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„Semiquantum Correlations and Indefinite Causal Structures for Two Laboratories“

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1 Introduction

The apparent *incompatibility of quantum physics and the theory of relativity* has been the source of an immense amount of theoretical research over the past decades. The search for a quantum theory of gravitation (a result of intrinsic properties of spacetime itself) is one main research interest. The concept of causality (a consequence of a finite signaling speed) and quantum mechanics have also been of interest from very early on.

The famous argument between Einstein and Bohr about the completeness of quantum mechanics, see [1, 2], was due to the apparent instantaneous action at a distance in entangled quantum systems. The violation of Bell's inequalities, quantum teleportation and (especially delayed choice) entanglement swapping, [3, 4, 5] all seem to evade the constraints of causality in some way. One can, however, not make use of this, for example use entanglement to send signals faster than at the speed of light. The seemingly instantaneous action at a distance in the effects mentioned above, however, is an *interpretational problem*. It can be resolved, if one regards quantum theory as the knowledge of the observer rather than a description of the actual system.

A situation in which the *finite signaling speed* should in fact be observable in a *quantum scenario* was first considered by E. Fermi in 1932 [6]. While Fermi's initial (approximative) calculations produced the expected result, they were contradicted later [7]. A correct description of Fermi's two atom scenario, i.e. one which does respect causality, remained an open problem for quite a long time, see for example [8, 9].

A different kind of approach to quantum theory and causality was the application of *quantum information theory to causal structures*. For example David Deutsch, [10], found a quantum theoretical description of *closed timelike curves*. Causal structures containing such curves are possible solutions of Einstein's equations but, classically, are excluded for logical reasons. When describing them quantum mechanically, however, the classical logical paradoxa do not appear. The quantum effects postulated by this model are very different from standard quantum mechanics, for example they include nonlinear evolution or the possibility of cloning quantum systems, but logically consistent.

Another, rather recent, quantum mechanical consideration of causal structures by O. Oreshkov, F. Costa and Č. Brukner led to an even more striking discovery, namely the fact that *signaling quantum correlations* can be *incompatible* with the concept of a *global causal structure* [11]. The most general quantum correlations between two closed laboratories, in which quantum mechanics is valid, can be described by so called *processes*. A *causal inequality* gives an upper bound for a probability associated with a certain communication task. The inequality cannot be violated if the agents in this task are embedded in a causal structure. There

are, however, processes that yield probabilities, which exceed this upper bound and are therefore not compatible with the concept of a global causal structure. This is similar to Bell's inequalities, which give an upper bound for a combination of expectation values and cannot be violated if the respective quantities are compatible with the idea of local realism.

In this thesis *semiquantum correlations* will be analyzed in terms of their compatibility with a global causal structure, making use of the formalism derived in [11]. It is structured as follows.

Chapter 2 gives a brief overview of the formalism of standard quantum mechanics, see 2.1, and the concept of quantum signaling via quantum channels. The classification of quantum channels as *semiquantum*, more precisely *classical-quantum* and *quantum-classical*, is introduced in chapter 2.2. As discussed in chapter 2.3, which is about the operational approach to quantum mechanics, these classifications can be extended to quantum correlations in general. Semiquantum correlations lie between classical and (truly) quantum ones, since one of the two correlated parties can be described classically, while the other one is quantum. Since it sets the framework for the analysis that follows, the contents of the original paper [11] are presented in detail in chapter 3. Processes between two closed laboratories can be represented by *bilinear functions of completely positive, trace non increasing maps* and can be represented by *process matrices*, see 3.2. The possibility to violate the causal inequality is linked to a property of process matrices called *causal separability*. If the two laboratories are related via a causally separable process, they cannot violate the causal inequality regardless of their local operations and communication strategies. One important result of [11] is that all processes between two *classical laboratories* are causally separable and therefore compatible with a global causal structure. The proof of that is presented in detail in 3.4. In chapter 4 the different cases of semiquantum correlations are analyzed. At first correlations between two semiquantum laboratories are considered, see 4.1 - 4.3. *Semiquantum laboratories* can be realized by limiting the operations the experimenters can perform. In a *classical-quantum* laboratory the *input* states have to be measured with a *fixed set of orthogonal projectors*, while the output states are arbitrary. An experimenter in a *quantum-classical* laboratory, on the other hand, can measure the input with any positive operator-valued measure, but has to prepare *compatible outputs*, which means all output states have to commute. Furthermore the correlations between *one classical and one quantum laboratory* are also semiquantum. Finally a semiquantum laboratory and a classical one can be semiquantum or classically correlated. All these scenarios are considered in 4.4. A summary and discussion of the results obtained is presented in chapter 5. It turns out that the *information gained from the input* determines, whether the

process matrices of interest are causally separable. If this information is (quasi) *classical* in both laboratories, the two agents *cannot violate the causal inequality*. Even correlations between a classical and q-c laboratory are not causally separable in general. In fact the example in [11] proves that the causal inequality can indeed be violated in that case. It is therefore doubtful whether signaling correlations, that are incompatible with a global causal structure, are a quantum effect.

2 Basic Notations and Concepts

2.1 Quantum Formalism

The facts in this chapter can be found in any standard textbook about quantum mechanics and quantum information, for example [12, 13], and are mentioned here mainly to clarify the notation used during the rest of this thesis.

A *quantum state* ρ can be expressed as an element of the *space of hermitian, linear operators on some Hilbert space* \mathcal{H}

$$\rho \in \mathcal{L}(\mathcal{H})$$

satisfying the following conditions:

$$\rho \geq 0 \tag{1}$$

$$\text{Tr}(\rho) = 1 \tag{2}$$

$$\text{Tr}(\rho\rho^\dagger) \leq 1 \tag{3}$$

If $\text{Tr}(\rho\rho^\dagger) = 1$ the state is *pure*, which means it can be written as $\rho = |\psi\rangle\langle\psi|$ with some $|\psi\rangle \in \mathcal{H}$. If, however, $\text{Tr}(\rho\rho^\dagger) < 1$ the state is said to be *mixed* and $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $|\psi_i\rangle \in \mathcal{H}$, $0 < p_i < 1$ and $\sum_i p_i = 1$.

In the case of a finite dimensional system, with $\dim(\mathcal{H}) = d$, this corresponds to positive semidefinite $d \times d$ - matrices with unit trace. Hence ρ is also called *density matrix* or *density operator*.

One way of writing ρ is in terms of an *orthonormal basis* (ONB) of \mathcal{H} :

$$\rho = \sum_{i,j=1}^d \rho_{ij} |i\rangle\langle j| \tag{4}$$

$$\text{with } |\phi\rangle = \sum_{i=1}^d \phi_i |i\rangle \quad \forall \phi \in \mathcal{H} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}$$

Another convenient expression for ρ is in the *Hilbert-Schmidt-basis* $\{\sigma_i\}$, where $i = 0 \dots d^2 - 1$, $\sigma_0 = \mathbb{1}_{d \times d}$ and σ_i , for $i > 0$, are the infinitesimal generators of $su(d)$. This means that $\text{Tr}(\sigma_i) = 0$ and $\text{Tr}(\sigma_i \sigma_j) = d \delta_{ij}$ for $i, j > 0$.

$$\rho = \sum_{i=0}^{d^2-1} x_i \sigma_i \quad (5)$$

where $x_i \in \mathbb{R}$ and $\sum_i x_i^2 \leq 1$.

Observables of a quantum system are also hermitian, linear operators on the Hilbert space \mathcal{H} .

$$A \in \mathcal{L}(\mathcal{H})$$

Possible *measurement results* of the observable are elements of the *spectrum* of A . In the finite dimensional case, which will be considered from now on, the observables are self adjoint $d \times d$ - matrices and their spectra are the sets of their *eigenvalues*, $\{a_i\}$. The eigenvectors of an observable, $A|i\rangle = a_i|i\rangle$, form an ONB of \mathcal{H} . The spectral decomposition of A is therefore given by

$$A = \sum_{i=1}^d a_i |i\rangle\langle i| \quad (6)$$

where the $|i\rangle\langle i|$ are a complete set of orthogonal projectors, which means they satisfy

$$\sum_{i=1}^d |i\rangle\langle i| = \mathbb{1}, \quad (7)$$

$$|i\rangle\langle i|j\rangle\langle j| = \delta_{ij} |i\rangle\langle j|. \quad (8)$$

The probability of measuring a non degenerate eigenvalue a_j of observable A in a state ρ is given by

$$P_\rho(a_j) = \text{Tr}(|j\rangle\langle j|\rho) = \langle j|\rho|j\rangle = \rho_{jj}. \quad (9)$$

A single measurement of observable A on a quantum system ρ giving the result a_j projects the system onto the corresponding eigenstate $|j\rangle$:

$$\rho \xrightarrow[\text{result: } a_j]{A} |j\rangle\langle j| = \frac{1}{P_\rho(a_j)} |j\rangle\langle j|\rho|j\rangle\langle j|^\dagger \quad (10)$$

When performing measurement series on identical systems the measurement results a_i occur with the respective probabilities $P_i = \rho_{ii}$. The expectation value of A in the state ρ is therefore

$$\langle A \rangle_\rho = \text{Tr}(A\rho) = \sum_{i=1}^d a_i \rho_{ii} \quad (11)$$

and the statistical dispersion of measurements is given by the standard deviation

$$\Delta A_\rho = \sqrt{\langle A^2 \rangle_\rho - \langle A \rangle_\rho^2} = \sqrt{\text{Tr}(A^2\rho) - \text{Tr}(A\rho)^2}. \quad (12)$$

From this probabilistic point of view it is clear that equations (1) and (2) express the facts that probabilities should be non negative and the probability of the quantum system to be observed at all should be unity.

The time evolution of a quantum system is given by the von-Neumann-equation

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \quad (13)$$

where H is the *Hamiltonian* of the system. Generally this leads to some *unitary operator* $U(t) \in \mathcal{L}^0(\mathcal{H})$, where $\mathcal{L}^0(\mathcal{H})$ is the space of linear operators on \mathcal{H} . The state of the system at a time $t > 0$ arises from the state of the system at time $t = 0$ in the following way:

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \quad (14)$$

If the Hamiltonian is time independent, the unitary operator of the time evolution is given by $U(t) = \exp\left(-\frac{i}{\hbar}tH\right)$.

A more general type of measurement involving combinations with ancilla systems and unitary transformations of the joint system is described by *positive operator-valued measures*, POVM's.

A POVM is any *complete set of positive operators* $\{M_i\}$, depending on possible measurement results i ,

$$M_i = E_i^\dagger E_i \quad ; \quad \sum_i M_i = \mathbb{1} \quad (15)$$

such that the following properties are satisfied:

- The probability of measuring result i on a system ρ is

$$P_\rho(i) = \text{Tr}\left(E_i \rho E_i^\dagger\right) \quad . \quad (16)$$

- After measuring result i the system is in the state

$$\frac{1}{P_\rho(i)} E_i \rho E_i^\dagger . \quad (17)$$

The measurement of an observable A , as described above, is then a special case of a POVM-measurement, where the operators M_i in (15) are the orthogonal projectors from the spectral decomposition, $E_i = |i\rangle\langle i|$. Such measurements are therefore also called *projective measurements*.

2.2 Signaling and Quantum Channels

A signal is the transfer of information from a sender to a receiver, which allows the establishment of correlations between events that would otherwise be uncorrelated. If the sender and the receiver can each freely choose some random variable, creating statistical correlations between these random variables can be regarded as the definition of *signaling*.

In quantum mechanics information is transmitted from a sender to a receiver via quantum channels, see [14, 15] for a nice overview. A quantum channel transforms the *input* state (of the sender) to the *output* state (of the receiver).

If one accounts for noise by considering interaction with the environment, figure 1 depicts a quantum channel. While some input state $\rho_{in} \in \mathcal{H}^A$ is sent through the channel it interacts with the environment $\rho_e \in \mathcal{H}^E$. This process is described by a unitary transformation on the joint system. After that the output state $\rho_{out} \in \mathcal{H}^B$ emerges from the channel. The environment is then in state ρ_f , which might even be an element of a different Hilbert space \mathcal{H}^F , because of the interaction with the transmitted system. The output state can be computed using *decoherence formalism*, see for example [16],

$$\rho_{out} = \text{Tr}_F (U \rho_{in} \otimes \rho_e U^\dagger) , \quad (18)$$

where $\text{Tr}_F(\cdot) = \sum_f \langle f| \cdot |f\rangle$ is the partial trace, with $\{|f\rangle\}$ an ONB of the Hilbert space of the environment \mathcal{H}^F after signal transmission. This can be written in terms of Kraus-operators [17] as follows:

$$\rho_{out} = \sum_i K_i \rho_{in} K_i^\dagger \quad (19)$$

Kraus-operators are maps from the input- to the output-space, $K : \mathcal{H}^A \rightarrow \mathcal{H}^B$, satisfying the relation

$$\sum_i K_i K_i^\dagger = \mathbb{1}^A . \quad (20)$$

They are not unique since

$$\sum_f \langle f|U\rho_{in} \otimes \rho_e U^\dagger|f\rangle = \sum_f K_f \rho_{in} K_f^\dagger$$

they depend on the choice of basis for \mathcal{H}^E . Especially if $\mathcal{H}^E = \mathcal{H}^F$ and $\rho_e = |e\rangle\langle e|$, they are given by $K_f = \langle f|U|e\rangle$. This, of course, does not introduce any ambiguity to ρ_{out} . Note that the operators E_i in (15) are Kraus-operators as well and decoherence can be regarded as a special case of measurements where the ancilla system is the environment and $\mathcal{H}^A = \mathcal{H}^B$.

In general a quantum channel can be identified with (19). It is a map from the input Hilbert space to the output Hilbert space, $\mathcal{C}^{AB} : \mathcal{L}(\mathcal{H}^A) \rightarrow \mathcal{L}(\mathcal{H}^B)$, which satisfies the following conditions:

- **Linearity:**
 Since a channel should be able to transform mixed states in accordance with standard quantum mechanics. That is, if $\rho_{in} = \sum_i P_i \rho_i$, ρ_{out} should be given by $\mathcal{C}(\sum_i P_i \rho_i) = \sum_i P_i \mathcal{C}(\rho_i)$.
- **Preservation of the trace:**
 As a channel maps (normalized) states into (normalized) states, it is required that $\text{Tr}(\rho_{in}) = \text{Tr}(\mathcal{C}(\rho_{in}) = \rho_{out}) = 1$.
- **Complete positivity:**
 A linear map from states to states should, of course, be positive. Complete positivity of the map \mathcal{C} follows from the requirement that one should be able to regard the input state as part of a larger system $\rho \in \mathcal{H}^A \otimes \mathcal{H}^R$, where only the subsystem $\rho_{in} = \text{Tr}_R(\rho)$ is affected by the channel. Thus $\mathcal{C} \otimes \mathcal{I}^R$ should still be a positive map for arbitrary dimensions of \mathcal{H}^R .

One kind of quantum channels, which will be of special importance in this thesis, was discussed and classified by Holevo [18]. They are of the form

$$\mathcal{C}(\rho) = \sum_k \tilde{\rho}_k \text{Tr}(M_k \rho) \quad (21)$$

where $\tilde{\rho}_k$ are quantum states and $\{M_k\}$ is a POVM.

Such channels are called (*quasi*) *classical*, if the POVM consists of orthogonal projectors $M_k = |k\rangle\langle k|$, with $\{|k\rangle\}$ an ONB, and all the $\tilde{\rho}_k$ commute, $[\tilde{\rho}_i, \tilde{\rho}_j] = 0$. A quasi classical quantum channel is equivalent to a classical information channel mapping classical letters $\{k\}$ to classical probability distributions $\{P_{ki}\}$, where $\tilde{\rho}_k = \sum_i P_{ki} |i\rangle\langle i|$ for all k . If $M_k = |k\rangle\langle k|$ but the $\tilde{\rho}_k$ are arbitrary, the channel is

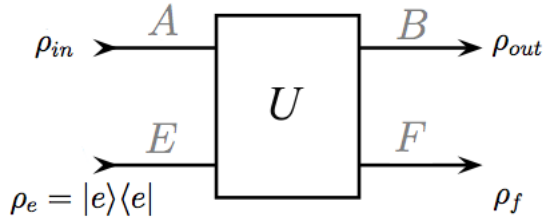


Figure 1: A quantum channel can be regarded as a unitary transformation of the input state $\rho_{in} \in \mathcal{L}(\mathcal{H}^A)$ and the environment $\rho_e \in \mathcal{L}(\mathcal{H}^E)$ leading to an output state $\rho_{out} \in \mathcal{L}(\mathcal{H}^B)$, while the environment is changed to $\rho_f \in \mathcal{L}(\mathcal{H}^F)$. It is assumed without loss of generality that the environment is initially in a pure state. This is possible since one can always purify a mixed state $\rho_{e,mix}$ using an ancilla system $\rho_r \in \mathcal{L}(\mathcal{H}^R)$, that does not interact with ρ_{in} , such that $\rho_{e,mix} = \text{Tr}_R(|e\rangle\langle e|)$ and $U = U' \otimes \mathbb{1}^R$.

called *classical-quantum* (c-q). Such a channel corresponds to a map of a classical input alphabet to quantum states, $\mathcal{C} : k \rightarrow \tilde{\rho}_k$. If, however, $\tilde{\rho}_k = |k\rangle\langle k|$ while the M_k are arbitrary, the channel is called *quantum-classical* (q-c). In this case, the channel is equivalent to a map from quantum states $\tilde{\rho}_k$ to classical probability distributions on letters $\{k\}$, $P(k) = \text{Tr}(M_k \rho)$.

2.3 Operational Concepts

A very practical approach to quantum mechanics is the operational one. There the main objects of interest are the possible *transformations and measurements of quantum systems*, while the *state* of the system is associated with some *preparation procedure*. The latter is significant, because the correct *probabilities for possible measurements* can be deduced from it.

On the one hand this provides a unified formulation of all possible operations experimenters can perform on a quantum system [19], as well as quantum channels, when retaining the Hilbert space representation of states, see for example [20]. All possible operations, like quantum channels, can be regarded as transformations from states to states. Analyzing these transformations provides a new view on the behavior of quantum systems and the correlations between them.

On the other hand the operational approach has also lead to new insights concerning the foundations of quantum mechanics. In [21, 22] quantum physics is considered as a probability theory, which can be derived from simple axioms. The authors start from a general experiment consisting of preparation, transformation

and measurement. The quantum state is a minimal set of probabilities, from which the probabilities of all possible measurements can be deduced. In each of the approaches [21, 22] one applies 5 axioms and arrives at the probability theory implied by the trace formula (9) and the representation of quantum mechanics discussed in 2.1.

The concept of a *quantum instrument* was introduced by E. B. Davis and J. T. Lewis [19] as a generalization of observables and operations. A quantum instrument or quantum operation is a *completely positive, trace non-increasing* (CP) map between an input space $\mathcal{L}(\mathcal{H}^{X_1})$ and an output space $\mathcal{L}(\mathcal{H}^{X_2})$, which might depend on a set of measurement outcomes $\{j\}$.

$$\mathcal{M}_j^X : \mathcal{L}(\mathcal{H}^{X_1}) \rightarrow \mathcal{L}(\mathcal{H}^{X_2}) \quad (22)$$

For an input state $\rho \in \mathcal{L}(\mathcal{H}^{X_1})$ this map can be written as

$$\mathcal{M}_j^X(\rho) = \sum_{k=1}^{d_{X_1} d_{X_2}} E_{jk} \rho E_{jk}^\dagger \quad (23)$$

where the $E_{jk} : \mathcal{H}^{X_1} \rightarrow \mathcal{H}^{X_2}$ are maps satisfying $\sum_k E_{jk} E_{jk}^\dagger \leq \mathbb{1}^{X_1}$. Note that $\mathcal{M}_j^A(\rho) \in \mathcal{L}(\mathcal{H}^{X_2})$ is the (not necessarily normalized) state after the operation has been performed.

Hence, the probability of a certain outcome j is given by

$$P(\mathcal{M}_j^X) = \text{Tr}(\mathcal{M}_j^X(\rho)). \quad (24)$$

Moreover summing over all possible outcomes should give a *completely positive, trace preserving* (CPTP) map, which means that $\sum_j E_{jk}$ are Kraus-operators, and

$$\sum_{jk} E_{jk} E_{jk}^\dagger = \mathbb{1}^{X_1}.$$

The probability of a CPTP- map should be unity

$$P\left(\sum_j \mathcal{M}_j^X\right) = 1$$

since this is the probability of observing any outcome at all.

Comparing (23) to (17) and (19) shows that the formalism of generalized measurements described by POVM's as well as quantum channels are captured in this description. It is, however, more general since the possibility of $\sum_k E_{jk} E_{jk}^\dagger < \mathbb{1}^{X_1}$

allows for the description of filtering operations, where $\text{Tr}(\mathcal{M}_j^X(\rho)) < 1$.

CP- maps given by (22) are related to *positive semidefinite matrices* $M_j^{X_1 X_2} \in \mathcal{L}(\mathcal{H}^{X_1} \otimes \mathcal{H}^{X_2})$ via the Choi-Jamiolkowski isomorphism [23, 24]

$$M_j^{X_1 X_2} := (\mathcal{I} \otimes \mathcal{M}_j^X (|\Phi^+\rangle\langle\Phi^+|))^T = \sum_{ij} (|i\rangle\langle j| \otimes \mathcal{M}_j^X (|i\rangle\langle j|))^T \quad (25)$$

where \mathcal{I} is the identity map and $|\Phi^+\rangle = \sum_i |i\rangle \otimes |i\rangle$ is an unnormalized, maximally entangled state in $\mathcal{H}^{X_1} \otimes \mathcal{H}^{X_1}$ with $\{|i\rangle\}$ being some ONB of \mathcal{H}^{X_1} . The transposition was introduced in [11] for convenience.

In this thesis, operations where a measurement with state $|\phi_1\rangle$ is performed and a state $|\phi_2\rangle$ is prepared, will be of special importance. From (25) it is easy to see that such an operation is described by the following CJ- matrix, if the ONB $\{|i\rangle\}$ of \mathcal{H}^{X_1} is chosen to be real ($|i\rangle^T = |i\rangle$),

$$\begin{aligned} M_j^{X_1 X_2} &= \sum_{ij} (|i\rangle\langle j| \otimes \mathcal{M}_j^X (|i\rangle\langle j|))^T = \\ &= \sum_{ij} |j\rangle\langle i| \otimes \langle\phi_1|i\rangle\langle j|\phi_1\rangle^{X_1} \cdot |\phi_2\rangle\langle\phi_2|^{T X_2} = \\ &= \sum_{ij} |j\rangle\langle j|\phi_1\rangle\langle\phi_1|i\rangle\langle i|^{X_1} \otimes |\phi_2\rangle\langle\phi_2|^{T X_2} = \\ &= |\phi_1\rangle\langle\phi_1|^{X_1} \otimes |\phi_2\rangle\langle\phi_2|^{T X_2}. \end{aligned} \quad (26)$$

Note, that by no means all operations can be written that way. The CJ- matrices of unitary transformations (and combinations of them), for example, are not of the form (26).

Moreover the CJ- matrix of a CPTP- map, $\mathcal{M}^X = \sum_j \mathcal{M}_j^X$, is characterized by

$$\text{Tr}_2(M^{X_1 X_2}) = \mathbb{1}^{X_1}$$

since $\text{Tr}(\mathcal{M}^X (|i\rangle\langle j|)) = \text{Tr}(|i\rangle\langle j|)$.

The originally defined CJ-matrix, (25) without the transposition, corresponds to a bipartite quantum state. Moreover, collections of quantum instruments can be regarded as quantum channels. The Choi-Jamiolkowski isomorphism is, therefore, also referred to as channel-state duality. As discussed in [25] properties of quantum channels and properties of bipartite quantum states are directly related via the isomorphism (25). For example, all channels of the form (21) are entanglement breaking, which means that $\mathcal{C}(\rho)$ is separable for all ρ , see [26]. The CJ-matrices corresponding to such channels have the structure of separable states.

Moreover the classification of channels as *(quasi) classical*, *classical-quantum* and *quantum classical* in 2.2 can be transferred to *correlations between quantum states*. Two quantum systems A and B are said to be (quasi) classically (classical-quantum, quantum-classically) correlated if the channel equivalent to their bipartite quantum state is (quasi) classical (classical-quantum, quantum classical).

This definition of classical, c-q and q-c correlations turns out to be equivalent to the one proposed by S. Luo [27]. While most correlation measures based on quantum entropy and mutual information (see [28]) define all correlations that are not due to entanglement as classical, for example [29], this classification is purely via the effect of measurements. It can also be formulated in terms of quantum discord, as introduced by H. Ollivier and W.H. Zurek [30].

A classical bipartite state ρ , with reduced states $\rho^A = \text{Tr}_B(\rho) = \sum_i P_i^a |i\rangle\langle i|$ and $\rho^B = \text{Tr}_A(\rho) = \sum_j P_j^b |j\rangle\langle j|$, is of the following form:

$$\rho = \sum_{ij} P_{ij} |i\rangle\langle i|^A \otimes |j\rangle\langle j|^B \quad (27)$$

Here $\{|i\rangle\}$ and $\{|j\rangle\}$ are ONB's of \mathcal{H}^A and \mathcal{H}^B and $\{P_{ij}\}$ are classical joint probabilities such that the probabilities for the subsystems are the respective marginals, $\sum_j P_{ij} = P_i^a$ and $\sum_i P_{ij} = P_j^b$. For a classical state there exists a combined measurement of A and B , which leaves the system undisturbed, namely $\mathcal{M} = \{|i\rangle\langle i|^A \otimes |j\rangle\langle j|^B\}$, where $\{|i\rangle\}$ and $\{|j\rangle\}$ are the same ONB's as in (27).

$$\rho \xrightarrow{\mathcal{M}} \sum_{ij} \frac{1}{P_{ij}} |i\rangle\langle i|^A \otimes |j\rangle\langle j|^B \rho |i\rangle\langle i|^A \otimes |j\rangle\langle j|^B = \rho$$

Such states have zero quantum discord with respect to both subsystems A and B . Analogously a classical-quantum state is one with zero quantum discord with respect to subsystem A , but not B . This means that there exists a local measurement on A , which does not perturb the overall state. A c-q state is of the form

$$\rho = \sum_{ij} P_i |i\rangle\langle i|^A \otimes \rho_j^B. \quad (28)$$

A quantum-classical state has zero quantum discord with respect to subsystem B and is given by

$$\rho = \sum_{ij} \rho_i^A \otimes P_j |j\rangle\langle j|^B. \quad (29)$$

Classical-quantum and quantum-classical states, channels and quantum instruments in general are classical in one subsystem in the sense that some local measurement on that subsystem can be carried out without perturbing the overall state. They are therefore called *semiquantum*.

3 Quantum Correlations with no Causal Order

In their paper [11] O. Oreshkov et. al. address the question, whether local validity of quantum mechanics and a global causal structure, such as spacetime, are always compatible with each other.

Considering *signaling correlations* between two *closed laboratories*, where locally standard quantum mechanics is valid, they derive a *causal inequality* for a certain kind of communication task between these laboratories. This inequality cannot be violated, if one assumes the two laboratories to be embedded in a global causal structure.

Starting from an operational approach, they arrive at a formalism of bilinear functions of positive semidefinite matrices via the Choi-Jamiołkowski isomorphism. This introduces so called *process matrices*, which can be analyzed in terms of compatibility with a global causal structure. A certain class of process matrices called *causally separable* can never violate the causal inequality. While it can be shown that *classical operations* always lead to causally separable processes, for *quantum mechanical operations* one can find process matrices that *violate* the causal inequality. This suggests that there are quantum mechanical correlations that are *not* compatible with the concept of a global causal structure and that the latter might emerge during the transition from quantum to classical behavior.

3.1 Causal Structure and Causal Inequality

Two events in spacetime, A and B , are always related in one of the following ways:

- B is in the future light cone of A
- A is in the future light cone of B
- A and B are spacelike separated

In the first case, event A can in principle cause event B , in the second case, B can cause A and in the third case the two events cannot (causally) influence one another. Spacetime is therefore called a causal structure.

A more general definition of a causal structure uses the concept of signaling, see 2.2:

"A causal structure ... is a set of event locations equipped with a partial order \preceq that defines the possible directions of signaling." (O. Oreshkov, F. Costa, Č. Brukner, 2012)

If the sender and receiver of the signal, A and B , are embedded in a causal structure, exactly one of the following statements is true.

-
- A can send a signal to B
 - B can send a signal to A
 - A and B cannot send signals to each other

This is equivalent with the three relations in the beginning of this section, if A or B are point events in spacetime. Then $A \preceq B$ means that A is in the causal past of (and therefore can send a signal to) B , and vice versa for $B \preceq A$. The case where signaling between A and B is not possible (A and B are spacelike separated) will be denoted $A \not\preceq B$.

A closed laboratory can receive a physical system (input) and send a physical system (output), but is otherwise isolated. If two such laboratories (A and B) are embedded in a causal structure, A 's output can be correlated with B 's input, if and only if $A \preceq B$. The existence of a global causal structure therefore puts certain restrictions on the communication between two laboratories, A and B , which can be used to formulate a *causal inequality*, when considering the following kind of communication task.

Let each laboratory receive its input, corresponding to spacetime events A_1 and B_1 , and then obtain a random bit, a and b . Moreover they each produce a guess of the bit obtained in the other laboratory, which will be denoted by x for A and y for B . One party (say B) has to obtain another random bit b' , which determines whether the guess of A 's is of interest ($b' = 0$) or that of B ($b' = 1$). The events of sending the output system will be denoted A_2 and B_2 .

In this setup the *probability of success* is defined as

$$P_{succ} := \frac{1}{2} (P(x = b|b' = 0) + P(y = a|b' = 1)) \quad (30)$$

where $P(i = k|b')$ is the conditional probability of guess i correctly stating bit k given bit b' .

There is an upper bound to this expression, if one makes the following assumptions:

1. A and B , and therefore all events in the communication task, are embedded in a *causal structure*.

This particularly means that $P(A_1 \preceq B_1) + P(B_1 \preceq A_1) + P(A_1 \not\preceq B_1) = 1$, as described in the beginning of this chapter.

2. The three random bits a, b and b' are *chosen freely* and therefore can only be correlated with events in their causal future.

Together with the first assumption this implies that a, b and b' are independent of each other and independent of the causal relations between A_1 and B_1 (e.g. $P(A_1 \preceq B_1|b) = P(A_1 \preceq B_1)$)

-
3. The two laboratories are *closed*, which means that events happening in A (B) can be correlated with events in B (A ; e.g. bit b), if the latter happen in the causal past of the input entering A (B ; e.g. $b \preceq A_1$).

The probability of success can then be written as follows:

$$\begin{aligned}
P_{succ} &= \left(\frac{1}{2}P(x = b|b' = 0; A_1 \preceq B_1) + \frac{1}{2}P(y = a|b' = 1; A_1 \preceq B_1) \right) P(A_1 \preceq B_1) \\
&+ \left(\frac{1}{2}P(x = b|b' = 0; B_1 \preceq A_1) + \frac{1}{2}P(y = a|b' = 1; B_1 \preceq A_1) \right) P(B_1 \preceq A_1) \\
&+ \left(\frac{1}{2}P(x = b|b' = 0; A_1 \not\preceq B_1) + \frac{1}{2}P(y = a|b' = 1; A_1 \not\preceq B_1) \right) P(A_1 \not\preceq B_1)
\end{aligned} \tag{31}$$

One finds that for $A_1 \preceq B_1$ $P(x = b|b' = 0; A_1 \preceq B_1) = \frac{1}{2}$, and for $B_1 \preceq A_1$ $P(y = a|b' = 1; B_1 \preceq A_1) = \frac{1}{2}$. In the case of $A_1 \not\preceq B_1$ $P(x = b|b' = 0; A_1 \not\preceq B_1) = P(y = a|b' = 1; A_1 \not\preceq B_1) = \frac{1}{2}$, see supplementary information of [11] for more details.

This gives the causal inequality

$$\begin{aligned}
P_{succ} &= \left(\frac{1}{4} + \frac{1}{2}P(y = a|b' = 1; A_1 \preceq B_1) \right) P(A_1 \preceq B_1) \\
&+ \left(\frac{1}{2}P(x = b|b' = 0; B_1 \preceq A_1) + \frac{1}{4} \right) P(B_1 \preceq A_1) \\
&+ \left(\frac{1}{4} + \frac{1}{4} \right) P(A_1 \not\preceq B_1) \leq \frac{3}{4}
\end{aligned} \tag{32}$$

which cannot be violated, if two closed laboratories A and B are embedded in a definite global causal structure and can choose the bits a, b and b' freely.

3.2 Processes and Process Matrices

If two closed laboratories A and B , as in 3.1, are described by standard quantum mechanics, their in- and outputs are quantum systems. These are related via quantum operations (22) performed in the laboratories. Figure 2 shows a schematic picture of the situation considered. From an operational point of view a laboratory *is* all the operations that can be performed in it. This means that a closed laboratory is fully described by its set of possible quantum instruments $\{\mathcal{M}_i^X\}$ with $X = A, B$.

Correlations between two laboratories depend on the joint probabilities of pairs of

outcomes i, j denoted as $P(\mathcal{M}_i^A, \mathcal{M}_j^B)$. A list $\{P(\mathcal{M}_i^A, \mathcal{M}_j^B)\}$ of such joint probabilities from which the probabilities of all other possible pairs of outcomes can be determined is called a *process*.

According to [21, 22] probabilities for outcomes have to be linear functions of the states involved. By similar argument probabilities of such transformations should be linear functions of the transformations. This implies that the probabilities $P(\mathcal{M}_i^A, \mathcal{M}_j^B)$ have to be *bilinear functions* of the CJ-matrices (25) corresponding to \mathcal{M}_i^A and \mathcal{M}_j^B and can therefore be written as

$$P(\mathcal{M}_i^A, \mathcal{M}_j^B) = \text{Tr} (W^{A_1 A_2 B_1 B_2} (M_i^{A_1 A_2} \otimes M_j^{B_1 B_2})) \quad (33)$$

where $W^{A_1 A_2 B_1 B_2} \in \mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$ is called a *process matrix*. It is assumed that probabilities are non negative even when the parties share non-signaling resources (e.g. entangled inputs). This implies that process matrices have to be positive semidefinite:

$$W^{A_1 A_2 B_1 B_2} \geq 0. \quad (34)$$

Moreover two CPTP- maps should, of course, give unit probability for any valid process matrix.

$$\begin{aligned} \text{Tr}[W^{A_1 A_2 B_1 B_2} (M^{A_1 A_2} \otimes M^{B_1 B_2})] &= 1 \\ \forall M^{A_1 A_2}, M^{B_1 B_2} \geq 0; \text{Tr}_2(M^{A_1 A_2}) = \mathbb{1}^{A_1}, \text{Tr}_2(M^{B_1 B_2}) = \mathbb{1}^{B_1}. \end{aligned} \quad (35)$$

The normalization is such that $\text{Tr}(W^{A_1 A_2 B_1 B_2}) = d_{A_2} d_{B_2}$.

According to (33) process matrices correlate the input- and output-spaces of the two laboratories A and B and can be characterized by that. The type of an expression in $W^{A_1 A_2 B_1 B_2}$ gives the spaces on which $W^{A_1 A_2 B_1 B_2}$ does act non-trivially and, therefore, defines which spaces are correlated. For example an expression $\mathbb{1}^{A_1} \otimes W^{A_2 B_1} \otimes \mathbb{1}^{B_2}$ correlates the output of A with the input of B . It is called an $A_2 B_1$ -type term and can be interpreted as a quantum channel from A to B .

From now on $\mathbb{1}$ will be omitted in the tensor products to shorten the expressions. As shown in the supplementary information of [11] any valid process matrix, fulfilling (34) and (35) is of the form

$$W^{A_1 A_2 B_1 B_2} = \frac{1}{d_{A_1} d_{B_1}} (\mathbb{1} + \sigma^{B \preceq A} + \sigma^{A \preceq B} + \sigma^{A \not\preceq B}) \quad (36)$$

where

$$\begin{aligned}
\sigma^{B \leq A} &:= \sum_{ij > 0} c_{ij} \sigma_i^{A_1} \otimes \sigma_j^{B_2} + \sum_{ijk > 0} d_{ijk} \sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes \sigma_k^{B_2} \\
\sigma^{A \leq B} &:= \sum_{ij > 0} e_{ij} \sigma_i^{A_2} \otimes \sigma_j^{B_1} + \sum_{ijk > 0} f_{ijk} \sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B_1} \\
\sigma^{A \not\leq B} &:= \sum_{i > 0} x_i \sigma_i^{A_1} + \sum_{i > 0} y_i \sigma_i^{B_1} + \sum_{ij > 0} g_{ij} \sigma_i^{A_1} \otimes \sigma_j^{B_1}
\end{aligned} \tag{37}$$

and $c_{ij}, d_{ijk}, e_{ij}, f_{ijk}, x_i, y_i, g_{ij} \in \mathbb{R}$. Illustrations of the different summands are depicted in figure 3.

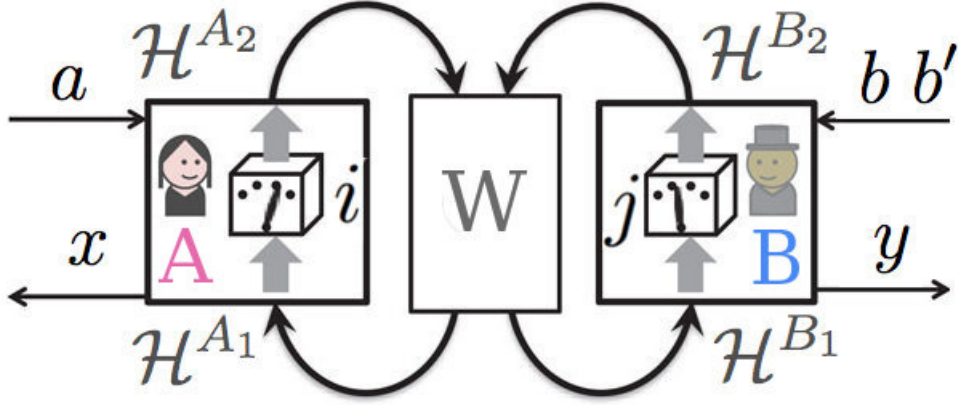


Figure 2: Two laboratories A and B are described by maps from the input Hilbert space, \mathcal{H}^{X_1} with $X = A, B$, and output Hilbert space, \mathcal{H}^{X_2} with $X = A, B$. They perform the following communication task. A and B obtain random bits a, b and guesses x, y of the other's bit. B 's additional bit b' determines whose guess is of interest. If the two laboratories are embedded in a global causal structure, they can correctly guess the other parties bit with a probability of $P_{succ} \leq \frac{3}{4}$. Picture taken from [11], altered.

The term $\sigma^{B \leq A}$ is compatible with A being in the causal future of B and describes quantum channels and quantum channels with memory from B to A . This is reflected in the fact that the summands in $\sigma^{B \leq A}$ correlate the output of B with either the input of A alone, or with both inputs.

Similarly $\sigma^{A \preceq B}$ describes quantum channels, with or without memory, from A to B and can be understood as B being in the causal future of A .

The term $\sigma^{A \not\preceq B}$ corresponds to non signaling correlations between A and B and does not require one to be in the causal future of the other. The first two summands in (37) are independent input states of the two laboratories, while the last expression corresponds to a shared input of A and B . Note that this last term contains all correlations involving entanglement, which are non signaling.

There are also types of expressions that do not appear in process matrices. One example are $A_1 A_2$ -type terms of the general form $\sum_{ij>0} h_{ij} \sigma_i^{A_1} \otimes \sigma_j^{A_2}$, which would correlate the output of A with its input. Such a correlation can be interpreted as a causal loop, which is logically problematic. Other non-appearing terms and their possible interpretations are shown in figure 4.

Note that the possible terms in (37) are a consequence of the conditions (34) and (35) alone, yet there are no expressions appearing that would directly lead to logical contradictions. Local loops, for example, can be interpreted as information channels back in time, which represent a version of the famous grandfather paradox [10]. Such expressions are excluded by these conditions, which are requirements derived solely from the assumption that locally standard quantum mechanics is valid.

$B \not\preceq A$	$A_1, B_1, A_1 B_1$	$A_2 B_1$	$A_1 A_2 B_1$
$A \not\preceq B$		$A_1 B_2$	$A_1 B_1 B_2$
Causal order	States	Channels	Channels with memory

Figure 3: Expressions generally appearing in a process matrix $W^{A_1 A_2 B_1 B_2}$. The type of an expression gives the spaces which are correlated by it (e.g. $\mathbb{1}^{A_1} \otimes W^{A_2 B_1} \otimes \mathbb{1}^{B_2}$ is an $A_2 B_1$ -type expression). Below each type interpretations and illustrations for each type are given; taken from [11].

The communication task considered in the derivation of the causal inequality in 3.1 requires the agents in the laboratories, Alice and Bob, to obtain random bits a, b and b' and make guesses x and y . The quantum operations \mathcal{M}_i^A and \mathcal{M}_j^B of interest therefore have to depend on those bits and guesses. The input states can

be used to obtain x, y , while information about a, b can be encoded in the output states, which are sent to the other laboratory. Bit b' will determine the particular protocol applied in laboratory B . The probabilities necessary to calculate the probability of success are given by equation (33).

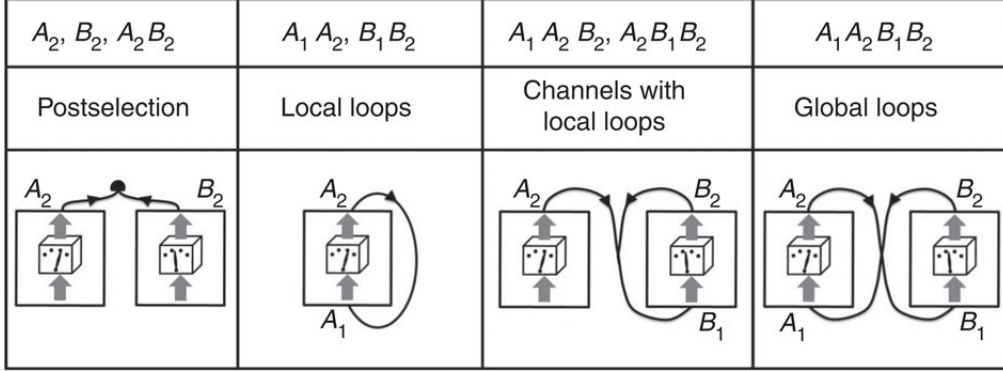


Figure 4: Types of expressions not appearing in a process matrix $W^{A_1 A_2 B_1 B_2}$. Below each type possible interpretations and their illustrations are given; taken from [11].

From (37) and the interpretations of these expressions it is clear that process matrices containing no $\sigma^{A \leq B}$ -terms, which will be denoted as $W^{A \not\leq B}$, are always compatible with a global causal structure. All such process matrices agree with the concept that A cannot signal to B . This can be understood as B being in the spacetime complement of the future light cone of A . Hence all process matrices $W^{A \not\leq B}$ are compatible with a global causal structure. Thus, they cannot be expected to violate the causal inequality for any pair of CP-maps.

The same is, of course, true for process matrices $W^{B \not\leq A}$, which contain no $\sigma^{B \leq A}$ -terms. Moreover any probabilistic mixture of such process matrices is compatible with a global causal structure as well. A process matrix is of that form

$$W^{A_1 A_2 B_1 B_2} = p W^{B \not\leq A} + (1 - p) W^{A \not\leq B} \quad (38)$$

where $p \in [0, 1]$ is called *causally separable* and can never yield probabilities which violate the causal inequality.

3.3 Violation of the Causal Inequality

In [11] an example for the violation of the causal inequality is presented for the case of the inputs and outputs of both laboratories being qubits. This means that

$\mathcal{L}(\mathcal{H}^{A_1}) = \mathcal{L}(\mathcal{H}^{A_2}) = \mathcal{L}(\mathcal{H}^{B_1}) = \mathcal{L}(\mathcal{H}^{B_2}) = \text{span}\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$, where σ_i are the Pauli matrices.

Consider the following strategy: Alice measures her input qubit in the z -basis making her guess depending on the outcome, with $x = 0$ for $|z_+\rangle$ and $x = 1$ for $|z_-\rangle$. She then obtains her random bit a , encodes it in the z -basis ($a = 0 \leftrightarrow |z_+\rangle$, $a = 1 \leftrightarrow |z_-\rangle$) and sends it away.

The CJ- matrix of this operation is given by:

$$M_{a,x}^{A_1 A_2} = \frac{1}{4} (\mathbb{1} + (-1)^x \sigma_z)^{A_1} \otimes (\mathbb{1} + (-1)^a \sigma_z)^{A_2} \quad (39)$$

Bob, on the other hand, obtains his bits b, b' and does the following: If $b' = 1$ he measures his input in z -basis to receive full information about Alice's bit. His guess is then $y = 0$ for $|z_+\rangle$ and $y = 1$ for $|z_-\rangle$ and the output state ρ he prepares is arbitrary, since Alice's guess is not of importance in this case. If, however, $b' = 0$, Bob measures his input in the x -basis and encodes his other random bit b in the z -basis depending on the measurement outcome. If the result is $|x_+\rangle$, b is encoded by $b = 0 \leftrightarrow |z_+\rangle$ and $b = 1 \leftrightarrow |z_-\rangle$. But, if he measures $|x_-\rangle$, he encodes b by $b = 0 \leftrightarrow |z_-\rangle$ and $b = 1 \leftrightarrow |z_+\rangle$. These operations correspond to the CJ- matrix

$$M_{b,b',y}^{B_1 B_2} = b' M_1^{B_1 B_2} + (b' \oplus 1) M_2^{B_1 B_2} \quad (40)$$

where \oplus is the sum modulo 2 and

$$M_1^{B_1 B_2} = \frac{1}{2} (\mathbb{1} + (-1)^y \sigma_z)^{B_1} \otimes \rho^{B_2}$$

$$M_2^{B_1 B_2} = \frac{1}{4} (\mathbb{1} + (-1)^y \sigma_x)^{B_1} \otimes (\mathbb{1} + (-1)^{b+y} \sigma_z)^{B_2}.$$

For the two CP-maps (39) and (40) the process matrix

$$W_{\star}^{A_1 A_2 B_1 B_2} = \frac{1}{4} \left(\mathbb{1}^{A_1 A_2 B_1 B_2} + \frac{1}{\sqrt{2}} (\sigma_z^{A_2} \otimes \sigma_z^{B_1} + \sigma_z^{A_1} \otimes \sigma_x^{B_1} \otimes \sigma_z^{B_2}) \right) \quad (41)$$

violates the causal inequality, since

$$P(y = a | b' = 1) = \frac{1}{2} \left(1 + \frac{(-1)^{y+a}}{\sqrt{2}} \right) \leq \frac{2 + \sqrt{2}}{4}$$

and

$$P(x = b | b' = 0) = \frac{1}{2} \left(1 + \frac{(-1)^{x+b}}{\sqrt{2}} \right) \leq \frac{2 + \sqrt{2}}{4}$$

which gives a probability of success of

$$P_{succ} = \frac{1}{2} (P(x = b | b' = 0) + P(y = a | b' = 1)) \leq \frac{2 + \sqrt{2}}{4} > \frac{3}{4}. \quad (42)$$

This example shows that general signaling correlations between two closed laboratories, which are described by standard quantum mechanics, are not compatible with the concept of a definite global causal structure.

3.4 Classical Processes

One part of the supplementary information of [11] is the proof that for any *classical process* the process matrices of interest can be written as (38) and are therefore *causally separable* and compatible with a global causal structure. As this proof will be of crucial importance when considering semi-quantum processes, see 4, it will be discussed in detail here.

Analogous to (21) a collection of *classical instruments* acts on an input state ρ as follows:

$$\mathcal{M}^X(\rho) = \sum_i \tilde{\rho}_i \text{Tr}(|i\rangle\langle i|\rho) \quad (43)$$

with $\{|i\rangle\}$ being some ONB of the input Hilbert space \mathcal{H}^{X_1} and $[\tilde{\rho}_i, \tilde{\rho}_{i'}] = 0$. This means the density matrices $\tilde{\rho}$ are jointly diagonalizable and (43) can be written as

$$\mathcal{M}^X(\rho) = \sum_{ki} P(k|i) |k\rangle\langle k| \text{Tr}(|i\rangle\langle i|\rho). \quad (44)$$

where $\{P(k|i)\}$ are classical conditional probabilities for the output k given input i . This can be understood as measuring the input with a certain complete set of orthogonal projectors, which corresponds to a *fixed measurement basis*. For each measurement $|i\rangle\langle i|$ a certain state $\tilde{\rho}_i$ is prepared as output. The fact that the different reprepared states all commute ensures that the output can be understood in terms of classical probabilities. One may speak of a *fixed output reparation basis*. In general the mapping $|i\rangle\langle i| \rightarrow \tilde{\rho}_i$ between the input measurement and the preparation of the output (and the probabilities $P(k|i)$) will also depend on the measurement result j .

Classical operations are, therefore, characterized by CJ-matrices which are all diagonal in a so called *pointer basis*, compare (26) and (44)

$$M_j^{X_1 X_2} = \sum_{ki} M_j^{ki} |i\rangle\langle i|^{X_1} \otimes |k\rangle\langle k|^{X_2} \quad (45)$$

where $X_{1,2}$ denote the input and output spaces of the closed laboratories ($X = A, B$), and $M_j^{ki} = P(k, j|i)$ is then the conditional probability of the result j being measured and output state k being prepared, when the input state is i . A laboratory will be called classical if all the operations performed in it are classical, i.e.

given by (45).

In the case of qubits, say *spin- $\frac{1}{2}$ -particles*, the input spin is always measured along direction \vec{n} and, therefore, $|i\rangle = |n\rangle$, where $\sigma_{\vec{n}}|n\rangle = \lambda_n|n\rangle$, with $\lambda_{\pm} = \pm 1$ and $\sigma_{\vec{n}} = \sum_i n_i \sigma_i$. Moreover $|k\rangle = |r\rangle$, with $\sigma_{\vec{r}}^T|r\rangle = \lambda_r|r\rangle$ and $\lambda_{\pm} = \pm 1$, which means the output is spin-polarized along direction \vec{r} , $\sigma_{\vec{r}} = \sum_i r_i \sigma_i$.

One is interested in bilinear functions of $M_i^{A_1 A_2} \otimes M_j^{B_1 B_2}$ given by (33). Since the trace is basis independent one can calculate it in the basis $\{|i\rangle\langle i'|^{A_1} \otimes |j\rangle\langle j'|^{A_2} \otimes |k\rangle\langle k'|^{B_1} \otimes |l\rangle\langle l'|^{B_2}\}$, which is a combination of the two pointer bases of A and B :

$$\begin{aligned} & \text{Tr} (W^{A_1 A_2 B_1 B_2} (M_a^{A_1 A_2} \otimes M_b^{B_1 B_2})) = \\ & = \text{Tr} \left(W^{A_1 A_2 B_1 B_2} \cdot \sum_{ijkl} M_a^{ij} M_b^{kl} |i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \right) \end{aligned} \quad (46)$$

In this basis a general process matrix is of the form

$$W^{A_1 A_2 B_1 B_2} = \sum_{\substack{ii'jj' \\ kk' ll'}} w_{ii'jj'kk' ll'} |i\rangle\langle i'|^{A_1} \otimes |j\rangle\langle j'|^{A_2} \otimes |k\rangle\langle k'|^{B_1} \otimes |l\rangle\langle l'|^{B_2}.$$

Restricting the possible CP-maps of the two laboratories to classical transformations, however, will limit the possible correlations in (46). This can be expressed in terms of *effective process matrices* W_{eff} , which have to fulfill:

$$\begin{aligned} & \text{Tr} \left(W^{A_1 A_2 B_1 B_2} \cdot \sum_{ijkl} M_a^{ij} M_b^{kl} |i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \right) = \\ & \stackrel{!}{=} \text{Tr} \left(W_{eff} \cdot \sum_{ijkl} M_a^{ij} M_b^{kl} |i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \right) \end{aligned} \quad (47)$$

The left hand side can be calculated straight forwardly

$$\begin{aligned} & \text{Tr} \left(\sum_{\substack{ii'jj'j'' \\ kk'kk''ll'l''}} w_{ii'jj'j''kk'kk''ll'l''} M_a^{ij} M_b^{kl} \dots \right. \\ & \left. \dots |i'\rangle \underbrace{\langle i''|i\rangle\langle i|^{A_1}}_{=\delta_{i''i}} \otimes |j'\rangle \underbrace{\langle j''|j\rangle\langle j|^{A_2}}_{=\delta_{j''j}} \otimes |k'\rangle \underbrace{\langle k''|k\rangle\langle k|^{B_1}}_{=\delta_{k''k}} \otimes |l'\rangle \underbrace{\langle l''|l\rangle\langle l|^{B_2}}_{=\delta_{l''l}} \right) = \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{i'i' \\ jj' \\ kk' \\ ll'}} w_{i'i'jj'kk'll'} M_a^{ij} M_b^{kl} \operatorname{Tr} \left(|i'\rangle\langle i|^{A_1} \otimes |j'\rangle\langle j|^{A_2} \otimes |k'\rangle\langle k|^{B_1} \otimes |l'\rangle\langle l|^{B_2} \right) = \\
&= \sum_{\substack{i'i' \\ jj' \\ kk' \\ ll'}} w_{i'i'jj'kk'll'} M_a^{ij} M_b^{kl} \underbrace{\langle i|i'\rangle}_{=\delta_{ii'}} \underbrace{\langle j|j'\rangle}_{=\delta_{jj'}} \underbrace{\langle k|k'\rangle}_{=\delta_{kk'}} \underbrace{\langle l|l'\rangle}_{=\delta_{ll'}} = \sum_{i,j,k,l} w_{iijjkkll} M_a^{ij} M_b^{kl} = \quad (48) \\
&= \sum_{i,j,k,l} w_{ijkl} M_a^{ij} M_b^{kl},
\end{aligned}$$

where $w_{ijkl} := w_{iijjkkll}$. From this it is easy to see that

$$W_{eff} = \sum_{ijkl} w_{ijkl} |i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \quad (49)$$

gives the same expression for the trace as (48):

$$\begin{aligned}
&\operatorname{Tr} \left(W_{eff} \cdot \sum_{ijkl} M_a^{ij} M_b^{kl} |i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \right) = \\
&= \operatorname{Tr} \left(\sum_{\substack{i'i' \\ jj' \\ kk' \\ ll'}} w_{i'i'jj'kk'll'} M_a^{ij} M_b^{kl} \delta_{i'i} \delta_{j'j} \delta_{k'k} \delta_{l'l} |i'\rangle\langle i|^{A_1} \otimes |j'\rangle\langle j|^{A_2} \otimes |k'\rangle\langle k|^{B_1} \otimes |l'\rangle\langle l|^{B_2} \right) = \\
&= \sum_{ijkl} w_{ijkl} M_a^{ij} M_b^{kl} \operatorname{Tr} \left(|i\rangle\langle i|^{A_1} \otimes |j\rangle\langle j|^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes |l\rangle\langle l|^{B_2} \right) = \sum_{ijkl} w_{ijkl} M_a^{ij} M_b^{kl} \quad (50)
\end{aligned}$$

This means that in the case of classical operations it is sufficient to consider process matrices that are diagonal in the combined pointer basis. Such *process matrices* (and *processes*) will also be called *classical*, since they correlate two classical laboratories. Generally, a process matrix is of the form

$$W^{A_1 A_2 B_1 B_2} = \frac{1}{d_{A_1} d_{B_1}} (\mathbb{1} + \sigma^{B \not\leftarrow A} + \sigma^{A \not\leftarrow B}) \quad (51)$$

where $\sigma^{B \not\leftarrow A}$ and $\sigma^{A \not\leftarrow B}$ are not unique since terms of $\sigma^{A \not\leftarrow \not\leftarrow B}$ may be contained in any one of the two. Let λ_i be the eigenvalues of $\sigma^{B \not\leftarrow A} + \sigma^{A \not\leftarrow B}$. Since $W^{A_1 A_2 B_1 B_2}$ is positive semidefinite ($1 + \lambda_i \geq 0 \ \forall i$) and $\sigma^{B \not\leftarrow A} + \sigma^{A \not\leftarrow B}$ is traceless ($\operatorname{Tr}(\sigma^{B \not\leftarrow A} + \sigma^{A \not\leftarrow B}) = \sum_i \lambda_i = 0$), the *minimal eigenvalue* λ_0 lies between -1 and 0 ($\lambda_0 \in [-1, 0]$).

One can therefore define

$$\begin{aligned}\kappa^{A_1 A_2 B_1} &:= -\lambda_0 \mathbb{1} + \sigma^{B_2^A} \\ \kappa^{A_1 B_1 B_2} &:= \sigma^{A_2^B}\end{aligned}\tag{52}$$

giving

$$W^{A_1 A_2 B_1 B_2} = \frac{1}{d_{A_1} d_{B_1}} ((1 + \lambda_0) \mathbb{1} + \kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2})\tag{53}$$

where $\kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2} \geq 0$, since any eigenvalue $\lambda(\kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2})$ of $\kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2}$ fulfills $\lambda(\kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2}) = -\lambda_0 + \lambda_i \geq -\lambda_0 + \lambda_0 = 0$.

The effective process matrices can, of course, also be written as in (53) with $\kappa^{A_1 A_2 B_1}$ and $\kappa^{A_1 B_1 B_2}$ being diagonal in the combined pointer basis.

Let then $m_1(i, j, k, l)$ be the eigenvalues of $\kappa^{A_1 A_2 B_1}$ and $m_2(i, j, k, l)$ that of $\kappa^{A_1 B_1 B_2}$. For every pair (i, k) , define

$$\begin{aligned}m'_1(i, k) &:= \min_{j, l} \{m_1(i, j, k, l)\} = \min_j \{m_1(i, j, k)\} \\ m'_2(i, k) &:= \min_{j, l} \{m_2(i, j, k, l)\} = \min_l \{m_2(i, k, l)\}.\end{aligned}\tag{54}$$

Since $\kappa^{A_1 A_2 B_1}$ and $\kappa^{A_1 B_1 B_2}$ act trivially on B_2 and A_2 , $m_1(i, j, k, l)$ do not depend on l and $m_2(i, j, k, l)$ do not depend on j . This allows the following construction of two positive semidefinite matrices $\bar{\kappa}^{A_1 A_2 B_1}$ and $\bar{\kappa}^{A_1 B_1 B_2}$ such that $\kappa^{A_1 A_2 B_1} + \kappa^{A_1 B_1 B_2} = \bar{\kappa}^{A_1 A_2 B_1} + \bar{\kappa}^{A_1 B_1 B_2}$. For simplicity the notation $\kappa_1 = \kappa^{A_1 A_2 B_1}$ and $\kappa_2 = \kappa^{A_1 B_1 B_2}$ will be used from now on.

For a given (i, k) the *minimal eigenvalue* of $\kappa_1 + \kappa_2$ is $m'_1(i, k) + m'_2(i, k)$ and since $\kappa_1 + \kappa_2$ is positive semidefinite

$$m'_1(i, k) + m'_2(i, k) \geq 0.\tag{55}$$

Either both m'_1 and m'_2 are larger than zero or one of them is negative.

1. If $m'_1(i, k) > 0$ and $m'_2(i, k) > 0$ one does not modify κ_1 and κ_2 for this pair (i, k) .
2. If $m'_1(i, k) < 0$ one modifies κ_1 and κ_2 , defining

$$\begin{aligned}\kappa'_1 &:= \kappa_1 - m'_1(i, k) |i\rangle\langle i|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes \mathbb{1}^{B_2} \\ \kappa'_2 &:= \kappa_2 + m'_1(i, k) |i\rangle\langle i|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes \mathbb{1}^{B_2}\end{aligned}\tag{56}$$

and, thus, leaving $\kappa_1 + \kappa_2$ unchanged. The eigenvalues of these operators are then

$$m(\kappa'_1) = m_1(i, j, k) - m'_1(i, k) \geq m'_1(i, k) - m'_1(i, k) = 0$$

and

$$m(\kappa'_2) = m_2(i, k, l) + m'_1(i, k) \geq m'_2(i, k) + m'_1(i, k) \geq 0.$$

3. If on the other hand $m'_2(i, k) < 0$ one defines

$$\begin{aligned}\kappa'_1 &:= \kappa_1 + m'_2(i, k)|i\rangle\langle i|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes \mathbb{1}^{B_2} \\ \kappa'_2 &:= \kappa_2 - m'_2(i, k)|i\rangle\langle i|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |k\rangle\langle k|^{B_1} \otimes \mathbb{1}^{B_2}\end{aligned}\tag{57}$$

instead, giving eigenvalues

$$\begin{aligned}m(\kappa'_1) &= m_1(i, j, k) + m'_2(i, k) \geq m'_1(i, k) + m'_2(i, k) \geq 0 \\ &\text{and} \\ m(\kappa'_2) &= m_2(i, k, l) - m'_2(i, k) \geq m'_2(i, k) - m'_2(i, k) = 0.\end{aligned}$$

Repeating step 1. 2. or 3. for all (i, k) , one arrives at two matrices $\bar{\kappa}_1$ and $\bar{\kappa}_2$ such that

$$\kappa_1 + \kappa_2 = \bar{\kappa}_1 + \bar{\kappa}_2 \geq 0 \quad \text{and} \quad \bar{\kappa}_1 \geq 0, \bar{\kappa}_2 \geq 0.$$

The process matrix can therefore be written as

$$W_{eff} = \frac{1}{d_{A_1}d_{B_1}}(\rho^{A_1A_2B_1} + \rho^{A_1B_1B_2})\tag{58}$$

$$\text{with } \rho^{A_1A_2B_1} := (1 + \lambda_0)\mathbb{1} + \bar{\kappa}_1 \geq 0 \quad \text{and} \quad \rho^{A_1B_1B_2} := \bar{\kappa}_2 \geq 0$$

Since $\rho^{A_1A_2B_1}$ contains only terms of the type $A \preceq B$ and $A \not\preceq B$ and is positive semidefinite the following expression is a valid process matrix of the $B \not\preceq A$ -form described in 3.2.

$$W^{B \not\preceq A} = \frac{d_{A_2}d_{B_2}}{\text{Tr}(\rho^{A_1A_2B_1})} \cdot \rho^{A_1A_2B_1}$$

Moreover

$$W^{A \not\preceq B} = \frac{d_{A_2}d_{B_2}}{\text{Tr}(\rho^{A_1B_1B_2})} \cdot \rho^{A_1B_1B_2}$$

is also a valid process matrix containing only $B \preceq A$ - and $A \not\preceq B$ -terms. As for any process matrix $\text{Tr}(W) = d_{A_2}d_{B_2}$ one gets

$$\underbrace{\frac{\text{Tr}(\rho^{A_1A_2B_1})}{d_{A_1}d_{A_2}d_{B_1}d_{B_1}}}_{=:p} + \underbrace{\frac{\text{Tr}(\rho^{A_1B_1B_2})}{d_{A_1}d_{A_2}d_{B_1}d_{B_1}}}_{=:1-p} = 1.$$

Hence in the case of *classical operations*, W_{eff} can be written as

$$W_{eff} = pW^{B \not\preceq A} + (1 - p)W^{A \not\preceq B}$$

and is therefore causally separable.

In this proof one constructs $W^{B \not\perp A}$ and $W^{A \not\perp B}$ from the general form of a process matrix. To be able to do that in the way described above some conditions have to be fulfilled. Note that any process matrix W can always be written as in (53) with κ_1 and κ_2 given by (52) and $\kappa_1 + \kappa_2 \geq 0$.

The question whether some W is causally separable is equivalent to the question whether the positive semidefinite sum of the κ_1 and κ_2 can be written as a sum of two positive semidefinite matrices $\bar{\kappa}_1 \geq 0$ and $\bar{\kappa}_2 \geq 0$ such that

$$\kappa_1 + \kappa_2 = \bar{\kappa}_1 + \bar{\kappa}_2.$$

This is generally not the case if κ_1 and κ_2 are given by (52) but otherwise arbitrary, i.e. hermitian matrices consisting of the respective expressions in (37). The following conditions, however, allow the construction described above.

Since κ_1 and κ_2 should eventually lead to process matrices $W^{B \not\perp A}$ and $W^{A \not\perp B}$, the only freedom one has in modifying them, is by adding and subtracting expressions

$$X = \sum_{i,j} k_{ij} A_i^{A_1} \otimes \mathbb{1}^{A_2} \otimes B_j^{B_1} \otimes \mathbb{1}^{B_2}, \quad (59)$$

where A_i and B_j are hermitian matrices, such that $\kappa_1 + \kappa_2$ is left unchanged. These manipulations have to change the eigenvalues of κ_1 and κ_2 in a systematic way. In order to be able to do that the eigenvalues $m(\cdot)$ of κ_1 , κ_2 and $\kappa_1 + \kappa_2$ have to fulfill

$$m(\kappa_1 + \kappa_2) = m_1(\kappa_1) + m_2(\kappa_2), \quad (60)$$

which is the case, if and only if κ_1 and κ_2 commute.

$$[\kappa_1, \kappa_2] = 0. \quad (61)$$

Furthermore the X -terms, given by (59), have to commute with both these terms,

$$[\kappa_1, X] = [\kappa_2, X] = 0, \quad (62)$$

since only then does one get

$$m(\kappa'_i) = m_i \pm \lambda(X) \quad \text{for } i = 1, 2 \quad (63)$$

where $\lambda(X)$ are the eigenvalues of X given by (59) and κ'_i are the modified κ_i , $i = 1, 2$. Another important requirement for the construction above, assuming

(60) is fulfilled, is that the minimization over the sum of the eigenvalues $m_1 + m_2$ is achieved by minimizing over m_1 and m_2 separately.

$$\min\{m(\kappa_1 + \kappa_2)\} = \min\{m_1 + m_2\} = \min\{m_1\} + \min\{m_2\} \geq 0 \quad (64)$$

which is the case, if the joint eigenvectors of κ_1 and κ_2 are product states on the space $\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$, see later (chapter 4.1).

Only if the three requirements (61), (62) and (64) are fulfilled the manipulations (56) or (57) will systematically create positive eigenvalues for the modified operators κ'_1 and κ'_2 . Altogether (61), (62) and (64) are sufficient conditions for the proof of causal separability described earlier in this section to work. To see that they are by no means necessary, consider a causally separable process matrix with σ -terms generally given by (37):

$$\begin{aligned} W^{A_1 A_2 B_1 B_2} &= pW^{B \not\leq A} + (1-p)W^{A \not\leq B} = \\ &= \frac{p}{d_{A_1} d_{B_1}} \left(\mathbb{1} + \sigma_1^{A \leq B} + \sigma_1^{A \not\leq B} \right) + \frac{1-p}{d_{A_1} d_{B_1}} \left(\mathbb{1} + \sigma_2^{B \leq A} + \sigma_2^{A \not\leq B} \right) = \\ &= \frac{1}{d_{A_1} d_{B_1}} \left(\mathbb{1} + \underbrace{p\sigma_1^{A \leq B}}_{=\sigma^{A \leq B}} + \underbrace{(1-p)\sigma_2^{B \leq A}}_{=\sigma^{B \leq A}} + \underbrace{p\sigma_1^{A \not\leq B} + (1-p)\sigma_2^{A \not\leq B}}_{=\sigma^{A \not\leq B}} \right) = \quad (65) \\ &= \frac{1}{d_{A_1} d_{B_1}} \left(\mathbb{1} + \sigma^{B \not\leq A} + \sigma^{A \not\leq B} \right) \end{aligned}$$

Since $[\kappa_1, \kappa_2] \propto [\sigma_1^{A \leq B}, \sigma_2^{B \leq A}]$ it will not vanish in general. For the simple counterexample, considering qubits,

$$\sigma_1^{A \leq B} = e \sigma_x^{A_2} \otimes \sigma_z^{B_1} + f \sigma_z^{A_1} \otimes \sigma_z^{A_2} \otimes \sigma_x^{B_1} \quad , \quad \sigma_2^{B \leq A} = c \sigma_x^{A_1} \otimes \sigma_x^{B_1} \otimes \sigma_z^{B_2}$$

$$\text{and } \sigma_1^{A \not\leq B} = \sigma_2^{A \not\leq B} = 0.$$

The commutator in (61) is given by

$$\begin{aligned} [\kappa_1, \kappa_2] &= [-\lambda_0 \mathbb{1} + p \sigma_1^{A \leq B}, (1-p) \sigma_2^{B \leq A}] = \\ &= (p-p^2) (ce \sigma_x^{A_1} \otimes \sigma_x^{A_2} \otimes [\sigma_z, \sigma_x]^{B_1} \otimes \sigma_z^{B_2} + cf [\sigma_z, \sigma_x]^{A_1} \otimes \sigma_z^{A_2} \otimes \sigma_z^{B_2}) \\ &\neq 0. \end{aligned}$$

The relation (64) is not fulfilled, since $m(\kappa_1 + \kappa_2) \neq m_1 + m_2$ for non-commuting κ_1 and κ_2 . Moreover any X of the form (59) would have to consist of matrices A_i and B_j that commute with both σ_x and σ_z in order to fulfill (62). This is the case, if and only if $A_i \propto \mathbb{1}$ and $B_j \propto \mathbb{1} \forall i, j$, leaving only trivial X -terms, which cannot modify κ_1 and κ_2 . So none of the sufficient conditions is fulfilled although

$$W^{A_1 A_2 B_1 B_2} = \frac{1}{d_{A_1} d_{B_1}} (p(\mathbb{1} + \sigma_1^{A \preceq B}) + (1-p)(\mathbb{1} + \sigma_2^{B \preceq A}))$$

was chosen to be causally separable to begin with.

4 Semiquantum Processes and Causal Separability

In this chapter *semiquantum processes*, which describe semiquantum correlations between two closed laboratories, will be analyzed.

Since laboratories are sets of quantum instruments, they can, analogous to quantum channels and bipartite states, see 2.2 and 2.3, be classified as *classical-quantum* and *quantum-classical*. The processes for two semiquantum laboratories are then also called semiquantum. One may also regard *processes*, which correlate a *truly quantum* to a *classical* laboratory as semiquantum since the correlations in this case are semiquantum as well. Finally, the correlations between a *classical* and a *semiquantum* laboratory are either classical or semiquantum and will be considered in the end of this chapter.

The limitation of possible correlations between the two laboratories can, as in the classical case, be expressed in terms of an *effective process matrix* W_{eff} characterizing semiquantum processes. These will then be analyzed in terms of causal separability.

A *c-q operation* is characterized by a CJ- matrix

$$M_i^{X_1 X_2} = \sum_n P(i|n) |n\rangle \langle n|^{X_1} \otimes X_i^{X_2} \quad (66)$$

where $P(i|n)$ is the (classical) probability of result i being measured, if the input is n , and X_i is some hermitian matrix depending on the measurement result i .

Any *laboratory* in which only operations like (66) are performed, is therefore also called *classical-quantum*. It corresponds to a laboratory, where the *input* state is always measured in a *fixed measurement basis*. If, for example, the input and output of the laboratory are spin- $\frac{1}{2}$ -particles, the spin is always measured along the same direction.

Q - c operation, on the other hand, have CJ- matrices of the following form

$$M_i^{X_1 X_2} = \sum_r X_i^{X_1} \otimes P(r|i) |r\rangle \langle r|^{X_2} \quad (67)$$

where again X_i is some hermitian matrix and $P(r|i)$ is the probability of output r given the measurement result i . The term *quantum-classical*, again, describes the whole *laboratory*, if only operations as in (67) are possible. In a q-c laboratory all the possible output states commute with one another. In the case of spin- $\frac{1}{2}$ -particles this means that the output states are spin polarized along a fixed direction.

4.1 Classical-Quantum Laboratories

If both laboratories, A and B , are c-q, see figure 5, the inputs in each of them are measured in a fixed basis. The corresponding CP-maps are represented by the following kind of tensor product of CJ-matrices

$$M_i^{A_1 A_2} \otimes M_j^{B_1 B_2} = \sum_{nm} P_n |n\rangle \langle n|^{A_1} \otimes A_i^{A_2} \otimes P_m |m\rangle \langle m|^{B_1} \otimes B_j^{B_2}. \quad (68)$$

The dependance of the operations on the measurement results is given by the probabilities $P_n = P(i|n)$ and $P_m = P(m|j)$, but will no longer be stated explicitly, since only the general structure of operations, which is the same for all i and j , is of interest. Tensor products, which contain diagonal matrices in at least one of the subspaces, will further be referred to as *partially diagonal*.

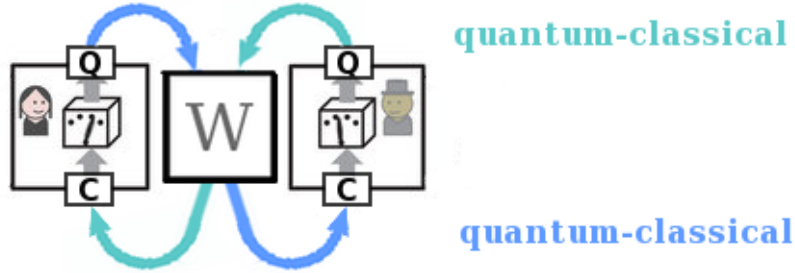


Figure 5: Correlations between two classical-quantum laboratories. The input measurements are quasi classical, while the outputs are arbitrary quantum states. All possible signaling channels are quantum-classical. Picture taken from [11], altered.

Consider the basis $\{|n\rangle\langle n|^{A_1} \otimes |i\rangle\langle i|^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes |j\rangle\langle j|^{B_2}\}$, where $\{|i\rangle^{A_2}\}$ and $\{|j\rangle^{B_2}\}$ are arbitrary ONB's of \mathcal{H}^{A_2} and \mathcal{H}^{B_2} , while $\{|n\rangle\}$ and $\{|m\rangle\}$ are those ONB's of the input spaces, out of which the orthogonal projectors in (66) are composed.

The effective process matrices of interest have to fulfill the following equation:

$$\begin{aligned} & \text{Tr} \left(W \cdot \sum_{\substack{ni' \\ mj'}} P_n |n\rangle\langle n|^{A_1} \otimes a_{ii'} |i\rangle\langle i|^{A_2} \otimes P_m |m\rangle\langle m|^{B_1} \otimes b_{jj'} |j\rangle\langle j|^{B_2} \right) = \\ & \stackrel{!}{=} \text{Tr} \left(W_{eff} \cdot \sum_{\substack{ni' \\ mj'}} P_n |n\rangle\langle n|^{A_1} \otimes a_{ii'} |i\rangle\langle i|^{A_2} \otimes P_m |m\rangle\langle m|^{B_1} \otimes b_{jj'} |j\rangle\langle j|^{B_2} \right) \end{aligned} \quad (69)$$

From that one can find W_{eff} like in 3.4:

$$\begin{aligned} & \text{Tr} \left(\sum_{\substack{nn'n'' \\ mm'm'' \\ ii'jj' \\ kk'll'}} w_{n'n''kk'm'm''ll'} P_n a_{ii'} P_m b_{jj'} \dots \right. \\ & \quad \left. \dots |n'\rangle\langle n''|n\rangle\langle n|^{A_1} \otimes |k\rangle\langle k'|i\rangle\langle i'|^{A_2} \otimes |m'\rangle\langle m''|m\rangle\langle m|^{B_1} \otimes |l\rangle\langle l'|j\rangle\langle j'|^{B_2} \right) = \\ & = \sum_{\substack{nn'mm'k \\ ii'jj'l}} w_{n'nkim'mlj} P_n a_{ii'} P_m b_{jj'} \text{Tr} \left(|n'\rangle\langle n|^{A_1} \otimes |k\rangle\langle i'|^{A_2} \otimes |m'\rangle\langle m|^{B_1} \otimes |l\rangle\langle j'|^{B_2} \right) = \\ & = \sum_{\substack{nn'mm'k \\ ii'jj'l}} w_{n'nkim'mlj} P_n a_{ii'} P_m b_{jj'} \underbrace{\langle n|n'\rangle}_{=\delta_{nn'}} \underbrace{\langle i'|k\rangle}_{=\delta_{i'k}} \underbrace{\langle m|m'\rangle}_{=\delta_{mm'}} \underbrace{\langle j'|l\rangle}_{=\delta_{j'l}} = \\ & = \sum_{\substack{ni' \\ mj'}} w_{nni'immj'j} P_n a_{ii'} P_m b_{jj'} \end{aligned} \quad (70)$$

With $w_{nni'immj'j}^{eff} := w_{nni'immj'j}$, the effective process matrices are given by

$$W_{eff} = \sum_{\substack{ni' \\ mj'}} w_{nni'immj'j}^{eff} |n\rangle\langle n|^{A_1} \otimes |i\rangle\langle i'|^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes |j\rangle\langle j'|^{B_2} \quad (71)$$

since

$$\begin{aligned}
& \text{Tr} \left(W_{eff} \cdot \sum_{\substack{ni' \\ mj'j'}} P_n |n\rangle \langle n|^{A_1} \otimes a_{ii'} |i\rangle \langle i'|^{A_2} \otimes P_m |m\rangle \langle m|^{B_1} \otimes b_{jj'} |j\rangle \langle j'|^{B_2} \right) = \\
& = \text{Tr} \left(\sum_{\substack{n,n',k,k' \\ m,m',l,l' \\ i,i',j,j'}} w_{n'kk'm'l'}^{eff} P_n a_{ii'} P_m b_{jj'} \delta_{n'n} \delta_{k'i} \delta_{m'm} \delta_{l'j} \dots \right. \\
& \quad \left. \dots |n'\rangle \langle n|^{A_1} \otimes |k\rangle \langle i'|^{A_2} \otimes |m'\rangle \langle m|^{B_1} \otimes |l\rangle \langle j'|^{B_2} \right) = \\
& = \sum_{\substack{nkii' \\ mljj'}} w_{nkimlj}^{eff} P_n a_{ii'} P_m b_{jj'} \underbrace{\langle n|n'\rangle}_{=\delta_{nn'}} \underbrace{\langle i'|k\rangle}_{=\delta_{i'k}} \underbrace{\langle m|m'\rangle}_{=\delta_{mm'}} \underbrace{\langle j'|l\rangle}_{=\delta_{j'l}} = \\
& = \sum_{\substack{ni' \\ mj'j'}} w_{nni'immj'j} P_n a_{ii'} P_m b_{jj'}.
\end{aligned} \tag{72}$$

Like the respective tensor product of the CJ- matrices the effective process matrix is partially diagonal in the basis $\{|n\rangle \langle n'|^{A_1} \otimes |i\rangle \langle i'|^{A_2} \otimes |m\rangle \langle m'|^{B_1} \otimes |j\rangle \langle j'|^{B_2}\}$. One can now analyze W_{eff} given by (71) in terms of causal separability. As the bases of \mathcal{H}^{A_2} and \mathcal{H}^{B_2} were arbitrary consider W_{eff} in the basis $\{|n\rangle \langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle \langle m'|^{B_1} \otimes \sigma_j^{B_2}\}$. In that case the terms defined in (37) are

$$\begin{aligned}
\sigma_{eff}^{B \prec A} &= \sum_{ni} c_{ni} |n\rangle \langle n|^{A_1} \otimes \sigma_i^{B_2} + \sum_{nmi} d_{nmi} |n\rangle \langle n|^{A_1} \otimes |m\rangle \langle m|^{B_1} \otimes \sigma_i^{B_2} \\
\sigma_{eff}^{A \prec B} &= \sum_{im} e_{im} \sigma_i^{A_2} \otimes |m\rangle \langle m|^{B_1} + \sum_{nim} f_{nim} |n\rangle \langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle \langle m|^{B_1} \\
\sigma_{eff}^{A \not\prec B} &= \sum_n x_n |n\rangle \langle n|^{A_1} + \sum_m y_m |m\rangle \langle m|^{B_1} + \sum_{nm} g_{nm} |n\rangle \langle n|^{A_1} \otimes |m\rangle \langle m|^{B_1}
\end{aligned} \tag{73}$$

where the sums are taken over $i, j = 1 \dots d_{X_2}^2 - 1$ and $n, m = 1 \dots d_{X_1}$, with $X = A, B$. For such W_{eff} one can start from the following definitions

$$\begin{aligned}
\kappa_{eff}^{A_1 A_2 B_1} &= \kappa_{eff}^1 := -\lambda_0 \mathbb{1} + \sigma_{eff}^{A \prec B} + \sum_m y_m |m\rangle \langle m|^{B_1} \\
\kappa_{eff}^{A_1 B_1 B_2} &= \kappa_{eff}^2 := \sigma_{eff}^{B \prec A} + \sum_n x_n |n\rangle \langle n|^{A_1} + \sum_{n,m} g_{nm} |n\rangle \langle n|^{A_1} \otimes |m\rangle \langle m|^{B_1}
\end{aligned} \tag{74}$$

where λ_0 is the minimal eigenvalue of $\kappa_{eff}^1 + \kappa_{eff}^2$.

In order to check whether one can modify κ_{eff}^1 and κ_{eff}^2 to arrive at $\bar{\kappa}_{eff}^1 \geq 0$ and $\bar{\kappa}_{eff}^2 \geq 0$ such that $\bar{\kappa}_{eff}^1 + \bar{\kappa}_{eff}^2 = \kappa_{eff}^1 + \kappa_{eff}^2$, one has to verify the conditions (61), (62) and (64).

First note that the commutator of κ_{eff}^1 and κ_{eff}^2 vanishes:

$$\begin{aligned} [\kappa_{eff}^1, \kappa_{eff}^2] &= -\lambda_0 [\mathbb{1}, \kappa_{eff}^2] + \sum_{im} e_{im} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, \kappa_{eff}^2] \\ &\quad + \sum_{nim} f_{nim} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, \kappa_{eff}^2] \\ &\quad + \sum_m y_m [|m\rangle\langle m|^{B_1}, \kappa_{eff}^2] = 0 \end{aligned}$$

To see this, consider each type of commutator separately:

1. $[\mathbb{1}, \kappa_{eff}^2] = 0$
2.
$$\begin{aligned} [|m\rangle\langle m|^{B_1}, \kappa_{eff}^2] &= \sum_{n'j} c_{n'j} [|m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_j^{B_2}] \\ &\quad + \sum_{n'm'j} d_{n'm'j} [|m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2}] \\ &\quad + \sum_{n'} x_{n'} [|m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1}] \\ &\quad + \sum_{n'm'} g_{n'm'} [|m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1}] = \\ &= \sum_{n'm'j} d_{n'm'j} (|n'\rangle\langle n'|^{A_1} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2} \dots \\ &\quad \dots - |n'\rangle\langle n'|^{A_1} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1} \otimes \sigma_j^{B_2}) \\ &\quad + \sum_{n'm'} g_{n'm'} (|n'\rangle\langle n'|^{A_1} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} - |n'\rangle\langle n'|^{A_1} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1}) = 0 \end{aligned}$$
3.
$$\begin{aligned} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, \kappa_{eff}^2] &= \sum_{n'j} c_{n'j} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_j^{B_2}] \\ &\quad + \sum_{n'm'j} d_{n'm'j} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2}] \dots \end{aligned}$$

$$\begin{aligned}
& \dots + \sum_{n'} x_{n'} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1}] \\
& + \sum_{n'm'} g_{n'm'} [\sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1}] = \\
& = \sum_{n'm'j} d_{n'm'j} (|n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2} \dots \\
& \qquad \qquad \qquad \dots - |n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1} \otimes \sigma_j^{B_2}) \\
& + \sum_{n'm'} g_{nm'} (|n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} \dots \\
& \qquad \qquad \qquad \dots - |n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1}) = 0
\end{aligned}$$

$$4. [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, \kappa_{eff}^2] =$$

$$\begin{aligned}
& = \sum_{n'j} c_{n'j} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_j^{B_2}] \\
& + \sum_{n'm'j} d_{n'm'j} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2}] \\
& + \sum_{n'} x_{n'} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1}] \\
& + \sum_{n'm'} g_{n'm'} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1}] = \\
& = \sum_{n'j} c_{n'j} (\delta_{n,n'} |n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes \sigma_j^{B_2} \dots \\
& \qquad \qquad \qquad \dots - \delta_{n'n} |n'\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes \sigma_j^{B_2}) \\
& + \sum_{n'm'j} d_{n'm'j} (\delta_{nn'} |n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} \otimes \sigma_j^{B_2} \dots \\
& \qquad \qquad \qquad \dots - \delta_{n'n} |n'\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1} \otimes \sigma_j^{B_2}) \\
& + \sum_{n'} x_{n'} (\delta_{n,n'} |n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1} - \delta_{n',n} |n'\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m|^{B_1}) \\
& + \sum_{n'm'} g_{n'm'} (\delta_{nn'} |n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{mm'} |m\rangle\langle m'|^{B_1} \dots \\
& \qquad \qquad \qquad \dots - \delta_{n'n} |n'\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \delta_{m'm} |m'\rangle\langle m|^{B_1}) = 0
\end{aligned}$$

Secondly, since κ_{eff}^1 and κ_{eff}^2 , like W_{eff} , are partially diagonal, one might modify them using X -terms of the form

$$X_{(n,m)} = |n\rangle\langle n|^{A_1} \otimes \mathbb{1}^{A_2} \otimes |m\rangle\langle m|^{B_1} \otimes \mathbb{1}^{B_2} \quad (75)$$

analogous to the case of classical operations. It is easy to see that $[\kappa_{eff}^1, X_{(n,m)}] = [\kappa_{eff}^2, X_{(n,m)}] = 0$.

$$\begin{aligned} [\kappa_{eff}^1, X_{(n,m)}] &= -\lambda_0 [\mathbb{1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{im'} e_{im'} [\sigma_i^{A_2} \otimes |m'\rangle\langle m'|^{B_1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{n'im'} f_{n'im'} [|n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes |m'\rangle\langle m'|^{B_1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{m'} y_{m'} [|m'\rangle\langle m'|^{B_1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] = \\ &= \sum_{im'} e_{im'} (\delta_{m'm} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m'\rangle\langle m|^{B_1} - \delta_{mm'} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m'|^{B_1}) \\ &+ \sum_{n'im'} f_{n'im'} (\delta_{n'n} \delta_{m'm} |n'\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes |m'\rangle\langle m|^{B_1} \dots \\ &\dots - \delta_{nn'} \delta_{mm'} |n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes |m\rangle\langle m'|^{B_1}) \\ &+ \sum_{m'} y_{m'} (\delta_{m'm} |n\rangle\langle n|^{A_1} \otimes |m'\rangle\langle m|^{B_1} - \delta_{mm'} |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m'|^{B_1}) = 0 \end{aligned}$$

and

$$\begin{aligned} [\kappa_{eff}^2, X_{(n,m)}] &= \sum_{n'i} c_{n'i} [|n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{n'm'i} d_{n'm'i} [|n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1} \otimes \sigma_i^{B_2}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{n'} x_{n'} [|n'\rangle\langle n'|^{A_1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] \\ &+ \sum_{n'm'} g_{n'm'} [|n'\rangle\langle n'|^{A_1} \otimes |m'\rangle\langle m'|^{B_1}, |n\rangle\langle n|^{A_1} \otimes |m\rangle\langle m|^{B_1}] = 0. \end{aligned}$$

This means that there exists an ONB of joint eigenvectors $\{|\psi^{n,m}\rangle\}$ of $X_{(n,m)}$, κ_{eff}^1 and κ_{eff}^2 . Let $\{|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle\}$ be some ONB of $\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$. Since the

$X_{(n,m)}$ are orthogonal projectors they have only eigenvalues 0 and 1. Moreover, a joint eigenvector $|\psi^{n,m}\rangle$ of $X_{(n,m)}$, κ_{eff}^1 and κ_{eff}^2 with $X_{(n,m)}|\psi^{n,m}\rangle = |\psi^{n,m}\rangle$ fulfills $X_{(n',m')}|\psi^{n,m}\rangle = 0$ for $(n',m') \neq (n,m)$. Consider now such an eigenvector $|\psi^{n,m}\rangle$:

$$\begin{aligned}
X_{(n,m)}|\psi^{n,m}\rangle &= X_{(n,m)} \sum_{ijkl} c_{ijkl} |i\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |k\rangle^{B_1} \otimes |l\rangle^{B_2} \\
&= \sum_{ijkl} c_{ijkl} |n\rangle \langle n|i\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle \langle m|k\rangle^{B_1} \otimes |l\rangle^{B_2} \\
&= \sum_{jl} c_{njml} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |l\rangle^{B_2} \\
&\stackrel{!}{=} \sum_{ijkl} c_{ijkl} |i\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |k\rangle^{B_1} \otimes |l\rangle^{B_2} \\
\Rightarrow |\psi^{n,m}\rangle &= \sum_{jl} c'_{jl} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |l\rangle^{B_2}
\end{aligned} \tag{76}$$

Note that for a given pair (n,m) there have to be $d_{A_1} d_{A_2}$ linearly independent eigenvectors $|\psi^{n,m}\rangle$ of $X_{(n,m)}$ with eigenvalue 1.

$$\begin{aligned}
\kappa_{eff}^1 |\psi^{n,m}\rangle &= \left(-\lambda_0 \mathbb{1} + \sigma_{eff}^{A \leq B} + \sum_{m'} y_{m'} |m'\rangle \langle m'|^{B_1} \right) |\psi^{n,m}\rangle \\
&= -\lambda_0 |\psi^{n,m}\rangle + \sum_{im'jl} c'_{jl} e_{im'} |n\rangle^{A_1} \otimes \sigma_i |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes |l\rangle^{B_2} \\
&\quad + \sum_{\substack{n'm' \\ ijl}} c'_{jl} f_{n'im'} |n'\rangle^{A_1} \underbrace{\langle n'|n\rangle}_{=\delta_{n'n}} \otimes \sigma_i |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes |l\rangle^{B_2} \\
&\quad + \sum_{jlm'} c'_{jl} y_{m'} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes |l\rangle^{B_2} \\
&= \sum_{jl} c'_{jl} |n\rangle^{A_1} \otimes \underbrace{\left((-\lambda_0 + y_m) \mathbb{1} + \sum_i e_{im} \sigma_i + \sum_i f_{nim} \sigma_i \right)}_{=: A_{(n,m)}} |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |l\rangle^{B_2} \\
&\stackrel{!}{=} m_1 \sum_{jl} c'_{jl} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |l\rangle^{B_2}
\end{aligned} \tag{77}$$

$$\begin{aligned}
\kappa_{eff}^2 |\psi^{n,m}\rangle &= \left(\sigma_{eff}^{B \prec A} + \sum_{n'} x_{n'} |n'\rangle \langle n'|^{A_1} + \sum_{n'm'} g_{n'm'} |n'\rangle \langle n'|^{A_1} \otimes |m'\rangle \langle m'|^{B_1} \right) |\psi^{n,m}\rangle \\
&= \sum_{n'ijl} c'_{jl} c_{n',i} |n'\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes \sigma_i |l\rangle^{B_2} \\
&+ \sum_{\substack{n'm' \\ ijl}} c'_{jl} d_{n'im'} |n'\rangle^{A_1} \underbrace{\langle n'|n\rangle}_{=\delta_{n'n}} \otimes |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes \sigma_i |l\rangle^{B_2} \\
&+ \sum_{jln'} c'_{jl} x_{n'} |n'\rangle^{A_1} \underbrace{\langle n'|n\rangle}_{=\delta_{n'n}} \otimes |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \otimes |l\rangle^{B_2} \\
&+ \sum_{n'm'} g_{n'm'} |n'\rangle^{A_1} \underbrace{\langle n'|n\rangle}_{=\delta_{n'n}} \otimes |j\rangle^{A_2} \otimes |m'\rangle^{B_1} \underbrace{\langle m'|m\rangle}_{=\delta_{m'm}} \otimes |l\rangle^{B_2} \\
&= \sum_{jl} c'_{jl} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes \underbrace{\left((g_{n,m} + x_n) \mathbb{1} + \sum_i c_{ni} \sigma_i + \sum_i d_{nim} \sigma_i \right)}_{=: B_{(n,m)}} |l\rangle^{B_2} \\
&\stackrel{!}{=} m_2 \sum_{jl} c'_{jl} |n\rangle^{A_1} \otimes |j\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |l\rangle^{B_2}
\end{aligned} \tag{78}$$

Together (77) and (78) are equivalent to

$$A_{(n,m)}^{A_2} \otimes \mathbb{1}^{B_2} \left(\sum_{jl} c'_{jl} |j\rangle^{A_2} \otimes |l\rangle^{B_2} \right) = \bar{a}_{(n,m)} \sum_{jl} c'_{jl} |j\rangle^{A_2} \otimes |l\rangle^{B_2}$$

and

$$\mathbb{1}^{A_2} \otimes B_{(n,m)}^{B_2} \left(\sum_{jl} c'_{jl} |j\rangle^{A_2} \otimes |l\rangle^{B_2} \right) = \bar{b}_{(n,m)} \sum_{jl} c'_{jl} |j\rangle^{A_2} \otimes |l\rangle^{B_2},$$

which means the joint eigenvectors $|\psi^{n,m}\rangle$ of $X_{(n,m)}$, κ_{eff}^1 and κ_{eff}^2 are of the form

$$|\psi^{n,m}\rangle = |n\rangle^{A_1} \otimes |a\rangle^{A_2} \otimes |m\rangle^{B_1} \otimes |b\rangle^{B_2}, \tag{79}$$

where $A_{(n,m)}|a\rangle^{A_2} = \bar{a}_{(n,m)}|a\rangle^{A_2}$ and $B_{(n,m)}|b\rangle^{B_2} = \bar{b}_{(n,m)}|b\rangle^{B_2}$.

Since all summands appearing in their definitions are hermitian matrices, so are

$A_{(n,m)}$ and $B_{(n,m)}$. Hence there are d_{A_2} orthogonal eigenvectors $|a\rangle$ and d_{B_2} orthogonal eigenvectors $|b\rangle$, giving $d_{A_1}d_{A_2}$ linearly independent $|\psi^{n,m}\rangle$ for a given pair (n, m) , as required.

Repeating this for all possible pairs (n, m) one arrives at an ONB of joint eigenvectors $|\psi^{n,m}\rangle$ in which κ_{eff}^1 , κ_{eff}^2 and all $X_{(n,m)}$ are diagonal. The eigenvalues $m_1(n, a, m, b)$ and $m_2(n, a, m, b)$ of κ_{eff}^1 and κ_{eff}^2 therefore add up to give the eigenvalues of $\kappa_{eff}^1 + \kappa_{eff}^2$, as required in (60). Furthermore the $\psi^{n,m}$ are product vectors and one can therefore apply the same argument as in the classical case concerning the minimal eigenvalues. Namely, since $m_1(n, a, m, b)$ does not depend on b and $m_2(n, a, m, b)$ does not depend on a , the minimal eigenvalue for a given pair (n, m) is again

$$\begin{aligned} \min_{a,b} \{m_1(n, a, m) + m_2(n, m, b)\} &= \\ &= \min_a \{m_1(n, a, m)\} + \min_b \{m_2(n, m, b)\} =: m'_1(n, m) + m'_2(n, m) \geq 0 \quad . \end{aligned} \quad (80)$$

All the necessary conditions for the construction in 3.4 are fulfilled and one can, like in the case of classical processes, modify κ_{eff}^1 and κ_{eff}^2 by adding and subtracting the necessary $X_{(n,m)}$ -terms for every (n, m) .

This will ultimately lead to $\bar{\kappa}_{eff}^1 \geq 0$ and $\bar{\kappa}_{eff}^2 \geq 0$ such that

$$\begin{aligned} W_{eff} &= \frac{1}{d_{A_1}d_{B_1}}((1 + \lambda_0)\mathbb{1} + \bar{\kappa}_{eff}^1 + \bar{\kappa}_{eff}^2) = \frac{1}{d_{A_1}d_{B_1}}(\rho^{A_1A_2B_1} + \rho^{A_1B_1B_2}) = \\ &= pW^{B \not\leq A} + (1 - p)W^{A \not\leq B}. \end{aligned} \quad (81)$$

analogous to the classical case, compare (58).

This means that correlations between two *classical-quantum laboratories* are also always *compatible with a global causal structure*. In the case of fixed input measurement bases, the *signaling correlations* between the laboratories are *quantum-classical*. Nevertheless they cannot violate the causal inequality (32).

4.2 Quantum-Classical Laboratories

Two q-c laboratories, see figure 6, correspond to the output states of each being prepared in the same basis in all operations. Generally one is interested in process matrices acting on tensor products

$$M_i^{A_1A_2} \otimes M_j^{B_1B_2} = \sum_{r,s} A_i^{A_1} \otimes P_r|r\rangle\langle r|^{A_2} \otimes B_j^{B_1} \otimes P_s|s\rangle\langle s|^{B_2}. \quad (82)$$

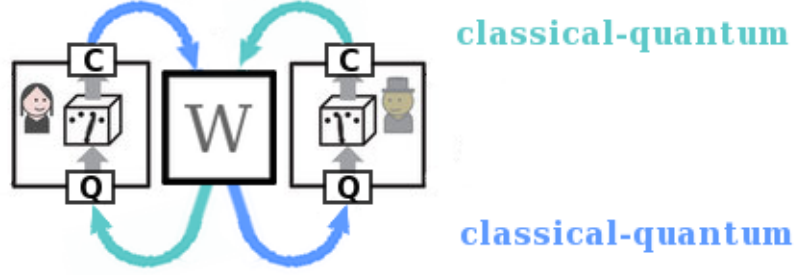


Figure 6: Correlations between two quantum-classical laboratories. The input measurements are arbitrary, while the output states of each laboratory commute. All possible signaling channels are classical-quantum. Picture taken from [11], altered.

Again, one can determine the effective process matrices for this limitation of CP-maps. The trace is calculated in the basis $\{|i\rangle\langle i'|^{A_1} \otimes |r\rangle\langle r'|^{A_2} \otimes |j\rangle\langle j'|^{B_1} \otimes |s\rangle\langle s'|^{B_2}\}$, where $\{|i\rangle^{A_1}\}$ and $\{|j\rangle^{B_1}\}$ are arbitrary ONB's of \mathcal{H}^{A_1} and \mathcal{H}^{B_1} .

$$\begin{aligned}
& \text{Tr} \left(\sum_{\substack{rr'r''ss's'' \\ ii'jj'kk'll'}} w_{kk'r'r''ll's's''} a_{ii'} P_r b_{jj'} P_s \dots \right. \\
& \quad \dots |k\rangle \underbrace{\langle k'|i\rangle\langle i'|^{A_1}}_{=\delta_{k'i}} \otimes |r'\rangle \underbrace{\langle r''|r\rangle\langle r|^{A_2}}_{=\delta_{r''r}} \otimes |l\rangle \underbrace{\langle l'|j\rangle\langle j'|^{B_1}}_{=\delta_{l'j}} \otimes |s'\rangle \underbrace{\langle s''|s\rangle\langle s|^{B_2}}_{=\delta_{s''s}} \Big) = \quad (83) \\
& = \sum_{\substack{rr'k \\ ss'l \\ ii'jj'}} w_{kir'rljs's} a_{ii'} P_r b_{jj'} P_s \delta_{i'k} \delta_{rr'} \delta_{j'l} \delta_{ss'} = \sum_{\substack{rii' \\ sjj'}} w_{ii'rrjj'ss} a_{ii'} P_r b_{jj'} P_s
\end{aligned}$$

With $w_{ii'rrjj's}^{eff} = w_{ii'rrjj'ss}$, the effective process matrices are of the form

$$W_{eff} = \sum_{\substack{r,i,i' \\ s,j,j'}} w_{ii'rrjj's}^{eff} |i\rangle\langle i'|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes |j\rangle\langle j'|^{B_1} \otimes |s\rangle\langle s|^{B_2}, \quad (84)$$

which in the basis $\{\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}\}$ consist of the following σ_{eff} -terms:

$$\begin{aligned}
\sigma_{eff}^{B \prec A} &= \sum_{is} c_{is} \sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2} + \sum_{ijs} d_{ijs} \sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
\sigma_{eff}^{A \prec B} &= \sum_{ri} e_{ri} |r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1} + \sum_{irj} f_{irj} \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1} \\
\sigma_{eff}^{A \not\prec B} &= \sum_i x_i \sigma_i^{A_1} + \sum_i y_i \sigma_i^{B_1} + \sum_{ij} g_{ij} \sigma_i^{A_1} \otimes \sigma_j^{B_1}
\end{aligned} \tag{85}$$

To see whether the construction in 3.4 works for such effective process matrices, consider the commutators between these terms.

$$\begin{aligned}
1. \quad [\sigma_{eff}^{B \prec A}, \sigma_{eff}^{A \prec B}] &= \sum_{is} c_{is} [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \prec B}] \\
&\quad + \sum_{ijs} d_{ijs} [\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \prec B}] \neq 0
\end{aligned}$$

in general. The first sum contains the following expressions

$$\begin{aligned}
[\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \prec B}] &= \sum_{rj} e_{rj} [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}] \\
&+ \sum_{jrk} f_{jrk} [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1}] = \\
&= \sum_{jrk} f_{jrk} [\sigma_i, \sigma_j]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1} \otimes |s\rangle\langle s|^{B_2} \quad ,
\end{aligned}$$

while the commutators in the second sum give

$$\begin{aligned}
[\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \prec B}] &= \sum_{rk} e_{rk} [\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1}] \\
&+ \sum_{krl} f_{krl} [\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_k^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_l^{B_1}] = \\
&= \sum_{rk} e_{rk} \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
&+ \sum_{krl} f_{krl} \left(\sigma_i \sigma_k^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j \sigma_l^{B_1} \otimes \sigma_m^{B_1} \otimes |s\rangle\langle s|^{B_2} \dots \right. \\
&\quad \left. \dots - \sigma_k \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_l \sigma_j^{B_1} \otimes \sigma_m^{B_1} \otimes |s\rangle\langle s|^{B_2} \right) = \dots
\end{aligned}$$

$$\begin{aligned}
\cdots &= \sum_{rk} e_{rk} \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
&+ \sum_{krl} \frac{2f_{irl}}{d_{A_1}} |r\rangle\langle r|^{A_2} \otimes [\sigma_j, \sigma_l]^{B_1} \otimes |s\rangle\langle s|^{B_2} + \frac{2f_{krj}}{d_{B_1}} [\sigma_i, \sigma_k]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes |s\rangle\langle s|^{B_2} \\
&+ \sum_{krl} 2if_{krl} \sum_{n,m} (\eta_{ikn}\beta_{jlm} + \gamma_{kin}\nu_{ljm}) \sigma_n^{A_1} \otimes \sigma_j^{B_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_m^{B_1} \otimes |s\rangle\langle s|^{B_2},
\end{aligned}$$

where η_{ijk} and ν_{ijk} are the totally antisymmetric and γ_{ijk} and β_{ijk} are the totally symmetric structure constants of $su(d_{A_1})$ and $su(d_{B_1})$.

Altogether one gets

$$\begin{aligned}
[\sigma_{eff}^{B \leq A}, \sigma_{eff}^{A \leq B}] &= \sum_{\substack{ijk \\ rs}} c_{is} f_{jrk} [\sigma_i, \sigma_j]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
&+ \sum_{\substack{ijkl \\ rsnm}} d_{ijs} (e_{rk} \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} \otimes |s\rangle\langle s|^{B_2} \dots \\
&\cdots + \frac{2}{d_{A_1}} f_{irl} |r\rangle\langle r|^{A_2} \otimes [\sigma_j, \sigma_l]^{B_1} \otimes |s\rangle\langle s|^{B_2} \dots \\
&\cdots + \frac{2}{d_{B_1}} f_{krj} [\sigma_i, \sigma_k]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes |s\rangle\langle s|^{B_2} \dots \\
&\cdots + 2if_{krl} (\eta_{ikn}\beta_{jlm} + \gamma_{kin}\nu_{ljm}) \sigma_n^{A_1} \otimes \sigma_j^{B_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_m^{B_1} \otimes |s\rangle\langle s|^{B_2}) \neq 0 \quad .
\end{aligned}$$

$$\begin{aligned}
2. \quad [\sigma_{eff}^{B \leq A}, \sigma_{eff}^{B \not\leq A}] &= \sum_{is} c_{is} [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \not\leq A}] \\
&+ \sum_{ijs} d_{ijs} [\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \not\leq A}]
\end{aligned}$$

Consider, again, the types of commutators in the two sums separately:

$$\begin{aligned}
[\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \not\leq A}] &= \sum_j x_j [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{A_1}] \\
&+ \sum_j y_j [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{B_1}] + \sum_{jk} g_{jk} [\sigma_i^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{A_1} \otimes \sigma_k^{B_1}] = \\
&= \sum_j x_j [\sigma_i, \sigma_j]^{A_1} \otimes |s\rangle\langle s|^{B_2} + \sum_{jk} g_{jk} [\sigma_i, \sigma_j]^{A_1} \otimes \sigma_k^{B_1} \otimes |s\rangle\langle s|^{B_2}
\end{aligned}$$

and

$$\begin{aligned}
& \left[\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \neq A} \right] = \sum_k x_k \left[\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_k^{A_1} \right] \\
& + \sum_k y_k \left[\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_k^{B_1} \right] + \sum_{kl} g_{kl} \left[\sigma_i^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_k^{A_1} \otimes \sigma_l^{B_1} \right] = \\
& = \sum_k x_k \left[\sigma_i, \sigma_k \right]^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2} + \sum_k y_k \sigma_i^{A_1} \otimes \left[\sigma_j, \sigma_k \right]^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
& + \sum_{kl} g_{l,k} \left(\sigma_i \sigma_k^{A_1} \otimes \sigma_j \sigma_l^{B_1} \otimes |s\rangle\langle s|^{B_2} - \sigma_k \sigma_i^{A_1} \otimes \sigma_l \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2} \right).
\end{aligned}$$

Therefore, in general, this commutator does not vanish, either.

$$\begin{aligned}
& \left[\sigma_{eff}^{B \neq A}, \sigma_{eff}^{B \neq A} \right] = \sum_{ijsk} c_{is} \left(x_j \left[\sigma_i, \sigma_j \right]^{A_1} \otimes |s\rangle\langle s|^{B_2} + g_{jk} \left[\sigma_i, \sigma_j \right]^{A_1} \otimes \sigma_k^{B_2} \otimes |s\rangle\langle s|^{B_2} \right) \\
& \sum_{\substack{ij \\ kls}} d_{ijs} \left(x_k \left[\sigma_i, \sigma_k \right]^{A_1} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s|^{B_2} + y_k \sigma_i^{A_1} \otimes \left[\sigma_j, \sigma_k \right]^{B_1} \otimes |s\rangle\langle s|^{B_2} \dots \right. \\
& \dots + \frac{2}{d_{A_1}} g_{ki} \left[\sigma_j, \sigma_l \right]^{B_1} \otimes |s\rangle\langle s|^{B_2} + \frac{2}{d_{B_1}} g_{jl} \left[\sigma_i, \sigma_k \right]^{A_1} \otimes |s\rangle\langle s|^{B_2} \dots \\
& \left. \dots + \sum_{nm} 2i g_{k,l} (\eta_{ikn} \beta_{jlm} + \gamma_{kin} \nu_{ljm}) \sigma_n^{A_1} \otimes \sigma_m^{B_1} \otimes |s\rangle\langle s|^{B_2} \right) \neq 0
\end{aligned}$$

The last commutator one has to consider is

$$\begin{aligned}
3. \quad & \left[\sigma_{eff}^{A \leq B}, \sigma_{eff}^{B \neq A} \right] = \sum_{r,i} e_{ri} \left[|r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1}, \sigma_{eff}^{B \neq A} \right] \\
& + \sum_{i,r,j} f_{irj} \left[\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B \neq A} \right],
\end{aligned}$$

with

$$\begin{aligned}
& \left[|r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1}, \sigma_{eff}^{B \neq A} \right] = \sum_j x_j \left[|r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1}, \sigma_j^{A_1} \right] \\
& + \sum_j y_j \left[|r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1}, \sigma_j^{B_1} \right] + \sum_{jk} g_{jk} \left[|r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1}, \sigma_j^{A_1} \otimes \sigma_k^{B_1} \right] = \\
& = \sum_j y_j \left[|r\rangle\langle r|^{A_2} \otimes \left[\sigma_i, \sigma_j \right]^{B_1} \right] + \sum_{jk} g_{jk} \left[\sigma_i, \sigma_j \right]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1}
\end{aligned}$$

and

$$\begin{aligned}
& \left[\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B \neq A} \right] = \sum_k x_k \left[\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}, \sigma_k^{A_1} \right] \\
& + \sum_k y_k \left[\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}, \sigma_k^{B_1} \right] + \sum_{kl} g_{kl} \left[\sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1}, \sigma_k^{A_1} \otimes \sigma_l^{B_1} \right] = \\
& = \sum_k x_k \left[\sigma_i, \sigma_k \right]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1} + \sum_k y_k \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \left[\sigma_j, \sigma_k \right]^{B_1} \\
& + \sum_{kl} g_{l,k} \left(\sigma_i \sigma_k^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j \sigma_l^{B_1} - \sigma_k \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_l \sigma_j^{B_1} \right).
\end{aligned}$$

Altogether this gives

$$\begin{aligned}
& \left[\sigma_{eff}^{B \neq A}, \sigma_{eff}^{B \neq A} \right] = \sum_{rijk} e_{ir} \left(y_j |r\rangle\langle r|^{A_2} \otimes \left[\sigma_i, \sigma_j \right]^{B_1} + g_{jk} \left[\sigma_i, \sigma_j \right]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_k^{B_1} \right) \\
& + \sum_{\substack{ij \\ klr}} f_{irj} \left(x_k \left[\sigma_i, \sigma_k \right]^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_j^{B_1} + y_k \sigma_i^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \left[\sigma_j, \sigma_k \right]^{B_1} \dots \right. \\
& \quad \dots + \frac{2}{d_{A_1}} g_{ki} |r\rangle\langle r|^{A_2} \otimes \left[\sigma_j, \sigma_l \right]^{B_1} + \frac{2}{d_{B_1}} g_{jl} \left[\sigma_i, \sigma_k \right]^{A_1} \otimes |r\rangle\langle r|^{A_2} \dots \\
& \quad \left. \dots + \sum_{mn} 2i g_{k,l} (\eta_{ikn} \beta_{jlm} + \gamma_{kin} \nu_{ljm}) \sigma_n^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_m^{B_1} \right) \neq 0 \quad .
\end{aligned}$$

Since none of $\sigma_{eff}^{B \neq A}$, $\sigma_{eff}^{A \neq B}$ and $\sigma_{eff}^{B \neq A}$ commute, neither will any κ_{eff}^1 and κ_{eff}^2 one can define. This means that (61) is not satisfied and one cannot construct $\bar{\kappa}_{eff}^1$ and $\bar{\kappa}_{eff}^2$ to arrive at an expression for W_{eff} as in (58).

Process matrices describing correlations between *two q-c laboratories* can, therefore, not be shown to be causally separable by means of the construction described in 3.4. In general they will not be and therefore it should be possible for two q-c laboratories, which communicate via *c-q channels*, to *violate the causal inequality*.

4.3 A Classical-Quantum and a Quantum-Classical Laboratory

The third possible combination of two semiquantum laboratories is one c-q (say A) and one q-c laboratory (B), see figure 7. This means that in A the input is always measured in the same basis, while in B the output state is prepared in the same basis every time. The products of the CJ- matrices of interest are all of the following form:

$$M_i^{A_1 A_2} \otimes M_j^{B_1 B_2} = \sum_{n,s} P_n |n\rangle\langle n|^{A_1} \otimes A_i^{A_2} \otimes B_j^{B_1} \otimes P_s |s\rangle\langle s|^{B_2} \quad (86)$$

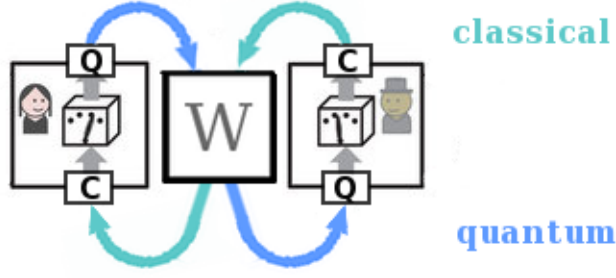


Figure 7: Correlations between a c-q (A) and a q-c laboratory (B). The input measurements in A are quasi classical, while the outputs are arbitrary quantum states. In B the input measurements are arbitrary, while all output states commute. Possible signaling channels are either truly quantum (from A to B) or quasi classical (from B to A). Picture taken from [11], altered.

This time, the effective process matrices W_{eff} are calculated in the basis $\{|n\rangle\langle n'|^{A_1} \otimes |i\rangle\langle i'|^{A_2} \otimes |j\rangle\langle j'|^{B_1} \otimes |s\rangle\langle s'|^{B_2}\}$, where $\{|i\rangle^{A_2}\}$ and $\{|j\rangle^{B_1}\}$ are, again, arbitrary ONB's of \mathcal{H}^{A_1} and \mathcal{H}^{B_2} .

$$\begin{aligned}
& \text{Tr} \left(\sum_{\substack{nn'n''ss's'' \\ ii'jj'kk'll'}} w_{n'n''kk'll's's''} P_n a_{ii'} b_{jj'} P_s \dots \right. \\
& \quad \left. \dots |n'\rangle \underbrace{\langle n''|n\rangle}_{=\delta_{n''n}} \langle n|^{A_1} \otimes |k\rangle \underbrace{\langle k'|i\rangle}_{=\delta_{k'i}} \langle i'|^{A_2} \otimes |l\rangle \underbrace{\langle l'|j\rangle}_{=\delta_{l'j}} \langle j'|^{B_1} \otimes |s'\rangle \underbrace{\langle s''|s\rangle}_{=\delta_{s''s}} \langle s|^{B_2} \right) = \\
& = \sum_{\substack{nn'kss' \\ ii'jj'l}} w_{n'n''kk'll's's''} P_n a_{ii'} b_{jj'} P_s \underbrace{\langle n|n'\rangle}_{=\delta_{nn'}} \underbrace{\langle i'|k\rangle}_{=\delta_{i'k}} \underbrace{\langle j'|l\rangle}_{=\delta_{j'l}} \underbrace{\langle s|s'\rangle}_{=\delta_{ss'}} \\
& = \sum_{\substack{ni' \\ sj'}} w_{nnii'jj'ss} P_n a_{i'i} b_{j'j} P_s
\end{aligned} \tag{87}$$

For $w_{nnii'jj'ss}^{eff} := w_{nnii'jj'ss}$ this gives effective process matrices

$$W_{eff} = \sum_{\substack{n,i,i' \\ s,j,j'}} w_{nnii'jj'ss}^{eff} |n\rangle\langle n|^{A_1} \otimes |i\rangle\langle i'|^{A_2} \otimes |j\rangle\langle j'|^{B_1} \otimes |s\rangle\langle s'|^{B_2}, \tag{88}$$

which in the basis $\{|n\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1} \otimes |s\rangle\langle s'|^{B_2}\}$ consist of the following σ_{eff} -terms:

$$\begin{aligned}
\sigma_{eff}^{B \leq A} &= \sum_{ns} c_{ns} |n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2} + \sum_{nis} d_{nis} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2} \\
\sigma_{eff}^{A \leq B} &= \sum_{ij} e_{ij} \sigma_i^{A_2} \otimes \sigma_j^{B_1} + \sum_{nij} f_{nij} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1} \\
\sigma_{eff}^{A \not\leq B} &= \sum_n x_n |n\rangle\langle n|^{A_1} + \sum_i y_i \sigma_i^{B_1} + \sum_{ni} g_{ni} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1}.
\end{aligned} \tag{89}$$

The commutators between these terms are

$$\begin{aligned}
1. \quad [\sigma_{eff}^{B \leq A}, \sigma_{eff}^{A \leq B}] &= \sum_{n,s} c_{ns} [|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \leq B}] \\
&\quad + \sum_{n,i,s} d_{nis} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \leq B}]
\end{aligned}$$

with

$$\begin{aligned}
[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \leq B}] &= \sum_{i,j} e_{ij} [|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_i^{A_2} \otimes \sigma_j^{B_1}] \\
+ \sum_{n',i,j} f_{n'ij} &[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}] = 0
\end{aligned}$$

but

$$\begin{aligned}
[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{A \leq B}] &= \sum_{j,k} e_{jk} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{A_2} \otimes \sigma_k^{B_1}] \\
+ \sum_{n',j,k} f_{n'jk} &[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_j^{A_2} \otimes \sigma_k^{B_1}] = \\
= \sum_{j,k} (e_{jk} + f_{n,j,k}) &|n\rangle\langle n|^{A_1} \otimes \sigma_j^{A_2} \otimes [\sigma_i, \sigma_k]^{B_1} \otimes |s\rangle\langle s|^{B_2}.
\end{aligned}$$

Therefore, in general, one gets

$$[\sigma_{eff}^{B \leq A}, \sigma_{eff}^{A \leq B}] = \sum_{\substack{n,i,s \\ j,k}} d_{nis} (e_{jk} + f_{n,j,k}) |n\rangle\langle n|^{A_1} \otimes \sigma_j^{A_2} \otimes [\sigma_i, \sigma_k]^{B_1} \otimes |s\rangle\langle s|^{B_2} \neq 0$$

for the first commutator. Moreover,

$$\begin{aligned}
2. \quad [\sigma_{eff}^{B \leq A}, \sigma_{eff}^{B \not\leq A}] &= \sum_{ns} c_{ns} [|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \not\leq A}] \\
&\quad + \sum_{nis} d_{nis} [|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B \not\leq A}].
\end{aligned}$$

Again

$$\begin{aligned}
& \left[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B_2 \not\prec A} \right] = \sum_{n'} x_{n'} \left[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \right] \\
& + \sum_i y_i \left[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_i^{B_1} \right] \\
& + \sum_{n'i} g_{n'i} \left[|n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_i^{B_1} \right] = 0,
\end{aligned}$$

but

$$\begin{aligned}
& \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_{eff}^{B_2 \not\prec A} \right] = \sum_{n'} x_{n'} \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \right] \\
& + \sum_j y_j \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, \sigma_j^{B_1} \right] \\
& + \sum_{n'j} g_{n'j} \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_j^{B_1} \right] = \\
& = \sum_j (y_j + g_{n,j}) |n\rangle\langle n|^{A_1} \otimes [\sigma_i, \sigma_j]^{B_1} \otimes |s\rangle\langle s|^{B_2},
\end{aligned}$$

which means that

$$\left[\sigma_{eff}^{B_2 \prec A}, \sigma_{eff}^{B_2 \not\prec A} \right] = \sum_{\substack{ni \\ sj}} d_{nis} (y_j + g_{n,j}) |n\rangle\langle n|^{A_1} \otimes [\sigma_i, \sigma_j]^{B_1} \otimes |s\rangle\langle s|^{B_2} \neq 0.$$

Finally, the third commutator is

$$\begin{aligned}
3. \left[\sigma_{eff}^{B_2 \prec A}, \sigma_{eff}^{B_2 \not\prec A} \right] &= \sum_{ij} e_{ij} \left[\sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B_2 \not\prec A} \right] \\
&+ \sum_{nij} f_{nij} \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B_2 \not\prec A} \right]
\end{aligned}$$

with

$$\begin{aligned}
& \left[\sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B_2 \not\prec A} \right] = \sum_n x_n \left[\sigma_i^{A_2} \otimes \sigma_j^{B_1}, |n\rangle\langle n|^{A_1} \right] + \sum_k y_k \left[\sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_k^{B_1} \right] \\
& + \sum_{nk} g_{nk} \left[\sigma_i^{A_2} \otimes \sigma_j^{B_1}, |n\rangle\langle n|^{A_1} \otimes \sigma_k^{B_1} \right] = \\
& = \sum_k y_k \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} + \sum_{nk} g_{nk} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1}
\end{aligned}$$

and

$$\begin{aligned}
& \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_{eff}^{B \not\leq A} \right] = \sum_{n'} x_{n'} \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}, |n'\rangle\langle n'|^{A_1} \right] \\
& + \sum_k y_k \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}, \sigma_k^{B_1} \right] \\
& + \sum_{n'/k} g_{n'/k} \left[|n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes \sigma_j^{B_1}, |n'\rangle\langle n'|^{A_1} \otimes \sigma_k^{B_1} \right] = \\
& = \sum_k (y_k + g_{nk}) |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1}.
\end{aligned}$$

So altogether this gives

$$\begin{aligned}
& \left[\sigma_{eff}^{A \leq B}, \sigma_{eff}^{B \not\leq A} \right] = \sum_{nijk} e_{ij} (y_k \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} + g_{nk} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1}) \\
& + \sum_{\substack{n,i \\ j,k}} f_{nij} (y_k + g_{n,k}) |n\rangle\langle n|^{A_1} \otimes \sigma_i^{A_2} \otimes [\sigma_j, \sigma_k]^{B_1} \neq 0,
\end{aligned} \tag{90}$$

which means that, like in 4.2, none of the σ_{eff} -terms, and therefore none of the possible κ_{eff}^1 and κ_{eff}^2 , commute and W_{eff} , again, will not be causally separable in general.

Note that quantum channels between a c-q (A) and a q-c laboratory (B) are either *truly quantum*, if they are from A to B , or *classical*, if they are from B to A . Using these two types of communication together, two laboratories should also be able to *violate the causal inequality*.

4.4 Other Semiquantum Correlations

Semiquantum correlations also arise between a classical laboratory and a truly quantum or even a semiquantum one, see figure 8. In the latter situations the correlations are either classical or semiquantum. All these cases will be discussed here.

The combination of a classical laboratory (say A) and a truly quantum one (B) can be realized by fixing input measurement and output reparation bases in A , while in B both are arbitrary. The respective combinations of CJ- matrices have the following structure

$$M_i^{A_1 A_2} \otimes M_j^{B_1 B_2} = \sum_{nr} P_{n,r} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes A_j^{B_1} \otimes B_j^{B_2} \tag{91}$$

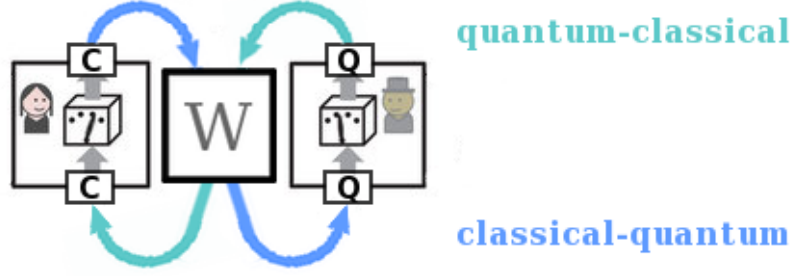


Figure 8: Correlations between a quasi classical laboratory (A) and a quantum one (B). In A the input measurements and the prepared outputs are quasi classical, while in B both are arbitrary. Possible signaling channels are either classical-quantum (from A to B) or quantum-classical (from B to A). Picture taken from [11], altered.

and lead to effective process matrices

$$W_{eff} = \sum_{\substack{ni \\ rj}} w_{nri'jj'} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes |i\rangle\langle i'|^{B_1} \otimes |j\rangle\langle j'|^{B_2}. \quad (92)$$

These W_{eff} consist of the following terms:

$$\begin{aligned} \sigma_{eff}^{B \prec A} &= \sum_{ni} c_{ni} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_2} + \sum_{nij} d_{nij} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes \sigma_j^{B_2} \\ \sigma_{eff}^{A \prec B} &= \sum_{ri} e_{ri} |r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1} + \sum_{nri} f_{nri} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1} \\ \sigma_{eff}^{A \not\prec B} &= \sum_n x_n |n\rangle\langle n|^{A_1} + \sum_i y_i \sigma_i^{B_1} + \sum_{ni} g_{ni} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \end{aligned} \quad (93)$$

Also these three terms do not commute with one another

$$[\sigma_{eff}^{B \prec A}, \sigma_{eff}^{A \prec B}] = \sum_{nrijk} d_{nij} (e_{rk} + f_{nrk}) |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_i, \sigma_k]^{B_1} \otimes \sigma_j^{B_2} \neq 0$$

$$[\sigma_{eff}^{B \prec A}, \sigma_{eff}^{B \not\prec A}] = \sum_{nij} d_{nij} (y_k + g_{nk}) |n\rangle\langle n|^{A_1} \otimes [\sigma_i, \sigma_k]^{B_1} \otimes \sigma_j^{B_2} \neq 0$$

$$\begin{aligned}
\left[\sigma_{eff}^{A \leq B}, \sigma_{eff}^{B \not\leq A} \right] &= \sum_{nri j} e_{ri} (y_j |r\rangle \langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1} \dots \\
&\quad \dots + g_{nj} |n\rangle \langle n|^{A_1} \otimes |r\rangle \langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1}) \\
&+ \sum_{nri j} f_{nri} (y_j + g_{nj}) |n\rangle \langle n|^{A_1} \otimes |r\rangle \langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1} \neq 0,
\end{aligned}$$

which again means that one cannot deduce that W_{eff} is causally separable in general.

Processes for two semiquantum laboratories are only compatible with a global causal structure if the two laboratories are classical-quantum, with q-c signaling between them. If the input measurement bases are fixed in both laboratories, every quantum system A and B receive appears classical. This means that any information they gain from the other laboratory is (seemingly) classical. In that case no violation of the causal inequality is possible.

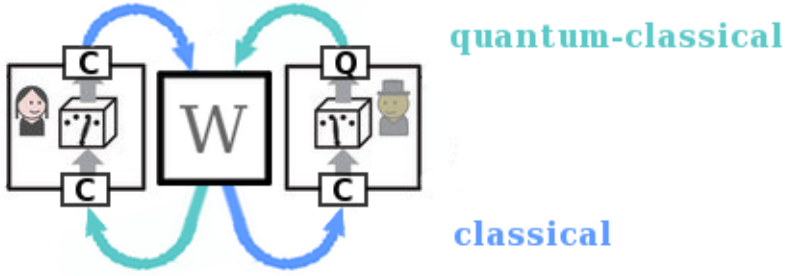


Figure 9: Correlations between a quasi classical (A) and a classical-quantum laboratory (B). While in A both input measurement and output preparation are quasi classical, in B only the input measurement basis is fixed. Possible signaling channels are either classical (from A to B) or quantum-classical (from B to A). Picture taken from [11], altered.

To see that for all the other cases a violation of the causal inequality is indeed possible, consider correlations between a classical laboratory (A) and a semiquantum one (B).

If B is c-q, see figure 9, the effective process matrices of interest are, of course, causally separable. This case lies between the purely classical case and that of two c-q laboratories. It means that in equation (79) $|a\rangle = |r\rangle$, where $\sigma_r^T |r\rangle = P_r |r\rangle$ and the construction of $\bar{\kappa}_{eff}^1$ and $\bar{\kappa}_{eff}^2$ can be carried out as described in 3.4. If,

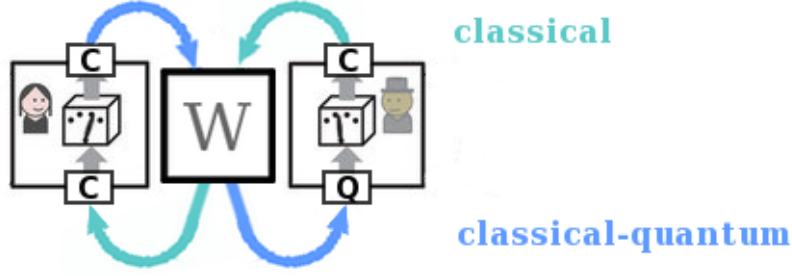


Figure 10: Correlations between a quasi classical (A) and a quantum-classical laboratory (B). While in A both input measurement and output preparation are quasi classical, in B only the prepared output states commute. Possible signaling channels are either classical-quantum (from A to B) or classical (from B to A). Picture taken from [11], altered.

however, B is q-c, see figure 10, the effective process matrices are

$$W_{eff} = \sum_{\substack{nrs \\ ii'}} w_{nrii's}^{eff} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes |i\rangle\langle i'|^{B_1} \otimes |s\rangle\langle s|^{B_2} \quad (94)$$

with $w_{nrii's}^{eff} = w_{nnrrrii'ss}$. They consist of σ_{eff} -terms.

$$\begin{aligned} \sigma_{eff}^{B \leq A} &= \sum_{ns} c_{ns} |n\rangle\langle n|^{A_1} \otimes |s\rangle\langle s|^{B_2} + \sum_{nis} d_{nis} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1} \otimes |s\rangle\langle s|^{B_2} \\ \sigma_{eff}^{A \leq B} &= \sum_{ri} e_{ri} |r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1} + \sum_{nri} f_{nri} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes \sigma_i^{B_1} \\ \sigma_{eff}^{A \not\leq B} &= \sum_n x_n |n\rangle\langle n|^{A_1} + \sum_i y_i \sigma_i^{B_1} + \sum_{ni} g_{ni} |n\rangle\langle n|^{A_1} \otimes \sigma_i^{B_1}. \end{aligned} \quad (95)$$

giving the following commutation relations.

$$[\sigma_{eff}^{B \leq A}, \sigma_{eff}^{A \leq B}] = \sum_{nirs} d_{nis} (e_{r,j} + f_{nrj}) |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1} \otimes |s\rangle\langle s|^{B_2} \neq 0$$

$$[\sigma_{eff}^{B \leq A}, \sigma_{eff}^{B \not\leq A}] = \sum_{nirs} d_{nis} (y_j + g_{nj}) |n\rangle\langle n|^{A_1} \otimes [\sigma_i, \sigma_j]^{B_1} \otimes |s\rangle\langle s|^{B_2} \neq 0$$

$$\begin{aligned} [\sigma_{eff}^{A \leq B}, \sigma_{eff}^{B \not\leq A}] &= \sum_{nirs} e_{ri} (y_j |r\rangle\langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1} + g_{nj} |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1}) \\ &\quad + f_{nri} (y_j + g_{nj}) |n\rangle\langle n|^{A_1} \otimes |r\rangle\langle r|^{A_2} \otimes [\sigma_i, \sigma_j]^{B_1} \neq 0 \end{aligned}$$

Even if one laboratory is (quasi) classical and the other only quantum-classical, which means that signaling correlations are either classical or c-q, W_{eff} cannot be shown to be causally separable. Moreover, the example $W_{\star}^{A_1 A_2 B_1 B_2}$ in (41) of a process, that actually *violates the causal inequality*, is exactly of the form (94) when considered in the basis $\{|z\rangle\langle z'|^{A_1} \otimes |z\rangle\langle z'|^{A_2} \otimes \sigma_i^{B_1} \otimes |z\rangle\langle z'|^{B_2}\}$, with $\sigma_z|\pm z\rangle = \pm|z\rangle$. The (quasi) classical operations in A can be regarded as a special case of both a c-q and a q-c laboratory. Hence $W_{\star}^{A_1 A_2 B_1 B_2}$ also shows that correlations between *two q-c laboratories*, see 4.2, and between *a c-q and a q-c laboratory*, 4.3, are incompatible with the concept of a global causal structure.

5 Conclusion

The most general quantum correlations between two closed laboratories are not compatible with the concept of a global causal structure in which they are embedded. If, however, the laboratories, and therefore the correlations between them, are classical, communication between them always respects the constraints of a background causal structure. A closed quantum laboratory corresponds to a set of CP- maps from the input Hilbert space to the output Hilbert space. A (quasi) classical laboratory can then be regarded as a special case, where the input states are measured with a fixed, complete set of orthogonal projectors and all possible output states commute with one another. If only one of these two properties is fulfilled, the laboratory is *semiquantum*. Semiquantum correlations, therefore, lie in between classical and quantum ones. In a c-q laboratory the input measurements can be described classically while the output states are arbitrary, when in a q-c laboratory the input measurements are arbitrary while all possible outputs commute. Correlations between two closed laboratories can always be understood in terms of a *global causal structure*, only if the information gained from the *inputs* can be described *classically*. This is the case if the laboratories are quasi classical or c-q. Note that this provides an outside view of the laboratories, which is useful when considering correlations between them. The inside of the laboratories is, per definition, described by quantum mechanics. Therefore experimenters can still observe quantum behavior, even in a quasi classical laboratory. They can, for example, use the quantum states after the input measurement, subject it to some non trivial time evolution and observe a violation of a Leggett inequality, [31].

The fact that two *c-q laboratories* are unable to violate the causal inequality seems to agree with the original conjecture that compatibility with a global causal structure arises with classicality. Restricting A and B to c-q operations limits the information they can receive from the other laboratory to classical information. It

is therefore not surprising that their success in the task of guessing events in the other laboratory does not exceed the classical limit. The example $W_{\star}^{A_1 A_2 B_1 B_2}$ in (41), however, raises serious doubts about the quantum character of correlations being responsible for possible incompatibility with a global causal structure. There are no truly quantum signaling correlations between a classical and a q-c laboratory. Actually, in the case of three or more parties purely classical laboratories can be correlated in ways that are not compatible with a global causal structure, see [32]. Indefinite causal structures, like the ones described by $W_{\star}^{A_1 A_2 B_1 B_2}$, do not depend on quantum correlations to be observed. Hence definite causal structures like spacetime are not an effect of a quantum-to-classical transition as first proposed. Still, the quantum theoretical description allows for these exotic causal structures. The question of how they might be observed motivates to further investigate the respective formalisms.

Generally, the quantum informational approach to causality might give rise to new approaches to a unification of quantum physics and the theory of relativity. Hopefully the formulation in terms of correlations and processes will further contribute to that. Moreover causality considerations could lead to further insights (and new formulations) of quantum physics itself, see for example [33]. It is currently one of the most promising (and, in my opinion, interesting) fields of theoretical research.

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Abstract

The existence of a definite global spacetime is usually regarded as a fundamental feature of nature. Also in relativistic quantum mechanics or quantum gravity this is always the case. Considering causal structures more generally and assuming local validity of standard quantum mechanics, however, allows for indefinite global causal structures. While locally everything can be understood as happening in spacetime, globally things like superpositions of different causal structures might exist, making such a notion impossible.

A causal structure can be defined in terms of signaling. If two parties A and B are embedded in a definite global causal structure, signaling between them is at most unidirectional. This means that either A can signal to B or B can signal to A or they cannot signal to one another. This fact can be expressed via a certain communication task between two closed laboratories, which are described by standard quantum mechanics. There is an upper bound to the probability of successfully performing this task, if one assumes a definite global causal structure. A quantum theoretical description, however, allows for higher success probabilities than this upper bound. The laboratories correspond to the possible quantum operations that can be performed in them. Correlations between them are described by so called processes from which the probabilities of interest can be calculated. When restricting the operations in the laboratories to (quasi) classical ones, the correlations between them can again be understood in terms of a definite global causal structure. One might, therefore, suggest that a definite global causal structure is an effect of the quantum-to-classical-transition, since in the quantum case indefinite global causal structures are possible.

To better understand this, semiquantum laboratories and correlations are considered in this thesis. In general, semiquantum objects consist of two subparts (e.g. input and output system), one of which can be understood classically. They are an in-between case of classical and quantum objects. It turns out that compatibility with the concept of a definite global causal structure is always given when the information gained from the input is (quasi) classical. Otherwise indefinite global causal structures are always possible. This is the case even for one classical laboratory and another one, where the output can be described classically. There are no truly quantum signaling correlations between two such laboratories. Yet they can perform the communication task mentioned above with a success that cannot be reached in a definite global causal structure, like spacetime. Apparently it is not the quantum character of the correlations that allows for an observation of indefinite causal structures. It has been found recently that between more than two parties, such causal structures are (in principle) observable, even if all these parties are classical. It is therefore rather unlikely, that the existence of a definite global causal structure should be an effect of a quantum-to-classical transition.

Zusammenfassung

Die Existenz einer eindeutigen globalen Raumzeit wird allgemein als fundamentale Natureigenschaft betrachtet, so auch im Bereich der relativistischen Quantenphysik und Quantengravitation. Betrachtet man jedoch kausale Strukturen allgemeiner und setzt nur lokale Gültigkeit der Quantenphysik voraus, kann man Situationen beschreiben, in denen die globale kausale Struktur nicht eindeutig ist. Obwohl Vorgänge lokal als Ereignisse in einer Raumzeit verstanden werden können, sind global unbestimmte kausale Strukturen, wie Superpositionen, denkbar.

Eine kausale Struktur kann man mittels der Möglichkeit Signale zu senden definieren. Sind zwei Parteien A und B Teil einer globalen kausalen Struktur kann ein Signal in maximal eine Richtung gesendet werden. Das bedeutet, dass entweder A ein Signal zu B senden kann oder umgekehrt oder die beiden kein Signal an den jeweils anderen schicken können. Dieser Umstand kann in Form einer gemeinsamen Aufgabe, zu deren Lösung Agenten in zwei Labors A und B kommunizieren müssen, ausgedrückt werden. Die Wahrscheinlichkeit, die entsprechende Aufgabe erfolgreich zu erfüllen, ist nach oben hin beschränkt, wenn A und B in einer eindeutigen kausalen Struktur eingebettet sind. In einer quantenphysikalischen Beschreibung ergeben sich jedoch teilweise höhere Erfolgswahrscheinlichkeiten, als die erwähnte Schranke erlauben würde. Die Labors entsprechen möglichen Quantenoperationen, die Korrelationen zwischen den Labors werden durch sogenannte Prozesse, mit deren Hilfe man die Wahrscheinlichkeiten von Interesse berechnen kann, beschrieben. Limitiert man die Operationen in den Labors auf (quasi)klassische, ergibt sich allerdings immer die Möglichkeit der Beschreibung durch eine eindeutige globale kausale Struktur. Man könnte annehmen, dass Letztere eben keine fundamentale Eigenschaft ist, sondern vielmehr ein Effekt des Übergangs von quantenphysikalisch zu klassisch, da im Fall von Quantenkorrelationen auch unbestimmte kausale Strukturen möglich sind.

Um dies besser zu verstehen, werden in dieser Arbeit Semiquanten-Labors und -Korrelationen betrachtet. Ein Semiquanten-System besteht aus zwei Teilen, von denen einer klassisch, der andere quantenmechanisch beschrieben werden kann (z.B. Eingangs- und Ausgangszustand eines Quantenkanals). Eine Beschreibung durch eine eindeutige globale kausale Struktur ist immer dann möglich, wenn die Information, die in den Labors aus den Inputs gewonnen wird, (quasi) klassisch ist. Anderenfalls können unbestimmte globale kausale Strukturen auftreten, sogar im Fall eines quasi klassischen Labors und einem, in dem der Output klassisch beschrieben werden kann. Obwohl zwischen zwei solchen Labors keine echte Quantenkommunikation stattfindet, können sie im Fall einer unbestimmten globalen kausalen Struktur die oben erwähnte Aufgabe erfolgreicher lösen, als es der Fall wäre, wenn die Labors Teil einer eindeutigen Raumzeit wären. Es ist daher anscheinend nicht der Quantencharakter der Korrelationen zwischen den Labors, der

eine Beobachtung einer unbestimmten kausalen Struktur ermöglicht. Im Fall von mehr als zwei Labors ist es (prinzipiell) sogar möglich solche kausalen Strukturen zu beobachten, wenn alle beteiligten Labors klassisch sind. Das Auftreten unbestimmter globaler kausaler Strukturen ist daher wohl kein Effekt des Übergangs von quantenphysikalisch zu klassisch.

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