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"Structure of Toeplitz C^*-algebras for pseudoconvex Reinhardt domains"

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Introduction

In this thesis, we study the C^*-algebra generated by Toeplitz operators with continuous symbols on the Bergman space over domains in $\mathbb{C}^n$. The primary goal will be to analyze the situation for bounded pseudoconvex Reinhardt domains in $\mathbb{C}^2$. In this introduction, we give a rough outline of the thesis.

Fix a bounded domain $\Omega \subseteq \mathbb{C}^n$. We denote by $A^2(\Omega)$ the vector space of square-integrable holomorphic functions on Ω. Thus, $A^2(\Omega)$ is a subspace of $L^2(\Omega)$ and we will see that this is a Hilbert space. We denote the orthogonal projection (the *Bergman projection*) by $P\colon L^2(\Omega) \to A^2(\Omega)$. If $M_f\colon L^2(\Omega) \to L^2(\Omega)$ denotes the multiplication operator with symbol $f \in L^\infty(\Omega)$, then $T_f := PM_f|_{A^2(\Omega)}\colon A^2(\Omega) \to A^2(\Omega)$ is called the *Toeplitz operator with symbol f*. The basic properties of the Bergman space and of Toeplitz operators can be found in sections 1.1 and 1.2, respectively. The primary object of study in this thesis is the *Toeplitz C^*-algebra*

$$\mathcal{T}(\Omega) := C^*(\{T_f : f \in C(\overline{\Omega})\}),$$

which is the C^*-subalgebra of the bounded linear operators $\mathscr{L}(A^2(\Omega))$ generated by the Toeplitz operators with *continuous* symbols.

In section 1.4 we study the Toeplitz C^*-algebra for Ω a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. The most important tool for this analysis is the solution to the $\overline{\partial}$-Neumann problem, which is the generalization of the Cauchy-Riemann equations to several complex variables. The crucial fact is that, under the assumptions on Ω, the operator which gives the solution to the $\overline{\partial}$-equation with minimal L^2-norm is compact. The main result about the Toeplitz C^*-algebra is then:

THEOREM ([**Rae78**, Theorem 3.1]). *Let $\Omega \subseteq \mathbb{C}^n$ with $n \geq 2$ be a bounded, strictly pseudoconvex domain with smooth (C^∞) boundary. The commutator ideal of $\mathcal{T}(\Omega)$ is precisely the ideal $\mathscr{K}(A^2(\Omega))$ of compact operators on the Bergman space. Moreover, there is an isomorphism of C^*-algebras*

$$\sigma\colon \mathcal{T}(\Omega)/\mathscr{K}(A^2(\Omega)) \xrightarrow{\cong} C(\partial\Omega)$$

such that $\sigma(T_f + \mathscr{K}(A^2(\Omega))) = f|_{\partial\Omega}$ for all $f \in C(\overline{\Omega})$.

The main focus of this thesis will be to arrive at a similar result for the case when Ω is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Thus, the assumptions of strict pseudoconvexity and smoothness of the boundary are dropped, but Ω is now required to have some symmetry. As we will see, the analysis for this case will get considerably more complicated, and it was carried out by Salinas, Sheu and Upmeier in [**SSU89**] using, among others, results from Curto and Muhly [**CM85**]. The idea is the following:

(i) Given Ω, a groupoid $\mathcal{G}_\Omega$ is constructed which contains geometric information about Ω. This information will be in terms of the *faces* of the logarithmic base Λ_Ω, see below.
(ii) It is shown that $\mathcal{T}(\Omega)$ is isomorphic to (a subalgebra of) the *groupoid C^*-algebra* of $\mathcal{G}_\Omega$.
(iii) The groupoid C^*-algebra is studied using their general theory while keeping track of the geometric data encoded in $\mathcal{G}_\Omega$.

For this reason, chapter 2 deals with the general theory of groupoid C^*-algebras, which was developed by Renault in [Ren80]. A *groupoid* is a set G together with some algebraic structure, namely an associative multiplication and a way to take the inverse of elements. In contrast to groups, the multiplication of a groupoid may not be defined on the whole of $G \times G$, so that we may talk about *composable pairs* of elements. Associated to a groupoid is a special subset $G^{(0)} \subseteq G$, called the *unit space*, which is the set of elements that are their own inverse. The most important examples of groupoids in these thesis will be obtained from actions of groups on sets, the so-called *transformation groupoids*.

We will mostly consider groupoids which are *locally compact*. By this we mean that G has a locally compact, second countable Hausdorff topology such that the multiplication and inversion are continuous. In order to do analysis on locally compact groupoids, one introduces the notion of a *(left) Haar system*, generalizing the Haar measure of groups, which is a system of (Radon) measures μ^u on G, where $u \in G^{(0)}$ runs through the unit space of G, that is continuous and left-invariant in a certain sense. Using such a Haar system, one can introduce a convolution operation on $C_c(G)$, the space of continuous functions on G with compact support, defined by

$$(f * g)(x) := \int_G f(xy)g(y^{-1})\, d\mu^{d(x)}(y),$$

where $d(x) := x^{-1}x$. Together with the involution $f^*(x) := \overline{f(x^{-1})}$, the space $C_c(G)$ becomes an involutive algebra and the norm

$$\|f\|_I := \max\left\{ \sup_{u \in G^{(0)}} \int_G |f|\, d\mu^u, \sup_{u \in G^{(0)}} \int_G |f^*|\, d\mu^u \right\}$$

is compatible with these operations in the sense that $\|f * g\|_I \leq \|f\|_I \|g\|_I$ and $\|f^*\|_I = \|f\|_I$ for all $f, g \in C_c(G)$. Using this structure, $C_c(G)$ can be equipped with a C^*-norm (that is, a norm satisfying $\|f * f\| = \|f\|^2$) by setting

$$\|f\|_{C^*(G,\mu)} := \sup\{\|\pi(f)\| : \pi \text{ is a } I\text{-norm continuous representation of } C_c(G)\}$$

for $f \in C_c(G)$. The completion of $C_c(G)$ with respect to this norm is called the *full groupoid C^*-algebra* and is denoted by $C^*(G, \mu)$ or simply $C^*(G)$. There is also a *reduced* groupoid C^*-algebra, $C^*_{\mathrm{red}}(G, \mu)$, which is obtained by only considering certain special representations in the above norm. Both have their advantages and it will turn out that they coincide for the groupoids we will be interested in.

Having dealt with the generalities of the theory of groupoid C^*-algebra, we move to chapter 3 to apply these techniques to the study of the Toeplitz C^*-algebra. The groupoid $\mathcal{G}_\Omega$ will be obtained by studying the character space (that is, the set of non-zero complex-valued homomorphisms) of the C^*-subalgebra $\mathcal{D}_\Omega$ of $\ell^\infty(\mathbb{Z}^2)$ generated by the *shifted weights*

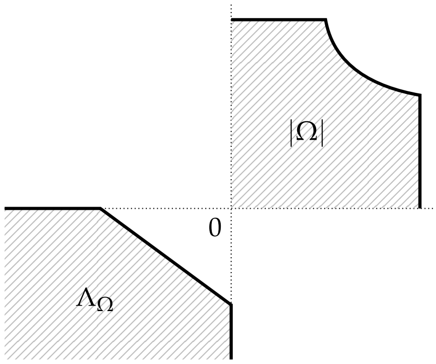
$$(\tau_\beta \chi_\gamma)(\alpha) := \begin{cases} \|z^{\alpha-\beta+\gamma}\|_{L^2(\Omega)} / \|z^{\alpha-\beta}\|_{L^2(\Omega)}, & \text{if } \alpha \in \mathbb{Z}^2_{\geq} \\ 0, & \text{if } \alpha \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq} \end{cases}$$

for $\beta \in \mathbb{Z}^2$ and $\gamma \in \mathbb{Z}^2_{\geq}$, where $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2}$. The reason why these weights are relevant is the following: by the assumptions on Ω, the monomials $e_\alpha := z^\alpha / \|z^\alpha\|_{L^2(\Omega)}$ form an orthonormal basis for $A^2(\Omega)$ and the Toeplitz operator with symbol z^β has the form $T_{z^\beta} e_\alpha = (\|z^{\alpha+\beta}\| / \|z^\alpha\|) e_{\alpha+\beta}$ with respect to this basis.

The character space $\mathrm{Spec}(\mathcal{D}_\Omega)$ of $\mathcal{D}_\Omega$ carries geometric information about Ω: as a Reinhardt domain, Ω is uniquely determined by its *absolute domain*, which is given by $|\Omega| := \{(|z_1|, |z_2|) : (z_1, z_2) \in \Omega\} \subseteq \mathbb{R}^2$. Consider the *logarithmic base*,

$$\Lambda_\Omega := \{(\log(y_1), \log(y_2)) : (y_1, y_2) \in |\Omega| \setminus \{0\}\}.$$

As Ω is pseudoconvex, the closure $\Lambda := \overline{\Lambda_\Omega} \subseteq \mathbb{R}^2$ is necessarily convex and we can consider the *faces*, that is, the convex subsets F of Λ which have the property that they contain the endpoints of every line segment whose interior intersects F.



The *proper* faces of Λ (that is, those which are not equal to Λ or $\emptyset$) are contained in the boundary of Λ and, since we work in two dimensions, are either points or line segments. Each proper face of Ω gives rise to certain characters of $\mathcal{D}_\Omega$, and we will denote these by $\mathrm{Spec}_F(\Omega) \subseteq \mathrm{Spec}(\mathcal{D}_\Omega)$. It will turn out that the sets $\mathrm{Spec}_F(\Omega)$ decompose (parts of) the character space of $\mathcal{D}_\Omega$ in the same way as the faces themselves decompose the boundary of Λ. Using this information and the general theory of groupoid C^*-algebra, the main result, Theorem 3.5.7, reads as the following:

THEOREM ([**SSU89**, Theorem 4.4]). *The C^*-algebra $\mathcal{T}(\Omega)$ contains ideals*

$$0 \subseteq \mathcal{J}_1 = \mathscr{K} \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_3 := C^*(\mathcal{G}_\Omega) \cong \mathcal{T}(\Omega)$$

such that

$$\mathcal{J}_3/\mathcal{J}_2 \cong C(\partial_2\Omega) \quad \text{and} \quad \mathcal{J}_2/\mathcal{J}_1 \cong \sum_F^{\oplus} C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}),$$

where F ranges over all one-dimensional faces of Λ and $\partial_2\Omega$ is the closure in $\mathbb{C}^2$ of the set of all pairs (se^x, te^y) such that $(s,t) \in \mathbb{T}^2$ and $\{(x,y)\}$ is a singleton face of Λ. Moreover,

$$C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}) \cong \begin{cases} C(\mathbb{T}) \otimes \mathscr{K}, & u_2/u_1 \in \mathbb{Q} \text{ or } u \in \{e_1, e_2\} \\ \mathcal{A}_{u_2/u_1} \otimes \mathscr{K}, & u_2/u_1 \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Here, $u \in \mathbb{R}^2$ is the outward pointing unit normal to F and $\mathscr{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. Finally, $\mathcal{A}_\alpha$ for irrational $\alpha \in \mathbb{R}$ is the irrational rotation algebra.

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CHAPTER 1

Toeplitz operators

This chapter introduces an important class of linear operators defined on the Bergman space $A^2(\Omega)$, which is the Hilbert space of holomorphic square integrable functions on an open subset Ω of $\mathbb{C}^n$. These so-called *Toeplitz operators* are defined by their *symbol*, which is a function in $L^\infty(\Omega)$. We will be specifically interested in the Toeplitz operators whose symbol is continuous on $\overline{\Omega}$ and we will study the C^*-subalgebra of the algebra of bounded operators on $A^2(\Omega)$ which they generate.

Also in this chapter, we will take a closer look at the Toeplitz C^*-algebras of bounded domains which are both strictly pseudoconvex and have smooth boundary, while the main chapter of this thesis, chapter 3, will deal with the more complicated case of Toeplitz operators on bounded pseudoconvex Reinhardt domains.

1.1. Bergman spaces

In this first section, we will look at the Bergman space of holomorphic functions on an open subset $\Omega \subseteq \mathbb{C}^n$ which are square-integrable with respect to Lebesgue measure. This space has several interesting properties, the most important one being its *reproducing property*. The Toeplitz operators which we will study in the following sections will be bounded linear operators on the Bergman space.

DEFINITION 1.1.1. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω with the topology of uniform convergence on compact subsets of Ω. More precisely, this is the topology on $\mathcal{O}(\Omega)$ induced by the seminorms $f \mapsto \sup_{z\in K} |f(z)|$ for $K \subseteq \Omega$ compact.

It is a basic fact that $\mathcal{O}(\Omega)$ is a Fréchet space (that is, a complete Hausdorff locally convex topological vector space whose topology is generated by a countable family of seminorms). In fact, $\mathcal{O}(\Omega)$ is a closed subspace of the Fréchet space $C(\Omega)$ of continuous functions on Ω by the Weierstrass convergence theorem.

DEFINITION 1.1.2. The *Bergman space over* Ω is the subspace of $L^2(\Omega)$ consisting of holomorphic functions,

$$A^2(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega) = \left\{ f\colon \Omega \to \mathbb{C} : f \text{ is holomorphic and } \int_\Omega |f|^2 \, d\lambda < \infty \right\},$$

where λ is the Lebesgue measure on $\mathbb{C}^n$. We put on $A^2(\Omega)$ the topology (and inner product $\langle f, g\rangle := \int_\Omega f\overline{g}\, d\lambda$) inherited from $L^2(\Omega)$.

By definition, the Bergman space is a subspace of $L^2(\Omega)$, even topologically. The next result shows that $A^2(\Omega)$ is also continuously included in $\mathcal{O}(\Omega)$.

PROPOSITION 1.1.3. *The inclusion of the Bergman space $A^2(\Omega)$ into the Fréchet space $\mathcal{O}(\Omega)$ is continuous, that is, for every compact subset $K \subseteq \Omega$ there is $c(K) > 0$ such that for all $f \in A^2(\Omega)$,*

$$\sup_{z\in K} |f(z)| \leq c(K)\|f\|_{L^2(\Omega)}. \tag{1.1.1}$$

PROOF. If $K \subseteq \Omega$ is compact, there exists $r(K) > 0$ such that $B(z, r(K)) \subseteq \Omega$ for all $z \in K$, where $B(z, r)$ is the open ball around z with radius r. Note that for any $f \in A^2(\Omega)$, the real and imaginary parts of f are harmonic. By the mean value property for harmonic functions and the Cauchy-Schwarz inequality, we get

$$|f(z)| = \frac{1}{\lambda(B(z,r(K)))}\left|\int_{B(z,r(K))} f\,d\lambda\right| \leq \lambda(B(z,r(K)))^{1/2}\|f\|_{L^2(B(z,r(K)))} \leq c(K)\|f\|_{L^2(\Omega)},$$

for all $z \in K$, where $c(K) := \lambda(B(z, r(K)))^{1/2} = \lambda(B(0, r(K)))^{1/2}$ depends only on K. ∎

COROLLARY 1.1.4. *For any compact subset $K \subseteq \Omega$, the restriction map $A^2(\Omega) \to C(K)$, $f \mapsto f|_K$ is a compact operator with respect to the L^2 norm on $A^2(\Omega)$ and the supremum norm on $C(K)$.*

PROOF. The statement follows from the Arzelà-Ascoli theorem if we can show that the image of the unit ball of $A^2(\Omega)$ under the restriction map,

$$\mathcal{F} := \{f|_K \colon f \in A^2(\Omega) \text{ and } \|f\|_{L^2(\Omega)} \leq 1\} \subseteq C(K),$$

is bounded and equicontinuous. Since K is compact, there is another compact subset L of Ω with $K \subseteq L$ and $d := \operatorname{dist}(K, \partial L) > 0$. Proposition 1.1.3 gives a number $c(L) > 0$ such that $\sup_{z \in L} |f(z)| \leq c(L)\|f\|_{L^2(\Omega)}$ for all $f \in A^2(\Omega)$. In particular, this shows that $\mathcal{F}$ is bounded. For $z, w \in K$ such that $|z - w| \leq d/2$ we have $[z, w] := \{tz + (1 - t)w : t \in [0, 1]\} \subseteq L$, since for $\xi \in [z, w]$ we have $d \leq \operatorname{dist}(z, \partial L) \leq \operatorname{dist}(z, \xi) + \operatorname{dist}(\xi, \partial L) \leq d/2 + \operatorname{dist}(\xi, \partial L)$, which implies $\operatorname{dist}(\xi, \partial L) \geq d/2$. It follows that the the ball $B_{d/2}(\xi)$ around ξ with radius $d/2$ is contained in L and hence so is the polydisc $P(\xi, ({}^{d}/_{2n}, \ldots, {}^{d}/_{2n}))$, see Example 3.1.2. By the mean value theorem, we get

$$|f(z) - f(w)| \leq |z - w| \sup_{\xi \in [z,w]} |Df(\xi)|$$

and the Cauchy estimates for the derivative of holomorphic functions (see [Ran86, Theorem 1.6]) give

$$|Df(\xi)| = \left(\sum_{i=1}^{n}\left(\left|\frac{\partial f}{\partial x_i}(\zeta)\right|^2 + \left|\frac{\partial f}{\partial y_i}(\zeta)\right|^2\right)\right)^{1/2} = 2\left(\sum_{i=1}^{n}\left|\frac{\partial f}{\partial z_i}(\zeta)\right|^2\right)^{1/2} \leq$$

$$\leq 2n\,\frac{2n}{d} \sup_{\zeta \in P(\xi, ({}^{d}/_{2n}, \ldots, {}^{d}/_{2n}))} |f(\zeta)| \leq \frac{4n^2}{d}\sup_{\zeta \in L}|f(\zeta)|$$

for $\xi \in [z, w]$. Altogether, we obtain

$$|f(z) - f(w)| \leq \frac{4n^2\,c(L)}{d}|z - w|\|f\|_{L^2(\Omega)}$$

for all $f \in A^2(\Omega)$ and $z, w \in K$ with $|z - w| \leq d/2$, which immediately implies equicontinuity of $\mathcal{F}$. ∎

THEOREM 1.1.5. *The Bergman space $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and, in particular, a Hilbert space.*

PROOF. Let $(f_n)_{n\geq 1}$ be a sequence in $A^2(\Omega)$ which converges to $g \in L^2(\Omega)$. Since $A^2(\Omega) \hookrightarrow \mathcal{O}(\Omega)$ is continuous by Proposition 1.1.3, we obtain the limit $f_n \to h$ as $n \to \infty$ in $\mathcal{O}(\Omega)$, with h holomorphic on Ω. By L^2-convergence, there exists a subsequence of $(f_n)_n$ which converges pointwise almost everywhere to g. Because the convergence in $\mathcal{O}(\Omega)$ implies pointwise convergence (everywhere) of that same subsequence, we conclude $g = h$ almost everywhere, hence also $f_n \to h$ in $L^2(\Omega)$ and $h \in A^2(\Omega)$. ∎

DEFINITION 1.1.6. The *Bergman projection* $P = P^{\Omega}$ is the orthogonal projection of $L^2(\Omega)$ onto its closed subspace $A^2(\Omega)$.

By (1.1.1), the point evaluations $\mathrm{ev}_z\colon A^2(\Omega) \to \mathbb{C}$, $f \mapsto f(z)$, for $z \in \Omega$ are continuous linear functionals. Hence, by the Riesz representation theorem, there is a unique holomorphic function $k_z \in A^2(\Omega)$ with $f(z) = \langle f, k_z \rangle$ for all $f \in A^2(\Omega)$.

DEFINITION 1.1.7. The function $K_\Omega\colon \Omega \times \Omega \to \mathbb{C}$, $K_\Omega(z,w) := \overline{k_z(w)}$ is called the *Bergman kernel for* Ω.

By construction, the Bergman kernel has the *reproducing property* for $A^2(\Omega)$, that is,

$$f(z) = \int_\Omega K_\Omega(z,w) f(w)\, d\lambda(w) \tag{1.1.2}$$

for all $f \in A^2(\Omega)$. More generally, we have

$$(Ph)(z) = \langle Ph, k_z \rangle = \langle h, Pk_z \rangle = \langle h, k_z \rangle = \int_\Omega K_\Omega(z,w) h(w)\, d\lambda(w)$$

for all $h \in L^2(\Omega)$. Hilbert spaces of functions for which point evaluations are continuous are called *reproducing kernel Hilbert spaces*. Note that by construction, the Bergman kernel is antiholomorphic in the second argument, since $\overline{K_\Omega(z,\cdot)} = k_z \in A^2(\Omega)$. Moreover, by the reproducing property (1.1.2),

$$\overline{K_\Omega(z,w)} = \int_\Omega K_\Omega(w,\cdot)\overline{K_\Omega(z,\cdot)}\, d\lambda = \overline{\int_\Omega K_\Omega(z,\cdot)\overline{K_\Omega(w,\cdot)}\, d\lambda} = \overline{\overline{K_\Omega(w,z)}} = K_\Omega(w,z),$$

so the Bergman kernel is conjugate symmetric and therefore holomorphic in the first argument. We record that the conjugate symmetry implies

$$(Ph)(z) = \langle h, \overline{K_\Omega(z,\cdot)} \rangle = \langle h, K_\Omega(\cdot,z) \rangle \tag{1.1.3}$$

for all $h \in L^2(\Omega)$ and $z \in \Omega$. Actually, these properties already fix the Bergman kernel completely:

PROPOSITION 1.1.8. *Any function $K\colon \Omega \times \Omega \to \mathbb{C}$ that reproduces $A^2(\Omega)$, is conjugate symmetric and such that $K(\cdot,w) \in A^2(\Omega)$ for every $w \in \Omega$ must coincide with K_Ω.*

PROOF. We have

$$K_\Omega(z,w) = \overline{K_\Omega(w,z)} = \int_\Omega K(z,\cdot)\overline{K_\Omega(w,\cdot)}\, d\lambda = \overline{\int_\Omega K_\Omega(w,\cdot)\overline{K(z,\cdot)}\, d\lambda} = \overline{\overline{K(z,w)}} = K(z,w). \qquad \blacksquare$$

Finally, we will need the following result on the diagonal behavior of the Bergman kernel:

LEMMA 1.1.9. *For every $z \in \Omega$, we have*

$$K_\Omega(z,z) = \|K_\Omega(\cdot,z)\|^2_{A^2(\Omega)} = \sup\{|f(z)|^2 : f \in A^2(\Omega) \text{ and } \|f\|_{A^2(\Omega)} \le 1\}.$$

In particular, since Ω is bounded, $K_\Omega(z,z) > 0$ for all $z \in \Omega$.

PROOF. By the reproducing property (1.1.2) and conjugate symmetry,

$$K_\Omega(z,z) = \langle K_\Omega(\cdot,z), \overline{K_\Omega(z,\cdot)} \rangle = \|K_\Omega(\cdot,z)\|^2_{A^2(\Omega}.$$

Next,

$$|f(z)|^2 = |\langle f, \overline{K_\Omega(z,\cdot)} \rangle|^2 \le \|f\|^2_{A^2(\Omega)} \langle \overline{K_\Omega(z,\cdot)}, \overline{K_\Omega(z,\cdot)} \rangle = \|f\|^2_{A^2(\Omega)} K_\Omega(z,z),$$

so $K_\Omega(z,z) \ge \sup\{|f(z)|^2 : f \in A^2(\Omega)$ and $\|f\|_{A^2(\Omega)} \le 1\}$. Now if $K(z,z) = \|K_\Omega(\cdot,z)\|^2_{A^2(\Omega)} = 0$, then by Cauchy-Schwarz

$$|f(z)| = |\langle f, K_\Omega(\cdot,z) \rangle| \le \|f\|_{A^2(\Omega)} \|K_\Omega(\cdot,z)\|_{A^2(\Omega)} = 0$$

for every $f \in A^2(\Omega)$ and the claim holds trivially. If $K_\Omega(z,z) \ne 0$, we can consider the function $\eta(\zeta) := \frac{K_\Omega(\zeta,z)}{\sqrt{K_\Omega(z,z)}}$ which is in $A^2(\Omega)$ and satisfies

$$|\eta(z)|^2 = K_\Omega(z,z) \quad \text{and} \quad \|\eta\|_{A^2(\Omega)} = \frac{\|K_\Omega(\cdot,z)\|_{A^2(\Omega)}}{\sqrt{K_\Omega(z,z)}} = 1.$$

Finally, because Ω is bounded, the constant function 1 is in $A^2(\Omega)$, so that $K_\Omega(z,z) \geq \lambda(\Omega)^{-2} > 0$ for all $z \in \Omega$. ∎

1.2. Toeplitz operators on bounded domains

We are now ready to introduce the Toeplitz operators. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, that is, Ω is open and connected. Under this assumption, all *multiplication operators* $M_f\colon L^2(\Omega) \to L^2(\Omega)$, $M_f h := fh$, with symbol $f \in L^\infty(\Omega)$ are bounded linear operators. For any Hilbert space H, we will denote the C^*-algebra of bounded linear operators on H by $\mathscr{L}(H)$, and we write $\mathscr{K}(H)$ for the algebra of compact operators on H.

DEFINITION 1.2.1. For $f \in L^\infty(\Omega)$, the *Toeplitz operator with symbol* f is the operator on $A^2(\Omega)$ defined by

$$T_f := PM_f|_{A^2(\Omega)}\colon A^2(\Omega) \to A^2(\Omega),$$

We also say that T_f is the *compression* of M_f to $A^2(\Omega)$.

Since, for $f \in L^\infty(\Omega)$, the multiplication operator M_f is bounded (with operator norm equal to $\|f\|_{L^\infty(\Omega)}$) and the Bergman projection P is a contraction, we see that T_f is bounded with operator norm $\|T_f\| \leq \|f\|_{L^\infty(\Omega)}$.

REMARK 1.2.2. By (1.1.2), we may view T_f as an integral operator,

$$(T_f\varphi)(z) = \int_\Omega K_\Omega(z,\cdot) f\varphi \, d\lambda, \quad \varphi \in A^2(\Omega).$$

REMARK 1.2.3. While we will only consider Toeplitz operators on the Bergman space, Toeplitz operators are perhaps more commonly known on the *Hardy space*. Specifically, if $\mathbb{T} := S^1$ is the boundary of the unit disc in $\mathbb{C}$, then the Hardy space $H^2(\mathbb{T})$ may be defined as the space of square-integrable functions on $\mathbb{T}$ whose negative Fourier coefficients vanish. The Hardy space is a closed subspace of $L^2(\mathbb{T})$ and if $Q\colon L^2(\mathbb{T}) \to H^2(\mathbb{T})$ denotes the orthogonal projection, then $T_f := QM_f|_{H^2(\mathbb{T})}$ is called the Toeplitz operator with symbol $f \in L^\infty(\mathbb{T})$. While the torus is the best known case, it is also possible to consider Hardy spaces and Toeplitz operators on more general and higher dimensional domains than the torus. We refer to [**Upm96**] for a survey.

DEFINITION 1.2.4. For $f, g \in L^\infty(\Omega)$, the *semi-commutator* of the Toeplitz operators T_f and T_g is the operator $T_fT_g - T_{fg}$ on $A^2(\Omega)$.

Note that for all $f, g \in L^\infty(\Omega)$, we can express the semi-commutator of the corresponding Toeplitz operators as

$$T_fT_g - T_{fg} = PM_fPM_g|_{A^2(\Omega)} - PM_{fg}|_{A^2(\Omega)} = PM_f(P-I)M_g|_{A^2(\Omega)}. \tag{1.2.1}$$

LEMMA 1.2.5. *Let H be a Hilbert space and $K \subseteq H$ a closed subspace. If $Q\colon H \to K$ denotes the orthogonal projection, then*

$$(T|_K)^* = QT^*\colon H \to K \quad \textit{and} \quad (QT|_K)^* = QT^*|_K\colon K \to K$$

for all operators $T \in \mathscr{L}(H)$.

The point of the Lemma is that the adjoints occurring are taken to be with respect to different Hilbert spaces. For example, in $(T|_K)^*$ we mean the adjoint operation from $\mathscr{L}(H,K)$ to $\mathscr{L}(K,H)$, where of course $\mathscr{L}(H,K)$ denotes the space of bounded linear operators from H to K.

PROOF. For $k \in H$ and $h \in H$, we have $\langle (T|_K)^*h, k\rangle = \langle h, T|_K k\rangle = \langle T^*h, k\rangle = \langle QT^*h, k\rangle$, so $(T|_K)^* = QT^*$. If $k, k' \in K$, then $\langle QTk, k'\rangle = \langle TQk, Qk'\rangle = \langle k, QT^*Qk'\rangle$, so $(QT|_K)^* = QT^*Q|_K = QT^*|_K$. ∎

Obviously, the map $L^\infty(\Omega) \to \mathscr{L}(A^2(\Omega))$ given by $f \mapsto T_f$ is linear. By Lemma 1.2.5, we see that

$$T_f^* = (PM_f|_{A^2(\Omega)})^* = PM_{\overline{f}}|_{A^2(\Omega)} = T_{\overline{f}} \tag{1.2.2}$$

for all $f \in L^\infty(\Omega)$, hence $f \mapsto T_f$ preserves the involutions. If $\varphi \in A^\infty(\Omega) := L^\infty(\Omega) \cap \mathcal{O}(\Omega)$, we have $T_f T_\varphi - T_{f\varphi} = 0$ by (1.2.1), since $(P - I)M_\varphi|_{A^2(\Omega)} = 0$. But of course, this rule will fail in the case where φ is not holomorphic. Thus, the semi-commutators will in general be non-zero, and the assignment $f \mapsto T_f$ is not an algebra homomorphism.

An interesting problem is the question of which symbols give rise to compact Toeplitz operators. For Toeplitz operators on the Hardy space over $\mathbb{T}$, see Remark 1.2.3, this is the case if and only if the symbol is trivial (meaning it vanishes almost everywhere), see for instance [**Mur90**, Theorem 3.5.8]. The next result shows that this is certainly not the case for the Bergman space:

PROPOSITION 1.2.6. *If $f \in C(\overline{\Omega})$ and $f|_{\partial\Omega} = 0$, then T_f is a compact operator on $A^2(\Omega)$.*

PROOF. We first consider the case where f is bounded and measurable but has compact support $K := \operatorname{supp}(f)$. The restriction to $A^2(\Omega)$ of the multiplication operator M_{χ_K} on $L^2(\Omega)$ with the characteristic function χ_K can be written as $M_{\chi_K}|_{A^2(\Omega)} = \iota_K \circ \rho_K$, where $\iota_K \colon C(K) \to L^2(\Omega)$ is defined by

$$\iota_K(g)(z) := \begin{cases} g(z), & z \in K \\ 0, & z \notin K \end{cases}$$

and $\rho_K \colon A^2(\Omega) \to C(K)$ is the restriction map. Obviously, ι_K is continuous and by Corollary 1.1.4, ρ_K is compact. Hence, $M_f|_{A^2(\Omega)} = M_{f\chi_K}|_{A^2(\Omega)} = M_f M_{\chi_K}|_{A^2(\Omega)}$ is also compact and so is the Toeplitz operator $T_f = PM_f|_{A^2(\Omega)}$.

Now let $f \in C(\overline{\Omega})$ satisfy $f|_{\partial\Omega} = 0$. We exhaust Ω by compact subsets K_m, $m \in \mathbb{N}$, such that $K_m \subseteq K_{m+1}^\circ$ and $\Omega = \bigcup_{m\in\mathbb{N}} K_m$. Fix $\varepsilon > 0$. Because f is uniformly continuous on the compact set $\overline{\Omega}$, we find $\delta > 0$ such that $|f(z) - f(w)| \leq \varepsilon$ if $|z - w| \leq \delta$. Let $M \in \mathbb{N}$ be such that $\operatorname{dist}(K_M, \partial\Omega) < \delta$. If $m \geq M$ and $z \in \overline{\Omega} \setminus K_m$, then there is $w \in \partial\Omega$ such that $|z - w| \leq \delta$ and hence $|f(z) - f(z)\chi_{K_m}(z)| = |f(z)| = |f(z) - f(w)| \leq \varepsilon$ since f vanishes on $\partial\Omega$. This implies that

$$\|f - f\chi_{K_m}\|_{L^\infty(\Omega)} \leq \varepsilon \quad \text{for } m \geq M.$$

But then $\|T_f - T_{f\chi_{K_m}}\|_{\mathscr{L}(A^2(\Omega))} \leq \|f - f\chi_{K_m}\|_{L^\infty(\Omega)} \to 0$ as $m \to \infty$ and hence T_f is the operator norm limit of compact operators and therefore itself compact. ■

In Corollary 1.4.17 we will see that the converse also holds provided that Ω is strictly pseudoconvex and with smooth boundary.

DEFINITION 1.2.7. The *(Bergman-) Toeplitz C^*-algebra over Ω* is the C^*-subalgebra of $\mathscr{L}(A^2(\Omega))$ generated by Toeplitz operators with symbols in $C(\overline{\Omega})$,

$$\mathcal{T}(\Omega) := C^*(\{T_f : f \in C(\overline{\Omega})\}).$$

For an overview of the basics of C^*-algebras, see Appendix C. Since $T_1 = \mathrm{id}_{A^2(\Omega)}$, where 1 is the constant function on Ω, we see that $\mathcal{T}(\Omega)$ is a unital C^*-algebra. The remainder of this thesis will ultimately be devoted to studying the structure of this C^*-algebra. The first observation is that, instead of considering T_f for *all* $f \in C(\overline{\Omega})$, the C^*-algebra $\mathcal{T}(\Omega)$ is already generated by the Toeplitz operators with monomial symbols:

PROPOSITION 1.2.8. *For $\alpha \in \mathbb{Z}_\geq^n$, let $T_\alpha := T_{z^\alpha}$ be the Toeplitz operator with the monomial symbol z^α, where $\mathbb{Z}_\geq^n := (\mathbb{Z}_\geq)^n$ and $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Then the Toeplitz C^*-algebra is generated by the T_α,*

$$\mathcal{T}(\Omega) = C^*(\{T_\alpha : \alpha \in \mathbb{Z}_\geq^n\}).$$

PROOF. Let $f \in C(\overline{\Omega})$. By the Stone-Weierstrass theorem, f is the uniform limit of functions of the form $z \mapsto \sum_{i=1}^{k} \overline{p_i(z)} q_i(z)$ with p_i and q_i polynomials. It then follows that T_f is the operator norm limit of the operators $\sum_{i=1}^{k} T_{p_i}^* T_{q_i}$ from which the claim follows. ∎

We have already observed in Proposition 1.2.6 that certain Toeplitz operators are compact. Our final goal for this section is to proof that all compact operators are contained in $\mathcal{T}(\Omega)$. A crucial ingredient for this is the following Proposition:

PROPOSITION 1.2.9. *The identity representation $\mathcal{T}(\Omega) \hookrightarrow \mathscr{L}(A^2(\Omega))$ of the Toeplitz C^*-algebra is irreducible.*

PROOF. We have to show that if $Q \in \mathscr{L}(A^2(\Omega))$ is an orthogonal projection with $QT_f = T_f Q$ for all $f \in C(\overline{\Omega})$, then $Q = 0$ or $Q = \mathrm{id}_{A^2(\Omega)}$, see Remark C.5.1. Let $g := Q(c_1)$, where c_1 denotes the constant function 1 on Ω. If $h, k \in A^2(\Omega)$, then

$$\langle g, \overline{h}k \rangle = \langle hg, k \rangle = \langle T_h g, k \rangle = \langle T_h Q(c_1), k \rangle = \\ = \langle QT_h(c_1), k \rangle = \langle Qh, k \rangle = \langle h, Qk \rangle = \langle h, T_k Q(c_1) \rangle = \langle h, kg \rangle = \langle \overline{g}, \overline{h}k \rangle.$$

In particular, $g - \overline{g}$ is orthogonal to every function of the form $z \mapsto \sum_{i=1}^{k} \overline{p_i(z)} q_i(z)$ with p_i and q_i polynomials. By the Stone-Weierstrass theorem and since $C(\overline{\Omega})$ is dense in $L^2(\Omega)$, we see that $g - \overline{g} = 0$. As g is a holomorphic function, this implies $g = c_a$ for some $a \in \mathbb{R}$. If $a \neq 0$, then

$$g = Q(c_1) = a^{-1} Q(c_a) = a^{-1} Q^2(c_1) = a^{-1} Q(c_1) = a^{-1} g = c_1.$$

Thus, either $g = 0$ or $g = 1$. Moreover,

$$Q(h) = Q(T_h(c_1)) = T_h Q(c_1) = T_h g$$

for all $h \in A^2(\Omega)$, which shows that either $Q = 0$ (in case $g = 0$) or $Q = \mathrm{id}_{A^2(\Omega)}$ (in case $g = 1$). ∎

COROLLARY 1.2.10. *For every bounded domain $\Omega \subseteq \mathbb{C}^n$, we have*

$$\mathscr{K}(A^2(\Omega)) \subseteq \mathrm{Comm}(\mathcal{T}(\Omega)) \subseteq \mathcal{T}(\Omega),$$

where $\mathrm{Comm}(\mathcal{T}(\Omega))$ is the commutator ideal of $\mathcal{T}(\Omega)$, that is, $\mathrm{Comm}(\mathcal{T}(\Omega))$ is the closed ideal of $\mathcal{T}(\Omega)$ generated by commutators of elements of $\mathcal{T}(\Omega)$.

PROOF. It follows from Proposition 1.2.9 and Theorem C.5.4 that $\mathscr{K}(A^2(\Omega)) \subseteq \mathcal{T}(\Omega)$ if we can produce a non-zero compact operator contained in $\mathcal{T}(\Omega)$. Let $z_0 \in \Omega$. From Lemma 1.1.9 we obtain an open subset $U \subseteq \Omega$ such that $K_\Omega(z_0, z) > 0$ for all $z \in U$. Let $f \in C_c(\overline{\Omega}, [0,1])$ be such that $\mathrm{supp}(f) \subseteq U$ and $f(z_0) = 1$, see Lemma A.2.3. By Proposition 1.2.6, T_f is compact since $f|_{\partial\Omega} = 0$ and using Remark 1.2.2, we get

$$(T_f 1)(z_0) = \int_\Omega K_\Omega(z_0, z) f(z)\, d\lambda(z) > 0$$

and hence $T_f \neq 0$, so $\mathscr{K}(A^2(\Omega)) \subseteq \mathcal{T}(\Omega)$.

Now let $\varphi\colon \mathcal{T}(\Omega) \to \mathbb{C}$ be any non-zero homomorphism. The restriction $\varphi|_{\mathscr{K}(A^2(\Omega))}$ is a homomorphism on the algebra of compact operators. The kernel of $\varphi|_{\mathscr{K}(A^2(\Omega))}$ is a closed ideal in $\mathscr{K}(A^2(\Omega))$, and any such ideal must be trivial, that is, either $\varphi|_{\mathscr{K}(A^2(\Omega))} = 0$ or $\varphi|_{\mathscr{K}(A^2(\Omega))}$ is injective.[1] The latter case can be dismissed since then $\dim(A^2(\Omega)) = 1$ which is not possible as $A^2(\Omega)$ contains the monomials as Ω is bounded.[2] Thus, $\mathscr{K}(A^2(\Omega)) \subseteq \ker(\varphi)$ for every homomorphism $\varphi\colon \mathcal{T}(\Omega) \to \mathbb{C}$ and hence $\mathscr{K}(A^2(\Omega)) \subseteq \mathrm{Comm}(\mathcal{T}(\Omega))$ follows from Lemma C.4.5. ∎

[1] In other words, the C^*-algebra $\mathscr{K}(H)$ is *simple*. We refer to [Mur90, Example 3.2.2] for a proof.

[2] There are, however, unbounded domains for which the Bergman space is nontrivial and finite dimensional, see for example [Wie84].

1.3. Toeplitz operators and biholomorphisms

In this section, we investigate how Toeplitz operators and the Toeplitz C^*-algebra transform when changing the domain by a biholomorphism. For two bounded domains Ω_1 and Ω_2 in $\mathbb{C}^n$, we denote by

$$P^{\Omega_1}\colon L^2(\Omega_1) \to A^2(\Omega_1) \quad \text{and} \quad P^{\Omega_2}\colon L^2(\Omega_2) \to A^2(\Omega_2)$$

the Bergman projections for Ω_1 and Ω_2, respectively. The corresponding Toeplitz operators will be denoted by

$$T_f^{\Omega_1} := P^{\Omega_1} M_f|_{A^2(\Omega_1)} \quad \text{and} \quad T_g^{\Omega_2} := P^{\Omega_2} M_g|_{A^2(\Omega_2)}$$

for $f \in L^\infty(\Omega_1)$ and $g \in L^\infty(\Omega_2)$. To investigate the transformation behavior of Toeplitz operators under biholomorphisms, we need the corresponding transformation formulas for the Bergman kernels and projections, the proofs of which are taken from [**Ran86**].

For a holomorphic map $F\colon \Omega_1 \to \Omega_2$ and $z_0 \in \Omega_1$, we denote by $F'(z_0)$ the matrix representation (with respect to the standard basis) of the complex linear derivative $DF(z_0)\colon \mathbb{C}^n \to \mathbb{C}^n$ of F at z_0. Thus, $F'(z_0)$ is the matrix

$$F'(z_0) = \begin{pmatrix} \frac{\partial F_1}{\partial z_1}(z_0) & \cdots & \frac{\partial F_1}{\partial z_n}(z_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial z_1}(z_0) & \cdots & \frac{\partial F_n}{\partial z_n}(z_0) \end{pmatrix}.$$

Viewing $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, and writing $z_k = x_k + iy_k$ for $1 \leq k \leq n$, we may also consider the matrix representation of $DF(z_0)$ with respect to the standard basis of $\mathbb{R}^{2n}$. We denote this real Jacobian matrix by $\mathcal{J}_{\mathbb{R}} F(z_0)$ and it is of course given by

$$\mathcal{J}_{\mathbb{R}} F(z_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(z_0) & \frac{\partial F_1}{\partial y_1}(z_0) & \cdots & \frac{\partial F_1}{\partial x_n}(z_0) & \frac{\partial F_1}{\partial y_n}(z_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial F_n}{\partial x_1}(z_0) & \frac{\partial F_n}{\partial y_1}(z_0) & \cdots & \frac{\partial F_n}{\partial x_n}(z_0) & \frac{\partial F_n}{\partial y_n}(z_0) \end{pmatrix}$$

The relation between the determinants of $\mathcal{J}_{\mathbb{R}} F(z)$ and $F'(z)$ is given by

$$\det \mathcal{J}_{\mathbb{R}} F(z) = |\det F'(z)|^2 \tag{1.3.1}$$

for all $z \in \Omega_1$, as can be seen by simple manipulations involving the Cauchy-Riemann equations, see [**Ran86**, Lemma I.2.1] for the details.

LEMMA 1.3.1. *For any biholomorphism $\Psi\colon \Omega_1 \to \Omega_2$, the map*

$$\Psi^\sharp\colon L^2(\Omega_2) \to L^2(\Omega_1), \quad g \mapsto (g \circ \Psi) \det \circ \Psi'$$

is unitary with inverse $(\Psi^\sharp)^{-1} = (\Psi^{-1})^\sharp$, and restricts to an unitary operator from $A^2(\Omega_2)$ to $A^2(\Omega_1)$.

PROOF. By the transformation formula for integrals and (1.3.1),

$$\int_{\Omega_2} |g|^2 \, d\lambda = \int_{\Omega_1} |g \circ \Psi|^2 |\det \circ \Psi'|^2 \, d\lambda = \int_{\Omega_1} |\Psi^\sharp g|^2 \, d\lambda, \quad (g \in L^2(\Omega_2))$$

we see that $\Psi^\sharp$ is an isometry which is obviously bijective, hence an isometric isomorphism from $L^2(\Omega_2)$ onto $L^2(\Omega_1)$. Because $(g \circ \Psi) \det \circ \Psi'$ is holomorphic as soon as g is, $\Psi^\sharp$ maps $A^2(\Omega_2)$ onto $A^2(\Omega_1)$. ∎

PROPOSITION 1.3.2. *We have $P^{\Omega_1} \circ \Psi^\sharp = \Psi^\sharp \circ P^{\Omega_2}$. More explicitly, for all $g \in L^2(\Omega_2)$,*

$$P^{\Omega_1}((g \circ \Psi) \det \circ \Psi') = (P^{\Omega_2}(g) \circ \Psi) \det \circ \Psi'.$$

PROOF. We first derive the corresponding transformation formula for the Bergman kernels K_{Ω_1} and K_{Ω_2}. By the reproducing property (1.1.2) of K_{Ω_2} and the fact that $\Psi^\sharp$ is an isometry, we have

$$\langle f, \Psi^\sharp K_{\Omega_2}(\cdot, w)\rangle = \langle (\Psi^{-1})^\sharp f, K_{\Omega_2}(\cdot, w)\rangle = ((\Psi^{-1})^\sharp f)(w) = f(\Psi^{-1}(w)) \det(\Psi'(\Psi^{-1}(w)))^{-1}$$

for all $f \in A^2(\Omega_1)$ and $w \in \Omega_2$. Writing $w = \Psi(z)$ with $z \in \Omega_1$, this implies

$$f(z) = \langle f, \Psi^\sharp K_{\Omega_2}(\cdot, \Psi(z))\rangle \det(\Psi'(z)) = \langle f, \det(\Psi'(\cdot))K_{\Omega_2}(\Psi(\cdot), \Psi(z))\overline{\det(\Psi'(z))}\rangle$$

for all $f \in A^2(\Omega_1)$. Therefore, the function $(\zeta, z) \mapsto \det(\Psi'(\zeta))K_{\Omega_2}(\Psi(\zeta), \Psi(z))\overline{\det(\Psi'(z))}$ has the reproducing property, is conjugate symmetric and holomorphic in the first argument. By the uniqueness of the Bergman kernel (Proposition 1.1.8), we therefore have

$$K_{\Omega_1}(\zeta, z) = \det(\Psi'(\zeta))K_{\Omega_2}(\Psi(\zeta), \Psi(z))\overline{\det(\Psi'(z))} \tag{1.3.2}$$

for all $\zeta, z \in \Omega_1$.

Using this and (1.1.3), we get

$$\begin{aligned} P^{\Omega_1}(\Psi^\sharp g)(z) &= \langle \Psi^\sharp g, K_{\Omega_1}(\cdot, z)\rangle = \langle \Psi^\sharp g, \Psi^\sharp (K_{\Omega_2}(\cdot, \Psi(z))\overline{\det(\Psi'(z))}\rangle = \\ &= \det(\Psi'(z))\langle g, K_{\Omega_2}(\cdot, \Psi(z))\rangle = \det(\Psi'(z))P^{\Omega_2}(g)(\Psi(z)) = \Psi^\sharp(P^{\Omega_2}(g))(z) \end{aligned}$$

for all $g \in L^2(\Omega_2)$, as desired. ∎

THEOREM 1.3.3. *The Toeplitz C^*-algebra is a biholomorphic invariant. More precisely, if Ω_1 and Ω_2 are two bounded domains such that there exists a biholomorphism $\Psi\colon \Omega_1 \to \Omega_2$, then the Toeplitz C^*-algebras $\mathcal{T}(\Omega_1)$ and $\mathcal{T}(\Omega_2)$ are isomorphic.*

PROOF. Using Proposition 1.3.2, the Toeplitz operators on $A^2(\Omega_2)$ can be written as

$$T_g^{\Omega_2} h = P^{\Omega_2}(gh) = (\Psi^\sharp)^{-1} P^{\Omega_1} \Psi^\sharp (gh) = (\Psi^\sharp)^{-1} P^{\Omega_1}\big((g \circ \Psi)\Psi^\sharp h\big) = (\Psi^\sharp)^{-1} T_{g\circ\Psi}^{\Omega_1} \Psi^\sharp h \tag{1.3.3}$$

for every symbol $g \in L^\infty(\Omega_2)$ and all $h \in A^2(\Omega_2)$. The unitary operator $\Psi^\sharp\colon L^2(\Omega_2) \to L^2(\Omega_1)$ gives rise to the isomorphism of C^*-algebras

$$\mathrm{conj}_{\Psi^\sharp}\colon \mathscr{L}(L^2(\Omega_2)) \xrightarrow{\cong} \mathscr{L}(L^2(\Omega_1)), \quad \mathrm{conj}_{\Psi^\sharp}(S) := \Psi^\sharp S (\Psi^\sharp)^{-1}.$$

By (1.3.3), we have

$$\mathrm{conj}_{\Psi^\sharp}(T_g^{\Omega_2}) = T_{g\circ\Psi}^{\Omega_1}$$

for every $g \in C(\overline{\Omega_2})$, so $\mathrm{conj}_{\Psi^\sharp}$ maps $\mathcal{T}(\Omega_2)$ into $\mathcal{T}(\Omega_1)$. Moreover, any $f \in C(\overline{\Omega_1})$ is of the form $g \circ \Psi$ for some $g \in C(\overline{\Omega_2})$ (simply take $g = f \circ \Psi^{-1}$), so $\mathrm{conj}_{\Psi^\sharp}$ bijectively maps the generators of $\mathcal{T}(\Omega_2)$ onto those of $\mathcal{T}(\Omega_1)$ and hence restricts to the desired isomorphism. ∎

1.4. The Toeplitz C^*-algebra of a strictly pseudoconvex domain

In this final section of this chapter, we will study the structure of the Toeplitz C^*-algebra for the case when Ω is a bounded strictly pseudoconvex domain with smooth boundary. The assumption of pseudoconvexity allows us to use the results on the $\overline{\partial}$-Neumann problem as presented below. It will turn out that the Toeplitz C^*-algebra has a relatively simple structure: its commutator ideal is precisely the ideal $\mathscr{K}(A^2(\Omega))$ of compact operators on the Bergman space (compare with Corollary 1.2.10) and the quotient C^*-algebra $\mathcal{T}(\Omega)/\mathscr{K}(A^2(\Omega))$ is isomorphic to $C(\partial\Omega)$, the algebra of continuous functions on the boundary of Ω.

1.4.1. The $\overline{\partial}$-Neumann problem. We now briefly introduce a (system of) partial differential equations which is important in complex analysis and illustrate how it can be solved using techniques from functional analysis. The following material can be found, for example, in [**Str10**] and [**Has14**].

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. We denote by $\mathcal{E}^{(p,q)}(\Omega)$ the space of smooth differential (p,q)-forms on Ω. Specializing to the case $p = 0$, any $\omega \in \mathcal{E}^{(0,q)}$ can be uniquely decomposed as

$$\omega = \sum_{1 \le j_1 < \cdots < j_q \le n} \omega_{j_1,\dots,j_q}\, d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \tag{1.4.1}$$

with smooth functions $\omega_{j_1,\dots,j_q}\colon \Omega \to \mathbb{C}$. Let $J = (j_1,\dots,j_q) \in \mathbb{N}^q$ satisfy $1 \le j_1 < \cdots < j_q \le n$ and denote the set of all such q-tuples by $\mathcal{I}_q$. Then we will write ω_J for $\omega_{j_1,\dots,j_q}$ and $d\overline{z}^J$ for $d\overline{z}_{j_1} \wedge \cdots \wedge \overline{z}_{j_q}$. Using these conventions, (1.4.1) becomes $\omega = \sum_{J \in \mathcal{I}_q} \omega_J d\overline{z}^J$. The expression

$$(\omega, \eta) := \sum_{J \in \mathcal{I}_q} \int_\Omega \omega_J \overline{\eta_J}\, d\lambda,$$

with λ being Lebesgue measure on $\mathbb{C}^n$, defines an inner product on $\mathcal{E}^{(0,q)}(\Omega)$ and we denote its Hilbert space completion by $L^2_{(0,q)}(\Omega)$.

DEFINITION 1.4.1. For $\omega \in \mathcal{E}^{(o,q)}(\Omega)$ a compactly supported smooth $(0,q)$-form, we set

$$\overline{\partial}\omega := \sum_{j=1}^{n} \sum_{J \in \mathcal{I}_q} \frac{\partial \omega_J}{\partial \overline{z}_j}\, d\overline{z}_j \wedge d\overline{z}_J$$

and we denote by $\overline{\partial}\colon \mathrm{dom}(\overline{\partial}) \subseteq L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ its maximal distributional extension, that is,

$$\mathrm{dom}(\overline{\partial}) = \{\omega \in L^2_{(0,q)}(\Omega) : \overline{\partial}\omega \in L^2_{(0,q+1)}(\Omega) \text{ in the sense of distributions}\}.$$

Because differential operators give continuous maps when extended to distributions, $\overline{\partial}$ is a closed, densely defined operator on $L^2_{(0,q)}(\Omega)$.

REMARK 1.4.2. Since $\overline{\partial}$ is a closed operator, it has a Hilbert space adjoint

$$\overline{\partial}^*\colon \mathrm{dom}(\overline{\partial}^*) \subseteq L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega),$$

with domain given by

$$\mathrm{dom}(\overline{\partial}^*) = \{\eta \in L^2_{(0,q+1)}(\Omega) : \omega \mapsto (\overline{\partial}\omega, \eta) \text{ is a continuous linear functional on } \mathrm{dom}(\overline{\partial})\}.$$

Moreover, again by closedness, $\ker(\overline{\partial}\colon L^2(\Omega) \to L^2_{(0,1)}(\Omega))$ is a closed subspace of $L^2(\Omega)$, which certainly contains the Bergman space $A^2(\Omega)$. Moreover, one can show that any distribution u satisfying $\overline{\partial}u = 0$ is actually a holomorphic function, see [**BG91**, Section 3.6], which implies that the kernel of $\overline{\partial}$ on functions (that is, $(0,0)$-forms) is precisely the Bergman space.

The goal is to solve the *inhomogeneous Cauchy-Riemann equations*, namely

$$\overline{\partial}\omega = \eta \tag{1.4.2}$$

for given $\eta \in L^2_{(0,q+1)}(\Omega)$. Since $\overline{\partial}^2 = 0$, a necessary condition for solvability of (1.4.2) is $\overline{\partial}\eta = 0$, that is, η is $\overline{\partial}$-closed. As it turns out, it is fruitful to consider the corresponding Laplacian,

$$\Box_q := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial},$$

defined on its natural domain

$$\mathrm{dom}(\Box_q) = \{\omega \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) : \overline{\partial}\omega \in \mathrm{dom}(\overline{\partial}^*) \text{ and } \overline{\partial}^*\omega \in \mathrm{dom}(\overline{\partial})\} \subseteq L^2_{(0,q)}(\Omega).$$

Using tools from the theory of unbounded operators, one shows that $\Box_q$ is a positive self-adjoint operator.

The idea now is to invert $\Box_q$ and use this inverse to construct solutions to (1.4.2). This problem of inverting $\Box_q$ is known as the $\bar{\partial}$-*Neumann problem*. Unfortunately, this is not possible in general and requires us to make certain geometric assumptions on the domain Ω.

DEFINITION 1.4.3. An open set $U \subseteq \mathbb{C}^n$ is called *pseudoconvex* if there is a strictly plurisubharmonic function $\varphi \in C^2(U, \mathbb{R})$ such that, for all $c \in \mathbb{R}$, the set $\varphi^{-1}((-\infty, c))$ is relatively compact in U. Such a function is called an *exhaustion function* for U.

Here, a C^2 function $\varphi\colon U \to \mathbb{R}$ is called *strictly plurisubharmonic* if

$$L_p(\varphi, t) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_k}(p) t_j \overline{t_k} > 0$$

for all $p \in U$ and all $t \in \mathbb{C}^n$, $t \neq 0$. There are several different conditions for an open set $U \subseteq \mathbb{C}^n$ to be pseudoconvex. If ∂U is of class C^2, then it can be shown that pseudoconvexity is equivalent to *Levi pseudoconvexity*. This means that if $r\colon U \to \mathbb{R}$ is a C^2 (global) defining function,[3] then $L_p(r, t) \geq 0$ for all $p \in U$ and all $t \in T_p^{\mathbb{C}}(\partial U)$, where $T_p^{\mathbb{C}}(\partial U)$ is the complex tangent space to U at $p \in U$, given by those $t \in \mathbb{C}^n$ with $\sum_{j=1}^n \frac{\partial r}{\partial z_j}(p) t_j = 0$. This condition is independent of the defining function r. Another very important characterization is that a domain in $\mathbb{C}^n$ is pseudoconvex if and only if it is a domain of holomorphy. For the proofs of these statements and more, we refer to textbooks on complex analysis in several variables, for example [Ran86] or [Kra01].

THEOREM 1.4.4. *If $\Omega \subseteq \mathbb{C}^n$ with $n \geq 2$ is a bounded pseudoconvex domain, then $\Box_q$ has a bounded, self-adjoint inverse N_q, called the* $\bar{\partial}$-Neumann operator. *Moreover,*

$$\bar{\partial} N_q \omega = N_{q+1} \bar{\partial} \omega \ (q \leq n-1) \quad \textit{and} \quad \bar{\partial}^* N_q \eta = N_{q-1} \bar{\partial}^* \eta \ (q \geq 2) \tag{1.4.3}$$

for all $\omega \in \operatorname{dom}(\bar{\partial})$ and $\eta \in \operatorname{dom}(\bar{\partial}^*)$ and, if $P_q\colon L^2_{(0,q)}(\Omega) \to \ker(\bar{\partial})$ denotes the orthogonal projection,

$$P_{q-1}|_{\operatorname{dom}(\bar{\partial})} = I - \bar{\partial}^* N_q \bar{\partial}. \tag{1.4.4}$$

The proof of Theorem 1.4.4 relies on the existence of a constant $C > 0$ such that

$$\|\omega\|^2 \leq C(\|\bar{\partial}\omega\|^2 + \|\bar{\partial}^*\omega\|^2)$$

for all $\omega \in \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*)$. This *basic estimate* implies that $\Box_q$ is injective and has dense range. The general theory of self-adjoint operators then implies that the Laplacian has a bounded self-adjoint inverse. We also remark that N_q is given by $j_q \circ j_q^*$, where $j_q\colon \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*) \hookrightarrow L^2_{(0,q)}(\Omega)$ is the embedding and $\operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*)$ is equipped with the inner product $Q(\omega, \eta) := (\bar{\partial}\omega, \bar{\partial}\eta) + (\bar{\partial}^*\omega, \bar{\partial}^*\eta)$, which yields a Hilbert space by the above estimate.

REMARK 1.4.5. The operator from (1.4.4) can be uniquely extended to a bounded operator on $L^2_{(0,q)}(\Omega)$. In fact, for $\omega \in \operatorname{dom}(\bar{\partial})$ we have by (1.4.3) that $\bar{\partial}^* N_q \bar{\partial} \omega = \bar{\partial}^* \bar{\partial} N_{q-1} \omega$. Since $\omega = \Box_{q-1} N_{q-1} \omega = \bar{\partial} \bar{\partial}^* N_{q-1} \omega + \bar{\partial}^* \bar{\partial} N_{q-1} \omega$ is an orthogonal decomposition, we have $\|\bar{\partial}^* N_q \bar{\partial} \omega\| = \|\bar{\partial}^* \bar{\partial} N_q \omega\| \leq \|\omega\|$ for all $\omega \in \operatorname{dom}(\bar{\partial})$, hence we can extend $\bar{\partial}^* N_q \bar{\partial}$ to $L^2_{(0,q-1)}(\Omega)$ and it follows that

$$I - P_{q-1} = \bar{\partial}^* N_q \bar{\partial} \tag{1.4.5}$$

as operators in $\mathscr{L}(L^2_{(0,q-1)}(\Omega))$.

[3] By this we mean that $U = \{z \in \mathbb{C}^n : r(z) < 0\}$ and $Dr(z) \neq 0$ for all $z \in U$. If U is bounded and with C^2 boundary, then such global defining functions always exist, see [Ran86, Lemma II.2.6].

COROLLARY 1.4.6. *Let $\Omega \subseteq \mathbb{C}^n$ with $n \geq 2$ be a bounded pseudoconvex domain. If $\eta \in \ker(\overline{\partial}_q)$, then $\overline{\partial}^* N_q \eta$ is the unique solution ω to (1.4.2) such that $\omega \perp \ker(\overline{\partial})$. In other words, $\overline{\partial}^* N_q \eta$ is the solution with minimal $L^2_{(0,q-1)}(\Omega)$ norm.*

PROOF. Indeed, since $\eta = \overline{\partial}\overline{\partial}^* N_q \eta + \overline{\partial}^* \overline{\partial} N_q \eta$, it remains to show that $\overline{\partial}^* \overline{\partial} N_q \eta = 0$. But $\overline{\partial}(\overline{\partial}^* \overline{\partial} N_q \eta) = \overline{\partial}\eta = 0$ and $\operatorname{im}(\overline{\partial}^*) = \ker(\overline{\partial})^\perp$, so $\overline{\partial}^* \overline{\partial} N_q \eta \in \ker(\overline{\partial}) \cap \ker(\overline{\partial})^\perp = \{0\}$. ■

DEFINITION 1.4.7. The operator $S_q := \overline{\partial}^* N_q|_{\ker(\overline{\partial})} \colon \ker(\overline{\partial}) \cap L^2_{(0,q)}(\Omega) \to L^2_{(0,q-1)}(\Omega)$ is called the *canonical solution operator* $\overline{\partial}$.

REMARK 1.4.8. As the composition of a closed and a bounded operator, S_q is again closed. By the closed graph theorem, S_q is therefore a bounded operator from the Hilbert space $\ker(\overline{\partial}) \cap L^2_{(0,q+1)}(\Omega)$ to $L^2_{(0,q)}(\Omega)$.

We will now turn to the question of when the operators N_q and S_q are compact. This will be important in the sequel as it will imply compactness of commutators of certain Toeplitz operators. The first result states that compactness of N_q is closely tied to compactness of the canonical solution operators:

PROPOSITION 1.4.9. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded pseudoconvex domain with $n \geq 2$. Then N_q is a compact operator if and only if both S_q and S_{q+1} are.*

While there seems to be no full characterization of compactness of N_q in terms of properties of Ω itself, there are some situations for which compactness has been shown.

DEFINITION 1.4.10. A bounded domain $\Omega \subseteq \mathbb{C}^n$ is called *strictly pseudoconvex* if there exists a neighborhood U of $\partial\Omega$ and a strictly plurisubharmonic function $\rho \colon U \to \mathbb{R}$ with $\Omega \cap U = \{z \in U : \rho(z) < 0\}$.

PROPOSITION 1.4.11. *If Ω is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary and $n \geq 2$, then the $\overline{\partial}$-Neumann operator N_q is compact for all $1 \leq q \leq n$.*

The reason for Proposition 1.4.11 is that under those assumptions, Ω satisfies a more general condition, namely Catlin's property (P), which has been shown to imply compactness. For a survey of these results, check [**BS99**] or [**Str10**].

1.4.2. The structure of the Toeplitz C^*-algebra. We now return to the topic of Toeplitz operators. The first step is to show that the commutators of Toeplitz operators with continuous symbols are compact. It is helpful to introduce the following class of operators, which are closely related to the Toeplitz operators:

DEFINITION 1.4.12. For $f \in L^\infty(\Omega)$, the operator

$$H_f := (I - P)M_f|_{A^2(\Omega)} \colon A^2(\Omega) \to L^2(\Omega)$$

is called the *Hankel operator* with symbol f.

LEMMA 1.4.13. *For every $f, g \in L^\infty(\Omega)$, we have $T_{\overline{f}} T_g - T_{\overline{f}g} = H_f^* H_g$.*

PROOF. Using Lemma 1.2.5, we can compute $H_f^* = (M_f|_{A^2(\Omega)})^*(I - P) = PM_{\overline{f}}(I - P)$, hence $H_f^* H_g = PM_{\overline{f}}(I - P)M_g|_{A^2(\Omega)}$, which by (1.2.1) is precisely the semi-commutator $T_{\overline{f}} T_g - T_{\overline{f}g}$ of Toeplitz operators. ■

The next Proposition relates compactness of the Hankel operators to both compactness of commutators of Toeplitz operators and the canonical solution operator to $\overline{\partial}$. It is a combination of results from [**SSU89**, Lemma 1.2] and [**Has08**, Theorem 3].

PROPOSITION 1.4.14. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Then the following statements are equivalent:*

(i) The Hankel operators $H_{\overline{z}_i}$ are compact for all $i \in \{1, \dots, n\}$.
(ii) The Hankel operators H_f are compact for all $f \in C(\overline{\Omega})$.
(iii) The semi-commutators $T_f T_g - T_{fg}$ are compact for all $f, g \in C(\overline{\Omega})$.
(iv) The commutators $[T_f, T_g] = T_f T_g - T_g T_f$ are compact for all $f, g \in C(\overline{\Omega})$.
(v) For all $i \in \{1, \dots, n\}$, the Toeplitz operators with symbol z_i are essentially normal, *that is, the* self-commutators *$[T_{z_i}^*, T_{z_i}]$ are compact.*[4]

If, in addition, $n \geq 2$ and Ω is pseudoconvex, then the above are also equivalent to

(vi) The operator $S_1|_{A^2_{(0,1)}(\Omega)} = \overline{\partial}^* N_1|_{A^2_{(0,1)}(\Omega)}$ is compact, where $A^2_{(0,1)}(\Omega)$ denotes the space of $(0,1)$-forms with coefficients in $A^2(\Omega)$.

PROOF. By Lemma 1.4.13, we have

$$[T_{z_i}^*, T_{z_i}] = T_{\overline{z}_i} T_{z_i} - T_{z_i} T_{\overline{z}_i} = T_{\overline{z}_i z_i} - T_{z_i} T_{\overline{z}_i} = -H_{\overline{z}_i}^* H_{\overline{z}_i}.$$

Consequently, the self-commutator $[T_{z_i}^*, T_{z_i}]$ is compact if and only if the Hankel operator $H_{\overline{z}_i}$ is, so (i) and (v) are equivalent. Next, suppose that $H_{\overline{z}_i}$ is compact for all $i \in \{1, \dots, n\}$ and let $f \in C(\overline{\Omega})$. Observe that

$$\begin{aligned} H_{gh} = (I-P)M_f M_g|_{A^2(\Omega)} &= (I-P)M_g P M_h|_{A^2(\Omega)} + \\ &+ (I-P)M_g(I-P)M_h|_{A^2(\Omega)} = H_g T_h + (I-P)M_g H_h \end{aligned} \tag{1.4.6}$$

for $g, h \in L^\infty(\Omega)$. Taking $g(z) = \overline{z}_i$ and $h(z) = \overline{z}_j$, we see that $H_{\overline{z}_i \overline{z}_j}$ is compact and the same holds for $H_{\overline{z}^\alpha}$ with $\overline{z}^\alpha := (\overline{z}_1)^{\alpha_1} \cdots (\overline{z}_n)^{\alpha_n}$ for $\alpha \in \mathbb{Z}^n_\geq$ by induction. Moreover,

$$H_{\overline{z}^\alpha z^\beta} = H_{\overline{z}^\alpha} T_{z^\beta}$$

for $\alpha, \beta \in \mathbb{Z}^n_\geq$ with $z^\beta := (z_1)^{\beta_1} \cdots (z_n)^{\beta_n}$, which follows from (1.4.6) and the fact that the Hankel operators with holomorphic symbol are zero. By the Stone-Weierstrass theorem, f can be uniformly approximated by linear combinations of $\overline{z}^\alpha z^\beta$, and using $\|H_f\| \leq \|f\|_{L^\infty}$ we conclude that H_f is also compact. This shows that (i) implies (ii). That (ii) implies (iii) follows from Lemma 1.4.13. If the semi-commutators $T_f T_g - T_{fg}$ are compact for all $f, g \in C(\overline{\Omega})$, then so are the commutators

$$[T_f, T_g] = (T_f T_g - T_{fg}) - (T_g T_f - T_{gf}).$$

Thus, (iii) implies (iv) and the step from (iv) to (v) is trivial since $T_{z_i}^* = T_{\overline{z}_i}$.

Now suppose that Ω is pseudoconvex. From (1.4.5), we know that

$$H_{\overline{z}_i} h = (I-P)(\overline{z}_i h) = \overline{\partial}^* N_1 \overline{\partial}(\overline{z}_i h) = \overline{\partial}^* N_1 (h\, d\overline{z}_i)$$

for all $h \in A^2(\Omega)$. Therefore,

$$\overline{\partial}^* N_1(\omega) = \sum_{i=1}^n H_{\overline{z}_i} \omega_i$$

for all $\omega = \sum_{i=1}^n \omega_i\, d\overline{z}_i \in A^2_{(0,1)}(\Omega)$. We conclude that that the restriction of $\overline{\partial}^* N_1$ to $A^2_{(0,1)}(\Omega)$ is compact if and only if all $H_{\overline{z}_i}$, $i \in \{1, \dots, n\}$, are. This shows that (i) and (vi) are equivalent. ∎

As the last ingredient before we can state the main result of this section, we need the following result from several complex variables, the proof of which can be found in [Ran86, Corollary VI.1.14]. For $K \subseteq \mathbb{C}^n$ compact, we denote by $\mathcal{O}(K)$ the space of functions which are defined and holomorphic on some open neighborhood of K.

[4] Equivalently, the projections of the T_{z_i} to the Calkin algebra $\mathcal{L}(A^2(\Omega))/\mathcal{K}(A^2(\Omega))$ are normal, hence the term "essentially normal."

LEMMA 1.4.15. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, strictly pseudoconvex domain with C^2 boundary. Then for every $z_0 \in \partial\Omega$ there exists a holomorphic function $\varphi \in \mathcal{O}(\overline{\Omega})$ such that $\varphi(z_0) = 1$ and $|\varphi(w)| < 1$ for all $w \in \overline{\Omega} \setminus \{z_0\}$. The function φ is called a* holomorphic peaking function for z_0 on $\overline{\Omega}$.

THEOREM 1.4.16 ([Rae78, Theorem 3.1]). *Let $\Omega \subseteq \mathbb{C}^n$ with $n \geq 2$ be a bounded, strictly pseudoconvex domain with smooth (C^∞) boundary. The commutator ideal of $\mathcal{T}(\Omega)$ is precisely the ideal $\mathcal{K}(A^2(\Omega))$ of compact operators on the Bergman space. Moreover, there is an isomorphism of C^*-algebras*

$$\sigma\colon \mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega)) \xrightarrow{\cong} C(\partial\Omega)$$

such that $\sigma(T_f + \mathcal{K}(A^2(\Omega))) = f|_{\partial\Omega}$ for all $f \in C(\overline{\Omega})$.

PROOF. It follows from Corollary 1.2.10 that $\mathcal{K}(A^2(\Omega)) \subseteq \mathrm{Comm}(\mathcal{T}(\Omega))$. By Propositions 1.4.11 and 1.4.9, the canonical solution operator S_1 to $\bar\partial$ is compact and Proposition 1.4.14 then implies that $\mathrm{Comm}(\mathcal{T}(\Omega)) \subseteq \mathcal{K}(A^2(\Omega))$ also holds. Now consider the map

$$\tau\colon C(\overline{\Omega}) \to \mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega)), \quad f \mapsto \pi(T_f),$$

where $\pi\colon \mathcal{T}(\Omega) \to \mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega))$ is the canonical projection. By Proposition 1.4.14, the semi-commutators of Toeplitz operators are compact, so that τ is a homomorphism of C^*-algebras which evidently is surjective. Let $\rho\colon C(\overline{\Omega}) \to C(\partial\Omega)$ denote the restriction map, $f \mapsto f|_{\partial\Omega}$. Since T_f is compact if $f|_{\partial\Omega} = 0$ (see Proposition 1.2.6), we see that $\ker(\rho) \subseteq \ker(\tau)$. Moreover, any continuous function on $\partial\Omega$ may be extended to a continuous function on Ω by the Tietze extension theorem, so ρ is surjective. By Lemma C.4.2, τ descends to a surjective C^*-homomorphism $\bar\tau\colon C(\partial\Omega) \to \mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega))$ with $\tau = \bar\tau \circ \rho$.

$$\begin{tikzcd}
C(\overline{\Omega}) \arrow[r, "\rho"] \arrow[d, "f \mapsto T_f"'] \arrow[dr, "\tau"] & C(\partial\Omega) \arrow[d, dotted, "\bar\tau"] \\
\mathcal{T}(\Omega) \arrow[r, "\pi"'] & \mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega))
\end{tikzcd}$$

It remains to show that $\bar\tau$ is injective, because then $\sigma := \bar\tau^{-1}$ has the desired properties. Since $\tau = \bar\tau \circ \rho$ and ρ is surjective, the kernel of $\bar\tau$ is $\rho(\ker(\tau))$. Thus, it suffices to show that, for $f \in C(\overline{\Omega})$, compactness of T_f implies $f|_{\partial\Omega} = 0$. Suppose that T_f is compact but there is $z_0 \in \partial\Omega$ with $f(z_0) \neq 0$. Since Ω is assumed to be strictly pseudoconvex with smooth boundary, there exists a holomorphic function $\varphi \in \mathcal{O}(\overline{\Omega})$ such that $\varphi(z_0) = 1$ and $|\varphi(w)| < 1$ for all $w \in \overline{\Omega} \setminus \{z_0\}$ by Lemma 1.4.15. Pick a countable neighborhood base $\{U_m\}_{m\in\mathbb{N}}$ of $\overline{\Omega}$ at z_0. By adding small enough constants to φ, we obtain holomorphic functions $\psi_m \in \mathcal{O}(\overline{\Omega})$, $m \in \mathbb{N}$, such that $|\psi_m(z_0)| > 1$ and $|\psi_m| < 1$ on $\overline{\Omega} \setminus U_m$. More precisely, set

$$\psi_m := \varphi + c_m \quad \text{with} \quad c_m := \frac{1}{2} \sup_{w \in \overline{\Omega}\setminus U_m} (1 - |\varphi(w)|) > 0.$$

Then $\psi_m(z_0) = 1 + c_m > 1$ and $|\psi_m| \leq |\varphi| + c_m < 1$ on $\overline{\Omega} \setminus U_m$. Next, choose natural numbers k_m with the property that $|\psi_m^{k_m}| < 1/m$ on $\overline{\Omega} \setminus U_m$ and $\|\psi_m^{k_m}\| \geq 1$, where we use the norm of $L^2(\Omega)$. Put $\varphi_m := \psi_m^{k_m}/\|\psi_m^{k_m}\|$. Given $\varepsilon > 0$ and an open neighborhood $U \subseteq \overline{\Omega}$ of z_0, we find $M \geq 1$ such that $U_M \subseteq U$ and $|\psi_M^{k_M}| \leq \varepsilon$ on $\overline{\Omega} \setminus U \subseteq \overline{\Omega} \setminus U_M$. Then

$$\sup_{w \in \overline{\Omega}\setminus U} |\varphi_m(w)| \leq \sup_{w \in \overline{\Omega}\setminus U} |\psi_m^{k_m}(w)| < \varepsilon$$

for $m \geq M$, so we see that $\varphi_m \to 0$ uniformly on the complement of any open neighborhood of z_0. For $\delta > 0$ it follows that

$$\begin{aligned}\|T_f\varphi_m - f(z_0)\varphi_m\|^2 &= \|P(f\varphi_m - f(z_0)\varphi_m)\|^2 \leq \|f\varphi_m - f(z_0)\varphi_m\|^2 = \\ &= \int_{V_\delta} |f - f(z_0)|^2|\varphi_m|^2\, d\lambda + \int_{\Omega\setminus V_\delta} |f - f(z_0)|^2|\varphi_m|^2\, d\lambda \leq \\ &\leq \delta + \|f - f(z_0)\|^2 \sup_{w\in\overline{\Omega}\setminus V_\delta} |\varphi_m(w)|\end{aligned}$$

where $V_\delta := \{z \in \Omega : |f(z) - f(z_0)| < \delta\}$ and we have used that $P\varphi_m = \varphi_m$. Therefore, $T_f\varphi_m \to f(z_0)\varphi_m$ as $m \to \infty$. Since $\|\varphi_m\| = 1$, by the Banach-Alaoglu theorem and after possibly passing to a subsequence, there is $\eta \in A^2(\Omega)$ with $\|\eta\| \leq 1$ such that $\varphi_m \to \eta$ weakly. By compactness, the image sequence under T_f converges in norm, $T_f\varphi_m \to T_f\eta$ in $A^2(\Omega)$. This gives

$$\langle T_f\eta - f(z_0)\eta, h\rangle = \lim_{m\to\infty}(\langle T_f\varphi_m, h\rangle - \langle f(z_0)\varphi_m, h\rangle) = \lim_{m\to\infty}\langle T_f\varphi_m - f(z_0)\varphi_m, h\rangle = 0$$

for every $h \in A^2(\Omega)$, so $T_f\eta = f(z_0)\eta$. Using this, we get

$$|f(z_0)|\|\varphi_m - \eta\| = \|f(z_0)\varphi_m - T_f\eta\| \leq \|f(z_0)\varphi_m - T_f\varphi_m\| + \|T_f\varphi_m - T_f\eta\|,$$

hence $\|\varphi_m - \eta\| \to 0$ as $m \to \infty$ since $f(z_0) \neq 0$. As a consequence, we find that $\|\eta\| = 1$. But a subsequence of $(\varphi_m)_m$ has to converge pointwise almost everywhere to η, meaning $\eta = 0$ in $A^2(\Omega)$, a contradiction. ∎

COROLLARY 1.4.17. *Under the assumptions of Theorem 1.4.16, the Toeplitz operator T_f with $f \in C(\overline{\Omega})$ is compact if and only if $f|_{\partial\Omega} = 0$.*

PROOF. This is just the injectivity of $\bar{\tau}$ together with Proposition 1.2.6. ∎

Let H and K be Hilbert spaces. We recall that a bounded linear operator $T \in \mathscr{L}(H, K)$ is called *Fredholm* if $\ker(T) \subseteq H$ is finite dimensional and the image $\operatorname{im}(T)$ has finite codimension in K. The *essential spectrum* of an operator $T \in \mathscr{L}(H)$ is the set

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator}\}.$$

The content of Atkinson's theorem (see for instance [**Mur90**, Theorem 1.4.16]) is that an operator $S \in \mathscr{L}(H)$ is Fredholm if and only if $S + \mathscr{K}(H)$ is invertible in the *Calkin algebra* $\mathscr{L}(H)/\mathscr{K}(H)$. Therefore, if $\pi\colon \mathscr{L}(H) \to \mathscr{L}(H)/\mathscr{K}(H)$ denotes the canonical projection, we have

$$\sigma_{\mathrm{ess}}(T) = \sigma_{\mathscr{L}(H)/\mathscr{K}(H)}(\pi(T)). \tag{1.4.7}$$

COROLLARY 1.4.18. *Under the assumptions of Theorem 1.4.16, the Toeplitz operator T_f with $f \in C(\overline{\Omega})$ has essential spectrum $\sigma_{\mathrm{ess}}(T_f) = f(\partial\Omega)$. It follows that T_f is Fredholm if and only if $f(z) \neq 0$ for all $z \in \partial\Omega$.*

PROOF. Let $H := A^2(\Omega)$ and denote by $\pi\colon \mathscr{L}(H) \to \mathscr{L}(H)/\mathscr{K}(H)$ the projection. Then by Theorem 1.4.16 and (1.4.7),

$$\sigma_{\mathrm{ess}}(T_f) = \sigma_{\mathscr{L}(H)/\mathscr{K}(H)}(\pi(T_f)) = \sigma_{\mathcal{T}(\Omega)/\mathscr{K}(H)}(\pi(T_f)) = \sigma_{C(\partial\Omega)}(f|_{\partial\Omega}) = f(\partial\Omega)$$

since $\mathcal{T}(\Omega)/\mathscr{K}(H)$ is a C^*-subalgebra of $\mathscr{L}(H)/\mathscr{K}(H)$ and we may calculate the spectrum of $\pi(T_f) \in \mathcal{T}(\Omega)/\mathscr{K}(H)$ with respect to this subalgebra, see Proposition C.2.3. By the definition of the essential spectrum, T_f is Fredholm if and only if, $0 \notin \sigma_{\mathrm{ess}}(T_f)$. By the above, this is equivalent to $f(z) \neq 0$ for all $z \in \partial\Omega$. ∎

Notes for chapter 1

The material from section 1.1 about Bergman spaces and the Bergman kernel function can largely be found in [**Kra01**, Chapter 1.4] and [**Ran86**]. The same holds for the transformation laws of P^{Ω} and K_{Ω} from section 1.3. Corollary 1.1.4 is from [**Upm96**, Proposition 4.1.17].

With exception to Corollary 1.2.10, all results of section 1.2 are taken from [**Upm96**]. While we are not aware of who first showed Corollary 1.2.10, it is proved in [**SSU89**, Theorem 1.4] using some tools from multivariable operator theory (namely the Taylor joint spectrum), whereas our proof relies on the diagonal behavior of the Bergman kernel in order to provide a non-zero compact operator in $\mathcal{T}(\Omega)$. The part of the proof concerning the commutator ideal was inspired by [**Con91**, Lemma 12.9]. Another proof can be found in [**Rae78**] under the assumption that Ω strictly pseudoconvex and with smooth boundary. It uses the fact that the commutators of Toeplitz operators are then compact, see Proposition 1.4.14.

The treatment of the $\overline{\partial}$-Neumann problem is based on [**Str10**] and [**Has14**]. Corollary 1.4.18 is from [**Upm96**, Corollary 4.1.30].

CHAPTER 2

Groupoid C^*-algebras

In the previous chapter, we have seen that the Toeplitz C^*-algebra for a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ with smooth boundary has a very simple structure: the commutators of Toeplitz operators (with continuous symbols) are all compact and the quotient algebra $\mathcal{T}(\Omega)/\mathcal{K}(A^2(\Omega))$ is isomorphic to $C(\partial\Omega)$. This was due to the fact that, under these assumptions, there exists a compact solution operator to a certain system of partial differential equations, namely the $\overline{\partial}$-equation, as well as holomorphic peaking functions for every boundary point. In what follows, we will concentrate on the situation for *Reinhardt domains*, but we will waive the assumption on *strict* pseudoconvexity and we will allow for domains with non-smooth boundary. It turns out that the resulting Toeplitz C^*-algebra has a much more complicated structure, which we will probe by showing that it is isomorphic to a (subalgebra of a) C^*-algebra associated to a groupoid that will be constructed from the domain Ω. The general theory of groupoid C^*-algebras will then allow us to make statements about the Toeplitz C^*-algebra.

The application to Toeplitz operators will be done in chapter 3, while in the present chapter, we will develop the basics of the theory of groupoids. In simple terms, a groupoid generalizes the notion of a group, in the sense that it comes with multiplication and inversion maps, the difference being that the multiplication need not be defined for *every* pair of elements. Moreover, a groupoid is not required to have a unit element, although cancellation laws still apply. Consequently, the usual group identities like associativity need only hold whenever they make sense in the groupoid setting.

In order to do analysis on groupoids, we will introduce generalizations of Haar measures of groups, the so-called *Haar systems*, which consist of an entire family of measures on the groupoid, indexed by its *unit space*. This will make it possible to associate first a convolution algebra and then C^*-algebras to groupoids, similarly to the notion of a group C^*-algebra.

2.1. Groupoids

We begin this chapter with a section on the more algebraic aspects of groupoids. Besides the elementary definitions, a large portion of it will be devoted to the most important examples of groupoids which will be of importance later on.

DEFINITION 2.1.1. A *groupoid* is a set G together with maps $\operatorname{inv}\colon G \to G$, $x \mapsto x^{-1}$, and $G^{(2)} \to G$, $(x,y) \mapsto xy$, where $G^{(2)}$ is a subset of $G \times G$, such that

(i) if $(x,y), (y,z) \in G^{(2)}$, then $(xy,z), (x,yz) \in G^{(2)}$ and $(xy)z = x(yz)$,
(ii) $(x^{-1},x) \in G^{(2)}$ for all $x \in G$ and if $(x,y) \in G^{(2)}$, then $x^{-1}(xy) = y$,
(iii) $(x,x^{-1}) \in G^{(2)}$ for all $x \in G$ and if $(z,x) \in G^{(2)}$, then $(zx)x^{-1} = z$.

Property (i) is called the *law of associativity* and (ii), (iii) are called the *left and right cancellation laws*, respectively. Elements of $G^{(2)}$ are called *composable pairs*, the map $(x,y) \mapsto xy$ is the *multiplication* or *product map* and $\operatorname{inv}\colon x \mapsto x^{-1}$ is the *inversion*.

We call G a *locally compact groupoid* if G is, in addition, a locally compact Hausdorff space (see Appendix A) such that the multiplication and inversion maps are continuous. In addition, we will always assume locally compact groupoids to be second countable. Here, $G^{(2)}$ carries the subspace topology with respect to $G \times G$.

REMARK 2.1.2. Note that in (ii) and (iii), the expressions $x^{-1}(xy)$ and $(zx)x^{-1}$ are well-defined by the associativity property (i).

Hence, groupoids can be seen as "generalized" groups, where the multiplication is replaced by a map which is not necessarily everywhere defined and where there is no distinguished unit element. Note that the inversion $x \mapsto x^{-1}$ is still defined on the whole groupoid.

PROPOSITION 2.1.3. *Let G be a groupoid and $x, y \in G$.*

(i) We have $(x^{-1})^{-1} = x$.
(ii) If xy is defined (that is, $(x,y) \in G^{(2)}$), then so is $y^{-1}x^{-1}$ and the equation $(xy)^{-1} = y^{-1}x^{-1}$ holds.

In particular, by (i), the inversion map is bijective. For a locally compact groupoid, it is therefore a homeomorphism.

PROOF. Using associativity and the cancellation laws, we calculate

$$(x^{-1})^{-1} = (x^{-1})^{-1}x^{-1}(x^{-1})^{-1} = (x^{-1})^{-1}x^{-1}xx^{-1}(x^{-1})^{-1} = x.$$

Similarly, if xy is defined, then so is $(xy)^{-1}(xy)$, hence $(xy)^{-1}xyy^{-1}$ is also defined (by associativity, because $(y, y^{-1}) \in G^{(2)}$). Since $xyy^{-1} = x$, we can form $xyy^{-1}x^{-1}$ and conclude

$$(xy)^{-1} = (xy)^{-1}xx^{-1} = (xy)^{-1}xyy^{-1}x^{-1} = y^{-1}x^{-1}$$

by canceling xy in the last step. ■

REMARK 2.1.4. Alternatively, groupoids may be defined as small categories (that is, its objects form a *set* and not a proper class) in which every morphism is invertible. In fact, if $\mathcal{G}$ is such a category, let G be the disjoint union

$$G := \bigsqcup_{a,b \in \mathrm{Ob}(\mathcal{G})} \mathrm{Hom}_{\mathcal{G}}(a,b),$$

where $\mathrm{Ob}(\mathcal{G})$ denotes the set of objects of $\mathcal{G}$ and $\mathrm{Hom}_{\mathcal{G}}(a,b)$ is the set of morphisms from a to b. Composition and inversion of morphisms then endows G with the structure of a groupoid as in Definition 2.1.1. While this approach is easier to handle in some circumstances (for instance, Proposition 2.1.3 becomes quite trivial), a treatment of Haar systems seems to favor the algebraic definition above.

DEFINITION 2.1.5. Let G be a groupoid and $x \in G$. The expressions

$$d(x) := x^{-1}x \quad \text{and} \quad r(x) := xx^{-1}$$

are called the *domain* (or *source*) and *range* (or *target*) of x, respectively. The set

$$G^{(0)} := d(G) = r(G)$$

is the *unit space* of G. Moreover, we set

$$G^u := r^{-1}(\{u\}), \quad G_v := d^{-1}(\{v\}) \quad \text{and} \quad G^u_v := G^u \cap G_v.$$

LEMMA 2.1.6. *We have $G^{(2)} = \{(x,y) \in G \times G : d(x) = r(y)\}$ and*

$$G^{(0)} = \{x \in G : d(x) = x\} = \{x \in G : r(x) = x\} \tag{2.1.1}$$

$$= \{x \in G : (x,x) \in G^{(2)} \text{ and } x^2 = x\} = \{x \in G : x^{-1} = x\}. \tag{2.1.2}$$

In particular, if G is a locally compact groupoid, then $G^{(2)}$ and $G^{(0)}$ are closed in G. Moreover, elements of $G^{(0)}$ are units in the sense that if $x, y \in G$ and $u \in G^{(0)}$ are such that $xu = y$ or $ux = y$, then $x = y$.

PROOF. If x and y are composable, then $x^{-1}(xy) = y$ and hence $(x^{-1}(xy))y^{-1} = yy^{-1}$. Since by associativity, both $(x^{-1}, xy) \in G^{(2)}$ and $(xy, y^{-1}) \in G^{(2)}$, we have $x^{-1}((xy)y^{-1}) = (x^{-1}(xy))y^{-1}$, again by using associativity. But then

$$d(x) = x^{-1}x = x^{-1}((xy)y^{-1}) = (x^{-1}(xy))y^{-1} = yy^{-1} = r(y).$$

Conversely, if $d(x) = r(y)$, then $(x^{-1}x, y) \in G^{(2)}$ and since also $(x, x^{-1}x) \in G^{(2)}$, we get by associativity that $(x, y) = (x(x^{-1}x), y) \in G^{(2)}$.

If $u \in G^{(0)}$, then $u = xx^{-1} = y^{-1}y$ for some $x, y \in G$. It follows that $d(u) = u^{-1}u = y^{-1}yxx^{-1} = y^{-1}y = u$ by the cancellation rule and similarly $r(u) = u$. The converse inclusions in (2.1.1) are self-evident.

Next, if $u \in G^{(0)}$, then $u = d(x)$ for some $x \in G$. Hence, $u^{-1} = (x^{-1}x)^{-1} = x^{-1}x = d(x) = u$ and therefore also $(u, u) = (u, u^{-1}) \in G^{(2)}$ and $u^2 = u^{-1}u = d(u) = u$. Conversely, if $(u, u) \in G^{(2)}$ and $u^2 = u$, then $u^{-1}u = u^{-1}uu = u$, so $u \in G^{(0)}$. This shows that $G^{(0)}$ is the set of idempotents. Similarly, $u^{-1} = u$ also implies $u \in G^{(0)}$ since then $u^{-1}u = uu = u$. This concludes the proof of (2.1.2).

Finally, if $x, y \in G$ and $u \in G^{(0)}$ such that $xu = y$, then $x = xuu^{-1} = xu^2 = xu = y$. The case $ux = y$ is treated similarly. ■

LEMMA 2.1.7. *The relation $u \sim v :\iff G^u_v \neq \emptyset$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called* orbits *and we denote the orbit of $u \in G^{(0)}$ by $[u]$. Therefore, $v \in [u]$ if and only if there exists $x \in G$ such that $r(x) = u$ and $d(x) = v$. We have $[u] = d(r^{-1}(\{u\})) = r(d^{-1}(\{u\}))$.*

PROOF. We have $u \in G^u_u$ because $r(u) = u = d(u)$ for all $u \in G^{(0)}$ by Lemma 2.1.6, so $\sim$ is reflexive. Next, if $x \in G^u_v$, then $x^{-1} \in G^v_u$, showing that $\sim$ is symmetric. Finally, if $x \in G^u_v$ and $y \in G^v_w$, then $r(xy) = xyy^{-1}x^{-1} = r(x) = u$ and $d(xy) = y^{-1}x^{-1}xy = d(y) = v$, implying $xy \in G^u_w$, which proves that $\sim$ is also transitive, hence an equivalence relation.

Clearly, $v \in [u]$ is equivalent to the existence of $x \in r^{-1}(\{u\}) \cap d^{-1}(\{v\})$. If such an x exists, then $v = d(x) \in d(r^{-1}(\{u\}))$, so $[u] \subseteq d(r^{-1}(\{u\}))$. Conversely, if $v \in d(r^{-1}(\{u\}))$, then there is $x \in r^{-1}(\{u\})$ with $v = d(x)$, so $x \in d^{-1}(\{v\}) \cap r^{-1}(\{u\})$. Therefore, we also have $[u] \supseteq d(r^{-1}(\{u\}))$. Similarly, one shows that $[u] = r(d^{-1}(\{u\}))$. ■

2.1.1. Groupoid homomorphisms. As with other algebraic objects, we can consider structure preserving maps between groupoids. As is expected, homomorphisms are required to commute with the multiplications, although of course only if they make sense:

DEFINITION 2.1.8. Let G and H be groupoids. A map $\varphi\colon G \to H$ is called a *groupoid homomorphism* if $(x, y) \in G^{(2)}$ implies that $(\varphi(x), \varphi(y)) \in H^{(2)}$ and $\varphi(xy) = \varphi(x)\varphi(y)$. A bijective groupoid homomorphism is called a *groupoid isomorphism*.

If G and H are locally compact groupoids, we will require homomorphisms to be continuous and isomorphisms to be homeomorphisms. A groupoid isomorphism which is also a homeomorphism will be called a *groupoid homeomorphism*.

REMARK 2.1.9. Note that in the category theory picture of Remark 2.1.4, a groupoid homomorphism $\varphi\colon G \to H$ is nothing but a covariant functor.

PROPOSITION 2.1.10. *Let G and H be groupoids and $\varphi\colon G \to H$ a groupoid homomorphism.*

(i) If $u \in G^{(0)}$, then $\varphi(u) \in H^{(0)}$, so φ preserves the unit spaces.
(ii) For all $x \in G$, we have $\varphi(x^{-1}) = \varphi(x)^{-1}$.
(iii) We have $d_H \circ \varphi = \varphi \circ d_G$ and $r_H \circ \varphi = \varphi \circ r_G$, where d_H and d_G are the domain maps of H and G, respectively, and similarly for the range maps r_H and r_G.

PROOF. If $u \in G^{(0)}$, then $u^2 = u$ by Lemma 2.1.6. It follows that $d(\varphi(u)) = \varphi(u)^{-1}\varphi(u^2) = \varphi(u)^{-1}\varphi(u)\varphi(u) = \varphi(u)$, so $\varphi(u) \in H^{(0)}$ by (2.1.1), which shows (i). Now let $x \in G$. By the homomorphism property, we have $\varphi(xx^{-1}) = \varphi(x)\varphi(x^{-1})$, which implies $\varphi(x)^{-1}\varphi(xx^{-1}) = \varphi(x)^{-1}\varphi(x)\varphi(x^{-1}) = \varphi(x^{-1})$ by left cancellation. Since $xx^{-1} \in G^{(0)}$, we have $\varphi(xx^{-1}) \in H^{(0)}$ by (i) and so we can cancel $\varphi(xx^{-1})$ by Lemma 2.1.6 to obtain $\varphi(x)^{-1} = \varphi(x^{-1})$. This proves (ii). Finally, we have $d_H(\varphi(x)) = \varphi(x)^{-1}\varphi(x) = \varphi(x^{-1}x) = \varphi(d_G(x))$ and the same holds for the range maps. ∎

2.1.2. Reductions and invariant subsets. Any groupoid may be "shrunk" in order to produce a new groupoid with prescribed unit space. This process is called *reduction*:

DEFINITION 2.1.11. For a groupoid G and a subset E of its unit space $G^{(0)}$, we set

$$G|_E := r^{-1}(E) \cap d^{-1}(E) = \{x \in G : r(x) \in E \text{ and } d(x) \in E\}$$

and call it the *reduction* of G by E.

PROPOSITION 2.1.12. *Let G be a groupoid and $E \subseteq G^{(0)}$. By restricting the product to $G^{(2)} \cap (G|_E \times G|_E)$ and the inversion to $G|_E$, the reduction becomes a groupoid on its own. Moreover, its unit space is $(G|_E)^{(0)} = E$ and for $u \in E$ we have $(G|_E)^u = G^u \cap G|_E$ and $(G|_E)_u = G_u \cap G|_E$.*

PROOF. For the first part we only need to show that $G|_E$ is closed under the algebraic operations, the cancellation rules and associativity then follow immediately. Let $x, y \in G^{(2)} \cap (G|_E \times G|_E)$. Then $r(xy) = xyy^{-1}x^{-1} = xx^{-1} = r(x) \in E$ and $d(xy) = y^{-1}x x^{-1} y = y^{-1}y = d(y) \in E$ and hence $xy \in G|_E$. For $x \in G|_E$ we obviously have $r(x^{-1}) = d(x) \in E$ and $d(x^{-1}) = r(x) \in E$, so x^{-1} is also in $G|_E$.

The source and range maps of $G|_E$ are just the restrictions of d and r to $G|_E$. We have $(G|_E)^{(0)} = d(G|_E) = d(r^{-1}(E) \cap d^{-1}(E)) \subseteq d(d^{-1}(E)) \subseteq E$. Conversely, if $x \in E \subseteq G^{(0)}$, then $d(x) = r(x) = x \in E$ by Lemma 2.1.6, so $x \in d(G|_E) = (G|_E)^{(0)}$. The remaining formulas are easy to verify. ∎

REMARK 2.1.13. Note that, if $E_1 \subseteq E_2 \subseteq G^{(0)}$, then $(G|_{E_2})|_{E_1} = G|_{E_1}$.

While any subset of the unit space of a groupoid can be used to reduce the original groupoid, there is a family of subsets for which the reduction will turn out to be particularly useful. These are the so-called *invariant* subsets.

DEFINITION 2.1.14. Let G be a groupoid. A subset $E \subseteq G^{(0)}$ of the unit space of G is called *invariant (under G, or G-invariant)* if $E = r(d^{-1}(E))$.

PROPOSITION 2.1.15. *A subset $E \subseteq G^{(0)}$ is G-invariant if and only if $d^{-1}(E) = r^{-1}(E)$. In this case,*

$$G|_E = r^{-1}(E) = d^{-1}(E) \quad \text{as well as} \quad (G|_E)^u = G^u \quad \text{and} \quad (G|_E)_u = G_u$$

for all $u \in E$. In particular, arbitrary unions and intersections of invariant sets are again invariant, as are their complements (relative $G^{(0)}$). Moreover,

$$G^{(2)} \cap (G \times G|_E) \subseteq (G|_E)^{(2)} \quad \text{and} \quad G^{(2)} \cap (G|_E \times G) \subseteq (G|_E)^{(2)}$$

for any invariant subset $E \subseteq G^{(0)}$. This means that elements of $G|_E$ can only be composed with other elements of $G|_E$.

PROOF. If $d^{-1}(E) = r^{-1}(E)$, then $r(d^{-1}(E)) = r(r^{-1}(E)) = E$ because $r\colon G \to G^{(0)}$ is surjective, so E is invariant. Now assume that E is invariant. Clearly, $G|_E \subseteq d^{-1}(E)$. By invariance of E, we have $r^{-1}(E) = r^{-1}(r(d^{-1}(E))) \supseteq d^{-1}(E)$. Therefore, $G|_E = r^{-1}(E) \cap d^{-1}(E) \supseteq d^{-1}(E) \cap d^{-1}(E) = d^{-1}(E)$, which shows $G|_E = d^{-1}(E)$. Next, we have

$$G|_E = d^{-1}(E) \supseteq d^{-1}(E) \cap r^{-1}(E) = G|_E,$$

which implies $d^{-1}(E) = d^{-1}(E) \cap r^{-1}(E) \subseteq r^{-1}(E)$ and hence $G|_E = d^{-1}(E) = r^{-1}(E)$. Finally, if $u \in E$, then $G^u = r^{-1}(\{u\}) \subseteq r^{-1}(E) = G|_E$, so $(G|_E)^u = G^u \cap G|_E = G^u$ by Proposition 2.1.12, and similarly for $(G|_E)_u$. The statement about unions, intersections and complements now follows from the compatibility of the preimage with these operations.

If $(x,y) \in G^{(2)} \cap (G \times G|_E)$, then $d(x) = r(y)$ by Lemma 2.1.6, so $x \in d^{-1}(\{r(y)\}) \subseteq d^{-1}(E) = G|_E$. Therefore, $(x,y) \in G^{(2)} \cap (G|_E \times G|_E) = (G|_E)^{(2)}$. The second statement is done analogously. ■

REMARK 2.1.16. It follows that any partition of $G^{(0)}$ into invariant sets gives rise to a partition of G into disjoint groupoids. In particular, for every invariant set $E \subseteq G^{(0)}$ we have the disjoint union $G|_E \cup G|_{G^{(0)} \setminus E}$. As a sanity check, note that groups have only the trivial invariant subsets $G^{(0)} = \{e\}$ and $\emptyset$, which is consistent with the fact that groups can *not* be written as disjoint unions of subgroups (every subgroup needs a unit element).

REMARK 2.1.17. Note that the G-invariant subsets are precisely the unions of orbits from Lemma 2.1.7. Indeed, if $u \in G^{(0)}$, then $[u] = r(d^{-1}(\{u\})) = r(d^{-1}([u]))$, so (unions of) orbits are invariant. Conversely, if $E \subseteq G^{(0)}$ is invariant and $u \in E$, then also $[u] = r(d^{-1}(\{u\})) \subseteq r(d^{-1}(E)) = E$, so $E = \bigcup_{u \in E}[u]$.

2.1.3. Examples. We will now present some examples of groupoids which will be needed in later sections of this thesis. While these examples are very basic, they will nevertheless be sufficient for our purposes.

EXAMPLE 2.1.18 (Groups). Every group G is a groupoid with everywhere defined multiplication (meaning $G^{(2)} = G \times G$) and unit space $G^{(0)} = \{e\}$, where e is the identity element of the group. It follows that $G^e = G_e = G$. ◊

EXAMPLE 2.1.19 (Cotrivial groupoids). Any set X can be made into a groupoid. Let $G := X$ and $G^{(2)} := \mathrm{diag}(X) := \{(x,x) : x \in X\}$, the diagonal of $X \times X$. Multiplication and inversion are defined by $xx := x$ and $x^{-1} := x$. One easily verifies that this construction gives a groupoid, the *cotrivial groupoid over* X. For the cotrivial groupoid we have $G^{(0)} = X$ and $G^x = \{x\} = G_x$. ◊

EXAMPLE 2.1.20 (Trivial groupoids). Given a set X, there is another way to obtain a groupoid with unit space X. Take $G := X \times X$ and $G^{(2)} := \{((x_1,x_2),(y_1,y_2)) \in G \times G : x_2 = y_1\}$ with multiplication $(x,y)(y,z) := (x,z)$ and inversion $(x,y)^{-1} := (y,x)$. This gives the *trivial (or matrix) groupoid over* X. For this groupoid, we have $r(x,y) = (x,x)$ and $d(x,y) = (y,y)$. Its unit space is $G^{(0)} = \{(x,x) : x \in X\}$, the diagonal of $X \times X$, as can be seen by using any of the formulas from (2.1.1) or (2.1.2). We may therefore identity $G^{(0)}$ with X itself. Moreover, $G^x = r^{-1}(\{x\}) = \{x\} \times X$ and $G_x = d^{-1}(\{x\}) = X \times \{x\}$.

Now let $E \subseteq X \cong G^{(0)}$ be a subset. We have $r^{-1}(E) = E \times X$ and $d^{-1}(E) = X \times E$. The reduction $G|_E$ is therefore given by

$$(X \times X)|_E = r^{-1}(E) \cap d^{-1}(E) = E \times E. \tag{2.1.3}$$

It follows that the reduction of a trivial groupoid is again a trivial groupoid. Note that E is invariant if and only if $E \times X = X \times E$ which is the case if and only if $E = X$ or $E = \emptyset$. ◊

EXAMPLE 2.1.21 (Equivalence relations). Let X be a set and $R \subseteq X \times X$ an equivalence relation over X. Defining $R^{(2)} := \{((x_1,y_1),(x_2,y_2)) \in R \times R : y_1 = x_2\}$, the set R becomes a groupoid with composition $(x_1,y_1)(y_1,y_2) := (x_1,y_2)$ and inverse $(x,y)^{-1} := (y,x)$. For the special case $R = X \times X$, we thus obtain the trivial groupoid over X. The other extreme is $R = \mathrm{diag}(X)$, which gives rise (up to the identification of $\mathrm{diag}(X)$ with X) to the cotrivial groupoid over X. ◊

EXAMPLE 2.1.22 (Transformation groupoids). Let Γ be a group acting on a set S from the right by $\alpha\colon \Gamma \to \mathrm{Aut}(S)$, written as $s \cdot g := \alpha(g)(s)$. We define the *transformation groupoid* (or *action groupoid*) associated to this action in the following way: as a set, we let $G := S \times \Gamma$. The set of composable pairs is

$$G^{(2)} := \{((s,g),(t,h)) \in S \times \Gamma : t = s \cdot g\}$$

and the operations are given as

$$(s,g)(s \cdot g, h) := (s, gh) \quad \text{and} \quad (s,g)^{-1} := (s \cdot g, g^{-1}).$$

It is elementary to check that this defines a groupoid which we will denote by $S \rtimes_\alpha \Gamma := G$. The domain and range maps of this groupoid are given by

$$d(s,g) = (s \cdot g, g^{-1})(s,g) = (s \cdot g, e) \quad \text{and} \quad r(s,g) = (s,g)(s \cdot g, g^{-1}) = (s,e).$$

If Γ is a second countable locally compact Hausdorff group, S a second countable locally compact Hausdorff space and the action $\Gamma \times S \to S$ is continuous, then this evidently defines a locally compact groupoid.

We note that the unit space $G^{(0)}$ of this groupoid may be identified with S by the mapping $(e,s) \mapsto s$, where e is the unit in Γ, since

$$G^{(0)} = r(G) = \{(s,e) : s \in S\} \cong S. \tag{2.1.4}$$

For $u = (s,e) \in G^{(0)}$, we have

$$G^u = r^{-1}(\{u\}) = \{(t,h) \in G : (t,e) = (s,e)\} = \{s\} \times \Gamma \tag{2.1.5}$$

and similarly

$$G_u = d^{-1}(\{u\}) = \{(t,h) \in G : (t \cdot h, e) = (s,e)\} = \{(t,h) \in G : t \cdot h = s\}. \tag{2.1.6}$$

Now let $E \subseteq S$ be a subset. Then the reduction of $S \rtimes_\alpha \Gamma$ to E is, as a set, given by

$$(S \rtimes_\alpha \Gamma)|_E = r^{-1}(E) \cap d^{-1}(E) = \{(t,h) \in E \times \Gamma : t \cdot h \in E\}, \tag{2.1.7}$$

where we have used (2.1.5) and (2.1.6).

We now take a look at invariant subsets of this groupoid. Let $E \subseteq S \cong G^{(0)}$ be arbitrary. By (2.1.6), we have $d^{-1}(E) = \{(t,h) \in S \times \Gamma : t \cdot h \in E\}$. Since $r\colon G \to G^{(0)} \cong S$ is just the map $(t,g) \mapsto t$, we have

$$r(d^{-1}(E)) = \{t \in S : \exists h \in \Gamma \text{ with } t \cdot h \in E\} = \{t \in S : (t \cdot \Gamma) \cap E \neq \varnothing\}, \tag{2.1.8}$$

where $t \cdot \Gamma$ denotes the orbit of t under Γ. Evidently, $r(d^{-1}(E)) \supseteq E$. If $E \subseteq r(d^{-1}(E))$, then any orbit that intersects E must fully belong to E. We conclude that the invariant subsets of G are precisely the subsets of S which are unions of group orbits.[1] A consequence of this is that the reduction of a transformation groupoid to an invariant subset is again a transformation groupoid,

$$(S \rtimes_\alpha \Gamma)|_E = E \rtimes_{\alpha|_E} \Gamma,$$

where $\alpha|_E$ denotes the restricted action of Γ on E. ◊

EXAMPLE 2.1.23. Let Γ be a group, acting on itself by right translations $\rho_h\colon \Gamma \to \Gamma$, $\rho_h(g) := gh$, so we can form the transformation groupoid $G \rtimes_\rho G$. Consider the map $\varphi\colon \Gamma \times \Gamma \to \Gamma \times \Gamma$, $\varphi(g,h) := (g, gh)$. Obviously, φ is bijective. We claim that φ establishes a groupoid isomorphism between $\Gamma \rtimes_\rho \Gamma$ and the trivial groupoid $\Gamma \times \Gamma$. Indeed, if $((g,h),(g',h')) \in (\Gamma \rtimes_\rho \Gamma)^{(2)}$, then $g' = gh$ and $(\varphi(g,h), \varphi(g',h')) = ((g,gh),(g',g'h')) = ((g,gh),(gh,g'h')) \in (\Gamma \times \Gamma)^{(2)}$. Moreover,

$$\varphi((g,h) \circ (g',h')) = \varphi(g, hh') = (g, ghh') = (g,gh)\square(g',g'h') = \varphi(g,h)\square\varphi(g',h'),$$

[1] This comes as no surprise as the group orbits are precisely the orbits in the groupoid sense as defined in Lemma 2.1.7.

where we denote by $\circ$ the multiplication in the transformation groupoid, and by $\square$ the operation in the trivial groupoid for clarity. Therefore, φ is indeed a groupoid isomorphism. In the case of a locally compact group, this is even a homeomorphism. Using (2.1.3) and Proposition 2.1.10, we also see

$$(\Gamma \rtimes_\rho \Gamma)|_E \cong E \times E$$

for any subset $E \subseteq \Gamma$ via the restriction of φ to $(\Gamma \rtimes_\rho \Gamma)|_E$. Indeed, since $d \circ \varphi = \varphi \circ d$ and $r \circ \varphi = \varphi \circ r$, we have

$$(\Gamma \rtimes_\rho \Gamma)|_{E \times \{e\}} = d^{-1}(E \times \{e\}) \cap r^{-1}(E \times \{e\}) = \varphi^{-1}\big((\Gamma \times \Gamma)|_{\varphi(E \times \{e\})}\big)$$

Since $\varphi(E \times \{e\}) = \{(g,g) : g \in E\}$ is the diagonal, the claim follows after identifying the unit spaces of both groupoids with Γ in the usual way. $\Diamond$

DEFINITION 2.1.24 (Product of groupoids). Let G_i, $i \in I$ be a family of groupoids. Then the Cartesian product

$$G := \prod_{i \in I} G_i \quad \text{with} \quad G^{(2)} := \prod_{i \in I} G_i^{(2)}$$

and the componentwise operations is a groupoid, called the *product* of the G_i. If I is countable and all G_i are second countable, locally compact Hausdorff spaces, then so is G when equipped with the product topology. As usual, the product of finitely many groupoids $G_1, \dots, G_n$ will be denoted by $G_1 \times \cdots \times G_n$.

EXAMPLE 2.1.25 (Trivial action groupoid). Let S be a set. If Γ is any group, then there is a trivial action of Γ on S, namely $s \cdot g := s$ for all $s \in S$ and $g \in \Gamma$. Moreover, the identity map $\mathrm{id}_{S \times \Gamma} \colon S \rtimes_{\mathrm{triv}} \Gamma \to S \times \Gamma$ is a groupoid isomorphism between the *trivial action groupoid* $S \rtimes_{\mathrm{triv}} \Gamma$ and the product groupoid $S \times \Gamma$, where S is the cotrivial groupoid over S and Γ is seen as a groupoid. Indeed, if (s,g) and (t,h) are composable in $S \rtimes_{\mathrm{triv}} \Gamma$, then $t = s \cdot g = s$ and hence $(s,g)(s,h) = (s,gh)$, which is also the multiplication in the product groupoid. $\Diamond$

DEFINITION 2.1.26 (Disjoint union of groupoids). Let G_i, $i \in I$, be a family of groupoids. Then the disjoint union

$$G := \bigsqcup_{i \in I} G_i \quad \text{together with} \quad G^{(2)} := \bigsqcup_{i \in I} G_i^{(2)}$$

and the obvious multiplication and inversion is a groupoid, called the *disjoint union* of the G_i. Thus, two elements x and y of $\bigsqcup_{i \in I} G_i$ are composable if and only if there exists $i \in I$ with $x, y \in G_i$ and xy is then the just product in G_i. If I is countable and all G_i are locally compact groupoids, then so is G when equipped with the disjoint union topology. Here, the countability assumption is necessary because we require locally compact groupoids to be second countable. The disjoint union groupoid satisfies

$$G^{(0)} = \bigsqcup_{i \in I} (G_i)^{(0)}, \quad G^u = (G_i)^u \quad \text{and} \quad G_u = (G_i)_u \quad \text{if } u \in (G_i)^{(0)}.$$

EXAMPLE 2.1.27. Let G be a groupoid and suppose that $E_i \subseteq G^{(0)}$, $i \in I$, is partition of $G^{(0)}$ into invariant subsets, $\bigcup_{i \in I} E_i = G^{(0)}$. Then G can, as a set, be identified with the disjoint union $\bigsqcup_{i \in I} G|_{E_i}$ (because the E_i are pairwise disjoint and $G|_{E_i} = d^{-1}(E_i)$ by Proposition 2.1.15). Moreover, the evident map $\iota \colon G \to \bigsqcup_{i \in I} G|_{E_i}$ is a groupoid isomorphisms. Indeed, if $(x,y) \in G^{(2)}$, then $d(x) = r(y) \in G^{(0)}$ by Lemma 2.1.6 and there exists $i \in I$ such that $d(x) = r(y) \in E_i$. It follows that $x \in d^{-1}(E_i) = G|_{E_i}$ and $y \in r^{-1}(E_i) = G|_{E_i}$ by invariance of E_i, see Proposition 2.1.15. Therefore, $(x,y) \in G^{(2)} \cap (G|_{E_i} \times G|_{E_i}) = (G|_{E_i})^{(2)}$, meaning x and y are composable in the reduction $G|_{E_i}$. But then $(\iota(x), \iota(y)) \in (\bigsqcup_{i \in I} G|_{E_i})^{(2)}$ and $\iota(xy) = \iota(x)\iota(y)$.

If G is a locally compact groupoid and $U_i \subseteq G^{(0)}$, $i \in I$, is a partition of $G^{(0)}$ into *open* invariant subsets, then G is homeomorphic to the disjoint union $\bigsqcup_{i\in I} G|_{U_i}$ with its disjoint union topology. Every $G|_{U_i}$ is then clopen in G, hence a union of connected components. $\diamondsuit$

2.2. Analysis on groupoids: Haar systems

Let G be a locally compact groupoid. We seek to generalize the notion of the Haar measure of groups to the groupoid setting. This generalization will be in the form of an entire family of Radon measures on G, a *Haar system*, parametrized by the unit space $G^{(0)}$ and with appropriate continuity and invariance properties. For the basics on Radon measures, see Appendix A. We will denote the set of Radon measures on G by $\mathcal{M}_+(G)$.

DEFINITION 2.2.1. A *left Haar system* for G is a map $\mu\colon G^{(0)} \to \mathcal{M}_+(G)$, $u \mapsto \mu^u$, such that

(i) $\operatorname{supp}(\mu^u) = G^u$ for all $u \in G^{(0)}$,
(ii) for any fixed $f \in C_c(G)$, the map $G^{(0)} \ni u \mapsto \int_G f\, d\mu^u \in \mathbb{C}$ is continuous, and
(iii) for all $x \in G$ and $f \in C_c(G)$, we have $\int_G f(xy)\, d\mu^{d(x)}(y) = \int_G f(y)\, d\mu^{r(x)}(y)$.

Similarly, a *right Haar system* for G is a map $\mu\colon G^{(0)} \to \mathcal{M}_+(G)$, $u \mapsto \mu_u$, such that

(i') $\operatorname{supp}(\mu_u) = G_u$ for all $u \in G^{(0)}$,
(ii') for any fixed $f \in C_c(G)$, the map $G^{(0)} \ni u \mapsto \int_G f\, d\mu_u \in \mathbb{C}$ is continuous, and
(iii') for all $x \in G$ and $f \in C_c(G)$, we have $\int_G f(y)\, d\mu_{d(x)}(y) = \int_G f(yx)\, d\mu_{r(x)}(y)$.

Property (iii) is called *left-invariance*, and property (iii') is called *right-invariance*.

We note that the integrals in this definition make sense: for $x \in G$ we have $\operatorname{supp}(\mu^{d(x)}) = G^{d(x)} = r^{-1}(d(x))$. For $y \in \operatorname{supp}(\mu^{d(x)})$ it follows that $yy^{-1} = x^{-1}x$ which by Lemma 2.1.6 means that $(x, y) \in G^{(2)}$ so that $f(xy)$ as in the integral occurring in the left-invariance formula for a left Haar system is defined. Similarly, one shows that the corresponding formula for the right Haar system is consistent.

PROPOSITION 2.2.2. *Given a left Haar system for G, the assignment $\mu_u := \mathrm{inv}_*(\mu^u)$, $u \in G^{(0)}$, defines a right Haar system for G, where $\mathrm{inv}_*(\mu^u)$ is the image measure of μ^u under inv, see Example A.4.11.*

PROOF OF PROPOSITION 2.2.2. Defining μ_u as above, we see that its support is $\operatorname{supp}(\mu_u) = \mathrm{inv}(\operatorname{supp}(\mu^u)) = \mathrm{inv}(G^u)$. Since $G^u = r^{-1}(u) = \{y \in G : yy^{-1} = u\}$, we have $\mathrm{inv}(G^u) = d^{-1}(u) = G_u$, implying $\operatorname{supp}(\mu_u) = G_u$, as required. Continuity is clear, since $\int_G f\, d\mu_u$ is just $\int_G f \circ \mathrm{inv}\, d\mu^u$ and $f \circ \mathrm{inv}$ is again in $C_c(G)$. Finally, for $x \in G$ and $f \in C_c(G)$,

$$\int_G f(y)\, d\mu_{d(x)}(y) = \int_G f(y^{-1})\, d\mu^{d(x)}(y) = \int_G f((xy)^{-1}x)\, d\mu^{d(x)}(y) = \\ = \int_G f(y^{-1}x)\, d\mu^{r(x)}(y) = \int_G f(yx)\, d\mu_{r(x)}(y),$$

where we have used Proposition 2.1.3. ■

REMARK 2.2.3. In contrast to the situation for groups, a Haar system on a locally compact groupoid need not necessarily exist (and even if it does, it need not be unique). In fact, the existence of a Haar system forces the range and domain maps $r, d : G \to G^{(0)}$ to be open maps, see Corollary 2.2.7 below. An example of a compact Hausdorff groupoid which does not admit a Haar system (and for which r and d are not open) was given in [**Sed86**]. Some authors also require locally compact groupoids to admit a Haar system by definition, for example [**Pat99**].

LEMMA 2.2.4. *Let G be a locally compact groupoid with left Haar system μ and let $x \in G$. Then*

$$\int_G f(xy)\, d\mu^{d(x)}(y) = \int_G f(y)\, d\mu^{r(x)}(y) \tag{2.2.1}$$

continues to hold for all positive Borel-measurable functions $f\colon G \to \mathbb{R} \cup \{+\infty\}$ *and for all* $\mu^{r(x)}$*-integrable* $f\colon G \to \mathbb{C}$.

PROOF. Consider first the case where $f = \chi_K$ is the characteristic function of a compact subset $K \subseteq X$. Then $K = \bigcap_{n\geq 1} U_n$ for a sequence U_n of open subsets[2] of X and taking $f_n \in C_c(G, [0,1])$ with $f_n|_K = 1$ and $f_n|_{G\setminus U_n} = 0$, we have $f_n \searrow \chi_K$. Since $|f_n| \leq f_1$, the dominated convergence theorem implies that (2.2.1) holds for χ_K. Any open subset of G is the countable union of compact sets since the open subsets are again second countable, locally compact and Hausdorff, hence have an exhaustion of compact subsets by Lemma *A*.3.1. It follows that (2.2.1) holds for χ_U with $U \subseteq G$ open. Let

$$\mathcal{A} := \{A \in \mathcal{B}(G) : (2.2.1) \text{ holds for } \chi_A\},$$

where $\mathcal{B}(G)$ denotes the Borel sets of G. By the above, $\mathcal{A}$ contains the open sets. Moreover, $\chi_{G\setminus A} = \chi_G - \chi_A$ for all $A \in \mathcal{A}$, hence $\mathcal{A}$ is closed under complements by linearity of (2.2.1). The monotone convergence theorem then shows that $\mathcal{A}$ is a σ-algebra, so that $\mathcal{A} = \mathcal{B}(G)$.

Now the claim for the positive Borel functions follows since those are pointwise monotone limits of simple functions. The case where f is integrable with respect to $\mu^{r(x)}$ follows by decomposing $f = (f_1 - f_2) + i(f_3 - f_4)$ with positive Borel functions f_i, $1 \leq i \leq 4$. ∎

LEMMA 2.2.5. *Let G be a locally compact groupoid with left Haar system μ. For each $f \in C_c(G)$, the function $G^{(0)} \ni u \mapsto \int_G f\,d\mu^u \in \mathbb{C}$ has compact support. Moreover, the map*

$$\widetilde{\mu}\colon C_c(G) \to C_c(G^{(0)}), \quad \widetilde{\mu}(f)(u) := \int_G f\,d\mu^u$$

is continuous with respect to the inductive limit topologies (see Appendix B).

PROOF. Let $f \in C_c(G)$ and set $K := \operatorname{supp}(f)$. Then $r(K)$ is a compact subset of $G^{(0)}$, and if $u \in G^{(0)} \setminus r(K)$, then $\int_{G^u} f\,d\mu^u = \int_{r^{-1}(\{u\})} f\,d\mu^u = 0$ because $r^{-1}(\{u\}) \subseteq r^{-1}(G^{(0)} \setminus r(K)) \subseteq G \setminus K$. This shows that $\operatorname{supp}(\widetilde{\mu}(f)) \subseteq r(K)$ is a compact subset of $G^{(0)}$.

To show continuity of $\widetilde{\mu}$, let f_i be a net in $C_c(G)$ such that $f_i \to f$ uniformly for some $f \in C_c(G)$ and $\operatorname{supp}(f_i) \subseteq K$ for some fixed compact subset $K \subseteq G$. Then $\operatorname{supp}(\widetilde{\mu}(f_i)) \subseteq r(K)$ for all i. Let $\psi \in C_c(G)$ be such that $\psi \geq \chi_K$, where χ_K is the characteristic function of K. We then have

$$|\widetilde{\mu}(f_i)(u) - \widetilde{\mu}(f)(u)| \leq \int_G |f_i - f|\,d\mu^u \leq \int_G |f_i - f|\psi\,d\mu^u \leq \|f_i - f\|_\infty \widetilde{\mu}(\psi)(u),$$

so $\|\widetilde{\mu}(f_i) - \widetilde{\mu}(f)\|_\infty \leq \|f_i - f\|_\infty \|\widetilde{\mu}(\psi)\|_\infty$ and hence $\widetilde{\mu}(f_i) \to \widetilde{\mu}(f)$, which shows that $\widetilde{\mu}$ is continuous for the inductive limit topologies, see Remark *B*.3. ∎

REMARK 2.2.6. One can show that Lemma 2.2.5 continues to hold for Borel functions. More precisely, if $f\colon G \to \mathbb{C}$ is a bounded Borel function which vanishes outside a compact subset, then $\widetilde{\mu}(f)$ is a bounded Borel function on $G^{(0)}$, again vanishing outside some compact set. We refer to [**Pat99**, Corollary 2.2.1] for a proof.

COROLLARY 2.2.7. *Let G be a locally compact groupoid and suppose that G admits a left Haar system μ. Then the maps $r, d\colon G \to G^{(0)}$ are open.*

PROOF. Take $U \subseteq G$ open, $x \in U$ and $f \in C_c(U, [0,1])$ such that $f(x) > 0$. By Lemma 2.2.5, we have $\widetilde{\mu}(f) \in C_c(G^{(0)})$ and $\widetilde{\mu}(f)(r(x)) = \int_G f\,d\mu^{r(x)} > 0$ as $x \in r^{-1}(\{r(x)\}) = \operatorname{supp}(\mu^{r(x)})$, see also Remark *A*.4.5. Moreover, $\operatorname{supp}(\widetilde{\mu}(f)) \subseteq r(U)$ by examining the proof of Lemma 2.2.5. It follows that $r(U)$ contains the open neighborhood $\{u \in G^{(0)} : \widetilde{\mu}(f)(u) > 0\}$ of $r(x)$ and is therefore open. Since $d = r \circ \operatorname{inv}$, the same conclusion holds for the domain map. ∎

[2] In other words, K is a G_δ set, meaning a countable intersection of open sets. In a metric space, all closed sets are G_δ and our groupoid G is metrizable by Theorem *A*.3.8.

COROLLARY 2.2.8. *If G is a locally compact groupoid with left Haar system μ and $E \subseteq G^{(0)}$ is an invariant subset, then also the interior G° and the closure $\overline{G}$ are invariant.*

PROOF. By Proposition 2.1.15, E is invariant if and only if $r^{-1}(E) = d^{-1}(E)$. It suffices to note that $r^{-1}(E^\circ) = (r^{-1}(E))^\circ$ and $r^{-1}(\overline{E}) = \overline{r^{-1}(E)}$ because r is continuous and open by Corollary 2.2.7, and similarly for d. ∎

2.2.1. Examples of Haar systems. We will now provide Haar systems for the various types of groupoids that have been discussed in section 2.1.3.

EXAMPLE 2.2.9 (Locally compact groups). Let G be a locally compact (Hausdorff, second countable) group and choose a left Haar measure μ on G (this is unique up to a multiplicative constant). Then evidently $\mu^e := \mu$ defines a left Haar system on the groupoid G. ◊

EXAMPLE 2.2.10 (Cotrivial groupoids). If X is a locally compact, second countable Hausdorff space and G the cotrivial groupoid over X, see Example 2.1.19, then $G^x = \{x\}$, so the only Haar systems are families of point masses, and we will always choose $\mu^x := \delta_x$ for $x \in X$. ◊

EXAMPLE 2.2.11 (Trivial groupoids). Let X be a locally compact space and consider the trivial groupoid $G = X \times X$ over X, see Example 2.1.20. Take a measure $\nu \in \mathcal{M}_+(X)$ with $\operatorname{supp}(\nu) = X$ and, for $x \in X \cong G^{(0)}$, set $\mu^x := \delta_x \times \nu$, where δ_x is the Dirac measure at $x \in X$. This is a Radon measure by Example *A*.4.8. Then μ is a left Haar system for G. Indeed, $\operatorname{supp}(\mu^x) = \{x\} \times \operatorname{supp}(\nu) = G^x$, so the measures have the right support. For $f \in C_c(X \times X)$, we have $\int_{X\times X} f\, d\mu^x = \int_X f(x,y)\, d\nu(y)$, which is continuous in x. Finally,

$$\begin{aligned}\int_G f((x,y)(x',y'))\, d\mu^{d(x,y)}(x',y') &= \int_{X\times X} f((x,y)(x',y'))\, d(\delta_y \times \nu)(x',y') \\ &= \int_X f(x,y')\, d\nu(y') = \int_{X\times X} f(x',y')\, d(\delta_x \times \nu)(x',y') = \int_G f(x',y')\, d\mu^{r(x,y)}(x',y')\end{aligned}$$

for $x, y \in X$ shows left invariance. ◊

EXAMPLE 2.2.12 (Transformation groupoids). Consider again a locally compact transformation groupoid $S \rtimes_\alpha G$ from Example 2.1.22 associated to a continuous group action $\alpha\colon S \times G \to S$, $s \cdot g := \alpha(s,g)$. Choose a left Haar measure λ on Γ and define

$$\mu^s := \mu^{(s,e)} := \delta_s \times \lambda, \tag{2.2.2}$$

where δ_s is the Dirac measure at $s \in S$. This is a Radon measure by Example *A*.4.8. Note that the parameter s of the measures has values in S since we made the identification (2.1.4).

We claim that this defines a left Haar system on G. Clearly, the support of this measure is $\{s\} \times \Gamma = G^{(e,s)}$ (see (2.1.5)), and the map $(s,e) \mapsto \int_S \int_\Gamma f(t,h)\, d\delta_s(t)\, d\lambda(h) = \int_\Gamma f(s,h)\, d\lambda(h)$ is continuous for fixed $f \in C_c(G)$. Finally, for $x = (s,g) \in G$ and $f \in C_c(S \times \Gamma)$,

$$\begin{aligned}\int_G f(xy)\, d\mu^{d(x)}(y) &= \int_S \int_\Gamma f((s,g)(t,h))\, d\lambda(h)\, d\delta_{s\cdot g}(t) = \int_\Gamma f(s,gh)\, d\lambda(h) = \\ &= \int_\Gamma f(s,h)\, d\lambda(h) = \int_S \int_\Gamma f(t,h)\, d\lambda(h)\, d\delta_s(t) = \int_G f(t,h)\, d\mu^s(t,h) = \int_G f(y)\, d\mu^{r(x)}(y),\end{aligned}$$

where we have used left-invariance of λ and, in the last step, that $r(x) = (s,e)$. Unless otherwise specified, we will always use this Haar system for transformation groupoids. ◊

2.2.2. Constructions. Finally, we will give some basic methods on how to construct new Haar systems from existing ones. First, we wish to restrict Haar systems to the reduction of a groupoid. Some care has to be taken here, as the reduction need not have a locally compact topology and even if it does, the restriction of the Haar system may have the wrong support, see the example in [**MR82**, 2.5].

PROPOSITION 2.2.13. *Let G be a locally compact groupoid, equipped with a left Haar system μ, and suppose that $E \subseteq G^{(0)}$ is*

(i) locally closed[3] and invariant, or

(ii) open.

Then the reduction $G|_E$ is a locally compact groupoid and $\mu|_E$ is a left Haar system for $G|_E$, where $(\mu|_E)^u := \mu^u|_{G|_E}$ for $u \in E$ is the restricted measure, see Example A.4.7.

PROOF. Our proof follows [**Goe09**, Proposition 5.17]. By Proposition 2.1.12, $G|_E$ is a groupoid. Because E is locally closed in both cases, there are subsets $C, O \subseteq G^{(0)}$, with C closed and O open, such that $E = C \cap O$, see Lemma A.1.1. It follows that $G|_E = r^{-1}(E) = r^{-1}(C) \cap r^{-1}(O)$ is the intersection of a closed and an open subset of G, so $G|_E$ is a locally compact subspace of G by Lemma A.1.2. In particular, we can restrict the measures μ^u to $G|_E$, see Example A.4.7 for the details on this restriction and on its support. We obtain $\operatorname{supp}(\mu^u|_{G|_E}) \subseteq \operatorname{supp}(\mu^u) \cap G|_E = G^u \cap G|_E = (G|_E)^u$ for all $u \in E$.

By Proposition 2.1.15, we have $\operatorname{supp}(\mu^u) = G^u = (G|_E)^u$ in the case where E is invariant. In particular, $\operatorname{supp}(\mu^u) \subseteq G|_E$ and therefore $\operatorname{supp}(\mu^u|_{G|_E}) = \operatorname{supp}(\mu) = G^u = (G|_E)^u$, so $\mu|_{G|_E}$ has the correct support. If E is open, then so is $G|_E$ and hence the support of the Radon measure $\mu^u|_{G|_E}$ is $\operatorname{supp}(\mu^u) \cap G|_E = G^u \cap G|_E = (G|_E)^u$, also as required.

Finally, take $f \in C_c(G|_E)$ and extend f to a function g in $C_c(G)$ by Corollary A.3.7. In the case where E is invariant, we then have

$$\int_{G|_E} f \, d(\mu^u|_{G|_E}) = \int_{G^u} f \, d(\mu^u|_{G|_E}) = \int_{G^u} f \, d\mu^u = \int_G g \, d\mu^u. \tag{2.2.3}$$

Now continuity and left-invariance follow from the corresponding properties of the Haar system μ. If E is open, then f can be extended to G by zero outside $G|_E$ and these properties also follow automatically. ∎

EXAMPLE 2.2.14. Let $G = S \rtimes_\alpha \Gamma$ be a locally compact transformation groupoid and let $E \subseteq S \cong G^{(0)}$ be open or both locally closed and invariant. By Example 2.1.22, invariance means that E is the union of Γ-orbits. If $\mu^s = \delta_s \times \lambda$ denotes the canonical Haar system from Example 2.2.12, then Proposition 2.2.13 implies that $G|_E$ is a locally compact groupoid with Haar system $\mu|_E$. For $s \in E$, the support of $(\mu|_E)^s = \mu^s|_{G|_E}$ is then

$$\operatorname{supp}((\mu|_E)^s) = (G|_E)^s = \{(s, g) : g \in \Gamma \text{ and } s \cdot g \in E\}$$

by using (2.1.7) and hence

$$\int_{G|_E} f \, d(\mu|_E)^s = \int_{\{h \in \Gamma : s \cdot h \in E\}} f(s, h) \, d\lambda(h)$$

for all $f \in C_c(G|_E)$. In the case where E is invariant, this of course equals $\int_\Gamma f(s, h) \, d\lambda(h)$. ◊

Next, we will transport Haar systems along groupoid homeomorphisms from one groupoid to the other, and this construction will be compatible with reductions. This construction will be important later on when we want to determine if the groupoid C^*-algebras of two groupoids coincide, provided that the groupoids themselves are already isomorphic.

DEFINITION 2.2.15. Let G and H be locally compact groupoids and suppose μ is a left Haar system on G. If $\varphi\colon G \to H$ is a groupoid homeomorphism, then we define

$$(\varphi_* \mu)^v(A) := \mu^{\varphi^{-1}(v)}(\varphi^{-1}(A))$$

for all $v \in H^{(0)}$ and all Borel sets $A \subseteq H$. We call $\varphi_* \mu$ the *pushforward* or *image* of μ by φ. In other words, $(\varphi_* \mu)^v$ is the image measure $\varphi_*(\mu^{\varphi^{-1}(v)})$ on H, see Example A.4.11.

[3] We refer to section A.1 in the appendix for the basics on locally closed subsets.

LEMMA 2.2.16. *The pushforward $\varphi_*\mu$ of a left Haar system on G is a left Haar system on H. Moreover, if $E \subseteq G^{(0)}$ is open or both locally closed and invariant, then $\varphi|_{G|_E} \colon G|_E \to H_{\varphi(E)}$ is a groupoid homeomorphism and*

$$(\varphi|_{G|_E})_*(\mu|_E) = (\varphi_*\mu)|_{\varphi(E)}$$

as left Haar systems on $H|_{\varphi(E)}$.

PROOF. We have $\operatorname{supp}((\varphi_*\mu)^v) = \varphi(\operatorname{supp}(\mu^{\varphi^{-1}(v)})) = \varphi(G^{\varphi^{-1}(v)}) = H^v$ for $v \in H^{(0)}$ by Proposition 2.1.10, so $\varphi_*\mu$ has the right support. Concerning continuity,

$$\int_H f(y)\, d(\varphi_*\mu)^v(y) = \int_G f(\varphi(x))\, d(\mu^{\varphi^{-1}(v)})(x)$$

is clearly continuous in v for fixed $f \in C_c(H)$ by continuity of μ and because $f \circ \varphi \in C_c(G)$. Finally, we verify left-invariance:

$$\int_H f(xy)\, d(\varphi_*\mu)^{d(x)}(y) = \int_G f(x\varphi(z))\, d(\mu^{\varphi^{-1}(d(x))})(z) = \int_G f(x\varphi(z))\, d(\mu^{d(\varphi^{-1}(x))})(z) =$$
$$= \int_G f(\varphi(z))\, d(\mu^{r(\varphi^{-1}(x))})(z) = \int_G f(\varphi(z))\, d(\mu^{\varphi^{-1}(r(x))})(z) = \int_H f(y)\, d(\varphi_*\mu)^{r(x)}(y),$$

where we have used left-invariance of μ and $x\varphi(z) = \varphi(\varphi^{-1}(x)z)$ for $z \in \operatorname{supp}(\mu^{d(\varphi^{-1}(x))})$ which holds since $(\varphi^{-1}(x), z) \in G^{(2)}$ by Lemma 2.1.6 since $z \in G^{d(\varphi^{-1}(x))}$.

Now let $E \subseteq G^{(0)}$ be open or both locally closed and invariant. Then $\varphi(E) \subseteq H^{(0)}$ also has these properties, hence we have the Haar systems $\mu|_E$ and $(\varphi_*\mu)|_{\varphi(E)}$ on $G|_E$ and $H|_{\varphi(E)}$, respectively, by Proposition 2.2.13. Note that $\varphi(G|_E) = d_H^{-1}(\varphi(E)) \cap r_H^{-1}(\varphi(E)) = H|_{\varphi(E)}$, so $\varphi|_{G|_E} \colon G|_E \to H|_{\varphi(E)}$ is a groupoid homeomorphism and we can form $(\varphi|_{G|_E})_*(\mu|_E)$. If $v \in \varphi(E) = (H|_{\varphi(E)})^{(0)}$ and $A \subseteq H|_{\varphi(E)}$ is any Borel set, then $\varphi^{-1}(A) \subseteq G|_E$ and

$$((\varphi|_{G|_E})_*(\mu|_E))^v(A) = \mu^{\varphi^{-1}(v)}|_{G|_E}(\varphi^{-1}(A) \cap G|_E) =$$
$$= \mu^{\varphi^{-1}(v)}(\varphi^{-1}(A)) = (\varphi_*\mu)^v|_{H|_{\varphi(E)}}(A) = ((\varphi_*\mu)|_{\varphi(E)})^v(A)$$

is as claimed. ∎

Recall that, given any family of groupoids, we can form their disjoint union, see Definition 2.1.26. Similarly, given two groupoids G and H, the product $G \times H$ from Definition 2.1.24 is again a groupoid. We now produce Haar systems for these constructions from Haar systems on the original groupoids.

PROPOSITION 2.2.17 (Haar systems on the disjoint union of groupoids). *Let G_i, $i \in I$, be a countable family of locally compact groupoids with Haar systems μ_i and let $G := \bigsqcup_{i \in I} G_i$ be the disjoint union groupoid. Then $G^{(0)} = \bigsqcup_{i \in I} (G_i)^{(0)}$ and*

$$\mu^u(A) := (\mu_i)^u(A \cap G_i) \quad \textit{if } u \in G_i^{(0)}$$

for every Borel set $A \subseteq G$ defines a left Haar system for G, which we will denote by $\sqcup_{i \in I}\mu_i$.

PROOF. Note that μ^u is the disjoint union measure $\sqcup_{i \in I}\nu_i$, see Example A.4.10, where $\nu_i = \mu_i^u$ if $u \in (G_i)^{(0)}$ and $\nu_i = 0$ otherwise. Therefore, $\operatorname{supp}(\mu^u) = \operatorname{supp}((\mu_i)^u) = (G_i)^u = G^u$ for $u \in G_i^{(0)}$. If $f \in C_c(G)$, then

$$G_i^{(0)} \to \mathbb{C}, \quad u \mapsto \int_G f\, d\mu^u = \int_{G_i} f\, d(\mu_i)^u$$

is continuous for every $i \in I$, and so the function $G^{(0)} \to \mathbb{C}$, $u \mapsto \int_G f\, d\mu^u$ is continuous on the disjoint union $G^{(0)} = \bigsqcup_{i \in I} G_i^{(0)}$. Finally, for $x \in G_i \subseteq G$, we have $d(x) \in G_i^{(0)}$ and

$$\int_G f(xy)\, d\mu^{d(x)}(y) = \int_{G_i} f|_{G_i}(xy)\, d(\mu_i)^{d(x)}(y) = \int_{G_i} f|_{G_i}(y)\, d(\mu_i)^{r(x)}(y) = \int_G f(y)\, d\mu^{r(x)}(y)$$

since $f|_{G_i} \in C_c(G_i)$. ■

PROPOSITION 2.2.18 (Haar systems on the product of groupoids). *Let G and G be locally compact groupoids with Haar systems μ and ν, respectively. Then*

$$\mu \times \nu \colon (G \times H)^{(0)} = G^{(0)} \times H^{(0)} \to \mathcal{M}_+(G \times H), \quad (\mu \times \nu)^{(u,v)} := \mu^u \times \nu^v$$

is a Haar system on the product groupoid $G \times H$.

PROOF. The product measures satisfy $\operatorname{supp}(\mu^u \times \nu^v) = \operatorname{supp}(\mu^u) \times \operatorname{supp}(\nu^v)$, so that $\mu \times \nu$ has the right support properties. Next, if $f \in C_c(G \times H)$, then

$$\int_{G \times H} f\, d(\mu \times \nu)^{(u,v)} = \int_G \int_H f(x,y)\, d\nu^v(y)\, d\mu^u(x) = \int_H \int_G f(x,y)\, d\mu^u(x)\, d\nu^v(y)$$

is continuous both in u and in v since $x \mapsto \int_H f(x,y)\, d\nu^v(y)$ and $y \mapsto \int_G f(x,y)\, d\mu^u(x)$ are continuous with compact support. Concerning left invariance,

$$\begin{aligned} &\int_{G \times H} f((x,y)(w,z))\, d(\mu \times \nu)^{d(x,y)}(w,z) = \\ &= \int_G \int_H f(xw, yz)\, d\mu^{d_G(x)}(w)\, d\nu^{d_H(y)}(z) = \int_G \int_H f(w,z)\, d\mu^{r_G(x)}(w)\, d\nu^{r_H(y)}(z) = \\ &= \int_{G \times H} f(w,z)\, d(\mu \times \nu)^{r(x,y)}(w,z) \end{aligned}$$

for $(x,y) \in G \times H$ and we are done. ■

2.3. Convolution algebras associated to locally compact groupoids

Fix a locally compact groupoid G and a left Haar system μ on it. In this section, we wish to equip the space $C_c(G)$ of all continuous complex-valued functions on G with compact support with certain algebraic structures that are compatible with the Haar system.

DEFINITION 2.3.1. For $f, g \in C_c(G)$ and $x \in G$ let

$$(f * g)(x) := \int_G f(xy) g(y^{-1})\, d\mu^{d(x)}(y) \quad \text{and} \quad f^*(x) := \overline{f(x^{-1})}.$$

We call $f * g$ the *convolution* of f and g and $f \mapsto f^*$ the *involution* of $C_c(G)$. To emphasize the dependence on the Haar system μ, we shall also write $C_c(G, \mu)$ for $C_c(G)$ together with these operations.

REMARK 2.3.2. (i) Note that convolution may also be written as

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, d\mu^{r(x)}(y), \tag{2.3.1}$$

since $\int_G f(y) g(y^{-1}x)\, d\mu^{r(x)}(y) = \int_G f(xy) g((xy)^{-1}x)\, d\mu^{d(x)}(y)$ by left-invariance of μ and the fact that $(xy)^{-1}x = y^{-1}x^{-1}x = y^{-1}$ by (i) of Proposition 2.1.3.

(ii) The following principle will be convenient later: if $F \in C_c(G \times G)$, $x \in G$ and $u \in G^{(0)}$, then

$$\int_G \int_G F(x,y)\, d\mu^{r(x)}(y)\, d\mu^u(x) = \int_{G^u} \int_{G^u} F(x,y)\, d\mu^u(y)\, d\mu^u(z) = \int_G \int_G F(x,y)\, d\mu^{r(y)}(x)\, d\mu^u(y). \tag{2.3.2}$$

Indeed, if $x \in \operatorname{supp}(\mu^u) = G^u$, then $r(x) = u$, showing the first equality. The second is just an application of Fubini's theorem and applying the same reasoning as in the first step.

PROPOSITION 2.3.3. *With convolution and the involution $f \mapsto f^*$, the space $C_c(G, \mu)$ is a *-algebra, see Definition C.1.1.*

PROOF. First, note that by the continuity assumptions on the Haar system and by continuity of the source map d, the convolution $f * g$ of two functions $f, g \in C_c(G, \mu)$ is continuous on G. For $f, g \in C_c(G, \mu)$ we have that $\operatorname{supp}(f * g)$ is compact, since

$$\begin{aligned}\operatorname{supp}(f * g) &\subseteq \{x \in G : \exists y \in G^{d(x)} \text{ such that } f(xy) \neq 0 \text{ and } g(y^{-1}) \neq 0\} \\ &\subseteq \{x \in G : \exists y \in G^{d(x)} \text{ such that } xy \in \operatorname{supp}(f) \text{ and } y^{-1} \in \operatorname{supp}(g)\}\end{aligned}$$

and for x and y as in the last line we have

$$x = x(yy^{-1}) = (xy)y^{-1} \in \{wz : (w, z) \in G^{(2)} \cap (\operatorname{supp}(f) \times \operatorname{supp}(g))\},$$

and the latter set is compact. Therefore, $f * g \in C_c(G, \mu)$. Moreover, $\operatorname{supp}(f^*) = \operatorname{supp}(f)^{-1}$, so $C_c(G, \mu)$ is closed under these operations.

We now show that convolution is associative. Let $f, g, h \in C_c(G, \mu)$ and compute for $x \in G$

$$\begin{aligned}(f * (g * h))(x) &= \int_G f(xy)(g * h)(y^{-1})\, d\mu^{d(x)}(y) \\ &= \int_G \int_G f(xy)g(y^{-1}z)h(z^{-1})\, d\mu^{r(y)}(z)\, d\mu^{d(x)}(y) \\ &= \int_G \int_G f(xy)g(y^{-1}z)h(z^{-1})\, d\mu^{r(z)}(y)\, d\mu^{d(x)}(z) \\ &= \int_G \int_G f(xzy)g(y^{-1})h(z^{-1})\, d\mu^{d(z)}(y)\, d\mu^{d(x)}(z) \\ &= \int_G (f * g)(xz)h(z^{-1})\, d\mu^{d(x)}(z) = ((f * g) * h)(x),\end{aligned}$$

where we have used $d(y^{-1}) = r(y)$ in the second line, (2.3.2) in the third, left-invariance in the fourth and finally $d(z) = d(xz)$ in the fifth so that the inner integral equals $(f * g)(xz)$.

Obviously $f \mapsto f^*$ is conjugate linear and involutive. Hence it remains to show that $(f * g)^* = g^* * f^*$ for all $f, g \in C_c(G, \mu)$. But

$$(f * g)^*(x) = \int_G \overline{f(x^{-1}y)g(y^{-1})}\, d\mu^{r(x)}(y) = \int_G \overline{f(y)}g^*(xy)\, d\mu^{d(x)}(y) = (g^* * f^*)(x)$$

for all $x \in G$ by left-invariance. ∎

REMARK 2.3.4. It is not hard to show (see [**Ren80**, II, Proposition 1.1] for the calculations) that $C_c(G, \mu)$ becomes a topological *-algebra with respect to the inductive limit topology in this way, meaning that both convolution and involution are continuous with respect to this topology.

EXAMPLE 2.3.5. Let $G = S \rtimes_\alpha \Gamma$ be a locally compact transformation groupoid, see Example 2.1.22, equipped with its canonical Haar system μ from Example 2.2.12. Then for $f, g \in C_c(G, \mu)$, their convolution is given by

$$(f * g)(s, g) = \int_S \int_\Gamma f((s, g)(t, h))g(t \cdot h, h^{-1})\, d\lambda(h)\, d\delta_{s \cdot g}(t) = \int_\Gamma f(s, gh)g(s \cdot gh, h^{-1})\, d\lambda(h)$$

because $d(s, g) = s \cdot g$. If $E \subseteq S$ is an open or both locally closed and invariant subset, then Example 2.2.14 implies that the convolution on the reduction $G|_E$ is given by

$$(f * g)(s, g) = \int_{\{h \in \Gamma : s \cdot gh \in E\}} f(s, gh)g(s \cdot gh, h^{-1})\, d\lambda(h)$$

for all $f, g \in C_c(G|_E, \mu|_E)$, where $\mu|_E$ is the Haar system from Proposition 2.2.13. ◊

The Haar system allows us to define a norm on $C_c(G,\mu)$ in the following way:

$$\|f\|_I := \max\left\{ \sup_{u\in G^{(0)}} \int_G |f|\,d\mu^u,\ \sup_{u\in G^{(0)}} \int_G |f|\,d\mu_u \right\}, \tag{2.3.3}$$

where we again set $\mu_u := \mathrm{inv}_*(\mu^u)$, the image measure under the inversion (see Proposition 2.2.2). Evidently, $\|\cdot\|_I$ is a seminorm on $C_c(G,\mu)$. If $f\in C_c(G,\mu)$ is non-zero, then there is $u\in G^{(0)}$ such that $f|_{G^u}\neq 0$ (the sets G^u form a partition of G), and therefore $\int_G |f|\,d\mu^u \neq 0$ since $\mathrm{supp}(\mu^u) = G^u$. Alternatively, we may write (2.3.3) as

$$\|f\|_I = \max\left\{ \sup_{u\in G^{(0)}} \|f\|_{L^1(G,\mu^u)},\ \sup_{u\in G^{(0)}} \|f^*\|_{L^1(G,\mu^u)} \right\}.$$

DEFINITION 2.3.6. The completion of $C_c(G,\mu)$ with respect to the I-norm is denoted by $L^I(G)$ or $L^I(G,\mu)$.

PROPOSITION 2.3.7. *The norm $\|\cdot\|_I$ is a *-algebra norm on $C_c(G,\mu)$, that is, $\|f*g\|_I \le \|f\|_I\|g\|_I$ and $\|f^*\|_I = \|f\|_I$ for all $f,g\in C_c(G,\mu)$. In particular, convolution and involution extend to continuous operations on $L^I(G,\mu)$ which then becomes a Banach *-algebra.*

PROOF. By (2.3.1), (2.3.2) and left-invariance, we have

$$\begin{aligned}
\int_G |f*g|\,d\mu^u &\le \int_G\int_G |f(y)||g(y^{-1}x)|\,d\mu^{r(x)}(y)\,d\mu^u(x)\\
&= \int_G\int_G |f(y)||g(y^{-1}x)|\,d\mu^{r(y)}(x)\,d\mu^u(y)\\
&= \int_G |f(y)| \int_G |g(x)|\,d\mu^{d(y)}(x)\,d\mu^u(y)\\
&\le \left(\sup_{v\in G^{(0)}} \int_G |g(x)|\,d\mu^v(x)\right) \int_G |f(y)|\,d\mu^u(y) \le \|g\|_I\|f\|_I
\end{aligned}$$

for all $f,g\in C_c(G)$ and $u\in G^{(0)}$. In the same way, one shows that $\int_G |f*g|\,d\mu_u \le \|g\|_I\|f\|_I$ and hence we have $\|f*g\|_I \le \|f\|_I\|g\|_I$ for all $f,g\in C_c(G,\mu)$.

Finally, the involution is isometric with respect to $\|\cdot\|_I$ since $\int_G |f^*|\,d\mu^u = \int_G |f\circ\mathrm{inv}|\,d\mu^u = \int_G |f|\,d\mu_u$ by the definition of the right Haar system $u\mapsto\mu_u$. ∎

LEMMA 2.3.8. *The topology induced on $C_c(G,\mu)$ by the I-norm is coarser than the inductive limit topology.*

PROOF. We have to check that the identity map $(C_c(G,\mu),\tau_{\text{ind. limit}}) \to (C_c(G,\mu),\|\cdot\|_I)$ is continuous. Denoting by $\widetilde{\mu}\colon C_c(G,\mu)\to C_c(G^{(0)})$ the continuous map $\widetilde{\mu}(f)(u) := \int_G f\,d\mu^u$ from Lemma 2.2.5, we have

$$\|f\|_I = \max\left\{\|\widetilde{\mu}(|f|)\|_\infty, \|\widetilde{\mu}(|f^*|)\|_\infty\right\},$$

so for a net $f_i\in C_c(G,\mu)$ with $f_i\to 0$ in the inductive limit topology we also have $\|f_i\|_I\to 0$ because the involution of $C_c(G,\mu)$ is also continuous for the inductive limit topology, see Remark 2.3.4. ∎

The following technical Lemma will be useful later on. The proof is quite involved, so we will only reference it.

LEMMA 2.3.9 ([**MRW87**, Corollary 2.11]). *The algebra $C_c(G,\mu)$ has a two-sided approximate identity for the inductive limit topology. More precisely, there exists a net $\iota\mapsto\varepsilon_\iota\in C_c(G,\mu)$ such that $\varepsilon_\iota * f\to f$ and $f*\varepsilon_\iota\to f$ in the inductive limit topology for every $f\in C_c(G,\mu)$.*

REMARK 2.3.10. The above result from [**MRW87**] applies to locally compact groupoids G with paracompact unit space $G^{(0)}$. Since our groupoids are all assumed to be second countable, $G^{(0)}$ is also second countable, locally compact (since it is closed in G, see Example *A*.1.3) and Hausdorff. Proposition *A*.3.3 then implies that $G^{(0)}$ is paracompact. We point out that it has already been shown in [**Ren80**, Proposition 1.9] that $C_c(G, \mu)$ has one-sided approximate identities, without requiring $G^{(0)}$ to be paracompact.

2.4. The full and reduced groupoid C^*-algebras

In the previous section, we have equipped the space $C_c(G)$ for a locally compact groupoid G with Haar system μ with algebraic operations and a norm such that the completion $L^I(G, \mu)$ with respect to this norm is a Banach *-algebra. We now want to go one step further and obtain C^*-algebras from $C_c(G)$, which allows us to use the tools of C^*-algebra theory (see Appendix C) in our applications.

2.4.1. The full groupoid C^*-algebra. In this subsection, we will introduce the *full* groupoid C^*-algebra, while the next subsection will deal with the *reduced* groupoid C^*-algebra. Both of these C^*-algebras have their pros and cons, and depending on the situation, one may be easier to handle than the other.

DEFINITION 2.4.1. The *(full) groupoid C^*-algebra of G* (with respect to the Haar system μ), denoted by $C^*(G)$ or $C^*(G, \mu)$, is the completion of $C_c(G, \mu)$ with respect to the *universal norm*

$$\|f\|_{C^*(G,\mu)} := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm continuous representation of } C_c(G, \mu)\}.$$

It will be shown in Proposition 2.4.18 below that this really defines a norm. Note that any representation π of the *-algebra $C_c(G, \mu)$ which is I-norm continuous extends to a representation of $L^I(G, \mu)$. Since this is a Banach *-algebra, we have $\|\pi(f)\| \leq \|f\|_I$ for all $f \in C_c(G, \mu)$ by Proposition C.2.2, so π is actually non-increasing for the I-norm.

REMARK 2.4.2. By Lemma 2.3.8, any representation of $C_c(G, \mu)$ as above is also continuous with respect to the inductive limit topology. Moreover, one can show that these representations define the same norm as the universal norm, that is,

$$\|f\|_{C^*(G,\mu)} = \sup_{\pi} \|\pi(f)\|,$$

where the supremum is taken over all representations of $C_c(G, \mu)$ which are continuous with respect to the inductive limit topology. The proof of this fact is rather involved and can, for example, be found in [**Goe09**, Corollary 4.20].

LEMMA 2.4.3. *Let A be a C^*-algebra and $\psi \colon C_c(G, \mu) \to A$ be a *-homomorphism that is continuous with respect to the inductive limit topology or the I-norm. Then $\|\psi(f)\|_A \leq \|f\|_{C^*(G,\mu)}$ for all $f \in C_c(G, \mu)$ and hence ψ extends to a *-homomorphism from $C^*(G, \mu)$ to A.*

PROOF. The proof is an adaptation of [**Wil07**, Corollary 2.47]. If ψ is continuous for the I-norm on $C_c(G, \mu)$, then also with respect to the inductive limit topology by Lemma 2.3.8. Let $L \colon A \to \mathscr{L}(H)$ be a faithful (that is, injective) representation of A, see Proposition C.5.3. Then $L \circ \psi \colon C_c(G, \mu) \to \mathscr{L}(H)$ is a representation of $C_c(G, \mu)$ which is continuous for the inductive limit topology. Since injective *-homomorphisms between C^*-algebras are isometries by Proposition C.3.8, we obtain

$$\|\psi(f)\|_A = \|L(\psi(f))\|_{\mathscr{L}(H)} \leq \|f\|_{C^*(G,\mu)}$$

by applying Remark 2.4.2. ∎

LEMMA 2.4.4. *If G is a locally compact groupoid with left Haar system μ, then the topology on $C_c(G, \mu)$ as a subspace of $C^*(G, \mu)$ is coarser than the inductive limit topology.*

PROOF. This follows immediately from Lemma 2.3.8 and $\|f\|_{C^*(G,\mu)} \le \|f\|_I$ for all $f \in C_c(G,\mu)$, which implies that the topology of $C_c(G,\mu)$ as a subspace of $C^*(G,\mu)$ is coarser than its topology as a subspace of $L^I(G,\mu)$. ■

We now investigate which isomorphisms of groupoids give rise to isomorphisms of the corresponding full groupoid C^*-algebras. Corollary 2.4.7 below is the basis for all the examples that will follow. Instead of directly proving it, we will find it as an immediate consequence of the next Proposition. While it does not seem to be part of the standard literature, Proposition 2.4.5 is a slightly more illustrative way of looking at this isomorphism, although the proof is virtually the same as that of Corollary 2.4.7 and is based on [**Wil07**, Corollary 2.48].

PROPOSITION 2.4.5. *Let G and H be locally compact groupoids, μ a left Haar system on G and $\varphi\colon G \to H$ a groupoid homeomorphism. Then the map*

$$\varphi^\sharp\colon C_c(H) \to C_c(G), \quad \varphi^\sharp(f) := f \circ \varphi,$$

extends to an isomorphism of $C^*(H,\varphi_*\mu)$ onto $C^*(G,\mu)$, where $\varphi_*\mu$ is the pushforward of μ by φ, see Definition 2.2.15.

REMARK 2.4.6. It seems to be unknown whether all Haar systems on a fixed groupoid give rise to isomorphic C^*-algebras. If this were the case, then any groupoid homeomorphism between locally compact groupoids with Haar systems would induce an isomorphism of the corresponding C^*-algebras by Proposition 2.4.5. What has been shown is that the C^*-algebras corresponding to different Haar systems are strongly Morita equivalent, see [**MRW87**, Theorem 2.8].

PROOF OF PROPOSITION 2.4.5. We first show that $\varphi^\sharp\colon C_c(H,\varphi_*\mu) \to C_c(G,\mu)$ is a *-homomorphism. Take $f,g \in C_c(H)$ and compute, using Proposition 2.1.10,

$$\begin{aligned}\varphi^\sharp(f*g)(x) = (f*g)(\varphi(x)) = \int_H f(\varphi(x)y)g(y^{-1})\,d(\varphi_*\mu)^{d(\varphi(x))}(y) = \\ = \int_G f(\varphi(xy))g(\varphi(y^{-1}))\,d\mu^{d(x)}(y) = (\varphi^\sharp(f)*\varphi^\sharp(g))(x)\end{aligned}$$

and

$$(\varphi^\sharp(f))^*(x) = \overline{\varphi^\sharp(f)(x^{-1})} = \overline{f(\varphi(x^{-1}))} = \overline{f(\varphi(x)^{-1})} = f^*(\varphi(x)) = \varphi^\sharp(f^*)(x)$$

for all $x \in G$. Evidently, $\varphi^\sharp$ is continuous for the inductive limit topologies. By Lemma 2.4.4, $\varphi^\sharp$ can be viewed as a continuous map from $C_c(H,\varphi_*\mu)$ to $C^*(G,\mu)$ and by Lemma 2.4.3 we can extend it to a *-homomorphism $C^*(H,\varphi_*\mu) \to C^*(G,\mu)$. In the same way, the inverse φ^{-1} gives rise to a *-homomorphism $C^*(G,\mu) = C^*(G,(\varphi^{-1})_*(\varphi_*\mu)) \to C^*(H,\varphi_*\mu)$ which is inverse to this extension of $\varphi^\sharp$ on a dense subspace, hence everywhere. ■

COROLLARY 2.4.7 ([**MR82**, Proposition 2.10]). *Let G and H be two locally compact groupoids, equipped with left Haar systems μ and ν, respectively, and assume there is a groupoid homeomorphism $\varphi\colon G \to H$ such that*

$$\int_G f \circ \varphi \, d\mu^u = \int_H f \, d\nu^{\varphi(u)} \tag{2.4.1}$$

for all $f \in C_c(H)$ and all $u \in G^{(0)}$. Then the C^*-algebras $C^*(G,\mu)$ and $C^*(H,\nu)$ are isomorphic. If $E \subseteq G^{(0)}$ is open or both locally closed and invariant, then also $C^*(G|_E,\mu|_E) \cong C^*(H|_{\varphi(E)},\nu|_{\varphi(E)})$.

PROOF. The condition (2.4.1) is just a restatement of $\nu = \varphi_*\mu$, so the first claim follows from Proposition 2.4.5. If $E \subseteq G^{(0)}$ is open or both locally closed and invariant, then $\varphi|_{G|_E}\colon G|_E \to H_{\varphi(E)}$ is a groupoid homeomorphism and $\nu|_{\varphi(E)} = (\varphi|_{G|_E})_*(\mu|_E)$ by Lemma 2.2.16. This shows the second claim. ■

As the last result in this section, we will state (parts of) Renault's Disintegration theorem for representations of $C_c(G, \mu)$. It will allow us to obtain representations of the full groupoid C^*-algebra starting from homomorphisms that need only satisfy relatively weak assumptions and which are easy to verify.

THEOREM 2.4.8 (Renault's Disintegration Theorem, [**Ren87**, Theorem 4.2]). *Let G be a locally compact groupoid with Haar system μ and let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Suppose that H_0 is a dense subspace of H and that Ψ is a homomorphism (that is, linear and multiplicative) from $C_c(G, \mu)$ to the algebra of linear maps on H_0 and such that*

(i) the set $\{\Psi(f)h : f \in C_c(G, \mu),\ h \in H_0\}$ is dense in H,

(ii) for every $h, k \in H_0$, the map

$$C_c(G, \mu) \to \mathbb{C}, \quad f \mapsto \langle\Psi(f)h, k\rangle$$

is continuous for the inductive limit topology, and

(iii) for every $f \in C_c(G, \mu)$ and $h, k \in H_0$,

$$\langle\Psi(f)h, k\rangle = \langle h, \Psi(f^*)k\rangle.$$

*Then the linear maps $\Psi(f)$ all extend to bounded operators $\widetilde{\Psi}(f)$ on H and $C_c(G, \mu) \to \mathscr{L}(H)$, $f \mapsto \widetilde{\Psi}(f)$ is an I-norm decreasing *-representation of $C_c(G, \mu)$.*

REMARK 2.4.9. Theorem 2.4.8 is actually only a part of what is known as Renault's Disintegration Theorem in the literature. A *representation* of a locally compact groupoid with Haar system is a triple $(\mu, G^{(0)} * H, L)$, where μ is a quasi-invariant measure on $G^{(0)}$ (see Definition 2.4.15 below) and a Borel Hilbert bundle $G^{(0)} * H$ over $G^{(0)}$, and $L\colon G \to \mathrm{Iso}(G^{(0)} * H)$ is a groupoid homomorphism from G into the *isomorphism groupoid* $\mathrm{Iso}(G^{(0)} * H)$. We refer to the literature, for example [**Muh97**, **Goe09**, **Pat99**], for the details on groupoid representation theory. One can integrate such a groupoid representation to a representation of $C_c(G, \mu)$, in a way similar to how one integrates covariant representations to obtain representations of the crossed product, see Example 2.6.4 below. In fact, this theory generalizes to the setting of *groupoid crossed products*, accounts of which can be found in [**MW08**, **Goe09**]. The integrated form is then a representation of $C_c(G, \mu)$ on the Hilbert space of square-integrable sections of the Hilbert bundle. Using this terminology, it can be shown that the representation $\widetilde{\Psi}$ from Theorem 2.4.8 is unitarily equivalent to the integrated form of such a groupoid representation, hence the name "Disintegration Theorem". Because the proof of the Disintegration Theorem uses—among other things—the theory of Borel Hilbert bundles, we refer to [**Muh97**, Theorem 3.29] for a proof.

2.4.2. Special representations and the reduced groupoid C^*-algebra. As announced, we will now introduce another C^*-algebra obtained from G and μ, the *reduced groupoid C^*-algebra*. This C^*-algebra will again be a completion of $C_c(G, \mu)$, but now with respect to a smaller norm than that of the full groupoid C^*-algebra.

For a positive Radon measure λ on the unit space $G^{(0)}$, consider on $C_c(G)$ the positive linear functional

$$f \mapsto \int_{G^{(0)}} \int_G f(x)\, d\mu^u(x)\, d\lambda(u).$$

Note that this makes sense as $u \mapsto \int_G f\, d\mu^u$ is in $C_c(G^{(0)})$ by Lemma 2.2.5. By the Riesz representation theorem, there is a unique Radon measure $\lambda \circ \mu$ on G representing it and we formally write

$$\lambda \circ \mu = \int_{G^{(0)}} \mu^u\, d\lambda(u)$$

for this measure. We call $\lambda \circ \mu$ the *measure induced by* λ, and we set $(\lambda \circ \mu)^{-1} := \mathrm{inv}_*(\lambda \circ \mu)$. If there is no confusion as to which measure is meant, we will put $\nu := \lambda \circ \mu$ for brevity.

REMARK 2.4.10. (i) It follows that

$$\int_G f \, d(\lambda \circ \mu) = \int_{G^{(0)}} \int_G f(x) \, d\mu^u(x) \, d\lambda(u)$$

holds for all $f \in C_c(G)$. By [**Bou04**, Theorem 1, V.24], this formula even remains valid *for all positive Borel-measurable and all $(\lambda \circ \mu)$-integrable functions f*. See [**Goe09**, Proposition 3.109] for an alternative proof. Note that

$$\int_G f \, d(\lambda \circ \mu)^{-1} = \int_{G^{(0)}} \int_G f(x^{-1}) \, d\mu^u(x) \, d\lambda(u) = \int_{G^{(0)}} \int_G f(x) \, d\mu_u(x) \, d\nu(u),$$

so we may write $(\lambda \circ \mu)^{-1} = \int_{G^{(0)}} \mu_u \, d\lambda(u)$.

(ii) If $A \subseteq G$ is a $(\lambda \circ \mu)$-null set, then the above implies that A is μ^u-null for λ-a.e. $u \in G^{(0)}$. Similarly, $(\lambda \circ \mu)^{-1}(A) = 0$ implies $\mu_u(A) = 0$ for λ-a.e. $u \in G^{(0)}$.

(iii) Let $E \subseteq G^{(0)}$ be λ-null. We argue that $d^{-1}(E)$ is a ν^{-1}-null set, where $\nu := \lambda \circ \mu$. Indeed, we have

$$\nu^{-1}(d^{-1}(E)) = \int_{G^{(0)}} \int_{G_u} \chi_{d^{-1}(E)}(x) \, d\mu_u(x) \, d\lambda(u),$$

where $\chi_{d^{-1}(E)}$ is the characteristic function of $d^{-1}(E)$. If $u \in G^{(0)} \setminus E$, then $\int_{G_u} \chi_{d^{-1}(E)} \, d\mu_u = \int_{d^{-1}(\{u\})} \chi_{d^{-1}(E)} \, d\mu_u = 0$. Therefore, $\nu^{-1}(d^{-1}(E)) = 0$ since $\lambda(E) = 0$. In the same way, one shows that $r^{-1}(E)$ is a ν-null set.

(iv) If $E \subseteq G^{(0)}$ is a λ-null set, then it follows from (iii) that the reduction $G|_E = d^{-1}(E) \cap r^{-1}(E)$ satisfies $\nu(G|_E) = \nu^{-1}(G|_E) = 0$. In particular, if $F := \operatorname{supp}(\lambda)$, then $\lambda(G^{(0)} \setminus F) = 0$, so that $\nu(G \setminus G|_F) = 0$. This shows that $\operatorname{supp}(\nu) \subseteq G|_F$, and similarly $\operatorname{supp}(\nu^{-1}) \subseteq G|_F$.

PROPOSITION 2.4.11. *For $f, \psi \in C_c(G)$ and $x \in G$, we define*

$$(\operatorname{Ind}_\lambda(f)\psi)(x) := (f * \psi)(x) = \int_G f(xy)\psi(y^{-1}) \, d\mu^{d(x)}(y). \tag{2.4.2}$$

Then $\operatorname{Ind}_\lambda(f)$ extends to a bounded operator on $L^2(G, (\lambda \circ \mu)^{-1})$ with $\|\operatorname{Ind}_\lambda(f)\| \le \|f\|_I$ for all $f \in C_c(G)$. Moreover, $f \mapsto \operatorname{Ind}_\lambda(f)$ is a representation of $C_c(G, \mu)$ on $L^2(G, (\lambda \circ \mu)^{-1})$.

REMARK 2.4.12. By the Proposition, $\operatorname{Ind}_\lambda$ extends to a representation of both $L^I(G, \mu)$ and $C^*(G, \mu)$. We call $\operatorname{Ind}_\lambda$ the *representation induced off the unit space by the measure λ*.

PROOF. Put $\nu := \lambda \circ \mu$. We first show that $\operatorname{Ind}_\lambda(f)$ is bounded with respect to the $L^2(G, \nu^{-1})$-norm for all $f \in C_c(G, \mu)$. For every $\psi \in C_c(G)$, we have

$$\int_G |\operatorname{Ind}_\lambda(f)\psi|^2 \, d\nu^{-1} \le \int_G \left(\int_G |f(x^{-1}y)\psi(y^{-1})| \, d\mu^{r(x)}(y) \right)^2 d\nu(x) =$$
$$= \int_{G^{(0)}} \int_G \int_G \int_G |f(x^{-1}y)f(x^{-1}z)\psi(y^{-1})\psi(z^{-1})| \, d\mu^{r(x)}(y) \, d\mu^{r(x)}(z) \, d\mu^u(x) \, d\lambda(u).$$

By (2.3.2), this equals

$$\int_{G^{(0)}} \int_{G^u} \int_{G^u} \int_{G^u} |f(x^{-1}y)f(x^{-1}z)\psi(y^{-1})\psi(z^{-1})| \, d\mu^u(y) \, d\mu^u(z) \, d\mu^u(x) \, d\lambda(u)$$

and we can use Fubini's theorem to exchange the order of integration. Using $|\psi(y^{-1})\psi(z^{-1})| \le \frac{1}{2}(|\psi(y^{-1})|^2 + |\psi(z^{-1})|^2)$ and noting that both terms contribute to the integrals in the same way, we obtain

$$\|\operatorname{Ind}_\lambda(f)\psi\|^2_{L^2(G,\nu^{-1})} \le \int_{G^{(0)}} \int_{G^u} \int_{G^u} \int_{G^u} |f(x^{-1}y)f(x^{-1}z)\psi(y^{-1})^2| \, d\mu^u(z) \, d\mu^u(x) \, d\mu^u(y) \, d\lambda(u).$$

We first consider the z-integral. Note that $r(x) = u$ for $x \in G^u = \operatorname{supp}(\mu^u)$, hence

$$\int_{G^{r(x)}} |f(x^{-1}y)f(x^{-1}z)\psi(y^{-1})^2| \, d\mu^{r(x)}(z) = \\ = |f(x^{-1}y)\psi(y^{-1})^2| \int_{G^{d(x)}} |f(z)| \, d\mu^{d(x)}(z) \leq \|f\|_I \, |f(x^{-1}y)\psi(y^{-1})^2|$$

by left-invariance. Now the x-integral contributes another factor $\|f\|_I$ since $r(y) = u$ for $y \in G^u$ and

$$\int_{G^{r(y)}} |f(x^{-1}y)\psi(y^{-1})^2| \, d\mu^{r(y)}(x) = |\psi(y^{-1})^2| \int_{G^{d(y)}} |f(x^{-1})| \, d\mu^{d(y)}(x) \leq \|f\|_I \, |\psi(y^{-1})^2|.$$

The remaining integral $\int_{G^{(0)}} \int_{G^u} |\psi(y^{-1})|^2 \, d\mu^u(y) \, d\lambda(u)$ is precisely $\|\psi\|^2_{L^2(G,\nu^{-1})}$. Summarizing, we have

$$\|\mathrm{Ind}_\lambda(f)\psi\|^2_{L^2(G,\nu^{-1})} \leq \|f\|_I^2 \|\psi\|^2_{L^2(G,\nu^{-1})},$$

and density of $C_c(G)$ in $L^2(G, \nu^{-1})$ (see Proposition A.4.12) shows that $\mathrm{Ind}_\lambda(f)$ extends to a bounded operator on $L^2(G, \nu^{-1})$ with norm $\|\mathrm{Ind}_\lambda(f)\| \leq \|f\|_I$. Next,

$$\mathrm{Ind}_\lambda(f * g)\psi = (f * g) * \psi = f * (g * \psi) = \mathrm{Ind}_\lambda(f)\mathrm{Ind}_\lambda(g)\psi$$

for $f, g, \psi \in C_c(G, \mu)$ by associativity of convolution and this continues to hold for $\psi \in L^2(G, \nu^{-1})$. Finally, for $\psi, \varphi \in C_c(G)$, we have

$$\begin{aligned}
\langle \mathrm{Ind}_\lambda(f^*)\psi, \varphi \rangle &= \int_G \int_G \overline{f((xy)^{-1})}\psi(y^{-1})\overline{\varphi(x)} \, d\mu^{d(x)}(y) \, d\nu^{-1}(x) \\
&= \int_{G^{(0)}} \int_{G^{r(x)}} \int_{G^u} \overline{f(y^{-1}x)}\psi(y^{-1})\overline{\varphi(x^{-1})} \, d\mu^{r(x)}(y) \, d\mu^u(x) \, d\lambda(u) \\
&= \int_{G^{(0)}} \int_{G^{r(y)}} \int_{G^u} \overline{f(y^{-1}x)}\psi(y^{-1})\overline{\varphi(x^{-1})} \, d\mu^{r(y)}(x) \, d\mu^u(y) \, d\lambda(u) \\
&= \int_G \int_G \psi(x)\overline{f(xy)\varphi(y^{-1})} \, d\mu^{d(x)}(y) \, d\nu^{-1}(x) = \langle \psi, \mathrm{Ind}_\lambda(f)\varphi \rangle,
\end{aligned}$$

where we have used (2.3.2) to interchange the x and y integrals. This shows that $\mathrm{Ind}_\lambda(f^*) = \mathrm{Ind}_\lambda(f)^*$ and that Ind_λ is a representation. ∎

DEFINITION 2.4.13. The *reduced groupoid C^*-algebra of G* (with respect to the Haar system μ), denoted by $C^*_{\mathrm{red}}(G, \mu)$ or $C^*_{\mathrm{red}}(G)$, is the C^*-completion of $C_c(G)$ with respect to the norm

$$\|f\|_{C^*_{\mathrm{red}}(G,\mu)} := \sup_{\lambda \in \mathcal{M}_+(G^{(0)})} \|\mathrm{Ind}_\lambda(f)\|_{\mathscr{L}(L^2(G,\nu^{-1}))}, \tag{2.4.3}$$

where λ ranges over all positive Radon measures on $G^{(0)}$.

REMARK 2.4.14. As $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} \leq \|f\|_I$, the topology of $C_c(G, \mu)$ as a subspace of $C^*_{\mathrm{red}}(G, \mu)$ is coarser than the inductive limit topology, compare with Lemmas 2.3.8 and 2.4.4. In other words, the inclusion $C_c(G, \mu) \hookrightarrow C^*_{\mathrm{red}}(G, \mu)$ is continuous.

We now want to show that $\|\cdot\|_{C^*_{\mathrm{red}}(G,\mu)}$ indeed defines a norm on $C_c(G, \mu)$. To this end, we consider special measures on $G^{(0)}$. Recall that two measures (on the same measurable space) are called *equivalent* if they are mutually absolutely continuous, meaning they have the same null sets. We will denote this equivalence relation by $\sim$.

DEFINITION 2.4.15. A Radon measure $\lambda \in \mathcal{M}_+(G^{(0)})$ is called *quasi-invariant* if ν and ν^{-1} are equivalent measures, where $\nu = \int_{G^{(0)}} \mu^u \, d\lambda(u)$ is the induced measure.

The idea is to show that there exist "enough" quasi-invariant measures on $G^{(0)}$. Let $\lambda \in \mathcal{M}_+(G^{(0)})$ be a Radon measure. A calculation similar to that of Lemma 2.4.16 below then shows that the image measure $d_*\nu$, where $\nu = \int_{G^{(0)}} \mu^u \, d\lambda(u)$ is the induced measure, gives rise to a measure which satisfies $(d_*\nu) \circ \mu \sim ((d_*\nu) \circ \mu)^{-1}$. A technical problem is that $d_*\nu$ need not be Radon (which would be the case if d were a proper map, see Example A.4.11). To this end, we use the process of *saturation* from [Ren80, p. 25]. Note that ν is σ-finite since G is assumed to be second countable, hence σ-compact by Lemma A.3.1. By Lemma A.4.14, ν is equivalent to a finite Radon measure $\widetilde{\nu}$. The resulting image measure $d_*\widetilde{\nu}$ is then also finite, hence Radon by Proposition A.4.6. In the language of [Bou04], $d_*\widetilde{\nu}$ is a *quasi-image* of ν.

LEMMA 2.4.16. *For every measure $\lambda \in \mathcal{M}_+(G^{(0)})$, the image measure $d_*\widetilde{\nu}$ on $G^{(0)}$ under the domain map $d\colon G \to G^{(0)}$ is quasi-invariant, where $\widetilde{\nu}$ is a finite Radon measure on G with $\nu \sim \widetilde{\nu}$ and $\nu = \int_{G^{(0)}} \mu^u \, d\lambda(u)$ is the induced measure.*

PROOF. We have to show that $(d_*\widetilde{\nu}) \circ \mu \sim ((d_*\widetilde{\nu}) \circ \mu)^{-1}$. Let $f := \chi_A$ for a Borel set $A \subseteq G$ and denote by $h := d\widetilde{\nu}/d\nu \in L^1(G,\nu)$ the Radon-Nikodym derivative. We can choose (a representative of) h such that $0 < h(x) < \infty$ for all $x \in G$. Then

$$\begin{aligned}
\int_G f \, d((d_*\widetilde{\nu}) \circ \mu) &= \int_G \int_G f(x) \, d\mu^{d(y)}(x) \, d\widetilde{\nu}(y) \\
&= \int_{G^{(0)}} \int_G \int_G f(x) \, d\mu^{d(y)}(x) \, h(y) \, d\mu^v(y) \, d\lambda(v) \\
&= \int_{G^{(0)}} \int_G \int_G f(y^{-1}x) h(y) \, d\mu^{r(y)}(x) \, d\mu^v(y) \, d\lambda(v) \\
&= \int_{G^{(0)}} \int_G \int_G f(y^{-1}x) h(y) \, d\mu^v(x) \, d\mu^v(y) \, d\lambda(v) \\
&= \int_{G^{(0)}} \int_G \int_G f((x^{-1}y)^{-1}) h(y) \, d\mu^v(y) \, d\mu^v(x) \, d\lambda(v) \\
&= \int_G \int_G f(y^{-1}) h(xy) \, d\mu^{d(x)}(y) \, d\nu(x),
\end{aligned}$$

where we have used left-invariance and the fact that $r(y) = v$ for $y \in \operatorname{supp}(\mu^v) = r^{-1}(\{v\})$ and Fubini's theorem to interchange the order of the μ^v integrals. Therefore, $((d_*\widetilde{\nu}) \circ \mu)(A) = 0$ if and only if for ν-almost every x and $\mu^{d(x)}$-almost every y we have $f(y^{-1})h(xy) = 0$. Because h is positive everywhere, this is equivalent to $f(y^{-1})h(x) = 0$ for these x and y. But this is equivalent to

$$0 = \int_G \int_{G^{d(x)}} f(y^{-1}) \, d\mu^{d(x)}(y) \, h(x) \, d\nu(x) = \int_G f(y^{-1}) \, d((d_*\widetilde{\nu}) \circ \mu)(y)$$

and we are done. ∎

LEMMA 2.4.17. *For every $u \in G^{(0)}$ there exists a quasi-invariant measure λ on $G^{(0)}$ such that $[u] \subseteq \operatorname{supp}(\lambda)$, where $[u] = d(r^{-1}(\{u\}))$ is the orbit of u, see Lemma 2.1.7.*

PROOF. Consider the Dirac measure δ_u and its induced measure $\int_{G^{(0)}} \mu^v \, d\delta_u(v) = \mu^u$. An application of Lemma 2.4.16 yields the quasi-invariant measure $\lambda := d_*\widetilde{\mu^u}$, where $\widetilde{\mu^u}$ is a finite measure on G such that $\mu^u \sim \widetilde{\mu^u}$. By Example A.4.11, we get $[u] = d(r^{-1}(\{u\})) = d(\operatorname{supp}(\mu^u)) = d(\operatorname{supp}(\widetilde{\mu^u})) \subseteq \operatorname{supp}(\lambda)$. ∎

PROPOSITION 2.4.18. *Both the reduced norm $\|\cdot\|_{C^*_{\mathrm{red}}(G,\mu)}$ and the universal norm $\|\cdot\|_{C^*(G,\mu)}$ are C^*-norms on $C_c(G,\mu)$.*

PROOF. It only remains to show that if $f \in C_c(G)$ with $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = 0$, then $f = 0$. The same conclusion for the universal norm then follows from $\|\cdot\|_{C^*_{\mathrm{red}}(G,\mu)} \leq \|\cdot\|_{C^*(G,\mu)}$. We first show that

$\ker(\mathrm{Ind}_\lambda) = \{f \in C_c(G,\mu) : f \text{ vanishes on } \mathrm{supp}(\nu)\}$ for every quasi-invariant measure λ on $G^{(0)}$, where $\nu := \lambda \circ \mu$. If $f|_{\mathrm{supp}(\nu)} = 0$, then (2.3.1) and (2.3.2) give

$$\int_G |\mathrm{Ind}_\lambda(f)\xi| \, d\nu \leq \int_G \int_G |f(y)||\xi(y^{-1}x)| \, d\mu^{r(y)}(x) \, d\nu(y) = 0$$

and hence $\mathrm{Ind}_\lambda(f)\xi = 0$ almost everywhere for ν for all $\xi \in C_c(G)$, so $f \in \ker(\mathrm{Ind}_\lambda)$. Conversely, if $\mathrm{Ind}_\lambda(f) = 0$, then $f * \varepsilon_\iota = 0$ in $L^2(G, \nu^{-1})$ for a right approximate identity ε_ι of $C_c(G,\mu)$, see Lemma 2.3.9. But then $f = \lim_\iota(f * \varepsilon_\iota) = 0$ in $L^2(G,\nu^{-1})$, so $f = 0$ on $\mathrm{supp}(\nu^{-1}) = \mathrm{supp}(\nu)$ because λ is quasi-invariant (equivalent measures have the same support).

Now let $f \in C_c(G)$ be such that $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = 0$, so that $f \in \ker(\mathrm{Ind}_\lambda)$ for every quasi-invariant measure λ. Take $x \in G$ and let $u := r(x)$. Pick a quasi-invariant measure λ such that $[u] \subseteq \mathrm{supp}(\nu)$, see Lemma 2.4.17. We have $x \in \mathrm{supp}(\nu)$, for if $g \in C_c(G,[0,1])$ satisfies $g(x) > 0$, then $\widetilde{\mu}(g) \colon v \mapsto \int_{G^v} g \, d\mu^v$ is in $C_c(G^{(0)})$ by Lemma 2.2.5 and satisfies $\widetilde{\mu}(g)(u) > 0$, because $\mathrm{supp}(\mu^u) = G^u$ and $x \in G^u$. But then $\int_G g \, d\nu = \int_{G^{(0)}} \widetilde{\mu}(g)(v) \, d\lambda(v) > 0$ because $u \in [u] \subseteq \mathrm{supp}(\lambda)$. This shows that $x \in \mathrm{supp}(\nu)$ and hence $f(x) = 0$ by the above. ■

REMARK 2.4.19. Consider for $u \in G^{(0)}$ the Dirac measure δ_u on $G^{(0)}$. Then the induced measure is $\nu := \int_{G^{(0)}} \mu^v \, d\delta_u(v) = \mu^u$ and $\nu^{-1} = \mu_u$, and we may consider the representation $\mathrm{Ind}_u := \mathrm{Ind}_{\delta_u}$ of $C_c(G)$ on $L^2(G,\mu_u) = L^2(G,\nu^{-1})$. Explicitly,

$$\mathrm{Ind}_u(f)\psi(x) = \int_G f(xy)\psi(y^{-1}) \, d\mu^u(y) \quad (x \in G) \tag{2.4.4}$$

for $f, \psi \in C_c(G)$ since $d(x) = u$ for $x \in G_u = \mathrm{supp}(\mu_u)$. The reduced norm $\|\cdot\|_{C^*_{\mathrm{red}}(G,\mu)}$ may alternatively be described using the representations Ind_u. In fact, it holds that

$$\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = \sup_{u \in G^{(0)}} \|\mathrm{Ind}_u(f)\|_{\mathscr{L}(L^2(G,\mu_u))}. \quad (f \in C_c(G)) \tag{2.4.5}$$

The proof of this result can be found in [**Pat99**, Proposition 3.1.2], where it is shown that even

$$\|\mathrm{Ind}_\lambda(f)\| = \sup_{u \in \mathrm{supp}(\lambda)} \|\mathrm{Ind}_u(f)\| \tag{2.4.6}$$

holds for all measures $\lambda \in \mathcal{M}_+(\Omega)$. In particular, $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = \|\mathrm{Ind}_\lambda(f)\|$ for measures λ of full support, so that the representation Ind_λ of $C^*_{\mathrm{red}}(G,\mu)$ is faithful. By Lemma *A*.4.13, such measures always exist. Note that (2.4.5) and (2.4.6) continue to hold for $f \in C^*_{\mathrm{red}}(G,\mu)$ by a simple continuity argument.

As for the full groupoid C^*-algebra, groupoid homeomorphisms induce isomorphisms between the reduced groupoid C^*-algebras. In fact, we have literally the same result as for the full algebra, see Proposition 2.4.5:

PROPOSITION 2.4.20. *Let G and H be locally compact groupoids, μ a left Haar system on G and $\varphi \colon G \to H$ a groupoid homeomorphism. Then the map*

$$\varphi^\sharp \colon C_c(H) \to C_c(G), \quad \varphi^\sharp(f) := f \circ \varphi,$$

extends to an isomorphism of $C^*_{\mathrm{red}}(H, \varphi_*\mu)$ onto $C^*_{\mathrm{red}}(G,\mu)$, where $\varphi_*\mu$ is the pushforward of μ by φ, see Definition 2.2.15. Consequently, Corollary 2.4.7 continuous to hold for the reduced C^*-algebras.

PROOF. For every $u \in G^{(0)}$, the map $\xi \mapsto \xi \circ \varphi$ is an isomorphism from $L^2(H, (\varphi_*\mu)_{\varphi(u)})$ to $L^2(G, \mu_u)$ because

$$\int_G (\xi \circ \varphi)\overline{(\eta \circ \varphi)} \, d\mu_u = \int_G \xi(\varphi(x^{-1}))\overline{\eta(\varphi(x^{-1}))} \, d\mu^u(x) =$$
$$= \int_H \xi(y^{-1})\overline{\eta(y^{-1})} \, d(\varphi_*\mu)^{\varphi(u)}(y) = \int_H \xi\overline{\eta} \, d(\varphi_*\mu)_{\varphi(u)}$$

holds for all $\xi, \eta \in L^2(H, (\varphi_*\mu)_{\varphi(u)})$. Similarly as in the computation in Proposition 2.4.5, it follows that

$$((\mathrm{Ind}_{\varphi(u)}(f)\xi) \circ \varphi)(x) = \int_H f(\varphi(x)y)\xi(y^{-1})\, d(\varphi_*\mu)^{\varphi(u)}(y) =$$
$$= \int_G f(\varphi(xy))\xi(\varphi(y^{-1}))\, d\mu^u(y) = \mathrm{Ind}_u(\varphi^\sharp(f))(\xi \circ \varphi)(x) \quad (x \in G)$$

for all $f \in C_c(H)$ and $\xi \in C_c(H)$, hence also for $\xi \in L^2(H, (\varphi_*\mu)_{\varphi(u)})$. With (2.4.5), this implies

$$\|\varphi^\sharp(f)\|_{C^*_{\mathrm{red}}(G,\mu)} = \sup_{v \in H^{(0)}} \|\mathrm{Ind}_v(\varphi^\sharp(f))\|_{\mathscr{L}(L^2(H,(\varphi_*\mu)_v))} =$$
$$= \sup_{u \in G^{(0)}} \|\mathrm{Ind}_u(f)\|_{\mathscr{L}(L^2(G,\mu_u))} = \|f\|_{C^*_{\mathrm{red}}(H,\varphi_*\mu)}$$

and $\varphi^\sharp$ extends to a *-isomorphism from $C^*_{\mathrm{red}}(H, \varphi_*\mu)$ to $C^*_{\mathrm{red}}(G, \mu)$. ■

2.4.3. Equality of the full and reduced groupoid C^*-algebras. In what follows, we will be interested in determining the C^*-algebras of the most important types of groupoids. For some applications, the full algebra will be easier to handle, while for others it might be advantageous to determine the reduced groupoid C^*-algebra. In order to reap the benefits of both worlds, it is therefore useful to ask for conditions on the groupoid G such that $C^*(G) = C^*_{\mathrm{red}}(G)$. A general condition for equality of the C^*-algebras is that of *amenability*, the primary authority on which is the book [**ADR00**]. For this section, we will omit the proofs of most results, as they would be too long to incorporate into this thesis.

DEFINITION 2.4.21 ([**ADR00**, Proposition 2.2.13]). A locally compact groupoid G with left Haar system μ is called *(topologically) amenable* if there exists a net $(f_\iota)_\iota$ in $C_c(G)$ with $f_\iota \geq 0$ and

(i) $\int_G f_\iota \, d\mu^u \leq 1$ for all $u \in G^{(0)}$ and all ι,
(ii) $\lim_\iota \int_G f_\iota \, d\mu^u = 1$ uniformly in u on compact subsets of $G^{(0)}$, and
(iii) $\lim_\iota \int_G |f_\iota(x^{-1}y) - f_\iota(y)| \, d\mu^{r(x)}(y) = 0$ uniformly in x on compact subsets of G.

THEOREM 2.4.22 ([**ADR00**, Proposition 6.1.8]). *For any locally compact amenable groupoid G with left Haar system μ, we have $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = \|f\|_{C^*(G,\mu)}$ for all $f \in C_c(G,\mu)$. In particular,*

$$C^*(G, \mu) = C^*_{\mathrm{red}}(G, \mu).$$

REMARK 2.4.23. The above Definition 2.4.21 is not the original definition from [**ADR00**], but it is the one best suited for our purposes. There are also more general notions of amenability for groupoids, for instance *measurewise amenability* which also generalizes to so-called *Borel groupoids*. Theorem 2.4.22 continuous to hold for measurewise amenable locally compact groupoids.

One important observation is that amenability is preserved when taking reductions by nice enough subsets of the unit space:

PROPOSITION 2.4.24 ([**ADR00**, Proposition 5.1.1]). *If G is a locally compact amenable groupoid with left Haar system μ and $E \subseteq G^{(0)}$ is open or both locally closed and invariant, then $G|_E$ with its Haar system $\mu|_E$ from Proposition 2.2.13 is also amenable.*

REMARK 2.4.25. We point out that in the statement of [**ADR00**, Proposition 5.1.1], H is supposed to be a locally closed subgroupoid of G such that $r|_H, d|_H$ are open maps onto $H^{(0)}$. However, by our assumptions in Proposition 2.4.24, $G|_E$ has a left Haar system so that the maps $r|_{G|_E}$ and $d|_{G|_E}$ are open by Corollary 2.2.7.

In our applications, we will be mostly interested in locally compact transformation groupoids. For this, we will need the concept of an amenable group.

DEFINITION 2.4.26. A locally compact group Γ with left Haar measure λ is called *amenable* if there is a state m on the C^*-algebra $L^\infty(\Gamma, \lambda)$ (that is, a positive linear functional with $m(1) = 1$) such that $m(s \cdot f) = m(f)$ for all $f \in L^\infty(\Gamma, \lambda)$ and all $s \in \Gamma$, where $(s \cdot f)(t) := f(s^{-1}t)$. The state m is then called a *left-invariant mean* on Γ.

EXAMPLE 2.4.27. All compact groups are amenable. Indeed, if we take λ to be normalized, then $m \colon L^\infty(\Gamma, \lambda) \to \mathbb{C}$, $m(f) := \int_\Gamma f \, d\lambda$ is a left-invariant mean. $\Diamond$

EXAMPLE 2.4.28. All locally compact abelian groups are amenable. We give a rough sketch of the argument from [Wil07]. If G is abelian, then it is shown in [Gre69, Theorem 1.2.1] that the discrete group G_d, which is just G equipped with the discrete topology, is amenable. Thus, there is a left-invariant mean on $L^\infty(G_d) = \ell^\infty(G)$, and this mean restricts to a left-invariant mean on the C^*-subalgebra $C_b(G)$ of $\ell^\infty(G)$. (Here, functions in $C_b(G)$ are of course supposed to be continuous with respect to the original topology of G.) Moreover, one can show that the existence of a left-invariant mean on $C_b(G)$ already implies the existence of a mean on $L^\infty(G)$, see [Wil07, Theorem A.13], so that G is amenable. $\Diamond$

As is the case for groupoids, there are many equivalent conditions for the amenability of a group. The following one from [Wil07, Proposition A.16 (d)] is best suited for our next goal of showing that the corresponding transformation groupoids are then amenable:

LEMMA 2.4.29. *A locally compact group Γ is amenable if and only if there exists a net $g_\iota \in C_c(\Gamma)$ with $g_i \geq 0$ such that*

(i) $\int_\Gamma g_\iota \, d\lambda = 1$ for all ι, and

(ii) $\lim_\iota \int_\Gamma |g_\iota(s^{-1}t) - g_\iota(t)| \, d\lambda(t) = 0$ uniformly in s on compact subsets of Γ.

With this result, the following Proposition is now no surprise.

PROPOSITION 2.4.30. *If $G = X \rtimes_\alpha \Gamma$ is a locally compact transformation groupoid with amenable group Γ, then G is an amenable groupoid with respect to its canonical Haar system $\mu^x := \delta_x \times \lambda$, $x \in X$, where λ is the Haar measure of Γ. In particular,*

$$C^*(X \rtimes_\alpha \Gamma) = C^*_{\mathrm{red}}(X \rtimes_\alpha \Gamma).$$

PROOF. Let $g_\iota \in C_c(\Gamma)$, $\iota \in \mathcal{I}$, be the net from Lemma 2.4.29. For every compact subset $K \subseteq X$, let $h_K \in C_c(X, [0,1])$ be such that $h_K|_K = 1$, see Lemma A.2.3. If $\mathcal{K}$ denotes the collection of compact subsets of X, then we can make $\mathcal{K} \times \mathcal{I}$ into a directed set by letting

$$(K_1, \iota_1) \leq (K_2, \iota_2) \text{ if and only if } K_1 \subseteq K_2 \text{ and } \iota_1 \leq \iota_2.$$

Put $f_{(K,\iota)}(x,t) := h_K(x) g_\iota(t)$. Then $f_{(K,\iota)}$ is a net in $C_c(G)$ with $f_{(K,\iota)} \geq 0$ and if $K_0 \subseteq X$ is compact, then

$$\int_G f_{(K,\iota)}(y,s) \, d\mu^x(y,s) = \int_\Gamma h_K(x) g_\iota(s) \, d\lambda(s) = 1$$

for all $x \in K_0$ if $K \subseteq X$ is compact with $K_0 \subseteq K$. This shows $\lim_{(K,\iota)} \int_G f_{(K,\iota)} \, d\mu^x = 1$ uniformly in x on compact subsets of $G^{(0)} \cong X$.

If $L \subseteq G = X \times \Gamma$ is compact, then $\mathrm{pr}_1(L)$ and $\mathrm{pr}_2(L)$ are both compact in X and Γ, respectively, and

$$\begin{aligned} \int_G &|f_{(K,\iota)}((x,t)^{-1}(y,s)) - f_{(K,\iota)}(y,s)| \, d\mu^{r(x,t)}(y,s) = \\ &= \int_\Gamma |f_{(K,\iota)}(x \cdot t, t^{-1}s) - f_{(K,\iota)}(x,s)| \, d\lambda(s) = \int_\Gamma |h_K(x \cdot t) g_\iota(t^{-1}s) - h_K(x) g_\iota(s)| \, d\lambda(s) = \\ &\qquad = \int_\Gamma |g_\iota(t^{-1}s) - g_\iota(s)| \, d\lambda(s) \end{aligned}$$

for all $(x,t) \in L$ if K is such that $\mathrm{pr}_1(L) \cdot (\mathrm{pr}_2(L) \cup \{e\}) \subseteq K$, because then $h_K(x \cdot t) = h_K(x) = 1$. By the assumptions on g_ι, this converges to zero uniformly for $t \in \mathrm{pr}_2(L)$. Thus, $f_{(K,\iota)}$ also satisfies the final requirement of Definition 2.4.21 and $X \rtimes_\alpha \Gamma$ is therefore amenable. ■

While we will not make much use of it, the requirements of Definition 2.4.21 are very easy to verify for the cotrivial and trivial groupoids, so that we mention them in the following two examples:

EXAMPLE 2.4.31. Let X be a locally compact, second countable Hausdorff space and let $G := X$ be the cotrivial groupoid over X, see Example 2.1.19. By Example 2.2.10, $\mu^x := \delta_x$ for $x \in X$ is a left Haar system for G. For every compact subset $K \subseteq X$, let $f_K \in C_c(X, [0,1])$ be such that $f_K|_K = 1$. Ordering the compact subsets of X by inclusion, we obtain a net in $C_c(X)$ which satisfies all the requirements of Definition 2.4.21. In fact, $\int_X |f_K(x^{-1}y) - f_K(y)| \, d\delta_x(y)$ is even identically zero because $x^{-1}x = x$. Thus, G is amenable and $C^*(X,\mu) = C^*_{\mathrm{red}}(X,\mu)$. ◊

EXAMPLE 2.4.32. Let X be a locally compact, second countable Hausdorff space and let $G := X \times X$ be the trivial groupoid over X, see Example 2.1.20. By Example 2.2.11, any Radon measure ν on X with $\mathrm{supp}(\nu) = X$ gives a left Haar system for G by $\mu^x := \delta_x \times \nu$ for $x \in X$. For every compact subset $K \subseteq X$, let $g_K \in C_c(X, [0,1])$ be such that $g_K|_K = 1$ and put

$$f_K \colon X \times X \to \mathbb{C}, \quad f_K(x,y) := \frac{g_K(x) g_K(y)}{\|g_K\|_{L^1(X,\nu)}}.$$

If the compact subsets of X are ordered by inclusion, f_K is a nonnegative net in $C_c(G)$ and while we will omit the details, it is straightforward to verify that it satisfies the requirements of Definition 2.4.21, so that G is amenable and $C^*(X \times X, \mu) = C^*_{\mathrm{red}}(X \times X, \mu)$. ◊

2.5. Invariant subsets and ideals

Suppose that G is a locally compact groupoid equipped with a left Haar system μ. In this section, we will associate ideals of $C^*(G,\mu)$ to open invariant subsets of the unit space $G^{(0)}$. The quotient of $C^*(G,\mu)$ by this ideal is then isomorphic to the groupoid C^*-algebra of a suitable reduction of G. The result will also allow us to describe the kernel of the representation Ind_λ of $C^*_{\mathrm{red}}(G,\mu)$ induced by a measure λ on $G^{(0)}$, as defined in Proposition 2.4.11.

DEFINITION 2.5.1. For $U \subseteq G^{(0)}$ open and invariant we let $\mathrm{Ex}(U)$ be the closure in $C^*(G,\mu)$ of the set

$$I_c(U) := \{f \in C_c(G,\mu) : \mathrm{supp}(f) \subseteq G|_U\},$$

where $G|_U$ is the reduction of G to U, see Definition 2.1.11, which we equip with the left Haar system $\mu|_U$ from Proposition 2.2.13.

LEMMA 2.5.2. *For each open invariant subset $U \subseteq G^{(0)}$, the set $\mathrm{Ex}(U)$ is a two-sided ideal in $C^*(G,\mu)$.*

PROOF. For $f \in I_c(U)$ and $g \in C_c(G,\mu)$ we found in the proof of Proposition 2.3.3 that the support of $f * g$ is contained in

$$\{zw : (z,w) \in G^{(2)} \cap (\mathrm{supp}(f) \times \mathrm{supp}(g))\}.$$

By Proposition 2.1.15, this is a subset of $(G|_U)^{(2)}$ since $\mathrm{supp}(f) \subseteq G|_U$. Therefore, $(z,w) \in G|_U \times G|_U$ and since $G|_U$ is a groupoid, $zw \in G|_U$. It follows that $f * g \in I_c(U)$ and the same argument can be used to show $g * f \in I_c(U)$. Moreover, $I_c(U)$ is *-invariant since $x \notin G|_U$ implies $x^{-1} \notin G|_U$, so $f^*(x) = \overline{f(x^{-1})} = 0$ in this case. Thus, we have shown that $I_c(U)$ is an ideal in $C_c(G,\mu)$, so the closure $\mathrm{Ex}(U)$ is an ideal in $C^*(G,\mu)$. ■

Let $U \subseteq G^{(0)}$ be an open invariant subset. Consider the map

$$j_0 \colon C_c(G|_U, \mu|_U) \to C_c(G,\mu), \quad j_0(f)(x) = \begin{cases} f(x), & x \in G|_U \\ 0, & x \notin G|_U \end{cases}$$

which extends functions by zero outside the open set $G|_U$. Then j_0 is a *-homomorphism. Indeed, take $f, g \in C_c(G|_U)$. By Lemma 2.5.2 we know that $j_0(f) * j_0(g)$ is zero outside $G|_U$. For $x \in G|_U$, we have $G^{d(x)} = (G|_U)^{d(x)} = G|_U$ by invariance of U and hence

$$\begin{aligned}(j_0(f) * j_0(g))(x) &= \int_{G^{d(x)}} j_0(f)(xy) j_0(g)(y^{-1})\, d\mu^{d(x)}(y) \\ &= \int_{G^{d(x)}} f(xy)g(y^{-1})\, d\mu^{d(x)}(y) = \int_{G|_U} f(xy)g(y^{-1})\, d(\mu|_U)^{d(x)}(y) = (f*g)(x)\end{aligned}$$

by using (2.2.3). We conclude that $j_0(f) * j_0(g) = j_0(f*g)$ and obviously $j_0(f^*) = j_0(f)^*$, so j_0 is a *-homomorphism. Moreover, j_0 is clearly continuous for the inductive limit topologies, hence extends to a *-homomorphism $j \colon C^*(G|_U, \mu|_U) \to C^*(G,\mu)$ by Lemmas 2.4.3 and 2.4.4.

Now let $F \subseteq G^{(0)}$ be closed and invariant. Consider the restriction map

$$\rho_0 \colon C_c(G,\mu) \to C_c(G|_F, \mu|_F), \quad f \mapsto f|_{G|_F}.$$

Then ρ_0 is a *-homomorphism, for if $f, g \in C_c(G)$ and $x \in G|_F$, then again $G^{d(x)} = (G|_F)^{d(x)} \subseteq G|_F$ and

$$(\rho_0(f) * \rho_0(g))(x) = \int_{G|_F} f|_{G|_F}(xy) g|_{G|_F}(y^{-1})\, d(\mu|_F)^{d(x)}(y) = \int_{G^{d(x)}} f(xy)g(y^{-1})\, d\mu^{d(x)}(y)$$

by (2.2.3), which is nothing but $\rho_0(f*g)(x)$. It is also clear that $\rho_0(f^*) = \rho_0(f)^*$, so ρ_0 is a *-homomorphism which as above extends to a *-homomorphism $\rho \colon C^*(G,\mu) \to C^*(G|_F, \mu|_F)$.

THEOREM 2.5.3. *Let G be a locally compact groupoid with left Haar system μ. Suppose $U \subseteq G^{(0)}$ is open and invariant and set $F := G^{(0)} \setminus U$. Then we have the short exact sequence of C^*-algebras*

$$0 \to C^*(G|_U, \mu|_U) \xrightarrow{j} C^*(G,\mu) \xrightarrow{\rho} C^*(G|_F, \mu|_F) \to 0.$$

Moreover, $\operatorname{im}(j) = \ker(\rho) = \operatorname{Ex}(U)$ *and, in particular,*

$$\operatorname{Ex}(U) \cong C^*(G|_U, \mu|_U) \quad \textit{and} \quad C^*(G,\mu)/\operatorname{Ex}(U) \cong C^*(G|_F, \mu|_F).$$

PROOF. We adapt the proof of [**Goe09**, Theorem 5.22] to our situation, where the same result is shown for the groupoid crossed products. Note that j_0 maps $C_c(G|_U, \mu|_{G|_U})$ onto $I_c(U)$, so we have $\operatorname{im}(j) = \operatorname{Ex}(U)$ because *-homomorphisms between C^*-algebras have closed range, see Proposition C.4.4. We now want to show that j is isometric, hence an isomorphism onto its image $\operatorname{Ex}(U)$. To this end, we will use Renault's Disintegration Theorem, see Theorem 2.4.8. Take a faithful, non-degenerate representation $\pi \colon C^*(G|_U) \to \mathscr{L}(H)$. These always exist by Proposition C.5.3. Let

$$H_0 := \operatorname{span}(\{\pi(g)h : g \in C_c(G|_U, \mu|_U),\ h \in H\})$$

and note that H_0 is dense in H because π is non-degenerate. For $f \in C_c(G)$ and $h = \sum_{i=1}^n \pi(g_i)h_i \in H_0$, we define

$$\Psi(f)h := \sum_{i=1}^n \pi\big((f * j_0(g_i))|_{G|_U}\big)h_i,$$

which makes sense since $f * j_0(g_i)$ has support in $G|_U$ by Lemma 2.5.2.

We now show that Ψ is a well-defined homomorphism from $C_c(G)$ to the space of linear operators on H_0. Evidently, Ψ is linear. Suppose that $h = \sum_{i=1}^n \pi(g_i)h_i = 0$. Let $\iota \mapsto \varepsilon_\iota \in$

$C_c(G|_U)$ be an approximating identity for the algebra $C_c(G|_U)$, see Lemma 2.3.9, which implies that $\varepsilon_\iota * g_i \to g_i$ for the inductive limit topology. Then

$$\Psi(f)h = \sum_{i=1}^{n} \pi\big((f * \lim_\iota j_0(\varepsilon_\iota * g_i))|_{G|_U}\big)h_i =$$

$$= \lim_\iota \sum_{i=1}^{n} \pi\big(((f * j_0(\varepsilon_\iota)) * j_0(g_i))|_{G|_U}\big)h_i = \lim_\iota \sum_{i=1}^{n} \pi\big((f * j_0(\varepsilon_\iota))|_{G|_U}\big)\pi(g_i)h_i = 0$$

because π is a homomorphism and convolution as well as π and j_0 are continuous with respect to the inductive limit topology, see Remark 2.3.4 and Lemma 2.4.4. In the second to last equality, we have used that[4]

$$((f * j_0(\varepsilon_\iota)) * j_0(g_i))|_{G|_U} = (j_0(\widetilde{f_\iota}) * j_0(g_i))|_{G|_U} = j_0(\widetilde{f_\iota} * g_i)|_{G|_U} = \widetilde{f_\iota} * g_i,$$

where $\widetilde{f_\iota} := (f * j_0(\varepsilon_\iota))|_{G|_U} \in C_c(G|_U)$ satisfies $j_0(\widetilde{f_\iota}) = f * j_0(\varepsilon_\iota)$. Therefore, $\Psi(f)h = 0$ for all $f \in C_c(G)$, which shows that Ψ is well-defined. Moreover, Ψ is multiplicative. Indeed, observe first that

$$\Psi(f * g)h = \sum_{i=1}^{n} \pi\big(((f * g) * j_0(g_i))|_{G|_U}\big)h_i = \sum_{i=1}^{n} \pi\big((f * (g * j_0(g_i)))|_{G|_U}\big)h_i$$

and

$$\Psi(f)(\Psi(g)h) = \Psi(f)\sum_{i=1}^{n} \pi\big((g * j_0(g_i))|_{G|_U}\big)h_i = \sum_{i=1}^{n} \pi\big((f * j_0(\widetilde{g}_i))|_{G|_U}\big)h_i$$

with $\widetilde{g}_i := (g * j_0(g_i))|_{G|_U} \in C_c(G|_U)$. But from $\operatorname{supp}(g * j_0(g_i)) \subseteq G|_U$ we get $j_0(\widetilde{g}_i) = g * j_0(g_i)$ and therefore $\Psi(f * g)h = \Psi(f)(\Psi(g)h)$. Next, we have

$$\Psi(j_0(f))h = \sum_{i=1}^{n} \pi\big((j_0(f) * j_0(g_i))|_{G|_U}\big)h_i = \sum_{i=1}^{n} \pi\big(j_0(f * g)|_{G_U}\big)h_i = \pi(f)\sum_{i=1}^{n} \pi(g_i)h_i = \pi(f)h$$

for all $f \in C_c(G|_U)$ because $j_0(f * g)|_{G|_U} = f * g$ and π is a homomorphism. This shows that $(\Psi \circ j_0)(f) = \pi(f)|_{H_0}$ for $f \in C_c(G|_U)$.

We will now verify the requirements of the Disintegration Theorem. If $\pi(f)h \in H_0$ with $f \in C_c(G|_U)$ and $h \in H$, then the net $\iota \mapsto j_0(\varepsilon_\iota)$ satisfies

$$\Psi(j_0(\varepsilon_\iota))(\pi(f)h) = \pi(\varepsilon_\iota * f)h \to \pi(f)h,$$

so we have that H_0 is contained in the closure of $\{\Psi(f)h : f \in C_c(G),\ h \in H_0\}$, and the latter set is therefore dense in H. Next,

$$\langle \Psi(f)(\pi(g)h), \pi(g')k\rangle = \big\langle \pi\big((f * j_0(g))|_{G|_U}\big)h, \pi(g')k\big\rangle$$

for $f \in C_c(G)$ and this is continuous in f with respect to the inductive limit topology on $C_c(G)$ because π and convolution are.[5] Therefore, $f \mapsto \langle \Psi(f)x, y\rangle$ is continuous for all pairs $x, y \in H_0$, because those are linear combinations of elements of the form $\pi(g)h$. Finally, we consider

$$\langle \Psi(f)(\pi(g)h), \pi(g')k\rangle = \langle \pi((f * j_0(g))|_{G|_U})h, \pi(g')k\rangle = \langle h, \pi(((f * j_0(g))|_{G_U})^* * g')k\rangle.$$

---

[4] What we really want here is that "the map of restricting to $G|_U$ is a *-homomorphism." But of course, $f|_{G|_U}$ need not be in $C_c(G|_U)$ for arbitrary $f \in C_c(G)$, so this is not well-defined. However, the functions which we take the restriction of have support in $G|_U$, so this works out nicely and it just needs some extra care to make precise.

[5] Also note that $f * j_0(g)$ has support contained in $G|_U$ and $\operatorname{supp}(f * j_0(g))$ is compact in G and hence also in $G|_U$. From this it immediately follows that $f \mapsto (f * j_0(g))|_{G|_U}$ is also continuous for the inductive limit topologies on $C_c(G)$ and $C_c(G|_U)$. Indeed, if f_ι is a net with $\operatorname{supp}(f_\iota) \subseteq K$ for all ι and $f_\iota \to f$ uniformly on G, then an argument similar to that in Lemma 2.5.2 shows that $\operatorname{supp}(f_\iota * j_0(g)) \subseteq \{wz : (w, z) \in G^{(2)} \cap (K \times \operatorname{supp}(g))\}$, a fixed compact subset of $G|_U$, and $f_\iota * j_0(g) \to f * j_0(g)$ uniformly on G, hence also on $G|_U$.

Now let $\widetilde{f} := (f * j_0(g))|_{G|_U} \in C_c(G|_U)$ and $\widetilde{f}' := (f^* * j_0(g'))|_{G|_U} \in C_c(G|_U)$. Then $j_0(\widetilde{f}) = f * j_0(g)$ and $j_0(\widetilde{f}') = f^* * j_0(g')$ and we have

$$\begin{aligned}((f * j_0(g))|_{G|_U})^* * g' &= \widetilde{f}^* * g' = j_0(\widetilde{f}^* * g')|_{G|_U} = (j_0(\widetilde{f})^* * j_0(g'))|_{G|_U} = \\ &= ((f * j_0(g))^* * j_0(g'))|_{G|_U} = (j_0(g^*) * (f^* * j_0(g')))|_{G|_U} = j_0(g^* * \widetilde{f}')|_{G|_U} = g^* * \widetilde{f}'.\end{aligned}$$

Therefore,

$$\langle \Psi(f)(\pi(g)h), \pi(g')k\rangle = \langle \pi(g)h, \pi\big((f^* * j_0(g'))|_{G|_U}\big)k\rangle = \langle \pi(g)h, \Psi(f^*)(\pi(g')k)\rangle$$

and the third and last requirement is also fulfilled.

By Theorem 2.4.8, Ψ extends to an I-norm bounded *-representation $C_c(G,\mu) \to \mathscr{L}(H)$ and to a representation $\widetilde{\Psi}$ of $C^*(G,\mu)$ on H by Lemma 2.4.3. Moreover, the identity $\Psi \circ j_0 = \pi(\cdot)|_{H_0}$ now implies $\widetilde{\Psi} \circ j = \pi$, so

$$\|f\|_{C^*(G|_U, \mu|_U)} = \|\pi(f)\| = \|\widetilde{\Psi}(j(f))\| \leq \|j(f)\|$$

and we obtain that j is isometric.

By Corollary *A*.3.5, ρ_0 is surjective and so is ρ as its image is closed. It remains to show that $\ker(\rho) = \mathrm{im}(j)$. Clearly, $\rho \circ j|_{C_c(G|_U)} = 0$, so that $\mathrm{im}(j) \subseteq \ker(\rho)$. Conversely, we now show that $\ker(\rho) \subseteq \mathrm{Ex}(U) = \mathrm{im}(j)$. Let ℓ be a faithful representation of $C^*(G,\mu)/\mathrm{Ex}(U)$ and set $L := \ell \circ \pi$, where $\pi\colon C^*(G,\mu) \to C^*(G,\mu)/\mathrm{Ex}(U)$ is the canonical projection. Then $\ker(L) = \mathrm{Ex}(U)$.

$$\begin{tikzcd}
C^*(G,\mu)/\mathrm{Ex}(U) \arrow[rr, dotted, "\cong"'] \arrow[ddr, bend right, "\ell"'] & & C^*(G|_F, \mu|_F) \arrow[ddl, dotted, bend left, "L_F"] \\
& C^*(G,\mu) \arrow[ul, "\pi"'] \arrow[ur, "\rho"] \arrow[d, "L"] & \\
& \mathscr{L}(H) &
\end{tikzcd}$$

Define $L_F\colon C_c(G|_F) \to \mathscr{L}(H)$ by $L_F(\rho(f)) := L(f)$. We now verify that this is well-defined. Let $f, g \in C_c(G)$ such that $\rho(f) = \rho(g)$ and let $K_\varepsilon := \{x \in G : |f(x) - g(x)| \geq \varepsilon\}$ for $\varepsilon > 0$. Then K_ε is a compact subset of G which is disjoint from $G|_F$ because $(f-g)|_{G|_F} = 0$. Let W be a relatively compact open neighborhood of K_ε with $K_\varepsilon \subseteq W \subseteq G|_U$ and choose $\varphi_\varepsilon \in C_c(G, [0,1])$ such that $\varphi_\varepsilon|_{K_\varepsilon} = 1$ and $\varphi_\varepsilon|_{G\setminus W} = 0$. Then $\mathrm{supp}(\varphi_\varepsilon(f-g)) \subseteq G|_U$, so $\varphi_\varepsilon(f-g) \in \mathrm{Ex}(U) = \ker(L)$. Moreover,

$$\sup_{x\in G} |\varphi_\varepsilon(x)(f(x) - g(x)) - (f(x) - g(x))| = \sup_{x \in G}(1 - \varphi_\varepsilon(x))|f(x) - g(x)| \leq \varepsilon,$$

so the sequence $(\varphi_{1/n}(f-g))_n$ converges uniformly on G to $f - g$ and satisfies $\mathrm{supp}(\varphi_{1/n}(f - g)) \subseteq \mathrm{supp}(f-g)$, so $\varphi_{1/n}(f-g) \to f - g$ in the inductive limit topology. It follows that $f - g \in \mathrm{Ex}(U)$ and therefore $L(f) = L(g)$, showing that L_F is well-defined.

Let $f \in C_c(G)$, $v \in C := (r(\mathrm{supp}(f)) \cup d(\mathrm{supp}(f))) \cap F$ and fix $\varepsilon > 0$. By the definition of a Haar system, the function $u \mapsto \int_G |f|\, d\mu^u$ on $G^{(0)}$ is continuous, so there is a relatively compact open neighborhood O_v of v such that

$$\int_G |f|\, d\mu^w \leq \int_G |f|\, d\mu^v + \varepsilon = \int_{G|_F} |\rho(F)|\, d(\mu|_F)^v + \varepsilon \leq \|\rho(f)\|_I + \varepsilon$$

for all $w \in O_v$, where we have used (2.2.3) and the I-norm is the one on $C_c(G|_F)$. Similarly, there is a relatively compact open neighborhood V_v of v such that

$$\int_G |f|\, d\mu_w \leq \int_G |f|\, d\mu_v + \varepsilon \leq \|\rho(f)\|_I + \varepsilon$$

for all $w \in V_v$. The family $\{O_v \cap V_v : v \in C\}$ is an open covering of the compact set C and hence there is a finite subcover $\{O_{v_k} \cap V_{v_k} : k \in \{1, \dots, N\}\}$ for some $v_k \in C$ and $N \in \mathbb{N}$. Put

$O := \bigcup_{k=1}^{N}(O_{v_k} \cap V_{v_k})$ and let $\varphi \in C_c(G^{(0)}, [0,1])$ be such that $\varphi|_C = 1$ and $\varphi|_{G^{(0)} \setminus O} = 0$. Let $g := (\varphi \circ r)(\varphi \circ d) f \in C_c(G)$. Then for $v \in G^{(0)}$,

$$\int_G |g| \, d\mu^v \leq \int_{G^v} |\varphi(r(x)) f(x)| \, d\mu^v(x) = \varphi(v) \int_G |f(x)| \, d\mu^v(x) \begin{cases} \leq \|\rho(f)\|_I + \varepsilon, & v \in O \\ = 0, & v \notin O \end{cases}$$

and similarly

$$\int_G |g| \, d\mu_v \leq \int_{G_v} |\varphi(d(x)) f(x)| \, d\mu_v(x) = \varphi(v) \int_G |f(x)| \, d\mu_v(x) \begin{cases} \leq \|\rho(f)\|_I + \varepsilon, & v \in O \\ = 0, & v \notin O, \end{cases}$$

so in any case $\|g\|_I \leq \|\rho(f)\|_I + \varepsilon$.

Note that $L(g) = L(f)$. Indeed, $\varphi(r(G|_F \cap \operatorname{supp}(f))) = \varphi(r(r^{-1}(F) \cap \operatorname{supp}(f))) \subseteq \varphi(C) = \{1\}$ and likewise $\varphi(d(G|_F \cap \operatorname{supp}(f))) \subseteq \{1\}$ because F is invariant, so $g - f = 0$ on $G|_F \cap \operatorname{supp}(f)$ and therefore also on the whole of $G|_F$. Thus, $g - f \in I_c(U) \subseteq \operatorname{Ex}(U)$ and $L(g) = L(f)$ follows from $\ker(L) = \operatorname{Ex}(U)$. This gives

$$\|L_F(\rho(f))\| = \|L(f)\| = \|L(g)\| \leq \|g\| \leq \|g\|_I \leq \|\rho(f)\|_I + \varepsilon$$

and since ε was arbitrary, we obtain that L_F is continuous with respect to the I-norm. It follows that L_F extends to a representation of $C^*(G|_F, \mu|_F)$ and since $L_F \circ \rho = L$ holds on the dense subspace $C_c(G, \mu) \subseteq C^*(G, \mu)$, it must hold everywhere, so $\ker(\rho) \subseteq \ker(L) = \operatorname{Ex}(U)$. ∎

THEOREM 2.5.4 ([MR82, Proposition 2.17]). *Let G be a locally compact groupoid with left Haar system μ and suppose that G is amenable, see Definition 2.4.21. Let λ be a Radon measure on $G^{(0)}$ and consider the representation $\operatorname{Ind}_\lambda \colon C^*_{\mathrm{red}}(G, \mu) \to L^2(G, \nu^{-1})$ with $\nu = \int_{G^{(0)}} \mu^u \, d\lambda(u)$ from Proposition 2.4.11. Then*

$$\ker(\operatorname{Ind}_\lambda) = \operatorname{Ex}(U) \cong C^*_{\mathrm{red}}(G|_U, \mu|_U),$$

where U is the largest open invariant subset of $G^{(0)}$ with $\lambda(U) = 0$.

REMARK 2.5.5. Note that this largest open invariant set from Theorem 2.5.4 really exists. Indeed, if $\mathfrak{U}_\lambda$ denotes the set of all open invariant $V \subseteq G^{(0)}$ such that $\lambda(V) = 0$, then $U := \bigcup \mathfrak{U}_\lambda$ is open and invariant, and because U is second countable, its open cover $\mathfrak{U}_\lambda$ has a countable subcover $\{U_i\}_{i \geq 1} \subseteq \mathfrak{U}_\lambda$, so that $U = \bigcup_{i \geq 1} U_i$. But then $\lambda(U) \leq \sum_{i \geq 1} \lambda(U_i) = 0$ and evidently U is the largest set with this property. We also point out that $G^{(0)} \setminus U$ contains the support of λ.

PROOF. By the definition of the reduced norm, $\operatorname{Ind}_\lambda$ defines a representation of $C^*_{\mathrm{red}}(G, \mu)$ on $L^2(G, \nu^{-1})$. If $f \in I_c(U)$, then $\operatorname{Ind}_\lambda(f)\xi = f * \xi \in I_c(U)$ for $\xi \in C_c(G)$ by Lemma 2.5.2. Since $\lambda(U) = 0$, we have $\nu^{-1}(G|_U) = 0$ by (iv) of Remark 2.4.10, which means $\operatorname{Ind}_\lambda(f)\xi = 0$ in $L^2(G, \nu^{-1})$ and hence $f \in \ker(\operatorname{Ind}_\lambda)$. It follows that $\operatorname{Ex}(U) \subseteq \ker(\operatorname{Ind}_\lambda)$ and $\operatorname{Ind}_\lambda \colon C^*_{\mathrm{red}}(G, \mu) \to \mathscr{L}(L^2(G, \nu^{-1}))$ descends to a representation

$$L \colon C^*_{\mathrm{red}}(G|_F, \mu|_F) \cong C^*_{\mathrm{red}}(G, \mu)/\operatorname{Ex}(U) \to \mathscr{L}(L^2(G, \nu^{-1}))$$

with $F = G^{(0)} \setminus U$, see Corollary C.4.3, where $C^*_{\mathrm{red}}(G|_F, \mu|_F)$ is isomorphic to $C^*_{\mathrm{red}}(G, \mu)/\operatorname{Ex}(U)$ by Theorems 2.5.3 and 2.4.22 since G is amenable. Let

$$\operatorname{Ind}_{\lambda|_F} \colon C^*_{\mathrm{red}}(G|_F, \mu|_F) \to \mathscr{L}(L^2(G|_F, (\lambda|_F \circ \mu|_F)^{-1}))$$

be the representation induced by the restricted measure $\lambda|_F$ on $(G|_F)^{(0)} = F$, where $\lambda|_F \circ \mu|_F = \int_F \mu^u|_{G|_F} \, d\lambda|_F(u)$ is the induced measure. We now want to argue that L is really nothing but

$\mathrm{Ind}_{\lambda|_F}$. Note that $(\lambda|_F \circ \mu|_F)^{-1}$ coincides with $\nu^{-1}|_{G|_F}$ since

$$\int_{G|_F} g\, d(\lambda|_F \circ \mu|_F)^{-1} = \int_F \int_{G^u} g(x^{-1})\, d(\mu|_F)^u(x)\, d(\lambda|_F)(u) = \\ = \int_F \int_{G^u} g(x^{-1})\, d\mu^u(x)\, d\lambda(u) = \int_{G|_F} g\, d\nu^{-1} = \int_{G|_F} g\, d(\nu^{-1}|_{G|_F})$$

for all $g \in C_c(G|_F)$, where we have used (2.2.3) and the fact that $\mathrm{supp}(\lambda) \subseteq F$ and $\mathrm{supp}(\nu^{-1}) \subseteq G|_F$ by (iv) of Remark 2.4.10, see also Example *A*.4.7. Thus, $\mathrm{Ind}_{\lambda|_F}$ is a representation on the Hilbert space $L^2(G|_F, \nu^{-1}|_{G|_F})$. The restriction map $L^2(G, \nu^{-1}) \to L^2(G|_F, \nu^{-1}|_{G|_F})$, $\xi \mapsto \xi|_{G|_F}$ is surjective, so that the map

$$\iota\colon \mathscr{L}(L^2(G|_F, \nu^{-1}|_{G|_F})) \to \mathscr{L}(L^2(G, \nu^{-1})), \quad \iota(T)(\xi)(x) := \begin{cases} T(\xi|_{G|_F})(x), & x \in G|_F, \\ 0, & x \notin G|_F, \end{cases}$$

is injective. Now consider the following diagram, where $\pi\colon C^*_{\mathrm{red}}(G,\mu) \to C^*_{\mathrm{red}}(G,\mu)/\mathrm{Ex}(U)$ is the canonical projection, $\widetilde{\rho}$ is the isomorphism from Theorem 2.5.3, and $\widetilde{\mathrm{Ind}_\lambda}\colon C^*_{\mathrm{red}}(G,\mu)/\mathrm{Ex}(U) \to \mathscr{L}(L^2(G,\nu^{-1}))$ is the map induced by Ind_λ on the quotient.

$$\begin{tikzcd}
 & & C^*_{\mathrm{red}}(G|_F, \mu|_F) \arrow[dd, "L"] \arrow[dr, "\mathrm{Ind}_{\lambda|_F}"] & \\
C^*_{\mathrm{red}}(G,\mu)/\mathrm{Ex}(U) \arrow[urr, bend left, "\widetilde{\rho}"] \arrow[drr, bend right, "\widetilde{\mathrm{Ind}_\lambda}"'] & C^*_{\mathrm{red}}(G,\mu) \arrow[l, "\pi"'] \arrow[ur, "\rho"] \arrow[dr, "\mathrm{Ind}_\lambda"] & & \mathscr{L}(L^2(G|_F, \nu^{-1}|_{G|_F})) \arrow[dl, "\iota"] \\
 & & \mathscr{L}(L^2(G, \nu^{-1})) &
\end{tikzcd}$$

By construction of L, the left half of the diagram commutes. Indeed, $L \circ \widetilde{\rho} = \widetilde{\mathrm{Ind}_\lambda}$ implies $L \circ \rho = L \circ \widetilde{\rho} \circ \pi = \widetilde{\mathrm{Ind}_\lambda} \circ \pi = \mathrm{Ind}_\lambda$. Now consider, for $f, \xi \in C_c(G,\mu)$ and $x \in G$,

$$\iota(\mathrm{Ind}_{\lambda|_F}(\rho(f)))(\xi)(x) = \begin{cases} \mathrm{Ind}_{\lambda|_F}(\rho(f))(\xi|_{G|_F})(x), & x \in G|_F \\ 0, & x \in G|_U. \end{cases}$$

We want to show that this equals $\mathrm{Ind}_\lambda(f)(\xi)(x)$. For $x \in G|_F$, we have

$$\mathrm{Ind}_{\lambda|_F}(\rho(f))(\xi|_{G|_F})(x) = \int_{G|_F} f|_{G|_F}(xy)\xi|_{G|_F}(y^{-1})\, d(\mu^{d(x)}|_{G|_F})(y) = \\ = \int_{G^{d(x)}} f(xy)\xi(y^{-1})\, d\mu^{d(x)}(y) = \int_G f(xy)\xi(y^{-1})\, d\mu^{d(x)}(y) = \mathrm{Ind}_\lambda(f)(\xi)(x),$$

and $G|_U$ is a ν-null set by (iv) of Remark 2.4.10, so that $\mathrm{Ind}_\lambda(f)(\xi) = \iota(\mathrm{Ind}_{\lambda|_F}(\rho(f)))(\xi)$ in $L^2(G,\nu^{-1})$. This shows that $\iota \circ \mathrm{Ind}_{\lambda|_F} \circ \rho = \mathrm{Ind}_\lambda$. It follows that

$$\iota \circ \mathrm{Ind}_{\lambda|_F} \circ \widetilde{\rho} \circ \pi = \iota \circ \mathrm{Ind}_{\lambda|_F} \circ \rho = \mathrm{Ind}_\lambda = \widetilde{\mathrm{Ind}_\lambda} \circ \pi$$

and we can cancel π in this equation since it is surjective. Therefore,

$$\iota \circ \mathrm{Ind}_{\lambda|_F} = \widetilde{\mathrm{Ind}_\lambda} \circ \widetilde{\rho}^{-1} = L,$$

showing that the right half of the diagram also commutes. Thus, the whole diagram is commutative and it suffices to show that $\mathrm{Ind}_{\lambda|_F}$ is injective, for then L is injective because ι also is, and then $\ker(\mathrm{Ind}_\lambda) = \ker(\rho) = \mathrm{Ex}(U)$ by Theorem 2.5.3, which is what we wanted to show.

Note that F is, by definition, the smallest closed invariant subset of $G^{(0)}$ such that $\mathrm{supp}(\lambda) \subseteq F$. In particular, $\mathrm{supp}(\lambda|_F) = \mathrm{supp}(\lambda)$, see Example *A*.4.7, which implies that $(G|_F)^{(0)} = F$ is the smallest closed $G|_F$-invariant subset containing $\mathrm{supp}(\lambda|_F)$. Therefore, we can assume for convenience and without loss of generality that $G = G|_F$ and $U = \emptyset$ and we have to show that Ind_λ is injective. By (2.4.6), we have

$$\|\mathrm{Ind}_\lambda(f)\| = \sup_{u \in \mathrm{supp}(\lambda)} \|\mathrm{Ind}_u(f)\|$$

for all $f \in C^*_{\mathrm{red}}(G,\mu)$, where Ind_u is the representation of $C^*_{\mathrm{red}}(G,\mu)$ on $L^2(G,\mu_u)$ induced by the measure δ_u. Now let $u, v \in G^{(0)}$ belong to the same orbit, see Lemma 2.1.7. Then there exists $z \in G$ with $r(z) = u$ and $d(z) = v$. For $f \in C_c(G)$, set

$$(Uf)(x) := \begin{cases} f(xz), & d(x) = u \\ 0, & d(x) \neq u. \end{cases}$$

Note that the product xz makes sense by Lemma 2.1.6 as $d(x) = u = r(z)$. We have

$$\int_{G_u} (Uf)(x)\overline{(Ug)(x)}\,d\mu_u(x) = \int_{G_u} f(xz)\overline{g(xz)}\,d\mu_u(x) = \int_{G_v} f(x)\overline{g(x)}\,d\mu_v(x)$$

for $f, g \in C_c(G)$ by right-invariance of the right Haar system $u \mapsto \mu_u$, see Proposition 2.2.2. Thus, U extends to a unitary operator from $L^2(G,\mu_v)$ to $L^2(G,\mu_u)$. Moreover,

$$(U \circ \mathrm{Ind}_v(f))(\xi)(x) = \int_{G^v} f(xzy)\xi(y^{-1})\,d\mu^v(y) =$$
$$= \int_{G^u} f(xy)\xi(y^{-1}z)\,d\mu^u(y) = (\mathrm{Ind}_v(f) \circ U)(\xi)(x)$$

for $\xi \in C_c(G)$ and $x \in G_v$ by left-invariance, which shows that Ind_u and Ind_v are unitarily equivalent. It follows that

$$\|\mathrm{Ind}_\lambda(f)\| = \sup\left\{\|\mathrm{Ind}_u(f)\| : u \in \textstyle\bigcup_{v \in \mathrm{supp}(\lambda)} [v]\right\} \tag{2.5.1}$$

for all $f \in C^*_{\mathrm{red}}(G,\mu)$. Clearly, the closure of $S := \bigcup_{v \in \mathrm{supp}(\lambda)} [v]$ is the smallest closed invariant subset of $G^{(0)}$ containing $\mathrm{supp}(\lambda)$, see Remark 2.1.17, so that $G^{(0)} = \overline{S}$. If now $f \in \ker(\mathrm{Ind}_\lambda)$, then $f \in \ker(\mathrm{Ind}_u)$ for all $u \in S$ by (2.5.1) and Lemma 2.5.6 below gives that $f \in \ker(\mathrm{Ind}_u)$ for all $u \in \overline{S} = G^{(0)}$. But then (2.4.6) gives $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = 0$ and we are done. ■

LEMMA 2.5.6. *Let G be a locally compact groupoid with left Haar system μ. If $S \subseteq G^{(0)}$ is any subset, then*

$$\bigcap\{\ker(\mathrm{Ind}_u) : u \in S\} = \bigcap\{\ker(\mathrm{Ind}_u) : u \in \overline{S}\},$$

where $\mathrm{Ind}_u \colon C^*_{\mathrm{red}}(G,\mu) \to \mathscr{L}(L^2(G,\mu_u))$ is the representation induced by δ_u, see Remark 2.4.19.

PROOF. It is clear that the right hand side is contained in the left hand side, so it remains to show that

$$\bigcap\{\ker(\mathrm{Ind}_u) : u \in S\} \subseteq \ker(\mathrm{Ind}_v) \quad \text{for all } v \in \overline{S}.$$

So let $v \in \overline{S}$ and take $\xi \in C_c(G)$. Let $(V_n)_{n \geq 1}$ be a countable neighborhood base of $v \in G^{(0)}$ with V_n open relatively compact and such that $\overline{V_{n+1}} \subseteq V_n$. We pick $\varphi_n \in C_c(G^{(0)}, [0,1])$ with $\varphi_n(\overline{V_{n+1}}) = \{1\}$ and $\mathrm{supp}(\varphi_n) \subseteq V_n$. Then $\xi_n := (\varphi_n \circ d)\xi \in C_c(G)$ satisfies

$$\langle \mathrm{Ind}_u(F)\xi_n, \xi_n\rangle = \int_G \int_G F(xy)\xi_n(y^{-1})\overline{\xi_n(x)}\,d\mu^u(y)\,d\mu_u(x) =$$
$$= \int_G \int_G F(xy)\varphi_n(r(y))\varphi_n(d(x))\xi(y^{-1})\overline{\xi(x)}\,d\mu^u(y)\,d\mu_u(x) = \varphi_n(u)^2\langle \mathrm{Ind}_u(F)\xi, \xi\rangle$$

for all $F \in C_c(G)$. The function $u \mapsto \langle \mathrm{Ind}_u(F)\xi, \xi\rangle$ belongs to $C_c(G^{(0)})$. Indeed, for $x \in G_u = \mathrm{supp}(\mu_u)$ we have $d(x) = u$ and hence

$$\langle \mathrm{Ind}_u(F)\xi, \xi\rangle = \int_G \int_G F(xy)\xi(y^{-1})\,d\mu^{d(x)}(y)\,\overline{\xi(x)}\,d\mu_u(x) = \int_G (F * \xi)(x^{-1})\overline{\xi(x^{-1})}\,d\mu^u(x)$$

and the claim follows from Lemma 2.2.5. Therefore, given $\varepsilon > 0$, there exists $N \geq 1$ such that

$$|\langle \mathrm{Ind}_u(F)\xi, \xi\rangle - \langle \mathrm{Ind}_v(F)\xi, \xi\rangle| \leq \varepsilon$$

for all $n \geq N$ and $u \in V_n$. Choose $u_n \in V_{n+1} \cap S$ for every n. Then

$$|\langle \mathrm{Ind}_{u_n}(F)\xi_n, \xi_n\rangle - \langle \mathrm{Ind}_v(F)\xi, \xi\rangle| \leq \varepsilon$$

for all $n \geq N$ and all $F \in C_c(G)$ by combining the above estimates and since $\varphi_n(u_n) = 1$. By continuity of the induced representations, this inequality remains valid for $F \in C^*_{\mathrm{red}}(G,\mu)$. Now take $f \in C^*_{\mathrm{red}}(G,\mu)$ with $f \in \ker(\mathrm{Ind}_u)$ for all $u \in S$. Then $\mathrm{Ind}_{u_n}(f^* * f) = \mathrm{Ind}_{u_n}(f)^*\mathrm{Ind}_{u_n}(f) = 0$ for all $n \geq 1$ and we obtain $\|\mathrm{Ind}_v(f)\xi\|^2 = |\langle \mathrm{Ind}_v(f^* * f)\xi, \xi\rangle| \leq \varepsilon$ for every $\varepsilon > 0$ and all $\xi \in C_c(G)$. This shows that $f \in \ker(\mathrm{Ind}_v)$. ■

REMARK 2.5.7. Lemma 2.5.6 shows that the two families $\{\mathrm{Ind}_u : u \in S\}$ and $\{\mathrm{Ind}_u : u \in \overline{S}\}$ of representations of the C^*-algebra $C^*_{\mathrm{red}}(G,\mu)$ are *weakly equivalent*, see [**Dix77**, p. 77] for more details on this concept. In the proof we have constructed, to each positive form[6] $\omega\colon F \mapsto \langle \mathrm{Ind}_v(F)\xi, \xi\rangle$ on $C^*_{\mathrm{red}}(G,\mu)$, a sequence of positive forms associated to a representation in $\{\mathrm{Ind}_u : u \in S\}$ such that ω is the weak-* limit of this sequence. The final argument that this implies $f \in \ker(\mathrm{Ind}_v)$, provided $\mathrm{Ind}_u(f) = 0$ for all $u \in S$, is then just [**Dix77**, Theorem 3.4.4, (ii)⇒(i)]. The statement of Lemma 2.5.6 is found without proof in [**MR82**, Proof of Proposition 2.17] and we have adapted the proof of [**Pat99**, Proposition 3.1.2] for the construction of the above sequence of positive forms on $C^*_{\mathrm{red}}(G,\mu)$.

2.6. Examples

In this section, we will compute the groupoid C^*-algebras of some of the types of groupoids that have been introduced in the previous sections.

EXAMPLE 2.6.1 (The C^*-algebra of a cotrivial groupoid). Let X be a locally compact, second countable Hausdorff space and G the cotrivial groupoid over X as in Example 2.1.19. We will equip G with the Haar system from Example 2.2.10, that is, $\mu^x = \delta_x$ for $x \in X$. For the convolution product of $f, g \in C_c(G,\mu)$ we get

$$(f * g)(x) = \int_G f(xy)g(y^{-1})\, d\mu^{d(x)}(y) = f(x\, d(x))g(d(x)^{-1}) = f(x)g(x)$$

and the involution is just complex conjugation. Hence, $C_c(G,\mu)$ is just $C_c(X)$ with pointwise multiplication and conjugation. Moreover, $\|f\|_I = \sup_{x\in X}|f(x)|$ is just the uniform norm, so that $L^I(G,\mu) = C_0(X)$. Since $C_0(X)$ is already a C^*-algebra, there is a faithful, hence isometric, representation of $C_0(X)$, so that $\|\cdot\|_{C^*(G,\mu)} = \|\cdot\|_{L^I(G,\mu)} = \|\cdot\|_{C_0(X)}$ and therefore

$$C^*(G,\mu) = C^*_{\mathrm{red}}(G,\mu) = C_0(X),$$

where we have used amenability of G, see Example 2.4.31. ♢

PROPOSITION 2.6.2 (The C^*-algebra of a trivial groupoid, [**Muh97**, Proposition 2.37]). *Let X be a locally compact, second countable Hausdorff space and let $G = X \times X$ be the trivial groupoid over X, see Example 2.1.20, equipped with a Radon measure ν on X of full support. If $\mu^x := \delta_x \times \nu$ for $x \in X$ is the Haar system from Example 2.2.11, then*

$$C^*(G,\mu) = C^*_{\mathrm{red}}(G,\mu) \cong \mathscr{K}(L^2(X,\nu)) \tag{2.6.1}$$

is the C^*-algebra of compact operators on $L^2(X,\nu)$.

[6] In the terminology of [**Dix77**], this form is called *associated* to Ind_v.

PROOF. The equality $C^*(G,\mu) = C^*_{\mathrm{red}}(G,\mu)$ follows from Example 2.4.32. We will show the rest of the Proposition by using the reduced groupoid C^*-algebra. For all $u \in G^{(0)}$, we have $G_u = X \times \{u\}$ and $\mu_u = (\delta_u \times \nu)^{-1} = \nu \times \delta_u$, so the map $U\colon L^2(G,\mu_u) \to L^2(X,\nu)$, $(U\psi)(x) := \psi(x,u)$ is unitary and satisfies

$$\begin{aligned}(U\,\mathrm{Ind}_u(f)\psi)(x) &= \int_G f((x,u)(w,z))\varphi(z,w)\,d(\delta_u \times \nu)(w,z)\\ &= \int_X f(x,z)(U\psi)(z)\,d\nu(z), \quad (f,\psi \in C_c(X\times X))\end{aligned}$$

hence all Ind_u, $u \in G^{(0)}$, are unitarily equivalent to the representation

$$\pi\colon C_c(G,\mu) \to \mathscr{L}(L^2(X,\nu)), \quad (\pi(f)\varphi)(x) := \int_X f(x,z)\varphi(z)\,d\nu(z),$$

and we therefore have $\|f\|_{C^*_{\mathrm{red}}(G,\mu)} = \|\pi(f)\|_{\mathscr{L}(L^2(X,\nu))}$.

We now argue that the range of π is dense in the algebra $\mathscr{K}(L^2(X,\nu))$ of compact operators on $L^2(X,\nu)$ for the operator norm, which then finishes the proof because π then induces an isomorphism $C^*_{\mathrm{red}}(G,\mu) \xrightarrow{\cong} \overline{\mathrm{im}(\pi)} = \mathscr{K}(L^2(X,\nu))$. Taking $f(x,y) = g(x)h(y)$ for $g,h \in C_c(X)$, we have $\pi(f)\varphi(x) = g(x)\int_X h(y)\varphi(y)\,d\nu(y)$, so $\mathrm{im}(\pi)$ contains the space of rank 1 operators, hence $\overline{\mathrm{im}(\pi)} \supseteq \mathscr{K}(L^2(X,\nu))$. Now letting $f \in C_c(X\times X)$ be arbitrary, we need to show that $\pi(f)$ is a compact operator. The family $\mathcal{A}$ of functions of the form $g \otimes h\colon (x,y) \mapsto g(x)h(y)$ with $g,h \in C_c(X)$ is a subalgebra of $C_0(X\times X)$ which separates the points of $X \times X$ as well as vanishes nowhere (that is, for every $(x,y) \in X\times X$, there is $g,h \in C_c(X)$ such that $g(x)h(y) \neq 0$). Therefore, by the Stone-Weierstrass theorem (for locally compact spaces, see [**Fol99**, Theorem 4.52]), $C_0(X\times X)$ is the uniform closure of $\mathcal{A}$. It follows that there are sequences $g_n, h_n \in C_c(X)$ with $g_n \otimes h_n \to f$ uniformly on $X \times X$.

Now for $\varphi \in L^2(X,\nu)$ with $\|\varphi\|_{L^2(X,\nu)} \leq 1$, we have

$$\int_X |f(x,y) - g_n(x)h_n(y)||\varphi(y)|\,d\nu(y) \leq \left(\int_X |f(x,y) - g_n(x)h_n(y)|^2\,d\nu(y)\right)^{1/2} \|\varphi\|_{L^2(X,\nu)}$$

by Hölder's inequality and therefore

$$\begin{aligned}\|\pi(f)\varphi - \pi(f_n)\varphi\|^2_{L^2(X,\nu)} &\leq \int_X \left(\int_X |f(x,y) - g_n(x)h_n(y)||\varphi(y)|\,d\nu(y)\right)^2 d\nu(x) \leq\\ &\leq \|f - g_n \otimes h_n\|^2_{L^2(X\times X,\nu\times\nu)}\|\varphi\|^2_{L^2(X,\nu)} \leq \|f - g_n\otimes h_n\|^2_{L^2(X\times X,\nu\times\nu)} \to 0\end{aligned}$$

as $n \to \infty$ since $g_n \otimes h_n \to f$ also in $L^2(X\times X, \nu\times\nu)$. This implies $\pi(f_n) \to \pi(f)$ in operator norm, and since $\pi(f_n)$ has rank 1, we see that $\pi(f)$ is compact, so $\overline{\mathrm{im}(\pi)} \subseteq \mathscr{K}(L^2(X,\nu))$ because the compact operators are closed in $\mathscr{L}(L^2(X,\nu))$. ∎

REMARK 2.6.3. One can also directly show that $C^*(G,\mu) \cong \mathscr{K}(L^2(X,\nu))$, without invoking amenability. The proof can be found in [**Pat99**, Theorem 3.1.2] and it uses the theory of groupoid representations, which makes it quite a lot more involved than the above proof for the reduced C^*-algebra.

EXAMPLE 2.6.4 (The C^*-algebra of a transformation groupoid). Let $X \rtimes_\alpha \Gamma$ be a locally compact transformation groupoid, equipped with its canonical Haar system from Example 2.2.12. Then

$$C^*(X \rtimes_\alpha \Gamma) \cong C_0(X) \rtimes_{\tilde{\alpha}} \Gamma \tag{2.6.2}$$

is a *crossed product*, where Γ acts on $C_0(X)$ by $\tilde{\alpha}_g(\varphi) := \varphi \circ \alpha_g$.

We will not make direct use of crossed products, so that we will only briefly describe what they are and refer to [**Wil07**] for an extensive treatment. A C^**-dynamical system* is a triple (A,Γ,β), where A is a C^*-algebra, Γ is a locally compact group, and $\beta\colon \Gamma \to \mathrm{Aut}(A)$, $s \mapsto \beta_s$, is a continuous group homomorphism in the sense that, for each $a \in A$, the map $\Gamma \to A$, $s \mapsto \beta_s(a)$, is continuous. Given a C^*-dynamical system (A, Γ, β), the vector space $C_c(\Gamma, A)$ of continuous A-valued functions on Γ with compact support can be made into a *-algebra by letting

$$(f * g)(s) := \int_\Gamma f(t)\beta_t(g(t^{-1}s))\, d\lambda(t) \quad \text{and} \quad f^*(s) := \Delta(s^{-1})\beta_s(f(s^{-1})^*), \tag{2.6.3}$$

where λ is a left Haar measure on Γ and $\Delta\colon \Gamma \to \mathbb{R}_+$ is the *modular function*, which is defined by the requirement $\Delta(s) \int_\Gamma f(ts)\, d\lambda(t) = \int_\Gamma f(t)\, d\lambda(t)$ for all $f \in C_c(\Gamma)$ and $s \in \Gamma$. Note that some care has to be taken as the integral in (2.6.3) is A-valued.[7] The operations from (2.6.3) satisfy $\|f^*\|_{L^1} = \|f\|_{L^1}$ and $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$, where $\|f\|_{L^1} := \int_\Gamma \|f(t)\|\, d\lambda(t)$. Similarly as in Lemma 2.3.8, the inductive limit topology on $C_c(\Gamma, A)$ is finer than the topology induced by the L^1-norm.

The crossed product of (A, Γ, β) will be obtained by completing $C_c(\Gamma, A)$ with respect to a norm which is obtained by certain representations, as was the case for the groupoid C^*-algebras. A *covariant representation* is a pair (ρ, U), where $\rho\colon A \to \mathscr{L}(H)$ is a representation and $U\colon \Gamma \to \mathscr{U}(H)$ is a unitary representation, meaning that U is a homomorphism to the set $\mathscr{U}(H)$ of unitary operators on H and U is continuous for the strong operator topology on $\mathscr{U}(H)$, and such that

$$U(s)\rho(a)U(s)^{-1} = \rho(\beta_s(a))$$

for all $a \in A$ and $s \in \Gamma$. Note that both A and Γ are represented on the *same* Hilbert space H. Given a covariant representation (ρ, U) of (A, Γ, β), one obtains a L^1-norm decreasing *-representation of $C_c(\Gamma, A)$ by

$$(\rho \rtimes U)(f) := \int_\Gamma \rho(f(t))U(t)\, d\lambda(t).$$

Finally, the *crossed product* $A \rtimes_\beta \Gamma$ of (A, Γ, β) is the completion of $C_c(\Gamma, A)$ with respect to the norm

$$\|f\| := \sup\{\|(\rho \rtimes U)(f)\| : (\rho, U) \text{ is a covariant representation of } (A, \Gamma, \beta)\}.$$

One can show that every (non-degenerate) representation of $A \rtimes_\beta \Gamma$ is of the form $\rho \rtimes U$ for some covariant representation (ρ, U), and that

$$\begin{aligned} \|f\| &= \sup\{\|\pi(f)\| : \pi \text{ is a } L^1\text{-norm decreasing representation}\} \\ &= \sup\{\|\pi(f)\| : \pi \text{ is continuous for the inductive limit topology}\}, \end{aligned}$$

which is similar to our definition of the full groupoid C^*-algebra and to Remark 2.4.2.

Finally, we point out that the isomorphism from (2.6.2) is given by extending the map

$$\iota\colon C_c(X \times \Gamma) \to C_c(\Gamma, C_0(X)), \quad \iota(f)(s)(x) := \Delta(s)^{-1/2} f(x, s),$$

see [**Goe09**, Propositions 4.4, 4.38] for a detailed argument. $\Diamond$

EXAMPLE 2.6.5 (Group C^*-algebras). If Γ is a locally compact (second countable Hausdorff) group, then we may view Γ as a groupoid with Haar system $\mu^e = \lambda$, where λ is a left Haar measure on Γ, see Examples 2.1.18 and 2.2.9. The convolution of $C_c(\Gamma)$ is given by the familiar expression

$$(f * g)(s) = \int_\Gamma f(t)g(t^{-1}s)\, d\lambda(t).$$

The C^*-algebra $C^*(\Gamma) := C^*(\Gamma, \mu)$ is called the *(full) group C^*-algebra* of Γ. If $\{*\}$ denotes the one-pointed space, then there is groupoid homeomorphism $\{*\} \rtimes \Gamma \cong \Gamma$, given by $(*, g) \mapsto g$.

[7] For a nice introduction to the theory of vector-valued integration, especially in the context of C^*-algebra, see [**Wil07**].

If the transformation groupoid $\{*\} \rtimes \Gamma$ is equipped with its canonical Haar system from Example 2.2.12, then this isomorphism obviously satisfies the requirements of Corollary 2.4.7, so we have

$$C^*(\Gamma) \cong C^*(\{*\} \rtimes \Gamma) \cong C_0(\{*\}) \rtimes \Gamma \cong \mathbb{C} \rtimes \Gamma$$

by invoking Example 2.6.4 and using that $C_0(\{*\}) \cong \mathbb{C}$ as C^*-algebras in the obvious way. We therefore obtain an alternative definition of $C^*(\Gamma)$ as a crossed product, which is used [Wil07].

As a special case, we will mention the situation for locally compact *abelian* groups. In this case, one can extend the Fourier transform to an isomorphism from $C^*(\Gamma)$ to $C_0(\widehat{\Gamma})$, see [Wil07, Proposition 3.1].[8] Here, $\widehat{\Gamma}$ denotes the *dual group* (or *character group*) of Γ, meaning the set of continuous group homomorphisms from Γ to the torus $\mathbb{T}$. The topology on $\widehat{\Gamma}$ is the one of uniform convergence on compact subsets. For example, we have $\widehat{\mathbb{R}} \cong \mathbb{R}$, so $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, and $\widehat{\mathbb{Z}} \cong \mathbb{T}$, so $C^*(\mathbb{Z}) \cong C_0(\mathbb{T}) = C(S^1)$ since the unit circle $S^1 \subseteq \mathbb{C}$ is compact. $\diamondsuit$

EXAMPLE 2.6.6 (Irrational rotation algebras). Let $\theta \in [0,1]$ and consider the action $\alpha\colon \mathbb{Z} \times \mathbb{T} \to \mathbb{T}$ given by $\alpha(n)z := e^{2\pi i n\theta}z$. The corresponding transformation groupoid C^*-algebra

$$\mathcal{A}_\theta := C^*(\mathbb{T} \rtimes_\alpha \Gamma) \cong C_0(\mathbb{T}) \rtimes_{\tilde{\alpha}} \mathbb{Z}$$

is called the *rational or irrational rotation algebra*, depending on whether θ is rational or irrational. For irrational θ, the algebra $\mathcal{A}_\theta$ is the "universal C^*-algebra" generated by isometries u, v such that $uv = e^{2\pi i\theta}vu$. This means that whenever U and V are unitaries in $\mathscr{L}(H)$ for a Hilbert space H with $UV = e^{2\pi i\theta}VU$, then there exists a representation $\pi\colon \mathcal{A}_\theta \to \mathscr{L}(H)$ such that $\pi(u) = U$, $\pi(v) = V$ and π is an isomorphism of $\mathcal{A}_\theta$ onto $C^*(U,V)$, the C^*-subalgebra of $\mathscr{L}(H)$ generated by U and V. For a proof, see [Wil07, Proposition 2.56]. A concrete realization of $\mathcal{A}_\theta$ as a subalgebra of $\mathscr{L}(L^2(\mathbb{T}))$ is given by the unitaries $U(f)(z) := zf(z)$ and $V(f)(z) := f(e^{-2\pi i\theta}z)$.

The dependence of $\mathcal{A}_\theta$ on the parameter θ is highly non-trivial. In fact, it holds that $\mathcal{A}_\theta \cong \mathcal{A}_{\theta'}$ if and only if $\theta = \theta' \mod 1$, see [Wil07, Remark 2.60] for further references and results. In [SSU89, Proposition 4.2], it is shown that if θ is irrational, then

$$\mathcal{A}_\theta \otimes \mathscr{K} \cong C_0(\mathbb{R}) \rtimes \mathbb{Z}^2,$$

where $\mathbb{Z}^2$ acts on $\mathbb{R}$ by $\mu \cdot x := x + \mu_1\theta - \mu_2$. This is done by showing that the crossed product $C_0(\mathbb{R}) \rtimes \mathbb{Z}^2$ is isomorphic to the *foliation C^*-algebra $C^*(\mathscr{F}_\theta)$* associated to the foliation of the torus $\mathbb{T}$ by lines with slope θ (the *Kronecker foliation*), and this foliation algebra is in turn isomorphic to $\mathcal{A}_\theta \otimes \mathscr{K}$, see [Con82]. $\diamondsuit$

EXAMPLE 2.6.7. Let Γ be a locally compact group and let $U \subseteq \Gamma$ be an open subset.[9] As in Example 2.1.23, we consider the transformation groupoid $\Gamma \rtimes_\rho \Gamma$ and the trivial groupoid $\Gamma \times \Gamma$. By Proposition 2.2.13, both $G := (\Gamma \rtimes_\rho \Gamma)|_U$ and $H := (\Gamma \times \Gamma)|_U = U \times U$ are locally compact groupoids which we equip with their canonical Haar systems μ and ν, respectively, see Examples 2.2.11 and 2.2.12, where we take the Haar measure λ of G as the measure on G with full support. From Example 2.1.23, we know that G and H are isomorphic via $\varphi\colon G \to H$, $\varphi(g,h) := (g, gh)$. For every $f \in C_c(H)$ and $g_0 \in \Gamma$, we have

$$\int_G f \circ \varphi \, d\mu^{(g_0,e)} = \int_\Gamma f(g_0, g_0 h)\, d\lambda(h) = \int_\Gamma f(g_0, h)\, d\lambda(h) = \int_H f\, d\nu^{(g_0,g_0)} = \int_H f\, d\nu^{\varphi(g_0,e)}.$$

Now Corollary 2.4.7 implies that $C^*((\Gamma \rtimes_\rho \Gamma)|_U) \cong C^*((\Gamma \times \Gamma)|_U) = C^*(U \times U)$. $\diamondsuit$

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[8] Note that this implies that $\widehat{\Gamma}$ is precisely the character space of the C^*-algebra $C^*(\Gamma)$.

[9] We do not consider invariant sets here because the only invariant subsets for the trivial groupoid $\Gamma \times \Gamma$ are Γ and $\emptyset$, see Example 2.1.20.

COROLLARY 2.6.8 (Stone-von Neumann theorem). *Let Γ be a locally compact, second countable Hausdorff group and denote by $\rho_g \colon \Gamma \to \Gamma$ for $g \in \Gamma$ the right translations, $\rho_g(h) := hg$. Then*

$$C_0(\Gamma) \rtimes_{\tilde{\rho}} \Gamma \cong C^*(\Gamma \rtimes_\rho \Gamma) \cong C^*(\Gamma \times \Gamma) \cong \mathscr{K}(L^2(\Gamma)).$$

PROOF. This is immediate from Examples 2.6.4 and 2.6.7 and Proposition 2.6.2. ∎

EXAMPLE 2.6.9. Let $G := X \rtimes_\alpha \Gamma$ and $H := Y \rtimes_\beta \Gamma$ be two locally compact transformation groupoids, equipped with their canonical Haar systems μ and ν, respectively, see Example 2.2.12. Let $\psi \colon X \to Y$ be a homeomorphism such that $\psi(x \cdot g) = \psi(x) \cdot g$ for all $x \in X$ and $g \in \Gamma$. We say that ψ is Γ-*equivariant*. Then the map

$$\varphi \colon G \to H, \quad \varphi(x,g) := (\psi(x), g)$$

is an isomorphism of locally compact groupoids. Indeed, φ is obviously a homeomorphism and if $((x,g),(y,h)) \in G^{(2)}$, then $y = x \cdot g$ and $((x,g),(y,h)) = (x, gh)$. Therefore,

$$(\varphi(x,g), \varphi(y,h)) = ((\psi(x),g),(\psi(x \cdot g),h)) = ((\psi(x),g),(\psi(x) \cdot g, h)) \in H^{(2)}$$

and

$$\varphi(x,g)\varphi(y,h) = (\psi(x),g)(\psi(y),h) = (\psi(x),gh) = \varphi(x,gh) = \varphi((x,g)(y,h)).$$

Moreover, φ fulfills the requirement of Corollary 2.4.7 since

$$\int_G f \circ \varphi \, d\mu^{(x_0,e)} = \int_\Gamma f(\psi(x_0),g)\, d\lambda(g) = \int_H f \, d\nu^{(\psi(x_0),e)} = \int_H f \, d\nu^{\varphi(x_0,e)}$$

for all $f \in C_c(H)$ and $x_0 \in X$. We conclude that $C^*(X \rtimes_\alpha \Gamma) \cong C^*(Y \rtimes_\beta \Gamma)$. ◊

EXAMPLE 2.6.10. We now slightly generalize Example 2.6.9. Let $G = X \rtimes_\alpha \Gamma$ and $H = Y \rtimes_\beta \Gamma$ be locally compact transformation groupoids. Let $U \subseteq X$ be open, so that $U \times \{e\}$ is open in $X \times \{e\} = G^{(0)}$. We can then form the reduction $G|_U = (X \rtimes_\alpha \Gamma)|_U$, where we identify $G^{(0)}$ with X as usual. Let $\psi \colon U \to Y$ be continuous and a homeomorphism onto its image, and such that

$$\psi(x \cdot g) = \psi(x) \cdot g \quad \text{for all } (x,g) \in U \times \Gamma \text{ with } x \cdot g \in U. \tag{2.6.4}$$

The point is of course that we may not have an equivariant map available on the whole of X, as was the case in Example 2.6.9. Then the map

$$\varphi \colon G|_U \to H, \quad \varphi(x,g) := (\psi(x), g)$$

is a homeomorphism onto its image $\varphi(G|_U) = H|_{\psi(U)}$, see the proof of Lemma 2.2.16. As in Example 2.6.9, φ is a groupoid homomorphism, so we have an isomorphism $\varphi \colon (X \rtimes_\alpha \Gamma)|_U \to (Y \rtimes_\beta \Gamma)|_{\psi(U)}$. Both groupoids come with canonical Haar systems, see Proposition 2.2.13, and the same calculation as in Example 2.6.9 shows that

$$\int_{G|_U} f \circ \varphi \, d\mu^{(x_0,e)} = \int_{H|_{\psi(U)}} f \, d\nu^{\varphi(x_0,e)}$$

for all $f \in C_c(H|_{\psi(U)})$ and all $x_0 \in G|_U$, so φ satisfies the requirements of Corollary 2.4.7, hence induces an isomorphism $C^*((X \rtimes_\alpha \Gamma)|_U) \cong C^*((Y \rtimes_\beta \Gamma)|_{\psi(U)})$.

In the case where $E \subseteq G^{(0)}$ is locally closed and Γ-invariant, $\psi|_E$ is equivariant by (2.6.4), so $\psi(E)$ is also invariant and we have $G|_E = E \rtimes_{\alpha|_E} \Gamma$ and $H|_{\psi(E)} = \psi(E) \rtimes_{\beta|_{\psi(E)}} \Gamma$, where $\alpha|_E$ and $\beta|_{\psi(E)}$ are the restricted actions. Moreover,

$$C^*((X \rtimes_\alpha \Gamma)|_E) = C^*(E \rtimes_{\alpha|_E} \Gamma) \cong C^*(\psi(E) \rtimes_{\beta|_{\psi(E)}} \Gamma) = C^*((Y \rtimes_\beta \Gamma)|_{\psi(E)})$$

holds by Example 2.6.9. ◊

EXAMPLE 2.6.11. Let $X \rtimes_{\mathrm{triv}} \Gamma$ be a locally compact trivial action groupoid, see Example 2.1.25. We already know that the identity map $\mathrm{id}_{X\times G}$ is a groupoid homeomorphism from $X \rtimes_{\mathrm{triv}} \Gamma$ to the product groupoid $X \times \Gamma$, where we view X as a cotrivial groupoid. The canonical Haar system $\mu^x = \delta_x \times \lambda$ on $X \rtimes_{\mathrm{triv}} \Gamma$, where λ is Haar measure on Γ, can also be viewed as the usual Haar system on the product groupoid $X \times \Gamma$, so we immediately see

$$C^*(X \rtimes_{\mathrm{triv}} \Gamma) \cong C^*(X \times \Gamma)$$

by using Proposition 2.4.5. $\Diamond$

Finally, we will determine the reduced C^*-algebra of a disjoint union of groupoids (see Definition 2.1.26) as well as of a Cartesian product (see Definition 2.1.24). It turns out that the disjoint union translates into the direct sum of the C^*-algebras, and the Cartesian product groupoid gives a tensor product of C^*-algebras. These results seem to be taken as a given in the literature, and are sometimes referenced for special cases, see [**OPP10**, p. 105] or [**Upm91**, Example 1.3]. We will show them here for general locally compact groupoids and for the *reduced* C^*-algebras. Keep in mind that all our groupoids are assumed to be second countable, a fact that will be crucial in the proof.

PROPOSITION 2.6.12 (C^*-algebra of the disjoint union of groupoids). *Let G_i, $i \in I$, be a countable family of locally compact groupoids with Haar systems μ_i. Then*

$$C^*_{\mathrm{red}}(\sqcup_{i\in I} G_i, \sqcup_{i\in I}\mu_i) \cong \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i, \mu_i)$$

is the direct sum of the C^*-algebras $C^*_{\mathrm{red}}(G_i, \mu_i)$, see Example C.1.11, where $\sqcup_{i\in I}\mu_i$ is the Haar system from Proposition 2.2.17. On the level of functions, this isomorphism is given by $\iota\colon C_c(\sqcup_{i\in I} G_i) \to \bigoplus_{i\in I} C_c(G_i)$, $\iota(f) := (f|_{G_i})_i$.

PROOF. We have remarked in Definition 2.1.26 that countable disjoint unions of locally compact groupoids are again locally compact. We put $G := \sqcup_{i\in I} G_i$ and $\mu := \sqcup_{i\in I}\mu_i$. By Remark 2.4.19, there are faithful representations

$$\varphi_i := \mathrm{Ind}_{\lambda_i}\colon C^*_{\mathrm{red}}(G_i, \mu_i) \to \mathscr{L}(L^2(G_i, \nu_i^{-1})), \quad \text{where} \quad \nu_i := \int_{G_i^{(0)}} \mu_i^u \, d\lambda_i(u)$$

with Radon measures λ_i on $G_i^{(0)}$ of full support. Similarly, the measure $\lambda := \sqcup_{i\in I}\lambda_i$ on $G^{(0)} = \sqcup_{i\in I} G_i^{(0)}$ has full support, so that

$$\mathrm{Ind}_\lambda\colon C^*_{\mathrm{red}}(G, \mu) \to \mathscr{L}(L^2(G, \nu^{-1}))$$

is a faithful representation, where $\nu := \int_{G^{(0)}} \mu^u \, d\lambda(u)$ is the induced measure. Note that ν is nothing but $\sqcup_{i\in I}\nu_i$ and $\nu^{-1} = \sqcup_{i\in I}\nu_i^{-1}$. The representations φ_i induce an injective *-homomorphism

$$\varphi := \sum_{i\in I}^{\oplus} \varphi_i \colon \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i, \mu_i) \to \sum_{i\in I}^{\oplus} \mathscr{L}(L^2(G_i, \nu_i^{-1})) \to \mathscr{L}\Big(\bigoplus\nolimits_{i\in I} L^2(G_i, \nu_i^{-1})\Big),$$

given by $\varphi((f_i)_i)((\xi_i)_i) := (\mathrm{Ind}_{\lambda_i}(f_i)\xi_i)_i$, see Examples C.1.11 and C.1.12. The Hilbert space $L^2(G, \nu^{-1})$ is canonically isomorphic to $\bigoplus_{i\in I} L^2(G_i, \nu_i^{-1})$ via the unitary operator $U(\xi) := (\xi|_{G_i})_i$, and hence $\mathrm{conj}_U\colon \mathscr{L}(L^2(G, \nu^{-1})) \to \mathscr{L}(\bigoplus_{i\in I} L^2(G_i, \nu_i^{-1}))$, $T \mapsto UTU^{-1}$, is an isomorphism of C^*-algebras. Now consider the following diagram:

$$\begin{tikzcd}
C_c(G) \arrow[r, hook] \arrow[d, "\iota"] & C^*_{\mathrm{red}}(G,\mu) \arrow[r, "\mathrm{Ind}_\lambda", "\cong"'] \arrow[d, dotted, "\tilde{\iota}"] & \mathrm{im}(\mathrm{Ind}_\lambda) \arrow[r, hook] \arrow[d, "\mathrm{conj}_U"] & \mathscr{L}(L^2(G,\nu^{-1})) \arrow[d, "\mathrm{conj}_U", "\cong"'] \\
\bigoplus_{i\in I} C_c(G_i) \arrow[r, hook] & \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i,\mu_i) \arrow[r, "\varphi", "\cong"'] & \mathrm{im}(\varphi) \arrow[r, hook] & \mathscr{L}(\bigoplus_{i\in I} L^2(G_i,\nu_i^{-1}))
\end{tikzcd}$$

Here, $\iota\colon C_c(G) \to \bigoplus_{i\in I} C_c(G_i)$ (algebraic direct sum) is the linear map $\iota(f) := (f|_{G_i})_i$. For $f, \xi \in C_c(G)$, we have

$$(\varphi(\iota(f)) \circ U)\xi = \varphi((f|_{G_i})_i)(\xi|_{G|_i}) = (\mathrm{Ind}_{\lambda_i}(f|_{G_i})(\xi|_{G_i}))_i.$$

If $x \in G_i$, then $d(x) \in G_i^{(0)}$ and hence

$$(\mathrm{Ind}_\lambda(f)\xi)|_{G_i}(x) = \int_G f(xy)\xi(y^{-1})\,d\mu^{d(x)}(y) = \int_{G_i} f|_{G_i}(xy)\xi|_{G_i}(y^{-1})\,d\mu_i^{d(x)}(y),$$

which shows $(\mathrm{Ind}_\lambda(f)\xi)|_{G_i} = \mathrm{Ind}_{\lambda_i}(f|_{G_i})(\xi|_{G_i}) \in L^2(G_i, \nu_i^{-1})$. Therefore,

$$(\varphi(\iota(f)) \circ U)\xi = ((\mathrm{Ind}_\lambda(f)\xi)|_{G_i})_i = (U \circ \mathrm{Ind}_\lambda(f))\xi$$

and the diagram commutes. In particular, $\mathrm{conj}_U(\mathrm{im}(\mathrm{Ind}_\lambda)) \subseteq \mathrm{im}(\varphi)$ and

$$\iota(f) = (\varphi^{-1} \circ \mathrm{conj}_U \circ \mathrm{Ind}_\lambda)(f)$$

for $f \in C_c(G)$. This immediately gives an extension of ι to an injective *-homomorphism

$$\tilde{\iota}\colon C^*_{\mathrm{red}}(G,\mu) \to \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i,\mu_i).$$

Moreover, the algebraic direct sum $\bigoplus_{i\in I} C^*_{\mathrm{red}}(G_i,\mu_i)$ is dense in $\sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i,\mu_i)$ and $C_c(G_i)$ is dense in $C^*_{\mathrm{red}}(G_i,\mu_i)$ for all $i \in I$, which implies that $\mathrm{im}(\iota)$ is a dense subspace of $\sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G_i,\mu_i)$ and hence that $\tilde{\iota}$ is also surjective as its range is necessarily closed by Proposition C.4.4. ■

COROLLARY 2.6.13. *Let G be a locally compact groupoid with left Haar system μ. Suppose that $U_i \subseteq G^{(0)}$, $i \in I$, is a partition of $G^{(0)}$ into open invariant subsets. Then*

$$C^*_{\mathrm{red}}(G,\mu) \cong \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G|_{U_i}, \mu|_{U_i}).$$

PROOF. By Example 2.1.27, $G \cong \bigsqcup_{i\in I} G|_{U_i}$. Without loss of generality, $U_i \neq \emptyset$ for all $i \in I$ so that I is at most countable. The canonical Haar system $\nu := \sqcup_{i\in I}(\mu|_{U_i})$ on $\bigsqcup_{i\in I} G|_{U_i}$ is given by

$$\nu^u(A) = (\mu|_{U_i})^u(A \cap G|_{U_i}) = \mu^u|_{G|_{U_i}}(A \cap G|_{U_i}) = \mu^u(A \cap G|_{U_i})$$

if $u \in U_i$. Let $j\colon \bigsqcup_{i\in I} G|_{U_i} \to G$ denote the obvious map. For $f \in C_c(G)$, the above implies

$$\int_{\bigsqcup_{i\in I} G|_{U_i}} f \circ j\,d\nu^u = \int_{G|_{U_i}} f\,d\mu^{j(u)} = \int_G f\,d\mu^{j(u)}$$

if $u \in U_i \subseteq \bigsqcup_{i\in I} U_i$ since $\mathrm{supp}(\mu^{j(u)}) = G^{j(u)} = (G|_{U_i})^{j(u)} \subseteq G|_{U_i}$. By Propositions 2.4.20 and 2.6.12, we conclude

$$C^*_{\mathrm{red}}(G,\mu) \cong C^*_{\mathrm{red}}(\sqcup_{i\in I} G|_{U_i}, \nu) \cong \sum_{i\in I}^{\oplus} C^*_{\mathrm{red}}(G|_{U_i}, \mu|_{U_i}).$$ ■

PROPOSITION 2.6.14 (C^*-algebra of a product of groupoids). *Let G_1 and G_2 be locally compact groupoids with Haar systems μ_1 and μ_2, respectively. Then*

$$C^*_{\mathrm{red}}(G_1 \times G_2, \mu_1 \times \mu_2) \cong C^*_{\mathrm{red}}(G_1,\mu_1) \otimes_* C^*_{\mathrm{red}}(G_2,\mu_2),$$

where $\otimes_*$ denotes the spatial, or minimal, tensor product of C^*-algebras and $\mu \times \nu$ is the Haar system from Proposition 2.2.18. On the level of functions, the isomorphism is given by $\iota\colon C_c(G_1) \odot C_c(G_2) \to C_c(G_1 \times G_2)$, $\iota(f \otimes g)(x,y) = f(x)g(y)$.

PROOF. The proof is similar to that of Proposition 2.6.12. From Remark 2.4.19, we know that any measure λ_i on $G_i^{(0)}$ with full support gives a faithful representation

$$\varphi_i := \mathrm{Ind}_{\lambda_i} \colon C^*_{\mathrm{red}}(G_i, \mu_i) \to \mathscr{L}(L^2(G_i, \nu_i^{-1})), \quad \text{where} \quad \nu_i := \int_{G_i^{(0)}} \mu_i^u \, d\lambda_i(u)$$

for $i \in \{1, 2\}$. By Theorem C.6.3 and the definition of the minimal tensor product, there exists a faithful representation

$$\varphi \colon C^*_{\mathrm{red}}(G_1, \mu_1) \otimes_* C^*_{\mathrm{red}}(G_2, \mu_2) \to \mathscr{L}(L^2(G_1, \nu_1^{-1}) \hat{\otimes} L^2(G_2, \nu_2^{-1}))$$

such that $\varphi(f \otimes g) = \mathrm{Ind}_{\lambda_1}(f) \hat{\otimes} \mathrm{Ind}_{\lambda_2}(g)$ for all $f \in C^*_{\mathrm{red}}(G_1, \mu_1)$ and $g \in C^*_{\mathrm{red}}(G_2, \mu_2)$. Because the product measure $\lambda := \lambda_1 \times \lambda_2$ on $(G_1)^{(0)} \times (G_2)^{(0)} = (G_1 \times G_2)^{(0)}$ also has full support, we similarly get a faithful representation

$$\mathrm{Ind}_\lambda \colon C^*_{\mathrm{red}}(G_1 \times G_2, \mu_1 \times \mu_2) \to \mathscr{L}(L^2(G_1 \times G_2, (\lambda \circ (\mu_1 \times \mu_2))^{-1})),$$

where $\lambda \circ (\mu_1 \times \mu_2) = \int_{G_1 \times G_2} (\mu_1^u \times \mu_2^v) \, d\lambda(u, v) = \nu_1 \times \nu_2$ is the induced measure. From

$$\int_{G_1 \times G_2} f((x,y)^{-1}) \, d(\nu_1 \times \nu_2)(x, y) = \int_{G_1} \int_{G_2} f(x^{-1}, y^{-1}) \, d\nu_1(x) \, d\nu_2(y)$$

it follows that $(\nu_1 \times \nu_2)^{-1} = \nu_1^{-1} \times \nu_2^{-1}$. By Example C.6.2, we have $L^2(G_1, \nu_1^{-1}) \hat{\otimes} L^2(G_2, \nu_2^{-1}) \cong L^2(G_1 \times G_2, \nu_1^{-1} \times \nu_2^{-1})$ via the map U which sends $f \otimes g$ to $U(f \otimes g)(x, y) = f(x) g(y)$. Consider the following diagram:

$$\begin{tikzcd}
C_c(G_1) \odot C_c(G_2) \arrow[r, hook] \arrow[d, "\iota"] & C^*_{\mathrm{red}}(G_1) \otimes_* C^*_{\mathrm{red}}(G_2) \arrow[r, "\pi", "\cong"'] \arrow[d, dotted, "\tilde{\iota}"] & \mathrm{im}(\varphi) \arrow[r, hook] \arrow[d, "\mathrm{conj}_U"] & \mathscr{L}(L^2(G_1) \hat{\otimes} L^2(G_2)) \arrow[d, "\mathrm{conj}_U", "\cong"'] \\
C_c(G_1 \times G_2) \arrow[r, hook] & C^*_{\mathrm{red}}(G_1 \times G_2) \arrow[r, "\mathrm{Ind}_\lambda", "\cong"'] & \mathrm{im}(\mathrm{Ind}_\lambda) \arrow[r, hook] & \mathscr{L}(L^2(G_1 \times G_2))
\end{tikzcd}$$

For all $f, \xi \in C_c(G_1)$ and $g, \eta \in C_c(G_2)$, we have

$$\begin{aligned}
U(\varphi(f \otimes g)(\xi \otimes \eta))(x, y) &= U(\mathrm{Ind}_{\lambda_1}(f) \hat{\otimes} \mathrm{Ind}_{\lambda_2}(g)(\xi \otimes \eta))(x, y) = \\
&= \mathrm{Ind}_{\lambda_1}(f)(\xi)(x) \mathrm{Ind}_{\lambda_2}(g)(\eta)(y) = \int_{G_1} f(xw) \xi(w^{-1}) \, d\mu_1^{d(x)}(w) \int_{G_2} g(yz) \eta(z^{-1}) \, d\mu_2^{d(y)}(z) = \\
&= \int_{G_1 \times G_2} \iota(f \otimes g)((x, y)(w, z)) U(\xi \otimes \eta)((w, z)^{-1}) \, d(\mu_1 \times \mu_2)^{d(x,y)}(w, z),
\end{aligned}$$

which shows $U \circ \varphi(f \otimes g) = \mathrm{Ind}_\lambda(\iota(f \otimes g)) \circ U$ as operators on $L^2(G_1, \nu_1^{-1}) \hat{\otimes} L^2(G_2, \nu_2^{-1})$, so the diagram commutes. In particular, $\mathrm{conj}_U(\mathrm{im}(\varphi)) \subseteq \mathrm{im}(\mathrm{Ind}_\lambda)$ and

$$\iota(f \otimes g) = (\mathrm{Ind}_\lambda^{-1} \circ \mathrm{conj}_U \circ \varphi)(f \otimes g)$$

for all $f \in C_c(G_1)$ and $g \in C_c(G_2)$. This gives an extension of ι to an injective *-homomorphism

$$\tilde{\iota} \colon C^*_{\mathrm{red}}(G_1, \mu_1) \otimes_* C^*_{\mathrm{red}}(G_2, \mu_2) \to C^*_{\mathrm{red}}(G_1 \times G_2, \mu_1 \times \mu_2).$$

By Lemma B.4, $\mathrm{im}(\iota)$ is dense in $C_c(G_1 \times G_2)$ for the inductive limit topology, and the dense inclusion $C_c(G_1 \times G_2) \hookrightarrow C^*_{\mathrm{red}}(G_1 \times G_2, \mu_1 \times \mu_2)$ is continuous by Remark 2.4.14, so that $\mathrm{im}(\iota)$ is also dense in $C^*_{\mathrm{red}}(G_1 \times G_2, \mu_1 \times \mu_2)$. This shows that $\tilde{\iota}$ is surjective. ∎

Notes for chapter 2

The study of groupoid C^*-algebras seems to have been initiated by Jean Renault, the primary reference being the monograph [**Ren80**], on which large portions of our treatment is based. In addition, we have used [**Pat99**], which gives a very detailed and modern account of the basics of groupoid C^*-algebras. The article [**Bun06**] provides a nice overview over the most important examples and results concerning groupoid C^*-algebras. If not otherwise specified, the results and examples of this chapter are taken from one of these references, or are simple consequences of statements therein. This is especially true for the first sections, which deal with the very basics of groupoids and Haar systems.

We have found that he measure-theoretic foundations are particularly well detailed in [**Pat99**], as Paterson doesn't use Bourbaki's theory of integration like Renault does. We also mention the dissertation of Goeff Goehle, [**Goe09**], which describes the more general theory of *groupoid crossed products* in very rich detail. Several of the proofs in this chapter have been adapted from there for our special case of groupoid C^*-algebras. In a similar manner, the book [**Wil07**] on (group) crossed products (see Example 2.6.4) was also very helpful, as some proofs and concepts are easily carried over to the groupoid setting.

Concerning the topological assumptions on locally compact groupoids, we have chosen to include second countability as a default since it makes the measure theoretical details a lot simpler, see Appendix A. As mentioned above, the theory of groupoid C^*-algebras originally used Radon measures in the sense of Bourbaki, that is, as (positive) functionals on $C_c(G)$. We also mention that it is sometimes useful to consider locally compact groupoids which are not Hausdorff. As an example, this arises when studying groupoid associated to foliated manifolds, and while it requires some additional work, it can be done, see for instance [**Con82**] or also [**Pat99**].

CHAPTER 3

Toeplitz C^*-algebras for Reinhardt domains

The goal of this chapter is to describe the structure of the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$ for a bounded, pseudoconvex Reinhardt domain $\Omega \subseteq \mathbb{C}^2$. This is done by associating to Ω a groupoid $\mathcal{G}_\Omega$ and showing that the groupoid C^*-algebra is isomorphic to the C^*-algebra generated by the polar decompositions of certain Toeplitz operators, which contains $\mathcal{T}(\Omega)$, see Theorem 3.5.4. In order to probe the finer structure of the Toeplitz algebra in terms of a composition series, we will apply the results on groupoid invariant subsets and associated ideals from section 2.5, and these invariant subsets will be obtained by parametrizing the unit space of the groupoid $\mathcal{G}_\Omega$ by the faces of the logarithmic base Λ_Ω as explained below.

This chapter is based on an article of Salinas, Sheu and Upmeier, [**SSU89**]. Along the way, we will encounter results from [**CM85**] due to Curto and Muhly.

3.1. Reinhardt domains

The domains that will be studied in the sequel are the Reinhardt domains. As the definition shows, they possess a several variables version of circular symmetry:

DEFINITION 3.1.1. A domain $\Omega \subseteq \mathbb{C}^n$ is called a *Reinhardt domain* (or a *polycircular* domain) if $(z_1, \dots, z_n) \in \Omega$ implies $(z_1t_1, \dots, z_nt_n) \in \Omega$ for every $(t_1, \dots, t_n) \in \mathbb{T}^n$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

EXAMPLE 3.1.2. Let $a \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n)$ with $r_i > 0$ for $1 \leq i \leq n$. The set

$$P(a, r) := \{z \in \mathbb{C}^n : |z_i - a_i| < r_i \text{ for all } k = 1, \dots, n\}$$

is called the (open) *polydisc* with multiradius r, centered at a. The polydiscs $P(0, r)$ with center $0 \in \mathbb{C}^n$ are evidently Reinhardt domains. We will denote the unit polydiscs by $\mathbb{D}^n := P(0, (1, \dots, 1))$. This is just the n-fold product of the open unit disc $\mathbb{D}$. Note that $P(a, r) \subseteq B(a, R)$ if $\sum_{i=1}^n r_i^2 < R^2$ and $B(a, R) \subseteq P(a, r)$ if $R < \min_{1 \leq i \leq n} r_i$, where $B(a, R) := \{z \in \mathbb{C}^n : |z - a| < R\}$ is the open ball around a with radius $R > 0$. $\Diamond$

DEFINITION 3.1.3. A Reinhardt domain Ω is called *complete* if, for every $z \in \Omega$, the polydisc $P(0, (|z_1|, \dots, |z_n|))$ is also contained in Ω. In particular, $0 \in \Omega$.

Thus, a complete Reinhardt domain is invariant not only under multiplication by elements of the torus, but also by elements of the entire polydisc. A Reinhardt domain is uniquely determined by its *absolute domain*

$$|\Omega| := \tau(\Omega) \subseteq \mathbb{R}^n_{\geq},$$

where $\mathbb{R}^n_{\geq} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$ and $\tau \colon \mathbb{C}^n \to \mathbb{R}^n_{\geq}$ is the map $\tau(z) := (|z_1|, \dots, |z_n|)$. This is really a domain (that is, open and connected) in $\mathbb{R}^n_{\geq}$, since the map τ is both open and continuous (hence its image is connected). For a subset S of $\mathbb{C}^n$ (or $\mathbb{R}^n$), we set

$$S^x := \{z = (z_1, \dots, z_n) \in S : z_1 \cdots z_n \neq 0\} = \{z \in S : z_j \neq 0 \text{ for all } j = 1, \dots, n\}.$$

DEFINITION 3.1.4. A Reinhardt domain $\Omega \subseteq \mathbb{C}^n$ is called *logarithmically convex* (or, shortly, *log-convex*) if

$$\Lambda_\Omega := \{(\log|x_1|, \dots, \log|x_n|) : (x_1, \dots, x_n) \in \Omega^x\}$$

is a convex subset of $\mathbb{R}^n$. The set Λ_Ω is called the *logarithmic base* of Ω.

The next Theorem shows that the logarithmic base of a Reinhardt domain may be used to check if it is a domain of holomorphy. We refer to [**Nar71**, p. 120] for a proof.

THEOREM 3.1.5. *A Reinhardt domain containing the origin is a domain of holomorphy (equivalently: pseudoconvex, see the discussion after Definition 1.4.3) if and only if it is logarithmically convex and complete.*

REMARK 3.1.6. Reinhardt domains in several variables play the same role as open discs in the one-dimensional case regarding the convergence of power series. Given any power series $\sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha$, its domain of convergence is a complete Reinhardt domain (which may be empty). Moreover, if Ω is a complete, logarithmically convex Reinhardt domain, then there exists a power series with domain of convergence precisely equal to Ω, see [**Ran86**, Theorem 3.28].

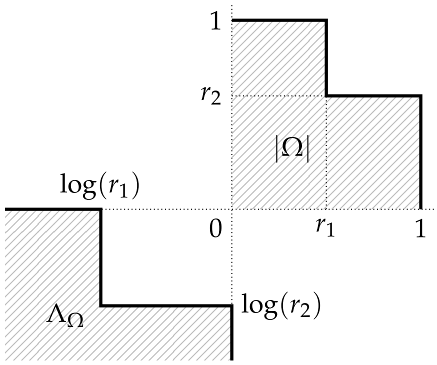
EXAMPLE 3.1.7. Let $\Omega := P(0,1) \subseteq \mathbb{C}^n$ be the unit polydisc. Then $|\Omega| = [0,1)^n$ and $|\Omega^x| = (0,1)^n$. Hence, the logarithmic base of Ω is $\Lambda_\Omega = (-\infty, 0)^n$ which is convex, implying that the polydisc is a domain of holomorphy. $\Diamond$

EXAMPLE 3.1.8. For $r_1, r_2 \in (0,1)$ consider the (complete) Reinhardt domain

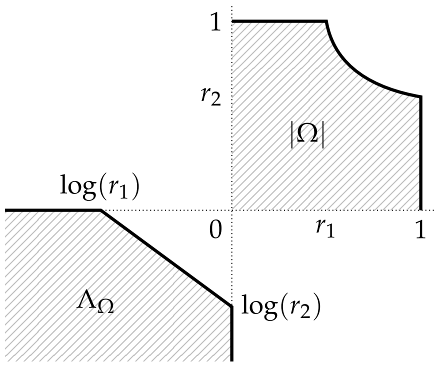
$$\Omega := \tau^{-1}(([0,r_1) \times [0,1)) \cup ([0,1) \times [0,r_2))) \subseteq \mathbb{C}^2.$$

Then $|\Omega^x|$ is simply U without its intersections with the coordinate axes, $|\Omega^x| = ((0,r_1) \times (0,1)) \cup ((0,1) \times (0,r_2))$. Consequently, the logarithmic base of Ω is the non-convex set

$$\Lambda_\Omega = ((-\infty, \log r_1) \times (-\infty, 0)) \cup ((-\infty, 0) \times (-\infty, \log r_2)).$$



Hence, Ω is not a domain of holomorphy. On the other hand, by taking the convex hull of Λ_Ω, we obtain the following pseudoconvex Reinhardt domain:

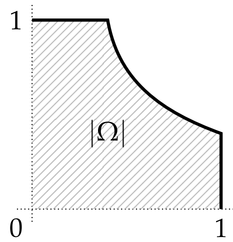


In fact, this Reinhardt domain is a simple prototype of the domains which will be studied in the rest of this chapter, and we will see that the Toeplitz C^*-algebra of Ω depends very much on the slope $-\log(r_2)/\log(r_1)$ of the line segment in $\partial\Lambda_\Omega$ and whether this is irrational or not. $\Diamond$

3.2. The algebra of shifted weights and its character space

Let $\Omega \subseteq \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain. In this chapter, we will define an auxiliary C^*-algebra whose character space will be important in constructing the groupoid which will allow us to analyze the structure of the Toeplitz C^*-algebra.

For technical reasons, we require Ω to be *normalized*, that is, $\Omega \subseteq \mathbb{D}^2$, the unit bidisk, and $V \subseteq \Omega$, where $V := \{(z_1, z_2) \in \mathbb{D}^2 : z_1 z_2 = 0\}$. In other words, $\Omega \cap V_i = \mathbb{D}$, where $V_i = \{(z_1, z_2) \in \mathbb{C}^2 : z_i = 0\}$. Thus, Ω has the same intersections with the coordinate axes as the bidisk and, in particular, $0 \in \Omega$. By Theorem 3.1.5, we see that Ω is logarithmically convex and complete. Since $\Omega \subseteq \mathbb{D}^2$, it follows that the logarithmic base Λ_Ω, see Definition 3.1.4, is contained in $\mathbb{R}^2_{\leq}$. Moreover, the boundary of Λ_Ω is the image of a continuous curve which has the negative coordinate axes as asymptotes since $(1,0), (0,1) \in \partial|\Omega|$.



Since our ultimate goal is to determine the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$ up to isomorphism, these requirements do not pose any restrictions as any bounded pseudoconvex Reinhardt domain may be put into this form by rescaling variables. Indeed, if $\rho_i := \sup\{|z_i| \colon z \in V_i\}$ for $i \in \{1,2\}$, then $\rho_i > 0$ because Ω is open and contains 0, and by the scaling $(z_1, z_2) \mapsto (z_1/\rho_1, z_2/\rho_2)$ we may assume $\rho_i = 1$ for $i = 1,2$. Therefore, $\Omega \cap V_i = \mathbb{D}$ and also $\Omega \subseteq \mathbb{D}^2$ because if there were $(z_1, z_2) \in \Omega$ with $|z_i| > 1$, then by completeness we see that $P(0, (|z_1|, |z_2|)) \subseteq \Omega$, a contradiction to $\rho_i = 1$. Therefore, Ω is normalized. By Theorem 1.3.3, this does not change the isomorphism class of the Toeplitz C^*-algebra.

We will denote by $\mathbb{Z}_{\geq} = \mathbb{N}_0 \subseteq \mathbb{Z}$ the set of non-negative integers, and by $\mathbb{Z}_{>} = \mathbb{N} \subseteq \mathbb{Z}$ the natural numbers. Similarly, we define $\mathbb{Z}_{\leq}$ and $\mathbb{Z}_{<}$ and we put $\mathbb{Z}^n_{\geq} := (\mathbb{Z}_{\geq})^n \subseteq \mathbb{Z}^n$ and likewise for $\mathbb{Z}_{>}$, $\mathbb{Z}_{\leq}$ and $\mathbb{Z}_{<}$. The same notation will also be employed for the corresponding subsets of $\mathbb{R}^n$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then we put $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

DEFINITION 3.2.1. For $\gamma \in \mathbb{Z}^2_{\geq}$, let χ_γ be the function on $\mathbb{Z}^2$ defined by

$$\chi_\gamma(\alpha) := \begin{cases} \|z^{\alpha+\gamma}\|_{L^2(\Omega)} / \|z^\alpha\|_{L^2(\Omega)}, & \text{if } \alpha \in \mathbb{Z}^2_{\geq} \\ 0, & \text{if } \alpha \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq}. \end{cases} \tag{3.2.1}$$

The conceptual reason why these functions will play an important role is the following: In Proposition *D.1* it is shown that the monomials $e_\alpha(z) := z^\alpha / \|z^\alpha\|_{L^2(\Omega)}$ with $\alpha \in \mathbb{Z}^2_{\geq}$ form an orthonormal basis for the Bergman space $A^2(\Omega)$. With respect to this basis, the Toeplitz operator $T_\beta := T_{z^\beta}$ with symbol z^β for $\beta \in \mathbb{Z}^2_{\geq}$ satisfies

$$T_\beta e_\alpha = \frac{z^{\alpha+\beta}}{\|z^\alpha\|} = \frac{\|z^{\alpha+\beta}\|}{\|z^\alpha\|} e_{\alpha+\beta}.$$

In other words, the Toeplitz operator T_β is a *weighted shift operator*.

By the assumption that Ω is contained in $\mathbb{D}^2$, we have $|z^{\alpha+\gamma}| = |z^\alpha||z^\gamma| \leq |z^\alpha|$ and hence χ_γ is bounded, meaning $\chi_\gamma \in \ell^\infty(\mathbb{Z}^2)$. For $\beta \in \mathbb{Z}^2$, let $\tau_\beta \colon \ell^\infty(\mathbb{Z}^2) \to \ell^\infty(\mathbb{Z}^2)$ be the translation operator by β,

$$(\tau_\beta h)(\alpha) := h(\alpha - \beta). \tag{3.2.2}$$

Evidently, τ_β is a bounded, even isometric, linear operator on $\ell^\infty(\mathbb{Z}^2)$. Moreover, the mapping $\beta \mapsto \tau_\beta$ is a group homomorphism from $\mathbb{Z}^2$ to $\mathrm{Aut}(\ell^\infty(\mathbb{Z}^2))$, where the latter denotes the group of (isometric) isomorphisms of $\ell^\infty(\mathbb{Z}^2)$. We say that $\mathbb{Z}^2$ *acts* on $\ell^\infty(\mathbb{Z}^2)$ by translations.

DEFINITION 3.2.2. The C^*-subalgebra of $\ell^\infty(\mathbb{Z}^2)$ generated by the translates $\tau_\beta \chi_\gamma$,

$$\mathcal{D}_\Omega := C^*(\{\tau_\beta \chi_\gamma : \beta \in \mathbb{Z}^2, \gamma \in \mathbb{Z}^2_{\geq}\}),$$

will be called the *C^*-algebra of shifted weights associated to Ω*.

REMARK 3.2.3. The translation action (3.2.2) of $\mathbb{Z}^2$ on $\ell^\infty(\mathbb{Z}^2)$ restricts to an action on the algebra of shifted weights, since $\mathcal{D}_\Omega$ is invariant under these translations as

$$\tau_\alpha(\tau_\beta\chi_\gamma) = \tau_{\alpha+\beta}\chi_\gamma$$

for all $\alpha, \beta \in \mathbb{Z}^2$ and $\gamma \in \mathbb{Z}^2_\geq$.

REMARK 3.2.4. In [**CM85**] and [**SSU89**], the C^*-algebra $\mathcal{D}_\Omega$ is defined to be generated by the functions $\tau_\beta w_i$ with $\beta \in \mathbb{Z}^2$ and $w_i := \chi_{e_i}$ for $i \in \{1, \ldots, n\}$, where e_i is the i-th standard unit vector of $\mathbb{Z}^2$. However, as also noted in those references, one may compute χ_γ from the w_i's and translations. In fact, $\chi_{\gamma+\gamma'}(\alpha) = \chi_\gamma(\gamma' + \alpha)\chi_{\gamma'}(\alpha) = ((\tau_{-\gamma'}\chi_\gamma)\chi_{\gamma'})(\alpha)$ holds for all $\gamma, \gamma' \in \mathbb{Z}^2_\geq$ and $\alpha \in \mathbb{Z}^2$ and allows to recursively compute χ_γ from w_i. Thus, the resulting C^*-algebras are identical.

Consider now, for functions $f \in \ell^1(\mathbb{Z}^2)$ and $g \in \ell^\infty(\mathbb{Z}^2)$, the convolution of f with g, given by $(f * g)(\alpha) := \sum_{\beta\in\mathbb{Z}^2} f(\alpha - \beta)g(\beta) = \sum_{\beta\in\mathbb{Z}^2} f(\beta)g(\alpha - \beta)$. Then we have $\|f * g\|_\infty \leq \|f\|_1\|g\|_\infty$, showing that $f * g$ is bounded on $\mathbb{Z}^2$. Letting $s_N(\alpha) := \sum_{|\beta|\leq N} f(\beta)g(\alpha - \beta) = \sum_{|\beta|\leq N} f(\beta)(\tau_\beta g)(\alpha)$, we see

$$|s_N(\alpha) - f * g(\alpha)| \leq \sum_{|\beta|>N} |f(\beta)||g(\alpha - \beta)| \leq \|g\|_\infty \sum_{|\beta|>N} |f(\beta)| \to 0$$

as $N \to \infty$. Therefore, we can approximate $f * g$ uniformly by linear combinations of the translates $\tau_\beta g$ and, by taking $g = \chi_\gamma$, we obtain an equivalent description of $\mathcal{D}_\Omega$:

$$\mathcal{D}_\Omega = C^*(\{f * \chi_\gamma : f \in \ell^1(\mathbb{Z}^2), \gamma \in \mathbb{Z}^2_\geq\}).$$

Since $\ell^\infty(\mathbb{Z}^2)$ is commutative, so is its subalgebra $\mathcal{D}_\Omega$. We consider the character space, or Gelfand spectrum, $\mathrm{Spec}(\mathcal{D}_\Omega)$ of $\mathcal{D}_\Omega$, which will be important in the following analysis. Recall that $\mathrm{Spec}(\mathcal{D}_\Omega)$ is the set of all *characters* of $\mathcal{D}_\Omega$, that is, the non-zero multiplicative linear functionals $\mathcal{D}_\Omega \to \mathbb{C}$ and we put on $\mathrm{Spec}(\mathcal{D}_\Omega)$ the relative weak-* topology. Then $\mathrm{Spec}(\mathcal{D}_\Omega)$ is a locally compact Hausdorff space and by the basic theory of commutative C^*-algebras, $\mathcal{D}_\Omega$ is isomorphic to $C_0(\mathrm{Spec}(\mathcal{D}_\Omega))$, see Theorem C.3.4. Note that $\mathcal{D}_\Omega$ is non-unital: any finite product and linear combination of functions of the form $\tau_\beta\chi_\gamma$ will be zero outside a set of the form $\beta_0 + \mathbb{Z}^2_\geq$ for some $\beta_0 \in \mathbb{Z}^2$. Thus, the constant function 1, which is the unit of $\ell^\infty(\mathbb{Z}^2)$, is not the uniform limit of such functions, and hence is not in the C^*-algebra generated by them. Therefore, $\mathrm{Spec}(\mathcal{D}_\Omega)$ is a locally compact Hausdorff space which is *not* compact, see Proposition C.3.5.

Let $\omega(\alpha)$ be the evaluation functional on $\ell^\infty(\mathbb{Z}^2)$ at $\alpha \in \mathbb{Z}^2$, that is,

$$\omega(\alpha) \colon \ell^\infty(\mathbb{Z}^2) \to \mathbb{C}, \quad \omega(\alpha)(\varphi) := \varphi(\alpha).$$

Then $\omega(\alpha)$ restricts to a character on $\mathcal{D}_\Omega$, so we can regard $\omega(\alpha)$ as an element of $\mathrm{Spec}(\mathcal{D}_\Omega)$. We will sometimes write ω_α instead of $\omega(\alpha)$ when it is convenient to do so. By the following Lemma, we may regard $\mathbb{Z}^2$ as a dense subset of the character space of $\mathcal{D}_\Omega$.

LEMMA 3.2.5 ([**CM85**, Lemma 2.1]). *The mapping $\omega\colon \mathbb{Z}^2 \to \mathrm{Spec}(\mathcal{D}_\Omega)$, $\alpha \mapsto \omega(\alpha)$, is injective, open, and its image is dense in* $\mathrm{Spec}(\mathcal{D}_\Omega)$.

In other words, the algebra $\mathcal{D}_\Omega \subseteq \ell^\infty(\mathbb{Z}^2)$ separates the points of $\mathbb{Z}^2$, that is, for every $\alpha_1, \alpha_2 \in \mathbb{Z}^2$ with $\alpha_1 \neq \alpha_2$ there exists $\varphi \in \mathcal{D}_\Omega$ such that $\varphi(\alpha_1) \neq \varphi(\alpha_2)$. Indeed, if $\varphi(\alpha_1) = \varphi(\alpha_2)$ for all $\varphi \in \mathcal{D}_\Omega$, then $\omega(\alpha_1) = \omega(\alpha_2)$ on $\mathcal{D}_\Omega$, a contradiction to the Lemma.

PROOF. We first show injectivity. Let $\alpha_1, \alpha_2 \in \mathbb{Z}^2$ be such that $\omega(\alpha_1)(\varphi) = \omega(\alpha_2)(\varphi)$ for all $\varphi \in \mathcal{D}_\Omega$. Taking $\varphi = \tau_\beta\chi_\gamma$, this means

$$\chi_\gamma(\alpha_1 - \beta) = \chi_\gamma(\alpha_2 - \beta)$$

for all $\gamma \in \mathbb{Z}^2_{\geq}$ and all $\beta \in \mathbb{Z}^2$. Our first task is to replace α_1 and α_2 in the above equality by elements of $\mathbb{N}^2$: we find $\mu, \nu \in \mathbb{N}^2$ and $\lambda \in \mathbb{Z}^2$ (depending on γ and β) such that

$$\alpha_1 - \alpha_2 = \mu - \nu \quad \text{and} \quad \beta = -\alpha_1 + \alpha_2 - \lambda = -\mu + \nu - \lambda.$$

Since $\alpha_1 + \nu - \mu = \alpha_2$, we have

$$\chi_\gamma(\alpha_1 + \mu - \nu + \lambda) = \chi_\gamma(\alpha_1 + \lambda)$$

for all $\gamma \in \mathbb{Z}^2_{\geq}$ and $\lambda \in \mathbb{Z}^2$. Substituting $\lambda = -\alpha_1 + \nu + \lambda'$, we see that

$$\chi_\gamma(\mu + \lambda') = \chi_\gamma(\nu + \lambda')$$

for all $\gamma \in \mathbb{Z}^2_{\geq}$ and $\lambda' \in \mathbb{Z}^2$.

Now assume that $\alpha_1 \neq \alpha_2$, which is equivalent to $\mu \neq \nu$. Suppose that $\mu_i < \nu_i$ for some $i \in \{1, 2\}$ and set $\lambda' := -(\mu_i + 1)e_i$. Then $\chi_\gamma(\mu + \lambda') = 0$ since $(\mu + \lambda')_i < 0$ implies $\mu + \lambda' \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq}$. But then also $\chi_\gamma(\nu + \lambda') = 0$ despite $\nu + \lambda'$ being in $\mathbb{Z}^2_{\geq}$, a contradiction to the definition of χ_γ. Thus, $\alpha_1 = \alpha_2$ and ω is injective.

To show density of $\mathbb{Z}^2$ in $\operatorname{Spec}(\mathcal{D}_\Omega)$, let $\Gamma_{\mathcal{D}_\Omega} \colon \mathcal{D}_\Omega \overset{\cong}{\to} C_0(\operatorname{Spec}(\mathcal{D}_\Omega))$ be the Gelfand representation, see Definition C.3.2. If $\overline{\omega(\mathbb{Z}^2)} \neq \operatorname{Spec}(\mathcal{D}_\Omega)$, then by Corollary *A*.2.5, there is $f \in C_0(\operatorname{Spec}(\mathcal{D}_\Omega))$ with $f|_{\omega(\mathbb{Z}^2)} = 0$ but $f \neq 0$. Since the Gelfand representation is an isomorphism, there is $0 \neq \varphi \in \mathcal{D}_\Omega$ with $\Gamma_{\mathcal{D}_\Omega}(\varphi) = f$. But then

$$\varphi(\alpha) = \omega(\alpha)(\varphi) = \Gamma_{\mathcal{D}_\Omega}(\varphi)(\omega(\alpha)) = f(\omega(\alpha)) = 0$$

for all $\alpha \in \mathbb{Z}^2$, a contradiction to $\varphi \neq 0$.

It remains to verify that $\omega \colon \mathbb{Z}^2 \to \operatorname{Spec}(\mathcal{D}_\Omega)$ is an open map. Since $\mathbb{Z}^2$ carries the discrete topology, it suffices to show that the singleton $\{\omega(\alpha)\}$ is open in $\operatorname{Spec}(\mathcal{D}_\Omega)$ for every $\alpha \in \mathbb{Z}^2$. By the above, for $\rho \in \operatorname{Spec}(\mathcal{D}_\Omega) \setminus \{\omega(\alpha)\}$, there exists a net $\iota \mapsto \alpha_\iota \in \mathbb{Z}^2$ such that $\omega(\alpha_\iota) \to \rho$ in $\operatorname{Spec}(\mathcal{D}_\Omega)$. Because $\operatorname{Spec}(\mathcal{D}_\Omega) \setminus \{\omega(\alpha)\}$ is open in $\operatorname{Spec}(\mathcal{D}_\Omega)$ (the character space is Hausdorff, so singletons are closed), we eventually have $\alpha_\iota \in \mathbb{Z}^2 \setminus \{\alpha\}$. Let $\psi \colon \mathbb{Z}^2 \to \mathbb{C}$ denote the characteristic function of the subset $\mathbb{Z}^2_{\geq} \subseteq \mathbb{Z}^2$. Note that $\psi \in \mathcal{D}_\Omega$ because ψ is just $\chi_{(0,0)}$ from (3.2.1). For $\beta \in \mathbb{Z}^2$ and $\psi_\alpha := \tau_\alpha \psi - \tau_{\alpha+(1,0)}\psi - \tau_{\alpha+(0,1)}\psi \in \mathcal{D}_\Omega$, we have

$$\psi_\alpha(\beta) = \begin{cases} +1, & \beta = \alpha \\ -1, & \beta \in (\alpha + (1,1)) + \mathbb{Z}^2_{\geq}, \\ 0, & \text{else.} \end{cases}.$$

It follows that $\Gamma_{\mathcal{D}_\Omega}(\psi_\alpha)(\rho) = \rho(\psi_\alpha) = \lim_\iota \psi_\alpha(\alpha_\iota) \leq 0$ and we may therefore describe $\{\omega(\alpha)\}$ as

$$\{\omega(\alpha)\} = \{\rho \in \operatorname{Spec}(\mathcal{D}_\Omega) : \Gamma_{\mathcal{D}_\Omega}(\psi_\alpha)(\rho) > 0\},$$

which is clearly open. ■

Since we want to use $\operatorname{Spec}(\mathcal{D}_\Omega)$ later to construct a (transformation) groupoid, we need to verify that its topology is second countable as we have made that assumption throughout chapter 2.

COROLLARY 3.2.6. *The character space* $\operatorname{Spec}(\mathcal{D}_\Omega)$ *of* $\mathcal{D}_\Omega$ *is separable and metrizable. In particular, its topology is second countable.*

PROOF. Lemma 3.2.5 shows that $\omega(\mathbb{Z}^2) \subseteq \operatorname{Spec}(\mathcal{D}_\Omega)$ is a countable dense subset which means that $\operatorname{Spec}(\mathcal{D}_\Omega)$ is separable. Because the C^*-algebra $\mathcal{D}_\Omega$ is generated by a countable number of elements (the $\tau_\beta \chi_\gamma$), we see that $\mathcal{D}_\Omega$ is a separable Banach space. By [**Con90**, Theorem V.5.1] this is equivalent to the unit ball X of the dual space ${\mathcal{D}_\Omega}^*$ being weak-* metrizable. Since characters of C^*-algebras are automatically continuous with operator norm bounded by one, we have

$\mathrm{Spec}(\mathcal{D}_\Omega) \subseteq X$ and so $\mathrm{Spec}(\mathcal{D}_\Omega)$ is also metrizable. Finally, a metric space is second countable if and only if it is separable. ■

3.3. Faces and associated characters

As in the previous section, we consider a normalized Reinhardt domain Ω in $\mathbb{C}^2$. In the above Lemma 3.2.5 we have identified $\omega(\mathbb{Z}^2)$ as a dense subset of the character space $\mathrm{Spec}(\mathcal{D}_\Omega)$ of $\mathcal{D}_\Omega$, the C^*-algebra of shifted weights. In this section, we will identify additional members of $\mathrm{Spec}(\mathcal{D}_\Omega)$ by associating certain characters of $\mathcal{D}_\Omega$ to the faces of the convex set $\overline{\Lambda_\Omega}$ from Definition 3.1.4. In this way, geometric information of Ω gets encoded into $\mathrm{Spec}(\mathcal{D}_\Omega)$ and, eventually, into the groupoid that will be constructed in the next section.

Consider now the proper faces of the closed convex set

$$\Lambda := \overline{\Lambda_\Omega} \subseteq \mathbb{R}^2,$$

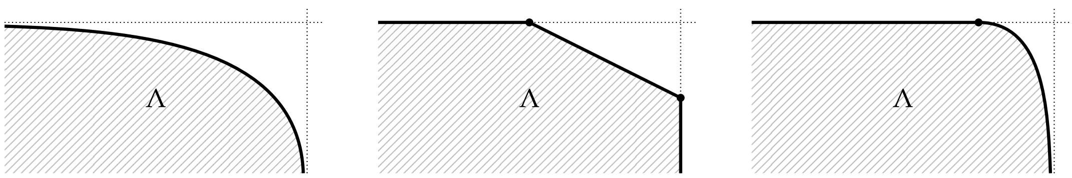
where Λ_Ω is the logarithmic base of Ω from Definition 3.1.4. For the basic definitions and properties related to faces of convex sets, see Appendix E. If F is a face of Λ, then we have the following two possibilities:

(i) The face F is *exposed*, that is, F is the intersection of Λ with a supporting hyperplane, see Definition E.8. In this case, there exists a unit vector $u \in \mathbb{R}^2$ such that

$$F = F_u = \{x \in \Lambda : x \cdot u = \sup_{y\in\Lambda} y \cdot u\},$$

where $v \cdot w := v_1 w_1 + v_2 w_2$ for $v, w \in \mathbb{R}^2$ is the standard inner product on $\mathbb{R}^2$. In other words, u is an outward pointing normal to F. While in the case $\dim(F) = 1$ (hence F is a line segment) the vector u is uniquely determined by the face, this does not necessarily have to be true for exposed singleton faces. By our assumptions on Ω, see the discussion at the beginning of section 3.2, we have that $\Lambda \subseteq \mathbb{R}^2_\leq$ and its boundary has the negative coordinate axes as asymptotes, so that $u \in \mathbb{R}^2_\geq$ and $u \in \{e_1, e_2\}$ corresponds to the case where F is a segment of one of the coordinate axes.

(ii) The face F is *not exposed*, implying that it is contained in a proper exposed face by Proposition E.11. Note that non-exposed faces of Λ necessarily consist of single points for dimensional reasons. More precisely, the non-exposed faces are the endpoints of line segments (that is, one-dimensional faces).



(A) All faces are exposed singletons. Both virtual faces are present, see Definition 3.3.5. All faces have uniquely determined outward unit vector $u \in \mathbb{R}^2_>$.

(B) All faces are exposed, including the two singletons. However, the unit vector u is not uniquely determined for the singletons. There are no virtual faces.

(C) The point is non-exposed, while all other points on the arc are exposed. The virtual face F_{e_1} is present.

A graphic illustrating several possibilities of the facial structure of the logarithmic base of a bounded pseudoconvex Reinhardt domain.

For a vector $u = (u_1, u_2) \in \mathbb{R}^2$ we define $u^\perp := (-u_2, u_1)$. Now let $F = F_u$ be a one-dimensional (exposed) face. Then we have

$$F - cu \subseteq \mathrm{span}(u^\perp) \quad \text{with} \quad c = \sup_{y\in\Lambda}(y \cdot u)$$

since $(p - cu) \cdot u = 0$ for all $p \in F$. If $u \in \mathbb{R}^2_>$, then F must be compact and there are $y^-, y^+ \in \operatorname{span}(u^\perp)$ such that

$$F - cu = [y^-, y^+] = \{y \in \operatorname{span}(u^\perp) : y^- \cdot u^\perp \leq y \cdot u^\perp \leq y^+ \cdot u^\perp\}. \tag{3.3.1}$$

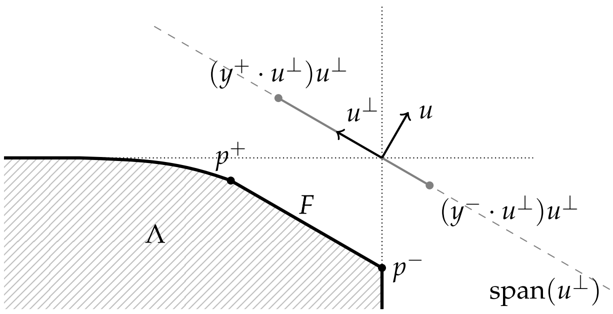
In other words, $p^\pm := y^\pm + cu$ are the endpoints of the line segment F. If $u = e_1$, then $F \subseteq \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\}$, in particular $c = 0$, and F has to be an unbounded interval in this coordinate axis. Thus, there is $p^+ = (0, p_2^+)$ such that

$$F = ((0, -\infty), p^+] = \{(0, y_2) : y_2 \leq p_2^+\}.$$

Similarly, if $u = e_2$, then there is $p^- = (p_1^-, 0)$ such that

$$F = ((-\infty, 0), p^-] = \{(y_1, 0) : y_1 \leq p_1^-\},$$

where the notation has been chosen such that it resembles (3.3.1) since $e_2^\perp = -e_1$. In other words, the endpoint p^+ is always the "counter-clockwise" one, as seen from the interior of Λ.



DEFINITION 3.3.1. For any one-dimensional face $F = F_u$ of Λ, we define its *endpoint faces* as

$$\partial_+ F := \begin{cases} \{p^+\}, & u \in \mathbb{R}^2_> \cup \{e_1\} \\ \emptyset, & \text{else} \end{cases} \quad \text{and} \quad \partial_- F := \begin{cases} \{p^-\}, & u \in \mathbb{R}^2_> \cup \{e_2\} \\ \emptyset, & \text{else} \end{cases}$$

so that the compact one-dimensional faces (those with $u \in \mathbb{R}^2_>$) have two endpoints $\partial_\pm F$, while the faces with $u \in \{e_1, e_2\}$ only have one endpoint. Note that $\partial_\pm F$ are faces of Λ by Lemma E.6.

We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ the extended real line and we put on it the usual topology (the order topology). More precisely, a basis for the topology of $\overline{\mathbb{R}}$ is given by the intervals (a, b), $[-\infty, b)$ and $(a, +\infty]$ for all $a, b \in \mathbb{R}$. Thus, $\overline{\mathbb{R}}$ is a compact Hausdorff space. For $\alpha \in \mathbb{Z}^2$, we denote by $\|\alpha\| := \sqrt{(\alpha_1)^2 + (\alpha_2)^2}$ the Euclidean norm of α. The following Lemma shows that it suffices to consider certain unbounded sequences in order to describe the closure $\operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp)$ of $\mathbb{Z}^2_> \cdot u^\perp = \{\alpha \cdot u^\perp : \alpha \in \mathbb{Z}^2_>\}$ in $\overline{\mathbb{R}}$.

LEMMA 3.3.2 ([**SSU89**, Lemma 3.1]). *For any unit vector $u \in \mathbb{R}^2_\geq$, we have*

$$\operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp) = \left\{ \lim_{k \to \infty} (\alpha_k \cdot u^\perp) \in \overline{\mathbb{R}} : \alpha_k \in \mathbb{Z}^2_> \text{ satisfies } \|\alpha_k\| \to \infty \text{ and } \frac{\alpha_k}{\|\alpha_k\|} \to u \right\}. \tag{3.3.2}$$

Explicitly, if $u \in \mathbb{R}^2_>$ and letting $\kappa := u_2/u_1 > 0$ be its slope, then

$$\operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp) = \begin{cases} (\|(p,q)\|^{-1}\mathbb{Z}) \cup \{+\infty, -\infty\}, & \kappa = p/q \in \mathbb{Q} \text{ with } p, q \in \mathbb{N} \text{ relatively prime} \\ \overline{\mathbb{R}}, & \kappa \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \tag{3.3.3}$$

while for the other cases $u = e_1 = (1, 0)$ and $u = e_2 = (0, 1)$ we have

$$\operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot e_1^\perp) = \mathbb{Z}_> \cup \{+\infty\} \quad \text{and} \quad \operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot e_2^\perp) = \mathbb{Z}_< \cup \{-\infty\}. \tag{3.3.4}$$

PROOF. Consider first $u \in \mathbb{R}^2_>$ and the case $\kappa := u_2/u_1 = p/q \in \mathbb{Q}$ with $p, q \in \mathbb{N}$ relatively prime. Then $v \cdot u^\perp = -v_1 u_2 + v_2 u_1$ for all $v \in \mathbb{R}^2$. We have $1 = u_1^2 + u_2^2 = (u_2 q/p)^2 + u_2^2 = (u_2/p)^2 \|(p,q)\|^2$ and similarly, $(u_1/q)^2\|(p,q)\|^2 = 1$. This implies

$$v \cdot u^\perp = -v_1 u_2 + v_2 u_1 = -v_1 u_1 \frac{p}{q} + v_2 u_2 \frac{q}{p} = (-v_1 p + v_2 q)\|(p,q)\|^{-1}.$$

By relative primality of p and q, the Diophantine equation $nq - mp = l$ has, for every $l \in \mathbb{Z}$, a solution $(m,n) \in \mathbb{Z}^2_>$. Therefore, any limit of $\alpha_k \cdot u^\perp$, where $(\alpha_k)_{k\geq 1}$ is a sequence in $\mathbb{Z}^2_>$, has to be of the form $(\|(p,q)\|^{-1}\mathbb{Z}) \cup \{+\infty, -\infty\}$. We now show that these limits can be obtained with sequences as in (3.3.2). Fix $l \in \mathbb{Z}$ and set $\alpha_k := (m,n) + k(q,p)$. Obviously, $\|\alpha_k\| \to \infty$ for $k \to \infty$, and since $\alpha_k \cdot u^\perp = (m,n) \cdot u^\perp = \|(p,q)\|^{-1} l$, we have $(\alpha_k/\|\alpha_k\|) \cdot u^\perp \to 0$ as $k \to \infty$. This implies $\alpha_k/\|\alpha_k\| \to u$. The infinite values are obtained by considering the sequences $\beta_k^\pm := k(m_\pm, n_\pm) + k^2(p,q)$, where $(m_\pm, n_\pm)$ is a solution of $nq - mp = \pm 1$. Indeed, $\beta_k^\pm \cdot u^\perp = \pm\|(p,q)\|^{-1} k \to \pm\infty$ and $\beta_k^\pm/\|\beta_k^\pm\| \to u$ as $k \to \infty$ as before.

If $\kappa \in \mathbb{R} \setminus \mathbb{Q}$, there are sequences $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ of natural numbers such that $p_k/q_k \to \kappa$. Note that this implies that at least one of $(p_k)_k$ or $(q_k)_k$ is unbounded, since if both were bounded sequences, we would have subsequences converging to integers, in which case κ would be rational. Therefore, $\|(p_k, q_k)\|^{-1} \to 0$ as $k \to \infty$. Fix a number $s \in \mathbb{R}$. Invoking the rational case and the Archimedean property of the real numbers, we see that for every $m \in \mathbb{N}$, there is $k_m \in \mathbb{N}$ and $l_m \in \mathbb{Z}$ such that

$$\|(p_{k_m}, q_{k_m})\|^{-1} l_m \leq s \leq \|(p_{k_m}, q_{k_m})\|^{-1}(l_m + 1),$$

and we choose $m \mapsto k_m$ to be increasing. Let $(\alpha_j^m)_{j\geq 1}$ be a sequence as in (3.3.2) with $\lim_j(\alpha_j^m \cdot u^\perp) = \|(p_{k_m}, q_{k_m})\|^{-1} l_m$. For $m \geq 1$ and $\varepsilon > 0$, choose j_m (increasing in m) such that

$$|\alpha_{j_m}^m \cdot u^\perp - \|(p_{k_m}, q_{k_m})\|^{-1} l_m| < \varepsilon/2$$

and set $\beta_m := \alpha_{j_m}^m$. Choosing $M \geq 1$ so large that $\big|\|(p_{k_m}, q_{k_m})\|^{-1} l_m - s\big| < \varepsilon/2$ for $m \geq M$, we see that

$$|\beta_m \cdot u^\perp - s| \leq |\alpha_{j_m}^m \cdot u^\perp - \|(p_{k_m}, q_{k_m})\|^{-1} l_m| + \big|\|(p_{k_m}, q_{k_m})\|^{-1} l_m - s\big| < \varepsilon$$

for $m \geq M$. Therefore, $s = \lim_{m\to\infty}(\beta_m \cdot u^\perp)$. Moreover, after discarding finitely many elements of each of the sequences $(\alpha_j^m)_{j\geq 1}$, we may assume $\|\alpha_{j_m}^m\| \geq \|\alpha_{j_m}^1\|$ for all $m \geq 1$ and hence $\|\beta_m\| \geq \|\alpha_{j_m}^1\| \to \infty$ as $m \to \infty$ by construction. As in the rational case, this also forces $\beta_m/\|\beta_m\| \to u$. The infinite values are again obtained in a way similar to the rational case.

For the remaining cases $u = e_i$, $i \in \{1,2\}$, the description (3.3.4) is trivial, and the argument for (3.3.2) is also not difficult: for $l \in \mathbb{Z}_>$, simply take $\alpha_k := l e_i^\perp + k e_i$, so that $\|\alpha_k\| \to \infty$ and $\alpha_k \cdot u^\perp = l$ is constant. ∎

To any exposed face F of Λ we now want to assign functions $\Phi_i(F) \colon \mathscr{D}(F) \to \mathbb{R}_\geq$, where $i \in \{1,2\}$ and $\mathscr{D}(F)$ is a certain subset of $\overline{\mathbb{R}}$. We will verify in Lemma 3.3.6 below that these definitions make sense.

DEFINITION 3.3.3. Let $F = F_u$ be an exposed face of Λ and with unit vector $u \in \mathbb{R}^2_\geq$. We define

$$\mathscr{D}(F) := \mathrm{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp), \tag{3.3.5}$$

see Lemma 3.3.2. Moreover, we specify functions $\Phi_i(F) \colon \mathscr{D}(F) \to \mathbb{R}_\geq$ for $i \in \{1,2\}$ as follows:

(i) If $F = \{p\}$ is a singleton, put

$$\Phi_i(F)(s) := \exp(2p_i),$$

independent of the argument s.

(ii) If F is a line segment and $u \in \mathbb{R}^2_>$, we set

$$\Phi_i(F)(s) := \frac{\int_{F-cu} \exp(2(s\,y \cdot u^\perp + y_i))\,dy}{\int_{F-cu} \exp(2s\,y \cdot u^\perp)\,dy} \exp(2cu_i), \quad s \in \mathbb{R}, \tag{3.3.6}$$

with $c := \sup_{y\in\Lambda}(y \cdot u)$ and

$$\Phi_i(F)(\pm\infty) := \exp(2p_i^\pm),$$

where $p^\pm$ are the endpoints of F, see the discussion above.

(iii) If F is a line segment with $u = e_1$, then $\mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\}$ by (3.3.4) and we put

$$\Phi_i(F)(s) := \begin{cases} 1, & i = 1 \\ \frac{s}{s+1} \exp(2p_2^+), & i = 2 \end{cases} \quad s \geq 1.$$

and

$$\Phi_i(F)(+\infty) := \exp(2p_i^+),$$

where $p^+ = (0, p_2^+)$ is the endpoint of F.

(iv) If F is a line segment with $u = e_2$, then $\mathscr{D}(F) = \mathbb{Z}_< \cup \{-\infty\}$ by (3.3.4) and we put

$$\Phi_i(F)(s) := \begin{cases} \frac{s}{s-1} \exp(2p_1^-), & i = 1 \\ 1, & i = 2 \end{cases} \quad s \leq -1$$

and

$$\Phi_i(F)(-\infty) := \exp(2p_i^-),$$

where $p^- = (p_1^-, 0)$ is the endpoint of F.

In (3.3.6), $\int_{F-cu} f(y)\,dy$ for $f \in C(F - cu)$ has to be understood as $\int_{y^-\cdot u^\perp}^{y^+\cdot u^\perp} f(tu^\perp)\,dt$ by parametrizing $F - cu$ as $[y^- \cdot u^\perp, y^+ \cdot u^\perp] \ni t \mapsto tu^\perp \in \operatorname{span}(u^\perp)$, see (3.3.1).

REMARK 3.3.4. Observe that while it may happen that F does not fix u uniquely, this only occurs in the case where F is a singleton. In this case, $\Phi_i(F)(s)$ is defined to be independent of s, so this discrepancy does not matter for our considerations.

In addition to the faces contained in $\partial\Lambda$, we are interested in the case where the boundary of Λ does not intersect one or more of the coordinate axes. This behavior is described with what is called "virtual faces" in [**SSU89**].

DEFINITION 3.3.5. If, for $j \in \{1, 2\}$, we have $\partial\Lambda \cap \operatorname{span}(e_j^\perp) = \emptyset$, then the set

$$F_{e_j} := \begin{cases} \{(0, -\infty)\}, & j = 1 \\ \{(-\infty, 0)\}, & j = 2 \end{cases}$$

is called the *virtual face* induced by e_j. For the virtual faces, we define the functions

$$\Phi_i(F_{e_j}) \colon \mathscr{D}(F) \to \mathbb{R}_\geq, \quad \Phi_i(F_{e_j})(s) := \delta_{ij},$$

where $\mathscr{D}(F) = \operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot e_j^\perp)$ is as in Lemma 3.3.2 and δ_{ij} is the Kronecker delta symbol.

LEMMA 3.3.6. *For every exposed or virtual face F of Λ, the functions $\Phi_i(F)$, $i = 1, 2$, are well-defined and continuous on $\mathscr{D}(F)$.*

PROOF. The claim is clear for the exposed singleton faces as well as for the virtual faces. Let $F = F_u$ be a line segment with $u \in \mathbb{R}^2_>$. First, note that the integrals from Definition 3.3.3 make sense since, in this case, $F - cu$ is a compact line segment, see (3.3.1). Clearly, $\Phi_i(F)$ is continuous on $\mathbb{R}$. Also by (3.3.1), we have with $u_i^\perp := (u^\perp)_i$ that

$$\int_{F-cu} \exp(2(s\,y \cdot u^\perp + y_i))\,dy = \int_{y^- \cdot u^\perp}^{y^+ \cdot u^\perp} \exp(2t(s + u_i^\perp))\,dt,$$

where we have parametrized $F - cu$ by $[y^- \cdot u^\perp, y^+ \cdot u^\perp] \ni t \mapsto tu^\perp$ as in Definition 3.3.3. Provided $s \neq -u_i^\perp$, this equals

$$\frac{1}{2(s + u_i^\perp)}\big(\exp(2(s\,y^+ \cdot u^\perp + y_i^+)) - \exp(2(s\,y^- \cdot u^\perp + y_i^-))\big)$$

by using $(y^\pm \cdot u^\perp)u_i^\perp = (y^\pm \cdot u^\perp)(u^\perp \cdot e_i) + (y^\pm \cdot u)(u \cdot e_i) = y^\pm \cdot e_i = y_i^\pm$ (note $y^\pm \perp u$). Doing a similar calculation for the integral in the denominator, we get

$$\Phi_i(F)(s) = \frac{s}{s + u_i^\perp}\,\frac{\exp(2(s\,y^+ \cdot u^\perp + y_i^+)) - \exp(2(s\,y^- \cdot u^\perp + y_i^-))}{\exp(2s\,y^+ \cdot u^\perp) - \exp(2s\,y^- \cdot u^\perp)}\,\exp(2cu_i) \tag{3.3.7}$$

for $s \in \mathbb{R} \setminus \{0, -u_i^\perp\}$. Because of $y^- \cdot u^\perp < y^+ \cdot u^\perp$, the first term in both the numerator and the denominator dominates as $s \to \infty$, while the second term wins for $s \to -\infty$. Hence, $\lim_{s \to \pm\infty} \Phi_i(F)(s) = \exp(2(y_i^\pm + cu_i)) = \exp(2p_i^\pm)$ and $\Phi_i(F)$ is continuous on $\overline{\mathbb{R}}$ and in particular on $\mathscr{D}(F) \subseteq \overline{\mathbb{R}}$.

In the case $u = e_1$, we have $\mathscr{D}(F) \subseteq [1, \infty]$ by Lemma 3.3.2, hence $\Phi_i(F)$ is defined on all of $\mathscr{D}(F)$ and obviously continuous on $[1, \infty]$. For $u = e_2$, we have $\mathscr{D}(F) \subseteq [-\infty, -1]$ and $\Phi_i(F)$ is continuous on this set. ∎

Before we can prove the main result of this section, we need some preparation about weak convergence of (probability) measures. Recall that a sequence of positive Radon measures $\mu_k \in \mathcal{M}_+(X)$ on a locally compact Hausdorff space X is called *weakly convergent* to $\mu \in \mathcal{M}_+(X)$ if

$$\int_X f\,d\mu_k \to \int_X f\,d\mu \quad (k \to \infty) \tag{3.3.8}$$

for all $f \in C_b(X)$, the space of bounded continuous (complex-valued) functions on X. Another notion of convergence for measures is that of *vague convergence*, that is, $\mu_k \to \mu$ if (3.3.8) holds for all $f \in C_c(X)$. Clearly, weak convergence is stronger than vague convergence. However, for a sequence of probability measures converging to another probability measure, these notions actually coincide, see for instance [**Bau01**, Corollary 30.9].

PROPOSITION 3.3.7 ([**SSU89**, Proposition 3.2]). *Let $F = F_u$ be an exposed or virtual face of Λ and let $s \in \mathscr{D}(F)$ with integer sequence $(\alpha_k)_{k\in\mathbb{N}}$ as in (3.3.2). Then the weak-* limit*

$$\Psi_F(s) := \lim_{k\to\infty} \omega_{\alpha_k - (1,1)}$$

exists in $\operatorname{Spec}(\mathcal{D}_\Omega)$ *and is given on the generators of $\mathcal{D}_\Omega$ by*

$$\Psi_F(s)(\tau_\beta w_i) = \begin{cases} \sqrt{\Phi_i(F)(s - \beta \cdot u^\perp)}, & s - \beta \cdot u^\perp \in \mathscr{D}(F) \\ 0, & s - \beta \cdot u^\perp \notin \mathscr{D}(F) \end{cases} \tag{3.3.9}$$

for every $\beta \in \mathbb{Z}^2$, where $w_i = \chi_{e_i}$ for $i \in \{1, 2\}$ is as defined in Remark 3.2.4. Moreover, the value 0 is only possible for the cases $u \in \{e_1, e_2\}$.

REMARK 3.3.8. In particular, the limit (3.3.9) only depends on the unit vector u and the number $s \in \mathscr{D}(F)$. Moreover, in the case where F does not determine u uniquely, $F = \{p\} \subseteq \Lambda$ is necessarily a singleton and the possible vectors u all belong to $\mathbb{R}^2_>$, because if $u \in \{e_1, e_2\}$, then either F_u is virtual or a line segment since the boundary of Λ has the coordinate axes as asymptotes. Hence, only the first case in (3.3.9) will occur and thus $\Psi_F(s)(\tau_\beta w_i) = \exp(p_i)$ for all $s \in \mathscr{D}(F)$, which is independent of u. We conclude that the limit $\Psi_F(s)$ really depends on the *face* F and not the vector u.

PROOF. Let $\mathbf{1} := (1,1)$. If $u \in \mathbb{R}^2_>$, then we have $\alpha_k - \beta \in \mathbb{Z}^2_>$ for large enough k, since otherwise $u = \lim_k(\alpha_k - \beta)/\|\alpha_k\| \in \{e_1, e_2\}$. It follows that $s - \beta \cdot u^\perp$ is always contained in $\mathscr{D}(F) = \mathrm{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp)$. For $u = e_1$, we have $\alpha_k - \beta \in \mathbb{Z}^2_>$ for large k if and only if

$$(\alpha_k - \beta) \cdot u^\perp \in \mathbb{Z}^2_> \cdot u^\perp = \mathbb{Z}_> \quad \text{for all } k \text{ large enough.} \tag{3.3.10}$$

Indeed, $(\alpha_k - \beta) \cdot u^\perp = (\alpha_k - \beta)_2$ and $(\alpha_k - \beta)_1$ is always positive for high k since $1 = (e_1)_1 = \lim_k(\alpha_k - \beta)_1/\|\alpha_k\|$ and $\|\alpha_k\| \to \infty$. Moreover, (3.3.10) is equivalent to $s - \beta \cdot u^\perp \in \mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\}$ since $\mathbb{Z}_> \subseteq \overline{\mathbb{R}}$ is discrete. The case $u = e_2$ is similar.

Summarizing, we have $s - \beta \cdot u^\perp \in \mathscr{D}(F)$ if and only if $\alpha_k - \mathbf{1} - \beta \in \mathbb{Z}^2_\geq$ for sufficiently large k and this is always the case if $u \in \mathbb{R}^2_>$. Therefore, by definition, we have

$$\Psi_F(s)(\tau_\beta \chi_{e_i}) = \lim_{k\to\infty} \chi_{e_i}(\alpha_k - \mathbf{1} - \beta) = \begin{cases} \lim\limits_{k\to\infty} \dfrac{\|z^{\alpha_k - \mathbf{1} - \beta + e_i}\|_{L^2(\Omega)}}{\|z^{\alpha_k - \mathbf{1} - \beta}\|_{L^2(\Omega)}}, & s - \beta \cdot u^\perp \in \mathscr{D}(F) \\ 0, & s - \beta \cdot u^\perp \notin \mathscr{D}(F) \end{cases}$$

and the second case can only occur if $u \in \{e_1, e_2\}$. We now concentrate on the case $s - \beta \cdot u^\perp \in \mathscr{D}(F)$. By using polar coordinates (see (D.2)), we have

$$\frac{\|z^{\alpha_k - \mathbf{1} - \beta + e_i}\|^2_{L^2(\Omega)}}{\|z^{\alpha_k - \mathbf{1} - \beta}\|^2_{L^2(\Omega)}} = \frac{\int_\Lambda e^{2(\alpha_k - \beta + e_i)\cdot x}\,dx}{\int_\Lambda e^{2(\alpha_k - \beta)\cdot x}\,dx} = \int_\Lambda e^{2e_i \cdot x}\,d\mu_k(x)$$

with the sequence of probability measures $\mu_k(A) := \int_A e^{2(\alpha_k - \beta)\cdot x}\,dx / \int_\Lambda e^{2(\alpha_k - \beta)\cdot x}\,dx$ on Λ. Note that these measures make sense (at least for large k) since, as observed above, $\alpha_k - \beta \in \mathbb{Z}^2_>$ for sufficiently high k and $\Lambda \subseteq \mathbb{R}^2_\leq$, so $x \mapsto e^{2(\alpha_k - \beta)\cdot x}$ is integrable over Λ. As the functions $x \mapsto e^{2e_i \cdot x}$ are in $C_b(\Lambda)$, the statement follows once we have verified that $(\mu_k)_k$ converges vaguely to a probability measure μ and computed this limit.

Again, we first consider the case $u \in \mathbb{R}^2_>$ and we distinguish between the different types of faces that may occur. Note that

$$\frac{\alpha_k}{\|\alpha_k\|} = \frac{\alpha_k \cdot u}{\|\alpha_k\|}\,u + \frac{\alpha_k \cdot u^\perp}{\|\alpha_k\|}\,u^\perp \to u$$

by the choice of the sequence α_k. Since u and $u^\perp$ are linearly independent, this gives $\alpha_k \cdot u / \|\alpha_k\| \to 1$ and $\alpha_k \cdot u^\perp / \|\alpha_k\| \to 0$. Therefore,

$$\frac{\alpha_k \cdot u^\perp}{\alpha_k \cdot u} \to 0. \tag{3.3.11}$$

CLAIM. If $F = F_u = \{p\}$ is a singleton face, then $\mu = \delta_p$. It follows that $\Psi_F(s)(\tau_\beta \chi_{e_i}) = \exp(p_i)$, which agrees with (3.3.9).

PROOF. Letting $c = \sup_{x\in\Lambda}(x \cdot u)$, we have $u \cdot x < c$ for all $x \in \Lambda \setminus \{p\}$ by the definition of F_u. Take $f \in C_c(\Lambda, \mathbb{R})$. Let $r \in \mathbb{R}$ satisfy $r > f(p)$ and set $U := \{x \in \Lambda : f(x) < r\}$. Since U is an open neighborhood of p in Λ, there exists $\rho > 0$ such that $u \cdot x - c \leq -2\rho$ for $x \in \Lambda \setminus U$. Define a function $\phi_k \colon \Lambda \to \mathbb{R}$ by

$$\phi_k(x) := u \cdot x - c + (\alpha_k \cdot u^\perp / \alpha_k \cdot u)\,u^\perp \cdot x.$$

By (3.3.11), we can choose k large enough such that $|(\alpha_k \cdot u^\perp / \alpha_k \cdot u)\,u^\perp \cdot x| \leq \rho$ for x in the compact set $\mathrm{supp}(f) \setminus U$ and hence

$$\sup\{\phi_k(x) : x \in \mathrm{supp}(f) \setminus U\} \leq -\rho$$

for large enough k. Since $u \cdot p - c = 0$, there is a compact neighborhood $V \subseteq \Lambda$ of p such that

$$\inf\{\phi_k(x) : x \in V\} \geq -\frac{\rho}{2}$$

for all k large enough. We set $h_k(x) := e^{2\alpha_k \cdot (x-cu)} e^{-2\beta \cdot x} = e^{2(\alpha_k \cdot u)\phi_k(x)} e^{-2\beta \cdot x}$, where we have used that $(\alpha_k \cdot u)(u \cdot x) + (\alpha_k \cdot u^{\perp})(u^{\perp} \cdot x) = \alpha_k \cdot x$. Then

$$\sup_{x \in \mathrm{supp}(f) \setminus U} h_k(x) \Big/ \inf_{x \in V} h_k(x) \leq e^{-(\alpha_k \cdot u)\rho} \sup_{x \in \mathrm{supp}(f) \setminus U} e^{-2\beta \cdot x} \Big/ \inf_{x \in V} e^{-2\beta \cdot x} = e^{-(\alpha_k \cdot u)\rho}\, C \to 0$$

for some constant $C > 0$ since $\alpha_k \cdot u \to \infty$. Moreover,

$$\int_{\Lambda} f \, d\mu_k = \frac{\int_{\Lambda} f h_k \, dx}{\int_{\Lambda} h_k \, dx} \leq \frac{\int_U f h_k \, dx}{\int_U h_k \, dx} + \frac{\int_{\Lambda \setminus U} f h_k \, dx}{\int_V h_k \, dx} \leq r + e^{-(\alpha_k \cdot u)\rho}\, C\, \lambda(V)^{-1} \|f\|_{L^1}$$

where λ is Lebesgue measure, and as this holds for every $r > f(p)$, we see $\limsup_{k\to\infty} \int_{\Lambda} f \, d\mu_k \leq f(p)$. By replacing f with $-f$, we obtain

$$\liminf_{k\to\infty} \int_{\Lambda} f \, d\mu_k = -\limsup_{k\to\infty} \int_{\Lambda} -f \, d\mu_k \geq -(-f(p)) = f(p)$$

and therefore $\lim_{k\to\infty} \int_{\Lambda} f \, d\mu_k = f(p)$, showing that $\mu_k \to \delta_p$ vaguely. ♦

CLAIM. If $F = F_u$ with $\dim(F) = 1$ and $s = \sigma\infty$ for $\sigma \in \{+, -\}$, then $\mu = \delta_{p^\sigma}$, the Dirac mass at the endpoint p^σ of F. It follows that $\Psi_F(\pm\infty)(\tau_\beta \chi_{e_i}) = \exp(p_i^{\pm})$, which agrees with (3.3.9).

PROOF. We treat the case $s = +\infty$ in detail, the case $s = -\infty$ is similar. Take $f \in C_c(\Lambda, \mathbb{R})$ and let $r > f(p^+)$. Set $U := \{x \in \Lambda : f(x) < r\}$. Then U is an open neighborhood of p^+ in Λ, so there is $\delta > 0$ such that $\{p \in \mathrm{aff}(F) : u^{\perp} \cdot (p - p^+) > -2\delta\} \cap \Lambda$ is contained in U.[1] Put $E_\delta := \mathbb{R}_{\leq} \cdot u + \{p \in \mathrm{aff}(F) : u^{\perp} \cdot (p - p^+) < -\delta\}$. We have that $E_\delta \cap \Lambda$ is an open neighborhood of $F \setminus U$ in Λ, so that $(U \cup E_\delta) \cap \Lambda$ is an open neighborhood of F. Because $u \cdot x = c = \sup_{y \in \Lambda}(u \cdot y)$ for $x \in F = F_u$, we get the existence of $\rho > 0$ such that $u \cdot x - c \leq -2\rho$ for $x \in \Lambda \setminus (U \cup E_\delta)$. Let $\phi_k \colon \Lambda \to \mathbb{R}$ be defined by

$$\phi_k(x) := u \cdot x - c + (\alpha_k \cdot u^{\perp} / \alpha_k \cdot u)\, u^{\perp} \cdot (x - p^+).$$

By (3.3.11), we can choose k large enough such that $|(\alpha_k \cdot u^{\perp} / \alpha_k \cdot u)\, u^{\perp} \cdot (x - p^+)| \leq \rho$ for x in the compact set $\mathrm{supp}(f) \setminus (U \cup E_\delta)$ and hence

$$\sup\{\phi_k(x) : x \in \mathrm{supp}(f) \setminus (U \cup E_\delta)\} \leq -\rho$$

for large enough k. Since $u \cdot p^+ - c = 0$, there is a compact neighborhood $V \subseteq \Lambda$ of p^+ such that

$$\inf\{\phi_k(x) : x \in V\} \geq -\frac{\rho}{2}$$

for large enough k. Let $h_k(x) := e^{2\alpha_k \cdot (x - cu - (p^+ \cdot u^{\perp})u^{\perp})} e^{-2\beta \cdot x} = e^{2(\alpha_k \cdot u)\phi_k(x)} e^{-2\beta \cdot x}$. Then

$$\sup_{x \in \mathrm{supp}(f) \setminus (U \cup E_\delta)} h_k(x) \Big/ \inf_{x \in V} h_k(x) \leq$$

$$\leq e^{-(\alpha_k \cdot u)\rho} \sup_{x \in \mathrm{supp}(f) \setminus (U \cup E_\delta)} e^{-2\beta \cdot x} \Big/ \inf_{x \in V} e^{-2\beta \cdot x} = e^{-(\alpha_k \cdot u)\rho}\, C \to 0$$

[1] Note that $u^{\perp} \cdot (p - p^+) \leq 0$ for $p \in F$ and this expression decreases when moving from p^+ to p^-

for a constant $C > 0$ since $\alpha_k \cdot u \to \infty$. Moreover,

$$\int_\Lambda f\,d\mu_k = \frac{\int_\Lambda f h_k\,dx}{\int_\Lambda h_k\,dx} \le \frac{\int_U f h_k\,dx}{\int_U h_k\,dx} + \frac{\int_{\mathrm{supp}(f)\setminus(U\cup E_\delta)} f h_k\,dx}{\int_V h_k\,dx} + \frac{\int_{E_\delta} f h_k\,dx}{\int_E h_k\,dx} \le$$
$$\le r + e^{-(\alpha_k\cdot u)\rho}\,C\,\lambda(V)^{-1}\|f\|_{L^1} + \|f\|_\infty \frac{\int_{E_\delta} e^{2\alpha_k\cdot x}e^{-2\beta\cdot x}\,dx}{\int_E e^{2\alpha_k\cdot x}e^{-2\beta\cdot x}\,dx},$$

where $E := \{tu + y : t \le 0,\ y \in F\}$. It remains to calculate the two integrals in the last expression. For the integral over E we parametrize $F - cu$ by $[y^-\cdot u^\perp, y^+\cdot u^\perp] \ni \tau \mapsto \tau u^\perp$ to obtain

$$\int_E e^{2\alpha_k\cdot x}e^{-2\beta\cdot x}\,dx =$$
$$= \int_{-\infty}^0 \int_{F-cu} e^{2(\alpha_k-\beta)\cdot(tu+y+cu)}\,dy\,dt = \int_{-\infty}^0 \int_{y^-\cdot u^\perp}^{y^+\cdot u^\perp} e^{2(\alpha_k-\beta)\cdot(tu+cu+\tau u^\perp)}\,d\tau\,dt =$$
$$= \frac{e^{2c(\alpha_k-\beta)\cdot u}}{(2(\alpha_k-\beta)\cdot u)(2(\alpha_k-\beta)\cdot u^\perp)}\left(e^{2(y^+\cdot u^\perp)(\alpha_k-\beta)\cdot u^\perp} - e^{2(y^-\cdot u^\perp)(\alpha_k-\beta)\cdot u^\perp}\right).$$

Similarly, since $E_\delta = \{tu + cu + \tau u^\perp : t \le 0,\ \tau \in (-\infty, y^+\cdot u^\perp - \delta)\}$, we get

$$\int_{E_\delta} e^{2\alpha_k\cdot x}e^{-2\beta\cdot x}\,dx = \int_{-\infty}^0 \int_{-\infty}^{y^+\cdot u^\perp-\delta} e^{2(\alpha_k-\beta)\cdot(tu+cu+\tau u^\perp)}\,d\tau\,dt =$$
$$= \frac{e^{2c(\alpha_k-\beta)\cdot u}}{(2(\alpha_k-\beta)\cdot u)(2(\alpha_k-\beta)\cdot u^\perp)}\,e^{2(y^+\cdot u^\perp-\delta)(\alpha_k-\beta)\cdot u^\perp},$$

where we have used that $\alpha_k \cdot u^\perp \to s = +\infty$ by assumption so that the τ-integral makes sense. Therefore,

$$\int_\Lambda f\,d\mu_k \le r + e^{-(\alpha_k\cdot u)\rho}\,C\,\lambda(V)^{-1}\|f\|_{L^1} + \frac{\|f\|_\infty\, e^{-2\delta(\alpha_k-\beta)\cdot u^\perp}}{1 - e^{2((y^- - y^+)\cdot u^\perp)(\alpha_k-\beta)\cdot u^\perp}}$$

and as $\alpha_k \cdot u \to +\infty$ and $\alpha_k \cdot u^\perp \to \infty$, we obtain $\limsup_{k\to\infty} \int_\Lambda f\,d\mu_k \le f(p^+)$. Again replacing f by $-f$ shows $\lim_{k\to\infty} \int_\Lambda f\,d\mu_k = f(p^+)$, so $\mu_k \to \delta_{p^+}$ vaguely. ♦

CLAIM. If $F = F_u$ with $\dim(F) = 1$ and $s \in \mathbb{R}$, then

$$\int_\Lambda \varphi\,d\mu = \frac{\int_F \varphi(y)\exp(2(su^\perp - \beta)\cdot y)\,dy}{\int_F \exp(2(su^\perp - \beta)\cdot y)\,dy}$$

for all $\varphi \in C_b(\Lambda)$. Shifting the domain of integration from F to $F - cu$ and using $\beta \cdot y = (\beta \cdot u^\perp)(y \cdot u^\perp)$ for $y \in F - cu \perp u$, this gives

$$\int_\Lambda \varphi\,d\mu = \frac{\int_{F-cu} \varphi(y+cu)\exp(2(s - \beta\cdot u^\perp)y\cdot u^\perp)\exp(-2c\beta\cdot u)\,dy}{\int_{F-cu} \exp(2(s - \beta\cdot u^\perp)y\cdot u^\perp)\exp(-2c\beta\cdot u)\,dy}$$

and this expression agrees with (3.3.9) after canceling the common factor $\exp(-2c\beta\cdot u)$ and evaluating at $\varphi(x) = \exp(2e_i \cdot x)$.

PROOF. Let $E := \{tu + y : t \le 0,\ y \in F\}$ and define $g_k \colon E \to \mathbb{R}$ by

$$g_k(x) := \exp(2(\alpha_k - \ell\mathbf{1})\cdot u\,(u\cdot x - c))$$

with $\ell \in \mathbb{Z}$. Note that if $x = tu + y \in E$, then $g_k(x) = \exp(2t(\alpha_k - \ell\mathbf{1})\cdot u)$ because $u\cdot(tu+y) = t + u\cdot y - c = 0$ as $u \cdot y = c$ by the definition of F_u. Therefore,

$$\int_E g_k(x)\,dx = \int_{\mathbb{R}_\le} \exp(2t(\alpha_k - \ell\mathbf{1})\cdot u)\,dt \int_F dy = \frac{1}{2(\alpha_k - \ell\mathbf{1})\cdot u}\int_F dy$$

for large enough k, since then $(\alpha_k - \ell\mathbf{1})\cdot u > 0$. Here, $\int_F f(y)\,dy$ is to be understood as the integral $\int_{F-cu} f(y+cu)\,dy$ for $f \in C(F)$.

We now show that for each $f \in C_c(E)$ it holds that

$$\lim_{k\to\infty} \frac{\int_E f(x)g_k(x)\,dx}{\int_E g_k(x)\,dx} = \frac{\int_F f(y)\,dy}{\int_F dy}. \tag{3.3.12}$$

The argument is similar to the proofs of the two previous Claims. First note that

$$\int_E f(x)g_k(x)\,dx = \int_{\mathbb{R}_\le}\int_F f(tu+y)g_k(t)\,dy\,dt = \int_{\mathbb{R}_\le} e^{2t(\alpha_k-\ell\mathbf{1})\cdot u}\int_F f(tu+y)\,dy\,dt.$$

Put $\eta(t) := \int_F f(tu+y)\,dy/\int_F dy$. Then $\eta \in C_c(\mathbb{R}_\le)$ and we have to show that

$$\lim_{k\to\infty}\frac{\int_{\mathbb{R}_\le}\eta(t)\widetilde{g}_k(t)\,dt}{\int_{\mathbb{R}_\le}\widetilde{g}_k(t)\,dt} = \eta(0), \tag{3.3.13}$$

where $\widetilde{g}_k(t) := \exp(2t(\alpha_k - \ell\mathbf{1})\cdot u)$. Let $r \in \mathbb{R}$ with $r > \eta(0)$ and set $U := \{t \in \mathbb{R}_\le : \eta(t) < r\}$. Then U is an open neighborhood of 0 in $\mathbb{R}_\le$, hence there is $\rho > 0$ such that $[-\rho, 0] \subseteq U$. We have

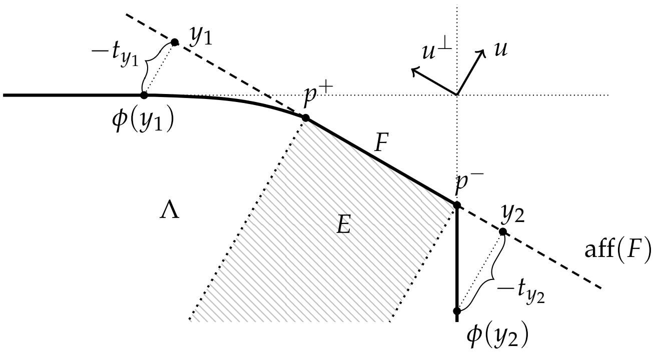
$$\sup_{t\in(-\infty,-\rho)}\widetilde{g}_k(t)\Big/\inf_{t\in[-\rho/2,0]}\widetilde{g}_k(t) \le e^{-\rho(\alpha_k-\ell\mathbf{1})\cdot u} \to 0$$

as $k \to \infty$ since $\alpha_k \cdot u \to \infty$. It follows that

$$\frac{\int_{\mathbb{R}_\le}\eta\widetilde{g}_k\,dt}{\int_{\mathbb{R}_\le}\widetilde{g}_k\,dt} \le \frac{\int_{[-\rho,0]}\eta\widetilde{g}_k\,dt}{\int_{[-\rho,0]}\widetilde{g}_k\,dt} + \frac{\int_{(-\infty,-\rho)}\eta\widetilde{g}_k\,dt}{\int_{[-\rho/2]}\widetilde{g}_k\,dt} \le r + e^{-\rho(\alpha_k-\ell\mathbf{1})\cdot u}\,(2/\rho)\|\eta\|_{L^1}$$

and therefore $\limsup_{k\to\infty}\int_{\mathbb{R}_\le}\eta\widetilde{g}_k\,dt/\int_{\mathbb{R}_\le}\widetilde{g}_k\,dt \le \eta(0)$ as $r > \eta(0)$ was arbitrary. By replacing f with $-f$ and therefore η with $-\eta$, we obtain (3.3.13). Note that (3.3.12) is still valid for all $f \in C_b(E)$ by the discussion prior to Proposition 3.3.7.

Next, let $\phi\colon \mathrm{aff}(F)\setminus F \to \mathbb{R}_<$ be defined as follows: consider the line $\{tu+y : t \in \mathbb{R}\}$ through $y \in \mathrm{aff}(F)\setminus F$. There exists a unique value $t_y \in \mathbb{R}_<$ such that $t_y u + y \in \partial\Lambda$ and we set $\phi(y) := t_y$. Since the boundary of Λ is continuous, it follows that ϕ is also continuous. Moreover, we have $\phi(y) \ge t$ whenever $tu + y \in \Lambda\setminus E$ with $y \in \mathrm{aff}(F)$ (in this case, $y \notin F$ is automatically satisfied).



Thus, the graph of ϕ is the boundary of Λ (without F), "as seen by looking from $\mathrm{aff}(F)$." We can use this function ϕ to parametrize integrals over $\Lambda\setminus E$:

$$\frac{\int_{\Lambda\setminus E} g_k(x)\,dx}{\int_\Lambda g_k(x)\,dx} \le \frac{\int_{\Lambda\setminus E} g_k(x)\,dx}{\int_E g_k(x)\,dx} = \frac{2(\alpha_k-\ell\mathbf{1})\cdot u}{\int_F dy}\int_{\mathrm{aff}(F)\setminus F}\int_{-\infty}^{\phi(y)}\exp(2t(\alpha_k-\ell\mathbf{1}))\,dt\,dy =$$

$$= \frac{1}{\int_F dy}\int_{\mathrm{aff}(F)\setminus F}\exp(2\phi(y)(\alpha_k-\ell\mathbf{1})\cdot u)\,dy \to 0$$

by dominated convergence as $k \to \infty$ since $\phi(y) < 0$ on $\mathrm{aff}(F) \setminus F$. Hence, for all $\varphi \in C_b(\Lambda)$,

$$\lim_{k\to\infty}\left(\frac{\int_\Lambda \varphi g_k\,dx}{\int_\Lambda g_k\,dx} - \frac{\int_E \varphi g_k\,dx}{\int_E g_k\,dx}\right) = \lim_{k\to\infty}\left(\frac{\int_{\Lambda\setminus E} \varphi g_k\,dx}{\int_\Lambda g_k\,dx} + \frac{\int_E \varphi g_k\,dx}{\int_\Lambda g_k\,dx} - \frac{\int_E \varphi g_k\,dx}{\int_E g_k\,dx}\right) =$$
$$= \lim_{k\to\infty}\left(\frac{\int_{\Lambda\setminus E} \varphi g_k\,dx}{\int_\Lambda g_k\,dx} - \frac{\int_E \varphi g_k\,dx}{\int_E g_k\,dx}\cdot\frac{\int_{\Lambda\setminus E} g_k\,dx}{\int_\Lambda g_k\,dx}\right) = 0$$

because $\int_{\Lambda\setminus E} g_k\,dx = \int_\Lambda g_k\,dx - \int_E g_k\,dx$. Combining this with (3.3.12), we have

$$\lim_{k\to\infty}\frac{\int_\Lambda \varphi g_k\,dx}{\int_\Lambda g_k\,dx} = \frac{\int_F \varphi(y)\,dy}{\int_F dy}$$

for all $\varphi \in C_b(\Lambda)$. We now let $\ell > 0$ be an integer such that $(\ell\mathbf{1}\cdot u)u + (\alpha_k\cdot u^\perp)u^\perp - \beta \in \mathbb{R}^2_>$ for large enough k which is possible since $\alpha_k \cdot u^\perp \to s \in \mathbb{R}$. The functions

$$h_k(x) := \exp\left(2((\ell\mathbf{1}\cdot u)u + (\alpha_k\cdot u^\perp)u^\perp - \beta)\cdot x\right) \quad (x\in\Lambda)$$

satisfy $|h_k(x)| \le 1$ by our choice of ℓ and because $\Lambda \subseteq \mathbb{R}^2_\le$. Moreover, $h_k \to h$ uniformly on the compact subsets of Λ, where

$$h(x) = \exp\left(2((\ell\mathbf{1}\cdot u)u + su^\perp - \beta)\cdot x\right). \quad (x\in\Lambda)$$

Let $K \subseteq \Lambda$ be a compact neighborhood of F and pick $\psi \in C_b(\Lambda)$ with $\psi \ge 0$, $\psi|_F = 0$ and $\psi|_{\Lambda\setminus K} = 1$. Then

$$\left|\frac{\int_{\Lambda\setminus K}\varphi g_k h_k\,dx}{\int_\Lambda g_k\,dx}\right| \le \|\varphi\|_\infty \frac{\int_{\Lambda\setminus K}\psi g_k\,dx}{\int_\Lambda g_k\,dx} \le \|\varphi\|_\infty\frac{\int_\Lambda \psi g_k\,dx}{\int_\Lambda g_k\,dx} \to \frac{\int_F \psi\,dy}{\int_F dy} = 0$$

for all $\varphi \in C_b(\Lambda)$ as $k \to \infty$, where we have used that $|h_k| \le 1$. Moreover,

$$\left|\frac{\int_K \varphi g_k h_k\,dx}{\int_\Lambda g_k\,dx} - \frac{\int_F \varphi h\,dy}{\int_F dy}\right| \le \|\varphi\|_\infty\frac{\int_K g_k|h_k - h|\,dx}{\int_\Lambda g_k\,dx} + \left|\frac{\int_K \varphi g_k h\,dx}{\int_\Lambda g_k\,dx} - \frac{\int_F \varphi h\,dy}{\int_F dy}\right| \to \infty$$

and therefore

$$\lim_{k\to\infty}\frac{\int_\Lambda \varphi g_k h_k\,dx}{\int_\Lambda g_k\,dx} = \lim_{k\to\infty}\frac{\int_K \varphi g_k h_k\,dx}{\int_\Lambda g_k\,dx} = \frac{\int_F \varphi(y)h(y)\,dy}{\int_F dy}.$$

for every $\varphi \in C_b(\Lambda)$. Note that

$$\begin{aligned} g_k(x)h_k(x) &= \exp 2\big((\alpha_k - \ell\mathbf{1})\cdot u(u\cdot x - c) + (\ell\mathbf{1}\cdot u)(u\cdot x) + (\alpha_k\cdot u^\perp)(u^\perp\cdot x) - \beta\cdot x\big) \\ &= \exp 2\big((\alpha_k\cdot u)(u\cdot x) + (\alpha_k\cdot u^\perp)(u^\perp\cdot x) - c\alpha_k\cdot u + c\ell\mathbf{1}\cdot u - \beta\cdot x\big) \\ &= \exp 2\big((\alpha_k - \beta)\cdot x\big)\exp 2\big(c(\ell\mathbf{1} - \alpha_k)\cdot u\big) \end{aligned}$$

and so, by canceling the x-independent factors,

$$\int_\Lambda \varphi\,d\mu_k = \frac{\int_\Lambda \varphi(x)e^{2(\alpha_k-\beta)\cdot x}\,dx}{\int_\Lambda e^{2(\alpha_k-\beta)\cdot x}\,dx} = \frac{\int_\Lambda \varphi g_k h_k\,dx}{\int_\Lambda g_k h_k\,dx} = \frac{\int_\Lambda \varphi g_k h_k\,dx}{\int_\Lambda g_k\,dx}\,\frac{\int_\Lambda g_k\,dx}{\int_\Lambda g_k h_k\,dx} \to \frac{\int_F \varphi(y)h(y)\,dy}{\int_F h(y)\,dy}$$

for all $\varphi \in C_b(\Lambda)$. Finally, for $y \in F$ we have $u\cdot y = c$, so we can cancel the factor $\exp(2((\ell\mathbf{1}\cdot u)(u\cdot y)))$, showing the claim. ◆

This finishes the proof for $u \in \mathbb{R}^2_>$. The cases $u \in \{e_1, e_2\}$ can be handled by similar methods. More precisely, the virtual case can be handled as in the first claim, and the non-virtual case is treated like the second and third claims, [**SSU89**, Proposition 3.2].

Thus, we have shown that the weak-* limit $\Psi_F(s) := \lim_{k\to\infty}\omega_{\alpha_k - \mathbf{1}}$ exists as a continuous linear functional on $\mathcal{D}_\Omega$ since the closed unit ball in the dual of $\mathcal{D}_\Omega$ is weak-* closed (even weak-* compact by the Banach-Alaoglu theorem). Clearly, $\Psi_F(s)$ is multiplicative as the $\omega_{\alpha_k-\mathbf{1}}$ are. Moreover, from its values (3.3.9) on the generators of $\mathcal{D}_\Omega$, it is clear that $\Psi_F(s) \ne 0$ for all $s \in \mathscr{D}(F)$, simply because at least one of the functions $\Phi_i(F)$, $i \in \{1,2\}$, is always positive, even for the virtual faces. This shows that $\Psi_F(s)$ is really a character of $\mathcal{D}_\Omega$. ∎

LEMMA 3.3.9 ([SSU89, Lemma 3.3]). *For any exposed or virtual face F, the map $\Psi_F \colon \mathscr{D}(F) \to \mathrm{Spec}(\mathcal{D}_\Omega)$ is continuous.*

PROOF. Because $\mathrm{Spec}(\mathcal{D}_\Omega)$ carries the weak-* topology and $\mathcal{D}_\Omega$ is generated by the elements $\tau_\beta w_i$ with $\beta \in \mathbb{Z}^2$ and $i \in \{1,2\}$, see Remark 3.2.4, it suffices to show that $\mathscr{D}(F) \ni s \mapsto \Psi_F(s)(\tau_\beta w_i) \in \mathbb{C}$ is continuous for all β and i. For $u \in \mathbb{R}^2_>$, we have

$$\Psi_F(s)(\tau_\beta w_i) = \sqrt{\Phi_i(F)(s - \beta \cdot u^\perp)}$$

by (3.3.9), so the statement follows from Lemma 3.3.6. If $u = e_1$, then the second case in (3.3.9) occurs when $s \leq \beta_2$ since $\mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\}$ in this case. Again, Lemma 3.3.6 immediately implies the assertion. The argument for the case $u = e_2$ is similar. ∎

LEMMA 3.3.10 (from the proof of [SSU89, Proposition 3.5]). *Let F be an exposed (non-virtual) face of Λ. For any $s \in \mathscr{D}(F) \cap \mathbb{R}$ there is $p^{(i)} = (p_1^{(i)}, p_2^{(i)}) \in \mathrm{relint}(F)$ such that*

$$\Psi_F(s)(w_i) = \exp\left(p_i^{(i)}\right) \tag{3.3.14}$$

for $i \in \{1,2\}$, where $\mathrm{relint}(F)$ denotes the relative interior of F, see Definition E.2.

PROOF. We have $\Psi_F(s)(w_i) = \sqrt{\Phi_i(F)(s)}$ by (3.3.9) and (3.3.4), so the proof reduces to verifying the claim for the functions $\Phi_i(F)$. If $F = \{p\}$ is a singleton, then the statement is evident from the definition of $\Phi_i(F)$: in this case, $\mathrm{relint}(F) = F$ and we may simply take $p^{(i)} = p$. If $\dim(F) = 1$ and $F = F_u$ with $u \in \mathbb{R}^2_>$, set

$$f(t) := \exp(2t(s + u_i^\perp)), \quad \text{and} \quad g(t) := \exp(2ts)$$

for $t \in [y^- \cdot u^\perp, y^+ \cdot u^\perp]$, where $y^\pm$ are as in (3.3.1). By (3.3.7) and Cauchy's mean value theorem, there exists $t_0 \in (y^- \cdot u^\perp, y^+ \cdot u^\perp)$ such that

$$\begin{aligned}\Phi_i(F)(s) = \frac{s}{s+u_i^\perp}\frac{f(y^+\cdot u^\perp) - f(y^-\cdot u^\perp)}{g(y^+\cdot u^\perp) - g(y^-\cdot u^\perp)}\exp(2cu_i) = \\ = \frac{s}{s+u_i^\perp}\frac{f'(t_0)}{g'(t_0)}\exp(2cu_i) = \exp(2(u_i^\perp t_0 + cu_i)).\end{aligned} \tag{3.3.15}$$

Since $t_0 u^\perp \in \mathrm{relint}(F) - cu$, the vector $p^{(i)} := t_0 u^\perp + cu$ belongs to $\mathrm{relint}(F)$ and satisfies (3.3.14). Finally, for $u = e_1$, we have

$$\Phi_1(F)(s) = 1 = \exp(0) \quad \text{and} \quad \Phi_2(F)(s) = \frac{s}{s+1}\exp(2p_2^+) = \exp 2\Big(\log\left(\frac{s}{s+1}\right) + p_2^+\Big)$$

for all $s \in \mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\}$. Since $\log(\frac{s}{s+1}) < 0$, we have $(0, \log(\frac{s}{s+1}) + p_2^+) \in \mathrm{relint}(F)$. A similar argument works for $u = e_2$. ∎

COROLLARY 3.3.11. *For any exposed or virtual face F of Λ and any $s \in \mathscr{D}(F)$ there is $p^{(i)} \in F$ such that $\Psi_F(s)(w_i) = \exp(p_i^{(i)})$ for $i \in \{1,2\}$, where in the virtual case we use the convention $\exp(-\infty) := 0$.*

PROOF. For F exposed and $s \in \mathscr{D}(F) \cap \mathbb{R}$, the claim follows from Lemma 3.3.10, and if $s = \pm\infty$, then this is obvious from the definition of $\Phi_i(F)$. The case of a virtual face also is immediate from the relevant definitions. ∎

3.4. The groupoid associated to a Reinhardt domain

In Proposition 3.3.7, we have associated certain characters $\Psi_F(s) \in \operatorname{Spec}(\mathcal{D}_\Omega)$ of the algebra $\mathcal{D}_\Omega$ to the exposed faces F of $\Lambda := \Lambda_\Omega$. The scope of the present section is to first associate to Ω a groupoid, denoted by $\mathcal{G}_\Omega$, whose groupoid C^*-algebra will help in determining the structure of the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$. We refer to chapter 2 for the details on groupoids and their associated C^*-algebras. The remainder of this section is then devoted to studying the unit space of this groupoid. It turns out that this unit space admits a parametrization in terms of the sets $\Psi_F(s)$, which is the content of the main result of this section, Proposition 3.4.9.

LEMMA 3.4.1 ([**SSU89**, Lemma 4.1 (i)]). *There is a constant $C > 0$ such that $\chi_{e_1}(\alpha)^2 + \chi_{e_2}(\alpha)^2 \geq C^2$ for all $\alpha \in \mathbb{Z}^2_{\geq}$.*

PROOF. Assume that the claim does not hold, so there is a sequence $(\alpha_k)_{k\geq 1}$ in $\mathbb{Z}^2_{>}$ with $\|\alpha_k\| \to \infty$ and $\chi_{e_i}(\alpha_k - \mathbf{1}) \to 0$ for $k \to \infty$ and $i \in \{1,2\}$, where $\mathbf{1} := (1,1)$. After passing to a subsequence, we may assume that $(\alpha_k/\|\alpha_k\|)_k$ converges to a unit vector $u \in \mathbb{R}^2_{\geq}$ and that $\alpha_k \cdot u^\perp \to s$ for some $s \in \operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_{>} \cdot u^\perp)$. This is because the spaces $S^1 \cap \mathbb{R}^2_{\geq}$ and $\operatorname{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_{>} \cdot u^\perp)$ are both compact. Let $F := F_u$ be the corresponding exposed or virtual face of Λ. Then

$$0 = \lim_{k\to\infty} \chi_{e_i}(\alpha_k - \mathbf{1}) = \Psi_F(s) = \exp(p_i^{(i)}) \quad (i = 1,2)$$

for some $p^{(i)} \in F$ by Corollary 3.3.11. But this is not possible even for the virtual faces, since in that case $\exp(p_i^{(i)})$ is zero only for one of $i = 1,2$ and not for both. ■

Using this, we can take another look at the mapping $\omega\colon \mathbb{Z}^2 \to \operatorname{Spec}(\mathcal{D}_\Omega)$ from Lemma 3.2.5. Recall that $\omega(\alpha)(\varphi) = \varphi(\alpha)$ for $\alpha \in \mathbb{Z}^2$ and $\varphi \in \mathcal{D}_\Omega$.

PROPOSITION 3.4.2 ([**CM85**, Lemma 2.3]). *The closure of $\omega(\mathbb{Z}^2_{\geq})$ is a compact and open subset of $\operatorname{Spec}(\mathcal{D}_\Omega)$. We define*

$$\operatorname{Spec}_\Lambda(\Omega) := \overline{\omega(\mathbb{Z}^2_{\geq})}.$$

REMARK 3.4.3. By Lemma 3.2.5, the restriction $\omega|_{\mathbb{Z}^2_{\geq}} : \mathbb{Z}^2_{\geq} \to \operatorname{Spec}_\Lambda(\Omega)$ is injective and a homeomorphism onto its image, hence a dense embedding of $\mathbb{Z}^2_{\geq}$ into $\operatorname{Spec}_\Lambda(\Omega)$. Because $\operatorname{Spec}_\Lambda(\Omega)$ is compact by the Proposition, we may therefore view $\operatorname{Spec}_\Lambda(\Omega)$ as a compactification of $\mathbb{Z}^2_{\geq}$.

PROOF. The proof is similar to that of Lemma 3.2.5. Consider the function $\varphi\colon \mathbb{Z}^2 \to \mathbb{C}$, $\varphi(\alpha) := \chi_{e_1}(\alpha)^2 + \chi_{e_2}(\alpha)^2$. Then we have $\varphi \in \mathcal{D}_\Omega$ and we may consider the Gelfand transform $\Gamma_{\mathcal{D}_\Omega}(\varphi) \in C_0(\operatorname{Spec}(\mathcal{D}_\Omega))$ of φ. For any $\alpha \in \mathbb{Z}^2_{\geq}$, we have $\Gamma_{\mathcal{D}_\Omega}(\varphi)(\omega(\alpha)) = \omega(\alpha)(\varphi) = \varphi(\alpha) \geq C^2 > 0$ by Lemma 3.4.1. Note that $\Gamma_{\mathcal{D}_\Omega}(\varphi) \geq C^2$ continues to hold on the closure $\operatorname{Spec}_\Lambda(\Omega) = \overline{\omega(\mathbb{Z}^2_{\geq})}$.

We claim that $\rho(\varphi) = 0$ holds for any character $\rho \in \operatorname{Spec}(\mathcal{D}_\Omega) \setminus \operatorname{Spec}_\Lambda(\Omega)$. Indeed, by Lemma 3.2.5, there is a net α_ι in $\mathbb{Z}^2$ such that $\omega(\alpha_\iota) \to \rho$ in $\operatorname{Spec}(\mathcal{D}_\Omega)$. In particular, this means $\rho(\varphi) = \lim_\iota \varphi(\alpha_\iota)$. By our choice of ρ and because $\operatorname{Spec}(\mathcal{D}_\Omega) \setminus \operatorname{Spec}_\Lambda(\Omega)$ is open, we see that α_ι must eventually lie in $\mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq}$. Since $\chi_{e_i}(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \mathbb{Z}^2_{\geq}$ by definition, this implies $\rho(\varphi) = 0$. Therefore,

$$\operatorname{Spec}_\Lambda(\Omega) = \{\rho \in \operatorname{Spec}(\mathcal{D}_\Omega) : \Gamma_{\mathcal{D}_\Omega}(\varphi)(\rho) \geq C^2\} = \{\rho \in \operatorname{Spec}(\mathcal{D}_\Omega) : \Gamma_{\mathcal{D}_\Omega}(\varphi)(\rho) > C^2/2\},$$

and we see that $\operatorname{Spec}_\Lambda(\Omega)$ is open (and obviously closed) in $\operatorname{Spec}(\mathcal{D}_\Omega)$. Moreover, since $\Gamma_{\mathcal{D}_\Omega}(\varphi)$ is a function in $C_0(\operatorname{Spec}(\mathcal{D}_\Omega))$, there must be a compact set $K \subseteq \operatorname{Spec}(\mathcal{D}_\Omega)$ such that $|\Gamma_{\mathcal{D}_\Omega}(\varphi)(\rho)| < C^2/2$ for $\rho \notin K$. By the above, we see $\operatorname{Spec}_\Lambda(\Omega) \subseteq K$, and hence $\operatorname{Spec}_\Lambda(\Omega)$ is also compact. ■

Consider again the character space $\operatorname{Spec}(\mathcal{D}_\Omega)$ of the C^*-algebra $\mathcal{D}_\Omega$ from Definition 3.2.2. By Remark 3.2.3, $\mathcal{D}_\Omega$ is invariant under the translations τ_α for $\alpha \in \mathbb{Z}^2$. By duality, we get an action of $\mathbb{Z}^2$ on $\operatorname{Spec}(\mathcal{D}_\Omega)$, defined by

$$\Theta\colon \operatorname{Spec}(\mathcal{D}_\Omega) \times \mathbb{Z}^2 \to \operatorname{Spec}(\mathcal{D}_\Omega), \quad (\omega, \alpha) \mapsto \omega \circ \tau_{-\alpha}. \tag{3.4.1}$$

We include the negative sign here so as to obtain the same action as in [**SSU89**].

DEFINITION 3.4.4. The groupoid $\mathcal{G}_\Omega$ associated to Ω is the reduction to $\operatorname{Spec}_\Lambda(\Omega)$ of the action groupoid corresponding to (3.4.1), that is,

$$\mathcal{G}_\Omega := (\operatorname{Spec}(\mathcal{D}_\Omega) \rtimes_\Theta \mathbb{Z}^2)|_{\operatorname{Spec}_\Lambda(\Omega)} = \{(\rho, \alpha) \in \operatorname{Spec}_\Lambda(\Omega) \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in \operatorname{Spec}_\Lambda(\Omega)\},$$

see Example 2.1.22. Consequently, the unit space $(\mathcal{G}_\Omega)^{(0)}$ of this groupoid is $\operatorname{Spec}_\Lambda(\Omega)$.

REMARK 3.4.5. The character space $\operatorname{Spec}(\mathcal{D}_\Omega)$ is a locally compact, second countable Hausdorff space by Corollary 3.2.6, so the groupoid $\mathcal{G}_\Omega$ satisfies all our assumptions on locally compact groupoids from Definition 2.1.1. By Proposition 3.4.2, $\operatorname{Spec}_\Lambda(\Omega)$ is an open subset of the unit space $\operatorname{Spec}(\mathcal{D}_\Omega)$ of the transformation groupoid $\operatorname{Spec}(\mathcal{D}_\Omega) \rtimes_\Theta \mathbb{Z}^2$. Therefore, Proposition 2.2.13 implies that $\mathcal{G}_\Omega$ is a locally compact groupoid with Haar system

$$\mu^\rho := (\delta_\rho \times \sigma)|_{\mathcal{G}_\Omega}, \quad \rho \in \operatorname{Spec}_\Lambda(\Omega),$$

where σ is the counting measure on $\mathbb{Z}^2$.

3.4.1. The unit space of $\mathcal{G}_\Omega$. In order to understand the groupoid $\mathcal{G}_\Omega$ better, we will take a closer look at its unit space $\operatorname{Spec}_\Lambda(\Omega)$. The goal is to parametrize this unit space in terms of the faces of the closed convex set $\Lambda = \overline{\Lambda_\Omega}$, where Λ_Ω is the logarithmic base of Ω.

LEMMA 3.4.6. *The subset $\omega(\mathbb{Z}^2_\geq) \subseteq (\mathcal{G}_\Omega)^{(0)}$ is open and invariant, in the sense of Definition 2.1.14.*

PROOF. Set $X := \omega(\mathbb{Z}^2_\geq)$. We know from Lemma 3.2.5 that X is open. Let r and d denote the range and domain maps of $\mathcal{G}_\Omega$, respectively. Then $r(\rho, \alpha) = (\rho, \alpha)(\rho, \alpha)^{-1} = (\rho, 0)$ and $d(\rho, \alpha) = (\rho \cdot \alpha, 0) = (\rho \circ \tau_{-\alpha}, 0)$, see Example 2.1.22. Therefore,

$$\begin{aligned} r^{-1}(X) &= \{(\rho, \alpha) \in \mathcal{G}_\Omega : \rho \in X\} = \{(\rho, \alpha) \in X \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in \operatorname{Spec}_\Lambda(\Omega)\} \quad \text{and} \\ d^{-1}(X) &= \{(\rho, \alpha) \in \mathcal{G}_\Omega : \rho \circ \tau_{-\alpha} \in X\} \\ &= \{(\rho, \alpha) \in \operatorname{Spec}_\Lambda(\Omega) \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in X \cap \operatorname{Spec}_\Lambda(\Omega) = X\}. \end{aligned}$$

If $(\rho, \alpha) \in r^{-1}(X)$, then $\rho = \omega(\beta)$ for some $\beta \in \mathbb{Z}^2$ and $\rho \circ \tau_{-\alpha} = \omega(\beta + \alpha) \in \operatorname{Spec}_\Lambda(\Omega)$. We show that $\beta + \alpha \in \mathbb{Z}^2_\geq$, which then implies that $(\rho, \alpha) \in d^{-1}(X)$. Assume that $\beta + \alpha \notin \mathbb{Z}^2_\geq$. Because $\operatorname{Spec}_\Lambda(\Omega) = \overline{\omega(\mathbb{Z}^2_\geq)}$, there is a sequence $\gamma_k \in \mathbb{Z}^2_\geq$ with $\omega(\beta + \alpha) = \lim_k \omega(\gamma_k)$.[2] Evaluating at $\chi_{(0,0)}$ from (3.2.1), which is just the characteristic function of $\mathbb{Z}^2_\geq$, this gives $0 = 1$, which is absurd. Therefore, $r^{-1}(X) \subseteq d^{-1}(X)$.

Conversely, if $(\rho, \alpha) \in d^{-1}(X)$, then $\rho \circ \tau_{-\alpha} = \omega(\beta)$ for some $\beta \in \mathbb{Z}^2_\geq$, hence $\rho = \omega(\beta) \circ \tau_\alpha = \omega(\beta - \alpha)$. Since also $\rho \in \operatorname{Spec}_\Lambda(\Omega)$, the same argument as before shows that $\beta - \alpha \in \mathbb{Z}^2_\geq$, so $\rho \in X$ and therefore $d^{-1}(X) \subseteq r^{-1}(X)$. Now Proposition 2.1.15 implies that X is invariant. ∎

DEFINITION 3.4.7. We set

$$\operatorname{Spec}^\partial_\Lambda(\Omega) := \operatorname{Spec}_\Lambda(\Omega) \setminus \omega(\mathbb{Z}^2_\geq) = \overline{\omega(\mathbb{Z}^2_\geq)} \setminus \omega(\mathbb{Z}^2_\geq)$$

[2] Note that it suffices to consider sequences to describe this closure, because $\operatorname{Spec}(\mathcal{D}_\Omega)$ is first countable by Corollary 3.2.6.

and, for every exposed or virtual face F of Λ, we define

$$\begin{aligned}\mathrm{Spec}_F(\Omega) &:= \{\Psi_F(s) : s \in \mathscr{D}(F)\} = \mathrm{im}(\Psi_F),\\ \mathrm{Spec}_F^\circ(\Omega) &:= \{\Psi_F(s) : s \in \mathscr{D}(F) \cap \mathbb{R}\},\\ \mathrm{Spec}_F^\partial(\Omega) &:= \mathrm{Spec}_F(\Omega) \setminus \mathrm{Spec}_F^\circ(\Omega),\end{aligned} \tag{3.4.2}$$

where $\Psi_F(s)$ for $s \in \mathscr{D}(F)$ are the characters from Proposition 3.3.7. If G is a non-exposed face, it must be the endpoint $\partial_\sigma F$ of a line segment F, where $\sigma \in \{-,+\}$, see Definition 3.3.1. In this case, we put

$$\mathrm{Spec}_G(\Omega) := \mathrm{Spec}_G^\circ(\Omega) := \{\Psi_F(\sigma\infty)\} \quad \text{and} \quad \mathrm{Spec}_G^\partial(\Omega) := \emptyset. \tag{3.4.3}$$

We equip all of these sets with the topology inherited from $\mathrm{Spec}(\mathcal{D}_\Omega)$, that is, the relative weak-* topology.

REMARK 3.4.8. (i) The sets $\mathrm{Spec}_G(\Omega)$ for non-exposed $G = \{p\}$ are well-defined, that is, they are independent of the face F which contains G as an endpoint. Indeed, if two one-dimensional faces F_1 and F_2 have G as an endpoint, say $\partial_+ F_1 = G = \partial_- F_2$, then

$$\Psi_{F_1}(+\infty)(\tau_\beta \chi_{e_i}) = \sqrt{\exp(2p_i)} = \Psi_{F_2}(-\infty)(\tau_\beta \chi_{e_i})$$

as claimed.

(ii) For any proper or virtual face F, the set $\mathrm{Spec}_F(\Omega)$ is the image under the continuous (see Lemma 3.3.9) map Ψ_F of a closed subset of the compact space $\overline{\mathbb{R}}$ and is therefore compact itself. Since $\mathrm{Spec}(\mathcal{D}_\Omega)$ is Hausdorff, $\mathrm{Spec}_F(\Omega)$ is also closed. Moreover, $\mathrm{Spec}_F^\circ(\Omega) \subseteq \mathrm{Spec}_F(\Omega)$ is a dense subset as

$$\overline{\mathrm{Spec}_F^\circ(\Omega)} = \overline{\Psi_F(\mathscr{D}(F) \cap \mathbb{R})} = \overline{\Psi_F(\mathrm{cl}_{\overline{\mathbb{R}}}(\mathscr{D}(F) \cap \mathbb{R}))} = \overline{\Psi_F(\mathscr{D}(F))} = \mathrm{Spec}_F(\Omega),$$

where we have used that $\mathscr{D}(F) \cap \mathbb{R}$ is dense in $\mathscr{D}(F)$ for all faces F.

(iii) Note that for any non-virtual singleton face F, the sets $\mathrm{Spec}_F(\Omega)$ and $\mathrm{Spec}_F^\circ(\Omega)$ agree and contain only one character. This is evident from the definition for the non-exposed faces, and the exposed zero-dimensional faces must have $u \in \mathbb{R}^2_>$, so $\Psi_F(s)$ is independent of $s \in \mathscr{D}(F)$.

(iv) Let $F = F_{e_1}$ be the proper or virtual face with $u = e_1$. We have

$$\Psi_F(s)(\tau_\beta \chi_{e_i}) = \begin{cases} \sqrt{\Phi_i(F)(s - \beta_2)}, & s > \beta_2 \\ 0, & s \le \beta_2 \end{cases} \quad \text{for all } s \in \mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\} \text{ and } i \in \{1,2\}.$$

In particular, the set $\mathrm{Spec}_F^\circ(\Omega)$ contains more than one character and differs from $\mathrm{Spec}_F(\Omega)$. Let $\rho \in \mathrm{Spec}_\Lambda^\partial(\Omega)$ be arbitrary. Then there is a sequence $(\alpha_k)_k$ in $\mathbb{Z}^2_\ge$ with $\|\alpha_k\| \to \infty$ such that $\rho = \lim_k \omega_{\alpha_k}$ in the weak-* topology. In particular, $\rho(\tau_\beta \chi_{e_i}) = \lim_k \chi_{e_i}(\alpha_k - \beta)$ exists for all $\beta \in \mathbb{Z}^2$ and $i \in \{1,2\}$. By Lemma 3.4.1, there exists a constant $C \in (0,1)$ such that

$$\tau_\beta(\rho) := \rho(\tau_\beta \chi_{e_1})^2 + \rho(\tau_\beta \chi_{e_2})^2 = \lim_{k\to\infty} \left(\chi_{e_1}(\alpha_k - \beta)^2 + \chi_{e_2}(\alpha_k - \beta)^2\right)$$

satisfies either $\tau_\beta(\rho) \ge C^2$ or $\tau_\beta(\rho) = 0$, depending on the asymptotic behavior of α_k. Note that the map $\mathrm{Spec}_\Lambda^\partial(\Omega) \to \mathbb{C}, \rho \mapsto \tau_\beta(\rho)$ is continuous for every $\beta \in \mathbb{Z}^2$ by the definition of the weak-* topology. We claim that for the face $F = F_{e_1}$ as above,

$$\mathrm{Spec}_F^\circ(\Omega) = \bigcup_{\ell \in \mathbb{Z}_\ge} \{\rho \in \mathrm{Spec}_\Lambda^\partial(\Omega) : \tau_{(0,\ell)}(\rho) = 0\} = \bigcup_{\ell \in \mathbb{Z}_\ge} \{\rho \in \mathrm{Spec}_\Lambda^\partial(\Omega) : \tau_{(0,\ell)}(\rho) < C^2\}, \tag{3.4.4}$$

and, in particular, $\mathrm{Spec}_F^\circ(\Omega)$ is open in $\mathrm{Spec}_\Lambda^\partial(\Omega)$. Indeed, by the above description of $\Psi_F(s)$, we have

$$\tau_{(0,\ell)}(\Psi_F(s)) = \Psi_F(s)(\tau_{(0,\ell)}\chi_{e_1})^2 + \Psi_F(s)(\tau_{(0,\ell)}\chi_{e_2})^2 = 0$$

for $\ell \geq s$. Conversely, let $\rho \in \mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$ satisfy $\tau_{(0,\ell)}(\rho) = 0$ for some $\ell \in \mathbb{Z}_{\geq}$. By Proposition 3.4.9 below, $\mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$ is the union of the sets $\mathrm{Spec}_G(\Omega)$ for the exposed and virtual faces G. Only the characters associated to the faces F_u with $u \in \{e_1, e_2\}$ can vanish at some of the generators of $\mathcal{D}_\Omega$ by Proposition 3.3.7. If $\rho = \Psi_{F_{e_2}}(s) \in \mathrm{Spec}_{F_{e_2}}(\Omega)$ for the proper or virtual face F_{e_2}, then

$$\rho(\tau_{(0,\ell)}\chi_{e_2}) = \sqrt{\Phi_2(F_{e_2})(s)} > 0$$

by (3.3.9), the definition of the functions $\Phi_2(F_{e_2})$ and $(0,\ell) \cdot u^\perp = -(0,\ell) \cdot e_1 = 0$. But this is a contradiction to $\tau_{(0,\ell)}(\rho) = 0$. Therefore, $\rho \in \mathrm{Spec}_F(\Omega)$ and ρ cannot agree with $\Psi_F(+\infty)$ since this character satisfies $\Psi_F(+\infty)(\tau_{(0,\ell)}\chi_{e_1}) > 0$. We conclude that $\rho \in \mathrm{Spec}^{\circ}_F(\Omega)$ and (3.4.4) holds. Similar considerations can be made for the virtual face F_{e_2}.

(v) We point out that in [**SSU89**], the sets $\mathrm{Spec}_F(\Omega)$ are only defined as in (3.4.2), which only makes sense for the exposed and virtual faces. This seems to be a minor oversight, as Proposition 3.4.9 requires them to be defined also for the non-exposed faces. Our definition of $\mathrm{Spec}_G(\Omega)$ for non-exposed G in (3.4.3) fixes this small discrepancy.

The next Proposition relates the sets of characters associated to the faces of the convex set Λ as defined in Definition 3.4.7. The main statement is that $\mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$ decomposes (up to exceptions concerning the virtual faces) into contributions from the faces contained in $\partial\Lambda$ in very much the same way as $\partial\Lambda$ itself is built up from the proper faces of Λ. In this analogy, $\mathrm{Spec}^{\circ}_F(\Omega)$ plays the role of the relative interior of F, see Remark 3.4.10 below.

PROPOSITION 3.4.9 ([**SSU89**, Proposition 3.5]). *The following relations hold between the sets from Definition 3.4.7:*

(i) We have

$$\mathrm{Spec}^{\partial}_{\Lambda}(\Omega) = \bigcup_{\substack{F \subseteq \Lambda \text{ exposed or} \\ \text{virtual face}}} \mathrm{Spec}_F(\Omega) = \bigcup_{\substack{F \subseteq \Lambda \text{ proper} \\ \text{face}}} \mathrm{Spec}^{\circ}_F(\Omega) \cup \bigcup_{\substack{F \text{ virtual} \\ \text{face}}} \mathrm{Spec}_F(\Omega) \tag{3.4.5}$$

where, in the second representation, all unions are disjoint.

(ii) If F is a one-dimensional face of Λ, then $\mathrm{Spec}^{\circ}_F(\Omega)$ is an open subset of $\mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$ and we have

$$\mathrm{Spec}^{\partial}_F(\Omega) = \mathrm{Spec}_{\partial_+ F}(\Omega) \cup \mathrm{Spec}_{\partial_- F}(\Omega),$$

where $\partial_\pm F$ are the endpoint faces of F, see Definition 3.3.1. Here, we put $\mathrm{Spec}_{\emptyset}(\Omega) := \emptyset$ in case F only has one endpoint.

(iii) For any virtual face F of Λ, the set $\mathrm{Spec}^{\circ}_F(\Omega)$ is open in $\mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$.

PROOF. By definition, $\mathrm{Spec}^{\partial}_{\Lambda}(\Omega)$ is the set of characters in $\mathrm{Spec}(\mathcal{D}_\Omega)$ which are weak-* limits $\rho = \lim_k \omega_{\alpha_k}$ with integer sequence $\alpha_k \in \mathbb{Z}^2_{\geq}$ satisfying $\|\alpha_k\| \to \infty$. Note that we may pass to subsequences without changing the weak-* limit ρ. Let $\beta_k := \alpha_k + (1,1) \in \mathbb{Z}^2_{>}$. The sequence $\beta_k / \|\beta_k\|$ has values in the compact space $\mathbb{R}^2_{\geq} \cap S^1$, hence there is a subsequence (without loss of generality: the sequence itself) which converges to some $u \in \mathbb{R}^2_{\geq}$ with $\|u\| = 1$. Since $\mathrm{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_{>} \cdot u^\perp)$ is also compact, we may assume that $\beta_k \cdot u^\perp \to s$ for some $s \in \mathrm{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_{>} \cdot u^\perp)$. This implies that $\rho = \lim_k \omega_{\beta_k - (1,1)} = \Psi_{F_u}(s) \in \mathrm{Spec}_{F_u}(\Omega)$. Conversely, any character in $\mathrm{Spec}_F(\Omega)$ is by construction the weak-* limit of certain characters $\omega_{\alpha_k - (1,1)}$ with $\|\alpha_k\| \to \infty$. This shows the first equality in (3.4.5).

Note that for one-dimensional faces F we have

$$\begin{aligned} \mathrm{Spec}_F(\Omega) &= \mathrm{Spec}^{\circ}_F(\Omega) \cup \{\Psi_F(s) : s \in \mathscr{D}(F) \setminus \mathbb{R}\} = \\ &= \mathrm{Spec}^{\circ}_F(\Omega) \cup \mathrm{Spec}_{\partial_+ F}(\Omega) \cup \mathrm{Spec}_{\partial_- F}(\Omega) = \mathrm{Spec}^{\circ}_F(\Omega) \cup \mathrm{Spec}^{\circ}_{\partial_+ F}(\Omega) \cup \mathrm{Spec}^{\circ}_{\partial_- F}(\Omega) \end{aligned} \tag{3.4.6}$$

with disjoint unions. This is clear from (3.3.9) and the definition of $\Phi_i(F)$ as well as the definition of $\operatorname{Spec}_{\partial_\pm F}(\Omega)$ in case one or both of $\partial_\pm F$ are non-exposed. Next, for any virtual face F, the set $\operatorname{Spec}_F(\Omega)$ has empty intersection with $\operatorname{Spec}_G(\Omega)$ for any proper face G of Λ since the characters in $\operatorname{Spec}_F(\Omega)$ have zeros at one or both of w_1 and w_2, which is not possible for the proper faces. Thus, we have shown the second equality in (3.4.5), but it still remains to be shown that this is a disjoint union.

Let ψ be the continuous function

$$\psi\colon \operatorname{Spec}^\partial_\Lambda(\Omega) \to \overline{\mathbb{R}}^2_\le, \quad \psi(\rho) := (\log\rho(w_1), \log\rho(w_2)),$$

where we set $\log 0 := -\infty$. (Note that $\rho(w_i) \in [0,1]$ for any $\rho \in \operatorname{Spec}^\partial_\Lambda(\Omega)$ since ρ is the weak-* limit of suitable ω_α.) For any subset $E \subseteq \overline{\mathbb{R}}^2$, let $B(E) := \operatorname{pr}_1(E) \times \operatorname{pr}_2(E) \subseteq \overline{\mathbb{R}}^2$.

By Lemma 3.3.10, ψ maps $\operatorname{Spec}^\circ_F(\Omega)$ into $B(\operatorname{relint}(F))$ for any exposed face F of Λ. Note that for all singleton faces (not necessarily exposed) we have $B(\operatorname{relint}(F)) = F$. Hence, if $G = \partial_\sigma F$ with $\sigma \in \{-,+\}$ is a non-exposed (singleton) face of Λ, then $\psi(\operatorname{Spec}^\circ_G(\Omega)) = \psi(\{\Psi_F(\sigma\infty)\}) = G = B(\operatorname{relint}(G))$. If $F = F_{e_j}$ is a virtual face, then $\Psi_F(s)(w_i) = \delta_{ij}$ for all $s \in \mathscr{D}(F)$ and hence also $\psi(\operatorname{Spec}_F(\Omega)) = \psi(\operatorname{Spec}^\circ_F(\Omega)) = F = B(\operatorname{relint}(F))$ if we define $\operatorname{relint}(F) := F$. Collecting the above observations and using (i), we have a continuous map

$$\psi\colon \operatorname{Spec}^\partial_\Lambda(\Omega) \to \mathcal{B} := \bigcup_{\substack{F\subseteq\Lambda \text{ proper or}\\ \text{virtual face}}} B(\operatorname{relint}(F)).$$

Moreover, we have $\operatorname{Spec}^\circ_F(\Omega) \subseteq \psi^{-1}(\psi(\operatorname{Spec}^\circ_F(\Omega))) \subseteq \psi^{-1}(B(\operatorname{relint}(F)))$ for any proper or virtual face F, and since the sets $B(\operatorname{relint}(F))$ are disjoint, we see that $\operatorname{Spec}^\circ_F(\Omega) \cap \operatorname{Spec}^\circ_{F'}(\Omega) = \emptyset$ for distinct faces F, F'. This, together with (3.4.6), completes the proof of the second equality in (3.4.5).

Let F be a proper (non-virtual) face and $\rho \in \psi^{-1}(B(\operatorname{relint}(F)))$. Then $\rho(w_i) = \exp(p_i)$ for some $p_i \in \operatorname{relint}(F)$ and all $i \in \{1,2\}$. Note that ρ cannot be in $\operatorname{Spec}_G(\Omega)$ for any virtual face G since those characters always have zeros at one of either w_1 or w_2, see Corollary 3.3.11. Thus, by (i), we must have $\rho \in \operatorname{Spec}^\circ_G(\Omega)$ for some proper face G. By Lemma 3.3.10 (in case G is exposed) and Definition 3.4.7 (in case G is non-exposed) there is $q \in \operatorname{relint}(G)$ such that $\rho(w_i) = \exp(q_i)$, $i \in \{1,2\}$. Because the sets $\operatorname{relint}(G)$ are disjoint, we must have $F = G$ and hence $\rho \in \operatorname{Spec}^\circ_F(\Omega)$. This shows that

$$\operatorname{Spec}^\circ_F(\Omega) = \psi^{-1}(B(\operatorname{relint}(F))).$$

For a one-dimensional face F, the set $B(\operatorname{relint}(F))$ is open in $\mathcal{B}$, hence we deduce that $\operatorname{Spec}^\circ_F(\Omega)$ is open in $\operatorname{Spec}^\partial_\Lambda(\Omega)$ from continuity of ψ. Finally, we have already shown in (iv) of Remark 3.4.8 that $\operatorname{Spec}^\circ_F(\Omega)$ is also open in $\operatorname{Spec}^\partial_\Lambda(\Omega)$ for every virtual face F. This concludes the proof. ∎

REMARK 3.4.10. In the absence of virtual faces, and if we define $\operatorname{Spec}^\circ_\Lambda(\Omega) := \omega(\mathbb{Z}^2_\ge)$ and $\operatorname{Spec}_\emptyset(\Omega) := \operatorname{Spec}^\circ_\emptyset(\Omega) := \emptyset$, we have

$$\operatorname{Spec}_\Lambda(\Omega) = \bigcup_{F\subseteq\Lambda \text{ face}} \operatorname{Spec}^\circ_F(\Omega)$$

and the union is disjoint, where now F ranges over *all* faces of Λ, including Λ itself and $\emptyset$. One should compare that to the decomposition of Λ into its faces, $\Lambda = \bigcup_{F\subseteq\Lambda \text{ face}} \operatorname{relint}(F)$. Similarly, for any one-dimensional face F, or—even more generally—*any* face F of Λ,

$$\operatorname{Spec}_F(\Omega) = \bigcup_{G\subseteq F \text{ face}} \operatorname{Spec}^\circ_G(\Omega).$$

This representation is similar to [**Upm96**, Proposition 3.4.27].

PROPOSITION 3.4.11 ([**SSU89**, Proposition 3.6]). *For the one-dimensional and virtual faces, the map $\Psi_F\colon \mathscr{D}(F) \to \operatorname{Spec}_F(\Omega)$ is a homeomorphism.*

PROOF. It suffices to show that Ψ_F is injective. By compactness of $\mathscr{D}(F)$ it then follows that Ψ_F is also open, hence a homeomorphism onto its image $\mathrm{im}(\Psi_F) = \mathrm{Spec}_F(\Omega)$. Consider first $u \in \mathbb{R}^2_>$ (so $F = F_u$ is non-virtual) and suppose $t, s \in \mathscr{D}(F) \cap \mathbb{R} = \mathrm{cl}_{\overline{\mathbb{R}}}(\mathbb{Z}^2_> \cdot u^\perp) \cap \mathbb{R}$, $t \neq s$, with $\Psi_F(s) = \Psi_F(t)$ and write $s = \lim_k(\alpha_k \cdot u^\perp)$, see (3.3.2). By (3.3.9), this means

$$\Phi_i(F)(s - \beta \cdot u^\perp) = \Phi_i(t - \beta \cdot u^\perp) \quad \text{for all } \beta \in \mathbb{Z}^2, i \in \{1,2\}$$

and hence

$$\Phi_i(F)(t - s - \beta \cdot u^\perp) = \lim_{k\to\infty} \Phi_i(F)(t - (\alpha_k + \beta) \cdot u^\perp) = \\ = \lim_{k\to\infty} \Phi_i(F)(s - (\alpha_k + \beta) \cdot u^\perp) = \Phi_i(F)(s - (\beta \cdot u^\perp + s)) = \Phi_i(F)(-\beta \cdot u^\perp)$$

for all $\beta \in \mathbb{Z}^2$ and $i \in \{1,2\}$. Without loss of generality, we may assume $t > s$. Therefore, the function $\Phi_i(F)$ is periodic with period $t - s > 0$. But by Definition 3.3.3 and Lemma 3.3.6, its limits $\Phi_i(F)(\pm\infty)$ exist and are not equal, a contradiction to periodicity. Thus, $t = s$ and Ψ_F is injective on $\mathscr{D}(F) \cap \mathbb{R}$ (with values in $\mathrm{Spec}^\circ_F(\Omega)$). By Proposition 3.4.9, we have

$$\mathrm{Spec}_{\partial_+ F}(\Omega) \cup \mathrm{Spec}_{\partial_- F}(\Omega) = \mathrm{Spec}^\circ_{\partial_+ F}(\Omega) \cup \mathrm{Spec}^\circ_{\partial_- F}(\Omega) = \mathrm{Spec}^\partial_F(\Omega) \subseteq \{\Psi_F(-\infty), \Psi_F(+\infty)\},$$

where the disjoint union on the left hand side contains (exactly) two elements. Thus, $\Psi_F(-\infty) \neq \Psi_F(+\infty)$ and both values do not lie in $\mathrm{Spec}^\circ_F(\Omega) = \mathrm{im}(\Psi_F|_{\mathscr{D}(F)\cap\mathbb{R}})$, which shows that Ψ_F is injective on the whole of $\mathscr{D}(F)$.

In the case of a proper face with $u = e_1$, we have

$$\Psi_F(s)(\tau_\beta w_1) = \begin{cases} 1, & s > \beta_2 \\ 0, & s \leq \beta_2 \end{cases} \quad \text{and} \quad \Psi_F(s)(\tau_\beta w_2) = \begin{cases} \left(\frac{s-\beta_2}{s-\beta_2+1}\right)^{1/2} \exp(p_2^+) & s > \beta_2 \\ 0, & s \leq \beta_2 \end{cases}$$

for all $s \in \mathscr{D}(F) = \mathbb{Z}_> \cup \{+\infty\}$ and all $\beta \in \mathbb{Z}^2$, see Proposition 3.3.7 and Definition 3.3.3, and all these characters are obviously different. A similar argument can be made for $u = e_2$ and the virtual faces. ∎

3.5. The structure of the Toeplitz C^*-algebra

The first step in describing the structure of the Toeplitz C^*-algebra is to show that the C^*-algebra generated by the polar decomposition of the Toeplitz operators T_{z_i} is isomorphic to the groupoid C^*-algebra associated to the groupoid $\mathcal{G}_\Omega$ as defined in Definition 3.4.4. Before doing that, we observe that

$$C^*(\mathcal{G}_\Omega) = C^*_{\mathrm{red}}(\mathcal{G}_\Omega)$$

by Propositions 2.4.30 and 2.4.24, because $\mathcal{G}_\Omega$ is the reduction of a locally compact transformation groupoid with abelian (hence amenable, see Example 2.4.28) group, and the same holds for the reductions of $\mathcal{G}_\Omega$ by open or both locally closed and invariant subsets.

DEFINITION 3.5.1. We denote by $\mathcal{W}(\Omega)$ the C^*-subalgebra of $\mathscr{L}(\ell^2(\mathbb{Z}^2_\geq))$ generated by the operators

$$U_\beta(h)(\alpha) := \begin{cases} h(\alpha - \beta), & \alpha - \beta \in \mathbb{Z}^2_\geq \\ 0, & \alpha - \beta \notin \mathbb{Z}^2_\geq \end{cases} \quad \text{and} \quad W_\gamma(h) := \chi_\gamma h \quad (h \in \ell^2(\mathbb{Z}^2_\geq), \alpha \in \mathbb{Z}^2_\geq)$$

for $\beta, \gamma \in \mathbb{Z}^2_\geq$, where $\chi_\gamma(\alpha) = \|z^{\alpha+\gamma}\|_{L^2(\Omega)} / \|z^\alpha\|_{L^2(\Omega)}$ are the weights from Definition 3.2.1.

REMARK 3.5.2. Note that W_γ is a self-adjoint operator on $\ell^2(\mathbb{Z}^2_\geq)$ and that U_β is a partial isometry, meaning that $U_\beta|_{\ker(U_\beta)^\perp}$ is an isometry. Indeed, if $\{\varepsilon_\alpha : \alpha \in \mathbb{Z}^2_\geq\}$ denotes the standard orthonormal basis of $\ell^2(\mathbb{Z}^2_\geq)$, then $\ker(U_\beta) = \mathrm{span}(\{\varepsilon_\alpha : \alpha - \beta \notin \mathbb{Z}^2_\geq\})$ and U_β is evidently an isometry on the orthogonal complement.

REMARK 3.5.3. We have already remarked after Definition 3.2.1 that the Toeplitz operator $T_\beta := T_{z^\beta}$ with symbol z^β for $\beta \in \mathbb{Z}^2_\geq$ satisfies

$$T_\beta e_\alpha = \frac{z^{\alpha+\beta}}{\|z^\alpha\|} = \frac{\|z^{\alpha+\beta}\|}{\|z^\alpha\|} e_{\alpha+\beta},$$

where $\{e_\alpha : \alpha \in \mathbb{Z}^2_\geq\}$ is the orthonormal basis $e_\alpha := z^\alpha / \|z^\alpha\|_{L^2(\Omega)}$ of the Bergman space $A^2(\Omega)$, see Proposition D.1. The choice of orthonormal basis induces an isomorphism $A^2(\Omega) \cong \ell^2(\mathbb{Z}^2_\geq)$. By the above, T_β corresponds to the operator $U_\beta W_\beta$ via this isomorphism. Indeed, if ε_α for $\alpha \in \mathbb{Z}^2_\geq$ again denotes the standard orthonormal basis of $\ell^2(\mathbb{Z}^2_\geq)$, we have

$$(U_\beta W_\beta \varepsilon_\alpha)(\alpha') = \begin{cases} \chi_\beta(\alpha' - \beta)\varepsilon_\alpha(\alpha' - \beta), & \alpha' - \beta \in \mathbb{Z}^2_\geq \\ 0, & \text{else} \end{cases}$$

for all $\alpha, \alpha' \in \mathbb{Z}^2_\geq$ and this is non-zero if and only if $\alpha' = \alpha + \beta$, so that

$$U_\beta W_\beta \varepsilon_\alpha = \chi_\beta(\alpha)\varepsilon_{\alpha+\beta} = (\|z^{\alpha+\beta}\| / \|z^\alpha\|)\varepsilon_{\alpha+\beta}.$$

Since this is evidently the polar decomposition of T_{z_i} (up to the above isomorphism), we may view $\mathcal{W}(\Omega)$ as the C^*-algebra generated by the polar decompositions of the Toeplitz operators with symbol z_α, $\alpha \in \mathbb{Z}^2_\geq$.

THEOREM 3.5.4 ([**CM85**, Theorem 2.5]). *The C^*-algebras $\mathcal{W}(\Omega)$ and $C^*_{\mathrm{red}}(\mathcal{G}_\Omega, \mu)$ are isomorphic, where $\mathcal{G}_\Omega$ is the groupoid from Definition 3.4.4 and μ is its canonical Haar system, see Remark 3.4.5.*

PROOF. Let $\lambda := \delta_{\omega(0)}$ be the Dirac measure on $\mathrm{Spec}_\Lambda(\Omega)$ with pole the evaluation functional $\omega(0)$. Then the measure $\nu := \int_{\mathrm{Spec}_\Lambda(\Omega)} \mu^\rho \, d\lambda(\rho)$ on $\mathcal{G}_\Omega$ is just $\mu^{\omega(0)} = (\delta_{\omega(0)} \times \sigma)|_{\mathcal{G}_\Omega}$ and its support is $\{\omega(0)\} \times \mathbb{Z}^2_\geq$, see Example 2.2.14, since $\omega(0) \cdot \alpha = \omega(\alpha) \in \mathrm{Spec}_\Lambda(\Omega)$ precisely for $\alpha \in \mathbb{Z}^2_\geq$. Hence, for every $f \in C_c(\mathcal{G}_\Omega)$,

$$\int_{\mathcal{G}_\Omega} f \, d\nu^{-1} = \int_{\mathcal{G}_\Omega} f((\rho,\alpha)^{-1}) \, d(\delta_{\omega(0)} \times \sigma)(\rho,\alpha) = \sum_{\alpha \in \mathbb{Z}^2_\geq} f(\omega(0) \circ \tau_{-\alpha}, -\alpha) = \sum_{\alpha \in \mathbb{Z}^2_\geq} f(\omega(\alpha), -\alpha), \tag{3.5.1}$$

which reveals that ν^{-1} is counting measure on the subset $\{(\omega(\alpha), -\alpha) : \alpha \in \mathbb{Z}^2_\geq\}$ of $\mathcal{G}_\Omega$.

Consider the linear map $V \colon L^2(\mathcal{G}_\Omega, \nu^{-1}) \to \ell^2(\mathbb{Z}^2_\geq)$ given by $(V\psi)(\alpha) := \psi(\omega(\alpha), -\alpha)$ and note that this is well-defined because ν^{-1} is just counting measure, see above. Moreover, V is actually an isomorphism (that is, unitary), since it preserves the inner product by (3.5.1) and has the inverse

$$(V^{-1}\varphi)(\rho,\alpha) := \begin{cases} \varphi(\beta), & \text{if } (\rho,\alpha) = (\omega(\beta), -\beta) \text{ for some } \beta \in \mathbb{Z}^2_\geq \\ 0, & \text{else} \end{cases}$$

for $\varphi \in \ell^2(\mathbb{Z}^2_\geq)$. Indeed, for all $\alpha \in \mathbb{Z}^2_\geq$, we have $(V^{-1}V\psi)(\omega(\alpha), -\alpha) = V\psi(\alpha) = \psi(\omega(\alpha), -\alpha)$ and $(VV^{-1}\varphi)(\alpha) = (V^{-1}\varphi)(\omega(\alpha), -\alpha) = \varphi(\alpha)$.

Now let $\mathrm{Ind}_\lambda = \mathrm{Ind}_{\omega(0)} \colon C^*_{\mathrm{red}}(\mathcal{G}_\Omega, \mu) \to \mathscr{L}(L^2(\mathcal{G}_\Omega, \nu^{-1}))$ be the representation induced off the unit space by λ, see Proposition 2.4.11 and (2.4.4). In the present case, this representation is given by

$$\begin{aligned} (\mathrm{Ind}_{\omega(0)}(f)\psi)(\omega(\alpha), -\alpha) &= \int_{\mathcal{G}_\Omega} f((\omega(\alpha), -\alpha)(\rho,\beta))\psi((\rho,\beta)^{-1}) \, d\mu^{\omega(0)}(\rho,\beta) \\ &= \sum_{\beta \in \mathbb{Z}^2_\geq} f(\omega(\alpha), \beta - \alpha)\psi(\omega(\beta), -\beta) \end{aligned}$$

for $f, \psi \in C_c(\mathcal{G}_\Omega)$ and $\alpha \in \mathbb{Z}^2_\geq$. The composition $\pi := \mathrm{conj}_V \circ \mathrm{Ind}_{\omega(0)}$, where

$$\mathrm{conj}_V \colon \mathscr{L}(L^2(\mathcal{G}_\Omega, \nu^{-1})) \to \mathscr{L}(\ell^2(\mathbb{Z}^2_\geq)), \quad T \mapsto VTV^{-1},$$

gives a representation of $C^*_{\mathrm{red}}(\mathcal{G}_\Omega, \mu)$ on $\ell^2(\mathbb{Z}^2_\geq)$, which takes the form

$$(\pi(f)\varphi)(\alpha) = \sum_{\beta \in \mathbb{Z}^2_\geq} f(\omega(\alpha), \beta - \alpha)\varphi(\beta)$$

for $f \in C_c(\mathcal{G}_\Omega)$, $\varphi \in \ell^2(\mathbb{Z}^2_\geq)$ and $\alpha \in \mathbb{Z}^2_\geq$.

We now show that π establishes an isomorphism from $C^*_{\mathrm{red}}(\mathcal{G}_\Omega, \mu)$ onto $\mathcal{W}(\Omega) \subseteq \mathscr{L}(\ell^2(\mathbb{Z}^2_\geq))$. For every $\beta \in \mathbb{Z}^2_\geq$, the function $T_\beta(\rho, \alpha) := \delta_{\beta,-\alpha}$ (Kronecker delta) for $\rho \in \mathrm{Spec}_\Lambda(\Omega)$ and $\alpha \in \mathbb{Z}^2$ belongs to $C_c(\mathcal{G}_\Omega)$ because $\mathrm{Spec}_\Lambda(\Omega)$ is compact by Proposition 3.4.2. Moreover,

$$(\pi(T_\beta)\varphi)(\alpha) = \sum_{\beta' \in \mathbb{Z}^2_\geq} \delta_{\beta, \alpha-\beta'}\varphi(\beta') = \begin{cases} \varphi(\alpha - \beta), & \alpha - \beta \in \mathbb{Z}^2_\geq \\ 0, & \alpha - \beta \notin \mathbb{Z}^2_\geq \end{cases}$$

for all $\varphi \in \ell^2(\mathbb{Z}^2_\geq)$ so that $\pi(T_\beta) = U_\beta$. Similarly, the function $M_\beta(\rho, \alpha) := \rho(\chi_\beta)\delta_{\alpha,0}$ is in $C_c(\mathcal{G}_\Omega)$ and satisfies

$$(\pi(M_\beta)\varphi)(\alpha) = \sum_{\beta' \in \mathbb{Z}^2_\geq} \omega(\alpha)(\chi_\beta)\delta_{\beta'-\alpha,0}\varphi(\beta') = \chi_\beta(\alpha)\varphi(\alpha) = W_\beta(\varphi)(\alpha),$$

which shows that $\pi(M_\beta) = W_\beta$ and hence that $\mathrm{im}(\pi) \supseteq \mathcal{W}(\Omega)$. For the converse inclusion, it suffices to show that the functions T_β and M_β generate a dense subalgebra of $C^*_{\mathrm{red}}(\mathcal{G}_\Omega)$. Using Example 2.3.5, we first compute the convolution $T_\alpha * M_\beta * T^*_{\widetilde{\alpha}}$. Let $(\rho, \gamma) \in \mathcal{G}_\Omega$ and, for $k \in \mathbb{Z}^2$, set $A_k := \{k' \in \mathbb{Z}^2 : \rho \cdot (k + k') \in \mathrm{Spec}_\Lambda(\Omega)\}$. Then if $\alpha, \beta \in \mathbb{Z}^2_\geq$ and $-\widetilde{\alpha} \in A_\gamma$, we have

$$\begin{aligned}
(T_\alpha * M_\beta * T^*_{\widetilde{\alpha}})(\rho, \gamma) &= \sum_{k \in A_\gamma} T_\alpha(\rho, \gamma + k)(M_\beta * T^*_{\widetilde{\alpha}})(\rho \cdot (\gamma + k), -k) \\
&= \sum_{k \in A_\gamma} \sum_{k' \in A_{-k}} T_\alpha(\rho, \gamma + k) M_\beta(\rho \cdot (\gamma + k), -k + k')\overline{T_{\widetilde{\alpha}}(\rho \cdot \gamma, k')} \\
&= \sum_{k \in A_\gamma} \sum_{k' \in A_{-k}} (\rho \cdot (\gamma + k))(\chi_\beta)\delta_{\alpha,-\gamma-k}\delta_{-k+k',0}\delta_{\widetilde{\alpha},-k'} \\
&= (\rho \cdot (\gamma - \widetilde{\alpha}))(\chi_\beta)\delta_{\alpha,-\gamma+\widetilde{\alpha}} \\
&= \begin{cases} \rho \cdot (-\alpha)(\chi_\beta) = \rho(\tau_\alpha\chi_\beta) = \Gamma_{\mathcal{D}_\Omega}(\tau_\alpha\chi_\beta)(\rho), & \gamma = \widetilde{\alpha} - \alpha \\ 0, & \text{else} \end{cases}
\end{aligned}$$

where $\Gamma_{\mathcal{D}_\Omega} \colon \mathcal{D}_\Omega \to C_0(\mathrm{Spec}(\mathcal{D}_\Omega))$ is the Gelfand representation. It follows that we can use $\widetilde{\alpha}$ to adjust the support of $T_\alpha * M_\beta * T^*_{\widetilde{\alpha}}$. The choice $\widetilde{\alpha} := \gamma + \alpha$ then satisfies $\rho \cdot (\gamma - \widetilde{\alpha}) = \rho \cdot \alpha \in \mathrm{Spec}_\Lambda(\Omega)$ because $\alpha \in \mathbb{Z}^2_\geq$ and gives a function supported "at γ".

Since the functions $\tau_\alpha\chi_\beta$ generate $\mathcal{D}_\Omega$ by definition, their Gelfand transforms $\Gamma_{\mathcal{D}_\Omega}(\tau_\alpha\chi_\beta)$ generate $C_0(\mathrm{Spec}(\mathcal{D}_\Omega))$. If $f \in C_c(\mathcal{G}_\Omega)$ and γ is fixed, then $\rho \mapsto f(\rho, \gamma)$ is a continuous function on the compact subset $\mathrm{pr}_1(\mathcal{G}_\Omega \cap (\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma\}))$ of $\mathrm{Spec}(\mathcal{D}_\Omega)$. By the Tietze extension theorem, it is therefore the restriction of a function in $C_c(\mathrm{Spec}(\mathcal{D}_\Omega))$ and hence can be approximated uniformly by functions of the form $\Gamma_{\mathcal{D}_\Omega}(\tau_\alpha\chi_\beta)$, so that $f|_{\mathcal{G}_\Omega \cap (\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma\})}$ can be approximated by functions of the form $(T_\alpha * M_\beta * T^*_{\gamma+\alpha})|_{\mathcal{G}_\Omega \cap (\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma\})}$. Now since $\mathbb{Z}^2$ carries the discrete topology, the support of f only intersects a finite number of sets of the form $\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma_i\}$ for certain $\gamma_i \in \mathbb{Z}^2$. By approximating each of the restrictions $f|_{\mathcal{G}_\Omega \cap (\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma_i\})}$ as above and taking sums, we get a sequence of compactly supported functions which converge uniformly to f on $\mathcal{G}_\Omega$. Moreover, the support of this sequence can obviously be chosen to be contained in $\bigcup_i (\mathcal{G}_\Omega \cap (\mathrm{Spec}_\Lambda(\Omega) \times \{\gamma_i\}))$, a fixed compact subset of $\mathcal{G}_\Omega$. This implies that the functions $T_\alpha * M_\beta * T^*_{\widetilde{\alpha}}$ generate a dense subalgebra of $C_c(\mathcal{G}_\Omega)$ for the inductive limit topology and hence the T_α and M_β with $\alpha, \beta \in \mathbb{Z}^2_\geq$ also generate a dense *-subalgebra of $C_c(\mathcal{G}_\Omega)$. Because the dense inclusion $C_c(\mathcal{G}_\Omega) \to C^*_{\mathrm{red}}(\mathcal{G}_\Omega)$ is continuous by Remark 2.4.14, we conclude that T_α and M_β also generate a dense subalgebra of the groupoid C^*-algebra. This finishes the proof of $\mathrm{im}(\pi) = \mathcal{W}(\Omega)$.

It remains to show that π is injective. Let $U \subseteq (\mathcal{G}_\Omega)^{(0)}$ be an open invariant subset. Then $U = r(d^{-1}(U))$ by the definition of invariance, and by the computations of Lemma 3.4.6, this means

$$U = \{\rho \in \operatorname{Spec}_\Lambda(\Omega) : \exists \alpha \in \mathbb{Z}^2 \text{ such that } \rho \circ \tau_{-\alpha} \in U\}.$$

Because U is open and $\omega(\mathbb{Z}^2_\geq)$ is dense in $\operatorname{Spec}_\Lambda(\Omega) = (\mathcal{G}_\Omega)^{(0)}$, there is $\alpha_0 \in \mathbb{Z}^2_\geq$ with $\omega(\alpha_0) \in U$. This implies $\omega(0) \in U$, for if $\omega(0) \notin U$, then $\omega(\alpha) = \omega(0) \circ \tau_{-\alpha} \notin U$ for all $\alpha \in \mathbb{Z}^2$, a contradiction to the definition of α_0. But this means that $\lambda(U) = \delta_{\omega(0)}(U) = 1 > 0$, so that the largest open invariant λ-null subset of $(\mathcal{G}_\Omega)^{(0)}$ is $\emptyset$. By Theorem 2.5.4, this implies $\ker(\operatorname{Ind}_\lambda) = 0$ and hence that π is injective. ∎

Thus, in order to understand $\mathcal{W}(\Omega)$ (and, ultimately, its subalgebra $\mathcal{T}(\Omega)$), we are led to study the (reduced) groupoid C^*-algebra associated to the groupoid $\mathcal{G}_\Omega$.

THEOREM 3.5.5 ([**SSU89**, Theorem 4.3]). *The C^*-algebra $\mathcal{W}(\Omega)$ contains ideals*

$$0 \subseteq \mathcal{J}_1 = \mathcal{K}(\ell^2(\mathbb{Z}^2_\geq)) \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_3 := C^*(\mathcal{G}_\Omega, \mu) \cong \mathcal{W}(\Omega) \tag{3.5.2}$$

such that

$$\mathcal{J}_3/\mathcal{J}_2 \cong C(\partial_2^+\Lambda \times \mathbb{T}^2) \quad \textit{and} \quad \mathcal{J}_2/\mathcal{J}_1 \cong \sum_F^\oplus C^*(\mathcal{G}_\Omega|_{\operatorname{Spec}_F^\circ(\Omega)}),$$

where F ranges over all one-dimensional and virtual faces, $\sum^\oplus$ denotes the direct sum of C^*-algebras, see Example C.1.11, and where

$$\partial_2^+\Lambda := \{p \in \overline{\mathbb{R}}^2 : \{p\} \textit{ is a singleton face of } \Lambda \textit{ or a virtual face}\}$$

is a closed, hence compact subspace of $\overline{\mathbb{R}}^2$. Moreover,

$$C^*(\mathcal{G}_\Omega|_{\operatorname{Spec}_{F_u}^\circ(\Omega)}) \cong \begin{cases} C(\mathbb{T}) \otimes \mathcal{K}, & u_2/u_1 \in \mathbb{Q} \textit{ or } u \in \{e_1, e_2\} \\ \mathcal{A}_{u_2/u_1} \otimes \mathcal{K}, & u_2/u_1 \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Here, $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space and $\mathcal{A}_\alpha$ for irrational $\alpha \in \mathbb{R}$ is the irrational rotation algebra from Example 2.6.6.

PROOF. Note that $\partial_2^+\Lambda$ really is closed in $\overline{\mathbb{R}}^2$ since it is given by $\overline{\partial\Lambda} \setminus \bigcup_F \operatorname{relint}(F)$, where the closure is in $\overline{\mathbb{R}}^2$ and F runs through all one-dimensional faces. Define $X_0 := \omega(\mathbb{Z}^2_\geq) \subseteq \operatorname{Spec}_\Lambda(\Omega)$ as well as

$$X_1 := \bigcup_{\substack{F \text{ one-dimensional} \\ \text{or virtual face}}} \operatorname{Spec}_F^\circ(\Omega) \quad \text{and} \quad X_2 := \bigcup_{\substack{F \subseteq \Lambda \text{ proper} \\ \text{singleton face}}} \operatorname{Spec}_F^\circ(\Omega) \cup \bigcup_{\substack{F \text{ virtual} \\ \text{face}}} \operatorname{Spec}_F^\partial(\Omega).$$

Thus, up to exceptions regarding the virtual faces and setting $\operatorname{Spec}_\Lambda^\circ(\Omega) := \omega(\mathbb{Z}^2_\geq)$ as in Remark 3.4.10, X_k is the union of the sets $\operatorname{Spec}_F^\circ(\Omega)$, where F runs through the faces of codimension k. By Proposition 3.4.9, we have $\operatorname{Spec}_\Lambda(\Omega) = X_0 \cup X_1 \cup X_2$ with disjoint unions.

CLAIM. The subset $X_2 \subseteq \operatorname{Spec}_\Lambda(\Omega) \cong (\mathcal{G}_\Omega)^{(0)}$ is closed and invariant.

PROOF. Every character in X_2 is translation invariant, hence X_2 is invariant for the transformation groupoid $\operatorname{Spec}_\Lambda(\Omega) \rtimes_\Theta \mathbb{Z}^2$ and therefore also for $\mathcal{G}_\Omega$.[3] Let $(\rho_k)_k$ be a sequence of characters in X_2 with $\rho_k \to \rho$ for $\rho \in \operatorname{Spec}_\Lambda(\Omega)$. Then for all $k \geq 1$, we have $\rho_k = \Psi_{\{p_k\}}(s_k)$ for points $p_k \in \partial_2^+\Lambda$ and $s_k \in \mathcal{D}(\{p_k\})$. Note that $s_k \in \{-\infty, +\infty\}$ if $\{p_k\}$ is virtual by the definitions of $\operatorname{Spec}_F^\partial(\Omega)$ and X_2, and that s_k is of course not important for the proper singleton faces. For every k, we have $\Psi_{\{p_k\}}(s_k)(\tau_\beta w_i) = \exp((p_k)_i)$ for $\beta \in \mathbb{Z}^2$ and $i \in \{1, 2\}$ by (3.3.9) (for the exposed

[3] If G is a groupoid and $E_1 \subseteq E_2 \subseteq G^{(0)}$ with E_1 invariant for G, then $d^{-1}(E_1) = r^{-1}(E_1)$ and hence also $(d|_{G|_{E_2}})^{-1}(E_1) = G|_{E_2} \cap d^{-1}(E_1) = G|_{E_2} \cap r^{-1}(E_1) = (r|_{G|_{E_2}})^{-1}(E_1)$, so E_1 is invariant for $G|_{E_2}$, see Proposition 2.1.15.

and virtual faces) and Definition 3.4.7 (for the non-exposed faces). Thus, $\rho(\tau_\beta w_i) = \exp(q_i)$ with $q := \lim_k p_k \in \partial_2^+\Lambda$ and it follows that $\rho = \Psi_{\{q\}}(s) \in X_2$ for some suitable $s \in \mathscr{D}(\{q\})$. Therefore, X_2 is closed. ♦

By Lemma 3.4.6, X_0 is an open invariant subset of $(\mathcal{G}_\Omega)^{(0)} \cong \operatorname{Spec}_\Lambda(\Omega)$, and the above Claim shows that $X_0 \cup X_1 = \operatorname{Spec}_\Lambda(\Omega) \setminus X_2$ is also open and invariant. Therefore, we can form the C^*-algebras of the reductions,

$$\mathfrak{J}_1 := C^*(\mathcal{G}_\Omega|_{X_0}), \quad \mathfrak{J}_2 := C^*(\mathcal{G}_\Omega|_{X_0\cup X_1}) \quad \text{and} \quad \mathfrak{J}_3 := C^*(\mathcal{G}_\Omega),$$

which can be seen as closed ideals of $C^*(\mathcal{G}_\Omega)$ by Lemma 2.5.2 and Theorem 2.5.3. Moreover,

$$\mathfrak{J}_2/\mathfrak{J}_1 \cong C^*(\mathcal{G}_\Omega|_{X_1}) \quad \text{and} \quad \mathfrak{J}_3/\mathfrak{J}_2 \cong C^*(\mathcal{G}_\Omega|_{X_2})$$

also by Theorem 2.5.3.

CLAIM. $\mathfrak{J}_1 \cong \mathscr{K}(\ell^2(\mathbb{Z}^2_\geq))$.

PROOF. By the proof of Lemma 3.4.6, we have $r^{-1}(X_0) = \{(\rho,\alpha) \in X_0 \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in X_0\}$. Since X_0 is invariant by the same Lemma,

$$\mathcal{G}_\Omega|_{X_0} = r^{-1}(X_0) = \{(\rho,\alpha) \in X_0 \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in X_0\} = (\omega(\mathbb{Z}^2) \rtimes_\Theta \mathbb{Z}^2)|_{X_0},$$

where Θ is the action from (3.4.1) (restricted to $\omega(\mathbb{Z}^2) \times \mathbb{Z}^2$) and the last equality is simply an application of (2.1.7). The map $\omega\colon \mathbb{Z}^2 \to \omega(\mathbb{Z}^2)$ is a homeomorphism by Lemma 3.2.5 and it satisfies

$$\Theta(\omega(\alpha),\beta) = \omega(\alpha) \circ \tau_{-\beta} = \omega(\alpha+\beta) = \omega(\rho_\beta(\alpha))$$

for all $\alpha,\beta \in \mathbb{Z}^2$, where $\rho_\beta(\alpha) := \alpha+\beta$ is (right) translation by β. Therefore, ω is a $\mathbb{Z}^2$-equivariant homeomorphism and we have

$$\mathfrak{J}_1 = C^*((\omega(\mathbb{Z}^2) \rtimes_\Theta \mathbb{Z}^2)|_{X_0}) \cong C^*((\mathbb{Z}^2 \rtimes_\rho \mathbb{Z}^2)|_{\mathbb{Z}^2_\geq}) \cong C^*(\mathbb{Z}^2_\geq \times \mathbb{Z}^2_\geq) \cong \mathscr{K}(\ell^2(\mathbb{Z}^2_\geq))$$

by applying Examples 2.6.9 and 2.6.7 and Proposition 2.6.2. ♦

CLAIM. $\mathfrak{J}_3/\mathfrak{J}_2 \cong C(\partial_2^+\Lambda \times \mathbb{T}^2)$.

PROOF. By the same argument as in the proof of Lemma 3.4.6, we see

$$r^{-1}(X_2) = \{(\rho,\alpha) \in X_2 \times \mathbb{Z}^2 : \rho \circ \tau_{-\alpha} \in \operatorname{Spec}_\Lambda(\Omega)\}.$$

As already mentioned, all characters in X_2 are translation invariant, so that $G|_{X_2} = r^{-1}(X_2) = X_2 \times \mathbb{Z}^2$ as a set. Moreover, the restriction of the action Θ on X_2 is trivial, $\Theta(\rho,\alpha) = \rho$ for all $\alpha \in \mathbb{Z}^2$ and $\rho \in X_2$. Therefore, $\mathcal{G}_\Omega|_{X_2} = X_2 \rtimes_{\mathrm{triv}} \mathbb{Z}^2$ is the trivial action groupoid. Moreover, X_2 is homeomorphic to $\partial_2^+\Lambda$: the map $\partial_2^+\Lambda \to X_2$, $p \mapsto \Psi_{\{p\}}(s)$ is a bijection and continuous,[4] because X_2 carries the weak-* topology and $\Psi_{\{p\}}(\tau_\beta w_i) = \exp(p_i)$ depends continuously on p. Compactness of $\partial_2^+\Lambda$ implies that this map is even a homeomorphism. By Example 2.6.11 and Proposition 2.6.14, we have

$$C^*(\mathcal{G}_\Omega|_{X_2}) \cong C^*(X_2 \times \mathbb{Z}^2) \cong C_0(X_2) \otimes C^*(\mathbb{Z}^2) \cong C(\partial_2^+\Lambda) \otimes C(\mathbb{T}^2) \cong C(\partial_2^+\Lambda \times \mathbb{T}^2),$$

where we have used $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$ (see Example 2.6.5). Also note that $C(\partial_2^+\Lambda)$ and $C(\mathbb{T}^2)$ are nuclear C^*-algebras, see Example C.6.9, so that there is only one possible tensor product. ♦

CLAIM. $\mathfrak{J}_2/\mathfrak{J}_1 \cong \sum_F^\oplus C^*(\mathcal{G}_\Omega|_{\operatorname{Spec}_F^\circ(\Omega)})$ is a C^*-algebraic direct sum, see Example C.1.11, where F ranges over the one-dimensional and virtual faces.

[4] Note that the dependence on s is of course artificial as $\Psi_{\{p\}}(s)$ is independent of s for the proper singleton faces, and only $s = \pm\infty$ occurs in X_2 for the virtual faces, see also the above proof of closedness of X_2.

PROOF. By Proposition 3.4.9, the sets $\mathrm{Spec}^\circ_F(\Omega)$ are open and pairwise disjoint in X_1 for all one-dimensional and virtual faces F of Λ. Moreover, the sets $\mathrm{Spec}^\circ_F(\Omega)$ are translation invariant, so that they are invariant subsets of the unit space $(\mathcal{G}_\Omega|_{X_1})^{(0)} = X_1$. By Corollary 2.6.13,

$$\mathcal{J}_2/\mathcal{J}_1 \cong C^*(\mathcal{G}_\Omega|_{X_1}) \cong \sum_F^{\oplus} C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}^\circ_F(\Omega)})$$

and the claim follows. ♦

It remains to study the groupoid C^*-algebras $C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}^\circ_F(\Omega)})$ for the one-dimensional and virtual faces F of Λ. By Proposition 3.4.11, there are homeomorphisms

$$\chi(F) := \Psi_F|_{\mathscr{D}(F)\cap\mathbb{R}} \colon \mathscr{D}(F)\cap\mathbb{R} \to \mathrm{Spec}^\circ_F(\Omega).$$

CLAIM. If $F = F_u$ with $u \in \{e_1, e_2\}$ is a one-dimensional or virtual face, then

$$C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}^\circ_F(\Omega)}) \cong \mathscr{K}(\ell^2(\mathbb{Z}_>)) \otimes C(\mathbb{T}).$$

PROOF. Consider the case $u = e_1$ and assume that F is non-virtual. We have $\mathscr{D}(F)\cap\mathbb{R} = \mathbb{Z}_>$ by Lemma 3.3.2. Define a right-action of $\mathbb{Z}^2$ on $\mathbb{Z}$ by $k\cdot\alpha := k + \alpha_2$. Note that in the proof of Proposition 3.4.11 we found that

$$\chi(F)(s)(\tau_\beta w_1) = \begin{cases} 1, & s > \beta_2 \\ 0, & s \le \beta_2 \end{cases} \quad \text{and} \quad \chi(F)(s)(\tau_\beta w_2) = \begin{cases} \left(\frac{s-\beta_2}{s-\beta_2+1}\right)^{1/2} \exp(p_2^+) & s > \beta_2 \\ 0, & s \le \beta_2 \end{cases}$$

for all $s \in \mathbb{Z}_>$ and all $\beta \in \mathbb{Z}^2$, which immediately gives $\chi(F)(k\cdot\alpha) = \chi(F)(k)\circ\tau_{-\alpha}$ for all $k \in \mathbb{Z}_>$ and $\alpha \in \mathbb{Z}^2$ such that $k\cdot\alpha \in \mathbb{Z}_>$. Therefore, $\chi(F)$ satisfies the requirements of Example 2.6.10 and induces an isomorphism

$$C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}^\circ_F(\Omega)}) = C^*((\mathrm{Spec}(\mathcal{D}_\Omega) \rtimes_\Theta \mathbb{Z}^2)|_{\mathrm{Spec}^\circ_F(\Omega)}) \cong C^*((\mathbb{Z}\rtimes\mathbb{Z}^2)|_{\mathbb{Z}_>}).$$

Now consider the map

$$\varphi\colon \mathbb{Z}\times\mathbb{Z}^2 \to (\mathbb{Z}\times\mathbb{Z})\times\mathbb{Z}, \quad (k,\alpha) \mapsto ((k,\alpha_2),\alpha_1).$$

Then φ is a groupoid isomorphism from the transformation groupoid $\mathbb{Z}\rtimes\mathbb{Z}^2$ to the product groupoid $(\mathbb{Z}\rtimes_\rho\mathbb{Z})\times\mathbb{Z}$, where $\mathbb{Z}$ acts on itself by right translation and the last $\mathbb{Z}$ factor is the group $\mathbb{Z}$ as a groupoid. Indeed,

$$\begin{aligned}\varphi((k,\alpha)(k\cdot\alpha,\beta)) &= \varphi(k,\alpha+\beta) = ((k,\alpha_2+\beta_2),\alpha_1+\beta_2) = \\ &= ((k,\alpha_2)(k+\alpha_2,\beta_2),\alpha_1+\beta_1) = ((k,\alpha_2),\alpha_1)((k+\alpha_2,\beta_2),\beta_1) = \varphi(k,\alpha)\varphi(k\cdot\alpha,\beta),\end{aligned}$$

where some care has to be taken regarding which multiplications are meant. Moreover, it is elementary to check that φ satisfies the requirements of Corollary 2.4.7, so that Corollary 2.4.7, Proposition 2.6.14 and Example 2.6.7 imply

$$C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}^\circ_F(\Omega)}) \cong C^*((\mathbb{Z}\rtimes\mathbb{Z}^2)|_{\mathbb{Z}_>}) \cong C^*((\mathbb{Z}\rtimes_\rho\mathbb{Z})|_{\mathbb{Z}_>}\times\mathbb{Z}) \cong C^*(\mathbb{Z}_>\times\mathbb{Z}_>)\otimes C^*(\mathbb{Z}),$$

where $\mathbb{Z}_>\times\mathbb{Z}_>$ is the trivial groupoid over $\mathbb{Z}_>$ and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ is the group C^*-algebra, see Example 2.6.5. Finally $C^*(\mathbb{Z}_>\times\mathbb{Z}_>) \cong \mathscr{K}(\ell^2(\mathbb{Z}_>))$ by Proposition 2.6.2 and this finishes the claim.

The same analysis can be carried out in the same way for $u = e_2$ and also for the virtual faces, as in this case the map $\chi(F)$ is even simpler. ♦

Now let $F = F_u$ be a one-dimensional face with $u \in \mathbb{R}^2_>$. As mentioned above, $\mathrm{Spec}^\circ_F(\Omega)$ is invariant under the $\mathbb{Z}^2$ action (3.4.1) on $\mathrm{Spec}(\mathcal{D}_\Omega)$, because $\mathrm{Spec}^\circ_F(\Omega) = \Psi_F(\mathscr{D}(F)\cap\mathbb{R})$ and

$$(\Psi_F(s)\circ\tau_{-\alpha})(\tau_\beta w_i) = \sqrt{\Phi_i(F)(s+\alpha\cdot u^\perp - \beta\cdot u^\perp)} = \Psi_F(s+\alpha\cdot u^\perp)(\tau_\beta w_i)$$

for $s \in \mathscr{D}(F) \cap \mathbb{R}$, $\beta \in \mathbb{Z}^2$ and $i \in \{1,2\}$ by Proposition 3.3.7. It follows that $\mathrm{Spec}_F^\circ(\Omega)$ is an invariant subset of the unit space $\mathrm{Spec}_\Lambda(\Omega)$ of the groupoid $\mathcal{G}_\Omega$ and therefore

$$\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)} = \mathrm{Spec}_F^\circ(\Omega) \rtimes_\Theta \mathbb{Z}^2,$$

see Example 2.1.22. The above also immediately shows that $\chi(F)$ is equivariant for the right action $s \diamond \alpha := s + \alpha \cdot u^\perp$ of $\mathbb{Z}^2$ on $\mathscr{D}(F) \cap \mathbb{R}$. Therefore, by Examples 2.6.9 and 2.6.4,

$$C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}) = C^*(\mathrm{Spec}_F^\circ(\Omega) \rtimes_\Theta \mathbb{Z}^2) \cong C^*((\mathscr{D}(F) \cap \mathbb{R}) \rtimes \mathbb{Z}^2) \cong C_0(\mathscr{D}(F) \cap \mathbb{R}) \rtimes \mathbb{Z}^2$$

is the crossed product of $C_0(\mathscr{D}(F) \cap \mathbb{R})$ by $\mathbb{Z}^2$.

Claim. If $\kappa := u_2/u_1$ is rational, then $C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}) \cong \mathscr{K}(\ell^2(\mathbb{Z})) \otimes C(\mathbb{T})$.

Proof. By (3.3.3), we have $\mathscr{D}(F) \cap \mathbb{R} = \|(p,q)\|^{-1}\mathbb{Z}$, where $\kappa = p/q$ with $p, q \in \mathbb{N}$ relatively prime. Note that

$$\alpha \cdot u^\perp = -\alpha_1 u_2 + \alpha_2 u_1 = -\alpha_1 u_1 \frac{p}{q} + \alpha_2 u_2 \frac{q}{p} = (-\alpha_1 p + \alpha_2 q)\|(p,q)\|^{-1}.$$

It follows that the map $\psi\colon \mathscr{D}(F) \cap \mathbb{R} \to \mathbb{Z}$, $s \mapsto \|(p,q)\|s$ is equivariant for the right action $k \diamond \alpha := k - \alpha_1 p + \alpha_2 q$ of $\mathbb{Z}^2$ on $\mathbb{Z}$. Indeed,

$$\psi(s \diamond \alpha) = \|(p,q)\|(s + \alpha \cdot u^\perp) = \|(p,q)\|s - \alpha_1 p + \alpha_2 q = \psi(s) \diamond \alpha.$$

By Example 2.6.9, $C^*((\mathscr{D}(F) \cap \mathbb{R}) \rtimes \mathbb{Z}^2) \cong C^*(\mathbb{Z} \rtimes \mathbb{Z}^2)$. Now let $a, b \in \mathbb{N}$ be such that $bq - ap = 1$. Then $b(q,p) - p(a,b) = (1,0)$ and $q(0,b) - a(0,p) = (0,1)$, which shows that $\{(a,b),(p,q)\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Moreover, $k \diamond (a,b) = k+1$ and $k \diamond (p,q) = k$. The change of basis $\vartheta\colon \mathbb{Z}^2 \to \mathbb{Z}^2$, $\vartheta(\alpha_1, \alpha_2) := \alpha_1(a,b) + \alpha_2(p,q)$ gives rise to the groupoid isomorphism

$$\varphi\colon (\mathbb{Z} \rtimes_\rho \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z} \rtimes \mathbb{Z}^2, \quad \varphi(k, \alpha_1, \alpha_2) := (k, \vartheta(\alpha_1, \alpha_2)),$$

where again ρ is right translation of $\mathbb{Z}$ on itself. This is indeed an isomorphism as

$$\varphi((k, \alpha_1, \alpha_2)(k + \alpha_1, \beta_1, \beta_2)) = \varphi(k, \alpha_1 + \beta_1, \alpha_2 + \beta_2) = (k, \vartheta(\alpha + \beta))$$

and

$$\begin{aligned} \varphi(k, \alpha_1, \alpha_2)\varphi(k + \alpha_1, \beta_1, \beta_2) &= (k, \vartheta(\alpha_1, \alpha_2))(k + \alpha_1, \vartheta(\beta_1, \beta_2)) = \\ &= (k, \vartheta(\alpha_1, \alpha_2))(k \diamond \vartheta(\alpha_1, \alpha_2), \vartheta(\beta_1, \beta_2)) = (k, \vartheta(\alpha_1, \alpha_1) + \vartheta(\beta_1, \beta_2)) = (k, \vartheta(\alpha + \beta)) \end{aligned}$$

and one easily verifies that this satisfies the assumptions of Corollary 2.4.7, so that

$$C^*(\mathbb{Z} \rtimes \mathbb{Z}^2) \cong C^*(\mathbb{Z} \rtimes_\rho \mathbb{Z}) \otimes C^*(\mathbb{Z}) \cong \mathscr{K}(\ell^2(\mathbb{Z})) \otimes C(\mathbb{T})$$

by Proposition 2.6.2 and Example 2.6.5, where we note that both $\mathscr{K}(\ell^2(\mathbb{Z}))$ and $C(\mathbb{T})$ are nuclear, see Examples C.6.8 and C.6.9, so there is only one possible tensor product. ♦

Claim. If $\kappa := u_2/u_1$ is irrational, then $C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}) \cong \mathcal{A}_\kappa \otimes \mathscr{K}$.

Proof. In this case $\mathscr{D}(F) \cap \mathbb{R} = \mathbb{R}$ by (3.3.3). The homeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto -u_1^{-1}s$, satisfies

$$\psi(s \diamond \alpha) = \psi(s + \alpha \cdot u^\perp) = -u_1^{-1}s + \alpha \cdot (\kappa, -1) = \psi(s) + \alpha \cdot (\kappa, -1).$$

But this means that ψ is equivariant for the action from Example 2.6.6, so that

$$C^*((\mathscr{D}(F) \cap \mathbb{R}) \rtimes \mathbb{Z}^2) \cong C_0(\mathbb{R}) \rtimes \mathbb{Z}^2 \cong \mathcal{A}_\kappa \otimes \mathcal{K}$$

is as claimed. ♦

The proof is now finished as all cases have been dealt with. ■

REMARK 3.5.6. The decomposition of $\mathcal{W}(\Omega)$ from (3.5.2) is an example of a *composition series*. For a general C^*-algebra A, a composition series is an increasing family $(\mathcal{J}_i)_{0\le i\le\alpha}$ of ideals of A with α an ordinal and such that $\mathcal{J}_0 = 0$, $\mathcal{J}_\alpha = A$, and if $\beta \le \alpha$ is a limit ordinal, then $\mathcal{J}_\beta = \overline{\bigcup_{\beta'<\beta}\mathcal{J}_{\beta'}}$. In the above case of the Toeplitz C^*-algebra, $\alpha = 3$ and the last condition is of course superfluous. Composition series occur in the study of *liminal* and *postliminal* C^*-algebras, see [**Dix77**].

Finally, we can use the result about $\mathcal{W}(\Omega)$ to obtain a composition series for the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$:

THEOREM 3.5.7 ([**SSU89**, Theorem 4.4]). *The Toeplitz C^*-algebra $\mathcal{T}(\Omega)$ of a normalized, pseudoconvex Reinhardt domain $\Omega \subseteq \mathbb{C}^2$ has the composition series*

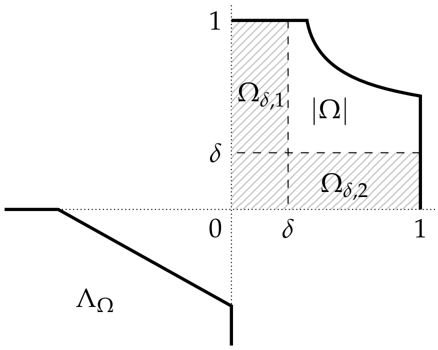
$$0 \subseteq \mathcal{J}_1 \cong \mathcal{K} \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_3 = \mathcal{T}(\Omega)$$

such that

$$\mathcal{J}_3/\mathcal{J}_2 \cong C(\partial_2\Omega) \quad \textit{and} \quad \mathcal{J}_2/\mathcal{J}_1 \cong \sum_F^{\oplus} C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)}),$$

where F ranges over all one-dimensional non-virtual faces of Λ and $\partial_2\Omega$ is the closure in $\mathbb{C}^2$ of the set of all pairs (se^x, te^y) such that $(s,t) \in \mathbb{T}^2$ and $\{(x,y)\}$ is a non-virtual singleton face of Λ. For explicit expressions of $C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)})$, see Theorem 3.5.5.

PROOF. We will only consider the case when no virtual faces are present, in which case we will find $\mathcal{W}(\Omega) \cong \mathcal{T}(\Omega)$. The full argument which includes the virtual case can be found in [**SSU89**]. By the absence of virtual faces, there exists $\delta \in (0,1)$ such that the set $\Omega_{\delta,i} := \{z \in \mathbb{D}^2 : |z_i| < \delta\}$ for $i = 1, 2$ is contained in Ω, where $\mathbb{D}^2$ denotes the unit polydisc.



We have $|z^{\alpha+e_i}|^2 \ge \delta^2|z^\alpha|^2$ for $z \in \Omega \setminus \Omega_{\delta,i}$ and

$$\int_{\Omega_{\delta,i}} |z^\beta|^2\, d\lambda(z) = (2\pi)^n \int_{|\Omega_{\delta,i}|} y^{2\beta_1+1} y^{2\beta_2+1}\, dy = \frac{\pi^2\delta^{2\beta_i+2}}{(\beta_1+1)(\beta_2+1)}$$

for all $\beta \in \mathbb{Z}^2_{\ge}$ by (D.1). This implies

$$\begin{aligned}
&\int_{\Omega_{\delta,i}} |z^{\alpha+e_i}|^2\, d\lambda(z) + \int_{\Omega\setminus\Omega_{\delta,i}} |z^{\alpha+e_i}|^2\, d\lambda(z) \ge \\
&\qquad \ge \frac{\delta^2(\alpha_1+1)(\alpha_2+1)}{((\alpha+e_i)_1+1)((\alpha+e_i)_2+1)}\left(\frac{\pi^2\delta^{2\alpha_i+2}}{(\alpha_1+1)(\alpha_2+1)} + \int_{\Omega_{\delta,i}} |z^\alpha|^2\, d\lambda(z)\right) \ge \\
&\qquad\qquad \ge \frac{\delta^2}{2}\int_\Omega |z^\alpha|^2\, d\lambda(z) = \frac{\delta^2}{2}\|z^\alpha\|^2
\end{aligned}$$

and hence $\chi_{e_i}(\alpha) = \|z^{\alpha+e_i}\|/\|z^\alpha\| \ge \delta/\sqrt{2} > 0$, so that the multiplication operator W_{e_i} has a bounded inverse. The same then holds true for W_β for all $\beta \in \mathbb{Z}^2_{\ge}$ by Remark 3.2.4. Now let $\widetilde{T}_\beta := U_\beta W_\beta$. By Remark 3.5.3, $\widetilde{T}_\beta$ corresponds to the Toeplitz operator with symbol z^β via the isomorphism $A^2(\Omega) \cong \ell^2(\mathbb{Z}^2_{\geq})$, which implies that $\widetilde{\mathfrak{T}} := C^*(\{\widetilde{T}_\beta : \beta \in \mathbb{Z}^2_{\geq}\})$ is isomorphic to the Toeplitz C^*-algebra $\mathfrak{T}(\Omega)$, see also Proposition 1.2.8. Because $U_\beta W_\beta$ is the polar decomposition of $\widetilde{T}_\beta$, we have $W_\beta = (\widetilde{T}_\beta^* \widetilde{T}_\beta)^{1/2}$ which is an element of $\widetilde{\mathfrak{T}}$ by the functional calculus, see Example C.3.7. But then also $U_\beta = \widetilde{T}_\beta W_\beta^{-1} \in \widetilde{\mathfrak{T}}$ and hence $\mathfrak{T}(\Omega) \cong \widetilde{\mathfrak{T}} \cong \mathcal{W}(\Omega)$. The claim now follows from Theorem 3.5.5 and by noting that $\partial_2^+ \Lambda \times \mathbb{T}^2 \cong \partial_2 \Omega$ via the map $((x,y),(s,t)) \mapsto (se^x, te^y)$. This settles the case when no virtual faces are present. ∎

We see that the Toeplitz C^*-algebra depends on the facial structure of the logarithmic base Λ. Moreover, this dependence is highly unstable, meaning that $\mathfrak{T}(\Omega)$ changes heavily when only slightly modifying the slopes of the faces. This is not only due to the contribution of each one-dimensional face changing depending on whether its slope is rational or irrational, but also to the fact that the irrational rotation algebra $\mathcal{A}_{u_2/u_1}$, which occurs in the expression for $C^*(\mathcal{G}_\Omega|_{\mathrm{Spec}_F^\circ(\Omega)})$ of a face with irrational slope, is almost never isomorphic to the rotation algebra for a different slope, see Example 2.6.6.

EXAMPLE 3.5.8 ([**SSU89**, Corollary 4.6]). Let Ω be a normalized pseudoconvex Reinhardt domain such that $\Lambda = \overline{\Lambda_\Omega}$ contains no one-dimensional faces. (Note that this implies the existence of both virtual faces.) From Theorem 3.5.7, we obtain $\mathfrak{J}_1 = \mathfrak{J}_2$, hence $\mathfrak{J}_2 \cong \mathscr{K}$ and

$$\mathfrak{T}(\Omega)/\mathscr{K} \cong C(\partial\Omega)$$

because $\partial_2\Omega = \partial\Omega$ in this case. Thus, $\mathfrak{T}(\Omega)$ has the same structure as for the domains from Theorem 1.4.16! Moreover, we see that the commutator ideal $\mathrm{Comm}(\mathfrak{T}(\Omega))$ is contained in $\mathscr{K}$, since otherwise the quotient could not be commutative. By Proposition 1.4.14, this implies that $S_1|_{A^2_{(0,1)}(\Omega)}$, the restriction of the canonical solution operator to $\bar{\partial}$ to the space of $(0,1)$-forms with holomorphic coefficients, is compact. We refer to section 1.4.1 for the definitions of S_1 and $\bar{\partial}$. ◊

REMARK 3.5.9. A description of the Toeplitz C^*-algebra in terms of a composition series as in Theorem 3.5.7 has also been given for domains other than the Reinhardt domains which were considered in this thesis. As an example, Upmeier in [**Upm84**] has studied the case where Ω is a bounded *symmetric domain* using *Jordan algebras*. The composition series for the Toeplitz C^*-algebra on the Hardy space there has a length depending on the *rank* of the symmetric domain. We also recommend looking at [**Upm96**] for a collection of similar results.

APPENDIX A

Locally compact Hausdorff spaces

A topological space X is called *locally compact* if every point $x \in X$ has a compact neighborhood, that is, there exists a compact set $V \subseteq X$ and an open set $U \subseteq X$ with $x \in U \subseteq K$. Although we do not require it in the definition of local compactness, we will mostly be interested in locally compact *Hausdorff* spaces. For a Hausdorff spaces, local compactness is equivalent to every point having a neighborhood base of relatively compact (equivalently: compact) subsets. Given $x \in X$ and a neighborhood U of x, there is even a relatively compact neighborhood V of x with $\overline{V} \subseteq U$.

A.1. Locally closed subsets

A subset $Y \subset X$ of a topological space X is called *locally closed* (in X) if every point of Y has an open neighborhood U in X such that $U \cap Y$ is closed in U.

LEMMA A.1.1. *Let X be a topological space. A subset $Y \subseteq X$ is locally closed if and only if there are $C, O \subseteq X$ with C closed and O open, such that $Y = C \cap O$.*

PROOF. If $Y = C \cap O$, then we can take $U := O$. It follows that U is an open neighborhood of every point of Y and $U \cap Y = C \cap U$ is closed in U. Conversely, if Y is locally closed in X, let $U_y \subseteq X$ be a neighborhood of $y \in Y$ such that $U_y \cap Y$ is closed in U_y. Set $C := \overline{Y}$ and $O := \bigcup_{y \in Y} U_y \supseteq Y$. Then

$$C \cap O = \overline{Y} \cap \bigcup_{y \in Y} U_y = \bigcup_{y \in Y} (\overline{Y} \cap U_y) = \bigcup_{y \in Y} (Y \cap U_y) = Y \cap O = Y. \qquad \blacksquare$$

LEMMA A.1.2. *Let X be a topological space and $Y \subseteq X$.*

(i) If X is Hausdorff and if Y is locally compact, then Y is locally closed in X.
(ii) If X is locally compact and if Y is locally closed, then Y is locally compact.

In particular, in locally compact Hausdorff spaces, the locally closed subsets are precisely the locally compact subspaces.

PROOF. Let X be Hausdorff and $Y \subseteq X$ locally compact. For $y \in Y$, let $U \subseteq X$ be an open neighborhood of y such that $U \cap Y$ is relatively compact in Y and denote its closure in Y by K. It follows that K is also compact in X and hence closed since X is Hausdorff. If x_ι is a net in $U \cap Y$ with $x_\iota \to x$ in U and $x \in U$, then also $x \in Y$ because $U \cap Y \subseteq K \subseteq Y$ and K is closed. Therefore, $U \cap Y$ is closed in U and Y is locally closed.

For the second assertion , let X be locally compact and $Y \subseteq X$ locally closed. Then $Y = C \cap O$ for some $C, O \subseteq X$ with C closed and O open by Lemma A.1.1. Fix $y \in Y$ and let $U \subseteq X$ be a relatively compact neighborhood of y in X with $\overline{U} \subseteq O$. Then $U \cap C = U \cap Y$ is an open neighborhood of y in Y and $\mathrm{cl}_Y(U \cap C) = \mathrm{cl}_Y(U \cap Y) = \mathrm{cl}_X(U) \cap Y = \mathrm{cl}_X(U) \cap C$ is compact, where $\mathrm{cl}_X(\cdot)$ and $\mathrm{cl}_Y(\cdot)$ denote the closure operations in X and Y, respectively. $\blacksquare$

EXAMPLE A.1.3. If Y is open or closed in the topological space X, then Y is also locally closed in X. Consequently, all closed (and all open) subspaces of a locally compact Hausdorff space are themselves locally compact Hausdorff spaces. $\Diamond$

A.2. Separation properties and partitions of unity

In the following, we will show that points in a locally compact Hausdorff space may be separated from closed subsets by functions. More precisely, we are dealing with completely regular spaces:

DEFINITION A.2.1. A topological space X is called *completely regular* if one-pointed sets are closed and if, for any point $x_0 \in X$ and any closed set $F \subseteq X$ with $x_0 \notin F$, there exists a continuous function $f\colon X \to [0,1]$ such that $f(x_0) = 1$ and $f|_F = 0$.

Thus, in a completely regular space, we can separate points from closed sets by continuous functions. In particular, such spaces are *regular*, meaning that points can be separated by closed sets by disjoint open sets. In a *compact* Hausdorff space, even more is true:

THEOREM A.2.2. *Every compact Hausdorff space is* normal, *that is, singletons are closed and any two disjoint closed sets can be separated by disjoint open sets.*

For a proof, see for instance [**Mun00**, Theorem 32.3]. Urysohn's lemma states that a topological space X is normal if and only if for every closed subset $F \subseteq X$ and every open subset $U \subseteq X$ with $F \subseteq U$, there exists a continuous function $f\colon X \to [0,1]$ with $f|_F = 1$ and $f|_{X\setminus U} = 0$. Equivalently, any two disjoint closed subsets may be separated *by a function*. We next generalize this result to the locally compact case. For a locally compact Hausdorff space, we denote by $C_c(X)$ the space of continuous complex-valued functions on X with compact support.

LEMMA A.2.3 (Urysohn's Lemma for locally compact spaces). *Let X be a locally compact Hausdorff space, $K \subseteq X$ compact and $U \subseteq X$ open with $K \subseteq U$. Then there exists $f \in C_c(X,[0,1])$ such that $f|_K = 1$ and $f|_{X\setminus U} = 0$. In particular, every locally compact Hausdorff space is completely regular.*

PROOF. Since X is Hausdorff, singletons are closed. By local compactness, we find a relatively compact open neighborhood V of K such that $\overline{V} \subseteq U$. Since $\overline{V}$ is both Hausdorff and compact, there is, by Theorem A.2.2, a continuous function $g\colon \overline{V} \to [0,1]$ with $g|_K = 1$ and $g|_{\overline{V}\setminus V} = 0$. Now define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} g(x) & \text{if } x \in \overline{V} \\ 0 & \text{if } x \in X \setminus V. \end{cases}$$

It follows that f is well-defined and continuous. Moreover, $f|_K = g|_K = 1$ and $f|_{X\setminus U} = 0$, as required. ∎

COROLLARY A.2.4. *Every locally compact Hausdorff space is completely regular.*

PROOF. If $x_0 \in X$ and $F \subseteq X$ is closed with $x_0 \notin F$, apply Lemma A.2.3 to $K = \{x_0\}$ and $U = X \setminus \{x_0\}$. ∎

We note that, in general, we cannot expect to be able to separate two arbitrary closed sets in the non-compact case: there are spaces that are both locally compact and Hausdorff, but fail to be normal, see [**SS95**].

COROLLARY A.2.5. *Let X be a locally compact Hausdorff space and assume $Y \subseteq X$ is not dense in X, that is, $\overline{Y} \neq X$. Then there is $f \in C_c(X)$ with $f|_Y = 0$ but $f \neq 0$.*

PROOF. Pick $x_0 \in X \setminus \overline{Y}$ and let $U \subseteq X$ be an open neighborhood of x_0 such that $\overline{U}$ is compact and $\overline{U} \subseteq X \setminus \overline{Y}$. By Corollary A.2.4, there is a continuous function $f \in C(X,[0,1])$ such that $f(x_0) = 1$ and $f|_{X\setminus U} = 0$. Since $Y \subseteq X \setminus \overline{U} \subseteq X \setminus U$, we have $f|_Y = 0$. Moreover, $\operatorname{supp}(f) \subseteq \overline{U}$ shows that f has compact support. ∎

We now turn to the existence of partitions of unity. A family $\mathcal{A}$ of subsets of a topological space X is called *locally finite* if every $x \in X$ has a neighborhood which only intersects finitely many elements of $\mathcal{A}$. Recall that a topological space X is called *paracompact* if every open cover $\mathcal{U}$ of X has a locally finite open refinement. This means that there is an open cover $\mathcal{V}$ which is locally finite and such that for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subseteq U$.

For a paracompact Hausdorff space X, there always exist *partitions of unity* subordinate to any given indexed open cover $\{U_i\}_{i\in I}$. By this we mean a family $\varphi_i \colon X \to [0,1]$, $i \in I$, of continuous functions such that

(i) $\operatorname{supp}(\varphi_i) \subseteq U_i$ for all $i \in I$,
(ii) $\{\operatorname{supp}(\varphi_i)\}_{i\in I}$ is locally finite, and
(iii) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in X$.

In Proposition *A*.3.3 below we will see that second countable, locally compact Hausdorff spaces are paracompact, hence partitions of unity always exist.

However, a general locally compact Hausdorff space need not be paracompact. As an example, we mention the ordinal space $[0,\omega_1)$, with ω_1 the first uncountable ordinal and equipped with the order topology, see [**SS95**, Counterexample 42]. If one restricts to *finite* coverings of a *compact* subset with *relatively compact* open sets, then local compactness is enough to guarantee existence of partitions of unity. The proof of this result can be found in [**Wil07**, Lemma 1.43].

PROPOSITION A.2.6. *Let X be a (not necessarily Hausdorff) locally compact topological space and $K \subseteq X$ a compact subspace. If $\{U_i\}_{i=1}^n$ is a finite cover of K with $\overline{U}_i$ compact for all $i \in \{1,\dots,n\}$, then there exist functions $\varphi_i \in C_c(X,[0,1])$ such that*

(i) $\operatorname{supp}(\varphi_i) \subseteq U_i$,
(ii) $\sum_{i=1}^n \varphi_i(x) = 1$ *for all* $x \in K$, *and*
(iii) $\sum_{i=1}^n \varphi_i(x) \le 1$ *for all* $x \notin K$.

A.3. Second countability and paracompactness

If X is a second countable, locally compact Hausdorff space, then X has a countable base of relatively compact open sets. Indeed, pick a countable base $\mathcal{V}$ of X and let $U \subseteq X$ be open and $x \in U$. Then there exists a compact neighborhood K of x such that $K \subseteq U$ and since $\mathcal{V}$ is a base, there is $V \in \mathcal{V}$ with $x \in V \subseteq K \subseteq U$. It follows that $\overline{V} \subseteq K$ is compact, so that $\{V \in \mathcal{V} : V \text{ is relatively compact}\}$ is a countable base of relatively compact open subsets.

Every second countable, locally compact Hausdorff space X is *σ-compact*. By this we mean that there is a countable family of compact subspaces which cover X. In fact, slightly more is true:

LEMMA A.3.1. *Let X be a second countable, locally compact Hausdorff space. Then there exists a sequence $(K_i)_{i\ge 1}$ of compact subsets $K_i \subseteq X$ such that $X = \bigcup_{i=1}^\infty K_i$ and $K_i \subseteq (K_{i+1})^\circ$. In particular, X is σ-compact.*

PROOF. By our assumptions, X has a countable base $\{U_i\}_{i=1}^\infty$ of relatively compact open subsets. Set $K_1 := \overline{U_1}$ and proceed by induction, assuming that $K_1,\dots,K_n$ have already been constructed such that $U_i \subseteq K_i$ for $i = 1,\dots,n$. Let $i_n \ge n+1$ be such that $K_n \subseteq U_1 \cup \cdots \cup U_{i_n}$ and put $K_{n+1} := \overline{U_1} \cup \cdots \cup \overline{U_{i_n}}$. Then K_{n+1} is compact and satisfies $(K_{n+1})^\circ \supseteq U_1 \cup \cdots \cup U_{i_n} \supseteq K_n$. Because also $U_{n+1} \subseteq K_{n+1}$, the constructed sequence $(K_i)_{i=1}^\infty$ covers X. ∎

DEFINITION A.3.2. A sequence $(K_i)_{i=1}^\infty$ of compact subsets of X as in Lemma *A*.3.1 is called an *exhaustion of X by compact subsets*.

PROPOSITION A.3.3. *Every second countable, locally compact Hausdorff space is paracompact.*

PROOF. Let $\mathcal{U}$ be an open cover of X and let $(K_i)_{i\geq 1}$ be an exhaustion of X by compact subsets, see Lemma A.3.1. Set $A_j := K_{j+1} \setminus (K_j)^\circ$ and $W_j := (K_{j+2})^\circ \setminus K_{j-1}$ for $j \geq 0$, where $K_0 := K_{-1} := \emptyset$. As a closed subset of K_{j+1}, we have that A_j is compact and contained in the open set W_j. For $x \in A_j$, let $U_x \in \mathcal{U}$ be such that $x \in U_x$. Put $V_x := U_x \cap W_j$. Then $\{V_x : x \in A_j\}$ is an open cover of A_j and hence has a finite subcover $\mathcal{V}_j$ of A_j. It follows that $\mathcal{V} := \bigcup_{j\geq 0} \mathcal{V}_j$ is an open cover of X which refines $\mathcal{U}$ by construction. Moreover, $\mathcal{V}$ is locally finite which follows from the fact that $W_j \cap W_k = \emptyset$ if $k \notin \{j-2, j-1, j, j+1, j+2\}$. ∎

COROLLARY A.3.4. *Every second countable, locally compact Hausdorff space is normal.*

PROOF. In fact, this is true for all paracompact Hausdorff spaces X: if $F \subseteq U \subseteq X$ are such that F is closed and U is open, then choosing a partition of unity subordinate to the open covering $\{U, X \setminus F\}$ gives a continuous function $\varphi\colon X \to [0,1]$ with $\varphi|_F = 1$ and $\varphi|_{X\setminus U} = 0$. ∎

COROLLARY A.3.5. *Let X be a second countable, locally compact Hausdorff space and $F \subseteq X$ a closed subset. If $f \in C_c(F)$, then there exists a function $g \in C_c(X)$ such that $g|_F = f$.*

PROOF. Because X is normal by Corollary A.3.4, we can apply the Tietze extension theorem to obtain a function $\widetilde{g} \in C(X)$ such that $\widetilde{g}|_F = f|_F$. Now let $U \subseteq X$ be a relatively compact neighborhood of $\operatorname{supp}(f)$ and let $\varphi \in C_c(X, [0,1])$ be such that $\varphi|_{\operatorname{supp}(f)} = 1$ and $\varphi|_{X\setminus U} = 0$, see Lemma A.2.3. Then $g := \varphi\,\widetilde{g}$ has compact support and extends f. ∎

REMARK A.3.6. Corollary A.3.5 remains true after dropping the assumption of second countability, see for instance [**Goe09**, Lemma 1.71]. We will be content with the above simpler version.

COROLLARY A.3.7. *Let X be a second countable, locally compact Hausdorff space and $Y \subseteq X$ a locally closed subset. If $f \in C_c(Y)$, then there exists a function $g \in C_c(X)$ such that $g|_Y = f$.*

PROOF. By Lemma A.1.1, Y is a closed subspace of an open subset O of X. Corollary A.3.5 furnishes an extension $g \in C_c(O)$ of f which we can extend by zero outside O to a function in $C_c(X)$. ∎

Finally, we mention that second countable, locally compact Hausdorff spaces are metrizable. In fact, this continuous to hold for second countable regular spaces. A proof of this result can be found in [**Mun00**, Theorem 34.1].

THEOREM A.3.8 (Urysohn metrization theorem). *Every second countable regular space is metrizable.*

A.4. Radon measures

Fix a locally compact Hausdorff space X and consider the space $C_c(X)$ of continuous complex-valued functions on X. A linear functional $\varphi\colon C_c(X) \to \mathbb{C}$ is called *positive* if $\varphi(f) \geq 0$ for all functions $f \geq 0$. Let μ be a locally finite Borel measure on X, meaning that every point of X has a neighborhood of finite measure. Because X is assumed to be locally compact, this is equivalent to μ being finite on compact subsets. It follows that $f \mapsto \int_X f\,d\mu$ defines a positive linear functional on $C_c(X)$. The content of the Riesz representation theorem is that all such functionals arise in this way, and the measures may be chosen to have certain regularity properties, that is, they are Radon measures.

REMARK A.4.1. One can show that every positive linear functional on $C_c(X)$ is already continuous with respect to the inductive limit topology from Proposition B.1, see [**Fol99**, Proposition 7.1].

DEFINITION A.4.2. A locally finite Borel measure μ on X is called a *Radon measure* if μ is outer regular on Borel sets and inner regular on open sets. More precisely,

$$\begin{aligned} \mu(A) &= \inf\{\mu(U) : U \text{ open with } E \subseteq U\} && \text{for all Borel } A \subseteq X, \text{ and} \\ \mu(U) &= \sup\{\mu(K) : K \text{ compact with } K \subseteq U\} && \text{for all open } U \subseteq X. \end{aligned}$$

The set of Radon measures on X will be denoted by $\mathcal{M}_+(X)$.

THEOREM A.4.3 (Riesz Representation Theorem for $C_c(X)$). *For any positive linear functional $\varphi\colon C_c(X) \to \mathbb{C}$, there exists a uniquely determined Radon measure μ on X such that*

$$\varphi(f) = \int_X f\,d\mu \quad \text{for all } f \in C_c(X).$$

Moreover, this measure satisfies

$$\begin{aligned} \mu(U) &= \sup\{\varphi(f) : f \in C_c(X,[0,1]) \text{ with } \operatorname{supp}(f) \subseteq U\}, \quad \text{and} \\ \mu(K) &= \inf\{\varphi(f) : f \in C_c(X),\ f \geq \chi_K\} \end{aligned} \tag{A.4.1}$$

for all open subsets $U \subseteq X$ and all compact $K \subseteq X$.

The measure μ from Theorem A.4.3 is constructed by first defining $\mu(U)$ for $U \subseteq X$ open by (A.4.1) and

$$\mu^*(E) := \inf\{\mu(U)\colon U \supseteq E,\ U \text{ open}\}$$

for every subset $E \subseteq X$. One then shows that μ^* is an outer measure and every open set is μ^* measurable, so that $\mu := \mu^*|_{\mathcal{B}(X)}$ is a Borel measure. We refer to [**Fol99**, Theorem 7.2] for the details. Note that for two Radon measures to be equal it is sufficient that the integrals of functions in $C_c(X)$ agree.

REMARK A.4.4. There are different regularity requirements for Radon measures in the literature. For example, [**Els05**] requires them to be locally finite and inner regular on all measurable sets. The point is that the Riesz Representation Theorem allows one to choose regularity of the measures to a certain extent while still maintaining uniqueness, see also the discussion in [**Bau01**, §29]. Moreover, any Radon measure μ is actually inner regular on all of its σ-finite sets, see [**Fol99**, Proposition 7.5], so if X is σ-compact, then μ is both inner and outer regular on all Borel sets. In particular, this is the case if X is second countable by Lemma A.3.1.

REMARK A.4.5. The *support* $\operatorname{supp}(\mu)$ of a Radon measure μ on X is the set of all $x \in X$ such that $\mu(U) > 0$ for all open neighborhoods U of x. It is easy to verify that $x \in \operatorname{supp}(\mu)$ if and only if $\int_X f\,d\mu > 0$ for all $f \in C_c(X,[0,1])$ with $f(x) > 0$.

If X is second countable, then $\mu(X \setminus \operatorname{supp}(\mu)) = 0$. Indeed, for $x \notin \operatorname{supp}(\mu)$ there is an open neighborhood U_x of x with $\mu(U_x) = 0$. The sets U_x cover $X \setminus \operatorname{supp}(\mu)$ and since X is second countable it is Lindelöf, so there is a countable subcover $\{U_{x_i}\}_{i\geq 1}$, which implies $\mu(X \setminus \operatorname{supp}(\mu)) \leq \sum_{i\geq 1} \mu(U_{x_i}) = 0$ by σ-subadditivity. In particular, we have

$$\int_X f\,d\mu = \int_{\operatorname{supp}(\mu)} f\,d\mu$$

for all positive Borel or integrable $f\colon X \to \mathbb{C}$.

The next result states that local finiteness is already sufficient for a Borel measure to be Radon, provided the space is nice enough. In particular, this will be true if X is second countable. For a proof, see [**Fol99**, Theorem 7.8].

PROPOSITION A.4.6. *Let X be a locally compact Hausdorff space such that every open subset of X is σ-compact. Then every locally finite Borel measure is regular, hence also Radon.*

EXAMPLE A.4.7 (Restriction of Radon measures). Let X be a second countable, locally compact Hausdorff space. One can restrict a Radon measure μ to a locally closed subset $Y \subseteq X$: by Lemma A.1.2, Y is itself locally compact and

$$\mu|_Y(A) := \mu(A)$$

for every Borel measurable $A \subseteq Y$ defines a measure on Y. Note that this is a Radon measure by Proposition A.4.6 as it is finite on compacta. The Lebesgue integral with respect to $\mu|_Y$ is given by

$$\int_Y f \, d\mu|_Y = \int_X \tilde{f} \, d\mu,$$

for all nonnegative measurable or integrable f, where $\tilde{f}$ is the extension of f to X by zero.

We have $\operatorname{supp}(\mu|_Y) \subseteq \operatorname{supp}(\mu) \cap Y$. Indeed, if $x \in \operatorname{supp}(\mu|_Y)$, then obviously $x \in Y$ and if $U \subseteq X$ is an open neighborhood of x, then $\mu(U) \geq \mu(U \cap Y) = \mu|_Y(U \cap Y) > 0$ as $U \cap Y$ is an open neighborhood of x in Y. We also point out that the converse inclusion does not hold in general: let $X = \mathbb{R}$, $Y = \{0\}$ and μ be Lebesgue measure on $\mathbb{R}$. Then $\mu|_Y = 0$ and hence $\operatorname{supp}(\mu|_Y) = \emptyset$, but $\operatorname{supp}(\mu) \cap Y = \{0\} \neq \emptyset$.

If, however, Y is open in X, then $\operatorname{supp}(\mu|_Y) = \operatorname{supp}(\mu) \cap Y$ holds. Indeed, if $x \in \operatorname{supp}(\mu) \cap Y$ and $U \subseteq Y$ is an open neighborhood of x, then U is also open in X and $\mu|_Y(U) = \mu(U) > 0$, so $x \in \operatorname{supp}(\mu|_Y)$.

Finally, if $\operatorname{supp}(\mu) \subseteq Y$, then $\operatorname{supp}(\mu|_Y) = \operatorname{supp}(\mu)$. For this it remains to show that $\operatorname{supp}(\mu)$ is contained in $\operatorname{supp}(\mu|_Y)$. Take $x \in \operatorname{supp}(\mu)$ and let $U \subseteq Y$ be an open neighborhood of x. Then there is an open neighborhood V of x in X such that $V \cap Y = U$. It follows that $\mu|_Y(U) = \mu(U) = \mu(V \cap Y) = \mu(V) > 0$ because $\mu(X \setminus \operatorname{supp}(\mu)) = 0$ by Remark A.4.5 and hence $x \in \operatorname{supp}(\mu|_Y)$. In this case, we have

$$\int_Y f \, d\mu|_Y = \int_Y f \, d\mu$$

for all nonnegative measurable or integrable f. $\diamondsuit$

EXAMPLE A.4.8 (Products of Radon measures). If μ and ν are two σ-finite Borel measures on topological spaces X and Y, respectively, then general measure theory allows the construction of a unique measure $\mu \times \nu$ on the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ such that $\mu \times \nu(A \times B) = \mu(A)\mu(B)$ for all measurable $A \subseteq X$ and $B \subseteq Y$. If X and Y are second countable, locally compact Hausdorff spaces and μ, ν are Radon measures, then this product is again a Radon measure (note that the measures are then σ-finite, because X and Y are σ-compact by Lemma A.3.1). To see this, note that second countability implies $\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$, so that $\mu \times \nu$ is a Borel measure. By Proposition A.4.6, it suffices to show that $\mu \times \nu$ is locally finite. But this is trivial, for if $K \subseteq X \times Y$ is compact and $\operatorname{pr}_X \colon X \times Y \to X$ and $\operatorname{pr}_Y \colon X \times Y \to Y$ denote the continuous projections, then $K \subseteq \operatorname{pr}_X(K) \times \operatorname{pr}_Y(K)$, so $\mu \times \nu(K) \leq \mu(\operatorname{pr}_X(K))\nu(\operatorname{pr}_Y(K)) < \infty$ as both $\operatorname{pr}_X(K)$ and $\operatorname{pr}_Y(K)$ are compact. $\diamondsuit$

REMARK A.4.9. There is also a way to construct products of Radon measures if second countability is not available. This product is obtained from $f \mapsto \int_X \int_Y f(x,y) \, d\mu(x) \, d\nu(y)$, which can be shown to be a positive linear functional on $C_c(X \times Y)$. We will not need this construction as we will always assume second countability in our applications and refer to [**Fol99**, 7.4].

EXAMPLE A.4.10 (Disjoint unions of Radon measures). Let X_i, $i \in I$, be a countable family of second countable, locally compact Hausdorff spaces. Then the disjoint union $X := \bigsqcup_{i \in I} X_i$ is again second countable, locally compact and Hausdorff when equipped with the disjoint union topology. This means that $U \subseteq X$ is open if and only if $U \cap X_i \subseteq X_i$ is open in X_i. Now suppose we are given Radon measures μ_i on X_i for all $i \in I$. Then

$$\mu(A) := \sum_{i \in I} \mu_i(A \cap X_i)$$

is a Borel measure on X which will be denoted by $\sqcup_{i\in I}\mu_i$. If $K \subseteq X$ is compact, then $K \cap X_i$ for only finitely many $i \in I$, so that $\mu(K) < \infty$. By Proposition A.4.6, μ is therefore a Radon measure. The support of μ is $\bigsqcup_{i\in I}\operatorname{supp}(\mu_i) \subseteq X$ and the integral with respect to μ satisfies

$$\int_X f\, d\mu = \sum_{i\in I}\int_{X_i} f|_{X_i}\, d\mu_i$$

for all nonnegative measurable or integrable $f\colon X \to \mathbb{C}$. ◊

EXAMPLE A.4.11 (Image measures). If (X, Σ_X) and (Y, Σ_Y) are measurable spaces, μ a measure on (X, Σ_X) and $f\colon X \to Y$ is measurable, then the *image measure* of μ under f is the measure $f_*\mu$ on Y with $f_*\mu(A) := \mu(f^{-1}(A))$ for all $A \in \Sigma_Y$. The Lebesgue integrals then satisfy $\int_Y g\, d(f_*\mu) = \int_X g \circ f\, d\mu$ for all positive extended real-valued measurable or $f_*\mu$-integrable g.

If X and Y are second countable, locally compact Hausdorff spaces, μ a Radon measure on X, and if f is continuous and proper (meaning preimages of compact sets under f are compact), then $f_*\mu$ is Radon. Indeed, $f_*\mu$ is a Borel measure, and $f_*\mu(K) = \mu(f^{-1}(K)) < \infty$ by properness of f and Proposition A.4.6 then implies that $f_*\mu$ is a Radon measure. Again, there are ways to construct Radon image measures without assuming second countability. We refer to [**Bou04**, V, §6, 1].

Now let μ be a Borel measure, not necessarily Radon. If $f\colon X \to Y$ is continuous, then $f(\operatorname{supp}(\mu)) \subseteq \operatorname{supp}(f_*\mu)$. Indeed, take $x \in \operatorname{supp}(\mu)$ and let $U \subseteq Y$ be any open neighborhood of $f(x)$. Since f is continuous, there is an open neighborhood $V \subseteq X$ of x such that $V \subseteq f^{-1}(U)$. This implies $f_*\mu(U) = \mu(f^{-1}(U)) \geq \mu(V) > 0$, so $f(x) \in \operatorname{supp}(f_*\mu)$. ◊

PROPOSITION A.4.12. *For a Radon measure μ on a locally compact Hausdorff space X, the space $C_c(X)$ is dense in $L^p(X,\mu)$ for all $p \in [1,\infty)$.*

PROOF. It is sufficient to approximate characteristic functions χ_A for Borel sets A with $\mu(A) < \infty$, since those are total in $L^p(X,\mu)$. We have remarked in Remark A.4.4 that μ is inner regular on σ-finite sets, so given $\varepsilon > 0$, there is $K \subseteq A$ compact and $U \supseteq A$ open with $\mu(U \setminus K) < \varepsilon$. Now use Lemma A.2.3 to find $\varphi \in C_c(X,[0,1])$ with $\varphi|_K = 1$ and $\varphi|_{X\setminus U} = 0$. Then $\|\chi_A - \varphi\|_{L^p(X,\mu)} \leq \mu(U \setminus K)^{1/p} < \varepsilon^{1/p}$, which finishes the proof. ■

LEMMA A.4.13. *Let X be a second countable, locally compact Hausdorff space. Then there exists a Radon measure on X with support equal to X.*

PROOF. Let $\{U_k\}_{k=1}^\infty$ be a countable basis for the topology of X. For every $k \geq 1$, there exists $f_k \in C_c(X,[0,1])$ such that $f_k \neq 0$ and $\operatorname{supp}(f_k) \subseteq U_k$. We may view f_k as a normal element of the C^*-algebra $C_0(X)$, and for such elements there always exist positive linear functionals φ_k on $C_0(X)$ with $\|\varphi_k\| = 1$ and such that $\|f_k\|_\infty = |\varphi_k(f_k)|$, see [**Mur90**, Theorem 3.3.6]. By Theorem A.4.3, there are Radon measures μ_k on X with $\varphi_k(g) = \int_X g\, d\mu_k$ for all $g \in C_c(X)$. Now set $\varphi(g) := \sum_{k=1}^\infty k^{-2}\varphi_k(g)$. Then φ is a positive linear functional on $C_c(X)$ and hence corresponds to a Radon measure μ. If $U \subseteq X$ is open, then there is $k_0 \geq 1$ with $U_{k_0} \subseteq U$ and hence

$$\mu(U) \geq \mu(U_{k_0}) \geq \varphi(f_{k_0}) \geq k_0^{-2}\varphi_{k_0}(f_{k_0}) > 0,$$

where we have used (A.4.1). It follows that $\operatorname{supp}(\mu) = X$ and we are done. ■

Recall that two measures (on the same measurable space) are called *equivalent* if they are mutually absolutely continuous, meaning they have the same null sets.

LEMMA A.4.14. *Let X be a σ-compact, locally compact Hausdorff space. Then every σ-finite Borel measure on X is equivalent to a finite Radon measure.*

PROOF. Let μ be a σ-finite Borel measure on X. We may assume that $\mu \neq 0$. Since μ is σ-finite, there is a sequence $(A_n)_{n\geq 1}$ of measurable subsets which cover X and such that $\mu(A_n) < \infty$ for $n \geq 1$. Without loss of generality, we may assume $A_n \subseteq A_{n+1}$ for all n. Put $a_n := 2^{-n}\mu(A_n)^{-1}$ if $\mu(A_n) > 1$ and $a_n := 2^{-n}$ otherwise. Define $\nu(A) := \sum_{n\geq 1} a_n \mu(A \cap A_n)$ for every Borel set $A \subseteq X$. Then ν is a Borel measure on X with $\nu(X) \leq 1$. If $\nu(A) = 0$, then $\mu(A \cap A_n) = 0$ for all $n \geq 1$ and by the continuity of measures from below, we get $\mu(A) = \lim_{n\to\infty} \mu(A \cap A_n) = 0$. Therefore, ν is equivalent to μ. By Proposition A.4.6 we see that ν is Radon. ■

REMARK A.4.15. Note that the essence of Lemma A.4.14 of course remains valid for arbitrary σ-finite measures, without assuming X to have any topology whatsoever.

Notes for Appendix A

The basics of locally compact (Hausdorff) spaces can be found in virtually any textbook on general topology, for example [**Mun00**]. Lemmas A.1.1 and A.1.2 are found in [**Bre93**, I.11.7] and [**Wil07**, Lemma 1.25], respectively. The proof of Urysohn's Lemma for locally compact spaces (Lemma A.2.3) stems from [**Wil07**, Lemma 1.41]. Lemma A.3.1 and Proposition A.3.3 have been taken from [**Lee10**], where they are Proposition 4.76 and Theorem 4.77. For the section on Radon measures, we have used [**Fol99**, Chapter 7] and [**Bau01**] as general references. Lemma A.4.13 was prompted by the need for faithful representations of the reduced groupoid C^*-algebra of a locally compact groupoid, see Propositions 2.6.12 and 2.6.14.

APPENDIX B

The inductive limit topology on $C_c(X, B)$

Let X be a locally compact Hausdorff space and $(B, \|\cdot\|_B)$ a Banach space. In this thesis, we frequently make use of the vector space $C_c(X, B)$ of continuous functions $f\colon X \to B$ with compact support $\operatorname{supp}(f)$. Our goal here is to equip this space with a suitable topology and to establish conditions which enable us to check continuity of maps defined on $C_c(X, B)$. As usual, we write $C_c(X)$ for $C_c(X, \mathbb{C})$.

For a compact subset $K \subseteq X$, let $C_K(X, B)$ denote the continuous functions $X \to B$ with compact support contained in K. Together with the supremum norm $\|f\|_K := \sup_{x \in K} \|f(x)\|_B$, the space $C_K(X, B)$ is a Banach space. Evidently, $C_K(X, B)$ is a subspace of $C_c(X, B)$ and we denote the inclusions by $\iota_K\colon C_K(X, B) \to C_c(X, B)$.

PROPOSITION B.1. *Let X be a locally compact Hausdorff space and B a Banach space. There exists a unique finest topology on $C_c(X, B)$ such that*

(i) $C_c(X, B)$ is a locally convex vector space with this topology, and
(ii) the inclusions $\iota_K\colon C_K(X, B) \to C_c(X, B)$ are continuous.

Moreover, this topology has the following universal property: for any other locally convex vector space F and any linear map $L\colon C_c(X, B) \to F$, we have that L is continuous if and only if $L \circ \iota_K\colon C_K(X, B) \to F$ is continuous for every compact subset $K \subseteq X$.

The proof can be found in [**FD88**, p. 138] (in the more general context of Banach bundles) or taken as a special case of final structures with respect to families of locally convex vector spaces, see [**Kri14**, Satz 3.3.2]. The locally convex structure is given by the seminorms p on $C_c(X, B)$ such that $p \circ \iota_K$ is a continuous seminorm on $C_K(X, B)$.

DEFINITION B.2. The topology on $C_c(X, B)$ from Proposition B.1 is called the *inductive limit topology*.

REMARK B.3. In the literature, the phrase "$f_i \to f$ in the inductive limit topology" for a net f_i in $C_c(X, B)$ is sometimes taken to mean that

(i) $f_i \to f$ uniformly on X and
(ii) there is a compact subset $K \subseteq X$ such that, eventually, $\operatorname{supp}(f_i) \subseteq K$.

Note that nets satisfying the above properties converge in the inductive limit topology, since they converge in $C_K(X, B)$ and the inclusion ι_K is continuous. The problem is that not *all* convergent nets in $C_c(X, B)$ have this property [**Wil07**, Remark 1.86]. However, by the universal property of the inductive limit topology, such nets are sufficient in order to describe continuity of maps $f\colon C_c(X, B) \to F$ for locally convex vector spaces F. Even more, since the $C_K(X, B)$ are Banach spaces, it suffices to consider *sequences* with these properties.

LEMMA B.4. *Let X, Y be locally compact Hausdorff spaces. Then*

$$\operatorname{span}(\{f \otimes g : f \in C_c(X), g \in C_c(Y)\}),$$

where $(f \otimes g)(x, y) := f(x)g(y)$, is dense in $C_c(X \times Y)$ for the inductive limit topology.

PROOF. Let $h \in C_c(X \times Y)$ and set $K := \operatorname{supp}(h)$. Fix a relatively compact neighborhood $W \subseteq X \times Y$ of K. For every $(x, y) \in K$ and $\varepsilon > 0$ there are, by continuity of h, relatively compact neighborhoods $U_x \subseteq X$ and $V_y \subseteq Y$ of x and y, respectively, such that $|f(x, y) - f(z, w)| < \varepsilon$ for all $(z, w) \in U_x \times V_y$. If necessary, we shrink these neighborhoods so that $U_x \times V_x \subseteq W$. Because K is compact, there are finitely many $x_1, \dots, x_N \in X$ and $y_1, \dots, y_N \in Y$ with $K \subseteq \bigcup_{i=1}^N U_{x_i} \times V_{y_i}$. Let $(\varphi_i)_{i=1}^N$ and $(\psi_i)_{i=1}^N$ with $\varphi_i \in C_c(X, [0, 1])$ and $\psi_i \in C_c(Y, [0, 1])$ be partitions of unity subordinate to $\{U_{x_i}\}_{i=1}^N$ and $\{V_{y_i}\}_{i=1}^N$, respectively, see Proposition A.2.6. Put

$$h_\varepsilon := \sum_{i=1}^N h(x_i, y_i)\, \varphi_i \otimes \psi_i.$$

Then $g \in C_c(X \times Y)$ and

$$|h(x, y) - h_\varepsilon(x, y)| \leq \sum_{i=1}^N \varphi_i(x)\psi_i(y)|h(x, y) - h(x_i, y_i)| \leq \sum_{i=1}^N \varphi_i(x)\psi_i(y)\varepsilon \leq \varepsilon \sum_{i=1}^N \varphi_i(x) \leq \varepsilon$$

because $\operatorname{supp}(\varphi_i \otimes \psi_i) \subseteq U_{x_i} \times V_{y_i}$. Moreover, $\operatorname{supp}(h_\varepsilon) \subseteq \overline{W}$, which implies $h_\varepsilon \to h$ in the inductive limit topology as $\varepsilon \to 0$. ∎

APPENDIX C

C^*-algebras

In this chapter we give the needed background on some of the basics of the theory of C^*-algebras. The content of this appendix is as follows: in the first section, we will state the basic definitions and look at the most important examples, as well as elementary constructions, such as the direct sum of C^*-algebras. After that, we will briefly look at the spectrum of elements and then move on to the topic of commutative C^*-algebras. There are also sections dealing with ideals and representations, and the final section will be a short introduction on the tensor products of C^*-algebras.

If not stated otherwise, all our results are taken from [**Mur90**]. We will only be able to cover a small selection of important results and most of them will be stated without proof. The interested reader is referred to the literature for an in-depth treatment, for example [**Dix77**], [**Mur90**] or [**KR83**].

C.1. Basic definitions and examples

By an *algebra*, we mean a vector space A together with a bilinear map $A \times A \to A$, $(x, y) \mapsto xy$, called the *multiplication*, such that $x(yz) = (xy)z$ for all $x, y, z \in A$. In other words, the multiplication is associative. Unless stated otherwise, we will always assume algebras to have $\mathbb{C}$ as their ground field. An algebra A is called *unital* if it contains a (necessarily unique) element $1 \in A$ such that $1x = x1 = x$ for all $x \in A$. If A is unital, then $x \in A$ is called *invertible* if there exists $y \in A$ with $xy = yx = 1$, and we call $x^{-1} := y$ the (necessarily unique) *inverse* of x.

DEFINITION C.1.1. An algebra A together with a map $^*\colon A \to A$ is called a **-algebra* (or *involutive algebra*) if, for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

(i) $(x + \lambda y)^* = x^* + \overline{\lambda} y^*$,
(ii) $(x^*)^* = x$,
(iii) $(xy)^* = y^* x^*$.

We say that A is *unital* if there is an element $1 \in A$ such that $x1 = 1x = x$ for all $x \in A$. By a **-subalgebra* of A, we mean a linear subspace which is closed under multiplication and under the involution. An element $x \in A$ is called *self-adjoint* if $x^* = x$, *normal* if $x^* x = xx^*$, and if A is unital, then x is called *unitary* if $x^* x = xx^* = 1$.

DEFINITION C.1.2. A *normed algebra* is an algebra A together with a norm $\|\cdot\|$ on A such that $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in A$. A *Banach algebra* is a complete normed algebra. A Banach algebra with involution as in Definition C.1.1 is called a *Banach *-algebra* if $\|x^*\| = \|x\|$ holds for all $x \in A$. A *C^*-algebra* is an involutive Banach algebra A in which the *C^*-identity* $\|x^* x\| = \|x\|^2$ holds for every $x \in A$.

DEFINITION C.1.3. Let A and B be involutive algebras. A **-homomorphism* (or a *homomorphism of *-algebras*) is a linear map $\varphi\colon A \to B$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ for all $x, y \in A$.

REMARK C.1.4. Note that for a C^*-algebra, the involution is automatically isometric as $\|x\|^2 = \|x^*x\| \leq \|x\|\|x^*\|$, so $\|x\| \leq \|x^*\|$ and applying this to x^* shows the reverse inequality. In particular, every C^*-algebra is a Banach *-algebra.

We will spend the rest of this introductory section on stating the most important examples and elementary constructions.

EXAMPLE C.1.5. For any topological space X, the space $C_b(X)$ of bounded complex-valued continuous functions on X is a unital C^*-algebra with pointwise complex conjugation as its involution and the norm $\|f\|_\infty := \sup_{x\in X} \|f(x)\|$. $\diamondsuit$

EXAMPLE C.1.6. (i) Let X be a locally compact Hausdorff space. Then the space $C_0(X)$ of continuous complex-valued functions which vanish at infinity is a C^*-subalgebra of $C_b(X)$. Here, a function $f\colon X \to \mathbb{C}$ is said to *vanish at infinity* if, for every $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Note that $C_0(X)$ is unital if and only if X is compact.

(ii) If A is a C^*-algebra, then the space $C_0(X, A)$ of A-valued continuous functions on X which vanish at infinity is also a C^*-algebra in the obvious way. $\diamondsuit$

EXAMPLE C.1.7. Let H be a Hilbert space. The algebra $\mathscr{L}(H)$ of bounded linear operators on H with composition as multiplication is a Banach algebra when equipped with the *operator norm* $\|T\| := \sup_{\|x\|\leq 1} \|Tx\|$. The mapping $T \mapsto T^*$, where T^* is the adjoint of T, is an involution which makes $\mathscr{L}(H)$ into a C^*-algebra. The algebra $\mathscr{K}(H)$ of compact operators is a C^*-subalgebra of $\mathscr{L}(H)$. While $\mathscr{L}(H)$ always contains the identity operator as a unit, $\mathscr{K}(H)$ only does so if H is finite-dimensional. $\diamondsuit$

EXAMPLE C.1.8. If (X, Σ, μ) is a measure space, then $L^\infty(X, \Sigma, \mu)$ is a unital C^*-algebra. Again, complex conjugation serves as the involution and the norm of $f \in L^\infty(X, \Sigma, \mu)$ is its essential supremum $\|f\|_{L^\infty(X,\Sigma,\mu)} := \operatorname{ess\,sup} |f|$. $\diamondsuit$

EXAMPLE C.1.9. Let A be a C^*-algebra. If $S \subseteq A$ is any subset, then we denote by $C^*(S)$ the smallest C^*-subalgebra of A containing S and we call $C^*(S)$ the *C^*-algebra generated by S*. In the case where $S = \{x_1, \ldots, x_n\}$ is finite, we will also write $C^*(x_1, \ldots, x_n) := C^*(S)$. $\diamondsuit$

EXAMPLE C.1.10. Let A be a *-algebra and suppose we are given a norm $\|\cdot\|$ on A such that $\|xy\| \leq \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$ hold for all $x, y \in A$. We call $\|\cdot\|$ a *C^*-norm* on A. Note that as a normed algebra, A can be uniquely completed to a Banach algebra $\widetilde{A}$. Because the involution is continuous with respect to $\|\cdot\|$, we can extend it to an involution on $\widetilde{A}$, which then becomes a C^*-algebra, called the *C^*-completion* of A. $\diamondsuit$

EXAMPLE C.1.11 (Direct sums and products). If A and B are *-algebras, then the direct sum of vector space $A \oplus B$ can be made into a *-algebra by defining the componentwise operations $(x, y)(x', y') := (xx', yy')$ and $(x, y)^* := (x^*, y^*)$. In the case where A and B are C^*-algebras, the norm $\|(x, y)\| := \max\{\|x\|, \|y\|\}$ is a C^*-norm on $A \oplus B$ and we call this C^*-algebra the *direct sum* of A and B.

If we wish to define the direct sum of infinitely many C^*-algebras, things get more involved as the algebraic direct sum will not be complete in general. Let A_i, $i \in I$, be any family of C^*-algebras and define

$$\prod_{i\in I}^{\oplus} A_i := \Big\{ (x_i)_i \in \prod_{i\in I} A_i : \sup_{i\in I} \|x_i\| < \infty \Big\}.$$

We equip $\prod_{i\in I}^{\oplus} A_i$ with the componentwise operations and the C^*-norm $\|(x_i)_i\| := \sup_{i\in I} \|x_i\|$. One easily verifies that this is indeed a C^*-algebra, called the *(C^*-algebraic) direct product* of the A_i. Another way to obtain a C^*-algebra is the *(C^*-algebraic) direct sum*,

$$\sum_{i\in I}^{\oplus} A_i := \Big\{ (x_i)_i \in \prod_{i\in I} A_i : \|x_i\| \to 0 \text{ as } i \to \infty \Big\},$$

where $\|x_i\| \to 0$ as $i \to \infty$ has to be understood as $i \mapsto \|x_i\|$ belonging to $C_0(I)$, where I is equipped with the discrete topology. This is equivalent to requiring that for every $\varepsilon > 0$ there are only finitely many $i \in I$ with $\|x_i\| \geq \varepsilon$. We equip the direct sum with the norm $\|(x_i)_i\| := \sup_{i\in I} \|x_i\|$ and this is again a C^*-algebra. It follows that $\sum_{i\in I}^{\oplus} A_i$ is a closed ideal in $\prod_{i\in I}^{\oplus} A_i$ and the algebraic direct sum is dense in $\sum_{i\in I}^{\oplus} A_i$ (but in general not in the direct product). We will reserve the notation $\bigoplus_{i\in I} A_i$ for when we talk about the *algebraic* direct sum, that is, elements of $\prod_{i\in I} A_i$ with only *finitely* many non-zero entries. Note that $\prod_{i\in I}^{\oplus} A_i$, $\sum_{i\in I}^{\oplus} A_i$ and $\bigoplus_{i\in I} A_i$ all agree if I is finite.

If $\varphi_i\colon A_i \to B_i$ are *-homomorphisms, then they induce homomorphisms

$$\prod_{i\in I}^{\oplus} \varphi_i \colon \prod_{i\in I}^{\oplus} A_i \to \prod_{i\in I}^{\oplus} B_i \quad \text{and} \quad \sum_{i\in I}^{\oplus} \varphi_i \colon \sum_{i\in I}^{\oplus} A_i \to \sum_{i\in I}^{\oplus} B_i$$

by mapping $(x_i)_i$ to $(\varphi_i(x_i))_i$. This follows from the fact that *-homomorphisms between C^*-algebras have operator norm $\|\varphi_i\| \leq 1$, see Proposition C.2.2 below. $\diamondsuit$

EXAMPLE C.1.12. Let H_i, $i \in I$, be a family of Hilbert spaces. Then their *Hilbert space direct sum* is

$$\bigoplus_{i\in I} H_i := \Big\{ (h_i)_i \in \prod_{i\in I} H_i : \sum_{i\in I} \|h_i\|^2 < \infty \Big\},$$

together with the inner product

$$\langle (h_i)_i, (k_i)_i \rangle := \sum_{i\in I} \langle h_i, k_i \rangle.$$

One shows that this is again a Hilbert space. Now consider the C^*-algebra $A := \prod_{i\in I}^{\oplus} \mathscr{L}(H_i)$. Then any element $(T_i)_i \in A$ defines a bounded operator on $\bigoplus_{i\in I} H_i$ by

$$T((h_i)_i) := (T_i h_i)_i$$

with operator norm $\|T\| = \sup_{i\in I} \|T_i\|$ and the assignment $\prod_{i\in I}^{\oplus} \mathscr{L}(H_i) \to \mathscr{L}(\bigoplus_{i\in I} H_i)$ with $(T_i)_i \mapsto T$ is an injective *-homomorphism. The operator T is then denoted by $\bigoplus_{i\in I} T_i$ and called the *direct sum* of the operators T_i, $i \in I$. $\diamondsuit$

EXAMPLE C.1.13 (Unitization). For any algebra A over $\mathbb{C}$, one obtains a unital algebra, called the *unitization* of A, by taking the vector space $\widetilde{A} := A \oplus \mathbb{C}$ with multiplication $(x,\lambda)(y,\mu) := (xy + \lambda y + \mu x, \lambda\mu)$. It is easily checked that $\widetilde{A}$ is a unital algebra with unit $(0,1)$. If A has a norm, then $\|(x,\lambda)\|_{\widetilde{A}} := \|x\|_A + |\lambda|$ defines a norm on the unitization. If A is a Banach algebra, then so is $\widetilde{A}$. In the case when A is a C^*-algebra, one needs to be careful if $\widetilde{A}$ is to be a C^*-algebra. For the involution, one sets $(x,\lambda)^* := (x^*, \overline{\lambda})$. In the case where A has no unit, the norm

$$\|(x,\lambda)\|_{\widetilde{A}} := \sup\left\{ \|(x+\lambda)y\|_A : y \in A,\ \|y\|_A \leq 1 \right\}$$

is a C^*-norm on $\widetilde{A}$ which has the same values on $A \subseteq \widetilde{A}$ as $\|\cdot\|_A$. If A already has unit $1 \in A$, it might still be of use to consider the unitization, in which case

$$\|(x,\lambda)\|_{\widetilde{A}} := \max\left\{ \|a + \lambda 1\|, \|\lambda\| \right\}$$

is a C^*-norm on $\widetilde{A}$. $\diamondsuit$

C.2. The spectrum of an element

If A is a unital algebra, the *spectrum* of an element $x \in A$ is the set

$$\sigma_A(x) := \{\lambda \in \mathbb{C} : x - \lambda \text{ is not invertible}\}.$$

In the case where A is non-unital, we set $\sigma_A(x) := \sigma_{\widetilde{A}}(x)$, where $\widetilde{A}$ is the unitization of A from Example C.1.13. The *spectral radius* of $x \in A$ is $r_A(x) := \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. If it is clear from the context which algebra A is meant, we will omit the subscripts in σ_A and r_A.

THEOREM C.2.1. *If A is a Banach algebra and $x \in A$, then $\sigma(x)$ is a compact, non-empty subset of $\mathbb{C}$. Moreover, $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \leq \|x\|$.*

The following Proposition shows that *-homomorphisms are the "correct" notion of maps between C^*-algebras, as they are automatically continuous.

PROPOSITION C.2.2. *Let A be a Banach *-algebra and B a C^*-algebra. Then every *-homomorphism $\varphi\colon A \to B$ has operator norm $\|\varphi\| \leq 1$.*

PROOF. We can always assume A and B to be unital and that φ satisfies $\varphi(1) = 1$, since otherwise we may consider $\widetilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$, $\widetilde{\varphi}(x,\lambda) := (\varphi(x),\lambda)$. First note that if y is self-adjoint, then $r(y) = \|y\|$ since $\|y^2\| = \|y\|^2$ and $r(a) = \lim_{n\to\infty} \|y^n\|^{1/n} = \lim_{n\to\infty} \|y^{2^n}\|^{1/2^n} = \|y\|$. For $x \in A$ we have $\sigma(\varphi(x)) \subseteq \sigma(x)$ since $\varphi(x)$ is invertible as soon as x is (recall that $\varphi(1) = 1$). But then $\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = r(\varphi(x^*x)) \leq \varphi(x^*x) \leq \|x^*x\| \leq \|x\|^2$. ∎

In applications, it is often easier to calculate the spectrum of some element x of a unital Banach algebra with respect to some closed subalgebra B that contains x and the unit of A. Clearly, one then has the inclusion $\sigma_A(x) \subseteq \sigma_B(x)$. One can show that $\sigma_B(x)$ results from $\sigma_A(x)$ by filling some of its *holes*, which are the bounded connected components of $\mathbb{C} \setminus \sigma_A(x)$. In particular, if $\sigma_A(x)$ has no holes, then $\sigma_B(x) = \sigma_A(x)$. If A is a C^*-algebra, then the situation is a lot simpler:

PROPOSITION C.2.3. *Let A be a unital C^*-algebra and B be a C^*-subalgebra which contains the unit of A. Then for every $x \in A$, we have $\sigma_A(x) = \sigma_B(x)$.*

C.3. Commutative algebras and the Gelfand transform

Let A be a commutative Banach algebra. A *character* of A is a non-zero algebra homomorphism $A \to \mathbb{C}$. We denote the set of characters of A by $\mathrm{Spec}(A)$. One can show, see, that any character $\varphi \in \mathrm{Spec}(A)$ is automatically continuous with operator norm $\|\varphi\| \leq 1$. If A is a C^*-algebra, then φ is even a *-homomorphism, $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$. In particular, we may view $\mathrm{Spec}(A)$ as a subset of the dual space A^* and we endow $\mathrm{Spec}(A)$ with the relative weak-* topology. This means that a net $\iota \mapsto \varphi_\iota$ in $\mathrm{Spec}(A)$ converges to $\varphi \in \mathrm{Spec}(A)$ if and only if $\varphi_\iota(x) \to \varphi(x)$ for all $x \in A$. The resulting topological space is called the *Gelfand spectrum* or *character space* of A.

PROPOSITION C.3.1. *For any commutative Banach algebra A, the character space* $\mathrm{Spec}(A)$ *is a locally compact Hausdorff space. Moreover, if A is unital, then* $\mathrm{Spec}(A)$ *is compact.*

DEFINITION C.3.2. Let A be a commutative Banach algebra such that $\mathrm{Spec}(A)$ is non-empty. The mapping
$$\Gamma_A\colon A \to C_0(\mathrm{Spec}(A)), \quad \Gamma_A(x)(\varphi) := \varphi(x)$$
is called the *Gelfand representation*. For $x \in A$, we call $\Gamma_A(x) \in C_0(\mathrm{Spec}(A))$ the *Gelfand transform* of x.

Note that $\Gamma_A(x)$ is continuous by the definition of the weak-* topology and if $\varepsilon > 0$, then $\{\varphi \in \mathrm{Spec}(A) : |\Gamma_A(x)(\varphi)| \geq \varepsilon\}$ is weak-* closed in the unit ball of A^* and hence compact by the Banach-Alaoglu theorem. Therefore, $\Gamma_A(x) \in C_0(\mathrm{Spec}(A))$.

THEOREM C.3.3. *For any commutative Banach algebra A with* $\mathrm{Spec}(A) \neq \emptyset$, *the Gelfand representation Γ_A is a norm-decreasing homomorphism. Moreover, $r(x) = \|\Gamma_A(x)\|_\infty$ and, if A is unital, then $\sigma(x) = \Gamma_A(x)(\mathrm{Spec}(A))$, whereas $\sigma(x) = \Gamma_A(x)(\mathrm{Spec}(A)) \cup \{0\}$ for non-unital A.*

THEOREM C.3.4 (Gelfand-Naimark). *If A is a commutative C^*-algebra, then the Gelfand representation Γ_A is an isomorphism of C^*-algebras.*

The Gelfand-Naimark theorem gives a complete description of all commutative C^*-algebras: any such algebra is of the form $C_0(X)$ for some locally compact Hausdorff space. Moreover, the C^*-algebras $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if X and Y are homeomorphic. This motivates the term "non-commutative topology" for the study of C^*-algebras.

Theorem C.3.4 together with the observation that $C_0(X)$ is unital if and only if X is compact gives the following stronger version of Proposition C.3.1 for C^*-algebras:

PROPOSITION C.3.5. *If A is a commutative C^*-algebra,*[1] *then the character space* $\mathrm{Spec}(A)$ *is compact if and only if A has a unit.*

Another important consequence of the Gelfand-Naimark theorem is the *functional calculus*. Let A be a unital C^*-algebra and $x \in A$ a normal element. If $C^*(1,x)$ denotes the C^*-subalgebra of A generated by x and the unit element of A, then $C^*(1,x)$ is clearly commutative and unital. By Theorem C.3.4, the Gelfand representation $\Gamma_{C^*(1,x)}$ is an isomorphism from $C^*(1,x)$ to $C(\mathrm{Spec}(C^*(1,x)))$. Consider now the Gelfand transform of x,

$$\mathrm{ev}_x := \Gamma_{C^*(1,x)}(x) \colon \mathrm{Spec}(C^*(1,x)) \to \mathbb{C}, \quad \mathrm{ev}_x(\varphi) = \varphi(x).$$

Then ev_x is injective, because $\varphi_1(x) = \varphi_2(x)$ implies $\varphi_1(p(x,x^*)) = \varphi_2(p(x,x^*))$ for all polynomials $p \in \mathbb{C}[z,\bar{z}]$ since φ_1, φ_2 are *-homomorphisms as remarked above. Because $\{p(x,x^*) : p \in \mathbb{C}[z,\bar{z}]\}$ is dense in $C^*(1,x)$ by construction, we have $\varphi_1 = \varphi_2$. Moreover,

$$\sigma(x) = \sigma_{C(\mathrm{Spec}(C^*(1,x)))}(\Gamma_{C^*(1,x)}(x)) = \mathrm{im}(\Gamma_{C^*(1,x)}(x)) = \mathrm{im}(\mathrm{ev}_x)$$

and ev_x is continuous because the character space is equipped with the weak-* topology. Finally, since $\mathrm{Spec}(C^*(1,x))$ is compact it follows that $\mathrm{ev}_x \colon \mathrm{Spec}(C^*(1,x)) \to \sigma(x)$ is a homeomorphism and hence induces a *-isomorphism $(\mathrm{ev}_x)^t$ of $C(\sigma(x))$ onto $C(\mathrm{Spec}(C^*(1,x)))$, given by $(\mathrm{ev}_x)^t(f) := f \circ \mathrm{ev}_x$.

DEFINITION C.3.6. Let A be a unital C^*-algebra and $x \in A$ a normal element. The *functional calculus* is the *-homomorphism $\psi \colon C(\sigma(x)) \to A$ defined by the composition

$$(\Gamma_{C^*(1,x)})^{-1} \circ (\mathrm{ev}_x)^t \colon C(\sigma(x)) \xrightarrow{\cong} C^*(1,x) \subseteq A.$$

It is the unique unit-preserving *-homomorphism from $C(\sigma(x))$ to A which maps $\mathrm{id}_{\sigma(x)}$ to x and one usually writes $f(x)$ instead of $\psi(f)$ for continuous functions $f \colon \sigma(x) \to \mathbb{C}$.

There is also a version of the functional calculus for C^*-algebras without unit, which can be derived by applying the above to the unitization. In this case, $f(x)$ is defined for all normal $x \in A$ and $f \in C(\sigma(x))$ with $f(0) = 0$.

EXAMPLE C.3.7. If $x \in A$ satisfies $\sigma(x) \subseteq [0,\infty)$, then x is called *positive*. This is the case if and only if $x = y^*y$ for some $y \in A$. In this case, the functional calculus gives the existence of the square root $\sqrt{x}$, which satisfies $\sqrt{x}\sqrt{x} = x$. Similarly, the absolute value $|x| := \sqrt{x^*x} \in A$ makes sense for any element of a C^*-algebra. $\lozenge$

As another application of the theory of commutative C^*-algebras, we mention the following sequel to Proposition C.2.2:

PROPOSITION C.3.8. *Every injective *-homomorphism between C^*-algebras is an isometry.*

[1] More generally, this holds for semisimple commutative Banach *-algebras. Here, a commutative Banach algebra A is called *semisimple* if its *radical* $\mathrm{rad}(A) := \bigcap_{\varphi \in \mathrm{Spec}(A)} \ker(\varphi)$ is trivial. We refer to [Kan09, Corollary 3.2.3].

C.4. Ideals and Quotients

Suppose that A is a normed algebra and $I \subseteq A$ is a closed ideal. We only consider *two-sided* ideals, meaning that $x \in I$ and $y \in A$ implies both $xy \in I$ and $yx \in I$. Then the quotient algebra A/I is again a normed algebra when equipped with the *quotient norm* $\|x + I\| := \inf_{y \in I} \|x + y\|$. As usual, the multiplication on A/I is $(x + I)(y + I) := xy + I$. If A is a Banach algebra, then so is A/I.

PROPOSITION C.4.1 (Quotients of C^*-algebras). *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal. Then I is self-adjoint, meaning that $x \in I$ implies $x^* \in I$, and the definition $(a + I)^* := a^* + I$ makes the Banach algebra A/I into a C^*-algebra.*

LEMMA C.4.2. *Let A, B, C be C^*-algebras and $\varphi\colon A \to C$, $\rho\colon A \to B$ be *-homomorphisms such that $\ker(\rho) \subseteq \ker(\varphi)$ and ρ is surjective. Then there exists a unique *-homomorphism $\widetilde{\varphi}\colon B \to C$ such that $\varphi = \widetilde{\varphi} \circ \rho$.*

PROOF. Define $\widetilde{\varphi}\colon B \to C$ by $\widetilde{\varphi}(b) := \varphi(a)$, where $a \in A$ is such that $\rho(a) = b$. Then $\widetilde{\varphi}$ is well-defined: if a_1 and a_2 are both in $\rho^{-1}(\{b\})$, then $a_1 - a_2 \in \ker(\rho) \subseteq \ker(\varphi)$ and hence $\varphi(a_1) = \varphi(a_2)$. The algebraic properties of $\widetilde{\varphi}$ follow directly from those of φ and ρ. Regarding uniqueness, suppose that $\psi\colon B \to C$ is another *-homomorphism such that $\varphi = \psi \circ \rho$. Then $\psi \circ \rho = \widetilde{\varphi} \circ \rho$ and uniqueness follows immediately from surjectivity of ρ. ∎

By the definition of the operations on the quotient, the canonical map $\pi\colon A \to A/I$ is a *-homomorphism. Applying Lemma C.4.2 to π, we obtain the universal property of the quotient C^*-algebra:

COROLLARY C.4.3. *Let A be a C^*-algebra and $I \subseteq A$ be a closed ideal. If B is another C^*-algebra and $\varphi\colon A \to B$ is a *-homomorphism such that $\varphi|_I = 0$, then there exists a unique *-homomorphism $\psi\colon A/I \to B$ such that $\varphi = \psi \circ \pi$*

PROPOSITION C.4.4. *Any *-homomorphism between C^*-algebras has closed range. In particular, the image of any such homomorphism is again a C^*-algebra.*

PROOF. If $\varphi\colon A \to B$ is such a homomorphism, then φ factors through $I := \ker(\varphi)$ to an injective *-homomorphism $\psi\colon A/I \to B$ such that $\varphi = \psi \circ \pi$. By Proposition C.3.8, ψ is isometric, hence has closed image. But $\operatorname{im}(\varphi) = \operatorname{im}(\psi)$ because π is surjective, so φ also has closed range. ∎

Next, we study the *commutator ideal* $\operatorname{Comm}(A)$ of a C^*-algebra A. By this we mean the closed ideal generated by the elements of the form $xy - yx$ for $x, y \in A$. Note that the quotient C^*-algebra $A/\operatorname{Comm}(A)$ is commutative. The proof of the following result can be found in [**Con91**, Lemma 12.2].

LEMMA C.4.5. *Let A be a unital C^*-algebra and denote by Φ_A the set of all nonzero homomorphisms $\varphi\colon A \to \mathbb{C}$.*[2] *Then*

$$\operatorname{Comm}(A) = \bigcap \{\ker(\varphi) : \varphi \in \Phi_A\}$$

and Φ_A can be identified with the character space of $A/\operatorname{Comm}(A)$.

Finally, we mention that while not every C^*-algebra has a unit, there always exist approximate ones:

THEOREM C.4.6. *Let A be a C^*-algebra. Then there exists a net $\iota \mapsto e_\iota$ of positive elements of A such that $\|e_\iota\| \leq 1$ for all ι and $x = \lim_\iota(e_\iota x) = \lim_\iota(x e_\iota)$ for all $x \in A$. Such a net is called a* two-sided approximate identity *for A.*

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[2] If A were commutative, then Φ_A would be the character space from section C.3.

More generally, one can show that every left ideal L of a C^*-algebra admits such an approximate identity, so that $x = \lim_\iota(e_\iota x) = \lim_\iota(xe_\iota)$ holds for all $x \in L$.

C.5. Representations

Let A be a *-algebra. A *(*-)representation* of A is a *-homomorphism $\rho\colon A \to \mathscr{L}(H)$ for some Hilbert space H. Such a representation of A is called

- *faithful* if the homomorphism ρ is injective,
- *non-degenerate* if $\rho(A)h = 0$ implies $h = 0$, and
- *irreducible* if 0 and H are the only closed invariant subspaces for ρ.

Here, a subspace $K \subseteq H$ is called *invariant* for ρ if $\rho(A)K \subseteq K$. The Hilbert space H is called the *representation space* and we will sometimes denote representations by the pair (ρ, H). Two representations (ρ, H) and (ρ', H') of A are called *unitarily equivalent* if there exists a unitary operator $U\colon H \to H'$ such that $U\rho(x) = \rho(x)U$ for all $x \in A$.

REMARK C.5.1. (i) A representation (ρ, H) of A is non-degenerate if and only if $\rho(A)H$ spans a dense subspace of H. This follows from the fact that density of $\operatorname{span}(\rho(A)H)$ in H if and only if $\langle\rho(x)k, h\rangle = 0$ for all $x \in A$ and $h, k \in H$ implies $h = 0$. Since ρ is a *-homomorphism, $\langle\rho(x)k, h\rangle = \langle k, \rho(x^*)h\rangle$, so that the above is equivalent to $\rho(A)h = 0$ forcing $h = 0$.

(ii) If (ρ, H) is a representation of A and if $K \subseteq H$ is a closed invariant subspace, then we obtain a representation $\rho_K\colon A \to \mathscr{L}(K)$ by letting $\rho_K(x) := \rho(x)|_K$.

(iii) For any representation (ρ, H) of A, the *essential subspace*

$$K := \overline{\operatorname{span}(\rho(A)H)}$$

is invariant, and the representation ρ_K is then non-degenerate. For this reason, not much is lost when restricting considerations to the non-degenerate case.

(iv) Note that, by Proposition C.2.2, every faithful representation of a C^*-algebra is isometric.

(v) A representation (ρ, H) of A is irreducible if and only if the only orthogonal projections contained in the commutant

$$\rho(A)' := \{T \in \mathscr{L}(H) : T\rho(x) = \rho(x)T \text{ for all } x \in A\}$$

are 0 and I (the identity operator). This follows from the fact that if P is an orthogonal projection, then $P \in \rho(A)'$ if and only if $\operatorname{im}(P)$ is an invariant subspace for ρ.

The next results state that every C^*-algebra can be realized as a C^*-subalgebra of bounded operators on a suitable Hilbert space. A *state* on a C^*-algebra A is a linear functional $\tau\colon A \to \mathbb{C}$ such that $\tau(x^*x) \geq 0$ for all $x \in X$ (positivity) and $\|\tau\| = 1$. If A is a unital C^*-algebra, this last condition is equivalent to $\tau(1) = 1$.

THEOREM C.5.2 (Gelfand-Naimark-Segal). *If A is a C^*-algebra and τ a state on A, then there exists a representation $\pi_\tau\colon A \to \mathscr{L}(H_\tau)$ on a Hilbert space H_τ, unique up to unitary equivalence, and a unit vector $\xi_\tau \in H$ such that*

$$\tau(x) = \langle\pi_\tau(x)\xi_\tau, \xi_\tau\rangle$$

for all $x \in A$. Moreover, ξ_τ is cyclic *for the representation π_τ, meaning that the set $\pi_\tau(A)\xi_\tau$ is dense in H_τ. The representation (π_τ, H_τ) is called the* GNS-representation *associated to τ.*

The idea to Theorem C.5.2 is the following: given the state τ, set $I_\tau := \{x \in A\colon \tau(x^*x) = 0\}$. This is a closed left ideal of A and one defines H_τ to be the completion of A/I_τ with respect to the sesquilinear form $\langle x + I_\tau, y + I_\tau\rangle := \tau(y^*x)$. The representation $\pi_\tau\colon A \to \mathscr{L}(H_\tau)$ is then obtained by left multiplication, $\pi_\tau(x)(y + I_\tau) := xy + I_\tau$.

PROPOSITION C.5.3. *Every C^*-algebra has a non-degenerate faithful representation.*

SKETCH OF PROOF. In fact, if $S(A)$ denotes the set of states on A, then the direct sum $H := \bigoplus_{\tau\in S(A)} H_\tau$ of Hilbert spaces and $\rho\colon A \to \mathscr{L}(H)$ with $\rho(x) := \bigoplus_{\tau\in S(A)} \pi_\tau(x)$ can be shown to define a faithful representation, where (π_τ, H_τ) is as in Theorem C.5.2. The representation ρ is called the *universal representation* of A.

Now let $K \subseteq H$ be the essential subspace of ρ from Remark C.5.1, $K := \overline{\operatorname{span}(\rho(A)H)}$. Then ρ_K is non-degenerate and if $\rho_K(x) = 0$, then $\rho(x)(\rho(y)h) = 0$ for all $y \in A$ and all $h \in H$. Since $\rho(x)\rho(y) = \rho(xy)$ and ρ is faithful, this implies $xy = 0$ for all $y \in A$. By using an approximate identity for A, see Theorem C.4.6, we see that $x = 0$, so ρ_K is also faithful. ∎

Concerning irreducible representations, we have the following relation with the compact operators on the representation space:

THEOREM C.5.4. *If $\rho\colon A \to \mathscr{L}(H)$ is an irreducible representation of a C^*-algebra with $\rho(A) \cap \mathscr{K}(H) \neq 0$, then $\mathscr{K}(H) \subseteq \rho(A)$.*

C.6. Tensor products

Given two C^*-algebras A and B, we can view them as complex vector spaces and form their algebraic tensor product $A \odot B$. We recall that the algebraic tensor product of two (complex) vector spaces V_1, V_2 has the following universal property: given a bilinear map $\varphi\colon V_1 \times V_2 \to W$, where W is another (complex) vector space, then there exists a unique linear map $\widetilde{\varphi}\colon V_1 \odot V_2 \to W$ such that $\widetilde{\varphi}(v_1 \otimes v_2) = \varphi(v_1, v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$. The tensor product $V_1 \odot V_2$ is uniquely determined by this universal property and is spanned by vectors of the form $x \otimes y$, with $x \in V_1$ and $y \in V_2$.

Any two linear maps $\varphi\colon V_1 \to W_1$ and $\psi\colon V_2 \to W_2$ give rise to a linear map

$$\varphi \odot \psi\colon V_1 \odot V_2 \to W_1 \odot W_2 \tag{C.6.1}$$

such that $\varphi \odot \psi(x \otimes y) = \varphi(x) \otimes \psi(y)$. This can be seen by applying the universal property to the bilinear map $V_1 \times V_2 \to W_1 \odot W_2$, $(x, y) \mapsto \varphi(x) \otimes \psi(y)$.

Returning to the case where A and B are C^*-algebras, there is always a unique multiplication on $A \odot B$ so that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for all $a, a' \in A$ and $b, b' \in B$. This can be seen by considering the bilinear map $\mu\colon A \times B \to \hom(A \odot B)$, $\mu(a, b) := \ell_a^A \odot \ell_b^B$, where $\hom(A \odot B)$ denotes the space of linear maps on $A \odot B$ and $\ell_a^A\colon A \to A$ is multiplication on A with a from the left and similarly for $\ell_b^B\colon B \to B$. By the above universal property, there exists a unique linear map $\widetilde{\mu}\colon A \odot B \to \hom(A \odot B)$ such that $\widetilde{\mu}(a \otimes b) = \ell_a^A \odot \ell_b^B$, meaning

$$\widetilde{\mu}(a \otimes b)(a' \otimes b') = (\ell_a^A \odot \ell_b^B)(a' \otimes b') = aa' \otimes bb',$$

so $\widetilde{\mu}$ has the required property and it is easy to see that this is really a multiplication on $A \odot B$.

Similarly, one shows that the involutions of A and B also give an involution on the algebraic tensor product, uniquely determined by requiring $(a \otimes b)^*$ to agree with $a^* \otimes b^*$. Thus, $A \odot B$ is made into a *-algebra, and for any two *-homomorphisms $\varphi\colon A_1 \to B_1$ and $\psi\colon A_2 \to B_2$, the induced map $\varphi \odot \psi$ from (C.6.1) is a *-homomorphism.

If we want to make $A \odot B$ into a C^*-algebra, the only thing missing is a C^*-norm with respect to which $A \odot B$ is complete. This is where things get interesting, as there may be more than one such norm on $A \odot B$.

REMARK C.6.1 (Hilbert space tensor product). If H and K are Hilbert spaces, then their algebraic tensor product $H \odot K$ carries a unique inner product with the property that $\langle x \otimes y, x' \otimes y' \rangle = \langle x, y \rangle \langle x', y' \rangle$ for all $x, x' \in H$ and all $y, y' \in K$. This also follows easily from the universal property of the tensor product. The Hilbert space completion of $H \odot K$ with respect to this inner product will be denoted by $H \hat{\otimes} K$. Moreover, for bounded operators $S \in \mathscr{L}(H)$ and $T \in \mathscr{L}(K)$, the linear map $S \odot T$ from (C.6.1) extends to a bounded operator which we denote by $S \hat{\otimes} T \in \mathscr{L}(H \hat{\otimes} K)$.

This construction satisfies $(S\hat{\otimes}T)(S'\hat{\otimes}T') = SS'\hat{\otimes}TT'$ and $(S\hat{\otimes}T)^* = S^*\hat{\otimes}T^*$ for $T, T' \in \mathscr{L}(H)$ and $S, S' \in \mathscr{L}(K)$.

EXAMPLE C.6.2. If (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) are σ-finite measure spaces, then

$$L^2(X_1, \mu_1)\hat{\otimes}L^2(X_2, \mu_2) \cong L^2(X_1 \times X_2, \mu_1 \times \mu_2),$$

see [KR83, Example 2.6.11]. In fact, the bilinear map

$$L^2(X_1, \mu_1) \times L^2(X_2, \mu_2) \to L^2(X_1 \times X_2, \mu_1 \times \mu_2), \quad (f, g) \mapsto ((x, y) \mapsto f(x)g(x))$$

gives a linear map $L^2(X_1, \mu_1) \odot L^2(X_2, \mu_2) \to L^2(X_1 \times X_2, \mu_1 \times \mu_2)$ which maps $f \otimes g$ to $(x, y) \mapsto f(x)g(y)$ for $f \in L^2(X_1, \mu_1)$ and $g \in L^2(X_2, \mu_2)$ and one shows that this extends to the desired isomorphism. $\diamondsuit$

THEOREM C.6.3. *Let A and B be C^*-algebras and $\varphi\colon A \to \mathscr{L}(H)$, $\psi\colon B \to \mathscr{L}(K)$ two *-representations of A and B, respectively. Then there exists exactly one *-homomorphism $\pi\colon A \odot B \to \mathscr{L}(H\hat{\otimes}K)$ with*

$$\pi(a \otimes b) = \varphi(a)\hat{\otimes}\psi(b)$$

for all $a \in A$ and $b \in B$. If, in addition, φ and ψ are faithful, then so is π.

SKETCH OF PROOF. Let $\widetilde{\varphi}\colon A \to \mathscr{L}(H\hat{\otimes}K)$ and $\widetilde{\psi}\colon B \to \mathscr{L}(H\hat{\otimes}K)$ be defined by

$$\widetilde{\varphi}(a) := \varphi(a)\hat{\otimes}\mathrm{id}_K \quad \text{and} \quad \widetilde{\psi}(b) := \mathrm{id}_H\hat{\otimes}\psi(b).$$

Then $\widetilde{\varphi}$ and $\widetilde{\psi}$ are *-homomorphism with commuting ranges, so that the bilinear map $A \times B \to \mathscr{L}(H\hat{\otimes}K)$, $(a, b) \mapsto \widetilde{\varphi}(a)\widetilde{\psi}(b)$, induces a *-homomorphism $\pi\colon A \odot B \to \mathscr{L}(H\hat{\otimes}K)$ which maps $a \otimes b$ to $\widetilde{\varphi}(a)\widetilde{\psi}(b) = \varphi(a)\hat{\otimes}\psi(b)$ and one shows that this homomorphism is injective. ■

We can use this result to show that there always is at least one C^*-norm on the algebraic tensor product. Let $\varphi\colon A \to \mathscr{L}(H)$ and $\psi\colon B \to \mathscr{L}(K)$ be faithful representations, see Proposition C.5.3. Then taking $\pi\colon A \odot B \to \mathscr{L}(H\hat{\otimes}K)$ as in Theorem C.6.3, we define a C^*-norm on $A \odot B$ by

$$\|c\|_* := \|\pi(c)\|,$$

called the *spatial (or minimal) C^*-norm*, and we denote the associated C^*-completion of $A \odot B$ by $A\otimes_* B$, the *spatial (or minimal) tensor product of A and B*. By construction, π then extends to a faithful representation of $A\otimes_* B$ on $H\hat{\otimes}K$. One can show that the spatial norm is independent of the faithful representations φ and ψ.

PROPOSITION C.6.4. *If A and B are C^*-algebras and γ is a C^*-norm on $A \odot B$, then $\gamma(a \otimes b) \leq \|a\|\|b\|$ for all $a \in A$ and all $b \in B$.*

The Proposition immediately gives access to another C^*-norm on the algebraic tensor product $A \odot B$ of two C^*-algebras A, B. Letting Γ be the set of all C^*-norms on $A \odot B$, we can form

$$\|c\|_{\max} := \sup\{\gamma(c) : \gamma \in \Gamma\} \quad (c \in A \odot B)$$

and call this the *maximal C^*-norm*. Indeed, the supremum makes sense because Γ is non-empty as it contains the spatial norm, and $\|c\|_{\max}$ is finite for all $c \in A \odot B$ by Proposition C.6.4 since c is the linear combination of elements of the form $a \otimes b$ with $a \in A$ and $b \in B$. The corresponding completion will be denoted by $A\otimes_{\max} B$, the *maximal tensor product* of A and B, and it has the following universal property:

THEOREM C.6.5. *If $\varphi\colon A \to C$ and $\psi\colon B \to C$ are *-homomorphisms between C^*-algebras such that the image of φ commutes with the image of ψ, then there exists a unique *-homomorphism $\pi\colon A\otimes_{\max} B \to C$ such that*

$$\pi(a \otimes b) = \varphi(a)\psi(b)$$

for all $a \in A$ and $b \in B$.

REMARK C.6.6. By construction, $\|\cdot\|_{\max}$ is the largest C^*-norm on the algebraic tensor product. It can be shown that the spatial norm is the *least* C^*-norm, meaning it is dominated by every other such norm, see [**Mur90**, Theorem 6.4.18].

DEFINITION C.6.7. A C^*-algebra A is called *nuclear* if there is only one C^*-norm on $A \odot B$ for every other C^*-algebra B. In this case, we will denote by $A \otimes B$ the only possible C^*-algebraic tensor product of A and B.

EXAMPLE C.6.8. All finite-dimensional C^*-algebras are nuclear and, more generally, the algebra of compact operators on a Hilbert space is also nuclear. $\Diamond$

EXAMPLE C.6.9. By a result due to Takesaki, all commutative C^*-algebras are nuclear, see [**Bla06**, II.9.4.4]. If $A = C_0(X)$ is commutative, then $A \otimes B \cong C_0(X, B)$ for every other C^*-algebra B. For commutative $B = C_0(Y)$, we have $C_0(X) \otimes C_0(Y) = C_0(X \times Y)$. $\Diamond$

APPENDIX D

An orthonormal basis of the Bergman space

Let Ω be a bounded, logarithmically convex and complete Reinhardt domain in $\mathbb{C}^n$. Thus, Ω contains the origin and is pseudoconvex by Theorem 3.1.5. We now want to find a suitable orthonormal basis of $A^2(\Omega)$. By using polar coordinates, the integral of an integrable function f on Ω can be expressed as

$$\int_\Omega f\,d\lambda = (2\pi)^n \int_{|\Omega|} \int_{\mathbb{T}^n} f(ys)\, y_1 \cdots y_n\, d\mu(s)\, d\widetilde{\lambda}(y) \tag{D.1}$$

$$= (2\pi)^n \int_{\Lambda_\Omega} \int_{\mathbb{T}^n} f(e^x s)\, e^{2x_1} \cdots e^{2x_n}\, d\mu(s)\, d\widetilde{\lambda}(x), \tag{D.2}$$

where μ is the normalized Haar measure of the group $\mathbb{T}^n$ (that is, arc length measure in every component) and $\widetilde{\lambda}$ is Lebesgue measure on $\mathbb{R}^n$. The details can be found in [**Upm96**, Proposition 1.4.24].

PROPOSITION D.1. *The monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with multiindices $\alpha \in \mathbb{Z}^n_{\geq}$ are pairwise orthogonal in $L^2(\Omega)$ and span a dense subspace of $A^2(\Omega)$. In particular, the functions*

$$e_\alpha := z^\alpha / \|z^\alpha\|_{L^2(\Omega)} \quad (\alpha \in \mathbb{Z}^n_{\geq}) \tag{D.3}$$

form an orthonormal basis of the Bergman space.

PROOF. Since Ω is bounded, all monomials are square-integrable. For $\alpha \neq \beta$ two multiindices we have, using (D.1),

$$\int_\Omega z^\alpha \overline{z}^\beta\, d\lambda(z) = \int_{|\Omega|} \int_{\mathbb{T}^n} \prod_{j=1}^n s_j^{\alpha_j - \beta_j}\, y_j^{\alpha_j + \beta_j + 1}\, d\mu(s)\, d\widetilde{\lambda}(y) = 0$$

since $\int_{\mathbb{T}} s_j^{\alpha_j - \beta_j}\, ds_j$ is zero for at least one $j \in \{1, \ldots, n\}$ because $\alpha \neq \beta$. Moreover, this shows that

$$\|z_\alpha\|^2 = (2\pi)^n \int_{|\Omega|} y^{2\alpha+1}\, d\widetilde{\lambda}(y), \tag{D.4}$$

where $2\alpha + 1$ is understood to be the multiindex $(2\alpha_1 + 1, \ldots, 2\alpha_n + 1)$.

To show that the orthonormal system (D.3) is complete, it suffices to show that $f \in A^2(\Omega)$ and $\langle f, e_\alpha \rangle = 0$ for all $\alpha \in \mathbb{Z}^n_{\geq}$ implies $f = 0$. Since Ω is Reinhardt and contains the origin, f admits a unique power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\partial^\alpha u(0)}{\alpha!} z^\alpha \quad (z \in \Omega)$$

with convergence in the topology of $\mathcal{O}(\Omega)$, meaning uniform convergence on compact subsets of Ω. Hence, using (D.1),

$$\langle f, z^\alpha \rangle = (2\pi)^n \int_{|\Omega|} \int_{\mathbb{T}^n} f(ys) s^{-\alpha} y^{\alpha+1}\, d\mu(s)\, d\widetilde{\lambda}(y) =$$

$$= (2\pi)^n \int_{|\Omega|} \left[\frac{1}{i^n} \int_{\mathbb{T}^n} \frac{f(ys)}{y^{\alpha+1} s^{\alpha+1}}\, i^n ys\, d\mu(s) \right] y^{2\alpha+1}\, d\widetilde{\lambda}(y) = (2\pi)^n \int_{|\Omega|} a_\alpha y^{2\alpha+1}\, d\widetilde{\lambda}(y)$$

by the Cauchy integral formula for the derivatives of f. Therefore, $\langle f, e_\alpha \rangle = 0$ implies $a_\alpha = 0$ and hence $f = 0$. ■

EXAMPLE D.2. We consider the one-dimensional case $\Omega = \mathbb{D}$, the unit disc in $\mathbb{C}$. Then we can compute the integral (D.4) explicitly,

$$\|z^k\|^2 = 2\pi \int_{[0,1)} y^{2k+1}\, d\widetilde{\lambda}(y) = 2\pi \int_0^1 y^{2k+1}\, dy = \frac{\pi}{k+1},$$

hence

$$\|f\|^2 = \pi \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}$$

for $f \in A^2(\mathbb{D})$ with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. ◊

APPENDIX E

Faces of convex sets

In this appendix chapter, we will study the *faces* of a convex subset $\Lambda \subseteq \mathbb{R}^n$, that is, those convex subsets F of Λ which have the property that they contain the endpoints of every line segment whose interior intersects F. The goal is to merely give an introduction into the basic concepts and definitions in order to supplement the geometric intuition, rather than to provide an in-depth treatment. The following material is taken from [**Sch93**], and we refer to this reference for more details on faces and convex sets in general.

DEFINITION E.1. For any non-empty subset $S \subseteq \mathbb{R}^n$ let $\mathrm{aff}(S)$ be its affine hull. Thus, $\mathrm{aff}(S)$ is the smallest affine subspace of $\mathbb{R}^n$ which contains S.

The affine hull of a subset $S \subseteq \mathbb{R}^n$ may be described as the set of *affine combinations* of points in S,

$$\mathrm{aff}(S) = \{\lambda_1 s_1 + \cdots + \lambda_n s_n : s_j \in S,\ \lambda_j \in \mathbb{R},\ j = 1, \ldots, n \text{ and } \lambda_1 + \cdots + \lambda_n = 1\}.$$

Any affine subspace of $A \subseteq \mathbb{R}^n$ is the affine hull of some finite number of affinely independent points, say $v_1, \ldots, v_{r+1}$. The *dimension* of the resulting affine subspace $A = \mathrm{aff}(\{v_1, \ldots, v_{r+1}\})$ is then, by definition, r, and this number only depends on A. If $C \subseteq \mathbb{R}^n$ is a convex subset, then its *dimension* is $\dim(C) := \dim(\mathrm{aff}(C))$.

DEFINITION E.2. For any subset $S \subseteq \mathbb{R}^n$, we let $\mathrm{relint}(S)$ be the interior of S as a subset of $\mathrm{aff}(S)$, where $\mathrm{aff}(S)$ is equipped with the subspace topology of $\mathbb{R}^n$.

For any $x, y \in \mathbb{R}^n$, we define the closed *line segment*

$$[x, y] := \{\theta x + (1-\theta) y : \theta \in [0, 1]\}.$$

Similarly, the open and half-open line segments (x, y), $(x, y]$ and $[x, y)$ are defined in the obvious manner. Note that $\mathrm{relint}([x, y]) = (x, y)$.

DEFINITION E.3. A *face* of a convex subset $\Lambda \subseteq \mathbb{R}^n$ is a convex subset $F \subseteq \Lambda$ with the property that $x, y \in \Lambda$ and $\theta x + (1-\theta) y \in F$ for some $\theta \in (0, 1)$ implies that $x, y \in F$. Expressed differently, a face $F \subseteq \Lambda$ is a convex subset such that $F \cap (x, y) \neq \emptyset$ for some $x, y \in \Lambda$ implies $x, y \in F$.

Since faces are convex by definition, any line segment in Λ with an interior point in F must automatically be completely contained in F. Obviously, $\emptyset$ and Λ are faces of Λ, called the *improper faces* of Λ, and all other faces are called *proper*.

EXAMPLE E.4. The cube $[0, 1]^3 \subseteq \mathbb{R}^3$ has, other than its improper faces, six two-dimensional faces (the sides), eight zero-dimensional faces (the vertices), and twelve one-dimensional faces (the edges). The closed ball $B := \{x \in \mathbb{R}^3 : |x| \leq 1\}$ has only zero-dimensional proper faces. In fact, every point in ∂B forms such a face. $\Diamond$

EXAMPLE E.5. Let $u \in \mathbb{R}^n$ be a non-zero vector. Then u defines a face F_u of Λ by

$$F_u := \left\{x \in \Lambda : x \cdot u = \sup\nolimits_{y \in \Lambda} y \cdot u\right\}.$$

To see this, let $x, y \in \Lambda$ and $\theta \in (0, 1)$ be such that $\theta x + (1-\theta) y \in F_u$. Hence, $\theta x \cdot u + (1-\theta) y \cdot u = c$, where $c := \sup_{z \in \Lambda} z \cdot u$. If $x \notin F_u$ or $y \notin F_u$, then $x \cdot u < c$ or $y \cdot u < c$ and therefore

$\theta x \cdot u + (1-\theta) y \cdot u < c$, a contradiction. Therefore, both x and y must belong to F_u. Obviously, F_u is convex. $\diamondsuit$

LEMMA E.6. *If F is a face of Λ and G is a face of F, then G is also a face of Λ.*

PROOF. Let $x, y \in \Lambda$ with $\theta x + (1-\theta) y \in G \subseteq \Lambda$. Because F is a face of Λ, we have $x, y \in F$, and since G is a face of F, this implies $x, y \in G$ so that G is a face of Λ. ■

LEMMA E.7. *For any proper face F of Λ, we have $F \cap \operatorname{relint}(\Lambda) = \emptyset$. In particular, open (in $\mathbb{R}^n$) convex sets do not have any proper faces.*

PROOF. Let $x \in F \cap \operatorname{relint}(\Lambda)$ and pick $y \in \Lambda \setminus F$ (this is possible since $F \neq \Lambda$ by assumption). As x is in the relative interior of Λ and $y \in \Lambda$, by convexity, there is $z \in \Lambda$ such that $x \in \operatorname{relint}([y, z])$. But F is a face, so $[y, z] \subseteq F$, a contradiction to $y \notin F$. ■

Let $H \subseteq \mathbb{R}^n$ be a *hyperplane*, that is, H is of the form $H = H_{u,c} := \{x \in \mathbb{R}^n : x \cdot u = c\}$ for some non-zero $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Put differently, H is the preimage of a point under a non-zero linear functional on $\mathbb{R}^n$. We consider the two closed *half-spaces* $H^+ := \{x \in \mathbb{R}^n : x \cdot u \geq c\}$ and $H^- := \{x \in \mathbb{R}^n : x \cdot u \leq c\}$ of $\mathbb{R}^n$. Let $S \subseteq \mathbb{R}^n$ be a subset. We say that H *supports* S *(at x)* if $x \in S \cap H$ and either $S \subseteq H^+$ or $S \subseteq H^-$. In this case, we call u a *supporting vector* of S at x. Note that we may assume that $S \subseteq H^-_{u,c}$ by replacing u by $-u$ and c by $-c$ if necessary. Moreover, if S is supported by $H_{u,c}$ and $S \subseteq H^-_{u,c}$, then necessarily $c = \sup_{y \in S} y \cdot u$. Indeed, $c \geq \sup_{y \in S} y \cdot u$ since $S \subseteq H^-_{u,c}$ and $x \cdot u = c$ for every $x \in S \cap H$. We set

$$H_{u,S} := \{x \in \mathbb{R}^n : x \cdot u = \sup_{y \in S} y \cdot u\}.$$

Note that F_u as in Example E.5 is precisely $\Lambda \cap H_{u,\Lambda}$. These faces are of special importance and hence they get their own name:

DEFINITION E.8. A face of Λ is called *exposed* if it is of the form $H \cap \Lambda$ for a supporting hyperplane H of Λ.

COROLLARY E.9. *Any exposed face of a closed convex set is closed as the intersection of two closed sets.*

REMARK E.10. If $F = \Lambda \cap H$ is an exposed face, then $\operatorname{aff}(F) \subseteq H$ since H contains F and is affine. Moreover, $F - cu \subseteq \{u\}^\perp$ if $F = F_u$ and u is a unit vector, which follows from $(x - cu) \cdot u = x \cdot u - c = 0$ for $x \in F$, where $c := \sup_{y \in \Lambda} y \cdot u$.

PROPOSITION E.11. *Every proper face of a convex set $\Lambda \subseteq \mathbb{R}^n$ is contained in a proper exposed face of Λ.*

PROOF. By Lemma E.7, $\operatorname{relint}(F) \cap \operatorname{relint}(\Lambda) \subseteq F \cap \operatorname{relint}(\Lambda) = \emptyset$, hence, by [Sch93, Theorem 1.3.8], F and Λ can be properly separated, that is, there exists a hyperplane H such that $F \subseteq H^+$, $\Lambda \subseteq H^-$ and F and Λ are not both contained in H. Since $F \subseteq \Lambda$, we necessarily also have $F \subseteq H$, concluding that $F \subseteq H \cap \Lambda =: F_0$, the latter being an exposed face of Λ by definition. Since $F \subseteq H$, we must have $\Lambda \subsetneq H$ and thus $F_0 \neq \Lambda$, showing that F_0 is proper. ■

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Abstract

In this thesis, we study the C^*-algebra generated by Toeplitz operators with continuous symbols. The primary focus will be on Toeplitz operators defined on Bergman spaces over bounded pseudoconvex Reinhardt domains Ω in $\mathbb{C}^2$ and the goal is to describe the structure of the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$ in terms of a composition series. This analysis was done by Salinas, Sheu and Upmeier in [**SSU89**] and it uses the theory of groupoid C^*-algebras as pioneered by Renault in [**Ren80**].

The most well known result on the structure of a Toeplitz C^*-algebra is the case of Toeplitz operators on the Hardy space over the circle S^1. In this case, $\mathcal{T}(S^1)$ contains the ideal of compact operators and the quotient is isomorphic to the C^*-algebra of continuous functions on S^1. In chapter 1 we first take a closer look at Toeplitz operators on the Bergman space in general and then concentrate on *strictly* pseudoconvex domains with smooth boundary. We will show that the situation there is similar to the one of the Hardy space over S^1. This is due to Raeburn, [**Rae78**]. The key ingredients here are methods from complex analysis, namely the theory of the $\overline{\partial}$-equation.

The rest of the thesis is then devoted to the aforementioned case of (bounded) pseudoconvex Reinhardt domains. Thus, we drop the assumptions on strict pseudoconvexity and smoothness of the boundary, but we require Ω to have some symmetries by only considering Reinhardt domains. As it turns out, the Toeplitz C^*-algebra for these domains has a much richer structure which we will be able to probe by associating to Ω a locally compact groupoid whose unit space in a way resembles the boundary geometry of Ω. The Toeplitz C^*-algebra is shown to be isomorphic to a (subalgebra of) the corresponding groupoid C^*-algebra. In more concrete terms, the faces of the logarithmic base Λ_Ω (a closed convex set in $\mathbb{R}^2$) give rise to invariant subsets of the unit space of the groupoid and this parametrization by faces allows to give a composition series of $\mathcal{T}(\Omega)$ in terms of geometric data. For the one-dimensional faces, this data will be the slopes of these faces and its contribution to the C^*-algebra will depend heavily on whether its slope is rational or irrational.

We will develop the theory of groupoid C^*-algebras in chapter 2, with a focus on the results and examples relevant to the analysis of $\mathcal{T}(\Omega)$. In chapter 3, these results are put to use in determining the structure of the Toeplitz C^*-algebra.

Zusammenfassung

In dieser Masterarbeit wird die von den Toeplitzoperatoren mit stetigem Symbol erzeugte C^*-Algebra untersucht. Der Hauptfokus ist dabei die Situation von Toeplitzoperatoren auf dem Bergmanraum über einem beschränkten pseudokonvexen Reinhardtgebiet Ω in $\mathbb{C}^2$ und das Ziel ist die Beschreibung der Toeplitz C^*-Algebra $\mathcal{T}(\Omega)$ durch eine Kompositionsreihe. Diese wurde von Salinas, Sheu und Upmeier in [**SSU89**] durchgeführt und verwendet die Theorie der Gruppoid C^*-Algebren, welche von Renault in [**Ren80**] pioniert wurde.

Das wohl bekannteste Resultat über die Struktur von Toeplitz C^*-Algebren is jenes für Toeplitzoperatoren auf dem Hardyraum über dem Kreis S^1. Hier enthält $\mathcal{T}(S^1)$ das Ideal der kompakten Operatoren und der Quotient ist isomorph zur C^*-Algebra der stetigen Funktionen auf S^1. In Kapitel 1 werden wir zuerst auf allgemeine Eigenschaften von Toeplitzoperatoren auf dem Bergmanraum eingehen und uns im Anschluss auf *streng* pseudokonvexe Gebiete mit glattem Rand konzentrieren. Wir werden zeigen dass die Situation hier ähnlich zu der vom Hardyraum über S^1 ist. Dieses Resultat stammt von Raeburn, [**Rae78**]. The wichtigsten Zutaten für den Beweis sind Methoden der komplexen Analysis, genauer: die Theorie des $\overline{\partial}$-Neumann Problems.

Der Rest der Masterarbeit befasst sich dann mit dem zuvor erwähnten Fall von (beschränkten) pseudokonvexen Reinhardtgebieten. Es werden also die Anforderungen an strenge Pseudokonvexität und die Glattheit des Randes fallen gelassen, aber Ω muss nun ein Reinhardtgebiet sein, also gewisse Symmetrien vorweisen. Es stellt sich heraus dass die Toeplitz C^*-Algebra für diese Gebiete eine viel reichhaltigere Struktur aufweist, welche wir untersuchen können, indem wir Ω einen lokalkompakten Gruppoid zuordnen, dessen Einheitenraum die Randgeometrie von Ω wiederspiegelt. Wir werden zeigen, dass die Toeplitz C^*-Algebra isomorph zu (einer Teilalgebra der) zugehörigen Gruppoid C^*-Algebra ist. Konkret geben die Gesichter der logarithmischen Basis Λ_Ω (eine abgeschlossene konvexe Teilmenge von $\mathbb{R}^2$) Anlass zu invarianten Teilmengen des Einheitenraums dieses Gruppoids und diese Parametrisierung durch die Gesichter erlaubt die Konstruktion einer Kompositionsreihe von $\mathcal{T}(\Omega)$ mittels geometrischer Information von Ω. Für die eindimensionalen Gesichter ist diese Information durch ihre Steigungen gegeben, und der Beitrag zur C^*-Algebra wird sehr davon abhängig sein, ob diese Steigungen rational oder irrational sind.

Wir werden in Kapitel 2 die Theorie der Gruppoid C^*-Algebren entwickeln und uns dabei auf jene Resultate und Beispiele konzentrieren, welche bei der Bestimmung von $\mathcal{T}(\Omega)$ relevant sind. In Kapitel 3 werden diese Resultate dann verwendet, um die Struktur der Toeplitz C^*-Algebra zu bestimmen.

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Education

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