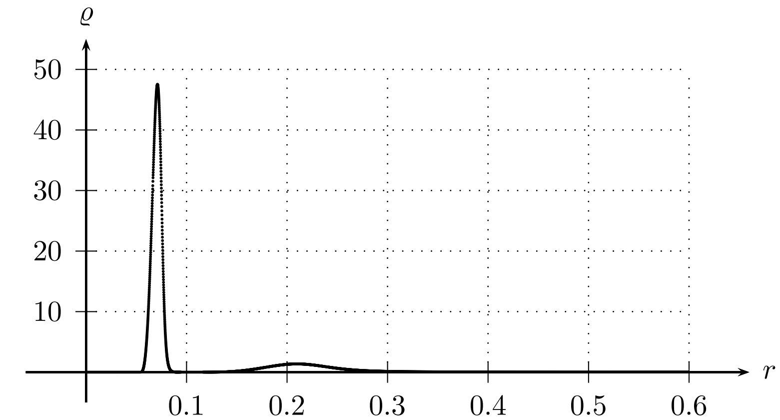

Figure 4: A shell with a tail, $k = 0$, $\ell = 1.5$, $L_0 = 0.2$, $y(0) = 0.12$. There is a vacuum region before the first peak and between the two peaks, as well. Before the support of the matter ends there is a quite large region with little matter, a so called tail. The energy density is supported on $(0.055, 0.092) \cup (0.118, 2.08)$.

Finally we construct a solution with a black hole of mass M_0 at the center. The existence of such solutions has been proven in subsection 4.3. There is a positive lower bound on L_0 given by the inequality $L_0 > 16M_0^2$. The left graph in figure 5 shows the metric coefficient μ. The right graph shows the corresponding energy density ϱ. On the support of the energy density, μ is obtained by a numerical integration of equation (4.40). In the vacuum regions, μ is merely given by the Schwarzschild metric with mass parameter M_0 or $M = M_0 + 4\pi \int r^2 \varrho(r) \mathrm{d}r$, respectively (see the dashed lines in figure 5). As mentioned in remark 4.8 either the vacuum solution inside or outside the matter shell is shifted with respect to the Schwarzschild metric. This shift represents only a rescaling of the time variable and is determined by the cutoff energy E_0 in the ansatz

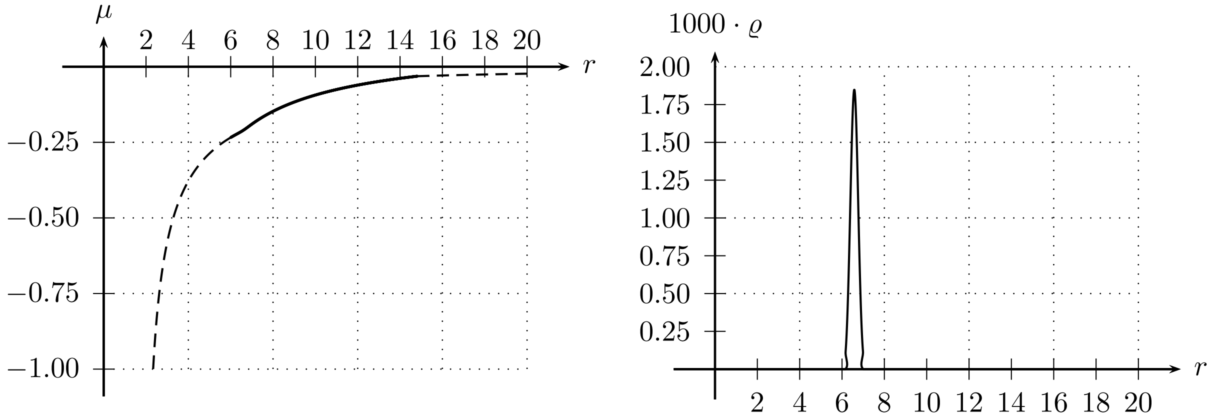

Figure 5: A spherically symmetric, static solution of the Einstein-Vlasov system, with a black hole of mass M_0 at the center. $M_0 = 1$, $k = 0$, $\ell = 1.5$, $L_0 = 18$. The left graph shows the metric coefficient μ and the right graph the energy density ϱ. The initial value for the integration is determined by the mass of the black hole. We have $E_0 = 0.969$ and the matter is supported in $(6.168, 6.99)$.

(3.39) of the matter distribution function f. In this example, E_0 has been chosen in a way that μ tends to zero as r goes to infinity.

5 Solutions with non-vanishing cosmological constant

This section is devoted to results concerning the Einstein-Vlasov system with non-vanishing cosmological constant. The presented results are the outcome of a joint work with Håkan Andréasson and David Fajman, published in [14].

5.1 Static, anisotropic globally regular solutions for $\Lambda > 0$

In this section we prove existence for globally regular, static solutions with $\Lambda > 0$. For this purpose, it is sufficient to consider equation (3.57),

$$y'(r) = -\frac{4\pi}{1 - \frac{\Lambda r^2}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s} \times \left(rH_\phi(r, y(r)) - \frac{r\Lambda}{12\pi} + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s \right), \tag{5.1}$$

because a solution to this equation already determines uniquely a solution to the full Einstein-Vlasov system (3.29) – (3.32). In the course of the proof we will refer to the corresponding solution of the Einstein-Vlasov system with $\Lambda = 0$ which we call a *background solution.* The existence, uniqueness and several other properties of this background solution have been proved and discussed in the previous section. In this context "corresponding" refers to a solution with the same initial value and the same ansatz for the matter distribution f. The boundary conditions may however differ since Λ changes the structure of the spacetime for large r. Quantities that belong to the solution with non-vanishing Λ will be subscripted with Λ, so for example we write y_Λ, ϱ_Λ and correspondingly. The proof has four steps. First we prove local existence using a contraction argument in lemma 5.1. Then, in lemma 5.2, we specify a continuation criterion for the local solution. In the next step we prove that for Λ small enough there will occur a vacuum region after a radius $R_{0\Lambda}$ and that $[0, R_{0\Lambda})$ is contained in the interval of existence of the solution μ_Λ of the Einstein-Vlasov system. Finally, we show that a globally defined C^1-solution can be obtained by continuing the metric with the Schwarzschild-de Sitter metric and the matter distribution with constant zero beyond $R_{0\Lambda}$.

So at first the following existence lemma is proven for small radii. This lemma corresponds to the first part of the proof of theorem 2.2 in [33] for the case $\Lambda = 0$.

Lemma 5.1. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let G_ϕ, H_ϕ be defined by equations (3.46) and (3.47), respectively. Then for every $y_0 \in \mathbb{R}$ and every $\Lambda > 0$ there is a $\delta > 0$ such that there exists a unique solution $y_\Lambda \in C^2([0, \delta])$ of equation (5.1) with initial value $y_\Lambda(0) = y_0$.*

Proof. We consider the equation (5.1) and integrate it using the initial condition $y_\Lambda(0) = y_0$. The following fixed point problem is obtained,

$$y_\Lambda(r) = (Ty_\Lambda)(r), \quad r \geq 0 \tag{5.2}$$

where the operator T is given by

$$(Tu)(r) := y_0 - \int_0^r \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma, u(\sigma))\mathrm{d}\sigma} \times \left(sH_\phi(s, u(s)) - \frac{s\Lambda}{12\pi} + \frac{1}{s^2}\int_0^s \sigma^2 G_\phi(\sigma, u(\sigma))\mathrm{d}\sigma \right) \mathrm{d}s. \tag{5.3}$$

This operator is considered on the set

$$M := \Big\{ u : [0, \delta] \to \mathbb{R} \mid u(0) = y_0, y_0 - 1 \leq u(r) \leq y_0 + 1, \frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, u(s))\mathrm{d}s \leq c < 1, r \in [0, \delta] \Big\}. \tag{5.4}$$

As carried out in detail in the appendix, section D, it is shown that T acts as a contraction on M. This implies (by the Banach fixed point theorem) that there exists $y_\Lambda \in M$ such that $Ty_\Lambda = y_\Lambda$. Differentiability of y_Λ follows from the structure of T. The differentiation with respect to r yields that y_Λ solves equation (5.1) on the interval $[0, \delta]$. Away from the singularity $r = 0$, standard existence and uniqueness results are applied to extend y_Λ to a maximal solution on an interval $[0, R_c)$. Obviously, the boundary condition at $r = 0$ is satisfied. The regularity of the functions G_ϕ and H_ϕ implies that $y_\Lambda \in C^2((0, R_c))$, (cf. [34]) and it can be shown that the second derivative continuously extends to $r = 0$ and $y'_\Lambda(0) = 0$. □

The characterization of the maximal existence interval of the local solutions constructed in lemma 5.1 is relevant to assure for the existence of solutions beyond the non-vacuum region proven in the next step. For this purpose a continuation criterion is proved in the following lemma 5.2. Define R_c as the maximal radius such that the unique local solution y_Λ of equation (5.1) can be continued to the interval $[0, R_c)$. If $R_c < \infty$ then either

$$\liminf_{r \to R_c} \left(1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s \right) = 0 \tag{5.5}$$

or

$$\limsup_{r\to R_c} |y_\Lambda(r)| = \infty, \tag{5.6}$$

which follows straightforward from equation (5.1). In the case $\Lambda = 0$ it can be shown that $R_c = \infty$, cf. [34]. In the case of a positive cosmological constant, R_c may be finite. However, the solution y_Λ exists at least as long as the denominator of the right hand side of equation 5.1 is strictly larger than zero. The following lemma formulates this assertion.

Lemma 5.2. *Let $y_0 \in \mathbb{R}$ and let $R_c > 0$ be the largest radius such that the unique local C^2-solution y_Λ of equation (5.1) with $y_\Lambda(0) = y_0$ exists on the interval $[0, R_c)$. Then there exists $R_D \leq R_c$ such that*

$$\liminf_{r\to R_D} \left(1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right) = 0. \tag{5.7}$$

Remark 5.3. *We can a priori not exclude the case $R_c = \infty$ which would however not occur due to the Λ term.*

Proof. Assume

$$1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s > 0 \tag{5.8}$$

for all $r \in [0, R_c)$. Otherwise $R_D < R_c$ (with R_D characterized as above) occurs due to the continuity of y_Λ and G_ϕ and the lemma follows. Assume now that the assertion of the lemma does not hold, i.e. there is a constant $a > 0$ such that

$$1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s \geq a \tag{5.9}$$

for all $r \in [0, R_c)$. First we show that this implies the existence of a $C > 0$ such that for all $r \in [0, R_c)$ we have $|y'_\Lambda(r)| \leq C$. Therefore we consider

$$|y'_\Lambda(r)| \leq \frac{4\pi}{a}\left(rH_\phi(r, y_\Lambda(r)) + \frac{r\Lambda}{12\pi} + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right). \tag{5.10}$$

Here it is used that H_ϕ and G_ϕ are positive. It is obvious that the second term, $\frac{r\Lambda}{12\pi}$, is bounded on the interval $[0, R_c)$. We show that the right hand side of (5.10) is bounded on this interval. Assume the opposite,

$$\limsup_{r\to R_c} H_\phi(r, y_\Lambda(r)) = \infty \quad \text{or} \quad \limsup_{r\to R_c} \int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s = \infty. \tag{5.11}$$

The second possibility implies $\limsup_{r\to R_c} G_\phi(r, y_\Lambda(r)) = \infty$. On the interval $[0, R_c)$ we have the upper bounds $H_\phi(r, y_\Lambda(r)) \leq H_\phi(R_c, y_\Lambda(r))$ and $G_\phi(r, y_\Lambda(r)) \leq G_\phi(R_c, y_\Lambda(r))$, cf. lemma 3.2, (ii). And since $H_\phi(r, y)$ and $G_\phi(r, y)$ are increasing functions in y (cf. lemma 3.2) this in turn implies

$$\limsup_{r\to R_c} y_\Lambda(r) = \infty. \tag{5.12}$$

It follows that for all $\varepsilon > 0$ sufficiently small there exists $r \in (R_c - \varepsilon, R_c)$ such that $y'_\Lambda(r) > 0$ which on the other hand implies

$$rH_\phi(r, y_\Lambda(r)) + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s < \frac{r\Lambda}{12\pi}, \tag{5.13}$$

by equation (5.1) for y'_Λ. This contradicts the assumption that either $H_\phi(r, y_\Lambda(r))$ or the integral $\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s$ diverge as the right hand side of (5.13) is bounded. Thus $|y'_\Lambda(r)|$ is bounded on $[0, R_c)$.

In the remainder of this proof it is shown that the solution can be continued beyond R_c which yields the desired contradiction. To achieve this, similar methods as in the proof of lemma 5.1 will be used. For $\delta, \varepsilon > 0$, $\delta > \varepsilon$ define $y_\varepsilon = y_\Lambda(R_c - \varepsilon)$, the interval I_ε containing R_c by $I_\varepsilon = [R_c - \varepsilon, R_c - \varepsilon + \delta]$, and

$$u_y(r) := \begin{cases} y_\Lambda(r); & r \in [0, R_c - \varepsilon] \\ u(r); & r > R_c - \varepsilon \end{cases}. \tag{5.14}$$

Consider the operator

$$\begin{aligned}(T_\varepsilon u)(r) = y_\varepsilon + \int_{R_c-\varepsilon}^r & \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma))\mathrm{d}\sigma} \\ & \times \left(sH_\phi(s, u(s)) - \frac{s\Lambda}{12\pi} + \frac{1}{s^2}\int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma))\mathrm{d}\sigma\right)\mathrm{d}s\end{aligned} \tag{5.15}$$

acting on the set

$$\begin{aligned} M_\varepsilon = \Big\{ u : I_\varepsilon \to \mathbb{R} \mid\ & u(R_c - \varepsilon) = y_\varepsilon, y_\varepsilon - 1 \le u(r) \le y_\varepsilon + 1, \\ & \frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, u_y(s))\mathrm{d}s \le c < 1, r \in I_\varepsilon \Big\}. \end{aligned} \tag{5.16}$$

Using (5.9) and $|y'_\Lambda(r)| < C$ on $[0, R_c)$ for a $C > 0$ one can prove that T_ε acts as a contraction on M_ε. In virtue of Banach's fixed point theorem the operator T_ε has a fixed point $y_\varepsilon \in M_\varepsilon$ such that $(y_\varepsilon)_y$ defined by (5.14) solves equation (5.1) on the interval $(0, R_c - \varepsilon + \delta)$. But this contradicts the definition of R_c and the lemma follows. □

In the next lemma we prove that within the interval of existence $[0, R_c)$ of the solution y_Λ there will be a radius $R_{0\Lambda}$ where the matter stops. Beyond this radius vacuum is on hand.

Lemma 5.4. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let y be the unique global C^1-solution of equation (5.1) in the case $\Lambda = 0$ where $y(0) = y_0 > 0$ (cf. section 4). As proved in [34], $\varrho(r) = G_\phi(r, y(r))$ has bounded support which is the interval $[0, R_0)$ where $y(R_0) = 0$ defines R_0 uniquely. Let y_Λ be the unique C^2-solution of equation (5.1) with $\Lambda > 0$ and $y_\Lambda(0) = y(0)$ that according to lemma 5.1 exists at least on an interval $[0, \delta]$ for a certain $\delta > 0$ and let ρ_Λ be the energy density corresponding to y_Λ.*

Then y_Λ exists at least on $[0, R_0 + \Delta R]$ and the support of ϱ_Λ is bounded by some $R_{0,\Lambda} < R_0 + \Delta R$ if Λ and $\Delta R > 0$ are chosen such that

$$0 < \Lambda < \min\left\{ \frac{|y(R_0 + \Delta R)|}{C_y(R_0 + \Delta R)}, \frac{\frac{1}{18}}{C_v(R_0 + \Delta R)} \right\} \tag{5.17}$$

holds. The constants $C_y(r)$ defined in equation (5.27) and $C_v(r)$ defined in equation (5.25) are fully determined by the background solution y.

Proof. We define

$$m(r) = 4\pi \int_0^r s^2 \varrho(s)\mathrm{d}s, \qquad m_\Lambda(r) = 4\pi \int_0^r s^2 \varrho_\Lambda(s)\mathrm{d}s, \tag{5.18}$$

$$v(r) = 1 - \frac{2m(r)}{r}, \qquad v_\Lambda(r) = 1 - \frac{r^2\Lambda}{3} - \frac{2m_\Lambda(r)}{r}. \tag{5.19}$$

Consider the continuous function v_Λ. Note that $v_\Lambda(0) = 1$. We define $r^* = r^*(\Lambda)$ as the smallest radius where $v_\Lambda(r) = \frac{1}{18}$. Lemma 5.2 assures that $r^* < R_c$, i.e. r^* is well defined. We can write

$$r^* = \inf\{r \in [0, R_c) \,|\, v_\Lambda(r) = 1/18\}. \tag{5.20}$$

In addition, we define $\tilde{r}$ as the radius until that $|y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|$. The right hand side of this inequality is given by the background solution y, which exists globally, i.e.

$$\tilde{r} = \sup\{r \in [0, R_c] \,|\, |y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|\}. \tag{5.21}$$

Note that $y(0) = y_\Lambda(0) = y_0$, so $0 < \tilde{r}$ by continuity of y and y_Λ. Let

$$\tilde{r}^* := \min\{r^*, \tilde{r}\}. \tag{5.22}$$

Choosing Λ s.t. (5.17) holds, we will show that $\tilde{r}^* > R_0 + \Delta R$. We assume the opposite, $\tilde{r}^* \leq R_0 + \Delta R$, and consider the sum $|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|$ on the interval $[0, \tilde{r}^*]$. By the mean value theorem we have

$$|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| = \left(\left|\partial_y G_\phi(r,y)\big|_{u_1}\right| + \left|\partial_y H_\phi(r,y)\big|_{u_2}\right| \right) |y_\Lambda(r) - y(r)| \tag{5.23}$$

where $u_1, u_2 \in [y(r), y_\Lambda(r)]$ are chosen appropriately. From the estimate (E.2) in Appendix B we have that for $r \leq \tilde{r}^*$

$$|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \leq \Lambda C_{gh}(\tilde{r}^*), \tag{5.24}$$

where C_{gh} is defined in (E.2). Note that $C_{gh}(r)$ is increasing in r. Still on $[0, \tilde{r}^*]$ we compute

$$\begin{aligned} |v(r) - v_\Lambda(r)| &\leq \frac{r^2\Lambda}{3} + \frac{2}{r}|m_\Lambda(r) - m(r)| = \frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2 |\varrho_\Lambda(s) - \varrho(s)|\mathrm{d}s \\ &\leq \left(\frac{(\tilde{r}^*)^2}{3} + \frac{8\pi}{3}(\tilde{r}^*)^2 C_{gh}(\tilde{r}^*) \right) \Lambda =: C_v(\tilde{r}^*)\Lambda \end{aligned} \tag{5.25}$$

Since we have $v(r) \geq \frac{1}{9}$ (Buchdahl inequality, cf. lemma 4.9) and $\Lambda < \frac{1/18}{C_v(R_0+\Delta R)}$ by choice of Λ we can conclude

$$v_\Lambda(r) \geq v(r) - \Lambda C_v(\tilde{r}^*) > \frac{1}{9} - \frac{1/18}{C_v(R_0 + \Delta R)} C_v(\tilde{r}^*) \geq \frac{1}{18} \tag{5.26}$$

on $[0, \tilde{r}^*]$ since $C_v(\tilde{r}^*) < C_v(R_0 + \Delta R)$ because $C_v(r)$ is increasing and $\tilde{r}^* \leq R_0 + \Delta R$ by assumption.

We also consider the distance between y and y_Λ on $[0, \tilde{r}^*]$. Following the procedure depicted in section E of the appendix one obtains

$$\begin{aligned}|y_\Lambda(r) - y(r)| \leq \Lambda \Bigg(3r^2 + 29\pi r^4 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\Bigg) \\ + 72\pi \left(r + 24\pi r^2 \left(H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0)\right)\right) \int_0^r C_{gh}(s) \Lambda \mathrm{d}s \\ =: C_y(r)\Lambda \leq C_y(\tilde{r}^*)\Lambda.\end{aligned} \tag{5.27}$$

Since $C_y(\tilde{r}^*) \leq C_y(R_0 + \Delta R)$ and $\Lambda < \frac{|y(R_0+\Delta R)|}{C_y(R_0+\Delta R)}$ on $[0, \tilde{r}^*]$ by assumption, the relation

$$|y_\Lambda(r) - y(r)| < |y(R_0 + \Delta R)| \tag{5.28}$$

already holds. Equations (5.26) and (5.28) state that $v_\Lambda(\tilde{r}^*) > \frac{1}{18}$ and $|y_\Lambda(\tilde{r}^*) - y(\tilde{r}^*)| < |y(R_0 + \Delta R)|$, respectively on the interval $[0, \tilde{r}^*]$, which is a contradiction to the definition of $\tilde{r}^*$. Thus we have $\tilde{r}^* > R_0 + \Delta R$ as desired.

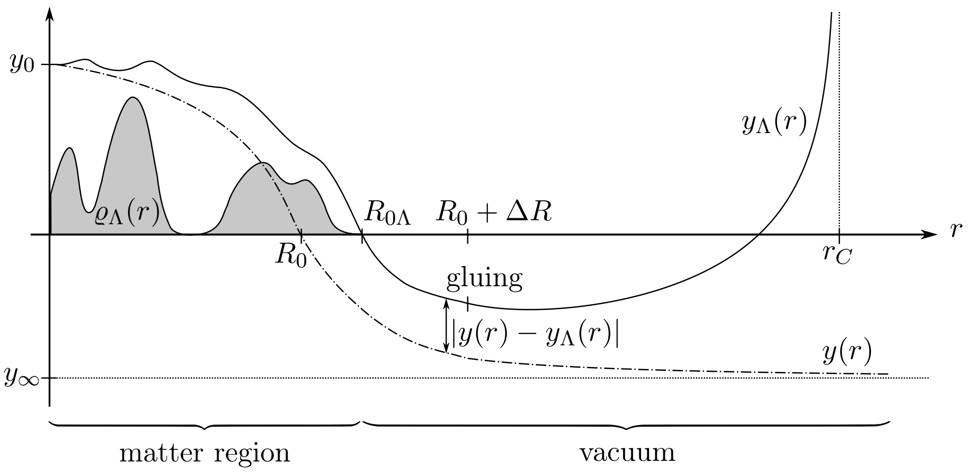

Figure 6: Qualitative sketch of the situation in the proof of lemma 5.4

We have shown that y_Λ exists at least on $[0, R_0+\Delta R]$ as the continuation criterion applies and from equation (5.28) we already know that $y_\Lambda(R_0+\Delta R) < 0$. Since y_Λ is continuous it has at least one zero at in the interval $(R_0, R_0 + \Delta R)$. In particular there exists an interval $(R_{0\Lambda}, R_0 + \Delta R)$ where y_Λ is strictly smaller than zero. $R_{0\Lambda}$ is the largest zero of y_Λ in $(R_0, R_0 + \Delta R)$. So the spatial support of f_Λ is contained in the interval $[0, R_{0,\Lambda})$ and this implies the assertion. □

In the last two subsections we have seen that for suitably chosen Λ there exists a unique solution y_Λ to equation (5.1) on the interval $[0, R_0 + \Delta R]$ for some $\Delta R > 0$. This solution uniquely induces a solution μ_Λ, λ_Λ of the equations (3.29), (3.30) on $[0, R_0 + \Delta R]$ whose energy density ϱ_Λ is of bounded support in space. By gluing a Schwarzschild-de Sitter metric to this solution one can construct a global static solution to the Einstein-Vlasov system.

Theorem 5.5. *Let $\Phi : \mathbb{R}^2 \to [0,\infty)$ be of the form (3.39). For every initial value $\mu_0 < 0$ there exists a constant $C = C(\mu_0, \phi) > 0$ such that for every $0 < \Lambda < C$ there exists a unique global solution $\mu_\Lambda, \lambda_\Lambda \in C^2([0,\infty))$, $f_\Lambda \in C^0([0,\infty))$ of the static, spherically symmetric Einstein-Vlasov system (3.29) – (3.32) with $\mu_\Lambda(0) = \mu_0$, and $\lambda_\Lambda(0) = 0$ such that the support of the distribution function f is bounded. This solution coincides with the Schwarzschild-de Sitter metric in the vacuum region.*

Proof. According to Lemma 5.1 there exists a C^2-solution y_Λ of equation (5.1) on a small interval $[0,\delta]$. In the proof of Lemma 5.4 we saw that this solution can be extended at least until $r = R_0 + \Delta R$ for any ΔR if one chooses Λ small enough. Beyond the support of ϱ_Λ and p_Λ, thus for $r \in [R_{0,\Lambda}, R_0 + \Delta R]$, equation (5.1) takes the form

$$y'_\Lambda(r) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right) \tag{5.29}$$

where $M = m_\Lambda(R_{0,\Lambda})$. This equation is solved by the (shifted) Schwarzschild-de Sitter metric, whose corresponding y-coefficient y_S is given by

$$y_S(r) = -\frac{1}{2}\ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right) - \ln\left(e^{-\lambda(R_{0,\Lambda})}\right). \tag{5.30}$$

The shift has been chosen such that y_Λ can be extended by y_S as a C^2-solution of equation (5.1) on $[0,\infty)$ using a modified ansatz for the matter distribution f_Λ. Namely, for $r > R_0 + \Delta R$ we drop the original ansatz Φ for f_Λ and continue f_Λ by the constant zero function, i.e.

$$f_\Lambda(x,v) = \begin{cases} [1-\varepsilon e^{-y}]_+^k\,[L-L_0]_+^\ell, & r \in [0, R_0+\Delta R] \\ 0, & r > R_0 + \Delta R \end{cases}. \tag{5.31}$$

Obviously f_Λ is continuous since $f_\Lambda(r) = 0$ already on $(R_{0,\Lambda}, R_0 + \Delta R)$ but $\frac{\mathrm{d}}{\mathrm{d}r} f_\Lambda(x,v)$ is not continuous in general.

Via $\mu_\Lambda = \ln(E_0) - y_\Lambda$ and

$$e^{-2\lambda_\Lambda} = 1 - \frac{r^2\Lambda}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s \tag{5.32}$$

one can construct a local solution $\mu_\Lambda, \lambda_\Lambda \in C^2([0,R_c))$ of the Einstein equations (3.29), (3.30), where $R_c > R_0 + \Delta R$. This solution fulfills the boundary conditions $\lambda_\Lambda(0) = 0$, $\mu_\Lambda(0) = \ln(E_0) - y_0$, $\lambda'_\Lambda(0) = \mu'_\Lambda(0) = 0$. We now see that $E_0 = e^{\mu(R_{0,\Lambda})}$ and continue μ_Λ and λ_Λ with the Schwarzschild-de Sitter coefficients μ_S, λ_S given by

$$e^{2\mu_S} = e^{-2\lambda_S} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{5.33}$$

in a continuous way beyond $R_0 + \Delta R$. From equation (5.29) we deduce that also the derivatives of μ_Λ and μ_S can be glued together in a continuous way. The functions μ_Λ, λ_Λ, and f_Λ solve the Einstein-Vlasov system (3.29) – (3.32) globally. □

Remark 5.6. *In the isotropic case, i.e. $L_0 = \ell = 0$ in the ansatz (3.39) for the distribution function f, the matter quantities ϱ and p are monotonically decreasing. This implies that their support in space is a ball. In the anisotropic case however, so called shell solutions occur, cf. section 4.5 or [1]. The support of such matter shells is in general not connected.*

5.2 Static, anisotropic, globally regular solutions for $\Lambda < 0$

In this section an existence lemma for $\Lambda < 0$ is stated for small radii. This lemma corresponds to the first part of the proof of theorem 2.2 in [33] for the case $\Lambda = 0$.

Lemma 5.7. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let G_ϕ, H_ϕ be defined by equations (3.46) and (3.47), respectively. Then for every $y_0 \in \mathbb{R}$ and every $\Lambda < 0$ there exists a $\delta > 0$ such that there exists a unique solution $y_\Lambda \in C^2([0, \delta])$ of equation* (5.1) *with initial value $y_\Lambda(0) = y_0$.*

Proof. The proof works in an exactly analogue way as in the case $\Lambda > 0$. □

For negative cosmological constants the global existence of solutions can be proved in an analogue way as done in [34] for the case $\Lambda = 0$. After establishing the local existence of solutions analog to the $\Lambda > 0$ case, we show that the metric components stay bounded for all $r \in \mathbb{R}_+$ with an energy estimate. This will yield the global existence of solutions of the Einstein-Vlasov system with negative cosmological constant. In the next step we show by virtue of a suitable choice of an ansatz Φ for the matter distribution f, that the matter quantities ϱ and p are of bounded support.

In the following theorem the existence on spatial intervals of the form $\mathbb{R} \setminus [0, r_0)$, for $r_0 > 0$ is included for the purpose of applying the same theorem to the construction of static spacetimes with Schwarzschild singularities in the center (cf. theorem 5.16 in subsection 5.3). The solutions of interest here are those where the radius variable takes values in all of $\mathbb{R}_+$.

Theorem 5.8. *Let $\Lambda < 0$ and let $\Phi : \mathbb{R}^2 \setminus [0, \infty)$ be of the form (3.39) and let G_ϕ and H_ϕ be defined by equations (3.29) and (3.30). Then for every $r_0 \geq 0$ and $\mu_0, \lambda_0 \in \mathbb{R}$ there exists a unique solution $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0, \infty))$ of the Einstein equations (3.29), (3.30) with $\mu_\Lambda(r_0) = \mu_0$ and $\lambda_\Lambda(r_0) = \lambda_0$. One has $\lambda_0 = 0$ if $r_0 = 0$.*

Proof. We use an energy argument similar to [34]. Let $y_\Lambda \in C^2([r_0, r_0 + \delta])$ be the local solution of equation (5.1) with $y_\Lambda(r_0) = \ln(E_0)e^{-\mu_0}$. If $r_0 = 0$ the existence of this local solution is established by lemma 5.1 and in the case $r_0 > 0$ the existence of a local solution follows directly from the regularity of the right hand sides of (3.29) and (3.30). Let $[r_0, R_c)$ be the maximal interval of existence of this solution. By $\mu_\Lambda = \ln(E_0) - y_\Lambda$ and

$$e^{-2\lambda_\Lambda} = 1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(\frac{r_0}{2}\left(1 - e^{-2\lambda_0}\right) + 4\pi \int_{r_0}^{r} s^2 G_\phi(s, y_\Lambda(s))\mathrm{d}s\right) \tag{5.34}$$

one constructs a local solution $\mu_\Lambda, \lambda_\Lambda \in C^2([r_0, R_c])$ of equations (3.29) and (3.30). We define

$$w_\Lambda(r) = -\frac{\Lambda}{12\pi} + \frac{1}{r^3}\left(-\frac{r_0^3\Lambda}{24\pi} + \frac{r_0}{8\pi}\left(1 - e^{-2\lambda_0}\right) + \int_{r_0}^{r} s^2 \varrho_\Lambda(s)\mathrm{d}s\right). \tag{5.35}$$

The Einstein equation (3.29) implies

$$\mu_\Lambda'(r) = 4\pi r e^{2\lambda_\Lambda(r)}\left(p_\Lambda(r) + w_\Lambda(r)\right). \tag{5.36}$$

By adding equations (3.29) and (3.30) we have

$$\left(\mu_\Lambda'(r) + \lambda_\Lambda'(r)\right) = 4\pi r e^{2\lambda_\Lambda(r)}(p_\Lambda(r) + \varrho_\Lambda(r)). \tag{5.37}$$

We assume $R_c < \infty$ and consider the quantity $e^{\mu_\Lambda+\lambda_\Lambda}(p_\Lambda + w_\Lambda)$ on the interval $\left[\frac{R_c}{2}, R_c\right)$. On this interval, in particular away from the origin, a differential inequality will be established that will allow us to deduce that both μ_Λ and λ_Λ are bounded on $\left[\frac{R_c}{2}, R_c\right)$. Using the TOV equation (3.58) (lemma 3.3) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\mu_\Lambda+\lambda_\Lambda}(p_\Lambda + w_\Lambda)\right) &= e^{\mu_\Lambda+\lambda_\Lambda}\left(-\frac{2p_\Lambda}{r} - \frac{3w_\Lambda}{r} - \frac{\Lambda}{4\pi r} + \frac{2p_{T\Lambda}}{r} + \frac{\varrho_\Lambda}{r}\right) \\ &\leq C_1 e^{\mu_\Lambda+\lambda_\Lambda} = \frac{C_1}{p_\Lambda + w_\Lambda}(p_\Lambda + w_\Lambda)\, e^{\mu_\Lambda+\lambda_\Lambda} \\ &=: C_2\,(p_\Lambda + w_\Lambda)\, e^{\mu_\Lambda+\lambda_\Lambda}. \end{aligned} \tag{5.38}$$

In the course of this estimate we have used that $\frac{\Lambda}{4\pi r}$, $p_{T\Lambda}(r)/r$, and $\varrho_\Lambda(r)/r$ stay bounded for $r \in \left[\frac{R_c}{2}, R_c\right)$. The constant C_2 is bounded since $w_\Lambda(r) > 0$ for negative Λ. It follows

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(e^{\mu_\Lambda+\lambda_\Lambda}(p_\Lambda + w_\Lambda)\right) \leq C_2 \quad \Rightarrow \quad \lambda_\Lambda + \mu_\Lambda < \infty. \tag{5.39}$$

Equation (5.36) implies that $\mu'_\Lambda(r) \geq 0$ and therefore $\mu_\Lambda(r) \geq \mu_0$. We also have

$$e^{-2\lambda_\Lambda} \leq 1 + \frac{r^2|\Lambda|}{3} \leq \frac{3 + R_c^2|\Lambda|}{3} < \infty. \tag{5.40}$$

This in turn implies $\lambda_\Lambda > -\infty$ and we deduce from equation (5.39) that both μ_Λ and λ_Λ are bounded on $\left[\frac{R_c}{2}, R_c\right)$. This allows to continue μ_Λ and λ_Λ as C^2-solutions of the Einstein equations beyond R_c which contradicts its definition. So $R_c = \infty$. □

We prove in the following that the matter sources in the solutions, constructed in the previous theorem, are compactly supported in r and thereby yield physically reasonable solutions.

Theorem 5.9. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) and let $\mu_0 \in \mathbb{R}$, $r_0 \geq 0$ and let $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0, \infty))$, $f(x, v) = \Phi(E, L)$ be the unique global-in-r solution of the Einstein-Vlasov system (3.29) – ((3.32)) with negative cosmological constant where $\mu_\Lambda(0) = \mu_0$ such that $y_0 = \ln(E_0)e^{-\mu_0} > 0$. Then there exists $R_0 \in (r_0, \infty)$ such that the spatial support of f_Λ is contained in the interval $[r_0, R_0)$.*

Proof. Due to lemma 3.2, (iii) we have vacuum, i.e. $\varrho_\Lambda(r) = G_\phi(r, y_\Lambda(r)) = 0$ and $p_\Lambda(r) = H_\phi(r, y_\Lambda(r)) = 0$, if $y_\Lambda(r) \leq 0$. By assumption we have $y_\Lambda(0) > 0$. In the following we show that $\lim_{r\to\infty} y_\Lambda(r) < 0$. Since y_Λ is continuous and monotonically decreasing, this implies that y_Λ possesses a single zero R_0 and the support of the matter quantities ϱ_Λ and p_Λ is contained in $[0, R_0)$. We define $y_{\mathrm{vac},\Lambda}$ by

$$y_{\mathrm{vac},\Lambda} = y_0 - \frac{1}{2}\ln\left(1 - \frac{r^2\Lambda}{3}\right). \tag{5.41}$$

So we have

$$y'_{\mathrm{vac},\Lambda}(r) = -\frac{4\pi}{1 - \frac{\Lambda r^2}{3}}\left(-\frac{r\Lambda}{12\pi}\right) \tag{5.42}$$

and $y_{\mathrm{vac},\Lambda}(0) = y_\Lambda(0) = y_0$. Furthermore, since $y'_\Lambda(r) < y'_{\mathrm{vac},\Lambda}(r)$ which can be seen immediately by means of equation (5.1) we have

$$y_\Lambda(r) \leq y_{\mathrm{vac},\Lambda}(r) = y_0 - \frac{1}{2}\ln\left(1 + \frac{r^2|\Lambda|}{3}\right) \overset{r\to\infty}{\longrightarrow} -\infty < 0 \tag{5.43}$$

and the theorem follows. □

Remark 5.10. *The solution coincides with Schwarzschild-Anti de Sitter for $r \geq R_0$ if the continuity condition*

$$\mu_\Lambda(R_0) = \ln(E_0) - y_\Lambda(R_0) = \frac{1}{2} \ln \left(1 - \frac{R_0^2 \Lambda}{3} - \frac{2M}{R_0} \right) \tag{5.44}$$

is fulfilled, where $M = 4\pi \int_0^{R_0} s^2 \varrho_\Lambda(s) \mathrm{d}s$. So if y_0 is given, the corresponding value of E_0 in the ansatz Φ for the matter distribution f can be read off.

5.3 Solutions with a black hole at the center

In this section we construct spherically symmetric, static solutions of the Einstein-Vlasov system with non-vanishing cosmological constant that contain a black hole at the center. This means that there is a vacuum region around the origin where the metric is given by the Schwarzschild-de Sitter or Schwarzschild-Anti de Sitter metric, respectively, with non vanishing mass parameter M_0. We will consider both positive and negative cosmological constants. The construction for the case $\Lambda > 0$ makes use of the corresponding solutions with vanishing Λ. We call this solution, where $\Lambda = 0$, a background solution like in subsection 5.1. As in the previous sections we distinguish between the notation for the Einstein-Vlasov system with $\Lambda = 0$ and the corresponding system with $\Lambda \neq 0$. Every quantity of the system with $\Lambda \neq 0$ is denoted by an additional index Λ, e.g. we write $y_\Lambda(r)$, $\varrho_\Lambda(r)$, and correspondingly.

5.3.1 Matter shells immersed in Schwarzschild-de Sitter spacetime

The construction of the solution with $\Lambda > 0$ can be outlined as follows. In the vacuum case, i.e. when the right hand sides of the Einstein equations (3.29) and (3.30) are zero, the solutions are given by

$$e^{2\mu(r)} = 1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}, \quad e^{2\lambda(r)} = \left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r} \right)^{-1}, \quad r > r_{B\Lambda} \tag{5.45}$$

where $r_{B\Lambda}$ is defined to be the black hole event horizon, i.e. the smallest positive zero of $1 - r^2\Lambda/3 - 2M_0/r$. If one chooses L_0 and M_0 appropriately and Λ sufficiently small the following configuration is on hand. For small $r > r_{B\Lambda}$ one sets $f(x, v) \equiv 0$ and the metric is given by Schwarzschild-de Sitter. Thus one has the coefficients (5.45). Increasing the radius r one reaches an interval $[r_{-\Lambda}, r_{+\Lambda}]$ where also an ansatz $f(x, v) = \Phi(E, L)$ of the form (3.39) yields vacuum, i.e. $G_\phi(r, y(r)) = H_\phi(r, y(r)) = 0$. In this interval it is possible to glue to the Schwarzschild-de Sitter solution (5.45) a non vacuum solution solving the Einstein-Vlasov system. It will be shown that the matter quantities ϱ_Λ and p_Λ of this solution have finite support. Beyond the support of the matter quantities the solution will be continued again by Schwarzschild-de Sitter. For negative cosmological constants, globally defined solutions can be constructed, too. Like in the upper case, the black hole is surrounded by a vacuum shell which is on its part surrounded by a shell containing matter. In the outer region, we again have vacuum.

In the course of the proof of theorem 5.13 we will need a continuation criterion for the solution of the Einstein equations, namely the following statement.

Lemma 5.11. *Let $\Lambda > 0$, $\mu_0 \in \mathbb{R}$ and $M_0, r_0 > 0$. Let G_ϕ and H_ϕ defined by equations (3.46) and (3.47). Then the equation*

$$\begin{aligned}\mu'_\Lambda = &\frac{1}{1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(M_0 + 4\pi \int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)} \\ &\times \left(4\pi r H_\phi(s, \mu_\Lambda(s)) - \Lambda\left(\frac{r}{3} + \frac{r_0^3}{6r^2}\right)\frac{1}{r^2}\left(M_0 + 4\pi\int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)\right)\end{aligned} \tag{5.46}$$

has a unique local C^2-solution μ_Λ with $\mu(r_0) = \mu_0$ with maximal interval of existence $[r_0, R_c)$, $R_c > 0$. Moreover, there exists $R_D \leq R_c$ such that

$$\liminf_{r \to R_D} \left(1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_0^3}{r}\right) - \frac{2}{r}\left(M_0 + 4\pi \int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s))\mathrm{d}s\right)\right) = 0. \tag{5.47}$$

Proof. The local existence of a C^2-solution of equation (5.46) follows from the regularity of the right hand side. Basically one is in the situation of lemma 5.2 in the $\Lambda > 0$ with regular center case except for the fact that there are additional terms containing r_0 and M_0. But on a finite interval $[r_0, R_c)$ these terms are bounded and well behaved, i.e. the proof can be carried out in an analogue way. ☐

Remark 5.12. *Lemma 5.11 implies that if there exists a solution μ_Λ of equation (5.46) and the denominator of the right hand side of equation (5.46) is strictly larger than zero on an interval $[r_0, r)$, then μ_Λ can be extended beyond r as a solution of (5.46).*

The following theorem states the existence of solutions for $\Lambda > 0$ with a Schwarzschild singularity at the center.

Theorem 5.13. *Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be of the form (3.39) with $E_0 = 1$ and let $L_0, M_0 \geq 0$ such that $L_0 > 16M_0^2$. Choose $\Lambda > 0$ sufficiently small. Then there exists a unique solution $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda}, \infty))$, $f \in C^0((r_{B\Lambda}, \infty))$ of the Einstein-Vlasov system (3.29) – (3.32). The support of the matter quantities ϱ_Λ and p_Λ is contained in a shell $\{r_{+\Lambda} < r < R_{0\Lambda}\}$. In the complement of this shell the solution of the Einstein equations is given by the Schwarzschild-de Sitter metric.*

Proof. In the first part of the proof we consider the black hole region and show that the chosen parameters lead to a convenient configuration with opportune properties, cf. figure 7. Then we make use of the existence of a background solution and construct the desired solution μ_Λ in a rigorous way.

We define the functions

$$a(r) = \sqrt{1 - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}, \tag{5.48}$$

$$a_\Lambda(r) = \sqrt{1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}. \tag{5.49}$$

Moreover, we define r_- and r_+ to be the first and second radius where $a(r) = 1$, respectively, and $r_B := 2M_0$ to be the event horizon of the Schwarzschild singularity. Since $L_0 > 16M_0^2$ we have $r_B < r_- < r_+$ (cf. [34]). Note also that $r_+ > 4M_0 > \frac{18}{5}M_0$.

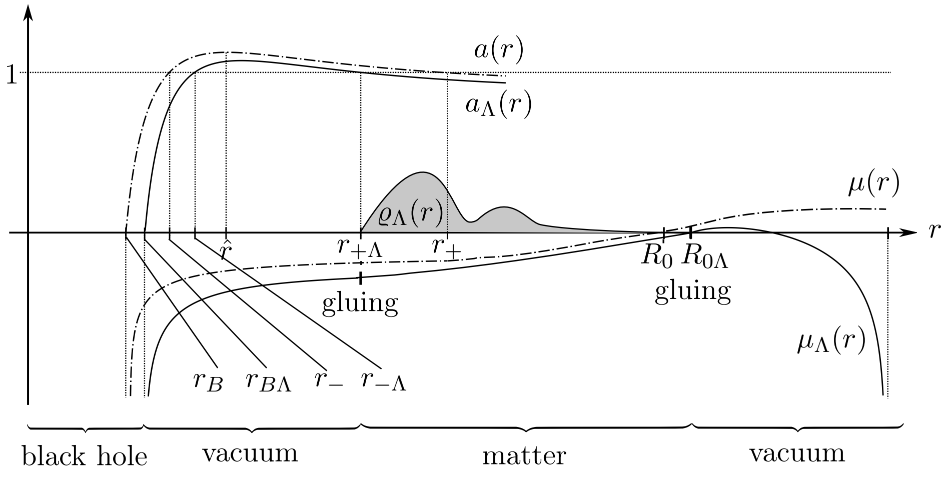

Figure 7: Qualitative sketch of a black hole configuration surrounded by a shell of matter

Since $9M_0^2\Lambda < 1$ by assumption (Λ is chosen to be small), there exists a black hole horizon $r_{B\Lambda}$ of the Schwarzschild-de Sitter metric with parameters M_0 and Λ. It can be calculated explicitly[1] by

$$r_{B\Lambda} = -\frac{2}{\sqrt{\Lambda}} \cos\left(\frac{1}{3}\arccos\left(-3M_0\sqrt{\Lambda}\right) + \frac{\pi}{3}\right). \tag{5.50}$$

Note that $r_B < r_{B\Lambda}$. We construct an upper bound to $r_{B\Lambda}$. Set $v(r) = 1 - \frac{2M_0}{r}$.

$$\begin{aligned} v(r_{B\Lambda}) &= \int_{r_B}^{r_{B\Lambda}} v'(s)\mathrm{d}s + \underbrace{v(r_B)}_{=0} \\ &\geq \int_{r_B}^{r_{B\Lambda}} \left(\inf_{s\in[r_B,r_{B\Lambda}]} v'(s)\right)\mathrm{d}s = (r_{B\Lambda} - r_B)v'(r_{B\Lambda}) \\ \Rightarrow \quad r_{B\Lambda} &\leq r_B + \frac{v(r_{B\Lambda})}{v'(r_{B\Lambda})} \end{aligned} \tag{5.51}$$

A short calculation yields $v(r_{B\Lambda}) = \frac{r_{B\Lambda}^2\Lambda}{3}$ and $v'(r_{B\Lambda}) = \frac{2M_0}{r_{B\Lambda}^2}$. One also checks by explicit calculation that $\frac{\mathrm{d}r_{B\Lambda}}{\mathrm{d}\Lambda} > 0$. So the distance

$$r_{B\Lambda} - r_B \leq \frac{r_{B\Lambda}^4\Lambda}{6M_0} \tag{5.52}$$

between the two horizons can be made arbitrarily small if Λ is chosen to be sufficiently small. In particular we need Λ to be small enough to assure for $r_{B\Lambda} < r_-$.

[1] To assure oneself of that one has chosen the right zero, using $\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$ one checks

$$\lim_{\Lambda\to 0} r_{B\Lambda} \overset{\text{l'Hôspital}}{=} \lim_{\Lambda\to 0} \frac{2\sin\left(\frac{1}{3}\arccos\left(-3M_0\sqrt{\Lambda}\right)+\frac{\pi}{3}\right)\frac{1}{3\sqrt{1-(3M_0\sqrt{\Lambda})^2}}\frac{3M_0}{2\sqrt{\Lambda}}}{1/\left(2\sqrt{\Lambda}\right)} = 2M_0.$$

Next we define $r_{-\Lambda}$ and $r_{+\Lambda}$ to be the first and second radius where $a_\Lambda(r) = 1$. Note that $a(r) > a_\Lambda(r)$ for all $r \in (r_{B\Lambda}, r_C)$, where r_C is the cosmological horizon of the vacuum solution, thus the second positive zero of $1 - r^2\Lambda/3 - 2M_0/r$. Between r_- and r_+ the function $a(r)$ has a unique maximum at $r = \hat{r}$, given by

$$\hat{r} = \frac{L_0 - \sqrt{L_0^2 - 12M_0^2 L_0}}{2M_0}. \tag{5.53}$$

We consider the distance between $a^2(r)$ and $a_\Lambda^2(r)$ at this radius $\hat{r}$:

$$|a^2(\hat{r}) - a_\Lambda^2(\hat{r})| = \Lambda\frac{\hat{r}^2 + L_0}{3}. \tag{5.54}$$

Choosing Λ sufficiently small one can attain $|a^2(\hat{r}) - a_\Lambda^2(\hat{r})| < a^2(\hat{r}) - 1$. This implies that $a_\Lambda(r) - 1$ has exactly two zeros in the interval (r_-, r_+). This in turn yields the desired configuration

$$2M_0 = r_B < r_{B\Lambda} < r_- < r_{-\Lambda} < \hat{r} < r_{+\Lambda} < r_+. \tag{5.55}$$

In the vacuum region $[r_{-\Lambda}, r_{+\Lambda}]$ the function $a_\Lambda(r)$ coincides with the expression $e^{-y_\Lambda(r)}\sqrt{1 + \frac{L_0}{r^2}}$. Lemma 3.2, (iii) implies that therefore for $r \in [r_{-\Lambda}, r_{+\Lambda}]$ also the ansatz Φ for the distribution function f yields $\varrho_\Lambda(r) = G_\phi(r, y_\Lambda(r)) = 0$ and $p_\Lambda(r) = H_\phi(r, y_\Lambda(r)) = 0$. So at $r = r_{+\Lambda}$ one can continue f by the ansatz Φ in a continuous way and for $r \geq r_{+\Lambda}$ the Einstein equations lead to the differential equation

$$\begin{aligned} \mu'_\Lambda = & \frac{1}{1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_{+\Lambda}^3}{r}\right) - \frac{2}{r}\left(\frac{r_{+\Lambda}}{2}(1 - e^{-2\lambda_0}) + 4\pi\int_{r_{+\Lambda}}^r s^2\varrho_\Lambda(s)\mathrm{d}s\right)} \\ & \times \left(4\pi r p - \Lambda\left(\frac{r}{3} + \frac{r_{+\Lambda}^3}{6r^2}\right) + \frac{r_{+\Lambda}}{2r^2}\left(1 - e^{-2\lambda_0}\right) + \frac{4\pi}{r^2}\int_{r_{+\Lambda}}^r s^2\varrho_\Lambda(s)\mathrm{d}s\right) \end{aligned} \tag{5.56}$$

where $\lambda_0 = \lambda(r_{+\Lambda})$.

There exists a background solution $\mu \in C^2((2M_0, \infty))$ to the Einstein equations with $\Lambda = 0$ (cf. [34]). For $r \in (2M_0, r_{+\Lambda}]$ this solution is given by the Schwarzschild metric and for $r > r_{+\Lambda}$ as a solution of equation (5.56) with $\Lambda = 0$. The background solution is continuous at $r_{+\Lambda}$ if

$$\frac{r_{+\Lambda}}{2}\left(1 - e^{2\lambda_0}\right) = M_0. \tag{5.57}$$

Furthermore, the background solution μ has the property that there exists $R_0 > 0$ such that the support of matter quantities ϱ and p is contained in the interval (r_+, R_0) (cf. [34]). In the remainder of the proof we show that using properties of this background solution μ one obtains a global solution μ_Λ of equation (5.56). We set

$$\mu_{0\Lambda} = \frac{1}{2}\ln\left(1 - \frac{r_{+\Lambda}^2\Lambda}{3} - \frac{2M_0}{r_{+\Lambda}}\right), \tag{5.58}$$

$$\mu_0 = \mu(r_{+\Lambda}) = \frac{1}{2}\ln\left(1 - \frac{2M_0}{r_{+\Lambda}}\right). \tag{5.59}$$

In the following we seek for a solution μ_Λ of equation (5.56) on on an interval beginning at $r = r_{+\Lambda}$ with the initial value $\mu_{0\Lambda}$ at given in (5.58) that we can glue to the vacuum solution on $(r_{B\Lambda}, r_{+\Lambda}]$. Note that $\mu_{0\Lambda} < 0$. Since there are no issues with an irregular center the local existence of μ_Λ on an interval $(r_{+\Lambda}, r_{+\Lambda} + \delta]$, $\delta > 0$ follows from the regularity of the right hand side of equation (5.56). So let $(2M_0, R_c)$ be the maximum interval of existence of μ_Λ. We define

$$v_{M_0}(r) = 1 - \frac{2}{r}\left(M_0 + 4\pi \int_{r_{+\Lambda}}^{r} s^2 \varrho(s)\mathrm{d}s\right), \tag{5.60}$$

$$v_{M_0\Lambda}(r) = 1 - \frac{\Lambda}{3}\left(r^2 - \frac{r_{+\Lambda}^3}{r}\right) - \frac{2}{r}\left(M_0 + 4\pi \int_{r_{+\Lambda}}^{r} s^2 \varrho_\Lambda(s)\mathrm{d}s\right) \tag{5.61}$$

as the denominator of the right hand side of equation (5.56). We set

$$\Delta v_0 := \frac{1}{18} v_{M_0,\Lambda}(r_{+\Lambda}) = \frac{1 - \frac{2M_0}{r_{+\Lambda}}}{18} \leq \frac{1}{18} \tag{5.62}$$

and define the radii

$$\begin{aligned} r^* &= \inf\left\{r \in (r_{+\Lambda}, R_c) \mid v_{M_0\Lambda}(r) = \Delta v_0\right\}, \\ \tilde{r} &= \sup\{r \in (r_{+\Lambda}, R_c) \mid |\mu_\Lambda(r) - \mu(r)| \leq \mu(R_0 + \Delta R)\}, \end{aligned} \tag{5.63}$$

and set $\tilde{r}^* := \min\{\tilde{r}, r^*\}$. We assume that $r \leq \tilde{r}^*$ and calculate $|\mu(r) - \mu_\Lambda(r)|$. To make calculations more convenient, we extend ϱ and p on $[0, 2M_0]$ as constant zero such that integrals of ϱ and p over (r_+, r) can be replaced by integrals over $(0, r)$. First we calculate

$$|\mu_0 - \mu_{0\Lambda}| = \frac{1}{2}\ln\left[1 + \frac{r_{+\Lambda}^2 \Lambda}{3}\left(1 - \frac{r_{+\Lambda}^2\Lambda}{3} - \frac{2M_0}{r_{+\Lambda}}\right)^{-1}\right] =: C_{0\Lambda}(r). \tag{5.64}$$

We write

$$\begin{aligned} &|\mu(r) - \mu_\Lambda(r)| \\ &\leq \int_{r_{+\Lambda}}^{r} \frac{1}{v_{M_0\Lambda}(s)}\Bigg[4\pi s|p_\Lambda(s) - p(s)| - \Lambda\left(\frac{s}{3} + \frac{r_{+\Lambda}^3}{s^2}\right) \\ &\qquad\qquad + \frac{4\pi}{s^2}\int_0^s \sigma^2 |\varrho_\Lambda(\sigma) - \varrho(\sigma)|\mathrm{d}\sigma\Bigg]\mathrm{d}s \\ &\quad + \int_{r_{+\Lambda}}^{r}\left(4\pi s p(s) + \frac{4\pi}{s^2}\int_0^s \sigma^2\varrho(\sigma)\mathrm{d}\sigma\right)\left|\frac{1}{v_{M_0\Lambda}(s)} - \frac{1}{v_{M_0}(s)}\right|\mathrm{d}s + C_{0\Lambda}(r) \end{aligned} \tag{5.65}$$

We would like to apply the generalized Buchdahl inequality (lemma 4.9) to the background solution μ on the interval $[r_{+\Lambda}, \infty)$. One easily sees that $r_{+\Lambda} > \hat{r} \geq 3M_0 > 9/4M_0$. The crucial condition is the existence of a vacuum region on $\left(2M_0, \frac{9}{4}M_0\right]$. But this is ensured by virtue of the assumption $L_0 > 16M_0^2$ which implies $r_+ > 4M_0$. So the difference $|\mu(r) - \mu_\Lambda(r)|$ can be further simplified and estimated. One obtains an inequality of the form

$$|\mu(r) - \mu_\Lambda(r)| \leq C_\Lambda(r) + C(r)\int_0^r \left(|p(s) - p_\Lambda(s)| + |\varrho(s) - \varrho_\Lambda(s)|\right)\mathrm{d}s \tag{5.66}$$

where $C(r)$ is increasing in r, $C_\Lambda(r)$ is increasing both in Λ and r and we have $C_\Lambda(r) = 0$ if $\Lambda = 0$. Note that the constants are fully determined by M_0, L_0, ϕ and μ.

In virtue of the mean value theorem, the sum $|p_\Lambda - p| + |\varrho_\Lambda - \varrho|$ is given by

$$|p_\Lambda(r) - p(r)| + |\varrho_\Lambda(r) - \varrho(r)| \leq C \cdot |\mu_\Lambda(r) - \mu(r)|, \tag{5.67}$$

where the constant C is determined by the derivatives of G_ϕ and H_ϕ. A Grönwall argument yields $|\mu_\Lambda(r) - \mu(r)| \leq C_{\mu\Lambda}(r)$ implying $|\varrho_\Lambda(r) - \varrho(r)| \leq C_{g\Lambda}(r)$ with certain constants $C_{g\Lambda}$ and $C_{\mu\Lambda}$.

One can choose Λ small enough such that for all $r \in (r_{+\Lambda}, R_0 + \Delta R]$ we have $|\mu_\Lambda(r) - \mu(r)| \leq \mu(R_0 + \Delta R)$. Moreover, we consider the difference

$$|v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{\Lambda}{3}\left| r^2 - \frac{r_{+\Lambda}^3}{r}\right| + \frac{8\pi r^2}{3} C_{g\Lambda}(r). \tag{5.68}$$

Lemma 4.9 implies $v_{M_0}(r) \geq \frac{1}{9}$ for all $r \in (r_{+\Lambda}, \infty)$. Choosing Λ sufficiently small, such that for all $r \in (r_{+\Lambda}, R_0 + \Delta R]$ we have $|v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{1}{18}$ one obtains $v_{M_0\Lambda} \geq \frac{1}{18}$ on $(r_{+\Lambda}, R_0 + \Delta R]$.

So altogether, one has deduced that $\tilde{r}^* \geq R_0 + \Delta R$ if Λ is chosen sufficiently small. This implies that μ_Λ exists at least on $[0, R_0 + \Delta R]$ by lemma 5.11 and also that $\mu_\Lambda(R_0 + \Delta R) > 0$. From the latter property one deduces that there exists a radius $R_{0\Lambda} > R_0$ (the zero of μ_Λ) such that for all $r \in [R_{0\Lambda}, R_0]$ we have $\varrho_\Lambda(r) = p_\Lambda(r) = 0$. On this interval, we can glue an appropriately shifted Schwarzschild-de Sitter metric to μ_Λ. This yields the desired solution defined on $(r_{B\Lambda}, \infty)$. □

Remark 5.14. *To see that the solutions constructed in theorem 5.13 are non-vacuum, one checks that for $r \geq r_{+\Lambda}$ one has*

$$\frac{\mathrm{d}}{\mathrm{d}r} a_\Lambda(r) < 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2} a_\Lambda(r) \leq 0. \tag{5.69}$$

Since $a_\Lambda(r)$ corresponds to $e^{-y_\Lambda(r)}$, this implies that for some $r > r_{+\Lambda}$ the quantity $e^{-y_\Lambda(r)}\sqrt{1 + \frac{L_0}{r^2}} < 1$ which in turn implies by lemma 3.2, (iii) that $\varrho_\Lambda(r), p_\Lambda(r) > 0$ for some $r > r_{+\Lambda}$.

Remark 5.15. *In contrary to the metric without a singularity at the center, the metric with a Schwarzschild singularity does not coincide with the not shifted Schwarzschild-de Sitter solution for $r > R_{0\Lambda}$. This can be seen as follows. We have*

$$\mu'_\Lambda(r) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r} \ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right). \tag{5.70}$$

Certainly, the mass parameter M of the vacuum solution, that is glued on in the outer region, is larger than M_0. This implies

$$1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r} > 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r} \tag{5.71}$$

for all $r \in (r_{B\Lambda}, r_C)$. So there is no ansatz Φ for the matter distribution that yields a metric component μ_Λ that connects the two vacuum solutions without any shift. But by suitable choice of Φ and E_0 one can determine whether the inner or the outer Schwarzschild-de Sitter metric is shifted. For the maximal analytic extension of the metric constructed in theorem 5.13 we will need the solution to coincide with the not shifted Schwarzschild-de Sitter metric for $r > R_{0\Lambda}$.

5.3.2 Matter shells immersed in Schwarzschild-Anti de Sitter spacetimes

We construct solutions of the Einstein-Vlasov system with a Schwarzschild singularity at the center for the case $\Lambda < 0$. The result is given in the following theorem.

Theorem 5.16. *Let $\Phi : \mathbb{R}^2 \to [0,\infty)$ be of the form (3.39) and let $L_0, M_0 \geq 0$ such that $L_0 < 16M_0^2$. Choose $\Lambda < 0$ such that $|\Lambda|$ is sufficiently small. Then there exists a unique solution $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda},\infty))$, $f \in C^0((r_{B\Lambda},\infty))$ of the Einstein-Vlasov system (3.29) – (3.32). The support of the matter quantities ϱ_Λ and p_Λ is contained in a shell, $\{r_{+\Lambda} < r < R_{0\Lambda}\}$. In the complement of this shell, the solution of the Einstein equations is given by the Schwarzschild-Anti de Sitter metric.*

Proof. We define $r_B := 2M_0$ to be the Schwarzschild black hole horizon of the background solution and $r_{B\Lambda}$ to be the black hole horizon for the Schwarzschild-Anti de Sitter with $\Lambda < 0$, i.e. the smallest positive zero of $1 - r^2\Lambda/3 - 2M_0/r$. Define also the functions

$$a(r) = \sqrt{1 - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}, \tag{5.72}$$

$$a_\Lambda(r) = \sqrt{1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}}\sqrt{1 + \frac{L_0}{r^2}}. \tag{5.73}$$

Moreover we define r_- and r_+ to be the first and second positive zero of $a(r) - 1$, respectively, as well as $r_{-\Lambda}$ and $r_{+\Lambda}$ to be the first and second positive zero of $a_\Lambda(r) - 1$. The assumption $L_0 < 16M_0^2$ assures that $r_B < r_- < r_+$ but a priori $r_{+\Lambda} = \infty$ and $r_{-\Lambda} = \infty$ are possible. However, we show that the configuration is

$$2M_0 = r_{B\Lambda} < r_{-\Lambda} < r_{+\Lambda} < \infty. \tag{5.74}$$

First, we observe that $a(r) < 1$ for all $r > r_+$ and also that $a_\Lambda(r) > a(r)$ for all $r \in \mathbb{R}_+$ since $\Lambda < 0$. So we have $r_{B_\Lambda} < r_{-\Lambda} < r_- < r_+ < r_{+\Lambda}$. It remains to show that $r_{+\Lambda} < \infty$. This is done by showing that for $|\Lambda|$ sufficiently small the functions a and a_Λ are sufficiently close at a radius $r_+ + \Delta r$, $\Delta r > 0$ such that $a_\Lambda(r_+ + \Delta r) < 1$. So we consider the difference $|a_\Lambda^2(r) - a^2(r)|$ at the radius $r_+ + \Delta r$:

$$|a_\Lambda^2(r_+ + \Delta r) - a^2(r_+ + \Delta r)| = |\Lambda|\frac{(r_+ + \Delta r)^2 + L_0}{3}. \tag{5.75}$$

Choosing $|\Lambda|$ small one attains this difference to be smaller than $a(r_+ + \Delta r) - 1$ which implies $r_{+\Lambda} < r_+ + \Delta r < \infty$.

Given this configuration (5.74) we construct a global solution of the Einstein-Vlasov system in the following manner. For $r \in (r_{B\Lambda}, r_{+\Lambda}]$ we set $f(x,v) \equiv 0$ and

$$\mu_\Lambda(r) = \frac{1}{2}\ln\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right). \tag{5.76}$$

For $r \geq r_{+\Lambda}$ we set $f(x,v) = \Phi(E,L)$. Since also $\Phi(E,L) = 0$ on the interval $(r_{-\Lambda}, r_{+\Lambda})$ the distribution function f is continuous and the metric coefficient μ_Λ is given by the ODE (5.56) with $\Lambda < 0$ for all $r \in (r_{-\Lambda},\infty)$. The initial value λ_0 of λ_Λ is determined by the continuity criterion

$$\frac{r_{+\Lambda}}{2}\left(1 - e^{2\lambda_0}\right) = M_0. \tag{5.77}$$

The last step of the proof is to assure for the existence of a solution with the desired properties of the Einstein-Vlasov system on $[r_{+\Lambda}, \infty)$ with initial values λ_0 given by (5.77) and $\mu_0 = \mu_\Lambda(r_{+\Lambda})$, given by equation (5.76). But this is already implied by the theorems 5.8 and 5.9. □

5.4 Solutions on $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2 \times \mathbb{R}$

In sections 5.1 and 5.3.1 we constructed spherically symmetric, static solutions of the Einstein-Vlasov system with small positive cosmological constant Λ. For small radii the Λ-term plays only a minor role. This was crucial for the method of proof. However, the global structure of the constructed spacetimes is substantially different when $\Lambda > 0$ and shows interesting properties. In particular, it allows for solutions with different global topologies.

The following theorem gives a class of new solutions to the non-vacuum field equations with non-trivial global topology. These solutions are constructed from pieces consisting of solutions constructed in theorems 5.5 and 5.13.

Theorem 5.17. *Let $\Lambda > 0$ be sufficiently small and let $\mathscr{M}_1 = \mathbb{R} \times S^3$ and $\mathscr{M}_2 = \mathbb{R} \times S^2 \times \mathbb{R}$. The following types of static metrics solving the Einstein-Vlasov system exist on these topologies.*

(i) There is a class of static metrics on $\mathscr{M}_1$, which is characterized by the diagram shown in figure 8. In regions I and IV a metric in this class coincides with two a priori different solutions of the type constructed in theorem 5.5 with identical total mass, but possibly different matter distributions and radii of the support of the matter quantities R_1 and R_2 and regular centers. The metric in regions II and III is vacuum.

(ii) There is a class of static metrics on $\mathscr{M}_1$, which is characterized in figure 9. A metric in this class consists of two regular centers with finitely extended matter distribution around each of the centers of equal mass but possible different matter distributions and radii R_1, R_2 of the type constructed in theorem 5.5. These two regions are connected by a chain of black holes of identical masses (the diagram shows the minimal configuration with one black hole).

(iii) There is a class of metrics on $\mathscr{M}_2$, which is characterized in figure 10. The spacetime consists of an infinite sequence of black holes, each surrounded by matter shells of possibly different radii and positions. In regions IV, VII, X and XIII these solutions coincide with those constructed in theorem 5.13. The necessary conditions on the masses are $M_\varrho^{A_1} = M_\varrho^{A_2}$, $M_\varrho^{B_1} = M_\varrho^{B_2}$ and $M_0^A + M_\varrho^{A_2} = M_\varrho^{B_1} + M_0^B$, where M_0^i, $i = A, B$, denote the mass parameter of the black holes and $M_\varrho^{i_j}$, $i = A, B$, $j = 1, 2$, denote the quasilocal mass of the matter shells defined in equation (5.88).

Remark 5.18.

(i) The $r = 0$ surfaces in the first an second class have the topology $\mathbb{R}$ and the topology of the $r = \infty$ surfaces is a cylinder, i.e. $\mathbb{R} \times S^2$. The $r = 0$ surfaces with singular metric in the second class have topology $\mathbb{R} \times S^2$.

(ii) The black hole masses in the third class of solutions in the previous theorem can be pairwise different. Only the total mass of black hole and matter shell have to agree pairwise (see condition in (iii) above).

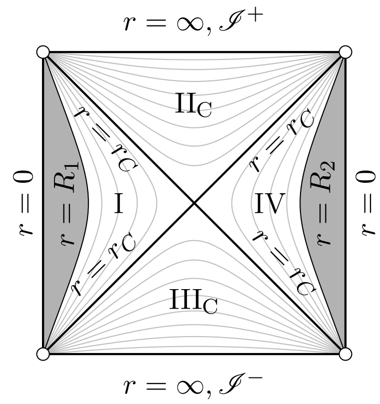

Figure 8: Penrose diagram of the maximal analytic extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. Region I corresponds to the region $0 < r < r_C$. The metric is extended in an analogue way to the standard extension of the de Sitter metric. The gray lines are surfaces of constant r.

(iii) Combinations of the classes 2 and 3 yield similar metrics on $\mathscr{M}_3 = \mathbb{R} \times \mathbb{R}^3$ with a regular center followed by an infinite sequence of black holes.

(iv) The second class of solutions could also be generalized by adding matter shells around the black holes. The mass parameters then have to be adjusted.

(v) When crossing the cosmological horizon or the event horizon of a black or white hole the Killing vector ∂_t changes from being timelike to spacelike. This means that the maximal extended spacetime contains both static and dynamic regions that are alternating. This holds for all constructed classes.

Proof. We outline now the construction of the spacetimes given in the previous theorem. For the first two classes of spacetimes we consider solutions of the Einstein-Vlasov system with a regular center. Let $(\mu_\Lambda, \lambda_\Lambda, f_\Lambda)$ be a static solution of the spherically symmetric Einstein-Vlasov system with positive cosmological constant Λ defined for $r \in [0, r_C)$ such that the support of the matter quantities is bounded by a radius $0 < R_{0\Lambda} < r_C$. The radius r_C denotes the cosmological horizon. On $[R_{0\Lambda}, r_C)$ there is vacuum on hand and the metric is given by the Schwarzschild-de Sitter metric (5.81) with the ADM mass M as mass parameter. The ADM mass M is then given by

$$M = 4\pi \int_0^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) \mathrm{d}s. \tag{5.78}$$

If $9M^2\Lambda < 1$, the polynomial $r^3 - \frac{3}{\Lambda}r + \frac{6M}{\Lambda}$ has one negative zero and two positive ones. The largest zero of this polynomial is defined to be the cosmological horizon r_C. Moreover, r_n is the negative zero, and $r_{B\Lambda}$ the smaller positive one. In terms of the ADM mass M and the cosmological constant Λ these zeros can be calculated explicitly.

Consider figure 8. This spacetime can be obtained in an analogue way to the standard procedure to compactify the de Sitter space as described for example in [23]. In the following, this procedure is carried out in detail. The metric is given as a non-vacuum solution of the

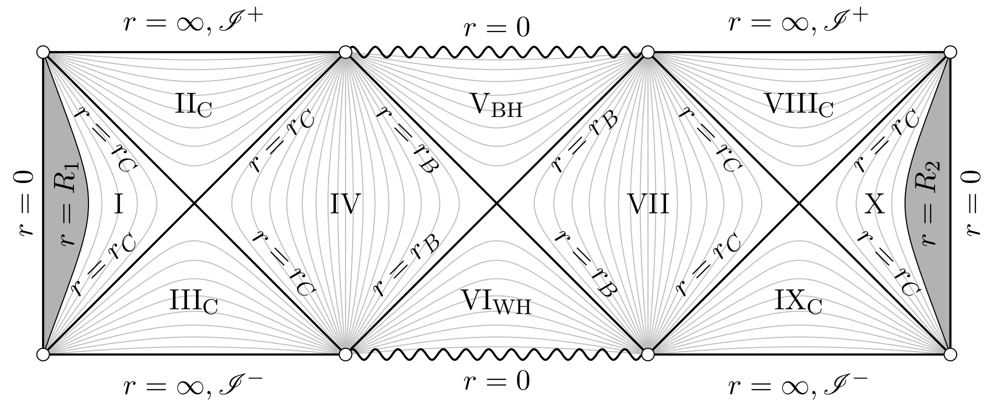

Figure 9: Penrose diagram of the maximal analytic extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. Region I corresponds to the region $0 < r < r_C$. In this region matter (represented by the shaded area) is present and the metric is regular. This metric is extended with the Schwarzschild-de Sitter metric that leads to a periodic solution. The periodic course stops when a matter region appears again preventing the metric from being singular at $r = 0$. The gray lines are surfaces of constant r.

Einstein-Vlasov system for $r \in [0, r_C)$, corresponding to region I in figure 8, as discussed in theorem 5.5. In this region we have

$$\mathrm{d}s^2 = -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2. \tag{5.79}$$

In the first step we introduce coordinates U_I, V_I that transform the region $\mathbb{R} \times [0, r_C) \times S^2$ into the left triangle (region I) in figure 8. The coordinates usually used to compactify the vacuum de Sitter metric as for example described in [23] will do. They are given by

$$U_\mathrm{I} = \sqrt{\frac{r_C - r}{r_C + r}}e^{-\frac{t}{r_C}}, \quad V_\mathrm{I} = -\sqrt{\frac{r_C - r}{r_C + r}}e^{\frac{t}{r_C}} \tag{5.80}$$

and can be compactified via the transformations $p_\mathrm{I} = \arctan(U_\mathrm{I})$, $q_\mathrm{I} = \arctan(V_\mathrm{I})$. The left part of figure 11 shows the transformed region $\mathbb{R} \times [0, r_C)$ in the p_I, q_I-coordinates.

The support of the matter (i.e. the matter distribution f) ends at a radius $R_{0\Lambda}$. For $r \geq R_{0\Lambda}$ the metric is merely given by the Schwarzschild-de Sitter metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}} + r^2\mathrm{d}\Omega^2, \quad R_{0\Lambda} \leq r < r_C. \tag{5.81}$$

At $r = r_C$ there is a coordinate singularity of the metric that we want to pass. For this purpose we express the metric in other coordinates that do not have a singularity at $r = r_C$. These coordinates are given by

$$U_C = \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}}e^{-\frac{t}{2\delta_C}} > 0, \quad V_C = -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}}e^{\frac{t}{2\delta_C}} < 0, \tag{5.82}$$

where $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$ and $\gamma = \frac{r_B}{(1 - \Lambda r_B^2)\delta_C}$, $0 < \gamma < 1$[2]. They are used in the standard

[2]The signs of these expressions can be checked by means of the equality $1 - \frac{r_C^2\Lambda}{3} - \frac{2M}{r_C} = 0$

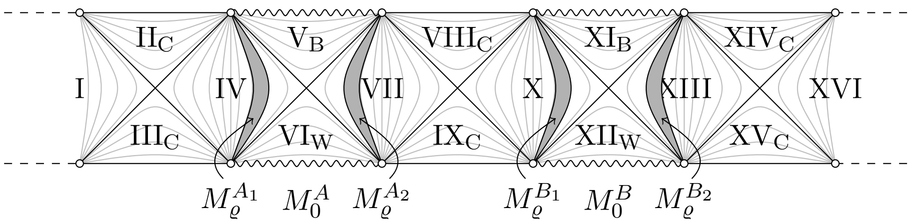

Figure 10: Penrose diagram of the maximal analytic extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. The solution coincides with the Schwarzschild-de Sitter spacetime in the vacuum regions and the black holes are surrounded by shells of Vlasov matter (gray shaded domains). Notably the black holes do not necessarily have the same mass. The gray lines are surfaces of constant r.

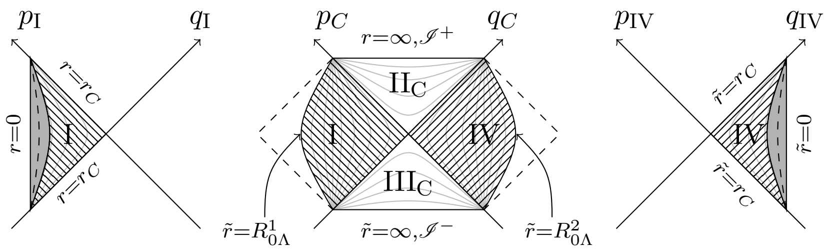

Figure 11: Construction of the spacetime shown in figure 8. We use three coordinate charts to compactify the spacetime. Regions that are shaded in the same orientation are covered by two of the coordinate charts simultaneously, thus there coordinates can be changed. The gray areas are matter regions and the dashed lines correspond to $r = r_{B\Lambda}$. We distinguish between r and $\tilde{r}$ to emphasize that there are different spacetime regions that cannot be covered by a single chart $(t, r, \vartheta, \varphi)$. All coordinates p and q take values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

compactification procedure of the Schwarzschild-de Sitter metric. For details, see [16] or [46]. In the new coordinates the line element of the Schwarzschild-de Sitter metric (5.81) reads

$$\mathrm{d}s^2 = -\frac{4\Lambda\delta_C^2}{3r}(r - r_n)^{2-\gamma}(r - r_B)^{1+\gamma}\mathrm{d}U_C\mathrm{d}V_C + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2, \quad r \geq R_{0\Lambda}. \qquad (5.83)$$

The coordinates U_C, V_C representing the region where $r \in [R_{0\Lambda}, r_C)$ (region I in the middle part of figure 11) only take values in $\{(u, v) \in \mathbb{R}^2 \,|\, u > 0, v < 0\}$. We extend them to $\mathbb{R}^2$. This extension goes beyond r_C. Again, the spacetime region covered by the coordinates U_C and V_C can be compactified using the transformation $p_C = \arctan(U_C)$, $q_C = \arctan(V_C)$. The middle part of figure 11 shows the region covered by U_C and V_C, taking values in $\mathbb{R}$ each, in the p_C, q_C-coordinates. The line element (5.83) can be extended to the whole area covered by U_C and V_C in an analytic way. In the region where $r \in [R_{0\Lambda}, r_C)$ the coordinates charts (5.80) and (5.82) overlap and one can change coordinates (the shaded areas in the left and middle part of figure

11). The transformation law is given by

$$\begin{aligned} U_C(U_{\mathrm{I}}) &= \sqrt{\frac{(r_C+r)(r-r_n)^{\gamma-1}}{(r-r_B)^{\gamma}}}\, e^{\frac{3-2\Lambda r_C^2}{r_C}} U_I, \\ V_C(V_{\mathrm{I}}) &= \sqrt{\frac{(r_C+r)(r-r_n)^{\gamma-1}}{(r-r_B)^{\gamma}}}\, e^{-\frac{3-2\Lambda r_C^2}{r_C}} V_I. \end{aligned} \tag{5.84}$$

Region IV in figure 8 corresponds to a second universe that also can be equipped with Schwarzschild coordinates $(\tilde{t},\tilde{r})$. We distinguish between r and $\tilde{r}$ to emphasize that the charts (t,r) and $(\tilde{t},\tilde{r})$ cover different regions of the spacetime. The vacuum parts of the regions are isometric and both contain matter balls around $r=0$. The latter property does not hold for the second class of spacetimes (ii). In the region $\tilde{r}\in[R_{0\Lambda},r_C)$ (region IV in the middle part of figure 11), in terms of the $\tilde{t}$, $\tilde{r}$-coordinates U_C and V_C are given by

$$U_C = -\sqrt{\frac{(r_C-\tilde{r})(\tilde{r}-r_-)^{\gamma-1}}{(\tilde{r}-r_B)^{\gamma}}}\, e^{-\frac{\tilde{t}}{2\delta_C}} < 0, \quad V_C = \sqrt{\frac{(r_C-\tilde{r})(\tilde{r}-r_-)^{\gamma-1}}{(\tilde{r}-r_B)^{\gamma}}}\, e^{\frac{\tilde{t}}{2\delta_C}} > 0. \tag{5.85}$$

To get a compactification of the whole region IV, including $\tilde{r}<r_B$, we introduce coordinates similar to (5.80), namely

$$U_{\mathrm{IV}} = -\sqrt{\frac{r_C-\tilde{r}}{r_C+\tilde{r}}}\, e^{-\frac{\tilde{t}}{r_C}}, \quad V_{\mathrm{IV}} = \sqrt{\frac{r_C-\tilde{r}}{r_C+\tilde{r}}}\, e^{\frac{\tilde{t}}{r_C}} \tag{5.86}$$

covering the region characterized by $\tilde{r}\in[0,r_C)$. This region can again be compactified via $p_{\mathrm{IV}} = \arctan(U_{\mathrm{IV}})$, $q_{\mathrm{IV}} = \arctan(V_{\mathrm{IV}})$. This yields the right part of figure 11. For $\tilde{r}\in[R_{0\Lambda},r_C)$ the coordinates can be changed using an analogue law to (5.85). On the spacetime region represented by the middle part of figure 11 the line element can be expressed by (5.83). Since in both regions I and IV the metric can be brought into the form (5.79) via coordinate transformations also the energy densities are identical in these regions. This of course implies that in both regions the mass parameter is equal.

Now we come to the spacetimes characterized by figure 9. For the construction of the analytic extension of the metric (5.79) at least five coordinate charts are necessary. Figure 12 illustrates this construction. We begin again with the region $r\in[0,r_C)$ where the metric is given by (5.79).

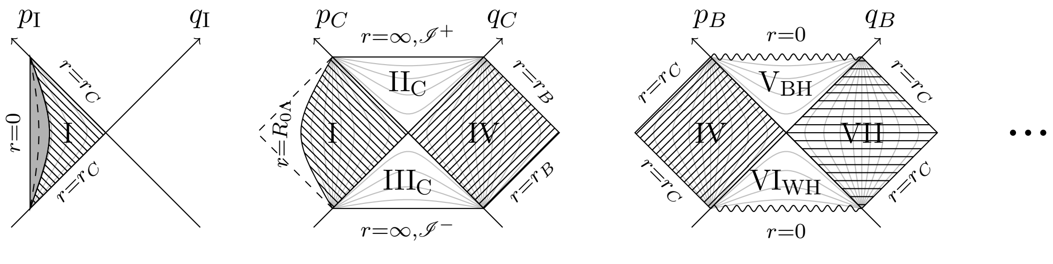

Figure 12: Construction of the spacetime shown in figure 9. On regions that are shaded in equal directions two coordinates are defined and one can change between them. All coordinates p, q take values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

In the same way as described above one expresses the line element in other coordinates p_C, q_C that avoid the singularity at $r = r_C$. The line element as given by (5.83) can be analytically extended onto the regions I – IV in figure 12 (middle part). From now on the procedure differs from the one above. Regions I and IV are not supposed to be identical but region IV shall be a vacuum region thus the metric will be given by Schwarzschild-de Sitter everywhere. Certainly, the line element (5.83) of the Schwarzschild-de Sitter metric given in terms of the coordinates U_C, V_C now shows a singularity at $r = r_{B\Lambda}$[3]. This coordinate singularity can be overcome by virtue of the coordinates

$$U_B = \sqrt{\frac{(r - r_B)(r - r_n)^{\beta - 1}}{(r_C - r)^\beta}} e^{\frac{t}{2\delta_B}}, \quad V_B = -\sqrt{\frac{(r - r_B)(r - r_n)^{\beta - 1}}{(r_C - r)^\beta}} e^{-\frac{t}{2\delta_B}}, \tag{5.87}$$

where $\delta_B = \frac{r_B}{1 - \Lambda r_B^2} > 0$ and $\beta = \frac{r_C}{(\Lambda r_C^2 - 1)\delta_B} > 1$. This is part of the standard compactification procedure of the Schwarzschild-de Sitter metric, cf. [16] or [46]. Alternating the coordinate charts (U_C, V_C) and (U_B, V_B) this procedure can be continued an arbitrary amount of times extending the spacetime to additional black hole and cosmological regions. This periodic extension stops if for $r < r_C$ the metric is not given by a vacuum solution of the Einstein equations but again by the solution (5.79) of the Einstein-Vlasov system. There is no coordinate singularity at $r = r_B$ and also a regular center at $r = 0$. So a regular expression of the line element by the coordinates (5.80) is possible again, leading to region X in figure 9. This region now is geometrically identical to region I in figure 9 (and also in figure 12). In the extension procedure just described the expressions for the coordinates (5.85) and (5.87) used to pass the coordinate singularities at $r = r_B$ and $r = r_C$ in the vacuum regions of the spacetime $\mathscr{M}_1$ depend on Λ and M. So the identification of corresponding regions in the different coordinate charts, e.g. I or IV in figure 12, is only possible if the parameters Λ and M are equal in all regions of $\mathscr{M}_1$. In terms of the notation of figure 9 this implies $M_1 = M_2$.

A maximal analytic extension of a solution to the Einstein-Vlasov system on the manifold $\mathscr{M}_2$ as characterized by figure 10, i.e. spacetimes in class (iii), can be obtained in a similar way. Starting point is the region $r_{B\Lambda} < r < r_C$. On this interval the existence of a unique solution to a given ansatz for f is established by theorem 5.13. The solution on hand can be understood as a Schwarzschild-de Sitter spacetime with an immersed shell of Vlasov matter supported on an interval $(r_{+\Lambda}, R_{0\Lambda})$. Two mass quantities are important. On the one hand one has the mass parameter M_0 of the black hole at the center. On the other hand M that is defined to be

$$M = M_0 + M_\varrho, \quad M_\varrho = 4\pi \int_{r_{+\Lambda}}^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) \mathrm{d}s. \tag{5.88}$$

This quantity represents the sum of the mass of the black hole and the shell of Vlasov matter. As constructed in theorem 5.13, for $r_{B\Lambda} < r \leq r_{+\Lambda}$ the metric is given by a shifted Schwarzschild-de Sitter metric

$$\mathrm{d}s^2 = -C\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{C\left(1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}\right)} + r^2\mathrm{d}\Omega^2, \quad r_{B\Lambda} < r \leq r_{+\Lambda} \tag{5.89}$$

with the mass M_0 of the black hole as mass parameter and the shift $C > 0$. For $R_{0\Lambda} \leq r < r_C$ the metric is given by the Schwarzschild-de Sitter metric (5.81) with mass parameter M.

[3]By abuse of notation we use r for the radius coordinate in every region of the spacetime $\mathscr{M}_1$.

The two critical horizons, $r_{B\Lambda}$ and r_C can be given explicitly as zeros of the expression $1-\frac{r^2\Lambda}{3}-\frac{2m(r)}{r}$. But it is important to note that the mass parameter $m(r)$ does not stay constant throughout the whole interval $(r_{B\Lambda}, R_{0\Lambda})$. The black hole horizon $r_{B\Lambda}$ is characterized by M_0 and the cosmological horizon r_C by M. This has to be kept in mind when choosing coordinates to construct an extension of the metric on $\mathscr{M}_2$ as illustrated in figure 13. We distinguish between

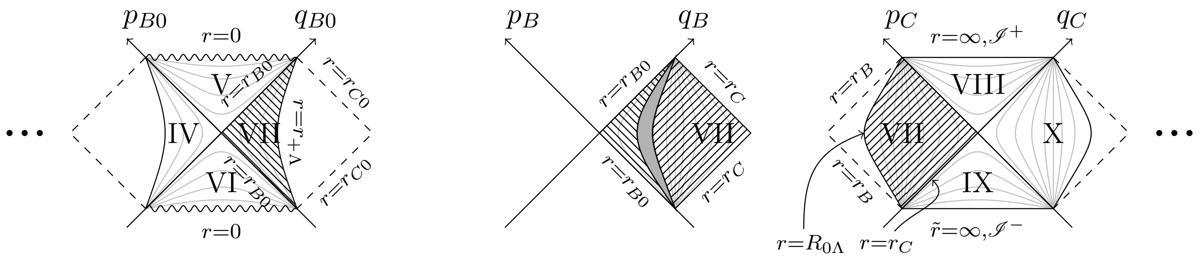

Figure 13: Construction of the spacetime shown in figure 10. The middle part shows a Schwarzschild-de Sitter spacetime with an immersed matter shell for $r_{B\Lambda} = r_{B0} < r < r_C$. The left and the right part show the adjacent vacuum region containing several coordinate singularities. On regions that are shaded in equal directions two coordinate charts are defined and one can change between them. All coordinates p, q take values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

the zeros of $1-\frac{r^2\Lambda}{3}-\frac{2m(r)}{r}$ when $m(r) \equiv M_0$ and $m(r) \equiv M$ and call them r_{B0}, r_{C0} or r_B, r_C, respectively. Note that $r_{B0} = r_{B\Lambda}$. Consider the metric on the region $r_{B0} < r < r_C$ being part of region VII in figure 10 or the middle part of figure 13. The metric shall be extended to the left (regions IV, $\mathrm{V_B}$, $\mathrm{VI_W}$) and to the right (regions $\mathrm{VIII_C}$, $\mathrm{IX_C}$, X) as a vacuum solution until the next matter shell appears. So the coordinate transformations have to be chosen with respect to the radii r_B (or r_{B0}) and r_C (or r_{C0}) belonging to the current mass parameter in the respective spacetime region. Three coordinate charts are needed to extend the metric beyond the black hole and the cosmological horizon. First we compactify the region $r_{B\Lambda} = r_{B0} < r < r_C$ using the coordinates

$$U_B = \sqrt{\frac{(r-r_{B0})(r-r_n)^{\beta-1}}{(r_C-r)^{\beta}}}e^{\frac{t}{2\delta_{B0}}}, \quad V_B = -\sqrt{\frac{(r-r_{B0})(r-r_n)^{\beta-1}}{(r_C-r)^{\beta}}}e^{-\frac{t}{2\delta_{B0}}}. \tag{5.90}$$

where $\delta_{B0} = \frac{r_{B0}}{1-\Lambda r_{B0}^2} > 0$ and $\beta = \frac{r_C}{(\Lambda r_C^2-1)\delta_{B0}} > 1$. These coordinates give rise to $p_B = \arctan(U_B)$ and $q_B = \arctan(V_B)$. This region is depicted in the middle part of figure 13. The spacetimes characterized by figure 10 show two types of connected vacuum regions. The first type is characterized by $r \leq r_{+\Lambda}$ (inside the matter shell) and the second one by $r \geq R_{0\Lambda}$ (beyond the matter shell). To extend the metric to the region inside the matter shell (and the black hole) one uses the coordinates

$$U_{B0} = \sqrt{\frac{(r-r_{B0})(r-r_n)^{\beta_0-1}}{(r_{C0}-r)_0^{\beta}}}e^{\frac{t}{2\delta_{B0}}}, \quad V_{B0} = -\sqrt{\frac{(r-r_{B0})(r-r_n)^{\beta_0-1}}{(r_{C0}-r)_0^{\beta}}}e^{-\frac{t}{2\delta_{B0}}}, \tag{5.91}$$

where $\delta_{B0} = \frac{r_{B0}}{1-\Lambda r_{B0}^2} > 0$ and $\beta_0 = \frac{r_{C0}}{(\Lambda r_{C0}^2-1)\delta_{B0}} > 1$, and the corresponding compactification $p_{B0} = \arctan(U_{B0})$, $q_{B0} = \arctan(V_{B0})$. The black hole horizon can be crossed using the usual arguments of the extension of the Schwarzschild-de Sitter metric as for example done in [23, 17, 46]. This is illustrated in the left part of figure 13. The region beyond the matter shell (and the cosmological horizon) can be reached via the coordinates

$$U_C = -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}} e^{-\frac{t}{2\delta_C}}, \quad V_C = \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}} e^{-\frac{t}{2\delta_C}}, \tag{5.92}$$

with $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$ and $\gamma = \frac{r_B}{(1-\Lambda r_B^2)\delta_C}$, $0 < \gamma < 1$. These coordinates extend the metric to the area shown in the right part of figure 13.

On the connected vacuum regions the metric is given by only one expression even though vacuum extends onto several regions of $\mathscr{M}_2$, e.g. regions VII, $\mathrm{VIII_C}$, $\mathrm{IX_C}$ and X. This implies that the coordinates U_{B0}, V_{B0} or U_C, V_C have to be given by the same expressions (5.91) or (5.92), respectively (modulo sign, cf. [23, 17, 46]) which in turn implies that the mass parameter has to stay the same on these connected vacuum regions. For the vacuum region with $r \geq R_{0\Lambda}$ this implies $M_0^A + M_\varrho^{A_2} = M_0^B + M_\varrho^{B_1}$ (notation of figure 10). On the region characterized by $r \leq r_{+\Lambda}$ this is always granted because the mass is entirely given by the black hole mass M_0. Finally the shift constants C of the vacuum metric have to coincide in this region (IV and VII in figure 10). They are determined by the matter shells surrounding the black hole and are equal in particular if these shells have the same shape which implies $M_\varrho^{A_1} = M_\varrho^{A_2}$. □

6 Concluding remarks

6.1 Achieved results

The first part of this work, section 1 to section 3, was devoted to a detailed introduction and an explicit derivation of the reduced Einstein-Vlasov system in spherical symmetry with non-vanishing cosmological constant. For this purpose, by virtue of a certain ansatz Φ for the matter distribution function f the Vlasov equation was solved by the method of characteristics. In addition, a physical motivation for the research on this system and the application of the Vlasov matter model was given. In section 4 results concerning spherically symmetric, static solutions of the Einstein-Vlasov system that already exist in the literature were addressed. This collection of results represents the current state of research and covers global in r existence of solutions as well as several properties of these solutions like boundedness of the support of the matter distribution function and bounds on the ratio $\frac{m(r)}{r}$ of quasilocal mass and radius. Furthermore, by explicit construction, the existence of non-vacuum solutions of the Einstein-Vlasov system that contain a black hole at the center was shown. A numerical investigation revealed that the static solutions of the Einstein-Vlasov system can be divided into several classes depending on the ansatz taken for the matter distribution. There are isotropic solutions that form the simplest case and show matter arranged in a ball around the origin. Moreover, the matter quantities are monotonically decreasing, thus there are large similarities to the behavior of an ideal fluid. So called polytropic solutions form a more diverse class. The matter is still arranged in a ball around the origin but the matter quantities, like energy density or pressure, are not monotonous but may show several peaks. Finally, we have anisotropic solutions that form the most general class of solutions that results are known for in the literature. The matter quantities belonging to these solutions may be supported on shells around the origin separated by vacuum. It is important to note that no existence result is known on the spherically symmetric, static Einstein-Vlasov system with non-vanishing cosmological constant. So far there are just results, like a Buchdahl type inequality, that assume the existence of these results without proving it [11]. This work provides first results bridging this gap. For the proofs on solutions of the system with non-vanishing cosmological constant Λ the existence of a corresponding solution with $\Lambda = 0$ and its properties is often crucial. So section 4 prepared the ground for the new results presented in this work.

Section 5 is concerned with the Einstein-Vlasov system with non-vanishing cosmological constant. As mentioned in the introduction, the notion of the cosmological constant gains more and more importance in the description of our universe. So the treatment of the Einstein-Vlasov system with positive cosmological constant is a worthwhile business. In section 5 new results are presented that are the outcome of a collaboration between Håkan Andréasson (Chalmers University of Technology, Gothenburg, Sweden), David Fajman (University of Vienna, Austria) and the author of this work. These results are published in [14]. The first result is the global in r existence of regular, spherically symmetric, static solutions to the Einstein-Vlasov system with small positive cosmological constants, cf. theorem 5.5. These solutions contain Vlasov matter supported on a ball around the origin. The support of the matter quantities is bounded by a certain radius $R_{0\Lambda}$ and beyond this radius the solution is given by the Schwarzschild-de Sitter solution. This result could be established for the anisotropic ansatz for the matter distribution f which is the most general one. This ansatz admits all kinds of matter arrangements mentioned above thus balls, shells, and multipeaked configurations. As already mentioned this result provides the footing for the Buchdahl type inequality shown in [11]. The existence of static solutions is also a helpful assumption at the discussion of the gravitational collapse for the time dependent Einstein-Vlasov system. In [8] the authors consider the time evolution of an initially given matter distribution function $\mathring{f} = \mathring{f}_s + \mathring{f}_m$ that leads to a gravitational collapse and the formation of a black hole. This initial data may consist of a static solution $\mathring{f}_s$ of the spherically symmetric Einstein-Vlasov system that is supported on a ball around the origin and, separated by vacuum, a shell of matter $\mathring{f}_m$ with ingoing particles. The existence of matter that stays inside a certain ball around the origin is essential for the proof of the gravitational collapse to work. Another interesting aspect of the constructed solutions with positive cosmological constant is the global structure of the obtained spacetimes. Section 5.4 provides an elaborate discussion of analytic extensions of these solutions onto the manifolds $\mathbb{R} \times S^3$, $\mathbb{R} \times S^2 \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{R}^3$. We constructed spacetimes resembling the analytic extension of the Schwarzschild-de Sitter metric but its periodic structure is interrupted when singularity formation at $r = 0$ is prevented by the appearance of regular matter regions.

Also for negative cosmological constants existence results for static solutions of the spherically symmetric Einstein-Vlasov system have been proved. It turned out that methods developed by Rein [34] in his proofs for the global existence of anisotropic solutions with vanishing cosmological constant and spatial boundedness of the support of the matter distribution could be applied also to the case with negative cosmological constant. Crucial for this to be possible is the fact that a negative cosmological constant changes the system not as profound as a positive one does. For example the property of the metric component μ of being monotonically increasing is preserved. The obtained solutions describe completely regular matter distributions of compact support that pass on to a Schwarzschild-Anti de Sitter solution for large radii. For both, positive and negative cosmological constants the existence of non-vacuum solutions that do not have a regular center but a Schwarzschild black hole at the origin has been proved. These proofs provide an explicit construction of these solutions showing the following configuration. The event horizon of the black hole is surrounded by a vacuum region that in turn is followed by a finitely extended shell of Vlasov matter. Beyond this shell there is again a vacuum region. These solutions could be considered as the final state of a gravitational collapse. For the case $\Lambda > 0$ such spacetimes can be considered as Schwarzschild-de Sitter spacetimes with immersed matter shells. An analytic extension of the metric on a periodic spacetime with alternating black holes and cosmological regions is possible just as for an ordinary Schwarzschild-de Sitter spacetime. But the matter shells surrounding the black holes effectuate that the black holes do not necessarily have to have the same mass.

6.2 Further questions

The proof of existence of solutions in the setting with $\Lambda > 0$ yields configurations with a non-vacuum solution containing a compactly supported matter formation that beyond the support of this matter a vacuum solution is glued to. At the gluing radius one essentially drops the ansatz for the matter distribution function f and continues it by constant zero. One might ask the question if this gluing procedure is necessary or if there exist also solutions of the Einstein-Vlasov system where f is given by a single ansatz Φ throughout the whole domain of definition of the solution. A qualitative analysis of the ODE (3.57) for y,

$$y'(r) = -\frac{4\pi}{1 - \frac{\Lambda r^2}{3} - \frac{8\pi}{r}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s} \times \left(r H_\phi(r, y(r)) - \frac{r\Lambda}{12\pi} + \frac{1}{r^2}\int_0^r s^2 G_\phi(s, y(s))\mathrm{d}s \right), \tag{6.1}$$

reveals that this is not the case. If the matter distribution is determined by an ansatz Φ of the form (3.39) the matter quantities ϱ and p are given by the functions G_ϕ and H_ϕ defined in (3.46) and (3.47). As stated by lemma 3.2 the functions are increasing both, in y and r. The dependence of G_ϕ and H_ϕ on the metric component y is crucial. First one notices that the denominator is monotonically decreasing and that there is a radius r_C such that it tends to zero when r goes to r_C. So the first factor of y' becomes very large as r_C is approached whereas the second one cannot become arbitrarily small. This implies that y tends to $+\infty$ or to $-\infty$. But both possibilities lead to a contradiction. If y tends to $+\infty$ then also $H_\phi(r, y)$ and $G_\phi(r, y)$ tend to $+\infty$ and the second factor of the right hand side in the ODE for y becomes positive. This implies that $y < 0$. Then again if y tends to $-\infty$ both functions, $G_\phi(r, y)$ and $H_\phi(r, y)$ go to zero. One can show that then for r arbitrarily close to r_C the expression for y' turns positive. This indicates that using an ansatz of the form (3.39) no further static solutions of the Einstein-Vlasov system can be obtained than those constructed in this work. This means they have a matter configuration supported on a finite ball around the origin and coincide with Schwarzschild-de Sitter outside. In this context it would be interesting to investigate for a given cosmological constant Λ what the maximum radius for the support of the matter quantities is.

Another interesting question is whether there exist static solutions for massless particles as well. Even for the system without cosmological constant this is not known. The Vlasov equation with $m = 0$ describes the dynamics of a photon gas or the limit of very fast (ultrarelativistic) particles. In the massless case particles show a greater tendency to fly apart. Numeric investigations and first analytic considerations indicate that the existence of static solutions with matter quantities of bounded support cannot be proved for massless particles with the methods that have been developed for the massive case. One always observes an infinitely extended tail of matter for large radii. A promising strategy seems to be taking an ansatz for the matter distribution with a large lower bound L_0 on the angular momenta of the particles. As suggested by the graphs for massive particles discussed in section 4.5 this might lead to matter configurations where vacuum separates several matter regions. In one of the vacuum regions one could glue a vacuum solution to the metric and continue the matter distribution with constant zero.

A Calculation of the Einstein tensor

In this section we show in a detailed way how to calculate the entries of the Einstein tensor G using the ansatz

$$\mathrm{d}s^2 = -e^{2\mu(r,t)}\mathrm{d}t^2 + e^{2\lambda(r,t)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2 \tag{A.1}$$

for the metric g.

At first one has to calculate the Christoffel symbols, the Ricci tensor, and the Ricci scalar. Since the metric is diagonal all Christoffel symbols with three different indices give zero. The non zero ones are:

$$\begin{array}{lll}
\Gamma^t_{tt} = \dot{\mu} & \Gamma^t_{tr} = \mu' & \Gamma^t_{rr} = \dot{\lambda}e^{2(\lambda-\mu)} \\
\Gamma^r_{tt} = \mu' e^{2(\mu-\lambda)} & \Gamma^r_{tr} = \dot{\lambda} & \Gamma^r_{rr} = \lambda' \\
\Gamma^r_{\vartheta\vartheta} = -e^{-2\lambda}r & \Gamma^r_{\varphi\varphi} = -e^{-2\lambda}\sin^2(\vartheta)r & \Gamma^\vartheta_{r\vartheta} = \frac{1}{r} \\
\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta & \Gamma^\varphi_{r\varphi} = \frac{1}{r} & \Gamma^\varphi_{\vartheta\varphi} = \frac{1}{\tan\vartheta}
\end{array}$$

In terms of the Christoffel symbols the Ricci tensor has the form

$$R_{\beta\delta} = \partial_\alpha\Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\nu\alpha}\Gamma^\nu_{\beta\delta} - \partial_\delta\Gamma^\alpha_{\beta\alpha} - \Gamma^\alpha_{\nu\delta}\Gamma^\nu_{\beta\alpha}. \tag{A.2}$$

The terms are calculated separately. The first summand will be:

$$\begin{aligned}
\partial_\alpha\Gamma^\alpha_{tt} &= \partial_t\Gamma^t_{tt} + \partial_r\Gamma^r_{tt} = \ddot{\mu} + 2(\mu' - \lambda')e^{2(\mu-\lambda)}\mu' + e^{2(\mu-\lambda)}\mu'' \\
\partial_\alpha\Gamma^\alpha_{rr} &= \partial_t\Gamma^t_{rr} + \partial_r\Gamma^r_{rr} = \left(2(\dot{\lambda} - \dot{\mu})\dot{\lambda} + \ddot{\lambda}\right)e^{2(\lambda-\mu)} + \lambda'' \\
\partial_\alpha\Gamma^\alpha_{\vartheta\vartheta} &= \partial_r\Gamma^r_{\vartheta\vartheta} = (2\lambda' r - 1)e^{-2\lambda} \\
\partial_\alpha\Gamma^\alpha_{\varphi\varphi} &= \partial_r\Gamma^r_{\varphi\varphi} + \partial_\vartheta\Gamma^\vartheta_{\varphi\varphi} = (2\lambda' r - 1)\sin^2\vartheta e^{-2\lambda} - \cos^2\vartheta + \sin^2\vartheta
\end{aligned}$$

For the second summand one obtains:

$$\begin{aligned}
\Gamma^\nu_{tt}\Gamma^\alpha_{\nu\alpha} &= \Gamma^t_{tt}\Gamma^t_{tt} + \Gamma^t_{tt}\Gamma^r_{tr} + \Gamma^r_{tt}\Gamma^t_{rt} + \Gamma^r_{tt}\Gamma^r_{rr} + \Gamma^r_{tt}\Gamma^\vartheta_{r\vartheta} + \Gamma^r_{tt}\Gamma^\varphi_{r\varphi} \\
&= \dot{\mu}(\dot{\mu} + \dot{\lambda}) + e^{2(\mu-\lambda)}\mu'\left(\mu' + \lambda' + \frac{2}{r}\right) \\
\Gamma^\nu_{rr}\Gamma^\alpha_{\nu\alpha} &= \Gamma^t_{rr}\Gamma^t_{tt} + \Gamma^t_{rr}\Gamma^r_{tr} + \Gamma^r_{rr}\Gamma^t_{rt} + \Gamma^r_{rr}\Gamma^r_{rr} + \Gamma^r_{rr}\Gamma^\vartheta_{r\vartheta} + \Gamma^r_{rr}\Gamma^\varphi_{r\varphi} \\
&= \dot{\lambda}e^{2(\lambda-\mu)}\left(\dot{\mu} + \dot{\lambda}\right) + \lambda'\left(\mu' + \lambda' + \frac{2}{r}\right) \\
\Gamma^\nu_{\vartheta\vartheta}\Gamma^\alpha_{\nu\alpha} &= \Gamma^r_{\vartheta\vartheta}\Gamma^t_{rt} + \Gamma^r_{\vartheta\vartheta}\Gamma^r_{rr} + \Gamma^r_{\vartheta\vartheta}\Gamma^\vartheta_{r\vartheta} + \Gamma^r_{\vartheta\vartheta}\Gamma^\varphi_{r\varphi} \\
&= -re^{-2\lambda}\left(\mu' + \lambda' + \frac{2}{r}\right) \\
\Gamma^\nu_{\varphi\varphi}\Gamma^\alpha_{\nu\alpha} &= \Gamma^r_{\varphi\varphi}\Gamma^t_{rt} + \Gamma^r_{\varphi\varphi}\Gamma^r_{rr} + \Gamma^r_{\varphi\varphi}\Gamma^\vartheta_{r\vartheta} + \Gamma^r_{\varphi\varphi}\Gamma^\varphi_{r\varphi} + \Gamma^\vartheta_{\varphi\varphi}\Gamma^\varphi_{\vartheta\varphi} \\
&= -e^{-2\lambda}\sin^2\vartheta r\left(\mu' + \lambda' + \frac{2}{r}\right) - \sin\vartheta\cos\vartheta\frac{1}{\tan\vartheta}
\end{aligned}$$

The third summand can be calculated to:

$$\begin{aligned}
\partial_t \Gamma^\alpha_{t\alpha} &= \partial_t \Gamma^t_{tt} + \partial_t \Gamma^r_{tr} = \ddot{\mu} + \ddot{\lambda} \\
\partial_r \Gamma^\alpha_{r\alpha} &= \partial_r \Gamma^t_{rt} + \partial_r \Gamma^r_{rr} + \partial_r \Gamma^\vartheta_{r\vartheta} + \partial_r \Gamma^\varphi_{r\varphi} = \mu'' + \lambda'' - \frac{2}{r^2} \\
\partial_\vartheta \Gamma^\alpha_{\vartheta\alpha} &= \partial_\vartheta \Gamma^\varphi_{\vartheta\varphi} = -\frac{1}{\sin^2\vartheta} \\
\partial_\varphi \Gamma^\alpha_{\varphi\alpha} &= 0
\end{aligned}$$

The fourth gives:

$$\begin{aligned}
\Gamma^\alpha_{\nu t}\Gamma^\nu_{t\alpha} &= \Gamma^t_{tt}\Gamma^t_{tt} + 2\Gamma^r_{tt}\Gamma^t_{tr} + \Gamma^r_{tr}\Gamma^r_{tr} \\
&= \dot{\mu}^2 + 2\mu'^2 e^{2(\mu-\lambda)} + \dot{\lambda}^2 \\
\Gamma^\alpha_{\nu r}\Gamma^\nu_{r\alpha} &= \Gamma^t_{rt}\Gamma^t_{rt} + 2\Gamma^r_{rt}\Gamma^t_{rr} + \Gamma^r_{rr}\Gamma^r_{rr} + \Gamma^\vartheta_{r\vartheta}\Gamma^\vartheta_{r\vartheta} + \Gamma^\varphi_{r\varphi}\Gamma^\varphi_{r\varphi} \\
&= \mu'^2 + 2\dot{\lambda}^2 e^{2(\lambda-\mu)} + \lambda'^2 + \frac{2}{r^2} \\
\Gamma^\alpha_{\nu\vartheta}\Gamma^\nu_{\vartheta\alpha} &= \Gamma^\varphi_{\vartheta\varphi}\Gamma^\varphi_{\vartheta\varphi} + 2\Gamma^\vartheta_{\vartheta r}\Gamma^r_{\vartheta\vartheta} = \frac{1}{\tan^2\vartheta} - 2e^{-2\lambda} \\
\Gamma^\alpha_{\nu\varphi}\Gamma^\nu_{\varphi\alpha} &= 2\Gamma^\varphi_{\varphi r}\Gamma^r_{\varphi\varphi} + 2\Gamma^\varphi_{\varphi\vartheta}\Gamma^\vartheta_{\varphi\varphi} = -2e^{-2\lambda}\sin^2\vartheta - 2\cos^2\vartheta
\end{aligned}$$

Now, we can calculate the diagonal entries of the Ricci tensor:

$$\begin{aligned}
R_{tt} &= \dot{\lambda}\dot{\mu} - \ddot{\lambda} - \dot{\lambda}^2 + e^{2(\mu-\lambda)}\left(-\lambda'\mu' + \mu'' + \mu'^2 + \frac{2\mu'}{r}\right) \\
R_{rr} &= \lambda'\mu' + \frac{2\lambda'}{r} - \mu'' - \mu'^2 + e^{2(\lambda-\mu)}\left(-\dot{\lambda}\dot{\mu} + \ddot{\lambda} + \dot{\lambda}^2\right) \\
R_{\vartheta\vartheta} &= e^{-2\lambda}(\lambda' r - \mu' r - 1) + 1 \\
R_{\varphi\varphi} &= \sin^2\vartheta\left(e^{-2\lambda}\left(\lambda' r - \mu' r - 1\right) + 1\right)
\end{aligned}$$

With this at hand we can also calculate the Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} = 2e^{-2\mu}\left(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\mu}\right) + 2e^{-2\lambda}\left(\lambda'\mu' - \mu'' - \mu'^2 - \frac{2\mu'}{r} + \frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{2}{r^2}. \quad \text{(A.3)}$$

And the first three diagonal entries of the Einstein tensor

$$\begin{aligned}
G_{tt} &= e^{2(\mu-\lambda)}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{2\mu}}{r^2}, && \text{(A.4)} \\
G_{rr} &= \frac{1}{r^2}\left(2r\mu' + 1 - e^{2\lambda}\right), && \text{(A.5)} \\
G_{\vartheta\vartheta} &= re^{-2\lambda}(\mu' + r\mu'^2 - \lambda'(1 + r\mu') + r\mu''). && \text{(A.6)}
\end{aligned}$$

B The Christoffel symbols in Cartesian coordinates

In this section the Christoffel symbols are calculated in Cartesian coordinates. The matrix representation of the metric is

$$(g_{\mu\nu}) = \begin{pmatrix} -e^{2\mu} & 0 & 0 & 0 \\ 0 & 1+\frac{e^{2\lambda}-1}{r^2}x^2 & \frac{e^{2\lambda}-1}{r^2}xy & \frac{e^{2\lambda}-1}{r^2}xz \\ 0 & \frac{e^{2\lambda}-1}{r^2}yx & 1+\frac{e^{2\lambda}-1}{r^2}y^2 & \frac{e^{2\lambda}-1}{r^2}yz \\ 0 & \frac{e^{2\lambda}-1}{r^2}zx & \frac{e^{2\lambda}-1}{r^2}zy & 1+\frac{e^{2\lambda}-1}{r^2}z^2 \end{pmatrix}. \tag{B.1}$$

The inverse metric reads

$$(g^{\mu\nu}) = \begin{pmatrix} -e^{-2\mu} & 0 & 0 & 0 \\ 0 & \frac{e^{-2\lambda}x^2+y^2+z^2}{r^2} & \frac{e^{-2\lambda}-1}{r^2}xy & \frac{e^{-2\lambda}-1}{r^2}xz \\ 0 & \frac{e^{-2\lambda}-1}{r^2}yx & \frac{x^2+e^{-2\lambda}y^2+z^2}{r^2} & \frac{e^{-2\lambda}-1}{r^2}yz \\ 0 & \frac{e^{-2\lambda}-1}{r^2}zx & \frac{e^{-2\lambda}-1}{r^2}zy & \frac{x^2+y^2+e^{-2\lambda}z^2}{r^2} \end{pmatrix}. \tag{B.2}$$

For the calculation of the Christoffel symbols, one will need the derivatives of the entries of $(G_{\mu\nu})$. We have

$$\begin{aligned} \partial_a g_{tt} &= -2\mu'\frac{a}{r}e^{2\mu} \\ \partial_t g_{ab} &= \frac{ab}{r^2}2\dot{\lambda}e^{2\lambda} \\ \partial_c g_{ab} &= \partial_c(ab)\frac{e^{2\lambda}-1}{r^2} + abc\frac{(2r\lambda'-2)e^{2\lambda}+2}{r^4} \end{aligned}$$

Now, we calculate all the Christoffel symbols:

$$\begin{aligned} \Gamma^x_{tt} &= e^{2(\mu-\lambda)}\frac{\mu' x}{r} \qquad \Gamma^x_{tx} = \frac{\dot{\lambda}x^2}{r^2} \qquad \Gamma^x_{ty} = \frac{\dot{\lambda}xy}{r^2} \qquad \Gamma^x_{tz} = \frac{\dot{\lambda}xz}{r^2} \\ \Gamma^x_{xx} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}x^3 + \frac{1-e^{-2\lambda}}{r^2}x \\ \Gamma^x_{xy} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}x^2y \\ \Gamma^x_{xz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}x^2z \\ \Gamma^x_{yy} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xy^2 + \frac{1-e^{-2\lambda}}{r^2}x \\ \Gamma^x_{yz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xyz \\ \Gamma^x_{zz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xz^2 + \frac{1-e^{-2\lambda}}{r^2}x \end{aligned}$$

$$\begin{aligned}
\Gamma^y_{tt} &= e^{2(\mu-\lambda)}\frac{\mu' y}{r} \qquad \Gamma^y_{tx} = \frac{\dot\lambda xy}{r^2} \qquad \Gamma^y_{ty} = \frac{\dot\lambda y^2}{r^2} \qquad \Gamma^y_{tz} = \frac{\dot\lambda yz}{r^2}\\
\Gamma^y_{xx} &= \frac{y}{r^4}\left((1-e^{-2\lambda})(y^2+z^2)+\lambda' x^2 r\right)\\
\Gamma^y_{xy} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xy^2\\
\Gamma^y_{xz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xyz\\
\Gamma^y_{yy} &= \frac{y}{r^4}\left(1-e^{-2\lambda})(x^2+z^2)+\lambda' y^2 r\right)\\
\Gamma^y_{yz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}y^2 z\\
\Gamma^y_{zz} &= \frac{y}{r^4}\left((1-e^{-2\lambda})(x^2+y^2)+\lambda' z^2 r\right)
\end{aligned}$$

$$\begin{aligned}
\Gamma^z_{tt} &= e^{2(\mu-\lambda)}\frac{\mu z}{r} \qquad \Gamma^z_{tx} = \frac{\dot\lambda xz}{r^2} \qquad \Gamma^z_{ty} = \frac{\dot\lambda yz}{r^2} \qquad \Gamma^z_{tz} = \frac{\dot\lambda z^2}{r^2}\\
\Gamma^z_{xx} &= \frac{z}{r^4}\left((1-e^{-2\lambda})(y^2+z^2)+\lambda' x^2 r\right)\\
\Gamma^z_{xy} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xyz\\
\Gamma^z_{xz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}xz^2\\
\Gamma^z_{yy} &= \frac{z}{r^4}\left(1-e^{-2\lambda})(x^2+z^2)+\lambda' y^2 r\right)\\
\Gamma^z_{yz} &= \frac{r\lambda'-1+e^{-2\lambda}}{r^4}yz^2\\
\Gamma^z_{zz} &= \frac{z}{r^4}\left((1-e^{-2\lambda})(x^2+y^2)+\lambda' z^2 r\right)
\end{aligned}$$

C Coordinate transformation of the Vlasov equation

At first we have to calculate the commutation functions defined by

$$[e_b, e_c] = \gamma^a_{bc} e_a. \tag{C.1}$$

It is easy to see that $[\partial_t, \partial_t]f = 0$, thus $\gamma^\alpha_{00} = 0$. Next we consider

$$[\partial_t, e_a] = -\dot\lambda \frac{e^{-\lambda}}{r^2} x_a x^i \partial_i. \tag{C.2}$$

We already see that $\gamma^0_{0b} = 0$. In order to find the γ^a_{0b}, we need to express the basis vectors ∂_i in terms of the e_j via coefficients: $\partial_i = c^j_i e_j$. (The coefficients for the other way round are already given: $e_i = e_i^j \partial_j$.) For the coefficients c_i^j one finds

$$\begin{aligned}
c_1^1 &= \frac{e^\lambda x^2 + y^2 + z^2}{r^2}, && \text{(C.3)}\\
c_1^2 &= c_2^1 = \frac{(e^\lambda - 1)xy}{r^2}, && \text{(C.4)}\\
c_1^3 &= c_3^1 = \frac{(e^\lambda - 1)xz}{r^2}, && \text{(C.5)}\\
c_2^2 &= \frac{x^2 + e^\lambda y^2 + z^2}{r^2}, && \text{(C.6)}\\
c_2^3 &= c_3^2 = \frac{(e^\lambda - 1)yz}{r^2}, && \text{(C.7)}\\
c_3^3 &= \frac{x^2 + y^2 + e^\lambda z^2}{r^2}. && \text{(C.8)}
\end{aligned}$$

Plugging this into equation (C.2) yields

$$\gamma_{0b}^a = -\dot{\lambda}\frac{e^{-\lambda}}{r^2} x^a x^b.$$

Note that always $\gamma_{bc}^a = -\gamma_{cb}^a$ holds and in our case also $\gamma_{0b}^a = \gamma_{0a}^b$.

Finally we consider

$$[e_a, e_b]f = \cdots = \underbrace{\frac{e^{-\lambda} - 1}{r^2}(x_j \delta_i^k - x_i \delta_j^k)}_{\gamma_{ij}^k} \underbrace{\left(\delta_k^c + \frac{e^{-\lambda} - 1}{r^2} x^c x_k\right)\partial_c f}_{e_k(f)}\,.$$

Now we are able to calculate the connection coefficients using the formula (3.21). The for the Vlasov equation relevant ones are

$$\begin{aligned}
\Gamma_{a00} &= \mu' e^{2\mu - \lambda}\frac{x^a}{r}; \qquad \Gamma_{ab0} = 0; \qquad \Gamma_{a0c} = \dot{\lambda}\frac{x^a x^c}{r^2};\\
\Gamma_{abc} &= \frac{e^{-\lambda} - 1}{r^2}(x_b \delta_{ac} - x_a \delta_{bc}).
\end{aligned}$$

Now we are able to write down the Vlasov equation in this frame. Keep in mind that $p^0 = e^{-\mu}\sqrt{1 + |v|^2}$ and $p^a = v^a \frac{e^{-\lambda} - 1}{r} v_r x^a$ and set $v^0 = p^0$.

$$\begin{aligned}
0 &= p^0 \frac{\partial f}{\partial x^0} + p^a \frac{\partial f}{\partial x^a} - \Gamma_{\beta\gamma}^\alpha v^\beta v^\gamma \frac{\partial f}{\partial v^a}\\
\Rightarrow \quad 0 &= \frac{\partial f}{\partial t} + e^\mu (1 + |v|^2)^{-1/2}\left(v^a + \frac{e^{-\lambda} - 1}{r} v_r x^a\right)\frac{\partial f}{\partial x^a}\\
&\quad - \left(e^{\mu - \lambda}\mu' \frac{x^a}{r}\sqrt{1 + |v|^2} + \dot{\lambda}\frac{v_r x^a}{r} + \frac{e^\mu}{\sqrt{1 + |v|^2}}\frac{e^{-\lambda} - 1}{r^2}(r v_r v^a - |v|^2 x^a)\right)\frac{\partial f}{\partial v^a}
\end{aligned}$$

If the distribution function f is spherically symmetric i.e. $f(t, \vec{x}, \vec{v}) = f(t, A\vec{x}, A\vec{v})$ for all $A \in$

$SO(3)$, the distribution function f merely depends on r, v_r, and $|v|$. It follows that

$$\begin{aligned}(|v|^2x^a - rv_rv^a)\frac{\partial f}{\partial v^a} &= (|v|^2x^a - rv_rv^a)\left(\frac{\partial |v|}{\partial v^a}\frac{\partial f}{\partial |v|} + \frac{\partial v_r}{\partial v^a}\frac{\partial f}{\partial v_r}\right)\\ &= (r|v|^2 - v_r^2)\frac{\partial f}{\partial v_r};\\ (r^2v^a - rv_rx^a)\frac{\partial f}{\partial x^a} &= (r^2v^a - rv_rx^a)\left(\frac{\partial r}{\partial x^a}\frac{\partial f}{\partial r} + \frac{\partial v_r}{\partial x^a}\frac{\partial f}{\partial v_r}\right)\\ &= (r|v|^2 - v_r^2)\frac{\partial f}{\partial v_r}.\end{aligned}$$

So

$$(r^2v^a - rv_rx^a)\frac{\partial f}{\partial x^a} = (|v|^2x^a - rv_rv^a)\frac{\partial f}{\partial v^a} \tag{C.9}$$

and the Vlasov equation becomes

$$\frac{\partial f}{\partial t} + \frac{e^{\mu-\lambda}}{\sqrt{1+|v|^2}}vb^a\frac{\partial f}{\partial x^a} - \frac{x^a}{r}\left(\dot{\lambda}v_r + e^{\mu-\lambda}\sqrt{1+|v|^2}\mu'\right)\frac{\partial f}{\partial v^a} = 0. \tag{C.10}$$

If one is only interested in the static case, one sets all time derivatives to zero and obtains the simplified equation

$$\frac{v^a}{\sqrt{1+|v|^2}}\frac{\partial f}{\partial x^a} - \sqrt{1+|v|^2}\mu'\frac{x^a}{r}\frac{\partial f}{\partial v^a} = 0. \tag{C.11}$$

D Proof that T acts as a contraction

In order to show that the operator T, defined in (5.3), acts as a contraction on the set M, defined in (5.4), one has to check

(i) $u \equiv y_0 \in M$,

(ii) $u \in M \Rightarrow Tu \in M$, and

(iii) $\exists a \in (0,1)\ \forall u, v \in M:\ \|Tu - Tv\|_{\infty,\delta} \leq a\|u - v\|_{\infty,\delta}$, where $\|\cdot\|_{\infty,\delta} = \sup_{r\in[0,\delta]}(.)$.

(i): Consider $u \equiv y_0$. Only the second critical condition

$$\frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2G_\phi(s, u(s))\mathrm{d}s \leq c \tag{D.1}$$

is relevant. We calculate

$$\frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2G_\phi(s, u(s))\mathrm{d}s \leq \frac{r^2\Lambda}{3} + \frac{8\pi r^2}{3}G_\phi(\delta, y_0) \leq \frac{\Lambda + 8\pi G_\phi(r, y_0)}{3}\delta^2 \leq c$$

for δ small enough.
(ii): We have to guarantee that $y_0 - 1 \leq (Tu)(r) \leq y_0 + 1$ and

$$\frac{r^2\Lambda}{3} + \frac{8\pi}{r}\int_0^r s^2G_\phi(s, Tu(s))\mathrm{d}s \leq c.$$

By choosing δ sufficiently small, one can achieve the domain of integration in T to become arbitrarily small and these properties follow.
(iii): We calculate

$$
\begin{aligned}
&\|Tu - Tv\|_{\infty,\delta} \\
&= \Bigg\| \int_0^r \Bigg[\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma, u(\sigma))\mathrm{d}\sigma} \Bigg(s(H_\phi(s,u(s)) - H_\phi(s,v(s))) \\
&\qquad + \frac{1}{s^2}\int_0^s \sigma^2 (G_\phi(\sigma,u(\sigma)) - G_\phi(\sigma,v(\sigma)))\mathrm{d}\sigma \Bigg) \\
&\quad + \Bigg(sH_\phi(s,v(s)) - \frac{s\Lambda}{12\pi} + \frac{1}{s^2}\int_0^s \sigma^2 G_\phi(\sigma,v(\sigma))\mathrm{d}\sigma \Bigg) \\
&\quad \times \Bigg(\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma,u(\sigma))\mathrm{d}\sigma} - \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma,v(\sigma))\mathrm{d}\sigma} \Bigg) \Bigg] \mathrm{d}s \Bigg\|_{\infty,\delta}.
\end{aligned}
$$

Since $G_\phi(r,u)$, $H_\phi(r,u)$, $\partial_u G_\phi(r,u)$, and $\partial_u H_\phi(r,u)$ are functions strictly increasing in u we have

$$
\begin{aligned}
\sup_{u\in[y_0-1,y_0+1]} H_\phi(r,u) &= H_\phi(r,y_0+1) =: H_{\sup}(r), \\
\sup_{u\in[y_0-1,y_0+1]} G_\phi(r,u) &= G_\phi(r,y_0+1) =: G_{\sup}(r), \\
\sup_{u\in[y_0-1,y_0+1]} |\partial_u H_\phi(r,u)| &= |\partial_u H_\phi(r,y_0+1)| =: G'_{\sup}(r), \\
\sup_{u\in[y_0-1,y_0+1]} |\partial_u G_\phi(r,u)| &= |\partial_u G_\phi(r,y_0+1)| =: H'_{\sup}(r).
\end{aligned}
$$

We can estimate the first summand in the following way:

$$
\begin{aligned}
&\int_0^r \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma,u(\sigma))\mathrm{d}\sigma} \\
&\quad \times \Big(s(H_\phi(s,u(s)) - H_\phi(s,v(s))) + \frac{1}{s^2}\int_0^s \sigma^2 (G_\phi(\sigma,u(\sigma)) - G_\phi(\sigma,v(\sigma)))\mathrm{d}\sigma \Big) \mathrm{d}s \\
&\le \frac{8\pi}{2(1-c)} \frac{\delta^2}{2} \left(H'_{\sup}(\delta) + \frac{1}{3} G'_{\sup}(\delta) \right) \|u - v\|_{\infty,\delta}.
\end{aligned}
$$

Next, we consider the second summand:

$$
\begin{aligned}
&\int_0^r \left(sH_\phi(s,v(s)) - \frac{s\Lambda}{12\pi} + \frac{1}{s^2}\int_0^s \sigma^2 G_\phi(\sigma,v(\sigma))\mathrm{d}\sigma \right) \\
&\quad \times \Bigg(\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma,u(\sigma))\mathrm{d}\sigma} - \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{8\pi}{s}\int_0^s \sigma^2 G_\phi(\sigma,v(\sigma))\mathrm{d}\sigma} \Bigg) \mathrm{d}s \\
&\le \int_0^r s \left(H_{\sup}(r) - \frac{\Lambda}{12\pi} + \frac{1}{3} G_{\sup}(r) \right) \frac{(8\pi)^2 s^2}{6(1-2c+c^2)} \mathrm{d}s \; G'_{\sup}(r) \; \|u - v\|_{\infty,\delta} \\
&\le \frac{(8\pi)^2 \delta^4}{24(1-2c+c^2)} \left(H_{\sup}(\delta) - \frac{\Lambda}{12\pi} + \frac{1}{3} G_{\sup}(\delta) \right) G'_{\sup}(\delta) \; \|u - v\|_{\infty,\delta}.
\end{aligned}
$$

So we get in total

$$\|Tu - Tv\|_{\infty,\delta} \leq \left(\frac{2\pi}{(1-c)} \left(H'_{\sup}(\delta) + \frac{1}{3} G'_{\sup}(\delta) \right) \delta^2 \right.$$
$$\left. + \frac{(8\pi)^2}{24(1-2c+c^2)} \left(H_{\sup}(\delta) - \frac{\Lambda}{12\pi} + \frac{1}{3} G_{\sup}(\delta) \right) G'_{\sup}(\delta)\delta^4 \right) \|u - v\|_{\infty,\delta}.$$

If one actually wants to calculate δ one can make use of the estimate

$$\begin{aligned} G_\phi(r,u) &= c_\ell r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^{\infty} \phi\left(1 - \varepsilon e^{-y}\right) \varepsilon^2 \left(\varepsilon^2 - \left(1 + \frac{L_0}{r^2}\right)\right)^{\ell+\frac{1}{2}} \mathrm{d}\varepsilon \\ &\leq c_\ell r^{2\ell} \int_1^\infty \phi\left(1 - \varepsilon e^{-y}\right) \varepsilon^2 \left(\varepsilon^2 - 1\right)^{\ell + \frac{1}{2}} \mathrm{d}\varepsilon \end{aligned} \tag{D.2}$$

and the analogue one for H_ϕ to obtain a polynomial in δ.

E Estimate of $|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|$

The following calculation is valid for $r \in [0, \tilde{r}^*]$ where we can take for granted $1 - \frac{2m(r)}{r} \geq \frac{1}{9}$ (Buchdahl inequality, cf. [2]), $1 - \frac{r^2\Lambda}{3} - \frac{2m_\Lambda(r)}{r} \geq \frac{1}{18}$ and $|y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|$, $\Delta R > 0$ where R_0 is defined to be the (first) zero of the background solution y. Since

$$\begin{aligned} &|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \\ &\quad \leq \left(\sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u G_\phi(r,u)| + \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u H_\phi(r,u)| \right) |y_\Lambda(r) - y(r)| \end{aligned} \tag{E.1}$$

we calculate

$$\begin{aligned} |y_\Lambda(r) - y(r)| &\leq \int_0^r |y'(s) - y'_\Lambda(s)| \mathrm{d}s \\ \leq \int_0^r \Bigg[& \underbrace{\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}}}_{\leq 72\pi} \\ & \times \Bigg(\left| -\frac{s\Lambda}{12\pi} \right| + s|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| \\ & \quad \underbrace{+ \frac{1}{s^2} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| \mathrm{d}\sigma}_{I_1} \Bigg) \\ & + \left(sH_\phi(s, y(s)) + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, y(\sigma)) \mathrm{d}\sigma \right) \underbrace{\left(\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}} - \frac{4\pi}{1 - \frac{2m(s)}{s}} \right)}_{I_2} \Bigg] \mathrm{d}s. \end{aligned}$$

We estimate I_1 and I_2 separately:

$$\begin{aligned} I_1 &= \int_0^r \frac{1}{s^2}\int_0^s \sigma^2|G_\phi(\sigma,y_\Lambda(\sigma))-G_\phi(\sigma,y(\sigma))|\mathrm{d}\sigma\mathrm{d}s \\ &\leq \int_0^r\int_0^r |G_\phi(\sigma,y_\Lambda(\sigma))-G_\phi(\sigma,y(\sigma))|\mathrm{d}\sigma\mathrm{d}s \\ &\leq r\int_0^r |G_\phi(\sigma,y_\Lambda(\sigma))-G_\phi(\sigma,y(\sigma))|\mathrm{d}\sigma, \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{4\pi}{1-\frac{s^2\Lambda}{3}-\frac{2m_\Lambda(s)}{s}}-\frac{4\pi}{1-\frac{2m(s)}{s}} \\ &\leq 4\pi\cdot 18\cdot 9\cdot\left(\frac{s^2\Lambda}{3}+\frac{8\pi}{s}\int_0^s \sigma^2|G_\phi(\sigma,y_\Lambda(\sigma))-G_\phi(\sigma,y(\sigma))|\mathrm{d}\sigma\right) \\ &\leq 648\pi\left(\frac{s^2\Lambda}{3}+8\pi s\int_0^s |G_\phi(\sigma,y_\Lambda(\sigma))-G_\phi(\sigma,y(\sigma))|\mathrm{d}\sigma\right). \end{aligned}$$

So using that y is decreasing we have

$$\begin{aligned} &|y_\Lambda(r)-y(r)| \\ &\leq \Lambda\int_0^r\left(6s+216\pi s^3\left(H_\phi(r,y_0)+\frac{1}{3}G_\phi(r,y_0)\right)\right)\mathrm{d}s \\ &\quad +72\pi r\int_0^r |H_\phi(s,y_\Lambda(s))-H_\phi(s,y(s))|\mathrm{d}s \\ &\quad +\left(72\pi r+5184\pi^2\frac{r^3}{3}\left(H_\phi(r,y_0)+\frac{1}{3}G_\phi(r,y_0)\right)\right)\int_0^r |G_\phi(s,y_\Lambda(s))-G_\phi(s,y(s))|\mathrm{d}s \\ &\leq \Lambda\left(3r^2+54\pi r^4\left(H_\phi(r,y_0)+\frac{1}{3}G_\phi(r,y_0)\right)\right) \\ &\quad +\left(72\pi r+1728\pi^2 r^3\left(H_\phi(r,y_0)+\frac{1}{3}G_\phi(r,y_0)\right)\right) \\ &\quad \times\int_0^r(|H_\phi(s,y_\Lambda(s))-H_\phi(s,y(s))|+|G_\phi(s,y_\Lambda(s))-G_\phi(s,y(s))|)\mathrm{d}s \\ &\leq \Lambda C_1(r)+C_2(r)\int_0^r\left(|p_\Lambda(s)-p(s)|+|\varrho_\Lambda(s)-\varrho(s)|\right)\mathrm{d}s \end{aligned}$$

The derivatives with respect to y of $G_\phi(r,y)$ and $H_\phi(r,y)$ are strictly increasing both in r and y. And since $|y_\Lambda(r)-y(r)|\leq|y(R_0+\Delta R)|$ we can write

$$\begin{aligned} &\left(\sup_{u\in[y_\Lambda(r),y(r)]}|\partial_u G_\phi(r,u)|+\sup_{u\in[y_\Lambda(r),y(r)]}|\partial_u H_\phi(r,u)|\right) \\ &\qquad\leq \left|\partial_u G_\phi(\tilde{r}^*,u)|_{y_0+|y(R_0+\Delta R)|}\right|+\left|\partial_u H_\phi(\tilde{r}^*,u)|_{y_0+|y(R_0+\Delta R)|}\right| =: C_3. \end{aligned}$$

So we have obtained that equation (E.1) is of the form

$$|p_\Lambda(s)-p(s)|+|\varrho_\Lambda(s)-\varrho(s)|\leq C_4(r)\Lambda+C_5(r)\int_0^r\left(|p_\Lambda(s)-p(s)|+|\varrho_\Lambda(s)-\varrho(s)|\right)\mathrm{d}s.$$

Note that $C_4(r)$ is strictly increasing. Grönwall's inequality yields

$$(|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|) \leq C_4(r)e^{\int_0^r C_5(r)\mathrm{d}s} = C_4(r)\Lambda e^{rC_5(r)} =: C_{gh}(r)\Lambda. \tag{E.2}$$

Note that $C_{gh}(r)$ is increasing when r is increasing.

References

[1] H. Andréasson, G. Rein, *On the steady states of the spherically symmetric Einstein-Vlasov system*, Class. Quantum Grav. **24**, 1809-1832 (2007)

[2] H. Andréasson, *Sharp bounds on 2m/r of general spherically symmetric static objects*, Jour. Diff. Eq., Vol. **245**, Issue 8, 2243–2266 (2008)

[3] H. Andréasson, *Sharp Bounds on the Critical Stability Radius for Relativistic Charged Spheres*, Commun. Math. Phys. **288**, 715-730 (2009)

[4] H. Andréasson & C. Böhmer, *Bounds on M/R for static objects with a positive cosmological constant*, Class. Quantum Grav. **26** (2009) 195007 (11pp)

[5] H. Andréasson, M. Kunze & G. Rein, *Global Existence for the Spherically Symmetric Einstein-Vlasov System with Outgoing Matter*, Commun. Part. Diff. Eqns. **33**, 656-668 (2008)

[6] H. Andréasson & G. Rein, *A numerical investigation of the steady states of the spherically symmetric Einstein-Vlasov-Maxwell system*, Class. Quantum Grav. **24** (2009) 145003 (20pp)

[7] H. Andréasson, M. Kunze & G. Rein, *Gravitational collapse and the formation of black holes for the spherically symmetric Einstein-Vlasov system*, Q.Appl.Math. **68**, 17-42 (2010)

[8] H. Andréasson & M. Kunze, *The formation of black holes in spherically symmetric gravitational collapse*, Math. Ann., **350**, 683-705 (2011)

[9] H. Andréasson, M. Kunze & G. Rein, *Existence of axially symmetric static solutions of the Einstein-Vlasov system*, Commun. Math. Phys. **308**, 23-47 (2011)

[10] H. Andréasson, *The Einstein-Vlasov System/Kinetic Theory*, Living Rev. Relativity **24**, 4-55 (2011)

[11] H. Andréasson, C. Böhmer & A. Mussa, *Bounds on M/R for charged objects with positive cosmological constant*, Class. Quantum Grav. **29** (2012) 095012 (10pp)

[12] H. Andréasson, M. Kunze & G. Rein, *Rotating, stationary, axially symmetric spacetimes with collisionless matter*, Commun. Math. Phys. **329**, 787-808 (2014)

[13] H. Andréasson & G. Rein, *On the rotation curves for axially symmetric disk solutions of the Vlasov-Poisson system*, arXiv:1406.7163v1 [astro-ph.GA] 27 Jun 2014

[14] H. Andréasson, D. Fajman & M. Thaller, *Static solutions to the Einstein-Vlasov system with non-vanishing cosmological constant*, in preparation

[15] J. Batt, W. Faltenbacher & E. Horst, *Stationary spherically symmetric models in stellar dynamics*, Arch. Ratioal. Mech. Anal. **93**, 159-183 (1980)

[16] S. L. Bażański & V. Ferrari, *Analytic Extension of the Schwarzschild-de Sitter Metric*, Il Nuovo Cimento Vol. **91** B, No.1 (1986)

[17] J. Binney & S. Tremaine, *Galactic Dynamics*, Princeton University Press, 1987

[18] A. Bosma, *21-cm line studies of spiral galaxies. II. The distribution and kinematics of neutral hydrogen in spiral galaxies of various morphological types*, Astrom. J. **86**, 1825-1846 (1981)

[19] S. Carroll, *The Cosmological Constant*, Living Rev. Relativity, **4**, 1-56 (2001)

[20] Y. Choquet-Bruhat, *Problème de Cauchy pour le système intégro différentiel d'Einstein Liouville*, Ann. Inst. Fourier, **21**, 181-201, (1971)

[21] G. F. R. Ellis, S. T. C. Siklos & J. Wainwright, *Dynamical Systems in Cosmology*, Cambridge University Press, 1997

[22] M. Fukugitam, C.J. Hogan & P.J.E. Peebles, *The Cosmic Baryon Budget*, Astrophys. J., **503**, 518-530 (1998)

[23] G. W. Gibbons, S. W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D, **15**, n° 10, 2738-2751 (1977)

[24] E. Horst, *On the asymptotic growth of the solutions of the Vlsov-Poisson system*, Math. Meth. Appl. Sci. **16**, 75-85 (1993)

[25] P. Karageorgis & J. Stalker, *Sharp bounds on $2m/r$ for static spherical objects*, Class. Quatnum Grav. **25** (2008) 195021 (14pp)

[26] A. Melchiorri, *A Measurement of Ω from the North American Test flight of Boomerang*, Astrophys. J. Lett., **536**, L63-L66 (2000)

[27] J.J. Mohr, B. Mathiesen & A.E. Evrard, *Properties of the Intracluster Medium in an Ensemble of Nearby Galaxy Clsuters*, Astrophys. J., **517**, 627-649 (1999)

[28] S. Perlmutter et al. (The Supernova Cosmology Project), *Measurement of Ω and Λ from 42 High-Redshift Supernovae*, Astrophys. J., **517**, 565-586 (1999)

[29] K. Pfaffelmoser, *Global Classical Solutions of the Vlasov-Poisson System in Three Dimensions for General Initial Data*, Jour. Dif. Eq. **95**, 281-303 (1992)

[30] T. Ramming & G. Rein, *Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic case – A simple proof for finite extension*, SIAM J. Math. Anal. **45** (2), 900-914 (2013)

[31] G. Rein & A. D. Rendall, *The Newtonian Limit of the Spherically Symmetric Vlasov-Esintein System*, Commun. Math. Phys. **150**, 585-591 (1991)

[32] G. Rein & A. D. Rendall, *Global Existence of Solutions of the Spherically Symmetric Vlasov–Einstein System with Small Initial Data*, Commun. Math. Phys. 150, 561-583 (1992)

[33] G. Rein & A. D. Rendall, *Smooth static solutions of the spherically symmetric Vlasov-Einstein system*, Annales de l'I.H.P., section A, tome 59, n° **4**, 383-397 (1993)

[34] G. Rein, *Static solutions of the spherically symmetric Vlasov-Einstein system*, Math. Proc. Camb. Phil. Soc. **115**, 559-570 (1994)

[35] G. Rein, A. D. Rendall & J. Schaeffer, *A Regularity Theorem for Solutions of the Spherically Symmetric Vlasov-Einstein System*, Commun. Math. Phys. **168**, 467-478 (1995)

[36] G. Rein, *Static shells for the Vlasov-Poisson and Vlasov-Einstein system*, Indiana University Mathematics Journal **48**, 335-346 (1999)

[37] A. D. Rendall & B. G. Schmidt, *Existence and properties of spherically symmetric static fluid bodies with a given equation of state*, Class. Quantum Grav. **8**, 985-1000 (1991)

[38] A.G. Riess et al., *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, Astrophys. J., **116**, 1009-1038 (1998)

[39] H. Ringström, *On the Topology and Future Stability od the Universe*, Oxford University Press, 978-0-19-968029-0, (2013)

[40] V. c. Rubin, J. Ford, W. K. Thonard & D. Burnstein, *Rotational properties of 23 SB galaxies*, Astron. Rep. **261**, 439-456 (1982)

[41] O. Sarbach & T. Zannias, *The geometry of the tangent bundle and the relativistic kinetic theory of gases*, Class. Quantum Grav. **31** 085013 (34pp) (2014)

[42] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*, Commun. Part. Diff. Eqns. **16**, 1313-1335 (1991)

[43] J. Schaeffer, *A Class of Counter examples to Jeans' Theorem for the Vlasov-Einstein System*, Commun. Math. Phys. **204**, 313-327 (1999)

[44] B.P. Schmidt et al., The High-Z Supernova Search: Measuring Cosmic Decelerationa and Global Curvature of the Universe Using Type Ia Supernovae, Astrophys. J. **507**, 46-63 (1998)

[45] *The Nobel Prize in Physics 2011.* Nobel Foundation. Retrieved 2011-10-04

[46] C. Stanciulescu, *Spherically Symmetric Solutions of the Vacuum Einstein Field Equations with Positive Cosmological Constant*, diploma thesis at the University of Vienna (1998)

Curriculum Vitae

Maximilian Thaller | Schönbrunner Straße 238/1/13 | A-1120 Wien
Telephone (mobile): +43 680 5589073
E-Mail (private): maximilian.thaller@arcor.de
Date of birth: 10.01.1990
Place of birth: Friedrichshafen (DE)
Citizenship: German

Education

Droste-Hülshoff Gymnasium Meersburg

- Majors: Mathematics, German, French, Physics, Religion
- 2008: Abitur, final score: 1,0

University of Constance

- Undergraduate courses in Mathematics, Physics and Informatics
- Bachelor thesis in statistical physics, grade: 1,0
- 2012: Bachelor's degree in Physics, final score: 1,9

University of Vienna

- Master studies in Physics, specialization in General Relativity
- Master thesis on kinetic theory in General Relativity
- Current overall average grade: 1,1

Scientific Experience

Publications

- H. Andréasson, D. Fajman & M. Thaller, *Static solutions to the Einstein-Vlasov system with non-vanishing cosmological constant,* in preparation (2014)

Summer schools and Conferences

- 2014 - Participation at the 4th Central European Seminar on Mathematical Relativity, ESI, Vienna
- 2014 - Summer school "Asymptotic Analysis in General Relativity", Institut Fourier, Grenoble
- 2014 - ESI-EMS-IAMP Summer school on Mathematical Relativity, Vienna

Practical Experience

Employments

- 2008 – 2009 - Civil service at the drug-rehab clinic “siebenzwerge”: accounting duties
- 2009 - Internship at IPcopter GmbH & Co. KG: programming tasks
- 2009 - Vacation work at ND SatCom GmbH, department for product verification
- 2010 – 2011 - Student assistant at the University of Constance: Writing of lecture notes
- 2013 - Figlmüller Hoher Markt 10 GmbH: Side job as waiter

Junior enterprise in business consulting: Bodensee Consulting e.V.

- 2011 – 2012 - Department manager “Sales”
- 2011 - Member of the project team “EVP am See GbR”: Execution of a survey among students on how they consider the customer enterprise as a potential future employer
- 2012 - Leader of the project team “Contentio GbR”: Competition analysis for the customer enterprise
- 2012 - Member of the project team “EcoFood GbR”: Elaborating the business plan for a startup enterprise in the catering sector
- Certified junior consultant by the JCNetwork

Awards and Recognitions

2007 - First place with the German team at the Quanta Competition for science, Mathematics, Mental Ability and Electronics in Lucknow, India

2008 - Ferry Porsche Price

2013 - “Leistungsstipendium” of the University of Vienna

Personal Interests and Skills

Languages

- German (native language)
- English (fluent)
- French (good)
- Latin (“Latinum”)

Software: LaTeX, MS Office, MS Project (basics), Wolfram Mathematica, Matlab, C++

Hobbies: Drawing, Literature, Music