DIPLOMARBEIT

Titel der Diplomarbeit

ENDS AND SPECTRA OF GRAPHS

verfasst von

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Dedicated to Melanie and Noah.

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Abstract

This thesis is separated into two major independent parts. In the first part (section 1) we will discuss some useful connections between two branches of mathematics: graph theory and linear algebra. In doing so, we will turn a special attention to spectral graph theory. There will be one (major) goal, which will be completely proven: *The Matrix-tree-theorem*. Moreover we will give several different formulations of this theorem. In the second part (section 2 to section 5) we will discuss connections between three different branches of mathematics: graph theory, group theory and topology. In doing so, we will present a compactification $\hat{X}$ for a locally finite graph X and prove a fruitful generalization of the well known *Freudenthal-Hopf-Theorem*. This motivates a new class of graphs, so-called *abelsian graphs*, which are defined by a group action where all orbits accumulate to each element of $\hat{X}\backslash VX$ generalizing almost transitive and hence transitive group actions on graphs. Moreover two conjectures about such graphs are stated. All statements are proven in full detail and elementary notions needed from linear algebra, topology and group theory are summarized in the appendix at the end of this thesis (Appendix A, B and C). One cornerstone of this thesis is its readability for mathematicians of any level.

Kurzbeschreibung

Die vorliegende Arbeit ist unterteilt in zwei voneinander unabhängige Hauptteile. Im ersten Teil (section 1) werden wir nützliche Verbindungen zwischen zwei Zweigen der Mathematik diskutieren: Graphentheorie und Lineare Algebra. Dabei werden wir im Speziellen die spektrale Graphentheorie besprechen. Es wird ein Hauptziel geben, welches vollständig bewiesen wird: *Das Matrix-Baum-Theorem*. Weiters werden wir mehrere verschiedene Formulierungen dieses Theorems geben. Im zweiten Teil (section 2 bis section 5) werden wir Verbindungen von drei verschiedenen Zweigen der Mathematik untersuchen: Graphentheorie, Gruppentheorie und Topologie. Dabei werden wir eine Kompaktifizierung $\hat{X}$ eines lokal endlichen Graphen X präsentieren und eine fruchtbare Verallgemeinerung des bekannten *Freudenthal-Hopf-Theorems* beweisen. Dies wird eine neue Klasse von Graphen, sogenannte *abelssche Graphen*, motivieren, welche durch eine Gruppenaktion definiert sind, in der sich alle Orbits gegen jedes Element von $\hat{X}\backslash VX$ häufen. Dies verallgemeinert fast transitive und somit transitive Gruppenaktionen auf Graphen. Weiters werden wir zwei Vermutungen über solche Graphen formulieren. Alle Aussagen werden in allen Einzelheiten bewiesen und elementare Begriffe der Linearen Algebra, Topologie und der Gruppentheorie sind zusammenfassend im Appendix am Ende der Arbeit zu finden (Appendix A, B und C). Ein Eckpfeiler dieser Arbeit ist ihre Lesbarkeit für Mathematiker eines jeden Grades.

Declaration

I do declare that I unassistedly wrote this thesis without additional help other than supervision and that I did not use other than quoted sources and knowledge from lectures.

Vienna, May 4, 2014

Manuel Raphael Urbina Moreano

Erklärung

Ich erkläre, dass ich diese Arbeit selbständig und abgesehen von der vorgesehen Betreuung ohne fremde Hilfe verfasst habe und dass ich keine außer den zitierten Quellen und Wissen aus Lehrveranstaltungen verwendet habe.

Vienna, May 4, 2014

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At first place I want to thank my supervisor *Bernhard Krön* for introducing me to the beautiful field of graph theory and its interactions with other branches of mathematics such as group theory and topology. It was always a joy to read and work upon this field.

Moreover I want to thank *Bernhard Krön* for his supervision per se.

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Manuel Raphael Urbina Moreano

1. The spectrum of a graph

We will now discuss some basic properties of the spectrum of matrices, which are associated to graphs. The aim is to translate properties of graphs into (linear) algebraic properties. This section conforms basically to [2] and to some extent to [11].

There are many ways to define graphs and well known notions such as *adjacency, neighbours, degree* and so on. In the first part of this thesis we will follow the definitions given in [2], since this is our main source for the spectral theory of graphs. Therefore a *general graph* X consists of three things: a set $\mathrm{V}(X)$, a set $\mathrm{E}(X)$ and an incidence relation, that is, a subset of $\mathrm{V}(X) \times \mathrm{E}(X)$. An element of $\mathrm{V}(X)$ is called a *vertex*, an element $\mathrm{E}(X)$ is called an *edge*, and the incidence relation is required to be such that an edge is incident with either one vertex (in which case it is a *loop*) or two vertices. If every edge is incident with two vertices (i.e. no loops), and no two edges are incident with the same pair of vertices (no multiedges), then X is said to be a *simple graph* or briefly, a *graph*. In this case, $\mathrm{E}(X)$ can be regarded as a subset of the set of unordered pairs of distinct vertices. If $e = \{v, w\} \in \mathrm{E}(X)$, then we say that v and w are *adjacent* or that w is a *neighbour* of v, and denote this by writing $v \sim w$. We also say that e *joins* v and w, and that v and w are the *ends* of e. The set of neighbours of a vertex x is denoted by $N(x) := \{y \in V(X) | \{x, y\} \in E(X)\}$, and $\deg(x) := |N(x)|$ is called the *degree* of x. A graph is said to be *regular* of degree k (or briefly *k-regular*) if each of its vertices has degree k. A vertex is *incident* with an edge if it is one of the two vertices of the edge.

It is often convenient, interesting, or attractive to represent a graph by a picture, with points for the vertices and lines for the edges, as in Figure 2.1. Strictly speaking, these pictures do not define graphs, since the vertex set is not specified. However, we may assign distinct integers arbitrarily to the points, and the edges can then be chosen to be ordered pairs. We emphasize that in a picture of a graph, the positions of the points and lines do not matter – the only information it conveys is which pairs of vertices are joined by an edge.

For the first part of this thesis about the spectrum of graphs we will only deal with *finite* graphs, i.e. finitely many vertices and hence finitely many edges (since we don't allow multiple edges). Therefore we will not explicitly state this assumption in the remainder of this section.

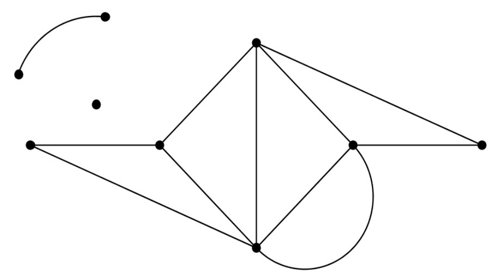

FIGURE 1. A simple graph
with 9 vertices and 11 edges.

A *subgraph* of a graph X is a graph Y such that

$$V(Y) \subseteq V(X) \quad \text{and} \quad E(Y) \subseteq E(X).$$

If $V(Y) = V(X)$, we call Y a *spanning subgraph* of X. Any spanning subgraph of X can be obtained by deleting some of the edges from X.
A subgraph Y of X is called an *induced subgraph* if two vertices of $V(Y)$ are adjacent in Y if and only if they are adjacent in X. Any induced subgraph of X can be obtained by deleting some of the vertices from X, along with any edges that contain a deleted vertex. Thus an induced subgraph is determined by its vertex set, say S, it will be denoted as $\langle S \rangle$. Of course, the number of induced subgraphs of X is equal to the number of subsets of $V(X)$. As an (ordinary) subgraph is determined by its edge set, say K, it will be denoted as $\langle K \rangle$.

1.1. **The adjacency matrix.** Suppose that X is a graph whose vertex-set V(X) is the set $\{v_1, v_2, \ldots, v_n\}$ and therefore E(X) as a set of two-element sets.

1.1.1. *Definition.* The *adjacency matrix* of X is the $n \times n$ - matrix $A = A(X)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & v_i \text{ and } v_j \text{ are } \textit{adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

It follows directly from the definition that A is a real symmetric matrix, and that $\mathrm{tr}(A) = 0$ (no loops).

1.1.2. *Lemma.* Let X be a graph. The following holds:

$$\sum_{v \in V(X)} \deg(v) = 2|E(X)|.$$

Proof. This can easily be shown by double counting. In particular you can extend this idea in terms of the adjacency matrix A. Considering the summation of the i-th column, one will observe that this is $\deg(v_i)$ and summing up over all columns gives us the left side of the equation. On the other hand, considering all entries a_{ij} with $i < j$ (in other words the 'underpart' of A), one easily sees, that summation over these entries gives us $|E|$ and by the property of symmetry we get the right side of the equation. □

Since the entries of A correspond to an arbitrary labelling of the vertices of X, we should be interested primarily in those properties of the adjacency matrix which are invariant under permutations of the rows and columns.

1.1.3. *Lemma.* Let X be a graph with vertex set $V(X) = \{v_1, v_2, \ldots, v_n\}$. Then, the adjacency matrix $A(Y)$ of a graph Y, obtained by permutating the vertex set of X, with respect to the edges, i.e. $v_i \sim v_j \Leftrightarrow v_{\sigma(i)} \sim v_{\sigma(j)}$, has got the same characteristic polynomial as $A(X)$.

Proof. Consider such a permutation $\sigma \in \mathrm{Sym}(V(X))$ and assume $\sigma(v_i) = v_j$. By Definition 1.1.1 the i-th column (row) of $A(X)$ is the j-th column (row) of $A(Y)$ and hence we get a permutation matrix P, such that $A(Y) = PA(X)P^T$. Since permutation matrices are orthogonal, $P^T = P^{-1}$, $A(X)$ and $A(Y)$ are similar and therefore A.2.5 gives the result. □

1.1.4. *Remark.* Such graphs X and Y as described above are called *isomorphic* and there is a lot of interesting material and open problems about isomorphic graphs, in particular deciding when two given finite graphs are isomorphic. Since this particular branch of graph theory will not be a subject of this thesis, we refer the interested reader to [2] Part 3 and [11] especially Chapter 1 to 4.

Considering the above lemma we are getting to the *spectral* properties of the adjacency matrix A. Recalling A, we know that an eigenvalue λ of A is real and furthermore the multiplicity of λ, as a root of the equation $\det(\lambda I - A) = 0$ (the characteristic polynomial), is equal to the dimension of the eigenspace of λ ([6] p.30, Satz 7.11).

1.1.5. *Definition.* The *spectrum* of a graph X is the set of eigenvalues of $A(X)$, together with their multiplicites. If the distinct eigenvalues of A are $\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1}$ and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\mathrm{Spec}(X) = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

1.1.6. *Remark.* We usually refer to the eigenvalues of $A = A(X)$ as the *eigenvalues* of X. Also, the characteristic polynomial $\det(\lambda I - A)$ will be referred to as the *characteristic polynomial* of X, and denoted by $\chi(X; \lambda)$. For

$$\chi(X; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_n$$

the coefficient $-c_1$ is the sum of the zeros, that is, the sum of the eigenvalues. This is also $\mathrm{tr}(A)$ which, as we have already noted, is zero (see Definition 1.1.1), and thus $c_1 = 0$. More generally, considering A.3.2, we know, that each coefficient can be expressed in terms of the principle minors of A. This yields to the following result:

1.1.7. *Proposition.* (see Proposition 2.3. in [2]) The coefficients of the characteristic polynomial of a graph X satisfy:

(1) $c_1 = 0$,
(2) $-c_2$ is the number of edges in X,
(3) $-c_3$ is twice the number of triangles (2-regular graphs with 3 vertices) in X.

Proof. From A.3.2 we know that for each $i \in \{1, 2, \ldots, n\}$, the number $(-1)^i c_i$ is the sum of the principle minors of A with i rows and columns. So we can argue as follows.

(1) Has been shown already (above remark)
(2) A principle minor of $A(X)$ with two rows and columns which has a non-zero entry must be of the form

$$\left| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right| = -1.$$

There is one such minor for each pair of adjacent vertices of X, and each has value -1. Hence $(-1)^2 c_2 = -|E(X)|$.

(3) There are essentially three possibilities (up to permutation) for non-zero principal minors with three rows and columns:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

The first and the second minors have determinant zero and the third has determinant 2. Considering this submatrix in terms of the underlying graph, this minor corresponds to three mutually adjacent vertices (i.e. a triangle) in X, and so we have the required description of c_3.

□

Those results indicate that the characteristic polynomial, as an algebraic construction, contains graphical information. The above proposition is just a pointer, and we shall soon obtain a more comprehensive result on the coefficients of the characteristic polynomial.
But first, let me introduce one more algebraic construction: Suppose A is the adjacency matrix of a graph X. Then the set of polynomials in A, with complex coefficients, forms an *algebra* under the usual matrix operations. This algebra has finite dimension as a complex vector space. Indeed, the Cayley-Hamilton theorem ($\chi(X; A) = 0$, see A.2.6) says that A satisfies its own characteristic equation, so the dimension is at most n, the number of vertices in X.

1.1.8. *Definition.* The *adjacency algebra* of a graph X is the algebra of polynomials in the adjacency matrix $A = A(X)$. We shall denote the adjacency algebra of X by $\mathbb{A}(X)$.

Since every element of $\mathbb{A}(X)$ is a linear combination of powers of A, we can obtain results about $\mathbb{A}(X)$ by studying these powers. We define a *walk* of length l in X, from v_i to v_j, to be a finite sequence of vertices of X,

$$v_i = u_0, u_1, \ldots, u_l = v_j$$

such that u_{t-1} and u_t are adjacent for $1 \leq t \leq l$. A walk is called *closed*, if $v_i = v_j$. Furthermore, a walk is called *path*, if $i \neq j$ implies $u_i \neq u_j$.

1.1.9. *Lemma.* (see Lemma 2.5 in [2] or Lemma 8.1.2 in [11]) The number of walks of length l in X, from v_i to v_j, is the ij-th entry of the matrix A^l.

Proof. The resultat is trivial for $l = 0$ and $l = 1$. Consider the ij-th entry of the matrix $A^2 = AA$. By the definition of the matrix product we have:

$$a_{ij}^{(2)} = \sum_k a_{ik}a_{kj}$$

One observes, that $a_{ik}a_{kj} \neq 0$ (hence $= 1$), if and only if there is a walk from v_i to v_k and v_k to v_j, which of course is a walk of length two. All in all the ij-th entry is the number of walks of length two from v_i to v_j. For the general induction step consider $A^{m+1} = A^m A$ and an analogous argument. □

1.1.10. *Corollary.* (see Corollary 8.1.3 in [11]) Let X be a graph with e edges and t triangles. If A is the adjacency matrix of X, then

(1) $tr(A) = 0$
(2) $tr(A^2) = 2e$
(3) $tr(A^3) = 6t$

Proof. Follows directly from the above lemma and double counting. □

1.1.11. *Definition.* A graph X is said to be *connected* if each pair of vertices is joined by a walk. The number of edges traversed in the shortest walk joining v_i and v_j is called the *distance* in X between v_i and v_j and is denoted by $d_X(v_i, v_j)$. This makes our connected graph X a ('discrete') metric space (VX, d_X).
The maximum value of this distance function in a connected graph X is called the *diameter* of X.
Alternatively, X is not connected, say *disconnected*, if we can partition its vertices into two non-empty sets, R and S, such that no vertex in R is adjacent to a vertex in S. In this case we say that X is the *disjoint union* of the two subgraphs induced by R and S. An induced subgraph of X that is maximal, subject to being connected, is called a *connected component*, or briefly a *component* of X.
Going back to the adjacency algebra $\mathbb{A}(X)$, we get the following

1.1.12. *Proposition.* (see Proposition 2.6 in [2]) Let X be a connected graph with adjacency algebra $\mathbb{A}(X)$ and diameter d. Then the dimension of $\mathbb{A}(X)$ is at least $d + 1$.

Proof. Let v and w be vertices of X such that $d_X(v, w) = d$, and suppose that

$$v = w_0, w_1, \ldots, w_d = w$$

is a walk of length d. Then, for each $i \in \{1, 2, \ldots, d\}$, there is at least one walk of lenght i, but no shorter walk, joining w_0 to w_i. Consequently, A^i has a non-zero entry in a position where the corresponding entries of $I, A, A^2, \ldots, A^{i-1}$ are zero. It follows that A^i is not linearly dependent on $\{I, A, \ldots, A^{i-1}\}$, and that $\{I, A, \ldots, A^d\}$ is a linearly independent set in $\mathbb{A}(X)$. Since this set has $d + 1$ members, the proposition is proved. □

1.1.13. *Remark.* There is a close connection between the adjacency algebra and the spectrum of X. If $A(X)$ has s distinct eigenvalues then, since it is a real symmetric matrix, its minimum polynomial (the monic polynomial of least degree which annihilates it) has degree s. Consequently the dimension of $\mathbb{A}(X)$ is equal to s. Thus we have the following bound for the number of distinct eigenvalues.

1.1.14. *Corollary.* (see Corollary 2.7 in [2]) A connected graph X with diameter d has at least $d+1$ distinct eigenvalues.

As it has been announced, I will present a more comprehensive result on the coefficients of the characteristic polynomial. But first, we need the definition of a special class of graphs.

1.1.15. *Definition.* An *elementary graph* is a simple graph, each component of which is regular of degree 1 or 2. In other words, each component is a single edge or a so-called *cycle*.

Let X be a graph with n vertices, m edges and c components. The *rank* of X and the *co-rank* of X are,

$$r(X) = n - c \quad \text{and} \quad s(X) = m - n + c \quad \text{respectively.}$$

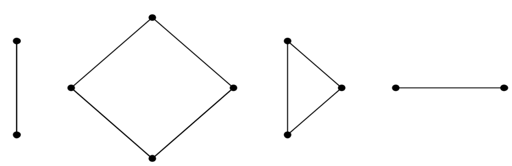

Figure 2. An elementary graph

1.1.16. *Remark.* The above figure shows an elementary graph X with 11 vertices 9 edges and 4 components. Hence $\mathrm{r}(X) = 7$ and $\mathrm{s}(X) = 2$. One may assume, that $\mathrm{s}(X)$ equals the number of cycles in the graph. Fortunately this statement is true for this special class of graphs and it will follow after Lemma 1.2.19. Nonetheless this can be proven directly.

Consider an elementary graph X with n vertices m edges and c components. Furthermore let c_1 be the number of 'single-edge'-components, c_2 be the number of 'cycle'-components and therefore $c_1 + c_2 = c$. It's clear that c_2 is also the number of cycles in X. Since the number of edges equals the number of vertices in each cycle (cycle-component) and each single-edge-component has two vertices and one edge it follows that

$$s(X) = m - n + c = c_1 - 2c_1 + \underbrace{c_1 + c_2}_{=c} = c_2.$$

1.1.17. *Theorem (Harary 1962).* (see Proposition 7.2 in [2]) Let A be the adjacency matrix of a graph X. Then

$$\det(A) = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation is over all spanning elementary subgraphs Λ of X.

Proof. Let $V(X) = [n]$ and consider a term $\operatorname{sgn}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$ of the formula for $\det(A)$. This term vanishes if, for some $i \in [n]$, $a_{i,\pi(i)} = 0$, that is, if $\{v_i, v_{\pi(i)}\}$ is not an edge of X. In particular, the term vanishes if π fixes any i. Thus, if the term corresponding to a permutation π is non-zero, then π can be expressed uniquely as the composition of disjoint cycles (in terms of permutations) of lenth at least two. Each cycle (ij) of length two corresponds to the factors $a_{ij}a_{ji}$, and signifies a single edge $\{i, j\}$ in $E(X)$. Each cycle $(i_1, i_2, \ldots, i_k)$ of length greater than two corresponds to the factors $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ and correspond to a cycle (in terms of graphs) $\{i_1, i_2, \ldots, i_k\}$ in X. Consequently, each non-vanishing term in the determinant expansion gives rise to an elementary subgraph Λ of X, with $V(\Lambda) = V(X)$.
The sign of a permutation π is $(-1)^{N_e}$, where N_e is the number of even cycles in π. If there are c_l cycles in π of length l, the equation $\sum lc_l = n$ implies that

$$n \equiv \sum_{l \text{ odd}} lc_l \pmod 2.$$

So that the congruence remains true for each substracted 'odd cycle' we need to add 1. In particular

$$n \underbrace{-3 \ldots -3}_{\#=c_3} \underbrace{+1 \ldots +1}_{\#=c_3} - \ldots \equiv \sum_{l \text{ odd}} lc_l - 3 \ldots - 3 + 1 \ldots + 1 - \ldots \pmod 2,$$

which directly brings us $n \equiv N_o \pmod 2$, where N_o denotes the number of odd cycles. Together with the fact that $c = N_o + N_e$, if c is the number of components of Λ, we get

$$r(\Lambda) = n - (N_o + N_e) \equiv N_e \pmod 2$$

and so the sign of π is equal to $(-1)^{r(\Lambda)}$.
Each elementary subgraph Λ with n vertices gives rise to several permutations π for which the corresponding term in the determinant expansion does not vanish. The number of such π arising from a given Λ is $2^{s(\Lambda)}$, since for each cycle-component in Λ (consider the above remark) there are two ways of choosing the corresponding cycle in π. Thus each Λ contributes $(-1)^{r(\Lambda)} 2^{s(\Lambda)}$ to the determinant and the assertion is proven. □

Example. Consider the *complete graph* K_n, that is, the graph with n vertices and all possible edges, which are $\binom{n}{2}$ (a triangle can also be written as K_3). Consider especially K_4: The adjacency matrix and hence the determinant of K_4 is given by

$$\det(A(K_4)) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -3.$$

There are just two 'kinds' of spanning elementary subgraphs: pairs of disjoint edges (for which $r = 2$ and $s = 0$) and 4-cycles (for which $r = 3$ and $s = 1$). There are three subgraphs of each kind so, according to the above theorem, we have

$$\det(A(K_4)) = 3(-1)^2 2^0 + 3(-1)^3 2^1 = -3.$$

Moreover the characteristic polynomial of K_4 is given by

$$p_{K_4} = (x+1)^3(x-3)$$

and hence

$$\mathrm{Spec}(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

With the help of A.2.7 and A.2.9 we generally obtain that

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

and hence

$$p_{K_n} = (x+1)^{(n-1)}(x-(n-1)).$$

Above we obtained a description of the first coefficients of the characteristic polynomial of a graph X. We shall now extend that result to all the coefficients. As before, let

$$\chi(X;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n.$$

1.1.18. *Proposition.* (see Proposition 7.3 in [2]) The coefficients of the characteristic polynomial of X are given by

$$(-1)^i c_i = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation is over all elementary subgraphs Λ of X with i vertices.

Proof. Again, by A.3.2, the number $(-1)^i c_i$ is the sum of all principal minors of $A(X)$ with i rows and columns. Each such minor is the determinant of the adjacency matrix of an induced subgraph of X with i vertices. Any elementary subgraph with i vertices is contained in precisely one of these induced subgraphs, and so, by applying the above theorem to each minor we obtain the required result. $\square$

The following results won't be proven, but mentioned however, because they are (as I think) very interesting. They can be found in [2], Part 1, Chapter 2, p. 11.

1.1.19. *A reduction formula for χ.* Suppose X is a graph with a vertex v_1 of degree 1, and let v_2 be the vertex adjacent to v_1. Let X_1 be the induced subgraph obtained by removing v_1, and X_{12} the induced subgraph obtained by removing $\{v_1, v_2\}$. Then

$$\chi(X;\lambda) = \lambda\chi(X_1;\lambda) - \chi(X_{12};\lambda).$$

1.1.20. *Corollary.* Let P_n be the *path graph* with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edges $\{v_i, v_{i+1}\}$ for $1 \le i \le n-1$. For $n \ge 3$ we have

$$\chi(P_n;\lambda) = \lambda\chi(P_{n-1};\lambda) - \chi(P_{n-2};\lambda).$$

Hence $\chi(P_n;\lambda) = U_n(\lambda/2)$, where U_n denotes the *Chebyshev polynomial of the second kind.*

1.1.21. *The derivative of* χ. Let X be a graph and for $i = 1, 2, \ldots, n$ let X_i denote the induced subgraph, obtained by removing v_i. Then

$$\chi'(X; \lambda) = \sum_{i=1}^{n} \chi(X_i; \lambda).$$

The reader, who is familiar with the theory of *species* related to *combinatorics*, should consider the *derivative of species*.

1.2. **The Laplacian matrix and the Matrix tree theorem.** In the following we will focus on a more practical matrix: the *Laplacian matrix*. To do so, we need the following definitions of other two 'graph' matrices.

Remember the definition of a general graph at the beginning of this section, especially the incidence relation and the notion *a vertex is incident with an edge*. For the sake of completeness of this thesis and, as I think, comprehensibility, we will introduce *directed graphs* (this gives rise to a second way to define graphs).

1.2.1. *Definition.* A *directed graph* or *digraph* X consists of a vertex set $V(X)$ and an *arc set* $A(X)$, where an arc, or *directed edge*, is an ordered pair of distinct vertices, and therefore $A(X) \subseteq V(X) \times V(X)$. In a drawing of a directed graph, the direction of an arc is indicated with an arrow, as in the Figure below. Most graph-theoretical concepts have intuitive analogues for directed graphs (such as Lemma 1.1.9). Indeed, for many applications a (simple) graph can equally well be viewed as a directed graph where (v, w) is an arc whenever (w, v) is an arc. The i, j-th entry of the (directed) adjacency matrix of a general directed graph is 1 if (v_i, v_j) is an arc and 0 otherwise. Hence it is *not* a symmetric matrix.

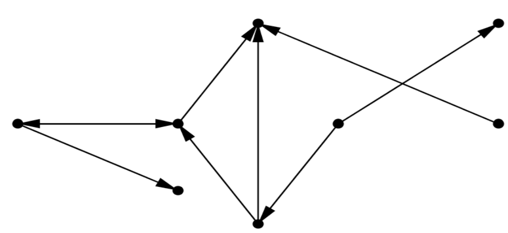

FIGURE 3. A directed graph
with 8 vertices and 9 arcs.

With this perspective, one considers a graph, as in [19], X as a quadruple $(V(X), E(X), o, t)$, where $V(X), E(X)$ are *arbitrary* sets and $o, t\colon E(X) \to V(X)$ functions. Edges e have an *origin vertex* $o(e)$, and a *terminal vertex* $t(e)$. Remembering the notion *v and w are the ends of e*, it is clear that an edge (or especially an arc) e can be interpreted as the pair $(o(e), t(e))$. According to different literature $t(e)$ and $o(e)$, are often called the *positive end* and the *negative end* of e or vice versa.

Therefore given a graph X we can view it as a directed graph by choosing for every edge $\{v, w\}$ a positive end and a negative end. The following results will be independent of this 'choice'. We refer to this 'choice' by saying that X has given an *orientation*.

1.2.2. *Definition.* The *incidence matrix* $D = D(X)$ of a graph X with $|V(X)| = n$ and $|E(X)| = m$, with respect to a given orientation of X, is the $n \times m$ matrix, whose entries are

$$d_{ij} = \begin{cases} +1 & \text{if } v_i \text{ is the positive end of } e_j \\ -1 & \text{if } v_i \text{ is the negative end of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

The rows of the incidence matrix correspond to the vertices of X, and its columns correspond to the edges of X; each column contains just two non-zero entries, $+1$ and -1, representing the positive and negative ends of the corresponding edge.

Before progressing to the Laplacian matrix, I would like to give the theoretical background for the definition of the rank and the co-rank of a graph X, which was given before the Theorem 1.1.17. Therefore we need two definitions.

1.2.3. *Definition.* The *vertex-space* $C_v(X)$ of a graph X is the vector space of all functions from $V(X)$ to $\mathbb{C}$. The *edge-space* $C_e(X)$ of X is the vector space of all functions from $E(X)$ to $\mathbb{C}$.

For $V(X) = \{v_1, v_2, \ldots, v_n\}$ and $E(X) = \{e_1, e_2, \ldots, e_m\}$, it follows that $\dim(C_v) = n$ and $\dim(C_e) = m$. Any function $\eta\colon V(X) \to \mathbb{C}$ can be represented by a column vector

$$y = (y_1, y_2, \ldots, y_n)^T,$$

where $y_i = \eta(v_i)$ $(1 \leq i \leq n)$. This representation corresponds to choosing as a basis for $C_v(X)$ the set of functions $\{\omega_1, \omega_2, \ldots, \omega_n\}$, defined by

$$\omega_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we may choose the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ for $C_e(X)$ defined by

$$\epsilon_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and hence represent a function $\xi\colon E(X) \to \mathbb{C}$ by a row vector $x = (x_1, x_2, \ldots, x_m)$ such that $x_i = \xi(e_i)$ $(1 \leq i \leq m)$. We shall refer to the basis $\{\omega_1, \omega_2, \ldots, \omega_n\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ as the *standard bases* for $C_v(X)$ and $C_e(X)$, respectively.

According to this, we remark that the incidence matrix $D(X)$ is the representation, with respect to the standard bases, of a linear mapping from $C_e(X)$ to $C_v(X)$. This mapping will be called the *incidence mapping*, and will be denoted by M. For each $\xi\colon E(X) \to \mathbb{C}$ the function $M\xi\colon V(X) \to \mathbb{C}$ is defined by

$$M\xi(v_i) = \sum_{j=1}^{m} d_{ij}\xi(e_j) \quad 1 \leq i \leq n.$$

1.2.4. *Proposition.* (see Proposition 4.3. in [2]) Let X be a graph with n vertices and c components. Then the incidence matrix D of X has rank $n-c$.

Proof. The incidence matrix can be written in the partitioned form

$$\begin{pmatrix} D^{(1)} & 0 & \dots & 0 \\ 0 & D^{(2)} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & D^{(c)} \end{pmatrix},$$

by a suitable labelling of the vertices and edges of X, where the matrix $D^{(i)}$ $(1 \leq i \leq c)$ is the incidence matrix of a component, say $X^{(i)}$, of X. We shall show that the rank of $D^{(i)}$ is $n_i - 1$, where $n_i = |V(X^{(i)})|$, from which the required result follows by addition.
Let d_j denote the row of $D^{(i)}$ corresponding to the vertex v_j of $X^{(i)}$. Since there is just one $+1$ and just one -1 in each column of $D^{(i)}$, it follows that $\sum d_j = \mathbf{0}$, and that the rank of $D^{(i)}$ is at most $n_i - 1$. Suppose we have a linear relation $\sum \alpha_j d_j = 0$, where the summation is over all rows of $D^{(i)}$, and not all the coefficients α_j are zero. Choose a row d_k for which $\alpha_k \neq 0$. This row has non-zero entries in those columns corresponding to the edges incident with v_k. For each such column, there is just one other row d_l with a non-zero entry in that column, and in order that the given linear relation should hold, we must have $\alpha_k = \alpha_l$. Thus, if $\alpha_k \neq 0$, then $\alpha_l = \alpha_k$ for all vertices v_l adjacent to v_k. Since $X^{(i)}$ is connected, it follows that all coefficients α_j are equal, and so the given linear relation is just a multiple of $\sum d_j = \mathbf{0}$. Consequently, the rank of $D^{(i)}$ is $n_i - 1$. □

1.2.5. *Remark.* According to this proposition the rank of a graph X is the rank of its incidence matrix $D(X)$.

1.2.6. *Corollary.* (see Theorem 4.5 in [2]) The kernel of the incidence mapping M of X is a vectorspace whose dimension is equal to the co-rank of X.

Proof. The above proposition shows that the rank of M is $n-c$, and the dimension of $C_e(X)$ is m. Counting the dimensions we obtain that the kernel of M has dimension $m - n + c = s(X)$. □

1.2.7. *Remark.* According to the above corollary we call the kernel of the incidence mapping M the *cycle-subspace* of X.
Motivating this definition, one considers Q to be a set of edges of a graph X, with m edges, such that the subgraph $\langle Q \rangle$ is a cycle (2-regular graph); the two possible cyclic orderings ('clockwise' or 'counter-clockwise') of the vertices of $\langle Q \rangle$ induce two possible *cycle-orientations* of the edges of Q. Let us choose one of these cycle-orientations, and define a function ξ_Q in $C_e(X)$ as follows:
We put $\xi_Q(e) = +1$ if e belongs to Q and its cycle-orientation coincides with its orientation in X, $\xi_Q(e) = -1$ if e belongs to Q and its cycle-orientation is the reverse of its orientation in X, while if e is not in Q we put $\xi_Q(e) = 0$. Now we can represent ξ_Q by a column vector $x_Q = (x_1, x_2, \dots, x_m)^T$ as above, i.e. $x_i = \xi_Q(e_i)$ $(1 \leq i \leq m)$.
We now take a look at the incidence matrix D of X, especially the i-th row, say d_i. If we want to calculate Dx_Q we consider $(Dx_Q)_i$, which is the inner product of d_i and x_Q.

(1) If v_i is not incident with some edges of Q, then this inner product is 0.
(2) If v_i is incident with some edges of Q, then it is incident with precisely two edges (because Q is 2-regular), say e_1 and e_2. If the cycle-orientations of these two edges coincides with the orientation of X, we have that $\xi_Q(e_1) = \xi_Q(e_2) = +1$, but the corresponding entries of d_i are +1 and -1, because for one edge, without loss of generality e_1, v_i is its negative end and for e_2 it must be its positive end and the inner product is zero. Analogues considerations for all possible compliances of the cycle-orientation and the orientation of X implie that the inner product is 0.

Thus $Dx_Q = \mathbf{0}$, and ξ_Q belongs to the kernel of D. A simple but important consequence is, that if the dimension of the cycle-subspace is zero, the underlying graph cannot have any such cycles. If additionally X is connected, X is, as we will define it later again, a *tree*.
There is a similar construction, called the *cut-subspace* of X, that is the orthogonal complement of the cycle-subspace in $C_e(X)$, with respect to the following inner product for two elements $\alpha, \beta \in C_e(X)$,

$$(\alpha, \beta) = \sum_{e \in E(X)} \alpha(e)\overline{\beta(e)}.$$

As the dimension of $C_e(X)$ and the cycle-subspace is m and $m-n+c$, respectively, the dimension of the *cut-subspace* is the rank of X. Let $V(X) = V_1 \cup V_2$ be a partition of $V(X)$ into non-empty disjoint subsets. If the set of edges, $H \subseteq E(X)$, which have one vertex in V_1 and one vertex in V_2 is non-empty, then we say that H is a *cut* in X. In an analogues way, one can define a function $\xi_H \in C_e(X)$ and show that it belongs to the cut-subspace. As there will be an interesting side note (see Remark 1.2.14 and Theorem 1.2.15) in matters of the Laplacian matrix, this will cross our way again.

1.2.8. *Definition.* Let X be a graph with $|V(X)| = n$. The *degree matrix* $Z = Z(X)$ is the $n \times n$ matrix whose entries are,

$$z_{ij} = \begin{cases} \deg(v_i) & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

1.2.9. *Definition.* Let X be a graph with n vertices. The *Laplacian matrix* $L = L(X)$ of X is the $n \times n$ matrix whose entries are,

$$l_{ij} = \begin{cases} \deg(v_i) & \text{for } i = j, \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

As it is clear from definition $L = Z - A$, but what will be important is its profitable relationship with the incidence matrix D.

1.2.10. *Proposition.* (see Proposition 4.8 in [2]) Let D be the incidence matrix (with respect to some orientation) of a graph X. Then the Laplacian matrix L satisfies

$$L = DD^T$$

In connection with the equation $L = Z - A$, consequently, DD^T is independent of the orientation given to X.

Proof. We know that $(DD^T)_{ij}$ is the inner product of the rows d_i and d_j of D.

(1) If $i \neq j$ then these rows have non-zero entries in the same column if and only if there is an edge joining v_i and v_j. In this case, the two non-zero entries are $+1$ and -1, so that $(DD^T)_{ij} = -1$.
(2) If $i = j$ then $(DD^T)_{ii}$ is the inner product of d_i with itself, and, since the number of entries ± 1 in d_i is equal to the degree of v_i, the result follows.

□

Before getting to the Matrix-tree-theorem, I would like to give some interesting results about the Laplacian matrix per se and its eigenvalues. These results can for instance be found in the lecture notes *Spectral graph theory and its Applications* and *Spectral graph theory* of Prof. Dan Spielman, Yale University, 2004 [30] and 2013 [31] respectively .

First of all, we observe from the definition that the Laplacian is symmetric. Furthermore, we can get another interesting fact about it. We'll begin by defining a new matrix K. First let $X_{1,2}$ be the graph on two vertices with one edge. We define

$$K_{X_{1,2}} := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Note that

$$x^T K_{X_{1,2}} x = (x_1 - x_2)^2 \quad \text{for} \quad x = (x_1, x_2)^T$$

In general, for the graph with n vertices and just one edge between the vertices v and w, we can define the matrix K similarly. For concreteness, we'll call the graph $X_{v,w}$ and define the matrix by

$$(K_{X_{v,w}})_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \in \{v, w\} \\ -1 & \text{if } i = v \text{ and } j = w, \text{ or vice versa} \\ 0 & \text{otherwise.} \end{cases}$$

For a graph X with edge set $E(X)$ we define

$$K(X) := \sum_{\{v,w\} \in E(X)} K_{X_{v,w}}.$$

Thus $K(X) = L(X)$. Many elementary properties follow from this definition. In particular, we see that $K_{X_{1,2}}$ has eigenvalues 0 and 2, and is therefore *positive semi-definite*, where we recall that a symmetric matrix M is *positive semi-definite* if all its eigenvalues are non-negative, which is equivalent to

$$x^T M x \geq 0, \text{ for all } x \in \mathbb{K}^n.$$

It follows immediately that the Laplacian matrix of every graph is positive semi-definite. One way to see this is to sum the above equation to get

$$x^T L(X) x = \sum_{\{v,w\} \in E(X)} (x_v - x_w)^2.$$

The following lemma follows immediately from the definition:

1.2.11. *Lemma.* Let X be a graph and let $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of its Laplacian matrix L. Then $\mu_0 = 0$.

Proof. Considering the 1-vector $x = (1, 1, \ldots, 1)^T$, one recomputes easily that $Lx = (0, 0, \ldots, 0)^T = \mathbf{0}$. Let l_i be the i-th row of L. We know by definition that the i-th entry of this vector is $\deg(v_i)$ and there are exactly that many -1's as entries of l_i. Therefore we get that $(Lx)_i = 0$ for all i which proves the assertion. $\square$

1.2.12. *Lemma.* (see Lemma 2.1.1 in [30]) Let X be a connected graph, and let $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of its Laplacian matrix L. Then $\mu_1 > 0$.

Proof. Let x be an eigenvector of eigenvalue 0. Then, we have

$$Lx = \mathbf{0}$$

and so

$$x^T Lx = \sum_{\{v,w\} \in E(X)} (x_v - x_w)^2 = 0.$$

Thus, for each pair of vertices (v, w) connected by an edge, we have $x_v = x_w$. As the graph is connected we have $x_v = x_u$ for all pairs of vertices v, u which implies that x is some constant times the 1-vector. Thus, the eigenspace of 0 has dimension 1. $\square$

1.2.13. *Corollary.* (see Corollary 2.1.2 in [30]) Let X be a graph, and let L be its Laplacian matrix. Then, the multiplicity of 0 as an eigenvalue of L equals the number of connected components of X.

Proof. This follows from the fact that the spectrum of a graph is just the union of the spectra of its components. Indeed as we have seen before we can permutate the matrix into a block diagonal matrix, such that each block represents a connected component. For any block diagonal matrix A the eigenvalues (and eigenvectors) of A are simply those of its blocks $A_1, A_2, \ldots A_n$ (combined), if

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

$\square$

1.2.14. *Remark.* The above Lemma led Miroslav Fiedler (Czech mathematician) to consider the magnitude of μ_1 (sometimes also λ_2) as a measure of how well-connected the graph is. Accordingly, it is often called the *Fiedler value* or the *algebraic connectivity* of a graph and the corresponding eigenvector as the *Fiedler vector.*

One of the reasons the Fiedler value is exciting is that it tells us how well you can *cut* a graph. Recall (see Remark 1.2.7) that a *cut* of a graph X is a subset H of edges that 'divides' the vertex set into two sets, say S and $\tilde{S}$. We usually want to find cuts consisting of as few edges as possible ('*cutting as few edges as possible*').

We let $E(S, \tilde{S})$ denote the set of edges whose vertices lie in S and in $\tilde{S}$ respectively. We define the *ratio* of the cut to be

$$\phi(S) = \frac{|E(S, \tilde{S})|}{\min(|S|, |\tilde{S}|)}.$$

The 'best' cut is the one of minimum ratio and its quality is the *isoperimetric number* of a graph X

$$\phi(X) = \min_S \phi(S).$$

We will skip the proof of the following theorem since it would need a lot of preparatory work. Allthough it is not in direct connection with the main aims of this thesis, I find it interesting to know anyhow.
According to this, a version of *Cheeger's inequality* says that the isoperimetric number is intimately related to the Fiedler value.

1.2.15. *Theorem.* (see Theorem 1.6.1 in [30]) Let X be a graph with Fiedler value μ_1. Then

$$\phi(X) \geq \mu_1 \geq \frac{\phi^2(X)}{2d_{\max}}$$

where $d_{\max}$ is the maximum degree of the vertices of X.

1.2.16. *Remark.* By Cheeger's inequality, the Fiedler value says something about every graph. If it is small, then it is possible to cut the graph into two pieces without cutting too many edges. If it is large, then every cut of the graph must cut many edges.

Finishing this excursion on the Laplacian matrix we are now getting ourselves near the Matrix-tree-theorem.

1.2.17. *Definition.* Let X be a graph. If X is connected and *acyclic*, i.e. the graph contains no cycles, it is called a *tree*. A *spanning tree* of another graph Y is spanning subgraph of Y, which is a tree.

A *tree* with n vertices has $n-1$ edges (this is an easy exercise). Sometimes a spanning tree of a connected graph X with n vertices is defined as a spanning subgraph of X which has $n-1$ edges and contains no cycles. By this definition it follows that a spanning tree is connected.

1.2.18. *Remark.* Recall Remark 1.1.16, where $s(X)$ also denoted the number of cycles in an elementary graph. Although this is false for an arbitrary graph X, it is true for a special kind of cycles.

Consider a graph X with c components. Each component has a spanning tree T_i with $1 \leq i \leq c$. It follows directly from the definition, that for every edge e, which is not contained in the edge sets of the spanning trees, there is a unique cycle consisting only of e and edges of T_i, where e is contained in the i-th component. These cycles are called *fundamental cycles* of X with respect to the set $\{T_i\}$.

The cycles in an elementary graph are exactly the fundamental cycles and so the consideration of Remark 1.1.16 would follow after the following lemma. The following proof is formulated by myself and so won't be found in the references.

1.2.19. *Lemma.* Let X be a graph with n vertices m edges and c components. Then the co-rank $s(X) = m - n + c$ is the number of fundamental cycles in X.

Proof. Let $\mathcal{C}$ be the set of components of X and $|\mathcal{C}| = c$. Let $C \in \mathcal{C}$ be a fixed component, and hence a subgraph of X, with n_C vertices and m_C edges and $s(C) = m_C - n_C + 1$ since C is connected. The connectness of C also implies that there exists at least one spanning tree, say T, of C. Furthermore T has got $n_C - 1$ edges and $n_C - 1 \leq m_C$ and $k := m_C - (n_C - 1) \geq 0$. This means that k is the number of 'remaining' edges of C, which are not in $E(T)$. As in the above remark each such edge e gives rise to a fundamental cycle in C (containing e and edges in T only). Hence we get that C has got k fundamental cycles and $k = m_C - n_C + 1 = s(C)$. Knowing that every fundamental cycle in X is exactly in one component of X, by summing up over all components we get,

$$s(X) = m - n + c = \sum_{C \in \mathcal{C}} m_C - \sum_{C \in \mathcal{C}} n_C + \sum_{C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} s(C)$$

which proves the assertion. $\square$

1.2.20. *Remark.* The fundamental cycles form a basis for the cycle-subspace.

We will now consider this classical result of algebraic graph theory, which shows that the number of spanning trees in a graph is determined by the Laplacian, the *Matrix tree theorem.*

1.2.21. *Definition.* The number of spanning trees of a graph X is its *tree-number*, denoted by $\tau(X)$.

Of course, if X is disconnected, then $\tau(X) = 0$. For the connected case the following results are versions of a formula for $\tau(X)$. As the Matrix tree theorem is one of the main aims of this thesis, I would like to give different versions of the formula. Nonetheless we will need a little preparatory work.

1.2.22. *Proposition (Poincaré 1901).* (see Proposition 5.3 in [2]) Any submatrix of the incidence matrix D of a connected graph X has determinant equal to 0 or $+1$ or -1.

Proof. Let S denote a square submatrix of D. If every column of S has two non-zero entries, then these entries must be $+1$ and -1 and so, since each column has sum zero, S is singular and $\det(S) = 0$. Also, if any column of S has only zero entries, then $\det(S) = 0$.
The remaining case occurs when a column of S has precisely one non-zero entry. In this case we can expand $\det(S)$ in terms of this column (see A.1.1) obtaining $\det(S) = \pm \det(\tilde{S})$, where $\tilde{S}$ has one row and column fewer than S. Continuing this process, we eventually arrive at either a zero determinant or a single entry of D, and so the result is proved. $\square$

1.2.23. *Proposition.* (see Proposition 5.4 in [2]) Let X be a connected graph with n vertices and let U be a subset of $E(X)$ with $|U| = n - 1$. Let D_U denote an $(n-1) \times (n-1)$ submatrix of the incidence matrix D, consisting of the intersection of those $n - 1$ columns of D corresponding to the edges in U and any set of $n - 1$ rows of D. Then D_U is invertible if and only if the subgraph $\langle U \rangle$ is a spanning tree of X.

Proof. Suppose that $\langle U\rangle$ is a spanning tree of X. Then a submatrix D_U consists of any $n-1$ rows of the $n\times(n-1)$ incidence matrix D' of $\langle U\rangle$. Since $\langle U\rangle$ is connected, the rank of D' is $n-1$, and so D_U is invertible.
Conversely, suppose that D_U is invertible. Then the incidence matrix D' of $\langle U\rangle$ has an invertible $(n-1)\times(n-1)$ submatrix, and consequently the rank of D' is $(n-1)$, means $\langle U\rangle$ is connected. Since $|U|=n-1$, this also leads to the cycle-subspace of $\langle U\rangle$, which has dimension zero, because $s(\langle U\rangle)=m-n+c=(n-1)-n+1=0$, hence $\langle U\rangle$ is a tree, in particular a spanning tree of X. □

1.2.24. *Lemma (Lemma 13.1.1 in* [11]*).* Let X be a graph with n vertices and c connected components. If L is the Laplacian matrix of X then its rank is $n-c$, denoted by $\mathrm{rk}(L)$.

Proof. Recall that D denotes the incidence matrix of an arbitrary orientation of X and $L=DD^T$. We shall show that $\mathrm{rk}(D)=\mathrm{rk}(D^T)=\mathrm{rk}(DD^T)$ and the result then follows ($\mathrm{rk}(D)=n-c$). If $z\in\mathbb{K}^n$ is a vector such that $DD^Tz=\mathbf{0}$, then $z^TDD^Tz=0$. But this is the squared length of the vector D^Tz and hence $D^Tz=\mathbf{0}$. Thus any vector in the kernel of DD^T is in the kernel of D^T. Conversely, suppose $D^Tz=\mathbf{0}$, $z\in\mathbb{K}^n$, then this implies $DD^Tz=\mathbf{0}$. All in all we have $\mathrm{rk}(DD^T)=\mathrm{rk}(D)$. □

1.2.25. *Remark.* Before reaching the final steps, one should recall the definition of the adjugate of a square matrix and some of its properties (see A.2.10 and A.2.11).

1.2.26. *Lemma.* (see Lemma 6.2 in [2]) Let D be the incidence matrix of a graph X, and let $L=DD^T$ be the Lapacian matrix. Then the adjugate of L is a multiple of J, the matrix each of whose entries is $+1$.

Proof. Let n be the number of vertices of X. If X is disconnected, then by the above lemma we have

$$\mathrm{rk}(L)=\mathrm{rk}(D)<n-1,$$

and so every cofactor of L is zero. That is, $\mathrm{adj}(L)=\mathbf{0}=0J$.
If X is connected, then the ranks of D and L are $n-1$. Since

$$L\ \mathrm{adj}(L)=\mathrm{adj}(L)L=\det(L)I=\mathbf{0},$$

it follows that each column of $\mathrm{adj}(L)$ belongs to the kernel of L. But this kernel is a one-dimensional space, spanned by $u=(1,1,\dots,1)^T$. Thus, each column of $\mathrm{adj}(L)$ is a multiple of u. Since L is symmetric, so is $\mathrm{adj}(L)$, and all the multipliers must be equal. Hence $\mathrm{adj}(L)$ is a multiple of J. □

1.2.27. *Matrix Tree Theorem.* (see Theorem 6.3 in [2]) Each cofactor of $L=L(X)$ is equal to the tree-number of X, that is

$$\mathrm{adj}(L)=\tau(X)J.$$

Proof. By the above Lemma, it is sufficient to show that one cofactor of L is equal to $\tau(X)$. Let D_0 denote the matrix obtained from the incidence matrix D by removing the last row; then $\det(D_0D_0^T)$ is a cofactor of L. This determinant can be expanded by the Binet Cauchy Theorem (see A.3.5) to

$$\det(D_0D_0^T)=\sum_{|U|=n-1}\det(D_U)\det(D_U^T),$$

where D_U denotes the square submatrix of D_0, whose $n-1$ columns correspond to the edges in a subset U of $E(X)$. Now, by Proposition 1.2.23, $\det(D_U)$ is non-zero if and only if the subgraph $\langle U \rangle$ is a spanning tree for X, and then $\det(D_U)$ takes the values ± 1 (Proposition 1.2.22 *Poincaré*). Since $\det(D_U^T) = \det(D_U)$ (see A.1.4), we have $\det(D_0 D_0^T) = \tau(X)$, and the result follows. $\square$

For the complete graph K_n we have $L(K_n) = nI - J$, which is the $n \times n$ matrix

$$L(K_n) = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{pmatrix}.$$

Using the above theorem, we delete the last row and the last column and calculate the determinante of the so obtained submatrix L_n. Denoting the rows by $v_1, \dots, v_{n-1}$ and $\omega := \sum_{i=1}^{n-1} v_i = (1, \dots, 1)$, we can write

$$\tau(K_n) = \det(v_1, \dots, v_{n-1}) = \det(\underbrace{\omega, v_2 + \omega, v_3 + \omega, \dots, v_{n-1} + \omega}_{:=A})$$

by the definition of the determinant as a multilinear function and obtain

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & n & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix}$$

and we realize, that it is a triangular matrix and therefore $\tau(K_n) = \det(L_n) = \det(A) = n^{n-2}$.

This result was first obtained, for small values of n, by *Cayley* (1889) and is also known as *Cayley's formula*.

We can dispense with the rather arbitrary procedure of removing one row and column from L, by means of the following result.

1.2.28. *Proposition (Temperley 1964).* (see Proposition 6.4 in [2]) The tree-number of a graph X with n vertices is given by the formula

$$\tau(X) = n^{-2} \det(J + L)$$

Proof. Since $nJ = J^2$ and $LJ = JL = \mathbf{0}$, we have the following equation:

$$(nI - J)(J + L) = \underbrace{nJ}_{=J^2} + nL - J^2 - \underbrace{JL}_{=0} = nL.$$

Thus, taking adjugates, using the *Matrix Tree Theorem* and *Cayley's formula*, we can argue as follows, where $\tau = \tau(X)$

$$\begin{aligned} \operatorname{adj}(J+L)\operatorname{adj}(\overbrace{nI - J}^{=L(K_n)}) &= \operatorname{adj}(nL) \\ \operatorname{adj}(J+L)n^{n-2}J &= n^{n-1}\operatorname{adj}(L) \\ \operatorname{adj}(J+L)J &= n\tau J \\ (J+L)\operatorname{adj}(J+L)J &= (J+L)n\tau J \\ \det(J+L)J = n\tau(J+L)J &= n\tau(\underbrace{J^2}_{nJ} + \underbrace{LJ}_{=0}) = n^2\tau J. \end{aligned}$$

It follows that $\det(J+L) = n^2\tau$, as required. $\square$

With this proposition, we can get the (for me) most interesting formula for $\tau(X)$, involving the spectrum of the Laplacian matrix.

1.2.29. *Corollary.* (see Corollary 6.5 in [2]) Let $0 = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_{n-1}$ be the spectrum of the Laplacian of a graph X with n vertices. Then

$$\tau(X) = \frac{\mu_1\mu_2\cdots\mu_{n-1}}{n}$$

Proof. As it was used in the proof of the above Proposition L and J are commutative, and as they are symmetric they are (unitary) diagonalizable. Therefore they are simultaniously diagonalizable (by the same unitary matrix) and as a consequence the eigenvalues of $J + L$ are the sums of corresponding eigenvalues of J and L. Let S be the unitary matrix such that, $SLS^{-1} = D$ and $SJS^{-1} = \tilde{D}$, with $D, \tilde{D}$ be the two diagonal matrices with the corresponding eigenvalues on the main diagonal. Hence we have $\tilde{D} + D = SJS^{-1} + SLS^{-1} = S(J + L)S^{-1}$.
The eigenvalues of J are $n, 0, \ldots, 0$ and so the eigenvalues of $J+L$ are $n, \mu_1, \mu_2, \ldots, \mu_{n-1}$. Since the determinant is the product of the eigenvalues, we have,

$$\tau(X) = n^{-2}\det(J+L) = n^{-2}n\mu_1\cdots\mu_{n-1} = \frac{\mu_1\mu_2\cdots\mu_{n-1}}{n}$$

$\square$

The fact, that if X is disconnected and therefore $\tau(X) = 0$, can now be seen in a pure technical view. Recall that if X has c components then $\mu_0 = \mu_1 = \cdots = \mu_{c-1} = 0$ and if X is disconnected ($c \geq 2$) at least $\mu_1 = 0$ and hence $\tau(X) = 0$.
I find it quite interesting that as a consequence of this last Corollary, $n \mid \mu_1\mu_2\cdots\mu_{n-1}$, which means that $\mu_1\mu_2\cdots\mu_{n-1} \in \mathbb{N}$ and by the fact that the sum of eigenvalues is the trace of a matrix, we also have that

$$\operatorname{tr}(L(X)) = \sum_{v\in V(X)} \deg(v) = 2|E(X)| = \sum_{i=1}^{n-1} \mu_i \in \mathbb{N}.$$

2. Locally finite graphs

As mentioned above this thesis is divided into two independent parts. We are now entering the second part concerning mostly the topology of graphs and groups acting on graphs and the correspondence between geometric properties of graphs and algebraic properties of groups acting on the graphs in question. Further we will associate a special graph to every group, the so-called *Cayley graph*, where we concentrate on finitely generated groups.

Being more precise our focus lies on infinite graphs, i.e. graphs with infinitely many vertices. We are especially interested in *ends of graphs*, which can be interpreted as *points at infinity*. The ends yield a topology on our graph X, the so called *end topology*, and further a *compactification* $\hat{X}$. With the construction of the *Cayley graph* of a finitely generated group we will get our connection between graph theory and group theory. The Cayley graph of a group turns out to be a strong tool, mostly regarded in the branch of mathematics called *geometric group theory*. Given a connected graph X we consider the natural *graph metric* d_X (shortest path between two vertices) and get a metric space (VX, d_X). Identifying our group with its Cayley graph we get a metric, called *word metric*. We would like to transfer properties of the Cayleygraph to our group. The Cayley graph of a finitely generated group G can 'look' quite different, if we consider different generating sets of G. But on the other hand two such Cayley graphs look pretty similar if we just 'step away far enough'. And fortunately this notion can be rigorosely formulated, known as a *quasi isometry* between two metric spaces. Hence one considers quasi isometric invariants as properties of the group G. There are many interesting invariants, e.g the *growth of a group*, *hyperbolicity* or *accessibility*. The ends of a Cayley graph are such invariants too, more precisely the *number of ends*. The *Freudenthal-Hopf-Theorem* states that a finitely generated infinite group can only have $1, 2$ or infinitely many ends thus one may ask the same question in graph theory: Which graphs determine having $1, 2$ or infinitely many ends? Answers to this question are *transitive graphs* and *almost transitive graphs*. Whereas this types of graphs are defined by having one or finitely many orbits respectively, the definition of our new graphs involves the end topology. These graphs are called *abelsian graphs*, named after *Herbert Abels* (see [1]). These graphs form a bigger class, containing the almost transitive graphs (in the locally finite case), where the set of ends contains at most 2 elements or is a *Cantor set*.

2.1. **Preliminaries.** For the theoretical procedure of this section we will mainly follow [20] and [19] respectively.

As above consider a (*undirected*) *graph* X as a pair (VX, EX) such that VX is any set and $EX \subseteq \{\{x, y\} \mid x, y \in VX\}$. The elements of VX are called *vertices* and the elements of EX are called *edges*. We call a graph X *infinite* if VX is infinite. If vertices x and y are *adjacent*, i.e. $\{x, y\}$ is an edge, we will denote this by $x \sim y$.

Let $C \subseteq VX$ be a set of vertices, then $NC := \{y \in VX \backslash C \mid y \sim x \text{ for some } x \in C\}$ is called the (*vertex-*) *boundary* of C. As in the first section we denote $N(x) := N\{x\}$ for the set of neighbours of the vertex x and recall that $\deg(x) := |N(x)|$ is

the *degree* of x. A graph is *locally finite* if all vertices have finite degree.
Moreover reall that a *path of length* $n \in \mathbb{N}_0$ from a vertex x to a vertex y is an $(n+1)$-tuple

$$(x = z_0, z_1, \ldots, z_n = y)$$

of distinct vertices such that $z_{i-1} \sim z_i$ for $1 \leq i \leq n$. The *distance* $d_X(x, y)$ between x and y is the length of the shortest path from x to y. Let A and B be non-empty sets of vertices. Then

$$d_X(A, B) = \min\{d_X(y, z) \mid y \in A, z \in B\},$$

and $d_X(x, A)$ is defined as $d_X(\{x\}, A)$. A path from x to y of length $d_X(x, y)$ is called *geodesic*. The path $(z_0, z_1, \ldots, z_n)$ is called *closed* if $z_0 = z_n$.

Moreover recall the notions of *connectness* and *trees*.

Since in *digraphs* (or *directed graphs*) the edges are not sets of vertices but ordered pairs of vertices we say that a digraph is *undirected* if (x, y) is an edge whenever (y, x) is an edge.

Recall that if X is a connected graph then (VX, d_X) is a metric space, where we will always say that X is the metric space. Throughout this part of the thesis we will be primarily interested in connected locally finite graphs.

2.1.1. *Example.* The figure below shows the so called *2-way infinite ladder graph* L_∞, an infinite connected locally finite 3-regular graph.

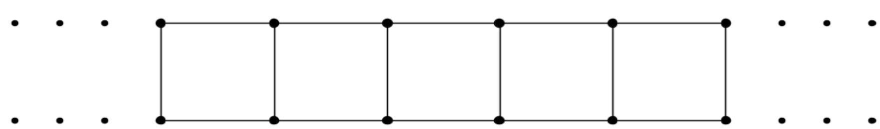

FIGURE 4. The 2-way infinite ladder graph L_∞

2.1.2. *Definition.* The *ball with center* $o \in VX$ and *radius* $r \in \mathbb{R}_{>0}$ in a graph X is the set

$$B_X(o, r) = \{x \in VX \mid d_X(o, x) \leq r\}.$$

The *diameter* of a set of vertices C is

$$\operatorname{diam}_X(C) = \max\{d_X(x, y) \mid x, y \in C\}.$$

The set C is *bounded* if its diameter is finite and *unbounded* otherwise.

2.1.3. *Lemma.* Let X be a connected locally finite graph. Then a set of vertices is bounded if and only if it is finite.

Proof. Let A be a set of vertices and $d = \mathrm{diam}_X(A)$. Suppose A is finite then it is clear that d is finite.
Conversely let A be bounded and let $o \in A$. Since $B_X(o,1)$ is exactly the union of o and its neighbours, it is finite by the assumption that X is locally finite. It follows by induction that $B_X(o,n)$ is finite for arbitrary $n \in \mathbb{N}$, since $B_X(o,n) = B_X(o,n-1) \cup NB_X(o,n-1)$. Since A is covered by $B(o,d)$ which is finite it follows that A is finite. $\square$

The property of being locally finite is crucial as the following example shows.

2.1.4. *Example.* Consider the *infinite star graph* S_∞, i.e. one center vertex adjacent to infinitely many vertices. Then the whole vertex set is bounded but it is not finite (see the figure below).

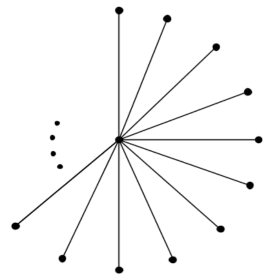

FIGURE 5. The infinite Star graph S_∞

2.1.5. *Remark.* We are now interested in describing how one can 'go to infinity' in a given infinite graph. The ladder graph L_∞ (Example 2.1.1) has two possibilites: one can go 'to the right' or 'to the left'. As an analogy one considers the compactification of $\mathbb{R}$ by two points $\{\pm\infty\}$, being precise $\hat{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Another similar example would be the stereographic projection, i.e. a homeomorphic map from $S^2 \backslash (1,0,0)$ to the complex plane. To get a homeomorphic extension to the whole sphere S^2 one has to add a 'point at infinity' to the complex numbers, hence $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Considering L_∞ again we want to define these 'points at infinity' in an rigorous way. In this sense *infinite paths* and a proper equivalence relation will be the right tool to work with.

2.1.6. *Definition.* A *ray* is an infinite path $(x_0, x_1, \dots)$ of distinct vertices. We will consider rays as sequences and sets of vertices at the same time. A set of vertices F *separates* sets of vertices A and B if any path from any vertex in A to any vertex in B contains an element of F. We say F is a A-B-*separator.* If we denote by $X \backslash F$, the induced graph of $VX \backslash F$, the former is equivalent to A and B lying in different components of $X \backslash F$.

That all infinite connected locally finite graphs contain a ray will follow from Lemma 2.2.5.

2.1.7. *Remark.* Considering again the infinite star graph of Example 2.1.4 we see that although it is infinite there exists no ray.

2.1.8. *Lemma.* Let X be a connected locally finite graph and R_1, R_2 be two rays in X. Then the following are equivalent:

(1) The rays R_1 and R_2 cannot be separated by a finite set of vertices.
(2) There exists a third ray R_3 which has infinitely many vertices in common with R_1 as well as with R_2.
(3) There exist infinitely many disjoint paths connecting vertices of R_1 with vertices of R_2.

Proof. (1) $\Rightarrow$ (3) Suppose the maximal number of disjoint paths connecting vertices of R_1 with vertices in R_2 is finite. This implies the existence of a finite R_1-R_2-separator, a contradiction.

(3) $\Rightarrow$ (1) Suppose there exists a finite R_1-R_2-separator, say F. Then by definition F contains an element of any path from any vertex in R_1 to any vertex in R_2. This contradicts the assumption that there exist infinitely many disjoint paths.

(2) $\Leftrightarrow$ (3) This equivalence is clear. □

2.1.9. *Definition.* We call two rays R_1, R_2 *equivalent*, denoted by $R_1 \sim R_2$, if one of the equivalent conditions of the above lemma holds. Note that this defines indeed a equivalence relation on the set of all rays.

(1) Reflexivity holds since for $R = R_1 = R_2$ we can choose $R = R_3$ obviously getting a proper third ray.
(2) Symmetry is trivial.
(3) Let $R_1 \sim R_2$ and $R_2 \sim R_3$. Suppose R_1 and R_3 can be separated by a finite set, say F. Hence R_1 and R_3 lie in different components of $X \backslash F$, say C_1 and C_2. Therefore R_2 either lies in C_1 or in C_2. Hence F is a finite R_1-R_2-separator or a finite R_2-R_3-separator, a contradiction. Therefore transitivity is proven.

The equivalence classes on the set of rays are called *ends* (or *vertex ends*). An end *lies in* a set of vertices C (or C *contains* ω) if all rays of ω eventually lie in C. The set of ends which lie in C is denoted by ΩC. We write ΩX instead of $\Omega V X$ for the set of all ends. A set of vertices F *seperates* ends ω_1 and ω_2 if F separates any ray in ω_1 from any ray in ω_2.

One should think of ends as 'points at infinity', which we add to our graph.

2.1.10. *Examples.*

(1) In Example 2.1.1 we introduced the laddergraph L_∞. If we consider a ray in this graph it is clear that it has to either 'go to the right' or 'to the left' since by definition a ray consists of distinct vertices. But each two rays going in the same direction are equivalent. Hence we have to ends, say ω_1 and ω_2, which we add to the picture of the graph as shown in the figure below.

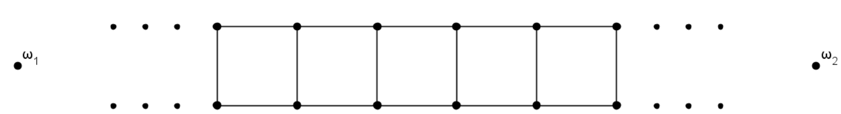

FIGURE 6. L_∞ together with its ends

(2) Another standard example is the *infinite grid graph* G_∞. If we choose any two rays in this graph it is clear that we can always find a third ray having infinitely many vertices in common with both rays. Hence we only have one end, say ω. One can compare this with the compactification of the complex plane, regarding the stereographic projection.

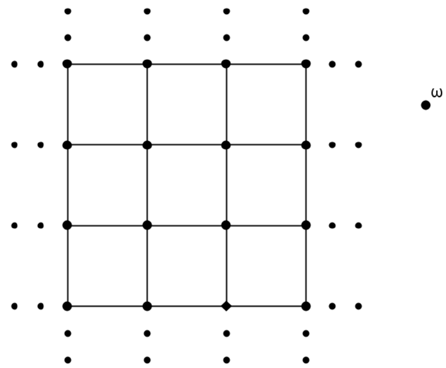

FIGURE 7. G_∞ together with its only end ω

2.1.11. *Lemma.* Let X be a connected locally finite graph and let C be a set of vertices with finite boundary. Then the following holds:

(1) If a ray R lies in C then the end which R belongs to lies in C.
(2) Each pair of distinct ends is separated by a finite set of vertices.
(3) The vertex set VX is at most countably infinite.

Proof. (1) Suppose S is a ray, equivalent with R which does not lie in C. The boundary of C is a finite R-S-separator, a contradiction.

(2) Follows directly from the definition.

(3) Let $o \in VX$ be fixed. We already know that $B_X(o,n)$ is finite for all $n \in \mathbb{N}$ (see proof of Lemma 2.1.3). Since $VX = \bigcup_{n\in\mathbb{N}} B_X(o,n)$ a countable union of finite sets, the assertion is proven. □

2.2. **The end topology.** We are now ready to investigate a proper topology on the graph induced by its ends. For the following Proposition we consider the set $\hat{X} := VX \cup \Omega X$.

2.2.1. *Proposition.* (Lemma 2 in [18]) The set

$$\mathcal{B}(X) = \{C \cup \Omega C \mid C \text{ has finite boundary}\}$$

is closed under finite intersection.

Proof. Let C, D be sets of vertices with finite boundaries and hence $C \cup \Omega C, D \cup \Omega D \in \mathcal{B}(X)$.

Claim 1: The set $C \cap D$ has finite boundary.

Let $x \in N(C \cap D)$, hence x is adjacent to some y in $C \cap D$. By definition $x \in VX \backslash (C \cap D)$ and hence x is either element of $VX\backslash C$ or element of $VX\backslash D$, where in the former $x \in NC$ and in the latter $x \in ND$. Therefore we get that $x \in NC \cup ND$ and consequently $N(C \cap D) \subseteq NC \cup ND$. Thus Claim 1 is proven.

Claim 2: It holds that $\Omega(C \cap D) = \Omega C \cap \Omega D$.

Let R be a ray which lies in $C \cap D$. Then R lies in C and in D. Conversely if R lies in C and in D then R lies in $C \cap D$. Thus Claim 2 is proven.

All in all we conclude,

$$(C \cup \Omega C) \cap (D \cup \Omega D) = (C \cap D) \cup (\Omega C \cap \Omega D) = (C \cap D) \cup \Omega(C \cap D),$$

where by Claim 1 the set $C \cap D$ has finite boundary and hence $(C \cup \Omega C) \cap (D \cup \Omega D)$ is an element of $\mathcal{B}(X)$. The assertion now follows by induction. □

2.2.2. *Definition.* The topology on $\hat{X}$ generated by the base $\mathcal{B}(X)$ is called (*vertex*) *end topology* of X. Furthermore if X is locally finite then the end topology is discrete on VX, since for $x \in VX$ the singleton $\{x\}$ is an element of the base by local finiteness and hence open.

2.2.3. *Theorem.* (Theorem 1 in [18] and Theorem 3 in [20]) The end topology of a locally finite graph is T_4 and has a countable base, i.e. $\hat{X}$ is second countable.

Proof. First we show that the base is countable. By Lemma 2.1.11 we know that VX is countable (note that local finiteness is crucial). Furthermore a countable set contains only countably many finite subsets. For each such finite subset, say F, there are only finitely many sets C with $NC = F$. In other words $VX \backslash F$ has only finitely many components. In particular $\mathcal{B}(X)$ is countable.

Recall that a topological space is T_4 if it is T_1 and *normal* (see B).

Claim 1: The end topology is T_1.

A topological space is a T_1-space if any two distinct points have a neighbourhood which does not contain the other point. Observe that this is equivalent with every singleton being closed (easy exercise).

Hence let $\alpha \in VX \cup \Omega X$. The singleton $\{\alpha\}$ is closed if and only if its complement is open. We will proof the latter by showing that for any other element $\beta \in VX \cup \Omega X$, $\beta \neq \alpha$, there is an element $U_\beta = C \cup \Omega C$ of the base $\mathcal{B}(X)$ which contains β but which does not contain α. It follows that the complement of $\{\alpha\}$, as the union of the open set U_β, $\beta \neq \alpha$, is open. Thus we have to consider three cases:

(1) If $\beta \in VX$ then we choose $U_\beta = \{\beta\}$ which is an element of $\mathcal{B}(X)$ since X is locally finite. Obviously U_β does not contain α, independent of α being a vertex or an end.
(2) If $\beta \in \Omega X$ and $\alpha \in VX$ then let C be the component of $VX \backslash \{\alpha\}$ which contains the end β. Then $U_\beta = C \cup \Omega C$ is an element of the base which contains β but not α.
(3) If $\beta, \alpha \in \Omega X$. By Lemma 2.1.11 α and β can be separated by a finite set of vertices F. Let C be the component of $VX \backslash F$ which contains β, then $U_\beta = C \cup \Omega C$ is an element of the base which contains β but not α.

In summary we obtain that all singletons are closed in the end topology and thus it is T_1.

Claim 2: The end topology is normal.

A topological space is *normal* if for two closed sets A, B there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$. In order to do so we will use the fact that a *regular* space with a countable base is already normal. Hence we only have to show that $\hat{X}$ is *regular*, that is for a closed set A and a single point x which is not contained in A there exist disjoint open sets U, V such that $A \subseteq U$ and $x \in V$.

Let therefore A be a closed set which does not contain some element $\alpha \in \hat{X}$. We have to distinguish two cases:

(1) If $\alpha \in VX$ then choose $U = \{\alpha\}$ and $V = \hat{X} \backslash \{\alpha\} = VX \backslash \{\alpha\} \cup \Omega(VX \backslash \{\alpha\}$, where V obviously contains A. Furthermore since X is locally finite both U and V are elements of the base and hence open.
(2) If $\alpha \in \Omega X$ consider the set A^c. By assumption it is open and hence it is a union of elements of the base. One of these elements, say $C \cup \Omega C$, contains α. We set $U = C \cup \Omega C$ and $V = U^c = VX \backslash C \cup \Omega(VX \backslash C)$. Since $U \subset A^c$ we have $A \subseteq U^c = V$. Since X is locally finite it follows directly that $VX \backslash C$ has finite boundary. Hence V is an element of the base and thus open. By construction U and V are disjoint.

Therefore $\hat{X}$ is regular and since it has a countable base it is normal. All in all we have proven that $\hat{X}$ is T_4. $\square$

2.2.4. *Definition.* A *subpath* of a path $(x_0, x_1, \dots, x_n)$ or a ray $(x_0, x_1, \dots)$ is a finite subsequence $(x_i, x_{i+1}, \dots, x_{i+k})$. A *geodesic ray* is a ray whose subpaths are all geodesic.

The components in the complement of a ball $B_X(o, r)$ are called *radial cuts* with center o and radius r. The set of all radial cuts with center o and radius r is denoted by $C_o(r)$. Given an end ω, we write $C_o(\omega, r)$ for the radial cut in $C_o(r)$ which contains ω.

Note that given o, r, and ω then $C_o(\omega, r)$ is well defined. But for a given radial cut o, r and ω are not necessarily uniquely determined.

In order to prove compactness of the end topology we need the following two lemmas.

2.2.5. *Lemma.* (Exercise 21.2 in [37] and Lemma 7 in [20]) Let X be a locally finite graph and let there be a descending sequence $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ of non-empty radial cuts C_r with center o and radius r. Then there is a geodesic ray R which originates in o and lies in all radial cuts C_n. The end which contains R is the only end which lies in all these radial cuts.

Proof. Let S_n be the set of geodesic paths from o to an element of NC_n. The paths in S_n have length n, since by definition their endpoints are contained in $NB_X(o, n-1)$. Furthermore S_n is contained in $B_X(o, n)$, which is finite for all n (see proof of Lemma 2.1.3), it follows that S_n is finite. Set $S = \bigcup_{n \geq 1} S_n$, then S is countably infinite. Hence there is a path π_1 in S_1 which is subpath of infinitely many elements of S. One of the finitely many paths in S_2, say π_2, which contains π_1 is itself again subpath of infinitely many elements of S. By induction we obtain a sequence of paths $(\pi_n)_{n \geq 1}$ such that $\pi_n \in S_n$ and π_n is a subpath of π_{n+1}. Set

$$R = \bigcup_{n \geq 1} \pi_n.$$

Then by construction R is a geodesic ray which lies in all radial cuts C_n. Let ω be the end which R belongs to.
Suppose there is another end η which lies in all these radial cuts C_n. Then there is a finite set F which seperates ω and η. Moreover there is a ball $B_X(o, m)$ which contains F. Hence η cannot lie in C_i, for all $i \geq m$, a contradiction. $\square$

2.2.6. *Lemma.* (Lemma 2.8 in [23]) Let C be a set of vertices such that $VX\backslash C$ is connected. If C contains the boundary of a set of vertices D then either $D \subseteq C$ or $VX\backslash D \subseteq C$.

Proof. Suppose there are vertices $x \in D\backslash C$ and $y \in (VX\backslash D)\backslash C$. Since $VX\backslash C$ is connected there is a path π from x to y which is completely contained in it. This path contains vertices of D and vertices of $VX\backslash D$. Thus π must contain a vertex of ND which is a contradiction to the assumption $ND \subseteq C$. □

2.2.7. *Theorem.* (see Lemma 5, Lemma 6 and Theorem 1 in [18]) Let X be a locally finite graph. Then every sequence of distinct elements of $VX \cup \Omega X$ has a subsequence which converges to an end. In particular the end topology of a locally finite graph is compact.

Proof. Let $\xi = (z_k)_{k\in\mathbb{N}}$ be a sequence of distinct elements of $\hat{X}$ and fix some vertex o. Since X is locally finite there are only finitely many radial cuts with center o and radius $r = 0$, i.e. components of $VX\backslash\{o\}$. Hence at least one of these radial cuts, say C_0 contains infinitely many z_k. Analogous there are only finitely many radial cuts with center o and radius $r = 1$ which are contained in C_0, whereat at least on of these, say C_1, contains infinitely many z_k. By induction we obtain an infinite descending sequence $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ of radial cuts C_r with center o and radius r, all of which contain infinitely many elements of ξ. Thus there is a subsequence $\eta = (z_{k_n})_{n\in\mathbb{N}}$ of ξ such that $z_{k_n} \in C_n \cup \Omega C_n$. By Lemma 2.2.5 there is exactly one end, say ω which lies in all C_n.
Let U be a neighbourhood of ω. Then U contains an element $D \cup \Omega D$ of the base such that $\omega \in \Omega D$. Since ND is finite there is an n_0 such that $ND \subseteq VX\backslash C_{n_0}$. Since the complement $VX\backslash(VX\backslash C_{n_0}) = C_{n_0}$ is connected we can apply the above lemma. Thus we obtain that

$$D \subseteq VX\backslash C_{n_0} \quad \text{or} \quad VX\backslash D \subseteq VX\backslash C_{n_0}.$$

Since ω lies in both C_{n_0} and D, the intersection $C_{n_0} \cap D$ is not empty and hence the first case, $D \subseteq VX\backslash C_{n_0}$ is impossible.

Thus we have $C_{n_0} \subseteq D$ and furthermore $\Omega C_{n_0} \subseteq \Omega D$. Thus $z_{k_n} \in U$ for $n \geq n_0$ and hence the subsequence converges to ω.

We already know that $\hat{X}$ is second countable and therefore the above consideration is equivalent with $\hat{X}$ being compact. □

2.2.8. *Corollary.* If X is locally finite then $\hat{X}$ is separable, every ray converges to its end and ΩX is compact.

Proof. By the above theorem it follows directly that every ray converges to its end and thus VX is dense in $\hat{X}$. Since X is locally finite VX is countable and hence $\hat{X}$ is separabel. Furthermore all singletons of VX are open and hence VX, as the union of its countably many elements, is open. Therefore its complement, namely ΩX, is closed and closed subsets of compact spaces are itself compact. □

2.2.9. *Remark.* Note that for some assertions of the above theorems the property of being locally finite is not only sufficient but often necessary. In the *non-locally finite* case some problems may occur, e.g. ΩX is in general not compact. Nonetheless we leave this as a sidenote and refer the interested reader to [18] where these problems are studied in more detail.

In the following subsection we start to connect graph theory and group theory, roughly speaking we look at the 'symmetries' of our graph. Moreover we will mention as a sidenote how ends can be used to classify these symmetries.

2.3. **Graph automorphisms.** For the following subsection one should remember basic notions of groups and groups acting on sets.

2.3.1. *Definition.* An *isomorphism* between two graphs X and Y is a bijective function $\alpha\colon VX \to VY$ such that

$$x \sim y \qquad \Longleftrightarrow \qquad \alpha(x) \sim \alpha(y).$$

Graphs X and Y are called *isomorphic* if there is an isomorphism between X and Y.

An isomorphism $\alpha\colon VX \to VX$ is called an *automorphism*. The set of all automorphisms together with the concatenation of functions forms a group, the *automorphism group* of X, denoted by $\mathrm{Aut}(X)$.

2.3.2. *Example.* Let C_n, for $n \geq 3$, be the *cycle graph*, that is a connected 2-regular graph with n vertices. One verifies easily that the automorphism group of C_n is isomorphic to D_n, the *dihedral group* (symmetry group of a regular n-gon). It is generated by a reflection and a rotation by $\frac{2\pi}{n}$.

2.3.3. *Definition.* A group G *acts on a graph* X if there is a homomorphism

$$\phi\colon G \to \mathrm{Aut}(X).$$

Instead of $\phi(g)$ we will usually write g if this causes no confusion. For $x \in VX$ we will often write $g(x) = g \cdot x$ or even omit the dot '$\cdot$' if no confusion can arise. A group G acts *freely* on X if the only element which fixes a vertex is the identity. This is equivalent with all stabilizers being trivial.

Note that usually one considers group actions as homomorphisms from the group to the *symmetry group* of a set M. Hence one considers all kind of permutations on the set M. But here we want the action to preserve the *structure* of our graph and hence we interpret group elements as only automorphisms of the graph.

2.3.4. *Example.* The group $(\mathbb{Z}, +)$ acts on the infinite ladder graph L_∞ (see Example 2.1.1). We map $1 \in \mathbb{Z}$ to the automorphism mapping each vertex to its 'right' neighbour ('right translation'). Since 1 generates $\mathbb{Z}$ we define our homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(L_\infty)$ by $\phi(n) := \phi(1)^n$, the n-time concatenation of $\phi(1)$ (if n is zero take the identity). This action is free.

2.3.5. *Remark.* As mentioned before we want to indicate how an action on the ends gives vital clues to how the automorphism acts (for a more detailed discussion see [27] and the original paper [13]). Consider $\alpha \in \mathrm{Aut}(X)$ and $\omega \in \Omega X$. It follows from definition that α maps rays to rays and hence we may get a natural action of $\mathrm{Aut}(X)$ on ΩX. Let therefore $R_1, R_2 \in \omega$ and consider $\alpha(R_1)$ and $\alpha(R_2)$. Since R_1 and R_2 are equivalent there are infinitely many disjoint paths $\pi_1, \pi_2, \ldots$ connecting vertices of R_1 with vertices of R_2. Since α is an automorphism, in particular bijective, $\alpha(\pi_1), \alpha(\pi_2), \ldots$ are again disjoint paths connection vertices of $\alpha(R_1)$ with vertices of $\alpha(R_2)$. Therefore $\alpha(R_1)$ and $\alpha(R_2)$ are equivalent and the action on ΩX

is well defined.

Halin showed in [13] how automorphisms of a graph X can be split up into three disjoint classes. We will prove the theorem below in section 5.

2.3.6. *Theorem.* (see also 1.2 in [27] or chapter 9 in [20]) Let $g \in \mathrm{Aut}(X)$. Then one of the following holds:

(1) g leaves invariant some non-empty finite subset of VX,
(2) g fixes precisely one end and does not satisfy (1),
(3) g fixes precisely two ends and does not satisfy (1).

Those automorphisms that satisfy (1) are called *elliptic*, those that satisfy (2) are called *parabolic* and those that satisfy (3) are called *hyperbolic*.

2.3.7. *Definition.* Let G act on a graph X. The set

$$G \cdot x := G(x) = \{g(x) \mid g \in G\}$$

is called the *orbit* of the vertex x. Note that orbits are either disjoint or equal and hence they form a partition of VX.

A group G acts *transitively* on a graph X if $G(x) = VX$ for some ($\Leftrightarrow$ every) vertex x of X or equivalently for each $x, y \in VX$ there exists $g \in G$ such that $gx = y$. We call a graph *transitive* if it has a transitive group action. Moreover a graph is transitive if and only if $\mathrm{Aut}(X)$ acts transitively on X.

A group G acts *almost transitively* on a graph X if it acts on X with finitely many orbits. We call a graph *almost transitive* if it has an almost transitive group action.

2.3.8. *Example.* The action of $\mathbb{Z}$ on L_∞ described in the previous example is not transitive but almost transitive. In particular there are two orbits, the 'upper line' and the 'lower line'. Nonetheless the action of $\mathrm{Aut}(L_\infty)$ on L_∞ is transitive, hence L_∞ is a transitive graph.

2.3.9. *Lemma.* (Lemma 1.3.1 in [11]) Let X be a graph, $x \in VX$ and $\alpha \in \mathrm{Aut}(X)$. Then $|N(x)| = |N(\alpha(x))|$ (can be infinite). In particular transitive locally finite graphs are k-regular for some k.

Proof. Let $y \in VX$ such that $x \sim y$. Then by definition this is equivalent to $\alpha(x) \sim \alpha(y)$ and since α is a bijection we have a bijective correspondence between the sets $N(x)$ and $N(\alpha(x))$. Thus the assertion is proven. $\square$

Hence an automorphism maps vertices to 'similar' vertices, or rather vertices with the 'same neighbourhood'.

We will now investigate the probably most famous class of transitive graphs, namely *Cayley graphs*. Therefore we will now focus our attention on group theory.

In the following section we will mainly be interested in finitely generated groups, which will correspond to locally finite graphs.

3.1. **Cayley graphs.**

3.1.1. *Definition.* Let G be a group and let S be a subset of G which does not contain the neutral element. Set $S^{\pm} = S \cup \{s^{-1} \mid s \in S\}$ (that is, $S^{\pm}$ is symmetric). The *undirected Cayley graph* $\mathrm{Cay}(G,S)$ has vertex set $VX = G$ and vertices (or group elements) x and y are adjacent if $x^{-1}y \in S^{\pm}$. The *directed Cayley graph* $\vec{\mathrm{Cay}}(G,S)$ has vertex set $VX = G$ and (x,y) is an edge if $x^{-1}y \in S$.

Vertices x and y are adjacent if and only if there is an s in $S^{\pm}$, or in S respectively, such that $xs = y$. Moreover each edge in $\mathrm{Cay}(G,S)$ is labeled by an element of S. A Cayley graph $\mathrm{Cay}(G,S)$ (directed or undirected) is connected if and only if S generates G.

The reason why the set S should not contain the neutral element is that in this case each vertex would have a loop. From now on we will always assume that the set S is symmetric and the term *Cayley graph* will always stand for the undirected case. Therefore if we picture a Cayley graph, as in the following examples, an edge between two vertices x and y should be viewed as two arrows, one from x to y the other from y to x.

3.1.2. *Remark.* The concept of Cayley graphs can be generalized for *compactly generated locally compact totally disconnected topological groups.* In particular the main problem is that many arguments fail if we consider Cayley graphs of non-finitely generated groups. Therefore in [22] there was introduced the so called *rough Cayley graph* which coincides with the *ordinary* Cayley graph if we consider finitely generated groups.

3.1.3. *Examples.*

(1) The infinite ladder graph L_∞ (see Example 2.1.1) is isomorphic to the Cayley graph of the group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generating set $S = \{(\pm 1, \bar{0}), (0, \bar{1})\}$, where the 'upper line' of L_∞ can be viewed as the elements of the form $(n, \bar{1})$ and the 'lower line' as the elements of the form $(n, \bar{0})$. Then the generators $(\pm 1, \bar{0})$ can be viewed as going 'left and right' and the generator $(0, \bar{1})$ as going 'up and down'.
(2) The infinite grid graph G_∞ (see Example 2.1.10) is isomorphic to the Cayley graph of the group $\mathbb{Z}^2$ with generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$.
(3) The infinite star graph S_∞ (see Example 2.1.4) cannot be isomorphic to the Cayley graph of any group (see 2.3.9 and the Lemma below).

3.1.4. *Lemma.* A group G acts on any of its Cayley graphs by left multiplication. Moreover G acts transitively and freely on $\mathrm{Cay}(G,S)$.

Proof. Let $X = \mathrm{Cay}(G,S)$ be a Cayley graph of G and let h be an element of G. The left multiplication l_h is the map

$$G \to G \quad \text{where} \quad g \mapsto hg.$$

This map is obviously injective and if g is any element of G then $l_h(h^{-1}g) = hh^{-1}g = g$ and hence l_h is bijective. We now have to proof that l_h preserves adjacency. Let therefore g_1, g_2 be two adjacent elements of G, i.e. there exists $s \in S$ such that $g_1 s = g_2$. If we apply l_h we get

$$l_h(g_1 s) = l_h(g_2),$$

$$hg_1 s = hg_2,$$

$$l_h(g_1)s = l_h(g_2),$$

and hence $l_h(g_1)$ is adjacent with $l_h(g_2)$. Since this equations can be read 'upwards' we get $g_1 \sim g_2 \Leftrightarrow l_h(g_1) \sim l_h(g_2)$.

Consider the action by left multiplication as described above. Let $g_1, g_2 \in G$. Then for $h = g_2 g_1^{-1}$ we get $l_h(g_1) = hg_1 = g_2 g_1^{-1} g_1 = g_2$ and hence the action is transitive.

Let $g, h \in G$ and $l_h(g) = g$, i.e. $hg = g$ we see that h is the neutral element of G. Thus the action is free. □

The 'converse' of the above statement is also true:

3.1.5. *Theorem of Sabidussi.* (Lemma 4 and Theorem 2 in [29]) A graph X is a Cayley graph of a group G if and only if there is a transitive and free action of G on X.

Proof. By the above lemma we only have to prove one direction. Let therefore X be a graph which admits a transitive and free group action of G. The idea of the proof is quite simple: Imagine $\mathrm{Cay}(G, S)$ for some generating set S then one obtains the set S as the neighbours of the identity. But the Cayley graph is transitive hence it 'looks the same' regardless on which vertex we focus.

First we need a bijection between VX and G (which is the vertex set of any Cayley graph of G). By the definition of transitivity we have

$$x, y \in VX \Rightarrow \exists g \in G : g \cdot x = y,$$

and together with the freeness of the action we directly obtain

$$x, y \in VX \Rightarrow \exists! g \in G : g \cdot x = y.$$

Hence we fix some vertex $o \in VX$ and map it to the identity $e \in G$. For an arbitrary vertex $x \in VX$ we consider the *unique* $g \in G$ such that $g \cdot o = x$. Thus we obtain a *well-defined* map $\phi : VX \to G$ which is obviously bijective. Let $\{x_i\}$ be the neighbours of o and define the set S as the elements s_i such that $s_i \cdot o = x_i$.

Suppose $x, y \in VX$ are adjacent. We get some unique $g, h \in G$ such that $g \cdot x = o$ and $h \cdot x = y$. We obtain that $x = g^{-1} \cdot o$ and $y = h \cdot x = hg^{-1} \cdot o$ thus $\phi(x) = g^{-1}$ and $\phi(y) = hg^{-1}$. Since G acts by automorphisms we get that $o = g \cdot x$ and $g \cdot y$ are adjacent, thus $g \cdot y = s \cdot o$ for some $s \in S \subseteq G$. A direct calculation using the freeness of the action shows that $s = ghg^{-1}$. We get

$$\phi(x)s = g^{-1}s = g^{-1}(ghg^{-1}) = hg^{-1} = \phi(y)$$

which means that $\phi(x)$ and $\phi(y)$ are adjacent in $\mathrm{Cay}(G, S)$.

Suppose conversely that $\phi(x)$ and $\phi(y)$ are adjacent in $\mathrm{Cay}(G, S)$, thus $\phi(x)s = \phi(y)$ for some $s \in S^{\pm}$. Without loss we can assume that $s \in S$ because otherwise we consider $\phi(x) = \phi(y)s^{-1}$.

Let $\phi(x) = g$ and $\phi(y) = h$, hence $gs = h$. Since o and $s \cdot o$ are adjacent in X we get that $g \cdot o = x$ and $gs \cdot o = h \cdot o = y$ are adjacent in X hence the assertion is proven. □

We now want to deduce properties of a given group G from the 'geometric' properties of its Cayley graphs. Unfortunately Cayley graphs with respect to different generating sets of a given group G can look quite different as the following example shows.

3.1.6. *Example.* Consider the group $\mathbb{Z}$ and the generating sets $S = \{\pm 1\}$ and $H = \{\pm 2, \pm 3\}$ respectively. The following figures show $\mathrm{Cay}(\mathbb{Z}, S)$ and $\mathrm{Cay}(\mathbb{Z}, H)$, where in the latter the shattered edges correspond to the generators ± 3 (see also [25] p.28).

FIGURE 8. $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$

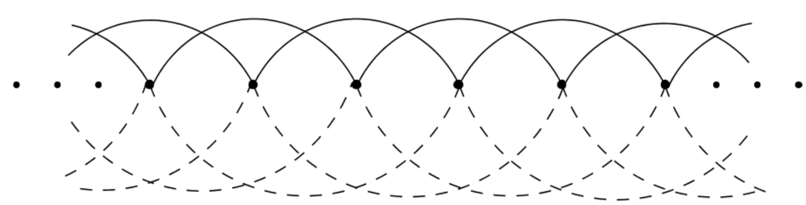

FIGURE 9. $\mathrm{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\})$

We see that these two graphs are quite different, although both Cayley graphs correspond to the same group. But somehow $\mathrm{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\}$ still seems to keep the basic 'long and skinny' shape of the real line as $\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$. One conceives 'stepping away' from $\mathrm{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\})$ it will 'look' more and more like $\mathrm{Cay}(\mathbb{Z}, \pm 1)$. Therefore roughly speaking these two graphs look the same from 'far away'. Fortunately there is a rigoros term that makes this notion precise, namely *quasi isometries* between metric spaces.

3.2. **Quasi isometries.**

3.2.1. *Definition.* Let (X, d_X) and (Y, d_Y) be two metric spaces. Then a *quasi isometry* between X and Y is a function $\phi \colon X \to Y$ which satisfies:

(i) $A^{-1} d_X(x, y) - A^{-1} B \leq d_Y(\phi(x), \phi(y)) \leq A d_X(x, y) + B \quad \forall x, y \in X$,

(ii) $d_Y(\bar{y}, \phi(X)) \leq B \quad \forall \bar{y} \in Y$, where $A \geq 1$ and $B \geq 0$ are constants. If a function ϕ only satisfies (i) then it is called a *quasi isometric embedding* and if ϕ only satisfies (ii) then ϕ is said to have a *quasi-dense image*. Hence a quasi isometry is a function which is a quasi-isometric embedding with a quasi-dense image. We write $X =_{QI} Y$ if there exists such a function ϕ and say X *and* Y *are quasi-isometric*.

3.2.2. *Examples.*

(1) Let $X = \{x\}$ be a single point and Y be a bounded non-trivial metric space with $\operatorname{diam}(Y) < \infty$. Then any function $\phi\colon \{x\} \to Y$ is a quasi isometry, since setting $A = 1$ and $B = \operatorname{diam}(Y)$, ϕ satisfies the required conditions, hence $\{x\} =_{QI} Y$.
(2) Let $X = \mathbb{Z}^2$ and $Y = \mathbb{R}^2$ both with the usual euclidian metric. Let ϕ be the canonical embedding of $\mathbb{Z}^2$ in $\mathbb{R}^2$ and choose $A = 1$ and $B = \frac{1}{\sqrt{2}}$. Then ϕ is a quasi isometry and therefore $\mathbb{Z}^2 =_{QI} \mathbb{R}^2$.

3.2.3. *Proposition.* Being quasi-isometric defines a equivalence relation on the class of metric spaces.

Proof. Let (X, d_X) be a metric space. If we choose $\phi = \mathrm{id}_X$ and $A = 1$, $B = 0$ then $\phi\colon X \to X$ is an quasi isometry and we obtain reflexivity.

Let now $X =_{QI} Y$, where ϕ denotes a quasi isometry. For each $y \in \phi(X)$ we fix $x \in \phi^{-1}(y)$ and define $\psi(y) := x$. Let $\bar{y} \in Y$ such that $d_Y(\bar{y}, y) \leq B$ and define $\psi(\bar{y}) := x$. By assumption ψ is now defined on the whole space Y. Roughly speaking we define ψ like a 'projection' of the $B - balls$ around the $\phi(x) = y \in Y$ for some $x \in X$. Set $\bar{A} = A$ and $\bar{B} = (2A+1)B \geq B$. Let now $x, y \in Y$. We will start with the first inequality of condition (i).

We have to show that $\bar{A}^{-1} d_Y(x,y) - \bar{A}^{-1}\bar{B} \leq d_X(\psi(x), \psi(y))$ which is equivalent to $d_Y(x,y) \leq d_X(\psi(x), \psi(y))A + (2A+1)B$. By construction of ψ we get

$$(*) \quad d_Y(x,y) \leq B + d((\phi \circ \psi)(x), (\phi \circ \psi)(y) + B.$$

By assumption from condition (i) relating to $X =_{QI} Y$ we know that

$$\begin{aligned} d_X(\psi(x), \psi(y))A + B + 2AB &\geq d_Y((\phi \circ \psi)(x), (\phi \circ \psi)(y)) + 2AB \\ &\overset{(*)}{\geq} d_Y(x,y) - 2B + 2AB \geq d_Y(x,y), \end{aligned}$$

and hence the first inequality is proven. With analogous argumentation we get

$$\begin{aligned} d_X(\psi(x), \psi(y)) &\leq d_Y((\phi \circ \psi)(x), (\phi \circ \psi)(y))A + B \\ &\leq (d_Y(x,y) + 2B)A + B = d_Y(x,y)\bar{A} + \bar{B}, \end{aligned}$$

and hence the second inequality is proven and ψ is a quasi-isometric embedding.

Let now $x \in X$. In order to show that ψ has a quasi-dense image we have to find $y \in Y$ such that $d_X(x, \psi(y)) \leq \bar{B}$. Therefore set $y = \phi(x)$ and hence

$$d_X(x, (\psi \circ \phi)(x)) \leq d_Y(\phi(x), \underbrace{(\phi \circ \psi \circ \phi)(x)}_{=\phi(x)})A + B \leq B \leq \bar{B}$$

All in all we have proven that $Y =_{QI} X$ and thus symmetry.

If we show that the concatenation of quasi isometries is again a quasi isometry then in particular we have transitivity. The former is obviously true and hence the assertion of the proposition is proven. $\square$

3.2.4. *Definition.* Let X be a graph and $\Gamma \leq \mathrm{Aut}(X)$. Furthermore let $\mathbb{O} = \{\Gamma x | x \in X\}$ the set of orbits. The *factor graph* of X with respect to Γ has vertex set $\mathbb{O}$ and $i, j \in \mathbb{O}$ are adjacent if and only if there are adjacent vertices $x, y \in X$ with $x \in i$ and $y \in j$. It will be denoted by X/Γ.

For an integer $k \geq 1$ the *k-fuzz* of X, denoted by $X^{(k)}$, is the graph with the same vertex set as X, where two vertices are adjacent if and only if $1 \leq d(x, y) \leq k$. Hence in the k-fuzz of X each ball $B_X(x, k)$ of radius k is complete for all $x \in VX$.

3.2.5. *Lemma.* Let X be a connected, locally finite graph and $\Gamma \leq \mathrm{Aut}(X)$. Then X/Γ is connected and locally finite.

Proof. Let $x \in VX$. By Lemma 2.3.9 we know that $N(x)$ and $N(\gamma(x))$ are in bijective correspondence for each $\gamma \in \Gamma$. In particular let two orbits, say Γ_1 and Γ_2, be adjacent, i.e there exist $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ with $x \sim y$. Thus Lemma 2.3.9 implies that for all $x \in \Gamma_1$ there exist $y \in \Gamma_2$ such that $x \sim y$ and vice versa. Hence we obtain that local finiteness of X implies local finiteness of X/Γ.

Suppose Γ_1, Γ_2 are two orbits which cannot be connected by a path in X/Γ. Let $x_1, x_2 \in VX$ be two vertices Γ_1 and Γ_2 respectively. It follows that x_1 and x_2 cannot be connected by a path in X, a contradiction. □

3.2.6. *Proposition.* (Proposition 3.9 in [37]) Every almost transitive graph is quasi-isometric to a transitive graph.

Proof. Let $\Gamma \leq \mathrm{Aut}(X)$ act almost transitively. Consider the orbits Γ_i, $i \in I = \{1, \ldots, N\}$ and the factor graph X/Γ. The latter is connected on N vertices and has diameter bounded by $N - 1$. Lifting back to X, we see that for every $x \in X$ there is $\bar{x} \in \Gamma_1$ such that $d(x, \bar{x}) \leq N - 1$, whereat we choose $\bar{x} = x$ when $x \in \Gamma_1$. Now consider the $(2N-1)$-fuzz $X^{(2N-1)}$. Let $\Gamma_1^{(2N-1)}$ denote the subgraph induced by Γ_1.

Claim 1: $\Gamma_1^{(2N-1)}$ *is connected.*

Let therefore $x, y \in \Gamma_1$ and consider a path $\pi = (x = x_0, x_1, \ldots, x_{k-1}, x_k = y)$ in X. Define $\bar{\pi} = (x = \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_k = y)$. In X we have $d(\bar{x}_{i-1}, \bar{x}_i) \leq d(\bar{x}_{i-1}, x_{i-1}) + 1 + d(x_i, \bar{x}_i) \leq 2N - 1$. Hence $\bar{\pi}$ is a path in $\Gamma_1^{(2N-1)}$, except the possibly $\bar{x}_{i-1} = \bar{x}_i$ for some i, in which case we simply delete $\bar{x}_i$. We obtain that $\Gamma_1^{(2N-1)}$ is connected and thus a metric space with the usual graph metric.

Claim 2: $\Gamma_1^{(2N-1)}$ *is transitive.*

Let $x, y \in \Gamma_1$. By definition there exist $z \in X$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x = \gamma_1 z$ and $y = \gamma_2 z$. Hence we obtain that $y = \gamma_2 z = \gamma_2 \gamma_1^{-1} x = (\gamma_2 \gamma_1^{-1}) x$ which proves transitivity.

Claim 3: X *is quasi isometric to* $\Gamma_1^{(2N-1)}$.

Define $\phi \colon X \to \Gamma_1^{(2N-1)}$ by $\phi(x) = \bar{x}$ as described above and set $A = 1$ and $B = 2(N - 1)$. We obtain

$$d(x, y) \leq \underbrace{d(x, \phi(x))}_{\leq N-1} + d(\phi(x), \phi(y)) + \underbrace{d(\phi(y), y)}_{\leq N-1} \leq d(\phi(x), \phi(y)) + B,$$

and hence the first inequality of condition (i) is satisfied. Analogously we get

$$d(\phi(x), \phi(y)) \leq d(\phi(x), x) + d(x, y) + d(y, \phi(y)) \leq d(x, y) + B,$$

hence we obtain that ϕ is a quasi-isometric embedding.

Since ϕ is surjective it has trivially a quasi-dense image. $\square$

The last theorem of this subsection has significant consequences. For that we will need one preparatory lemma. In the following we fix a vertex o and write $|x| = d(o, x)$. Furthermore we abbreviate $B_k = B_X(o, k)$.

3.2.7. *Lemma.* (Lemma 21.3 in [37]) Let X be a connected locally finite graph.

(1) Let $k \geq 1$ and $x, y \in X \backslash B_k$. If $|x| + |y| - d(x, y) \geq 2k$ then x and y belong to the same radial cut (component) of $X \backslash B_k$.

(2) Any sequence (x_n) of vertices satisfying $|x_n| + |x_{n+1}| - d(x_n, x_{n+1}) \to \infty$ converges to an end of X.

Proof. Let x, y be vertices that belong to different components of $X \backslash B_k$. Any path from x to y must pass through B_k. Hence there is $w \in B_k$ such that $d(x, y) = d(x, w) + d(w, y)$. Since $|x| = d(o, x) \leq d(o, w) + d(w, x) \leq k + d(w, x)$ and analogous $|y| \leq k + d(w, y)$ we obtain

$$d(x, y) \geq |x| + |y| - 2k,$$

and hence (1) is proven.

Let $|x_n| + |x_{n+1}| - d(x_n, x_{n-1}) \to \infty$. Since it holds that

$$|x_n| + |x_{n+1}| - d(x_n, x_{n-1} \leq |x_n| + |x_n| + d(x_n, x_{n+1}) - d(x_n, x_{n+y}) = 2|x_n|,$$

it follows that $|x_n| \to \infty$. Hence by Theorem 2.2.7 (x_n) has a subsequence converging to an end. Suppose that it has two distinct accumulation points in ΩX. Then there must be k and infinitely many n such that x_n and x_{n+1} lie in distinct components of $X \backslash B_k$. By (1) we obtain

$$2k \geq |x_n| + |x_{n+1}| - d(x_n, x_{n+1}),$$

a contradiction. $\square$

3.2.8. *Theorem.* (Lemma 21.4 in [37]) Let X and Y be quasi-isometric, connected locally finite graphs. Then the quasi isometry extends to a continuous mapping from $\hat{X}$ to $\hat{Y}$, whose restriction to ΩX is a homeomorphism onto ΩY.

Proof. Let $\phi\colon X \to Y$ be a quasi isometry with constants A, B. Furthermore for the sake of notation we write $d(.,.)$ for both the metrics of X and of Y, where this should not cause any confusion since they are both the usual graph metric. Let $o' = \phi(o)$ be our fixed vertex in Y and define $|y| = d(o', y)$ for $y \in Y$.

Let $\xi \in \Omega X$ be arbitrary and consider a ray $\pi = (x_0, x_1, \dots)$ representing ξ. Then $d(x_n, x_{n+1}) = 1$ and obviously $|x_n| \to \infty$. For the sequence $(\phi(x_n))$ we obtain

$$\begin{aligned} |\phi(x_n)| + |\phi(x_{n+1})| - d(\phi(x_n), \phi(x_{n+1})) \geq \\ \geq (|x_n| + |x_{n+1}| - 2B)A^{-1} - (A + B) \to \infty, \end{aligned}$$

and hence by the above lemma $(\phi(x_n))$ converges to an end $\tilde{\xi} \in \Omega Y$. We define $\phi(\xi) = \tilde{\xi}$.

In proving continuity, we shall see that this does not depend on the choice of the ray π.

Let (w_n) be a sequence in $\hat{X}$ converging to ξ. This means that for every k we can find n_k such that each w_n, $n \geq n_k$ is connected to π outside of B_k. That is, there are $x_{r(n)}$ and a path or ray $\pi_n = (y_0, y_1, \dots)$ in $X \backslash B_k$ such that $y_0 = x_{r(n)}$, and π_n terminates in w_n, if $w_n \in VX$ or represents w_n, if $w_n \in \Omega X$. Let us take a look at the set $\phi(\pi_n)$. We obtain

$$|\phi(y_i)| = d(\phi(o), \phi(y_i)) \geq (\underbrace{d(o, y_i)}_{\geq k} - B)A^{-1} \geq (k - B)A^{-1} =: c,$$

and hence all points of $\phi(\pi_n)$ lie outside of the ball B_c with radius c and center o'. Now, $\phi(\pi_n)$ is not necessarily a path or ray, but since $d(y_i, y_{i+1}) = 1$ we have $d(\phi(y_i), \phi(y_{i+1})) \leq A + B$. Therefore we can fill the gap between $\phi(y_i)$ and $\phi(y_{i+1})$ by a path in Y of length at most $A + B$. Hence we see that $\phi(w_n)$ is connected to $\phi(x_{r(n)}) = \phi(y_0)$ by a path or ray (respectively) in $Y \backslash B_l$, where $l = c - (A + B)$. If $k \to \infty$ then $l \to \infty$ and thus $\phi(w_n) \to \phi(\xi) = \tilde{\xi}$. Note that if (w_n) is a ray which belongs to ξ we have proven that ϕ is well defined. Moreover ϕ is continuous.

To prove the rest of the theorem we need a 'natural' property of the end compactification of locally finite graphs.

Claim 1: Let (x_n), (y_n) be sequences in X, then $x_n \to \xi \in \Omega X$ and $d(x_n, y_n) \leq M < \infty$ for all n implies $y_n \to \xi$.

Roughly speaking sequences which are 'close' to each other converge to the same end, if at least one of them converge to this end. Let therefore (x_n) and (y_n) be assumed as above. Let U be a neighbourhood of ξ and $C \cup \Omega C$ a base element which contains ξ. Since X is locally finite and C has finite boundary there exists k such that $NC \subseteq B_k$. Let $\tilde{C} = C_o(\xi, k)$ be the well defined radial cut of radius k and center o which conatins ξ. We obtain that $\tilde{C} \subseteq C$. Moreover $x_n \in \tilde{C}$ and $d(x_n, o) > k + M$ for all $n \geq n_k$. As (y_n) is connected to x_n by a path of length at most M, also $y_n \in \tilde{C} \subseteq C$ for all $n \geq n_k$. Therefore $y_n \to \xi$ and the claim is proven.

Consider the restriction of ϕ to ΩX. Then by definition the image of it is a subset of ΩY.

Let $\eta \in \Omega Y$ and $\pi = (y_1, y_2, \dots)$ be a ray which belongs to η. Since X and Y are quasi isometric there exist $x_i \in X$ such that

$$d(y_i, \phi(x_i)) \leq B \quad \text{for } i \geq 1.$$

Further we obtain that

$$\begin{aligned} d(x_i, x_{i+1}) &\leq d(\phi(x_i), \phi(x_{i+1}))A + B \leq \\ &\leq [d(\phi(x_i), y_i) + d(y_i, y_{i+1}) + d(y_{i+1}, \phi(x_{i+1}))]A + B \leq \\ &\leq (2B + 1)A + B =: a, \end{aligned}$$

and hence the distance of consecutive elements of (x_n) is bounded by a. It follows by Lemma 3.2.7 (2) that (x_n) converges to an end ξ. As described above it holds that $d(y_i, \phi(x_i)) \leq B$ and hence it follows by the above claim that $\phi(x_i) \to \eta$. Thus $\phi(\xi) = \eta$ and hence ϕ restricted to ΩX is surjective.

The following claim should be intuitively clear whereat ψ denotes a quasi isometry from Y to X with $\bar{A} \geq 1$ and $\bar{B} \geq 0$ the corresponding constants (see the proof of Proposition 3.2.3).

Claim 2: Suppose B_r separates the vertices x and y in X. Then B_{cr+d} separates the vertices $\phi(x)$ and $\phi(y)$, where c, d are some universal constants.

Let $\pi = (\phi(x) = y_0, \ldots, y_n = \phi(y))$ be a path from $\phi(x)$ to $\phi(y)$ in Y. In the proof of Proposition 3.2.3 we showed that for any $v \in X$ it holds that $d(\psi(\phi(v)), v) \leq \bar{B}$. Consider the vertex set $\psi(\pi)$. It holds that

$$d(\psi(y_i), \psi(y_{i+1})) \leq d(y_i, y_{i+1})\bar{A} + \bar{B} \leq \bar{A} + \bar{B}.$$

Now connect $\psi(y_i)$ with $\psi(y_{i+1})$ with geodesic paths of length at most $\bar{A} + \bar{B}$. We obtain a path, say $\tilde{\pi}$, from x to y in X (note that at first we obtain a walk but it clearly contains a path). Since by assumption B_r separates x and y there exists $z \in \tilde{\pi}$ such that $z \in B_r$. It follows by construction of $\tilde{\pi}$ that there exists $i(z)$ such that

$$d(\psi(y_{i(z)}), z) \leq \frac{\bar{A} + \bar{B}}{2} =: \bar{C}.$$

Moreover since $d(o, z) \leq r$ we obtain

$$d(\phi(o), \phi(z)) \leq d(o, z)A + B \leq rA + B.$$

We now only need that $\phi(z)$ is not 'too far away' from our path π. Therefore we calculate

$$\begin{aligned} d(\phi(z), y_{i(z)}) &\leq d(\psi(\phi(z)), \psi(y_{i(z)}))\bar{A} + \bar{B} \leq \\ &\leq [\underbrace{d(\psi(\phi(z)), z)}_{\leq \bar{B}} + \underbrace{d(z, \psi(y_{i(z)}))}_{\leq \bar{C}}]\bar{A} + \bar{B}, \end{aligned}$$

which implies the claim.

In particular, from the above claim it follows immediately that the restriction of ϕ to ΩX is injective.

So far we have proven that the restriction of ϕ to ΩX, for now denoted by f, is a bijective and continous function.

Let $A \subseteq \Omega X$ be a closed set. Since ΩX is compact A is compact and hence $f(A) \subseteq \Omega Y$ is compact. Moreover from Theorem 2.2.3 it follows that ΩY is Hausdorff and compact subsets of Hausdorff spaces are closed, i.e. $f(A)$ is closed. It follows that f is a homeomorphism from ΩX onto ΩY. □

3.3. **The word metric.** We will now consider a given group as a metric space, that is, the Cayley graph with respect to a finite generating set and its natural graph metric. Moreover we can describe this metric in pure group theoretical manners, the so-called *Word metric*.

3.3.1. *Definition.* Let G be a group and S be a finite generating set of G. Then for $g, h \in G$

$$d_S(g, h) = \min\{n \in \mathbb{N} \mid \exists s_1, s_2, \ldots, s_n \in S^{\pm} \text{ such that } g^{-1}h = s_1 s_2 \ldots s_n\}$$

defines a metric on G. We will call d_S the *word metric* with respect to S.

Note that d_S is exactly the graph metric ('shortest path') in $\mathrm{Cay}(G, S)$.

As it was indicated in 3.1.6, the following proposition should not be suprising.

3.3.2. *Proposition.* Let G be a group and S, H be two different finite symmetric generating sets for G. Then

$$\operatorname{Cay}(G,S) =_{QI} \operatorname{Cay}(G,H)$$

Proof. Let $H = \{h_1, \dots, h_r\}$ and $S = \{s_1, \dots, s_m\}$. We will now define proper constants A, B such that $\phi = \mathrm{id}_G$ a quasi isometry between the corresponding Cayley graphs. Therefore let $e \in G$ be the neutral element and set

$$\lambda_1 = \max\{d_H(e,s) \mid s \in S\},$$

hence λ_1 is the 'maximum length' of the elements of S represented as 'words' in H. Analogously we set

$$\lambda_2 = \max\{d_S(e,h) \mid h \in H\}.$$

Let $g, h \in G$ be arbitrary. If $d_S(g,h) = n$ then $g^{-1}h = s_{i_1} \cdots s_{i_n}$ where n is minimal. Since $d_H(e, s_{i_j}) \leq \lambda_1$ we obtain $d_H(g,h) \leq \lambda_1 n$, hence

$$d_H(g,h) \leq \lambda_1 d_S(g,h).$$

The same argument yields

$$d_S(g,h) \leq \lambda_2 d_H(g,h).$$

Therefore we set $A = \max(\lambda_1, \lambda_2)$ and obtain

$$A^{-1} d_S(g,h) \leq d_H(g,h) \leq A d_S(g,h)$$

Since the identity is surjective we can set $B = 0$ and hence $\phi = \mathrm{id}_G$ with constants A and B as above is a quasi isometry between $\operatorname{Cay}(G,S)$ and $\operatorname{Cay}(G,H)$. $\square$

The previous proposition motivates to search for *quasi isometry invariants*, i.e. properties which do not change under quasi isometries. Hence these will be properties of the group, sometimes called *geometric properties* and the theory behind it *large scale geometry of groups* (see e.g. [25]). As it was indicated in the introduction, ends (as defined in 2.1.9) will be of interest.

3.4. **Ends of groups.** As one may assume the *ends of a group* G will be the ends of one Cayley graph of G with respect to a finite generating set. In 2.1.9 we defined ends as equivalence classes of rays, that is infinite paths in the graph. Here the ends per se are not the quasi isometry invariant, but the *number* of ends. Therefore we will make another approach to the term *end*.

3.4.1. *Definition.* Let G be a group, S be a finite symmetric generating set for G and e be the neutral element of G. Moreover let $B_S(n) = B_S(e,n)$ be the ball of radius n, e as center element and the word metric with respect to S. Then

(1) $\|B_S(n)\|$ = number of *unbounded* components of $\operatorname{Cay}(G,S) \backslash B_S(n)$,
(2) $e_S = \lim_{n\to\infty} \|B_S(n)\|$.

Note that $(\|B_S(n)\|)_{n\in\mathbb{N}}$ is a non-decreasing sequence and e_S can be infinite.

The following lemma shows the connection between ends (in the sense of 2.1.9) and the above definition. Therefore we set

$$e_C = \sup\{\|C\| \mid C \text{ is a finite vertex subset of } \operatorname{Cay}(G,S)\}.$$

3.4.2. *Lemma.* (Lemma 11.17 in [25]) Let G, S and $B_S(n)$ be as above. Then e_S equals e_C.

Proof. Since G is finitely generated its Cayley graph is locally finite and hence all balls of finite radius are finite. Hence $e_S \leq e_C$.

Conversely, given any finite vertex set C, there is an $m \in \mathbb{N}$ such that $C \subseteq B_S(m)$. It follows that $e_C \leq e_S$. □

Since we know from Lemma 2.1.8 (1) that ends can be separated by finite sets of vertices, we see that e_S equals the number of ends in the graph $\mathrm{Cay}(G, S)$. The following proposition shows that the *number of ends* is a geometric property of a finitely generated group.

3.4.3. *Proposition.* (Lemma 11.22 and Theorem 11.23 in [25]) Let G be a group and S, H are two finite symmetric generating sets. Then

$$e_S = e_H.$$

Proof. This follows directly from Theorem 3.2.8 and Proposition 3.3.2.

Nonetheless we also want to prove this result with the notions defined above. Therefore we claim the following

Claim: There exist a constant $\mu \geq 1$ such that if g, h can be joined by a path outside of $B_S(\mu n + \mu)$ then they are outside of $B_H(n)$ and they can be joined by a path outside of $B_H(n)$.

Let A be as in Proposition 3.3.2 thus

$$d_H(g, h) \geq A^{-1} d_S(g, h).$$

Set $\mu = A^2 + 1$.

Let $(g = x_0, x_1, \ldots, x_n = h)$ be the path outside of $B_S(\mu n + \mu)$. Then

$$d_H(e, x_i) \geq A^{-1}(\mu n + \mu) > n + A$$

Thus each x_i is outside of $B_H(n + A)$. Moreover we have

$$d_H(x_i, x_{i+1}) \leq \underbrace{d_S(x_i, x_{i+1})}_{=1} A.$$

Hence, when we connect each x_i with x_{i+1} by a geodesic path we obtain a path connecting g and h which stays outside of $B_H(n)$. Therefore the Claim is proven.

If now two vertices of $\mathrm{Cay}(G, S)$ can be connected in the complement of $B_S(\mu n + \mu)$ then they can be connected in the complement of $B_H(n)$. Thus $\|B_S(\mu n + \mu\| \geq \|B_H(n)\|$. This implies

$$\lim_{n\to\infty} \|B_S(n)\| \geq \lim_{n\to\infty} \|B_H(n)\|,$$

that is $e_S \geq e_H$. Obviously we can switch S and H and we obtain $e_S = e_H$. □

The above proposition vindicates the following definition.

3.4.4. *Definition.* Let G be a finitely generated group. The *number of ends of G* is the number of ends of *any* Cayley graph of G with respect to some finite generating set. We will denote it by e_G.

3.4.5. *Lemma.* (Exercise 11.25 in [25]) Let G be a finitely generated group. Then

$$e_G = 0 \Leftrightarrow G \text{ is finite.}$$

Proof. ($\Rightarrow$) There exists $m \in \mathbb{N}$ such that $B_S(m)$ covers the whole Cayley graph of G. Since G is finitely generated the Cayley graph is locally finite and hence $B_S(m)$ is finite. Therefore G is finite.
($\Leftarrow$) Let s be the diameter of $\mathrm{Cay}(G, S)$. Then s is finite and $B_S(s)$ covers the graph and $\|B_S(s)\| = 0$. Since $(\|B_S(n)\|)_{n\in\mathbb{N}}$ is a non-decreasing sequence it follows that $e_G = 0$. □

3.4.6. *Remark.* We already know that $\mathbb{Z}^2$ has one end (see Example 3.1.3) and that $\mathbb{Z}$ (see Example 3.1.6) has two ends. The group $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (see Example 3.1.3), having $\mathbb{Z}$ as a finite index subgroup (anticipatively see Theorem 3.4.10) has also two ends. For the reason that we want a group with infinitely many ends the reader should recall the basic notions of *free groups* and also *group presentations* (see C). Moreover this will cross our way again in Definition 3.4.11 and in subsection 4.3.

3.4.7. *Remark.* As an aside I want to mention that the notion of group presentation opens the gate to understand so called *random groups*, which were invented by Gromov, on an intuitive level. Let $S = \{a_1, \dots, a_n\}$. Now consider reduced words in F_S and unify them in a set, say R. Now consider the group G which is defined by $\langle S|R\rangle$, hence $G \cong F_S/R^F$. Thus roughly speaking G is 'random' since we arbitrarily picked some words for R and work with a subject that has been constructed 'randomly'. For a rigorous definition of *random groups* we refer to [12], in particular for the above-mentioned Example 1.1.

Let $S = \{a_1, \dots a_n\}$ and consider the following group presentation

$$\langle a_1, \dots a_n | r_i = 1, r_i \in R\rangle\,.$$

The latter shows the connection to $X = \mathrm{Cay}(G, S)$, since the relations appear to be cycles in X starting and ending at the neutral element of G.

3.4.8. *Remark.* Consider the free group of rank n, $F = F_n$. 'By definition' F does not have any relations and hence one observes that $X = \mathrm{Cay}(F, S)$, where S is a base, has no cycles. Thus X is a tree, in particular an infinite $2n$ - regular tree. Consider B_1 and observe that NB_1 has $2n(2n-1)$ many elements. Inductively we obtain that NB_n has $2n(2n-1)^n$ many elements, where we observe that the latter equals the number $\|B_n\|$ and hence $e_F = \infty$.

We have seen groups with zero, one, two and infinitely many ends. It is a 'natural' question to ask which other values e_G can attain. Referring to this the following theorem, the so-called *Freudenthal-Hopf-Theorem* (also called *1-2-∞-Theorem*), gives an remarkable answer.

3.4.9. *Freudenthal-Hopf-Theorem.* (Satz I in [15], Theorem 11.27 in [25]) Let G be a finitely generated group. Then

$$e_G = 0, 1, 2 \text{ or } \infty.$$

Proof. In the previous remark we have summarized that there are groups with zero, one, two or infinitely many ends. Thus we need to establish that there is no group G where e_G is finite but $e_G \geq 3$.

Therefore assume to the contrary that the Cayley graph X (with respect to some finite generating set S) of a finitely generated group G has $k \geq 3$ ends. We will abbreviate $B(m) = B_S(m)$. By the above lemma $k \geq 3$ implies that G is an infinite group and that there exists $n \in \mathbb{N}$ such that the complement of $B(n)$ has k unbounded components.

Since G is infinite there is an element $g \in G$ such that $d_S(e, g) \geq 2n$ (e denotes the neutral element) and g is in an unbounded component of $X \backslash B(n)$. Now the idea is to 'send' the ball far away, namely by the action of left multiplication. In particular we have $gB(n) \cap B(n) = \emptyset$. Further since $gB(n)$ is contained in an unbounded component of $X \backslash B(n)$, it divides this component into at least k connected pieces, at least $k - 1$ of which are unbounded. Since G is finitely generated X is locally finite and thus the balls are finite. Thus $C = gB(n) \cup B(n)$ is a finite vertex set and the number of unbounded components of $X \backslash C$ is at least $2k - 2$. It follows that $e_C \geq 2k - 2$ (see Lemma 3.4.2). Thus $e_G \geq 2k - 2 > k$, since $k \geq 3$, a contradiction. $\square$

The Freudenthal Hopf Theorem gives us a partition of infinite finitely generated groups into three classes:

(1) 1 - ended groups,
(2) 2 - ended groups,
(3) infintely ended groups.

It turns out that the 2-ended groups are easy to understand and to characterize. The group $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is an indicator for the following theorem where we skip the proof, however it can be found in the references.

3.4.10. *Theorem.* (Theorem 3 in [27]) Let G be a finitely generated group. Then the following are equivalent:

(1) G has precisely two ends,
(2) G has an infinite cyclic subgroup of finite index,
(3) G has a finite normal subgroup N such that G/N is either isomorphic to the infinite cyclic group or to the infinite dihedral group.

Proof. See e.g. [25] Chapter 11.6 p.212-217. $\square$

It turns out that the 1-ended groups are the most difficult and that they are, together with finite groups, kind of 'building blocks' for infinite finitely generated groups. For the following definition (see [14] p.50 - 52) one reminds the notion of *group presentations* (see C.3.2). Nonetheless the following will just be a sidenote and one may also jump to Remark 3.4.15.

3.4.11. *Definition.* Let A, G_1 and G_2 be groups with two inclusions (injective homomorphisms) $\phi : A \hookrightarrow G_1$ and $\psi : A \hookrightarrow G_2$ (where we identify A with its image in G_1 and G_2 respectively).

The *free product of* G_1 *and* G_2 *with amalgamation over* A is denoted by $G_1 *_A G_2$ and one may identify G_1, G_2 with subgroups of it. This construction is determined by the following *universal property*:

For a group Γ and two homomorphisms $h_i\colon G_i \to \Gamma$ $(i = 1, 2)$ such that $h_1(a) = h_2(a)$ for all $a \in A$, there exists a *unique* homomorphism $h\colon G_1 *_A G_2 \to \Gamma$ such that $h(g_i) = h_i(g_i)$ for all $g_i \in G_i$ $(i = 1, 2)$.

The group $G_1 *_A G_2$ is finitely generated as long as G_1 and G_2 are. Moreover if $G_1 = \langle S_1 | R_1 \rangle$ and $G_2 = \langle S_2 | R_2 \rangle$ then

$$G_1 *_A G_2 = \langle S_1, S_2 | R_1, R_2, \phi(a) = \psi(a)\ \forall a \in A \rangle.$$

This is how one can prove the existence of the free product with amalgamation over A, by showing that the group defined by the above presentation indeed satisfies the above universal property. Uniqueness then follows in the usual manner of the universal property.

Let G be a group, A, B two subgroups and $\theta\colon A \to B$ an isomorphism. This provides a group Γ, an injection $G \hookrightarrow \Gamma$ and an element $t \in \Gamma$ such that $\theta(a) = t^{-1}at$ for all $a \in A$. The group Γ is called the *HNN-extension of G relative to θ*. It is generated by the image of G and by t, whereat usual notation is

$$\Gamma = \langle G, t | t^{-1}At = B \rangle \quad \text{or} \quad \Gamma = G *^A .$$

Moreover if $G = \langle S | R \rangle$ then

$$G*^A = \langle S, t | R, t^{-1}at = \theta(a)\ \forall a \in A \rangle.$$

A group Γ is said to be *split over a finite group* if it can be written in one of the forms:

(1) $\Gamma = G_1 *_A G_2$ with A a *finite* proper subgroup of both G_1 and G_2,
(2) $\Gamma = \langle G, t | t^{-1}At = B \rangle = G*^A$ with A, B *finite* subgroups of G.

3.4.12. *Remark.* The proof for the following celebrated theorem is far beyond the scope of this theses. However it should be stated since it is probably one of the most remarkable statemants in (geometric) group theory. Note that normally the following theorem is stated for finitely generated groups, but in [8] it has been proven that it is indeed true for any group, where one has to be careful about the notions of ends in non-finitely generated groups. One possibility is to define the notion of *more than one end* (see [8] for a detailed description). In [17] a short proof for the finitely generated case is given. The original proof of Stallings has been published in a series of papers (see [32], [33] and [34])

3.4.13. *Theorem of Stallings.* (see Chapter 9.2 in [8], especially Theorem 9.7) Let G be a any group. Then G has more than one end if and only if G splits over a finite group.

3.4.14. *Examples.* It may take a while to verify the following examples (see e.g. [14] p.51).

(1) $\mathbb{Z} = \{1\}*^{\{1\}}$
(2) $SL(2, \mathbb{Z}) = (\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$
(3) $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}*^{\mathbb{Z}/2\mathbb{Z}}$
(4) $\mathbb{Z}^2 = \mathbb{Z}*^{\mathbb{Z}}$

One may compare the above examples for HNN-extensions with the group presentations of Example C.3.3.

We will now get back to the main focus of this thesis.

3.4.15. *Remark.* Careful reading of the proof of the *Freudenthal Hopf Theorem* shows that the crucial argument is based on the Caley graph being transitive. Therefore it is a natural question to ask for the number of ends in other classes of graphs.

4. Abelsian graphs

As it was mentioned in the introduction and the last remark of the previous section we are now interested in ΩX for some connected locally finite graph X.

4.1. **Ends in transitive and almost transitive graphs.** For the following theorem we will skip the proof since it will be an immediate consequence of our main theorem, namely the *Theorem of Abels* in 4.2.8.

4.1.1. *Theorem.* Let X be a connected locally finite transitive graph. Then X has at most two or infinitely many ends.

Since an almost transitive graph is quasi isometric to a transitive graph and by Theorem 3.2.8 we get the following

4.1.2. *Corollary.* Let X be a connected locally finite almost transitive graph. Then X has at most two or infinitely many ends.

4.1.3. *Remark.* For both previous statements one can also prove that in the case ΩX is infinite, considering the end topology of X, it is a *Cantor set*, that is a *non-empty, totally disconnected, perfect, metrizable Hausdorff space* (see B). As mentioned above we will proof this for a more general class of graphs, the so-called *abelsian graphs*, named after *Herbert Abels.* Although Abels proved this statement (see [1]) in a 'group theoretic manner' we can adept this for a graph theoretic analogue.

4.2. **Ends as accumulation points and the Theorem of Abels.**

4.2.1. *Definition.* A group G *acts abelsian* on a graph X, if $\Omega X \subseteq \overline{G(x)}$ for every $x \in X$, where $\overline{G(x)}$ denotes the *topological closure* of the orbit $G(x)$ in the end topology.

We say a graph X is *abelsian* if it admits an abelsian action by some group. Note that in this case in particular $\mathrm{Aut}(X)$ acts abelsian on X.

4.2.2. *Remark.* If X is a connected locally finite transitive graph then X is abelsian, since by definition $VX = G(x)$ for every x and VX is dense in $\hat{X} = VX \cup \Omega X$. Hence each end is an accumulation point of $G(x)$. Nonetheless the converse is not true which is shown by the following example.

4.2.3. *Example.* One considers the *half grid.* Hence a graph 'looking' like the half of G_∞ (see Example 2.1.10 and the following figure). We conclude directly that this graph cannot be transitive since there are vertices with different degree. Moreover, it is not even almost transitive, since one sees that the orbits under the group of 'vertical translations' are precisely the 'horizontal lines'. Hence we obtain infinitely many orbits. Further each orbit obviously accumulates to the only end ω and hence the graph is abelsian. As an indicator for Conjecture 4.3.12 one checks that the corresponding factor graph has exactly one end.

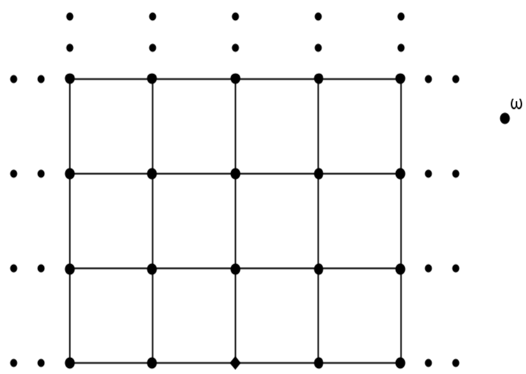

FIGURE 10. The half grid together with its only end ω.

The fact that a connected locally finite almost transitive graph X is abelsian is not that obvious as the transitive case above. For the following let ∂A denote the *topological boundary* of a subset of $\hat{X}$, that is $\partial A = \overline{A}\backslash A^\circ$, whereat A° denotes the *interior* of A.

4.2.4. *Lemma.* (Exercise 13 in [20]) Let X be a locally finite graph and let A be a subset of $\hat{X}$. Then $\partial A \subseteq \Omega X$.

Proof. Let $A = A_1 \cup A_2$ with $A_1 \subseteq VX$ and $A_2 \subseteq \Omega X$. The interior A° is the union of open sets contained in A and since the end topology is discrete on VX we know that A_1 is open and hence we can write $A^\circ = A_1 \cup R$, where $R \subseteq A_2 \subseteq \Omega X$. We conclude that

$$\partial A = \overline{A}\backslash(A_1 \cup R) = (\overline{A}\backslash A_1) \cap (\overline{A}\backslash R).$$

Since $\hat{X}$ is second countable it is first countable and therefore $\overline{A}$ is the set of limits of converging sequences contained in A. By Theorem 2.2.7 all sequences of distinct elements converge to an end and hence we obtain that $\overline{A}\backslash A_1 \subseteq \Omega X$. Thus $\partial A \subseteq \Omega X$. $\square$

4.2.5. *Lemma.* (Lemma 11 in [20]) Let a group G act on a connected locally finite graph X such that $\partial G(x) = \Omega X$ for some vertex x. Then $\partial G(y) = \Omega X$ for any vertex y.

Proof. Let $y \in VX$. By the above Lemma 4.2.4 we know that $\partial G(y) \subseteq \Omega X$ hence we have to show that $\Omega X \subseteq \partial G(y)$. Let therefore $\omega \in \Omega X$ and $C \cup \Omega C$ be an element of the base $\mathcal{B}(X)$ which contains ω. We have to show that $G(y)$ accumulates to ω. Set

$$C' = \{z \in C | d_X(z, VX\backslash C) > d_X(x, y)\}.$$

The set $C\backslash C'$ is bounded and since X is locally finite this is equivalent with $C\backslash C'$ being finite. Thus C' has finite boundary and $\Omega C' = \Omega C$ and hence $C' \cup \Omega C'$ is also an open neighbourhood of ω. Since by assumption $\partial G(x) = \Omega X$, there exists an element $g(x) \in C' \subseteq C$ with $g \in G$. Since g represents an automorphism of X and by the construction of C' we obtain

$$d_X(g(x), VX \backslash C) > d_X(x, y) = d_X(g(x), g(y)).$$

This implies $g(y) \in C$. □

4.2.6. *Proposition.* Let X be a connected locally finite almost transitive graph. Then X is abelsian.

Proof. Without loss of generality let X be infinite. Further let G be the group acting almost transitively on X. Since there are only finitely many orbits at least one Orbit, say $G(x)$ is infinite, where we fix $x \in VX$. By Lemma 4.2.4 we know that $\partial G(x) \subseteq \Omega X$. We claim the following:

Claim: There is an integer r such that $d_X(y, G(x)) \leq r$ for any $y \in VX$.

Let $y \in VX$ and let r be the number of orbits under G. This implies that the diameter in the factor graph X/G is at most $r-1$. Hence the distance between the orbit which y belongs to and $G(x)$ is at most $r-1$. Therefore consider the geodesic path between these two orbits. If we 'lift' this path back to X we obtain a path between y and some $g(x)$ of length at most $r-1$ and hence the claim is proven.

Let now $\omega \in \Omega$ and $\pi = (x_1, x_2, \dots)$ be a ray which belongs to ω. Then by the above claim there exist $g_i(x)$ which satisfy $d_X(x_i, g_i(x)) \leq r$ for $i \in \{1, 2, \dots\}$.

Let $C \cup \Omega C$ be an element of the base which contains ω. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ it holds that $x_n \in C$. Let $m_0 \geq n_0$ be such that $d_X(VX \backslash C, x_{m_0}) > r$, where since X is locally finite and C has finite boundary such an index m_0 exists. It follows that $g_{m_0}(x) \in C$. Thus $\partial G(x) = \Omega X$.

By the above Lemma 4.2.5 this holds for every orbit and hence X is abelsian. □

4.2.7. *Remark.* The above Remark 4.2.2, Example 4.2.3 and Proposition 4.2.6 show that the class of abelsian graphs is a proper extension of the class of almost transitive and hence transitive graphs (in the locally finite case).

We will now present one of the main theorems of this thesis which is a generalization of the Freudenthal Hopf Theorem (Theorem 3.4.9). It clearly has motivated the definition of abelsian graphs in the first place.

4.2.8. *Theorem of Abels.* (Satz A in [1] and Theorem 6 in [20]) Let X be a connected locally finite abelsian graph. Then ΩX has at most two elements or it is a Cantor set.

Proof. Without loss of generality let X be infinite, hence there exists at least one ray and therefore at least one end. Since we already know examples of abelsian graphs with one and two ends we have to consider the case when there exist more than two ends. Hence let ω_1, ω_2 and ω_3 be three distinct ends of X. Remember that a *Cantor set* is a *non-empty, totally disconnected, perfect* and *metrizable Hausdorff space*.

Claim 1: ΩX is perfect.

We have to show that there are no isolated points. Suppose the contrary, i.e. there exists an isolated end η thus $\{\eta\}$ is open in the relative topology of the subspace ΩX. Therefore there is an open and closed element $C \cup \Omega C$ of the base $\mathcal{B}(X)$ such that $\Omega C = \{\eta\}$. We can choose C such that $VX \backslash C$ is connected. Then clearly $C \neq VX$ and $NC \neq \emptyset$. By assumption there exists a finite connected set F

which separates ω_1, ω_2 and ω_3 from each other. Let D_i be the component of $VX\backslash F$ which contains ω_i for $i = 1, 2, 3$, hence $\omega_i \in \Omega D_i$. Note that the D_i's are pairwise disjoint. Now set

$$C' = \{z \in C | d_X(z, NC) \geq \mathrm{diam}_X F\}.$$

Let x be a vertex in F. Since X is abelsian there exists $g \in \mathrm{Aut}(X)$ such that $g(x) \in C'$. Since g is an automorphism it follows that $\mathrm{diam}F = \mathrm{diam}g(F)$ and this implies $g(F) \subseteq C$. Therefore also $g(ND_i) \subseteq C$ and obviously $g(ND_i) = Ng(D_i)$. The complements $VX\backslash D_i$ are connected, because F is connected. The same holds for $VX\backslash g(D_i)$ and $g(F)$. Therefore we can apply Lemma 2.2.6 three times. Hence we get that

$$\text{either} \quad g(D_i) \subseteq C \quad \text{or} \quad VX\backslash g(D_i) \subseteq C, \quad \text{for} \quad i = 1, 2, 3,$$

where the latter is equivalent to $VX\backslash C \subseteq g(D_i)$. Since η is the only end which lies in C the former inclusion $g(D_i) \subseteq C$ can only be satisfied for at most one i. Thus there are at least two distinct indices i_a and i_b such that $VX\backslash C \subseteq g(D_{i_a})$ and $VX\backslash C \subseteq g(D_{i_b})$. By assumption D_{i_a} and D_{i_b} are disjoint and therefore $g(D_{i_a})$ and $g(D_{i_b})$ are also disjoint. Hence we get a contradiction and the claim is proven.

Claim 2: ΩX is totally disconnected.

We have to prove that ΩX contains no non-trivial subset A which is connected in the topological sense. Assume the contrary then this is equivalent with A and $\emptyset$ being the only sets which are open and closed at once in the relative topology on A.

Let therefore ξ_1 and ξ_2 be any distinct ends in A. Then there is a finite set of vertices F such that ξ_1 and ξ_2 lie in different components, say C_1 and C_2 of $VX\backslash F$. Since X is locally finite and C_i has finite boundary it follows that $VX\backslash C_i$ has finite boundary and the complements of the base elements $\hat{C}_i = C_i \cup \Omega C_i$, namely $(VX\backslash C_i) \cup (\Omega X\backslash \Omega C_i)$ are also elements of the base. Hence we obtain that $\hat{C}_i$ are both open and closed and clearly disjoint in the end topology on $\hat{X} = VX \cup \Omega X$ and therefore open, closed and disjoint in the relative topology on ΩX. Hence $\Omega C_i \cap A$ are both open and closed and further disjoint subsets of A, a contradiction. Thus we obtain that ΩX is totally disconnected.

Claim 3: ΩX is Hausdorff.

We already know by Theorem 2.2.3 that $\hat{X}$ is T_4, hence it is Hausdorff and subsets of Hausdorff spaces are Hausdorff.

Claim 4: ΩX is metrizable.

Since X is locally finite we already know by Corollary 2.2.8 that ΩX is compact. By Claim 4, ΩX is Hausdorff and by Theorem 2.2.3 it has a countable base. By Urysohn's metrization theorem (see B.3.3) a compact Hausdorff space is metrizable if and only if it has a countable base.

In summary we have proven that if ΩX has at least three distinct ends it is a Cantor set. □

4.2.9. *Remark.* In [20] an explicit metric which induces the topology is described. Furthermore it is an *ultrametric* and hence the property of being totally disconnected would follow immediately. We do not need this metric and hence the interested reader is referred to [20], Chapter 7, p.13 - 15.

4.2.10. *Remark.* Considering the same statement, swapping *Cantor set* with *infinitely many ends*, one has another proof in [37], where it has a different approach. One combines in [37] Chapter IV, Proposition 20.7, Proposition 20.9 and Proposition 21.5. Moreover in [37] this proof chain is only applied to transitive and almost transitive graphs, where it could be immediately applied to our abelsian graphs.

4.3. **The commutator subgroup of F_n.** Recall Definition C.3.1. An interesting example for an abelsian group action on a graph is the *commutator subgroup* of $F_n = \langle a_1, \ldots, a_n \rangle$, the free group of rank n, acting on the Cayley graph of the latter with respect to a base (hence a $2n$-regular tree).

4.3.1. *Definition.* Let G be a group and $a, b \in G$. Then $a^{-1}b^{-1}ab$ is called *commutator* of a and b and is denoted by $[a, b]$. The subgroup of G which is generated by the set of commutators is called the *commutator subgroup* and is denoted by $[G : G]$.

4.3.2. *Definition.* A *long commutator* of G is an element $g \in G$ of which can be expressed in the form

$$a_1^{-1}a_2^{-1}\cdots a_n^{-1}a_1a_2\cdots a_n, \quad a_i \in G.$$

We call n the *length of the expression of g.*

4.3.3. *Theorem.* (Theorem 3 in [39]) (1) Any product of k commutators is a long commutator which has a realization of length $2k$.

(2) Any long commutator with a realization of length m, where $m = 2k$ or $m = 2k + 1$, is a product of k commutators.

Proof. (1) This is true for $k = 1$. Assume it is true for $k = n$ and let $c_1, \ldots, c_n$ and $u^{-1}v^{-1}uv$ be commutators. Then

$$u^{-1}v^{-1}uv(c_1\cdots c_n) = u^{-1}v^{-1}uv(a_1^{-1}\cdots a_{2n}^{-1}a_1\cdots a_{2n}) =$$
$$= (a_1v^{-1})^{-1}(uva_1^{-1})^{-1}(a_1v^{-1}u^{-1}v)^{-1}a_2^{-1}\cdots a_{2n}^{-1}(a_1v^{-1})(uva_1^{-1})(a_1v^{-1}u^{-1}v)a_2\cdots a_{2n} =$$
$$= b_1^{-1}\cdots b_{2n+2}^{-1}b_1\cdots b_{2n+2}.$$

(2) Since $a_1^{-1}a_2^{-1}a_1a_2$ and $a_1^{-1}a_2^{-1}a_3^{-1}a_1a_2a_3 = (a_2a_1)^{-1}(a_2a_3)^{-1}(a_2a_1)(a_2a_3)$ are both commutators, the statement is true for $m = 2$ and $m = 3$. Suppose it is true when $m = r$. Then

$$a_1^{-1}\cdots a_{r+2}^{-1}a_1\cdots a_{r+2} =$$
$$= a_1^{-1}\cdots a_r^{-1}a_1\cdots a_r(a_{r+1}a_1\cdots a_r)^{-1}(a_{r+1}a_{r+2})^{-1}(a_{r+1}, a_1\cdots a_r)(a_{r+1}a_{r+2}).$$

By the inductive hypothesis, the expression $a_1^{-1}\cdots a_r^{-1}a_1\cdots a_r$ may be written as $\frac{1}{2}r$ or $\frac{1}{2}(r-1)$ commutators, according as r is even or odd. The remaining expressions constitute a single commutator which proves the assertion. $\square$

We get the following

4.3.4. *Corollary.* (Corollary 5 in [20]) Let G be a group. Then $[G : G]$ is the set of long commutators.

From now on we consider for G the free group with base S, where $|S| = n$. We simultaneosly use the notation F_n and F_S. The above considerations immediately imply the following lemma.

4.3.5. *Lemma.* Let F_S be as above. Then $[F_S : F_S]$ is the set of all reduced words where the number of a's equals the number of a^{-1}'s for all $a \in S$.

4.3.6. *Examples.* By the above lemma it follows that the word $a^3b^{-2}a^{-1}b^2a^4$ is not in $[F_2, F_2]$. Further examples are b or aba^{-1}. However the words $a^3b^{-2}a^{-1}b^2a^{-2}$ and $ab^2a^{-1}b^{-2}$ are elements of $[F_2, F_2]$.

The following was announced at the beginning of this subsection.

4.3.7. *Theorem.* (Exercise 15 in [20]) Let F_S be as above. Then $[F_S : F_S]$ acts abelsian on $\mathrm{Cay}(F_S, S)$ by left multiplication if $|S| > 1$ and this action is not almost transitive.

Proof. First we observe that if $S = \{a\}$, i.e. F_S is cyclic, it is commutative. It is evident that $[F_S : F_S]$ is trivial and $F_S \cong \mathbb{Z}$. Hence we have infinitely many orbits but the action is not abelsian. Let therefore from now on $|S| > 1$. Let $G = [F_S : F_S]$ and e be the neutral element of F_S. Further let $X = \mathrm{Cay}(F_S, S)$. We have to prove two claims.

Claim 1: G acts on X with infinitely many orbits.

Suppose the contrary, i.e. G acts with finitely many orbits. Fix a set of representatives of the orbits, say $R = \{x_1, x_2, \dots, x_n\}$. It follows that $G \cdot R = \{g(x_i) | g \in G, i = 1, 2 \dots n\}$ covers the whole graph, namely $G \cdot R = VX$. Furthermore we find $r \in \mathbb{N}$ such that $R \subseteq B(e, r) = B$. Consequently $G \cdot B = VX$. In other words for all $x \in VX$ and for all i it holds that $d(x, G(x_i)) \leq 2r$. Hence for an arbitray vertex x we find elements of *each* orbit at a distance of at most $2r$ around x.

Consider j such that the corresponding orbit contains e, hence $G(x_j) = G(e) = [F_S : F_S]$. Let $s \in S$ and $n \in \mathbb{N}$. We now estimate $d(s^n, [F_S : F_S])$. By the above lemma an element of $[F_S; F_S]$ which is 'near' s^n must contain at least n times s^{-1}. Thus we obtain that $d(s^n, [F_S : F_S]$ is at least n. Thus for all $g \in [F_S : F_S]$ it holds that $d(s^n, g) \geq n$. We can choose n arbitrarily big, a contradiction.

Claim 2: G acts abelsian on X.

Consider again the orbit which contains e, hence $G(e) = [F_S : F_S]$. Let $\omega \in \Omega X$ and let $C \cup \Omega C$ be an element of the (topological) base which contains ω. Without loss of generality let $VX \backslash C$ be connected and $C \neq VX$. We have to find an element of $[F_S : F_S]$ which is contained in C. Set $d = \mathrm{diam} NC$ which is finite by assumption. Let $R = (e = x_0, x_1, x_2, \dots)$ be the ray starting at e which belongs to ω, then there exists an N such that for all $n \geq N$ it holds that $x_n \in C$, in particular $x_{N+d} \in C$. Consider an arbitrary reduced word $s_1 s_2 \cdots s_n$, $s_i \in S^{\pm}$, and let $a, b \in S$ with $a \notin \{b, b^{-1}\}$. Then it is clear that either

(1) $s_1 \cdots, s_n aba^{-1} s_{n-1}^{-1} \cdots s_1^{-1}$ is reduced or
(2) $s_1 \cdots, s_n a^{-1} ba s_{n-1}^{-1} \cdots s_1^{-1}$ is reduced.

Suppose (1) holds then either

(1a) $s_1 \cdots, s_n aba^{-1} s_{n-1}^{-1} \cdots s_1^{-1} ab^{-1}a^{-1}$ is reduced or
(1b) $s_1 \cdots, s_n aba^{-1} s_{n-1}^{-1} \cdots s_1^{-1} a^{-1}b^{-1}a$ is reduced.

Clearly the case (2) can be treated analogously. Hence we observe that in any case we can extend *any* reduced word to a reduced word in $[F_S : F_S]$. Now let D be the set of reduced words which begin with the word $x_{N+d} = s_1 s_2 \cdots s_n$. Since X

is a tree it holds that for two distinct vertices in the tree there is exactly one path between these two vertices. Because otherwise X would have a circle. We obtain that $ND = \{x_{N+d-1}\}$ which is contained in C, since $d \geq 1$ (we set $C \neq VX$). Hence we can apply Lemma 2.2.6 and obtain

$$\text{either} \quad D \subseteq C \quad \text{or} \quad VX \backslash D \subseteq C.$$

The latter is equivalent with $VX\backslash C \subseteq D$ which is obviously impossible since e.g x_{N-1} is an element of $VX\backslash C$ but it cannot be an element of D by definition. Thus we have $D \subseteq C$ and by construction we have an element of $[F_S : F_S]$ which is contained in C. Therefore $\partial G(e) = \Omega X$. By Lemma 4.2.5 all orbits accumulate to ΩX and thus G acts abelsian on X. □

The above theorem gives us another example of an abelsian but not almost transitive group action on a graph.

4.3.8. *Remark.* We want to close this section with a 'plausible' conjecture. Therefore consider the commutator subgroup. With Theorem 4.3.3 and the resulting corollary we get an 'explicit' expression of the elements of $[G : G]$. Moreover, $[G : G]$ is a normal subgroup of G. Considering the free group of rank n, F_n, one verifies easily that

$$F_n \big/ [F_n : F_n] \cong \mathbb{Z}^n$$

(compare with the presentation in Example C.3.3). The group $\mathbb{Z}^n$ has exactly one end. Considering the above theorem roughly speaking we factor out an abelsian action with infinitely many orbits and get an 'one ended structure'. One may ask if the same holds generally in a more graph theoretic manner. In particular let X be a connected locally finite graph which admits an abelsian action with infinitely many orbits (thus X is infinite) by some group G. Recalling Definition 3.2.4, i.e. the factor graph, one asks if

$$X \big/ G = Y$$

also has exactly one end. Since X is abelsian it has 1,2 or infinitely many ends. Therefore we might distinguish these three cases.

4.3.9. *Example.* Consider G_∞ (see Example 2.1.10), the infinite grid graph and the group G of vertical translations. One observes that every orbit under the action of G accumulates to the only end of G_∞ and hence it is an abelsian action. Nonetheless the corresponding factor graph is 2-ended. Hence for the one ended case the above question cannot be true.

In the case of X having two ends we even have the following

4.3.10. *Lemma.* Let X be a locally finite connected 2-ended graph. Then X is abelsian if and only if X is almost transitive.

Proof. Since we have already proven in Proposition 4.2.6 that an almost transitive graph is abelsian we only have to prove the other implication. Let therefore G act on X abelsian. Since X is 2-ended there exists a ball of radius n, say B, such that $X\backslash B$ has exactly two infinite connected components, say C and D. Note that we have a partition $VX = C \cup B \cup D$ and $ND \subseteq B$ and $NC \subseteq B$. Since G acts abelsian we find $g, h \in G$ such that $g(B) \subseteq C$ and $h(B) \subseteq D$ (compare with the proof of Claim 1 in Theorem 4.2.8). The following claim is based on the proof of

Theorem 4 in [7].

Claim: There exists $\tau \in G$ such that $\tau(B \cup C) \subseteq C$.

If $B \cap g(C) = \emptyset$ then $g(B \cup C)$ is a (connected) subset of $VX \backslash B$. Since in this case $g(B \cup C) \cap C \neq \emptyset$ we conclude that $g(B \cup C) \subseteq C$. We may then put $\tau := g$.

If $B \cap h(D) = \emptyset$ then $B \subseteq h(C)$. Since $B \cup C$ is a connected subgraph of $VX \backslash h(B)$ we have $B \cup C \subseteq h(C)$. We may then set $\tau := h^{-1}$.

If neither of the above holds, then $B \subseteq h(D) \cap g(C)$, because B is connected and avoids both $h(B)$ and $g(B)$. Then $D \cup B$ is a connected subset of $VX \backslash g(B)$ meeting $g(C)$. Hence we have $D \cup B \subseteq g(C)$. Similarly we conclude that $B \cup C \subsetneq h(D)$. Putting $\tau := g^{-1}h^{-1}$ we get

$$\tau(B \cup C) \subseteq g^{-1}(D) \subseteq g^{-1}(D \cup B) \subseteq C$$

as desired.

Note that the above claim implies that $\tau^{-1}(B \cup D) \subseteq D$. Further it implies that $\tau^k(C)$ has distance of at least $k+1$ in X from any vertex of $D \cup B$. As every vertex in C has finite distance from $D \cup B$, this implies in particular that

$$\bigcap_{k \in \mathbb{N}} \tau^k(C) = \emptyset \quad \text{and} \quad \bigcap_{k \in \mathbb{N}} \tau^{-k}(D) = \emptyset.$$

Consider the following

$$\bigcup_{n \in \mathbb{Z}} \tau^n(B) = \bigcup_{n \in \mathbb{Z}} (\tau^n(C \cup D))^c = [\bigcap_{n \in \mathbb{Z}} \tau^n(C \cup D)]^c.$$

Suppose that there exists $x \in \bigcap_{n \in \mathbb{Z}} \tau^n(C \cup D)$, i.e. $x \in \tau^n(C \cup D)$ for all $n \in \mathbb{Z}$. Hence x is an element of $\tau^n(C)$ or $\tau^n(D)$ for all $n \in \mathbb{Z}$, a contradiction. Therefore we have

$$\bigcup_{n \in \mathbb{Z}} \tau^n(B) = VX$$

which, since B is finite, implies that X is almost transitive. □

4.3.11. *Remark.* We conclude that the above question may only hold in the case when X has infinitely many ends. Being more precise if X is connected and locally finite we know that the same holds for $Y = X/G$. Since by assumption G acts with infinitely many orbits consequently Y is an infinite graph. In summary Y has at least one end. Moreover since G acts abelsian every orbit accumulates to each end of X. When we construct the factor graph each orbit 'collapses' to one vertex and since *each* orbit comes 'arbitrarily close' to *each* end one assumes that ΩX also 'collapses' to one point. This motivates the following

4.3.12. *Conjecture.* Let X be a connected locally finite graph with infinitely many ends. If a group of automorphisms Γ acts on X with infinitely many orbits and abelsian then X/Γ has exactly one end.

5. Structure of ends and automorphisms

In this last section we will focus on the *structure* of ends and in doing so define the *thickness* of ends. This notion of thickness will provide another geometric property of graphs (i.e. respected under quasi isometries). Moreover we will classify 'different' automorphisms of graphs (see Theorem 2.3.6).

5.1. **Thickness of ends.**

5.1.1. *Definition.* The *thickness on an end* ω is the smallest integer $\mu = \mu(\omega)$ such that there is a *descending* sequence of sets $C_n \subseteq VX$ which have *finite* boundary and contain ω for all $n \in \mathbb{N}$ and such that

$$\operatorname{diam}_X(NC_n) \leq \mu \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} C_n = \emptyset.$$

An end ω is called *thin* if $\mu(\omega) < \infty$ and *thick* otherwise.

5.1.2. *Examples.* The infinite grid graph G_∞ from Example 2.1.10 has one thick end. The same holds for the infinite half grid from Example 4.2.3. The infinite ladder graph L_∞ has two thin ends (again see Example 2.1.10).

To provide more examples we look at the following two constructions of graphs.

5.1.3. *Definition.* (Cartesian product) Let X and Y be two graphs. Then the *Cartesian product* $X \times Y$ is the graph given by

(1) $V(X \times Y) = VX \times VY$
(2) Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if

$$x_1 = x_2 \quad \text{and} \quad y_1 \sim y_2 \quad \text{or}$$

$$x_1 \sim x_2 \quad \text{and} \quad y_1 = y_2.$$

Observe that this construction is *not* the *product* of two graphs in the category of graphs.

5.1.4. *Definition.* (Free product) Let (X_i, o_i), $i \in \mathcal{I}$, be a family of sets with a distinguished element $o_i \in X_i$. For simplicity we may assume that $\mathcal{I}$ is *finite*. The *free poduct* $(X, o) = *_{i\in\mathcal{I}}(X_i, o_i)$ is defined as follows:

We identify all the o_i with o and write $\tilde{X}_i = X_i \backslash \{o_i\}$. If $x \in \tilde{X}_i$ we write $\alpha(x) = i$. One should think of X as all words with letters from the $\tilde{X}_i$ such that no two successive letters come from the same $\tilde{X}_i$. Thus we have

$$X = \{x_1 x_2 \dots x_n \mid n \geq 0,\ x_j \in \bigcup_i \tilde{X}_i,\ \alpha(x_j) \neq \alpha(x_{j-1})\},$$

where for $n = 0$ we intend the empty word o. In particular $X_i \subseteq X$. If $n \geq 1$ we say that the word is *non-empty.* We set X_i^{end} the set of non-empty words which do *not* have their last letter in $\tilde{X}_i$ together with the empty words o. Similarly we set X_i^{start} the set of non-empty words which do *not* start with a letter in $\tilde{X}_i$. If $u \in X_i^{\text{end}}$ and $x \in X_i^{\text{start}}$ then ux stands for their *concatenation* as words in X, in particular $uo = u$.

If the X_i are graphs (without loops and multiedges), then adjacency in X is given as follows:

if $x, y \in X_i$ are adjacent in X_i then $ux \sim uy$ in X for all $u \in X_i^{\mathrm{end}}$.

Thus one starts by gluing all X_i together by identifying all o_i. At each other point of each X_i we attach copies of *all* the X_j with $j \neq i$ by o_j and so on.

If the X_i are groups we obtain the *free product of groups* (see C.3.4).

The following proposition is a straightforward calculation.

5.1.5. *Proposition.* Let G_i be a group and S_i be a finite symmetric generating set for $i = 1, 2$. Then

$$\mathrm{Cay}(G_1 \times G_2, S_1 \times S_2) \cong \mathrm{Cay}(G_1, S_1) \times \mathrm{Cay}(G_2, S_2),$$

where $S_1 \times S_2 := \{(s, e_2), (e_1, t) \mid s \in S_1, t \in S_2\}$ and

$$\mathrm{Cay}(G_1 * G_2, S) \cong \mathrm{Cay}(G_1, S_1) * \mathrm{Cay}(G_2, S_2),$$

where $S := S_1 \cup S_2$.

In analogy to Proposition 5.1.5 and referring to Conjecture 4.3.12 we have the following lemma for factor graphs.

5.1.6. *Lemma.* Let G be a group, S a finite generating set and N a normal subgroup of G. Then

$$\mathrm{Cay}(G/N, S/N) \quad \cong \quad \mathrm{Cay}(G,S)/N,$$

where $S/N := \{gN \mid gN \cap S \neq \emptyset\}$.

Proof. The bijection is given by

$$\begin{aligned} \alpha\colon G/N \quad &\to \quad V \\ gN \quad &\mapsto \quad N(g), \end{aligned}$$

where V denotes the vertex set of the factor graph. Obviously this is bijection since the orbits under N are just right cosets.

Suppose that $gN \sim hN$ which is equivalent to $g^{-1}hN \in S/N$, i.e. $g^{-1}hN \cap S \neq \emptyset$. This is equivalent with the existance of $n \in N$ and $s \in S$ such that $g^{-1}hn = s$ thus $hn = gs$.

On the other hand if $N(g) \sim N(h)$, i.e the existance of $x_1 \in N(g)$ and $x_2 \in N(h)$ such that $x_1 \sim x_2$ in $\mathrm{Cay}(G, S)$. This is equivalent with the existance of $\tilde{s} \in S$ such that $x_1\tilde{s} = x_2$.

Hence we have to find such x_1 and x_2. Looking at the first paragraph we stopped with $hn = gs$. Obviously $g \in N(g)$ hence g is our x_1. Further hn should equal $x_2 \in N(h)$ but we are in general not abelian. But since N is a normal subgroup, i.e. $hN = Nh$ there exists $\tilde{n} \in N$ such that $hn = \tilde{n}h$ and since $\tilde{n}h \in N(h)$ we get $hn \in N(h)$ and thus hn is our x_2.

Combining all arguments the assertion is proven. $\square$

5.1.7. *Example.* If we take the free product of some finite groups we conclude that the Cayley graph of this free product looks 'roughly' like a tree. More generally this holds for the free product of arbitrary connected finite graphs (see Lemma below). Moreover the free product of two (or more) connected graphs has always infinitely many ends unless the graphs we are starting with have only two vertices (see [37] p. 232).

5.1.8. *Lemma.* Let $X_1, \dots X_n$ some finite connected graphs with distinguished vertices $o_i \in X_i$. Then the free product $(X, o) = *_{i\in\mathcal{I}}(X_i, o_i)$ is quasi isometric to a tree.

Proof. We define an appropriate tree as follows: For each upcoming copy of $X_i^{(l)}$ in the free product, with $l \in \mathcal{L}$ some index set, we take a vertex $a_i^{(l)}$. We connect $a_i^{(l)}$ with $a_j^{(l')}$ by an edge if $i \neq j$ and the correspondig $X_i^{(l)}$ and $X_j^{(l')}$ have been glued together by o_j or o_i $(l, l' \in \mathcal{L})$. The resulting graph T is obviously a tree (by definition of the free product). We obtain a canonical map $\phi\colon V((X, o)) \to VT$. With $A := \sum_{i=1}^{n} \mathrm{diam} X_i$ and $B := 0$ (since ϕ is surjective) it is a straightforward computation to verify that ϕ is a quasi isometry.

□

5.1.9. *Example.* As it has been noted already with the above Lemma 5.1.6 we see that the factor graph of $\mathrm{Cay}(F_n, S)$, where F_n denotes the free group with rank n and S a free basis, modulo the action of $[F_n, F_n]$ is indeed one ended.

5.1.10. *Examples.* With the above Proposition and earlier examples we see that the cartesian product of two graphs with two thin ends ($\mathrm{Cay}(\mathbb{Z}, \{\pm 1\})$) can be a graph with one thick end (G_∞). In that regard we have the following

5.1.11. *Lemma.* (see [37] p. 232) Let X, Y be two connected infinite locally finite graphs. Then $X \times Y$ has one end. Moreover this end is thick.

Proof. Since X and Y are infinte and locally finite so is $X \times Y$. Let B be a ball of finite radius r in $X \times Y$ and let $z = (x_1, y_1)$ and $w = (x_2, y_2)$ be contained in some *infinite* component of the complement of B. We want to find a path from z to w which avoids B.

Let π_X and π_Y be the canonical projections of $X \times Y$ onto X and Y respectively. Let $B_X = \pi_X(B)$ and $B_Y = \pi_Y(B)$ be the 'X and Y- coordinates' of elements of B. By local finiteness and since z is in an infinite component we can find a *geodesic* ray R starting at z, avoiding B and with unbounded distance of B. By definition of adjacency of the cartesian products the coordinates of the elements of R must be different infinitely often for at least one coordinate, say X. In other words $\pi_X(R)$ is a ray (while ignoring identical elements and deleting circles). But when we walk along R 'long enough' and stop at some vertex, say (x, y), we can walk in the 'Y-coordinates' arbitrary and always avoid B. We can do the same thing with w obtaining a ray $\tilde{R}$ which has infinitely many different X-coordinates (the other case is treated analogously). We get to some vertex, say $(\tilde{x}, \tilde{y})$, from where we can walk arbitrary in the 'Y-coordinates'. Hence we can find some $\bar{y} \in Y$ such that the path induced from the one from x to $\tilde{x}$ in X avoids B. Thus we obtain a path from z to w avoiding B.

All in all this proves that $X \times Y$ has only one end.

Since X is infinite and locally finite there exists a ray $R = (x_1, x_2, \dots)$ in X. The rays $S_y = (s_i)_{i\in\mathbb{N}}$ with $s_i = (x_i, y)$, $i \in \mathbb{N}$ are rays in $X \times Y$ for each $y \in Y$. Clearly S_y and S_z are disjoint for distinct elements $y, z \in Y$ and since Y is infinite we have infinitely many pairwise disjoint rays in $X \times Y$ which proves that its only end is thick. $\square$

Since bounded spaces are quasi isometric to a singleton the following should not be surprising.

5.1.12. *Lemma.* Let X be a connected locally finite infinite graph and Y a finite connected graph. Then $X \times Y$ is quasi isometric to X.

Proof. Consider the canonical injection $\pi_y \colon X \to X \times Y$, $\pi_y(x) = (x, y)$ for some fixed $y \in Y$. Set $A = 1$ and $B = \operatorname{diam}(Y)$. It is a straightforward calculation to show that π_y is a quasi isometry. $\square$

5.1.13. *Corollary.* Let X be a connected locally finite graph with one end and $Y_1, \dots Y_n$ some connected locally finite graphs. Then $X \times (Y_1 \times \dots \times Y_n)$ has one end.

5.1.14. *Example.* Let G be a finitely generated *abelian* group. Then $G \cong \mathbb{Z}^d \times T$ for some $d \in \mathbb{N}$ and T some finite group (the *torsion subgroup*). Hence G has one end. Moreover this end is thick (see the following proposition).

We now want to prove that the thickness of ends is a geometric property of graphs, where this will follow directly from the following lemma.

5.1.15. *Lemma.* (see Lemma 2.2 in [24] and Lemma 12 in [21]) Let $\phi \colon X \to Y$ be a quasi isometric embedding and $A_1, A_2, \dots$ be finite vertex sets and B a vertex set of X. Then

$$\operatorname{diam}(B) < \infty \quad \Leftrightarrow \quad \operatorname{diam}(\phi(B)) < \infty$$

and

$$\lim_{n\to\infty} \operatorname{diam}(A_n) = \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \operatorname{diam}(\phi(A_n)) = \infty.$$

Proof. Since ϕ is a quasi isometric embedding we get for $x, y \in B$

$$d(\phi(x), \phi(y)) \le A d(x, y) + B$$

and

$$d(x, y) \le A d(\phi(x), \phi(y)) + B,$$

which proves the first assertion.

Let $\lim_{n\to\infty} \operatorname{diam}(A_n) = \infty$ and $x, y \in A_n$. Similar to above we obtain

$$d(\phi(x), \phi(y)) \ge A^{-1} d(x, y) - B,$$

and because x, y were arbitrary we conclude

$$\operatorname{diam}(\phi(A_n)) \ge A^{-1} \operatorname{diam}(A_n) - B.$$

This shows that $\lim_{n\to\infty} \operatorname{diam}(\phi(A_n)) = \infty$.

The other direction is treated analogously and hence the assertion is proven. $\square$

For the following one may recall Theorem 3.2.8.

5.1.16. *Proposition.* Let X and Y be quasi isometric, locally finite connected graphs. Then the thickness of ends is preserved under the homeomorphism of ΩX onto ΩY which is induced by the quasi isometry.

Intuitively it is clear how to 'distinguish' a thick from a thin end if one has a 'satisfying picture' of a graph. In the next subsection we want to develop a technique how to 'detect' thick and thin ends rigorously with the help of group actions.

5.2. **Elliptic, parabolic and hyperbolic elements.** The following notions and statements were introduced in [13], where we will follow [20] and [27].

5.2.1. *Definition.* Let G act on a graph X. An element $g \in G$ is called *elliptic* if it fixes a finite set of vertices $F \subseteq VX$, i.e. $g(F) = F$. Note that g fixes F setwise and *not* pointwise.

Moreover if $x \in VX$ we call the set $\{g^i(x) \mid i \in \mathbb{Z}\}$ the *orbit of g in x.*

5.2.2. *Lemma.* Let G act on a graph X and $g \in G$.

(1) If g is elliptic, then g^j is elliptic for $j \in \mathbb{Z}$.
(2) If g is a torsion element, then g is elliptic.
(3) If the action is free and g is elliptic, then g is a torsion element.

Proof. Since g is elliptic there exists a finite set F such that $g(F) = F$, which is equivalent with $F = g^{-1}(F)$. Further we have

$$g^2(F) = g(g(F)) = g(F) = F$$

and by induction $g^n(F) = F$ for all $n \in \mathbb{N}$. Similarly we obtain $g^{-n}(F) = F$, hence (1).

Let g be a torsion element, i.e. there exists $n \in \mathbb{N}$ such that $g^n = e$. Let $x \in VX$ and $F = \{g^j(x) \mid 1 \leq j \leq n\}$ be the orbit of g then obviously $g(F) = F$, hence (2).

Let g be elliptic and $F = \{f_1, f_2, \ldots, f_n\}$ with $g(F) = F$. Like in (1) we have $g^j(F) = F$ for all $j \in \mathbb{N}$. Consider $g^j(f_1) = f_{k_j}$ for some $k_j \in \{1, \ldots, n\}$. By the pigeon-hole principle we find some *distinct* a, b such that $k_a = k_b$. Hence $g^a(f_1) = g^b(f_1)$ and since the action is assumed to be free we get $g^{b-a} = e$, thus (3). $\square$

From the proof of item (3) of Lemma 5.2.2 we immediately deduce

5.2.3. *Corollary.* Let G act on a graph and let $g \in G$ be elliptic. Then there exists $x \in VX$ and $n \in \mathbb{Z}$ such that $g^n \in \mathrm{Stab}(x)$.

5.2.4. *Remark.* From the above Lemma 5.2.2 we conclude that elliptic elements are exactly torsion elements if we consider the action of a group on one of its Cayley graphs. Moreover non-elliptic elements must have infinite order. In that regard we have the following

5.2.5. *Lemma.* (see Lemma 12 in [20]) Let G act on a connected locally finite graph X and $g \in G$. The following are equivalent:

(1) g is not elliptic,
(2) there is a vertex x such that the orbit of g in x is infinite, and
(3) for any vertex x the orbit of g in x is infinite.

Proof. (1) $\Rightarrow$ (3): Suppose there exists a vertex x such that the orbit of g in x is finite and denote this set by F. Then clearly $g(F) = F$.

(3) $\Rightarrow$ (2): This is trivial.

(2) $\Rightarrow$ (1): Let $F \subseteq VX$ be finite. Since the orbit of g in x is infinite we conclude from Lemma 2.1.3 that it is unbounded. Thus there exists an m such that $d_X(x, g^m(x)) > 2(d_X(x,F) + \mathrm{diam}_X(F)$. Thus we get

$$d_X(x, g^m(x)) \leq d_X(x,F) + \mathrm{diam}_X(F) + d_X(F, g^m(F)) + \\ + \underbrace{\mathrm{diam}_X(g^m(F))}_{=\mathrm{diam}_X(F)} + \underbrace{d_X(g^m(F), g^m(x))}_{=d_X(x,F)}.$$

Hence we conclude

$$d_X(F, g^m(F)) \geq d_X(x, g^m(x)) - 2(d_X(x,F) + \mathrm{diam}_X(F)) > 0,$$

which implies that F and $g^m(F)$ are disjoint and therefore F is not fixed by g. $\square$

5.2.6. *Corollary.* (see Corollary 7 in [13]) If g is non-elliptic then g^k for $k \in \mathbb{Z}$, $k \neq 0$ is non-elliptic.

5.2.7. *Lemma.* (see Proposition 13 in [13]) If g is non-elliptic and h is an arbitrary automorphism, then $h^{-1}gh$ is non-elliptic.

Proof. Suppose the contrary, i.e. the existence of a finite F such that $h^{-1}gh(F) = F$. But this implies that g fixes the finite set $h(F)$, a contradiction. $\square$

5.2.8. *Definition.* A two-sided infinite path $(\dots, x_{-1}, x_0, x_1, \dots)$ of distinct vertices is called a *line*. A non-elliptic automorphism g has a *periodic line L with period* $p \geq 1$ if $g^p(L) = L$. We will also say that L is *g-periodic* (with period p).

A non-elliptic automorphism g acts on a periodic line $L = (\dots, x_{-1}, x_0, x_1, \dots)$ with period p as a non-trivial translation in the sense that there is a positive integer k such that $g^p(x_i) = x_{i+k}$.

In [7] it is noted that the following theorem from [13] is stated for locally finite graphs, where the proof makes no use of this assumption.

5.2.9. *Theorem.* (see Theorem 7 in [13]) Every non-elliptic automorphism g of a connected graph X has a periodic line.

Proof. Let d be the minimal distance $d_X(x, g^n(x))$ for all vertices x and all integers $n \geq 1$. Choose a vertex x and an integer p such that this minimun is attained, i.e. $d_X(x, g^p(x)) = d$. Let π_0 be a (geodesic) path from x to $g^p(x)$ of length d and set $v_i = g^{ip}(x)$ for $i \in \mathbb{Z}$. Since g is non-elliptic we have $d > 0$ and $v_i \neq v_k$ for $i \neq k$. We set $\pi_i = g^{ip}(\pi_0)$, which is a (geodesic) path from v_i to v_{i+1} of length d and L the concatenation of all π_i for $i \in \mathbb{Z}$. Let us call for the moment the vertices v_i *boundary vertices* and all other vertices of L *inner vertices*. Suppose that $L = (\dots, x_{-1}, x_0, x_1, \dots)$ is not a line, i.e. there exist $i \neq j$ such that $x_i = x_j$. Since

g is non-elliptic x_i must be a boundary vertex and an inner vertex or an innervertex of two different π_l and π_m ($l < m$, because the π_i's are geodesics) at the same time. The first obviously gives a contradiction to the minimality of d. The latter gives $x_i \in \pi_m$ and by construction $g^{(m-l)p}(x_i) \in \pi_m$ which again contradicts minimality of d. Thus L is indeed a line. Finally $g^p(L) = L$ holds by construction. □

5.2.10. *Lemma.* (see Lemma 13 in [20] and Theorem 4 in [13]) Let G act on a connected locally finite graph X and $g \in G$ be non-elliptic. Further let R be a subray of a g-periodic line with period p such that $g^p(R) \subseteq R$. Then the end ω which contains R is invariant under g.

Proof. Let x be the initial vertex of R and $d = d_X(x, g(x))$. Since $g^p(R) \subseteq R$ we obtain $g^{ip}(x) \in R$ and $g^{ip+1}(x) = g(g^{ip}(x)) \in g(R)$ for all $i \geq 0$. Suppose the end ω which contains R is not invariant under g. Then there exists a ball $B(x, r$ which separates R and $g(R)$. Let C be the component which conatins ω and D be the component which contains $g(\omega)$ in the complement of $B(x, r)$. Therefore all but finitely many vertices $g^{ip}(x)$ are in C and all but finitely many vertices $g^{ip+1}(x)$ are in D. Since X is locally finite R and $g(R)$ are unbounded set, thus there is an integer $m \geq 1$ such that $g^{mp}(x) \in C \backslash B(x, d + r)$ and $g^{mp+1}(x) \in C \backslash B(x, d + r)$. Since every path from $g^{mp}(x)$ to $g^{mp+1}(x)$ must go through that ball, we conclude that $d_X(g^{mp}(x), g^{mp+1}(x)) > d = d_X(x, g(x))$, a contradiction □

5.2.11. *Definition.* Let G act on a connected locally finite graph X and let R be a ray such that $g^n(R) \subseteq R$ for some non-elliptic element of $g \in G$. Then we call the end $\mathcal{D}(g)$ which contains R the *direction* of g.

Moreover if $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ then g is called *parabolic*, and if $\mathcal{D}(g) \neq \mathcal{D}(g^{-1})$ then we call g *hyperbolic*.

Obviously it holds that $\mathcal{D}(g^k) = \mathcal{D}(g)$ for all $k > 0$.

The following lemma is a first way to detect hyperbolic elements and moreover thin ends.

5.2.12. *Lemma.* (see Lemma 14 in [20]) Let C have finite boundary and let g be an element such that $g(C \cup NC) \subseteq C$. Then g is hyperbolic, $\mathcal{D}(g)$ lies in C, $\mathcal{D}(g^{-1})$ lies in $VX \backslash C$ and $\mu(\mathcal{D}(g))$ and $\mu(\mathcal{D}(g^{-1}))$ are both less or equal $\text{diam}_X(NC)$, i.e. they are both thin.

Proof. We have $C \sqcup NC \supseteq C \supseteq g(C \sqcup NC) = g(C) \sqcup g(NC)$ since g is an automorphism. By induction we get

$$C \sqcup NC \supseteq C \supseteq g(C) \sqcup g(NC) \supseteq g(C) \supseteq g^2(C) \sqcup g^2(NC) \supseteq g^2(C) \supseteq \dots$$

and the sets $NC, g(NC), g^2(NC), \dots$ are pairwise disjoint. Since every path from a vertex x in NC to a vertex y in $g^m(NC)$ has to contain vertices of $g^j(NC)$ for $j = 0, \dots, m$ we obtain by induction

$$(*) \quad d_X(NC, g^m(NC)) \geq m,$$

which implies

$$(**) \quad d_X(NC, g^m(C \cup NC)) \geq m.$$

By $(*)$ for $x \in NC$, we have $d_X(x, g^m(x)) \geq m$, i.e. the orbit of g in x is infinite which implies that g is non-elliptic. Hence there exists a g-periodic line L with period p. Let y be an element of L and set $m = d_X(x,y)+1$. We get

$$d_X(g^m(x), NC) \geq d_X(x,y)+1 = d_X(g^m(x), g^m(y))+1,$$

thus $g^n(y) \in C$ for $n \geq m$. Let $i_0 \in \mathbb{Z}$ be such that $i_0 p \geq m$. Then $g^{ip}(y) \in C \cap L$, for $i \geq i_0$, i.e. C contains infinitely many elements of L. Since NC is finite and L is a line (consists of distinct vertices) we conclude that C contains a subray of L and $\mathcal{D}(g)$ lies in C.

Set

$$C_n = g^{np}(C \cup NC) \quad \text{for } n \in \mathbb{N}.$$

Let $z \in VX$ be arbitrary. Because of $(**)$ there is a positive integer m such that

$$d_X(NC, g^{mp}(C \cup NC)) > d_X(NC, z),$$

which implies that $z \notin g^{mp}(C \cup NC) = C_m$. We easily conclude that $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$ and $\mu(\mathcal{D}(g)) \leq \operatorname{diam}_X(NC)$.

The inclusion $g(C \cup NC) \subseteq C$ is equivalent with $g^{-1}(VX\backslash C) \subseteq VX\backslash(C \cup NC)$. If we set $D = VX\backslash(C \cup NC)$ we obtain $ND = NC$, $VX\backslash C = D \cup ND$ and $g^{-1}(D \cup ND) \subseteq D$.

By considering g^{-1} instead of g and D instead of C in the above considerations we obtain that $\mathcal{D}(g^{-1})$ lies in $D \subseteq VX\backslash C$ and $\mu(\mathcal{D}(g^{-1})) \leq \operatorname{diam}_X(ND) = \operatorname{diam}_X(NC)$. □

5.2.13. *Theorem.* (see Theorem 8 in [13]) Let X be a connected and locally finite graph and g be a non-elliptic element. Then there are no fixed ends under g other than $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$.

Proof. Let ω be an end which is fixed by g, which is different from $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ and L be a g-periodic line. Then there exists a finite connected set F which separates ω from $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ and contains an element of L (the case $\mathcal{D}(g) = \mathcal{D}(g^{-1})$ is not excluded). Let A be the component in the complement of F which contains ω. Since $F \cup A$ has only a finite intersection with L and g is non-elliptic there exists a k such that $g^k(F) \cap (F \cup A) = \emptyset$ (without loss $k > 0$). We set $C = VX\backslash(F \cup A)$. Since $F \cup A = VX\backslash C$ is connected and C contains $Ng^k(A) = g^k(NA) \subseteq g^k(F)$ we can apply Lemma 2.2.6. Therefore we have

$$g^k(A) \subseteq C \quad \text{or} \quad VX\backslash g(A) \subseteq C.$$

If $g^k(A) \subseteq C$ then g cannot fix ω. If $VX\backslash g^k(A) \subseteq C$ then $A \cup F = VX\backslash C \subseteq g^k(A)$ and thus $g^{-k}(A \cup F) \subseteq A$. Since F contains NA the above lemma implies that $\mathcal{D}(g^{-k})$ and hence $\mathcal{D}(g^{-1}$ must lie in A, which is impossible since A has only a finite intersection with L. □

5.2.14. *Remark.* The above theorem proves our earlier stated Theorem 2.3.6.

The above Lemma 5.2.12 stated that the directions of these certain hyperbolic elements are thin. Indeed this is true for any hyperbolic element and as one may assume the direction of a parabolic element is a thick end. The following theorem was proved in [13] in a stronger version including additional statemants on maximal sets of disjoint periodic lines.

5.2.15. *Theorem.* (see Theorem 9 in [13]) Let G act on a connected locally finite graph and $g \in G$ be non-elliptic. If g is hyperbolic then $\mathcal{D}(g)$ is thin and if g is parabolic then $\mathcal{D}(g)$ is thick .

Proof. (1): Suppose that g is hyperbolic and let F be a connected finite set which separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$. As before there exists a k such that $g^k(F) \cap F = \emptyset$, since g is non-elliptic. Let C be the component of $VX\backslash F$ which contains $g^k(F)$ (compare with the proof of Theorem 5.2.13). One of the directions $\mathcal{D}(g)$ and $\mathcal{D}(g^{-1})$ lies in C, the other in $VX\backslash C$. Since $VX\backslash C$ is connected and $Ng^k(C) = g^k(NC) \subseteq g^k(F) \subseteq C$ we can apply Lemma 2.2.6. Thus

$$g^k(C) \subseteq C \quad \text{or} \quad g^k(VX\backslash C) \subseteq C.$$

The latter is impossible since then g^k would not fix the direction which lies in $VX\backslash C$. Thus we can conclude that $g^k(NC \cup C) \subseteq g^k(F \cup C) = g^k(F) \cup g^k(C) \subseteq C$ and can therefore apply Lemma 5.2.12 which implies that $\mu(\mathcal{D}(g))$ and $\mu(\mathcal{D}(g^{-1})$ are both thin.

(2): Suppose the contrary, i.e. $\mathcal{D}(g)$ is thin. Let $(C_n)_{n\in\mathbb{N}}$ be a descending sequence of sets with finite boundary which contain $\mathcal{D}(g)$, such that the diameters of their boundaries are bounded by a finite number μ and $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$. Let L be a g-periodic line with period p. Since $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$, there is a set C_m which does not contain the whole line L. We have to distinguish two cases.

If $L\backslash C_m$ is infinite then $VX\backslash C_m$ must contain a subray of L, since NC_m is finite and L consists of distinct vetices. But then NC_m separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$, a contradiction.

If $L\backslash C_m$ is finite then set $d = \mathrm{diam}_X(L\backslash C_m)$. Since g is non-elliptic there is an integer i such that $d_X(g^{ip}(NC_m), NC_m) > d$. If $L\backslash g^{ip}(C_m)$ is infinite we conclude as above that $Ng^{ip}(C_m) = g^{ip}(NC_m)$ separates $\mathcal{D}(g)$ from $\mathcal{D}(g^{-1})$, a contradiction.

Let therefore $L\backslash g^{ip}(C_m)$ be finite. Let $z \in L\backslash C_m = A$ (note that this set cannot be empty) then every path from z to any point of $L\backslash A$ must pass through NC_m and/or $Ng^{ip}(C_m) = g^{ip}(NC_m)$. Thus since $d_X(g^{ip}(NC_m), NC_m) > d$ we conclude that $\mathrm{diam}_X(L\backslash g^{ip}(C_m)) > d$. On the other hand we have

$$\mathrm{diam}_X(L\backslash g^{ip}(NC_m)) = \mathrm{diam}_X(g^{-ip}(L\backslash g^{ip}(NC_m))) =$$

$$\mathrm{diam}_X(\underbrace{g^{-ip}(L)}_{=L}\backslash NC_m) = \mathrm{diam}_X(L\backslash NC_m),$$

which yields a contradiction and the theorem is proven. □

It is now a natural question to ask for the *existence* of non-elliptic elements, where the following theorem will give an answer. It is a strengthened version for locally finite graphs of Theorem 1 in [16]. Moreover in [27] and [7] it is noted that the crucial property of transitive group actions on which many arguments are based on is captured in the following

5.2.16. *Definition.* Let G act on a graph X. We say that G *shuffles* X if for every infinite set $C \subseteq VX$ with finite boundary and every finite set $F \subseteq VX$ there is an automorphism $g \in G$ such that $g(F) \subseteq C$.

If G acts transitive it is easy to see that G shuffles X. Moreover the same argument shows that it is true for abelsian group actions. Indeed we have already proved and used this fact (see the proof of Lemma 4.2.5 or the proof of Claim 1 in Theorem 4.2.8).

5.2.17. *Lemma.* Let G act on a connected locally finite graph X. Then G acts abelsian if and only if G shuffles X.

Proof. Suppose G acts abelsian and let F be finite and C be infinite with finite boundary, i.e. $C \cup \Omega C$ is an element of the base $\mathcal{B}(X)$. Set

$$C' = \{z \in C | d_X(z, NC) > \mathrm{diam}_X F\}.$$

If $z, w \in C \backslash C'$ we have

$$d_X(z, w) \leq d_X(z, NC) + d_X(NC, w) \leq 2\mathrm{diam}_X(F),$$

and since $\mathrm{diam}_X(F)$ is finite we obtain that the set $C \backslash C'$ is bounded and since X is locally finite this is equivalent with $C \backslash C'$ being finite. Thus C' has finite boundary and $\Omega C' = \Omega C$ and hence $C' \cup \Omega C'$ is also an element of the base $\mathcal{B}(X)$. Let $x \in F$ then by assumption there exists $g \in G$ such that $g(x) \in C'$ thus $g(F) \subseteq C$.

Suppose G shuffles X. Let $\omega \in \Omega X$ and $C \cup \Omega C$ be a base element which contains ω. Let $x \in VX$ be arbitrary. By assumption for every $F \subseteq VX$ finite there exists $g \in G$ such that $g(F) \subseteq C$. In particular this holds for $F := \{x\}$. $\square$

We have essentially used the same arguments for the proof of the following theorem in Lemma 4.3.10.

5.2.18. *Theorem.* (see Theorem 1 in [27] and compare with Theorem 1 in [16]) Let G act abelsian on a connected locally finite graph X. Suppose C is a infinite subset of VX with infinite complement and finite boundary. Then there exists a non-trivial $g \in G$ such that $g(C) \subseteq C$. Indeed we even get $g(NC \cup C) \subseteq C$, thus such a g is hyperbolic and $\mathcal{D}(g)$ lives in C.

Proof. Let F be finite connected set which contains NC and put $D = VX \backslash (C \cup F)$ which is infinite and has finite boundary. By assumption there exists $h \in G$ such that $h(F) \subseteq C$. If $h(C) \subseteq C$ we are done, hence let us suppose not. Similarly there exists $k \in G$ such that $k(F) \subseteq D$. If $k(D) \subseteq D$ then $(C \subseteq) C \cup F \subseteq k(C \cup F) = k(C) \cup k(F)$ and since $k(F)$ is disjoint from C we conclude that $k^{-1}(C) \subseteq C$ and we are done. Hence suppose

$$(*) \quad k(D) \not\subseteq D \quad \text{and} \quad h(C) \not\subseteq C.$$

Suppose that $F \cap h(C) = \emptyset$. By the assumption $(*)$ we conclude that $h(C) \subseteq D$ and we know that $h(F) \subseteq C$. But on the other hand since $F \cup C$ is connected (because F is and $NC \subseteq F$, note that C need not to be connected) and h is an automorphism $h(F \cup C) = h(F) \cup h(C)$ is connected, a contradiction. Thus we have $F \cap h(C) \neq \emptyset$. With the same arguments we conclude that $F \cap k(D) \neq \emptyset$. Moreover since F is connected and $h(F) \subset C$ and $k(F) \subset D$ and these sets disconnent X we even get $F \subseteq h(C)$ and $F \subseteq k(D)$, i.e. $F \subseteq h(C) \cap k(D)$.

Now similar to above $F \cup D$ is a connected subset of $VX \backslash h(F)$ (which is disconnected as above). Moreover since $F \cup D$ meets $h(C)$ we conclude that $F \cup D \subseteq h(C)$. Similarly we have $F \cup C \subseteq k(D)$. From these two inclusions we directly obtain

$$k^{-1}(C) \subseteq D \quad \text{and} \quad h^{-1}(D) \subseteq C.$$

Thus for $g := h^{-1}k^{-1} \in G$ we get

$$g(C) = h^{-1}(k^{-1}(C)) \subseteq h^{-1}(D) \subseteq C.$$

In each case we found $g \in G$ such that $g(C) \subseteq C$. In summary we had three cases. In the first case we assumed $h(C) \subseteq C$ and by assumption we had $h(F) \subseteq C$, i.e. for $g := h$ we conclude $g(F \cup C) \subseteq C$. In the second case we assumed $k(D) \subseteq D$ and obtained $k^{-1}(C) \subseteq C$. Moreover we get

$$F \subseteq C \cup F \subseteq k(C) \cup \underbrace{k(F)}_{\subseteq D},$$

and since F is disjoint from D we conclude $F \subseteq k(C)$. Thus for $g := k^{-1}$ we again obtain $g(F \cup C) \subseteq C$. In the third and last case we concluded $F \subseteq h(C) \cap k(D)$ and $h^{-1}(D) \subseteq C$. This however implies

$$(h^{-1}k^{-1})(F) \subseteq (h^{-1}k^{-1}h)(C) \cap \underbrace{h^{-1}(D)}_{\subseteq C} \subseteq C.$$

Thus for $g := h^{-1}k^{-1}$ we lastly obtain $g(F \cup C) \subseteq C$. Since $NC \subseteq F$ we can apply Lemma 5.2.12 which implies that g is hyperbolic and that $\mathcal{D}(g)$ lives in C. □

The following corollaries are immediate.

5.2.19. *Corollary.* Let X be connected locally finite abelsian graph with two ends. Then both ends are thin.

5.2.20. *Corollary.* Let G act abelsian on a connected locally finite graph with more than one end. Then G contains an element of infinite order.

Moreover we can prove the following (note however that in [25] the proof is a bit different)

5.2.21. *Lemma.* (cf Lemma 3.14 in [37] and Corollary 11.34 in [25]) Let G be a finitely generated group. Then G has two ends if and only if G contains $\mathbb{Z}$ as a finite index subgroup.

Proof. Suppose G contains an element h such that $H := \langle h \rangle \cong \mathbb{Z}$ and it has finite index in G. Thus H acts almost transitively on the Cayley graph of G with respect to some finite symmetric generating set. This implies that we find a ball B such that the H-translates of B cover the whole graph. One now easily constructs a quasi-isometry between H and G, i.e. G has two ends. (Indeed this is true for any subgroup of finite index.)

Conversely Let G have two ends. By the above corollary there exists $h \in G$ with infinite order. Moreover as in the proof of Lemma 4.3.10 we find a ball B such that $\bigcup_{i \in \mathbb{Z}} h^i(B)$ covers the whole graph. It is now immediate that $\langle h \rangle$ can only have at most finite index in G. □

A group G is called *torsion group* if all elements of G are torsion elements. In 1902, Willian Burnside asked the following natural question (see [25], p.120):

Does every finitely generated infinite group contain an element of infinite order?

Hence Burnside asked for the existence of a finitely generated infinite torsion group. Although it apperas to be an elementary question, it took over sixty years to resolve. For a detailed discussion of an explicit example see [25], Chapter 6.

5.2.22. *Corollary.* A finitely generated infinite torsion group cannot act abelsian on a connected locally finite graph with more than one end. In particular such groups have one end.

5.2.23. *Corollary.* Let X be connected locally finite abelsian graph with infinitely many ends. Then X has at least two thin ends.

Looking at the above corollary we can deduce even more from Theorem 5.2.18.

5.2.24. *Definition.* Let G act on a connected graph X. Then $\mathcal{H}(G)$ denotes the set of ends of X which are directions of some hyperbolic element of G.

5.2.25. *Corollary.* Let G act abelsian on a connected locally finite graph X with at least two ends. Then $\mathcal{H}(G)$ is dense in ΩX.

Proof. If X has two ends we already know that $\mathcal{H}(X) = \Omega X$. Hence let X have infinitely many ends. Let $C \cup \Omega C$ be a base element. Then NC has finite diameter and suppose $VX \backslash C$ is infinite (otherwise there is nothing to prove). Then by Theorem 5.2.18 there exists a hyperbolic element g such that $\mathcal{D}(g)$ lives in C,i.e. $\mathcal{D}(g) \in \Omega C$. Thus $\mathcal{H}(G)$ is dense in ΩX. □

We end this thesis with another two questions.

We already observed that for a connected graph X the automorphism group acts on ΩX. It is now natural to ask if we can identify the kernel (see also [38]).

5.2.26. *Definition.* An automorphism g of a graph X is said to be *bounded* if there is a natural number n such that $d_X(v, g(v)) \leq n$ for all $v \in VX$. The set of bounded automorphisms is denoted by $B(X)$.

It is straightforward to check that $B(X)$ forms a normal subgroup of $\mathrm{Aut}(X)$. Moreover one easily verifies that $B(X)$ lies in the kernel of the action of $\mathrm{Aut}(X)$ on ΩX. In [26] it was shown (see Theorem 6 in [26]) that the converse is true for locally finite connected transitive graphs with infinitely many ends. In [23] it is shown for connected (metrically) almost transitive graphs with more than one (metric) end (see Theorem 4.2 in [23] and the notion of *metric ends*). Since by Lemma 4.3.10 we know that 2-ended graphs are almost transitive if and only if they are abelsian we can formulate the following

5.2.27. *Conjecture.* Let X be a connected locally finite abelsian graph with infinitely many ends. Then $B(X)$ is the kernel of the action of $\mathrm{Aut}(X)$ on ΩX.

As it was mentioned in the introduction of section 2 Freudenthal was the first who introduced *ends of topological spaces*. It is now natural to ask 'to which extent' we can generalize the end theory for graphs for similar or analoguous theorems about arbitrary topological spaces, maybe getting some kind of Stalling's theorem for non-discrete groups acting on topological spaces. Maybe getting analoguous notions and results of Halin, such as thickness of ends and elliptic, parabolic and hyperbolic elements?

A. Linear Algebra

For the spectral properties of graphs (see section 1) we are only considering finite graphs and therefore finite matrices, thus it suffices to bear the lectures about linear algebra in mind. One should recall some basic results about *determinants, eigenvalues*, the *characteristic polynomial* and referring to this, *symmetric matrices*. This first part of the appendix should give a brief summary and refers basically to [6].

A.1. **Determinants.** Let $\mathbb{K}$ be a field. A function $F\colon M_n(\mathbb{K}) = \mathbb{K}^n \times \cdots \times \mathbb{K}^n \to \mathbb{K}$ is called *determinant function* if the following two conditions hold:

(1) F is linear in every column, i.e. for every $i = 1, \ldots, n$ and arbitrary fixed elements v_j, for $j \neq i$, $v_i \mapsto F(v_1, \ldots, v_i, \ldots, v_n)$ is a linear mapping $\mathbb{K}^n \to \mathbb{K}$.
(2) if $v_i = v_j$ for an $i \neq j$, then $F(v_1, \ldots, v_n) = 0$.

if additionally

(3) $F(I_n) = F(e_1, \ldots, e_n) = 1$

holds, F is called *normed determinant function* and for $A \in M_n(\mathbb{K})$ it will sometimes be denoted as $F(A) =: |A|$.

A.1.1. *Theorem of Existence.* For every field $\mathbb{K}$ and every $n \in \mathbb{N}$ there exists a normed determinant function $\det\colon M_n(\mathbb{K}) \to \mathbb{K}$.

Proof. $\det(A) := \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_j)$ proofs the assertion, whereat A_j denotes the submatrix of A by deleting the j-th column and row. $\square$

For $n \in \mathbb{N}$ let S_n denote the set of all bijections of the set $[n] := \{1, \ldots, n\}$ and for $\sigma \in S_n$, $\operatorname{sgn}(\sigma) \in \{-1, 1\}$ the well-defined *signum function*.

A.1.2. *Theorem of Uniqueness (The Leibniz formula).* Let $\mathbb{K}$ be a field, F be a determinant function and $I_n \in M_n(\mathbb{K})$ the identity matrix.

Then for $A = (a_{ij}) \in M_n(\mathbb{K})$ the following formula holds:

$$F(A) = F(I_n) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

In particular, $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$ and $F(A) = F(I_n) \det(A)$.

A.1.3. *Corollary.*

(1) A matrix $A \in M_n(\mathbb{K})$ is invertible, if and only if $\det(A) \neq 0$
(2) Let $A = (a_{ij}) \in M_n(\mathbb{K})$ be a triangular matrix. Then $\det(A) = a_{11} \cdots a_{nn}$.

A.1.4. *Theorem.* Let $\mathbb{K}$ be a field. Then

$$\det(A) = \det(A^T) \quad \text{and} \quad \det(AB) = \det(A)\det(B)$$

hold for every $A, B \in M_n(\mathbb{K})$, where A^T denotes the transpose of a matrix A.

A.2. **Eigenvalues, the characteristic polynomial and symmetric matrices.** Let $\mathbb{K}$ be a field, V a $\mathbb{K}$-vector space and $f\colon V \to V$ a linear mapping.

(1) An element $\lambda \in \mathbb{K}$ is called an *eigenvalue* of f if there exists a vector $v \in V$ with $v \neq 0$, such that $f(v) = \lambda v$.
(2) If λ is an eigenvalue of f, then $V_\lambda^f := \{v \in V : f(v) = \lambda v\}$ is called the *eigenspace* of λ.
(3) The linear mapping f is called *diagonalizable*, if there exists a basis $\{v_i\}$ of V, such that every v_i is an eigenvector of f.
(4) These terms are defined analogously for a matrix $A \in M_n(\mathbb{K})$, with the linear mapping $\mathbb{K}^n \to \mathbb{K}$, given by $x \mapsto Ax$.

A.2.1. *Proposition.* Let V be a finite-dimensional $\mathbb{K}$-vector space and $f\colon V \to V$ a linear mapping with its corresponding matrix $A \in M_n(\mathbb{K})$. Then $\lambda \in \mathbb{K}$ is an eigenvalue, if $\det(\lambda I_n - A) = 0$. If λ is an eigenvalue, then the eigenspace V_λ^f is the kernel of $\lambda\mathrm{id} - f$.

A.2.2. *Definition.* Let $\mathbb{K}$ be a field and $A \in M_n(\mathbb{K})$. Then $\det(tI_n - A) =: p_A(t)$ is called, the *characteristic polynomial* of A.

Motivating this definition by the above proposition, translating it into matrices: We know that $\lambda \in \mathbb{K}$ is an eigenvalue if $B = \lambda\mathbb{I} - A$, i.e. $b_{ii} = \lambda - a_{ii}$ and $b_{ij} = -a_{ij}$ for $i \neq j$, has determinant zero. Replace λ with a variable t: for every $i = 1, \dots, n$ we can regard $t - a_{ii}$ as a polynomial of degree 1 and for $i \neq j$, $a_{ij} \in \mathbb{K}$, as a polynomial of degree 0. Considering Theorem A.1.2, each summand is a product of polynomials of degree 1 or 0 and therefore a polynomial, all in all we observe that the characteristic polynomial is indeed a *polynomial*, i.e. $p_A \in \mathbb{K}[t]$.

A.2.3. *Corollary.* Let $A \in M_n(\mathbb{K})$ be an $n \times n$-matrix. The eigenvalues of the linear mapping $x \mapsto Ax$ are precisely the roots of the characteristic polynomial, i.e. $\lambda \in \mathbb{K}$ is an eigenvalue, if $p_A(\lambda) = 0$.

A.2.4. *Theorem.* Let $A \in M_n(\mathbb{K})$ with characteristic polynomial $p_A \in \mathbb{K}[t]$. Then $\deg(p_A) = n$, the leading coefficient of p_A is 1, the constant coefficient is $(-1)^n$ $\det(A)$ and the coefficient of t^{n-1} is $-(a_{11} + \cdots + a_{nn}) = -\mathrm{tr}(A)$ (where $\mathrm{tr}(A)$ denotes the so-called *trace* of the matrix A, that is, the sum of the diagonal elements).

Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Two matrices $A, B \in M_n(\mathbb{K})$ are called *similar*, if there exists an invertible matrix $T \in M_n(\mathbb{K})$, such that $B = TAT^{-1}$.

A.2.5. *Theorem.* Let $A, B \in M_n(\mathbb{K})$ be similar matrices. Then A and B have the same characteristic polynomial.

Proof. By Assumption there exists an invertible matrix $T \in M_n(\mathbb{K})$, such that $B = TAT^{-1}$. Let $\lambda \in \mathbb{K}$ be arbitrary, then the following simple equation holds: $T(\lambda I_n)T^{-1} = \lambda T I_n T^{-1} = \lambda I_n$. Therefore we get:

$$\lambda I_n - B = T(\lambda I_n)T^{-1} - TAT^{-1} = T(\lambda I_n - A)T^{-1}$$

Applying the *determinant* to the left and right side of the above equation and using Theorem A.1.4 (det is multiplicative), proofs the assertion. $\square$

A complex square matrix A is called *normal*, if it commutes with its conjugate transpose, which is denoted by A^*. If A is a real matrix then $A^* = A^T$. A square matrix A is called *symmetric*, if $A = A^T$. A symmetric matrix is normal because $A^T A = AA = AA^T$ and therefore it is diagonalizable by a unitary matrix. Furthermore, all eigenvalues of a symmetric matrix are *real*.

A.2.6. *Theorem of Cayley-Hamilton.* Let $\mathbb{K}$ be a field and $A \in M_n(\mathbb{K})$ with characteristic polynomial $p_A \in \mathbb{K}[t]$, then $p_A(A) = 0$.

For the following see [2], p.16.

A.2.7. *Definition.* Let $C \in M_n(\mathbb{K})$. Then C is called a *circulant matrix* if the i-th row of C is obtained from the first row of C by a cyclic shift of $i-1$ steps.

A.2.8. *Example.* The following matrix is circulant:

$$\begin{pmatrix} 1 & -6 & 5 & -4 \\ -4 & 1 & -6 & 5 \\ 5 & -4 & 1 & -6 \\ -6 & 5 & -4 & 1 \end{pmatrix}.$$

A.2.9. *Remark.* Let W be the circulant matrix with first row $(0,1,0,\ldots,0)$ and let C be an arbitrary circulant matrix with first row $(c_1, c_2, \ldots, c_n)$. Since W represents the cycle permutation $\begin{pmatrix} 1 & n & n-1 & \ldots & 3 & 2 \end{pmatrix}$ and powers of W correspond to concatenations of this permutation one easily obtains

$$C = \sum_{j=1}^{n} c_j W^{j-1}.$$

Since the eigenvalues of W are the roots of unity, hence $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega = \exp(\frac{2\pi i}{n})$, it follows that the eigenvalues of C are

$$\lambda_r = \sum_{j=1}^{n} c_j \omega^{(j-1)r}, \quad r = 0, 1, \ldots, n-1.$$

A.2.10. *Definition.* Let $A \in M_n(\mathbb{K})$ be a square matrix and M_{ij} the *minor* of A, i.e. the determinant of the matrix obtained from A by deleting the i-th row and j-th column. Now define the *cofactor matrix* C of A by $C_{ij} = (-1)^{i+j} M_{ij}$, whose entries are called *cofactors*. Then the *adjugate* of A is defined as $\operatorname{adj}(A) := C^T$.

A.2.11. *Proposition.* Let $A, B \in M_n(\mathbb{K})$ and m be a scalar. Then the following holds:

(1) $A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A) I_n$,
(2) $A^{-1} = \det(A)^{-1} \operatorname{adj}(A)$ if A is invertible,
(3) $\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A)$,
(4) $\operatorname{adj}(mA) = m^{n-1} \operatorname{adj}(A)$.

A.3. **The principle minors and Binet-Cauchy.** We close this first section of the appendix with two results, which, as I guess, haven't been shown in most lectures about linear algebra (at least not in mine). The second one, the *Theorem of Binet–Cauchy*, will play an essential role in the proof of the *Matrix-tree-theorem* (see 1.2.27).

A.3.1. *Definition.* Let $A \in M_n(\mathbb{K})$. A matrix B is called *submatrix of* A, if B is obtained from A by deleting a subset of the rows and the columns. A *principle minor* of A is the determinant of a submatrix B of A obtained by deleting a subset of the rows and the *same* subset of the columns (compare with the proof of Theorem A.1.1 and the notation A_j for deleting the j-th column and row from A).

A.3.2. *Proposition.* Let $A \in M_n(\mathbb{K})$ be a matrix and $p_A = t^n + c_1 t^{n-1} + \cdots + c_n$ its characteristic polynomial. For each $i \in \{1, 2, \ldots, n\}$, the number $(-1)^i c_i$ is the sum of those principle minors of A, which have i rows and columns.

Proof. Consider the Leibniz formula, $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$. Apply this formula to $p_B(t) = \det(B)$ with $B = tI - A$ and consider the entries of B. Therefore, getting c_i, the coefficient of t^{n-i}, one must take exactly $n-i$ times the '$t - a_{ii}$-entries' from the main diagonal of B. Looking at this formula, those entries fixed chosen and only considering those summands with the corresponding σ's, fixing those entries. Now $\prod(t - a_{jj})$ factors out all those summands, hence

$$\sum(\operatorname{sgn}(\sigma) b_{\sigma(l_1)l_1} \cdots b_{\sigma(l_i)l_i} \prod(t - a_{jj})) = \prod(t - a_{jj}) \sum(\operatorname{sgn}(\sigma) b_{\sigma(l_1)l_1} \cdots b_{\sigma(l_i)l_i}).$$

We want the coefficient of t^{n-i} hence expanding the product we are only interested in the several t's. Therefore we have

$$t^{n-i} \sum \operatorname{sgn}(\sigma) b_{\sigma(l_1)l_1} \cdots b_{\sigma(l_i)l_i},$$

where the sum represents the principle minor of B with i rows and columns, respectively. Looking at this submatrix one observes that there are still t's in the (sub) main diagonal. Calculating the determinant of this submatrix again with the above formula, one can split every summand into the '$-a_{jj}$ part' and the 't-part'. Because those summands with the 'extra t's' would change the exponent of our t^{n-i}, one only utilizes the corresponding principle minor of A for the coefficient of t^{n-i} (Account for the $(-1)^i$, which needs to factor out). Using this idea for all $n-i$ possible fixed choices for the '$t - a_{jj}$-entries' and the $(-1)^i$, which factors out, proofs the assertion. □

A.3.3. *Definition.* Let $A \in M_{n\times m}(\mathbb{K})$ be a matrix and let $D \subseteq [n]$ and $S \subseteq [m]$. Then $A_{D,S}$ denotes the submatrix of A containing only rows which are indexed by an element of D and columns which are indexed by an element of S. As a special case $A_{=,S}$ denotes the submatrix of A containing all rows of A and only columns which are indexed by S ($A_{D,=}$ is defined analogously).

A.3.4. *Example.* Consider $A \in M_{4\times 3}(\mathbb{R})$ and $D := \{1, 2, 4\}$ and $S := \{2, 3\}$. If

$$A = \begin{pmatrix} 2 & -4 & 8 \\ 9 & -1 & 5 \\ 6 & 7 & -3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{then} \quad A_{D,S} = \begin{pmatrix} -4 & 8 \\ -1 & 5 \\ 3 & 1 \end{pmatrix}.$$

A.3.5. *Theorem of Binet-Cauchy.* Let $m, n \in \mathbb{N}$ and $A \in M_{n\times m}(\mathbb{K})$ and $B \in M_{m\times n}(\mathbb{K})$ be matrices, then

$$\det(AB) = \sum_{\substack{S\subseteq\{1,2,\ldots,m\}\\ |S|=n}} \det(A_{=,S})\det(B_{S,=}).$$

Proof. Let $A = (a_{ij})$ and $B = (b_{ji})$ and $C = AB \in M_n(\mathbb{K})$. Then by definition of the matrix product we have $(c_{ij}) = \sum_{k=1}^m a_{ik}b_{kj}$. The determinant is multilinear, by definition, hence we get:

$$\det(C) = \begin{vmatrix} \sum_{k_1=1}^m a_{1k_1}b_{k_11} & \ldots & \sum_{k_n=1}^m a_{1k_n}b_{k_nn} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^m a_{nk_1}b_{k_11} & \ldots & \sum_{k_n=1}^m a_{nk_n}b_{k_nn} \end{vmatrix} =$$

$$= \sum_{k_1,k_2,\ldots,k_n=1}^m \begin{vmatrix} a_{1k_1}b_{k_11} & a_{1k_2}b_{k_22} & \ldots & a_{1k_n}b_{k_nn} \\ \vdots & \vdots & & \vdots \\ a_{nk_1}b_{k_11} & a_{nk_2}b_{k_22} & \ldots & a_{nk_n}b_{k_nn} \end{vmatrix} =$$

$$= \sum_{k_1,k_2,\ldots,k_n=1}^m b_{k_11}b_{k_22}\cdots b_{k_nn} \underbrace{\begin{vmatrix} a_{1k_1} & a_{1k_2} & \ldots & a_{1k_n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} & a_{nk_2} & \ldots & a_{nk_n} \end{vmatrix}}_{=:A^{\{k_1,\ldots,k_n\}}}.$$

If $k_i = k_j$ for $i \neq j$ $A^{\{k_1,\ldots,k_n\}} = 0$, hence we consider $D := (k_1, k_2, \ldots, k_n) \in [m]^n$ with $k_i \neq k_j$ for $i \neq j$. If $m < n$ such D cannot exist and we simply get $\det(C) = 0$. Therefore we are considering the case $n \leq m$.
Let $S := \{k_1, k_2, \ldots, k_n\}$ be a set and consider $D = (k_1, k_2, \ldots, k_n)$ fixed. Think of a n-tuple with the same entries as D, just on different positions. This n-tuple, say D', yields the same summand with a possibly different sign, because the sign might change after permuting the columns of the matrix. It's clear that an 'n-tuple with the same entries as D, just on different positions' can be described with a permutation, say $\sigma \in \mathrm{Sym}(S)$. The summand of D' has the same b_{k_jj}'s as the one of D and $A^{\{k_1,\ldots,k_n\}}$ is the same as $A^{\sigma(\{k_1,\ldots,k_n\})}$ just with a different sign, but this can clearly be written as $\mathrm{sgn}(\sigma)$. Therefore we can summarize all such summands and get

$$\sum_{\substack{(k_1,k_2,\ldots,k_n)\in[m]^n\\ k_i\neq k_j \text{ for } i\neq j}} b_{k_11}b_{k_22}\cdots b_{k_nn} \begin{vmatrix} a_{1k_1} & a_{1k_2} & \ldots & a_{1k_n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} & a_{nk_2} & \ldots & a_{nk_n} \end{vmatrix} =$$

$$= \sum_{\substack{\{k_1,k_2,\ldots,k_n\}=S\\ S\subseteq[m]}} \underbrace{\sum_{\sigma\in\mathrm{Sym}(S)} b_{\sigma(k_1)1}b_{\sigma(k_2)2}\cdots b_{\sigma(k_n)n}\mathrm{sgn}(\sigma)}_{=\det(B_{S,=})} \underbrace{\begin{vmatrix} a_{1k_1} & a_{1k_2} & \ldots & a_{1k_n} \\ \vdots & \vdots & & \vdots \\ a_{nk_1} & a_{nk_2} & \ldots & a_{nk_n} \end{vmatrix}}_{=\det(A_{=,S})}$$

which proofs the assertion. □

A.3.6. *Remark.* As it occurs in the above proof for $n > m$, $\det(AB) = 0$, and as a special case for $n = m$, we get $\det(AB) = \det(A)\det(B)$, so the *Theorem of Binet-Cauchy* is a generalization of the multiplication formula of the determinant.

A.3.7. *Example.* As this formula can be a little bit confusing, a simple example might help. Therefore let

$$C = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$$

We have $n = 2, m = 3$ and $\det(C) = 58 \cdot 154 - 64 \cdot 139 = 36$. By the above formula we could also calculate this with:

$$\det(C) = \sum_{\substack{S \subseteq \{1,2,3\} \\ |S|=2}} \det(A_{=,S}) \det(B_{S,=}) =$$

$$\det(A_{=,\{1,2\}}) \det(B_{\{1,2\},=}) + \det(A_{=,\{2,3\}}) \det(A_{\{2,3\},=}) + \det(A_{=,\{1,3\}}) \det(A_{\{1,3\},=})$$

$$= \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 9 & 10 \\ 11 & 12 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \begin{vmatrix} 7 & 8 \\ 11 & 12 \end{vmatrix}$$

$$= (-3) \cdot (-2) + (-3) \cdot (-2) + (-6) \cdot (-4) = 36.$$

B. Topology

In this second part of the appendix we present elementary notions of topology needed in this thesis. This is standard material which can probably be found in any book about topology and should be covered in an introductory course on topology. We will mainly follow [28] and [36]. See also [4] and [5] for a more geometric approach to topology.

B.1. **Topological spaces.**

B.1.1. *Definition.* A system $\mathcal{O}$ of subsets of a set X is called *topology* on X if the following holds:

(1) $X, \emptyset \in \mathcal{O}$.
(2) The union of sets of $\mathcal{O}$ is again in $\mathcal{O}$:

$$O_i \in \mathcal{O}, i \in \mathcal{I} \Rightarrow \bigcup_{i \in \mathcal{I}} O_i \in \mathcal{O}.$$

(3) The finite intersection of sets of $\mathcal{O}$ is again in $\mathcal{O}$:

$$O_1, \ldots, O_n \in \mathcal{O} \Rightarrow \bigcap_{i=1}^{n} O_i \in \mathcal{O}.$$

A *topological space* is a pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a topology on X. The elements of $\mathcal{O}$ are called *open sets* of $(X, \mathcal{O})$ and their complements are called *closed sets*. If no confusion can result about $\mathcal{O}$ we ommit it and say that X is a topological space.

B.1.2. *Definition.* Let $(X, \mathcal{O})$ be a topological space and $A \subseteq X$. Then the set $\{O \cap A \mid O \in \mathcal{O}\}$ defines a topology on A which is called *relative topologly* for A. Moreover in this case we call A a *subspace* of the topological space X.

B.1.3. *Definition.* The *interior* A° of a subset A of X is the biggest open set which is contained in A. Hence A° is the union of all open subsets of A. The *closure* $\overline{A}$ of A is the smallest closed set which contains A. Hence $\overline{A}$ is the intersection of all closed sets which contain A. The *(topological) boundary* ∂A of A is defined as $\partial A = \bar{A} \backslash A^\circ$. By definition ∂A is a closed set.

B.1.4. *Definition.* A subset A of a topological space X is called *dense* in X if $\overline{A} = X$. Note that A is dense if and only if for every $x \in X$ and any neighbourhood U of x there exists $a \in A$ such that $a \in U$. The space X is called *separable* if it contains a *countable, dense* subset.

B.1.5. *Examples.*

(1) The *indiscrete topology* on a set X is defined as $\mathcal{O}_{\text{ind}} := \{\emptyset, X\}$. The interior of any non-empty, proper subset is $\emptyset$ and the closure is X.
(2) The *discrete topology* on a set X is defined as $\mathcal{O}_{\text{dis}} := \mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the power set of X. Since by definition every set is open and closed we have $A = A^\circ = \overline{A}$ for any subset $A \subseteq X$.
(3) The *standard topology* or *euclidean topology* on $\mathbb{R}$ is defined as the union of all open intervals (a, b) with $a, b \in \mathbb{R}$. Consider the set $A = (a, b]$ for distinct $a, b \in \mathbb{R}$. Then $A^\circ = (a, b)$, $\overline{A} = [a, b]$ and $\partial A = \{a, b\}$. Moreover $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and dense in X.

The last example indicates the following

B.1.6. *Definition.* A system $\mathcal{B}$ of open sets of a topological space $(X, \mathcal{O})$ is called a *basis for the topology* $\mathcal{O}$ if any open set is a union of sets of $\mathcal{B}$, i.e. for any $x \in O$ with $O \in \mathcal{O}$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq O$.

B.1.7. *Proposition.* (see 2.4 in [28]) Let $(X, \mathcal{O})$ be a topological space and $\mathcal{B}$ be a basis.

(1) All topologies on X which have $\mathcal{B}$ as a basis coincide with $\mathcal{O}$.
(2) The basis $\mathcal{B}$ has the following properties:
    (a) The union of all sets of $\mathcal{B}$ is X.
    (b) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
(3) Conversely let $\mathcal{B}' \subseteq \mathcal{P}(X)$ with the two properties of item (2). Then $\mathcal{B}'$ is a basis for a unique topology on X.

B.1.8. *Definition.* Let $(X, \mathcal{O}_1)$ and $(Y, \mathcal{O}_2)$ be topological spaces.

A function $f\colon X \to Y$ is called *continuous* (with respect to $\mathcal{O}_1$ and $\mathcal{O}_2$) if $f^{-1}(O) \in \mathcal{O}_2$ for all $O \in \mathcal{O}_1$.

A function $f\colon X \to Y$ is called *homeomorphism* if f is bijective, continuous and its inverse is also continuous.

Obviously we have

B.1.9. *Lemma.* (cf 2.21 Satz in [28]) Let $(X, \mathcal{O}_1)$ and $(Y, \mathcal{O}_2)$ be topological spaces. A function $f\colon X \to Y$ is continuous if and only if for an arbitrary basis $\mathcal{B}$ of $\mathcal{O}_2$ it holds that $f^{-1}(B) \in \mathcal{O}_1$ for every $B \in \mathcal{B}$.

B.1.10. *Definition.* Let $(X, \mathcal{O})$ be a topological space and $x \in X$. A set $U \subseteq X$ is called *neighbourhood* of x if there exists an open set O such that $x \in O \subseteq U$. The set of all neighbourhoods of x is called *system of neighbourhoods* of x and is denoted by $\mathcal{U}(x)$.

The following lemma follows directly from the definitions.

B.1.11. *Lemma.* (see 2.7 Hilfssatz in [28]) The following are equivalent:

(1) O is open.
(2) O is a neighbourhood of all its points.
(3) for every $x \in O$ there exists $U \in \mathcal{U}(x)$ such that $U \subseteq O$.

B.1.12. *Definition.* Let $(X, \mathcal{O})$ be a topological space. A subsystem $\mathcal{B}(x)$ of $\mathcal{U}(x)$ is called a *basis of neighbourhoods* for x if for all $U \in \mathcal{U}(x)$ there exists $B \in \mathcal{B}(x)$ such that $B \subseteq U$.

B.1.13. *Definition.* A topological space $(X, \mathcal{O})$ is *first countable* if every point x admits a *countable* basis of neighbourhoods.

A topological space $(X, \mathcal{O})$ is called *second countable* if $\mathcal{O}$ admits a *countable* basis.

B.2. **Separation properties, convergence and covers.**

B.2.1. *Definition.* Let $(X, \mathcal{O})$ be a topological space.

(1) X is called T_0-*space* if one point out of any two distinct points has a neighbourhood which does not contain the other point.
(2) X is called T_1-*space* if any two distinct points have a neighbourhood which does not contain the other point.
(3) X is called T_2-*space* or *Hausdorff-space* if any two distinct points have disjoint neighbourhoods which do not contain the other point.
(4) X is called *regular* if whenever a set A is closed and a point x which is not in A there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. A regular T_1-space is called T_3-*space.*
(5) X is called *normal* if whenever A and B are closed sets there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. A normal T_1-space is called T_4-*space.*

The following lemma is an easy exercise.

B.2.2. *Lemma.* (cf 6.3 Satz in [28]) Let $(X, \mathcal{O})$ be a topological space and $x \in X$. Then X is a T_1-space if and only if every singleton $\{x\}$ is closed.

From the above lemma one immediately deduces the following implications:

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

B.2.3. *Definition.* A sequence $(x_n)_{n\in\mathbb{N}}$ of points in a topological space X is said to *converge to* $x \in X$ if for every neighbourhood U of x there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. The point x is called *limit point* and will be denoted

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x.$$

A point $x \in X$ is called *accumulation point* of a set $A \subseteq X$ if any neighbourhood U of x contains an element of A which is different from x.

B.2.4. *Definition.* A collection of subsets $(U_i)_{i\in\mathcal{I}}$ of a topological space X is called *open cover* if all U_i are open and

$$\bigcup_{i\in\mathcal{I}} U_i = X.$$

A subset of an open cover which is again an open cover is called *subcover*.

A topological space $(X, \mathcal{O})$ is called *Lindelöff* or *Lindelöff-space* if every open cover of X has a countable subcover.

A topological space $(X, \mathcal{O})$ is called *compact* if every open cover of X has a finite subcover, i.e.

$$\bigcup_{i\in\mathcal{I}} U_i = X \quad \text{and} \quad U_i \in \mathcal{O} \Rightarrow \exists \mathcal{I}' \subseteq \mathcal{I} \colon |\mathcal{I}'| < \infty, \bigcup_{i\in\mathcal{I}'} U_i = X.$$

Of course every compact space is Lindelöff.

B.2.5. *Proposition.* (see 8.6 Satz in [28])

(1) Every closed subset of a compact topological space is compact.
(2) Every compact subset of a Hausdorff-space is closed.

B.2.6. *Theorem.* (see 8.9 Satz and 8.10 Satz in [28])

(1) A compact Hausdorff-space is T_4.
(2) If X is compact and $f\colon X \to Y$ is continuous then $f(X)$ is compact.

B.2.7. *Theorem.* (see Theorem 16.9 in [36]) Let X be a second countable. Then X is Lindelöff and separable.

Proof. Let $\mathcal{B}$ be a countable basis and $(U_i)_{i\in\mathcal{I}}$ an open cover of X. Then for every x and U_i with $x \in U_i$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_i$. Then obviously the collections of these B's is a countable open subcover, i.e. X is Lindelöff.

For every $B \in \mathcal{B}$ choose a point $x_B \in B$. It is immediate that $\{x_B \mid B \in \mathcal{B}\}$ is a countable dense subset, i.e. X is separable. □

B.2.8. *Theorem.* (see Theorem 16.8 in [36]) A regular Lindelöff space is normal.

B.3. **Metric spaces.**

B.3.1. *Definition.* A *metric spaces* is a pair (M, d), where M is a set and d is a function from $M \times M$ to $\mathbb{R}$ such that for any $x, y, z \in M$:

(1) $d(x, x) = 0$ and $d(x, y) > 0$ for $x \neq y$,
(2) $d(x, y) = d(y, x)$ and
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Condition (1) is called *positive definiteness*, (2) *symmetry* and (3) is called the *triangle inequality*. The function d is called *metric*. A metric d is called *ultrametric* if it satisfies the *strong triangle inequality*, i.e.

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for } x, y, z \in M.$$

B.3.2. *Definition.* Let (M, d) be a metric space, $x \in M$ and $r > 0$ a real number The set

$$B_d(x, r) = \{y \in M \mid d(x, y) < r\}$$

is called the *open ball with center x and radius r*. It is evident that the set

$$\mathcal{O}_d = \{O \subseteq M \mid \forall x \in O \; \exists r > 0 \text{ such that } B_d(x, r) \subseteq O\}$$

defines a topology on M.

A topological space $(X, \mathcal{O})$ is called *metrizable* if there exists a metric d on X such that $\mathcal{O} = \mathcal{O}_d$.

B.3.3. *Urysohn's second Metrization theorem.* (see Theorem 10.13 in [28]) A compact Haudsorff space is metrizable if and only if it is second countable.

C. Group Theory

We end this appendix with elementary notions of group theory. This should be 'standard' material which can be found in many textbooks on this subject and are probably covered by any introductory course on group theory. We will mainly follow [25], [14] and [3].

C.1. **Groups and Group actions.**

C.1.1. *Definition.* A *group* $(G, e, \circ)$ is a set G, containing an element e, together with a function $\circ \colon G \times G \to G$, called *group operation*, such that for all $a, b, c \in G$

(1) $e \circ a = a \circ e = a$,
(2) $(a \circ b) \circ c = a \circ (b \circ c)$ and
(3) there exists $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

The element e is called *neutral element* and the element a^{-1} in item (3) is called the *inverse element* of a. An element $b \in G$ is called a *torsion element* if there exists $n \in \mathbb{N}$ such that $b^n := b \circ b \circ \cdots \circ b = e$.

Clearly the neutral element and the inverse element of an element a are unique. We will often just write ab instead of $a \circ b$. Moreover we will say that G is a group if no confusion can arise.

C.1.2. *Definition.* Let $(G, e, \circ)$ be a group. A subset H of G is called *subgroup* if $(H, e, \circ)$ is a group.

C.1.3. *Definition.* Let $(G, e, \circ)$ and $(H, f, \Box)$ be groups. A function $\phi \colon G \to H$ is called *homomorphism* if for all $a, b \in G$,

$$\phi(a \circ b) = \phi(a) \Box \phi(b).$$

It is called *isomorphism* if it is bijective and G and H are called *isomorphic*. The *kernel* of ϕ is defined as $\ker(\phi) := \phi^{-1}(f)$. Obviously $\ker(\phi)$ is subgroup of G and the image of ϕ, denoted by $\phi(G)$, is a subgroup of H.

C.1.4. *Examples.*

(1) Let M be a set. Then $\mathrm{Sym}(M)$, the set of all bijections $M \to M$ is a group with group operation the concatenation of functions and neutral element the identity id_M.
(2) The set $\mathrm{GL}_n(\mathbb{K})$ of all invertible $n \times n$ matrices with entries in $\mathbb{K}$ is a group with group operation the matrix multiplication and neutral element the identity matrix I_n.

C.1.5. *Definition.* An *action* of a group G on a set M is a homomorphism ϕ from G to $\mathrm{Sym}(M)$.

If $x \in M$ and $g \in G$ we will write $g \cdot x$ or $g(x)$ instead of $\phi(g)(x)$. Moreover we will call the set

$$\mathrm{Stab}(x) = \{g \in G \mid g(x) = x\}$$

the *stabilizer* of x. Trivially $\mathrm{Stab}(x)$ is a subgroup of G for any $x \in M$. We call an action *free* if all stabilizers are trivial, i.e. $\mathrm{Stab}(x) = e$ for all $x \in M$.

The set

$$G(x) = \{g(x) \mid g \in G\}$$

is called the *orbit* of x.

C.2. **Normal subgroups and quotients.** For the following definition observe that the intersection of subgroups is again a subgroup.

C.2.1. *Definition.* Let S be a subset of a group G. Then the intersection of all subgroups which contain S is the subgroup which is *generated* by S and will be denoted as $\langle S \rangle$. The set S is called *generating set* of G if it generates G. Moreover let S^{-1} be the set of all inverses of elements $s \in S$. The set S is called symmetric if $S = S^{\pm} := S \cup S^{-1}$, i.e. for every $s \in S$ it holds that $s^{-1} \in S$.

It is straightforward to show that

$$\langle S \rangle = \{s_1 s_2 \dots s_n \mid n \in \mathbb{N},\ s_i \in S^{\pm}\}.$$

Let $g \in G$ and H be a subgroup of G. Then $gH := \{gh \mid h \in H\}$ is called a *left coset* of H. Similarly one defines *right cosets* Hg. A subgroup H of G is called *normal*, denoted by $H \trianglelefteq G$, if its left and right cosets coincide, i.e. $gH = Hg$ for all $g \in G$. With the mapping $gH \mapsto Hg^{-1}$ one gets a bijection between the set of left cosets and the set of right cosets. The set of left cosets of H is denoted by G/H and its cardinality is called the *index* of H.

It is easily checked that the kernel of a homomorphism is a normal subgroup.

C.2.2. *Definition.* For subsets A and B of a group G we call

$$AB := \{ab \mid a \in A,\ b \in B\}$$

the *product of sets*. If H is normal subgroup it is easily checked that G/H becomes a group with group operation the product of sets and identity element H.

C.2.3. *Lemma.* Let $\phi\colon G \to H$ be a homomorphism of groups. Then

$$G/\ker(\phi) \cong \phi(G).$$

Proof. the isomorphism is given by the mapping $g\ker(\phi) \mapsto \phi(g)$. $\square$

C.3. **Free groups, Free products and Group presentations.**

C.3.1. *Definition.* Let S be a set. The *free group with base* S, denoted by F_S, is the set of all *reduced words* with letters in $S^{\pm}$. Therefore let $S = \{a_1, \dots, a_n\}$ then a word ω with letters in $S^{\pm}$ may look like $\omega = a_2 a_4^{-1} a_n a_{n-5}^{-1} a_1$ where a word is called *reduced* if it contains no subwords of the form $a_i a_i^{-1}$. Thus the above example shows a reduced word. With the *concatenation of words* (where we are regarding the property of the obtained word staying reduced in the obvious way) one easily shows that F_S is indead a group with S a generating set.

Moreover it is evident that every group G is isomorphic to a quotient group of a proper free group, in particular if S generates G we obtain $G \cong F_S/N$, where $N \trianglelefteq F_S$ is a normal subgroup. On can show (e.g. with abelianization) that free groups F_S and F_H are isomorphic if and only if $|S| = |H|$ and hence one speaks of *the free group of rank* n, where $n = |S|$, where it is denoted by F_n.

Let now $R \subseteq F := F_n$. Then the *normal closure* of R, denoted by R^F, is the smallest normal subgroup of F which conatins R. Hence we have

$$R^F = \bigcap_{\substack{N \trianglelefteq F,\\ R \subseteq N}} N.$$

As a sidenote we mention that it is a straightforward calculation to show that the normal closure is of the form

$$R^F = \{\prod_{i=1}^{k} f_i^{-1} r_i^{\pm 1} f_i | f_i \in F, r_i \in R, k \geq 0\}.$$

C.3.2. *Definition.* Let G be a group generated by the set $S \subseteq G$ and $R \subseteq F := F_S$. If $G \cong F/R^F$ we call R the set of *defining relations* or briefly *relations* of G. Moreover $\langle S|R\rangle$ is called a *presentation* of G. Roughly speaking G is the largest group that is generated by S and satisfies the relations.

C.3.3. *Examples.* The following examples should be intuitively clear, where it is not always obvious to show that the corresponding sets R are indeed sets of defining relations.

(1) $F_1 = \mathbb{Z} = \langle a|\rangle$ (compare Example 3.1.6).
(2) $\mathbb{Z}^2 = \langle a, b|aba^{-1}b^{-1} = 1\rangle$ (compare Example 3.1.3)
(3) For the *dihedral group* D_n we have $\langle a, b|a^2 = 1, b^2 = 1, (ab)^n = 1\rangle$.
(4) $\mathbb{Z}/n\mathbb{Z} = \langle a|a^n = 1\rangle$ (compare Example 2.3.2).
(5) $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle a, b|b^2 = 1, aba^{-1}b = 1\rangle$ (compare Example 3.1.3).

C.3.4. *Definition.* Let $(G_i)_{i\in\mathcal{I}}$ be a family of groups. Set $A = \bigsqcup_{i\in\mathcal{I}} G_i$ (disjoint union) and $W(A)$ the set of all words on A, i.e. finite sequences of elements of A. Let $\sim$ be the equivalence relation of $W(A)$ generated by

$$we_iw' \sim ww' \quad \text{where } e_i \text{ is the neutral element of } G_i \text{ for some } i \in \mathcal{I}$$

$$wabw' \sim wcw' \quad \text{whenever } a, b, c \in G_i \text{ with } c = ab \text{ for some } i \in \mathcal{I}$$

for all $w, w' \in W(A)$. It is straightforward to show that the quotient $W(A)/\sim$ obtains a group structure with group operation the concatenation of words and neutral element the empty word. It is called the *free product* of the groups G_i $(i \in \mathcal{I})$ and is denoted by

$$*_{i\in\mathcal{I}} G_i.$$

Since every element of the free product is represented by a unique reduced word we obtain

C.3.5. *Lemma.* (see Corollary 2 in [14]) Let $(G_i)_{i\in\mathcal{I}}$ and $G = *_{i\in\mathcal{I}} G_i$ be as above. For each $i_0 \in \mathcal{I}$ the canonical homomorphism

$$G_{i_0} \to G$$

is injective.

C.3.6. *Remark.* One can also define free groups as follows. Let X be a set. Then the *free group over* X is the free product of a family of copies of $\mathbb{Z}$ indexed by X.

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